Chaotic motion in the Solar System

Jack J. Lissauer

Space Science Division, MS 245-3, NASA Ames Research Center, Moffett Field, California 94035

The motion of planetary bodies is the archetypal clockwork system. Indeed, clocks and calendars were developed to keep track of the relative motions of the Earth, the Sun, and the Moon. However, studies over the past few decades imply that this predictable regularity does not extend to small bodies, nor does it apply to the precise trajectories of the planets themselves over long timescales. [S0034-6861(99)00603-0]

CONTENTS

I. INTRODUCTION

The mechanical model for the workings of the Universe developed by Isaac Newton and other natural philosophers of the Enlightenment was one of the greatest advances of knowledge in human history. Predictable, deterministic models were developed that were able to accurately describe an immense variety of superficially disparate phenomena observed in the heavens and on Earth. Technological advances made possible by this physical understanding have brought about tremendous material benefits for humankind. The philosophical implications of understanding and being able to predict the workings of nature have been equally profound, and have engendered major changes in values and beliefs.

Einstein's theory of relativity modified Newton's models for motion and gravity, but did not attack the principle of predictability of the Universe. The other great revolution of twentieth century physics, quantum mechanics, introduced the concept of fundamental uncertainties in the behavior of individual systems. Indeed, Einstein found certain aspects of quantum mechanics philosophically unsatisfying because of these unpredictabilities. The motion of the planets (as well as the motions of various other physical systems, including the double pendulum and the simple inclined billiard described below) is in principle deterministic, in the sense that given exact knowledge of the masses and initial conditions, the motion can be calculated as precisely as desired (neglecting quantum effects). However, some trajectories depend so sensitively on the system parameters that they are for practical purposes quite unpredictable.

Hence, the seemingly contradictory term *deterministic chaos* has been developed to describe this type of motion. The realization that deterministic chaos also can have profound effects on many macroscopic processes has provided another significant blow to the "clockwork" view of the Universe.

A single pendulum can be viewed as a rigid mass constrained to move in one dimension around a pivot. The motion of such a pendulum in a uniform gravitational field provides a classical example used to demonstrate regular, predictable motion. Suspending the pendulum from a point rather than from a rod doubles the number of degrees of freedom that the system has, but does not fundamentally alter the character of the motion. However, if instead one adds a second pendulum pivoting from near the base of the first, the resulting system exhibits a far richer range of physical behavior. At very low or very high energy, the motion of such a double pendulum is quite simple. However, at intermediate energies it can move wildly, following a sensitive and nearly impossible to predict "chaotic" trajectory (Shinbrot *et al.*, 1992).

The problem of two spherically symmetric bodies attracted to one another by Newtonian gravity is also quite regular and predictable. However, the addition of a third body allows a far richer range of phenomena, and systems of four or more mutually gravitating bodies are extremely difficult to analyze analytically. Indeed, Newton suspected that the mutual gravitational interactions of the planets ultimately would destabilize the Solar System, and hypothesized that divine intervention was occasionally needed to restabilize the orbits of the planets.

When a comet or an asteroid passes close to a planet, its orbital path is altered by the planet's gravity. A small change in its trajectory prior to the encounter will be magnified into a much larger divergence by the interaction. Planet-crossing trajectories can (except in a small fraction of special cases) suffer repeated close encounters, so exceedingly small variations in the initial parameters of the problem can result in large differences in the eventual trajectory and fate of the body (Fig. 1). Thus, close approaches lead to a "trivial" example of chaos. Some slow approaches can be very convoluted, even leading to temporary capture of planetary satellites, which make many close approaches to a planet within a brief interval of time (Fig. 2, cf. Petit and Hénon, 1986; Greenzweig and Lissauer, 1990; Kary and Dones, 1996). In such cases, neighboring trajectories can rapidly di-
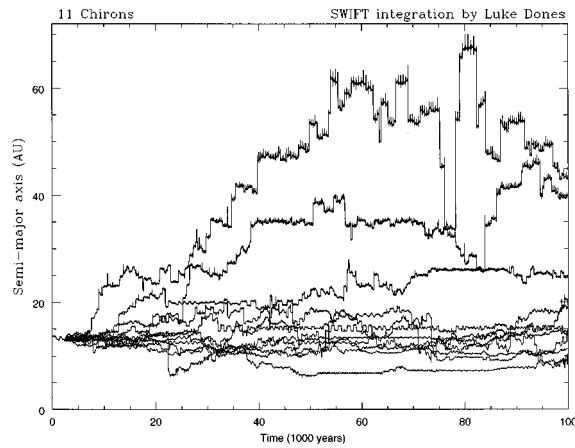
FIG. 1. Eleven simulations of the future orbital semimajor axis of comet 95P/Chiron, which currently crosses the orbits of both Saturn and Uranus. Initial orbital elements of the simulated bodies differed by about 1 part in 10^6, which is smaller than observational uncertainties. Note that the orbit is highly chaotic, with gross divergence of trajectories in less than 10^4 years. Diagrams courtesy L. Dones, who performed these simultions using the SWIFT integration package (Levison and Duncan, 1994).

verge in character. (Similar chaotic motion occurs in the "inclined simple billiard problem," where a point particle that is confined to move in a plane and subjected to a constant "gravity" force bounces elastically off two fixed disks; see Hénon, 1988. Trajectories of the particle can diverge rapidly for small changes in its initial position or velocity.) Backwards integrations of the orbit of comet D/Shoemaker-Levy 9, which collided with Jupiter in 1994, cannot be used to definitively determine this body's precapture heliocentric orbit because of this chaos (Chodas and Yeomans, 1996). Long-range, far weaker interactions among the planets can also lead to chaos for certain orbital configurations, although other configurations appear stable for extremely long periods of time.

In this article, I will summarize some of the types of chaotic motion which occur or may have occurred in our Solar System and other planetary systems, and discuss their implications for the stability and longevity of these systems. Section II covers the forces responsible for the motion of planetary bodies. Analytic and numerical calculation techniques are compared in Sec. III. The concept of chaotic and regular trajectories is introduced in Sec. IV. The relationship between resonances and chaos is discussed in Sec. V. Chaos has profound implications for the formation of planetary systems, as described in Sec. VI. Most planetary dynamical interactions are Hamiltonian processes, in which energy is conserved and trajectories are time reversible, but drag forces are important in some situations, and can lead to a fundamentally different type of chaos that is reviewed in Sec. VII. Section VIII contains a summary of the implications of chaotic motion for the stability of planetary systems. I conclude in Sec. IX by citing some types of chaotic motion not discussed elsewhere in this article. Some
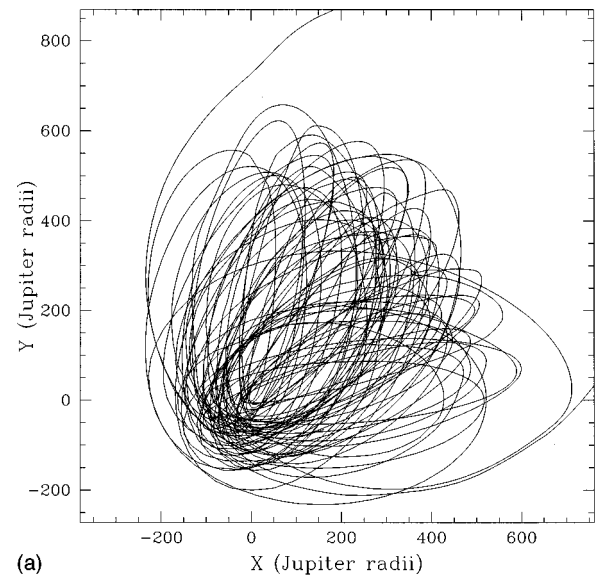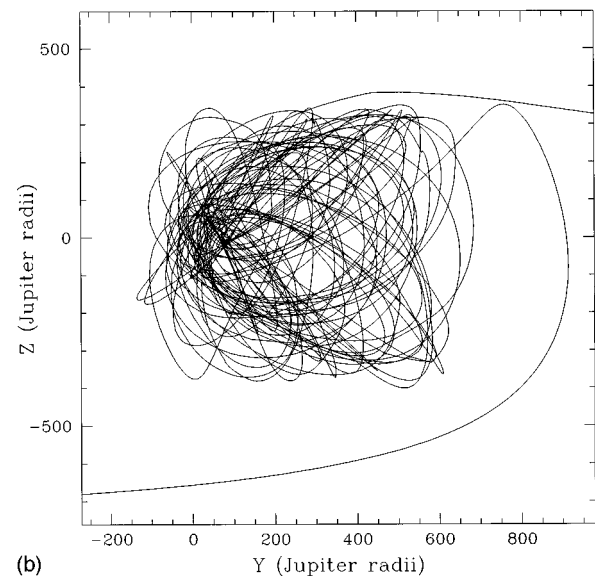
FIG. 2. Trajectory of a test particle initially orbiting the Sun that was temporarily captured into an unusually long-duration unstable orbit about Jupiter. (a) Projected into the plane of Jupiter's orbit about the Sun. (b) Projected into a plane perpendicular to Jupiter's orbit. Figure from Kary and Dones (1996).

of this material is covered in previous reviews of chaotic motion in the Solar System that were written by Wisdom (1987a, 1987b), Duncan and Quinn (1993) and Laskar (1996). Malhotra (1998) reviews various aspects of resonant behavior in the Solar System, including chaos. More general treatments of planetary dynamics can be found in Danby (1988) and C. D. Murray and Dermott (1999). A nontechnical discussion of chaotic motion in the Solar System is provided by Peterson (1993).

II. FORCES

Newtonian gravity provides an excellent approximation to the force acting upon most bodies in the Solar

System. The major departures from simple Keplerian motion of large objects generally result from asymmetries in the gravitational fields of the dominant forcing body or perturbations by other bodies. General relativistic corrections are usually quite small in comparison (even for the precession of Mercury's periapse, planetary perturbations are more than ten times as important as are general relativistic effects; Misner *et al.*, 1973), and quantum effects are minuscule. General relativity was included in some of the simulations of the Solar System discussed herein, and it may actually significantly destabilize the obliquity (also known as the axial tilt, i.e., the angle between the orbital angular momentum vector and the rotational angular momentum vector) of Mars (Touma and Wisdom, 1993), but this appears to be caused by the proximity of the system to a particular resonance rather than a fundamental change resulting from inclusion of a different type of force. Most of the results presented in this article have been derived assuming that the only force acting is Newtonian gravity.

Small particles are affected by radiation forces, and charged small particles can have their motions altered by planetary magnetic fields (Burns *et al.*, 1979; Burns, 1987). The trajectories of comets are perturbed by the rocketlike response to asymmetric loss of volatile material from their surfaces (Marsden *et al.*, 1973; Yeomans, 1994). Physical collisions change trajectories; however, such impacts are rare except in planetary ring systems; nonetheless, they may be important for moving asteroids and Kuiper belt objects (asteroid-sized solid bodies which orbit in a disk exterior to the planet Neptune) into major resonances. Gas drag affects bodies encountering planetary atmospheres. Both gas drag and collisions were much more important during the planet formation epoch, when many solid bodies of comparable sizes orbited the Sun within a gas-dominated disk. In contrast to the small and primarily quantitative effects of general relativity, the example of chaotic motion discussed in Sec. VII depends in a more basic way on the dissipative (non-Hamiltonian) nature of the gas drag force.

III. ANALYTIC RESULTS AND NUMERICAL TECHNIQUES

The mutual forces between the planets can be expanded into series whose terms contain increasing powers of planetary masses, inclinations, and eccentricities. However, although such perturbation expansions are done in powers of small parameters, the existence of resonances between the planets introduces small divisors into the expansion terms. These small divisors make some high order terms in the power series unexpectedly large, and Poincaré (1892) showed that these perturbation series often diverge and have validity only over finite time spans.

Mathematical stability (in the sense that planetary orbits will remain well-separated and the system will remain bound for infinite time) can be proven for a system of extremely small but nonetheless finite mass planets with orbits similar to those in our Solar System (Arnold, 1962). However, the set of initial conditions for which the proof *does not apply* is everywhere dense, i.e., there is always a point in phase space arbitrarily close to a given choice of initial conditions for which the proof does not guarantee stability. Thus, the system might not remain stable if it were subjected to perturbations, even if these perturbations were arbitrarily small. Mathematical theorems like this are of very limited use when discussing astronomical stability.

From an astronomical viewpoint, stability implies that the system will remain bound (no ejections) and that no mergers of planets will occur for the possibly long but finite period of interest, and that this result is robust against (most if not all) sufficiently small perturbations. In the remainder of this discussion, we shall only be concerned with stability in an astronomical sense.

Some chaotic regions in the three-body problem have been modeled analytically (cf. Sec. V). However, the analytic results thus far obtained for many-body gravitating systems are of limited applicability, and thus numerical experiments have proven extremely valuable. These experiments have benefited from the substantial advances in both computer hardware and integration algorithms over the past few decades. Algorithms for Solar System integrations typically differ substantially from the type of N-body codes used for stellar and galactic integrations. In Solar System problems, there are fewer bodies, one body usually dominates the potential, and integrations must be carried out for a substantially larger number of orbital periods, because the systems are dynamically much older (the dynamical age of a planetary-type system is its actual age divided by the shortest or most characteristic orbital period; this provides a nondimensional measure of the number of orbits that bodies within the system have completed).

Efficient schemes have been developed by separating the Hamiltonian of the system into the sum of a Keplerian part and an interaction part (Wisdom and Holman, 1991; Duncan *et al.*, 1998). The Keplerian portion can be solved analytically, and is equivalent to moving the body along a conic section. The interaction portion, which is much smaller, is also amenable to analytic solution. By alternate application of the two Hamiltonians, the system can be advanced in an accurate and efficient manner. Applying the operators serially is an approximation, but as the interaction Hamiltonian is small, good accuracy can be achieved even using time steps that are much longer than allowed by previously popular integration schemes.

IV. REGULAR VS CHAOTIC TRAJECTORIES

For some initial conditions, numerical experiments show that the trajectories are *regular* (quasiperiodic), with variations in their orbital elements that seem to be well described by readily calculable perturbation series, while, for other initial conditions, the trajectories are found to be *chaotic* and are not as confined in their motions (Fig. 3). The evolution of a system which is chaotic
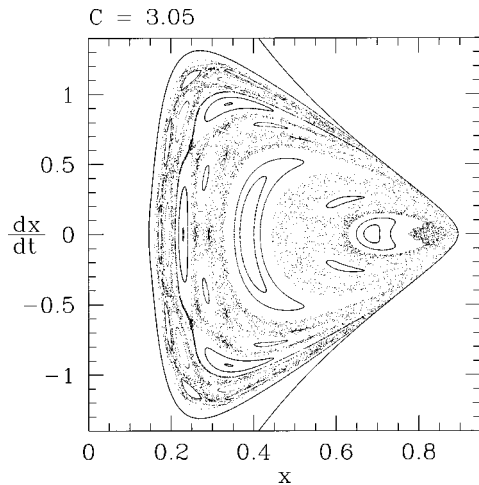
FIG. 3. Surface of section showing both chaotic and regular trajectories for test particles subject to the gravitational tugs of a planet and a star whose mass is 999 times as large as that of the planet. The frame rotates about the center of mass of the system such that the star and planet (which travel on a circular orbit) remain at $(-0.001, \ 0)$ and $(0.999, 0)$, respectively. Every test particle's x coordinate and its first derivative are plotted each time the particle crosses the x axis moving towards positive y. Particles on regular trajectories generate the smooth curves in the figure, whereas the scattered points were generated by a few particles on chaotic trajectories. Each test particle in the simulation has the same value of the Jacobi parameter, a combination of energy and angular momentum which is an integral of motion in the restricted circular three-body problem. Plot courtesy M. Holman.

depends so sensitively on the system's precise initial state that the behavior is in effect unpredictable, even though it is strictly determinate in a mathematical sense. Chaos can occur in a wide variety of physical systems; see Lichtenberg and Lieberman (1992) for a more comprehensive treatment of chaotic motion.

There is a key difference between regular and chaotic orbits that is often used as a definition of *chaos*: Two trajectories that begin arbitrarily close in phase space in a regular region will diverge from one another as a power (usually linear) of the elapsed time, whereas in a chaotic region two nearby trajectories will typically diverge exponentially in time. Within a given chaotic region, the time scale for this divergence typically does not depend on the precise values of the initial conditions. If one computes the distance $d(t)$ between two particles having an initially small separation $d(0)$, one finds that for regular orbits $d(t) - d(0)$ grows as a power of time t (usually linearly) whereas for chaotic orbits

$$d(t) \sim d(0)e^{\delta t}, \qquad (1)$$

where (in the limit of infinitesimal initial separation and as $t \rightarrow \infty$) δ is the *Lyapunov characteristic exponent* and δ^{-1} is the *Lyapunov time scale* (Fig. 4). The Lyapunov time scale thus represents the *e*-folding time for the divergence of neighboring trajectories. Note that the Lyapunov time scale is not well defined if close encounters can occur, because the time interval between successive close approaches varies irregularly (Whipple and
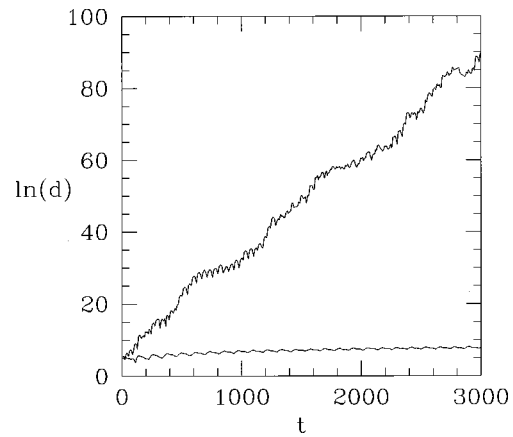

FIG. 4. Logarithm of the separation in phase space between two pairs of initially adjacent orbits. The lower curve represents a pair of particles that appear near (0.7, 0) in the surface of section shown in Fig. 3. The upper curve represents a pair that begins in the chaotic zone. Plot courtesy M. Holman.

Shields, 1993). Additionally, the criterion for chaos given by Eq. (1) is misleading if bodies are ejected from the system, as exponential growth no longer occurs, but nevertheless the trajectory does not behave in a manner that most researchers would consider regular.

The exponential divergence discussed above implies that chaotic orbits show such a sensitive dependence on initial conditions that the detailed long-term behavior of the orbits is lost within several Lyapunov time scales. Roughly speaking, one bit of accuracy is lost every Lyapunov time. Even a fractional perturbation as small as 10^{-10} in the initial conditions will result in a 100% discrepancy in ~ 25 Lyapunov times. Short Lyapunov times are indicative of eventual large-scale chaos. However, one of the interesting features of much of the chaotic behavior seen in simulations of the orbital evolution of bodies in the Solar System is that the time scale for large changes in the principal orbital elements (semimajor axis, eccentricity and inclination) is often many orders of magnitude larger than the Lyapunov time scale (Duncan and Quinn, 1993; Holman and Murray, 1996). Moreover, except in highly chaotic regions, the Lyapunov time scale is usually not a very accurate predictor for the time that a particular body requires in order to exhibit major orbital changes (N. Murray and Holman, 1997).

V. RESONANCES AND CHAOS

In dynamical systems like the Solar System, chaotic regions are not distributed at random. Chaotic trajectories are generally associated with resonances, i.e., locations at which the ratios of characteristic forcing and response frequencies of the system are sufficiently well approximated by quotients of (usually small) integers. In these locations, perturbations add constructively, and the effects of many small tugs can build up over time to create a large-amplitude, long-period response. Resonance forcing usually produces variations in orbital ec-

centricities and/or inclinations much larger than in semi-major axes. The simplest orbital resonances to visualize are the *mean-motion resonances*, in which the orbital periods of two bodies are commensurate. *Secular resonances*, which occur between the precession frequencies of planetary bodies, are also important for destabilizing orbits within the Solar System.

A. The resonance overlap criterion for chaos

Consider the case of a massive secondary which perturbs massless (test) particles. For nearly circular and coplanar orbits, the strongest mean motion resonances occur at locations where the ratio of the orbital period of the test particle to that of the secondary is of the form $N:(N\pm 1)$ where N is an integer. At these locations, conjunctions (closest approaches) always occur at the same phase in the test particle's orbit, and tugs add coherently. The strength and effective width of these first-order resonances increases as N grows, because the magnitude of the perturbations is larger when the closest approach distance is smaller. First-order resonances also become closer together near the orbit of the secondary. Sufficiently close to the secondary, resonance regions overlap. This leads to the onset of chaos as particles shift between the nonlinear perturbations of various resonances. The region of overlapping resonances is approximately symmetric about the planet's orbit, and has a half-width Δa_{ro} given by

$$\Delta a_{ro} \approx 1.5 \left(\frac{m}{M}\right)^{2/7} a, \qquad (2)$$

where a is the semimajor axis of the planet's orbit, m is the mass of the planet, and M is the mass of the star. The functional form of Eq. (2) has been derived analytically (Wisdom, 1980), whereas the coefficient is a numerical estimate based upon the behavior of integrated trajectories (Duncan *et al.*, 1989).

B. Resonances in the asteroid belt

There are obvious patterns in the distribution of asteroidal semimajor axes that are associated with mean-motion resonances with Jupiter (Fig. 5). At these resonances, a particle's period of revolution about the Sun is a small integer ratio multiplied by Jupiter's orbital period. In the outermost part of the belt, asteroids are concentrated near the 1:1, 3:2, and 4:3 resonances. In contrast, resonance regions are depleted at greater distances from Jupiter. The mechanisms believed to be responsible for producing this distribution are discussed below.

The Trojan asteroids travel in a 1:1 mean motion resonance with Jupiter; they execute small to moderate amplitude librations about the L_4 and L_5 triangular Lagrangian points 60° ahead of or behind Jupiter, thereby avoiding close approaches to Jupiter. (The L_4 and L_5 points of a planet on a circular orbit lie in the orbital plane, and each of them, together with the star and the planet, forms the vertices of an equatorial triangle. These two locations are stable equilibrium points viewed
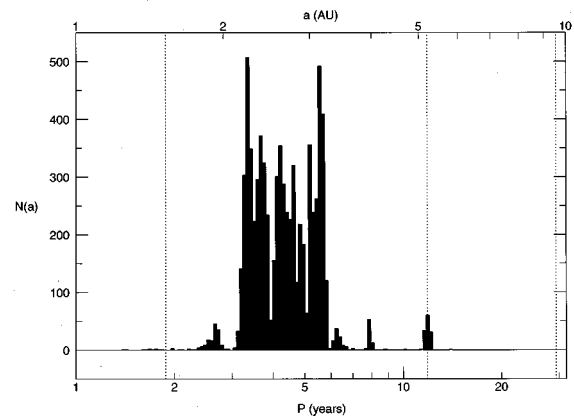

FIG. 5. Histogram of all numbered asteroids brighter than 15th magnitude versus orbital period (with corresponding semimajor axes shown on the upper scale); the scale of the abscissa is logarithmic. The dashed vertical lines represent from left to right the planets Mars, Jupiter, and Saturn. As asteroids with shorter orbital periods are better lit and pass closer to Earth, there is a strong observational bias favoring objects plotted towards the left. Note the prominent gaps in the distribution for orbital periods 1/2, 2/5, 3/7, and 1/3 that of Jupiter. Plot courtesy A. Dobrovolskis.

in a frame rotating with the angular velocity of the planet, provided the planet's mass is less than 1/27th that of the star.) Large amplitude librators are removed from resonance rapidly, and the current distribution is still being gradually "eroded" from its edges (Levison *et al.*, 1997). Another example of a protection mechanism provided by a resonance is the Hilda group of asteroids at Jupiter's 3:2 mean motion resonance and the asteroid 279 Thule at the 4:3 resonance. The Hilda asteroids have a libration about 0° of their critical argument, $3\lambda' - 2\lambda - \varpi$, where λ' is Jupiter's longitude, λ is the asteroid's longitude, and ϖ is the asteroid's longitude of perihelion (the place in its orbit where it is nearest the Sun). In this way, whenever the asteroid is in conjunction with Jupiter ($\lambda = \lambda'$), the asteroid is close to perihelion ($\lambda' \approx \varpi$) and well away from Jupiter. The stability of resonance locks is discussed in detail by Peale (1976). Calculations by Ferraz-Mello *et al.* (1996) suggest that the Hilda asteroids are not necessarily permanently confined, but may escape from resonance on a time scale comparable to the age of the Solar System.

Early investigations of Jupiter's 3:1 mean motion resonance (in the inner asteroid belt) found that most orbits starting with small eccentricity were regular and showed very little variation in eccentricity or semimajor axis over time scales of 5×10^4 yr, in agreement with the results of low-order perturbation expansions. Subsequently, Wisdom (1982) showed that an orbit near this resonance could maintain a low eccentricity ($e < 0.1$) for nearly a million years and then rapidly "jump" to $e > 0.3$ (Fig. 6). The outer boundaries of the chaotic zone as determined by Wisdom's (1985a) work coincide well with the observed boundaries of the 3:1 Kirkwood gap (Fig. 7). Since asteroids which begin on nearly circular orbits in the gap acquire sufficient eccentricities to cross
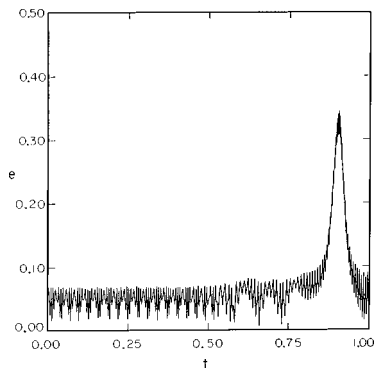
FIG. 6. Eccentricity vs time for the orbit of a test particle near Jupiter's 3:1 resonance. The time is measured in units of 20 000 orbital periods of Jupiter (about 240 000 years). From Wisdom (1982).

the orbit of Mars and in many cases that of the Earth, the perturbative effects of the terrestrial planets are capable of clearing out the 3:1 gap. Some particles are forced to such high eccentricity by the 3:1 resonance that they either collide with the Sun or are destroyed by solar heating or tides during close perihelion passages (Gladman *et al.*, 1997; Migliorini *et al.*, 1997). The 3:1 and other resonances in the inner asteroid belt provide a chaotic path for delivery of meteorites to Earth (Wisdom, 1985b; Gladman *et al.*, 1997). Resonances can account for several other gaps in the distribution of orbital elements of the asteroids (e.g., Holman and Murray, 1996), and variations in Jupiter's orbital elements caused by tugs from Saturn can further destabilize trajectories in resonance regions (N. Murray *et al.*, 1998).

The region immediately interior to Jupiter's orbit is cleared by resonance overlap, but the general depletion of the outer belt (and some of the Kirkwood gaps) has not been fully accounted for (Gladman and Duncan, 1990; Lecar *et al.*, 1992; Holman and Murray, 1996).
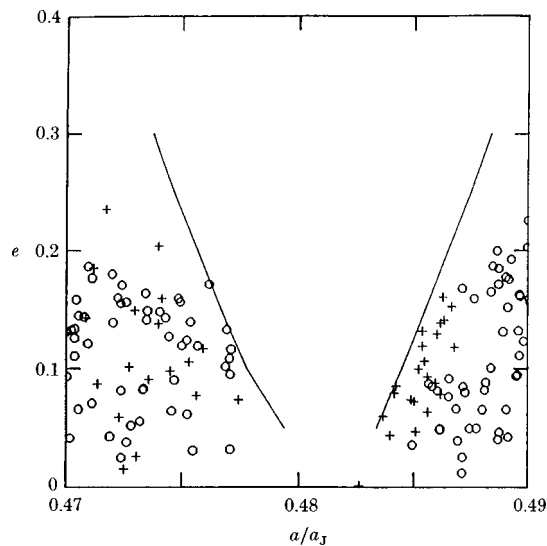

FIG. 7. Comparison of the distribution of observed asteroids with the calculated outer boundaries associated with the 3:1 resonance of Jupiter. From Wisdom (1987a).

These clear regions are probably the result of cosmogonic effects (i.e., those associated with the formation of the asteroids and Jupiter, cf. Torbett and Smoluchowski, 1980; Gomes, 1997; Lecar and Franklin, 1997; Liou and Malhotra, 1997) such as resonance sweeping by the migrating and/or growing Jupiter.

The Kuiper belt, which consists of icy bodies orbiting a bit farther from the Sun than does Neptune, is subjected to many of the same type of resonant perturbations as is the asteroid belt (Duncan *et al.*, 1995; Malhotra, 1996). These gravitational perturbations, along with collisionally induced orbital changes (Morbidelli, 1997), bring some of the bodies very close to Neptune. Ultimately, some of these objects become visible to us as short-period comets (Levison and Duncan, 1997).

VI. CHAOS AND PLANET FORMATION

Terrestrial planets are believed to grow by pairwise accretion of solid bodies known as "planetesimals" (Safronov, 1969). The initial stages of the formation of gas giant planets probably also involves binary agglomeration of planetesimals, with accretion of significant quantities of gas initiated once the heavy-element core had acquired enough mass to trap hydrogen and helium gravitationally (Pollack *et al.*, 1996). Although some characteristic properties of planetary systems are probably strongly correlated with the masses and sizes of the disks from which they accreted, chaos is a major factor in planetary growth (Lissauer, 1995). In the early and middle stages of planetary growth, large numbers of planetesimals travel on intersecting, highly chaotic orbits. Gravitational focusing by the largest bodies in the swarm allows them to double in mass more rapidly than can smaller bodies, via a process known as runaway accretion (Greenberg *et al.*, 1978; Wetherill and Stewart, 1989). When these large bodies accumulate enough of their neighbors, the remaining protoplanets are no longer in crossing orbits. However, the cumulative effects of forcing by many weak resonances can excite eccentricities sufficiently for orbits to cross and accretion to continue (Chambers *et al.*, 1996).

Accretion (augmented, if and when the planets grow massive enough to scatter debris onto very eccentric orbits, by collisions with the star and ejection to interstellar space) proceeds until a configuration that is stable for the lifetime of the star is formed. Resonances that may be able to destabilize orbital configurations occur at places where relevant periods are related by rational numbers. As the set of rational numbers is dense, there appear to be an infinite number of paths to chaos (moreover, there are often many bodies whose resonances may be important and secular resonances as well as mean motion resonances often come into play). However, some of these paths are much faster than others, so objects located near strong resonances can be accreted or ejected from the system very quickly, whereas other particles can remain in low eccentricity orbits for many orders of magnitude longer, if not indefinitely.

Changing orbits can be viewed as diffusion through phase space. However, this diffusion is not a purely random walk; rather, some paths are much faster than others. The lifetimes of the members of an ensemble of test particles on similar but not identical orbits are, at least in some cases, distributed such that similar numbers of particles become planet-crossing per decade of system evolution (Holman and Wisdom, 1993). That is, as the system ages, the expected time remaining until orbit crossing for the particles currently on well-behaved orbits increases. This is in contrast to many other natural systems such as radioactivity of a given isotope, which exhibit exponential decay, in which the expected future life of the remaining atoms does not change with time. Chaotic systems in which all periodic orbits are unstable generally exhibit exponential decay, whereas those that contain stable regions of nonzero area generally show algebraic decay of the survival probability (Yalcinkaya and Lai, 1995).

Mapping out the structure of the chaotic "sea" is easiest when the system contains a small number of massive bodies. Another extreme, that of many low mass bodies on neighboring orbits, exhibits predictable patterns in the time to orbit crossing despite the chaotic nature of the individual trajectories involved. Chambers *et al.* (1996) integrated systems of three or more (in most cases equal mass) planets that began on equally (or nearly equally) spaced coplanar circular orbits. They found that, for given masses, the time elapsed before the first pair of orbits crossed, t_c, is well approximated by the formula

$$t_c = b e^{c \Delta_o}, \tag{3}$$

where Δ_o signifies the initial separation in orbital radii of the planets and b and c are constants.

Duncan and Lissauer (1997) integrated systems with initial orbits identical to moons of Uranus, but with the masses of the secondaries all increased (or, in a few cases, decreased) from their "actual" values (in the case of Uranus's smaller moons, estimated values are quite uncertain) by the same factor, m_f. We found that the crossing time obeyed a relationship analogous to Eq. (3). In particular,

$$t_c = \beta m_f^{\alpha}, \tag{4}$$

where α and β are constants. These power law fits extend over seven orders of magnitude in crossing time (Fig. 8), with some systems exhibiting a steepening (greater sensitivity to changes in m_f) at the smallest masses (which is very frustrating, as these simulations require the most computer time, in some cases a few months of c.p.u. time on a fast workstation). The rms dispersion of individual systems about the best fit curve is only about a factor of two for the small inner Uranian moons, which travel on closely spaced orbits; systems with more massive and more separated secondaries exhibit greater scatter. The value of α varies from about -4 for Uranus's inner moons to approximately -10 for its outer satellites. The scalings given by Eqs. (3) and (4) are not yet understood theoretically.
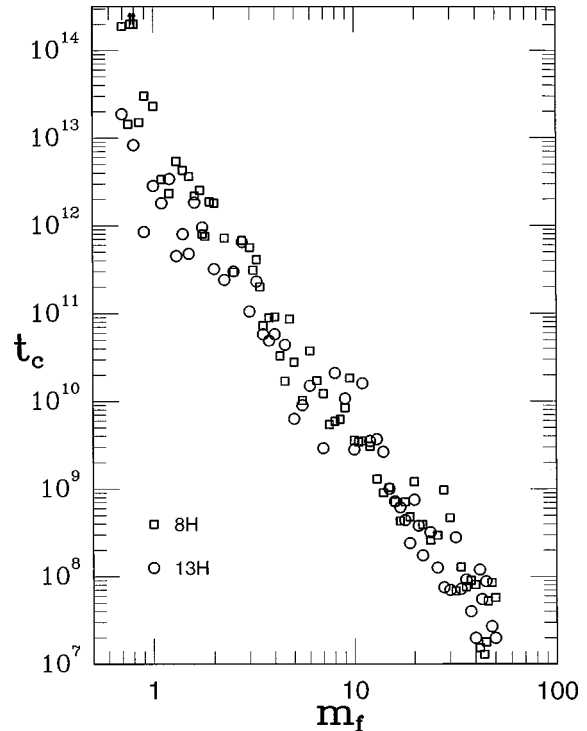

FIG. 8. The orbit crossing time t_c, in seconds, is shown as a function of the mass enhancement factor m_f, for various synthetic systems based on the Uranian satellite system. The square symbols indicate runs that included eight of the ten Uranian moons discovered by the Voyager spacecraft (Cordelia and Ophelia, the ϵ ring shepherds, were omitted). The circles denote results for 13 moon runs, which also included the larger classical satellites of Uranus: Miranda, Ariel, Umbriel, Titania and Oberon, whose orbits lie well exterior to those of the Voyager-discovered moons. From Duncan and Lissauer (1997).

The integrations of Duncan and Lissauer (1997) indicate that the orbits of Uranus's outer moons, taken as an isolated Hamiltonian dynamical system, are stable for a time scale much longer than the age of the Solar System. However, two or more of that planet's inner moons may become orbit crossing in a few million to several tens of millions of years. Other processes, such as tidal damping, may stabilize the system, or these moons may be much younger than the Solar System, having formed within the past $\sim 10^8$ years as the result of a catastrophic impact destroying a larger moon. Another interesting result of these integrations is that the merging of the least stable pair of moons increases the lifetime of the system by many orders of magnitude.

The irregularity of orbits increases the potential diversity of planetary systems. The combination of runaway accretion and chaos implies that microscopic changes in initial conditions can lead to substantially different outcomes. Indeed, protoplanetary disks very similar to the one from which our planetary system is believed to have formed can also produce 3 to 5 terrestrial planets in a variety of sizes and orbits (Wetherill, 1990).

Chaotic interactions leading to mergers and ejections of planets tend to increase orbital eccentricities and in-

clinations of the planets remaining in the system (Duncan and Lissauer, 1997). The nearly coplanar and circular orbits of the major planets in our Solar System (and of the major moons in most satellite systems) suggests that some damping process (e.g., accretion or ejection of many small planetesimals, interactions with gas) was important in determining the current configurations (Levison *et al.*, 1998). The eccentric orbits of the planets that were recently discovered in orbit about the stars 70 Virginis (Marcy and Butler, 1996) and 16 Cygni B (Cochran *et al.*, 1997) suggest that chaotic scattering may have played an even more profound role in determining the configurations of other planetary systems (Rasio and Ford, 1996; Weidenschilling and Marzari, 1996; see, however, Holman *et al.*, 1997, for an alternative explanation for the large eccentricity of 16 Cygni B's planetary companion). Calculations by Levison *et al.* (1998) demonstrate that some plausible giant planet systems can appear well behaved (orbits which are not crossing and slowly changing) for longer than the 10^7 year time scale characteristic of planetary growth (Lissauer, 1993), yet become wildly chaotic (mergers and/or ejections) in less than 10^9 years.

VII. DISSIPATION-INDUCED CHAOS

The characteristics of a dynamical system change when energy can be removed from the system. In particular, the system can damp down towards a stable state which is referred to as an *attractor*. A trivial physical example is an unforced pendulum, which ultimately damps to the state in which the weight is at the bottom. A not so trivial example with a single attractor is a resonance lock in which one moon on an eccentric orbit can settle into a state wherein it is always at the same radius whenever it passes the moon with which it is in resonance (Peale, 1976). Even very simple nonlinear dissipative systems can have more than one attractor for some parameter values and can transition from regular to chaotic motion as this parameter is varied. The simple logistical map given by

$$x_{n+1} = A x_n (1 - x_n), \tag{5}$$

where the parameter $0 < A < 4$, maps the interval $[0, 1)$ onto itself. For $A \leqslant 3$, this mapping converges towards a single value, but for larger A, the behavior is more complex. When the value of A is between 3 and approximately 3.45, the value of x asymptotically bounces back and forth between two attractors. At larger and successively closer values of A, the number of attractors doubles again and again with x bouncing between an ever greater number of attractors. This sequence at which these "period doublings" occur converges and the period becomes infinite, leading to a chaotic situation when A is near 3.57. For even higher values of A, either x bounces between a finite number of attractors or it varies chaotically. A more detailed description of the behavior of the map given in Eq. (5), as well as of simple physical systems (such as the forced damped nonlinear pendulum) which exhibit similar behavior, can be found

in textbooks on chaos theory (e.g., Bergé *et al.*, 1984; Schuster, 1989; Hilborn, 1994).

Planetesimals orbited the Sun on slightly perturbed Keplerian paths, whereas the gaseous component of the solar nebula orbited more slowly due to partial pressure support. Planetesimals thus experienced a headwind which caused their orbits to decay sunwards; the orbits of small planetesimals decayed most rapidly due to this gas drag (Adachi *et al.*, 1976). Such rapid radial transport of material could greatly increase the effective feeding zone of a planet, assuming this material could be efficiently captured as it spiraled by the planet on its sunward path. However, Weidenschilling and Davis (1985) found that many planetesimals were trapped in mean motion resonances exterior to the planet, and thus they could not be accreted.

Kary *et al.* (1993) integrated the orbits of numerous planetesimals subjected to gas drag and the perturbations of one large protoplanet on a circular orbit. As planetesimals approach the planet, the resonances that they encounter become stronger and stronger. For a given drag rate and protoplanet mass, the resonance farthest from the planet that is capable of trapping can be calculated by balancing the angular momentum supplied to the planetesimal by the planet with that removed by the gas. Resonances very near the planet, although individually quite strong, cannot trap planetesimals because their overlap induces chaos (Sec. V.A). As planetesimals have nonzero eccentricities when they approach the resonances capable of trapping, the initial torque that they are subjected to, and their fate within each resonance, depends on the phase of the approach, and as such trapping can be considered to be probabilistic. In most cases, Kary *et al.* found that particles were trapped in a range of resonances, with the outermost trapping resonance depending on both planetary mass and the drag parameter, while the inner one depended only on planetary mass, and was that given by the resonance overlap criterion [Eq. (2)]. However, for some drag parameters, several planetesimals were trapped in distant resonances, but no planetesimals were trapped in the resonances immediately exterior to the chaotic zone predicted by the resonance overlap criterion. This result was surprising, because these resonances were believed to be the ones most capable of trapping planetesimals.

Upon closer examination of the equilibrium position at conjunction of particles that were trapped in these "peculiar" resonances at lower drag parameters, Kary *et al.* (1993) found a period-doubling transition to chaos (Fig. 9). Particles trapped in most resonances settle into orbits in which they are always at the same distance from the Sun when they are passed by the planet, in analogy to the mapping given by Eq. (5) when $A \leqslant 3$. The planet is thus able to give them just enough energy and angular momentum in this encounter to balance the energy and angular momentum removed by gas drag since the previous passage. However, for some resonances, although such a stable single attractor exists when drag is small, when the gas drag is increased beyond a certain value no single equilibrium passage dis-
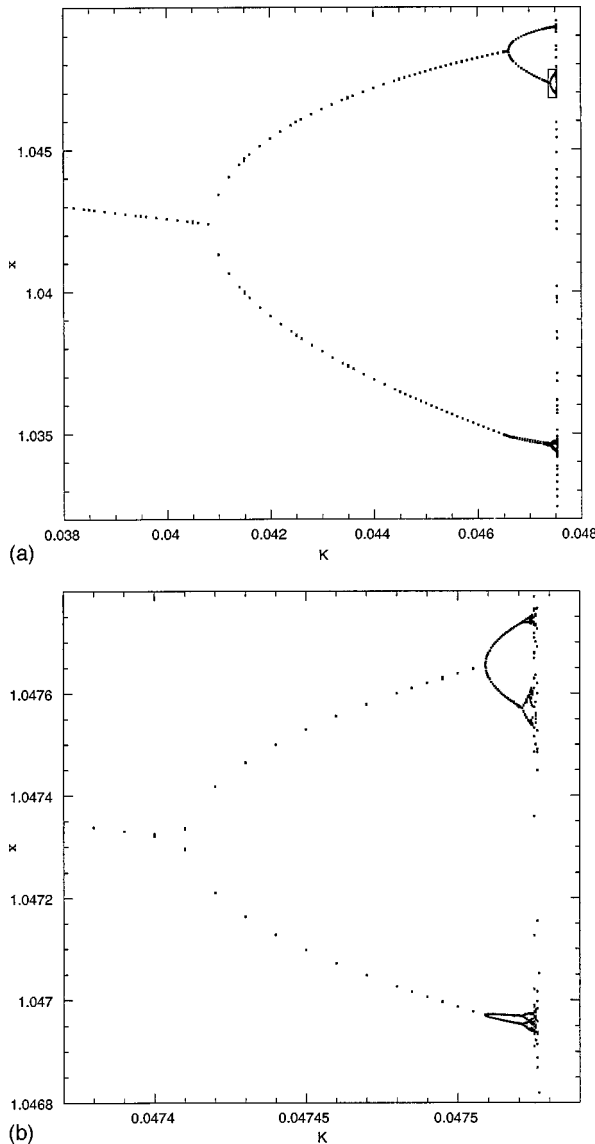
FIG. 9. Surfaces of section showing particle distance x from the star (measured in units of the distance between the planet and the star) at conjunction as a function of the drag parameter K, for particles trapped in the 20/21 mean motion resonance with a planet whose mass is 10^{-6} times that of the star. A similar "pitchfork" pattern is seen when the particle distance is replaced with semimajor axis, eccentricity, or eccentric anomaly at conjunction. At any given K, the particle cycles through all of the stable positions. The box in the upper right corner of (a) indicates the range of (b). From Kary *et al.* (1993).

tance exists. Rather, the particle alternates between two radii at conjunction. For even larger drag rates, the equilibrium locations bifurcate again and again, repeatedly doubling the time required by the particle to complete a cycle and return to its previous location relative to the planet and the star. Progressively smaller increases in drag rate are required for each successive period doubling, with the ratio of successive bifurcation distances approaching the universal value of 4.669 . . . for dissipative systems in which the functional description has quadratic maxima (Feigenbaum, 1978). The system becomes

chaotic (and the resonance is no longer capable of trapping planetesimals) at the convergence point of the sequence of period doublings.

To my knowledge, this is the only celestial mechanics problem yet analyzed to display such a period-doubling transition to chaos. However, it appears to be analogous to the well-studied problem of the periodically kicked damped rotator (Schuster, 1989; Kary, 1993).

VIII. LONG-TERM STABILITY OF PLANETARY ORBITS

Questions concerning the stability of our Solar System have been studied for over three centuries. Analytic calculations have largely been superseded by numerical integrations in recent decades. Integrations of the orbits of the four outer planets on million year time scales (Cohen *et al.*, 1973) compare well with perturbation calculations, showing quasiperiodic behavior for the four major outer planets. In addition, the angle $3\lambda - 2\lambda_N - \varpi$ (where λ and λ_N are the mean longitudes of Pluto and Neptune, respectively, and ϖ is the longitude of perihelion of Pluto) was found to be in libration with a period of 20,000 yr (Williams and Benson, 1971). This relationship, referred to as the 2:3 mean motion resonance, acts to prevent close encounters of Pluto with Neptune and hence protects the orbit of Pluto. However, numerical integrations show evidence for very long period changes in Pluto's orbital elements, with a Lyapunov exponent of $\sim (20 \text{ Myr})^{-1}$ (Sussman and Wisdom, 1988).

Laskar (1989, cf. Sussman and Wisdom, 1992) found that the orbits of the inner planets are also chaotic, with a Lyapunov exponent of only $\sim (5 \text{ Myr})^{-1}$. This chaos, combined with imperfect knowledge of initial conditions and planetary masses, implies that it may never be possible to accurately calculate the location of the Earth in its orbit 100 million years in the past or into the future (Duncan, 1994). Minor unmodeled perturbations, such as the gravitational tug of asteroids or of passing stars, could alter planetary longitudes substantially on this timescale, although they are unlikely to change the character of the motion (Tremaine, 1995).

Despite the observed chaos, it is likely that the Solar System is astronomically stable, in the sense that the 8 or 9 largest known planets will probably remain bound to the Sun in low eccentricity, low inclination orbits until the Sun exhausts the hydrogen fuel in its core and grows into a red giant (Duncan and Quinn, 1993; Laskar, 1997). Nonetheless, it is possible, although highly unlikely, that this chaos could lead Mercury to collide with either the Sun or Venus or be ejected from the Solar System prior to the end of the Sun's main sequence phase 6 billion years in the future (Laskar, 1994). Moreover, from a mathematical point of view, the Solar System is unlikely to be stable indefinitely even in the absence of solar evolution and external perturbations (Tremaine, 1995).

The mass-chaos relationship given by Eq. (4) also has implications for the ultimate fates of planetary systems. Sun-like stars lose a significant fraction of their mass during their post-main sequence evolution (Sackmann

et al., 1993). Duncan and Lissauer (1998) have integrated systems with orbital parameters the same as those of the planets, but with the masses of the planets increased relative to that of the Sun. They find similar relationships between masses and times to orbit crossing as for the Uranian moons, but the dispersion about the central fits is greater and the relationships show signs of breaking down at small mass enhancement factors, where integration times become too long to allow for definitive answers. For reasonable solar mass loss rates, Duncan and Lissauer find that the orbits of the giant planets are likely to remain stable for at least tens of billions of years subsequent to the Sun becoming a white dwarf. Integrations of the terrestrial planets also show a very stable system, although such computationally intensive simulations have only modeled evolution of the system for order of 10^9 years. The orbits of Pluto and of many asteroids may become unstable soon after the Sun becomes a white dwarf. However, the physical consequences of the Sun's luminous red giant phase may turn these smaller objects into cometary bodies which are subject to significant nongravitational forces resulting from asymmetric mass ejection.

IX. PLANETARY ROTATION, ETC.

The list of applications of chaos theory to the study of the motions of planetary bodies presented herein is by no means exhaustive. For instance: Several pairs of planetary satellites probably passed through chaotic zones on their path into or through mean motion resonances; in some cases such passages may have excited sufficient orbital eccentricities to induce substantial tidal heating (Dermott *et al.*, 1988; Tittemore, 1990; Showman and Malhotra, 1997). The obliquity of the planet Mars is believed to vary chaotically (Touma and Wisdom, 1993), and the obliquities of the other terrestrial planets may have suffered similar variations in the distant past (Laskar and Robutel, 1993). Saturn's moon Hyperion is apparently tumbling chaotically (Wisdom *et al.*, 1984; Klavetter, 1989; Black *et al.*, 1995). Many other moons probably briefly experienced chaotic rotation prior to becoming synchronously locked billions of years ago (Wisdom, 1987c). Orbits around irregularly shaped bodies (such as most asteroids) can be highly chaotic even in the absence of external perturbers (Petit *et al.*, 1997). The study of chaotic dynamics in the Solar System is developing rapidly, and many new examples and insights are likely to be discovered in the near future.

ACKNOWLEDGMENTS

REFERENCES

Adachi, I., C. Hayashi, and K. Nakazawa, 1976, Prog. Theor. Phys. **56**, 1756.

Arnold, V. I., 1962, Sov. Math. Dokl. **3**, 1008.

Bergé, P., Y. Pomeau, and C. Vidal, 1984, *Order Within Chaos, Towards a Determininsic Approach to Turbulence* (Wiley, New York).

Black, G. J., P. D. Nicholson, and P. C. Thomas, 1995, Icarus **117**, 149.

Burns, J. A., 1987, in *The Evolution of the Small Bodies of the Solar System* (Soc. Italiana di Fisica, Bologna, Italy), p. 252.

Burns, J. A., P. L. Lamy, and S. Soter, 1979, Icarus **40**, 1.

Chambers, J. E., G. W. Wetherill, and A. P. Boss, 1996, Icarus **119**, 261.

Chodas, P. W., and D. K. Yeomans, 1996, in *The Collision of Comet Shoemaker-Levy 9 and Jupiter*, IAU Colloquium Vol. 156 (Cambridge University Press, Cambridge, England), p. 1.

Cochran, W. D., A. P. Hatzes, R. P. Butler, and G. W. Marcy, 1997, Astrophys. J. **483**, 457.

Cohen, C. J., E. C. Hubbard, and C. Oesterwinter, 1973, Astron. Pap. Am. Ephem. (U.S. Printing Office, Washington), Vol. 22, Pt. I.

Danby, J. M. A., 1988, *Fundamentals of Celestial Mechanics* (Willmann-Bell, Richmond, VA).

Dermott, S. F., R. Malhotra, and C. D. Murray, 1988, Icarus **76**, 295.

Duncan, M. J., 1994, in Institut d'Astrophysique de Paris Astrophysics Meeting No. 10: *Circumstellar Dust Disks and Planet Formation*, edited by R. Ferlet and A. Vidal-Madjar (Editions Frontiers, Gif sur Yvette, France), p. 245.

Duncan, M. J., and J. J. Lissauer, 1997, Icarus **125**, 1.

Duncan, M. J., and J. J. Lissauer, 1998, Icarus **134**, 303.

Duncan, M., H. Levison, and M. Budd, 1995, Astrophys. J. **110**, 3073.

Duncan, M., H. Levison, and M. Lee, 1998, Astron. J. **116**, 2067.

Duncan, M. J., and T. Quinn, 1993, Annu. Rev. Astron. Astrophys. **31**, 265.

Duncan, M., T. Quinn, and S. Tremaine, 1989, Icarus **82**, 402.

Feigenbaum, M. J., 1978, J. Stat. Phys. **19**, 25.

Ferraz-Mello, S., J. C. Klafke, T. A. Michtchenko, and D. Nesvorny, 1996, Celest. Mech. Dyn. Astron. **64**, 93.

Gladman, B., and M. J. Duncan, 1990, Astron. J. **100**, 1669.

Gladman, B. J., *et al.*, 1997, Science **277**, 197.

Gomes, R. S., 1997, Astron. J. **114**, 396.

Greenberg, R., J. F. Wacker, W. K. Hartmann, and C. R. Chapman, 1978, Icarus **35**, 1.

Greenzweig, Y., and J. J. Lissauer, 1990, Icarus **87**, 40.

Hénon, M., 1988, Physica D **33**, 132.

Hilborn, R. C., 1994, *Chaos and Nonlinear Dynamics, An Introduction for Scientists and Engineers* (Oxford University Press, New York).

Holman, M. J., and N. W. Murray, 1996, Astron. J. **112**, 1276.

Holman, M. J., J. Touma, and S. Tremaine, 1997, Nature (London) **386**, 254.

Holman, M. J., and J. Wisdom, 1993, Astron. J. **105**, 1987.

Kary, D. M., 1993, Ph.D. thesis (State University of New York at Stony Brook).

Kary, D. M., and L. Dones, 1996, Icarus **121**, 207.

Kary, D. M., J. J. Lissauer, and Y. Greenzweig, 1993, Icarus **106**, 288.

Klavetter, J. J., 1989, Astron. J. **97**, 570.

Laskar, J., 1989, Nature (London) **338**, 237.

Laskar, J., 1994, Astron. Astrophys. **287**, L9.

Laskar, J., 1996, Celest. Mech. Dyn. Astron. **64**, 115.

Laskar, J., 1997, Astron. Astrophys. **317**, L75.

Laskar, J., and P. Robutel, 1993, Nature (London) **361**, 608.

Lecar, M., and F. Franklin, 1997, Icarus **129**, 134.

Lecar, M., F. Franklin, and M. Murison, 1992, Astron. J. **104**, 1230.

Levison, H. F., and M. J. Duncan, 1994, Icarus **108**, 18.

Levison, H. F., and M. J. Duncan, 1997, Icarus **127**, 13.

Levison, H. F., J. J. Lissauer, and M. J. Duncan, 1998, Astron. J. **116**, 1998.

Levison, H. F., E. M. Shoemaker, and C. S. Shoemaker, 1997, Nature (London) **385**, 42.

Lichtenberg, A. J., and M. A. Lieberman, 1992, *Regular and Chaotic Dynamics*, 2nd ed. (Springer-Verlag, New York).

Liou, J.-C., and R. Malhotra, 1997, Science **275**, 375.

Lissauer, J. J., 1993, Annu. Rev. Astron. Astrophys. **31**, 129.

Lissauer, J. J., 1995, Icarus **114**, 217.

Malhotra, R., 1996, Astron. J. **111**, 504.

Malhotra, R., 1998, in *Solar System Formation and Evolution,* edited by D. Lazzaro *et al.*, ASP Conference Series Vol. 149 (Astronomical Society of the Pacific, San Francisco), p. 37.

Marcy, G. W., and R. P. Butler, 1996, Astrophys. J. **464**, L147.

Marsden, B. G., Z. Sekanina, and D. K. Yeomans, 1973, Astron. J. **78**, 211.

Migliorini, F., *et al.*, 1997, Meteoritics Planet. Sci. **32**, 903.

Misner, C. W., K. S. Thorne, and J. A. Wheeler, 1973, *Gravitation* (Freeman, San Francisco).

Morbidelli, A., 1997, Icarus **127**, 1.

Murray, C. D., and S. F. Dermott, 1999, *Solar System Dynamics* (Cambridge University Press, Cambridge, England, in press).

Murray, N., and M. Holman, 1997, Astron. J. **114**, 1246.

Murray, N., M. Holman, and M. Potter, 1998, Astron. J. **116**, 2583.

Peale, S. J., 1976, Annu. Rev. Astron. Astrophys. **14**, 215.

Peterson, I., 1993, *Newton's Clock: Chaos in the Solar System* (Freeman, New York).

Petit, J.-M., D. D. Durda, R. Greenberg, T. A. Hurford, and P. E. Geissler, 1997, Icarus **130**, 177.

Petit, J.-M., and M. Henon, 1986, Icarus **66**, 536.

Poincaré, H., 1892, *Les Méthodes Nouvelles de la Méchanique Céleste* (Paris: Gauthiers-Villars, Paris). English translation: *New Methods of Celestial Mechanics* (AIP, New York, 1993).

Pollack, J. B., O. Hubickyj, P. Bodenheimer, J. J. Lissauer, M. Podolak, and Y. Greenzweig, 1996, Icarus **124**, 62.

Rasio, F. A., and E. B. Ford, 1996, Science **274**, 954.

Sackmann, I.-J., A. I. Boothroyd, and K. E. Kraemer, 1993, Astrophys. J. **418**, 457.

Safronov, V. S., 1969, *Evolution of the Protoplanetary Cloud and Formation of the Earth and Planets* (Nauka, Moscow) (in Russian). English translation: NASA TTF-677, 1972.

Schuster, H. G., 1989, *Deterministic Chaos—An Introduction*, 2nd ed. (VCH, New York).

Shinbrot, T., C. Grebogi, J. Wisdom, and J. A. Yorke, 1992, Am. J. Phys. **60**, 491.

Showman, A. P., and R. Malhotra, 1997, Icarus **127**, 93.

Sussman, G. J., and J. Wisdom, 1988, Science **241**, 433.

Sussman, G. J., and J. Wisdom, 1992, Science **257**, 56.

Tittemore, W. C., 1990, Science **250**, 263.

Torbett, M., and R. Smoluchowski, 1980, Icarus **44**, 722.

Touma, J., and J. Wisdom, 1993, Science **259**, 1294.

Tremaine, S., 1995, in *Frontiers of Astrophysics, Proceedings of the Rosseland Centenary Symposium*, edited by P. B. Lilje and P. Maltby (Institute of Theoretical Astrophysics, Oslo), p. 85.

Weidenschilling, S. J., and D. R. Davis, 1985, Icarus **62**, 16.

Weidenschilling, S. J., and F. Marzari, 1996, Nature (London) **384**, 619.

Wetherill, G. W., 1990, Annu. Rev. Earth Planet Sci. **18**, 205.

Wetherill, G. W., and G. R. Stewart, 1989, Icarus **77**, 330.

Whipple, A. L., and P. J. Shields, 1993, Icarus **105**, 408.

Williams, J. G., and G. S. Benson, 1971, Astron. J. **76**, 167.

Wisdom, J., 1980, Astron. J. **85**, 1122.

Wisdom, J., 1982, Astron. J. **87**, 577.

Wisdom, J., 1985a, Icarus **63**, 272.

Wisdom, J., 1985b, Nature (London) **315**, 731.

Wisdom, J., 1987a, Proc. R. Soc. London, Ser. A **413**, 109.

Wisdom, J., 1987b, Icarus **72**, 241.

Wisdom, J., 1987c, Astron. J. **94**, 1350.

Wisdom, J., and M. Holman, 1991, Astron. J. **102**, 1528.

Wisdom, J., S. J. Peale, and F. Mignard, 1984, Icarus **58**, 137.

Yalcinkaya, T., and Y.-C. Lai, 1995, Comput. Phys. **9**, 511.

Yeomans, D. K., 1994, *Asteroids, Comets, Meteors 1993*, Proceedings of the IAU Symposium No. 160 (Kluwer, Dordrecht), p. 241.