Built upon sand: Theoretical ideas inspired by granular flows

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Granulated materials, like sand and sugar and salt, are composed of many pieces that can move independently. The study of collisions and flow in these materials requires new theoretical ideas beyond those in the standard statistical mechanics or hydrodynamics or traditional solid mechanics. Granular materials differ from standard molecular materials in that frictional forces among grains can dissipate energy and drive the system toward frozen or glassy configurations. In experimental studies of these materials, one sees complex flow patterns similar to those of ordinary liquids, but also freezing, plasticity, and hysteresis. To explain these results, theorists have looked at models based upon inelastic collisions among particles. With the aid of computer simulations of these models they have tried to build a "statistical dynamics" of inelastic collisions. One effect seen, called *inelastic collapse*, is a freezing of some of the degrees of freedom induced by an infinity of inelastic collisions. More often some degrees of freedom are partially frozen, so that there can be a rather cold clump of material in correlated motion. Conversely, thin layers of material may be mobile, while all the material around them is frozen. In these and other ways, granular motion looks different from movement in other kinds of materials. Simulations in simple geometries may also be used to ask questions like When does the usual Boltzmann-Gibbs-Maxwell statistical mechanics arise?, What are the nature of the probability distributions for forces between the grains?, and Might the system possibly be described by uniform partial differential equations? One might even say that the study of granular materials gives one a chance to reinvent statistical mechanics in a new context. [S0034-6861(99)00701-1]

CONTENTS

I. INTRODUCTION: A BASIC DESCRIPTION OF STATISTICS AND FLOW

Granular materials show a wonderfully diverse set of behaviors.[1] Make a sand castle, and the material appears solid. Push on the castle and it can fall down in an avalanche-like pattern. Sometimes the avalanche moves the bulk of the material, sometimes it is confined to a thin layer on the surface. Shake up crushed ice in a martini shaker, and it moves like a gas. Try to pour salt through an orifice, and it has a characteristic tendency to choke up and clog the orifice. Gas, liquid, solid, plastic flow, glassy behavior—a granular material can mimic them all. In addition, the properties of a granular material can depend upon its history. Tamped sand is different from loose sand.

But in many ways, a granular material is like an ordinary fluid. Both types of material are composed of many small particles, and each has a bulk behavior that hides the material's graininess. It is thus natural to ask whether the same equations, concepts, and theories that work for molecular material also apply to the granular form of matter.

So let us go back to the fundamental statistical description of matter as developed by Boltzmann, Maxwell, Gibbs, and many others. The macroscopic state of a statistical system is described by a probability distribution, which contains a set of parameters like the temperature, chemical potential, and velocity of the system. The number of such parameters is equal to the number of independent conserved quantities. For the standard one-component fluid, the parameters can be taken to be the mass density ρ, the average velocity of the system $\mathbf{u}$, and the temperature T. We use the symbols ρ_α to denote the densities of the different conserved quantities, and the symbols μ_α to denote the macroscopic parameters.

In part, the conserved quantities and the parameters are observable in the thermodynamics of the system. They are also visible in the dynamics. We concentrate upon the slow dynamics that arise when the time variation within the material is very slow in comparison to the typical collision and relaxation time of the constituent particles. We also require that spatial variations be slow in comparison to mean free paths and to the size of any one of the constituent particles. One source of possible slow motion is the gradual flow of conserved quantities from one part of the system to another. These motions may be described by saying that each part of the system is in a local thermodynamic equilibrium and then

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[1] For recent reviews see Behringer and Jenkins (1997). For a more specialized view, close to the outlook of the present work, see Jaeger, Nagel, and Behringer (1996a and 1966b).

using the conservation laws as flow equations. Additional slow motions may arise when there is a broken symmetry in the system. We shall not worry much about the broken symmetries here. A less well understood source of slow relaxation is glassy behavior, with its partial freezing of degrees of freedom and its concomitant very slow relaxation to full equilibrium. We shall see a considerable glassiness in granular systems.

To derive equations for the slow flow, one starts from the conservation laws and uses the local equilibrium to derive what are described as the *hydrodynamic equations* for the system. For each conserved quantity there is a corresponding current $\mathbf{j}_\alpha(\mathbf{r},t)$ describing its flow. The hydrodynamic equations are the set of conservation laws:

$$\frac{\partial}{\partial t}\rho_\alpha(\mathbf{r},t)+\nabla\circ\mathbf{j}_\alpha(\mathbf{r},t)=0 \qquad (1)$$

supplemented by thermodynamic relations and constitutive equations that define how the density and current depend upon the thermodynamic parameters μ_α. We assume and assert that the densities are local functions of the parameters, e.g., that the energy density at $\mathbf{r},t$ depends only upon the mass density, temperature, and velocity at the same point in space-time. We further take the currents to be only functions of the local parameters and their gradients. For example, Galilean invariance gives the mass current as $\rho\mathbf{u}$, while a constitutive equation gives the heat current as $-\kappa\nabla T$, where κ is the thermal conductivity.

All this is classical and gives us the usual hydrodynamic equations as partial differential equations in space and time. There are no long-range effects, that is, no integrals over space. There are no memory effects, that is, no integrals over previous history. The entire description is in the few partial differential equations and their boundary conditions.

Along with this nonequilibrium theory, we inherit from the old masters an equilibrium theory in which the probability of anything is given by the Gibbs form of Maxwell-Boltzmann statistics, i.e., the probability is an exponential linear in such conserved quantities as the energy and momentum. The parameters μ_α are the adjustable constants in these probability functions, which are then used to set the average values of the various conserved quantities.

II. EXPERIMENTS ON SHAKEN SAND

Many authors have done studies of the effects of shaking a granular material, perhaps composed of glass beads, or grains of sand, or rice, or coal. [See, for example, Larouche, Douady, and Fauve (1989); Knight, Jaeger, and Nagel (1993); Pak, Doorn, and Behringer (1993); Melo, Umbanhowar, and Swinney (1994, 1995); Metcalf, Knight, and Jaeger (1997)]. In these experiments, there are many many individual constituents, each one in motion and bumping into its neighbors. The objects are small and the relaxations are rapid. At first sight it would appear to be quite easy to satisfy the con-

ditions for the derivation of hydrodynamic equations, which are mostly concerned with the slowness of variations in space and time. Maybe hydrodynamics does describe motion in granular materials. Indeed experiments in fluid and in granular materials look similar. Many people, going back to Faraday[2] (1831), have seen localized and delocalized excitations on the surface of a container of shaken sand. These patterns of motion look quite similar to the Faraday crispation patterns developed on the surface of a shaken fluid. Even the new localized sand excitations called *oscillons* (Umbanhower, Melo, and Swinney, 1996) look very similar to localized excitations seen in water by Lathrop and Putterman.[3]

However, there is a major difference between molecules and grains. An ordinary fluid conserves energy within the observed degrees of freedom. Thus, if one puts heat into a fluid, this heat contributes to the kinetic energy of each of the molecules as part of the process of raising the temperature of the fluid. This added energy is never lost, except perhaps by a slow leakage through the walls of the vessel. Thus a macroscopic flow, once started, is likely to continue for a long time. Moreover, the relative motion of the basic particles, called heat, will never die away.

In contrast, in a granular material, some energy can be lost to heat in each collision. Heat energy is stored in the (unobserved) molecules within the grains, and not as kinetic energy of the observed grains. From the point of view of the grains, the system dissipates energy very rapidly. If left alone, the system would get stuck in a solid or glassy configuration and relative motion would come to a virtual or complete halt. This complete relaxation might happen in one region of the material and not in a neighboring one. Thus sand may never show the relaxation to overall uniform equilibrium that is required in the usual derivations of hydrodynamic equations (Chapman and Cowling, 1990). Because of this failure, we cannot be at all sure that hydrodynamic equations describe the behavior of granular materials in any general way.

But it would be nice if a set of stable hydrodynamics equations did apply. We would like to be able to describe the granular material by a small set of local variables. Perhaps the variables would be a local velocity, a density, and an effective temperature, as in an ordinary one-component fluid. We would further wish that we could write a set of local equations connecting the local values of these quantities. These local equations would, in our dreams, be partial differential equations, maybe very closely similar to those of ordinary hydrodynamics. Then the equations would be supplemented by boundary conditions, and we would have a complete description of the space and time development of the system (Haff, 1983; Jenkins and Savage, 1983; Jenkins, 1992;

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[2]Faraday (1831) himself credits Chladni for the discovery of sand patterns. More information on Chladni can be found in Waller (1961).

[3]Private communications.

Hayakawa, Yue, and Hong, 1995; Sela and Goldhirsch, 1996; Grossman, Zhou, and Ben-Naim, 1997). Are our dreams connected to reality?

Before we look at the experimental answers to this question we should contrast hydrodynamic behavior with the possible alternative behaviors that might arise. The hydrodynamic situation occurs when the system is fully defined by a set of partial differential equations and their boundary conditions. We also have to specify the values of some small set of state variables at some initial time. Except for the description of walls that define the container, there is no explicit coordinate dependence in the equations. All coefficients in the equations are completely independent of time. These hydrodynamic equations remain uniformly valid throughout space and time. Unless the partial differential equations themselves develop troubles in the form of singularities or infinities (Kadanoff, 1997), the equations will hold everywhere. We call the situation in which the system is described by uniform and uniformly valid partial differential equations *hydrodynamic* behavior, or more precisely *uniform and local hydrodynamics*.

In the next level of complexity, instead of partial differential equations, the system is described by integral equations. If there are time integrals, we say that the system has a history dependence; if there are space integrals, we say that it is nonlocal. Either way, the equations are more complex than in the hydrodynamic situation. Another kind of complexity arises when the system is quite sensitive to local fluctuations. In this situation, small fluctuations can give large and unpredictable deviations from uniformity. These nonuniform cases can be found in turbulence, flame fronts, and other chaotic situations. When, as in the case of granular materials, there is an inherent nonuniformity in the system itself, any mechanism for magnifying fluctuations can give us all kinds of complications. On large scales, the equations describing the system may end up as complex as a set of nonlinear partial differential equations containing stochastic coefficients and/or forcing. [See, for example, Shinbrot (1997).] That kind of situation can be as complex as anything in quantum field theory. So ask once more, What kind of problem is posed by a granular material?

III. SHAKEN MOTION: SOME EVIDENCE AGAINST SIMPLE HYDRODYNAMICS

In this section, I look at some of the evidence that suggests that granular materials might not be describable by uniform and local hydrodynamic equations. I am going to follow over some of the ground covered by recent review papers (Jaeger, Nagel, and Behringer, 1996a, 1996b) but emphasize different aspects of the data.

There is a very interesting series of experiments in which sand is brought into motion by successive vertical shakes (Jaeger, Knight, Liu, and Nagel, 1994). An acceleration of more than 1 g sends the sand near the top of the container flying. One then asks two questions:
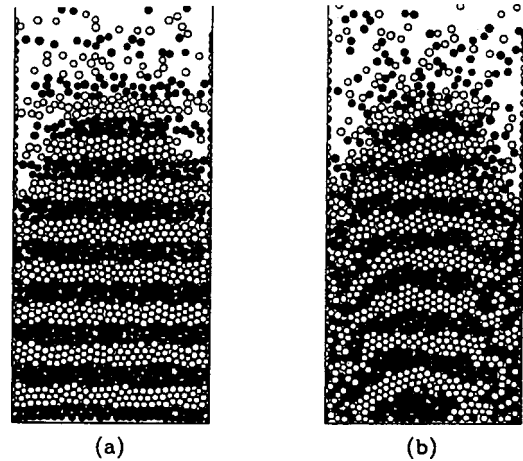

FIG. 1. The positions of grains of sand at the same phase in the motion of two successive shakes. In plate (a), the particles are colored in black or white depending on their position at that time. These same particles are depicted in plate (b). Notice the downward motion in a boundary layer very near the wall.

1. What is the pattern of the grains during a shake?
2. What is the long-term pattern of motion? Specifically, can one give a qualitative description of how the various grains will move over the long term? Naturally, the long-term motion is an expression of the composite effects of the short-term motion.

A. Rapid variation

We have information about this situation from both experiments (Jaeger and Nagel, 1992) and simulations (Grossman, 1997a, 1997b). We look in detail at the results of a simulation done by Grossman. This simulation shows many of the same features as the experiment it was intended to model. We want to know whether the motion during a shake and the long-term motion are hydrodynamic in character.

Figure 1 shows sand flying at the maximum of two successive shakes. The balls are colored in black and white so that one can follow the net motion integrated over one full shake. Note that the top is flying free and that there is considerable motion in boundary layers at the side walls. Integrated over the entire shake, there is little net motion in the central bulk of the material. Recall that the derivation of hydrodynamic equations requires slow variation in space. However, the "convection experiment" of Fig. 1 involves motion in a thin boundary layer, where—in fact—the motion is most rapid (Hayakawa, Yue, and Hong, 1995). In both the real experiment and the simulation, the layer is only a few grains thick. This thickness is not suitable for the development of a uniform, hydrodynamic description. It may well be that, in most granular flows, thin layers of distinctive behavior will dominate the flow of hydrodynamic materials and ruin the possibility of a uniform hydrodynamic theory, [for further work on shaking and fluidization, see Clément and Rajchenbach (1991); Gallas, Hermann, and Sokolowski (1992); Esipov and
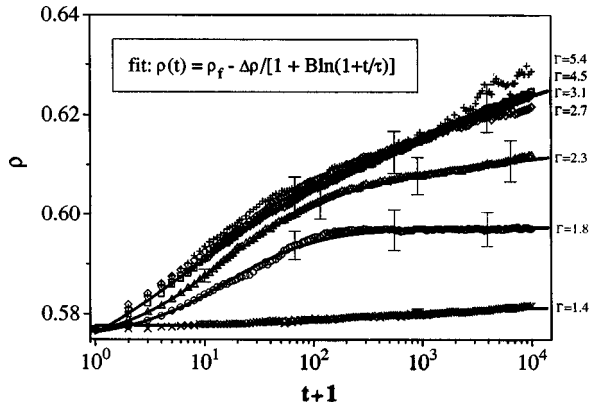
FIG. 2. Compaction of a granular material. See Nowak *et al.* (1995). A granular material is initially "fluffed up" by shaking it with a stream of nitrogen gas. Then it is shaken with a maximum acceleration Γ, measured in g's. As the shaking goes on it compacts more and more. The data points show how the density ρ depends upon the number of shakes *t*. The curves are constructed as the best fit to Eq. (2).

Pöschel (1997)]. Such boundary layers can also appear on free surfaces [see de Gennes (1997) and Boutreus and de Gennes (1997)].

The time dependence also shows rapid variation. Figure 1 shows that the grains are thrown into the air where they move freely for a time comparable with their relaxation time at that density. Is such free-particle motion hydrodynamic? This free motion further casts doubt on the validity of hydrodynamic theory for this situation.

In fact, the possibility of hydrodynamic breakdown via short-wavelength behavior is well known in another context. If you shear a granular material, or indeed an ordinary solid, sufficiently hard it can produce thin layers of weakly bound material (Oda, Iwasbhita, and Kakiuchi, 1997). These so-called *shear bands* have been conjectured to play a role in earthquake fault zones.

B. History dependence

In addition, there are classical experiments on granular materials which show that the state of these materials is quite dependent on their past history. Such history dependence is possible for glasses or spin glasses or frozen materials, but it is not consistent with equilibrium statistical mechanics. I describe one of these experiments here (Knight, Fandrich, Lau, Jaeger, and Nagel, 1995). Imagine that you fill a tall glass vessel with granular material and measure the density of the material by placing the glass tube inside a set of capacitor plates. The capacitance gives a direct measurement of the relative volume of interior filled by air and by the grains. You start out by fluffing up the granular material by swirling it around with compressed gas, and then you shake the tube in a vertical motion many, many times. The entire experiment is described by two variables: Γ the maximum acceleration relative to *g* and *t* the number of shakes.

Figure 2 shows typical pictures of the density as a

function of the number of shakes. The system starts with a low density. For a very long period, including many shakes, the density slowly rises. In between the shakes, the material does not move. So time-independent behavior is observed at many different densities. The densities are determined by the prior history of the sample. We see then that the system has a history dependence in which slow changes over long periods can have a marked effect. This dependence is particularly important because the material's flow properties are very sensitive to the density, with higher densities producing enhanced resistance to flow (Ristow, Strassburger, and Rehberg, 1997).

The data in Fig. 2 are fit by curves described by Eq. (2) (Nowak, Knight, Ben-Naim, Jaeger, and Nagel, 1995; Ben-Naim, Knight, Nowak, Jaeger, and Nagel, 1998). This fit shows that in the long run the density will rise to an "equilibrium" value which depends on the acceleration Γ. The slow rise and the Γ dependence show that the density does not just depend upon the conditions at one time. Instead, it has an important history dependence. This kind of dependence is not found in any ordinary liquid.

Ben-Naim developed a theory[4] of automobile parking that effectively explains the shake dependence of Fig. 2 (Nowak, Knight, Ben-Naim, Jaeger, and Nagel, 1995). In this theory a parking spot appears via the collection of many bits and pieces of empty space until, by a random process, sufficient space is assembled to make a full parking spot. Thus, for high density, the time to park varies exponentially with the available space. Conversely, the density varies in a logarithmic law involving time. When translated into the granular-compaction process, this model suggests a law for compaction which gives the density as a function of the number of shakes *t* (or equivalently the time since the material was last fluffed up) as

$$\rho(\infty) - \rho(t) = A/\log(1 + t/\tau), \qquad (2)$$

where A, $\rho(\infty)$, and τ are all parameters that can depend upon Γ. As one can see from Fig. 2, the phenomenological theory does fit the observed facts. A little later Gavrilov (1997)[5] developed a parallel theory based upon highway congestion, which measures the number of clusters of different sizes. His theory gave a result like Eq. (2) and also described the fluctuations in density after the system reached steady state (Gavrilov, 1998).

This and analogous results leave the theory in a very uncomfortable position. On the one hand, we would like a hydrodynamic theory that is local in space and time. On the other hand, we see that the system can come to a

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[4]Ben-Naim's theory was developed in Boston, where one can have ample time to speculate on mechanisms for producing suitable parking spots. A competitive theory was developed by Gavrilov (1997).

[5]Gavrilov, a graduate student at Chicago, developed his (1997) theory of clustering by observing traffic patterns from a train window. He then applied this theory to granular materials (unpublished.)

stop with a wide range of different densities. Hence the density, at least, is history dependent. It is hard to make a system of partial differential equations that comes to a stop in this way. Thus it is quite possible that no differential equation theory will apply to this system. In the models, the relaxation of a shaken system comes from the filling in of defects and vacancies in the material. To describe processes like these, we need some statistical theory of defect production and removal (Shinbrot *et al.*, 1997; Venkataramani, 1998). Hydrodynamic equations for a uniform system do not provide for these kinds of statistical processes. The theories of Ben-Naim and Gavrilov thus suggest that we have to reach beyond hydrodynamics to include stochastic processes as part of the basic equations of the system. When such processes act in space and time, they can produce very rich behavior, so that the resulting theory may be extremely difficult.

IV. STATISTICAL DYNAMICS

A. Bunching

Let us start again. Let us look at simulations of isolated systems containing a relatively large number of particles. For the moment, we take these particles to have center-of-mass motion but no spinning motion. They all have equal masses. (In this paper, we shall take the masses of all particles to be equal to one.) The peculiar properties of granular materials are represented by having the particles lose a little energy in each collision. Specifically take the overall collision to conserve momentum and to conserve for each particle the component of the velocity perpendicular to the line of centers. In an elastic collision, described in the center-of-mass system, the components of each particle's velocity along the line of centers is reversed. In our case, we shall take the velocity to be diminished by a factor r, called the coefficient of restitution, so that the new normal component is related to the old, v_n, by

$$v'_n = -rv_n \quad -1 \leqslant r \leqslant 1. \tag{3}$$

The last condition is required to make the energy continually decrease. The usual elastic case is $r=1$.

Figure 3 shows two simulations due to McNamara and Young (1996), in which they look at inelastic collisions among 1024 particles in a two-dimensional box with periodic boundary conditions. The simulation is of the "event-driven" variety (Rapaport, 1980). The program looks ahead to the next collisions that can occur, picks the one that actually occurs first, readjusts the velocities of the two colliding particles, and then looks ahead once again with the appropriately adjusted set of possible collisions. In the simulation, the particles start out with randomly chosen positions and velocities picked from a Gaussian ensemble. As time goes on the particles slow down considerably, but this slowing does not compromise the effectiveness of the computer program. Compare the simulations. Both pictures show the situation after a large number of collisions. In both cases, the par-
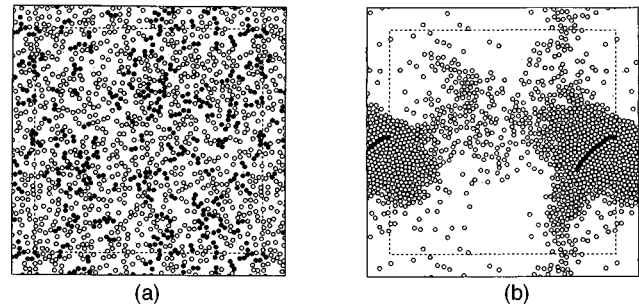

FIG. 3. Simulations of granular motion. Taken from McNamara and Young (1994). Here there are 1024 inelastic disks with periodic boundary conditions. The cases are (a) $r=0.99$ and (b) $r=0.6$. In both cases the system is run from random initial data until there are a large number of collisions per particle. The particles marked in black have taken part in the last 200 collisions. Note the clumping of particles and collisions.

ticles shown in black are those which have participated in one of the last 200 collisions.

In case (a), the system is almost elastic. Only weak correlations develop among the particles and the recent collisions. In case (b) there is more inelasticity and the particles are more bunched up in space. Figure 4 shows a simulation with considerably more particles carried out by Goldhirsch and Zanetti (1993). It has the same r value as in Fig. 3(b). Notice once more the very considerable bunching up of the particles. In fact, these authors argue that the bunching is the result of a hydrodynamic instability in which regions of reduced temperature have reduced pressure, which then fills them with more particles so as to equalize the pressure. But then these regions have more frequent collisions, which reduces the temperature still further, and so forth until some small region has a very high density. Since hydrodynamics fails when the system's dynamics produces very rapid variations in space or time, one can
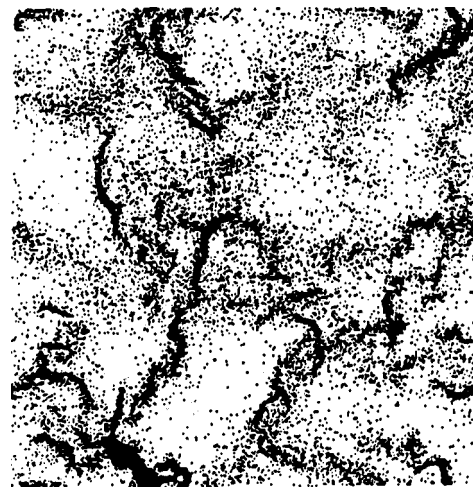

FIG. 4. Another picture of granular motion. From Goldhirsch and Zanetti (1993). Here there are 40 000 particles with $r = 0.6$. Notice the very evident clustering.

worry that the bunching shown in Figs. 3(b) and 4 might invalidate any hydrodynamic-style theory of the granular material.

B. Inelastic collapse

There is another, independent, symptom of the indefinite bunching of particles and collisions. As pointed out by Haff (1983), Bernu and Mazighi (1990), and McNamara and Young (1992), subsets of the particles in this hard-sphere granular system can undergo an infinite number of collisions in a finite period of time. One can in fact see this phenomenon in Fig. 3(b). Notice how the dark particles, those which have participated in the last 200 collisions, form a weakly curved connected region. These particles will soon collide an infinite number of times, so that all their motion in the lateral directions will go to zero. In the meantime, however, the motion in the transverse directions will continue, and after a time the particles will move apart. However, before this happens much of the energy of their relative motion will have been lost. This phenomenon of infinitely repeated collisions is called *inelastic collapse.*

There has been some controversy about the importance of such collapse for real granular materials.[6] We shall discuss this further below. However, before passing judgment upon its significance, we should try to understand the collapse by studying it in the simplest possible situations.

Imagine a ball in vertical motion. It is pulled down by gravity and is bounced up by partially inelastic collisions with a table top. This is a high-school physics problem, with a tiny modification for inelasticity. Say that the speed just after the mth bounce is s_m. The speed on the next bounce is assumed to be given by an expression like that used in Eq. (3), namely,

$$s_{m+1} = r s_m . \tag{4}$$

Here the coefficient of restitution r is restricted to be between zero and one. The speeds approach zero as a geometric series, $s_m = s_0 r^m$, while the heights of the nth bounce obey $h_m = s_m^2/(2g) \sim r^{2m}$ and so the time between collisions, being proportional to h_m/s_m goes to zero as r^m. Thus we have an infinite number of collisions

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[6]For example, a referee of this paper said: "The issue of inelastic collapse is in one sense a computational artifact, in that nothing physical happens. Rather, computations grind to a halt as the number of collisions grows," I neither fully hold to nor fully deny this view. On the one hand, for dimensions greater than one, there remains an unquenched component of velocity perpendicular to the line of collisions. Thus, as stated, motion continues after collapse. Further, collapse does not occur in the same way if the particles have a soft—but dissipative—interaction. On the other hand, the collapse time is an instant of extremely singular velocity correlations. Mathematicians are interested in questions about how singularities can arise from classical mechanics. In addition, the collapse signals the onset of strong correlations, which may be robust even if the collapse is not.

in a finite period of time. In this model, after a finite time the ball comes to rest on the surface of the table. Thus inelastic collapse does indeed occur in a very simple everyday situation.

Another case open to analysis involves three particles with equal mass which are free to move on a line (Constantin, Grossman, and Mungan, 1995). Their velocities are, in order of increasing x coordinate, u, v, and w. Their entire motion can be described in terms of the ratio of the relative velocities, $\eta = (u\text{-}v)/(w\text{-}v)$. The subsequent math is very easy once one sees how to set up the problem. Assume that initially the two outer particles approach the inner one. The subsequent process is one in which the outer particles collide successively with the inner particle. If the particles with velocities u and v collide, the subsequent value of η is

$$1/\eta' = \frac{1+r}{2r} - \frac{1}{r\,\eta} .$$

If the other two particles collide, the new value of η is given by

$$\eta' = \frac{1+r}{2r} - \frac{\eta}{r} .$$

Given these relations, one can follow the entire subsequent motion and figure out the range of r and initial values of η that will produce one collision, or two, or ten, or an infinite number. The last is the case of collapse. The net result is very simple: collapse can occur for suitable values of η only when r stands in the range

$$0 \leqslant r \leqslant 7 - 4\sqrt{3} = 0.073 \ldots . \tag{5}$$

The entire collapse is a kind of approach to a "fixed point" in which the value of η which appears after the nth u-v collision relaxes to an n-independent value for large n. Thus, in the long run, the process simply repeats itself.

As this calculation is extended to higher dimensions, one finds that there are additional variables which describe the state of the system at the point of collapse (Zhou and Kadanoff, 1996). First there is the angle θ between the particles' lines of centers at the moment of collapse. For each value of r, there is a range of θ values that will permit collapse. This range is shown in Fig. 5. Notice that only small values of r permit collapse and that the permissible range of θ gets smaller and smaller as r is increased.

A second set of new variables are the transverse components of the momentum. In one region of Fig. 5, labeled stable collapse, nonzero values of these variables will not upset the approach to a fixed point in which the motion is repeated again and again over decreasing scales of distance and lateral velocity. Another region, labeled unstable collapse, requires these values to be set to zero. If these transverse components are not set to zero, the particles can have many collisions but will eventually veer away from one another, preventing inelastic collapse.

Note that Fig. 5 in part explains why the collapsing particles in Fig. 3(b) essentially lie on a straight line. For
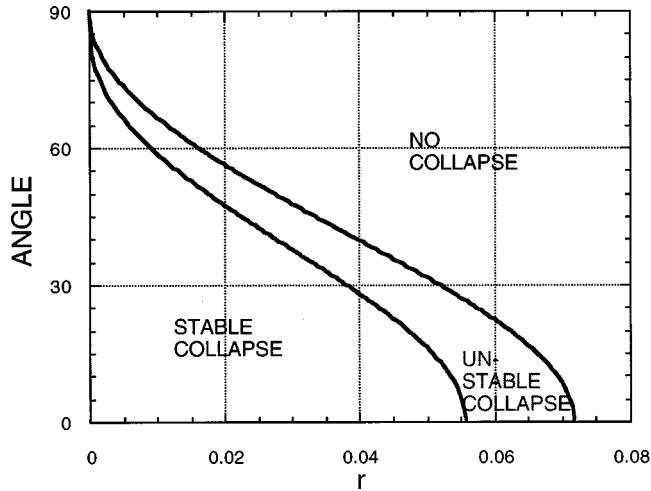
FIG. 5. Permitted values of r and θ for inelastic collapse. See Zhou and Kadanoff (1996).

three particles, we see that the collapse occurs over the largest range of r, when θ is small. Thus collapse is favored when θ is small. This requirement apparently extends itself to the many-particle case and produces the roughly collinear observed behavior.

All this describes a situation in which the particles cannot rotate. Schörghofer and Zhou (1996) extended this calculation to include rotational motion. They found that, when the particles rotate, high-θ collapse configurations become possible—and indeed quite probable—for three particles. When they simulate collapse for a many-particle configuration (see Fig. 6), they find that the collapsing particles form a zig-zag pattern, reflecting a high-θ value during the collapse.

These and a variety of other calculations show that collapse can be understood, and is indeed mostly understood. However, this understanding does not contribute much to our knowledge about either the nature of granular motion or the overall effect of the collapse upon that motion. Several workers in the field have argued that the collapse phenomenon is so dependent upon the hard-sphere system that it is essentially unconnected with the sorts of slowdowns and collapse that will occur in real granular materials. If collapse is not very robust, it will probably offer little insight into the interesting things that happen in real materials. On the other hand, collapse might be a robust symptom of the bunching and slowing down that certainly occurs in a granular system. It might produce isolated regions in space-time which are so frozen that they form defects in the hydrodynamics and, in fact, produce regions in which additional variables or information are necessary to complete the hydrodynamics. The last word has not been said about the importance or unimportance of this collapse.

V. LOW-DIMENSIONAL SYSTEMS

To go beyond the question of collapse, let us look at what is happening to very simple examples of dissipative
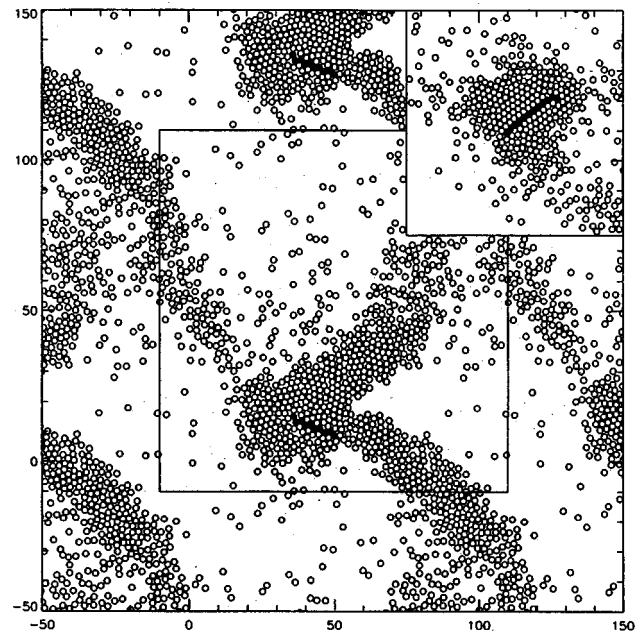
FIG. 6. Collapse with rotation. From Schörghofer and Zhou (1996). A simulation of elastic collision among particles that are free to rotate, in a two-dimensional box with periodic boundary conditions. The inset in the upper right-hand corner shows a similar simulation without rotation. Notice that the collapsed particles (in black) form a zig-zag pattern in the rotating case and a much straighter pattern in the absence of rotation. This result indicates large values of θ in the rotating case and smaller ones in the absence of rotation.

systems. Start with a large number N of particles moving along a line (Du, Li, and Kadanoff, 1995). In this one-dimensional example, we shall pick the inelasticity, $\varepsilon = 1 - r$, to be a small number so that the system may be expected to have a substantial similarity to a set of fully elastic particles. The left-hand wall of the system will be hot, defined so that any particle which hits it comes off with a perpendicular component of velocity chosen from the Maxwell-Boltzmann distribution[7] $v \exp(-v^2/2)$ (see Fig. 7). The right-hand wall of the system is taken to be a reflecting wall. A particle that hits it is reflected without changing its speed.

Before we studied the simulation, we expected that the system would relax to a situation like that in the usual equilibrium statistical mechanics: a uniform distribution in space with a Gaussian distribution in each component of each particle's velocity. The expectation was that, as ε got smaller, the fit to the usual statistical mechanics would get better and better. Instead something entirely different happened. In the limiting situation in which N is big and $N\varepsilon$ is of the order of unity, the

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[7]This probability distribution is not itself the Gaussian distribution connected with the names of Maxwell and Boltzmann. However, this distribution of wall-induced velocities produces the appropriate distribution in phase space for the particles moving away from the wall, prior to their first collision with other particles.
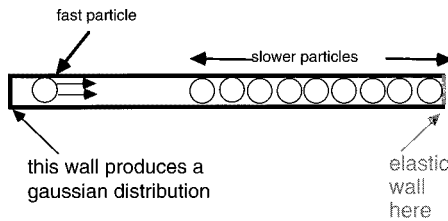
FIG. 7. A group of slightly inelastic particles in one dimension. After Du, Li, and Kadanoff (1995). The particles end up in a situation in which one particle moves fast and the others sit almost still near the elastic wall.

system can do one of two things. It can undergo inelastic collapse. We put those collapse behaviors aside and study the noncollapse behavior. In this situation, the system splits into two parts. One part is composed of a single fast-moving particle near the hot wall; the other is a group of slower particles huddled together near the reflecting wall.

This situation is maintained through the collisions. When the fast-moving particle, with speed of order one, hits the first member of the clump, it transfers all its momentum, save a fraction ε, to the slower particle and is itself slowed down to a speed of order ε. This process happens again and again, with the fast momentum being reduced by a factor $(1-\varepsilon)^n \approx \exp(-n\varepsilon)$, after n collisions. The momentum reaches the far wall, is reflected, and returns by the same process. Finally, one particle with momentum reduced by a factor $\exp(-2N\varepsilon)$ moves leftward out of the pack. It is still hot, but leaves behind it particles with speeds of order ε. Thus the equilibrium state of the system is one in which there is a clumping of particles, but one particle is left out of the clump. This behavior could not have been predicted from any hydrodynamic style of analysis.

One might have expected that the one-dimensional problem would cause trouble. One can see this trouble directly from the Boltzmann equation analysis of McNamara and Young (1992) or of Grossman and Roman (1996) or from a consideration of the basics of statistical mechanics. Recall that in statistical mechanics the equilibrium state of any system is described as an exponential of linear combinations of conserved quantities. In this exponential, there is one undetermined parameter for each conserved quantity. For one dimension and $\varepsilon = 0$, each collision simply interchanges the velocities of the colliding particles. Therefore the number of particles with any given value of the momentum is conserved. McNamara and Young (1992) set up a Boltzmann equation to describe this situation and then said that the equilibrium solution of this Boltzmann equation is composed of two pieces: any spatial distribution of particles at rest and, on top of this, any distribution of particles in velocity, all of these velocities being uniformly distributed through space. The first piece corresponds to our clump; the second to our single particle in motion. Thus the $\varepsilon \rightarrow 0$ limit does reduce to statistical mechanics, but to a very nonstandard statistical situation.

Next consider the same situation in two dimensions. Once again there is a hot left wall, a reflecting right wall,

and reflecting walls on the top and the bottom. There are two simple instructive cases, the first is a "pipe" that is only a bit wider than a single particle, and the second is a square box. The pipe is substantially different from the one-dimensional case because the structure of the equilibrium situation is changed entirely. As soon as there is a transverse component of the velocity sharing energy in a scattering event, the system loses all the extra conservation laws that dominated the one-dimensional situation.

Simulations by Grossman, Zhou, and Ben-Naim (1997) show that both the pipe and the square box have a much smoother behavior than that of the one-dimensional case. No longer is there a single particle that moves much faster than all of the others. Instead there is a gradual and continuous falloff of temperature as one moves away from the hot wall. It seems that this situation can, in fact, be described by using quasi-hydrodynamic equations connecting the density ρ, average velocity $\mathbf{u} = (u, 0)$, and temperature T. As usual, the hydrodynamics is given in terms of conservation laws (Haff, 1983; Savage, 1988), which form equations for the time derivatives of densities of conserved quantities ρ, momentum density $= \rho \mathbf{u}$, and energy density $\rho T + \rho \mathbf{u}^2/2$. In a steady situation the equation for mass conservation can be satisfied if the average velocity $\mathbf{u}$ is zero. The equation for momentum conservation will be true if the pressure is independent of position. The final equation, for energy conservation, has the form in which net heat flow from conduction is balanced against energy dissipation:

$$\nabla(\kappa \nabla T) = -(\text{energy dissipation rate}). \qquad (6)$$

Here κ is the thermal conductivity. Grossman, Zhou, and Ben-Naim (1997) use kinetic theory to obtain a determination of the unknown functions in the theory, namely, the dependence of the pressure, the dissipation rate, and the thermal conductivity upon ρ and u. The simulations show that neither T nor ρ is position independent, with the first one rising and the other falling near the hot wall. The equation of state will determine the pressure as a function of T and ρ, and the constancy of the pressure will give one relationship between ρ and T. These quantities will be fully determined when the T equation [Eq. (6)] is solved with boundary conditions provided by the known temperature on the hot surface and a no-current-flow condition on the reflecting surface. A few parameters are needed to generate kinetic theory results for the pressure, the thermal conductivity, and the energy dissipation via collisions. The net result is a temperature that falls off as $1/\cosh(\alpha n \varepsilon^{1/2})$ away from the hot plate, where n is the number of molecules counted from the hot plate. Here ε is the inelasticity defined here as $(1-r^2)/2$ while α is a coefficient of order unity. Correspondingly, as n increases the density rises and then saturates. These results fit the simulational data quite well.

The probability distributions for the velocity are not of Gaussian form. They are skewed by the large energy current that flows through the system. However, they
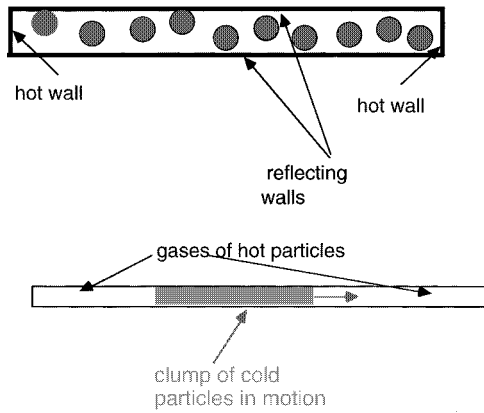
FIG. 8. Behavior of inelastic spheres confined between two hot walls. From Zhou and Kadanoff (1998 unpublished). The top picture shows the geometry and a disordered initial state. The bottom picture is a sketch of the configuration reached after a long time. The particles near the wall produce a hot gas, with the temperature declining away from the hot walls. In the central region, there is a clump of material with very low relative velocities. This clump moves as a unit and acts as a Brownian particle, which has been formed in the midst of an inelastic gas.

obey a sort of scaling law so that the velocity divided by $\sqrt{T}$ has the same probability distribution throughout the sample. Experiments by Kudrolli and Gollub (1997) and Kudrolli, Wolpert, and Gollub (1997), conducted with balls on a slanted table, show a qualitatively similar behavior to that found in the simulation. Thus, in a very simple situation, hydrodynamics seems to work quite well.[8] However, there is still room for some doubt. The authors forced the agreement by using two adjustable parameters, one for each wall. This adjustment might be necessary precisely because the system is far from an equilibrium steady state, so that the usual conditions for the derivation of hydrodynamic equations do not hold.

Clumping can ruin the hydrodynamic description. In very recent work, Zhou (1998) have been studying a pipe in which both right and left walls are hot. In this system, the majority of the particles clump up in a mass, which is separated from these walls by a smaller number of fast-moving particles (see Fig. 8). This clump seems to undergo Brownian motion, fed by momentum fluctuations that are provided by the particles added at the hot wall. The speed of the correlated motion can be very much greater than the thermal velocity describing the relative motion of the particles at the center of the clump. The clump contains most of the particles, which can move all together and quite rapidly. We can use kinetic theory to calculate the motion of the clump, but this calculation includes the clump's Brownian motion. That motion is left out in a purely hydrodynamic analy-

sis of the sort done in the problem with one hot wall. We are far from sure that any hydrodynamic description will work even in this very simple situation. Thus we are left with a distrust of uniform hydrodynamics as a possible description for a granular system.

VI. CONCLUSIONS

Throughout this paper, our main question has been Can a granular material be described by hydrodynamic equations, most specifically those equations which apply to an ordinary fluid? It seems to me that the answer is "no!" Glassy behavior is familiar, but it is not fully described by any simple set of partial differential equations. It is also not fully understood. In fact, there are a rich variety of familiar but weakly understood nonequilibrium problems. These range from plastic flow to crack propagation to charge density waves and to ordinary and spin glasses. Granular materials form a particularly rich example of this kind of system, with behaviors which are, at this moment, not fully understood.

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[8]For another, earlier, investigation which supports two-dimensional hydrodynamics, see Louge, Jenkins, and Hopkins (1993). A parallel publication, Louge (1994), suggests that there are some difficulties with boundary conditions in the hydrodynamic formulation.

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