Renormalization group theory: Its basis and formulation in statistical physics*

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The nature and origins of renormalization group ideas in statistical physics and condensed matter theory are recounted informally, emphasizing those features of prime importance in these areas of science in contradistinction to quantum field theory, in particular: critical exponents and scaling, relevance, irrelevance and marginality, universality, and Wilson's crucial concept of flows and fixed points in a large space of Hamiltonians.

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FOREWORD

"*In March 1996 the Departments of Philosophy and of Physics at Boston University cosponsored a Colloquium 'On the Foundations of Quantum Field Theory.' But in the full title, this was preceded by the phrase 'A Historical Examination and Philosophical Reflections,' which set the aims of the meeting. The participants were mainly high-energy physicists, experts in field theories, and interested philosophers of science.*[1] *I was called on to speak, essentially in a service role, presumably because I had witnessed and had some hand in the development of renormalization group concepts and because I have played a role in applications where these ideas really mattered. It is hoped that this article, based on the talk I presented in Boston, may prove of interest to a wider audience.*"

*Based on a lecture presented on 2 March 1996 at the Boston Colloquium for the Philosophy of Science: "A Historical Examination and Philosophical Reflections on the Foundations of Quantum Field Theory," held at Boston University 1–3 March 1996.

[1]The proceedings of the conference are to be published under the title *Conceptual Foundations of Quantum Field Theory* (Cao, 1998): for details see the references collected in the Selected Bibliography.

I. INTRODUCTION

It is held by some that the "Renormalization Group"—or, better, renormalization group*s* or, let us say, *Renormalization Group Theory* (or RGT) is "one of the underlying ideas in the theoretical structure of Quantum Field Theory." That belief suggests the potential value of a historical and conceptual account of RG theory and the ideas and sources from which it grew, as viewed from the perspective of statistical mechanics and condensed matter physics. Especially pertinent are the roots in the theory of critical phenomena.

The proposition just stated regarding the significance of RG theory for Quantum Field Theory (or QFT, for short) is open to debate even though experts in QFT have certainly invoked RG ideas. Indeed, one may ask: How far is some concept only instrumental? How far is it crucial? It is surely true in physics that when we have ideas and pictures that are extremely useful, they acquire elements of reality in and of themselves. But, philosophically, it is instructive to look at the degree to which such objects are purely instrumental—merely useful tools—and the extent to which physicists seriously suppose they embody an essence of reality. Certainly, many parts of physics are well established and long precede RG ideas. Among these is statistical mechanics itself, a theory *not* reduced and, in a deep sense, *not* directly reducible to lower, more fundamental levels without the introduction of specific, new postulates.

Furthermore, statistical mechanics has reached a stage where it is well posed mathematically; many of the basic theorems (although by no means all) have been proved with full rigor. In that context, I believe it is possible to view the renormalization group as merely an instrument or a computational device. On the other hand, at one extreme, one might say: "Well, the partition function itself is really just a combinatorial device." But most practitioners tend to think of it (and especially its logarithm, the free energy) as rather more basic!

Now my aim here is not to instruct those field theorists who understand these matters well.[2] Rather, I hope to convey to nonexperts and, in particular, to any with a philosophical interest, a little more about what Renor-

[2]Such as D. Gross and R. Shankar (see Cao, 1998, and Shankar, 1994). Note also Bagnuls and Bervillier (1997).

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malization Group Theory is[3]—at least in the eyes of some of those who have earned a living by using it! One hopes such information may be useful to those who might want to discuss its implications and significance or assess how it fits into physics more broadly or into QFT in particular.

II. WHENCE CAME RENORMALIZATION GROUP THEORY?

This is a good question to start with: I will try to respond, sketching the foundations of RG theory in the *critical exponent relations* and crucial *scaling concepts*[4] of Leo P. Kadanoff, Benjamin Widom, and myself developed in 1963–66[5]—among, of course, other important workers, particularly Cyril Domb[6] and his group at King's College London, of which, originally, I was a member, George A. Baker, Jr., whose introduction of Padé approximant techniques proved so fruitful in gaining quantitative knowledge,[7] and Valeri L. Pokrovskii and A. Z. Patashinskii in the Soviet Union who were, perhaps, the first to bring field-theoretic perspectives to bear.[8] Especially, of course, I will say something of the genesis of the full RG concept—the systematic integrating out of appropriate degrees of freedom and the resulting RG flows—in the inspired work of Kenneth G. Wilson[9] as I saw it when he was a colleague of mine and Ben Widom's at Cornell University in 1965–1972. And I must point also to the general, clarifying formulation of RG theory by Franz J. Wegner (1972a) when he was associated with Leo Kadanoff at Brown University: their focus on *relevant, irrelevant* and *marginal* 'operators' (or perturbations) has played a central role.[10]

But, if one takes a step back, two earlier, fundamental theoretical achievements must be recognized: the first is the work of Lev D. Landau, who in reality, is the founder of systematic *effective field theories*, even though he might not have put it that way. It is Landau's *invention*—as it may, I feel, be fairly called—of the *order parameter* that is so important but often underappreciated.[11] To assert that there exists an order parameter in essence says: "I may not understand the microscopic phenomena at all" (as was historically, the case for superfluid helium), "but I recognize that there is a microscopic level and I believe it should have certain general, overall properties especially as regards locality and symmetry: those then serve to govern the most characteristic behavior on scales greater than atomic." Landau and Ginzburg (a major collaborator and developer of the concept[12]) misjudged one or two of the important general properties, in particular the role of fluctuations and singularity; but that does not alter the deep significance of this way of looking at a complex, condensed matter system. Know the nature of the order parameter—suppose, for example, it is a complex number and like a wave function—then one knows much about the macroscopic nature of a physical system!

Significantly, in my view, Landau's introduction of the order parameter exposed a novel and unexpected *foliation* or level in our understanding of the physical world. Traditionally, one characterizes statistical mechanics as directly linking the *microscopic* world of nuclei and atoms (on length scales of 10^{-13} to 10^{-8} cm) to the *macroscopic* world of say, millimeters to meters. But the order parameter, as a dynamic, fluctuating object in many cases intervenes on an intermediate or *mesoscopic* level characterized by scales of tens or hundreds of angstroms up to microns (say, $10^{-6.5}$ to $10^{-3.5}$ cm). The advent of Wilson's concept of the renormalization group gave more precise meaning to the effective ("coarse-grained") Hamiltonians that stemmed from the work of Landau and Ginzburg. One now pictures the LGW—for Landau-Ginzburg-Wilson—Hamiltonians as true but significantly renormalized Hamiltonians in which finer microscopic degrees of freedom have been integrated-out. (See below for more concrete and explicit expressions.) Frequently, indeed, in modern condensed matter theory one *starts* from this intermediate level with a physically appropriate LGW Hamiltonian *in place* of a true (or, at least, more faithful or realistic) microscopic Hamiltonian; and *then* one brings statistical mechanics

[3]It is worthwhile to stress, at the outset, what a "renormalization group" is *not*! Although in many applications the particular renormalization group employed may be invertible, and so constitute a continuous or discrete, group of transformations, it is, in general, only a *semigroup*. In other words a renormalization group is not necessarily invertible and, hence, cannot be 'run backwards' without ambiguity: in short it is *not* a "group." The misuse of mathematical terminology may be tolerated since these aspects play, at best, a small role in RG theory. The point will be returned to in Secs. VIII and XI.

[4]Five influential reviews antedating renormalization-group concepts are Domb (1960), Fisher (1965, 1967b), Kadanoff *et al.* (1967) and Stanley (1971). Early reviews of renormalization group developments are provided by Wilson and Kogut (1974b) and Fisher (1974): see also Wilson (1983) and Fisher (1983). The first texts are Pfeuty and Toulouse (1975), Ma (1976), and Patashinskii and Pokrovskii (1979). The books by Baker (1990), Creswick *et al.* (1992), and Domb (1996) present retrospective views.

[5]See Essam and Fisher (1963), Widom (1965a, 1965b), Kadanoff (1966), and Fisher (1967a).

[6]Note Domb (1960), Domb and Hunter (1965), and the account in Domb (1996).

[7]See Baker (1961) and the overview in Baker (1990).

[8]The original paper is Patashinskii and Pokrovskii (1966); their text (1979), which includes a chapter on RG theory, appeared in Russian around 1975 but did not then discuss RG theory.

[9]Wilson (1971a, 1971b), described within the QFT context in Wilson (1983).

[10]Note the reviews by Kadanoff (1976) and Wegner (1976).

[11]See Landau and Lifshitz (1958) especially Sec. 135.

[12]In particular for the theory of superconductivity: see V. L. Ginzburg and L. D. Landau, 1959, "On the Theory of Superconductivity," Zh. Eksp. Teor. Fiz. **20**, 1064; and, for a personal historical account, V. L. Ginzburg, 1997, "Superconductivity and Superfluidity (What was done and what was not)," Phys. Usp. **40**, 407–432.

to bear in order to understand the macroscopic level. The derivation and validity of the many types of initial, LGW Hamiltonians may then be the object of separate studies to relate them to the atomic level.[13]

Landau's concept of the order parameter, indeed, brought light, clarity, and form to the general theory of phase transitions, leading eventually, to the characterization of multicritical points and the understanding of many characteristic features of ordered states.[14] But in 1944 a bombshell struck! Lars Onsager, by a mathematical *tour de force*, deeply admired by Landau himself,[15] computed exactly the partition function and thermodynamic properties of the simplest model of a ferromagnet or a fluid.[16] This model, the *Ising model*, exhibited a sharp critical point: but the explicit properties, in particular, the nature of the critical singularities disagreed profoundly—as I will explain below—with essentially all the detailed predictions of the Landau theory (and of all foregoing, more specific theories). From this challenge, and from experimental evidence pointing in the same direction,[17] grew the ideas of *universal* but nontrivial *critical exponents*,[18] special *relations* between different *exponents*,[19] and then, *scaling descriptions* of the region of a critical point.[20] These insights served as stimulus and inspiration to Kenneth Wilson in his pursuit of an understanding of quantum field theories.[21] Indeed, once one understood the close mathematical analogy between doing statistical mechanics with effective Hamiltonians and doing quantum field theory (especially with the aid of Feynman's path integral) the connections seemed almost obvious. Needless to say, however, the realization of the analogy did not come overnight: in fact, Wilson himself was, in my estimation, the individual who first understood clearly the analogies at the deepest levels. And they are being exploited, to mutual benefit to this day.

In 1971, then, Ken Wilson, having struggled with the problem for four or five years,[22] was able to cast his renormalization group ideas into a conceptually effective framework—effective in the sense that one could do certain calculations with it.[23] And Franz Wegner, very soon afterwards,[24] further clarified the foundations and exposed their depth and breadth. An early paper by Kadanoff and Wegner (1971) showing when and how universality could *fail* was particularly significant in demonstrating the richness of Wilson's conception.

So our understanding of "anomalous," i.e., nonLandau-type but, in reality, standard critical behaviour was greatly enhanced. And let me stress that my *personal aim* as a theorist is to *gain* insight and understanding: What that may truly mean is, probably, a matter for deep philosophical review: After all, "What constitutes an *explanation*?" But, on the other hand, if you work as a theoretical physicist in the United States, and wish to publish in The Physical Review, you had better *calculate* something concrete and interesting with your new theory pretty soon! For *that* purpose, the *epsilon expansion*, which used as a small, perturbation parameter the deviation of the spatial dimensionality, d, from four dimensions, namely, $\epsilon=4-d$, provided a powerful and timely tool.[25] It had the added advantage, if one wanted to move ahead, that the method looked something like a cookbook—so that "any fool" could do or check the calculations, whether they really understood, at a deeper level, what they were doing or not! But in practice that also has a real benefit in that a lot of calculations do get done, and some of them turn up new and interesting things or answer old or new questions in instructive ways. A few calculations reveal apparent paradoxes and problems which serve to teach one and advance understanding since, as Arthur Wightman has observed, one asks: "Maybe we should go back and think more carefully about what we are actually doing in implementing the theoretical ideas?" So that, in outline, is what I want to convey in more detail, in this exposition.

III. WHERE STANDS THE RENORMALIZATION GROUP?

Beyond sketching the origins, it is the breadth and generality of RG theory that I wish to stress. Let me, indeed, say immediately that the full RG theory should no more be regarded as based on QFT perturbative expansions—despite that common claim—than can the magnificent structure of Gibbsian statistical mechanics be viewed as founded upon ideal classical gases, Boltzmannian kinetic theory, and the virial and cluster expansions for dilute fluids! True, this last route was still frequently retravelled in textbooks more than 50 years after Gibbs' major works were published; but it deeply misrepresents the power and range of statistical mechanics.

The parallel mischaracterizations of RG theory may be found, for example, in the much cited book by Daniel Amit (1978), or in Chapter 5 of the later text on *Statis-*

[13]These issues have been discussed further by the author in "Condensed Matter Physics: Does Quantum Mechanics Matter?" in *Niels Bohr: Physics and the World*, edited by H. Feshbach, T. Matsui and A. Oleson, 1988 (Harwood Academic, Chur), pp. 177–183.

[14]See Landau and Lifshitz (1958).

[15]As I know by independent personal communications from Valeri Pokrovskii and from Isaak M. Khalatnikov.

[16]Onsager (1944), Kaufman and Onsager (1949), Onsager (1949).

[17]See, e.g. Fisher (1965), Stanley (1971).

[18]Domb (1960, 1996) was the principal pioneer in the identification and estimation of critical exponents: see also the preface to Domb (1996) by the present author.

[19]Advanced particularly in Essam and Fisher (1963).

[20]Widom (1965a, 1965b), Domb and Hunter (1965), Kadanoff (1966), and Patashinskii and Pokrovskii (1966).

[21]Wilson (1971a, 1971b; 1983).

[22]See below and the account in Wilson (1983).

[23]As we will explain: see Wilson (1971a, 1971b).

[24]Wegner (1972a, 1972b).

[25]Wilson and Fisher (1972).

tical Field Theory by Itzykson and Drouffe (1989), or, more recently, in the lecture notes entitled *Renormalization Group* by Benfatto and Gallavotti (1995), ‘‘dedicated to scholars wishing to reflect on some details of the foundations of the modern renormalization group approach.’’ There we read that the authors aim to expose how the RG looks to them as physicists, namely: ‘‘this means the achievement of a coherent perturbation theory based on second order (or lowest-order) calculations.’’ One cannot accept that! It is analogous to asking ‘‘What does statistical mechanics convey to a physicist?’’ and replying: ‘‘It means that one can compute the second-virial coefficient to correct the ideal gas laws!’’ Of course, historically, that is not a totally irrelevant remark; but it is extremely misleading and, in effect, insults one of America’s greatest theoretical physicists, Josiah Willard Gibbs.

To continue to use Benfatto and Gallavotti as strawmen, we find in their preface that the reader is presumed to have ‘‘some familiarity with classical quantum field theory.’’ That surely, gives one the impression that, somehow, QFT is necessary for RG theory. Well, it is totally unnecessary![26] And, in particular, by implication the suggestion overlooks entirely the so-called ‘‘real space RG’’ techniques,[27] the significant Monte Carlo RG calculations,[28] the use of *functional* RG methods,[29] etc. On the other hand, if one wants to do certain types of calculation, then familiarity with quantum field theory and Feynmann diagrams can be very useful. But there is *no necessity*, even though many books that claim to tell one about renormalization group theory give that impression.

I do not want to be unfair to Giovanni Gallavotti, on whose lectures the published notes are based: his book is insightful, stimulating and, accepting his perspective as a mathematical physicist[30] keenly interested in field theory, it is authoritative. Furthermore, it forthrightly acknowledges the breadth of the RG approach citing as examples of problems implicitly or explicitly treated by RG theory:[31]

(i) The KAM (Kolmogorov-Arnold-Moser) theory of Hamiltonian stability
(ii) The constructive theory of Euclidean fields
(iii) Universality theory of the critical point in statistical mechanics
(iv) Onset of chaotic motions in dynamical systems (which includes Feigenbaum’s period-doubling cascades)
(v) The convergence of Fourier series on a circle
(vi) The theory of the Fermi surface in Fermi liquids (as described by Shankar (1994; and in Cao, 1998))

To this list one might well add:

(vii) The theory of polymers in solutions and in melts
(viii) Derivation of the Navier-Stoker equations for hydrodynamics
(ix) The fluctuations of membranes and interfaces
(x) The existence and properties of ‘critical phases’ (such as superfluid and liquid-crystal films)
(xi) Phenomena in random systems, fluid percolation, electron localization, etc.
(xii) The Kondo problem for magnetic impurities in nonmagnetic metals.

This last problem, incidentally, was widely advertised as a significant, major issue in solid state physics. However, when Wilson solved it by a highly innovative, *numerical* RG technique[32] he was given surprisingly little credit by that community. It is worth noting Wilson’s own assessment of his achievement: ‘‘This is the most exciting aspect of the renormalization group, the part of the theory that makes it possible to solve problems which are unreachable by Feynman diagrams. The Kondo problem has been solved by a nondiagrammatic computer method.’’

Earlier in this same passage, written in 1975, Wilson roughly but very usefully divides RG theory into four parts: (a) the formal theory of fixed points and linear and nonlinear behavior near fixed points where he especially cites Wegner (1972a, 1976), as did I, above; (b) the diagrammatic (or field-theoretic) formulation of the RG for critical phenomena[33] where the ϵ expansion[34] and its many variants[35] plays a central role; (c) QFT methods, including the 1970–71 Callan-Symanzik equations[36] and the original, 1954 Gell-Mann-Low RG theory—restricted to systems with only a single, marginal

[26]See, e.g., Fisher (1974, 1983), Creswick, Farach, and Poole (1992), and Domb (1996).

[27]See the reviews in Niemeijer and van Leeuwen (1976), Burkhardt and van Leeuwen (1982).

[28]Pioneered by Ma (1976) and reviewed in Burkhardt and van Leeuwen (1982). For a large scale calculation, see: Pawley, Swendsen, Wallace, and Wilson (1984).

[29]For a striking application see: Fisher and Huse (1985).

[30]The uninitiated should note that for a decade or two the term ‘mathematical physicist’ has meant a theorist who provides *rigorous proofs* of his main results. For an account of the use of the renormalization group *in* rigorous work in mathematical physics, see Gawȩdski (1986).

[31]Benfatto and Gallavotti (1995), Chap. 1.

[32]Wilson (1975); for the following quotation see page 776, column 1.

[33]Wilson (1972), Brézin, Wallace, and Wilson (1972), Wilson and Kogut (1974), Brézin, Le Guillou and Zinn-Justin (1976).

[34]Wilson and Fisher (1972), Fisher and Pfeuty (1972).

[35]Especial mention should be made of $1/n$ expansions, where n is the number of components of the vector order parameter (Abe, 1972, 1973; Fisher, Ma, and Nickel, 1972; Suzuki, 1972; and see Fisher, 1974, and Ma, 1976a) and of coupling-constant expansions in fixed dimension: see Parisi (1973, 1974); Baker, Nickel, Green, and Meiron (1976); Le Guillou and Zinn-Justin (1977); Baker, Nickel, and Meiron (1978): For other problems, dimensionality expansions have been made by writing $d=8-\epsilon$, $6-\epsilon$, $4+\frac{1}{2}m-\epsilon$ $(m=1,2,\cdots)$, $3-\epsilon$, $2+\epsilon$, and $1+\epsilon$.

[36]The Callan-Symanzik equations are described, e.g., in Amit (1978) and Itzykson and Drouffe (1989). The coupling-constant expansions in fixed dimension (Parisi, 1973, 1974; Baker *et al.*, 1976) typically use these equations as a starting point and are usually presented purely formally in contrast to the full Wilson approach (b).

variable[37]—from which Wilson drew some of his inspiration and which he took to name the whole approach.[38] Wilson characterizes these methods as efficient calculationally—which is certainly the case—but applying only to Feynman diagram expansions and says: "They completely hide the physics of many scales." Indeed, from the perspective of condensed matter physics, as I will try to explain below, the chief drawback of the sophisticated field-theoretic techniques is that they are safely applicable only when the basic physics is already well understood. By contrast, the general formulation (a), and Wilson's approach (b), provide insight and understanding into quite fresh problems.

Finally, Wilson highlights (d) "the construction of nondiagrammatic RG transformations, which are then solved numerically." This includes the real-space, Monte Carlo, and functional RG approaches cited above and, of course, Wilson's own brilliant application to the Kondo problem (1975).

IV. EXPONENTS, ANOMALOUS DIMENSIONS, SCALE INVARIANCE AND SCALE DEPENDENCE

If one is to pick out a single feature that epitomizes the power and successes of RG theory, one can but endorse Gallavotti and Benfatto when they say "it has to be stressed that the *possibility of nonclassical critical indices* (i.e., *of nonzero anomaly* η) *is probably the most important achievement of the renormalization group*."[39] For nonexperts it seems worthwhile to spend a little time here explaining the meaning of this remark in more detail and commenting on a few of the specialist terms that have already arisen in this account.

To that end, consider a locally defined microscopic variable which I will denote $\psi(\mathbf{r})$. In a ferromagnet this might well be the local magnetization, $\vec{M}(\mathbf{r})$, or spin vector, $\vec{S}(\mathbf{r})$, at point $\mathbf{r}$ in ordinary d-dimensional (Euclidean) space; in a fluid it might be the deviation $\delta\rho(\mathbf{r})$, of the fluctuating density at $\mathbf{r}$ from the mean density. In QFT the local variables $\psi(\mathbf{r})$ are the basic *quantum fields* which are 'operator valued.' For a magnetic system, in which quantum mechanics was important, $\vec{M}(\mathbf{r})$ and $\vec{S}(\mathbf{r})$ would, likewise, be operators. However, the distinction is of relatively minor importance so that we may, for ease, suppose $\psi(\mathbf{r})$ is a simple classical variable. It will be most interesting when ψ is closely related to the order parameter for the phase transition and critical behavior of concern.

By means of a scattering experiment (using light, x rays, neutrons, electrons, etc.) one can often observe the corresponding *pair correlation function* (or basic 'two-point function')

$$G(\mathbf{r}) = \langle \psi(\mathbf{0})\psi(\mathbf{r})\rangle, \tag{1}$$

where the angular brackets $\langle\cdot\rangle$ denote a statistical average over the thermal fluctuations that characterize all equilibrium systems at nonzero temperature. (Also understood, when $\psi(\mathbf{r})$ is an operator, are the corresponding quantum-mechanical expectation values.)

Physically, $G(\mathbf{r})$ is important since it provides a direct measure of the influence of the leading microscopic fluctuations at the origin $\mathbf{0}$ on the behavior at a point distance $r = |\mathbf{r}|$ away. But, almost by definition, in the vicinity of an appropriate critical point—for example the Curie point of a ferromagnet when $\psi \equiv \vec{M}$ or the gas-liquid critical point when $\psi = \delta\rho$—a strong "ordering" influence or correlation spreads out over, essentially, macroscopic distances. As a consequence, precisely *at* criticality one rather generally finds a *power-law decay*, namely,

$$G_c(\mathbf{r}) \approx D/r^{d-2+\eta} \quad \text{as} \quad r\to\infty, \tag{2}$$

which is characterized by the *critical exponent* (or critical *index*) $d-2+\eta$.

Now all the theories one first encounters—the so-called 'classical' or Landau-Ginzburg or van der Waals theories, etc.[40]—predict, quite unequivocally, that η *vanishes*. In QFT this corresponds to the behavior of a free massless particle. Mathematically, the reason underlying this prediction is that the basic functions entering the theory have (or are assumed to have) a smooth, *analytic, nonsingular* character so that, following Newton, they may be freely differentiated and, thereby expanded in Taylor series with positive integral powers[41] even at the critical point. In QFT the classical exponent value $d-2$ (implying $\eta=0$) can often be determined by naive dimensional analysis or 'power counting': then $d-2$ is said to represent the 'canonical dimension' while η, if nonvanishing, represents the 'dimensional anomaly.' Physically, the prediction $\eta=0$ typically results from a neglect of fluctuations or, more precisely as Wilson emphasized, from the assumption that only fluctuations on much smaller scales can play a significant role: in such circumstances the fluctuations can be safely incorporated into *effective* (or *renormalized*) parameters (masses, coupling constants, etc.) with no change in the basic character of the theory.

[37] See Wilson (1975), page 796, column 1. The concept of a "marginal" variable is explained briefly below: see also Wegner (1972a, 1976), Fisher (1974, 1983), and Kadanoff (1976).

[38] See Wilson (1975, 1983).

[39] See Benfatto and Gallavotti (1995) page 64.

[40] Note that 'classical' here, and in the quote from Benfatto and Gallavotti above means 'in the sense of the ancient authors'; in particular, it is *not* used in contradistinction to 'quantal' or to allude in any way to quantum mechanics (which has essentially no relevance for critical points at nonzero temperature: see the author's article cited in Footnote 13).

[41] The relevant expansion variable in scattering experiments is the square of the scattering wave vector, $\mathbf{k}$, which is proportional to $\lambda^{-1}\sin\frac{1}{2}\theta$ where θ is the scattering angle and λ the wavelength of the radiation used. In the description of near-critical thermodynamics, Landau theory assumes (and mean-field theories lead to) Taylor expansions in powers of $T-T_c$ and $\Psi = \langle\Psi(\mathbf{r})\rangle$, the equilibrium value of the order parameter.

But a power-law dependence on distance implies a *lack* of a definite length scale and, hence, a *scale invariance*. To illustrate this, let us rescale distances by a factor b so that

$$\mathbf{r} \Rightarrow \mathbf{r}' = b\mathbf{r}, \tag{3}$$

and, at the same time, rescale the order parameter ψ by some "covariant" factor b^{ω} where ω will be a critical exponent characterizing ψ. Then we have

$$\begin{aligned} G_c(\mathbf{r}) = \langle \psi(\mathbf{0})\psi(\mathbf{r})\rangle_c \Rightarrow \\ G'_c(b\mathbf{r}) &= b^{2\omega}\langle \psi(\mathbf{0})\psi(b\mathbf{r})\rangle_c \\ &\approx b^{2\omega} D / b^{d-2+\eta} r^{d-2+\eta}. \end{aligned} \tag{4}$$

Now, observe that if one has $\omega = \frac{1}{2}(d-2+\eta)$, the factors of b drop out and the form in Eq. (2) is recaptured. In other words $G_c(\mathbf{r})$ is *scale invariant* (or covariant): its variation reveals no characteristic lengths, large, small, or intermediate!

Since power laws imply scale invariance and the *absence* of well separated scales, the classical theories should be suspect at (and near) criticality! Indeed, one finds that the "anomaly" η does *not* normally vanish (at least for dimensions d less than 4, which is the only concern in a condensed matter laboratory!). In particular, from the work of Kaufman and Onsager (1949) one can show analytically that $\eta = \frac{1}{4}$ for the $d=2$ Ising model.[42] Consequently, the analyticity and Taylor expansions presupposed in the classical theories are *not* valid.[43] Therein lies the challenge to theory! Indeed, it proved hard even to envisage the nature of a theory that would lead to $\eta \neq 0$. The power of the renormalization group is that it provides a conceptual and, in many cases, a computational framework within which anomalous values for η (and for other exponents like ω and its analogs for all local quantities such as the energy density) arise naturally.

In applications to condensed matter physics, it is clear that the power law in Eq. (2) can hold only for distances relatively large compared to atomic lengths or lattice spacings which we will denote a. In this sense the scale invariance of correlation functions is only *asymptotic*—hence the symbol $\approx$, for "asymptotically equals," [44] and the proviso $r \to \infty$ in Eq. (2). A more detailed description would account for the effects of nonvanishing a, at least in leading order. By contrast, in QFT the microscopic distance a represents an "ultraviolet" cutoff which, since it is in general unknown, one normally wishes to remove from the theory. If this removal is not done with surgical care—which is what the renormalization program in QFT is all about—the theory remains plagued with infinite divergencies arising when $a \to 0$, i.e., when the "cutoff is removed." But in statistical physics one always anticipates a short-distance cutoff that sets certain physical parameters such as the value of T_c; infinite terms *per se* do not arise and certainly do *not* drive the theory as in QFT.

In current descriptions of QFT the concept of the *scale-dependence of parameters* is often used with the physical picture that the typical properties of a system measured at particular length (and/or time) scales change, more-or-less slowly, as the scale of observation changes. From my perspective this phraseology often represents merely a shorthand for a somewhat simplified view of RG *flows* (as discussed generally below) in which only one variable or a single trajectory is followed,[45] basically because one is interested only in one, unique theory—the real world of particle physics. In certain condensed matter problems something analogous may suffice or serve in a first attack; but in general a more complex view is imperative.

One may, however, provide a more concrete illustration of scale dependence by referring again to the power law Eq. (2). If the exponent η vanishes, or equivalently, if ψ has its canonical dimension, so that $\omega = \omega_{\text{can}} = \frac{1}{2}(d-2)$, one may regard the amplitude D as a fixed, measurable parameter which will typically embody some real physical significance. Suppose, however, η does *not* vanish but is nonetheless relatively *small*: indeed, for many ($d=3$)-dimensional systems, one has $\eta \simeq 0.035$.[46] Then we can introduce a "renormalized" or "scale-dependent" parameter

$$\tilde{D}(R) \approx D/R^{\eta} \quad \text{as} \quad R \to \infty, \tag{5}$$

and rewrite the original result simply as

$$G_c(r) = \tilde{D}(r)/r^{d-2}. \tag{6}$$

Since η is small we see that $\tilde{D}(R)$ varies slowly with the scale R on which it is measured. In many cases in QFT the dimensions of the field ψ (*alias* the order parameter) are subject only to marginal perturbations (see below) which translate into a $\log R$ dependence of the renormalized parameter $\tilde{D}(R)$; the variation with scale is then still weaker than when $\eta \neq 0$.

V. THE CHALLENGES POSED BY CRITICAL PHENOMENA

It is good to remember, especially when discussing theory and philosophy, that physics is an experimental science! Accordingly, I will review briefly a few experimental findings[47] that serve to focus attention on the

[42] Fisher (1959): see also Fisher (1965, Sec. 29; 1967b, Sec. 6.2), Fisher and Burford (1967).

[43] Precisely the same problem undermines applications of catastrophe theory to critical phenomena; the assumed expressions in powers of $(T-T_c)$ and $\Psi = \langle\psi\rangle$ are simply not valid.

[44] See the Appendix for a discussion of appropriate conventions for the symbols $\simeq$, $\approx$, and $\sim$.

[45] See below and, e.g., Wilson and Kogut (1974), Bagnuls and Bervillier (1997).

[46] See, e.g., Fisher and Burford (1967), Fisher (1983), Baker (1990), and Domb (1996).

[47] Ideally, I should show here plots of impressive experimental data and, in particular, dramatic color pictures of carbon dioxide passing though its critical point. [See Stanley (1971) for black and white photographs.] It is not, however, feasible to reproduce such figures here; instead the presentation focuses on the conclusions as embodied in the observed power laws, etc.

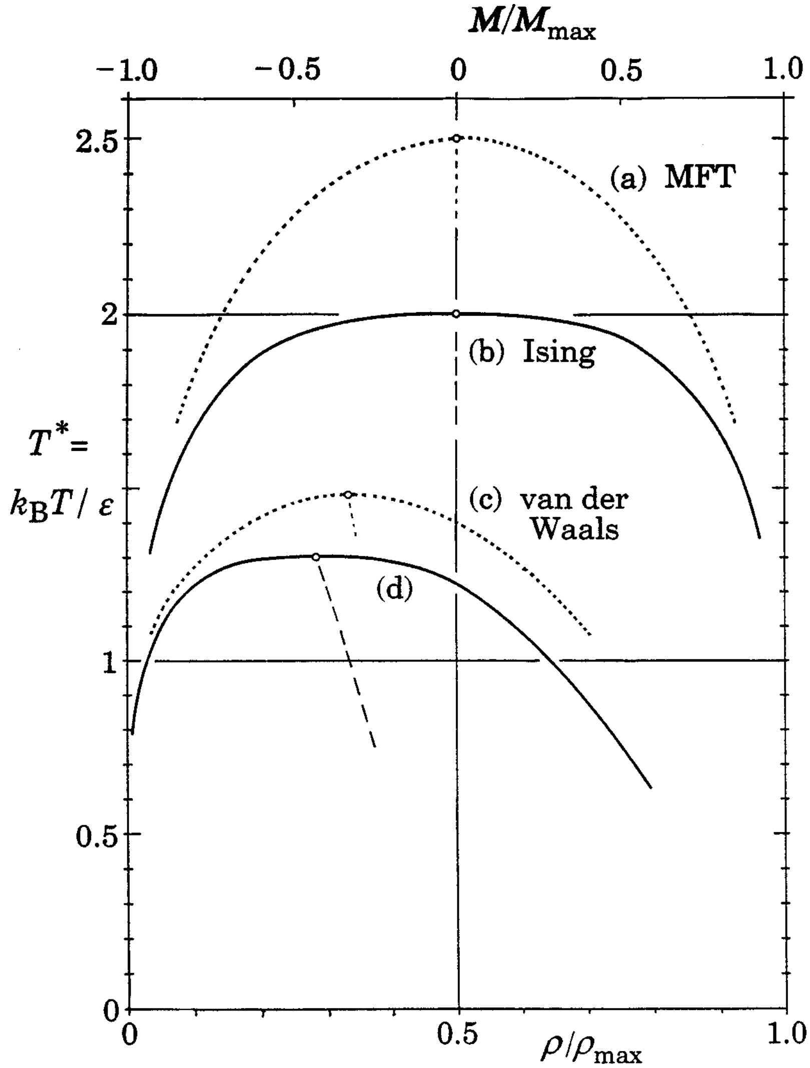

FIG. 1. Temperature variation of gas-liquid coexistence curves (temperature, T, *versus* density, ρ) and corresponding spontaneous magnetization plots (magnetization, M, *versus* T). The solid curves, (b) and (d), represent (semiquantitatively) observation and modern theory, while the dotted curves (a) and (c) illustrate the corresponding "classical" predictions (mean-field theory and van der Waals approximation). These latter plots are parabolic through the critical points (small open circles) instead of obeying a power law with the universal exponent $\beta \simeq 0.325$: see Eqs. (9) and (11). The energy scale ε, and the maximal density and magnetization, ρ_{max} and M_{max}, are nonuniversal parameters particular to each physical system; they vary widely in magnitude.

principal theoretical challenges faced by, and rather fully met by RG theory.

In 1869 Andrews reported to the Royal Society his observations of carbon dioxide sealed in a (strong!) glass tube at a mean overall density, ρ, close to 0.5 gm cm^{-3}. At room temperatures the fluid breaks into two phases: a liquid of density $\rho_{liq}(T)$ that coexists with a lighter vapor or gas phase of density $\rho_{gas}(T)$ from which it is separated by a visible meniscus or interface; but when the temperature, T, is raised and reaches a sharp critical temperature, $T_c \simeq 31.04\,°\mathrm{C}$, the liquid and gaseous phases become identical, assuming a common density $\rho_{liq} = \rho_{gas} = \rho_c$ while the meniscus disappears in a "mist" of "critical opalescence." For all T above T_c there is a complete "continuity of state," i.e., no distinction whatsoever remains between liquid and gas (and there is no meniscus). A plot of $\rho_{liq}(T)$ and $\rho_{gas}(T)$—as illustrated somewhat schematically in Fig. 1(d)—represents the so-called gas-liquid *coexistence curve*: the two halves, $\rho_{liq} > \rho_c$ and $\rho_{gas} < \rho_c$, meet smoothly at the *critical point* (T_c, ρ_c)—shown as a small circle in Fig. 1: the dashed line below T_c represents the *diameter* defined by $\overline{\rho}(T) = \frac{1}{2}[\rho_{liq}(T) + \rho_{gas}(T)]$.

The same phenomena occur in all elemental and simple molecular fluids and in fluid mixtures. The values of T_c, however, vary widely: e.g., for helium-four one finds 5.20 K while for mercury $T_c \simeq 1764$ K. The same is true for the critical densities and concentrations: these are thus "nonuniversal parameters" directly reflecting the atomic and molecular properties, i.e., the physics on the scale of the cutoff a. Hence, in Fig. 1, ρ_{max} (which may be taken as the density of the corresponding crystal at low T) is of order $1/a^3$, while the scale of $k_B T_c$ is set by the basic microscopic potential energy of attraction denoted ε. While of considerable chemical, physical, and engineering interest, such parameters will be of marginal concern to us here. The point, rather, is that the *shapes* of the coexistence curves, $\rho_{liq}(T)$ and $\rho_{gas}(T)$ *versus* T, become asymptotically *universal* in character as the critical point is approached.

To be more explicit, note first an issue of symmetry. In QFT, symmetries of many sorts play an important role: they may (or must) be built into the theory but can be "broken" in the physically realized vacuum state(s) of the quantum field. In the physics of fluids the opposite situation pertains. There is no real physical symmetry between coexisting liquid and gas: they are just different states, one a relatively dense collection of atoms or molecules, the other a relatively dilute collection—see Fig. 1(d). However, if one compares the two sides of the coexistence curve, gas and liquid, by forming the ratio

$$R(T)=[\rho_c-\rho_{\mathrm{gas}}(T)]/[\rho_{\mathrm{liq}}(T)-\rho_c], \tag{7}$$

one discovers an extraordinarily precise *asymptotic* symmetry. Explicitly, when T approaches T_c from below or, introducing a convenient notation,

$$t\equiv(T-T_c)/T_c\rightarrow 0-, \tag{8}$$

one finds $R(T)\rightarrow 1$. This simply means that the physical fluid builds for itself an exact mirror symmetry in density (and other properties) *as* the critical point is *approached*. And this is a universal feature for all fluids near criticality. (This symmetry is reflected in Fig. 1(d) by the high, although *not* absolutely perfect, degree of asymptotic *linearity* of the coexistence-curve diameter, $\overline{\rho(T)}$—the dashed line described above.)

More striking than the (asymptotic) symmetry of the coexistence curve is the universality of its shape close to T_c—visible in Fig. 1(d) as a flattening of the graph relative to the parabolic shape of the corresponding classical prediction—see plot (c) in Fig. 1, which is derived from the famous van der Waals equation of state. Rather generally one can describe the shape of a fluid coexistence curve in the critical region via the power law

$$\Delta\rho\equiv\tfrac{1}{2}[\rho_{\mathrm{liq}}(T)-\rho_{\mathrm{gas}}(T)]\approx B|t|^{\beta} \quad \text{as} \quad t\rightarrow 0-, \tag{9}$$

where B is a *non*universal amplitude while the critical exponent β takes the *universal* value

$$\beta\simeq 0.325, \tag{10}$$

(in which the last figure is uncertain). To stress the point: β is a nontrivial number, not known exactly, but it is the *same* for all fluid critical points! This contrasts starkly with the classical prediction $\beta=\frac{1}{2}$ [corresponding to a parabola: see Fig. 1(c)]. The value in Eq. (10) applies to $(d=3)$-dimensional systems. Classical theories make the same predictions for all d. On the other hand, for $d=2$, Onsager's work (1949) on the square-lattice Ising model leads to $\beta=\frac{1}{8}$. This value has since been confirmed experimentally by Kim and Chan (1984) for a "two-dimensional fluid" of methane (CH_4) adsorbed on the flat, hexagonal-lattice surface of graphite crystals.

Not only does the value in Eq. (10) for β describe many types of fluid system, it also applies to anisotropic *magnetic materials*, in particular to those of Ising-type with one "easy axis." For that case, in vanishing magnetic fields, H, below the Curie or critical temperature, T_c, a ferromagnet exhibits a spontaneous magnetization and one has $M=\pm M_0(T)$. The sign, + or −, depends on whether one lets H approach zero from positive or negative values. Since, in equilibrium, there is a full, natural physical symmetry under $H\Rightarrow -H$ and $M\Rightarrow -M$ (in contrast to fluid systems) one clearly has $M_c=0$: likewise, the asymptotic symmetry corresponding to Eq. (7) is, in this case exact for all T: see Fig. 1, plots (a) and (b). Thus, as is evident in Fig. 1, the *global shape* of a spontaneous magnetization curve does not closely resemble a normal fluid coexistence curve. Nevertheless, in the asymptotic law

$$M_0(T)\approx B|t|^{\beta} \quad \text{as} \quad t\rightarrow 0-, \tag{11}$$

the exponent value in Eq. (10) still applies for $d=3$: see Fig. 1(b); the corresponding classical "mean-field theory" in plot (a), again predicts $\beta=\frac{1}{2}$. For $d=2$ the value $\beta=\frac{1}{8}$ is once more valid!

And, beyond fluids and anisotropic ferromagnets many other systems belong—more correctly their critical behavior belongs—to the "*Ising universality class*." Included are other magnetic materials (antiferromagnets and ferrimagnets), binary metallic alloys (exhibiting order-disorder transitions), certain types of ferroelectrics, and so on.

For each of these systems there is an appropriate order parameter and, via Eq. (2), one can then define (and usually measure) the correlation decay exponent η which is likewise universal. Indeed, essentially any measurable property of a physical system displays a universal critical singularity. Of particular importance is the exponent $\alpha\simeq 0.11$ (Ising, $d=3$) which describes the divergence to infinity of the specific heat via

$$\mathcal{C}(T)\approx A^{\pm}/|t|^{\alpha} \quad \text{as} \quad t\rightarrow 0\pm, \tag{12}$$

(at constant volume for fluids or in zero field, $H=0$, for ferromagnets, etc.). The amplitudes A^+ and A^- are again *non*universal; but their *dimensionless ratio*, A^+/A^-, is *universal*, taking a value close to 0.52. When $d=2$, as Onsager (1944) found, $A^+/A^-=1$ and $|t|^{-\alpha}$ is replaced by $\log|t|$. But classical theory merely predicts a jump in specific heat, $\Delta\mathcal{C}=\mathcal{C}_c^- - \mathcal{C}_c^+>0$, for all d!

Two other central quantities are a divergent isothermal compressibility $\chi(T)$ (for a fluid) or isothermal susceptibility, $\chi(T)\propto(\partial M/\partial H)_T$ (for a ferromagnet) and, for all systems, a *divergent correlation length*, $\xi(T)$, which measures the growth of the 'range of influence' or of correlation observed say, via the decay of the correlation function $G(R;T)$—see Eq. (1) above—to its long-distance limit. For these functions we write

$$\chi(T)\approx C^{\pm}/|t|^{\gamma} \quad \text{and} \quad \xi(T)\approx\xi_0^{\pm}/|t|^{\nu}, \tag{13}$$

as $t\rightarrow 0\pm$, and find, for $d=3$ Ising-type systems,

$$\gamma\simeq 1.24 \quad \text{and} \quad \nu\simeq 0.63 \tag{14}$$

(while $\gamma=1\frac{3}{4}$ and $\nu=1$ for $d=2$).

As hinted, there are other universality classes known theoretically although relatively few are found experimentally.[48] Indeed, one of the early successes of

[48]See e.g., the survey in Fisher (1974b) and Aharony (1976).

RG theory was delineating and sharpening our grasp of the various important universality classes. To a significant degree one found that only the vectorial or tensorial character of the relevant order parameter (e.g., scalar, complex number *alias* two-component vector, three-component vector, etc.) plays a role in determining the universality class. But the whys and the wherefores of this self-same issue represent, as does the universality itself, a prime challenge to any theory of critical phenomena.

VI. EXPONENT RELATIONS, SCALING AND IRRELEVANCE

By 1960–62 the existence of universal critical exponents disagreeing sharply with classical predictions may be regarded as well established theoretically and experimentally.[49] The next theoretical step was the discovery of *exponent relations*, that is, simple algebraic equations satisfied by the various exponents *independently* of the universality class. Among the first of these were[50]

$$\gamma=(2-\eta)\nu \quad \text{and} \quad \alpha+2\beta+\gamma=2. \tag{15}$$

As the reader may check from the values quoted above, these relations hold *exactly* for the $d=2$ Ising models and are valid when $d=3$ to within the experimental accuracy or the numerical precision (of the theoretical estimates[51]). They are even obeyed exactly by the classical exponent values (which, today, we understand[52] as valid for $d>4$).

The first relation in Eq. (15) pertains just to the basic correlation function $G(\mathbf{r};T)=\langle\psi(\mathbf{0})\psi(\mathbf{r})\rangle$ as defined previously. It follows from the assumption,[53] supported in turn by an examination of the structure of Onsager's matrix solution to the Ising model[54] *that in the critical region all lengths* (much larger than the lattice spacing a) *scale like the correlation length* $\xi(T)$—introduced in Eq. (13). Formally one expresses this principle by writing, for $t\to 0$ and $r\to\infty$,

$$G(T;\mathbf{r})\approx\frac{D}{r^{d-2+\eta}}\,\mathcal{G}\left(\frac{r}{\xi(T)}\right), \tag{16}$$

where, for consistency with (2), the *scaling function*, $\mathcal{G}(x)$, satisfies the normalization condition $\mathcal{G}(0)=1$. Integrating $\mathbf{r}$ over all space yields the compressibility/susceptibility $\chi(T)$ and, thence, the relation $\gamma=(2-\eta)\nu$. This *scaling law* highlights the importance of the correlation length ξ in the critical region, a feature later stressed and developed further, especially by Widom

[49]This retrospective statement may, perhaps, warrant further comment. First, the terms "universal" and "universality class" came into common usage only after 1974 when (see below) the concept of various types of renormalization-group fixed point had been well recognized (see Fisher, 1974b). Kadanoff (1976) deserves credit not only for introducing and popularizing the terms but especially for emphasizing, refining, and extending the concepts. On the other hand, Domb's (1960) review made clear that all (short-range) Ising models should have the same critical exponents irrespective of lattice structure but depending strongly on dimensionality. The excluded-volume problem for polymers was known to have closely related but *distinct* critical exponents from the Ising model, depending similarly *on* dimensionality but *not* lattice structure (Fisher and Sykes, 1959). And, as regards the Heisenberg model—which possesses what we would now say is an ($n=3$)-component vector or O(3) order parameter—there were strong hints that the exponents were again different (Rushbrooke and Wood, 1958; Domb and Sykes, 1962). On the experimental front matters might, possibly be viewed as less clear-cut: indeed, for ferromagnets, nonclassical exponents were unambiguously revealed only in 1964 by Kouvel and Fisher. However, a striking experiment by Heller and Benedek (1962) had already shown that the order parameter of the *anti*ferromagnet MnF_2, namely, the *sublattice magnetization* $M_0^\dagger(T)$, vanishes as $|t|^\beta$ with $\beta\simeq 0.33_5$. Furthermore, for fluids, the work of the Dutch school under Michels and the famous analysis of coexistence curves by Guggenheim (1949) allowed little doubt—see Rowlinson (1959), Chap. 3, especially, pp. 91–95—that all reasonably simple atomic and molecular fluids displayed the same but *non*classical critical exponents with $\beta\simeq\frac{1}{3}$: And, also well before 1960, Widom and Rice (1955) had analyzed the *critical isotherms* of a number of simple fluids and concluded that the corresponding critical exponent δ (see, e.g., Fisher, 1967b) took a value around 4.2 in place of the van der Waals value $\delta=3$. In addition, evidence was in hand showing that the *consolute point* in binary fluid mixtures was similar (see Rowlinson, 1959, pp. 165–166).

[50]See Fisher (1959; 1962; 1964, see Eq. (5.7); 1967b) for the first relation here; the second relation was advanced in Essam and Fisher (1963) where the now generally accepted notation for the thermodynamic critical exponents was also introduced. See, in addition, Fisher (1967a) based on a lecture given in March 1965. Actually the initial proposal was written as $\alpha'+2\beta+\gamma'=2$, where the primes denote exponents defined *below* T_c. This distinction, although historically important, is rarely made nowadays since, in general, scaling (see below) implies the $T\gtrless T_c$ equalities $\alpha'=\alpha$, $\gamma'=\gamma$, $\nu'=\nu$, etc. [also mentioned in Essam and Fisher and Fisher (1967a)]. Moved by the suggested thermodynamic exponent equality, Rushbrooke (1963) quickly showed that for magnetic systems (with $H\Rightarrow -H$ symmetry) the positivity of specific heats implied by the Second Law of Thermodynamics could be used to prove rigorously the *inequality* $\alpha'+2\beta+\gamma'\geqslant 2$. His proof was soon extended to fluid systems (Fisher 1964), see Eq. (2.20). Corresponding to the first equality in Eq. (15), the inequality $\gamma\leqslant(2-\eta)\nu$ was proven rigorously in (Fisher, 1969). Other valuable exponent inequalities encompassing "scaling laws" for the exponents as the limiting case of equality were proved by Griffiths (1965, 1972) for thermodynamic exponents and Buckingham and Gunton (1969) for correlation exponents.

[51]See e.g., Fisher (1967b), Baker (1990), Domb (1996).

[52]See Wilson and Fisher (1972), Wilson and Kogut (1974), Fisher (1974, 1983).

[53]See Fisher (1959, 1962).

[54]Onsager (1944), Kaufman and Onsager (1949).

(1965), Kadanoff (1966, 1976), and Wilson (1983).[55] It is worth remarking that in QFT the inverse correlation-length ξ^{-1}, is basically equivalent to the *renormalized mass* of the field ψ: *masslessness* then equates with *criticality* since $\xi^{-1}\to 0$.

The next theoretical question was: "How can one construct an *equation of state* for a system which has nonclassical critical exponents?" The "equation of state"—for concreteness let us say, for a ferromagnet—is an equation relating the magnetization, M, the temperature T, the magnetic field, H, and perhaps, some further variable, say P, like, for example, the overall pressure or, more interestingly, the strength of the direct electromagnetic, dipole-dipole couplings. More generally, one wants to know the free energy $F(T,H,P)$ from which all the thermodynamic properties follow[56]—or, better still, the full correlation function $G(\mathbf{r};T,H,P)$ (where previously we had supposed $H=0$ and $P=P_0$, fixed) since this gives more insight into the "structure" of the system.

The equation of state is crucial knowledge for any applications but, at first sight, the question appears merely of somewhat technical interest. Classical theory provides a simple answer—basically just a power series expansion in $(T-T_c)$, $(M-M_c)$, and $(P-P_c)$, etc.; but that always *enforces* classical exponent values! It transpires, therefore, that the mathematical issues are much more delicate: For convenience, let us focus on the *singular part* of the free energy density, namely,[57]

$$f_s(t,h,g)\equiv -\Delta F(T,H,P)/Vk_BT, \tag{17}$$

as a function of the physically appropriate reduced variables

$$t=(T-T_c)/T_c, \quad h=\mu_B H/k_BT, \quad g=P/k_BT. \tag{18}$$

Now, not only must $f(t,h,g)$ reproduce all the correct critical singularities when $t\to 0$ (for $h=0$, etc.), it must *also* be *free* of singularities, i.e. "analytic," *away* from the critical point (and the phase boundary $h=0$ below T_c).

The solution to this problem came most directly via Widom's (1965b) *homogeneity* or, as more customarily now called, *scaling hypothesis* which *embodies* a minimal number of the critical exponents. This may be written

$$f_s(t,h,g)\approx |t|^{2-\alpha}\mathcal{F}\left(\frac{h}{|t|^{\Delta}},\frac{g}{|t|^{\phi}}\right), \tag{19}$$

where α is the specific heat exponent introduced in Eq. (12) while the new exponent, Δ, which determines *how h scales with t*, is given by

$$\Delta=\beta+\gamma. \tag{20}$$

Widom observed, incidentally, that the classical theories themselves obey scaling: one then has $\alpha=0$, $\Delta=1\frac{1}{2}$, $\phi=-\frac{1}{2}$.

The second new exponent, ϕ, did *not* appear in the original critical-point scaling formulations;[58] neither did the argument $z=g/|t|^{\phi}$ appear in the *scaling function* $\mathcal{F}(y,z)$. It is really only with the appreciation of RG theory that we know that such a dependence should in general be present and, indeed, that a full spectrum $\{\phi_j\}$ of such higher-order exponents with $\phi\equiv\phi_1>\phi_2>\phi_3>\cdots$ must normally appear![59]

But how could such a spectrum of exponents be overlooked? The answer—essentially as supplied by the general RG analysis[60]—is that g and all the higher-order "coupling constants," say g_j, are *irrelevant* if their associated exponents ϕ_j are *negative*. To see this, suppose, as will typically be the case, that $\phi\equiv\phi_1=-\theta$ is negative (so $\theta>0$). Then, on approach to the critical point we see that

$$z=g/|t|^{\phi}=g|t|^{\theta}\to 0. \tag{21}$$

Consequently, $\mathcal{F}(y,z)$, in Eq. (19) can be replaced simply by $\mathcal{F}(y,0)$ which is a function of just a *single variable*. Furthermore, asymptotically when $T\to T_c$ we get the *same* function *whatever* the actual value of g — clearly[61] this is an example of *universality*.

Indeed, within RG theory this is the general mechanism of universality: in a very large (generally infinitely large) space of Hamiltonians, parametrized by t, h, and all the g_j, there is a controlling critical point (later seen to be a *fixed point*) about which each variable enters with a characteristic exponent. All systems with Hamiltonians differing only through the values of the g_j (within suitable bounds) will exhibit the *same critical behavior* determined by the same free-energy scaling function $\mathcal{F}(y)$, where now we drop the irrelevant argu-

[55]See also Wilson and Kogut (1974).

[56]Thus, for example, the equation of state is given by $M=-(\partial F/\partial H)_{T,P}$; the specific heat is $C=-T(\partial^2F/\partial T^2)_{H=0,P}$.

[57]The "singular part," ΔF in Eq. (17), is found by subtracting from F analytic terms: $F_0(T,H,P)=F_c+F_1(T-T_c)+F_2H+\cdots$. In Eq. (17) the volume V of the physical system is shown but a conceptually crucial theoretical issue, namely the *taking* of the *thermodynamic limit*, $V\to\infty$, has, for simplicity, been ignored. In Eq. (18), μ_B denotes the Bohr magneton, so that h is dimensionless.

[58]Widom (1965), Domb and Hunter (1965), Kadanoff (1966), Patashinskii and Pokroskii (1966); and see Fisher (1967b) and Stanley (1971).

[59]See Wilson (1971a) and, for a very general exposition of scaling theory, Fisher (1974a).

[60]Wegner (1972, 1976), Fisher (1974a), Kadanoff (1976).

[61]Again we slide over a physically important detail, namely, that T_c for example, will usually be a function of any irrelevant parameter such as g. This comes about because, in a full scaling formulation, the variables t, h, and g appearing in Eq. (19) must be replaced by *nonlinear scaling fields* $\widetilde{t}(t,h,g)$, $\widetilde{h}(t,h,g)$ and $\widetilde{g}(t,h,g)$ which are smooth functions of t, h, and g (Wegner, 1972, 1976; Fisher, 1983). By the same token it is usually advantageous to introduce a prefactor A_0 in Eq. (19) and "metrical factors" E_j in the arguments $y\equiv z_0$ and z_j (see, e.g., Fisher, 1983).

ment(s). Different universality classes will be associated with different controlling critical points in the space of Hamiltonians with, once one recognizes the concept of RG *flows*, different "domains of attraction" under the flow. All these issues will be reviewed in greater detail below.

In reality, the expectation of a general form of scaling[62] is frequently the most important consequence of RG theory for the practising experimentalist or theorist. Accordingly, it is worth saying more about the meaning and implications of Eq. (19). First, (i) it very generally *implies* the thermodynamic exponent relation Eq. (15) connecting α, β and γ; and (ii) since all leading exponents are determined entirely by the two exponents α and Δ $(=\beta+\gamma)$, it predicts similar exponent relations for any other exponents one might define—such as δ specified *on* the critical isotherm[63] by $H \sim M^{\delta}$. Beyond that, (iii) if one fixes P (or g) and similar parameters and observes the free energy or, in practice, the equation of state, the data one collects amount to describing a function, say $M(T,H)$, of *two variables*. Typically this would be displayed as *sets of isotherms*: i.e., many plots of M vs. H at various closely spaced, fixed values of T near T_c. But according to the scaling law Eq. (19) if one plots the *scaled variables* $f_s/|t|^{2-\alpha}$ or $M/|t|^{\beta}$ vs. the scaled field $h/|t|^{\Delta}$, for appropriately chosen exponents and critical temperature T_c, one should find that all these data "collapse" (in Stanley's (1971) picturesque terminology) onto a single curve, which then just represents the scaling function $x=\mathcal{F}(y)$ itself!

Indeed, this dramatic collapse is precisely found in fitting experimental data. Furthermore, the same "collapse" occurs for different systems since the scaling function $\mathcal{F}(y)$ itself, *also* proves to be *universal* (when properly normalized), as first stressed by Kadanoff (1976). A particularly striking example of such data collapse yielding the same scaling function for a range of irrelevant parameter values, may be found in the recent work by Koch *et al.* (1989).[64] They studied a quite different physical problem, namely, the proposed "vortex-glass" transition in the high-T_c superconductor YBCO. There the voltage drop, E, across the specimen, measured over 4 or 5 decades, plays the role of M; the current density J, measured over a similar range, stands in for h, while the external magnetic field, H, acting on the sample, provides the irrelevant parameter P. The scaling function was finally determined over 10 decades in value and argument and seen to be universal!

[62]Allowing for irrelevant variables, nonlinear scaling fields, and universality, as indicated in Eq. (19) and the previous footnote.

[63]See also Footnote 49 above.

[64]The scaling function, as plotted in this reference, strikes the uninitiated as two distinct functions, one for $T \geqslant T_c$, another for $T \leqslant T_c$. However, this is due just to the presentation adopted: scaling functions like $\mathcal{F}(y)$ in Eq. (19) are typically single functions *analytic through* $T=T_c$ for $y<\infty$ (i.e., $h\neq 0$) and can be re-plotted in a way that exhibits that feature naturally and explicitly.

VII. RELEVANCE, CROSSOVER, AND MARGINALITY

As mentioned, the scaling behavior of the free energy, the equation of state, the correlation functions, and so on, always holds only in some *asymptotic sense* in condensed matter physics (and, indeed, in most applications of scaling). Typically, scaling becomes valid when $t \sim (T-T_c)$ becomes small, when the field H is small, and when the microscopic cut-off a is much smaller than the distances of interest. But one often needs to know: "How small is small enough?" Or, put in other language, "What is the nature of the leading corrections to the dominant power laws?" The "extended scaling" illustrated by the presence of the second argument $z=g/|t|^{\phi}$ in Eq. (19) provides an answer via Eq. (21)—an answer that, phenomenologically, can be regarded as independent of RG theory *per se*[65] but which, in historical fact, essentially grew from insights gained via RG theory.[66]

Specifically, if the physical parameter $P \propto g$ is irrelevant then, by definition, $\phi=-\theta$, is negative and, as discussed, $z=g|t|^{\theta}$ becomes small when $|t|\to 0$. Then one can, fairly generally, hope to expand the scaling function $\mathcal{F}(y,z)$ in powers of z. From this one learns, for example, that the power law Eq. (11) for the spontaneous magnetization of a ferromagnet should, when t is no longer very small, be modified to read

$$M_0(T)=B|t|^{\beta}(1+b_{\theta}|t|^{\theta}+b_1 t+\cdots), \tag{22}$$

where b_{θ} $(\propto g)$ and b_1 are nonuniversal.[67] The exponent θ is often called the *"correction-to-scaling" exponent*—of course, it is universal.[68] It is significant because when θ is smaller than unity and b_{θ} is of order unity, the presence of such a singular correction hampers the reliable estimation of the primary exponent, here β, from experimental or numerical data.

Suppose, on the other hand, that ϕ is *positive* in the basic scaling law Eq. (19). Then when $t \to 0$ the scaled variable $z=g/|t|^{\phi}$ grows larger and larger. Consequently the behavior of $\mathcal{F}(y,z)$ for z small or vanishing becomes of less and less interest. Clearly, the previous discussion of asymptotic scaling fails! When that happens one says that the physical variable P represents a *relevant perturbation* of the original critical behavior.[69] Two possibilities then arise: *Either* the critical point may be *destroyed* altogether! This is, in fact, the effect of the magnetic field, which must itself be regarded as a relevant perturbation since $\phi_0 \equiv \Delta=\beta+\gamma>0$. *Alternatively*, when z grows, the true, asymptotic critical behavior may

[65]See Fisher (1974a).

[66]See Wegner (1972) and Fisher (1974).

[67]See Wegner (1972, 1976) and Fisher (1974, 1983).

[68]For $d=3$ Ising-type systems one finds $\theta \simeq 0.54$: see Chen *et al.* (1982), Zinn and Fisher (1996).

[69]Wegner (1972, 1976), Kadanoff (1976): see also Fisher (1983).

crossover[70] to a new, quite *distinct* universality class with different exponents and a new asymptotic scaling function, say, $\mathcal{F}_\infty(y)$.[71]

The crossover scenario is, in fact, realized when the physical system is a ferromagnet with microscopic spin variables, say $\vec{S}(r)$, coupled by *short-range* "exchange" interactions while P measures the strength of the additional, *long-range* magnetic dipole-dipole coupling mediated by the induced electromagnetic fields.[72] Interested theorists had *felt* intuitively that the long-range character of the dipole-dipole coupling *should* matter, i.e., P should be *relevant*. But theoretically there seemed no feasible way of addressing the problem and, on the other hand, the experimentally observed critical exponents (for an important class of magnetic materials) seemed quite independent of the dipole-dipole coupling P.

The advent of RG theory changed that: First, it established a general framework within which the relevance or irrelevance of some particular perturbation P_j could be judged—essentially by the positive or negative sign of the associated exponent ϕ_j, with especially interesting *nonscaling* and *nonuniversal* behavior likely in the *marginal* case $\phi_j = 0$.[73] Second, for many cases where the $P_j=0$ problem was well understood, RG theory showed how the *crossover exponent* ϕ could be determined exactly or perturbatively. Third, the ϵ expansion allowed calculation of ϕ *and* of the new critical behavior to which the crossover occurred.[74] The dipole-dipole problem for ferromagnets was settled via this last route: the dipole perturbation is *always* relevant; *however*, the new, dipolar critical exponents for typical ferromagnets like iron, nickel and gadolinium are numerically so close in value to the corresponding short-range exponents[75] that they are almost indistinguishable by experiment (or simulation)!

On the other hand, in the special example of *anisotropic*, easy-axis or Ising-type ferromagnets in $d = 3$ dimensions the dipolar couplings behave as *marginal* variables at the controlling, *dipolar* critical point.[76] This leads to the prediction of *logarithmic* modifications of the classical critical power laws (by factors diverging as $\log|T - T_c|$ to various powers). The predicted logarithmic behavior has, in fact, been verified experimentally by Ahlers *et al.* (1975). In other cases, especially for $d=2$, marginal variables lead to continuously variable exponents such as $\alpha(g)$, and to quite different thermal variation, like $\exp(\tilde{A}/|t|^{\tilde{\nu}})$; such results have been checked both in exactly solved statistical mechanical models and in physical systems such as superfluid helium films.[77]

I have entered into these relatively detailed and technical considerations—which a less devoted reader need only peruse—in order to convey something of the flavor of how the renormalization group is used in statistical physics and to bring out those features for which it is so valued; because of the multifaceted character of condensed matter physics these are rather different and more diverse than those aspects of RG theory of significance for QFT.

VIII. THE TASK FOR RENORMALIZATION GROUP THEORY

Let us, at this point, recapitulate briefly by highlighting, from the viewpoint of statistical physics, what it is one would wish RG theory to accomplish. First and foremost, (i) it should explain the ubiquity of power laws at and near critical points: see Eqs. (2), (9), (11)–(13). I sometimes like to compare this issue with the challenge to atomic physics of explaining the ubiquity of sharp spectral lines. Quantum mechanics responds, crudely speaking, by saying: "Well, (a) there is some wave—or a *wave function* ψ—needed to describe electrons in atoms, and (b) to fit a wave into a confined space the wave length must be quantized: hence (c) only certain definite energy levels are allowed and, thence, (d) there are sharp, spectral transitions between them!"

Of course, that is far from being the whole story in quantum mechanics; but I believe it captures an important essence. Neither is the first RG response the whole story: but, *to anticipate*, in Wilson's conception RG theory crudely says: "Well, (a) there is a *flow* in some *space*, $\mathbb{H}$, *of Hamiltonians* (or "coupling constants"); (b) the critical point of a system is associated with a *fixed point* (or stationary point) of that flow; (c) the flow operator—technically the *RG transformation*,[78] $\mathbb{R}$—can

[70]See the extensive discussion of crossover in Fisher (1974b) and Aharony (1976).

[71]Formally, one might write $\mathcal{F}_\infty(y)=\mathcal{F}(y, z\to z_\infty)$ where z_∞ is a critical value which could be ∞; but a more subtle relationship is generally required since the exponent α in the prefactor in Eq. (19) changes.

[72]A "short-range" interaction potential, say $J(\mathbf{r})$, is usually supposed to decay with distance as $\exp(-r/R_0)$ where R_0 is some microscopic range, but certainly must decay *faster* than $1/r^{d+2}$; the dipole-dipole potential, however, decays more slowly, as $1/r^d$, and has a crucially important angular dependence as well.

[73]See the striking analysis of Kadanoff and Wegner (1971).

[74]Fisher and Pfeuty (1972), Wegner (1972b).

[75]Fisher and Aharony (1973).

[76]Aharony (1973, 1976).

[77]See Kadanoff and Wegner (1971) and, for a review of the extensive later developments—including the Kosterlitz-Thouless theory of two-dimensional superfluidity and the Halperin-Nelson-Kosterlitz-Thouless-Young theory of two-dimensional melting—see Nelson (1983).

[78]As explained in more detail in Secs. XI and XII below, a specific renormalization transformation, say $\mathbb{R}_b$, acts on some 'initial' Hamiltonian $\mathcal{H}^{(0)}$ in the space $\mathbb{H}$ to transform it into a new Hamiltonian, $\mathcal{H}^{(1)}$. Under repeated operation of $\mathbb{R}_b$ the initial Hamiltonian "flows" into a sequence $\mathcal{H}^{(l)}$ $(l=1,2,\cdots)$ corresponding to the iterated RG transformation $\mathbb{R}_b\cdots\mathbb{R}_b$ (l times) which, in turn, specifies a new transformation $\mathbb{R}_{b^l}$. These "products" of repeated RG operations serve to define a *semigroup* of transformations that, in general, does *not* actually give rise to a group: see Footnote 3 above and the discussion below in Sec. XI associated with Eq. (35).

be *linearized* about that fixed point; and (d) typically, such a linear operator (as in quantum mechanics) has a spectrum of discrete, but nontrivial eigenvalues, say λ_k; then (e) each (asymptotically independent) exponential term in the flow varies as $e^{\lambda_k l}$, where l is the *flow* (or renormalization) *parameter* and corresponds to a physical power law, say $|t|^{\phi_k}$, with critical exponent ϕ_k proportional to the eigenvalue λ_k.'' How one may find suitable transformations $\mathbb{R}$ and why the flows matter, are the subjects for the following chapters of our story.

Just as quantum mechanics does much more than explain sharp spectral lines, so RG theory should also explain, at least in principle, (ii) the values of the leading thermodynamic and correlation exponents, α, β, γ, δ, ν, η, and ω (to cite those we have already mentioned above) and (iii) clarify why and how the classical values are in error, including the existence of borderline dimensionalities, like $d_\times=4$, above which classical theories become valid. Beyond the leading exponents, one wants (iv) the correction-to-scaling exponent θ (and, ideally, the higher-order correction exponents) and, especially, (v) one needs a method to compute crossover exponents, ϕ, to check for the relevance or irrelevance of a multitude of possible perturbations. Two central issues, of course, are (vi) the understanding of universality with nontrivial exponents and (vii) a derivation of *scaling*: see (16) and (19).

And, more subtly, one wants (viii) to understand the *breakdown* of universality and scaling in certain circumstances—one might recall continuous spectra in quantum mechanics—and (ix) to handle effectively logarithmic and more exotic dependences on temperature, etc.

An important further requirement as regards condensed matter physics is that RG theory should be firmly related to the science of statistical mechanics as perfected by Gibbs. Certainly, there is no need and should be no desire, to replace standard statistical mechanics as a basis for describing equilibrium phenomena in pure, homogeneous systems.[79] Accordingly, it is appropriate to summarize briefly the demands of statistical mechanics in a way suitable for describing the formulation of RG transformations.

We may start by supposing that one has a set of microscopic, fluctuating, mechanical variables: in QFT these would be the various quantum fields, $\psi(\mathbf{r})$, defined—one supposes—at all points in a Euclidean (or Minkowski) space. In statistical physics we will, rather, suppose that in a physical system of volume V there are N discrete ''degrees of freedom.'' For classical fluid systems one would normally use the coordinates $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N$ of the constituent particles. However, it is simpler mathematically—and the analogies with QFT are closer—if we consider here a set of ''*spins*'' $s_\mathbf{x}$ (which could be vectors, tensors, operators, etc.) associated with discrete lattice sites located at uniformly spaced points $\mathbf{x}$. If, as before, the lattice spacing is a, one can take $V=Na^d$ and the density of degrees of freedom in d spatial dimensions is $N/V=a^{-d}$.

In terms of the basic variables $s_\mathbf{x}$, one can form various ''local operators'' (or ''physical densities'' or ''observables'') like the local magnetization and energy densities

$$M_\mathbf{x}=\mu_B s_\mathbf{x}, \quad \mathcal{E}_\mathbf{x}=-\tfrac{1}{2}J\sum_{\boldsymbol{\delta}} s_\mathbf{x} s_{\mathbf{x}+\boldsymbol{\delta}}, \quad \cdots, \tag{23}$$

(where μ_B and J are fixed coefficients while $\boldsymbol{\delta}$ runs over the nearest-neighbor lattice vectors). A physical system of interest is then specified by its *Hamiltonian* $\mathcal{H}[\{s_\mathbf{x}\}]$—or energy function, as in mechanics—which is usually just a spatially uniform sum of local operators. The crucial function is the *reduced Hamiltonian*

$$\bar{\mathcal{H}}[s;\, t,\, h,\, \cdots,\, h_j, \cdots]=-\mathcal{H}[\{s_\mathbf{x}\};\, \cdots,\, h_j, \cdots]/k_B T, \tag{24}$$

where s denotes the set of all the microscopic spins $s_\mathbf{x}$ while $t, h, \cdots, h_j, \cdots$ are various ''*thermodynamic fields*'' (in QFT—the coupling constants): see Eq. (18). We may suppose that one or more of the thermodynamic fields, in particular the temperature, can be controlled directly by the experimenter; but others may be ''given'' since they will, for example, embody details of the physical system that are ''fixed by nature.''

Normally in condensed matter physics one thus focuses on some specific form of $\bar{\mathcal{H}}$ with at most two or three variable parameters—the Ising model is one such particularly simple form with just two variables, t, the reduced temperature, and h, the reduced field. An important feature of Wilson's approach, however, is to regard any such ''physical Hamiltonian'' as merely specifying a subspace (spanned, say, by ''coordinates'' t and h) in a very large space of possible (reduced) Hamiltonians, $\mathbb{H}$: see the schematic illustration in Fig. 2. This change in perspective proves crucial to the proper formulation of a renormalization group: in principle, it enters also in QFT although in practice, it is usually given little attention.

Granted a microscopic Hamiltonian, statistical mechanics promises to tell one the thermodynamic properties of the corresponding macroscopic system! First one must compute the partition function

$$Z_N[\bar{\mathcal{H}}]=\mathrm{Tr}_N^s\{e^{\bar{\mathcal{H}}[s]}\}, \tag{25}$$

where the *trace operation*, $\mathrm{Tr}_N^s\{\cdot\}$, denotes a summation

[79]One may, however, raise legitimate concerns about the adequacy of customary statistical mechanics when it comes to the analysis of random or impure systems—or in applications to systems far from equilibrium or in metastable or steady states—e.g., in fluid turbulence, in sandpiles and earthquakes, etc. And the use of RG ideas in chaotic mechanics and various other topics listed above in Sec. III, clearly does *not* require a statistical mechanical basis.

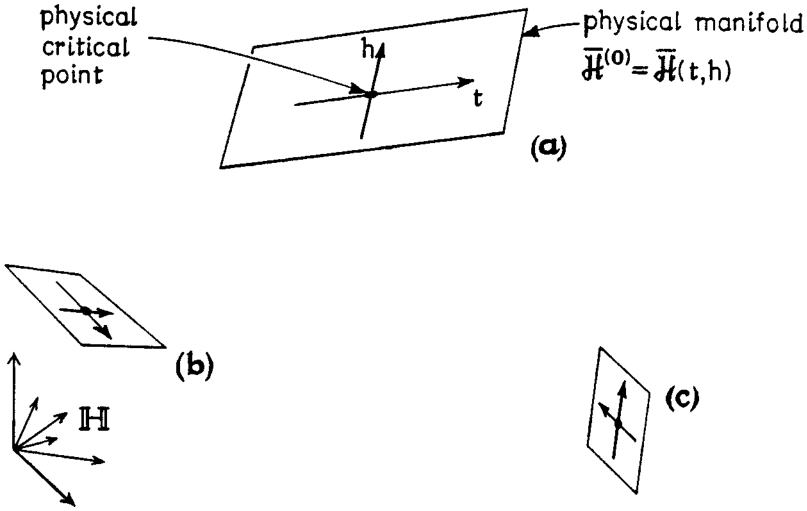

FIG. 2. Schematic illustration of the space of Hamiltonians, $\mathbb{H}$, having, in general, infinitely many dimensions (or coordinate axes). A particular physical system or model representing, say, the ferromagnet, iron, is specified by its reduced Hamiltonian $\overline{\mathcal{H}}(t,h)$, with $t=(T-T_c)/T_c$ and $h=\mu_B H/k_B T$ defined for *that* system: but in $\mathbb{H}$ this Hamiltonian specifies only a submanifold—the physical manifold, labelled (a), that is parametrized by the 'local coordinates' t and h. Other submanifolds, (b), $\cdots$ (c), $\cdots$ located elsewhere in $\mathbb{H}$, depict the physical manifolds for Hamiltonians corresponding to other particular physical systems, say, the ferromagnets nickel and gadolinium, etc.

or integration[80] over all the possible values of all the N spin variables $s_{\mathbf{x}}$ in the system of volume V. The *Boltzmann factor*, $\exp(\overline{\mathcal{H}}[s])$, measures, of course, the probability of observing the microstate specified by the set of values $\{s_{\mathbf{x}}\}$ in an equilibrium ensemble at temperature T. Then the thermodynamics follow from the total free energy density, which is given by[81]

$$f[\overline{\mathcal{H}}] \equiv f(t,\, h,\, \cdots h_j \cdots) = \lim_{N,V\to\infty} V^{-1} \log Z_N[\overline{\mathcal{H}}]; \tag{26}$$

this includes the singular part $f_s[\overline{\mathcal{H}}]$ near a critical point of interest: see Eq. (17). Correlation functions are defined similarly in standard manner.

To the degree that one can actually perform the trace operation in Eq. (25) for a particular model system and take the "thermodynamic limit" in Eq. (26) one will obtain the precise critical exponents, scaling functions, and so on. This was Onsager's (1944) route in solving the $d=2$, spin-$\frac{1}{2}$ Ising models in zero magnetic field. At first sight one then has no need of RG theory. That surmise, however, turns out to be far from the truth. The issue is "simply" one of understanding! (Should one ever achieve truly high precision in simulating critical systems on a computer—a prospect which still seems some decades away—the same problem would remain.) In short, while one knows for sure that $\alpha = 0\,(\log)$, $\beta=\frac{1}{8}$, $\gamma=1\frac{3}{4}$, $\nu=1$, $\eta=\frac{1}{4}, \cdots$ for the planar Ising models one does not know *why* the exponents have these values or *why* they satisfy the exponent relations Eqs. (15) or why the scaling law Eq. (16) is obeyed. Indeed, the seemingly inevitable mathematical complexities of solving even such physically oversimplified models exactly[82] serve to conceal almost all traces of general, underlying mechanisms and principles that might "explain" the results. Thus it comes to pass that even a rather crude and approximate solution of a two-dimensional Ising model by a real-space RG method can be truly instructive.[83]

IX. KADANOFF'S SCALING PICTURE

The year from late-1965 through 1966 saw the clear formulation of scaling for the thermodynamic properties in the critical region and the fuller appreciation of scaling for the correlation functions.[84] I have highlighted Widom's (1965) approach since it was the most direct and phenomenological—a bold, new thermodynamic hypothesis was advanced by generalizing a particular feature of the classical theories. But Domb and Hunter (1965) reached essentially the same conclusion for the thermodynamics based on analytic and series-expansion considerations, as did Patashinskii and Pokrovskii (1966)

[80]Here, for simplicity, we suppose the $s_{\mathbf{x}}$ are classical, commuting variables. If they are operator-valued then, in the standard way, the trace must be defined as a sum or integral over diagonal matrix elements computed with a complete basis set of N-variable states.

[81]In Eq. (26) we have explicitly indicated the thermodynamic limit in which N and V become infinite maintaining the ratio $V/N = a^d$ fixed: in QFT this corresponds to an infinite system with an ultraviolet lattice cutoff.

[82]See the monograph by Rodney Baxter (1982).

[83]See Niemeijer and van Leeuwen (1976), Burkhardt and van Leeuwen (1982), and Wilson (1975, 1983) for discussion of real-space RG methods.

[84]Although one may recall, in this respect, earlier work (Fisher, 1959, 1962, 1964) restricted (in the application to ferromagnets) to zero magnetic field.

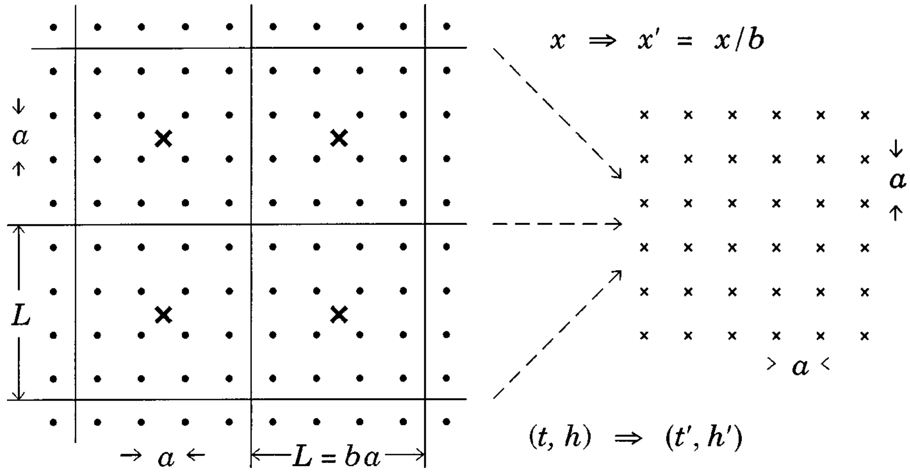

FIG. 3. A lattice of spacing a of Ising spins $s_{\mathbf{x}} = \pm 1$ (in $d=2$ dimensions) marked by solid dots, divided up into Kadanoff blocks or cells of dimensions $(L=ba)\times(L=ba)$ each containing a block spin $s'_{\mathbf{x}'} = \pm 1$, indicated by a cross. After a rescaling, $\mathbf{x} \Rightarrow \mathbf{x}' = \mathbf{x}/b$, the lattice of block spins appears identical with the original lattice. However, one supposes that the temperature t, and magnetic field h, of the original lattice can be renormalized to yield appropriate values, t' and h', for the rescaled, block-spin lattice: see text. In this illustration the spatial rescaling factor is $b = 4$.

using a more microscopic formulation that brought out the relations to the full set of correlation functions (of all orders).[85]

Kadanoff (1966), however, derived scaling by introducing a completely new concept, namely, the *mapping* of a critical or near-critical system onto itself by a reduction in the effective number of degrees of freedom.[86] This paper attracted much favorable notice since, beyond obtaining all the scaling properties, it seemed to lay out a direct route to the actual *calculation* of critical properties. On closer examination, however, the implied program seemed—as I will explain briefly—to run rapidly into insuperable difficulties and interest faded. In retrospect, however, Kadanoff's scaling picture embodied important features eventually seen to be basic to Wilson's conception of the full renormalization group. Accordingly, it is appropriate to present a sketch of Kadanoff's seminal ideas.

For simplicity, consider with Kadanoff (1966), a lattice of spacing a (and dimensionality $d>1$) with $S=\frac{1}{2}$ Ising spins $s_{\mathbf{x}}$ which, by definition, take only the values $+1$ or -1: see Fig. 3. Spins on nearest-neighbor sites are coupled by an energy parameter or coupling constant, $J>0$, which favors parallel alignment [see, e.g., Eq. (23) above]. Thus at low temperatures the majority of the spins point "up" ($s_{\mathbf{x}}=+1$) or, alternatively, "down" ($s_{\mathbf{x}}=-1$); in other words, there will be a spontaneous magnetization, $M_0(T)$, which decreases when T rises until it vanishes at the critical temperature $T_c>0$: recall (11).

Now divide the lattice up into (disjoint) blocks, of dimensions $L\times L\times\cdots\times L$ with $L=ba$ so that each block contains b^d spins: see Fig. 3. Then associate with each block, $\mathcal{B}_{\mathbf{x}'}$ centered at point $\mathbf{x}'$, a new, effective *block spin*, $s'_{\mathbf{x}'}$. If, finally, we *rescale* all spatial coordinates according to

$$\mathbf{x} \Rightarrow \mathbf{x}' = \mathbf{x}/b, \tag{27}$$

the new lattice of block spins $s'_{\mathbf{x}'}$ looks just like the original lattice of spins $s_{\mathbf{x}}$. Note, in particular, the density of degrees of freedom is unchanged: see Fig. 3.

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[85]It was later seen (Kiang and Stauffer, 1970; Fisher, 1971, Sec. 4.4) that thermodynamic scaling with general exponents (but particular forms of scaling function) was embodied in the "droplet model" partition function advanced by Essam and Fisher (1963) from which the exponent relations $\alpha'+2\beta+\gamma'=2$, etc., were originally derived. (See Eq. (15), Footnote 49, and Fisher, 1967b, Sec. 9.1; 1971, Sec. 4.)

[86]Novelty is always relative! From a historical perspective one should recall a suggestive contribution by M. J. Buckingham, presented in April 1965, in which he proposed a division of a lattice system into cells of geometrically increasing size, $L_n = b^n L_0$, with controlled intercell couplings. This led him to propose "the *existence* of an asymptotic 'lattice problem' such that the description of the nth order in terms of the $(n-1)$th is the same as that of the $(n+1)$th in terms of the nth." This is practically a description of "scaling" or "self similarity" as we recognize it today. Unfortunately, however, Buckingham failed to draw any significant, correct conclusions from his conception and his paper seemed to have little influence despite its presentation at the notable international conference on *Phenomena in the Neighborhood of Critical Points* organized by M. S. Green (with G. B. Benedek, E. W. Montroll, C. J. Pings, and the author) and held at the National Bureau of Standards, then in Washington, D.C. The Proceedings, complete with discussion remarks, were published, in December 1966, under the editorship of Green and J. V. Sengers (1966). Nearly all the presentations addressed the rapidly accumulating experimental evidence, but many well known theorists from a range of disciplines attended including P. W. Anderson, P. Debye, C. de Dominicis, C. Domb, S. F. Edwards, P. C. Hohenberg, K. Kawasaki, J. S. Langer, E. Lieb, W. Marshall, P. C. Martin, T. Matsubara, E. W. Montroll, O. K. Rice, J. S. Rowlinson, G. S. Rushbrooke, L. Tisza, G. E. Uhlenbeck, and C. N. Yang; but B. Widom, L. P. Kadanoff, and K. G. Wilson are *not* listed among the participants.

But if this appearance is to be more than superficial one must be able to relate the new or "renormalized" coupling J' between the block spins to the original coupling J, or, equivalently, the renormalized temperature deviation t' to the original value t. Likewise one must relate the new, renormalized magnetic field h' to the original field h.

To this end, Kadanoff supposes that b is large but less than the ratio, ξ/a, of the *correlation length*, $\xi(t,h)$, to the lattice spacing a; since ξ diverges at criticality—see Eq. (13)—this allows, asymptotically, for b to be chosen *arbitrarily*. Then Kadanoff notes that the total coupling of the magnetic field h to a block of b^d spins is equivalent to a coupling to the average spin

$$\overline{s_{\mathbf{x}'}} \equiv b^{-d} \sum_{\mathbf{x}\in\mathcal{B}_{\mathbf{x}'}} s_{\mathbf{x}} \cong \zeta(b) s'_{\mathbf{x}'}, \tag{28}$$

where the sum runs over all the sites $\mathbf{x}$ in the block $\mathcal{B}_{\mathbf{x}'}$, while the "asymptotic equivalence" to the new, Ising block spin $s'_{\mathbf{x}'}$ is, Kadanoff proposes, determined by some "spin rescaling or renormalization factor" $\zeta(b)$. Introducing a similar thermal renormalization factor, $\vartheta(b)$, leads to the *recursion relations*

$$t' \approx \vartheta(b) t \quad \text{and} \quad h' \approx \zeta(b) h. \tag{29}$$

Correspondingly, the basic correlation function—compare with Eqs. (1), (4), and (16)—should renormalize as

$$G(\mathbf{x};\, t,\, h) \equiv \langle s_{\mathbf{0}}\, s_{\mathbf{x}} \rangle \approx \zeta^2(b) G(\mathbf{x}';\, t', h'). \tag{30}$$

In summary, under a spatial scale transformation and the integration out of all but a fraction b^{-d} of the original spins, the system asymptotically *maps back into itself* although at a renormalized temperature and field! However, the map is *complete* in the sense that *all* the statistical properties should be related by similarity.

But how should one choose—or, better, determine—the renormalization factors ζ and ϑ? Let us consider the basic relation Eq. (30) *at* criticality, so that $t=h=0$ and, by Eq. (29), $t'=h'=0$. Then, if we accept the observation/expectation Eq. (2) of a power law decay, i.e., $G_c(\mathbf{x}) \sim 1/|\mathbf{x}|^{d-2+\eta}$ one soon finds that $\zeta(b)$ must be just a power of b. It is natural, following Kadanoff (1966), then to propose the forms

$$\zeta(b) = b^{-\omega} \quad \text{and} \quad \vartheta(b) = b^{\lambda}, \tag{31}$$

where the two exponents ω and λ characterize the critical point under study while b is an essentially unrestricted *scaling parameter*.

By capitalizing on the freedom to choose b as $t,\, h \to 0$, or, more-or-less equivalently, by *iterating* the recursion relations Eqs. (29) and (30), one can, with some further work, show that all the previous scaling laws hold, specifically, Eqs. (15), (16), and (19) although with $g \equiv 0$. Of course, all the exponents are now determined by ω and λ: for example, one finds $\nu = 1/\lambda$ and $\beta = \omega\nu$. Beyond that, the analysis leads to new exponent relations, namely, the so-called *hyperscaling laws*[87] which explicitly involve the spatial dimensionality: most notable is[88]

$$d\nu = 2 - \alpha. \tag{32}$$

But then Kadanoff's scaling picture is greatly strengthened by the fact that this relation holds *exactly* for the $d=2$ Ising model! And also for all other exactly soluble models when $d<4$.[89]

Historically, the careful numerical studies of the $d=3$ Ising models by series expansions[90] for many years suggested a small but significant deviation from Eq. (32) as allowed by pure scaling phenomenolgy.[91] But, in recent years, the accumulating weight of evidence critically reviewed has convinced even the most cautious skeptics![92]

Nevertheless, all is not roses! Unlike the previous exponent relations (all being independent of d) hyperscaling fails for the classical theories unless $d=4$. And since one knows (rigorously for certain models) that the classical exponent values are valid for $d>4$, it follows that hyperscaling cannot be generally valid. Thus something is certainly missing from Kadanoff's picture. Now, thanks to RG insights, we know that the breakdown of hyperscaling is to be understood via the second argument in the "fuller" scaling form Eq. (19): when d exceeds the appropriate borderline dimension, $d_\times$, a "dangerous irrelevant variable" appears and must be allowed for.[93] In essence one finds that the scaling function limit $\mathcal{F}(y,\, z \to 0)$, previously accepted without question, is no longer well defined but, rather, diverges as a power of z: asymptotic scaling survives but $d^* \equiv (2-\alpha)/\nu$ sticks at the value 4 for $d>d_\times=4$.

However, the issue of hyperscaling was *not* the main road block to the analytic development of Kadanoff's picture. The principal difficulties arose in explaining the *power-law* nature of the rescaling factors in Eqs. (29)–(31) and, in particular, in justifying the idea of a *single*, effective, renormalized coupling J' between adjacent block spins, say $s'_{\mathbf{x}'}$ and $s'_{\mathbf{x}'+\boldsymbol{\delta}'}$. Thus the interface between two adjacent $L\times L\times L$ blocks (taking $d=3$ as an

[87]See (Fisher, 1974a) where the special character of the hyperscaling relations is stressed.

[88]See Kadanoff (1966), Widom (1965a), and Stell (1965, unpublished, quoted in Fisher, 1969, and 1968).

[89]See, e.g., Fisher (1983) and, for the details of the exactly solved models, Baxter (1982).

[90]For accounts of series expansion techniques and their important role see: Domb (1960, 1996), Baker (1961, 1990), Essam and Fisher (1963), Fisher (1965, 1967b), and Stanley (1971).

[91]As expounded systematically in (Fisher, 1974a) with hindsight enlightened by RG theory.

[92]See Fisher and Chen (1985) and Baker and Kawashima (1995, 1996).

[93]See Fisher in (Gunton and Green, 1974, p. 66) where a "dangerous irrelevant variable" is characterized as a "hidden relevant variable;" and (Fisher, 1983, App. D).

example) separates two block faces each containing b^2 strongly interacting, original lattice spins $s_{\mathbf{x}}$. Well below T_c all these spins are frozen, ''up'' or ''down,'' and a single effective coupling could well suffice; but at and above T_c these spins must fluctuate on many scales and a single effective-spin coupling seems inadequate to represent the inherent complexities.[94]

One may note, also that Kadanoff's picture, like the scaling hypothesis itself, provides no real hints as to the origins of universality: the rescaling exponents ω and λ in Eq. (31) might well change from one system to another. Wilson's (1971a) conception of the renormalization group answered both the problem of the ''lost microscopic details'' of the original spin lattice and provided a natural explanation of universality.

X. WILSON'S QUEST

Now because this account has a historical perspective, and since I was Ken Wilson's colleague at Cornell for some twenty years, I will say something about how his search for a deeper understanding of quantum field theory led him to formulate renormalization group theory as we know it today. The first remark to make is that Ken Wilson is a markedly independent and original thinker and a rather private and reserved person. Secondly, in his 1975 article, in Reviews of Modern Physics, from which I have already quoted, Ken Wilson gave his considered overview of RG theory which, in my judgement, still stands well today. In 1982 he received the Nobel Prize and in his Nobel lecture, published in 1983, he devotes a section to ''Some History Prior to 1971'' in which he recounts his personal scientific odyssey.

He explains that as a student at Caltech in 1956–60, he failed to avoid ''the default for the most promising graduate students [which] was to enter elementary-particle theory.'' There he learned of the 1954 paper by Gell-Mann and Low ''which was the principal inspiration for [his] own work prior to Kadanoff's (1966) formulation of the scaling hypothesis.'' By 1963 Ken Wilson had resolved to pursue quantum field theories as applied to the strong interactions. Prior to summer 1966 he heard Ben Widom present his scaling equation of state in a seminar at Cornell ''but was puzzled by the absence of any theoretical basis for the form Widom wrote down.'' Later, in summer 1966, on studying Onsager's solution of the Ising model in the reformulation of Lieb, Schultz, and Mattis,[95] Wilson became aware of analogies with field theory and realized the applicability of his own earlier RG ideas (developed for a truncated version of fixed-source meson theory[96]) to critical phenomena. This gave him a scaling picture but he discovered that he ''had been scooped by Leo Kadanoff.'' Thereafter Ken Wilson amalgamated his thinking about field theories on a lattice and critical phenomena learning, in particular, about Euclidean QFT[97] and its close relation to the transfer matrix method in statistical mechanics—the basis of Onsager's (1944) solution.

That same summer of 1966 I joined Ben Widom at Cornell and we jointly ran an open and rather wide-ranging seminar loosely centered on statistical mechanics. Needless to say, the understanding of critical phenomena and of the then new scaling theories was a topic of much interest. Ken Wilson frequently attended and, perhaps partially through that route, soon learned a lot about critical phenomena. He was, in particular, interested in the series expansion and extrapolation methods for estimating critical temperatures, exponents, amplitudes, etc., for lattice models that had been pioneered by Cyril Domb and the King's College, London group.[98] This approach is, incidentally, still one of the most reliable and precise routes available for estimating critical parameters. At that time I, myself, was completing a paper on work with a London University student, Robert J. Burford, using high-temperature series expansions to study in detail the correlation functions and scattering behavior of the two- and three-dimensional Ising models.[99] Our theoretical analysis had already brought out some of the analogies with field theory revealed by the transfer matrix approach. Ken himself undertook large-scale series expansion calculations in order to learn and understand the techniques. Indeed, relying on the powerful computer programs Ken Wilson developed and kindly made available to us, another one of my students, Howard B. Tarko, extended the series analysis of the Ising correlations functions to temperatures below T_c and to all values of the magnetic field.[100] Our results have lasted rather well and many of them are only recently being revised and improved.[101]

Typically, then, Ken Wilson's approach was always ''hands on'' and his great expertise with computers was ever at hand to check his ideas and focus his thinking.[102]

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[94]In hindsight, we know this difficulty is profound: in general, it is *impossible* to find an adequate single coupling. However, for certain special models it does prove possible and Kadanoff's picture goes through: see Nelson and Fisher (1975) and (Fisher, 1983). Further, in defense of Kadanoff, the condition $b \ll \xi/a$ was supposed to ''freeze'' the original spins in each block sufficiently well to justify their replacement by a simple block spin.

[95]See Schultz *et al.* (1964).

[96]See Wilson (1983).

[97]As stressed by Symanzik (1966) the Euclidean formulation of quantum field theory makes more transparent the connections to statistical mechanics. Note, however, that in his 1966 article Symanzik did not delineate the special connections to critical phenomena *per se* that were gaining increasingly wide recognition; see, e.g., Patashinskii and Pokrovskii (1966), Fisher (1969, Sec. 12) and the remarks below concerning Fisher and Burford (1967).

[98]See the reviews Domb (1960), Fisher (1965, 1967b), Stanley (1971).

[99]Fisher and Burford (1967).

[100]Tarko and Fisher (1975).

[101]See Zinn and Fisher (1996), Zinn, Lai, and Fisher (1996), and references therein.

[102]See his remarks in Wilson (1983) on page 591, column 1.

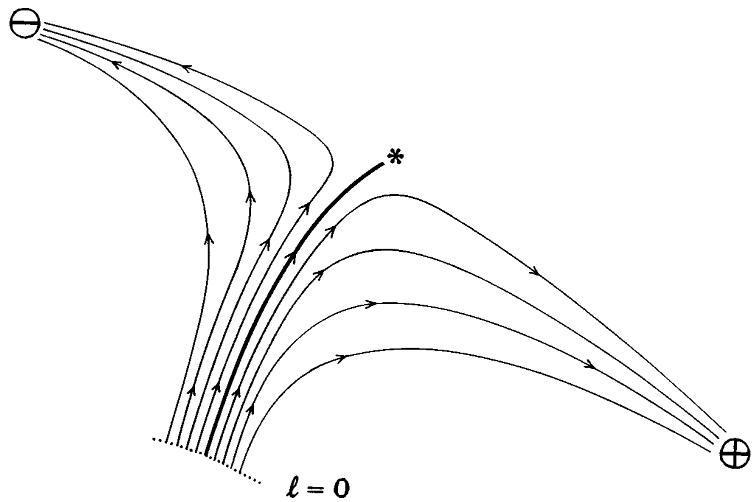

FIG. 4. A "vision" of flows in some large space inspired by a seminar of K. G. Wilson in the period 1967–1970. The idea conveyed is that initially close, smoothly connected points at the start of the flow—the locus $l=0$ — can eventually separate and run to far distant regions representing very different "final" physical states: the essence of a phase transition. In modern terms the flow is in the space $\mathbb{H}$ of Hamiltonians; the intersection of the separatrix, shown bolder, with the initial locus ($l=0$) represents the physical critical point; $\ast$ denotes the controlling fixed point, while $\oplus$ and $\ominus$, represent asymptotic high-T, disordered states and low-T, ordered states, respectively.

From time to time Ken would intimate to Ben Widom or myself that he might be ready to tell us where his thinking about the central problem of explaining scaling had got to. Of course, we were eager to hear him speak at our seminar although his talks were frequently hard to grasp. From one of his earlier talks and the discussion afterwards, however, I carried away a powerful and vivid picture of *flows*—flows in a large space. And the point was that at the initiation of the flow, when the "time" or "flow parameter" l, was small, two nearby points would travel close together; see Fig. 4. But as the flow developed a point could be reached—a bifurcation point (and hence, as one later realized, a stationary or fixed point of the flow)—beyond which the two originally close points could separate and, as l increased, diverge to vastly different destinations: see Fig. 4. At the time, I vaguely understood this as indicative of how a sharp, nonanalytic phase transition could grow from smooth analytic initial data.[103]

But it was a long time before I understood the nature of the space—the space $\mathbb{H}$ of Hamiltonians—and the *mechanism* generating the flow, that is, a renormalization group transformation. Nowadays, when one looks at Fig. 4, one sees the locus of initial points, $l=0$, as identifying the manifold corresponding to the original or 'bare' Hamiltonian (see Fig. 2) while the trajectory leading to the bifurcation point represents a locus of critical points; the two distinct destinations for $l\to\infty$ then typically, correspond to a high-temperature, fully disordered system and to a low-temperature fully ordered system: see Fig. 4.

In 1969 word reached Cornell that two Italian theorists, C. Di Castro and G. Jona-Lasinio, were claiming[104] that the "multiplicative renormalization group," as expounded in the field-theory text by Bogoliubov and Shirkov (1959), could provide "a microscopic foundation" for the scaling laws (which, by then, were well established phenomenologically). The formalism and content of the field-theoretic renormalization group was totally unfamiliar to most critical-phenomena theorists: but the prospect of a microscopic derivation was clearly exciting! However, the articles[105] proved hard to interpret as regards concrete progress and results. Nevertheless, the impression is sometimes conveyed that Wilson's final breakthrough was somehow anticipated by Di Castro and Jona-Lasinio.[106]

Such an impression would, I believe, be quite misleading. Indeed, Di Castro was invited to visit Cornell where he presented his ideas in a seminar that was listened to attentively. Again I have a vivid memory: walking to lunch at the Statler Inn after the seminar I checked my own impressions with Ken Wilson by asking: "Well, did he really say anything new?" (By "new" I meant some fresh insight or technique that carried the field forward.) The conclusion of our conversation was "No". The point was simply that none of the problems then outstanding—see the "tasks" outlined above (in Section VIII)—had been solved or come under effective attack. In fairness, I must point out that the retrospective re-

[103]See the (later) introductory remarks in Wilson (1971a) related to Fig. 1 there.

[104]The first published article was Di Castro and Jona-Lasinio (1969).

[105]See the later review by Di Castro and Jona-Lasinio (1976) for references to their writings in the period 1969–1972 prior to Wilson's 1971 papers and the ϵ-expansion in 1972.

[106]See, for example, Benfatto and Gallavotti (1995) on page 96 in *A Brief Historical Note*, which is claimed to represent only the authors' personal "cultural evolution through the subject."

view by Di Castro and Jona-Lasinio themselves (1976) is reasonably well balanced: One accepted a scaling hypothesis and injected that as an ansatz into a general formalism; then certain insights and interesting features emerged; but, in reality, only scaling theory had been performed; and, in the end, as Di Castro and Jona-Lasinio say: "Still one did not see how to perform explicit calculations." Incidentally, it is also interesting to note Wilson's sharp criticism[107] of the account presented by Bogoliubov and Shirkov (1959) of the original RG ideas of Stueckelberg and Petermann (who, in 1953, coined the phrase "groupes de normalization") and of Gell-Mann and Low (1954).

One more personal anecdote may be permissible here. In August 1973 I was invited to present a tutorial seminar on renormalization group theory while visiting the Aspen Center for Physics. Ken Wilson's thesis advisor, Murray Gell-Mann, was in the audience. In the discussion period after the seminar Gell-Mann expressed his appreciation for the theoretical structure created by his famous student that I had set out in its generality, and he asked: "But tell me, what has all that got to do with the work Francis Low and I did so many years ago?"[108] In response, I explained the connecting thread and the far-reaching intellectual inspiration: certainly there is a thread but—to echo my previous comments—I believe that its length is comparable to that reaching from Maxwell, Boltzmann, and ideal gases to Gibbs' general conception of ensembles, partition functions, and their manifold inter-relations.

XI. THE CONSTRUCTION OF RENORMALIZATION GROUP TRANSFORMATIONS: THE EPSILON EXPANSION

In telling my story I have purposefully incorporated a large dose of hindsight by emphasizing the importance of viewing a particular physical system—or its reduced Hamiltonian, $\bar{\mathcal{H}}(t,h,\cdots)$: see Eq. (24)—as specifying only a relatively small manifold in a large space, $\mathbb{H}$, of possible Hamiltonians. But why is that more than a mere formality? One learns the answer as soon as, following Wilson (1975, 1983), one attempts to implement Kadanoff's scaling description in some concrete, computational way. In Kadanoff's picture (in common with the Gell-Mann-Low, Callan-Symanzik, and general QFT viewpoints) one *assumes* that after a "rescaling" or "renormalization" the new, renormalized Hamiltonian (or, in QFT, the Lagrangean) has the *identical form* except for the renormalization of a single parameter (or coupling constant) or—as in Kadanoff's picture—of at most a small *fixed* number, like the temperature t and field h. That assumption is the dangerous and, unless one is especially lucky,[109] the *generally false* step! Wilson (1975, p. 592) has described his "liberation" from this straight jacket and how the freedom gained opened the door to the systematic design of RG transformations.

To explain, we may state matters as follows: Gibbs' prescription for calculating the partition function—see Eq. (25)—tells us to sum (or to integrate) over the allowed values of *all* the N spin variables $s_{\mathbf{x}}$. But this is very difficult! Let us, instead, adopt a strategy of "divide and conquer," by separating the set $\{s_{\mathbf{x}}\}$ of N spins into two groups: first, $\{s_{\mathbf{x}}^<\}$, consisting of $N'=N/b^d$ spins which we will leave as untouched fluctuating variables; and, second, $\{s_{\mathbf{x}}^>\}$ consisting of the remaining $N-N'$ spin variables over which we will integrate (or sum) so that they drop out of the problem. If we draw inspiration from Kadanoff's (or Buckingham's[110]) block picture we might reasonably choose to integrate over all but one central spin in each block of b^d spins. This process, which Kadanoff has dubbed "decimation" (after the Roman military practice), preserves translational invariance and clearly represents a concrete form of "coarse graining" (which, in earlier days, was typically cited as a way to derive, "in principle," mesoscopic or Landau-Ginzburg descriptions).

Now, after taking our partial trace we must be left with some new, *effective Hamiltonian*, say, $\bar{\mathcal{H}}_{\mathrm{eff}}[s^<]$, involving only the preserved, unintegrated spins. On reflection one realizes that, in order to be faithful to the original physics, such an effective Hamiltonian must be defined via its Boltzmann factor: recalling our brief outline of statistical mechanics, that leads directly to the explicit formula

$$e^{\bar{\mathcal{H}}_{eff}[s^<]}=\mathrm{Tr}_{N-N'}^{s^>}\{e^{\bar{\mathcal{H}}[s^<\cup s^>]}\}, \tag{33}$$

where the 'union', $s^<\cup s^>$, simply stands for the full set of original spins $s\equiv\{s_{\mathbf{x}}\}$. By a spatial rescaling, as in Eq. (27), and a relabelling, namely, $s_{\mathbf{x}}^<\Rightarrow s'_{\mathbf{x}'}$, we obtain the "renormalized Hamiltonian," $\bar{\mathcal{H}}'[s']\equiv\bar{\mathcal{H}}_{eff}[s^<]$. Formally, then, we have succeeded in defining an *explicit renormalization transformation*. We will write

$$\bar{\mathcal{H}}'[s']=\mathbb{R}_b\{\bar{\mathcal{H}}[s]\}, \tag{34}$$

where we have elected to keep track of the spatial rescaling factor, b, as a subscript on the RG operator $\mathbb{R}$.

Note that if we complete the Gibbsian prescription by taking the trace over the renormalized spins we simply get back to the desired partition function, $Z_N[\bar{\mathcal{H}}]$. (The formal derivation for those who might be interested is set out in the footnote below.[111]) Thus nothing has been lost: the renormalized Hamiltonian retains all the ther-

[107]See, especially, Wilson (1975) on page 796, column 1, and Footnote 10 in Wilson (1971a).

[108]That is, in Gell-Mann and Low (1954).

[109]See Footnote 94 above and Nelson and Fisher (1975) and Fisher (1983).

[110]Recall Footnote 86 above.

[111]We start with the definition Eq. (33) and recall Eq. (25) to obtain

$$Z_{N'}[\bar{\mathcal{H}}']\equiv\mathrm{Tr}_{N'}^{s'}\{e^{\bar{\mathcal{H}}[s']}\}=\mathrm{Tr}_{N'}^{s^<}\{e^{\bar{\mathcal{H}}_{eff}[s^<]}\}$$
$$=\mathrm{Tr}_{N'}^{s^<}\,\mathrm{Tr}_{N-N'}^{s^>}\{e^{\bar{\mathcal{H}}[s^<\cup s^>]}\}=\mathrm{Tr}_{N}^{s}\{e^{\bar{\mathcal{H}}[s]}\}=Z_N[\bar{\mathcal{H}}],$$

from which the free energy $f[\bar{\mathcal{H}}]$ follows via Eq. (26).

modynamic information. On the other hand, experience suggests that, rather than try to compute Z_N directly from $\mathcal{H}'$, it will prove more fruitful to *iterate* the transformation so obtaining a sequence, $\overline{\mathcal{H}}^{(l)}$, of renormalized Hamiltonians, namely,

$$\overline{\mathcal{H}}^{(l)} = \mathbb{R}_b[\overline{\mathcal{H}}^{(l-1)}] = \mathbb{R}_{b^l}[\overline{\mathcal{H}}], \tag{35}$$

with $\overline{\mathcal{H}}^{(0)} \equiv \overline{\mathcal{H}}$, $\overline{\mathcal{H}}^{(1)} = \overline{\mathcal{H}}'$. It is these iterations that give rise to the *semigroup* character of the RG transformation.[112]

But now comes the crux: thanks to the rescaling and relabelling, the microscopic variables $\{s'_{\mathbf{x}'}\}$ are, indeed, completely equivalent to the original spins $\{s_{\mathbf{x}}\}$. However, when one proceeds to determine the nature of $\mathcal{H}_{eff}$, and thence of $\mathcal{H}'$, by using the formula (33), one soon discovers that one *cannot* expect the original form of $\mathcal{H}$ to be reproduced in $\mathcal{H}_{eff}$. Consider, for concreteness, an initial Hamiltonian, $\mathcal{H}$, that describes Ising spins ($s_{\mathbf{x}} = \pm 1$) on a square lattice in zero magnetic field with just nearest-neighbor interactions of coupling strength $K_1 = J_1/k_B T$: in the most conservative Kadanoff picture there must be *some* definite recursion relation for the renormalized coupling, say, $K'_1 = \mathcal{T}_1(K_1)$, embodied in a definite function $\mathcal{T}(\cdot)$. But, in fact, one finds that $\mathcal{H}_{eff}$ must actually contain *further* nonvanishing spin couplings, K_2, between second-neighbor spins, K_3, between third-neighbors, and so on up to *indefinitely* high orders. Worse still, four-spin coupling terms like $K_{\square 1} s_{\mathbf{x}_1} s_{\mathbf{x}_2} s_{\mathbf{x}_3} s_{\mathbf{x}_4}$ appear in $\mathcal{H}_{eff}$, again for *all* possible arrangements of the four spins! And also six-spin couplings, eight-spin couplings, $\cdots$. Indeed, for any given set Q of $2m$ Ising spins on the lattice (and its translational equivalents), a nonvanishing coupling constant, K'_Q, is generated and appears in $\mathcal{H}'$!

The only saving grace is that further iteration of the decimation transformation Eq. (33) cannot (in zero field) lead to anything worse! In other words the space $\mathbb{H}_{\mathrm{Is}}$ of Ising spin Hamiltonians in zero field may be specified by the infinite set $\{K_Q\}$, of all possible spin couplings, and is *closed* under the decimation transformation Eq. (33). Formally, one can thus describe $\mathbb{R}_b$ by the full set of *recursion relations*

$$K'_P = \mathcal{T}_P(\{K_Q\}) \quad (\text{all } P). \tag{36}$$

Clearly, this answers our previous questions as to what becomes of the complicated across-the-faces-of-the-block interactions in the original Kadanoff picture: They actually carry the renormalized Hamiltonian *out* of the (too small) manifold of nearest-neighbor Ising models and introduce (infinitely many) further couplings. The resulting situation is portrayed schematically in Fig. 5: the renormalized manifold for $\mathcal{H}'(t',h')$ generally has no overlap with the original manifold. Further iterations, and *continuous* [see Eq. (40) below] as against discrete RG transformations, are suggested by the flow lines or "trajectories" also shown in Fig. 5. We will return to some of the details of these below.

In practice, the naive decimation transformation specified by Eq. (33) generally fails as a foundation for useful calculations.[113] Indeed, the design of effective RG transformations turns out to be an art more than a science: there is no standard recipe! Nevertheless, there are guidelines: the general philosophy enunciated by Wilson and expounded well, for example, in a recent lecture by Shankar treating fermionic systems,[114] is to attempt to *eliminate* first those microscopic variables or degrees of freedom of "least direct importance" to the macroscopic phenomenon under study, while *retaining* those of most importance. In the case of ferromagnetic or gas-liquid critical points, the phenomena of most significance take place on long length scales—the correlation length, ξ, diverges; the critical point correlations, $G_c(\mathbf{r})$, decay slowly at long-distances; long-range order sets in below T_c.

Thus in his first, breakthrough articles in 1971, Wilson used an ingenious "phase-space cell" decomposition for continuously variable scalar spins (as against ± 1 Ising spins) to treat a lattice Landau-Ginzburg model with a general, single-spin or 'on-site' potential $V(s_{\mathbf{x}})$ acting on each spin, $-\infty < s_{\mathbf{x}} < \infty$. Blocks of cells of the smallest spatial extent were averaged over to obtain a single, renormalized cell of twice the linear size (so that $b=2$). By making sufficiently many simplifying approximations Wilson obtained an explicit *nonlinear, integral recursion relation* that transformed the l-times renormalized potential, $V^{(l)}(\cdot)$, into $V^{(l+1)}(\cdot)$. This recursion relation could be handled by computer and led to a *specific numerical estimate* for the exponent ν for $d=3$ dimensions that was *quite different* from the classical value $\frac{1}{2}$ (and from the results of any previously soluble models like the spherical model[115]). On seeing that result, I knew that a major barrier to progress had been overcome!

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[112]Thus successive decimations with scaling factors b_1 and b_2 yield the quite general relation

$$\mathbb{R}_{b_2}\mathbb{R}_{b_1} = \mathbb{R}_{b_2 b_1},$$

which essentially defines a unitary *semigroup* of transformations. See Footnotes 3 and 78 above, and the formal algebraic definition in MacLane and Birkhoff (1967): a unitary semigroup (or 'monoid') is a set M of elements, $u, v, w, x, \cdots$ with a binary operation, $xy = w \in M$, which is associative, so $v(wx) = (vw)x$, and has a unit u, obeying $ux = xu = x$ (for all $x \in M$)—in RG theory, the unit transformation corresponds simply to $b=1$. Hille (1948) and Riesz and Sz.-Nagy (1955) describe semigroups within a continuum, functional analysis context and discuss the existence of an infinitesimal generator when the flow parameter l is defined for continuous values $l \geqslant 0$: see Eq. (40) below and Wilson's (1971a) introductory discussion.

[113]See Kadanoff and Niemeijer in Gunton and Green (1974), Niemeijer and van Leeuwen (1976), Fisher (1983).

[114]See R. Shankar in Cao (1998) and Shankar (1994).

[115]For accounts of the critical behavior of the spherical model, see Fisher (1966a), where long-range forces were also considered, and, e.g., Stanley (1971), Baxter (1982), and Fisher (1983).

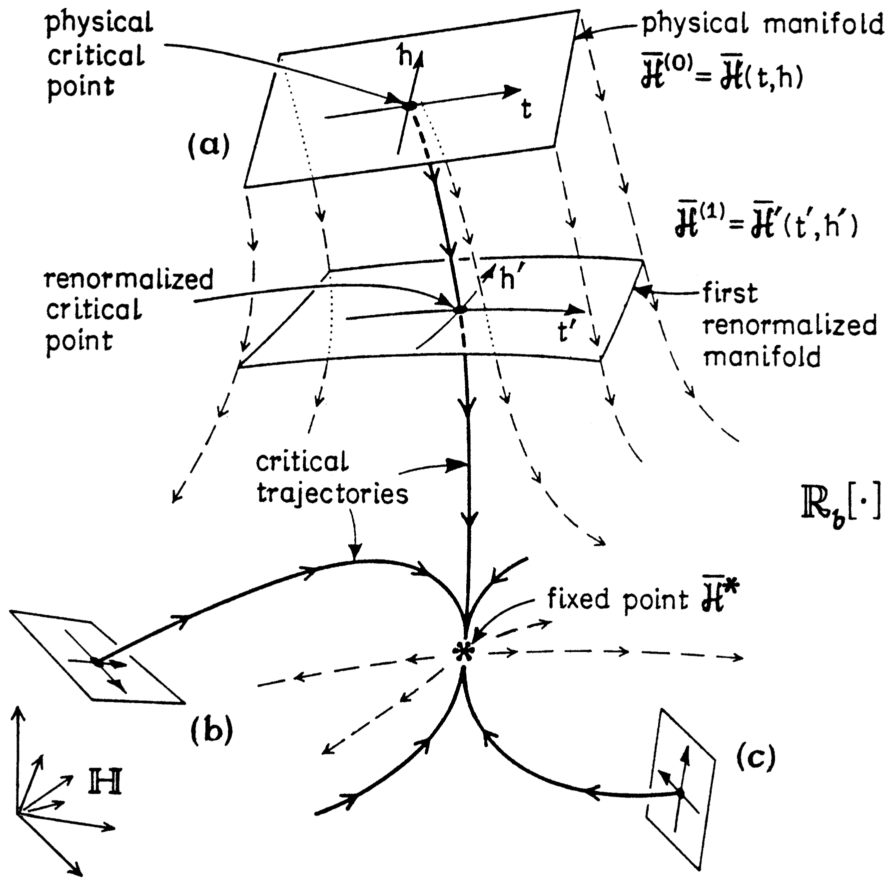

FIG. 5. A depiction of the space of Hamiltonians $\mathbb{H}$ — compare with Fig. 2—showing initial or physical manifolds [labelled (a), (b), $\cdots$, as in Fig. 2] and the flows induced by repeated application of a discrete RG transformation $\mathbb{R}_b$ with a spatial rescaling factor b (or induced by a corresponding continuous or differential RG). Critical trajectories are shown bold: they all terminate, in the region of $\mathbb{H}$ shown here, at a fixed point $\bar{\mathcal{H}}^*$. The full space contains, in general, other nontrivial, critical fixed points, describing multicritical points and distinct critical-point universality classes; in addition, trivial fixed points, including high-temperature "sinks" with no outflowing or relevant trajectories, typically appear. *Lines of fixed points* and other more complex structures may arise and, indeed, play a crucial role in certain problems. [After Fisher (1983).]

I returned from a year's sabbatic leave at Stanford University in the summer of 1971, by which time Ken Wilson's two basic papers were in print. Shortly afterwards, in September, again while walking to lunch as I recall, Ken Wilson discussed his latest results from the nonlinear recursion relation with me. Analytical expressions could be obtained by expanding $V^{(l)}(s)$ in a power series:

$$V^{(l)}(s) = r_l\, s^2 + u_l\, s^4 + v_l\, s^6 + \cdots . \tag{37}$$

If truncated at quadratic order one had a soluble model—the Gaussian model (or free-field theory)—and the recursion relation certainly worked *exactly* for that! But to have a nontrivial model, one had to start not only with r_0 (as, essentially, the temperature variable) but, as a minimum, one also had to include $u_0 > 0$: the model then corresponded to the well known $\lambda\varphi^4$ field theory. Although one might, thus, initially set $v_0 = w_0 = \cdots = 0$, all these higher order terms were immediately generated under renormalization; furthermore, there was no reason for u_0 to be small and, for this reason and others, the standard field-theoretic perturbation theories were ineffective.

Now, I had had a long-standing interest in the effects of the *spatial dimensionality* d on singular behavior in various contexts:[116] so that issue was raised for Ken's recursion relation. Indeed, d appeared simply as an explicit parameter. It then became clear that $d=4$ was a special case in which the leading order corrections to the Gaussian model vanished. Furthermore, above $d=4$ dimensions classical behavior reappeared in a natural way (since the parameters u_0, v_0, ... all then became irrelevant). These facts fitted in nicely with the known special role of $d=4$ in other situations.[117]

For $d=3$, however, one seemed to need the infinite set of coefficients in Eq. (37) which all coupled together

[116]Fisher and Gaunt (1964), Fisher (1966a, 1966b; 1967c; 1972).

[117]See references in the previous footnote and Larkin and Khmel'nitskii (1969), especially Appendix 2.

under renormalization. But I suggested that maybe one could treat the dimensional deviation, $\epsilon = 4-d$, as a small, *nonintegral* parameter in analyzing the recursion relations for $d<4$. Ken soon showed this was effective! Furthermore, the recursion relations proved to be *exact* to leading order in ϵ (so that if one replaced $b=2$ by a general value, the expected universal results were indeed, independent of b). A paper, entitled by Ken, "Critical Exponents in 3.99 Dimensions" was shortly written, submitted, and published:[118] it contained the first general formula for a nonclassical exponent, namely, $\gamma = 1 + \frac{1}{6}\epsilon + O(\epsilon^2)$.

It transpired, however, that the perturbation parameter ϵ provided more—namely, a systematic way of ordering the infinite set of discrete recursion relations not only for the expansion coefficients of $V^{(l)}(s)$ but also for further terms in the appropriate full space $\mathbb{H}$, involving spatial gradients or, equivalently but more usefully, the momenta or wave vectors $\mathbf{q}_i$ labelling the spin variables $\hat{s}_{\mathbf{q}}$, now re-expressed in Fourier space. With that facility in hand, the previous approximations entailed in the phase-space cell analysis could be dispensed with. Wilson then saw that he could precisely implement his *momentum-shell renormalization group*[119]—subsequently one of the most-exploited tools in critical phenomena studies!

In essence this transformation is like decimation[120] except that the division of the variables in Eq. (33) is made in momentum space: for ferromagnetic or gas-liquid-type critical points the set $\{\hat{s}_q^<\}$ contains those 'long-wavelength' or 'low-momentum' variables satisfying $|\mathbf{q}| \leq q_\Lambda/b$, where $q_\Lambda = \pi/a$ is the (ultraviolet) momentum cutoff implied by the lattice structure. Conversely, the 'short-wavelength', 'high-momentum' spin components $\{\hat{s}_{\mathbf{q}}^>\}$ having wave vectors lying in the momentum-space *shell*: $q_\Lambda/b < |\mathbf{q}| \leq q_\Lambda$, are integrated out. The spatial rescaling now takes the form

$$\mathbf{q} \Rightarrow \mathbf{q}' = b\mathbf{q}, \tag{38}$$

as follows from Eq. (27); but in analogy to $\zeta(b)$ in Eq. (28), a *nontrivial spin rescaling factor* ("multiplicative-wave function renormalization" in QFT) is introduced via

$$\hat{s}_{\mathbf{q}} \Rightarrow \hat{s}'_{\mathbf{q}'} = \hat{s}_{\mathbf{q}}/\hat{c}[b,\overline{\mathcal{H}}]. \tag{39}$$

The crucially important rescaling factor $\hat{c}$ takes the form $b^{d-\omega}$ and must be *tuned* in the critical region of interest [which leads to $\omega = \frac{1}{2}(d-2+\eta)$: compare with Eq. (4)]. It is also worth mentioning that by letting $b \to 1+$, one can derive a *differential* or continuous flow RG and rewrite the recursion relation Eq. (34) as[121]

$$\frac{d}{dl}\overline{\mathcal{H}} = \mathbb{B}[\overline{\mathcal{H}}]. \tag{40}$$

Such continuous flows are illustrated in Figs. 4 and 5. (If it happens that $\mathcal{H}$ can be represented, in general only approximately, by a single coupling constant, say, g, then $\mathbb{B}$ reduces to the so-called beta-function $\beta(g)$ of QFT.)

For deriving ϵ expansions on the basis of the momentum shell RG, Feynman-graph perturbative techniques as developed for QFT prove very effective.[122] They enter basically because one can take $u_0 = O(\epsilon)$ small and they play a role both in efficiently organizing the calculation and in performing the essential integrals (particularly for systems with simple propagators and vertices).[123] Capitalizing on his field-theoretic expertise, Wilson obtained, in only a few weeks after submitting the first article, *exact expansions* for the exponents ν, γ, and ϕ to order ϵ^2 (and, by scaling, for all other exponents).[124] Furthermore, the anomalous dimension—defined in Eq. (2) at the beginning of our story—was calculated exactly to order ϵ^3: I cannot resist displaying this striking result, namely,

$$\eta = \frac{(n+2)}{2(n+8)^2}\epsilon^2 + \frac{(n+2)}{2(n+8)^2}\left[\frac{6(3n+14)}{(n+8)^2} - \frac{1}{4}\right]\epsilon^3 + O(\epsilon^4), \tag{41}$$

where the symmetry parameter n denotes the number of components of the microscopic spin vectors, $\vec{s}_{\mathbf{x}} \equiv (s_{\mathbf{x}}^\mu)_{\mu=1,\ldots,n}$, so that one has just $n=1$ for Ising

[118] Wilson and Fisher (1972). The first draft was written by Ken Wilson who graciously listed the authors in alphabetical order.

[119] See Wilson and Fisher (1972) Eq. (18) and the related text.

[120] A considerably more general form of RG transformation can be written as

$$\exp(\overline{\mathcal{H}}'[s']) = \mathrm{Tr}_N^s\{\mathcal{R}_{N',N}(s';s)\exp(\overline{\mathcal{H}}[s])\},$$

where the trace is taken over the full set of original spins s. The $N' = N/b^d$ renormalized spins $\{s'\}$ are introduced via the RG kernel $\mathcal{R}_{N',N}(s';s)$ which incorporates spatial and spin rescalings, etc., and which should satisfy a trace condition to ensure the partition-function-preserving property (see Footnote 111) which leads to the crucial free-energy flow equation: see Eq. (43) below. The decimation transformation, the momentum-shell RG, and other transformations can be written in this form.

[121] See Wilson (1971a) and Footnote 112 above: in this form the RG semigroup can typically be extended to an Abelian group (MacLane and Birkhoff, 1967); but as already stressed, this fact plays a negligible role.

[122] See Wilson (1972), Brézin, Wallace, and Wilson (1972), Wilson and Kogut (1974), the reviews Brézin, Le Guillou, and Zinn-Justin (1976), and Wallace (1976), and the texts Amit (1978) and Itzykson and Drouffe (1989).

[123] Nevertheless, many more complex situations arise in condensed matter physics for which the formal application of graphical techniques without an adequate understanding of the appropriate RG structure can lead one seriously astray.

[124] See Wilson (1972) which was received on 1 December 1971 while Wilson and Fisher (1972) carries a receipt date of 11 October 1971.

spins.[125] Over the years these expansions have been extended to order ϵ^5 (and ϵ^6 for η)[126] and many further related expansions have been developed.[127]

XII. FLOWS, FIXED POINTS, UNIVERSALITY AND SCALING

To complete my story—and to fill in a few logical gaps over which we have jumped—I should explain how Wilson's construction of RG transformations in the space $\mathbb{H}$ enables RG theory to accomplish the "tasks" set out above in Sec. VIII. As illustrated in Fig. 5, the recursive application of an RG transformation $\mathbb{R}_b$ induces a *flow* in the space of Hamiltonians, $\mathbb{H}$. Then one observes that "sensible," "reasonable," or, better, "well-designed" RG transformations are *smooth*, so that points in the original physical manifold, $\overline{\mathcal{H}}^{(0)}=\overline{\mathcal{H}}(t,h)$, that are close, say in temperature, remain so in $\overline{\mathcal{H}}^{(1)}\equiv\overline{\mathcal{H}}'$, i.e., under renormalization, and likewise as the flow parameter l increases, in $\overline{\mathcal{H}}^{(l)}$. Notice, incidentally, that since the spatial scale renormalizes via $\mathbf{x}\Rightarrow\mathbf{x}'=b^l\mathbf{x}$ one may regard

$$l=\log_b(|\mathbf{x}'/|\mathbf{x}|), \tag{42}$$

as measuring, logarithmically, the scale on which the system is being described—recall the physical *scale dependence of parameters* discussed in Sec. IV; but note that, in general, the *form* of the Hamiltonian is also changing as the "scale" is changed or l increases. Thus a partially renormalized Hamiltonian can be expected to take on a more-or-less generic, mesoscopic form: Hence it represents an appropriate candidate to give meaning to a Landau-Ginzburg or, now, LGW effective Hamiltonian: recall the discussion of Landau's work in Sec. II.

Thanks to the smoothness of the RG transformation, if one knows the free energy $f_l\equiv f[\overline{\mathcal{H}}^{(l)}]$ at the l-th stage of renormalization, then one knows the *original* free energy $f[\mathcal{H}]$ and its critical behavior: explicitly one has[128]

$$f(t,h,\cdots)\equiv f[\overline{\mathcal{H}}]=b^{-dl}f[\overline{\mathcal{H}}^{(l)}]\equiv b^{-dl}f_l(t^{(l)},h^{(l)},\cdots). \tag{43}$$

Furthermore, the smoothness implies that all the universal critical properties are preserved under renormalization. Similarly one finds[129] that the critical point of $\overline{\mathcal{H}}^{(0)}\equiv\overline{\mathcal{H}}$ maps on to that of $\overline{\mathcal{H}}^{(1)}\equiv\overline{\mathcal{H}}'$, and so on, as illustrated by the bold flow lines in Fig. 5. Thus it is instructive to follow the *critical trajectories* in $\mathbb{H}$, i.e., those RG flow lines that emanate from a physical critical point. In principle, the topology of these trajectories could be enormously complicated and even chaotic: in practice, however, for a well-designed or "apt" RG transformation, one most frequently finds that the critical flows terminate—or, more accurately, come to an asymptotic halt—at a *fixed point* $\overline{\mathcal{H}}^*$, of the RG: see Fig. 5. Such a fixed point is defined, via Eqs. (34) or (40), simply by

$$\mathbb{R}_b[\overline{\mathcal{H}}^*]=\overline{\mathcal{H}}^* \quad \text{or} \quad \mathbb{B}[\overline{\mathcal{H}}^*]=0. \tag{44}$$

One then searches for fixed-point solutions: the role of the fixed-point equation is, indeed, roughly similar to that of Schrödinger's Equation $\mathcal{H}\Psi=E\Psi$, for stationary states Ψ_k of energy E_k in quantum mechanics.

Why are the fixed points so important? Some, in fact, are *not*, being merely *trivial*, corresponding to *no interactions* or to *all spins frozen*, etc. But the *non*trivial fixed points represent critical states; furthermore, the nature of their criticality, and of the free energy in their neighborhood, must, as explained, be *identical* to that of all those distinct Hamiltonians whose critical trajectories converge to the same fixed point! In other words, a particular fixed point *defines* a *universality class* of critical behavior which "governs," or "attracts" all those systems whose critical points eventually map onto it: see Fig. 5.

Here, then we at last have the natural explanation of *universality*: systems of quite different physical character may, nevertheless, belong to the domain of attraction of the *same* fixed point $\overline{\mathcal{H}}^*$ in $\mathbb{H}$. The distinct sets of inflowing trajectories reflect their varying physical content of associated irrelevant variables and the corresponding nonuniversal rates of approach to the asymptotic power laws dicated by $\mathcal{H}^*$: see Eq. (22).

From each critical fixed point, there flow at least two "unstable" or outgoing trajectories. These correspond to one or more *relevant* variables, specifically, for the case illustrated in Fig. 5, to the temperature or thermal field, t, and the magnetic or ordering field, h. See also Fig. 4. If there are further relevant trajectories then, as discussed in Sec. VII, one can expect *crossover* to different critical behavior. In the space $\mathbb{H}$, such trajectories will then typically lead to distinct fixed points describing (in general) completely new universality classes.[130]

[125] See, e.g., Fisher (1967b, 1974b, 1983), Kadanoff *et al.* (1967), Stanley (1971), Aharony (1976), Patashinskii and Pokrovskii (1979).

[126] See Gorishny, Larin, and Tkachov (1984) but note that the $O(\epsilon^5)$ polynomials in n are found accurately but some coefficients are known only within uncertainties.

[127] Recall Footnote 35.

[128] Recall the partition-function-preserving property set out in Footnote 111 above which actually implies the basic relation Eq. (43).

[129] See Wilson (1971a), Wilson and Kogut (1974), and Fisher (1983).

[130] A skeptical reader may ask: "But what if no fixed points are found?" This can well mean, as it has frequently meant in the past, simply that the chosen RG transformation was poorly designed or "not apt." On the other hand, a fixed point represents only the simplest kind of asymptotic flow behavior: other types of asymptotic flow may well be identified and translated into physical terms. Indeed, near certain types of trivial fixed point, such procedures, long ago indicated by Wilson (1971a, Wilson and Kogut, 1974), *must* be implemented: see, e.g., Fisher and Huse (1985).

But what about *power laws* and *scaling*? The answer to this question was already sketched in Sec. VIII; but we will recapitulate here, giving a few more technical details. However, trusting readers or those uninterested in the analysis are urged to *skip to the next section*!

That said, one must start by noting that the smoothness of a well-designed RG transformation means that it can always be expanded locally—to at least some degree—in a Taylor series.[131] It is worth stressing that it is this very property that fails for free energies in a critical region: to regain this ability, the large space of Hamiltonians is crucial. Near a fixed point satisfying Eq. (43) we can, therefore, rather generally expect to be able to *linearize* by writing

$$\mathbb{R}_b[\overline{\mathcal{H}}^* + g\mathcal{Q}] = \overline{\mathcal{H}}^* + g\mathbb{L}_b\mathcal{Q} + o(g) \tag{45}$$

as $g \to 0$, or in differential form,

$$\frac{d}{dl}(\overline{\mathcal{H}}^* + g\mathcal{Q}) = g\mathbb{B}_1\mathcal{Q} + o(g). \tag{46}$$

Now $\mathbb{L}_b$ and $\mathbb{B}_1$ are *linear operators* (albeit acting in a large space $\mathbb{H}$). As such we can seek eigenvalues and corresponding "eigenoperators", say $\mathcal{Q}_k$ (which will be "partial Hamiltonians"). Thus, in parallel to quantum mechanics, we may write

$$\mathbb{L}_b\mathcal{Q}_k = \Lambda_k(b)\mathcal{Q}_k \quad \text{or} \quad \mathbb{B}_1\mathcal{Q}_k = \lambda_k\mathcal{Q}_k, \tag{47}$$

where, in fact, (by the semigroup property) the eigenvalues must be related by $\Lambda_k(b) = b^{\lambda_k}$. As in any such linear problem, knowing the spectrum of eigenvalues and eigenoperators or, at least, its dominant parts, tells one much of what one needs to know. Reasonably, the $\mathcal{Q}_k$ should form a basis for a general expansion

$$\overline{\mathcal{H}} \cong \overline{\mathcal{H}}^* + \sum_{k \geq 1} g_k\mathcal{Q}_k. \tag{48}$$

Physically, the expansion coefficient g_k ($\equiv g_k^{(0)}$) then represents the thermodynamic field[132] conjugate to the "critical operator" $\mathcal{Q}_k$ which, in turn, will often be close to some combination of *local* operators. Indeed, in a characteristic critical-point problem one finds two *relevant operators*, say $\mathcal{Q}_1$ and $\mathcal{Q}_2$ with $\lambda_1, \lambda_2 > 0$. Invariably, one of these operators can, say by its symmetry, be identified with the local energy density, $\mathcal{Q}_1 \cong \mathcal{E}$, so that $g_1 \cong t$ is the thermal field; the second then characterizes the order parameter, $\mathcal{Q}_2 \cong \Psi$ with field $g_2 \cong h$. Under renormalization each g_k varies simply as $g_k^{(l)} \approx b^{\lambda_k l} g_k^{(0)}$.

Finally,[133] one examines the flow equation (43) for the free energy. The essential point is that the degree of renormalization, b^l, can be *chosen* as large as one wishes. When $t \to 0$, i.e., in the critical region which it is our aim to understand, a good choice proves to be $b^l = 1/|t|^{1/\lambda_1}$, which clearly diverges to ∞. One then finds that Eq. (43) leads to the *basic scaling relation* Eq. (19) which we will rewrite here in greater generality as

$$f_s(t, h, \cdots, g_j, \cdots) \approx |t|^{2-\alpha}\mathcal{F}\left(\frac{h}{|t|^{\Delta}}, \cdots, \frac{g_j}{|t|^{\phi_j}}, \cdots\right). \tag{49}$$

This is the essential result: recall, for example, that it leads to the "collapse" of equation-of-state data as described in Sec. VI.

Now, however, the critical exponents can be expressed directly in terms of the RG eigenexponents λ_k (for the fixed point in question). Specifically one finds

$$2-\alpha = \frac{d}{\lambda_1}, \quad \Delta = \frac{\lambda_2}{\lambda_1}, \quad \phi_j = \frac{\lambda_j}{\lambda_1}, \quad \text{and} \quad \nu = \frac{1}{\lambda_1}. \tag{50}$$

Then, as already explained in Secs. VI and VII, the sign of a given ϕ_j and, hence, of the corresponding λ_j determines the *relevance* (for $\lambda_j > 0$), *marginality* (for $\lambda_j = 0$), or *irrelevance* (for $\lambda_j < 0$) of the corresponding critical operator $\mathcal{Q}_j$ (or "perturbation") and of its conjugate field g_j: this field might, but for most values of j will *not*, be under direct experimental control. As explained previously, all exponent relations (15), (20), etc., follow from scaling, while the first and last of the equations (50) yield the *hyperscaling relation* Eq. (32).

When there are no marginal variables and the least negative ϕ_j is larger than unity in magnitude, a simple scaling description will usually work well and the Kadanoff picture almost applies. When there are *no* relevant variables and only one or a few *marginal variables*, field-theoretic perturbative techniques of the Gell-Mann-Low (1954), Callan-Symanzik[134] or so-called "parquet diagram" varieties[135] may well suffice (assuming the dominating fixed point is sufficiently simple to be well understood). There may then be little incentive for specifically invoking general RG theory. This seems, more or less, to be the current situation in QFT and it applies also in certain condensed matter problems.[136]

XIII. CONCLUSIONS

My tale is now told: following Wilson's 1971 papers and the introduction of the ϵ-expansion in 1972 the significance of the renormalization group approach in

[131]See Wilson (1971a), Wilson and Kogut (1974), Fisher (1974b), Wegner (1972, 1976), Kadanoff (1976).

[132]Reduced, as always, by the factor $1/k_BT$: see e.g., Eq. (18).

[133]See references in Footnote 131.

[134]See Wilson (1975), Brézin *et al.* (1976), Amit (1978), Itzykson and Drouffe (1989).

[135]Larkin and Khmel'nitskii (1969).

[136]See, e.g., the case of dipolar Ising-type ferromagnets in $d=3$ dimensions investigated experimentally by Ahlers, Kornblit, and Guggenheim (1975) following theoretical work by Larkin and Khmel'nitskii (1969) and Aharony (see 1976, Sec. 4E).

statistical mechanics was soon widely recognized[137] and exploited by many authors interested in critical and multicritical phenomena and in other problems in the broad area of condensed matter physics, physical chemistry, and beyond. Some of these successes have already been mentioned in order to emphasize, in particular, those features of the full RG theory that are of general significance in the wide range of problems lying beyond the confines of quantum field theory and fundamental high-energy physics. But to review those developments would go beyond the mandate of this Colloquium.[138]

A further issue is the relevance of renormalization group concepts to quantum field theory. I have addressed that only in various peripheral ways. Insofar as I am by no means an expert in quantum field theory, that is not inappropriate; but perhaps one may step back a moment and look at QFT from the general philosophical perspective of understanding complex, interacting systems. Then, I would claim, statistical mechanics is a central science of great intellectual significance—as just one reminder, the concepts of "spin-glasses" and the theoretical and computational methods developed to analyze them (such as "simulated annealing") have proved of interest in physiology for the study of neuronal networks and in operations research for solving hard combinatorial problems. In that view, even if one focuses only on the physical sciences, the land of statistical physics is broad, with many dales, hills, valleys and peaks to explore that are of relevance to the real world and to our ways of thinking about it. Within that land there is an island, surrounded by water: I will not say "by a moat" since, these days, more and broader bridges happily span the waters and communicate with the mainland! That island is devoted to what was "particle physics" and is now "high-energy physics" or, more generally, to the deepest lying and, in that sense, the "most fundamental" aspects of physics. The reigning theory on the island is quantum field theory—the magnificent set of ideas and techniques that inspired the symposium[139] that lead to this Colloquium. Those laboring on the island have built most impressive skyscrapers reaching to the heavens!

Nevertheless, from the global viewpoint of statistical physics—where many degrees of freedom, the ever-present fluctuations, and the diverse spatial and temporal scales pose the central problems—quantum field theory may be regarded as describing a rather special set of statistical mechanical models. As regards applications they have been largely restricted to $d=4$ spatial dimensions [more physically, of course to (3+1) dimensions] although in the last decade *string theory* has dramatically changed that! The practitioners of QFT insist on the preeminence of some pretty special symmetry groups, the Poincaré group, $SU(3)$, and so on, which are not all so "natural" at first sight—even though the role of guage theories as a unifying theme in modeling nature has been particularly impressive. But, of course, we know these special features of QFT are not matters of choice—rather, they are forced on us by our explorations of Nature itself. Indeed, as far as we know presently, there is only one high-energy physics; whereas, by contrast, the ingenuity of chemists, materials scientists, and of Life itself, offers a much broader, multifaceted and varied panorama of systems to explore both conceptually and in the laboratory.

From this global standpoint, renormalization group theory represents a theoretical tool of depth and power. It first flowered luxuriantly in condensed matter physics, especially in the study of critical phenomena. But it is ubiquitous because of its potential for linking physical behavior across disparate scales; its ideas and techniques play a vital role in those cases where the fluctuations on many different physical scales truly interact. But it provides a valuable perspective—through concepts such as 'relevance,' 'marginality' and 'irrelevance,' even when scales are well separated! One can reasonably debate how vital renormalization group concepts are for quan-

[137]Footnote 86 drew attention to the first international conference on critical phenomena organized by Melville S. Green and held in Washington in April 1965. Eight years later, in late May 1973, Mel Green, with an organizing committee of J. D. Gunton, L. P. Kadanoff, K. Kawasaki, K. G. Wilson, and the author, mounted another conference to review the progress in theory in the previous decade. The meeting was held in a Temple University Conference Center in rural Pennsylvania. The proceedings (Gunton and Green, 1974) entitled *Renormalization Group in Critical Phenomena and Quantum Field Theory*, are now mainly of historical interest. The discussions were recorded in full but most papers only in abstract or outline form. Whereas in the 1965 conference the overwhelming number of talks concerned experiments, now only J.M.H. (Anneke) Levelt Sengers and Guenter Ahlers spoke to review experimental findings in the light of theory. Theoretical talks were presented, in order, by P. C. Martin, Wilson, Fisher, Kadanoff, B. I. Halperin, E. Abrahams, Niemeijer (with van Leeuwen), Wegner, Green, Suzuki, Fisher and Wegner (again), E. K. Riedel, D. J. Bergman (with Y. Imry and D. Amit), M. Wortis, Symanzik, Di Castro, Wilson (again), G. Mack, G. Dell-Antonio, J. Zinn-Justin, G. Parisi, E. Brézin, P. C. Hohenberg (with Halperin and S.-K. Ma) and A. Aharony. Sadly, there were no participants from the Soviet Union but others included R. Abe, G. A. Baker, Jr., T. Burkhardt, R. B. Griffiths, T. Lubensky, D. R. Nelson, E. Siggia, H. E. Stanley, D. Stauffer, M. J. Stephen, B. Widom and A. Zee. As the lists of names and participants illustrates, many active young theorists had been attracted to the area, had made significant contributions, and were to make more in subsequent years.

[138]Some reviews already mentioned that illustrate applications are Fisher (1974b), Wilson (1975), Wallace (1976), Aharony (1976), Patashinskii and Pokrovskii (1979), Nelson (1983), and Creswick *et al.* (1992). Beyond these, attention should be drawn to the notable article by Hohenberg and Halperin (1977) that reviews dynamic critical phenomena, and to many articles on further topics in the Domb and Lebowitz series *Phase Transitions and Critical Phenomena*, Vols. 7 and beyond (Academic, London, 1983).

[139]See Cao (1998).

tum field theory itself. Certain aspects of the full theory do seem important because Nature teaches us, and particle physicists have learned, that quantum field theory is, indeed, one of those theories in which the different scales are connected together in nontrivial ways via the intrinsic quantum-mechanical fluctuations. However, in current quantum field theory, only certain facets of renormalization group theory play a pivotal role.[140] High energy physics did not have to be the way it is! But, even if it were quite different, we would still need renormalization group theory in its fullest generality in condensed matter physics and, one suspects, in further scientific endeavors in the future.

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APPENDIX. ASYMPTOTIC BEHAVIOR

In specifying critical behavior (and asymptotic variation more generally) a little more precision than normally used is really called for. Following well-established custom, I use $\simeq$ for "approximately equals" in a rough and ready sense, as in $\pi^2 \simeq 10$. But to express "$f(x)$ varies like x^λ when x is small and positive," i.e., just to specify a critical exponent, I write:

$$f(x) \sim x^\lambda \quad (x \to 0+). \tag{A1}$$

Then the precise implication is

$$\lim_{x\to 0+}[\ln|f(x)|/\ln x] = \lambda; \tag{A2}$$

see Fisher (1967b, Sec. 1.4; 1983, Sec. 2.4). To give more information, specifically a *critical amplitude* like D in Eq. (2), I define $\approx$ as "asymptotically equals" so that

$$f(x) \approx g(x) \tag{A3}$$

as $x \to 0+$ implies

$$\lim_{x\to 0+} f(x)/g(x) = 1. \tag{A4}$$

Thus, for example, one has

$$(1-\cos x) \approx \tfrac{1}{2}x^2 \sim x^2, \tag{A5}$$

when $x \to 0$. See Fisher (1967b, Secs. 6.2, 6.3, and 7.2) but note that in Eqs. (6.2.6)–(6.3.5) the symbol $\simeq$ should read $\approx$; note also De Bruijn's (1958) discussion of $\approx$ in his book *Asymptotic Methods in Analysis*. The AIP and APS "strong recommendation" to use $\approx$ as "approximately equals" is to be, and has been strongly decried![142] It may also be remarked that few physicists, indeed, use $\sim$ in the precise mathematical sense originally introduced by Poincaré in his pioneering analysis of asymptotic series: see, e.g., Jeffreys and Jeffreys (1956) Secs. 17·02, 23·082, and 23·083. De Bruijn and the Jeffreys also, of course, define the $O(\cdot)$ symbol which is frequently misused in the physics literature: thus $f = O(g)$ $(x \to 0)$, should mean $|f| < c|g|$ for some constant c and $|x|$ small enough so that, e.g. $(1-\cos x) = O(x)$ is correct even though less informative than $(1-\cos x) = O(x^2)$.

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The reader is cautioned that this article is not intended as a systematic review of renormalization group theory and its origins. Likewise, this bibliography makes no claims of completeness; however, it includes those contributions of most significance in the personal view of the author. The reviews of critical phenomena and RG theory cited in Footnote 4 above contain many additional references. Further review articles appear in the series *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (later replaced by J. L. Lebowitz) and published by Academic Press, London (1972): some are listed below. Introductory accounts in an informal lecture style are presented in Fisher (1965, 1983).

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[140] It is interesting to look back and read in Gunton and Green (1973) pp. 157–160, Wilson's thoughts in May 1973 regarding the "Field Theoretic Implications of the Renormalization Group" at a point just before the ideas of *asymptotic freedom* became clarified for non-Abelian gauge theory by Gross and Wilczek (1973) and Politzer (1973).

[141] Most recently under Grant CHE 96-14495.

[142] More recently, an internal AIP/APS memorandum (H-3, revised 6/94) states: "*Approximate equality:* $\simeq$ is preferred." Unfortunately, however, the latest published guidelines (e.g., for Rev. Mod. Phys. as revised October 1995) do not recognize this well advised change of policy. [Ed. Note: the third edition of the RMP Style Guide, currently in press, reflects this and other style changes.]

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