Classical monopoles: Newton, NUT space, gravomagnetic lensing, and atomic spectra

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This article reviews the dynamics and observational signatures of particles interacting with monopoles, beginning with a scholium in Newton's *Principia*. The orbits of particles in the field of a gravomagnetic monopole, the gravitational analog of a magnetic monopole, lie on cones; when the cones are slit open and flattened, the orbits are the ellipses and hyperbolas that one would have obtained without the gravomagnetic monopole. The more complex problem of a charged, spinning sphere in the field of a magnetic monopole is then discussed. The quantum-mechanical generalization of this latter problem is that of monopolar hydrogen. Previous work on monopolar hydrogen is reviewed and details of the predicted spectrum are given. Protons around uncharged monopoles have a bound continuum. Around charged ones, electrons have levels *and* decaying resonances, so magnetic monopoles can grow in mass by swallowing both electrons and protons. In general relativity, the spacetime produced by a gravomagnetic monopole is NUT space, named for Newman, Tamborino, and Unti (1963). This space has a nonspherical metric, even though a mass with a gravomagnetic monopole is spherically symmetric. All geodesics in NUT space lie on cones, and this result is used to discuss the gravitational lensing by bodies with gravomagnetic monopoles. [S0034-6861(98)00402-4]

CONTENTS

I. INTRODUCTION

Poincaré (1895) seems to have been the first to discuss the interesting motion of a charged particle in the field of a magnetic monopole. Goddard and Olive (1978), in their fine review of monopoles in gauge-field theories, derive his result in vector form whereas Hautot (1972) discusses the motion about a charged monopole by separation of variables in r,θ,ϕ coordinates. He extends his results to relativistic motion. Schwinger *et al.* (1976) discuss the motion of dyons, charged monopoles in each other's fields. Goldstein (1980), in the second edition of his book, poses an interesting example of motion in the field of monopoles. However, no-one previously has pointed out Newton's interesting insights relating to this problem in his *Principia*. Our interest arose because one of us was asked to review Chandrasekhar's (1995) book on Newton's *Principia* (1686) for *Notes and Records of the Royal Society* (Lynden-Bell, 1996). This led to his reading passages of Cajori's translation of the *Principia* (1934). In his first proposition Newton shows that motion under the influence of a central force will be in a plane and that equal areas will be swept by the radius vector in equal times. In his second proposition he shows that if a radius from a point S to a body sweeps out equal areas in equal times then the force is central. There follows this scholium: "A body may be urged by a centripetal force compounded of several forces; in which case the meaning of the proposition is that the force which results out of all tends to the point S. But if any force acts continually in the direction of lines perpendicular to the *described surface*, this force will make the body to deviate from the plane of its motion; but it will neither augment nor diminish the area of the *described surface* and is therefore to be neglected in the composition of forces."

What does this mean?

The words *described surface* have been translated from a Latin word that carries the extra connotation of a surface described by its edge. We shall take this to be the surface swept out by the radius vector to the body that is now describing the non-coplanar path. A force normal to this surface at the body must be perpendicular to $\mathbf{r}$ and $\mathbf{v}$, which are both within the surface, so Newton is considering extra forces of the form $Nm_0\mathbf{r}\times\mathbf{v}$ where N may depend on $\mathbf{r}$, $\mathbf{v}$, t, etc. We write the equation of motion

$$m_0 d^2\mathbf{r}/dt^2 = -V'(r)\hat{\mathbf{r}} + N\mathbf{L}, \tag{1.1}$$

where $V(r)$ is the potential for the central force, $\hat{\mathbf{r}}$ is the unit radial vector, and

$$\mathbf{L} = m_0\mathbf{r}\times\mathbf{v}. \tag{1.2}$$

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Taking the cross product $\mathbf{r}\times$Eq. (1.1) we have

$$d\mathbf{L}/dt=N\mathbf{r}\times\mathbf{L}, \tag{1.3}$$

from which it follows either geometrically *à la* Newton or by dotting with $\mathbf{L}$ that

$$|\mathbf{L}|=\text{const.} \tag{1.4}$$

Now if φ is the angle measured within the described surface between a fixed half line ending at S and the radius vector,

$$\frac{1}{2}r^2\dot{\varphi}=\frac{1}{2}|\mathbf{L}|/m_0, \tag{1.5}$$

so equal areas are swept out in equal times just as Newton says. To see this angle more precisely it is perhaps worthwhile to work in axes that are continually tilting to keep up with the plane of the motion. In any axes rotating with angular velocity $\mathbf{\Omega}(t)$, the apparent acceleration $\ddot{\mathbf{r}}$ is related to the absolute acceleration $d^2\mathbf{r}/dt^2$ by

$$d^2\mathbf{r}/dt^2=\ddot{\mathbf{r}}+2\mathbf{\Omega}\times\dot{\mathbf{r}}+\dot{\mathbf{\Omega}}\times\mathbf{r}+\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}). \tag{1.6}$$

We shall apply this formula to axes that are always tilting about $\hat{\mathbf{r}}$ such that in these axes the motion appears as planar. Thus putting $\mathbf{\Omega}=\Omega\hat{\mathbf{r}}$ in Eq. (1.6), we find that

$$d^2\mathbf{r}/dt^2=\ddot{\mathbf{r}}+\Omega r^{-1}\mathbf{L}/m_0. \tag{1.7}$$

Inserting this into Eq. (1.1) and choosing

$$\Omega=rN, \tag{1.8}$$

we recover in these axes the equation we would have had in inertial axes had Newton's extra force$\propto N$ been absent, i.e.,

$$m_0\ddot{\mathbf{r}}=-V'\hat{\mathbf{r}}. \tag{1.9}$$

Thus relative to these moving axes $\mathbf{r}\times m_0\dot{\mathbf{r}}=\mathbf{L}$ is constant not only in magnitude but also in direction, and

$$|\mathbf{r}\times\dot{\mathbf{r}}|=r^2\dot{\varphi}=L/m_0, \tag{1.10}$$

where φ is the angle at S between some line fixed in the moving axes and the current radial line (this is of course equal to the earlier angle, since this moving plane is "rolling" on the described surface about the common radius vector).

We now return to the inertial axes in which the direction of $\mathbf{L}$ varies in accord with Eq. (1.3). Dotting Eq. (1.1) with $\mathbf{v}=d\mathbf{r}/dt$, we find that the N term goes out so the energy equation is left unchanged and we have, remembering that L^2 is constant,

$$\frac{m_0}{2}\mathbf{v}^2+V=\frac{m_0}{2}\left[\dot{r}^2+\left(\frac{L}{m_0}\right)^2r^{-2}\right]+V=E. \tag{1.11}$$

Here $\dot{r}$ is the same in fixed or rotating axes, since this r is scalar. Equation (1.11) demonstrates that the radial motion $r(t)$ is precisely that which would have occurred had N been zero. Furthermore Eqs. (1.9) and (1.10) demonstrate that within the tilting axes, or [using Eq. (1.5)] within the described surface, the solution $r(\varphi)$ is precisely the same function that we would have found for the truly planar motion that occurs with N absent. Although this extension of Newton's theorem is not in the *Principia*, it would surprise us if Newton had not seen and understood it. There is interesting historical research to be done here on Newton's surviving manuscripts. We know from Whiteside that this scholium was not in the first draft of Newton's *De Motu Corporum* written in Autumn of 1684 but appears in its revision, which is probably dated to the Spring of 1685.

Although Eqs. (1.5) and (1.11) are sufficient for the solution of the motion within the described surface, we need to find that surface by solving Eq. (1.3) for a complete description of the motion. This is not particularly simple, and to do it we need to prescribe how N depends on $\mathbf{r}$, $\mathbf{v}$, t, etc. However, $d\mathbf{L}/dt$ and $d\hat{\mathbf{r}}/dt$ are always parallel, since both are perpendicular to $\mathbf{r}$ and $\mathbf{L}$. This led us to consider under what circumstances they might be proportional. In particular,

$$d\hat{\mathbf{r}}/dt=-\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{v})/r=-\mathbf{r}\times\mathbf{L}/(r^3m_0), \tag{1.12}$$

so in full generality we have from Eq. (1.3)

$$d\mathbf{L}/dt=-(m_0Nr^3)d\hat{\mathbf{r}}/dt. \tag{1.13}$$

This demonstrates that when $m_0Nr^3=Q_*=\text{const}$ we have a beautifully simple solution to Eq. (1.13), to wit,

$$\mathbf{L}+Q_*\hat{\mathbf{r}}=\mathbf{j}=\text{const.} \tag{1.14}$$

Here $\mathbf{j}$ is the vector constant of integration; notice that Q_* has the same dimensions as L. Since $\mathbf{L}$ and $\hat{\mathbf{r}}$ are perpendicular we deduce, dotting with $\hat{\mathbf{r}}$,

$$\mathbf{j}\cdot\hat{\mathbf{r}}=Q_*, \tag{1.15}$$

which shows that the angle between $\mathbf{j}$ and $\hat{\mathbf{r}}$ is constant so $\hat{\mathbf{r}}$ moves on a cone whose axis is along $\mathbf{j}$. Similarly dotting Eq. (1.14) with $\mathbf{L}$, we find $L^2=\mathbf{j}\cdot\mathbf{L}$, so likewise $\mathbf{L}$ moves on another cone with $\mathbf{j}$ as its axis. If this cone has semi-angle χ, then $L/|\mathbf{j}|=\cos\chi$, but $\hat{\mathbf{r}}$ and $\mathbf{L}$ are orthogonal and by Eq. (1.14) they are coplanar with $\mathbf{j}$, so we may choose [cf. Eq. (1.15)]

$$Q_*/|\mathbf{j}|=\sin\chi. \tag{1.16}$$

Thus the angle between $\hat{\mathbf{r}}$ and $\mathbf{j}$ is $\pi/2-\chi$, as shown in Fig. 1. Notice from Eq. (1.16) that the angle of the cone is determined completely from $|\mathbf{L}|$ and the force constant Q_*. Orbits with larger $|\mathbf{j}|$ have smaller χ, so the angular momentum then moves around a narrow cone and $\hat{\mathbf{r}}$ then moves around a very open one. For $|\mathbf{j}|\gg Q_*$ that cone is almost planar. Figure 1 illustrates two circular orbits moving in opposite senses about the same axis. Notice that the one moving right-handedly about the upward-pointing axis is displaced above the center, sitting like a halo about it, while that moving left-handedly is displaced below the center like an Elizabethan ruff below the head. One might have supposed that for $j\gg Q_*$ these two circular orbits would approach the central plane, but although the cone becomes much flatter and more open the displacement between the direct and retrograde orbits actually increases. For circular orbits at distance a from S we have, for a Newtonian potential, $L^2=GMam_0^2$ and the displacement is

$$2\hat{\mathbf{j}}\cdot\mathbf{r}=a/\sqrt{GMam_0^2Q_*^{-2}+1}\rightarrow m_0^{-1}Q_*\sqrt{a/(GM)}.$$

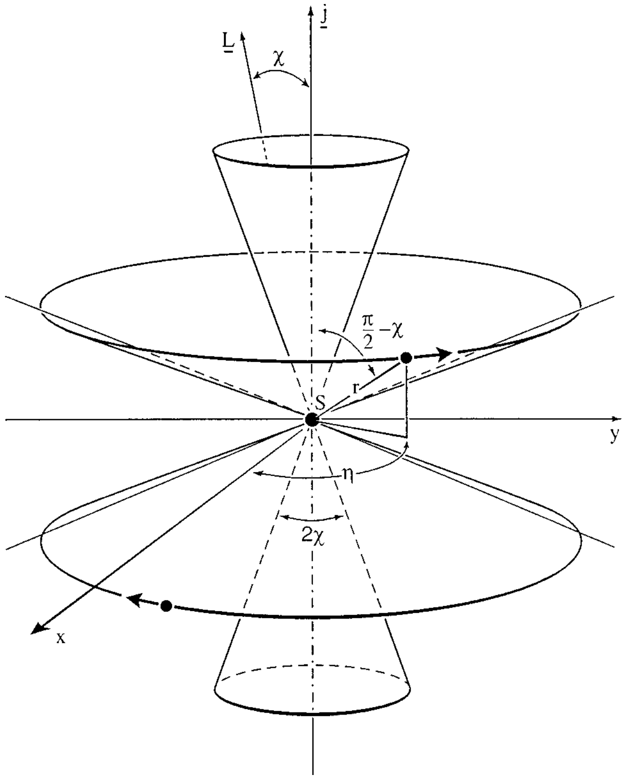

FIG. 1. The circular orbits about a central potential endowed with a monopole. The orbits in opposite senses are displaced above and below the center of force. For a Newtonian potential their vertical separation gradually increases as their radii are increased. The orbits with a given angular momentum $|\mathbf{L}|$ lie on cones with opening angle $\cos^{-1}(Q_*/|\mathbf{j}|)$.

We have been led to the case $m_0 N r^3 = Q_* = \text{const}$ for reasons of mathematical simplicity, but this case is more than a mathematical curiosity because

(1) Of all the forces of Newton's N type [see Eq. (1.1)] only those of the form $-\mathbf{v}\times\hat{\mathbf{r}}r^{-2}Q_*(\theta,\phi)$ derive from a Lagrangian. For a monopole Q_* is constant.

(2) We may rewrite this force in the form

$$N\mathbf{L} = -Q_*\mathbf{v}\times\mathbf{r}/r^3 = \frac{m_0}{c}\,\mathbf{v}\times\mathbf{B}_g\,, \tag{1.17}$$

where

$$\mathbf{B}_g = -Q\hat{\mathbf{r}}/r^2, \quad Q = Q_* c/m_0. \tag{1.18}$$

We have introduced the velocity of light c to make the analogy with magnetic forces even more obvious. $\mathbf{B}_g$ is clearly the field of a magnetic monopole of strength Q, but since this sort of magnetism acts not on moving charges but rather on moving masses, it is a gravomagnetic field. Such fields are well known in general relativity [see Landau and Lifshitz (1966) §89, problem 1]. They are position-dependent Coriolis forces associated with what relativists less helpfully call the dragging of inertial frames. The field $\mathbf{B}_g$ as we have defined it has the same dimensions as $\mathbf{g}$, the acceleration due to gravity, and Q/G has the dimensions of mass. In electricity, like charges repel, while in gravity, like masses attract. It is the same with like magnetic monopoles. They repel while the gravomagnetic monopoles of like sign attract one another; hence the negative sign in Eq. (1.18) is best left there rather than combined into a new definition of the pole strength Q. We may find the Lagrangian corresponding to the force (1.17) by analogy with the electrodynamic case. There we add a term $q\mathbf{v}\cdot\mathbf{A}/c$ where q is the charge and $\mathbf{A}$ is the vector potential. For any poloidal axisymmetric magnetic field one may choose $\mathbf{A}$ to be of the form $A\nabla\phi$ where ϕ is the azimuth around the axis. We require

$$-Q\hat{\mathbf{r}}/r^2 = \mathbf{B}_g = \boldsymbol{\nabla}\times(A\nabla\phi) = \nabla A\times\nabla\phi \tag{1.19}$$

from which one readily finds that $A = Q(1+\cos\theta)$ gives the right $\mathbf{B}_g$. Thus a Lagrangian for Eq. (1.1) is

$$\mathcal{L} = \tfrac{1}{2}m_0 v^2 - m_0 V(r) + Q_*(1+\cos\theta)\mathbf{v}\cdot\nabla\phi. \tag{1.20}$$

Although the dynamic system is spherically symmetric, the Lagrangian is not and cannot be made so. The only spherically symmetric vector fields are $f(r)\mathbf{r}$. If $\mathbf{A}$ were of this form its curl would be zero and therefore could not be the field of a monopole. Of course we can choose any axis we like and measure θ and ϕ appropriately from it. The $\mathbf{A}$ field will then be quite different but it will give the same $\mathbf{B}_g$ field by construction. Thus the difference between any two such $\mathbf{A}$ fields will have zero curl, showing that $\mathbf{A}' = \mathbf{A} + \nabla\chi$, i.e., a gauge transformation. The Lagrangian (1.20) is neither spherically symmetric nor gauge invariant but it is a member of a whole class of equivalent Lagrangians with different axes which are related by gauge transformations. Whereas none of these is individually spherical, the class of all of them is spherically symmetric. The moral is that it can be restrictive to impose symmetry on a single member of the class if the member is not gauge invariant.

So far everything holds for any spherical potential $V(r)$. We could, for example, choose it to be Henon's (1959) isochrone potential $2aV_0/(a+\sqrt{r^2+a^2})$ or its better known limits the simple harmonic oscillator $a \gg r$ or the Newtonian potential $a \ll r$. For all isochrones the orbits can be solved using only trigonometric functions (see, for example, Lynden-Bell, 1963; Evans *et al.*, 1990). Here we shall stick to the Newtonian potential $V/m_0 = -GM/r$. We have already shown that the motion lies on a cone whose semi-angle is given by $\cos^{-1}(Q_*/|\mathbf{j}|)$; furthermore, if we slit that cone along $\varphi = 0$ and flatten it, the orbit will be exactly what it would have been in the absence of N, i.e., a conic section. Of course when we slit and flatten the orbit's cone a gap appears whose angle is $\gamma = 2\pi(1 - L/\sqrt{L^2+Q^2})$; see Fig. 2. An ellipse with focus at S and apocenter at $\varphi = 0$ would get back to apocenter at $\varphi = 2\pi$ but unfortunately the gap intervenes. On the cone we identify $\varphi = 0$ not with $\varphi = 2\pi$ but rather with $\varphi = 2\pi - \gamma$. Thus on the cone the ellipse will precess forwards by an angle γ in each radial period (Fig. 3). This angle γ is an angle like φ measured at S within the cone's surface. It is perhaps more natural to measure angles η around the axis of the cone; these angles are related through $\dot{\eta} = \dot{\varphi}/\cos\chi = L/(m_0 r^2 \cos\chi)$, so $\eta = \varphi\sec\chi = \varphi|\mathbf{j}|/L$.

In these terms the precession per radial period is

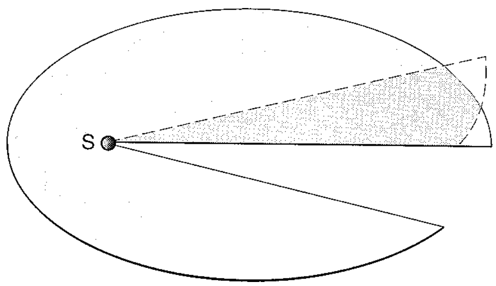

FIG. 2. When one of the cones is slit and flattened a gap opens along the slit. On the cone itself the sides of this gap are identified. Orbits that close on a plane will not close on the cone because of the gap. As a result they precess.

$$\Delta\eta = 2\pi(|\mathbf{j}|/L - 1). \tag{1.21}$$

Newton in his proposition on revolving orbits showed that the addition of an inverse cube force led to an orbit of exactly the same shape, but traced relative to axes that rotate at a rate proportional to $\dot{\phi}$ in the original orbit. It is natural to ask whether such an additional force can stop the precession around the cone of an orbit in the monopolar problem and so yield an orbit that closes on itself in fixed axes. Wonderfully a simple change in $V(r)$ does this, not just for one orbit but for all orbits at once. We thus obtain a new superintegrable system in which all bound orbits close. By analogy with Hamilton's derivation of his eccentricity vector (Hamilton, 1847) we take the cross product of the equation of motion (1.1) with $\mathbf{j} = \mathbf{L} + Q_*\hat{\mathbf{r}}$. On the right-hand side two terms are zero and the remaining two are multiples of $d\hat{\mathbf{r}}/dt$ [cf. Eq. (1.12)] so we find

$$m_0 \mathbf{j} \times d^2\mathbf{r}/dt^2 = -(m_0 r^2 V' + Q_*^2 r^{-1}) d\hat{\mathbf{r}}/dt. \tag{1.22}$$

This will integrate vectorially if the bracket is constant. Calling it GMm_0^2 we find the potential must be of the form

$$V/m_0 = -GMr^{-1} + \frac{1}{2}\frac{Q_*^2}{m_0^2} r^{-2}. \tag{1.23}$$

Evidently the required inverse cube repulsive force is proportional to the square of the monopole moment Q. Integrating Eq. (1.22) we have

$$d\mathbf{r}/dt \times \mathbf{j} = GMm_0(\hat{\mathbf{r}} + \mathbf{e}) \tag{1.24}$$

where $\mathbf{e}$ is the vector constant of integration. Dotting Eq. (1.24) with $\hat{\mathbf{r}}$ we have

$$\ell_*/r = (1 + \mathbf{e}\cdot\hat{\mathbf{r}}) \tag{1.25}$$

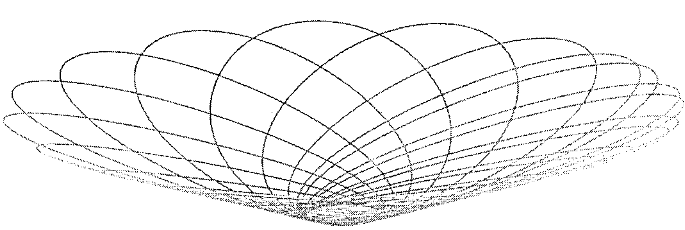

FIG. 3. An ellipse precessing around a cone of semi-angle 70°, making a rosette orbit on it.

where $\ell_* = \mathbf{L}\cdot\mathbf{j}/(GMm_0^2) = \text{const}$. Equation (1.25) is the equation of a conic section of eccentricity $\mathbf{e}$ which defines the direction to pericenter. But we have not yet proved that the orbit lies in a plane so Eq. (1.25) actually defines a prolate spheroid, paraboloid, or hyperboloid. Nick Manton, by analogy with his work on monopoles in Euclidean Taub space (Gibbons and Manton, 1986), showed us that the motion is in fact planar; multiplying Eq. (1.25) by $Q_* r$ and making use of Eq. (1.15), we obtain

$$Q_*\ell_* = (\mathbf{j} + Q_*\mathbf{e})\cdot\mathbf{r}.$$

This demonstrates that the orbit lies on a plane whose normal is $\mathbf{j} + Q_*\mathbf{e}$. As $\mathbf{r}$ also lies on a cone this provides another proof that the motion lies along a conic section.

Notice that the vector integral $\mathbf{e}$ in Eq. (1.24), together with the integral $\mathbf{j}$, appears to provide six integrals of the motion. However, they are not all independent because $-\mathbf{e}\cdot\mathbf{j} = \hat{\mathbf{r}}\cdot\mathbf{j} = Q_*$, so they provide five independent integrals. Thus we have a new superintegrable dynamic system in which the bound orbits exactly close (cf. Evans, 1990, 1991).

It was the beauty and simplicity of these results for monopoles in classical mechanics that led us to believe that a similar simplicity might well be discernible both in quantum mechanics and in general relativity. We were not disappointed; both had already attracted attention. For motion in special relativity $\mathbf{j}$ is still conserved provided $\mathbf{L}$ is interpreted as $m_0\mathbf{r}\times d\mathbf{r}/d\tau = m_0\mathbf{r}\times\mathbf{v}/\sqrt{1 - v^2/c^2}$ (Hautot, 1972).

For comparison with quantum theory it is most revealing to consider first the classical motion of a small spinning charged sphere in the field of a monopole that may or may not have a charge Ze. We call the spin angular momentum of the sphere $\mathbf{S}$ and suppose it to have magnetic moment $g^*\mathbf{S}$ where g^* is constant.

The monopole's magnetic field is $\mathbf{B} = Q\hat{\mathbf{r}}r^{-2} = \text{Curl } \mathbf{A} = \text{Curl}[-Q(1+\cos\theta)\nabla\Phi]$. The Hamiltonian is (Corben and Stehle, 1960)

$$H = (2m_0)^{-1}(\mathbf{p} - qc^{-1}\mathbf{A})^2 + q\Phi + (2I)^{-1}[(\mathbf{S} - g^*I\mathbf{B})^2 - \mathbf{S}^2], \tag{1.26}$$

where I is the sphere's moment of inertia, q its charge, and $\Phi = Ze/r$.

Writing $\mathbf{L} = \mathbf{r}\times m_0\mathbf{v} = \mathbf{r}\times(\mathbf{p} - qc^{-1}\mathbf{A})$, we find that the conserved total angular momentum is

$$\mathbf{j} = \mathbf{L} - Q_*\hat{\mathbf{r}} + \mathbf{S}, \tag{1.27}$$

where $Q_* = qQ/c$ and the sign change from Eq. (1.14) reflects the repulsion of like charges *vis à vis* the attraction of like masses. In Sec. II.D the $-Q_*\hat{\mathbf{r}}$ term is shown to be the angular momentum in the Poynting vector of the electromagnetic field that accompanies the charged sphere. The spin magnetic moment reacts to the magnetic field, so

$$\dot{\mathbf{S}} = g^*\mathbf{S}\times\mathbf{B}. \tag{1.28}$$

Evidently $\mathbf{S}^2$ is constant. For this reason we subtracted $(2I)^{-1}\mathbf{S}^2$ from the Hamiltonian so as to make our energy zero-point the spinning sphere at ∞. Substituting for $\mathbf{S}$ on the right via Eq. (1.27) and using $\mathbf{B}=Q\hat{\mathbf{r}}r^{-2}$, we have

$$\dot{\mathbf{S}}=g^*(\mathbf{j}-\mathbf{L})\times\mathbf{B}. \tag{1.29}$$

But

$$\mathbf{L}\times\mathbf{B}=Qm_0(\hat{\mathbf{r}}\times\dot{\mathbf{r}}/r)\times\hat{\mathbf{r}}=d/dt(Qm_0\hat{\mathbf{r}}), \tag{1.30}$$

so that term can be taken to the left in Eq. (1.29). If we write $\mathbf{S}^*=\mathbf{S}+\frac{1}{2}gQ_*\hat{\mathbf{r}}$ where $g=2m_0cg^*/q$, then Eq. (1.29) reads

$$d\mathbf{S}^*/dt=g^*\mathbf{j}\times\mathbf{B}. \tag{1.31}$$

Since $\mathbf{j}$ is constant there is a new constant of the motion,

$$\mathbf{S}^*\cdot\mathbf{j}=C_1. \tag{1.32}$$

Conservation of angular momentum (1.27) may now be rewritten

$$\mathbf{j}-\mathbf{S}^*=\mathbf{L}-(\tfrac{1}{2}g+1)Q_*\hat{\mathbf{r}} \tag{1.33}$$

so

$$(\mathbf{j}-\mathbf{S}^*)^2=\mathbf{L}^2+(\tfrac{1}{2}g+1)^2Q_*^2. \tag{1.34}$$

The energy (1.26) may now be rewritten putting $\mathbf{S}^{*2}$ in terms of $\mathbf{S}\cdot\hat{\mathbf{r}}$ in Eq. (1.34):

$$E=\tfrac{1}{2}m_0\dot{r}^2+qZer^{-1}+(2m_0r^2)^{-1}[\mathbf{L}^2-gQ_*(\mathbf{S}\cdot\hat{\mathbf{r}})] + g^2Q_*^2\,Im_0^{-2}r^{-4}/8 \tag{1.35}$$

$$=\tfrac{1}{2}m_0\dot{r}^2+qZer^{-1}+(2m_0r^2)^{-1}[\mathbf{j}^2+\mathbf{S}^2-2C_1 -(g+1)Q_*^2]+g^2Q_*^2Im_0^{-2}r^{-4}/8. \tag{1.36}$$

The coefficient of r^{-2} can now be seen to be constant, $C_2/(2m_0)$ say.

The final term is of order $I/(m_0r^2)$ times the one before it, so when r is greater than the classical radius of the electron it may be ignored. The problem then reduces to classical motion in a Coulomb potential but with $\mathbf{L}^2$ replaced by the square bracket of Eq. (1.36) or (1.35). However, there can be an important difference because the new square bracket can be negative. For example, if we start our sphere with $\mathbf{L}$ small and $\mathbf{S}$ along $Q_*\hat{\mathbf{r}}$ then the square bracket C_2 starts negative and, since it is constant, it remains so. This occurs when the attraction of the monopole for the spin magnetic moment overcomes any centrifugal barrier, so there is nothing to stop the sphere from falling onto the monopole unless the r^{-4} term we neglected is resurrected. Such attractive r^{-2} potentials also cause trouble in quantum theory, as we shall see. Using Eqs. (1.27) and (1.32), one may evaluate $\mathbf{L}\cdot\mathbf{S}=C_1+\frac{1}{2}gQ_*^2-\mathbf{S}^2+(1-\frac{1}{2}g)Q_*\hat{\mathbf{r}}\cdot\mathbf{S}$. Thus for the very special case $g=2$—e.g., the Dirac electron—we find that $\mathbf{L}\cdot\mathbf{S}$ is constant as a result of the integral (1.32). This allows a very simple treatment of the Dirac equation (Sec. II.E).

II. DIRAC'S MONOPOLE AND THE SPECTRA OF MONOPOLAR ATOMS

A. Introduction, gauge transformations, and Dirac's quantized monopole

Dirac (1934, 1948), in developing his theory of quantized magnetic monopoles and quantized charges, solved Schrödinger's equation in the field of a magnetic monopole and showed that it had no bound state. Stimulated by this Harish-Chandra (1948) asked whether the magnetic moment of the spinning electron could alter this result. By a clever but involved separation of the Dirac equation he showed that Dirac's result was unchanged—no bound state exists.

Hautot (1973) realized that Dirac's equation was still separable in the field of a charged monopole and rapidly extended Harish-Chandra's result to give what we believe to be the first derivation of the formula for the energy levels of the bound states. Goldhaber (1965) had already discussed the scattering of fast particles by magnetic monopoles, and papers by Kazama and Yang (1976) and Kazama *et al.* (1977) showed this subject had considerable theoretical interest.

More recently work has extended to cover the rather strange behavior of the energy levels when an Aharanov-Bohm string of confined magnetic flux is added to the problem, which is then still separable. (Hoang *et al.*, 1992). We believe that the flat bottom to Fig. 4 helps the understanding of that behavior but it is not considered here. Villalba (1994), while in general agreement with that work, finds further solutions of the hypergeometric equation that obey the normalization conditions so there is incomplete agreement. To elucidate the problem he turned to the simpler case of the Schrödinger equation for a nonspinning electron in orbit about a charged monopole with an Aharanov-Bohm string (Villalba, 1995). We, too, initially found difficulties with the multitude of special cases of the hypergeometric function, but we chose the problem without the added string and by following the old precepts of the book by Pauling and Wilson (1935) we found no ambiguities. With the energy levels known since Hautot's 1973 work, it is somewhat surprising that no discussion of the allowed transitions, nor any drawn-out diagrams of the energy levels, nor any tables of allowed transitions are to be found. Although such calculations are simple to perform, experimenters and observers are far more likely to look for accurately predicted spectral lines if they are presented with such diagrams and tables, so one of the purposes of this paper is to stimulate interest in such searches by performing this service. The spectra are therefore worked out in detail and tabulated here. The dipole transitions follow a selection rule unlike that of normal hydrogen, which is $\Delta\ell=\pm1$. The nearest equivalent to that selection rule in monopolar atoms is $\Delta J=\pm1$ or 0. Thus transitions directly down the $J=0$ states are allowed (as are those down $J=1$). These may be considered as analogous to the $\ell=0$ (and $\ell=1$) states of normal hydrogen down which dipole

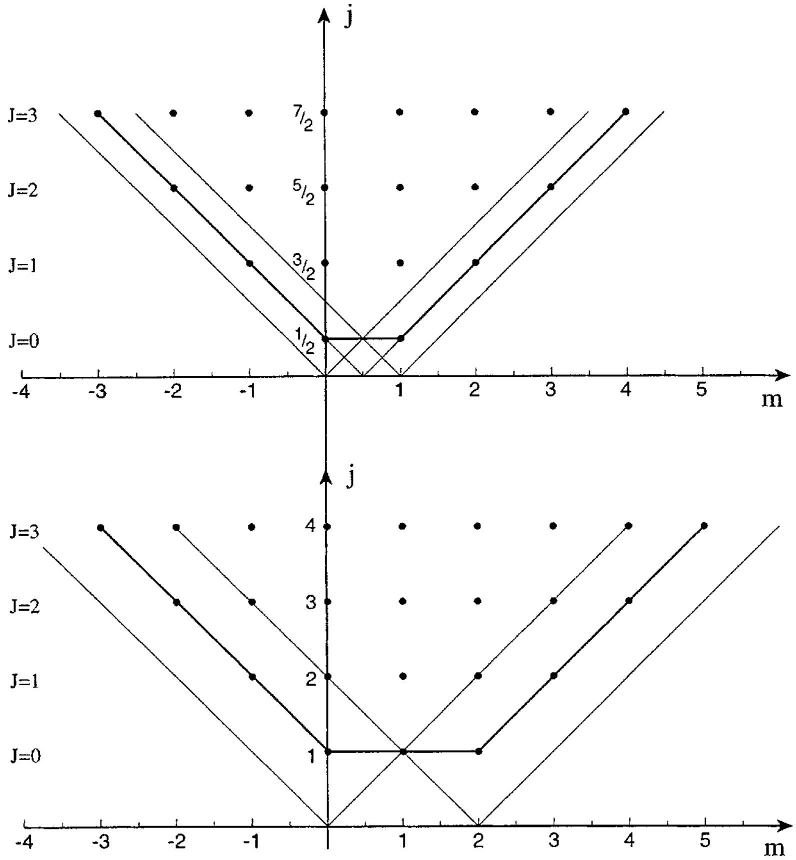

FIG. 4. j values allowed by the conditions $j \geqslant \frac{1}{2}(|m|+|m-N|)$. j cannot be less than the average of the two faint V lines in the diagram. Unit monopole $N = 1$ above, $N = 2$ below.

transitions are forbidden. The quantum number J above, while containing contributions from the Poynting vector of the electromagnetic field where the electric field of the electron crosses the magnetic flux of the magnetic monopole, is (initially) for a nonspinning electron and thus obeys the above Wigner-Eckhart theorem selection rule even when there is no contribution from electron spin.

Those with a knowledge of the old quantum theory and the correspondence principle will expect that the precession of the orbits around cones found in Sec. I will split the energy levels by $h\times$ (precession frequency); indeed, the degeneracy of the $s,p,d,f\ldots$ levels so characteristic of normal hydrogen is broken by this precession rate (which depends on the angular momentum). When such splittings are added to the extra freedom in the dipole selection rules we find an unexpectedly rich spectrum for monopolar hydrogen with two Lyman series, five Balmer series, eight Paschen series, etc. It might be thought that the magnetic field of the monopole would not have a major influence on the energy levels, but it has a profound influence on angular momentum [see Eq. (1.14)] and in practice even one Dirac monopole halves the binding energy of the ground state.

While we consider that it ought to be possible to argue solely from the hypergeometric function solutions of the Schrödinger equation that the magnetic monopole strength Q must be an integral multiple, N, of $\frac{1}{2}\hbar c/e$, the Dirac monopole, we have not found a way to do this directly. Probably this reflects our inadequate knowledge of hypergeometric functions. However, this leads us to give our version of the arguments that led Dirac to his quantized magnetic monopoles and to relate them to the Aharanov-Bohm effect. Schwinger's (1966) investigations led *him* to believe that the true unit of monopole strength was twice the Dirac unit. We have therefore calculated the spectra both for one and for two Dirac monopoles attached to the nucleus.

We give the derivation of the energy levels and spectra not merely because we had to rederive them to understand fully what others did earlier. We also believe that use of the angular momentum operators and their commutation relations circumvents the difficult part of Harish-Chandra's work and therefore gives an easier path to the energy levels. We give the solution of the no-spin Schrödinger equation first as this is much more easily understood and helps to isolate the effects of the monopole on the nucleus from the effects of the electron's spin and magnetic moment. We hope also that others, who do not use quantum mechanics on a daily basis, may enjoy an elementary refresher course with the added interest of the "new" twist given by the monopole on the nucleus.

The Lagrangian for a particle of mass m_0 and charge $-e$ in an electromagnetic field is

$$\mathcal{L}=\tfrac{1}{2}m_0\dot{\mathbf{r}}^2-e\dot{\mathbf{r}}\cdot\mathbf{A}/c+e\Phi(r),$$

where Φ is the electrostatic potential and $\mathbf{B}=\text{Curl }\mathbf{A}$ is the magnetic field. The momentum conjugate to $\mathbf{r}$ is

$$\mathbf{p}=\partial\mathcal{L}/\partial\dot{\mathbf{r}}=m_0\dot{\mathbf{r}}-e\mathbf{A}/c,$$

which is not a gauge-invariant quantity. However the particle's momentum $m_0\dot{\mathbf{r}}=\mathbf{p}+e\mathbf{A}/c$ is gauge invariant and therefore has greater physical significance. The Hamiltonian is given by

$$H=\mathbf{p}\cdot\mathbf{v}-\mathcal{L}=(2m_0)^{-1}(\mathbf{p}+e\mathbf{A}/c)^2-e\Phi. \tag{2.1}$$

For Schrödinger's equation we replace $\mathbf{p}$ by $-i\hbar\nabla$ and solve $H\psi=E\psi$ for the wave function of a steady state. A given magnetic field $\mathbf{B}$ can be described by many different vector potentials $\mathbf{A}$ related by gauge transformations $\mathbf{A}'=\mathbf{A}+\nabla\chi$. Each will give us a different Hamiltonian. Let us first see how the different wave functions corresponding to these are related. Define a new function ψ' such that

$$\psi=\exp[ie\chi/(\hbar c)]\psi'. \tag{2.2}$$

Then $(-i\hbar\nabla+e\mathbf{A}/e)\psi=\exp[ie\chi/(\hbar c)](-i\hbar\nabla+e\mathbf{A}/c+e\nabla\chi/c)\psi'$ and the combination $\mathbf{A}'=\mathbf{A}+\nabla\chi$ has appeared. Applying the above operator twice, we see that

$$H\psi=\exp[ie\chi/(\hbar c)]H'\psi'$$

where H' is H with $\mathbf{A}$ replaced by $\mathbf{A}'$. It follows that Schrödinger's equation $H\psi=E\psi$ implies $H'\psi'=E\psi'$, so under gauge transformation ψ transforms to ψ' given by Eq. (2.2).

This is a perfectly good wave function whenever χ is single valued, but following Aharonov and Bohm (1959) we now consider the wave function of a particle outside a small impenetrable cylinder $R=a$. If we take $\nabla\chi=\nabla(F\phi/2\pi)$, $R\geqslant a$ where ϕ is the azimuth, this corresponds to the same magnetic field outside the cylinder but a different magnetic flux within it because

$$\int \mathbf{B}'\cdot d\mathbf{S}=\oint \mathbf{A}'\cdot d\ell=\oint(\mathbf{A}+\nabla\chi)\cdot d\ell$$

$$=\int \mathbf{B}\cdot d\mathbf{S}+F,$$

which identifies the constant F as the extra flux threading the cylinder. If we adopt our transformation of wave function for a gauge transformation we get the phase factor

$$\exp[-ieF\phi/(hc)], \tag{2.3}$$

which is only single valued when the flux takes the special values

$$F=N(hc/e), \tag{2.4}$$

where N is an integer (positive, negative, or zero). (This N is not the force coefficient of Sec. I.) Thus while we get the correct wave function for those particular values of F, we need to solve the problem anew with the correct boundary condition that ψ' must be periodic in ϕ whenever F is not an integer multiple of the flux quantum hc/e. Indeed, when it is not, there is interference between the two parts of a beam of electrons that pass on either side of such a cylinder just because their phases differ by $e\oint\Delta\mathbf{A}\cdot d\ell/(hc)=eF/(hc)$. It was just this phase shift that was observed in the experiments demonstrating the Aharanov-Bohm effect of the magnetic flux even when the electron beams were untouched by the magnetic field. There is an intimate connection of this result with Dirac's (1931) earlier quantum of a magnetic monopole from which one flux unit [Eq. (2.4)] emanates. This comes about because in the presence of a monopole $\oint\mathbf{A}\cdot d\ell$ is itself multivalued.

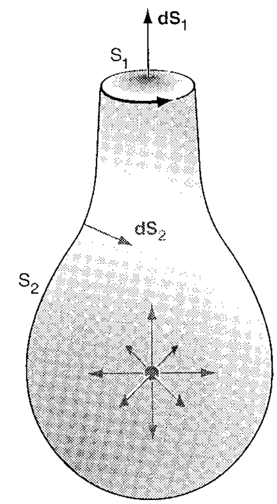

FIG. 5. A monopole and its $\mathbf{B}$ field showing the surfaces S_1, S_2 and $S_3\equiv S_1-S_2$.

Consider the integral $\oint\mathbf{A}\cdot d\ell$ around a small loop; this is clearly the flux of $\mathbf{B}$ through the loop but such a flux is ambiguous in the presence of a monopole, since it depends on whether the surface spanning the loop is chosen to pass above or below the monopole, i.e., S_1 or S_2 in Fig. 5. The difference between these two estimates is just $\int_{S_1-S_2}\mathbf{B}\cdot d\mathbf{S}=4\pi Q$ by Gauss's theorem. Inserting this $\Delta\int\mathbf{A}\cdot d\ell$ in place of F in Eq. (2.3) we see that the wave function will only have an unambiguous phase provided

$$4\pi Q=N(hc/e), \tag{2.5}$$

i.e., provided that the monopole strength is quantized in Dirac units of $\frac{1}{2}\hbar c/e\approx 137e/2$.

The quantum of magnetic flux, Eq. (2.4), is inversely proportional to the charge. Quanta of half this size are observed in the Josephson effect in superconductivity, where the effect is due to paired electrons. There is some evidence for the larger unit, Eq. (2.4), in ordinary conductors at low temperatures (Umbach *et al.*, 1986).

Returning to Schrödinger's equation (2.1) and using the vector potential [cf. under Eq. (1.19)]

$$\mathbf{A}=-Q(1+\cos\theta)\nabla\phi \tag{2.6}$$

we have the correct magnetic field for a monopole of strength Q, but we notice that A is singular along the line $\theta=0$ although it is regular along $\theta=\pi$. Near the singular line $\mathbf{A}\to-2Q\nabla\phi$, which is the vector potential of a tube carrying a flux $4\pi Q$ downwards. Thus Eq. (2.6) represents the vector potential of a magnetic monopole fed its flux by the singular half line $\theta=0$. This half line gives an unobservable Aharanov-Bohm effect provided $4\pi Q=Nhc/e$, that is, provided the monopole is a multiple of the Dirac (1931, 1948) unit. Extra interest in his monopole comes from Dirac's argument that it can also be read as a reason for charge quantization, because, if Q is the least monopole, then e must be a multiple of $\hbar c/Q$; thus, in his picture, charge quantization and monopole quantization spring from the same source. It is of interest that A in Eq. (2.6) is single valued. It has avoided the multivaluedness alluded to above by having the singular string at $\theta=0$ down to the monopole. This plays the role of the cut in multivalued functions in the complex plane (Wat and Yang, 1976). An interesting historical remark is that Schrödinger in 1922 saw that quantum conditions in the old quantum theory led to $\Gamma\equiv e/c[\oint\Phi dt-\mathbf{A}\cdot d\mathbf{x}]=nh$, while Weyl's gauge theory led him to consider $\exp(-\Gamma/\gamma)$ with γ as yet unspecified. He realized that the identification $\gamma=-i\hbar$ would lead naturally to such quantum numbers and, after de Bröglie (1925), he built on this idea to invent his wave mechanics in 1926 (Schrödinger, 1926) (see Yang, 1987).

B. Solution of Schrödinger's equation

Written in spherical polar coordinates, Schrödinger's equation is

$$-\frac{\hbar^2}{2m_0r^2}\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\psi}{\partial\mu}\right]+\frac{1}{1-\mu^2}\left[\frac{\partial^2\psi}{\partial\phi^2}-iN(\mu+1)\frac{\partial\psi}{\partial\phi}-\tfrac{1}{4}N^2(\mu+1)^2\psi\right]\right\}-e\Phi\psi=E\psi. \tag{2.7}$$

Here μ has been written for $\cos\theta$ and N is the number of Dirac monopoles on the nucleus. ϕ only occurs as $\partial/\partial\phi$ in the above equation, so we may take one Fourier component with $\psi\propto\exp(im\phi)$ and m an integer positive, negative, or zero in order that ψ be single valued. On multiplication by $-2m_0r^2/(\hbar^2\psi)$ we then find the separated equation

$$\frac{1}{\psi}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\psi}{\partial\mu}\right]-\frac{[m-N\frac{1}{2}(\mu+1)]^2}{1-\mu^2}=-C=-\frac{1}{\psi}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)-\frac{r^2 2m_0}{\hbar^2}(E+e\Phi). \tag{2.8}$$

Writing $\psi=\psi_r(r)\psi_\mu(\mu)$, we see that the left-hand side is a function of μ alone and the right-hand side is a function of r alone, so both must be a constant, which we call $-C$. The resultant equation for ψ_μ has regular singular points at $\mu=\pm1$. The indicial equations for the series solutions about $\mu=\pm1$ have regular solutions behaving as $(1-\mu)^{1/2|m-N|}$ and $(1+\mu)^{1/2|m|}$, respectively, so we remove those factors by writing

$$\psi_\mu=(1-\mu)^{1/2|m-N|}(1+\mu)^{1/2|m|}F(\mu). \tag{2.9}$$

After some algebra the equation for F takes the form

$$(1-\mu^2)F''+[(|m|+1)(1-\mu)-(|m-N|+1)\times(1+\mu)]F'+\tfrac{1}{2}[2C-m(m-N)-|m||m-N|-|m-N|-|m|]F=0. \tag{2.10}$$

We now write $z=\frac{1}{2}(1+\mu)$, so $z(1-z)=(1-\mu^2)/4$ and $dz=\frac{1}{2}d\mu$, which reduces the above equation into the standard form for the hypergeometric equation, i.e.,

$$z(1-z)d^2F/dz^2+[c-(a+b+1)z]dF/dz-abF=0 \tag{2.11}$$

where

$$c=|m|+1, \tag{2.12}$$

$$a+b=|m|+|m-N|+1, \tag{2.13}$$

and $-2ab$ is the final square bracket in Eq. (2.10).

The hypergeometric function finite at $\mu=-1$, $z=0$ diverges like $(1-\mu)^{c-a-b}$ at $\mu=1$, that is, twice as fast as the first factor in Eq. (2.9) converges, so in order to get convergence the hypergeometric series must terminate. This occurs only if a or b is a negative integer or zero. W.l.g. taking $b=-K$ we find that F reduces to a Jacobi polynomial $P_K^{\alpha\beta}(\mu)$ so that ψ_μ takes the form

$$\psi_\mu=C_{kmn}(1-\mu)^{1/2|m-N|}(1+\mu)^{1/2|m|}P_K^{|m-N|,|m|}. \tag{2.14}$$

Here $\int_{-1}^{+1}\psi_\mu^2d\mu=1$ and C_{kmn} is the normalization

$$\left[\frac{(2K+|m-N|+|m|+1)K!(K+|m-N|+|m|)!}{2^{|m-N|+|m|+1}(K+|m-N|)!(K+|m|)!}\right]^{1/2}.$$

The condition that $b=-K$ gives

$$-2ab=2K(|m|+|m-N|+1+K)$$

and hence [noticing that $K=0$ leaves Eq. (2.14) finite] we have

$$C=K(K+1)+K(|m|+|m-N|)+\tfrac{1}{2}[m(m-N)+|m||m-N|+|m-N|+|m|]. \tag{2.15}$$

If we write $j=K+\frac{1}{2}(|m|+|m-N|)$, then we notice that j is a positive half integer and $j\geqslant\frac{1}{2}(|m|+|m-N|)$,

$$C=j(j+1)-N^2/4. \tag{2.16}$$

Thus C and j are integers only when N is an *even* integer. When N is odd, C and j are an integer $\pm\frac{1}{2}$. For given j and $N\geqslant0$, $m-N/(2)$ takes the $2j+1$ values from $-j$ to $+j$ in steps of 1. For $N=1$ the ground state has $j=\frac{1}{2}$ and $C=\frac{1}{2}$ rather than the values 0 familiar from the normal hydrogen atom. The $j=\frac{1}{2}$ states with $m=1$ and $m=0$ are degenerate (see Fig. 4). Note that $j(j+1)h^2$ are the eigenvalues of the conserved $|\mathbf{j}|^2$ defined in Eq. (1.14). See Sec. II.D for its identification with angular momentum.

With the value of Eq. (2.16) for C we now turn to the radial equation for ψ_r, Eq. (2.8). Here the treatment is very close to the classical case clearly laid out by Pauling and Wilson (1935). We take $\Phi=Ze/r$, Ze being the nuclear charge, and E negative. We write

$$\alpha^2=-2m_0E/\hbar^2 \tag{2.17}$$

$$\zeta=m_0Ze^2h^{-2}\alpha^{-1} \tag{2.18}$$

and use a normalized radius $\tilde{r}=2\alpha r$. As all the radii in the rest of this section are so normalized we shall forget the tilde and take it as read. Equation (2.8) now takes the form

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi_r}{dr}\right)+\left(-\frac{1}{4}-\frac{C}{r^2}+\frac{\zeta}{r}\right)\psi_r=0. \tag{2.19}$$

The asymptotic form of this equation for large r is $\psi_r''=\psi_r/4$ so $\psi_r\to\exp(r/2)$ or $\exp-(r/2)$. Of these only the second is acceptable so we write

$$\psi_r=\exp(-r/2)r^s f(r), \tag{2.20}$$

where $f(r)$ may be expanded in series about the origin in the form

$$\sum_{p=0}^{\infty} a_p r^p$$

and s is chosen so that $a_0\neq 0$. The indicial equation found by substitution of the series from Eq. (2.20) into Eq. (2.19) is

$$[s(s+1)-C]a_0=0,$$

but by hypothesis $a_0\neq 0$, so using Eq. (2.16) we obtain s by

$$(s+\tfrac{1}{2})^2=\tfrac{1}{4}+s(s+1)=\tfrac{1}{4}+C=(j+\tfrac{1}{2})^2-\tfrac{1}{4}N^2. \tag{2.21}$$

The recurrence relation for general p is then

$$p(p+2s+1)a_p=(s+p-\zeta)a_{p-1}$$

and the asymptotic form for large p is $a_p\to a_{p-1}/p$, which shows that $f\to e^r$. In that case ψ_r would diverge at large r. This is unacceptable, so the series must terminate. Thus there must be a positive integer $p=n'+1$ such that

$$\zeta=p+s=n'+s+1 \tag{2.22}$$

with s given by Eq. (2.21). Returning to Eqs. (2.18) and (2.17), we see that this gives the eigenvalues for the energy in the form

$$E=-\frac{m_0Z^2e^4}{2\hbar^2}\frac{1}{(n'+s+1)^2}=-\frac{m_0Z^2e^4}{2\hbar^2}\frac{1}{(n+\Delta)^2} \tag{2.23}$$

where $n=n'+J+1$, and J replaces the usual ℓ, where J takes values 0,1,2, . . . ,

$$J=j-\tfrac{1}{2}|N|\geqslant 0.$$

Here N is the number of Dirac monopoles on the nucleus and

$$\Delta=\sqrt{(J+\tfrac{1}{2})(J+\tfrac{1}{2}+|N|)}-(J+\tfrac{1}{2})\geqslant 0. \tag{2.24}$$

Notice that Δ depends on J as well as $|N|$ and is only zero when $N=0$. For large $J/|N|$,

$$\Delta\to\tfrac{1}{2}|N|\left[1-\frac{1}{4}\frac{|N|}{J+\frac{1}{2}}+\frac{1}{8}\left(\frac{|N|}{J+\frac{1}{2}}\right)^2-\cdots\right],$$

while for the ground state $J=0$

$$\Delta=\tfrac{1}{2}(\sqrt{2|N|+1}-1),$$

which becomes $\frac{1}{2}(\sqrt{3}-1)$ for $N=1$. So Δ is *not* small. For a spinless electron, the degeneracy of a state of given J and n is $2j+1=2J+1+|N|$ with $m-\frac{1}{2}|N|$ taking values from $-j$ to $+j$. Notice that the ground state $J=0$, $n=1$ is a doublet for $N=1$ and has j value $\frac{1}{2}$ with $m=0$ and $m=+1$ states even *before* we have allowed for further degeneracy due to electron spin. A single Dirac monopole thus gives some effects reminiscent of spin $\frac{1}{2}$ particles (Goldhaber, 1976).

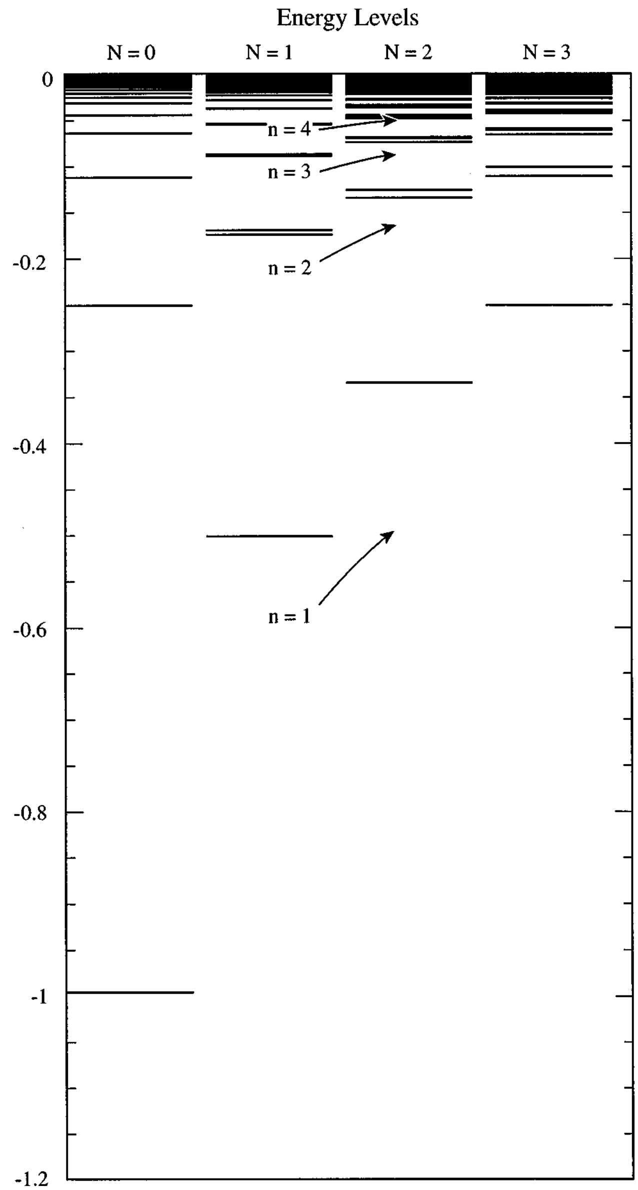

FIG. 6. Energy levels for a spinning electron in hydrogen with 0, 1, 2, or 3 Dirac monopoles on its nucleus. Excepting "isotopic" shifts due to the changed nuclear mass and relativistic corrections, the energy levels of the ground states are in the ratio $1:\frac{1}{2}:\frac{1}{3}:\frac{1}{4}$. These levels do not include the unstable resonances which occur for $N=1$ at the same levels as the s states of common or garden hydrogen ($N=0$). Such resonances also occur for $N=3,5,\ldots$, etc.

The dependence of Δ upon J lifts the degeneracy of the different J states (ℓ states) that occurs in normal hydrogen. The energy levels are near to those for an atom with a true spinning electron, as laid out in Figs. 6, 7, 8 and Tables I and II. It is because the precession rates around the cones in Sec. I depend on J that we find this J dependence of the energy levels in the quantized

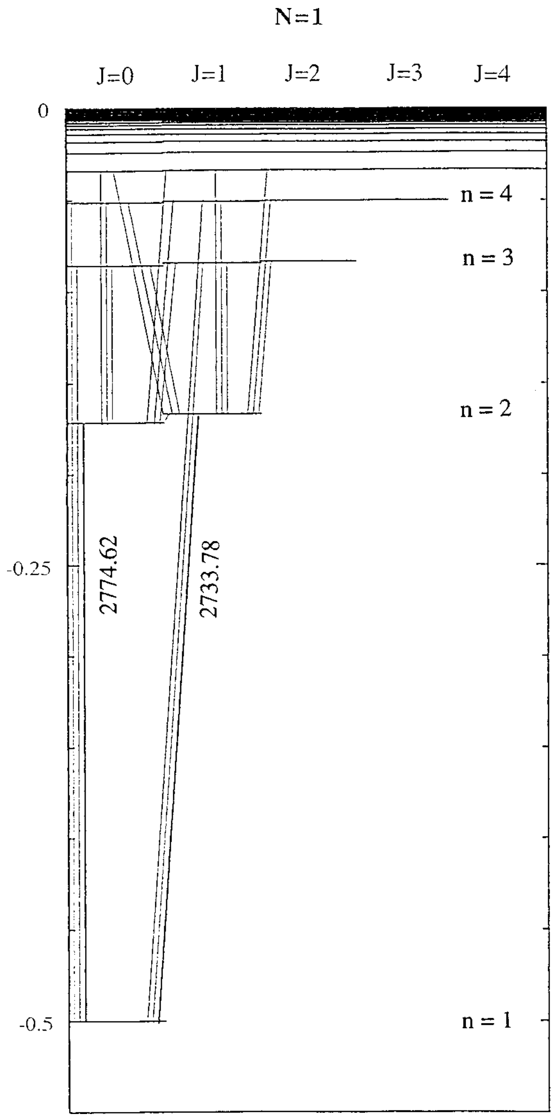

FIG. 7. Energy-level diagram $E(n,J)$ for $N=1$, hydrogen with one Dirac monopole on its nucleus. The nucleus has been assumed to be fixed. These levels do not include the unstable resonances alluded to under Fig. 6.

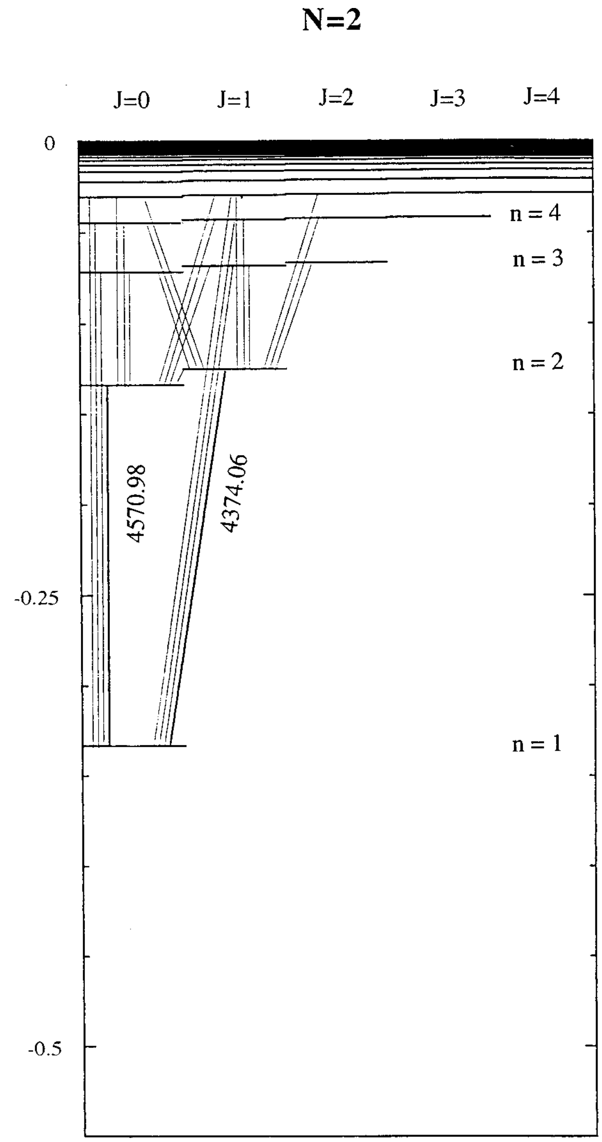

FIG. 8. Energy-level diagram $E(n,J)$ for $N=2$, hydrogen with two Dirac monopoles on its nucleus.

atom. The degeneracy would return if the extra repulsive potential $\frac{1}{2}Q^2/(m_0r^2c^2)$ of Eq. (1.23) were added to the problem. Then the precessions would stop and the $-\frac{1}{4}N^2$ in Eq. (2.21) would be cancelled, so s would become equal to j as in the normal hydrogen atom (where j is ℓ). In Sec. I we saw that orbits with positive z angular momentum were displaced up the axis; it is this displacement that gives the asymmetry in the wave functions with a definite m which makes the Jacobi polynomials replace the Legendre polynomials. Thus the change of selection rule can be traced to the lack of reflection symmetry of the wave functions of definite m in the plane $z=0$.

C. Selection rules

The string to the monopole makes it look nonspherical but this is not truly the case, as putting the string in any other direction can be achieved by a mere gauge transformation. Therefore without loss of generality we may evaluate transition moments by taking the displacement in the z direction, in which case we get

$$R_{ab}=\int \psi_a^* r\mu\psi_b d^3r$$

$$=2\pi\delta_{m_a m_b}\int_0^\infty \psi_{ra}\psi_{rb}r^3 dr\int_{-1}^{+1}\psi_{\mu a}\mu\psi_{\mu b}d\mu.$$

The radial integral is that for normal hydrogen, but the scales have changed since s in Eq. (2.22) is no longer ℓ but is given instead by Eq. (2.21). We shall concentrate on the important change in selection rules given by the final integral.

Whereas for the Legendre polynomials in normal hydrogen wave functions we have

$$\mu P_\ell(\mu)=\frac{\ell+1}{2\ell+1}P_{\ell+1}+\frac{\ell}{2\ell+1}P_{\ell-1},$$

TABLE I. Wavelengths in Angstroms of the two Lyman series, the five Balmer series, and the eight Paschen series of hydrogen with ($N=1$) one Dirac monopole attached to the proton (see Fig. 7). The wavelengths after the dots are those of the series limits. These wavelengths do not include those of transitions to or from the unstable resonances.

Lyman Series: $n\to1$		Fine structure transiton $n=3-3$ 521911.0 Paschen Series. $n\to3$	
2774.62	2733.78	26439.6	25832.0
2199.98	2190.74	17635.2	17486.2
2030.99	2027.32	14821.9	14758.3
1955.97	1954.13	13481.2	13447.0
1915.65	1914.58	12716.1	12695.2
1891.33	1890.66	⋯	⋯
1875.50	1875.05	10622.4	10622.4
⋯	⋯	J=0−0	J=1−0
1822.52	1822.52		
$J=0-0$	$J=1-0$		

Fine structure transiton $n=2-2$ 185695.00 Balmer Series $n\to2$		Fine structure transition $n=3-3$ 1287750.0 Paschen Series. $n\to3$		
10622.40	10410.50	27850.5	27177.1	26912.1
7577.90	7527.15	18251.9	18092.4	18028.0
6629.30	6608.14	15255.2	15187.7	15160.3
6187.80	6176.68	13838.6	13802.6	13787.9
5941.13	5934.48	13033.6	13011.7	13002.8
5787.67	5783.34	⋯	⋯	⋯
⋯	⋯	10845.1	10845.1	10845.1
5311.20	5311.20	$J=0-1$	$1-1$	$2-1$
$J=0-0$	$J=1-0$			

Balmer Series $n\to2$			Paschen Series. $n\to3$		
11266.90	11028.80	10935.20	27763.0	27486.5	27338.0
7900.30	7845.16	7822.92	18350.2	18284.0	18248.1
6874.73	6851.97	6842.71	15369.0	15340.9	15325.6
6401.10	6389.20	6384.34	13952.2	13937.1	13928.9
6137.50	6130.40	6127.49	13144.6	13135.4	13130.4
5973.86	5969.25	5967.30	⋯	⋯	⋯
⋯	⋯	⋯	10935.2	10935.2	10935.2
5467.58	5467.58	5467.58	$J=1-2$	$2-2$	$3-2$
$J=0-1$	$1-1$	$2-1$			

so that $\int_{-1}^{+1}P_{\ell'}\mu P_{\ell}d\mu$ is only nonzero when $\ell'-\ell=\pm1$, for the Jacobi polynomials in monopolar hydrogen

$$\mu P_K^{\alpha\beta}=(a_1P_{K+1}^{\alpha\beta}+a_2P_K^{\alpha\beta}+a_4P_{K-1}^{\alpha\beta})/a_3,$$

where

$$a_1=2(K+1)(K+\alpha+\beta+1)(2K+\alpha+\beta),$$

$$a_2=(2K+\alpha+\beta+1)(\alpha^2-\beta^2),$$

$$a_3=(2K+\alpha+\beta)(2K+\alpha+\beta+1)(2K+\alpha+\beta+2),$$

$$a_4=2(K+\alpha)(K+\beta)(2K+\alpha+\beta+2),$$

so that $\int_{-1}^{+1}(1-\mu)^\alpha(1+\mu)^\beta P_{K'}^{\alpha\beta}\mu P_K^{\alpha\beta}d\mu$ will be nonzero when $K'-K=\pm1$ or 0. (The 0 term is only absent when $a_2=0$, i.e., $\alpha\equiv|m-N|=\beta\equiv|m|$. This occurs for $N=0$ always, for $N=2$ when $m=1$, but never for $N=1$.)

Thus there is a significant change in the selection rules for electric dipole transitions. Some might imagine that magnetic dipole transitions should be important, but the magnetic monopole is on a heavy nucleus and barely responds to an oscillating magnetic field, so it is still the electric dipole transitions of the electron that are important. Since m is unchanged for a dipole along the z axis, $\Delta K=\pm1$ or 0 leads directly to Δj and hence $\Delta J=\pm1$ or 0 for such transitions.

TABLE II. Wavelengths in angstroms of the Lyman, Balmer, and Paschen series of hydrogen with two Dirac monopoles attached to the proton (see Figs. 6 and 8).

Lyman Series $n \to 1$		Fine Structure Transition 255290.0 Paschen Series $n \to 3$	
4570.98	4374.06	33578.1	31527.5
3484.27	3437.35	22031.7	21512.5
3156.71	3137.52	18323.6	18098.5
3008.48	2998.60	16547.3	16425.2
2927.58	2921.78	15528.9	15453.9
2878.22	2874.50	$J=0-0$	$J=1-0$
2845.76	2843.23		
$J=0-0$	$J=1-0$		

Fine Structure Transition 101532.00 Balmer Series $n \to 2$			Fine Structure Transition 583813.0 Paschen Series $n \to 3$	
14655.70	13860.00	38663.5	35969.7	34894.2
10202.60	10004.90	24112.7	23492.1	23225.7
8801.13	8717.08	19740.4	19479.4	19364.7
8142.84	8098.08	17694.2	17554.7	17492.6
7772.09	7745.05	16534.6	16449.6	16411.6
7539.83	7522.11	$J=0-1$	$J=1-1$	$J=2-1$
$J=0-0$	$J=1-0$			

Balmer Series $n \to 2$			Paschen Series $n \to 3$		
17128.10	16051.20	15621.70	38331.4	37112.4	36450.6
11342.40	11098.50	10994.00	24477.1	24188.0	24025.4
9636.45	9535.78	9491.59	20151.8	20029.1	19959.2
8852.83	8799.96	8776.47	18098.9	18032.9	17995.2
8416.35	8384.65	8370.47	16926.6	16886.2	16863.1
8144.66	8123.98	8114.68	$J=1-2$	$J=2-2$	$J=3-2$
$J=0-1$	$J=1-1$	$J=2-1$			

Even order-of-magnitude estimates show that the interaction of the electron spin's magnetic moment with unit monopole gives not a delicate fine structure but significant changes in the eigenvalues. Thus to find the true eigenvalues the Dirac equation is a necessity. Before treating it we clear up some details. We took the form of Eq. (2.2) for $\mathbf{A}$ corresponding to a monopole with a string along $\mu=+1$. For $|N|\geqslant 2$ we could have taken two or more inwardly directed strings of flux. Are these different string configurations really different monopoles or do they all give the same eigenvalues? The effect of such a change is to add a unit flux string along the z axis. It is simple to show that this is equivalent to adding one to m everywhere that it occurs. Provided we do that also to m in the definition of j under Eq. (2.15), the final spectrum remains unchanged. What does change are the K and m values associated with a given j value.

A second detail is the value of m_0, which for $N=0$ would be the reduced mass of the electron, so for hydrogen it is $m_0 = m_e m_p/(m_e+m_p)$.

Particle physicists expect a heavy mass for any monopole, so any monopolar hydrogen will have a nucleus much heavier than the proton, and m_e should be substituted for m_0 in predicting spectra. A third detail for later reference is the energy spectrum of the relativistic Klein-Gordon equation. Here we follow Schiff's treatment and obtain, writing $\alpha_z = Ze^2/(\hbar c)$,

$$E = m_0 c^2 \left\{ \left[1 + \frac{\alpha_z^2}{(n+\Delta_1)^2} \right]^{1/2} - 1 \right\} \tag{2.25}$$

where

$$\Delta_1 = \sqrt{(J+\tfrac{1}{2})(J+\tfrac{1}{2}+|N|) - \alpha_z^2} - (J+\tfrac{1}{2}). \tag{2.26}$$

D. Angular momentum

Returning to the classical conserved quantity of Eq. (1.14) we see the conserved quantity is not the particle's angular momentum $\mathbf{L}=\mathbf{r}\times m_0\mathbf{v}$ but rather that supplemented by $eQc^{-1}\hat{\mathbf{r}}$. The physics behind this supplement lies in the Poynting vector of the electromagnetic field, which carries an angular momentum

$$\frac{1}{4\pi c}\int \mathbf{r}'\times\left(\mathbf{E}\times\frac{Q}{r'^2}\,\hat{\mathbf{r}}'\right)d^3r'$$
$$=\frac{Q}{4\pi c}\int(\mathbf{E}\cdot\nabla)\hat{\mathbf{r}}'d^3r'=\frac{-Q}{4\pi c}\int\hat{\mathbf{r}}'\nabla\cdot\mathbf{E}d^3r'$$
$$=+\frac{eQ}{c}\,\hat{\mathbf{r}}=+\frac{1}{2}\,N\hbar\hat{\mathbf{r}}$$

where $\nabla\cdot\mathbf{E}=-e4\pi\delta^3(\mathbf{r}'-\mathbf{r})$. The total angular momentum is thus $\mathbf{j}=\mathbf{L}+eQc^{-1}\hat{\mathbf{r}}$.

As we saw above in Eq. (2.1), $m_0\mathbf{v}=\mathbf{p}+e\mathbf{A}/c$ in the presence of a magnetic field, so the operator representing $\mathbf{j}$ is $\mathbf{r}\times(-i\hbar\nabla+e\mathbf{A}/c)+\frac{1}{2}N\hbar\hat{\mathbf{r}}$. The commutators $[-i\hbar\partial_j+eA_j/c,-i\hbar\partial_k+eA_k/c]=-i\hbar ec^{-1}\varepsilon_{jkl}B_l=-i\hbar eQc^{-1}\varepsilon_{jkl}x^l/r^3$ and $[-i\hbar\partial_j+eA_j/c,x^k]=-i\hbar\partial_j^k$ enable one to derive the commutator

$$[j_j,j_k]=i\hbar\varepsilon_{jkl}j_l,$$

which demonstrates that $\mathbf{j}$ obeys the angular momentum algebra of the rotation group. One may also demonstrate that $\mathbf{j}^2$ commutes with $\mathbf{j}$ and that $j_\pm=j_x\pm ij_y$ are the raising and the lowering operators for j_z. From which it follows by the usual argument that the eigenvalues of j_z are $-j\hbar$ to $+j\hbar$ and that the eigenvalues of $\mathbf{j}^2$ are $j(j+1)\hbar^2$. But $|\mathbf{j}|^2=|\mathbf{L}|^2+\frac{1}{4}N^2\hbar^2$, so the eigenvalues of $|\mathbf{L}|^2$ are $[j(j+1)-\frac{1}{4}N^2]\hbar^2$. Now looking at our separation of variables expression [Eq. (2.8)] we see that the left-hand side is just $-\hbar^{-2}|\mathbf{L}^2|$ by construction and hence $C=(j(j+1)-\frac{1}{4}N^2)$, which agrees with Eq. (2.16) and identifies the j defined there with the generalized angular momentum eigenvalue defined in this section. Note that for a single Dirac monopole and a nonspinning electron we showed [Eq. (2.16)] that j took half odd-integer values.

In the next section we look at the Dirac equation for a spinning electron. There the correct generalization of the total angular momentum is $\mathbf{j}=\mathbf{L}+\frac{1}{2}N\hbar\hat{\mathbf{r}}+\frac{1}{2}\hbar\boldsymbol{\sigma}$.

This new $\mathbf{j}$ obeys the angular momentum algebra of the rotation group, but now its eigenvalues are $j(j+1)\hbar^2$ with j taking integer (or half integer) values $\geqslant\frac{1}{2}||N|-1|$, depending on whether N is odd or even.

E. Dirac equation

The Dirac equation may be written in standard notation

$$H\psi=[-c\boldsymbol{\alpha}\cdot(\mathbf{p}+e\mathbf{A}/c)-\beta m_0c^2+V]\psi=E\psi.$$

With the newly defined $\mathbf{j}$ we find $d\mathbf{j}/dt=[\mathbf{j},H]=0$, so that each component of this generalized $\mathbf{j}$ commutes with the Hamiltonian, always provided that $\mathbf{A}$ is a vector potential for the monopole. Following Schiff's treatment (1955) we define $p_r=r^{-1}(\mathbf{r}\cdot\mathbf{p}-i\hbar)$ and $\alpha_r=r^{-1}(\boldsymbol{\alpha}\cdot\mathbf{r})$ and $\hbar k=\beta[\boldsymbol{\sigma}'\cdot(\mathbf{r}\times(\mathbf{p}+e\mathbf{A}/c))+\hbar]$. No $\mathbf{A}$ term is needed in p_r since $\mathbf{r}\cdot\mathbf{A}=0$ for our monopole.

The Hamiltonian is now written

$$H=-c\alpha_rp_r-i\,\frac{\hbar c}{r}\,\alpha_r\beta k-\beta m_0c^2+V$$

and as before α_r, β, and p_r all commute with k. The eigenvalues of k follow by squaring the definition and using $\mathbf{L}\times\mathbf{L}=i\hbar(\mathbf{L}-ec^{-1}\mathbf{rr}\cdot\mathbf{B})=i\hbar(\mathbf{L}-\frac{1}{2}N\hbar\hat{\mathbf{r}})$:

$$\hbar^2k^2=(\boldsymbol{\sigma}'\cdot\mathbf{L})^2+2\hbar\boldsymbol{\sigma}'\cdot\mathbf{L}+\hbar^2$$
$$=\mathbf{L}^2+i\boldsymbol{\sigma}'\cdot\mathbf{L}\times\mathbf{L}+2\hbar\boldsymbol{\sigma}'\cdot\mathbf{L}+\hbar^2$$
$$=\mathbf{j}^2+\tfrac{1}{4}\hbar^2(1-N^2),$$

so k^2 has eigenvalues $(j+\frac{1}{2})^2-\frac{1}{4}N^2\geqslant 0$. Notice that for $N=1$ one of these eigenvalues is $k=j=0$. Save for this change of k the usual separation of the Dirac equation goes through unscathed and, following Schiff, one obtains the energy levels

$$E=m_0c^2\left\{\left[1+\frac{\alpha_z^2}{(s+n')^2}\right]^{-1/2}-1\right\}$$

where $s=(k^2-\alpha_z^2)^{1/2}$ and $\alpha_z=Ze^2/(\hbar c)$ and if $n'=0$, $k<0$. Notice there is trouble in s if $k=0$.

Here n' is the radial quantum number. Inserting our eigenvalues $k^2=(j+\frac{1}{2})^2-\frac{1}{4}N^2$ with $j=J+\frac{1}{2}(|N|+1)$ and $J=0,1,2$, etc., we have

$$E=m_0c^2\left\{\left[1+\frac{\alpha_z^2}{(n+\Delta)^2}\right]^{-1/2}-1\right\}$$

where $n=n'+J+1$ and

$$\Delta=\sqrt{(J+1)(J+1+|N|)-\alpha_z^2}-(J+1). \tag{2.27}$$

These energy levels were first derived by Hautot (1972, 1973). We have drawn the bound energy levels that result in Figs. 6, 7, and 8 and derived the wavelengths of the lines of "monopolar hydrogen" with one or two Dirac monopoles attached to the nucleus in Tables I and II. Schwinger (1966) suggested that the unit monopole should have the strength of two Dirac monopoles. With colleagues he also calculated the motions of charged monopoles, dyons, under their mutual attraction (Schwinger *et al.*, 1976). While monopoles may seem esoteric, it is worthwhile looking for lines of monopolar hydrogen in the spectra of exotic astronomical objects.

We now return to the strange history of the $k=0$ levels that do not give real values of s. In 1951 Malkus solved for the energy levels of the Pauli equation, which is the nonrelativistic approximation to the Dirac equation. He found these singlet s levels which in our notation would have $n'=1,2\ldots$ and $J=-1$. They have exactly the same energy levels as the s states of hydrogen without a monopole, and he gives the wave functions of these states. However, the α_z^2 that causes the trouble is neglected in the Pauli approximation because it is of order 10^{-4}. When we make that same approximation we find $s=0$ instead of a complex value, and we get Malkus's extra levels. Why is their existence or nonexistence so sensitive to such a small correction? The answer lies at the end of Sec. I. From our quantum numbers for $\mathbf{j}$, $\mathbf{S}$ and $\mathbf{L}\cdot\mathbf{S}$, we can find C_1 and putting $g=2$ and $Q_*=N\hbar/2$ we can evaluate the coefficient C_2 of the effective r^{-2} term. It is $\hbar^2(k^2 \pm k)$, that is, zero for the $j=0$ states just as it is for $\ell=0$ states of normal hydrogen, so the normal hydrogen s-state spectrum is found, as Malkus (1951) says. But when we move to the relativistic Dirac equation for the monopole, C_2 becomes negative. Without spin we have two ground-state levels with $m-\frac{1}{2}N=\pm\frac{1}{2}$ and an $\mathbf{L}+\frac{1}{2}N\hat{\mathbf{r}}$ of $\frac{1}{2}$. With spin we get a triplet $j=1$ state, but the singlet $j=0$ state with $n'=1$ has $k=0$ and is weakly nonstationary because the energy level has $(k^2-\alpha_z^2)^{1/2}$. The $-\alpha_z^2$ term comes from what Schift calls the "relativistic correction to potential." These states are replaced by propagating waves that take the electron into the monopole. Only the intervention of some term like the final one in our classical expression can stop this, and no term due to the finite moment of inertia of the electron is in the Dirac equation (should it be amended?). If some such term were included there would be a whole hierarchy of extra levels, with the electron very strongly bound by nearly its rest mass. Banderet (1946) already pointed out that because the proton has a larger g factor it likewise has a continuum of negative-energy states that propagate in. We should therefore regard monopolar hydrogen as an atom with an unstable resonance below its lowest truly stationary state, and from this resonance it will decay by propagation into the monopole. Once a monopole has swallowed a proton it becomes charged, and so any other proton will be repelled at large distances. However, electrons are attracted and they can likewise be swallowed. Since the monopolar magnetic field is not altered by this, monopoles may grow continually into surprisingly massive nuclei which, due to the magnetic fields in space, may acquire very large momenta. Perhaps the suggestion that the highest-energy cosmic rays may contain monopoles is not as far fetched as it first seemed. It is important to recognize that, although there are *no* stationary states of a proton in the field of an uncharged monopole, nevertheless there is a bound continuum of propagating states, which would give highly bound levels if the final term in the classical energy were included. Dirac electrons do not fall into uncharged monopoles. Protons only fall in because their magnetic moments give $C_2 \propto -(g-2)$ and so an r^{-3} attraction. Electrons can propagate onto *charged* monopoles because the relativistic correction to the potential energy is attractive.

III. GRAVOMAGNETIC MONOPOLES IN GENERAL RELATIVITY, NUT SPACE

A. NUT space, the general spherically symmetric gravity field

In Sec. I we followed up Newton's remarks and were led to introduce gravomagnetic forces and gravomagnetic monopoles. However, gravomagnetic forces are already part of general relativity, as explained by Landau and Lifshitz, although they are more usually referred to by relativists (who eschew the idea of gravitational forces) as the dragging of inertial frames. Rindler (1997) has recently emphasized that dragging of inertial frames is a poor way of understanding the phenomena and that gravomagnetism gives a better picture. Here by giving the names $\mathbf{E}_g$ and $\mathbf{B}_g$ to the gravitational fields that one finds in any stationary spacetime we emphasize that the electromagnetic analogy gives useful insights even when very strong gravitational fields are present. To define $\mathbf{E}_g$ and $\mathbf{B}_g$ we need a timelike Killing vector so, for the present at least, these fields are only defined for stationary states.

Taub (1951), in his general studies of metrics with several symmetries, discovered the interesting empty cosmological spacetime that is still named after him. By proceeding through its horizon and relabeling appropriately one arrives in NUT space, which was discovered by another method by Newman, Tamburino, and Unti (1963) and is named appropriately both for the authors and for its strange properties. Ehlers (1957) had earlier given his transformation by which one may readily find it. In spite of its symmetries (four Killing vectors) the metric of NUT space while stationary is only axially symmetric. The relationship of NUT space to Taub space and many of its paradoxical properties have been beautifully illuminated by Misner and Taub (1969). NUT space was early recognized as a generalization of Schwarzschild's metric which contained a second parameter besides the mass—the so-called NUT parameter. The fact that NUT space was really the spacetime produced by a mass which had also a gravomagnetic monopole was found by Demiansky and Newman (1966), who also found the three-parameter NUT version of Kerr space which has angular momentum mass and the NUT parameter, which is of course the gravomagnetic monopole strength. Dowker and Roche (1967) independently rediscovered this interpretation of the NUT parameter (as indeed we did 30 years later). A mass endowed with a gravomagnetic monopole is a spherically symmetrical object. Why then is its metric not spherically symmetrical? The reason lies in our common-sense intuition that time is absolute, which we know is untrue. Spherically symmetrical objects do not always have any coordinate system in which the metric has spherical symmetry. They do have to possess many equivalent coordinate systems in which any apparent asymmetry can be made to point in any direction we choose. Just as the vector potential of a magnetic monopole cannot be chosen to be spherically symmetric but can be chosen to have any axis, so it is with the metric in the space of a gravomagnetic monopole. Similar considerations for cylindrical symmetry have been explored by one of us elsewhere (Nouri-Zonoz, 1997). Here we give a new and more elementary derivation of NUT space based on the spherical symmetry of its gravelectric and gravomagnetic fields $\mathbf{E}_g$ and $\mathbf{B}_g$ and the spherical symmetry of its spatial metric γ_{ab}. All these are "gauge" invariant under position-dependent changes in the zero point of the time coordinate. We then prove that, as in Sec. I, all geodesics of NUT space lie on spatial cones and use this interesting theorem to determine the gravitational lensing properties of NUT space. Light rays are not merely bent but twisted as they pass the gravomagnetic monopole lens. The differential twisting produces a characteristic spiral shear in lensed objects. Such spiral lensing is peculiar to gravomagnetic monopoles; it is not produced by the Kerr metric. To complete our survey of classical monopoles we have brief sections on Dirac quantization of monopoles and its implications of a periodicity in time, as well as on the observability of monopoles.

Zelmanov (1956) and Landau and Lifshitz (1966), in developing their very physical approach to general relativity, consider stationary spacetimes and put the metric in the form

$$ds^2=e^{-2\nu}(dx^0-A_\alpha dx^\alpha)^2-\gamma_{\alpha\beta}dx^\alpha dx^\beta \tag{3.1}$$

where $\nu\geqslant 0$ and A_α and $\gamma_{\alpha\beta}$ are independent of $x^0=ct$. (Our ν is $-\frac{1}{2}\nu$ of Landau and Lifshitz.)

However, this form is not unique since a transformation of time zero $x'^0=x^0+\chi(x^\alpha)$ leads to

$$ds^2=e^{-2\nu}(dx'^0-A'_\alpha dx^\alpha)^2-\gamma_{\alpha\beta}dx^\alpha dx^\beta$$

where $A'_\alpha=A_\alpha+\nabla_\alpha\chi$, so under such a change $\mathbf{A}$ undergoes a gauge transformation. Landau and Lifshitz also show that $\gamma_{\alpha\beta}$ can be regarded as a metric of space, i.e., the quotient space V^4/L^1 (Geroch, 1971)—as opposed to spacetime. They show that test bodies following geodesics of spacetime depart from the geodesics of space as if acted on by gravitational forces, which in our notation take the form

$$\mathbf{f}=\frac{m_0}{\sqrt{1-v^2/c^2}}\left(\mathbf{E}_g+\frac{\mathbf{v}}{c}\times e^{-\nu}\mathbf{B}_g\right) \tag{3.2}$$

where the gravitational field

$$\mathbf{E}_g=c^2\nabla\nu \tag{3.3}$$

and

$$\mathbf{B}_g=c^2\mathrm{Curl}\ \ \mathbf{A}. \tag{3.4}$$

The conserved energy of the particle in motion is

$$\varepsilon=m_0c^2e^{-\nu}(1-v^2/c^2)^{-1/2}, \tag{3.5}$$

where $e^{-\nu}<1$ is the redshift factor by which energy is degraded.

Rewriting Landau and Lifshitz's form of Einstein's equations (§95 problem), we find

$$\mathrm{div}\ \mathbf{B}_g=0, \tag{3.6}$$

$$\mathrm{Curl}\ \ \mathbf{E}_g=0, \tag{3.7}$$

$$\mathrm{div}\ \mathbf{E}_g=-c^{-2}\left[4\pi G\,\frac{(\rho c^2+3p)+v^2/c^2(\rho c^2-p)}{1-v^2/c^2}-\frac{1}{2}\,e^{-2\nu}\mathbf{B}_g^2-\mathbf{E}_g^2\right] \tag{3.8}$$

where ρ is the energy density in the rest frame of the fluid, $3p$ is the trace of its pressure tensor, and v its velocity defined locally by local time synchronized along the fluid's motion. For nonrelativistic velocities this equation reduces to Poisson's equation with the primary term on the right being $4\pi G\rho$. The remaining term has the form of a negative-energy density contributed by the gravity fields. The next equation takes the form

$$\mathrm{Curl}(e^{-\nu}\mathbf{B}_g)=-c^{-3}(16\pi G\mathbf{j}_g-2c\mathbf{E}_g\times e^{-\nu}\mathbf{B}_g). \tag{3.9}$$

Notice that $e^{-\nu}\mathbf{B}_g$ occurs also in that combination in the expression for the force. It is attractive to regard the final term as an energy current corresponding to a Poynting vector flux of gravitational field energy. Here $\mathbf{j}_g$, the matter energy current, is given by

$$\mathbf{j}_g=\frac{\rho c^2+p}{1-v^2/c^2}\,\mathbf{v}.$$

The final Landau and Lifshitz equation for the three stress tensor is

$$P^{\alpha\beta}-E_g^{\alpha;\beta}=\left(T^{\alpha\beta}+\tfrac{1}{2}T\gamma^{\alpha\beta}\right)+e^{-2\nu}(B_g^\alpha B_g^\beta-B_g^2\gamma^{\alpha\beta})+E_g^\alpha E_g^\beta, \tag{3.10}$$

where $P^{\alpha\beta}$ is the three-dimensional Ricci tensor constructed from the metric $\gamma^{\alpha\beta}$. Those familiar with the Maxwell stresses of magnetic and electric fields in, say, magnetohydrodynamics will find some interest in the field terms on the right. The matter terms may be rewritten as physical quantities for an isotropic fluid in motion,

$$T^{\alpha\beta}+\frac{1}{2}\,T\gamma^{\alpha\beta}=\frac{8\pi G}{c^4}\left[\frac{(p+\rho c^2)v^\alpha v^\beta}{c^2-v^2}+\frac{1}{2}\,(\rho c^2-p)\,\gamma^{\alpha\beta}\right].$$

It should be stressed that all these equations hold good even when spacetime is strongly curved. Unlike some treatments they are not restricted to nearly flat space, but it is assumed that the spacetime is stationary.

To find the general spherically symmetric solution for empty space we take $dl^2=\gamma_{\alpha\beta}dx^\alpha dx^\beta=e^{2\lambda}dr^2+r^2d\hat{\mathbf{r}}^2$ where $\hat{\mathbf{r}}$ is the unit Cartesian vector $(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$. Then $d\hat{\mathbf{r}}^2=d\theta^2+\sin^2\theta\, d\phi^2$, but the advantage of the vector notation is that no axis for θ,ϕ need be taken. In spherical symmetry $\mathbf{B}_g$ must be radial and divergenceless, so Gauss's theorem gives $|\mathbf{B}_g|r^2=Q=\mathrm{const}$ which is the field of a gravomagnetic monopole,

$$B_g^r=-Qe^{-\lambda}/r^2. \tag{3.11}$$

Reinserting $\mathbf{E}_g=c^2\nabla\nu$ into Eq. (3.8) we have

$$R_{00}=-\nu''+\nu'^2-2\nu'/r+\lambda'\nu'+\tfrac{1}{2}e^{2(\lambda-\nu)}Q^2(cr)^{-4}=0. \tag{3.12}$$

To form P^α_β we need the three-dimensional Christoffel symbols

$$\lambda^\sigma_{\mu\nu}=\tfrac{1}{2}\gamma^{\sigma\eta}(\gamma_{\eta\mu,\nu}+\gamma_{\eta\nu,\mu}-\gamma_{\mu\nu,\eta}), \tag{3.13}$$

which are

$$\lambda^{\sigma}_{\mu\sigma}=\tfrac{1}{2}\gamma\ \gamma_{,\mu}\quad \lambda^{\sigma}_{\phi\phi}=-\tfrac{1}{2}\gamma^{\sigma\eta}\gamma_{\phi\phi,\eta}$$

$$\lambda^{\sigma}_{rr}=\begin{cases}0 & \sigma\neq r\\ \tfrac{1}{2}\gamma^{rr}\gamma_{rr,r} & \sigma=r\end{cases}$$

$$\lambda^{\sigma}_{\theta\theta}=\begin{cases}0 & \sigma\neq r\\ -\tfrac{1}{2}\gamma^{rr}\gamma_{\theta\theta,r} & \sigma=r\end{cases}$$

$$\lambda^{\sigma}_{\phi\sigma}=0\quad \lambda^{\sigma}_{r\tau}=\tfrac{1}{2}\gamma^{\sigma\eta}\gamma_{\eta\tau,r}. \tag{3.14}$$

Equations (3.6), (3.7), and (3.9) are identically satisfied. The surviving equations of Eq. (3.10) are

$$R^{rr}=-\nu''+\nu'^2+\lambda'\nu'-2\lambda'/r=0 \tag{3.15}$$

and

$$R^{\theta\theta}=R^{\phi\phi}=\lambda' e^{-2\lambda}-\frac{e^{-2\lambda}}{r}+\frac{1}{r}+\frac{1}{2}\,e^{-2\nu}Q^2c^{-4}r^{-3}$$
$$+\nu' e^{-2\lambda}=0. \tag{3.16}$$

Equations (3.12), (3.15), and (3.16) must be solved for ν and λ. Eliminating ν'' from Eqs. (3.12) and (3.15) we find

$$2(\lambda'-\nu')+\tfrac{1}{2}e^{2(\lambda-\nu)}Q^2c^{-4}r^{-3}=0, \tag{3.17}$$

which integrates on division by $e^{2(\lambda-\nu)}$, giving

$$e^{-2(\lambda-\nu)}=-q^2r^{-2}+C \tag{3.18}$$

where $q=Q/2c^2$ which has the dimensions of a length. Multiplying Eq. (3.16) by $e^{2\nu}$ and using Eqs. (3.17) and (3.18) we have

$$(C-q^2r^{-2})(r^{-1}-2\nu')=q^2r^{-3}+e^{+2\nu}/r.$$

Dividing by $e^{2\nu}(C-q^2r^{-2})$ we obtain

$$(e^{-2\nu})'+\frac{1}{r}\left(\frac{Cr^2-2q^2}{Cr^2-q^2}\right)e^{-2\nu}-\frac{r}{Cr^2-q^2}=0,$$

which is linear in $e^{-2\nu}$ and readily solved by integrating factors to give

$$e^{-2\nu}=\frac{1}{C}-\frac{2q^2}{C^2r^2}+\frac{2\bar{C}}{r^2}\sqrt{Cr^2-q^2} \tag{3.19}$$

where C and $\bar{C}$ are both constants. It follows from Eq. (3.18) that

$$\gamma_{rr}=e^{2\lambda}=(C-q^2r^{-2})^{-1}e^{+2\nu}. \tag{3.20}$$

To get $e^{-2\nu}$ and $\gamma_{\alpha\beta}$ asymptotically of Schwarzschild form we need $C=1$ and $\bar{C}=-\tilde{m}$, the asymptotic mass GM/c^2. Thus we find

$$g_{00}=e^{-2\nu}=1-2r^{-2}(q^2+\tilde{m}\sqrt{r^2-q^2}), \tag{3.21}$$

$$\gamma_{rr}=(1-q^2r^{-2})e^{+2\nu}, \tag{3.22}$$

which are the metric components of NUT space. Notice that when $Q=2qc^2=0$ this reduces to Schwarzschild's metric. The metric is completed by taking a vector potential A_α for the gravomagnetic field $\mathbf{B}_g$; any one will do, since they are connected by gauge transformation which merely changes the zero point of time. As we saw in Sec. I, it is impossible to choose a spherically symmetric vector potential, but this does not affect the spherical symmetry of the physics. A suitable vector potential is that given in Eq. (1.19), which gives us the metric

$$ds^2=e^{-2\nu}[c\,dt-2q(1+\cos\theta)d\phi]^2$$
$$-(1-q^2/r^2)e^{+2\nu}dr^2-r^2d\hat{\mathbf{r}}^2, \tag{3.23}$$

where $e^{-2\nu}$ is given by Eq. (3.21). This metric is more commonly written in terms of the radial variable $\tilde{r}=\sqrt{r^2-q^2}$ because the square roots disappear, leaving an analytic expression. However, we have preferred the variable that makes the surface area of the sphere $4\pi r^2$ as in Schwarzschild space. Of course the metric of Eq. (3.23) appears to have a preferred axis, but this is illusory because we can switch it into any direction we like by a gauge transformation; see the discussion under Eq. (1.20). The horizon where g_{00} changes sign is given by $\tilde{r}=\tilde{m}+\sqrt{q^2+\tilde{m}^2}$, at which point γ_{rr} changes sign, as in Schwarzschild space.

B. Orbits and gravitational lensing by NUT space

The geodesics of NUT space may be determined from $\delta\int ds=0$ using the metric in the form of Eq. (3.1). When $ds^2\neq 0$ we write τ for the proper time and when $ds^2=0$ we replace it by an affine parameter (which we also call τ). Varying $\dot{t}=dt/d\tau$ and using the fact that the metric is stationary we have

$$e^{-2\nu}(c\dot{t}-A_\alpha\dot{x}^\alpha)=\varepsilon=\text{const.} \tag{3.24}$$

Varying x^α we find

$$\delta x^\alpha\left\{\frac{d}{d\tau}\,[e^{-2\nu}(c\dot{t}-A_\beta\dot{x}^\beta)A_\alpha+\gamma_{\alpha\beta}\dot{x}^\beta]\right.$$
$$\left.+\frac{1}{2}\,\frac{\partial}{\partial x^\alpha}\,[e^{-2\nu}(c\dot{t}-A_\beta\dot{x}^\beta)^2-\gamma_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma]\right\}. \tag{3.25}$$

Using Eq. (3.24) and transferring the $\varepsilon dA_\alpha/d\tau=\varepsilon\dot{x}^\beta\partial_\beta A_\alpha$ term into the second bracket we find the equation of motion, in which A occurs only through $\partial_\alpha A_\beta-\partial_\beta A_\alpha=\eta_{\alpha\beta\gamma}B_g^\gamma$ where $\eta_{\alpha\beta\gamma}$ is the antisymmetric tensor, $\sqrt{\gamma}$ times the alternating symbol,

$$\delta x^\alpha\left[\frac{d}{d\tau}\,(\gamma_{\alpha\beta}\dot{x}^\beta)+\frac{1}{2}\,\frac{\partial}{\partial x^\alpha}\,(e^{-2\nu})\varepsilon^2e^{+4\nu}\right.$$
$$\left.-\frac{1}{2}\,\gamma_{\beta\gamma,\alpha}\dot{x}^\beta\dot{x}^\gamma-\varepsilon\,\eta_{\alpha\beta\gamma}B_g^\gamma\dot{x}^\beta\right]=0. \tag{3.26}$$

We now write $\gamma_{\alpha\beta}$ in the form involving the unit Cartesian vector $\hat{\mathbf{r}}$,

$$\gamma_{\alpha\beta}dx^\alpha dx^\beta=e^{2\lambda}dr^2+r^2(d\hat{\mathbf{r}})^2.$$

Here $\delta\hat{\mathbf{r}}$, the variation of $\hat{\mathbf{r}}$, is an arbitrary small vector perpendicular to $\hat{\mathbf{r}}$. Thus making variations with r fixed we deduce from Eq. (3.26), using Eq. (3.11) for B_g^r,

$$\delta\hat{\mathbf{r}}\cdot\left[\frac{d}{d\tau}\left(r^2\,\frac{d\hat{\mathbf{r}}}{d\tau}\right)+\varepsilon\,\frac{d\hat{\mathbf{r}}}{d\tau}\times Q\hat{\mathbf{r}}\right]=0.$$

Since $\delta\hat{\mathbf{r}}$ is an arbitrary vector perpendicular to $\hat{\mathbf{r}}$ we deduce that

$$\hat{\mathbf{r}}\times\frac{d}{d\tau}\left(r^2\frac{d\hat{\mathbf{r}}}{d\tau}\right)=\frac{d\mathbf{L}}{d\tau}=-\frac{d}{d\tau}(\varepsilon Q\hat{\mathbf{r}}),$$

$$\mathbf{L}+\varepsilon Q\hat{\mathbf{r}}=\mathbf{j}=\text{const}. \tag{3.27}$$

Except for the factor ε, which reduces to m_0c^2 in the nonrelativistic case, we see that this is precisely the vector integral (1.14). Dotting Eq. (3.27) with $\hat{\mathbf{r}}$, we find $\mathbf{j}\cdot\hat{\mathbf{r}}=\varepsilon Q$, showing that $\hat{\mathbf{r}}$ lies on a cone. Similarly $\mathbf{L}\cdot\mathbf{j}=L^2=\mathbf{j}^2-\varepsilon^2Q^2=\text{const}$, so $\mathbf{L}$ moves around a cone. The radial equation of motion is redundant, since we may use the energy and the equation $(ds/d\tau)^2=U=1$ or 0 instead. U is 1 for timelike geodesics and 0 for lightlike ones.

This gives us

$$\varepsilon^2e^{+2\nu}-\dot{r}^2e^{2\lambda}-L^2r^{-2}=U. \tag{3.28}$$

To see the geometry of the trajectory we introduce the curvilinear angle φ of Sec. I measured around the cone's surface. Then $r^2\dot{\varphi}=L$, so Eq. (3.1) can be integrated by quadrature,

$$\varphi-\varphi_0=\int\frac{Lr^{-2}dr}{\sqrt{\varepsilon^2e^{-2(\lambda-\nu)}-(U+L^2r^{-2})e^{-2\lambda}}}. \tag{3.29}$$

In general this integral cannot be performed explicitly for the λ and ν of NUT space even after substitution in terms of $\tilde{r}$ to make it more analytic. We therefore turn to the $r^2\gg q^2+\tilde{m}^2$ limit well away from the event horizon. This is the important case in all gravitational lenses observed to date. In that limit the q^2/r^2 term in the effective potential is attractive and therefore of the wrong sign to give the nonprecessing orbits of Sec. I. The precession around the cones is faster than in the classical Kepler+monopole problem by a factor 3/2. To the first order in $\tilde{m}/b$, where b is the impact parameter, we find a bending angle measured like φ of $\Delta\varphi=4\tilde{m}/b$ just as for the Schwarzschild metric; however, the difference is that this angle is measured around a cone, not in a plane. Again to first order we can find the effect of the gravomagnetic field by integrating the momentum transfer along the unperturbed straight-line path. This gives an out-of-plane bending of $4q/b$; a result that is confirmed by the full NUT-space calculation [Nouri-Zonoz and Lynden-Bell (1997)]. Thus the major effect of the gravomagnetic monopole Q is to twist the rays that pass it. While the bending angle is proportional to b^{-1}, the effect is exaggerated when looking down the line toward the NUT lens by the factor D_L/b, so the twist around the lens is $4qD_L/b^2$. Here D_L is the distance from the observer to the lens. The same exaggeration factor occurs for the normal gravitational bending, so for a source at infinity and an image at (b,θ) in the plane of the sky at the lens distance, the apparent position of the source is

$$(b_s,\theta_s)=\left[b\left(1-\frac{4\tilde{m}D_L}{b^2}+\frac{8q^2D_L^2}{b^4}\right),\theta-\frac{4qD_L}{b^2}\right].$$

This expression defines a map from image to source. From this map one can work out both the shear and the magnification of a NUT lens in the large-impact-parameter régime. The magnification of area and thus luminosity is, dropping the tilde on m,

$$db^2/db_s^2=[1-16b^{-4}D_L^2(m^2+q^2)]^{-1}. \tag{3.30}$$

A small circular source will be imaged into an ellipse of axial ratio

$$\frac{b^2+4D_L(m+\sqrt{m^2+q^2})}{b^2+4D_L(m-\sqrt{m^2+q^2})},$$

with the short axis of the ellipse inclined to the radius at the angle (see Fig. 9):

$$\tan^{-1}\left(\frac{q}{m+\sqrt{m^2+q^2}}\right).$$

This is 45° for $q\gg m$ and 13° for $Q=2qc^2=mc^2$. This spiral conformation of the images about a NUT lens is very characteristic. It is not displayed in normal gravitational imaging, and the gravomagnetic lens due to a rotating object seen pole on does not show it because the twist of the ray as it approaches such a lens is canceled by the opposite twist as it recedes. Thus the discovery of a spiral shear field about a lens would indicate the presence of a gravomagnetic monopole. Such effects should be looked for by those studying gravitational lenses. The expectation must be small, but the reward might be an amazing discovery.

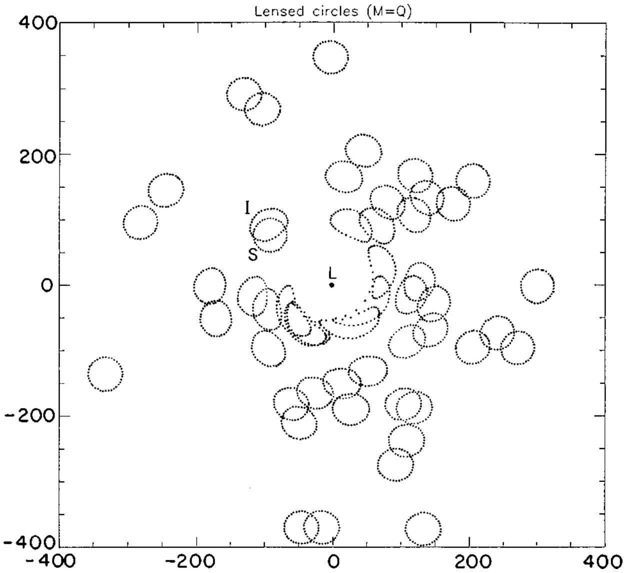

FIG. 9. Gravitational lensing by NUT space of a small circular source at S appears as an inclined ellipse at the image I. Many such images make a spiral effect around the NUT lens L.

C. Quantization of gravomagnetic monopoles and their classical physics

By analogy with Dirac's argument for the quantization of magnetic monopoles and charges, Dowker and Roche (1967), Dowker (1974), Hawking (1979), and Zee (1985) have suggested quantization of gravomagnetic monopoles and energy. Corresponding to Dirac's $Q_m e = \frac{1}{2}N\hbar c$ for magnetic Q_m, they have $Qm_0 = \frac{1}{2}N\hbar c$ for a gravomagnetic monopole Q. This implies that both Q and mass m_0 are quantized in conjugate units Q_1 and m_1 obeying $Q_1 m_1 = \frac{1}{2}\hbar c$. Whereas such ideas are naturally attractive, they do not naturally lead to a self-consistent relativistic theory. For instance, looking at the Klein-Gordon equation in NUT space and separating variables with $\psi \propto e^{i(m\phi+\omega t)}$, one finds an eigenvalue equation for ω. The Dirac monopole quantization condition, $Q(\hbar\omega/c^2) = \frac{1}{2}N\hbar c$ with N an integer, shows us that the only possible eigenvalues ω are integer multiples of $\frac{1}{2}c^3/Q$ and the corresponding energy $\hbar\omega$ is the total energy of the "orbit" including rest mass. However, this condition conflicts with the energies of the bound states,[1] which are not integer multiples of any unit (Mueller and Perry, 1986). Thus if such ideas are viable at all a more radical change in basic theory is needed. In $+++-$ NUT space it does not appear to be possible to build a consistent quantum theory like Dirac's magnetic monopole theory. This is what Ross (1983) concluded and is related to Misner's (1963) finding that NUT space contains closed timelike lines, with time being periodic every $8\pi q/c$. For a discussion of the energy levels in $++++$ Taub-NUT space, the reader is referred to the paper by Gibbons and Manton (1986). This space was shown to be relevant to the interactions of monopoles by Atiyah and Hitchin (1985a, 1985b). If magnetic monopoles exist, Maxwell's equations must be changed to include $\operatorname{div}\mathbf{B} = 4\pi\rho_m$, $\operatorname{Curl}\mathbf{E} + (1/c)\times(\partial\mathbf{B}/\partial t) = 4\pi\mathbf{j}_m$, where ρ_m is the monopole density and $\mathbf{j}_m$ is the monopole current density. Such modified Maxwell equations do not come with a vector potential. It is natural to ask how general relativity must be modified to allow for gravomagnetic monopole densities and currents. While this is not so obvious, we conjecture the generalization will be to spaces with unsymmetric affine connections which have nonzero torsion. It would be interesting to demonstrate this conjecture, as it could introduce a greater degree of physical understanding of those spaces.

IV. OBSERVABILITY

Following Kibble's (1980) suggestion that magnetic monopoles would be a natural consequence of the Big Bang, they have long been sought.

We have concentrated on the spectra of monopolar atoms and the lensing properties of gravomagnetic monopoles since these are ways in which, at least in principle, monopoles might be discovered observationally. Spectroscopically one may argue that the best place to look is in the spectra of supernovae, quasars, or active galactic nuclei, where the basic Lyman α lines of Tables I or II might be seen as very weak absorption lines in very-high-resolution spectra. Quasars have the advantage that these lines will be shifted into the visible. We have looked at IUE ultraviolet spectra of Supernova 1987A and seen no lines at the wavelengths 2774.62 or 2733.78. More supernovae and stacked high-resolution spectra of quasars should be pursued, although in regions of observed magnetic fields the limits obtained spectroscopically will fall far short of the Parker (1970) bound. While the nature of the dark matter that constitutes most mass in the universe remains unknown, such esoteric possibilities are worth pursuit.

Searches on Earth have produced one unrepeatable event and a monopolar observatory under the Gran Sasso that has so far found no monopoles in cosmic rays. There has been a speculative suggestion, Kephart and Weiler (1996), that the leveling up on the numbers of cosmic rays at the highest energies might be due to monopoles, but there is no confirmation of that idea. To date the best limit on the numbers of monopoles in interstellar space comes from the Parker (1970, 1971a, 1971b) bound. This arises from the idea that too many magnetic monopoles would "short out" the galactic magnetic fields that are observed. A good general discussion of such limits may be found in the book of Kolb and Turner (1991). For more recent work on monopoles in field theory see reviews by Olive (1996, 1997) and the papers by Sen (1994) and by Seiberg and Witten (1994). More details of the fundamental work on monopoles in field theory by 't Hooft (1974) and by Polyakov (1974) can be found in the review by Goddard and Olive (1978).

[1]To get definite bound states one must impose a potential barrier, so that the black hole is not reached.

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REFERENCES

Aharanov, Y., and D. Bohm, 1959, Phys. Rev. **115**, 485.
Atiyah, M. F., and N. J. Hitchin, 1985a, Phys. Lett. A **107**, 21.
Atiyah, M. F., and N. J. Hitchin, 1985b, Philos. Trans. R. Soc. London, Ser. A **315**, 459.
Banderet, P. P., 1946, Helv. Phys. Acta **19**, 503.
Cajori, F., 1934, *Newton's Principia, Motte's Translation Revised* (University of California, Berkeley).
Chandrasekhar, S., 1995, *Newton's Principia for the Common Reader* (Oxford University Press, London).
Corben, H. C., and Stehle, P., 1960, *Classical Mechanics*, 2nd Ed. (Dover, New York), p. 259, Eq. 78.6.
de Bröglie, L., 1925, Ann. Phys. (Paris) **3**, 22.
Demianski, M., and E. T. Newman, 1966, Bull. Acad. Pol. Sci. Ser. Math. Astron. Phys. XIV, 653.

Dirac, P. A. M., 1931, Proc. R. Soc. London **133**, 60.
Dirac, P. A. M., 1948, Phys. Rev. **74**, 817.
Dowker, J. S., 1974, Gen. Relativ. Gravit. **5**, 603.
Dowker, J. S., and J. A. Roche, 1967, Proc. Phys. Soc. London **92**, 1.
Ehlers, J., 1957, dissertation (University of Hambourg).
Evans, N. W., 1990, Phys. Rev. A **41**, 5666.
Evans, N. W., 1991, J. Math. Phys. **32**, 3369.
Evans, N. W., P. T. de Zeeuw, and D. Lynden-Bell, 1990, Mon. Not. R. Astron. Soc. **244**, 111.
Geroch, R., 1971, J. Math. Phys. **12**, 918.
Gibbons, G. W., and N. S. Manton, 1986, Nucl. Phys. B **274**, 183.
Goddard, P., and D. I. Olive, 1978, Rep. Prog. Phys. **41**, 1357.
Goldhaber, A. S., 1965, Phys. Rev. B **140**, 1407.
Goldhaber, A. S., 1976, Phys. Rev. Lett. **36**, 1122.
Goldstein, H., 1980, *Classical Mechanics, 2nd Ed.* (Addison-Wesley, Reading, MA), p. 124, example 23.
Hamilton, W. R., 1847, in Proc. R. Irish Acad. III, Appendix, p. 36.
Harish-Chandra, H., 1948, Phys. Rev. **74**, 883.
Hautot, A., 1972, J. Math. Phys. **13**, 710.
Hautot, A., 1973, J. Math. Phys. **14**, 201.
Hawking, S. W., 1979, *General Relativity, an Einstein Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England), p. 746.
Henon, M., 1959, Annals d'Astrophys. **22**, 126 and **23**, 474.
Hoang, L. V., L. X. Hai, L. I. Komarov, and J. S. Romanova, 1992, J. Phys. A **25**, 6461.
Kazama, Y., and C. N. Yang, 1976, Phys. Rev. D **15**, 2300.
Kazama, Y., C. N. Yang, and A. S. Goldhaber, 1977, Phys. Rev. D **15**, 2287.
Kephart, T. J., and T. W. Weiler, 1996, Astropart. Phys. **4**, 271.
Kibble, T. W. B., 1980, Phys. Rep. **67**, 183.
Kolb, E. W., and M. S. Turner, 1991, *The Early Universe* (Addison-Wesley, New York).
Landau, L. D., and E. M. Lifshitz, 1966, *Classical Theory of Fields* (Pergamon, New York), §89 problem 1, §95 problem.
Lynden-Bell, D., 1963, Observatory **83**, 23.
Lynden-Bell, D., 1996, Notes and Records of the Royal Society, London **50**, 253.
Malkus, W. V. R., 1951, Phys. Rev. **83**, 899.
Misner, C. W., 1963, J. Math. Phys. **4**, 924.
Misner, C. W., and A. H. Taub, 1969, Sov. Phys. JETP **28**, 122.
Mueller, M., and M. J. Perry, 1986, Class. Quantum Grav. **3**, 65.
Newman, E. T., L. Tamburino, and T. Unti, 1963, J. Math. Phys. **4**, 915.
Newton, I., 1686, *Philosophiae Naturalis Principia Mathematicia* (Royal Society, London).
Nouri-Zonoz, M., 1997, Class. Quantum Grav. **14**, 3123.
Nouri-Zonoz, M., and D. Lynden-Bell, 1997, Mon. Not. R. Astron. Soc. **292**, 714.
Olive, D. I., 1996, Nucl. Phys. B, Proc. Suppl. **45**, 88.
Olive, D. I., 1997, *Roy. Soc. Dirac Commemoration Symposium*.
Parker, E. N., 1970, Astrophys. J. **160**, 383.
Parker, E. N., 1971a, Astrophys. J. **163**, 225.
Parker, E. N., 1971b, Astrophys. J. **166**, 295.
Pauling, L., and E. B. Wilson, 1935, *Introduction to Quantum Mechanics* (McGraw-Hill, New York), Chap. V.
Poincaré, H., 1896, C. R. Acad. Sci. **123**, 530.
Polyakov, A. M., 1974, JETP Lett. **20**, 194.
Rindler, W., 1997, preprint.
Ross, D. K., 1983, J. Math. Phys. **24**, 1814.
Schiff, L. I., 1955, *Quantum Mechanics, 2nd Edition* (McGraw-Hill, New York), §44, p. 331.
Schrödinger, E., 1922, Z. Phys. **12**, 13.
Schrödinger, E., 1926, Ann. Phys. (Leipzig) **79**, 361, 489, 734.
Schwinger, J., 1966, Phys. Rev. **144**, 1087.
Schwinger, J., K. A. Milton, W.-Y. Tsai, L. L. De Raad, and D. C. Clark, 1976, Ann. Phys. (N.Y.) **101**, 451.
Seiberg, N., and E. Witten, 1994, Nucl. Phys. B **431**, 484.
Sen, A., 1994, Phys. Lett. B **329**, 217.
Taub, A. H., 1951, Ann. Math. 53, 472.
't Hooft, G., 1974, Nucl. Phys. B **79**, 276.
Umbach, C. P., C. Van Haesendonck, R. B. Laibowitz, S. Washburn, and R. A. Webb, 1986, Phys. Rev. Lett. **56**, 386.
Villalba, V. M., 1994, Phys. Lett. **193**, 218.
Villalba, V. M., 1995, J. Math. Phys. **36**, 3332.
Wat, S., and C. N. Yang, 1976, Nucl. Phys. B **107**, 365.
Weyl, H., 1918, *Berlin Sitzungsberichte, 30th May*, p. 465.
Whiteside, D. T., 1967–80, *Mathematical Papers of Isaac Newton VI 1674–1691* (Cambridge University Press, Cambridge, England).
Yang, C. N., 1987, in *Schrödinger Centenary*, edited by C. W. Kilminster (Cambridge University Press, Cambridge, England), p. 57.
Zee, A., 1985, Phys. Rev. Lett. **55**, 2379.
Zelmanov, A. L., 1956, Astron. Zh. **33**, 227.