Quasinormal-mode expansion for waves in open systems

E. S. C. Ching, P. T. Leung, A. Maassen van den Brink, W. M. Suen,* S. S. Tong, and K. Young

Department of Physics, The Chinese University of Hong Kong, Hong Kong, China

An open system is not conservative because energy can escape to the outside. As a result, the time-evolution operator is not Hermitian in the usual sense and the eigenfunctions (factorized solutions in space and time) are no longer normal modes but quasinormal modes (QNMs) whose frequencies ω are complex. Qausinormal-mode analysis has been a powerful tool for investigating open systems. Previous studies have been mostly system specific, and use a few QNMs to provide approximate descriptions. Here the authors review developments that lead to a unifying treatment. The formulation leads to a mathematical structure in close analogy to that in conservative, Hermitian systems. Hence many of the mathematical tools for the latter can be transcribed. Emphasis is placed on those cases in which the QNMs form a complete set and thus give an exact description of the dynamics. In situations where the QNMs are not complete, the "remainder" is characterized. Applications to optics in microspheres and to gravitational waves from black holes are given as examples. The second-quantized theory is sketched. Directions for further development are outlined.
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I. INTRODUCTION

A. Physical motivation

Many concepts in physics rely on the idea of the normal modes of a conservative system. Thus one speaks of energy eigenstates, molecular orbitals, energy bands, transitions between states, and excitation energies (at the first-quantized level) or of propagators in a Feynman diagram (at the second-quantized level). These concepts for interacting quantum fields are rooted in eigenfunction expansions in terms of the normal modes of free classical fields; e.g., the photon propagator in QED depends on the free classical electromagnetic (EM) field being expanded in plane waves. Such expansions are possible because conservative systems are associated with Hermitian operators.

On the other hand, if energy can escape to the outside, the system would be open and nonconservative, and the associated mathematical operators would not be Hermitian in the usual sense. The eigenfunctions are then quasinormal modes (QNMs) with complex frequencies ω. Figure 1(a) shows schematically the EM spectrum observed outside a linear optical cavity of length a, due to emission by a broadband source, e.g., fluorescent dye molecules. The resonances or QNMs, spaced by $\Delta\omega \approx \pi c/a$ (where c is the speed of light), are characteristic of the cavity rather than of the source. The resonance width γ is determined (in the absence of absorption) by the amount of leakage. In the limit of zero leakage, these QNMs reduce to the normal modes of a conservative cavity. It would be both interesting and useful if the physics of open systems could be discussed in terms of the discrete QNMs, providing an eigenfunction expansion for the dynamics.

This task is nontrivial because there are very few known results for eigenfunction expansions in nonconservative, non-Hermitian systems. Yet interestingly, as we shall show in this article, the familiar formalism, with very little alteration, applies to a wide class of them.

The problem of EM waves escaping from an open cavity is directly relevant for cavity QED phenomena: in Fabry-Perot cavities (De Martini *et al.*, 1987, Heinzen *et al.*, 1987), in superconducting microwave cavities (Kleppner, 1981, Goy *et al.*, 1983), in semiconductor heterostructures Yokoyama *et al.*, 1990) and in microspheres that confine glancing rays by total internal reflection (see, for example, Chang and Campillo, 1996; Sandoghdar *et al.*, 1996). First, the spectrum is dominated by the resonances. Figure 1(b) shows an experimental spectrum observed from a microsphere, similar in essence to Fig. 1(a). Second, the decay rate of an excited atom or molecule is enhanced if the emitted radiation falls on the resonances and suppressed if the emitted radiation falls away from the resonances (Kleppner, 1981; De Martini *et al.*, 1987; Heinzen *et al.*, 1987; Barnes *et al.*, 1996). This phenomenon is intriguing as the atom or the molecule of dimension ~0.1 nm "knows" about its environment, on a scale of ~1 μm, even before any photon is emitted.

*Permanent address: McDonnell Center for the Space Sciences, Department of Physics, Washington University, St. Louis, MO 63130.

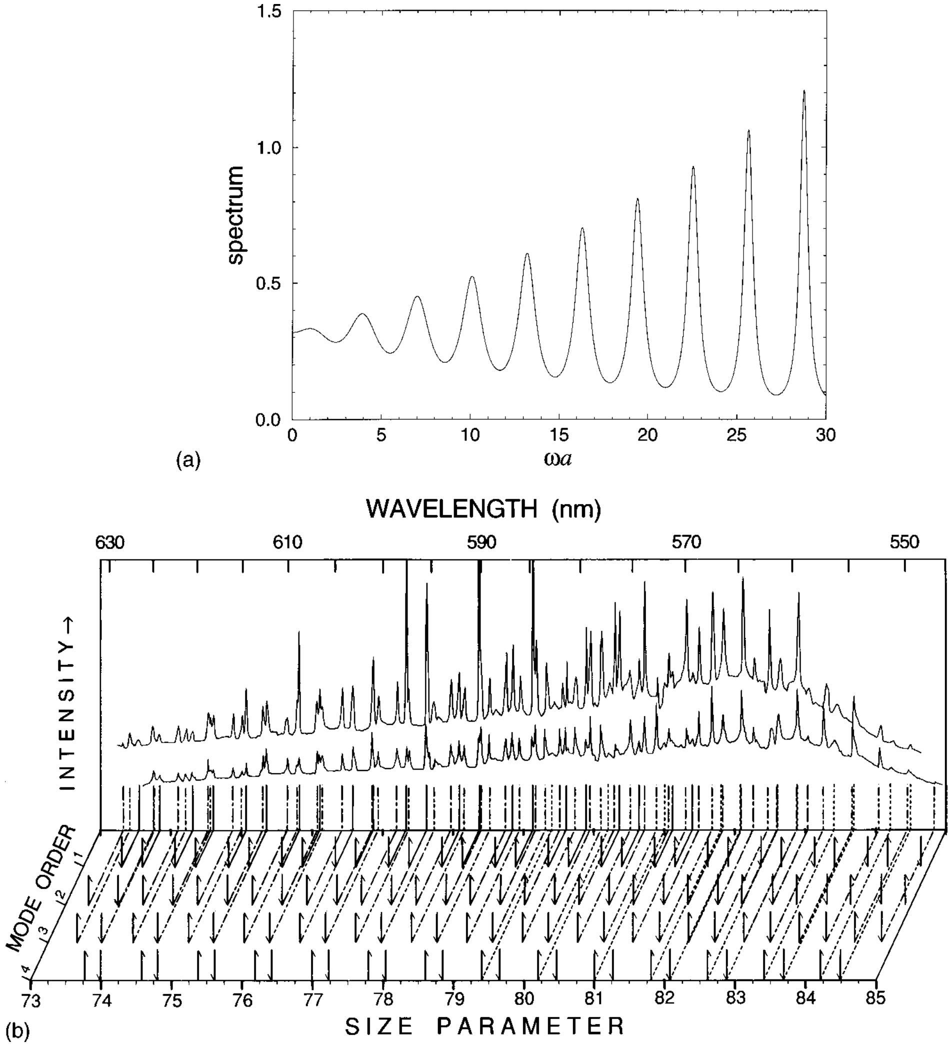

FIG. 1. Electromagnetic spectrum observed outside a cavity. (a) Electromagnetic spectrum (schematic) observed outside a linear optical cavity of length a, due to a broadband source inside the cavity. (b) Fluorescent spectrum (upper) of a dye-doped ethanol microsphere. Resonance peaks correlate well with the computed resonances (lower). Figure 1(b) provided by A. J. Campillo and J. D. Eversole.

Another example concerns gravitational waves, which can be produced by matter falling into a black hole and may soon be observed (see, for example, Abramovici *et al.*, 1992). The intervening space is curved, leading to nontrivial wave propagation and scattering, which can be described by an effective potential, from which the gravitational waves eventually escape. In this regard, this region of space is much like an optical cavity, which likewise causes nontrivial EM wave propagation and scattering, and from which the EM waves eventually escape. Numerical simulations (Vishveshwara, 1970; Detweiler and Szedenits, 1979; Smarr, 1979; Stark and Piran, 1985; Anninos *et al.*, 1993) show that the amplitude is dominated, at intermediate times, by a ringing signal $\Sigma a_j e^{-i\omega_j t}$ (Fig. 2). Each term, labeled by an index j, corresponds to a QNM, and the complex frequencies ω_j contain information about the background geometry rather than about the emitting source.

Open systems are a special class of dissipative systems in which the loss of energy is due to interaction with the surroundings or a bath (Feynman and Vernon, 1963; Ullersma, 1966; Caldeira and Leggett, 1983). The proper treatment is first to include the bath degrees of freedom and then to eliminate them from the path integral or equations of motion. Similarly, an open cavity can be embedded in a "universe" (linear dimension $\Lambda \to \infty$), with the space outside the cavity being the bath. The totality of cavity plus bath is conservative, with the modes of the universe forming a continuum ($\Delta\omega \propto \Lambda^{-1} \to 0$). A rigorous theory of lasing in 1D cavities has been developed using the modes of the universe (Lang, Scully, and Lamb, 1973; Lang and Scully, 1973), and the ideas can be generalized to higher dimensions, e.g., to optics in microspheres (see, for example, Ching, Lai, and Young, 1987a; 1987b).

The concept of resonance domination for cavity QED was first discussed by Purcell (1946). He proposed that, in the Fermi golden rule, the density of states per unit volume $d_0(\omega) = \omega^2/(\pi^2 c^3)$ should be replaced by $d(\omega) \sim \mathcal{N}/(2\gamma V)$ for an $\mathcal{N}$-fold degenerate QNM of width γ

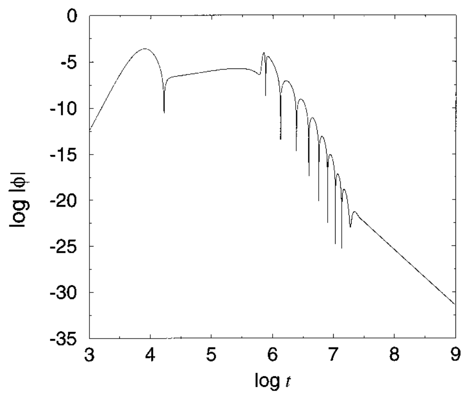

FIG. 2. Numerical simulation of the amplitude ϕ of linearized gravitational waves propagating on a static, spherically symmetric black hole background as a function of time. The ringing signal at intermediate times is due to the quasinormal modes.

in a cavity of volume V. This leads to an enhancement factor of $K=d/d_0 \sim (1/8\pi)\mathcal{N}Q(\lambda^3/V)$ for spontaneous emission, where λ is the wavelength of light emitted and Q is the quality factor of the cavity. Purcell's ideas have been demonstrated explicitly using the modes of the universe (Ching, Lai, and Young, 1987a; 1987b). The key result is that the photon modes are redistributed: accumulated at the resonance and depleted away from the resonances.

A more natural way of stating these ideas is to use the QNMs of the open system alone, rather than the modes of the universe—in other words to eliminate the bath degrees of freedom. This would be possible as an exact statement only if the QNMs are complete. Completeness relates to two issues. First of all, can any function $\phi(x)$ be expanded as

$$\phi(x)=\sum_j a_j f_j(x), \tag{1}$$

where the $f_j(x)$ are the QNMs? More importantly, do the resonances represent the dynamics exactly

$$\Phi(x,t)=\sum_j a_j f_j(x) e^{-i\omega_j t}, \tag{2}$$

for all $t \geqslant 0$ and all x in the cavity? One would wish to establish conditions under which these expansions are valid and, in circumstances where they are not, to characterize the remainder.

One should also be able to determine the coefficients a_j from $\phi(x)$ via a projection formula, i.e., some sort of inner product. Then, initial-value problems become formally trivial: take the initial data, project out a_j, and evolve by $a_j \mapsto a_j e^{-i\omega_j t}$. Moreover, to the extent that similarities can be established with the conservative case, one should be able to transcribe the tools of mathematical physics and to establish a parallel formalism. A particularly important goal, at the second-quantized level, is to develop Feynman rules for cavity QED, in which the field propagator is labeled by the *discrete* QNM index j rather than by a continuous momentum p—not only for computational convenience (sums rather than integrals over closed loops), but also for conceptual clarity (each term corresponds to a resonance).

This article reviews progress towards these goals.

B. The mathematical problem

Waves in open systems may be governed by the Schrödinger equation (e.g., nuclear physics), the wave equation (e.g., optics), or the Klein-Gordon equation with a potential V [e.g., each angular momentum sector of linearized gravitational waves (Chandrasekhar, 1991)]. The wave equation can be mapped exactly onto the Klein-Gordon equation, and as far as time-independent problems are concerned, the Klein-Gordon equation can be mapped onto the Schrödinger equation by relabeling $\omega^2 \mapsto \omega$. Thus many properties are similar, and we shall focus on the scalar wave equation, on a certain domain $\mathcal{R}$,

$$D\,\Phi(\mathbf{r},t) \equiv [\rho(\mathbf{r})\partial_t^2 - \nabla^2]\Phi(\mathbf{r},t)=0. \tag{3}$$

This describes the scalar model of electromagnetism (ρ =dielectric constant, and henceforth $c=1$) or elastic vibrations ($\rho=$ density). In this article, we shall primarily consider the 1D version ($\nabla^2 \mapsto \partial_x^2$) restricted to a half line $0 \leqslant x < \infty$; the "cavity" is the interval $\mathcal{R}=[0,a]$. Generalizations are cited later. A node is imposed at the origin, and waves escape through the point $x=a$ to the rest of the "universe" in (a,∞). This model describes a laser cavity with a totally reflecting mirror at $x=0$ and a partially transmitting mirror at $x=a$ (Lang, Scully, and Lamb, 1973), the radial problem of gravitational radiation from a stellar object of radius a (Price and Husain, 1992), or the transverse vibrations of a string clamped at one end $x=0$ and loaded with a point mass at $x=a$ (Dekker, 1985; Lai, Leung, and Young, 1987).

For conservative systems, one usually imposes $\Phi=0$ on the boundary of $\mathcal{R}$ ($x=a$ in the 1D case).[1] The eigenfunctions or normal modes are factorized solutions $\Phi(\mathbf{r},t)=f(\mathbf{r})e^{-i\omega t}$. The nodal condition on the boundary implies that $\mathcal{R}$ is *closed*, so that energy cannot escape. Mathematically, the operator $-\rho(\mathbf{r})^{-1}\nabla^2$ is Hermitian; thus ω is real and the eigenfunctions $\{f\}$ form a complete orthogonal set.

In contrast, if Φ satisfies the outgoing wave condition on the boundary of $\mathcal{R}$, the system is *open*. The eigenfunctions satisfy (in 1D) $\partial_x^2 f(x)=-\omega^2\rho(x)f(x)$, with $\mathrm{Im}\,\omega<0$. The question is then whether these $\{f\}$ are complete in the sense of Eqs. (1) and (2). The differential equation by itself admits two symmetries: if the frequency ω and the function f is a solution, then so is that obtained by (a) $\omega \mapsto -\omega$, $f \mapsto f$ and (b) $\omega \mapsto \omega^*$, $f \mapsto f^*$. However, in order to satisfy the outgoing boundary condition as well, only the combination of these operations, i.e., $\omega \mapsto -\omega^*$, $f \mapsto f^*$, would map an allowed solution to

[1]The Dirichlet condition can be replaced by the Neumann condition.

another. Thus if ω is an eigenfrequency, then so is $-\omega^*$; but, since $\omega^2 \neq (-\omega^*)^2$, the two eigenvalues are distinct and the corresponding eigenfunctions f and f^* are linearly independent. Thus the modes are doubled compared with the conservative case. Hence we shall order the eigenfrequencies ω_j according to increasing real parts, such that $\omega_{-j} = -\omega_j^*$. There could be QNMs with $\mathrm{Re}\,\omega = 0$; these would not be paired and will be called zero modes and generically labeled as ω_0.

II. COMPLETENESS

The entire formalism hinges on a central result (Leung, Liu, and Young, 1994a): the QNMs are complete on $\mathcal{R}=[0,a]$ provided two conditions are satisfied. (a) *The discontinuity condition*: $\rho(x)$ must have a step or stronger discontinuity at $x=a$ so as to provide a natural demarcation of a finite interval. (b) *The no-tail condition*: $\rho(x)=1$ (or other constant value) for $x>a$, such that outgoing waves are not scattered back. The no-tail condition is especially natural in the case of optics, where typically one has vacuum outside of the finite cavity. It does not matter whether the inside of the cavity has ρ larger or smaller than the constant outside value.

The proof of completeness will also provide the framework for understanding other contributions when the QNMs are not complete. The Fourier transform of the Green's function $G(x,y;t)$ [defined by $DG(x,y;t) = \delta(x-y)\delta(t)$, with $G=0$ for $t\leqslant 0$] is given explicitly by $f(\omega,x)g(\omega,y)/W(\omega)$ for $0\leqslant x\leqslant y$. Here f and g are homogeneous solutions at a frequency ω, with f satisfying the left nodal condition $[f(\omega,0)=0]$ and g satisfying the right outgoing wave condition $[g(\omega,x)\propto e^{i\omega x}$ for $x\rightarrow\infty]$; W is their Wronskian.

For $t\geqslant 0$, $G(x,y;t)$ is evaluated by the inverse Fourier transform with the contour of integration closed by a large semicircle in the lower half ω plane. In general there are three different contributions:

(a) *Large semicircle—prompt response*. The large semicircle ($|\omega|\rightarrow\infty$) gives the short-time or prompt response. For large t, the factor $e^{-i\omega t}$ provides sufficient damping in the lower half ω plane to control the asymptotic behavior so that the prompt response always vanishes for $t>t_p(x,y)$, where $t_p(x,y)$ is the geometric optics transit time between x and y (Bachelot and Motet-Bachelot, 1993; Leung, Liu, and Young, 1994a).

If the discontinuity condition is satisfied, one has a stronger result: by a WKB estimate, this contribution vanishes for all $t\geqslant 0$. It is natural that the $|\omega|\rightarrow\infty$ behavior should be sensitive to short spatial scales, in particular a discontinuity.

(b) *Singularities in g—late-time tail*. The function $f(\omega,x)$ is obtained by integrating the time-independent wave equation, in which ω appears analytically, through a finite distance from 0 to x. Crudely speaking, this is a finite combination of analytic functions of ω, so $f(\omega,x)$ is analytic in ω (Newton, 1960). But $g(\omega,x)$ could have discontinuities in ω (typically a cut on the negative $\mathrm{Im}\,\omega$ axis), because one has to integrate from $x\rightarrow\infty$.

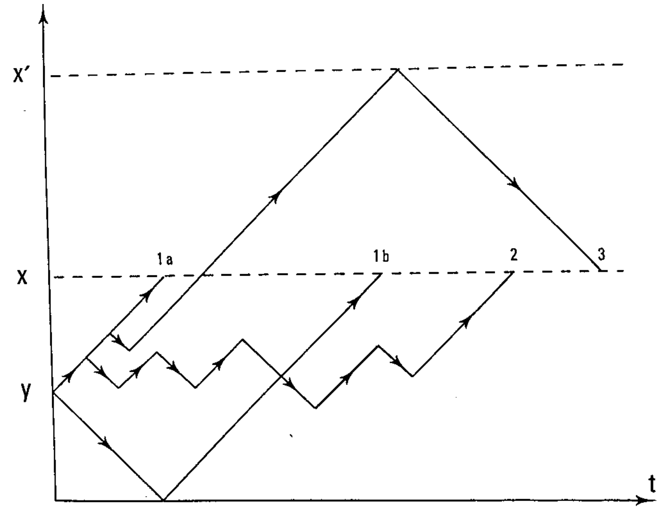

FIG. 3. Spacetime diagram illustrating the three different contributions to the Green's function: "direct" propagation without scattering (rays $1a$ and $1b$, prompt response); repeated scatterings at finite x' (ray 2, quasinormal-mode contributions); waves scattered at asymptotic x' by $V(x')$ (ray 3, late-time tail).

However, if the "no-tail" condition is satisfied, then one can impose the outgoing condition for $g(\omega,x)$ at $x=a^+$ and again integrate only through a finite distance. Thus g is guaranteed to be analytic as well, and this contribution is removed.

(c) *Zeros of W—QNM contributions*. Finally, there are the zeros of $W(\omega)$. At a zero ω_j of W, the functions f and g are linearly dependent: $f_j(x)\equiv f(\omega_j,x) = C_j g(\omega_j,x)$; thus f_j satisfies *both* the left nodal and the right outgoing boundary conditions and is an eigenfunction or QNM.

In general all three contributions are present and correspond nicely with the main features of the time-domain signal (Fig. 3). The large semicircle in the ω plane leads to the initial transients—waves propagating directly from the source point y to the observation point x (rays $1a$,$1b$). The zeros of W give rise to the QNM ringing at intermediate times, due to repeated scattering from the potential (ray 2). The cut in the ω plane, especially its tip near $\omega=0$, then gives the late-time behavior: waves propagate from the source point y to a distant point x', are scattered by $\rho(x')$, and return to the observation point x (ray 3).

We now focus on the case where both the discontinuity and no-tail conditions are satisfied. At each zero ω_j, the residue is related to $dW(\omega_j)/d\omega$, which can be evaluated using the defining equations for f and g to be

$$-C_j\frac{dW(\omega_j)}{d\omega} = 2\omega_j\int_0^{a^+}\rho(x)f_j(x)^2dx + if_j(a)^2 \equiv \langle f_j|f_j\rangle. \tag{4}$$

Using the normalization (and phase) convention[2] $\langle f_j|f_j\rangle = 2\omega_j$, one then obtains

$$G(x,y;t)=\frac{i}{2}\sum_j \frac{1}{\omega_j} f_j(x) f_j(y) e^{-i\omega_j t}. \tag{5}$$

The initial conditions on G and $\partial_t G$ lead to[3]

$$\frac{i}{2}\sum_j \frac{1}{\omega_j} f_j(x) f_j(y)=0, \tag{6}$$

$$\frac{\rho(x)}{2}\sum_j f_j(x) f_j(y)=\delta(x-y) \tag{7}$$

for $x,y\in\mathcal{R}$. The identity (7) has an obvious analog in conservative systems; the factor 1/2 accounts for the doubling of modes. On the other hand, Eq. (6) becomes vacuous in the limit of zero leakage.

The heart of the QNM expansion is contained in Eqs. (5), (6), and (7). The identity (7) obviously leads to Eq. (1), while Eq. (5) leads to Eq. (2). We stress that under the two conditions stated at the beginning of this section, this expansion and the results that follow are exact and in no way limited to weak dissipation. In particular, the dynamics in Eq. (5) applies to overdamped oscillations (represented by zero modes) as well.

III. ANALOGY WITH CONSERVATIVE SYSTEMS

The expansion of G leads to a description of the dynamics formally analogous to conservative systems. Time-dependent problems require two sets of initial data: $\phi\equiv\Phi(x,t=0)$ and $\hat{\phi}\equiv\rho(x)\partial_t\Phi(x,t=0)$ (Footnote 4). One is thus led to consider, from Eq. (2), the simultaneous expansion of a pair of functions,

$$\begin{pmatrix}\phi(x)\\ \hat{\phi}(x)\end{pmatrix}=\sum_j a_j\begin{pmatrix}1\\ -i\omega_j\rho(x)\end{pmatrix} f_j(x), \tag{8}$$

using the *same* coefficients a_j for both components. For a QNM, the second component is explicitly $\hat{f}_j(x)=-i\omega_j\rho(x)f_j(x)$, where ω_j is the eigenvalue.

The factor $\rho(x)$ turns the second component into the conjugate momentum. The use of two components is natural (Feshbach and Villars, 1957; Unruh, 1974; Ford, 1975), but here the outgoing wave condition[5] $\hat{\phi}(x=a^+)=-\phi'(x=a^+)$ links them in a novel way. Compared with the case of normal modes, twice the degrees of freedom (a_j and a_{-j}) are used to satisfy double the conditions (ϕ and $\hat{\phi}$).

The two-component expansion (8) is proved by starting with the Green's-function solution to the initial-value problem

$$\Phi(x,t)=\int_0^\infty [G(x,y;t)\hat{\phi}(y) + \partial_t G(x,y;t)\rho(y)\phi(y)]dy. \tag{9}$$

Inserting the expansion (5), one then obtains Eq. (8) and also Eq. (2) with a_j given by

$$a_j=\frac{i}{2\omega_j}\left\{\int_0^{a^+}[f_j(y)\hat{\phi}(y)+\hat{f}_j(y)\phi(y)]dy + f_j(a)\phi(a)\right\}. \tag{10}$$

All reference to the outside of the cavity is removed, because the integral on (a,∞) can be collapsed to the surface term by making use of the outgoing condition on the initial data and the retarded condition on G. The elimination of the outside is the crucial step in obtaining a self-contained description of the cavity.

Consider the linear space of outgoing function pairs, denoted as ket vectors $|\boldsymbol{\phi}\rangle=(\phi,\hat{\phi})^{\mathrm{T}}$. Time evolution can be written in Schrödinger form $\partial_t|\boldsymbol{\phi}\rangle=-i\mathcal{H}|\boldsymbol{\phi}\rangle$, where

$$\mathcal{H}=i\begin{pmatrix}0 & \rho(x)^{-1}\\ \partial_x^2 & 0\end{pmatrix}. \tag{11}$$

The first component of the Schrödinger equation reproduces the identification of $\hat{\phi}$ as $\rho(x)\partial_t\Phi$. The QNMs can now be defined simply by $\mathcal{H}|\mathbf{f_j}\rangle=\omega_j|\mathbf{f_j}\rangle$.

The projection (10) suggests a generalized inner product between two vectors $|\boldsymbol{\phi}\rangle$ and $|\boldsymbol{\psi}\rangle$,

$$\langle\boldsymbol{\psi}|\boldsymbol{\phi}\rangle=i\left[\int_0^{a^+}(\psi\hat{\phi}+\hat{\psi}\phi)\,dx+\psi(a)\phi(a)\right], \tag{12}$$

which is symmetric and linear in both the bra and ket vectors (rather than conjugate linear in the bra vector). The inner product of a QNM with itself reproduces the generalized norm (4). Moreover, the coefficients a_j can now be written compactly as

$$a_j=\langle\mathbf{f}_j|\boldsymbol{\phi}\rangle/(2\omega_j). \tag{13}$$

Under the generalized inner product, the Hamiltonian is symmetric: $\langle\boldsymbol{\psi}|\{\mathcal{H}|\boldsymbol{\phi}\rangle\}=\langle\boldsymbol{\phi}|\{\mathcal{H}|\boldsymbol{\psi}\rangle\}\equiv\langle\boldsymbol{\psi}|\mathcal{H}|\boldsymbol{\phi}\rangle$. The proof of this statement requires an integration by parts; surface terms are incurred because the functions do not vanish at the end points, but these are compensated exactly by the surface terms in the definition of the inner product.

There is thus an almost complete parallel with conservative systems, and the mathematical structure is in place to carry over essentially all the familiar tools based on eigenfunction expansions. For example, the symmetry of $\mathcal{H}$ (similar to hermiticity in the conservative case)

[2]For higher-order poles, which cannot occur for 1D closed systems, $\langle f_j|f_j\rangle=0$. This situation can be analyzed by letting simple poles coalesce.

[3]Subject to the validity of term-by-term differentiation and the limit $t\to 0^+$.

[4]These are assumed to vanish at infinity, representing waves emitted at a finite time in the past.

[5]The last condition is specified at a^+ because the system may have discontinuities at $x=a$. However, ϕ itself is continuous since the positivity of ρ limits its singularity at $x=a$ to at most a δ function; this has already been used in the last term of Eq. (4).

immediately leads to the orthogonality of the QNMs and hence to the uniqueness of the expansion (8). The only exception is the lack of a positive-definite norm, and with it a simple probability interpretation—hardly surprising since probability (or energy) is not conserved in $\mathcal{R}$.

Incidentally, the generalized inner product $\langle \boldsymbol{\psi} | \boldsymbol{\phi} \rangle$ can be put into the context of non-Hermitian Hamiltonians by introducing a duality transformation $\mathcal{D}(\phi_1,\phi_2)^{\mathrm{T}} \equiv -i(\phi_2^*,\phi_1^*)^{\mathrm{T}}$. Then, $\langle \boldsymbol{\psi} | \boldsymbol{\phi} \rangle$ is just the conventional inner product between $|\boldsymbol{\phi}\rangle$ and $\mathcal{D}|\boldsymbol{\psi}\rangle$. Moreover, $\{|\mathbf{f}_j\rangle\}$ and $\{\mathcal{D}|\mathbf{f}_j\rangle\}$ are seen to constitute a biorthogonal basis (Leung, Suen, Sun, and Young, 1998).

In the above, we have described the formalism for a half-line problem with a nodal condition at one end ($x=0$) and a discontinuity at the other ($x=a$), with the outgoing wave condition just outside the discontinuity. It is straightforward to generalize the formalism to a full line, with at least two discontinuities ($x=a_1,a_2$), and the outgoing wave conditions just outside these discontinuities. The completeness relationship then holds on the interval (a_1,a_2), and the generalized inner product contains two surface terms. Likewise, the formalism can be generalized from the wave equation to the Klein-Gordon equation [the "no-tail" condition is that $V(x)=0$ on the "outside" (Ching *et al.*, 1995b, 1996)], and to the presence of absorption (provided that it satisfies the usual Kramers-Kronig dispersion relation); Leung, Liu, and Young, 1994b). Much the same methods also apply in 3D, though in the presence of a centrifugal barrier the surface terms in the inner product become progressively more complicated with increasing angular momentum l.

IV. TIME-INDEPENDENT PROBLEMS

The simplest and most useful applications concern time-independent problems, especially time-independent perturbation theory.

A. Perturbation

Let $\rho(x)^{-1}=\rho_0^{-1}(x)[1+\mu V(x)]$, where $|\mu|\ll 1$ and $V(x)$ is nonzero only in the cavity. The system with $\rho_0(x)$ is exactly solvable with a complete set of QNMs. The task is to develop a *discrete* representation for the exact frequency $\omega_j=\omega_j^{(0)}+\mu\omega_j^{(1)}+\mu^2\omega_j^{(2)}+\cdots$. With the formalism described in the last section, the perturbative formulas can be obtained by simply transcribing the textbook derivation for conservative systems, evaluating the pertinent matrix elements $\langle \mathbf{f}_j|\Delta\mathcal{H}|\mathbf{f}_k\rangle$ using Eqs. (11) and (12); e.g., the first-order shift is

$$\omega_j^{(1)}=\frac{\omega_j^{(0)}}{2}V_{jj}, \tag{14}$$

with $V_{jk}=\int_0^a f_j^{(0)}\rho_0 V f_k^{(0)}dx$. However, these formulas are now *complex* and give the shifts in both the resonance positions and their widths; thus they contain twice the information of their superficially similar conservative counterparts.

The first-order correction has been known for a long time for the Schrödinger equation, but here the normalization [see Eq. (4)] involves a surface term rather than a regularization (Zeldovich, 1960), the latter being less convenient computationally. The first-order result has been generalized to the EM case (Lai *et al.*, 1990), and in fact applies also to systems without discontinuities.

More importantly, the transcription also gives the higher-order shifts, e.g.,

$$\omega_j^{(2)}=\sum_{k\neq j}\frac{\omega_j^{(0)}\omega_k^{(0)}V_{jk}^2}{4(\omega_j^{(0)}-\omega_k^{(0)})}. \tag{15}$$

The (slow) convergence of Eq. (15) is readily accelerated using identities such as (6) (Leung, Tong, and Young, 1997b). The wave functions can be calculated in a parallel way (Leung *et al.*, 1994).

B. Physical examples

Some interesting applications relate to optics, where the interface between two media gives a discontinuity and the vacuum outside the system naturally satisfies the no-tail condition. Consider a microsphere of radius a and refractive index n. Glancing rays suffer total internal reflection and are confined, but evanescent waves cause some leakage, which makes the microsphere an open system. In practice, experiments are often done on droplets falling in air; the droplets are slightly oblate (ellipticity $e\sim 10^{-3}-10^{-2}$) due to viscous drag. The breaking of spherical symmetry lifts the degeneracy among the $2l+1$ members of a multiplet. This splitting has been calculated numerically by brute force (Barber and Hill, 1988), but the perturbative formalism allows an analytic expression (Lai *et al.*, 1990),

$$\frac{\omega_m^{(1)}}{\omega^{(0)}}=-\frac{e}{6}\left[1-\frac{3m^2}{l(l+1)}\right], \tag{16}$$

in which m is the azimuthal quantum number, and $e=(r_p-r_e)/a$, where r_p and r_e are the polar and equatorial radii, respectively, and $a=(r_p r_e^2)^{1/3}$. Equation (16) has been used to interpret spectroscopic measurements on the splitting between neighboring lines (m and $m+1$); Chen *et al.*, 1991), on the total spread of the multiplet from $m=0$ to $m=\pm l$ (Chen *et al.*, 1993), and on time-domain variations in intensity arising from the beating among different m's, which can also be regarded as a precession of the photon orbit (Swindal *et al.*, 1993).

A levitated droplet is also distorted when subjected to a quadrupole field E_Q, with $e\propto E_Q/s$, where s is the surface tension. Determination of e from the splitting then allows s to be found, down to length scales of tens of μm (Arnold, Spock, and Folan, 1990). (Previously, surface tension was known only macroscopically, at length scales of 1 mm or more.)

On a very different length scale, the perturbation results have also been applied to black holes perturbed by interactions with their surroundings (e.g., a massive accretion disk). The first-order shifts in the QNM frequen-

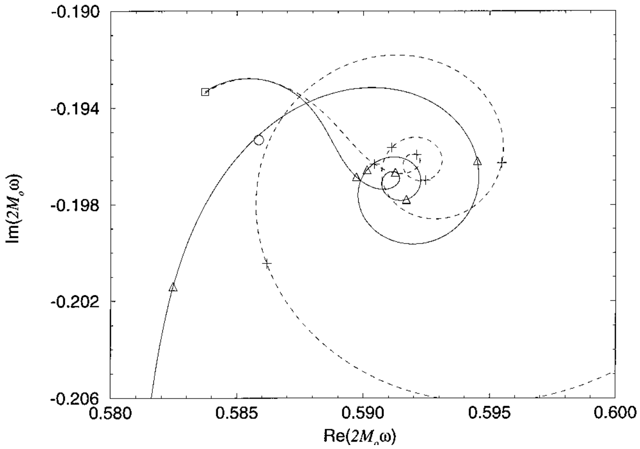

FIG. 4. The locus of the lowest quasinormal mode for $l=1$ scalar waves propagating on a Schwarzschild background, as the shell of magnitude $\mu=0.01$ is placed at different positions measured by r_s; solid line, the exact results; dashed line, the first-order perturbative results; the square, a shell position of $r_s/M_a=2.22$ (the extreme value that satisfies the dominant-energy condition); the triangle (on the solid line) and the crosses (on the dashed line), to r_s/M_a from 6 to 60 at intervals of 6. The first-order result is seen to capture the qualitative features.

cies of gravitational waves are expressed by the Klein-Gordon analog to Eq. (14), further generalized to allow for a "tail" (Leung *et al.*, 1997).[6] Consider, for example, a Schwarzschild black hole perturbed by a thin static shell of matter. Figure 4 shows, in the complex ω plane, the locus of the lowest QNM for $l=1$ scalar waves, as a light shell (0.01 times the mass of the hole) is placed at different positions r_s. The first-order perturbative results (dashed lines) capture the qualitative features of the exact results (solid lines). Since these shifts depend on the perturbation, they may provide a useful probe of the intervening spacetime curvature. The possible applications in this direction have yet to be fully exploited.

V. SECOND QUANTIZATION

An important goal of the formalism is second quantization (Ho, 1997; Leung, Maassen van den Brink, and Young, 1997; Ho *et al.*, 1998). First of all, the fields ϕ and $\hat{\phi}$ may be regarded as dynamical variables of the entire universe—which is conservative and Hermitian, and therefore can be quantized in the usual way; thus these fields are promoted to operators by imposing the canonical equal-time commutators $[\phi(x),\hat{\phi}(y)]=i\delta(x-y)$. Since the coefficients a_j are given by projecting these operators onto the c-number functions f_j in the manner of Eq. (10), one immediately obtains their commutation relations,

$$[a_k,a_j^+]=\frac{\omega_k+\omega_j^*}{4\omega_j^*\omega_k}I_{jk}, \tag{17}$$

where the overlap integral is

$$I_{jk}=\int_0^{a^+}\rho f_j^* f_k\,dx\approx\delta_{jk}, \tag{18}$$

in which the last approximate expression refers to $j,k>0$ and is valid in the limit where the leakage is negligible. Thus these are (apart from normalizing factors $\sqrt{2\omega_k}$, etc.) the analogs of the creation and annihilation operators for the field quanta in a particular mode.

More importantly, the Feynman propagator can be written as

$$\tilde{G}^{\mathrm{F}}(x,y;\omega)=\sum_j f_j(x)\Delta_j(\omega)f_j(y). \tag{19}$$

The factors $f_j(x)$ and $f_j(y)$ represent the coupling to the mode j at x and y, while the propagator for the field quanta is now labeled by a discrete index j and given by

$$\Delta_j(\omega)=[2\omega_j(|\omega|-\omega_j)]^{-1} \tag{20}$$

for real ω; contrast the analogous quantity $\Delta(p,\omega)$ for the propagation of a quantum with continuous momentum p in the modes of the universe approach. With Eq. (19), one has achieved the goal set out at the beginning: to obtain cavity Feynman diagrams for cavity QED.

As an illustration, consider a stationary atom with two levels $\pm\sigma/2$ placed in the cavity at position x. The energy and decay rate of the upper state are obtained by locating the poles Ω_i of the retarded-atom propagator $[\omega-\sigma/2-\tilde{\Sigma}_\uparrow(\omega)]^{-1}$ [continued from $\omega>0$ (Abrikosov *et al.*, 1975)]. The self-energy of the excited level $\tilde{\Sigma}_\uparrow(\omega)$ is calculated using only the lowest-order diagram (i.e., emission and subsequent absorption of a virtual "photon") generated by $H_{\mathrm{int}}=\lambda\phi(x)|\downarrow\rangle\langle\uparrow|+\mathrm{H.c.}$, and assuming domination by the resonance ω_j in the photon line (Ho *et al.*, 1998). In terms of $\eta\equiv|f_j(x)f_j(a)|^2/4|\omega_j|^2|\mathrm{Im}\omega_j|$ (Ho *et al.*, 1997) (with $\eta\propto|\mathrm{Im}\omega_j|^0$ in the closed limit), two cases must be considered:

[(a)] $\lambda^2\eta\sigma\ll|\sigma-\omega_j|^2$, a broad resonance (i.e., weak coupling). One pole represents a transient, while for the other $-2\mathrm{Im}\Omega_2=2\lambda^2\eta\sigma|\mathrm{Im}\omega_j|/|\sigma-\omega_j|^2$ is the resonance-enhanced (the vacuum rate being λ^2/σ) golden-rule decay rate (Purcell, 1946) also obtainable using the modes of the universe.

[(b)] $\lambda^2\eta\sigma\gg|\sigma-\omega_j|^2$, a narrow resonance (strong coupling). One finds $\Omega_{1,2}=\omega_j/2\pm\lambda\sqrt{\eta\sigma}$, which represents revivals (Rabi, 1937) with the atomic frequency $\lambda\sqrt{\eta\sigma}$ and decay with the much slower cavity leakage rate $|\mathrm{Im}\omega_j|$.

The "unit weight" of narrow resonances here emerges as a result and need not be assumed (Lai, Leung, and Young, 1988); see also (Ho *et al.*, 1998).

Besides using the Feynman rules directly as above, one can calculate the vacuum fluctuations of the fields and relate physical processes to them, e.g., spontaneous

[6]With a "tail," the QNMs are not complete, but clearly this is irrelevant for the first-order shift.

emission being "stimulated" by the vacuum fluctuations (Ho, 1997); or one can deal with the density of states, which is the imaginary part of the Green's function. In any of these equivalent methods, the quantities of interest are expressible exactly as a sum over QNMs. This then provides a clean justification for many semiempirical approaches based on dominance by one or a few resonances. Furthermore the decay of a two-level atom coupled to an open cavity with Lorentzian resonances can be solved exactly in the rotating-wave approximation (Lai, Leung, and Young, 1988; Garraway, 1997a, 1997b). Thus the completeness of QNMs, and their use in a second-quantized theory, serve as powerful tools to analyze resonance-dominated optical phenomena in open cavities.

VI. NONRESONANT CONTRIBUTIONS

For cases in which the QNMs are not complete, there are nonresonant contributions. Interesting examples are gravitational waves and waves governed by the Schrödinger equation. Gravitational waves from Schwarzschild black holes are described by the Klein-Gordon equation; the potential $V(x)$, related to the background metric, is continuous and goes asymptotically as $x^{-3}\log x$ as $x\rightarrow+\infty$ (apart from a centrifugal barrier), violating both the discontinuity and the no-tail condition. We here indicate how this situation affects the dynamics (Ching *et al.*, 1995a; 1995c).

First, without a discontinuity, there is a prompt signal. Second, the asymptotic part of $V(x)$ leads to a cut in the Green's function on the negative $\mathrm{Im}\omega$ axis, the tip of which near $\omega=0$ determines the late-time dynamics. Generically, if $V(x)\sim x^{-\alpha}(\log x)^{\beta}$, then $\phi(x,t)\sim t^{-\mu}(\log t)^{\nu}$, where μ and ν are determined by α, β, and the angular momentum l. For example, if $\beta=0$ or 1, in general $\mu=2l+\alpha$ and $\nu=\beta$. However, there is an exception when $\alpha<2l+3$ is an integer; in that case the leading term vanishes, and the late-time dynamics is dominated by the next leading term, which is $\sim t^{-(2l+\alpha)}$ for $\beta=1$ and $\sim t^{-(2l+\alpha+1)}$ for $\beta=0$. It turns out, interestingly, that the case of the Schwarzschild black hole ($\alpha=3$, $\beta=1$) belongs to this exceptional category, and the late-time behavior is a pure power. Figure 2 shows precisely such a power-law tail. However, the signal at intermediate times would still be dominated by the QNMs, so that in practice a restricted and approximate notion of completeness still holds.

For the Schrödinger equation, outgoing waves are defined by $\phi(x)\sim e^{ikx}$ as $x\rightarrow\infty$, where $k=\sqrt{2\omega}$ instead of $k=\omega$. Thus a cut in the lower half ω plane is unavoidable, and the system decays by a power-law $t^{-\alpha}$ at large times, due to the contribution near threshold ($k\approx0$), even though $V(x)$ vanishes exactly outside some interval. This phenomenon was first noted by Khalfin (1957) and Winter (1961), with a $t^{-3/2}$ tail being generic. However, if the potential is unbounded from below, no finite threshold exists, and the large-t behavior is again given by the QNMs (Suen and Young, 1991). This applies to some models of minisuperspace, in which x is essentially the scale factor of the universe (Suen and Young, 1989), and there is the intriguing possibility that the wave function of the universe at the end of the quantum era is given by the lowest QNM, independent of cosmological initial conditions.

The power-law decay in the case of the Schrödinger equation comes from the threshold, whereas the power law in the Klein-Gordon case comes from the spatial tail of $V(x)$. The origins of the two are entirely different.

Despite these complications, in time-independent problems, one is free to relabel $\omega^2\mapsto\omega$, so the Schrödinger case should not be different from the Klein-Gordon case. An application in this restricted time-independent sense is given in Leung and Young (1991).

VII. CONCLUSION

We have considered linear, classical waves in open systems of a certain type, and sketched a formalism in terms of a QNM expansion, in a manner analogous to the description of conservative systems by their normal modes. Dissipation is contained in these discrete QNMs themselves, especially in $\mathrm{Im}\omega_j$. The analogy with conservative systems is achieved by the introduction of a generalized inner product, in terms of which the time-evolution operator is symmetric. Apart from this modification, almost nothing needs to be changed, and the familiar tools for Hermitian systems can be carried over. These results are nontrivial, in that if either the discontinuity condition or the no-tail condition is violated, the QNMs would not be complete. The most significant feature in these cases is a power-law tail in the long-time behavior.

It is useful to place the present formalism in the general context of dissipative systems (Feynman and Vernon, 1963; Ullersma, 1966; Caldeira and Leggett, 1983). In all cases, one starts with a system-plus-bath description. The key result is that reference to this bath (here the "outside") can be eliminated. For example, in Eq. (10) the outside contribution has been collapsed to a surface term. The no-tail condition ensures that earlier data are not scattered back, so that the system $\mathcal{R}$ does not have memory; this is analogous to taking an ohmic bath spectrum, without which the conventional treatments would likewise exhibit memory. These rigorous treatments differ fundamentally from those introducing irreversibility by coarse graining or by assuming random phase.

The wave systems under discussion differ from conventional models in two interesting respects. First, in the conventional models, the system (say Q_i) and the bath coordinates (q_n) are coupled through a potential $\lambda V(Q_i,q_n)$ (bilinear in the simplest cases). Such couplings are "soft" and can be switched off ($\lambda\rightarrow0$); thus the overall Hilbert space is the same as that for $\lambda=0$, i.e., the product $\Omega_S\otimes\Omega_B$ of the system and bath spaces. In the wave systems, coupling is achieved through boundary conditions on $\Phi(x,t)$ and its derivative across $x=a$. Such couplings are "hard" in that they cannot be turned off, and the Hilbert space is not $\Omega_S\otimes\Omega_B$. Factorization occurs only in the decoupled limit (obtained by imposing a node at $x=a$), where one loses one degree of freedom, namely $\Phi(a,t)$.

Second, conventional models typically involve few Q_i, $i\leqslant N$. In contrast, the region $\mathcal{R}$ has an infinite number of degrees of freedom, and thus their optimal representation becomes an issue. The QNM expansion replaces the continuum $\Phi(x,t)$ with the set $\{a_j(t)\}$. Each a_j represents a resonance, so that the formalism is especially suitable when one resonance dominates. Moreover, it closely parallels the conservative case.

By using the tools presented in this review, a beginning has been made in dealing with cavity QED, such that field propagators are labeled by the discrete QNM basis, opening the way to a simpler and more transparent description.

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