Quantum noise in photonics

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The basic equations governing noise phenomena are derived from first principles and applied to examples in optical communications. Quantum noise arises from two sources, the momentum fluctuations of electrons at optical frequencies and the uncertainty-related fluctuations of the electromagnetic field. Shot noise results from the beating of the noise sources with the signal field. In high-gain amplifiers, the spontaneous-emission noise dominates shot noise and results in a noise figure of at least 3 dB. It is shown explicitly how, at high power, both the laser field and the laser noise source become classical. The increase in noise in lasers with open cavities is not due to enhanced spontaneous emission as once thought, but to single-pass amplification. The noise fields and spontaneous currents have Gaussian distributions, while nonlasing modes have exponential photon-number distributions. Low-frequency intensity fluctuations arise from the electric current driving the laser and can be sub-Poissonian, in contrast to shot noise, which has a Poissonian distribution. The calculational tools are a wave equation for the field operator and a rate equation for the carrier-number operator, each containing spontaneous current noise sources. The correlation functions of these sources are determined by the fluctuation-dissipation theorem.
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CONTENTS

SYMBOLS AND ABBREVIATIONS

$\hat{\mathcal{A}}$	vector potential
A	slowly varying amplitude of vector potential
$\hat{A}_{\rm St}(t)$	positive frequency part of vector potential at Stokes frequencies
A_ω	frequency component of vector potential
a_ω, a_0	attenuation coefficient
a_I, a_E	coefficient of loss internal and external to laser
$\hat{a}^\dagger_{uk}, \hat{a}_{uk}$	creation and annihilation operators
$b_{\rm sig}$	signal-field amplitude
$\hat{b}_H$	Hermitian combination of noise-field amplitudes
$\hat{b}_0, \hat{b}_\omega$	noise-field amplitude
$\hat{b}_{\rm in}, \hat{b}_{\rm out}, \hat{b}_{\rm em}$	amplitude of noise field entering amplifier, leaving amplifier, and emitted without amplifier
$\hat{b}^\dagger_q, \hat{b}_q$	creation and annihilation operators of optical phonons
$C, \mathcal{C}$	current, current per unit volume
D_0, D_n	normalization coefficient
D_q	coefficient for Raman scattering by mode q
$\frac{dN}{d\omega_k}, \frac{dN}{d\omega_q}$	density of states
$\hat{\mathcal{E}}$	electric field
$\hat{\mathcal{F}}_g, \hat{\mathcal{F}}_a$	Langevin force per unit volume associated with gain and loss
$\hat{F}_N(t)$	Langevin force for electron-number fluctuations
$G_0(\mathbf{x}, \mathbf{x}')$	Green's function for the mode 0
$\tilde{G}$	integral operator relating field and spontaneous current density
G	gain of two-level system, amplification of optical amplifier
g, g_ω, g_0	gain coefficient
$g(z, z')$	z-dependent Green's-function solution
$\hat{\mathcal{H}}$	magnetic field
$\hat{H}_e, \hat{H}_{\rm int}, \hat{H}_{\rm rad}$	electron, interaction, and radiation Hamiltonians
$\hat{H}_L$	optical-phonon Hamiltonian
$\hat{J}(t)$	slowly varying amplitude of spontaneous current density
$\hat{J}_{\rm ind}(t), \hat{J}_{\rm tot}(t)$	slowly varying amplitude of induced and total current density
$\hat{j}(t), \hat{j}_\omega$	positive-frequency part and frequency component of spontaneous current density
$\hat{j}_R(t), \hat{j}_{R\omega}$	positive-frequency part and frequency component of spontaneous current for Raman amplification
$\hat{j}_k$	current-density operator associated with transitions at wave vector k
$\hat{j}_G, \hat{j}_\Gamma$	spontaneous current associated with gain and loss
$\hat{j}_{\omega\,\rm eff}$	effective spontaneous current density in waveguide with scattering
K	Petermann noise enhancement factor
k_0, k_ω	propagation constant
$\hat{m}$	number of photons detected in a time interval
N	average occupation of upper level of set of two-level states
$\hat{\mathcal{N}}, \hat{\mathcal{N}}$	electron number and density in conduction band
$\frac{\partial \mathcal{N}_{\rm spont}}{\partial t}$	spontaneous-emission rate
n_{ck}, n_{vk}	conduction- and valence-band occupation factors
n_{sp}	spontaneous-emission factor
$\bar{n}_\omega$	photon mode occupation number in equilibrium with electrons
$\bar{n}_q$	phonon occupation number of mode q
$\hat{O}(t), \hat{O}_S, \hat{O}_I(t)$	general quantum-mechanical operator in Heisenberg, Schrödinger, and interaction representations
$\hat{O}_N, \hat{O}_A$	normal and antinormal orderings of general operator, applied to energy density, energy flux, and carrier transition rate
$\hat{p}, \hat{\mathbf{P}}$	electron momentum, total momentum
$\hat{P}, \hat{\mathcal{P}}$	photon number and photon or energy density
p_k	momentum matrix element at wave vector k
$\hat{Q}$	energy flux
r_u, r_d	contributions of upward and downward transitions to gain coefficient
$\hat{\mathbf{S}}$	energy flux density
V	volume containing electrons, volume of laser cavity, and volume of dielectric inhomogeneity
v_g	group velocity
$\mathcal{V}$	bias voltage
x	both transverse dimensions
$\chi(\omega)$	complex linear susceptibility
ϵ_ω	dielectric function
$\phi_0(x), \Phi_0(\mathbf{x})$	transverse mode 0 and laser mode 0
ω_k	transition function
ω_S	signal or typical optical frequency
ΔG_0	deviation of gain from threshold of mode 0
$\Delta\epsilon_\omega$	inhomogeneity in dielectric function
Γ	loss rate of two-level system
τ	carrier lifetime
$\varepsilon_{\rm ck}, \varepsilon_{\rm vk}$	electron energy in conduction band and valence band at wave vector k
ψ	electron wave function
$(\cdots)_\Omega$	integral over volume Ω

I. INTRODUCTION

A. What is photonics?

One answer to this question can be found by looking at the subjects covered by *IEEE Photonics Technology Letters*. They include lasers, optical amplifiers, photodetectors, communication systems, integrated optoelectronics, integrated optics, and sensors. Quantum noise, the subject of this paper, is not essential to all of these fields, but it is important for applications related to the information industry. We are interested in basic physical phenomena associated with noise in photonics. These are best illustrated by considering noise in lasers, in traveling-wave amplifiers, in optical transmission, and in photodetection.

In considering the interaction of light and matter, we shall use a model of carriers in a semiconductor. We do this because lasers and detectors in photonics are made of semiconductors. However, the results that we obtain are quite general and are not tied to specific models. For example, our description of an optical amplifier applies also to Er-doped fiber amplifiers, which have become very important in optical communications.

Noise enters into photonics applications in some important ways. Laser phase noise limits the sensitivity of coherent optical communications and sensors based on interference (Garret and Jacobsen, 1986). The laser linewidth itself is due to phase fluctuations (Lax, 1967a; Henry, 1982). Analog modulation used in cable TV transmission strongly depends on achieving very low laser-amplitude noise (Darcie and Bodeep, 1990; Petermann, 1991). Low bit-error rates in high-speed long-distance digital transmission systems require stable single-mode operation without jumping even rarely to other modes. This jumping is usually referred to as mode-partition noise (Ogawa, 1982; Link *et al.*, 1985) For a specific bit-error rate, the minimum photon number per bit is determined by shot noise (Henry, 1985). In practice, detection of a shot-noise-limited signal requires low-noise optical preamplification. After amplification, the minimum signal is determined by both shot noise and signal-spontaneous emission beat noise (Olsson, 1989; Tonguz and Kazovsky, 1991).

Despite the growing maturity of this field, it is still difficult for the large population of scientists and engineers working in photonics to obtain a fundamental understanding of how these optical processes take place and how to think about the light involved. This is because of the dual wave and particle behavior of light. In most situations, light can be thought of as a classical wave. This concept is useful in describing modes, diffraction, interference, etc. However, noise phenomena cannot be described in the framework of classical waves. Shot noise in detection and laser-intensity noise, for example, are often explained in terms of a photon gas.

The contemplation of quantum noise associated with the generation, amplification, attenuation, and detection of light raises fundamental questions. For example, what causes quantum noise? How is quantum noise related to the uncertainty principle? A wave description of light accounts for propagation, modes, interference, and other properties of light. Can it also account for noise? Are vacuum fluctuations needed? Are they amplified and attenuated? We expect the fields of lasers and amplifiers to behave classically when the number of photons per mode is large. How does the transition to classical behavior come about when the sources of these fields are quantum in origin and characterized by parameters equivalent to about one photon per mode?

In this paper, we attempt to give a clear and unified description of quantum noise that answers these and other questions. We were motivated to do this to clarify our own understanding of this subject and by the belief that many of our colleagues working in photonics would find a clear and rigorous method of picturing and calculating quantum-noise phenomena useful.

B. Practical treatments of quantum noise

The practical way of treating noise is to add Langevin forces to the classical wave and rate equations. Once the correlation functions of these noise sources, called diffusion coefficients (Lax, 1960), are established, the equations are solved as if the light were a fluctuating classical wave. There are a number of different procedures for finding these diffusion coefficients. The diffusion coefficient for the autocorrelation of field amplitudes can be set to account for the known spontaneous-emission rate into a mode (Henry, 1986a). The diffusion coefficients associated with photon number and minority-carrier number can be obtained by applying recombination-generation statistics to the photon-electron system (Lax, 1960, 1968; Henry, 1986b). An approach favored by engineers familiar with noise in microwave circuits is to make the analogy between lasers and transmission lines with Nyquist noise sources (Lowery, 1988; Nilsson *et al.*, 1991; Nilsson, 1994). Another way is to adapt the diffusion coefficients found in rigorous quantum treatments of gas lasers to the semiconductor case (Henry, 1983; Marcenac and Carroll, 1993). After a quantum-classical correspondence procedure, these treatments reduce the description of the laser to a set of classical rate equations with random noise sources called Langevin forces (Gordon, 1967; Lax, 1967b, 1968; Haken, 1985).

Much work has been done in treating various noise phenomena in semiconductor lasers during the last 15 years, since single-mode semiconductor lasers were developed. We mention here some of the basic results associated with quantum noise. The anomalously broad linewidths of semiconductor lasers (Fleming and Mooradian, 1982) were explained (Henry, 1982). The lineshape was found to be not a simple Lorentzian but structured with side peaks (Diano *et al.*, 1983, Vahala and Yariv, 1983). This was explained in detail (Henry, 1983; Vahala and Yariv, 1983). It was found that the linewidth of a semiconductor laser could be narrowed from several tens of megahertz to a few kilohertz by external feedback (Wyatt and Devlin, 1983; Link and Pollack, 1986; Olsson and Van der Ziel, 1987). Modification of the Langevin rate equations of a laser to account for the effect of feedback led to a thorough understanding of this narrowing (Patzak *et al.*, 1983; Kazarinov and Henry, 1987).

The early semiconductor lasers had no optical lateral confinement but were gain guided. When this was remedied by the development of index-guided lasers, it was found that the mode spectrum diminished from many longitudinal modes to only a few longitudinal modes. This surprising result was first explained by Petermann as enhanced spontaneous emission in gain-guided lasers (Petermann, 1979; Streifer *et al.*, 1982). The physical nature of this phenomenon has been clarified only recently

(Deutch *et al.*, 1991; Goldberg *et al.*, 1991) and will be discussed later in this paper.

Another important noise phenomenon is mode-partition noise in nearly single-mode lasers. It was found that even when a laser had only one dominant mode, as a rare event another mode could become intense, robbing strength from the lasing mode (Ogawa, 1982; Abbas and Yee, 1985; Link *et al.*, 1985). This was explained by calculation of the distribution of intensities of weak nonlasing modes stemming from fluctuations in spontaneous emission (Henry *et al.*, 1984; Miller and Marcuse, 1984; see review of Liu, 1991). Suppression of mode-partition noise is essential for high-bit-rate transmission. This need led to the development of distributed-feedback lasers in which nonlasing modes have very low intensities, even when the laser is modulated.

The behavior of distributed-feedback lasers is extremely complex owing to the change of mode suppression with increasing optical power in the cavity. This change is brought about by axial-spatial hole burning of the gain resulting from the nonuniform axial light intensity distribution (and stimulated emission) in the cavity. To understand these lasers, modeling must take into account changes in the field and carrier density along the length of the cavity. This is a departure from most laser modeling, in which the distributions of carrier density and field are regarded as fixed. The description of noise in these lasers requires solving coupled ordinary differential equations with random noise sources. This has been done analytically (Olesen *et al.*, 1993; Tromborg *et al.*, 1994) and by computer simulation (Lowery and Novak, 1994; Marcenac and Carroll, 1994).

The development of optical-fiber amplifiers doped with erbium is resulting in practical receivers that are limited by quantum noise instead of the thermal noise of electrical amplifiers (Desurvire, 1994; Park and Granlund, 1994). The amplifier adds a broadband amplified spontaneous-emission noise field to the signal, which must be reduced by narrow-band optical filtering before detection. A quantitative model of this noise, developed by P. S. Henry for semiconductor optical amplifiers, applies equally well to optical-fiber amplifiers (Olsson, 1989; Tonguz and Kazovsky, 1991).

Despite these successes, there are a number of displeasing aspects to the phenomenological description of light fluctuations. Rate equations with Langevin forces were used earlier to describe the generation of radio waves and thermal noise in electrical circuits (Van der Pol, 1934; Van der Ziel, 1954). However, going from $\hbar\omega \ll kT$ to $\hbar\omega \gg kT$ requires the introduction of noise sources as quantum-mechanical operators. Replacing operators by classical Langevin forces may lead to significant errors. Furthermore, finding the correlation functions of Langevin forces by these procedures does not identify the physical nature of quantum noise.

In addition, the arbitrary switching between wave and particle pictures to describe light in different situations is physically unsatisfying. For example, the noise detected after optical amplification is treated as the beating of signal and noise waves generated in spontaneous emission. However, a shot-noise contribution is also added, which is interpreted as noise associated with detection of photons. This is not noise added in the detection process because, for an opaque detector, all noise is associated with the optical field. Either a wave or photon description is permissible, but it is desirable that one description explain all results and, ideally, the explanation should be a solution of basic equations.

Shot noise is in fact explainable by a wave picture. Slusher and Yurke (1990) explain how shot noise is introduced in beam splitters through the beating of a noise field of vacuum fluctuations with the signal field. This concept of a noise field of vacuum fluctuations is introduced into quantum optics to explain excess noise in photodetectors of less than unit quantum efficiency by Yuen and Shapiro (1980) and in beam splitters by Yurke (1985). A similar noise field is used by Yamamoto and Imoto (1986) to explain how shot noise occurs in the intensity noise spectrum of lasers. These explanations of shot noise are indications that all forms of quantum noise can be explained from a wave picture provided that additional noise fluctuations associated with quantum uncertainties are added to the classical description.

C. Prior theories of quantum noise in lasers

The theory that gives a consistent description of the dual particle and wave behavior of light is the quantum theory of radiation. It is used in the classic papers on noise fluctuations of lasers written in the 1960s, principally by Haken, Lamb, Lax, and their co-workers (McCumber, 1966; Gordon, 1967; Lax, 1968; Louisell, 1974; Sargent *et al.*, 1974; Haken, 1984). These theories are also summarized in a number of textbooks (Marcuse, 1980; Haken, 1981, 1985; Loudon, 1983; Meystre and Sargent, 1990; Gardiner, 1991; Chow *et al.*, 1994; Mandel and Wolf, 1995). The major achievements of these works are in providing explanations of how the laser linewidth narrows and how low-frequency intensity fluctuations become reduced as a single-mode laser goes from below to above threshold. These theories also determine the photocount distributions of laser light from modes that are both below and above threshold. The theoretical results are in excellent agreement with experiments on gas lasers (Arecchi *et al.*, 1966; Freed and Haus, 1966; Gerhardt *et al.*, 1972; Lax and Zwanziger, 1973).

However, these approaches were limited to the description of lasers with closed cavities. These papers were written with gas lasers in mind, the well-behaved lasers of the 1960s, whereas noise is most important for communications lasers, which are semiconductor lasers. Gas lasers have highly reflecting end mirrors forming a nearly closed cavity. It was tempting to regard the cavity as completely closed and to quantize the modes of the lossless cavity. Today's semiconductor lasers and optical amplifiers have facet reflections that range from more than 90% to nearly zero and must be regarded as open structures. This requires a quantum treatment that, like the classical one, is not restricted by the need to start with cavity modes.

Yamamoto and Imoto (1986) pointed out that a correct description of shot noise in the intensity noise spectrum of lasers requires taking into account coupling of the laser cavity to outside optical modes. Yamamoto and co-workers (Yamamoto *et al.*, 1986) also predicted the suppression of the low-frequency part of the intensity spectrum to below the shot-noise limit when the laser is pumped with a quiet current source (verified by Richardson *et al.*, 1991). However, this treatment was still limited by weak coupling. Working in the related field of quantum optics, Collett and Gardiner (1984; Gardiner and Collett, 1985) also developed theories of cavities coupled to input and output modes, and Carmichael (1993a, 1993b) developed "quantum trajectory theory," a general theory of weakly coupled quantized systems. The theory of Gardiner and Collett is not restricted to weak coupling and has been applied to model a distributed photodetector, with an exponentially decaying optical field (Gardiner, 1991).

The exponential growth of the optical field is an important feature of semiconductor lasers with low facet reflectivities and travling-wave amplifiers. The problem of a one-dimensional description of a laser with arbitrary facet reflectivities was studied in detail by Ujihara in a series of papers in which the laser field is expanded in the quantized modes of a lossless dielectric that extends to infinity (see, for example, Ujihara, 1977, 1984). While this pioneering work obtained interesting results, such as the enhancement of laser linewidth in lasers of low facet reflectivity, the complexity of Ujihara's analysis illustrates the cumbersome nature of this expansion.

More recently, a number of authors have approached this problem in a manner that is closer to the one presented here. Maxwell's equations for propagation in one dimension are solved for a field operator propagating in an amplifying medium described by a complex dielectric function. The wave equation also contains a noise-source operator as an inhomogeneous term (Prasad, 1992; Tromborg *et al.*, 1994; Marani and Lax, 1995). Parametric-amplifier experiments have been analyzed in a similar manner (Gardiner and Savage, 1984; Prasad, 1994a, 1994b).

D. Our perspective

In this paper, we extend the quantum theory of lasers and amplifiers to arbitrary geometries ranging from closed cavities to traveling-wave amplifiers. We believe that this theory will appeal to those working in photonics because it has a form very similar to that of classical electromagnetic theory.

There are two fundamental sources of quantum noise in a system of coupled electrons and electromagnetic fields. One source is spontaneous currents resulting from momentum fluctuations at optical frequencies of electrons localized by atomic or crystalline fields. The other source is field fluctuations caused by the quantum-mechanical uncertainty of electric and magnetic fields.

The first part of this paper is devoted to delineating the properties of the electrons and the spontaneous and induced currents. Later, we show by a number of examples how our theory can be used to describe the quantum-noise phenomena encountered in photonics.

We describe optical noise from the point of view of fluctuations of electromagnetic fields, without the need to switch from a wave to a particle description of light. In this description, quantum-noise phenomena are governed by two equations: a scalar wave equation for the optical field and a carrier rate equation. Both equations contain spontaneous current-density noise sources. The beating of the optical field with spontaneous currents alters the rate of optical transitions and gives rise to recombination-generation noise of the carriers. The spontaneous current-density source in the wave equation results in the emission of a noise field. In electron systems with populated upper states, this is the field of spontaneous emission. In "cold" systems with unpopulated upper states, the noise field contributes to vacuum or zero-point fluctuations that are associated with the uncertainty of the electric and magnetic field of the optical wave (Loudon, 1983). The current of a photodetector is quadratic in the fields and results from the mutual beating of the signal and noise fields.

The problems of quantum noise in amplifying or absorbing media differ from those traditionally considered in the quantum theory of radiation and require different methods of solution. The conventional quantum treatment of radiation is the theory of Dirac (1927), which is beautifully presented in the review of Fermi (1932). This theory considers an atom interacting with modes of radiation in free space or with modes of a closed cavity. The mode amplitudes are taken to be operators. A radiation Hamiltonian and operator commutation relations are chosen that reproduce Maxwell's equations. An electron-field interaction is then introduced and absorption, emission, and scattering processes can then be calculated by perturbation theory. This can be extended phenomenologically to lossless dielectrics by including a dielectric function in the Hamiltonian (Carniglia and Mandel, 1971; Ujihara, 1975; Glauber and Lewenstein, 1991; Vogel and Welsch, 1994). However, the Dirac theory is not useful in describing the propagation of light in photonics. The modes of a dielectric extending over all space are not of direct interest and do not explicitly take into account the exponential growth and decay during propagation that accompanies gain and loss.

Instead, we assume as a first principle that the field and the current-density operators obey Maxwell's equations in free space. The precedents for making this assumption are discussed below. These equations can be expressed through the vector potential, in the Coulomb gauge, as a vector wave equation with a current-density source. To simplify our procedures, we approximate the vector wave equation by a scalar wave equation, an approximation that is often made in photonics applications and that is usually a good approximation when effects of polarized light can be neglected.

We expand the current-density operator in powers of the field, keeping the field-independent and linear terms. Averaging the linear term over the electrons results in the Kubo formula, which describes the induced current density in terms of a susceptibility. We show that the correlation functions of spontaneous current operators are related to the imaginary part of this susceptibility in accordance with the fluctuation-dissipation theorem. The derivation covers the case of a biased semiconductor, including one with an inverted population, and it extends the validity of the fluctuation-dissipation theorem to the case in which the electrons in the valence and conduction bands are separately in equilibrium and characterized by different quasi-Fermi levels.

If we express the current density as composed of a spontaneous part and a part induced by the field, the wave equation becomes one describing propagation in a complex dielectric medium with spontaneous current-density sources. This is the same equation that occurs in classical electrodynamics, but this quantum treatment differs from a classical one in that the fields are noncommuting Heisenberg operators with spontaneous current-density operator sources, which do not exist in a classical treatment.

Only averages of operators can be related to measurable quantities. The averages of a pair of current-density operators are given by the fluctuation-dissipation theorem. We show that averages of higher-order products of current-density operators can be determined in terms of averages of operator pairs in a manner reminiscent of Wick's theorem in quantum field theory. The higher moments are useful in calculating field and photon-number distributions in specific cases.

In most problems of quantum noise, a linear relation exists between the field and the spontaneous current-density sources. Proceeding exactly the same way as in classical electromagnetic problems, a solution of the wave equation can be found relating the field to the sources. The field-operator averages can then be determined in terms of averages of the spontaneous current-density operators.

An energy conservation equation is derived from the wave equation. From this equation we are able to find operators for the energy density and energy flux density. We show that whenever the averages of the energy density or energy flux density in a given spectral range become large compared to these densities for vacuum fluctuations, the noncommuting parts of these averages become negligible and the field can be treated as classical. The noise field in an amplifier and the noise sources in lasers (Langevin forces) are noncommuting operators related to both loss and gain. In the transition to classical fields, the noncommuting parts of the two sources of noise nearly cancel, and the effective noise source can be treated as classical. This treatment of the transition from quantum to classical behavior of the field is, in our view, physically transparent and differs from that given in prior work.

E. The Lifshitz method

The approach outlined above was used earlier to treat quantum effects of the electromagnetic field in complex geometries. As far as we know, it was first used by Lifshitz (1956) to find the force between closely spaced dielectric surfaces. As limiting cases, he obtained Casimir's formula for a force between uncharged metal surfaces (Casimer, 1948) and London's formula for the Van der Waals force between molecules. The Casimir force is attributed to the pressure of vacuum fluctuations, which is altered when the two surfaces approach each other (see Milonni, 1994). Further work on this problem by Schwinger *et al.* (1978) confirmed the validity of Lifshitz's results. Lifshitz's formula was confirmed experimentally, as discussed by Milonni (1994). Landau and Lifshitz (1960) described this method in their book, *Electrodynamics of Continuous Media*, and applied it to fluctuations of blackbody radiation and van der Waals forces. A related approach was taken by Agarwal in a series of papers discussing the influence of dielectric and metal surfaces on quantum-electrodynamic phenomena (Agarwal, 1974a–1974c).

The Lifshitz method resembles the semiclassical treatment of radiation found in textbooks. Schiff (1955) shows that the spontaneous-emission rate of an atom is correctly given by solving Maxwell's equations with a current-density source. Earlier, Blatt and Weisskopf (1952) derived general formulas for the decay rates of nuclei by radiative multipole transitions in the same manner. In carrying out these calculations, one makes the transition to quantum mechanics by replacing the squared integral over the current density by a squared matrix element of the current-density operator between the initial and final states. These same squared matrix elements enter into our expressions for the averages of the spontaneous current-density sources.

A number of authors justify the use of an operator wave equation with operator noise sources to describe light propagation in inhomogeneous, lossy, and dispersive dielectric media. Glauber and Lewenstein (1991) consider light propagation in an inhomogeneous dielectric, without dispersion or loss, and show that the vector-potential operator satisfies the classical vector wave equation. Caves and Crouch (1987) show that loss, dispersion, and noise can be included in a traveling-wave description of electromagnetic radiation by adding fictitious beam splitters, which create loss and dispersion, and by coupling vacuum fluctuations into the beam, which act as noise sources (see also Jeffers *et al.*, 1993). Hunter and Barnett (1992) derive the operator wave equation with a noise source, loss, and dispersion from a microscopic model of light propagation in a dielectric (see also Kupiszewska, 1992; Grunner and Welsh, 1995). Matloob *et al.* (1995) show that the noise operators are compatible with both the expected equal-time commutation rules of the field and the expected mean-square average field of vacuum fluctuations.

The organization of this paper is shown in the Table of Contents. For the convenience of the reader, we have listed specific results in Tables I and II along with equation numbers, indicating where they can be found in the text.

II. THE ORIGIN OF QUANTUM NOISE

A. Spontaneous current fluctuations

Classical noise arises from fluctuations in the motion of particles and in the number of particles within a given volume. Examples of this noise are thermal noise and shot noise. Quantum noise arises from uncertainty concerning the position and momentum of quantum-mechanical objects. For example, consider an electron in a potential well in its ground state, shown in Fig. 1. With a classical description, there would be no fluctuations, but the uncertainty principle does not allow a confined particle to be motionless. The operator

$$\hat{j}=\frac{e}{m}\hat{p} \tag{2.1}$$

is proportional to the current. We are interested in the frequency spectrum of the fluctuations of $\hat{j}$. In noise theory, this is given by the Fourier transform of the correlation function $\langle \hat{j}(t)\hat{j}(0)\rangle$.

In quantum mechanics, $\hat{j}$ and $\hat{p}$ are operators, and the time dependence of an operator is described by the Heisenberg representation. In this representation, the wave function ψ_o is time independent and describes only initial conditions. More familiar, perhaps, is the Schrödinger representation where the average value of $\hat{O}(t)$ is given by

$$\langle \hat{O}(t)\rangle=\langle \psi(t)|\hat{O}_S|\psi(t)\rangle, \tag{2.2}$$

where the subscript S indicates a time-independent Schrödinger operator. The formal solution of the Schrödinger equation for the wave function is

$$\psi(t)=\exp\left(-\frac{i}{\hbar}\hat{H}(t-t_0)\right)\psi_o, \tag{2.3}$$

where $\psi_o=\psi(t_0)$. Using this solution, the average becomes

$$\langle \hat{O}(t)\rangle=\langle \psi_o|\hat{O}(t)|\psi_o\rangle, \tag{2.4}$$

where the Heisenberg operator $\hat{O}(t)$ is

$$\hat{O}(t)\equiv\exp\left(\frac{i}{\hbar}\hat{H}(t-t_0)\right)\hat{O}_S\exp\left(-\frac{i}{\hbar}\hat{H}(t-t_0)\right) \tag{2.5}$$

and $\hat{H}$ is the Hamiltonian. Equations (2.4) and (2.5) define the Heisenberg representation.

We can write $\langle \hat{j}(t)\hat{j}(0)\rangle$ as a sum over a complete set of states, which can be chosen as energy eigenstates. Then $\langle \hat{j}(t)\hat{j}(0)\rangle$ becomes

$$\langle \hat{j}(t)\hat{j}(0)\rangle=\left(\frac{e}{m}\right)^2\sum_n \exp[-i(E_n-E_o)t/\hbar] \times|\langle \psi_n|\hat{p}|\psi_o\rangle|^2. \tag{2.6}$$

We see that the spectrum of current fluctuations is just the optical excitation spectrum and the spectral density is just the modulus squared of the transition-matrix elements. This spectrum has only positive frequencies, that is, frequencies corresponding to transitions from ground states to excited states. If excited states are also occupied, both positive and negative frequencies will appear in the spectrum.

Heisenberg operators of the same quantity at two different times do not commute. For example, with $\hat{j}(t)$ and $\hat{j}(0)$ interchanged in Eq. (2.6), the resulting correlation function with the electron in the ground state would have negative frequencies instead of positive frequencies. This clearly shows that currents at two different times are noncommuting operators:

$$\langle \hat{j}(t)\hat{j}(0)\rangle\neq\langle \hat{j}(0)\hat{j}(t)\rangle. \tag{2.7}$$

Since the electron is a charged particle, the momentum or velocity fluctuations are a source of electromagnetic radiation and hence a source of quantum noise.

B. Electromagnetic-field uncertainty

The other source of quantum noise is the quantum uncertainty between electric and magnetic fields. Consider the simple example of the electromagnetic field in a closed, lossless cavity. The field in the cavity can be decomposed into modes, which are characterized by specific spatial distributions and resonant frequencies. The energy stored in a mode oscillates between electric and magnetic forms similar to the way it oscillates between the kinetic and potential forms for a mechanical oscillator. This analogy motivated Dirac (1927) to regard the electric and magnetic fields of the mode as the generalized coordinates and momenta of an oscillator and to apply quantum mechanics to describe it [see Pais (1986) and Schweber (1994) for a summary of Dirac's work and earlier work by Born, Heisenberg, and Jordan]. This treatment leads to quantization of the mode energies, which are known as photons, see Fig. 2(a). As a quantum harmonic oscillator, the mode has an equidistant energy spectrum with level separations $\hbar\omega$ and a ground-state energy of $\hbar\omega/2$:

$$E_n=(n+\tfrac{1}{2})\hbar\omega. \tag{2.8}$$

Just as in the case of an electron in a potential well, the finite energy of the ground state is caused by momentum and coordinate fluctuations associated with the uncertainty principle. The fluctuating electromagnetic field in the ground state is referred to as the field of vacuum fluctuations. The electric and magnetic fields of a mode satisfy a similar uncertainty principle as the coordinates and momenta in quantum mechanics. The uncertainty principle follows from the well-known commutation rules between coordinate and momentum operators (Bohm, 1951). From the correspondence between coordinates and momenta and the electric and magnetic fields, Dirac concluded that the electric and

TABLE I. Summary of results.

Result	Equation
Uncertainty-related field fluctuations	
Emission from a cold opaque source mimics vacuum fluctuations.	(7.10)
The energy flux of the uncertainty-related field fluctuations is neither attenuated nor amplified.	(A.8), Sec. VI.D
The correlation function of vacuum fluctuations in an opaque medium decays with distance because attenuation is accompanied by uncorrelated generation with energy conserved.	(B.8 - B.9), after (7.10)
Vacuum fluctuations excite no optical transitions of electrons from lower levels.	before (6.15), after (8.31)
Shot-noise fluctuations of the photon flux of amplified and attenuated signals are the beating of the uncertainty-related field fluctuations with the signal field	(7.26),(7.40)
Spontaneous currents	
Spontaneous currents are momentum fluctuations at optical frequencies of confined electrons. They persist in cold systems, having only lower-level occupation.	(2.6)
In Raman amplifiers, spontaneous currents result from the mixing of lattice fluctuations and the optical field of the pump.	(7.45)
The correlation functions of spontaneous current pairs are related to loss and gain coefficients by the fluctuation-dissipation theorem.	(4.38),(4.42)
Shot noise in a nearly transparent photodetector is a beating of the signal and spontaneous currents within the detector, which modulates the rate of optical absorption.	(7.32)
The fluctuation-dissipation theorem also holds for scattering loss, where the noise source is vacuum fluctuations scattered into the guided modes.	(7.50)
Generation and propagation of noise fields	
Maxwell's equations for the field and current operators can be approximated as a scalar wave equation for field-operator propagation in a complex dielectric with spontaneous current-density operator sources.	(6.5)
When gain is time independent, a Green's function linearly relates the noise field to spontaneous current-density sources.	(6.7)
Noise fields emitted from an opaque source account for Planck radiation and light-emitting diode emission.	(7.9)
Narrow-band noise in high-gain amplifiers is the beating of the signal field with the field of amplified spontaneous emission. The minimum amplifier noise figure is 3 dB.	(7.40)
Spontaneous emission and internally generated and external vacuum fluctuations contribute to the Langevin force of a laser mode.	(8.9)

TABLE II. Summary of results.

Result	Equation
Optical transition noise and spontaneous emission	
A general equation for the rate of optical transitions of carriers is derived which accounts for absorption, stimulated and spontaneous emission, and noise.	(5.10 - 5.11)
The beating of an external optical field with spontaneous currents modulates the rate of stimulated emission and absorption, causing recombination-generation noise.	(5.13)
Spontaneous emission into a cavity mode is weighted by the spatial distribution of the squared mode field and goes to zero at nodes in this field.	(8.32)
The rate of spontaneous emission into a laser mode is not enhanced in open cavities by the Petermann factor as once thought. Instead, the enhancement of noise and mode intensity is due to single-pass ampflication.	(8.32),(8.44)
Dirac's formula for the rates of optical absorption and stimulated and spontaneous emission holds in the limit of lossless modes.	(8.50)
Probability distributions of observables inferred from higher-order correlation functions	
Higher-order spontaneous current correlation functions are calculated in terms of pair-correlation functions, with rules resembling Wick's theorem.	after (4.62)
The real and imaginary parts of the noise fields and the spontaneous currents have Gaussian distributions.	(7.19)
Shot noise in an opaque detector has a Poisson distribution.	(7.26 - 7.27)
Nonlasing modes near threshold have an exponential photon-number distribution. This distribution has high-intensity fluctuations causing mode-partition noise.	(8.19)

magnetic mode field amplitudes are also operators satisfying similar commutation rules (Loudon, 1983).

Since a general electromagnetic field can always be decomposed into modes, it can be regarded as a quantum-mechanical system with an infinite number of degrees of freedom. Just as in the classical case, the electromagnetic field satisfies Maxwell's equations with current sources. The only change is that, in the quantum-mechanical case, both the field and the current are operators. This description of the field interacting with the current is called quantum electrodynamics, or the quantum theory of radiation when restricted to the transverse part of the field associated with radiation (Fermi, 1932).

There are two independent sources of quantum noise in a system of coupled electrons and electromagnetic fields. One results from the Heisenberg uncertainties between position and momentum of electrons, and it leads to spontaneous current fluctuations. The other is the Heisenberg uncertainty between electric and magnetic fields, which results in quantum field fluctuations. (Another noise source, essential for the description of Raman amplifiers, will be discussed in Sec. VII.F.) Ideally, in situations where the field is weakly interacting with electrons absorbing and emitting light, we can regard the light and electrons as two systems, each having its own uncertainty-related and thermal fluctuations. In reality, the two systems are coupled, and this coupling causes each system to induce fluctuations and dissipations in the other.

Consider a closed cavity with heated walls that can both absorb and emit radiation. The field in thermal equilibrium can be regarded as both damped and generated by fluctuating currents in the walls. It is convenient

TABLE II. (*Continued.*)

Photons interacting with carriers in a closed lossless cavity have a Bose-Einstein distribution of photon number.	(8.22 - 8.23)
Energy conservation of narrow-band radiation interacting with carriers	
The quantum wave equation and the quantum carrier rate equations are consistent with conservation of the energy stored in the electron and photon systems.	(6.12)
Photon-flux density and photon-number density operators are identified from terms in the energy conservation equation.	(6.13 - 6.14)
The signal of an ideal opaque photodetector measures photon flux received, with no additional noise.	(6.16)
The low-frequency fluctuations of the photon flux emitted from a laser cavity follow the fluctuations in electrical current. These fluctuations are sub-Poissonian because of the low electrical current noise.	(8.57)
Classical behavior at high fields	
The uncertainty-related field fluctuations are not amplified and become negligible at high fields.	(A.7 - A.8), Sec. VI.D
As laser power increases, the noncommuting contributions to the Langevin force associated with spontaneous currents and vacuum fluctuations cancel, leaving a Langevin rate equation that can be treated as classical.	(4.56),(8.12)
The noncommuting contributions to the energy flux of amplified spontaneous emission from spontaneous currents and induced by vacuum fluctuations cancel. Consequently, the noise field of a high-gain amplifier can be treated as classical.	(7.35b),(7.36)

to extend the process of attenuation and emission by the walls to zero temperature as a way of describing the ground state of the combined system. This is the approach that is followed in this paper. Thus we treat vacuum fluctuations as emitted and absorbed like thermal radiation. We do this despite the fact that vacuum fluctuations cannot be detected directly (see end of Sec. VI.C). We find that the calculated emission from opaque walls at zero temperature mimics the flux of vacuum fluctuations from empty space (see Sec. VII.A). Therefore we shall not distinguish the noise field emitted from a cold absorber and the noise field coming from empty space. The emission from cold absorbers is also useful in describing shot noise (see Sec. VII.C) and noise in lasers (see Secs. IV.H and VIII.A).

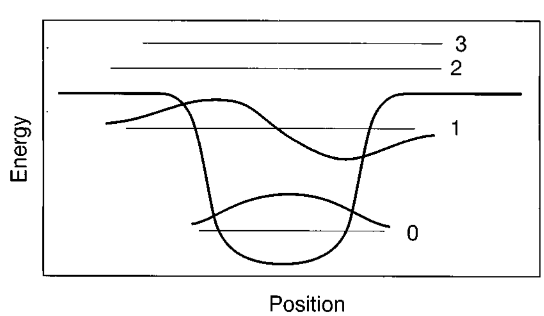
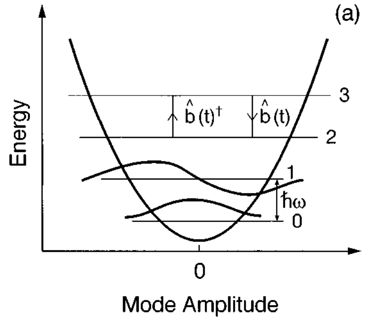


FIG. 1. Origin of spontaneous currents. The potential well has bound electron states. There are momentum fluctuations associated with the confinement. The frequencies of the fluctuations of a bound electron are the frequencies of transitions to other levels. Momentum fluctuations of charged particles are spontaneous currents.

III. SEMICONDUCTOR MODEL

In this section, we introduce fluctuating electron currents, which we regard as a principal source of quantum noise. For clarity of description, we shall use a model of noninteracting electrons and holes in a semiconductor. Our results are, however, general and not restricted to this specific model.



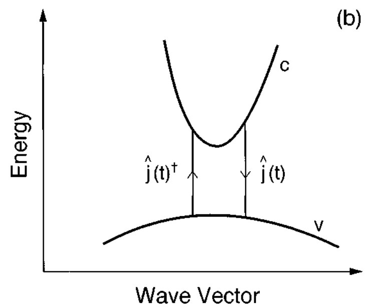


FIG. 2. Analogy between photon creation and annihilation operators and spontaneous current operators: (a) The energy of an optical mode vs its amplitude. The quantized field states have equidistant energy levels, which are photons. The transitions between the equidistant levels 1, 2, 3, ... are described by creation and annihilation operators $\hat{b}^\dagger(t)$ and $\hat{b}(t)$, which oscillate at the frequency corresponding to the level separation. (b) A simplified band diagram of a semiconductor with conduction band and valence band. Spontaneous current-density operators are associated with the upward and downward transitions and oscillate at frequencies corresponding to the level separations.

A. Second-quantization description of electrons

It is convenient in describing a many-body system to use the second quantization of the electron field. Clear discussions of second quantization can be found in the books of Landau and Lifshitz (1958a) and Schrieffer (1964). Let $|uk\rangle$ represent the semiconductor one-electron states, where $u=c,v$ and refers to the conduction or valence band and where k is the wave vector of the state [see Fig. 2(b)]. In the second-quantization description, the wave function of the system is described in terms of occupation numbers n_{uk} of one-electron states, where n_{uk} can only take the values of 0 and 1 because of the Pauli principle:

$$\psi=|n_{ck_1},n_{ck_2},\ldots n_{vk_1},n_{vk_2},\ldots\rangle. \tag{3.1}$$

The alternative is to describe the many-electron wave function of the system as a Slater determinant of one-electron wave functions, but this is very cumbersome.

Creation and annihilation operators $\hat{a}^\dagger_{uk}$ and $\hat{a}_{uk}$ are introduced that alter these occupation numbers. The creation operator $\hat{a}^\dagger_{uk_1}$ acts to raise the occupation number of state uk_1 from zero to one:

$$\hat{a}^\dagger_{uk_1}|0_{uk_1},n_{uk_2},\ldots\rangle=|1_{uk_1},n_{uk_2},\ldots\rangle. \tag{3.2}$$

Similarly, the annihilation operator $\hat{a}_{uk_1}$ lowers the occupation number:

$$\hat{a}_{uk_1}|1_{uk_1},n_{uk_2},\ldots\rangle=|0_{uk_1},n_{uk_2},\ldots\rangle. \tag{3.3}$$

Because of the Pauli principle (each state cannot be occupied by more than one electron) and the indistinguishability of electrons (a wave function can change only phase when two electrons are interchanged), it can be shown that the wave function changes sign when two electrons are interchanged. When combined with Eqs. (3.2) and (3.3), this leads to the anticommutation rules of the operators:

$$\hat{a}^\dagger_{uk}\hat{a}_{u'k'}+\hat{a}_{u'k'}\hat{a}^\dagger_{uk}=\delta_{u,u'}\delta_{k,k'}, \tag{3.4a}$$

$$\hat{a}_{uk}\hat{a}_{u'k'}+\hat{a}_{u'k'}\hat{a}_{uk}=\hat{a}^\dagger_{uk}\hat{a}_{u'k'}{}^\dagger+\hat{a}_{u'k'}{}^\dagger\hat{a}^\dagger_{uk}=0. \tag{3.4b}$$

These equations imply that pairs of Hermitian-conjugate operators associated with the same state are related by $\hat{a}_{uk}\hat{a}^\dagger_{uk}=1-\hat{a}^\dagger_{uk}\hat{a}_{uk}$ and all other pairs of operators anticommute.

Consider an operator acting on a system of identical electrons. For example, consider the total momentum, which is the sum of momentum of all electrons,

$$\hat{\mathbf{P}}=\sum_i \hat{\mathbf{p}}_i. \tag{3.5}$$

The matrix elements of the momentum between two many-body wave functions (Slater determinants) can be shown to simplify to

$$\langle\psi_A|\hat{\mathbf{P}}|\psi_B\rangle=\sum_{uk}\sum_{u'k'}\langle uk|\hat{\mathbf{p}}|u'k'\rangle\langle\psi_A|\hat{a}^\dagger_{uk}\hat{a}_{u'k'}|\psi_B\rangle. \tag{3.6}$$

Since the momentum operator is diagonal between states of wave vector, but has nondiagonal components between bands, Eq. (3.6) simplifies to

$$\langle\psi_A|\hat{\mathbf{P}}|\psi_B\rangle=\sum_k\langle ck|\hat{\mathbf{p}}|vk\rangle\langle\psi_A|\hat{a}^\dagger_{ck}\hat{a}_{vk}|\psi_B\rangle+\sum_k\langle vk|\hat{\mathbf{p}}|ck\rangle\langle\psi_A|\hat{a}^\dagger_{vk}\hat{a}_{ck}|\psi_B\rangle. \tag{3.7}$$

These equations show that the total-momentum matrix element can be written in terms of creation and annihilation operators acting on occupation factors, such as the wave function in Eq. (3.1). This applies to any operator acting on a single-electron coordinate. This description can be extended to particle interactions that involve two-electron coordinates, but we do not need this for our discussion here.

B. Hamiltonian and current-density operators

The interaction of the electrons of a semiconductor with the electromagnetic field involves the current-density operator $\hat{\mathcal{J}}$. The time dependence of all operators is determined by the Hamiltonian. To proceed, we need to express both the Hamiltonian and the current density as second-quantized operators.

The Hamiltonian operator $\hat{H}_e$ has diagonal matrix elements and is given by

$$\hat{H}_e=\sum_k [\varepsilon_{ck}\hat{a}_{ck}^\dagger\hat{a}_{ck}+\varepsilon_{vk}\hat{a}_{vk}^\dagger\hat{a}_{vk}], \tag{3.8}$$

where ε_{ck} and ε_{vk} are the one-electron energies in the conduction and valence bands. We need consider only currents associated with interband optical transitions. These components of the current are diagonal with respect to wave vector. The current-density operator associated with these transitions, for the current in a particular direction, is

$$\hat{\mathcal{J}}_S=\frac{e}{Vm}\sum_k [\langle vk|\hat{p}|ck\rangle\hat{a}_{vk}^\dagger\hat{a}_{ck}+\langle ck|\hat{p}|vk\rangle\hat{a}_{ck}^\dagger\hat{a}_{vk}]$$
$$=\hat{j}_S+\hat{j}_S^\dagger, \tag{3.9}$$

where e is the electron charge (negative number) and V is the volume normalization. We see that $\hat{j}_S$ is associated with downward transitions and $\hat{j}_S^\dagger$ is associated with upward transitions. The energy bands and transitions are illustrated in Fig. 2(b), which shows the analogy of these currents and the creation and annihilation operators for photons. Strictly speaking, the current operator has an additional term proportional to the vector potential. However, this term does not cause optical transitions and other effects that we consider, so we shall skip it.

We shall consider V to be a volume small in scale compared to a wavelength of light but still containing thousands of carrier states in each band. The interaction of the subsystem of carriers in volume V with an external electromagnetic field described by the vector potential $\hat{\mathcal{A}}(\mathbf{x},t)$ is given by

$$\hat{H}_{\rm int}=-\frac{V}{c}\hat{\mathcal{J}}_S\cdot\hat{\mathcal{A}}(t), \tag{3.10}$$

where $\hat{\mathcal{A}}$ is assumed to be uniform within the small volume V. For brevity, throughout this paper we shall refer to the vector potential $\hat{\mathcal{A}}$ and its frequency components as the "optical field" or simply as the "field." In describing the electromagnetic field, we use Gaussian units, the units used by several classic texts on electromagnetic theory (Landau and Lifshitz, 1960; Jackson, 1968). The Hamiltonian of the radiation is given by the familiar expression

$$\hat{H}_r=\int d\mathbf{x}\frac{\hat{\mathcal{E}}^2+\hat{\mathcal{H}}^2}{8\pi}, \tag{3.11}$$

where $\hat{\mathcal{E}}=-\dot{\hat{\mathcal{A}}}/c$ and $\hat{\mathcal{H}}=\nabla\times\hat{\mathcal{A}}$. In the Schrödinger representation, the operators $\hat{\mathcal{J}}_S$ and $\hat{\mathcal{A}}_S$ act on different sets of occupation-number states and therefore commute.

IV. TIME-DEPENDENT CURRENT-DENSITY OPERATORS

Here we introduce time-dependent current-density operators, which are usually referred to as operators in the Heisenberg representation. We expand these operators in powers of the electromagnetic field, keeping only the field-independent and linear terms. The field-independent operator is the spontaneous current, and the operator linear in the field is the induced current. The induced current is described in terms of a susceptibility that is related to the averages of the commutator of the spontaneous current-density operators. This is known as the Kubo formula. We derive the fluctuation-dissipation theorem, which relates correlation functions of the spontaneous current-density operators to the imaginary part of the susceptibility. The fluctuation-dissipation theorem is extended to the case of an electron-hole system with different quasi-Fermi levels, which allows us to describe how the correlation functions depend on the level of population inversion. We also develop a procedure for evaluating higher-order correlation functions of the spontaneous current-density operators.

A. Time-dependent operators

The operators $\hat{H}_e$ and $\hat{\mathcal{J}}_S$ defined by Eqs. (3.8) and (3.9) are regarded as operators in the Schrödinger representation. In this representation, the wave function depends on time, but the operators are time independent. The time dependence of physical quantities results from the time dependence of the wave function. There is another way to calculate the time-dependent averages of physical quantities, and that is to use the Heisenberg representation as was done in Eqs. (2.3)–(2.5). Consider a time-independent operator $\hat{O}_S$ in the Schrödinger representation. The matrix element between states A and B is

$$\langle\psi_A(t)|\hat{O}_S|\psi_B(t)\rangle\equiv\langle\psi_A(t_0)|\hat{O}(t)|\psi_B(t_0)\rangle. \tag{4.1}$$

On the right side of this expression, $\hat{O}(t)$ is the Heisenberg operator [Eq. (2.5)], which acts on the initial states.

We can now find a formal solution of the Schrödinger equation:

$$i\hbar\dot{\psi}_S=[\hat{H}_e+\hat{H}_{\rm int}(t)]\psi_S. \tag{4.2}$$

In the case of an electromagnetic interaction, the interaction is small and can be treated by perturbation theory. We shall follow a procedure that was developed for quantum field theory and introduce another representation, the interaction representation (Abrikosov *et al.*, 1963). In this representation, the wave function ψ_I is related to ψ_S by

$$\psi_S(t)=\exp\left(-\frac{i}{\hbar}\hat{H}_e t\right)\psi_I(t). \tag{4.3}$$

The purpose of this substitution is to remove the large term $\hat{H}_e$ from the Schrödinger equation. Substituting Eq. (4.3) into Eq. (4.2), we see that ψ_I obeys a Schrödinger equation with an effective time-dependent but small Hamiltonian $\hat{H}_I(t)$:

$$i\hbar\dot{\psi}_I(t)=\hat{H}_I(t)\psi_I(t), \tag{4.4a}$$

$$\hat{H}_I(t) \equiv \exp\left(\frac{i}{\hbar}\hat{H}_e t\right)\hat{H}_{\text{int}}(t)\exp\left(-\frac{i}{\hbar}\hat{H}_e t\right). \tag{4.4b}$$

Substituting Eq. (4.3) into the expression for the average value of operator $\hat{O}_S$ given in Eq. (4.1), we have

$$\langle \psi_A(t)|\hat{O}_S|\psi_B(t)\rangle = \langle \psi_{AI}(t)|\hat{O}_I(t)|\psi_{BI}(t)\rangle, \tag{4.5}$$

where

$$\hat{O}_I(t) \equiv \exp\left(\frac{i}{\hbar}\hat{H}_e t\right)\hat{O}_S\exp\left(-\frac{i}{\hbar}\hat{H}_e t\right) \tag{4.6}$$

is an operator in the interaction representation.

We can solve the Schrödinger equation for $\psi_I(t)$ [Eq. (4.4)], the time evolution of the wave function, in the interaction representation. The well-known formal solution of this equation is

$$\psi_I(t) = \hat{S}(t,t_0)\psi_I(t_0), \tag{4.7a}$$

$$\hat{S}(t,t_0) = \hat{T}\left[\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}_I(u)du\right]\right], \tag{4.7b}$$

where $\hat{T}$ is a time-ordering operator (not needed in what follows) and $\hat{S}(t,t_0)$ is the evolution operator (Abrikosov *et al.*, 1963). If we take t_0 to be an early time $-\infty$, when there is no interaction, and then turn on the interaction slowly, the initial states $\psi_S(t_0)$ and $\psi_I(t_0)$ are the same. We can now express the matrix element at time t in terms of the initial states:

$$\langle \psi_{AI}(t)|\hat{O}_I(t)|\psi_{BI}(t)\rangle$$
$$= \langle \psi_A(t_0)|\hat{S}(t,t_0)^{-1}\hat{O}_I(t)\hat{S}(t,t_0)|\psi_B(t_0)\rangle. \tag{4.8}$$

Comparing this equation with the definition of the Heisenberg operator in Eq. (4.1), we see that

$$\hat{O}(t) = S(t,t_0)^{-1}\hat{O}_I S(t,t_0). \tag{4.9}$$

Expanding the evolution operator [Eq. (4.7b)], we find

$$\hat{S}(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t}\hat{H}_I(u)du + \cdots, \tag{4.10}$$

and keeping only the zeroth- and first-order terms, we arrive at an approximate expression for the Heisenberg operator $\hat{O}(t)$,

$$\hat{O}(t) \simeq \hat{O}_I(t) + \frac{i}{\hbar}\int_{t_0}^{t}[\hat{H}_I(u),\hat{O}_I(t)]du. \tag{4.11}$$

B. Spontaneous and induced current

We call the current density existing in the absence of a field the spontaneous current density $\hat{\mathcal{J}}(t)$ associated with a small volume V. The Heisenberg operator for a current density in the presence of a field is given by substituting $\hat{H}_{\text{int}}$ of Eq. (3.10) into the formula for constructing the approximate Heisenberg operator [Eq. (4.11)]:

$$\hat{\mathcal{J}}_{\text{tot}}(t) = \hat{\mathcal{J}}(t) - \frac{iV}{\hbar c}\int_{-\infty}^{t}[\hat{\mathcal{J}}(u),\hat{\mathcal{J}}(t)]\hat{\mathcal{A}}(u)du, \tag{4.12}$$

where $\hat{\mathcal{A}}(t)$ is the Heisenberg operator for the external field. The meaning of the external field is that it excludes that part of the field emitted from within the volume V. The external field originates in other volumes. We call this operator $\hat{\mathcal{J}}_{\text{tot}}(t)$ because it is the sum of the spontaneous current density and a current density induced by an external field. The first term is the spontaneous current, a current not induced by the field. It is the source of the noise field and can be regarded as a quantum Langevin force.

The second term in Eq. (4.12) is proportional to the commutator of the spontaneous current at different times. It also has a fluctuating part, but, in the spirit of low-order perturbation theory, we shall neglect a Langevin force proportional to the electromagnetic field and replace the commutator in the second term by its average value over the electron states. With this assumption, and substituting $u = t - \tau$ into Eq. (4.12), we find $\hat{\mathcal{J}}_{\text{tot}}(t)$ has the form

$$\hat{\mathcal{J}}_{\text{tot}}(t) = \hat{\mathcal{J}}(t) + \hat{\mathcal{J}}_{\text{ind}}(t), \tag{4.13a}$$

$$\hat{\mathcal{J}}_{\text{ind}}(t) = -\frac{iV}{\hbar c}\int_0^{\infty}\langle[\hat{\mathcal{J}}(t-\tau),\hat{\mathcal{J}}(t)]\rangle\hat{\mathcal{A}}(t-\tau)d\tau. \tag{4.13b}$$

We see that $\hat{\mathcal{J}}_{\text{tot}}(t)$ is the sum of a spontaneous current term $\hat{\mathcal{J}}(t)$ that is independent of field and an induced current $\hat{\mathcal{J}}_{\text{ind}}(t)$ that is proportional to the external field.

C. Time dependence of the spontaneous current

We want to find the time dependence of $\hat{\mathcal{J}}(t)$ in the interaction representation. Applying Eq. (4.6) for the time dependence of an operator in the interaction representation to the operator for the current density, we find

$$\hat{\mathcal{J}}(t) = \exp\left(\frac{i}{\hbar}\hat{H}_e t\right)\hat{\mathcal{J}}_S\exp\left(-\frac{i}{\hbar}\hat{H}_e t\right)$$
$$= \frac{e}{mV}\sum_k p_k\exp\left(\frac{i}{\hbar}\hat{H}_e t\right)\hat{a}_{vk}^{\dagger}\hat{a}_{ck}\exp\left(-\frac{i}{\hbar}\hat{H}_e t\right)$$
$$+\text{H.c.},$$
$$= \frac{e}{mV}\sum_k [p_k\hat{a}_{vk}^{\dagger}(t)\hat{a}_{ck}(t) + p_k\hat{a}_{ck}^{\dagger}(t)\hat{a}_{vk}(t)], \tag{4.14}$$

where $\hat{a}_{ck}(t) = \exp[(i/\hbar)\hat{H}_e t]\hat{a}_{ck}\exp[-(i/\hbar)\hat{H}_e t]$, etc., and $p_k \equiv \langle vk|p|ck\rangle$.

The time dependence of $\hat{a}_{ck}(t)$ is readily found by differentiating this operator and applying the anticommutation rules, Eq. (3.4). We find

$$\frac{d\hat{a}_{ck}}{dt} = \frac{i}{\hbar}[\hat{H}_e,\hat{a}_{ck}] = \frac{i}{\hbar}\varepsilon_{ck}[\hat{a}_{ck}^{\dagger}\hat{a}_{ck},\hat{a}_{ck}] = -\frac{i}{\hbar}\varepsilon_{ck}\hat{a}_{ck}, \tag{4.15}$$

with the solution

$$\hat{a}_{ck}(t)=\exp(-i\varepsilon_{ck}t/\hbar)\hat{a}_{ck}. \tag{4.16a}$$

Similarly, we write

$$\hat{a}_{vk}(t)=\exp(-i\varepsilon_{vk}t/\hbar)\hat{a}_{vk}. \tag{4.16b}$$

Substituting the time-dependent creation and destruction operators into the equation for $\hat{\mathcal{J}}(t)$, we have

$$\hat{\mathcal{J}}(t)=\sum_k \hat{j}_k\exp(-i\omega_k t)+\hat{j}_k^\dagger\exp(i\omega_k t),$$

$$\equiv\hat{j}(t)+\hat{j}(t)^\dagger, \tag{4.17a}$$

where

$$\hat{j}_k=\frac{ep_k}{mV}\hat{a}_{vk}^\dagger\hat{a}_{ck} \tag{4.17b}$$

and

$$\omega_k=\frac{\varepsilon_{ck}-\varepsilon_{vk}}{\hbar} \tag{4.17c}$$

is the optical transition frequency (shown in Fig. 3).

We see that $\hat{\mathcal{J}}(t)$ is the sum of two terms having positive and negative frequencies, where $\exp(-i\omega_k t)$ is defined as a positive-frequency component. The positive-frequency current $\hat{j}(t)$ is associated with downward transitions. It destroys a conduction-band electron and creates a valence-band electron. The negative-frequency current $\hat{j}(t)^\dagger$ is associated with upward transitions. The analogy between these operators and the photon creation and annihilation operators is illustrated in Fig. 2.

The order of a pair of operators $\hat{j}(t_1)$ and $\hat{j}^\dagger(t_2)$ is important because these operators do not commute. We will refer to the pair $\hat{j}^\dagger(t_2)\hat{j}(t_1)$ with the positive frequency current density to the right as "normally ordered" and to the pair $\hat{j}(t_1)\hat{j}^\dagger(t_2)$ with the positive frequency current density to the left as "antinormally ordered." This notation will also be used in referring to pairs of positive and negative frequency components of the field.

D. Kubo's formula for the induced current

We can simplify the formula for the induced current, Eq. (4.13b), by separating the positive and negative frequency terms using Eq. (4.17). We can also write the vector potential $\hat{\mathcal{A}}(t)$ as a sum of positive- and negative-frequency components:

$$\hat{\mathcal{A}}(t)=\int_0^\infty\frac{d\omega}{\sqrt{2\pi}}(\hat{A}_\omega e^{-i\omega t}+\hat{A}_\omega^\dagger e^{i\omega t}). \tag{4.18}$$

The current induced by a single positive-frequency Fourier component $\hat{A}_\omega\exp(-i\omega t)$ is given by Eq. (4.13) as

$$\hat{j}_{\text{ind}}(t)=-\frac{iV}{\hbar c}\int_0^\infty d\tau\langle[\hat{\mathcal{J}}(t-\tau),\hat{\mathcal{J}}(t)]\rangle e^{i\omega\tau}\hat{A}_\omega e^{-i\omega t}. \tag{4.19}$$

We make the assumption of "stationarity," that the average $\langle\hat{\mathcal{J}}(t-\tau)\hat{\mathcal{J}}(t)\rangle$ is independent of the absolute time t. Since, when interactions are included, the correlation function decays in a polarization relaxation time (Kazarinov *et al.*, 1982), which for a semiconductor is less than a picosecond, stationarity actually means that the electron distribution is not changing during this time. With this assumption, we write

$$\langle[\hat{\mathcal{J}}(t-\tau),\hat{\mathcal{J}}(t)]\rangle=\langle[\hat{\mathcal{J}}(0),\hat{\mathcal{J}}(\tau)]\rangle=-\langle[\hat{\mathcal{J}}(\tau),\hat{\mathcal{J}}(0)]\rangle. \tag{4.20}$$

The commutation rules in Eq. (3.4) show that $\hat{j}(t)$ and $\hat{j}(t')$ commute, so only the combinations $\langle[j(\tau),j^\dagger(0)]\rangle$ and $\langle[j(\tau)^\dagger,j(0)]\rangle$ contribute to the commutator. Only the first of these has a positive frequency $\exp(-i\omega_k\tau)$ capable of canceling $\exp(i\omega\tau)$ and giving a resonant term. Keeping only this term, we have

$$\hat{j}_{\text{ind}}(t)=\chi(\omega)\hat{A}_\omega e^{-i\omega t}, \tag{4.21}$$

where

$$\chi(\omega)\equiv\frac{iV}{\hbar c}\int_0^\infty\langle[\hat{j}(\tau),\hat{j}^\dagger(0)]\rangle e^{i\omega\tau}d\tau. \tag{4.22}$$

This is Kubo's formula relating the susceptibility to the average value of the spontaneous current commutator at two times (Kubo, 1966; Martin, 1968).

A formula for $\chi''(\omega)=(\chi(\omega)-\chi(\omega)^*)/2i$ is found by taking the complex conjugate of Eq. (4.22). The formula for $-\chi(\omega)^*$ is the same as $\chi(\omega)$, but with the limits of integration changed to $-\infty$ and 0, so the imaginary part of the susceptibility $\chi''(\omega)$ is given by

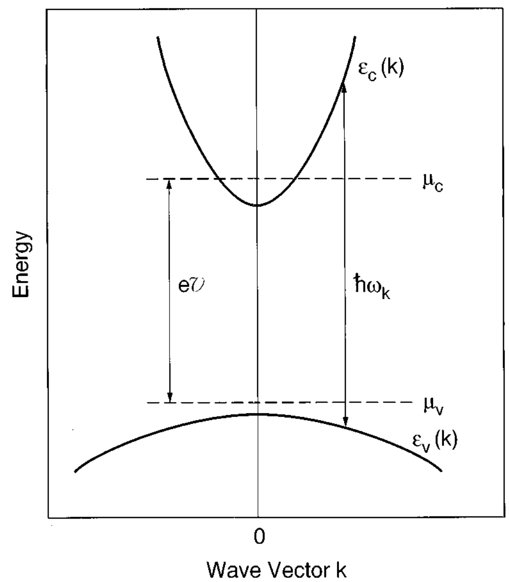


FIG. 3. Parameters of the semiconductor model. The energy diagram of a biased semiconductor is shown with the conduction- and valence-band energies ε_c and ε_v as functions of wave vector k. These energies are separated by $\hbar\omega_k$. The positions of the quasi-Fermi levels μ_c and μ_v of the carriers in the conduction and valence bands are shown by dashed lines. In semiconductor lasers, the quasi-Fermi levels are separated by $e\mathcal{V}$, where $\mathcal{V}$ is the bias voltage at the active layer.

$$\chi''(\omega)=\frac{V}{2\hbar c}\int_{-\infty}^{\infty}(\langle\hat{j}(\tau)\hat{j}^{\dagger}(0)\rangle-\langle\hat{j}^{\dagger}(0)\hat{j}(\tau)\rangle)e^{i\omega\tau}d\tau. \tag{4.23}$$

We can evaluate $\chi(\omega)$ for a system of noninteracting one-electron states. In a system with interacting states the correlation function $\langle j(\tau)j(0)^{\dagger}\rangle$ decays in about an electron-scattering time. We shall mimic this behavior in a noninteracting system by adding $e^{-\epsilon\tau}$ to the integrand in Eq. (4.22) for $\chi(\omega)$, where $\epsilon\to 0$ in our model of noninteracting electrons. This will ensure convergence of the integral. Using Eq. (4.17) to express $\hat{j}(t)$ in terms of frequency components and creation and destruction operators, we find

$$\chi(\omega)=\frac{ie^2}{\hbar cVm^2}\sum_k |p_k|^2\langle[\hat{a}_{vk}^{\dagger}\hat{a}_{ck},\hat{a}_{ck}^{\dagger}\hat{a}_{vk}]\rangle \times\int_0^{\infty}e^{i(\omega-\omega_k)\tau-\epsilon\tau}d\tau. \tag{4.24}$$

Evaluating the average of the commutator with the rules of Eq. (3.4) and then doing the integral leads to

$$\chi(\omega)=\frac{e^2}{\hbar cVm^2}\sum_k\frac{|p_k|^2(\bar{n}_{vk}(1-\bar{n}_{ck})-(1-\bar{n}_{vk})\bar{n}_{ck})}{\omega_k-\omega-i\epsilon}, \tag{4.25}$$

where $\bar{n}_{ck}\equiv\langle\hat{a}_{ck}^{\dagger}\hat{a}_{ck}\rangle$ is the average occupation of state ck, etc. Converting the sum to an integral with $\Sigma_k\to V\int(dN/d\omega_k)d\omega_k$ and noting that the imaginary part of $(\omega_k-\omega-i\epsilon)^{-1}$ is $\pi\delta(\omega_k-\omega)$, we can write

$$\chi''(\omega)=\frac{dN}{d\omega_k}\frac{\pi e^2}{\hbar cm^2}|p_k|^2[\bar{n}_{vk}(1-\bar{n}_{ck})-(1-\bar{n}_{vk})\bar{n}_{ck}] \tag{4.26}$$

evaluated at $\omega_k=\omega$. We see that the imaginary part of the susceptibility is the difference between contributions of the upward and downward optical transitions at ω and is weighted by the average occupation factors associated with these transitions.

E. Fluctuation-dissipation theorem

The imaginary part of the susceptibility [Eq. (4.23)] is the Fourier transform of the difference of antinormally and normally ordered correlation functions of spontaneous current-density operators at two different times. These same Fourier integrals occur when we seek the correlation functions of the frequency components of the spontaneous current-density operators. A positive-frequency component of the spontaneous current operator is given by

$$\hat{j}_{\omega}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dt\hat{j}(t)e^{i\omega t}. \tag{4.27}$$

The correlation function of the frequency components of the spontaneous current density is

$$\langle\hat{j}_{\omega}\hat{j}_{\omega'}^{\dagger}\rangle=\frac{1}{2\pi}\int_{-\infty}^{\infty}dte^{i(\omega-\omega')t}\int_{-\infty}^{\infty}d\tau\langle\hat{j}(t+\tau)\hat{j}^{\dagger}(t)\rangle e^{i\omega\tau}. \tag{4.28}$$

Due to stationarity, the average in this equation is independent of t, and the integral over t becomes a delta function:

$$\langle\hat{j}_{\omega}\hat{j}_{\omega'}^{\dagger}\rangle=\int_{-\infty}^{\infty}d\tau\langle\hat{j}(\tau)\hat{j}^{\dagger}(0)\rangle e^{i\omega\tau}\delta(\omega-\omega'). \tag{4.29a}$$

Similarly, we write

$$\langle\hat{j}_{\omega'}^{\dagger}\hat{j}_{\omega}\rangle=\int_{-\infty}^{\infty}d\tau\langle\hat{j}^{\dagger}(0)\hat{j}(\tau)\rangle e^{i\omega\tau}\delta(\omega-\omega'). \tag{4.29b}$$

The difference of the two correlation functions is related to $\chi''(\omega)$ by Eq. (4.23):

$$\langle\hat{j}_{\omega}\hat{j}_{\omega'}^{\dagger}\rangle-\langle\hat{j}_{\omega'}^{\dagger}\hat{j}_{\omega}\rangle=\frac{2\hbar c}{V}\chi''(\omega)\delta(\omega-\omega'). \tag{4.30}$$

By finding the ratio of $\langle\hat{j}_{\omega}\hat{j}_{\omega'}^{\dagger}\rangle$ and $\langle\hat{j}_{\omega'}^{\dagger}\hat{j}_{\omega}\rangle$, we can relate these averages individually to $\chi''(\omega)$. These relationships constitute the fluctuation-dissipation theorem.

We can find the ratio of these correlation functions by evaluating $\langle\hat{j}_{\omega}\hat{j}_{\omega'}^{\dagger}\rangle$ with our semiconductor model. The Fourier component is found by taking the Fourier transform of $\hat{j}(t)$ defined by Eq. (4.17):

$$\hat{j}_{\omega}=\frac{1}{\sqrt{2\pi}}\sum_k\hat{j}_k\int_{-\infty}^{\infty}dt\ \exp(i(\omega-\omega_k)t) =\sqrt{2\pi}\sum_k\hat{j}_k\delta(\omega-\omega_k). \tag{4.31}$$

The average product $\langle\hat{j}_{\omega}\hat{j}_{\omega'}^{\dagger}\rangle$ is found using $\langle j_kj_{k'}^{\dagger}\rangle = \langle j_kj_k^{\dagger}\rangle\ \delta_{kk'}$ and $\delta(\omega-\omega_k)\delta(\omega_k-\omega')=\delta(\omega-\omega_k)\delta(\omega-\omega')$. It is

$$\langle\hat{j}_{\omega}\hat{j}_{\omega'}^{\dagger}\rangle=2\pi\sum_k\langle j_kj_k^{\dagger}\rangle\delta(\omega-\omega_k)\delta(\omega-\omega'). \tag{4.32}$$

The $\Sigma_k\delta(\omega-\omega_k)$ can be replaced by $V(dN/d\omega_k)$ evaluated at $\omega_k=\omega$, and the average $\langle j_kj_k^{\dagger}\rangle$ can be evaluated using Eq. (4.17b). This results in

$$\langle\hat{j}_{\omega}\hat{j}_{\omega'}^{\dagger}\rangle=\frac{dN}{d\omega_k}\frac{2\pi e^2|p_k|^2}{m^2V}\bar{n}_{vk}(1-\bar{n}_{ck})\delta(\omega-\omega'), \tag{4.33a}$$

where the matrix element and the density of states are evaluated at $\omega_k=\omega$. Similarly, the other order of the correlation function is

$$\langle\hat{j}_{\omega'}^{\dagger}\hat{j}_{\omega}\rangle=\frac{dN}{d\omega_k}\frac{2\pi e^2|p_k|^2}{m^2V}(1-\bar{n}_{vk})\bar{n}_{ck}\delta(\omega-\omega'). \tag{4.33b}$$

Notice that the general relation between the difference in correlation functions and χ'' [Eq. (4.30)] is satisfied by the semiconductor model [see Eqs. (4.33) and (4.26)].

The ratio of correlation functions is just the ratio of occupation factors associated with the upward and downward rates:

$$\frac{\langle \hat{j}_\omega \hat{j}^\dagger_{\omega'} \rangle}{\langle \hat{j}^\dagger_{\omega'} \hat{j}_\omega \rangle} = \frac{\bar{n}_{vk}(1-\bar{n}_{ck})}{(1-\bar{n}_{vk})\bar{n}_{ck}}. \tag{4.34}$$

The ratio is determined by the level-occupation factors associated with upward and downward transitions. We show in Sec. IV.G that this ratio is more conveniently written in terms of $\bar{n}_\omega$, the average optical-mode occupation factor of photons in equilibrium with a biased semiconductor:

$$\frac{\langle \hat{j}_\omega \hat{j}^\dagger_{\omega'} \rangle}{\langle \hat{j}^\dagger_{\omega'} \hat{j}_\omega \rangle} = \frac{\bar{n}_\omega + 1}{\bar{n}_\omega}. \tag{4.35}$$

We use this ratio [Eq. (4.35)] and the difference in correlation functions [Eq. (4.30)] separately to evaluate the correlation functions in terms of $\chi''(\omega)$:

$$\langle \hat{j}_\omega \hat{j}^\dagger_{\omega'} \rangle = \frac{2\hbar c}{V} \chi''(\omega)\,\delta(\omega-\omega')(\bar{n}_\omega+1), \tag{4.36a}$$

$$\langle \hat{j}^\dagger_{\omega'} \hat{j}_\omega \rangle = \frac{2\hbar c}{V} \chi''(\omega)\,\delta(\omega-\omega')\bar{n}_\omega. \tag{4.36b}$$

These equations are precisely the fluctuation-dissipation theorem of Callen and Welton (1951). A derivation can also be found in Landau and Lifshitz (1958b). The derivation given here includes quasithermal equilibrium (discussed in Sec. IV.G) and correlations of noncommuting operators not considered in the earlier derivations. Another derivation of the fluctuation-dissipation theorem that includes these effects has been given by Marani and Lax (1995). The fluctuation-dissipation theorem is general and depends only on the form of the current-field interaction H_{int} given by Eq. (3.10). Notice that no details of the semiconductor model appear in Eq. (4.36).

The correlation functions of the spontaneous currents [Eq. (4.36)] are proportional to the rates of upward and downward optical transitions. However, $\chi''(\omega)$ and the dissipation rate are proportional to the net rate of upward transitions, which is the difference in the two rates. Thus expression of the individual correlation functions in terms of χ'' requires additional information. This is supplied by the factors $\bar{n}_\omega+1$ and $\bar{n}_\omega$.

The fluctuation-dissipation theorem can be generalized to correlations between operators at different spatial points. So far we have been considering a small volume V, with dimensions small compared to the wavelength of light, but large enough to contain thousands of carrier states. Suppose we divide the system of interest into many of these volumes. It is reasonable to assume that the spontaneous currents associated with carriers in different volumes are uncorrelated. Therefore we can generalize Eq. (4.36) to an equation for the correlations of spontaneous current densities at two points $\mathbf{x}$ and $\mathbf{x}'$ by replacing V^{-1} by $\delta(\mathbf{x}-\mathbf{x}')$.

A more thorough derivation of the spatial dependence is given by Martin (1968), who relates correlation functions to generalized susceptibilities. Applying his method, we find that the susceptibility determines the current induced at one point by an optical-frequency field applied at another point. The correlation function of the spontaneous current density falls off with spatial separation in the same way as this generalized susceptibility. Martin's result is derived in Appendix B for the special case of the correlation function of optical fields.

In deriving the wave equation, it will be shown in Eq. (6.4) that the imaginary part of the susceptibility is related to more commonly used optical constants,

$$\frac{4\pi}{c}\chi''(\omega) = k_\omega(\mathbf{x})a_\omega(\mathbf{x}) = \frac{\omega^2}{c^2}\epsilon''_\omega(\mathbf{x}), \tag{4.37}$$

where $k_\omega(\mathbf{x})$ is the real propagation constant, $a_\omega(\mathbf{x})$ is the attenuation coefficient, and $\epsilon''_\omega(\mathbf{x})$ is the imaginary part of the dielectric function. These constants are related to the real and imaginary parts of the refractive index by $k_\omega(\mathbf{x}) = \omega n'_\omega(\mathbf{x})/c$ and $a_\omega(\mathbf{x}) = 2\omega n''_\omega(\mathbf{x})/c$. When we use the first equality in this equation and the delta-function substitution for V, the fluctuation-dissipation theorem is

$$\left(\frac{4\pi}{c}\right)^2 \langle \hat{j}_\omega(\mathbf{x}) \hat{j}^\dagger_{\omega'}(\mathbf{x}') \rangle = 8\pi\hbar k_\omega(\mathbf{x})a_\omega(\mathbf{x})(\bar{n}_\omega(\mathbf{x})+1) \times \delta(\mathbf{x}-\mathbf{x}')\,\delta(\omega-\omega'), \tag{4.38a}$$

$$\left(\frac{4\pi}{c}\right)^2 \langle \hat{j}^\dagger_{\omega'}(\mathbf{x}') \hat{j}_\omega(\mathbf{x}) \rangle = 8\pi\hbar k_\omega(\mathbf{x})a_\omega(\mathbf{x})\bar{n}_\omega(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{x}') \times \delta(\omega-\omega'). \tag{4.38b}$$

F. Slowly varying envelope approximation

We are interested in describing time-dependent noise fields with frequency components near a particular frequency ω_S, which, for example, may be the signal frequency or the frequency of the lasing mode. For a small range of such components, the field and spontaneous current-density operators are

$$\hat{\mathcal{A}}(t) = \hat{A}(t)e^{-i\omega_S t} + \hat{A}^\dagger(t)e^{i\omega_S t}, \tag{4.39a}$$

$$\hat{\mathcal{J}}(t) = \hat{J}(t)e^{-i\omega_S t} + \hat{J}^\dagger(t)e^{i\omega_S t}. \tag{4.39b}$$

We can think of $\hat{A}$ and $\hat{J}$ as operators for the slowly varying envelopes of the field and current density. The spontaneous current densities $\hat{J}(t)$ and $\hat{J}^\dagger(t')$ have the correlation function

$$\langle \hat{J}(\mathbf{x},t)\hat{J}^\dagger(\mathbf{x}',t') \rangle = \frac{1}{2\pi}\int d\omega \int d\omega' \langle \hat{j}_\omega(\mathbf{x}) \hat{j}^\dagger_{\omega'}(\mathbf{x}') \rangle \times e^{-i(\omega-\omega_S)t + i(\omega'-\omega_S)t'}. \tag{4.40}$$

Using the fluctuation-dissipation theorem [Eq. (4.38)] to evaluate the correlation function, we have

$$\left(\frac{4\pi}{c}\right)^2 \langle \hat{J}(\mathbf{x},t)\hat{J}^\dagger(\mathbf{x}',t') \rangle = \frac{8\pi\hbar}{2\pi}\int k_\omega(\mathbf{x})a_\omega(\mathbf{x})(\bar{n}_\omega(\mathbf{x})+1) \times e^{-i(\omega-\omega_S)(t-t')} d\omega\,\delta(\mathbf{x}-\mathbf{x}'). \tag{4.41}$$

The integrand is highly oscillatory except near $\omega = \omega_S$. If $k_\omega(\mathbf{x})a_\omega(\mathbf{x})(\bar{n}_\omega(\mathbf{x})+1)$ is approximated as constant and removed from the integral, the correlation functions are

$$\left(\frac{4\pi}{c}\right)^2 \langle \hat{J}(\mathbf{x},t)\hat{J}^\dagger(\mathbf{x}',t')\rangle \cong 8\pi\hbar k_{\omega_S}(\mathbf{x}) a_{\omega_S}(\mathbf{x})(\bar{n}_{\omega_S}(\mathbf{x})+1) \times \delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t'), \tag{4.42a}$$

$$\left(\frac{4\pi}{c}\right)^2 \langle \hat{J}^\dagger(\mathbf{x},t')\hat{J}(\mathbf{x},t)\rangle \cong 8\pi\hbar k_{\omega_S}(\mathbf{x}) a_{\omega_S}(\mathbf{x})\bar{n}_{\omega_S}(\mathbf{x}) \times \delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t'). \tag{4.42b}$$

G. Equilibrium of photons and carriers in a biased semiconductor

In a biased semiconductor (see Fig. 3), there is only partial thermal equilibrium. The Fermi levels of each band are separated by $e\mathcal{V}$, where $\mathcal{V}$ is the bias voltage. The carriers of each band are separately in thermal equilibrium, and their one-electron states are occupied according to Fermi statistics:

$$\bar{n}_{ck,vk} = \frac{1}{\exp\left(\frac{\varepsilon_{ck,vk}-\mu_{c,v}}{kT}\right)+1}, \tag{4.43}$$

where $\mu_{c,v}$ are the chemical potentials or Fermi levels of each band. The Fermi levels are separated by

$$\mu_c - \mu_v = e\mathcal{V}. \tag{4.44}$$

For true thermal equilibrium $e\mathcal{V}=0$. Evaluation of the occupation factors in Eq. (4.34) shows that the ratio of the spontaneous current correlation functions is

$$\frac{\langle \hat{j}_\omega \hat{j}^\dagger_{\omega'}\rangle}{\langle \hat{j}^\dagger_{\omega'}\hat{j}_\omega\rangle} = \exp\left(\frac{\hbar\omega - e\mathcal{V}}{kT}\right). \tag{4.45}$$

It is useful to express this ratio in terms of another parameter $\bar{n}_\omega$, which is defined as the average photon occupation number per mode that would occur if the photons were in equilibrium with the carriers of a biased semiconductor. Such a hypothetical equilibrium can occur only if the photons are associated with an optical cavity that is closed and lossless, except for optical-absorption transitions between the semiconductor bands. The rate of upward transitions is determined by the level occupations and is proportional to $\bar{n}_{vk}(1-\bar{n}_{ck})\bar{n}_\omega$. Similarly, we expect the rate of downward transitions to be proportional to $\bar{n}_{ck}(1-\bar{n}_{vk})(\bar{n}_\omega+1)$, where 1 is added to $\bar{n}_\omega$ to include spontaneous emission (see Sec. VIII.F). In steady state, this ratio is unity and hence the ratio of correlation functions becomes

$$\frac{\langle \hat{j}_\omega \hat{j}^\dagger_{\omega'}\rangle}{\langle \hat{j}^\dagger_{\omega'}\hat{j}_\omega\rangle} = \frac{\bar{n}_\omega+1}{\bar{n}_\omega} = \exp\left(\frac{\hbar\omega - e\mathcal{V}}{kT}\right). \tag{4.46}$$

Using this equation, we find that

$$\bar{n}_\omega = \frac{1}{\exp\left(\frac{\hbar\omega - e\mathcal{V}}{kT}\right)-1}. \tag{4.47}$$

With $e\mathcal{V}=0$, this is the familiar mode occupation described by Planck's formula. With $e\mathcal{V} \neq 0$, it has the form of the occupation of states of a Bose gas, with $e\mathcal{V}$ playing the role of the chemical potential μ. The chemical potential is the free energy required to add an additional photon. However, for a Bose gas in thermal equilibrium, $\mu>0$ is not allowed because it results in negative occupation numbers, which are unphysical.

When the Fermi levels of the two bands are separated by $e\mathcal{V}$, we are dealing with a highly nonequilibrium system. To understand Eq. (4.47), consider a biased ideal semiconductor enclosed within a closed lossless optical cavity. Photons below the energy gap do not interact with the electrons and can be thought of as separately in equilibrium and as having a Planck distribution [Eq. (4.47)] with $\mu \equiv e\mathcal{V}=0$.

For this idealized case, photons above the energy gap E_g come into equilibrium with the electrons and have $\mu=e\mathcal{V}$. For $e\mathcal{V}<E_g$, the mode occupation number $\bar{n}_\omega$ is positive and described by Eq. (4.47). As $e\mathcal{V}$ approaches the energy of the first cavity mode having $\hbar\omega \geq E_g$, $\bar{n}_\omega \to \infty$. The physical reason for this singular behavior is explained in the next section [after Eq. (4.50)]. The mode becomes a lasing mode, and this change can be interpreted as Bose-Einstein condensation of the photons into this mode. Therefore $e\mathcal{V}$ cannot exceed this energy. This limitation applies to the steady-state equilibrium situation only.

In discussing Fig. 2, we pointed out the analogy between the current operators $\hat{j}^\dagger$ and $\hat{j}$ and the photon creation and annihilation operators $\hat{b}^\dagger$ and $\hat{b}$. This analogy extends to the operator correlation functions. The correlation functions of the $\hat{b}$ and $\hat{b}^\dagger$ operators for photons in thermal equilibrium are $\langle \hat{b}\hat{b}^\dagger\rangle = \bar{n}_\omega+1$ and $\langle \hat{b}^\dagger\hat{b}\rangle = \bar{n}_\omega$, showing the similarity with the fluctuation-dissipation theorem Eq. (4.36).

H. Dependence of correlation functions on population inversion

The spontaneous currents act as quantum Langevin forces. These noise sources are characterized by the correlation functions $\langle \hat{j}\hat{j}^\dagger\rangle$ and $\langle \hat{j}^\dagger\hat{j}\rangle$, whose inequality shows that these noise sources are nonclassical. These correlation functions as well as $\chi''(\omega)$ and $\bar{n}_\omega$ are all dependent on the occupation of the levels associated with transitions at optical frequency ω. To gain insight into how these quantities change with the level occupation, we consider a simple model: a cavity filled uniformly with two-level atoms. Either the ground state or the excited state of each atom is occupied. The levels are shown in Fig. 4(a).

Let N be the occupation probability of the upper level and let $1-N$ be that of the lower level. The plots in Fig. 4 are versus N, which has a range from 0 to 1. The value $N=0$ corresponds to a cold system, with the upper level unpopulated. The value $N=1/2$ corresponds to the point of transparency, when both levels are equally populated. The value $N=1$ corresponds to complete population inversion. The upward and downward transition rates depend on the populations and are propor-

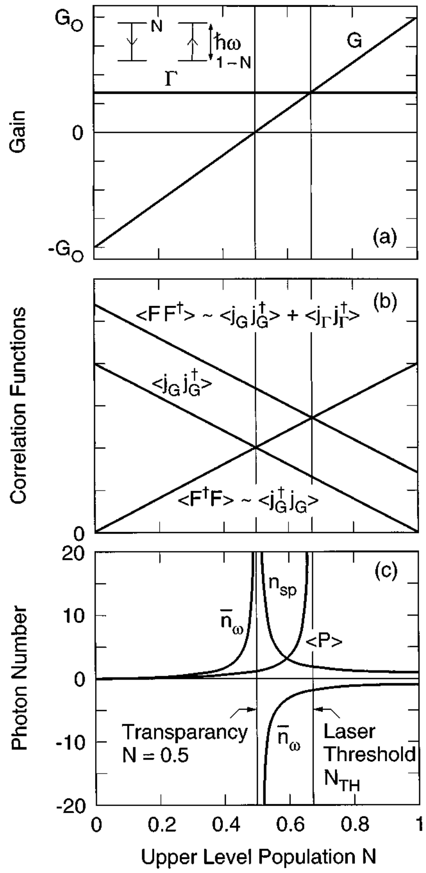


FIG. 4. Dependence of the gain, the correlation functions of Langevin forces, and the photon number on level population. The population of the upper level N ranges from 0 to 1. The correlation functions of the Langevin force and spontaneous currents are order independent (classical) at the values of N corresponding to laser threshold, shown by the light vertical lines for lossless and lossy cavities. At these values, the normally ordered and antinormally ordered correlation functions in (b) are equal. This classical condition is approached as the photon number increases. The plot of photon number has a range of 20. Actual lasers have photon numbers 3–4 orders of magnitude higher than this. Population inversion occurs at $N=1/2$. For $N>1/2$, $\bar{n}_\omega<0$ and the positive parameter $n_{sp}=-\bar{n}_\omega$ is used.

tional to $1-N$ and N, respectively. The imaginary susceptibility χ'' is proportional to the difference in the upward and downward rates, $\chi''\sim 1-2N$. The gain (rate of growth of a mode amplitude) is proportional to $-\chi''$ and can be written as

$$G=G_o(2N-1), \tag{4.48}$$

where G_o is the gain at full inversion. The gain is plotted in Fig. 4(a).

The spontaneous current correlation functions associated with gain are $\langle \hat{j}_G \hat{j}_G^\dagger\rangle$ and $\langle \hat{j}_G^\dagger \hat{j}_G\rangle$. The difference between the two correlation functions is proportional to G, and the ratio of two correlation functions is equal to the ratio of upward and downward rates. These two conditions are satisfied by

$$\langle \hat{j}_G^\dagger \hat{j}_G\rangle = aG_oN, \tag{4.49a}$$

$$\langle \hat{j}_G \hat{j}_G^\dagger\rangle = aG_o(1-N), \tag{4.49b}$$

where a is a constant of proportionality. These correlation functions are plotted in Fig. 4(b).

The ratio of the antinormally and normally ordered correlation functions is equal to the ratio of $(\bar{n}_\omega+1)/\bar{n}_\omega$ [Eq. (4.46)]. Using this relation and Eq. (4.49), we find

$$\bar{n}_\omega=\frac{N}{1-2N}. \tag{4.50}$$

This relation is plotted in Fig. 4(c). We see that $\bar{n}_\omega$ is zero for a cold system, becomes singularly large at the transparency point, and becomes negative when population inversion occurs.

For a lossless cavity, the photons of a cavity mode, with $\hbar\omega$ equal to the level separation, will come into equilibrium with the system of two-level atoms. Then, $\bar{n}_\omega$ equals the number of photons in the mode. This singular growth of the number of photons is a Bose-Einstein condensation. (In a semiconductor, this hypothetical situation takes place when $e\mathcal{V}\rightarrow\hbar\omega$.) The photon number $\bar{n}_\omega$ is determined by the balance of spontaneous emission and optical absorption. The latter goes to zero at the transparency point, resulting in the singular growth of $\bar{n}_\omega$. Population inversion cannot occur in the case of thermal equilibrium, so negative $\bar{n}_\omega$ and unphysical negative numbers of photons do not occur, as we discussed previously.

A real laser has cavity losses, and a true reversible equilibrium does not exist. Instead, there is a steady state, with the energy lost continually being resupplied, and population inversion is possible. It is customary to describe population inversion by a new positive parameter $n_{sp}\equiv-\bar{n}_\omega$. Accordingly, we write

$$n_{sp}=\frac{N}{2N-1}. \tag{4.51}$$

Suppose a uniform loss Γ, shown in Fig. 4(a), exists in the cavity. It will be shown in Sec. VIII.B that the average photon number of a laser mode, for a uniform closed cavity, is given by

$$\langle \hat{P}\rangle=\frac{Gn_{sp}}{(\Gamma-G)}. \tag{4.52}$$

Let N_{th} be the threshold population for which $G(N_{th})=\Gamma$, as shown in Fig. 4(a). From the above equations relating n_{sp} and G to N, we find

$$\langle \hat{P}\rangle=\frac{N}{2N_{th}-2N}, \tag{4.53}$$

which is plotted in Fig. 4(c). The photon number increases with N in a manner quite similar to that of Bose-Einstein condensation, as described by Eq. (4.50).

In practice, laser threshold current is found by plotting laser power versus current and extrapolating linearly back to the current of zero power. The gain is nearly pinned at the value corresponding to the threshold current and asymptotically approaches $G(N\mathrm{th})$ with increasing power.

Associated with the loss will be a spontaneous current $\hat{j}_\Gamma(t)$. The total Langevin force will then be

$$\hat{F}(t) \sim \hat{j}_G(t) + \hat{j}_\Gamma(t). \tag{4.54}$$

We shall assume that the loss comes from a cold absorber. The correlation functions are

$$\langle \hat{j}_\Gamma^\dagger \hat{j}_\Gamma \rangle = 0, \tag{4.55a}$$

$$\langle \hat{j}_\Gamma \hat{j}_\Gamma^\dagger \rangle = a\Gamma, \tag{4.55b}$$

where a is the same constant of proportionality used in estimating the correlation functions associated with gain.

In the absence of loss, $\hat{F}(t) \sim \hat{j}_G(t)$ and the two correlation functions $\langle \hat{F}^\dagger \hat{F} \rangle$ and $\langle \hat{F} \hat{F}^\dagger \rangle$ are linear functions of N that cross at $N = 1/2$, where the singular buildup in photons due to Bose-Einstein condensation occurs, as shown in Fig. 4. With the inclusion of loss, there is an additional contribution to $\langle \hat{F} \hat{F}^\dagger \rangle$, shifting the crossing point of the Langevin force correlation functions to that of laser threshold N_{th}, as shown in Fig. 4. The equality of the Langevin force correlation functions at laser threshold is easily established. Their difference is

$$\begin{aligned}\langle \hat{F}^\dagger \hat{F} \rangle - \langle \hat{F} \hat{F}^\dagger \rangle &= \langle \hat{j}_G^\dagger \hat{j}_G \rangle - \langle \hat{j}_G \hat{j}_G^\dagger \rangle - \langle \hat{j}_\Gamma \hat{j}_\Gamma^\dagger \rangle \\ &= aG_o(2N-1) - a\Gamma \\ &= a(G-\Gamma).\end{aligned} \tag{4.56}$$

At threshold, $G = \Gamma$ and the two correlation functions become equal. The two crossings of the normally and antinormally ordered correlation functions, shown in Fig. 4(b), occur where the photon number becomes infinite. It appears that whenever the photon number becomes large, the classical condition of commuting Langevin forces is approached, and the fields generated by these forces can be treated as classical. This will be shown more generally in Sec. VIII.A.

In a semiconductor, population inversion occurs for $e\mathcal{V} > \hbar\omega$, and a_ω and $\bar{n}_\omega$ become negative. It is conventional to work in terms of positive parameters $n_{\mathrm{sp}} = -\bar{n}_\omega$ and $g_\omega = -a_\omega$. The following changes are made in this case:

$$a_\omega \to -g_\omega, \tag{4.57a}$$

$$\bar{n}_\omega \to -n_{\mathrm{sp}}, \tag{4.57b}$$

$$a_\omega \bar{n}_\omega \to g_\omega n_{\mathrm{sp}}, \tag{4.57c}$$

$$a_\omega(\bar{n}_\omega + 1) \to g_\omega(n_{\mathrm{sp}} - 1). \tag{4.57d}$$

I. Evaluation of higher-order correlation functions

The average product of arbitrary numbers of spontaneous current operators $\hat{J}(t)$ and $\hat{J}^\dagger(t')$ can be calculated in terms of pair averages. These higher moments are used later to calculate the probability distribution of fluctuating fields and photon numbers. For example, we can find the distribution of photons in a nonlasing mode.

The spontaneous current for frequencies near ω_S can be expressed as

$$\hat{J}(t) = \sum_k \hat{j}_k e^{-i(\omega_k - \omega_S)t}, \tag{4.58}$$

where $\hat{j}_k = (ep_k/mV)\hat{a}_{vk}^\dagger \hat{a}_{ck}$. The average is $\langle \hat{J}(t) \rangle = 0$ since the operators $\hat{a}_{vk}^\dagger$ and $\hat{a}_{ck}$ act on states of different bands and their thermal averages among these states are zero. Similarly, averages of odd numbers of $\hat{J}(t)$ and $\hat{J}(t')^\dagger$ are zero.

Only pairs involving $\hat{a}_{ck}$ and $\hat{a}_{ck}^\dagger$ or $\hat{a}_{vk}$ and $\hat{a}_{vk}^\dagger$ are nonzero. Therefore

$$\begin{aligned}\langle \hat{J}(t)\hat{J}^\dagger(t') \rangle &= \sum_k \left(\frac{e|p_k|}{mV} \right)^2 \langle \hat{a}_{vk}^\dagger \hat{a}_{vk} \rangle \\ &\quad \times \langle \hat{a}_{ck} \hat{a}_{ck}^\dagger \rangle e^{-i(\omega_k - \omega_S)(t-t')} \\ &= \sum_k \langle \hat{j}_k \hat{j}_k^\dagger \rangle e^{-i(\omega_k - \omega_S)(t-t')}.\end{aligned} \tag{4.59a}$$

Similarly, we write

$$\langle \hat{J}^\dagger(t') \hat{J}(t) \rangle = \sum_k \langle \hat{j}_k^\dagger \hat{j}_k \rangle e^{-i(\omega_k - \omega_S)(t-t')}. \tag{4.59b}$$

The averages $\langle \hat{J}(t)\hat{J}(t') \rangle$ and $\langle \hat{J}(t)^\dagger \hat{J}(t')^\dagger \rangle$ are zero because the operator pairs do not contain pairs of creation and destruction operators of the same state.

We can develop a general rule for calculating higher-order averages by calculating the average of two $\hat{J}(t)$ and two $\hat{J}(t)^\dagger$ operators:

$$\begin{aligned}\langle \hat{J}(t_1)\hat{J}^\dagger(t_2)\hat{J}(t_3)\hat{J}^\dagger(t_4) \rangle &= \sum_{k_1} \sum_{k_2} \sum_{k_3} \sum_{k_4} \langle \hat{j}_{k_1} \hat{j}_{k_2}^\dagger \hat{j}_{k_3} \hat{j}_{k_4}^\dagger \rangle \\ &\quad \times e^{-i(\omega_{k_1} - \omega_S)t_1} \cdots .\end{aligned} \tag{4.60}$$

The average $\langle \hat{j}_{k_1} \hat{j}_{k_2}^\dagger \hat{j}_{k_3} \hat{j}_{k_4}^\dagger \rangle$ is nonzero only if the k_i's are equal in pairs or if all four k_i's are equal:

$$\begin{aligned}\langle \hat{J}(t_1)\hat{J}^\dagger(t_2)\hat{J}(t_3)\hat{J}^\dagger(t_4) \rangle &= \sum_{k_1} \langle \hat{j}_{k_1} \hat{j}_{k_1}^\dagger \rangle \sum_{k_3} \langle \hat{j}_{k_3} \hat{j}_{k_3}^\dagger \rangle e^{-i(\omega_{k_1} - \omega_S)(t_1 - t_2)} e^{-i(\omega_{k_3} - \omega_S)(t_3 - t_4)} \\ &\quad + \sum_{k_1} \langle \hat{j}_{k_1} \hat{j}_{k_1}^\dagger \rangle \sum_{k_3} \langle \hat{j}_{k_3}^\dagger \hat{j}_{k_3} \rangle e^{-i(\omega_{k_1} - \omega_S)(t_1 - t_4)} e^{-i(\omega_{k_3} - \omega_S)(t_3 - t_2)} \\ &\quad + \sum_{k_1} \langle \hat{j}_{k_1} \hat{j}_{k_1}^\dagger \hat{j}_{k_1} \hat{j}_{k_1}^\dagger \rangle e^{-i(\omega_{k_1} - \omega_S)(t_1 - t_2 + t_3 - t_4)}.\end{aligned} \tag{4.61}$$

The first two terms come from pairing $\hat{a}_c$ and $\hat{a}_c{}^\dagger$ operators and pairing $\hat{a}_v$ and $\hat{a}_v{}^\dagger$ operators that belong to the same pair of current operators. The last term comes from additional pairings of creation and annihilation operators when this is not the case. There is no need to discuss this last term because the case of all operators with the same k occupies a smaller region of phase space and can be neglected. If we neglect the last term, we have

$$\langle \hat{J}(t_1)\hat{J}^\dagger(t_2)\hat{J}(t_3)\hat{J}^\dagger(t_4)\rangle = \langle \hat{J}(t_1)\hat{J}^\dagger(t_2)\rangle\langle \hat{J}(t_3)\hat{J}^\dagger(t_4)\rangle + \langle \hat{J}(t_1)\hat{J}^\dagger(t_4)\rangle \times\langle \hat{J}^\dagger(t_2)\hat{J}(t_3)\rangle. \quad (4.62)$$

This simple result resembles Wick's theorem for averages of creation and annihilation operators at a finite temperature (Abrikosov *et al.*, 1963), and we shall refer to this result as "Wick's theorem." As applied to our case, it states that *an average of the product of an even number of* $\hat{J}(t)$ *and* $\hat{J}^\dagger(t')$ *operators can be expanded as a sum of the averages of all possible pairings of the operators.*

We can summarize the rules of obtaining averages: (1) averages of odd $\hat{J}$'s are zero; (2) averages need equal numbers of $\hat{J}$'s and $\hat{J}^\dagger$'s; (3) the average equals the sum of all possible pairings, which is similar to Wick's theorem in quantum field theory.

Similar rules were found by Senitsky (1960, 1961), who wrote pioneering papers discussing the quantized harmonic oscillator with loss and noise, and occur as well in the classical theory of Brownian motion (Wang and Uhlenbeck, 1945).

V. OPTICAL TRANSITIONS OF CARRIERS

In this section, we deduce an equation for the rate of optical transitions of carriers. Spontaneous currents contribute to the rate, causing recombination-generation noise and spontaneous emission. Let us calculate the rate of change $\hat{\dot{N}}(t)$ of electrons in the conduction band as a result of interaction with the electromagnetic field $\hat{\mathcal{A}}(t)$. In the Schrödinger representation, the operator $\hat{N}_S$ is

$$\hat{N}_S \equiv \sum_k \hat{a}^\dagger_{ck}\hat{a}_{ck}. \quad (5.1)$$

In the Heisenberg representation, this operator is

$$\hat{N}(t) = \exp\left(\frac{i}{\hbar}\hat{H}t\right)\hat{N}_S \exp\left(-\frac{i}{\hbar}\hat{H}t\right), \quad (5.2)$$

where the total Hamiltonian is $\hat{H}=\hat{H}_e+\hat{H}_{\text{int}}+\hat{H}_r$. The rate of change of $\hat{N}(t)$ can be found by differentiating this equation. This results in the term $[\hat{H},\hat{N}_S]$. The operators $\hat{H}_e$ and $\hat{N}_S$ are constructed from the same operators $\hat{a}^\dagger_{ck}\hat{a}_{ck}$ and therefore commute. The field operators in the Schrödinger representation do not affect carrier occupation numbers and, similarly, the electronic operators do not affect the photon occupation numbers, so these operators commute. That is, $[\hat{H}_r,\hat{N}_S]=0$. Thus only the contribution $[\hat{H}_{\text{int}},\hat{N}_S]$ is nonzero and

$$\frac{d\hat{N}(t)}{dt} = \frac{i}{\hbar}\exp(i\hat{H}t/\hbar)[\hat{H}_{\text{int}},\hat{N}_S]\exp(-i\hat{H}t/\hbar). \quad (5.3)$$

We can work out this commutator in the Schrödinger representation, where

$$\hat{H}_{\text{int}} = -\frac{V}{c}(\hat{j}_S+\hat{j}^\dagger_S)(\hat{A}_S+\hat{A}^\dagger_S). \quad (5.4)$$

It is easily shown, using the explicit forms of $\hat{N}_S$ in Eq. (5.1) and $\hat{j}_S$ and $\hat{j}^\dagger_S$ in Eq. (3.9) and applying the commutation rules in Eq. (3.4), that

$$[\hat{j}_S,\hat{N}_S]=\hat{j}_S, \quad [\hat{j}^\dagger_S,\hat{N}_S]=-\hat{j}^\dagger_S. \quad (5.5)$$

With these results, we find

$$\frac{d\hat{N}(t)}{dt} = -\frac{iV}{\hbar c}e^{i\hat{H}t/\hbar}(\hat{j}_S-\hat{j}^\dagger_S)(\hat{A}_S+\hat{A}^\dagger_S)e^{-i\hat{H}t/\hbar},$$
$$= -\frac{iV}{\hbar c}[\hat{J}_{\text{tot}}(t)e^{-i\omega_S t}-\hat{J}^\dagger_{\text{tot}}(t)e^{i\omega_S t}] \times[\hat{A}(t)e^{-i\omega_S t}+\hat{A}(t)^\dagger e^{i\omega_S t}], \quad (5.6)$$

where $\hat{J}_{\text{tot}}(t)$ is the sum of spontaneous and induced currents and $\hat{A}(t)$ is the amplitude of the field in the slowly varying envelope approximation. The dependence of $\hat{N}(t)$ on t is slow compared to an optical frequency, so there is no need to keep terms $\hat{J}(t)\hat{A}(t)e^{-2i\omega_S t}$ and $\hat{J}^\dagger(t)\hat{A}^\dagger(t)e^{2i\omega_S t}$, which are at twice the optical frequency. Averaging over an optical period nulls the contribution of these terms. Without these terms, $d\hat{N}(t)/dt$ reduces to

$$\frac{d\hat{N}(t)}{dt} = \frac{iV}{\hbar c}[\hat{J}_{\text{tot}}(t)^\dagger\hat{A}(t)-\hat{A}^\dagger(t)\hat{J}_{\text{tot}}(t)]. \quad (5.7)$$

We have rearranged the order of $\hat{A}(t)^\dagger$ and $\hat{J}(t)$. With this ordering, the operator is Hermitian and normally ordered. The operators $\hat{A}_S^\dagger$ and $\hat{J}_S$ commute in the Schrödinger representation and thus can be taken in either order. The commutation rules of operators do not change as we go from the Schrödinger to the Heisenberg representations. The order independence of $d\hat{N}(t)/dt$ will be discussed at greater length in Appendix A.

The operator $\hat{J}_{\rm tot}(t)$ can be written as

$$\hat{J}_{\rm tot}(t)=\frac{1}{\sqrt{2\pi}}\int d\omega\chi(\omega)\hat{A}_\omega e^{-i(\omega-\omega_S)t}+\hat{J}(t). \qquad (5.8)$$

For a narrow range of optical frequencies near ω_S, $\chi(\omega)\approx\chi(\omega_S)$ and

$$\hat{J}_{\rm tot}(t)=\chi(\omega_S)\hat{A}(t)+\hat{J}(t). \qquad (5.9)$$

Expressing $\chi(\omega_S)''$ in terms of the absorption and propagation constants [Eq. (4.37)], we obtain the rate of change of carrier number,

$$\frac{d\hat{N}(t)}{dt}=\frac{akV}{2\pi\hbar}\hat{A}^\dagger(t)\hat{A}(t)+\hat{F}_N(t), \qquad (5.10)$$

where

$$\hat{F}_N(t)=\frac{iV}{\hbar c}(\hat{J}^\dagger(t)\hat{A}(t)-\hat{A}^\dagger(t)\hat{J}(t)). \qquad (5.11)$$

The first term of Eq. (5.10) is the average rate of optical transitions given by the golden rule (Landau and Lifshitz, 1958a, Sec. 42). The second term of Eq. (5.10), $\hat{F}_N(t)$, is a Langevin force representing fluctuations in the carrier transition rate brought about by the field's interacting with the spontaneous current. [Subscript N refers to the variable $\hat{N}(t)$.] The average value of $\hat{F}_N(t)$ is not zero because a small part of the field is generated by spontaneous currents within V and contributes to a nonzero average value of $\hat{F}_N(t)$, which is the average spontaneous-emission rate. We shall discuss this in Sec. VIII.D.

The behavior of one or several variables driven by white noise is conveniently described by Langevin rate equations, such as Eq. (5.10). The Langevin rate equation expresses the rate of change of a variable as the sum of a steady change, the drift term, and a fluctuating change, the Langevin force. Langevin forces cause fluctuations, while drift terms restore the system to its steady state (Lax, 1960, 1968). The correlation functions of the random forces at two times are delta functions [see Eq. (5.13)]. For example, a semiconductor laser operating above threshold is often described by three variables: phase, photon number, and minority-carrier number. There are three coupled Langevin rate equations and a matrix of diffusion coefficients describing the correlations of the three noise sources (see, for example, Henry, 1983). Equation (5.10) represents only the rate of change of carriers due to optical transitions. Additional terms are needed to fully characterize the fluctuations of carriers in a laser and to determine the steady-state carrier density [see Eq. (8.51)].

The Fokker-Planck equation provides an alternative method of describing classical noise fluctuations. It is a partial differential equation for the probability distribution of a system as a function of the variables and time (Lax, 1968). The same drift and diffusion coefficients enter into both the Langevin rate equations and the Fokker-Planck equation. The name "diffusion coefficient" is appropriate because, in the absence of restoring forces (drift terms), the Fokker-Planck equation reduces to a diffusion equation. This is the case for the phase of a laser, and phase noise is often referred to as phase diffusion.

The fluctuations in the recombination rate occur because the field, whose source is external to V, beats with the spontaneous currents within V. This noise is characterized by the autocorrelation function of $\hat{F}_N(t)$. We calculate this quantity, regarding $\hat{A}(t)$ as external in origin and uncorrelated with the spontaneous current:

$$\langle\hat{F}_N(t)\hat{F}_N(t')\rangle=\frac{V^2}{\hbar^2c^2}[\langle\hat{J}^\dagger(t)\hat{J}(t')\rangle\langle\hat{A}(t)\hat{A}^\dagger(t')\rangle+\langle\hat{J}(t)\hat{J}^\dagger(t')\rangle\langle\hat{A}^\dagger(t)\hat{A}(t')\rangle]. \qquad (5.12)$$

Evaluating the averages of the spontaneous currents [Eq. (4.42)] results in

$$\langle\hat{F}_N(t)\hat{F}_N(t')\rangle=\frac{akV}{2\pi\hbar}[(\bar{n}_{\omega_S}+1)\langle\hat{A}^\dagger(t)\hat{A}(t)\rangle+\bar{n}_{\omega_S}\langle\hat{A}(t)\hat{A}^\dagger(t)\rangle]\delta(t-t'). \qquad (5.13)$$

The coefficient of $\delta(t-t')$ is the sum of two terms. In the usual notation, this coefficient is $2D_{NN}$. The two terms are the rates of induced upward and downward transitions. This can be seen as follows. In the case of an external classical field, we can neglect the difference of $\langle\hat{A}^\dagger\hat{A}\rangle$ and $\langle\hat{A}\hat{A}^\dagger\rangle$; then the difference of the two terms is just the net transition rate given by the first term in Eq. (5.10). The ratio of the two terms is $(\bar{n}_\omega+1)/\bar{n}_\omega$, which was shown to be the ratio of the upward and downward rates [Eq. (4.46)]. This is exactly what is found in carrier recombination-generation statistics, where $2D_{NN}$ is just the sum of the rate of generation and the rate of recombination (Lax 1960, 1968).

The equation for the rate of optical transitions of carriers can be understood as the rate of transfer of energy between an electromagnetic field and a system of carriers. Recall the well-known result of classical electrodynamics. The rate of work done by an electric field **E** per unit volume on a system of charges with current density **J** is given by **E**·**J**; see Jackson (1968). If we regard **E** and **J** as operators, express the field in terms of the vector potential, expand the current density in terms of induced and spontaneous contributions, and identify the work done as $[d\hat{N}(t)/dt]\hbar\omega_S$, we obtain the rate equation (5.10).

We illustrate the phase relations between the components of **E** and **J** in Fig. 5. There we consider the same transverse vector components of **E** and **J**, with positive

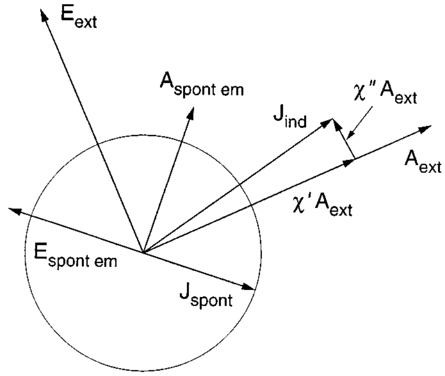


FIG. 5. Phase relations of current densities, vector potentials, and electric fields. The component of $J_{\rm ind}$ in phase with $E_{\rm ext}$ is determined by $\chi''(\omega)$. When $\chi''(\omega)$ is positive, energy is transferred from the field to the carriers, and the field is attenuated. The energy transfer is in the form of optical transitions. When $\chi''(\omega)$ is negative, energy flows from the carriers to the field, which is amplified. The phase between $J_{\rm spont}$ and $E_{\rm ext}$ can take any value and causes fluctuations in the rate of optical transitions. In addition, there is a fixed phase relation between the $J_{\rm spont}$ and $E_{\rm spont}$ consistent with transfer of energy to the field by spontaneous emission. The illustrated physical quantities are pictured as classical; however, operator equations of motion underlie this description.

frequency $\sim\exp(-i\omega_S t)$. The fields are illustrated as classical but are actually Heisenberg operators. The phase between the external signal field $\hat{E}_{\rm ext}$ and the induced current density $\hat{J}_{\rm ind}$ is constant, resulting in a constant rate of optical transitions, if the signal is noise free. This is what is predicted by the golden rule. The component of $\hat{J}_{\rm ind}$ in phase with $\hat{E}_{\rm ext}$ is determined by $\chi''(\omega)$. If $\chi''(\omega)$ is positive, the two operators are in phase, and energy is transferred from the field to the carriers. Conversely, if $\chi''(\omega)$ is negative, energy is transferred from the carriers to the field, which is amplified.

The phase relation between this signal and the spontaneous current is random and results in fluctuations in the rate of optical transitions. In addition, there is a spontaneously emitted field that has a constant phase relation relative to the spontaneous current.

VI. FIELD PROPAGATION

In this section, we introduce a wave equation for the field operator and discuss the approximations that we make in order to solve it in specific cases. This equation contains the spontaneous current-density operator as a source. An energy conservation equation is derived from the wave equation. We identify terms in this equation as the energy flux density and energy density operators. These operators have normally ordered and antinormally ordered forms, and we discuss their physical meaning. The difference in the two operator orderings is associated with uncertainty-related field fluctuations. In the case of high photon number, this difference becomes negligible, which we interpret as the transition to classical fields.

A. Wave equation with noise sources

In using the electric and magnetic fields $\mathcal{E}$ and $\mathcal{H}$ to describe radiation, it is convenient to write these fields in terms of vector and scalar potentials $\mathcal{A}$ and Φ. Maxwell's equations in a vacuum reduce to the vector wave equation for $\mathcal{A}$ with a current-density source $\mathcal{J}_{\rm tot}$. Only the transverse part of the current-density source enters the vector wave equation, which holds both classically and in the quantum theory as an operator identity (Glauber and Lewenstein, 1991). We assume there is no free charge, i.e., that the mobile charges are neutralized by a static background charge. For this assumption, we can choose $\Phi=0$ and express the fields as $\mathcal{E}=-\dot{\mathcal{A}}/c$ and $\mathcal{H}=\nabla\times\mathcal{A}$. In the Coulomb gauge, with $\nabla\cdot\mathcal{A}=0$, the vector wave equation reduces to a scalar wave equation for each transverse component (see Jackson, 1968, Sec. 6.5):

$$\nabla^2\hat{\mathcal{A}}(\mathbf{x},t)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\hat{\mathcal{A}}(\mathbf{x},t)=-\frac{4\pi}{c}\hat{\mathcal{J}}_{\rm tot}(\mathbf{x},t), \tag{6.1}$$

where $\hat{\mathcal{J}}_{\rm tot}(\mathbf{x},t)$ is the sum of the induced and spontaneous current densities.

We assume that this equation holds as an operator identity. The hats indicate that, in the quantum theory, $\hat{\mathcal{A}}(\mathbf{x},t)$ and $\hat{\mathcal{J}}_{\rm tot}(\mathbf{x},t)$ are Heisenberg operators acting on the wave function of the system consisting of the electromagnetic field and carriers. We cannot describe the wave function of the system of interacting carriers and light explicitly. What we do know is the equilibrium distribution of the electrons and holes. We restrict our descriptions to electromagnetic fields emitted by several systems of carriers separately in equilibrium.

Solutions of Eq. (6.1) will allow us to express $\hat{\mathcal{A}}(\mathbf{x},t)$ in terms of the sources $\hat{\mathcal{J}}_{\rm tot}(\mathbf{x},t)$. Both $\hat{\mathcal{A}}(\mathbf{x},t)$ and $\hat{\mathcal{J}}_{\rm tot}(\mathbf{x},t)$ can be written as integrals over their frequency components:

$$\hat{\mathcal{A}}(\mathbf{x},t)=\int_0^\infty\frac{d\omega}{\sqrt{2\pi}}[\hat{A}_\omega(\mathbf{x})e^{-i\omega t}+\hat{A}_\omega^\dagger(\mathbf{x})e^{i\omega t}], \tag{6.2a}$$

$$\hat{\mathcal{J}}_{\rm tot}(\mathbf{x},t)=\int_0^\infty\frac{d\omega}{\sqrt{2\pi}}[\hat{j}_{{\rm tot}\,\omega}(\mathbf{x})e^{-i\omega t}+\hat{j}_{{\rm tot}\,\omega}^\dagger(\mathbf{x})e^{i\omega t}]. \tag{6.2b}$$

The positive- and negative-frequency components are Hermitian conjugates, so their sum forms a Hermitian operator.

It follows from Eqs. (4.13) and (4.21) that $\hat{j}_{{\rm tot}\,\omega}(\mathbf{x})$ has spontaneous and induced contributions,

$$\hat{j}_{{\rm tot}\,\omega}(\mathbf{x})=\hat{j}_\omega(\mathbf{x})+\chi_\omega(\mathbf{x})\hat{A}_\omega(\mathbf{x}). \tag{6.3}$$

Let us substitute Eqs. (6.2) and (6.3) into the wave equation [Eq. (6.1)] and bring the induced term to the left side, where it contributes to the complex dielectric constant, leaving only the spontaneous current density on the right side. We shall write the term with the susceptibility as a squared complex propagation constant,

$$\frac{\omega^2}{c^2}+\frac{4\pi}{c}\chi(\mathbf{x},\omega)=\left[k_\omega(\mathbf{x})+i\frac{a_\omega}{2}(\mathbf{x})\right]^2,$$
$$\approx k_\omega(\mathbf{x})^2+ik_\omega(\mathbf{x})a_\omega(\mathbf{x}), \qquad (6.4)$$

where $k_\omega(\mathbf{x})$ is the real propagation constant and $a_\omega(\mathbf{x})$ is the attenuation coefficient. With these changes, the wave equation becomes

$$\nabla^2\hat{A}_\omega(\mathbf{x})+[k_\omega^2(\mathbf{x})+ik_\omega(\mathbf{x})a_\omega(\mathbf{x})]\hat{A}_\omega(\mathbf{x})=-\frac{4\pi}{c}\hat{j}_\omega(\mathbf{x}). \qquad (6.5)$$

When population inversion occurs, the attenuation coefficient changes sign and becomes gain. We shall make this change explicit by replacing the attenuation coefficient $a_\omega(\mathbf{x})$ with $-g_\omega(\mathbf{x})$, where $g_\omega(\mathbf{x})$ is the gain coefficient.

In deriving the wave equation, we have used the Kubo formula [Eqs. (4.21) and (4.22)] to replace a commutator of spontaneous current operators by its average value, which is the susceptibility. This procedure corresponds to the self-consistent-field approximation. Classically, it is the conventional procedure for accounting for the dielectric medium.

Equation (6.5) appears linear, but nonlinearities enter into the description of lasers and amplifiers through the dependence of the propagation constants on carrier density and temperature. The average value of the current-density commutator in the Kubo formula for the susceptibility [Eq. (4.22)] is a function of the carrier concentration and the carrier temperature, which are in turn dependent on optical power. The dependence of the carrier concentration on optical power can be found by solving rate equations. This procedure is part of the conventional theory of lasers and amplifiers and, for example, results in the pinning of the Fermi levels of the carriers in lasers above threshold. The dependence of the carrier temperature on optical power can also be found by solving kinetic equations; however, this part of semiconductor laser theory is not well developed and this nonlinearity is usually described phenomenologically by gain-saturation parameters. What is omitted here are the effects of optical mixing and second-harmonic generation as well as spectral hole-burning effects. However, we do cover noise phenomena in Raman amplifiers in Sec. VII.F; those phenomena are based on the mixing of the optical field of a pump with atomic vibrations.

The above description is an approximation that has replaced the vector wave equation with the scalar wave equation for each component of the field. The approximation can be traced to our assumption that the charge density is zero. In inhomogeneous dielectrics, charge exists at gradients and discontinuities of the dielectric function, such as at interfaces. This charge results in discontinuities in the field at these interfaces, which do not occur in solutions of the scalar wave equation. In most photonics applications, the field discontinuities are small, and the scalar wave equation is a good approximation that is often used in the calculation of modes and light propagation. This is the case when one or more of the following conditions exist: the dielectric discontinuity is small, the mode overlap with the interface is small, or the electric field is nearly parallel to the interface. It is with this in mind that we use the scalar wave equation in discussing quantum noise.

B. Green's-function solution

Without the source, Eq. (6.5) is a classical wave equation governing propagation of light in a medium. With $\hat{j}_\omega(\mathbf{x})$ on the right side, this is an equation for operators. The operator aspects of $\hat{j}_\omega(\mathbf{x})$ show up only when we take products of two or more operators. Then, the order of the operators makes a difference. In many situations, the solution is a linear relation between $\hat{A}_\omega(\mathbf{x})$ and $\hat{j}_\omega(\mathbf{x}')$. This relation can be found by solving the wave equation in exactly the same way as for classical fields. The solution is expressed through a classical Green's function $G_\omega(\mathbf{x},\mathbf{x}')$ satisfying

$$\nabla^2 G_\omega(\mathbf{x},\mathbf{x}')+[k_\omega^2(\mathbf{x})+ik_\omega(\mathbf{x})a_\omega(\mathbf{x})]G_\omega(\mathbf{x},\mathbf{x}')$$
$$=\delta(\mathbf{x}-\mathbf{x}'). \qquad (6.6)$$

The solution relating the field to the current density is

$$\hat{A}_\omega(\mathbf{x})=-\frac{4\pi}{c}\int G_\omega(\mathbf{x},\mathbf{x}')\hat{j}_\omega(\mathbf{x}')d\mathbf{x}'. \qquad (6.7)$$

Once $G_\omega(\mathbf{x},\mathbf{x}')$ is found, averages of products of $\hat{A}_\omega(\mathbf{x})$ operators can be expressed as averages of products of $\hat{j}_\omega(\mathbf{x})$. The latter averages are calculated using the fluctuation-dissipation theorem [Eq. (4.38)] and Wick's theorem for finding higher-order averages, given in Sec. IV.I.

The case of a laser above threshold is more complicated than the one just described. The Green's function has poles in the lower half of the complex ω plane that correspond to damped modes below threshold. As the laser approaches threshold, a pole associated with the lasing mode approaches the real ω axis and, for frequencies near this pole, the Green's function becomes resonantly large. The position of the pole is determined by the carrier number, so that the equation for carrier number must be solved simultaneously with the wave equation. Even in this case, the average steady-state laser field above threshold is described by the Green's function [Eq. (6.7)] with a fixed carrier density and gain. However, small fluctuations about the steady state require an approximate simultaneous solution of the wave equation and the carrier-density rate equation.

C. Energy conservation

In most photonics applications, we are concerned with only a small spread of optical frequencies about a central frequency ω_S, which may be that of the signal or the laser mode. Then we can write the field as an operator

$\hat{A}(\mathbf{x},t)$ describing the amplitude of the field that is slowly varying compared to an optical frequency [Eq. (4.39)]. The frequency range of interest is usually small compared to the spectral width of the material gain or loss. Over this spectral range, we shall consider loss and gain to be independent of the frequency and approximate the dispersion of the squared propagation constant as linear. Near ω_S, we can write

$$k_\omega(\mathbf{x})^2 = k^2(\mathbf{x}) + \frac{dk_\omega^2(\mathbf{x})}{d\omega}(\omega-\omega_S), \tag{6.8a}$$

$$a_\omega(\mathbf{x}) \approx a(\mathbf{x}). \tag{6.8b}$$

This expansion limits us to a linear dispersion described by a group velocity $v_g(\mathbf{x})$, where $dk_\omega^2(\mathbf{x})/d\omega = 2k(\mathbf{x})/v_g(\mathbf{x})$. If necessary, additional terms describing higher-order dispersion can be added to Eq. (6.8a).

We can find a wave equation for the slowly varying envelope of the field $\hat{A}(\mathbf{x},t)$ by Fourier transforming the wave equation for $\hat{A}_\omega(\mathbf{x})$ [Eq. (6.5)] with $\int d\omega/\sqrt{2\pi}\exp[-i(\omega-\omega_S)t]$ and substituting Eq. (6.8). We find

$$\nabla^2\hat{A}(\mathbf{x},t) + [k(\mathbf{x})^2 + ik(\mathbf{x})a(\mathbf{x})]\hat{A}(\mathbf{x},t) + i\frac{dk^2(\mathbf{x})}{d\omega}\frac{\partial}{\partial t}\hat{A}(\mathbf{x},t) = -\frac{4\pi}{c}\hat{J}(\mathbf{x},t). \tag{6.9}$$

The squared complex propagation constant in the wave equation (6.9) is shown as time independent. However, a careful derivation starting with the time-dependent wave equation would show that this quantity can change in time. The time dependence is due to changes in level occupation. The propagation constant and loss (or gain) follow these changes adiabatically provided they take place in a time long compared to an energy relaxation time (a few picoseconds). Then the wave equation is unchanged. When this equation is applied to a laser in steady-state operation, $a(\mathbf{x}) \to -g(\mathbf{x})$. For laser transients, the changes in both propagation and gain are usually written as $\Delta k(\mathbf{x},t) - i\Delta g(\mathbf{x},t)/2 \to (-i/2)\Delta g(\mathbf{x},t)(1-i\alpha)$, where α is known as the linewidth enhancement factor (Lax, 1967a; Henry, 1982).

The energy conservation law can be derived from the wave equation for $\hat{A}(\mathbf{x},t)$ [Eq. (6.9)] in a conventional way. This is done by multiplying this equation by $\hat{A}^\dagger(x,t)$ on the left side and subtracting from this the equation for $\hat{A}^\dagger(\mathbf{x},t)$ multiplied by $\hat{A}(\mathbf{x},t)$ on the right side. In so doing, we find that the term $k^2\hat{A}^\dagger\hat{A}$ cancels and the terms $\hat{A}^\dagger\nabla^2\hat{A} - \nabla^2\hat{A}^\dagger\hat{A}$ can be rewritten as a divergence. The result is

$$\nabla\cdot(\hat{A}^\dagger\nabla\hat{A} - \nabla\hat{A}^\dagger\hat{A}) + i\frac{dk^2}{d\omega}\frac{\partial}{\partial t}(\hat{A}^\dagger\hat{A}) + i2ak\hat{A}^\dagger\hat{A} - \frac{4\pi}{c}(\hat{J}^\dagger\hat{A} - \hat{A}^\dagger\hat{J}) = 0. \tag{6.10}$$

All terms in this equation are normally ordered, with the positive frequency current density and field operators on the right. Later, we shall consider similar terms that are antinormally ordered. To distinguish the two types of related operators, we shall use subscripts N and A to label normal and antinormal ordering. The physical meaning of the difference in the two operator orders will be discussed in Sec. VI.D.

In Sec. V, Eq. (5.10), we showed that the rate of change of the carrier density $\hat{\mathcal{N}}$ (in the conduction band) due to interaction with the field is given by

$$\frac{\partial\hat{\mathcal{N}}_N}{\partial t} = \frac{ak}{2\pi\hbar}\hat{A}^\dagger\hat{A} + \frac{i}{\hbar c}(\hat{J}^\dagger\hat{A} - \hat{A}^\dagger\hat{J}). \tag{6.11}$$

If we divide Eq. (6.10) by $4\pi\hbar i$, the last two terms of this equation are equal to $\partial\hat{\mathcal{N}}_N/\partial t$, and the conservation law becomes

$$\nabla\cdot\hat{\mathbf{S}}_N + \frac{\partial\hat{\mathcal{P}}_N}{\partial t} + \frac{\partial\hat{\mathcal{N}}_N}{\partial t} = 0, \tag{6.12}$$

where

$$\hat{\mathbf{S}}_N(\mathbf{x},t) = \frac{1}{4\pi\hbar i}[\hat{A}(\mathbf{x},t)^\dagger\nabla\hat{A}(\mathbf{x},t) - \nabla\hat{A}(\mathbf{x},t)^\dagger\hat{A}(\mathbf{x},t)] \tag{6.13}$$

and

$$\hat{\mathcal{P}}_N = \frac{dk^2}{d\omega}\frac{\hat{A}(\mathbf{x},t)^\dagger\hat{A}(\mathbf{x},t)}{4\pi\hbar}. \tag{6.14}$$

The natural interpretation of Eq. (6.12) is that, in the interaction between the field and the carriers, the sum of the number of photons and electrons in the conduction band is conserved. With this interpretation, $\hat{\mathcal{P}}_N(\mathbf{x},t)$ is the photon density operator and $\hat{\mathbf{S}}_N(\mathbf{x},t)$ is the photon flux density operator. We are considering only a small range of frequencies about ω_S. We cannot distinguish this interpretation from that of energy conservation. If we regard Eq. (6.12) as a law of energy conservation, $\hat{\mathbf{S}}_N\hbar\omega_S$ is the energy flux density and $\hat{\mathcal{P}}_N\hbar\omega_S$ is the field energy density.

In the remainder of the paper, we shall refer to $\hat{\mathbf{S}}_N$ both as the photon flux density and as the energy flux density of the optical field. It will be understood that the latter must be multiplied by $\hbar\omega_S$ to be in correct units. Similarly, we shall refer to $\hat{\mathcal{P}}_N$ both as the photon density and as the energy density of the optical field. Two names are needed because $\hat{\mathbf{S}}_A$ and $\hat{\mathcal{P}}_A$, discussed in Sec. VI.D, include the energy flux and energy density of the uncertainty-related field fluctuations, which is an additional energy not due to photons.

The operators $\hat{\mathcal{P}}_N(\mathbf{x},t)$ and $\hat{\mathbf{S}}_N(\mathbf{x},t)$ are zero for vacuum fluctuations (emission from cold sources). This can be demonstrated by applying the fluctuation-dissipation theorem and Wick's theorem. These operators are related through the Green's-function solution [Eq. (6.7)] to the normally ordered products of spontaneous current operators $\hat{j}_\omega^\dagger(\mathbf{x})\hat{j}_{\omega'}(\mathbf{x}')$. The fluctuation-dissipation theorem [Eq. (4.38)] shows that the average of this product is zero for cold sources. Furthermore, applying Wick's theorem shows that the averages of powers of $\hat{\mathcal{P}}_N$ and $\hat{\mathbf{S}}_N$ are also zero, because these averages decompose into products of pair averages and because these products contain at least one normally ordered pair average, which is zero.

Another useful operator is the energy flux passing though a surface,

$$\hat{Q}_N = \int \hat{\mathbf{S}}_N \cdot d\mathbf{a}. \tag{6.15}$$

We now show that it is $\hat{Q}_N$ that is measured in photodetection. The photodetector can be regarded as a cold system, with only the lower electron levels occupied. If opaque, such a system emits only vacuum fluctuations and has no normally ordered outward flux. The photocount $\hat{m}$ during time T is found by integrating $\partial \hat{N}_N/\partial t$ over the photodetector volume and time T. Applying the normally ordered energy conservation equation [Eq. (6.12)] and integrating shows that the photodetector count is just the difference of the incident normally ordered energy flux $\hat{Q}_{N\mathrm{in}}(t)$ integrated over time T and the change of the number of photons within the detector during time T. The latter can be neglected because the number of photons residing within the detector is very small compared to the number of photons received during time T. (This is so because the time photons spend within the photodetector is just the time to travel an absorption length, about 10^{-14} sec, for an absorption length of one micron, which is much less than the values of T that we need to consider.) We conclude that, to a good approximation, the photodetector measures the normally ordered energy flux:

$$\hat{m} = \int_0^T \hat{Q}_{N\mathrm{in}}(t)\,dt. \tag{6.16}$$

Since $\hat{Q}_N$ is zero for vacuum fluctuations and is measured by an ideally efficient opaque photodetector, we are justified in referring to it as the "photon flux."

D. Uncertainty-related field fluctuations

So far we have discussed energy conservation in terms of normally ordered operators. However, as discussed in Appendix A, the equation for energy conservation can also be written in terms of antinormally ordered operators for the energy density and energy flux density. The physical meaning of these antinormally ordered operators is that, in addition to the photon flux and photon density, they represent the energy flux and energy density associated with the uncertainty-related field fluctuations. In the absence of photons, they represent these quantities for the vacuum fluctuations.

As shown in Appendix A, the time derivative of $\hat{\mathcal{P}}_A - \hat{\mathcal{P}}_N$ and the divergence of $\hat{\mathcal{S}}_A - \hat{\mathcal{S}}_N$ are zero, but $\hat{\mathcal{P}}_A \neq \hat{\mathcal{P}}_N$ and $\hat{\mathcal{S}}_A \neq \hat{\mathcal{S}}_N$. These operators are formed from products of the field operators $\hat{A}$ and $\hat{A}^\dagger$ in different orders. These products depend on operator order because the creation and annihilation operators that compose the field operators do not commute. Associated with this operator noncommutation is an uncertainty in the physical quantities described by these operators. The normally ordered energy flux $\hat{Q}_N(t)$ can be measured directly by an opaque photodetector. The difference between $\hat{Q}_A(t)$ and $\hat{Q}_N(t)$ is a consequence of field uncertainty, and it contributes to the fluctuations of $\hat{Q}_N(t)$ as shot noise on a signal. The uncertainty-related field fluctuations are not affected by amplification or attenuation because the Heisenberg field uncertainty remains constant. Shot noise is the beating of the signal with these field fluctuations and thus is a fluctuation in $\hat{Q}_N(t)$ that is proportional to the signal field.

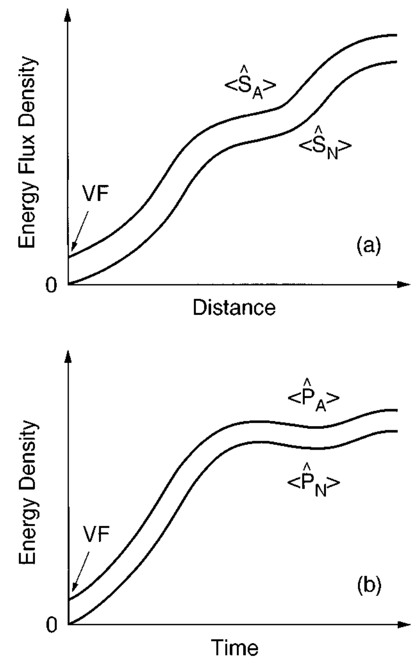


FIG. 6. Growth of normally and antinormally ordered energy density and energy flux density: (a) the energy flux in an optical amplifier vs propagation distance; (b) the energy density in a laser vs time. The constant differences between the curves are the corresponding values of the uncertainty-related field fluctuations. In the absence of radiation, these fluctuations are vacuum fluctuations (VF). This occurs at the input of the amplifier and for the laser at the initial time in our example.

This is a description of shot noise in the wave picture of light. A conventional explanation of shot noise is that $\hat{Q}_N(t)$ consists of a flux of uncorrelated particles, the photons. But the photon picture can be thought of as originating from the Heisenberg uncertainty principle itself. The photon picture is based on a quantum oscillator having equidistant discrete energy levels. This directly follows from the commutation rules of the coordinate and momentum of a harmonic oscillator (Landau and Lifshitz, 1958a). For an optical mode, the coordinate is the vector potential, which has the same phase as the magnetic field, and the momentum is the electric field. Thus the wave and particle pictures of shot noise have the same origin.

An important consequence of the order independence of the divergence of the energy density is illustrated in Fig. 6(a), which shows the energy flux propagating through an optical amplifier with no input signal. The average energy fluxes $\langle \hat{Q}_N \rangle$ and $\langle \hat{Q}_A \rangle$ are found by integrating $\hat{S}_N$ and $\hat{S}_A$ over the cross-sectional area of the amplifier and averaging. As a consequence of the equality of the divergences of energy flux densities of different operator orders [Eq. (A8)], the difference of $\langle \hat{Q}_N \rangle$ and $\langle \hat{Q}_A \rangle$ is a constant equal to $\langle \hat{Q}_A \rangle$ for the vacuum-fluctuation noise field at the input of the amplifier. (It was shown in Sec. VI.C that $\hat{Q}_N(t)=0$ for vacuum fluctuations.) It is clear from the figure that $\langle \hat{Q}_A - \hat{Q}_N \rangle$ is a constant, as we have already discussed. At high fields, this difference becomes negligible. When the average energy flux in a spectral range becomes large compared to that of the vacuum fluctuations, the fields can be treated as order independent and classical.

Similarly, Fig. 6(b) shows the growth of $\langle \hat{\mathcal{P}}_N \rangle$ and $\langle \hat{\mathcal{P}}_A \rangle$ with time for a laser, beginning at an initial time when the laser cavity contains only vacuum fluctuations. As a consequence of Eq. (A7), the two orders of energy density differ only by the energy density of the vacuum fluctuations, so when the energy density in a given spectral range becomes much larger than that of vacuum fluctuations, it can be treated as order independent and classical.

The above discussion anticipates results to be worked out in detail in the next two sections. Shot noise results from the beating of the uncertainty-related field fluctuations and the signal field, in both attenuators (Sec. VII.C) and amplifiers (Sec. VII.E). At high amplification, noise fields of an optical amplifier can be treated as commuting and classical (Sec. VII.E). As threshold is approached, the steady-state laser field can be treated as commuting and classical (Sec. VIII.A).

VII. QUANTUM NOISE IN OPTICAL WAVEGUIDE TRANSMISSION

We start our discussion of applications by considering light propagating in a single transverse mode of a waveguide. The noise field emitted by spontaneous currents from an opaque source is calculated. The real and imaginary parts of this field are shown to be Gaussian distributed. The noise field emitted from a cold opaque source mimics the field of vacuum fluctuations. For a source at higher temperature, the noise field is that of Planck radiation, and for a biased semiconductor it is light-emitting diode radiation. The beating of the noise field of vacuum fluctuations and a noise-free monochromatic signal field results in shot noise measured by an ideal opaque photodetector. The beating of an optical-signal field with spontaneous currents causes fluctuations in the rate of optical absorption. This would result in shot noise in a nearly transparent photodetector. The noise occurring in optical amplification is calculated. The dominant noise is amplified spontaneous emission, which at high gain can be regarded as a fluctuating classical wave field. In Raman amplification, spontaneous currents result from lattice fluctuations mixing with the optical field of the pump. Finally, we show that the fluctuation-dissipation theorem, derived earlier for absorption loss, also holds for scattering loss. These results can be found in the text at the places listed in Tables I and II.

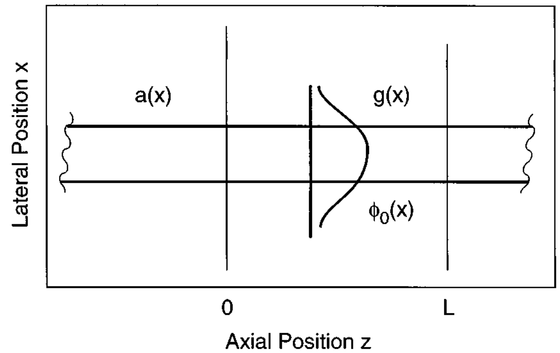


FIG. 7. Waveguide geometry used to discuss the emission from an opaque source and propagation in attenuating and amplifying waveguides. An unending waveguide is shown with absorption coefficient $a(x)$ for $z<0$ and gain $g(x)$ for $0<z<L$. The gain $g(x)$ is only nonzero for the case of an optical amplifier. The field distribution of transverse mode $\phi_0(x)$ is also shown. The vertical boundaries, indicating changes in loss or gain, are assumed to be nonreflecting.

A. Emission from an opaque source

Let us calculate the field emitted from an opaque portion of the waveguide shown in Fig. 7. The gain $g(x)$ in Fig. 7 will be used later in discussing the traveling-wave amplifier, but it is ignored here. The only difference between the absorbing region, with $z<0$, and the transparent region, with $z\geq 0$, is the presence of the absorption coefficient $a(x)$, where x represents both transverse coordinates. Reflections at interfaces, where the absorption (or gain) changes, can be ignored. We assume the waveguide is index guided and has a set of real orthonormal transverse modes $\phi_n(x)$. We shall only be concerned with the noise field of a single transverse mode $\phi_0(x)$. The field of the mode is

$$\hat{A}_\omega = \hat{b}_\omega(z)\phi_0(x). \tag{7.1}$$

In the transparent region the mode is given by $\phi_0(x)\exp(ik_0 z)$, where

$$\frac{\partial^2 \phi_0(x)}{\partial x^2} + k^2(x)\phi_0(x) = k_0^2 \phi_0(x). \tag{7.2}$$

In the opaque region, with optical absorption coefficient $a(x)$, $k^2(x) \to k^2(x) + ik(x)a(x)$, where we assume that $a(x) \ll k(x)$. A perturbation calculation shows that the propagation constant is $k_0 + ia_0/2$, where

$$k_0 a_0 = \int \phi_0^2(x) k(x) a(x) dx. \tag{7.3}$$

The Green's function for the generation of a field in mode 0 by sources is given by Morse and Feshbach (1953; see also Henry, 1986a):

$$G_0(\mathbf{x},\mathbf{x}')=g(z,z')\phi_0(x)\phi_0(x'), \tag{7.4a}$$

where, for the case of forward propagation $z \geqslant z'$ and $z' \leqslant 0$,

$$g(z,z')=\begin{cases} \dfrac{1}{2ik_0}e^{(ik_0-a_0/2)(z-z')}, & z\leqslant 0, \\ \dfrac{1}{2ik_0}e^{ik_0 z-(ik_0-a_0/2)z'}, & z>0 \end{cases} \tag{7.4b}$$

is the z-dependent Green's function. The complete Green's function, which is not needed here, is obtained by summing over the different transverse modes. We shall not consider the dependence of k_0 on ω. This neglects the effect of the delay in propagation, which is not important for this discussion.

Using the $G_0(\mathbf{x},\mathbf{x}')$ to express the field at $z>0$ in terms of the $\hat{j}_\omega$, we obtain the Green's-function solution [Eq. (6.7)]:

$$\hat{b}_\omega(z)=-\frac{4\pi}{c}\frac{1}{2ik_0}e^{ik_0 z}\int_{-\infty}^{0}dz'\,e^{-(ik_0-a_0/2)z'} \times\int dx'\,\phi_0(x')\hat{j}_\omega(x',z'). \tag{7.5}$$

A similar expression can be written for $\hat{A}_\omega^\dagger(\mathbf{x})$ by taking the Hermitian conjugate of this equation.

We can calculate $\langle \hat{b}_\omega \hat{b}_\omega^\dagger\rangle$ using the fluctuation-dissipation theorem [Eq. (4.38)] to determine $\langle \hat{j}_\omega \hat{j}_{\omega'}^\dagger\rangle$. We find

$$\langle \hat{b}_\omega(z)\hat{b}_{\omega'}^\dagger(z)\rangle=\frac{2\pi\hbar}{k_0^2}(\bar{n}_\omega+1)\delta(\omega-\omega') \times\int_{-\infty}^{0}dz'\exp(a_0 z') \times\int dx\,\phi_0(x)^2 a(x)k(x). \tag{7.6}$$

The first integral is a_0^{-1}, and the second integral is $a_0 k_0$ from Eq. (7.3). We see that the absorption coefficient a_0 drops out, leaving

$$\frac{k_0}{2\pi\hbar}\langle \hat{b}_\omega \hat{b}_{\omega'}^\dagger\rangle=\delta(\omega-\omega')(\bar{n}_\omega+1) \tag{7.7a}$$

and similarly

$$\frac{k_0}{2\pi\hbar}\langle \hat{b}_{\omega'}^\dagger \hat{b}_\omega\rangle=\delta(\omega-\omega')\bar{n}_\omega. \tag{7.7b}$$

The result is similar to that obtained in the theory of blackbody radiation, where the emission of an opaque body is independent of the absorption coefficient.

The photon flux [Eq. (6.15)] along z can be calculated using Eq. (6.13) and $\hat{A}(x,z,t)\sim\exp(ik_0 z)$. It is given by

$$\hat{Q}_N=\frac{k_0}{2\pi\hbar}\hat{b}^\dagger(t)\hat{b}(t). \tag{7.8}$$

We can calculate the average photon flux by Fourier-transforming Eqs. (7.7) with $1/2\pi\int d\omega\int d\omega'\exp[-i(\omega-\omega')t]$. The integral over ω' is removed by integrating the delta function. We restrict the integral over ω to a small range of optical frequencies $2\pi\Delta\nu$, then

$$\langle \hat{Q}_N\rangle=\frac{k_0}{2\pi\hbar}\langle \hat{b}^\dagger(t)\hat{b}(t)\rangle=\Delta\nu\bar{n}_\omega. \tag{7.9}$$

This is just what would have been obtained from a conventional treatment using quantized modes of propagation. For a waveguide of length L, the number of modes is $(L\Delta\omega)/(2\pi v_g)$, and the photon flux per mode is $\bar{n}_\omega v_g/L$.

A similar calculation shows that the energy flux of the uncertainty-related field fluctuations is given by

$$\langle \hat{Q}_A\rangle-\langle \hat{Q}_N\rangle=\frac{k_0}{2\pi\hbar}[\langle \hat{b}(t)\hat{b}^\dagger(t)\rangle-\langle \hat{b}^\dagger(t)\hat{b}(t)\rangle]=\Delta\nu. \tag{7.10}$$

When $\bar{n}_\omega$ represents a thermal mode occupation, $\langle \hat{Q}_N\rangle$ [Eq. (7.9)] expresses Planck's law of radiation (into a single transverse mode). The photon flux density $\langle \hat{Q}_N\rangle$ also describes the emission from an opaque biased semiconductor. For $\hbar\omega-e\mathcal{V}\gg kT$, the first term in the denominator of the formula for $\bar{n}_\omega$, Eq. (4.47), is dominant and $\bar{n}_\omega\approx\exp(e\mathcal{V}-\hbar\omega/kT)$. This agrees with the well-known result of Shockley and Queisser (1961) that the intensity of a light-emitting diode increases as $\exp(e\mathcal{V}/kT)$. Equation (7.10) also describes the flux of vacuum fluctuations with $\bar{n}_\omega=0$ and, consequently, $\langle \hat{Q}_N\rangle=0$. Vacuum fluctuations are not directly detectable by a photodetector. Nevertheless, vacuum fluctuations are an important source of noise. We find it convenient to represent external vacuum fluctuations as generated by spontaneous currents in cold absorbers surrounding devices. This is used in discussions of noise in traveling-wave amplifiers (Sec. VII.E, Fig. 7), noise associated with scattering loss in waveguides (Sec. VII.G, Fig. 12), and noise in lasers (Sec. VIII.A, Fig. 13).

The antinormally ordered energy flux and field correlation functions of a signal and vacuum fluctuations propagating in a cold absorbing waveguide are shown in Fig. 8. The correlation functions of the signal field and the field of vacuum fluctuations in this figure are described by Eqs. (B9) and (B11). With $z_1=z_2$, these equations also describe the energy flux. The energy flux of the incident fields decays to the flux of the vacuum fluctuations and remains constant thereafter. All fields, including those of vacuum fluctuations, obey the wave equation and are attenuated by optical absorption. The constancy of the energy flux in the mode results from emission of noise fields by spontaneous currents in the absorber, which are partially coupled into the waveguide mode. For vacuum fluctuations propagating in the waveguide mode, the added energy flux of these noise fields exactly balances the loss of optical absorption. The balance between absorption and emission of optical energy, which occurs for all optical modes coupled to the ab-

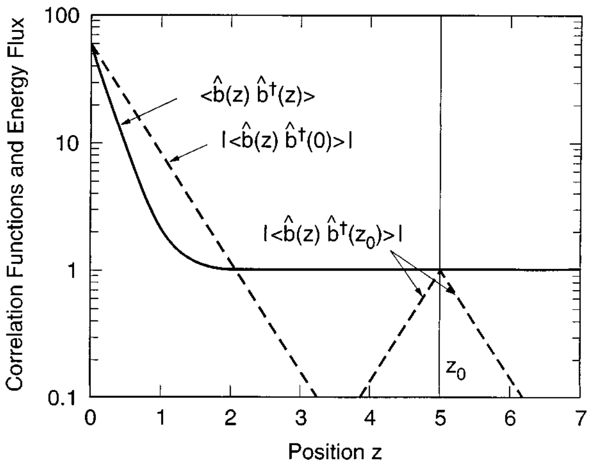


FIG. 8. Average energy flux and correlation functions of fields propagating in a cold absorber: solid curve, the average of the antinormal ordered energy flux of a signal and vacuum fluctuations, which decays to the energy flux of the vacuum fluctuations; dashed lines, the absolute value of the correlation function of the field at two values of z, which decays to zero with increasing separation. The decaying of the correlation function on either side of z_0 is interpreted to mean that the field of vacuum fluctuations is attenuated and emitted in an uncorrelated manner with energy conserved.

sorber, also preserves the stability of the atomic ground states of the absorbing atoms. This will be discussed further in Sec. VIII.D and in Appendix E.

The added noise field is uncorrelated with the initial field in the mode. This lack of correlation is consistent with the decay of the correlation function of the vacuum-fluctuation field at two points. This correlation function is given by Eq. (B9). It decays completely as shown in Fig. 8. The rate of decay is the same as that of the Green's function governing light propagating in a waveguide with optical attenuation.

The decay of the correlation function follows from a different form of the fluctuation-dissipation theorem (Martin, 1968), which shows that, for a system in thermal equilibrium, the field correlation function decays spatially in exactly the same way as the field decays in response to an external force. This falloff is described by the spatial decay of the classical Green's function. This should apply to all forms of radiation, including vacuum fluctuations. A derivation of this theorem for the specific case of a noise field generated by spontaneous current sources is given in Appendix B. We conclude that vacuum fluctuations propagating in an opaque waveguide are simultaneously attenuated and generated. These changes preserve the (antinormally ordered) energy flux but cause the field correlation function to decay with distance.

The absorption and emission of the field of vacuum fluctuations may bother some readers who are used to associating the vacuum field with the stationary ground states of quantized optical modes. The ground states of the optical field coupled to absorbing atoms is very complicated because each mode of the field is coupled to currents in the atoms that in turn are coupled to other modes. These ground states are stationary, but the field of a mode that we are interested in is not. If we follow the propagation of only this field, we shall find it simultaneously undergoing attenuation and generation.

B. Field distribution of thermal radiation and vacuum fluctuations

The $\hat{\mathcal{E}}$, $\hat{\mathcal{H}}$, and $\hat{\mathcal{A}}$ noise fields at a point $\mathbf{x}$ are all Hermitian linear combinations of $\hat{b}(t)$ and $\hat{b}(t)^\dagger$ given by

$$\hat{b}_H(t) \equiv c\hat{b}(t) + c^*\hat{b}^\dagger(t), \tag{7.11}$$

where $\hat{b}(t)$ and $\hat{b}^\dagger(t)$ are linearly related to $\hat{j}_\omega(x',z')$ and $\hat{j}^\dagger_\omega(x',z')$ by the Fourier transform of Eq. (7.5), and c is a complex number. This can be expressed symbolically as

$$\hat{b}_H = c\tilde{G}\hat{j} + c^*\tilde{G}^*\hat{j}^\dagger. \tag{7.12}$$

Let us calculate the lower moments of $\hat{b}_H(t)$. In Sec. IV I, we showed that averages of odd numbers of $\hat{j}(x,t)$ are zero. Since the fields are linearly related to these operators,

$$\langle \hat{b}_H \rangle = \langle \hat{b}_H^3 \rangle = \langle \hat{b}_H^5 \rangle \cdots = 0. \tag{7.13}$$

In calculating the powers of $\hat{b}_H(t)$, we shall neglect terms such as $\hat{b}(t)\hat{b}(t)$, which are at twice the optical frequency and also average to zero when calculated in terms of the spontaneous current-density operators. Therefore the second moment is just the sum of the two contributions given by the correlation functions (7.9) and (7.10),

$$\langle \hat{b}_H^2 \rangle = \frac{2\pi\hbar\Delta\nu}{k_0}|c|^2(2\bar{n}_\omega + 1). \tag{7.14}$$

In the notation of Eq. (7.12), the second moment is

$$\begin{aligned}\langle \hat{b}_H^2 \rangle &= \langle (\tilde{G}\hat{j} + \tilde{G}^*\hat{j}^\dagger)(\tilde{G}\hat{j} + \tilde{G}^*\hat{j}^\dagger) \rangle \\ &= \tilde{G}\tilde{G}^*\langle \hat{j}\hat{j}^\dagger \rangle + \tilde{G}^*\tilde{G}\langle \hat{j}^\dagger\hat{j} \rangle. \end{aligned}\tag{7.15}$$

The fourth moment is

$$\begin{aligned}\langle \hat{b}_H^4 \rangle = \langle &(\tilde{G}\hat{j} + \tilde{G}^*\hat{j}^\dagger)(\tilde{G}\hat{j} + \tilde{G}^*\hat{j}^\dagger)(\tilde{G}\hat{j} + \tilde{G}^*\hat{j}^\dagger) \\ &\times (\tilde{G}\hat{j} + \tilde{G}^*\hat{j}^\dagger) \rangle. \end{aligned}\tag{7.16}$$

According to Wick's theorem, this reduces to the product of all possible combinations of average pairs. There are three independent ways to pair the bracketed terms, and each results in $\langle \hat{b}_H^2 \rangle^2$, hence

$$\langle \hat{b}_H^4 \rangle = 3\langle \hat{b}_H^2 \rangle^2. \tag{7.17}$$

Similarly, there are five ways to form the first pair and three ways to form the second pair for $\langle \hat{b}_H^6 \rangle$, etc., hence

$$\langle \hat{b}_H^6 \rangle = 5\cdot 3\langle \hat{b}_H^2 \rangle^3, \tag{7.18a}$$

$$\langle \hat{b}_H^8 \rangle = 7\cdot 5\cdot 3\langle \hat{b}_H^2 \rangle^4. \tag{7.18b}$$

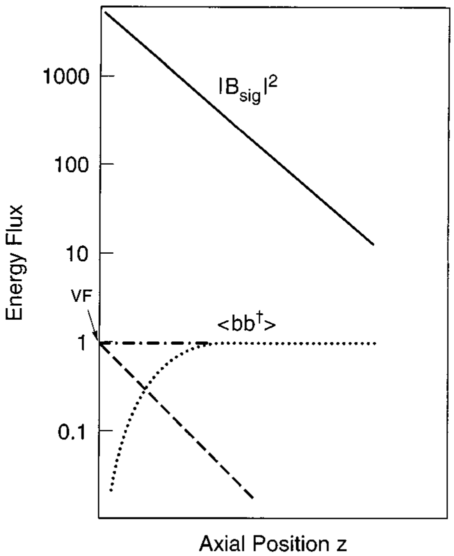


FIG. 9. Contributions to the average energy flux of vacuum fluctuations propagating in a cold absorber. The initial average energy flux of the vacuum fluctuations (dashed line) decays at the same rate as the signal flux (solid line). The total energy flux of vacuum fluctuations (dot-dashed line) remains constant because of a buildup from spontaneous current emission (dotted line).

These are just the moments of a Gaussian probability distribution for the values b_H of the operator $\hat{b}_H$:

$$\hat{\mathcal{P}}(b_H)\sim\exp\left(-\frac{b_H^2}{2\langle\hat{b}_H^2\rangle}\right). \tag{7.19}$$

We see that the noise fields $\hat{\mathcal{E}}$, $\hat{\mathcal{H}}$, and $\hat{\mathcal{A}}$ are Gaussian distributed.

The same arguments can be applied to the spontaneous current-density operator. Any Hermitian combination of $\hat{j}$ and $\hat{j}^\dagger$ is Gaussian distributed; i.e., it is a Gaussian random variable. This conclusion applies to $\hat{\mathcal{J}}$ [Eq. (4.39b)] and to the operators associated with the real and imaginary parts of $\hat{j}$. Consequently, the real and imaginary parts of the laser Langevin forces are Gaussian random variables. [Langevin forces were introduced in the discussion after Eq. (5.10).] The laser Langevin force is defined in Eqs. (8.5) and (8.6). The theories of laser linewidth and mode-partition noise make use of this important property (Lax, 1967a; Henry, 1982; Henry *et al.*, 1984). The Gaussian distribution follows from application of Wick's theorem, which in turn relies on the assumption that transitions between many different pairs of levels contribute to the averages of spontaneous currents (see derivation in Sec. IV.I).

C. Optical shot noise

Let us consider the propagation of a monochromatic noise-free signal and vacuum fluctuations in a cold absorbing waveguide. The attenuation of the signal and the contributions to the vacuum fluctuations are shown in Fig. 9. The incident energy flux of vacuum fluctuations is attenuated at the same rate as the signal energy flux. However, the combined effects of emission of vacuum fluctuations and attenuation result in a constant background of vacuum fluctuations with increasing distance along the waveguide. The field of vacuum fluctuations is not detectable, but the interference of this field and the signal field results in a modulation of the photon flux that is detectable. This modulation is shot noise.

As we have seen in Sec. VII.A, these vacuum fluctuations have a constant average flux given by Eq. (7.10). We represent the optical signal and noise fields after attenuation by

$$\hat{A}(\mathbf{x},t)=[b_{\rm sig}+\hat{b}(t)]\phi_0(x)\exp(-i\omega_S t), \tag{7.20}$$

where the signal amplitude $b_{\rm sig}$ is a complex classical c-number and the noise field amplitude $\hat{b}(t)$ is given by the Fourier transform of Eq. (7.5), with $\phi_0(x)$ and $\exp(-i\omega_S t)$ removed.

The correlation functions of $\hat{b}(t)$ at two times t_1 and t_2 are given by Fourier transforming Eq. (7.7) with $\bar{n}_{\omega_S}=0$. This results in

$$\frac{k_0}{2\pi\hbar}\langle\hat{b}(t_1)\hat{b}^\dagger(t_2)\rangle=\delta(t_1-t_2), \tag{7.21a}$$

$$\frac{k_0}{2\pi\hbar}\langle\hat{b}^\dagger(t_1)\hat{b}(\hat{t}_2)\rangle=0. \tag{7.21b}$$

The photon flux [Eq. (6.15)] in the transverse mode at time t is

$$\hat{Q}_N(t)=\frac{k_0}{2\pi\hbar}[b^*_{\rm sig}+\hat{b}^\dagger(t)][b_{\rm sig}+\hat{b}(t)]. \tag{7.22}$$

The total number of photons $\hat{m}$ passing a position during time T is

$$\hat{m}=\int_0^T dt\hat{Q}_N(t)$$
$$=\frac{k_0}{2\pi\hbar}|b_{\rm sig}|^2T+\frac{k_0}{2\pi\hbar}\int_0^T[b^*_{\rm sig}\hat{b}(t)+b_{\rm sig}\hat{b}^\dagger(t)]dt$$
$$+\frac{k_0}{2\pi\hbar}\int_0^T\hat{b}^\dagger(t)\hat{b}(t)dt. \tag{7.23}$$

The average of $\hat{m}$ is

$$\bar{m}=\frac{k_0}{2\pi\hbar}|b_{\rm sig}|^2T. \tag{7.24}$$

Combining the last two equations, the fluctuations of m are

$$\Delta\hat{m}=\hat{m}-\bar{m}$$
$$=\frac{k_0}{2\pi\hbar}\int_0^T dt[b^*_{\rm sig}\hat{b}(t)+b_{\rm sig}\hat{b}(t)^\dagger+\hat{b}^\dagger(t)\hat{b}(t)]. \tag{7.25}$$

Squaring this equation and averaging, with the help of Eq. (7.21), we find

$$\langle \Delta \hat{m}^2 \rangle = \frac{k_0}{2\pi\hbar} |b_{\rm sig}|^2 \int_0^T dt_1 \int_0^T dt_2 \delta(t_1 - t_2) = \bar{m}. \quad (7.26)$$

A similar calculation for $\langle \Delta \hat{m}^3 \rangle$ yields

$$\langle \Delta \hat{m}^3 \rangle = \frac{k_0}{2\pi\hbar} |b_{\rm sig}|^2 \int_0^T dt_1 \int_0^T dt_2 \int_0^T dt_3 \delta(t_1 - t_2) \times \delta(t_2 - t_3) = \bar{m}. \quad (7.27)$$

Both of these moments agree exactly with those of a Poisson distribution, which is a feature of shot noise (Mandel and Wolf, 1995). The Poisson distribution characterizes the noise expected when a stream of uncorrelated photons is detected. Unlike a Gaussian distribution, the Poisson distribution has a definite relation between the second moment and the average value, and it can have odd moments about its average value. The noise in the photon flux $\hat{Q}_N$ will be exactly what is detected by an ideal photodetector. Such a detector is opaque, and all photons received are converted into photoelectrons. There is no additional noise in this conversion process.

D. Shot noise in optical absorption

Now consider the opposite situation of a nearly transparent photodetector. There is additional noise associated with the optical absorption process. Suppose a strong noise-free light signal $b_{\rm sig}$ traveling in the fundamental mode passes through a length Δz of an optical waveguide. The number of photocarriers generated by optical absorption is given by Eq. (6.11) integrated over x and Δz,

$$\dot{\hat{N}}_N = |b_{\rm sig}|^2 \frac{k_0 a_0 \Delta z}{2\pi\hbar} + \frac{i}{\hbar c} \int dz\, dx\, \phi_0(x) (\hat{J}^\dagger b_{\rm sig} - b_{\rm sig}^* \hat{J}). \quad (7.28)$$

The average number of photocarriers generated in time T is given by the first term

$$\bar{m} = |b_{\rm sig}|^2 \frac{k_0}{2\pi\hbar} a_0 \Delta z T. \quad (7.29)$$

Integrating Eq. (7.28) over time T we have

$$\Delta \hat{m} = \hat{m} - \bar{m} = \frac{i}{\hbar c} \int_0^T dt \int dz\, dx\, \phi_0(x) (\hat{J}^\dagger b_{\rm sig} - b_{\rm sig}^* \hat{J}). \quad (7.30)$$

The average $\langle \Delta \hat{m}^2 \rangle$ is found by squaring and averaging this equation and keeping only the nonzero antinormally ordered term:

$$\langle \Delta \hat{m}^2 \rangle = \frac{|b_{\rm sig}|^2}{\hbar^2 c^2} \int_0^T dt_1 \int_0^T dt_2 \int dz_1 dx_1 \times \int dz_2 dx_2 \langle \hat{J}(\mathbf{x}_1, t_1) \hat{J}^\dagger(\mathbf{x}_2, t_2) \rangle \times \phi_0(x_1) \phi_0(x_2). \quad (7.31)$$

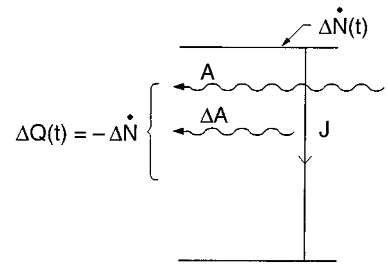


FIG. 10. How fluctuations of photon flux and fluctuations in the carrier transition rate combine to conserve energy. The fluctuation in the rate of optical transitions $\Delta \dot{N}$ is caused by the beating of the spontaneous current J and the field A. The fluctuation in the energy flux ΔQ is caused by the beating of spontaneously emitted field ΔA and the field A. There is a fixed phase difference between ΔA and J such that $\Delta \dot{N} = -\Delta Q$. Quantum operator equations underlie this classical description.

Using the fluctuation-dissipation theorem [Eq. (4.42)] to evaluate the spontaneous current-density correlation function, this reduces to

$$\langle \Delta \hat{m}^2 \rangle = |b_{\rm sig}|^2 \frac{k_0}{2\pi\hbar} a_0 \Delta z T = \bar{m}. \quad (7.32)$$

Therefore $\langle \Delta \hat{m}^2 \rangle$ is consistent with shot noise.

We demonstrated in the preceding section that absorbing regions generate vacuum fluctuations that beat with the transmitted signal, resulting in fluctuations in the photon flux $\hat{Q}_N(t)$. Conservation of energy [Eq. (6.12)] requires that a fluctuation in photon flux be accompanied by a correlated fluctuation in the rate of generation of photo-excited carriers. As we discussed at the end of Sec. VI.C, the photon energy stored in a small volume is negligible and, consequently, energy conservation reduces to $\Delta \hat{Q}_N(t) + d\hat{N}_N/dt = 0$. This conservation is possible because the changes in $\hat{Q}_N(t)$ and in $d\hat{N}_N/dt$ are controlled by the phase of the spontaneous current-density relative to that of the external field and are equal and opposite to one another. This is illustrated in Fig. 10. The phase relations are shown in Fig. 5.

The shot noise in a nearly transparent photodetector is due to fluctuations in the rate of optical absorption. This noise will be present in the currents of photodetectors with less than unit quantum efficiency. Excess noise in such photodetectors has been modeled by Yuen and Shapiro (1980) and Yurke (1985), by a different mechanism. They do not use spontaneous currents as just described. Instead, they represent the real photodetector as equivalent to an ideal opaque photodetector and a beam splitter in the incident optical field. The beam splitter reduces the detection efficiency and introduces vacuum fluctuations which add noise. The equivalence of these two models has not been established and needs further investigation.

E. Traveling-wave amplification

We now turn to noise generated in optical amplification. A traveling-wave amplifier of length L is shown in

Fig. 7. Suppose that at the input of the amplifier there is a monochromatic signal field $b_{\mathrm{sig\ in}}$ and a noise field of vacuum fluctuations $\hat{b}_{\mathrm{in}}(t)$. These fields propagate through the amplifier and change in phase and amplitude. In addition, the noise field emitted by spontaneous currents and subsequently amplified $\hat{b}_{\mathrm{em}}(t)$ also appears at the output. The z-dependent signal and noise fields at the output of the amplifier are

$$b_{\mathrm{sig}}+\hat{b}_{\mathrm{out}}(t)=\sqrt{G}b_{\mathrm{sig\ in}}\,e^{ik_0L}+[\sqrt{G}\hat{b}_{\mathrm{in}}(t)e^{ik_0L}+\hat{b}_{\mathrm{em}}(t)], \tag{7.33}$$

where $G=\exp(g_0L)$ is the amplification. The second term on the right side of Eq. (7.33) appears formally to be an "amplified field of vacuum fluctuations." However, the flux of uncertainty-related field fluctuations, as we discussed in Sec. VI.D, remains constant. We find that this is the case when both contributions to the noise field, given by the bracketed terms on the right side of this equation, are combined.

The internally emitted and amplified noise field $\hat{b}_{\mathrm{em}}(t)$ is calculated in the same way as the noise field emitted from an opaque waveguide [Eq. (7.5)]. The field at $z=L$ is given by the Fourier transform of

$$\hat{b}_{\mathrm{em}\omega}=-\frac{4\pi}{c}\frac{1}{2ik_0}\int_0^L dz'\,e^{(ik_0+g_0/2)(L-z')}\times\int dx'\,\phi_0(x')\hat{j}_\omega(x',z'). \tag{7.34}$$

The normally and antinormally ordered energy fluxes of this field are found using the same steps used previously in going from Eq. (7.5) to Eq. (7.9) and result in

$$\frac{k_0}{2\pi\hbar}\langle\hat{b}^\dagger_{\mathrm{em}}(t)\hat{b}_{\mathrm{em}}(t)\rangle=\Delta\nu(G-1)n_{\mathrm{sp}}, \tag{7.35a}$$

$$\frac{k_0}{2\pi\hbar}\langle\hat{b}_{\mathrm{em}}(t)\hat{b}^\dagger_{\mathrm{em}}(t)\rangle=\Delta\nu(G-1)(n_{\mathrm{sp}}-1), \tag{7.35b}$$

where n_{sp} was defined in Eq. (4.51).

The energy flux of the amplified vacuum fluctuations is that of the vacuum fluctuations [Eq. (7.10)] multiplied by the amplification G. Only the antinormally ordered energy flux of this noise field is nonzero. It can be written as the sum of two terms:

$$\frac{k_0}{2\pi\hbar}G\langle\hat{b}_{\mathrm{in}}(t)\hat{b}^\dagger_{\mathrm{in}}(t)\rangle=\Delta\nu(G-1)+\Delta\nu. \tag{7.36}$$

The second term on the right side of this equation is the energy flux of the uncertainty-related field fluctuations. It is equal to that of vacuum fluctuations at the amplifier input and remains constant along the length of the amplifier. We show in Appendix E that this constant flux of field fluctuations induces spontaneous emission that is amplified and results in the first term in Eq. (7.36).

The energy fluxes of the total noise field $\hat{b}_{\mathrm{out}}(t)=\sqrt{G}\hat{b}_{\mathrm{in}}(t)e^{ik_0L}+\hat{b}_{\mathrm{em}}(t)$ are found by combining Eqs. (7.36) and (7.35):

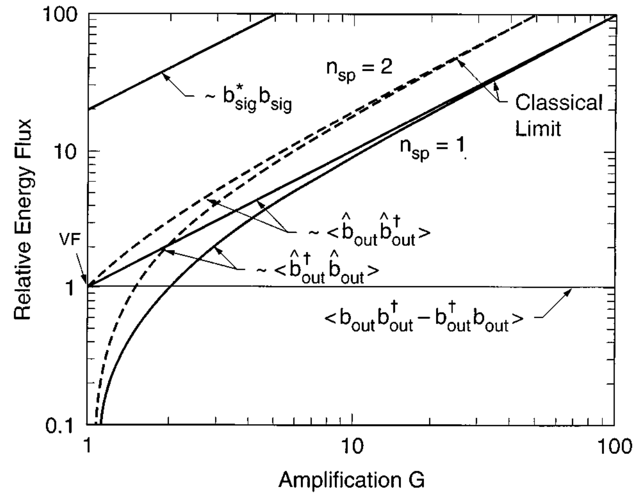


FIG. 11. Increase in the energy flux of the signal and noise fields with optical amplification. $\langle\hat{b}^\dagger_{\mathrm{out}}\hat{b}_{\mathrm{out}}\rangle$ is proportional to the energy flux of amplified spontaneous emission. $\langle\hat{b}_{\mathrm{out}}\hat{b}^\dagger_{\mathrm{out}}-\hat{b}^\dagger_{\mathrm{out}}\hat{b}_{\mathrm{out}}\rangle$ is proportional to the energy flux of the uncertainty-related field fluctuations, which remains unchanged with amplification. $\langle\hat{b}_{\mathrm{out}}\hat{b}^\dagger_{\mathrm{out}}\rangle$ is just the sum of these two contributions. At high amplification, the classical limit is approached when the energy flux of the uncertainty-related field fluctuations become negligible compared to that of amplified spontaneous emission. The cases of full population inversion, $n_{\mathrm{sp}}=1$, and partial population inversion, $n_{\mathrm{sp}}=2$, are shown. At a fixed amplification, the minimum amplified spontaneous-emission noise occurs for $n_{\mathrm{sp}}=1$. The amplifier noise figure is the increase in noise-to-signal ratio, $(\langle\hat{b}^\dagger_{\mathrm{out}}\hat{b}_{\mathrm{out}}\rangle+\langle\hat{b}_{\mathrm{out}}\hat{b}^\dagger_{\mathrm{out}}\rangle)/|b_{\mathrm{sig}}|^2$, with amplification. The curve for $\langle\hat{b}^\dagger_{\mathrm{out}}\hat{b}_{\mathrm{out}}\rangle$ approaches that for $\langle\hat{b}_{\mathrm{out}}\hat{b}^\dagger_{\mathrm{out}}\rangle$ at high amplification. For $n_{\mathrm{sp}}=1$, this results in a noise figure of 2, i.e., in a minimum 3 dB noise figure.

$$\frac{k_0}{2\pi\hbar}\langle\hat{b}^\dagger_{\mathrm{out}}(t)\hat{b}_{\mathrm{out}}(t)\rangle=\Delta\nu(G-1)n_{\mathrm{sp}}, \tag{7.37a}$$

$$\frac{k_0}{2\pi\hbar}\langle\hat{b}_{\mathrm{out}}(t)\hat{b}^\dagger_{\mathrm{out}}(t)\rangle=\Delta\nu[(G-1)n_{\mathrm{sp}}+1]. \tag{7.37b}$$

The term $\Delta\nu n_{\mathrm{sp}}(G-1)$ is the average photon flux of amplified spontaneous emission. This order-independent term comes from the amplification of both emission by spontaneous currents and emission induced by the uncertainty-related field fluctuations. The remaining term is the energy flux of the uncertainty-related field fluctuations. The average energy fluxes of the signal and the noise field are plotted in Fig. 11 for the ideal case of $n_{\mathrm{sp}}=1$ and for $n_{\mathrm{sp}}=2$. The maximum gain and spontaneous emission per unit length occur with complete population inversion, $n_{\mathrm{sp}}=1$. When the amount of population inversion is reduced, gain drops more rapidly than spontaneous emission. Consequently, a longer amplifier is required and more noise is generated to achieve the same amplification G. The best amplifier noise figure is achieved with $n_{\mathrm{sp}}=1$.

When G is large compared to unity, the difference between the normally and antinormally ordered energy flux is negligible, and the noise field can be treated as classical waves. In this case, the normal and antinormal ordered correlation functions of Eq. (7.37) can be approximated as equal. Then the higher-order moments of the energy flux are readily calculated using Wick's theorem. It can be shown that the instantaneous energy flux is exponentially distributed. A calculation of this type is made in Sec. VIII.B, where it is shown that the instantaneous photon number in a nonlasing mode near threshold is exponentially distributed [see Eq. (8.19)]. This distribution is also encountered in the description of polarized thermal or chaotic light (Loudon, 1983; Goodman, 1985).

An exponential distribution $dP(I)=\exp(-I/\bar{I})dI/\bar{I}$ for the intensity $I(t)$ has large fluctuations: $I(t)$ can drop to zero and increase to many times its average value with nonnegligible probability. Why is it that these fluctuations are not a problem when intense broadband sources such as light-emitting diodes are used in optical communications? The answer is that the fluctuations in a source of spectral width $\Delta\nu$ only remain correlated for times $\simeq\Delta\nu^{-1}$. [This can be shown by calculating the correlation function of $\hat{b}_{\text{out}}(t)$ at two times.] Thus the light intensity can drop to zero, but only for times of order $\Delta\nu^{-1}$. If the bit time T is much greater than $\Delta\nu^{-1}$, the distribution of intensity integrated over the bit time will be a convolution of many independent exponential distributions. Such a convolution results in a well-behaved Gaussian-like distribution with negligible probability of having very small or very large values of the integrated intensity (Goodman, 1985).

Let us calculate the noise in the photon flux that occurs when an amplified signal is present. The signal and noise fields are given by Eq. (7.33). The photon flux $\hat{Q}_N(t)$ is given by

$$\hat{Q}_N(t)=\frac{k_0}{2\pi\hbar}[b_{\text{sig}}^*+\hat{b}_{\text{out}}^\dagger(t)][b_{\text{sig}}+\hat{b}_{\text{out}}(t)]$$

$$=Q_{\text{sig}}+\Delta\hat{Q}(t)+\frac{k_0}{2\pi\hbar}\hat{b}_{\text{out}}^\dagger(t)\hat{b}_{\text{out}}(t), \qquad (7.38)$$

where $Q_{\text{sig}}=(k_0/2\pi\hbar)|b_{\text{sig}}|^2$ and

$$\Delta\hat{Q}(t)=\frac{k_0}{2\pi\hbar}[b_{\text{sig}}^*\hat{b}_{\text{out}}(t)+b_{\text{sig}}\hat{b}_{\text{out}}^\dagger(t)]. \qquad (7.39)$$

The amplifier output is composed of a signal field and a fluctuating noise field due to amplified spontaneous emission and the uncertainty-related field fluctuations. The noise power due to the beating of the signal field with the noise field of amplified spontaneous emission is called "signal-spontaneous beat noise." The noise power due to the beating of the amplified spontaneous-emission noise field with itself is called "spontaneous-spontaneous beat noise." In addition, there is shot noise, which is the beating of the signal field with the uncertainty-related field fluctuations.

To achieve low noise, the optical field must be passed through a narrow-band optical filter after amplification to suppress spontaneous-spontaneous beat noise. For simplicity, we assume that this has been done and neglect the term associated with $\hat{b}_{\text{out}}^\dagger(t)\hat{b}_{\text{out}}(t)$. The filtered energy flux at the output is $\hat{Q}_N(t)=Q_{\text{sig}}+\Delta\hat{Q}(t)$, where $\Delta\hat{Q}(t)$ is a Hermitian combination of $\hat{b}_{\text{out}}(t)$ and $\hat{b}_{\text{out}}^\dagger(t)$. Therefore, as discussed in Sec. VII.B, the fluctuations of $\Delta\hat{Q}(t)$ are Gaussian distributed.

Let $\Delta\nu_e$ equal the electronic amplifier bandwidth. Only the components of the noise field that are within $\Delta\nu_e$ of the signal, either above or below, will have beats that are amplified. Therefore, in applying Eq. (7.37) to calculate $\langle\hat{b}_{\text{out}}^\dagger\hat{b}_{\text{out}}\rangle$, we shall use a frequency range of $2\Delta\nu_e$. The mean-square fluctuation of photon flux is given by

$$\langle\Delta\hat{Q}^2\rangle=Q_{\text{sig}}\frac{k_0}{2\pi\hbar}[\langle\hat{b}_{\text{out}}^\dagger\hat{b}_{\text{out}}\rangle+\langle\hat{b}_{\text{out}}\hat{b}_{\text{out}}^\dagger\rangle]$$

$$=Q_{\text{sig}}2\Delta\nu_e[(G-1)2n_{\text{sp}}+1]. \qquad (7.40)$$

The first term is the signal-spontaneous beat noise and the second term is shot noise.

The amplifier noise figure is the increase in the noise to signal ratio with amplification, where this ratio is defined as $\langle\Delta\hat{Q}^2\rangle/Q_{\text{sig}}^2$ (Desurvire, 1994). This ratio is given by Eq. (7.40) as $[\langle\hat{b}_{\text{out}}^\dagger\hat{b}_{\text{out}}\rangle+\langle\hat{b}_{\text{out}}\hat{b}_{\text{out}}^\dagger\rangle]/|b_{\text{sig}}|^2$. At high amplification, the noise figure is given by $2n_{\text{sp}}$, which is $\times 2$ (or 3 dB) for the case of minimum noise, with $n_{\text{sp}}=1$. This is illustrated by the plots of energy flux in Fig. 11.

A thorough discussion of the practical aspects of amplifier noise is given by Olsson (1989; see also Tonguz and Kazovsky, 1991; Park and Granlund, 1994). He includes corrections for transmission losses, coupling losses in entering and leaving the amplifier, and spontaneous-spontaneous beat noise of filtered light. The effect of loss within the amplifier may also be important. It increases the noise and the effective value of n_{sp} (Henry, 1986a). Other nonidealities also enhance noise (Yamamoto, 1980; Mukai *et al.*, 1982; Simon, 1983).

F. Noise in Raman amplification

Traveling-wave Raman amplifiers are likely to become an important source of amplification in optical communications. The Raman amplification spectrum is broader than that of the Er-doped optical-fiber amplifier. Raman amplification occurs on the low-energy (Stokes) side of the pump frequency, where it is separated from the pump by the energy of the optical phonons taking part in the Raman scattering. Thus the Raman amplification band can be placed at any spectral position if a pump at the right wavelength is available.

Raman amplification in optical fibers has been extensively studied (Mochizuki *et al.*, 1986; Olsson and Hegarty, 1986; Aoki, 1988; Desurvire, 1994). However, until recently, the high pump power required was an obstacle to the use of this type of amplifier in communication systems. This has now been overcome by the use of multistripe semiconductor lasers to pump fibers with absorbing rare-earth doped single-mode cores (Grubb, 1996). The multimode pump energy is coupled into the optical-fiber cladding and is subsequently absorbed in the core and efficiently converted into stimulated emission in the fundamental transverse mode of the optical fiber. In this way, watts of single-mode pump power can be delivered to the fiber. This power can then be efficiently shifted to longer wavelengths by multiple stages of stimulated Raman scattering (Grubb, 1996; White and Grubb, in press).

The noise field in a Raman amplifier is due to amplified spontaneous Raman scattering. The optical-frequency spontaneous currents that are a source of this noise field have a different physical origin than we have encountered until now. They are due to fluctuations of atoms at optical-phonon frequencies mixed with the pump field. This mixing occurs as a result of the nonlinear interaction of the optical field with the atomic vibrations in the glass. The atomic motion is primarily due to zero-point fluctuations of the lowest vibrational levels and can be thought of as uncertainty-related momentum fluctuations of the atoms confined in the glass.

Let us assume that each optical-phonon mode q is executing simple harmonic motion described conventionally by quantized creation and annihilation operators $\hat{b}_q^\dagger$ and $\hat{b}_q$ that have commutation relations $[\hat{b}_q,\hat{b}_r^\dagger]=\delta_{q,r}$. The energies of the quantized vibrations are given by a Hamiltonian $\hat{H}_L=\Sigma_q\hbar\Omega_q(\hat{b}_q^\dagger\hat{b}_q+1/2)$, where Ω_q is the angular frequency of mode q. The modes are in thermal equilibrium with nonzero operator averages given by

$$\langle\hat{b}_q^\dagger\hat{b}_q\rangle=\bar{n}_q,\quad\langle\hat{b}_q\hat{b}_q^\dagger\rangle=\bar{n}_q+1, \tag{7.41}$$

where $\bar{n}_q=[\exp(\hbar\Omega_q/kT)-1]^{-1}$. In the interaction representation, these operators have a time dependence

$$\hat{b}_q(t)=\hat{b}_q e^{-i\Omega_q t},\quad\hat{b}_q^\dagger(t)=\hat{b}_q^\dagger e^{i\Omega_q t}, \tag{7.42}$$

which can be established from the operator equations of motion (see Sec. IV.C).

The interaction leading to Raman scattering can be written phenomenologically as a sum of products of the vibrational mode amplitudes ($\hat{b}_q$+H.c.), the signal field at the Stokes frequency $[\hat{A}_{\rm St}(t)+\text{H.c.}]$, and the pump field $[A_p\exp(-i\omega_p t)+\text{c.c.}]$, which will be regarded as classical and monochromatic. In calculating the spontaneous and induced current in a small volume V, we shall regard the optical fields as external in origin. Only resonant terms with products of the fields that can drive the lattice at an optical-phonon frequency need be kept. The interaction reduces to

$$\begin{aligned}\hat{H}_{\rm int}&=-\sum_q D_q\hat{b}_q^\dagger A_p e^{-i\omega_p t}\hat{A}_{\rm St}^\dagger(t)+\text{H.c.}\\&=-\frac{V}{c}\hat{j}_R\hat{A}_{\rm St}^\dagger(t)+\text{H.c.},\end{aligned} \tag{7.43}$$

where the coefficients D_q give the strength of the interaction. The second equality defines an effective current-density operator occurring in Raman scattering $\hat{j}_R$. Using this operator and Eq. (7.42), we obtain the time-dependent spontaneous current-density operator,

$$\hat{j}_R(t)=\frac{c}{V}\sum_q D_q\hat{b}_q^\dagger A_p e^{-i(\omega_p-\Omega_q)t}. \tag{7.44}$$

The frequency components $\hat{j}_{R\omega}$ are found by Fourier-transforming this equation [see Eq. (4.29)]. The correlation functions of $\hat{j}_{R\omega}$ are readily found using Eq. (7.41) and replacing Σ_q by $V\int(dN/d\Omega_q)d\Omega_q$:

$$\langle\hat{j}_{R\omega}\hat{j}_{R\omega'}^\dagger\rangle=\frac{2\pi c^2}{V}\frac{dN}{d\Omega_q}|A_p|^2|D_q|^2\delta(\omega-\omega')\bar{n}_q, \tag{7.45a}$$

$$\langle\hat{j}_{R\omega'}^\dagger\hat{j}_{R\omega}\rangle=\frac{2\pi c^2}{V}\frac{dN}{d\Omega_q}|A_p|^2|D_q|^2\delta(\omega-\omega')(\bar{n}_q+1), \tag{7.45b}$$

where the q-dependent terms are evaluated at $\Omega_q=\omega_p-\omega$. As we found in Sec. IV.D, the correlation functions are associated with rates of induced absorption and emission of photons of the signal field. Accompanying these processes are absorption and emission of phonons. The uncertainty-related vibrational fluctuations make phonon emission rates always greater than phonon absorption rates. This is reflected in the factors $\bar{n}_q$ and $\bar{n}_q+1$ in Eq. (7.45).

The Raman gain g_ω is related to the imaginary part of the susceptibility, Eq. (4.37), which, in turn, is given by Eq. (4.30), an equation derived from the Kubo formula, as proportional to the difference in the two correlation functions. Using these equations, we find

$$g_\omega k_\omega=\frac{4\pi^2}{\hbar}\frac{dN}{d\Omega_q}|D_q|^2|A_p|^2. \tag{7.46}$$

The gain is positive and proportional to the squared pump field $|A_p|^2$. Raman gain does not have a threshold, but net gain requires sufficient pump power to overcome attenuation loss in the optical fiber.

A similar calculation could be done at the anti-Stokes frequency. In this case, the spontaneous current operator is proportional to $\hat{b}_q$ instead of $\hat{b}_q^\dagger$. Formally, this change results in attenuation instead of gain at the anti-Stokes frequency. Physically, induced emission in the anti-Stokes case is accompanied by phonon absorption weighted by $\bar{n}_q$, whereas induced absorption is accompanied by phonon emission and is weighted by $\bar{n}_q+1$. Therefore the net induced change in the anti-Stokes field is one of attenuation.

The fluctuation-dissipation theorem, relating the spontaneous current correlation functions to gain, is found by combining Eqs. (7.45) and (7.46). In doing this, we shall also replace V^{-1} by $\delta(\mathbf{x}-\mathbf{x}')$.

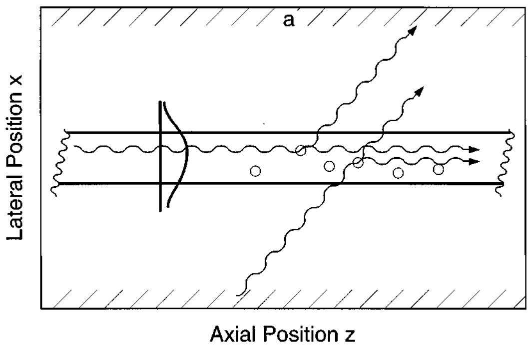


FIG. 12. Loss and noise in a waveguide with scattering. The guided light is scattered out of the waveguide, resulting in optical attenuation. Vacuum fluctuations are scattered into the waveguide and act as a source of noise. The vacuum fluctuations can be thought of as emitted from spontaneous currents in the absorbing boundary.

$$\left(\frac{4\pi}{c}\right)^2 \langle \hat{j}_{R\omega}(\mathbf{x})\hat{j}^{\dagger}_{R\omega'}(\mathbf{x}')\rangle = 8\pi\hbar k_\omega(\mathbf{x}) g_\omega(\mathbf{x})\bar{n}_q(\mathbf{x}) \times \delta(\mathbf{x}-\mathbf{x}')\,\delta(\omega-\omega'), \tag{7.47a}$$

$$\left(\frac{4\pi}{c}\right)^2 \langle \hat{j}^{\dagger}_{R\omega'}(\mathbf{x}')\hat{j}_{R\omega}(\mathbf{x})\rangle = 8\pi\hbar k_\omega(\mathbf{x}) g_\omega(\mathbf{x})(\bar{n}_q(\mathbf{x})+1) \times \delta(\mathbf{x}-\mathbf{x}')\,\delta(\omega-\omega'). \tag{7.47b}$$

If we assign $n_{\mathrm{sp}}=\bar{n}_q+1$, these equations agree identically with the fluctuation-dissipation theorem of a conventional amplifier. With this change our prior discussions of the traveling-wave amplifier in Sec. VII.E also apply to the Raman amplifier. This includes our calculations of the average energy flux, signal-spontaneous beat noise, the noise figure, the rate of spontaneous emission into the transverse mode [Eq. (E6)] (spontaneous Raman scattering into the mode, in this case), and the transition to classical fields at high amplification.

A typical Raman shift in fused silica is about 55 meV. At T=300 K, $\bar{n}_q \cong 0.12$. Thus n_{sp} is close to unity. The Raman amplifier should have a noise figure close to that of an ideal amplifier, provided that the gain is sufficiently high and that optical-fiber losses are negligible.

G. Fluctuation-dissipation relation for scattering loss

The fluctuation-dissipation theorem relates the correlation function of the spontaneous current densities to the loss coefficient [Eq. (4.38)]. This relation was derived under the assumption that the loss is due to optical absorption. Much of the loss encountered in optical fibers and waveguides results from scattering, not absorption. In Appendix C, we show that this form of loss leads to effective spontaneous currents with correlation functions related to optical attenuation by the same fluctuation-dissipation theorem relation. The vacuum fluctuations "emitted" by these currents are actually fields from external sources that are scattered into the bound modes while passing through the waveguide, as shown in Fig. 12.

The optical field obeys the wave equation

$$\nabla^2 \hat{A}_\omega(\mathbf{x}) + \frac{\omega^2}{c^2}[\epsilon_\omega(\mathbf{x}) + \Delta\epsilon_\omega(\mathbf{x})]\hat{A}_\omega(\mathbf{x}) = -\frac{4\pi}{c}\hat{j}_\omega(\mathbf{x}), \tag{7.48}$$

where $\Delta\epsilon_\omega(x)$ is associated with inhomogeneous scattered light and $\hat{j}_\omega(\mathbf{x})$ is the spontaneous current density at the absorbing walls in Fig. 12. Imitating open surroundings the absorption ensures that the scattered light does not reflect off the boundary.

In Appendix C, this equation is solved approximately by perturbation theory, and we arrive at an effective equation for the guided modes $\hat{A}_{\omega\,\mathrm{modes}}(\mathbf{x})$ given by

$$\nabla^2 \hat{A}_{\omega\,\mathrm{modes}}(\mathbf{x}) + [k_\omega{}^2(\mathbf{x}) + 2k_\omega(\mathbf{x})\Delta k_\omega(\mathbf{x}) + ik_\omega(\mathbf{x})a_\omega(\mathbf{x})_{\mathrm{eff}}]\hat{A}_{\omega\,\mathrm{modes}}(\mathbf{x}) = -\frac{4\pi}{c}\hat{j}_{\omega\,\mathrm{eff}}(\mathbf{x}). \tag{7.49}$$

The average attenuation coefficient of the guided field due to scattering loss is $a_{\omega\,\mathrm{eff}}(\mathbf{x})$. The change in the real part of the squared propagation constant caused by multiple scattering slightly altering the velocity of propagation is $2k_\omega(\mathbf{x})\Delta k_\omega(\mathbf{x})$. The operator $\hat{j}_{\omega\,\mathrm{eff}}(\mathbf{x})$ is the effective spontaneous current source resulting from vacuum fluctuations scattered into the propagating modes. The nonzero correlation function of $\hat{j}_{\omega\,\mathrm{eff}}$ is

$$\left(\frac{4\pi}{c}\right)^2 \langle \hat{j}_{\omega\,\mathrm{eff}}(\mathbf{x})\hat{j}^{\dagger}_{\omega\,\mathrm{eff}}(\mathbf{x}')\rangle = 8\pi\hbar k_\omega(\mathbf{x}) a_{\omega\,\mathrm{eff}}(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{x}'). \tag{7.50}$$

This is exactly the expected relation between the loss and spontaneous current density predicted by the fluctuation-dissipation theorem [Eq. (4.38)] with $\bar{n}_\omega=0$.

The fluctuation-dissipation theorem appears to hold regardless of the type of loss. This can be explained using a different formulation of the fluctuation-dissipation theorem given by Kubo (1966) and by Martin (1968). A derivation of this form of the fluctuation-dissipation theorem is given in Appendix B. When applied to noise fields, this version of the theorem shows that the correlation of the fields at two points is related to the imaginary part of the Green's function governing the response of the field at one point to a source at the other point. The decay of the Green's function depends on the field attenuation and is the same for any type of attenuation, regardless of its cause.

If the noise-field correlation functions are calculated from those of a source, then to have source-independent correlation functions, the source must have the same correlation functions regardless of its nature. The use of the fluctuation-dissipation theorem without reference to sources has been applied to quantum electrodynamic problems by Agarwal (1974a, 1974b, 1974c). We have not followed this approach because it does not illuminate the physical nature of the sources of quantum noise and because noise sources are useful in dealing with the nonequilibrium situations encountered in photonics applications.

The fact that different types of loss result in the same noise field is a common artifice used in the analysis of quantum optics experiments. For example, coupling and absorption losses are represented by fictitious beam splitters that alter beam propagation and introduce vacuum fluctuations (Caves and Crouch, 1987; Slusher and Yurke, 1990; Jeffers *et al.*, 1993).

VIII. QUANTUM NOISE OF LASERS

We now turn to quantum noise in lasers. For simplicity, we assume that only a single laser mode is near threshold. For the laser below threshold and in steady-state operation above threshold, we can take the gain to be constant in time. This allows us to use the simple methods employed thus far and to obtain the following results. A general formula for the rate of spontaneous emission into a cavity mode is derived. We calculate the distribution of photon-number fluctuations for modes below threshold. As laser threshold is approached, the sources of vacuum fluctuations and spontaneous emission combine to produce a field that can be treated as classical, just as they did for optical amplifiers. The noise and mode intensity of open-cavity lasers, e.g., gain-guided lasers and Fabry-Perot lasers with low reflecting facets, are enhanced by the Petermann factor (Petermann, 1979; Wang *et al.*, 1987). This is not caused by enhancement of spontaneous emission, as once thought, but is due to single pass amplification.

A laser above threshold is more difficult to analyze. The round-trip amplification is very sensitive to the gain in the laser cavity. This gain is dependent on carrier number and, for this reason, the description of fluctuations of a laser about the steady state requires a simultaneous solution for the field and carrier density. The situation is further complicated by the fact that the transient field can have a different spatial distribution than the steady-state field, so that a set of coupled differential equations with noise sources must be solved. We shall only consider one such effect, sub-Poissonian laser-intensity noise. This can be understood from energy conservation, without explicitly solving these coupled differential equations.

A. Langevin rate equation for the steady-state lasing mode

To describe a laser mode, we use the field equation in the slowly varying envelope approximation [Eq. (6.9)]. To emphasize the different origins of gain and loss, we shall replace $a(\mathbf{x})$ by $-g(\mathbf{x})+a_I(\mathbf{x})+a_E(\mathbf{x})$, where $g(\mathbf{x})$ is the gain, $a_I(\mathbf{x})$ is the internal loss, and $a_E(\mathbf{x})$ is the loss at the absorbing walls in Fig. 13. The latter loss is the source of external vacuum fluctuations. The field goes to zero on the surface of the enclosing box, a property that is used to establish mode orthogonality. The absorbing walls are necessary to avoid reflections and to ensure a continuous energy spectrum in the box, which mimics an infinite space. This equation provides a complete description of the laser in the steady state, where $g(\mathbf{x})$ can be considered independent of time.

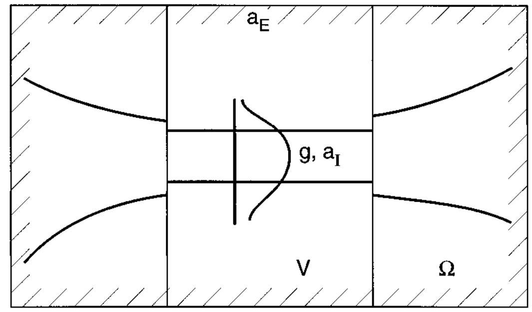


FIG. 13. Laser geometry. The laser cavity has a volume V between the facets and is embedded in a larger volume Ω, where the field goes to zero at the boundary. The absorber with coefficient a_E prevents reflections from the walls enclosing Ω. It is also a source of external vacuum fluctuations. The coefficients g and a_I are the gain and attenuation internal to the laser. These sources result in spontaneous emission and the emission of vacuum fluctuations, respectively.

Let us expand $\hat{A}(\mathbf{x},t)$ into modes by first considering the homogeneous version of this equation for mode $\Phi_n(\mathbf{x})\exp[-i(\tilde{\omega}_n-\omega_S)t]$ satisfying

$$\nabla^2\Phi_n(\mathbf{x})+\left(k^2-ik(g(\mathbf{x})-a_I(\mathbf{x})-a_E(\mathbf{x}))+\frac{dk^2}{d\omega}(\mathbf{x})(\tilde{\omega}_n-\omega_S)\right)\Phi_n(\mathbf{x})=0, \tag{8.1a}$$

$$\tilde{\omega}_n\equiv\omega_n+i\frac{\Delta G_n}{2}, \tag{8.1b}$$

where ω_n is the mode frequency, ω_S is a typical optical frequency, and ΔG_n is the gain or loss of the nth mode. At threshold, $\Delta G_n=0$, and below threshold, $\Delta G_n<0$. We obtain an orthogonality condition by multiplying the equation for Φ_n [Eq. (6.1)] by $\Phi_m(\mathbf{x})$ and subtracting a similar equation for $\Phi_m(\mathbf{x})$ multiplied by Φ_n. The mode fields go to zero at the boundary of the enclosing box of volume Ω, see Fig. 13. We obtain $(\tilde{\omega}_n-\tilde{\omega}_m)(dk^2/d\omega\Phi_n\Phi_m)_\Omega=0$, where we are abbreviating the integral over Ω, the total space of the box in Fig. 13, as $\int_\Omega d\mathbf{x}F(\mathbf{x})\equiv(F)_\Omega$. It follows that the mode orthogonality relation is

$$\left(\frac{dk^2}{d\omega}\Phi_n\Phi_m\right)_\Omega=\delta_{nm}\left(\frac{dk^2}{d\omega}\Phi_n^2\right)_\Omega. \tag{8.2}$$

Let us assume that $\Phi_n(\mathbf{x})$ are a complete set of modes and expand $\hat{A}(\mathbf{x},t)$ as

$$\hat{A}(\mathbf{x},t)=\sum_n \hat{A}_n\equiv\sum_n D_n\hat{b}_n(t)\Phi_n(\mathbf{x}), \tag{8.3}$$

where D_n are constants to be determined. Substituting this expansion into the wave equation [Eq. (6.9)], we find

$$\sum_n i\frac{dk^2}{d\omega}D_n\left[\frac{d\hat{b}_n}{dt}-\left(\omega_n+i\frac{\Delta G_n}{2}-\omega_S\right)\hat{b}_n\right]\Phi_n(\mathbf{x})$$
$$=-\frac{4\pi}{c}\hat{J}(\mathbf{x},t). \tag{8.4}$$

Multiplying this equation by $\Phi_0(\mathbf{x})$, integrating, and using the mode-orthogonality relation to select this mode, we find the rate equation for the mode amplitude

$$\frac{d\hat{b}_0}{dt}=\left[-i(\omega_0-\omega_S)+\frac{\Delta G_0}{2}\right]\hat{b}_0+\hat{F}_0(t), \tag{8.5}$$

where

$$\hat{F}_0(t)=\frac{4\pi i}{cD_0}\frac{(\hat{J}(\mathbf{x},t)\Phi_0)_\Omega}{\left(\frac{dk^2}{d\omega}\Phi_0^2\right)_\Omega} \tag{8.6}$$

is the Langevin force for the mode Φ_0. The contribution to the integral in the denominator from the volume outside the cavity is nearly negligible because of the oscillatory nature of the field propagating outside the cavity, whereas the field inside the cavity contains standing waves for which Φ_0^2 does not average to zero.

We can determine D_0 by requiring that the photon number within the laser cavity be equal to $\hat{b}_0^\dagger\hat{b}_0$. Using Eq. (8.3) to express $\hat{A}_0$ in terms of $\hat{b}_0$,

$$\hat{P}_0=\frac{\left(\frac{dk^2}{d\omega}\hat{A}_0^\dagger\hat{A}_0\right)_V}{4\pi\hbar}=\hat{b}_0^\dagger\hat{b}_0, \tag{8.7}$$

whence

$$D_0^2=\frac{4\pi\hbar}{\left(\frac{dk^2}{d\omega}|\Phi_0|^2\right)_V}. \tag{8.8}$$

The subscript V indicates an integral over the volume within the facets of the laser in Fig. 13.

The correlation functions for $\hat{F}_0(t)$ are found using the fluctuation-dissipation theorem [Eqs. (4.42) and (4.57)]. Using $dk^2/d\omega=2k/v_g$, where v_g is the group velocity, these reduce to

$$\langle\hat{F}_0(t_1)\hat{F}_0^\dagger(t_2)\rangle=K(\overline{gv_gn_{\mathrm{sp}}}+\overline{a_Iv_g}+\overline{a_Ev_g}-\overline{gv_g})\times\delta(t_1-t_2), \tag{8.9a}$$

$$\langle F_0^\dagger(t_2)F_0(t_1)\rangle=K\overline{gv_gn_{\mathrm{sp}}}\delta(t_1-t_2), \tag{8.9b}$$

where the bar represents the average of a quantity over the lasing mode

$$\bar{f}\equiv\frac{\left(\frac{dk^2}{d\omega}f|\Phi_0|^2\right)_\Omega}{\left(\frac{dk^2}{d\omega}|\Phi_0|^2\right)_V}, \tag{8.10}$$

and K, the Petermann enhancement factor (Petermann, 1979; Wang *et al.*, 1987), is

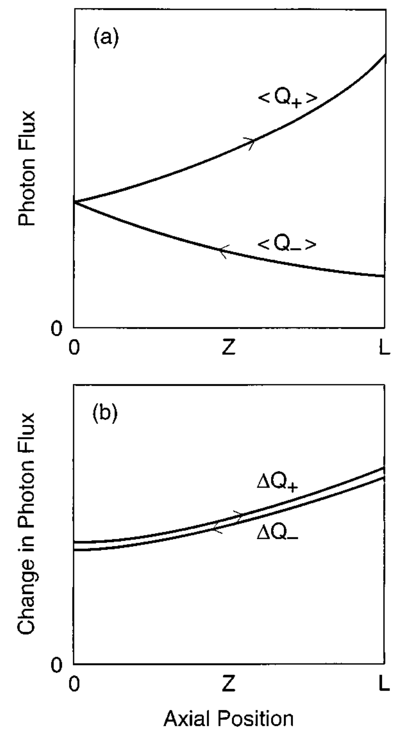


FIG. 14. The z dependence of the steady-state values and the fluctuations of the photon flux in a Fabry-Perot cavity that is open at one end: (a) The steady-state values, $\langle Q_+\rangle$ and $\langle Q_-\rangle$. The growth of flux with propagation is due to single-pass amplification that is needed to overcome facet losses. (b) Fluctuations $\langle\Delta Q_+\rangle$ and $\langle\Delta Q_-\rangle$, which illustrates the extreme case of nearly complete suppression of intensity fluctuations. The fluctuations ΔQ_+ and ΔQ_- are nearly equal, and the net flux emitted at the right facet is negligible. This limit is approached for the low-frequency fluctuations of a semiconductor laser with low internal loss and low current noise, which is operating far above threshold. Such a laser exhibits sub-Poissonian light intensity fluctuations. It has very different spatial distributions of the steady-state photon flux and fluctuations in photon flux.

$$K\equiv\frac{\left(\frac{dk^2}{d\omega}|\Phi_0|^2\right)_V^2}{\left|\left(\frac{dk^2}{d\omega}\Phi_0^2\right)_\Omega\right|^2}. \tag{8.11}$$

The Petermann enhancement factor is greater than 1 for open cavities. For closed cavities with uniform gain and loss, $\Phi_0(\mathbf{x})$ can be taken as real and then $K=1$.

At laser threshold, the net amplification associated with a single round trip is very close to unity. In an open cavity, end losses are made up by single-pass amplification. This is illustrated in Fig. 14(a). We mentioned at the start of Sec. VI.C that in describing lasers we would only be concerned with a relatively narrow spectrum of frequencies, which is small compared to the spectral width of the material gain and loss. Within this band of frequencies, the gain can be regarded as constant and enhancing noise. Any contribution to the laser noise field, such as spontaneous emission within the cavity or vacuum fluctuations entering the cavity at the facet, will be enhanced by single-pass amplification. This is the enhancement associated with the Petermann factor. We discuss the K factor further in Sec. VIII.E.

We can show that, as threshold is approached, the last three terms in Eq. (8.9a) cancel.

$$\overline{gv_g}-\overline{a_I v_g}-\overline{a_E v_g}=0. \tag{8.12}$$

The rate $\overline{a_E v_g}$ is proportional to the rate at which the flux is absorbed by the distant walls. The rate $\overline{gv_g}-\overline{a_I v_g}$ is proportional to the net rate of stimulated emission leaving the cavity. In addition to this flux, there is the contribution of spontaneous emission leaving the cavity. When the laser power significantly exceeds the rate of spontaneous emission into the mode, Eq. (8.12) is satisfied.

Therefore the sum of the last three terms in the antinormally ordered correlation function goes to zero as the laser power increases and spontaneous emission becomes a negligible contribution to the photon flux leaving the cavity. Thus the Langevin force correlation functions (8.9) become order independent as laser power increases. The steady-state laser field, which has its source in the Langevin forces, becomes classical as the photon number becomes large. Just as in the case of optical amplifiers, the noncommuting contributions to noise associated with spontaneous emission and vacuum fluctuations cancel as the photon number increases. This holds regardless of whether the source of vacuum fluctuations is internal loss or comes from the absorbing boundary in Fig. 13. A similar result was obtained for a closed laser cavity in Sec. IV.H.

B. Photon-number distribution for a nonlasing mode

The operator for the number of photons in a mode 0 is given by $\hat{P}_0=\hat{b}_0^\dagger(t)\hat{b}_0(t)$ [Eq. (8.7)]. We shall now work out the moments of $\hat{P}_0$. Our results apply only to nonlasing modes and to the average photon number for a mode above threshold. For the lasing mode, the reaction of the carriers to fluctuations in $\hat{P}_0$ must be included. This suppresses the low-frequency photon-number fluctuations and greatly alters the photon-number distribution (see the review of Liu, 1991; see also Hempstead and Lax, 1967; Risken and Vollmer, 1967; Henry *et al.*, 1984).

The field amplitude $b_0(t)$ can be found by solving the Langevin rate equation [Eq. (8.5)]. In doing this, we shall drop $(\omega_0-\omega_S)$. This term merely changes the phase of $\hat{F}_0(t)$ and does not affect the correlation functions. The solution of Eq. (8.5) is

$$\hat{b}_0(t)=\int_{-\infty}^{t} du\,\hat{F}_0(u)e^{\Delta G_0(t-u)/2}, \tag{8.13}$$

where ΔG_0 is less than zero for a mode below threshold and also for the steady-state lasing mode. The correlation functions $\langle\hat{b}_0(t)\hat{b}_0^\dagger(t)\rangle$ and $\langle\hat{b}_0^\dagger(t)\hat{b}_0(t)\rangle$ can be calculated using this equation and the Langevin force correlation functions [Eq. (8.9)]. In the approximation that Eq. (8.12) is satisfied for modes near threshold, the Langevin force correlation functions are order independent and

$$\langle\hat{P}_0\rangle=\langle\hat{b}_0^\dagger(t)\hat{b}_0(t)\rangle=\langle\hat{b}_0(t)\hat{b}_0^\dagger(t)\rangle=-K\frac{\overline{gv_g n_{\mathrm{sp}}}}{\Delta G}. \tag{8.14}$$

[A more careful calculation, taking into account that the steady-state operating point of the laser occurs with gain slightly below loss, would result in a small difference between $\langle\hat{b}_0(t)\hat{b}_0^\dagger(t)\rangle$ and $\langle\hat{b}_0^\dagger(t)\hat{b}_0(t)\rangle$.]

The higher-order moments can immediately be calculated from Wick's theorem. For example,

$$\langle\hat{P}^2\rangle=\langle\hat{b}_0(t)^\dagger\hat{b}_0(t)\hat{b}_0(t)^\dagger\hat{b}_0(t)\rangle,$$
$$=\int_{-\infty}^{t}du_1\int_{-\infty}^{t}du_2\int_{-\infty}^{t}du_3\int_{-\infty}^{t}du_4 e^{\Delta G(t-u_1)/2+\Delta G(t-u_2)/2+\Delta G(t-u_3)/2+\Delta G(t-u_4)/2}\langle\hat{F}_0^\dagger(u_1)\hat{F}_0(u_2)\hat{F}_0^\dagger(u_3)\hat{F}_0(u_4)\rangle. \tag{8.15}$$

Applying Wick's theorem, we expand the correlation function of the $\hat{F}_0$'s as the sum of all possible combinations of averages of pairs of $\hat{F}_0$'s. This results in a corresponding expansion of the $\hat{b}_0$'s:

$$\langle\hat{b}_0^\dagger\hat{b}_0\hat{b}_0^\dagger\hat{b}_0\rangle=\langle\hat{b}_0^\dagger\hat{b}_0\rangle\langle\hat{b}_0^\dagger\hat{b}_0\rangle+\langle\hat{b}_0^\dagger\hat{b}_0\rangle\langle\hat{b}_0\hat{b}_0^\dagger\rangle. \tag{8.16}$$

Using the order independence of the correlation functions, we find

$$\langle P_0^2\rangle=2\langle b_0^+b_0\rangle^2=2\langle P_0\rangle^2. \tag{8.17}$$

Applying the same arguments to higher moments, we find

$$\langle\hat{P}_0^n\rangle=n!\langle\hat{P}_0\rangle^n. \tag{8.18}$$

These are just the moments of an exponential probability distribution $\mathcal{P}(P_0)$ for photon number P_0 within the cavity:

$$d\mathcal{P}(P_0)=\frac{1}{\langle\hat{P}_0\rangle}\exp\left(-\frac{P_0}{\langle\hat{P}_0\rangle}\right)dP_0. \tag{8.19}$$

The exponential distribution for nonlasing modes is well known. It was observed and interpreted in early studies of gas lasers (Freed and Haus, 1966). The exponential distribution has an extensive tail compared to the Gaussian distribution normally encountered in noise theory. Consequently, weak nonlasing modes on rare occasions grow to intensities much greater than their average values, e.g., the probability of $P_0=20\langle\hat{P}_0\rangle$ is about 10^{-9}. This can result in occasional redistributing of energy between the lasing and nonlasing modes, known as mode-partition noise. This noise can cause errors in high bit rate transmission along dispersive optical fibers. The lasing and nonlasing modes differ in optical frequency and consequently have different delays in transmission, resulting in digital transmission errors. (See the review article of Liu, 1991; see also Henry *et al.*, 1984; Link *et al.*, 1985.) This has necessitated the use of distributed feedback lasers in which the nonlasing modes are highly suppressed.

The exponential distribution is easily understood on classical grounds. In Sec. VII.B, we showed that a field emitted from a spontaneous current source will have Gaussian distributions of its real and imaginary components. The superposition of fields from several sources will also be Gaussian distributed. Under these circumstances, the modulus squared of the field has an exponential distribution (Henry *et al.*, 1984). In a classical description, this squared modulus is proportional to the photon number.

C. Bose statistics of photons interacting with electrons

Photons in a closed lossless cavity interacting with electrons come into equilibrium with the carriers of a semiconductor, as discussed after Eq. (4.47). The photons should have a thermal distribution of a Bose gas (Landau and Lifshitz, 1958b). We shall show that this is a consequence of our theory.

For a closed lossless cavity with uniform gain g, $K\rightarrow 1$, $\Delta G\rightarrow gv_g$, and $\overline{gv_g n_{\rm sp}}=-gv_g\bar{n}_\omega$. With these changes, the average photon number [Eq. (8.14)] reduces to

$$\langle P_0\rangle=\langle b_0^\dagger(t)b_0(t)\rangle=\bar{n}_\omega. \tag{8.20}$$

By repeating the steps leading to Eq. (8.14) with $\overline{a_I v_g}=0$, we find the antinormally ordered correlation function to be

$$\langle b_0(t)b_0^\dagger(t)\rangle=\bar{n}_\omega+1, \tag{8.21}$$

where $\bar{n}_\omega$ is given by Eq. (4.47).

The higher moments of photon numbers can be calculated as in the last section, using Wick's theorem and the two correlation functions [Eqs. (8.20) and (8.21)]. We find

$$\langle P_0^2\rangle=2\bar{n}_\omega{}^2+\bar{n}_\omega, \tag{8.22}$$

$$\langle P_0^3\rangle=6\bar{n}_\omega{}^3+6\bar{n}_\omega{}^2+\bar{n}_\omega. \tag{8.23}$$

These are exactly the moments expected for a Bose gas. For such a gas, the probability of a mode's containing n photons is (Landau and Lifshitz, 1958b)

$$\mathcal{P}(n)=(1-e^{-(\hbar\omega-e\mathcal{V})/kT})e^{-n(\hbar\omega-e\mathcal{V})/kT}. \tag{8.24}$$

It is readily shown for such a distribution that

$$\langle P_0\rangle=\bar{n}_\omega, \tag{8.25a}$$

$$\langle P_0(P_0-1)\rangle=2!\bar{n}_\omega{}^2, \tag{8.25b}$$

$$\langle P_0(P_0-1)(P_0-2)\rangle=3!\bar{n}_\omega{}^3,\ \text{etc.} \tag{8.25c}$$

From this, $\langle P_0^2\rangle$ and $\langle P_0^3\rangle$ [Eqs. (8.22) and (8.23)] can be calculated.

Of course, the Bose distribution for thermal photons is well known, but here we have established it for photons interacting with electrons of a semiconductor having two bands with separated quasi-Fermi levels. Our result that averages of photon creation and annihilation operators can be found using Wick's theorem was shown by Louisell (1974) for noninteracting bosons in thermal equilibrium.

D. Average rate of spontaneous emission

Here we derive a formula for the average rate of spontaneous emission per unit volume and apply it to calculate the average rate of spontaneous emission into a laser mode. The spontaneous-emission rate depends on a pair of operators, and therefore the ordering of the operators is significant. We shall show that the same result is obtained regardless of which operator order is used.

We start with the rate equation for optical transitions of carriers, which was shown to be order independent in Appendix A. Let us begin with the normally ordered rate equation (6.11), with a replaced by $-g$,

$$\frac{\partial\hat{\mathcal{N}}_N}{\partial t}=-\frac{gk}{2\pi\hbar}\hat{A}^\dagger\hat{A}+\frac{i}{\hbar c}(\hat{J}^\dagger\hat{A}-\hat{A}^\dagger\hat{J}). \tag{8.26}$$

We can rewrite the first term in this equation as the photon density given by Eq. (6.14). We are interested only in average values, so we take the average of the resulting equation,

$$\left\langle\frac{\partial\hat{\mathcal{N}}_N}{\partial t}\right\rangle=-gv_g\langle\hat{\mathcal{P}}_N\rangle+\frac{i}{\hbar c}\langle\hat{J}^\dagger\hat{A}-\hat{A}^\dagger\hat{J}\rangle. \tag{8.27}$$

The only contribution of the field $\hat{A}(\mathbf{x},t)$ to $\langle\hat{J}^\dagger\hat{A}\rangle$ is the field emitted by the spontaneous current $\hat{J}(\mathbf{x},t)$, so the second term in Eq. (8.27) is independent of optical power. In his famous paper on spontaneous and stimulated emission, Einstein divided the optical-transition rate into a stimulated rate proportional to the average photon density and a rate of spontaneous emission that is independent of photon density (Einstein, 1917; summarized by Pais, 1982). Following this approach, we interpret the second term in Eq. (8.27) as the spontaneous rate.

We can repeat the argument leading to Eq. (8.27) beginning with the antinormally ordered carrier rate [Eq. (A2b)]:

$$\left\langle \frac{\partial \hat{\mathcal{N}}_A}{\partial t} \right\rangle = -g v_g \langle \hat{\mathcal{P}}_A \rangle + \frac{i}{\hbar c} \langle \hat{A} \hat{J}^\dagger - \hat{J} \hat{A}^\dagger \rangle. \tag{8.28}$$

This equation can be expressed in terms of the average photon-number density $\langle \hat{\mathcal{P}}_N \rangle$ and an additional term that is proportional to the average energy density of the uncertainty-related field fluctuations:

$$\left\langle \frac{\partial \hat{\mathcal{N}}_A}{\partial t} \right\rangle = -g v_g \langle \hat{\mathcal{P}}_N \rangle - g v_g \langle \hat{\mathcal{P}}_A - \hat{\mathcal{P}}_N \rangle + \frac{i}{\hbar c} \langle \hat{A} \hat{J}^\dagger - \hat{J} \hat{A}^\dagger \rangle. \tag{8.29}$$

The average uncertainty-related energy density $\langle \hat{\mathcal{P}}_A - \hat{\mathcal{P}}_N \rangle$ is independent of optical power [see Sec. VI.D and Fig. 6(b)], so it contributes to the spontaneous-emission rate. Therefore we identify the sum of the last two terms of this equation to be equal to the average spontaneous-emission rate $\langle \partial \hat{\mathcal{N}}_{\mathrm{spont}} / \partial t \rangle$. Since $\langle \partial \hat{\mathcal{N}}_N / \partial t \rangle = \langle \partial \hat{\mathcal{N}}_A / \partial t \rangle$, we have two expressions for $\langle \partial \hat{\mathcal{N}}_{\mathrm{spont}} / \partial t \rangle$, which must be equal:

$$\left\langle \frac{\partial \hat{\mathcal{N}}_{\mathrm{spont}}}{\partial t} \right\rangle = \frac{i}{\hbar c} \langle \hat{J}^\dagger \hat{A} - \hat{A}^\dagger \hat{J} \rangle, \tag{8.30a}$$

$$\left\langle \frac{\partial \hat{\mathcal{N}}_{\mathrm{spont}}}{\partial t} \right\rangle = -g v_g \langle \hat{\mathcal{P}}_A - \hat{\mathcal{P}}_N \rangle + \frac{i}{\hbar c} \langle \hat{A} \hat{J}^\dagger - \hat{J} \hat{A}^\dagger \rangle. \tag{8.30b}$$

We evaluate these expressions for $\langle \partial \hat{\mathcal{N}}_{\mathrm{spont}} / \partial t \rangle$ in Appendix E and show that both equations give the same result:

$$\left\langle \frac{\partial \hat{\mathcal{N}}_{\mathrm{spont}}}{\partial t}(\mathbf{x}) \right\rangle = \frac{2 g k n_{\mathrm{sp}}}{\pi} \int d\omega \; \mathrm{Im}[G_\omega(\mathbf{x},\mathbf{x})]. \tag{8.31}$$

Both the gain g and n_{sp} change sign when population inversion occurs, but the product $g n_{\mathrm{sp}} = r_d$ is always positive and equal to the contribution made by downward transitions to the gain g; see Eqs. (4.48) and (4.51).

The spontaneous-emission rate is unambiguous, but ambiguity occurs in trying to separate the contributions to spontaneous emission. Consider a set of carriers occupying only upper energy levels, so $n_{\mathrm{sp}}=1$. Then the antinormally ordered average of a pair of spontaneous current operators is zero and, consequently, so is the second term in Eq. (8.30b). Then, according to Eq. (8.30a), spontaneous emission is due to emission by spontaneous currents while, according to Eq. (8.30b), it is induced by the uncertainty-related field fluctuations.

What is the physical difference in the formulation of the carrier rate equation with different operator orders? Normal ordering of electromagnetic energy flux does not include the uncertainty-related field fluctuations while antinormal ordering includes it. As we have just shown, the spontaneous-emission rate is the same, regardless of operator order.

The confusing physical interpretation comes from trying to interpret spontaneous emission as only due to one of two sources: uncertainty-related field fluctuations or spontaneous currents. Both sources are present whenever spontaneous emission occurs. Both sources depend on noncommuting operator pairs. Only their combination results in order-independent and reasonable physical results. All that we can say is that spontaneous emission has two sources: emission from spontaneous currents and emission induced by the uncertainty-related field fluctuations. The dependence of the spontaneous-emission rate on operator order was discussed by Milonni (1994), who reached a similar conclusion.

Dalibard *et al.* (1982, 1984), working in atomic physics, have been able to determine how much each source of quantum-mechanical fluctuations contributes to spontaneous emission and to radiative corrections, such as the Lamb shift. They did this by constructing unique Hermitian operators for the changes associated with each source that are composed of Hermitian combinations of the field operators and of the atomic variables. They refer to the two sources as "radiative reaction," the interaction of an electron with its own electromagnetic field, and vacuum fluctuations. Energy loss of an atom by radiative reaction appears to be similar or identical to what we describe as spontaneous emission by spontaneous currents. They calculate the average rate of energy loss to the electromagnetic field and find equal contributions from both sources when the electron is in an excited state. When the electron is in the ground state, the rate of energy loss to the field induced by vacuum fluctuations is negative (absorption of energy by the atom) and the two contributions cancel, assuring stability of this state. Instability of the ground state and other inconsistencies occur if one of the two sources of quantum fluctuations is neglected.

We also find this cancellation, as discussed after Eq. (E4). The cancellation is necessary to explain two puzzling aspects of the interaction of light and matter: an accelerating electron in the ground state of an atom does not lose energy to radiation; and the optical frequency field of vacuum fluctuations will not excite an electron out of its ground state. We can make the interpretation that both of these effects take place together so that energy transfer is cancelled. The energy of the field and of the atom remain constant, but these changes cause the vacuum field to become uncorrelated during propagation in an absorbing medium, as discussed in Appendix B.

The average rate of spontaneous emission into a laser cavity mode near threshold can be found by expanding the Green's function as a sum over modes. This is done in Appendix F. Each mode near threshold makes a resonant contribution to the Green's function and to the rate of spontaneous emission given by Eq. (8.31). Evaluation of this equation in Appendix E shows that the rate of spontaneous emission into mode 0 is

$$\left\langle \frac{d\hat{N}_{\mathrm{spont}}}{dt} \right\rangle = -\mathrm{Re}\left[\frac{\left(g v_g n_{\mathrm{sp}} \dfrac{dk^2}{d\omega} \Phi_0^2 \right)_\Omega}{\left(\dfrac{dk^2}{d\omega} \Phi_0^2 \right)_\Omega} \right], \tag{8.32}$$

where $(\cdots)_\Omega$ is an integral over the space enclosing the laser (see Fig. 13).

The integral over Φ_0^2 has significant contributions only within the laser cavity, where there are standing waves. Thus Eq. (8.32) is essentially an average of $gv_g n_{sp}$ over the laser cavity. Notice that $\langle d\hat{N}_{spont}/dt\rangle$ has no Petermann enhancement [defined in Eq. (8.11)]. This conclusion holds for either index-guided or gain-guided laser cavities. This conclusion was also reached by Deutch *et al.* (1991) by different methods.

The contributions to spontaneous emission go to zero at the nodes of Φ_0. This can be understood with the aid of Fig. 15 showing the paths of single-pass propagation in a Fabry-Perot laser cavity. Consider emission by an atom near the perfectly reflecting mirror or cavity facet. The spontaneous current radiates in both positive and negative directions along z. After reflection, the two emitted fields interfere. This interference results in a cancellation of spontaneous emission into the mode from positions of nodes. The modification of spontaneous emission by laser cavities has been studied by Yamamoto *et al.* (1991).

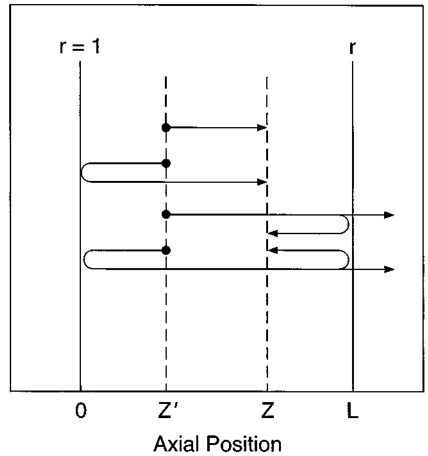


FIG. 15. Single-pass amplification of spontaneous emission in a Fabry-Perot cavity. The propagation paths of spontaneous emission at z' contributing to the field at z are shown. These single-pass paths do not include feedback due to round-trip propagation. In an open cavity, where the front facet reflectivity r is small compared to unity, the single-pass propagation is accompanied by amplification and leads to enhancement of the noise field described by the Petermann factor K. The existence of multiple paths of spontaneous emission explains the weighting of the rate of spontaneous emission into the mode by the square of the mode field. For example, waves emitted from nodal points in the forward and backward directions interfere destructively and cancel.

E. Interpretation of the Petermann factor

In the first semiconductor lasers used in optical communications, the optical field was confined in only one of the two transverse directions. Such lasers are called "gain-guided" because the diffraction losses of the optical field are made up for by gain. The spectra of these lasers are composed of many longitudinal modes. Gain-guided lasers were later supplanted by "index-guided" lasers, in which the transverse optical mode is completely confined. Surprisingly, index-guided lasers have only one or two intense longitudinal modes in their optical spectra.

This spectral change was explained by Petermann (1979) and Streifer *et al.* (1982). In a classical calculation, Petermann showed that emissions from point sources into the fundamental transverse mode are enhanced by K, given by Eq. (8.11), when the waveguide is gain guided. He interpreted this as an enhancement of spontaneous emission. Streifer *et al.* showed that the number of prominent modes in the spectrum is very sensitive to the value of K. This is because mode intensity is determined by very small differences in loss and gain $(\gamma_0 - g_0)$; see Eq. (8.37). When K increases, the difference of $\gamma_0 - g_0$ to achieve a given laser power increases. Then the relative differences in this parameter for different modes diminish and so does the mode selectivity. Later, Petermann's group established that the same enhancement in photon number by K occurs in index-guided Fabry-Perot lasers with low-reflecting facets (Wang *et al.*, 1987). It was also found that other noise phenomena, such as laser linewidth, are enhanced by K (Ujihara, 1984; Arnaud, 1986; Henry, 1986a; Hammel and Woerdman, 1989; Goldberg *et al.*, 1991; van Exeter *et al.*, 1991; Prasad, 1992).

The explanations of these phenomena have been of a mathematical nature. The K factor has been linked to wave-front curvature in gain-guided lasers (Petermann, 1979; Agrawal, 1984) and to the non-Hermitian nature of the electromagnetic wave interacting with the amplifying medium (Arnaud, 1983; Hammel and Woerdman, 1989; Siegman, 1989a, 1989b).

A more physical explanation for index-guided Fabry-Perot lasers was given by Goldberg *et al.* (1991), who attributed the K factor to single-pass amplification (see also Henry, 1991, Sec. 2.5.2). We believe that this is the correct explanation and that it applies to both index-guided and gain-guided lasers. Goldberg *et al.* (1991) also suggest this is the case. In gain-guided lasers, there is lateral loss, which is made up by single-pass lateral amplification. The Petermann factor can be large in gain-guided lasers. One may find it hard to understand how there could be much single-pass amplification as light laterally crosses a distance of typically only ≈ 5 μm. However, the light in a gain-guided laser is actually traveling at nearly glancing incidence and thus goes a long distance along the axis of the waveguide while crossing it laterally.

Although the K factor is referred to as an excess spontaneous-emission factor, we concluded in Sec. VIII.D that spontaneous emission is not enhanced by the K factor. The enhancement is actually single-pass amplification of the noise field following spontaneous emission. This amplification is broadband and in some ways mimics enhanced spontaneous emission.

In the remainder of this section, we demonstrate that this is the case by calculating the average photon number in an index-guided Fabry-Perot cavity and show that, when there is low facet reflectivity, the photon number is enhanced by K. Our calculation shows explicitly that this enhancement comes from single-pass amplification.

Consider an index-guided Fabry-Perot laser, shown in Fig. 15, that is open at one end. We shall take the mode propagation constant k_0, the modal gain g_0, the group velocity v_g, and $n_{\rm sp}$ all to be uniform along the length of the cavity. We shall also neglect internal losses. The threshold condition is

$$1=re^{g_0/L}\equiv e^{(g_0-\gamma_0)L}, \tag{8.33}$$

where γ_0 is the gain at threshold. The field at threshold is given by

$$\Phi_0(\mathbf{x})=\phi_0(x)[e^{ik_0+\ \gamma_0/2\ z}+e^{-(ik_0+\ \gamma_0/2)z}]. \tag{8.34}$$

The Petermann factor is

$$K=\frac{\left(|\Phi_0|^2\frac{d^2k}{d\omega}\right)_V^2}{\left|\left(\Phi_0^2\frac{d^2k}{d\omega}\right)_\Omega\right|^2}$$

$$=\frac{[\int_0^L(e^{\gamma_0 z}+e^{-\gamma_0 z}+\text{osc. terms})dz]^2}{|\int_0^L(2+\text{osc. terms})dz|^2}. \tag{8.35}$$

The oscillating terms contribute very little to the integrals. Neglecting them,

$$K=\left(\frac{e^{\gamma_0 L}-e^{-\gamma_0 L}}{2\gamma_0 L}\right)^2. \tag{8.36}$$

Note that for a closed cavity $\gamma_0\rightarrow 0$ and $K\rightarrow 1$.

The average photon number $\langle P_0\rangle$ [Eq. (8.14)], with $\Delta G=(g_0-\gamma_0)v_g$, is

$$\langle P_0\rangle=K\frac{g_0 n_{\rm sp}}{\gamma_0-g_0}. \tag{8.37}$$

To understand the physical origin of K, we can calculate $\langle P_0\rangle$ more directly, beginning with Eq. (6.14) defining the photon-number density. Writing this in terms of the frequency components of the fields, we find

$$\langle\hat{P}\rangle=\frac{1}{8\pi^2\hbar}\int_0^\infty d\omega\int_0^\infty d\omega'\left(\frac{dk^2}{d\omega}\langle\hat{A}_\omega{}^\dagger\hat{A}_{\omega'}\rangle\right)_V e^{-i(\omega-\omega')t}. \tag{8.38}$$

Using Eq. (6.7) relating $\hat{A}_\omega(\mathbf{x})$ to $\hat{j}_\omega(\mathbf{x}')$ by $G_\omega(\mathbf{x},\mathbf{x}')$ and the correlation function (4.38), we find

$$\langle\hat{P}\rangle=\int_0^\infty\frac{d\omega}{\pi}\int d\mathbf{x}\int d\mathbf{x}'\frac{dk^2(\mathbf{x})}{d\omega}\times|G_\omega(\mathbf{x},\mathbf{x}')|^2k(\mathbf{x}')g(\mathbf{x}')n_{\rm sp}(\mathbf{x}'). \tag{8.39}$$

Writing the Green's function in the form

$$G_\omega(\mathbf{x},\mathbf{x}')=g(z,z')\phi_0(x)\phi_0(x') \tag{8.40}$$

and using perturbation theory to replace $\int dx\,\phi_0(x)^2kg=k_0g_0$ and $\int dx\,\phi_0(x)^2dk^2/d\omega=dk_0^2/d\omega=2k_0/v_{g0}$ gives

$$\langle\hat{P}\rangle=\frac{2k_0^2g_0n_{\rm sp}}{v_{g0}\pi}\int_0^\infty d\omega\int_0^L dz\int_0^L dz'|g(z,z')|^2. \tag{8.41}$$

For a traveling-wave amplifier, the z-dependent Green's function $g(z,z')$ is given by Eq. (7.4). For the cavity, the Green's function $g(z,z')$ must take into account the various paths of single-pass propagation in going from z' to z (shown in Fig. 15) as well as a factor $[1-r\exp(2ik_0L+g_0L)]^{-1}$ that sums up the contributions from multiple round trips. This method of writing the Green's function as a sum of round-trip propagation in the waveguide has been used by Ackerman and Flynn (1990). For $z>z'$, this procedure results in

$$g(z,z')=\frac{e^{i\tilde{k}(z-z')}+e^{i\tilde{k}(z'+z)}+re^{2i\tilde{k}L-i\tilde{k}(z+z')}+re^{2i\tilde{k}L-i\tilde{k}(z-z')}}{2ik_0(1-r\exp(g_0L+2ik_0L))}, \tag{8.42}$$

where $\tilde{k}=k_0-ig_0/2$ is a complex propagation constant.

The denominator can be approximated at frequencies ω near the lasing mode ω_0 by using Eq. (8.33) to express r in terms of the threshold gain γ_0:

$$[1-r\exp(g_0L+2ik_0L)]$$

$$\approx-\frac{2iL}{v_{g0}}\left(\omega-\omega_0+i\frac{(\gamma_0-g_0)v_g}{2}\right). \tag{8.43}$$

The four terms in the numerator in Eq. (8.42) all have different phase factors. In calculating $|g(z,z')|^2$ in Eq. (8.41), we neglect the products of these terms, which are oscillating with changes in z and z', and also set $re^{2i\tilde{k}L}=1$ and $g_0=\gamma_0$, which are the threshold values of these quantities. With these changes, $\langle\hat{P}\rangle$ of Eq. (8.41) becomes

$$\langle\hat{P}\rangle=\frac{g_0n_{\rm sp}v_{g0}}{8\pi L^2}\int_0^\infty d\omega\int_0^L dz\int_0^L dz'\times\frac{e^{\gamma_0(z-z')}+e^{\gamma_0(z+z')}+e^{-\gamma_0(z+z')}+e^{-\gamma_0(z-z')}}{(\omega-\omega_0)^2+\left(\frac{\gamma_0-g_0}{2}\right)^2v_{g0}^2}. \tag{8.44}$$

The four terms in the integrals over z represent single-pass amplification of spontaneous emission associated with the four paths from z' to z in Fig. 15. The integrand numerator can be factored into $[\exp(\gamma_0 z)+\exp(-\gamma_0 z)][\exp(\gamma_0 z')+\exp(-\gamma_0 z')]$. Then the integrals are all independent of each other. Doing the integration again results in exactly the photon number enhanced by the Petermann factor found earlier and given by Eqs. (8.36) and (8.37). We conclude that the enhancement of $\langle P_0\rangle$ by K is due to single-pass amplification of spontaneous emission.

F. Dirac's formula for spontaneous and stimulated emission

Every textbook covering the quantum theory of radiation includes a discussion of the theory of Dirac (1927) for the rates of optical absorption, spontaneous emission, and stimulated emission. A good discussion is given by Fermi (1932). In this theory, the optical modes are usually chosen as those of a large box containing all of free space. We shall show that our approach, in which we formulate the carrier transition rate in terms of a quantum rate equation with spontaneous current-density sources, reduces to the theory of Dirac for lossless modes.

The average carrier recombination rate is given by Eq. (6.11) integrated over all space and averaged,

$$\left\langle \frac{d\hat{N}}{dt}\right\rangle = -\frac{(gk\langle \hat{A}^\dagger \hat{A}\rangle)_\Omega}{2\pi\hbar} + \frac{i}{\hbar c}(\langle \hat{J}^\dagger \hat{A}\rangle - \langle \hat{A}^\dagger \hat{J}\rangle)_\Omega . \quad (8.45)$$

We can expand $\hat{A}(\mathbf{x},t)$ as a set of orthogonal functions $\Phi_n(\mathbf{x})$ as in Eq. (8.3). For lossless modes, we can choose the spatial functions of the modes $\Phi_n(\mathbf{x})$ as real functions. With this expansion, the first term of Eq. (8.45) reduces to

$$\frac{gk\langle \hat{A}^\dagger \hat{A}\rangle}{2\pi\hbar} = \sum_n \sum_m \frac{\left(gv_g \dfrac{dk^2}{d\omega}\Phi_n\Phi_m\right)_\Omega \langle \hat{b}_n^\dagger \hat{b}_m\rangle}{\left(\dfrac{dk^2}{d\omega}\Phi_n^2\right)_\Omega^{1/2}\left(\dfrac{dk^2}{d\omega}\Phi_m^2\right)_\Omega^{1/2}} . \quad (8.46)$$

In cases of modes in thermal equilibrium or discrete modes having different frequencies, the mode amplitudes are uncorrelated,

$$\langle \hat{b}_n^\dagger \hat{b}_m\rangle = P_n \delta_{nm} , \quad (8.47)$$

where P_n is the number of photons of the nth mode. The second term of Eq. (8.45) was evaluated in Eq. (8.32). Substituting these equations into Eq. (8.45), we find that the average recombination rate reduces to

$$\left\langle \frac{d\hat{N}}{dt}\right\rangle = -\sum_n (\overline{gv_g}P_n + \overline{gv_g n_{\rm sp}}), \quad (8.48)$$

where the bar represents an average over the mode weighted by $dk^2/d\omega\Phi_n^2$ [Eq. (8.10)].

In Sec. IV.D [Eq. (4.26)], we showed that gain g is the difference of contributions of downward and upward stimulated transitions $g=r_d-r_u$. In deriving the fluctuation-dissipation theorem, it was established in Eq. (4.46) that the ratio of the upward and downward rates is given by $(n_{\rm sp}-1)/n_{\rm sp}$. From this it follows that

$$r_d = gn_{\rm sp}, \quad r_u = g(n_{\rm sp}-1). \quad (8.49)$$

Expressing g and $gn_{\rm sp}$ in terms of r_d and r_u, we obtain the recombination rate

$$\left\langle \frac{d\hat{N}}{dt}\right\rangle = -\sum_n [\overline{r_d v_g}(P_n+1) - \overline{r_u v_g}P_n], \quad (8.50)$$

a result consistent with the Dirac theory.

Our derivation began with the normally ordered form of the recombination rate [Eq. (8.45)]. Had we started with the antinormally ordered form, the only change would be that $P_n \to P_n+1$ and $n_{\rm sp}\to n_{\rm sp}-1$ in Eq. (8.48). Of course, this would not alter the recombination rate.

G. Sub-Poissonian intensity fluctuations

The laser field can be regarded as nearly monochromatic and classical. As we discussed in Sec. VII.C, when such a noise free classical signal field beats with the field of vacuum fluctuations, we obtain shot-noise fluctuations in the photon flux. This in fact is expected to occur for very high-frequency fluctuations in the laser output (Yamamoto *et al.*, 1986). However, this picture of laser intensity noise as a beating with vacuum fluctuations assumes that the laser does not react to the vacuum fluctuations outside of the laser cavity. Yamamoto *et al.* (1986) showed that, at low frequencies, the laser does react and this response leads to a suppression of noise below the shot-noise limit.

This phenomenon is known as sub-Poissonian intensity fluctuations. Sub-Poissonian phenomena were recently reviewed by Davidovich (1996) in this journal. Employing energy conservation, we show that, at low frequencies, fluctuations in the photon flux out of the laser follow fluctuations in electrical current. Then we argue that the latter fluctuations are much less than expected for shot noise.

We shall now demonstrate that at low frequencies the fluctuations in the photon flux $Q(t)$ out of the laser cavity follow fluctuations in the electrical current to the diode $C(t)$. To describe the fluctuations in the laser output, we need two equations: an equation for photon-number density and an equation for carrier-number density. The rate equation for the carrier-number density [Eq. (6.11)] and the energy conservation equation [Eq. (6.12)] hold both below and above laser threshold. The latter can be thought of as a rate equation for the photon-number density. The carrier-number density rate equation [Eq. (6.11)] only includes the rate of change due to spontaneous and stimulated emission into the lasing mode. We add other phenomenological rates of change to this description: a density of injected-carrier current $\mathcal{C}$ and the sum of radiative recombination into other modes and nonradiative recombination $\hat{\mathcal{N}}/\tau$, where τ is the lifetime for these processes. We also replace the rate of stimulated emission $2gk\hat{A}^\dagger\hat{A}/(4\pi\hbar)$

with $gv_g\hat{\mathcal{P}}(\mathbf{x},t)$, where $\hat{\mathcal{P}}$ is the photon density, Eq. (6.14). With these changes, the rate equation for the carriers is

$$\frac{\partial\hat{\mathcal{N}}}{\partial t}=\mathcal{C}-\frac{\hat{\mathcal{N}}}{\tau}-gv_g\hat{\mathcal{P}}-\hat{\mathcal{F}}_g, \tag{8.51}$$

and the equation for conservation of energy is

$$\frac{\partial\hat{\mathcal{P}}}{\partial t}=-\nabla\cdot\hat{\mathbf{S}}+gv_g\hat{\mathcal{P}}-av_g\hat{\mathcal{P}}+\hat{\mathcal{F}}_g-\hat{\mathcal{F}}_a, \tag{8.52}$$

where the Langevin forces due to spontaneous current densities $\hat{J}_g$ and $\hat{J}_a$, associated with gain and loss, are

$$\hat{\mathcal{F}}_g(\mathbf{x},t)=\frac{i}{\hbar c}[\hat{A}^\dagger(\mathbf{x},t)\hat{J}_g(\mathbf{x},t)-\hat{J}_g^\dagger(\mathbf{x},t)\hat{A}(\mathbf{x},t)], \tag{8.53a}$$

$$-\hat{\mathcal{F}}_a(\mathbf{x},t)=\frac{i}{\hbar c}[\hat{A}^\dagger(\mathbf{x},t)\hat{J}_a(\mathbf{x},t)-\hat{J}_a^\dagger(\mathbf{x},t)\hat{A}(\mathbf{x},t)]. \tag{8.53b}$$

The signs of these Langevin forces have been chosen to emphasize that the Langevin forces are fluctuations in the rates of stimulated emission and loss.

We make the assumption that $\hat{A}$ entering into the Langevin forces is a steady-state field, which can be treated as a classical field for a laser above threshold. The nonzero correlation functions of the Langevin forces are given by Eq. (5.13).

$$\langle\hat{\mathcal{F}}_g(\mathbf{x},t)\hat{\mathcal{F}}_g(\mathbf{x}',t')\rangle=gv_g(2n_{\mathrm{sp}}-1)\mathcal{P}_0(\mathbf{x},t)\times\delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t'), \tag{8.54a}$$

$$\langle\hat{\mathcal{F}}_a(\mathbf{x},t)\hat{\mathcal{F}}_a(\mathbf{x}',t')\rangle=av_g\mathcal{P}_0(\mathbf{x},t)\,\delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t'), \tag{8.54b}$$

where $\mathcal{P}_0(\mathbf{x},t)$ is the steady-state photon density.

We can regard the rate equations for $\partial\hat{\mathcal{P}}/\partial t$ and $\partial\hat{\mathcal{N}}/\partial t$ as a set of classical equations since the laser field can be treated as classical, and the nonzero Langevin force correlation functions [Eq. (8.54)] are not order dependent.

Let us add the equations for $\partial\mathcal{N}/\partial t$ and $\partial\mathcal{P}/\partial t$. The rates and Langevin forces associated with spontaneous and stimulated emission cancel out, leaving

$$\frac{\partial\mathcal{N}}{\partial t}+\frac{\partial\mathcal{P}}{\partial t}+\nabla\cdot\mathbf{S}+av_g\mathcal{P}+\mathcal{F}_a(t)+\frac{\mathcal{N}}{\tau}=\mathcal{C}. \tag{8.55}$$

If we integrate this equation over the volume V of the laser cavity and replace the integral over $\nabla\cdot\mathbf{S}$ by Q, the photon flux leaving the cavity, then

$$Q=C-av_gP-F_a(t)-\frac{N}{\tau}-\frac{dN}{dt}-\frac{dP}{dt}, \tag{8.56}$$

where C is the current to the laser (in electrons/second).

The average rate of recombination N/τ equals the threshold current C_{th}. Far above laser threshold, $C_{\mathrm{th}}\ll Q$. The fluctuations in N/τ are shot-noise fluctuations in the rate of recombination. We expect these fluctuations to be small compared to the shot noise in Q. We shall neglect them and replace N/τ by its average value C_{th}, the threshold current.

The relaxation oscillations in a laser rapidly damp out transient behavior, keeping P and N at their steady-state values. At low frequencies, dN/dt and dP/dt can be neglected. In practice, this holds for frequencies well below the relaxation-oscillation frequency and damping rate. In a laser with high internal efficiency, the internal loss av_gP+F_a can also be neglected in comparison to Q. This leaves

$$Q(t)\approx C(t)-C_{\mathrm{th}}. \tag{8.57}$$

Thus at frequencies small compared with the relaxation-oscillation damping rate and for operation far above threshold, where the other recombination processes are negligible, the photon flux out of the laser follows fluctuations in the laser current.

We expect that the noise in electrical current will be much less than the shot noise in $Q(t)$. One might think that, when current travels through a resistive wire, the scattering processes associated with resistance would cause ordered electrons entering the wire to emerge completely uncorrelated with shot-noise fluctuations. This is not the case because the Coulomb force between carriers and fixed charge is so strong that electrical neutrality must be maintained, at least for frequencies small compared to the frequencies of plasma oscillations, which are 10^{12} per second for semiconductors and 10^{16} per second for metals. Any scattering process that alters electrical neutrality immediately sets up an electrical potential that reestablishes it. The only effect of the scattering is to cause voltage fluctuations across the resistor. These fluctuations are the Johnson noise (Johnson, 1928) explained by the Nyquist theorem (Nyquist, 1928). Yamamoto *et al.* (1986) have shown that these voltage fluctuations are negligible for lasers having reasonable values of series resistance in the external circuit. Thus the current to the laser has sub-Poissonian fluctuations.

Since current fluctuations are negligible compared to shot noise, so are fluctuations in the net photon flux out of the cavity. For a laser such as shown in Fig. 14, where all the photons emerge from one facet, this sub-Poissonian intensity noise can be observed, provided that nearly all the light leaving the cavity is detected (Yamamoto *et al.*, 1991). Any division or attenuation of the emerging light or internal losses in the cavity will reduce this effect. As we have seen in Sec. VII.C, these attenuating effects act to restore the shot-noise limit. In summary, the sub-Poissonian output is a direct consequence of two effects: the output intensity following the current at frequencies well below those of relaxation oscillations and the negligible noise associated with current fluctuations as a consequence of the quasineutrality of carriers and fixed charge. An explanation of sub-Poissonian intensity noise, similar to that given here, was made by Nilsson (1994).

Traditionally, the Langevin rate equation for the laser-mode amplitude [Eq. (8.5)] together with the rate equation for the carrier number [Eq. (8.51)] (integrated over the cavity) have been used to describe transients of

Fabry-Perot semiconductor lasers. These equations were successfully applied to give simple analytical treatments of relaxation oscillations, relative intensity noise, line broadening, line narrowing with feedback, and other phenomena. However, in some cases, these equations do not correctly describe the spatial distribution of field fluctuations within the cavity. The description of sub-Poissonian intensity noise is an example where this treatment fails.

Observations of sub-Poissonian noise are made with Fabry-Perot lasers that have a high reflecting back facet and a low reflecting output facet. This arrangement allows nearly all of the emitted energy flux to be detected. The low reflecting facet also makes the output flux large compared to internal losses. The spatial distribution of steady-state energy flux for such a laser is plotted in Fig. 14(a), and the spatial distribution of the low-frequency fluctuations in energy flux are plotted in Fig. 14(b). We see that the fluctuation in energy flux has a very different spatial distribution from the steady-state energy flux. This example demonstrates that the rate equation for the mode amplitude [Eq. (8.5)] cannot be used to describe low-frequency fluctuations about the steady state because it assumes a spatial dependence $\hat{A}(\mathbf{x},t) \sim \Phi_0(\mathbf{x})$. This holds for the steady-state field but not for the fluctuations of the field about the steady-state solution, which has a different spatial distribution.

IX. CONCLUSION

There now exist world-wide optical communications networks. The practical point of view of light-wave engineers who deal with generation, amplification, and detection is that light is a fluctuating classical wave. The tools of their description are the classical wave equation and rate equations for carriers to which Langevin noise sources are added. Most noise phenomena of practical interest are adequately described by this procedure. However, the actual physical nature of this noise is often hidden in this description. What is missing is a quantum-mechanical basis for these descriptions that retains the advantages of the classical treatments: the ease of treating open structures having exponentially growing and decaying waves. In this paper, we provide a theoretical basis for a variety of physical phenomena related to the real problems of photonics that has a form as close as possible to the classical description and allows a clear physical picture of the quantum effects left out of the classical descriptions.

A. Summary

We summarize our results with emphasis on the principles governing quantum noise. Tables I and II should help the reader to locate where items are discussed in the paper.

1. Two noise sources: Field uncertainty and spontaneous currents

There are two sources of noise: uncertainty-related field fluctuations and optical-frequency spontaneous currents. In the semiconductor and rare-earth doped glass systems, spontaneous currents are related to momentum fluctuations due to electron localization. In Raman amplifiers, spontaneous currents result from the mixing of lattice fluctuations and the optical field of the pump. Neither of these sources has a classical counterpart. These fluctuations are independent of one another and would exist in the absence of electrical charge.

2. Attenuation and emission of vacuum fluctuations

In reality, light and carriers are coupled. This coupling makes it difficult to separate cleanly the contributions of the two noise sources. For example, while vacuum fluctuations are a consequence of field uncertainty, they also show up as emission from spontaneous currents in a cold opaque absorber. In propagation within a cold absorbing medium, the vacuum fluctuations become uncorrelated with distance. This can be interpreted as simultaneous attenuation and uncorrelated generation without violation of energy conservation. The field of vacuum fluctuations satisfies the operator wave equation and is generated and absorbed like other radiation.

3. Detection of pure field uncertainty noise and pure spontaneous current noise

However, for some specific cases, we can associate the noise with one of the sources. Vacuum fluctuations cannot be detected directly, but they show up in the presence of a signal field. Shot noise in an opaque photodetector is due to the field of vacuum fluctuations beating with the signal field. Because of energy conservation, field fluctuations are detected in an efficient opaque detector without additional noise. In the opposite case of a nearly transparent photodetector, which is insensitive to field fluctuations, the shot noise is due to spontaneous currents within the detector beating with the signal field, which causes fluctuations in the rate of optical absorption.

4. Fluctuation-dissipation theorem for systems in quasithermal equilibrium

To gain physical clarity, we use a model of carriers in a semiconductor. We find that the correlation functions for a pair of spontaneous current operators are related to the coefficient of optical loss (or gain), in agreement with the fluctuation-dissipation theorem. This derivation is an extension of the fluctuation-dissipation theorem to quasithermal equilibrium, where the electron and hole Fermi levels are separated, and population inversion is possible. The fluctuation-dissipation theorem is shown to hold for scattering loss in a waveguide, where effective spontaneous currents result from the scattering of vacuum fluctuations into the waveguide.

5. Field operators satisfy a scalar wave equation with spontaneous current-noise sources

The tools for describing quantum-noise phenomena are the wave equation for the field operator and the equation for the rate of optical transitions. The generation and propagation of noise and signal fields is approximately described by the scalar wave equation for the field operator. This description is especially convenient for dealing with open cavities with exponentially growing and decaying fields. Spontaneous currents enter this equation in the same way as Langevin forces enter the classical wave equation, but here they are well-defined operators. Most discussions of noise fields involve the solution of this equation. This is done to describe emission of radiation and vacuum fluctuations from an opaque source, propagation in amplifiers and attenuators, the formation of a quantum Langevin rate equation for the laser mode, and propagation in a waveguide with scattering loss. This equation can be solved in the same way as a classical wave equation with a source, by the Green's-function method. This solution linearly relates the field operators to the spontaneous current operator sources and allows the calculation of field operator averages in terms of spontaneous current operator correlation functions.

6. Probability distributions of observables from higher-order correlation functions

The semiconductor model provides a means for calculating higher-order correlation functions of spontaneous current operators. They are related to pair correlation functions in a manner reminiscent of Wick's theorem. By calculating a few of the higher-order correlation functions, we are able to draw conclusions about the probability distributions of measured quantities. The real and imaginary parts of the field and of spontaneous currents are Gaussian distributed. Shot noise of an opaque detector is Poisson distributed. The photon-number fluctuations of nonlasing modes near threshold are exponentially distributed, which leads to mode-partition noise. The number of photons in a closed lossless cavity, interacting with carriers in quasithermal equilibrium, is Bose-Einstein distributed.

7. Noise and spontaneous emission in optical transitions

The semiconductor model also provides a general equation for the rate of optical transitions. This equation contains spontaneous currents. Both spontaneous currents and uncertainty-related field fluctuations contribute to spontaneous emission. When only the ground state is occupied, these contributions cancel, ensuring stability of this state in the absence of radiation. In the case of a nearly lossless cavity mode, the rate equation reduces to the well-known formulas of Dirac for the rates of absorption and spontaneous and stimulated emission. A general formula for the average rate of spontaneous emission into a cavity mode is derived that is valid for open cavities with gain or loss. This rate is not enhanced for open cavities by the Petermann factor as once thought. Rather, the enhancement of noise and mode intensities in open cavities is the result of single-pass amplification following spontaneous emission. The Dirac formula and the rate of spontaneous emission are average rates. In addition, there are fluctuations in the rates of optical transitions caused by the beating of spontaneous currents with external optical fields. These fluctuations have the correlation function expected for recombination-generation noise of electrons and photons.

8. Conservation of stored energy

Just as in classical electromagnetic theory, the fields and currents obey an equation of energy conservation. From this equation, we identify operators for the photon flux density and photon-number density. Energy conservation is useful in showing that the low-frequency fluctuations of photon flux of a semiconductor laser follow the fluctuations in electrical current to the laser diode. These fluctuations are sub-Poissonian as a consequence of low electrical current noise. It also follows from energy conservation that the flux of vacuum fluctuations in a cold absorber is conserved. Another conclusion is that an ideal opaque photodetector measures the time-dependent photon flux, without contributing additional noise.

9. Classical limit at high amplification

The uncertainty-related field fluctuations are neither amplified nor attenuated. They become negligible when the amplified noise field becomes large compared to that of vacuum fluctuations, in a spectral range, as occurs in lasers and amplifiers. This means that the field becomes classical. At the same time, the quantum Langevin forces contributing to the steady-state field of a laser also become classical as threshold is approached and the photon number increases; analogously, the noise field of an amplifier becomes classical when the amplification becomes large. In both of these cases, the contributions of spontaneous current emission and vacuum fluctuations combine to give noise sources that can be treated as classical.

B. What is not included in this review

The purpose of this paper has been to derive the basic equations governing noise phenomena from first principles and to illustrate them with some examples. As we mentioned in the Introduction, there is an extensive literature covering the practical aspects of noise in optical communications. One such aspect is transmission noise, which includes the noise of amplification and detection. Another is the noise of the laser source. This includes laser linewidth and phase noise, intensity noise, mode-partition noise, and line narrowing with external feedback. These problems are treated reasonably well from a practical point of view, using phenomenological approaches such as adding Langevin forces to classical rate equations. However, there are also some noise phenomena that are not yet unambiguously explained. These include low-frequency behavior of noise known as $1/f$ noise, chaos and other noisy behavior that are brought about by reflections back into a laser, as well as the physical phenomena leading to rarely occurring errors in digital optical communications. A detailed account of these phenomena would require a separate review article.

We have not covered, moreover, other quantum-noise phenomena not directly related to optical communications, such as parametric amplification and the generation of optical squeezed states.

Theories of laser noise, whether phenomenological or quantum, are based on the assumption that the carriers in each band are in thermal equilibrium and are unaffected by the recombination process. This is the basis of the fluctuation-dissipation theorem and hence of most of the discussion in this paper. We do not know whether this assumption is correct at high rates of stimulated emission. Some observations of laser noise under these conditions, such as the linewidth floor, a halting of line narrowing with increased laser power, may require going beyond the existing theory.

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APPENDIX A: OPERATOR ORDER INDEPENDENCE OF THE ENERGY CONSERVATION EQUATION

The equation for energy conservation (6.12) is composed of normally ordered terms. This equation was derived by subtracting the wave equation for $\hat{A}^\dagger$ multiplied by $\hat{A}$ from the equation for $\hat{A}$ multiplied by $\hat{A}^\dagger$. The equation is normally ordered if $\hat{A}^\dagger$ is kept to the left of $\hat{A}$ and antinormally ordered if $\hat{A}^\dagger$ is kept to the right of $\hat{A}$. Thus we have two energy conservation equations,

$$\nabla\cdot\hat{\mathbf{S}}_N+\frac{\partial\hat{\mathcal{P}}_N}{\partial t}+\frac{\partial\hat{\mathcal{N}}_N}{\partial t}=0, \tag{A1a}$$

$$\nabla\cdot\hat{\mathbf{S}}_A+\frac{\partial\hat{\mathcal{P}}_A}{\partial t}+\frac{\partial\hat{\mathcal{N}}_A}{\partial t}=0, \tag{A1b}$$

where

$$\frac{\partial\hat{\mathcal{N}}_N}{\partial t}=\frac{i}{\hbar c}(\hat{J}^\dagger_{\text{tot}}\hat{A}-\hat{A}^\dagger\hat{J}_{\text{tot}}), \tag{A2a}$$

$$\frac{\partial\hat{\mathcal{N}}_A}{\partial t}=\frac{i}{\hbar c}(\hat{A}\hat{J}^\dagger_{\text{tot}}-\hat{J}_{\text{tot}}\hat{A}^\dagger), \tag{A2b}$$

$$\frac{\partial\hat{\mathcal{P}}_N}{\partial t}=\frac{1}{4\pi\hbar}\frac{dk^2}{d\omega}\frac{\partial}{\partial t}(\hat{A}^\dagger\hat{A}), \tag{A3a}$$

$$\frac{\partial\hat{\mathcal{P}}_A}{\partial t}=\frac{1}{4\pi\hbar}\frac{dk^2}{d\omega}\frac{\partial}{\partial t}(\hat{A}\hat{A}^\dagger), \tag{A3b}$$

and where subscripts N and A indicate normal and antinormal ordering.

The resolution of the ambiguity of having two energy conservation equations is that the terms in each equation are order independent, so the two equations are identical. Consider the carrier-recombination rate. The difference of the two orders can be expressed as a commutator of the field and current density:

$$\frac{\partial(\hat{\mathcal{N}}_A-\hat{\mathcal{N}}_N)}{\partial t}=\frac{i}{\hbar c}([\hat{A},\hat{J}^\dagger_{\text{tot}}]+[\hat{A}^\dagger,\hat{J}_{\text{tot}}]). \tag{A4}$$

In the Schrödinger representation, the operators $\hat{A}_S$ and $\hat{j}^\dagger_S$ act on different wave functions and commute. Going from the Schrödinger to the Heisenberg representation does not alter the equal-time commutation relations, so

$$\frac{\partial\hat{\mathcal{N}}_A}{\partial t}=\frac{\partial\hat{\mathcal{N}}_N}{\partial t}. \tag{A5}$$

We have verified this equality by showing that the lower moments of these two operators are equal.

We can also demonstrate that the time derivative of $\hat{\mathcal{P}}$ is order independent. The difference between $\partial\hat{\mathcal{P}}_A/\partial t$ and $\partial\hat{\mathcal{P}}_N/\partial t$ is

$$\frac{\partial(\hat{\mathcal{P}}_A-\hat{\mathcal{P}}_N)}{\partial t}=\frac{1}{4\pi\hbar}\frac{dk^2}{d\omega}\frac{\partial}{\partial t}[\hat{A}(x,t),\hat{A}^\dagger(x,t)]. \tag{A6}$$

This commutator is found by a canonical quantization procedure. Accordingly, the commutation rules of field operators at equal times may not be zero but are time-independent c numbers (for example, see Babiker and Loudon, 1983; Glauber and Lewenstein, 1991) so that

$$\frac{\partial\hat{\mathcal{P}}_A}{\partial t}=\frac{\partial\hat{\mathcal{P}}_N}{\partial t}. \tag{A7}$$

This equality can also be verified by showing that the lower moments of these two operators are equal.

We conclude that both $\partial\hat{\mathcal{N}}/\partial t$ and $\partial\hat{\mathcal{P}}/\partial t$ are order independent. The difference of the two energy conservation equations (A1) shows that

$$\nabla\cdot\hat{\mathbf{S}}_A-\nabla\cdot\hat{\mathbf{S}}_N=0, \tag{A8}$$

so all terms in the energy equation are order independent.

APPENDIX B: CORRELATION FUNCTION OF THE NOISE FIELD

For a system in equilibrium, the noise-field correlation function is independent of the noise source. It is related to $\text{Im}[G_\omega(\mathbf{x}_1,\mathbf{x}_2)]$, where $G_\omega(\mathbf{x}_1,\mathbf{x}_2)$ is the Green's function determining the field at one point due to a source at another point. This was established generally for a system in thermal equilibrium by Kubo (1966) and Martin (1968). It can be thought of as an alternative formulation of the fluctuation-dissipation theorem. Here we derive this result as a property of the solution of the scalar wave equation with spontaneous current-density sources. Our result is valid for quasiequilibrium, in which the quasi-Fermi levels of the carriers have a constant separation.

Consider a closed system occupying a volume Ω with the field going to zero at the enclosing surface. The surface can have walls coated with absorbers to prevent reflections, as in Figs. 12 and 13. The frequency components of the fields at two points are given by the Green's-function solution of the scalar wave equation (6.7). The solution relating the field to the current density is

$$\hat{A}_\omega(\mathbf{x}_1) = -\frac{4\pi}{c}\int_\Omega G_\omega(\mathbf{x}_1,\mathbf{x})\hat{j}_\omega(\mathbf{x})d\mathbf{x} \tag{B1}$$

and

$$\hat{A}^\dagger_{\omega'}(\mathbf{x}_2) = -\frac{4\pi}{c}\int_\Omega G_{\omega'}(\mathbf{x}_2,\mathbf{x}')^*\hat{j}^\dagger_{\omega'}(\mathbf{x}')d\mathbf{x}'. \tag{B2}$$

The correlation function of these two fields is found by taking the product of these two equations, averaging, and using the fluctuation-dissipation theorem [Eq. (4.38)] to evaluate the correlation function of the two spontaneous current-density operators. This results in

$$\langle \hat{A}_\omega(\mathbf{x}_1)\hat{A}^\dagger_{\omega'}(\mathbf{x}_2)\rangle = 8\pi\hbar\,\delta(\omega-\omega') \times\int_\Omega G_\omega(\mathbf{x}_1,\mathbf{x})G_{\omega'}(\mathbf{x}_2,\mathbf{x})^* k_\omega(\mathbf{x})a_\omega(\mathbf{x}) \times[\bar{n}_\omega(\mathbf{x})+1]d\mathbf{x}. \tag{B3}$$

If we assume the system to be in quasiequilibrium, then $\bar{n}_\omega(\mathbf{x})$ is independent of $\mathbf{x}$ and can be removed from the integral. The remaining integral can be evaluated using Eq. (D1). A similar result holds for the other order of the field operators with $\bar{n}_\omega+1$ replaced by $\bar{n}_\omega$. The two correlation functions are

$$\langle \hat{A}_\omega(\mathbf{x}_1)\hat{A}^\dagger_{\omega'}(\mathbf{x}_2)\rangle = -8\pi\hbar\ \mathrm{Im}[G_\omega(\mathbf{x}_1,\mathbf{x}_2)] \times\delta(\omega-\omega')(\bar{n}_\omega+1), \tag{B4a}$$

$$\langle \hat{A}^\dagger_{\omega'}(\mathbf{x}_2)\hat{A}_\omega(\mathbf{x}_1)\rangle = -8\pi\hbar\ \mathrm{Im}[G_\omega(\mathbf{x}_1,\mathbf{x}_2)] \times\delta(\omega-\omega')\bar{n}_\omega. \tag{B4b}$$

Let us use this result to investigate the correlation of vacuum fluctuations propagating in a single transverse mode of a waveguide with optical attenuation by a cold medium. The normally ordered correlation function is zero, so we shall only consider the antinormally ordered one. Expressing the z dependence of the field and the Green's function with Eqs. (7.1) and (7.4), we find that Eq. (B4) reduces to

$$\langle \hat{b}_\omega(z_1)\hat{b}^\dagger_{\omega'}(z_2)\rangle = -8\pi\hbar\ \mathrm{Im}[g(z_1,z_2)]\delta(\omega-\omega'), \tag{B5}$$

where

$$g(z_1,z_2) = \frac{1}{2ik_0}e^{(ik_0-a_0/2)|z_2-z_1|}. \tag{B6}$$

After evaluation of the imaginary part of the one-dimensional Green's function, we have

$$\langle \hat{b}_\omega(z_1)\hat{b}^\dagger_{\omega'}(z_2)\rangle = \frac{4\pi\hbar}{k_0}\delta(\omega-\omega')\cos[k_0(z_2-z_1)] \times e^{-a_0|z_2-z_1|/2}. \tag{B7}$$

The corresponding equal-time correlation functions of the z-dependent fields associated with a narrow band of frequencies $\Delta\nu$ can be found by multiplying Eq. (B7) by $\exp[-i(\omega-\omega')t]/2\pi$ and integrating ω and ω' over this frequency range, which results in

$$\langle \hat{b}(z_1,t)\hat{b}^\dagger(z_2,t)\rangle = \frac{4\pi\hbar\Delta\nu}{k_0}\cos[k_0(z_2-z_1)] \times e^{-a_0|z_2-z_1|/2}. \tag{B8}$$

This result is consistent with the vacuum fluctuations being absorbed and emitted in an opaque medium, with no correlation between the absorbed and emitted fields. The field of vacuum fluctuations is composed of positively and negatively propagating fields. Consider the positively propagating fields. According to Eq. (7.10), at $z_1=z_2$ the correlation function is $2\pi\hbar\Delta\nu/k_0$. For $z_1>z_2$, the propagation will result in an additional phase change and field attenuation $\exp[ik_0(z_1-z_2) -a_0|z_2-z_1|/2]$. Accompanying attenuation is a process of emission of an uncorrelated field, with conservation of energy. This field does not contribute to the correlation function. Repeating this argument for $z_1<z_2$ gives an identical result, so the correlation function is

$$\langle \hat{b}(z_1,t)\hat{b}^\dagger(z_2,t)\rangle = \frac{2\pi\hbar\Delta\nu}{k_0}e^{ik_0(z_1-z_2)}e^{-a_0|z_2-z_1|/2}. \tag{B9}$$

The correlation function for negative propagating vacuum fluctuations is the same, except $k_0\to-k_0$. The sum of the two contributions results in Eq. (B8).

A positively propagating signal is described by

$$b_{\mathrm{sig}}(z,t) = b_{\mathrm{sig}0}e^{(ik_0-a_0/2)z-i\omega_S t}. \tag{B10}$$

It adds

$$b_{\mathrm{sig}}(z_1,t)b_{\mathrm{sig}}(z_2,t)^* = |b_{\mathrm{sig}0}|^2 e^{ik_0(z_1-z_2)}e^{-a_0(z_1+z_2)/2} \tag{B11}$$

to the correlation function. The absolute values of the correlation functions, given by Eqs. (B9) and (B11), are plotted in Fig. 8.

APPENDIX C: EFFECTIVE WAVE EQUATION FOR A WAVEGUIDE WITH SCATTERING LOSS

Here we discuss the field propagating in a transparent waveguide containing inhomogeneities, shown in Fig. 12. We arrive at an effective wave equation for the propagating modes in which the effect of the inhomogeneities in the dielectric function is described by an attenuation coefficient. The attenuation is due to scattering loss. In addition, there is an effective spontaneous current density acting as a source of noise fields. The noise fields actually originate from vacuum fluctuations scattered into the propagating modes. We show that the correlation function of these currents is related to the attenuation coefficient by the fluctuation-dissipation theorem.

The fluctuations in the dielectric function $\Delta\epsilon_\omega(\mathbf{x})$ are characterized by a correlation function. Any nonzero averages of $\Delta\epsilon_\omega(\mathbf{x})$ can be included in $\epsilon_\omega(\mathbf{x})$, so we take $\langle\Delta\epsilon_\omega(\mathbf{x})\rangle=0$. We assume that the dielectric function of the inhomogeneities $\Delta\epsilon_\omega(\mathbf{x})$ is spatially correlated only over a small volume V and that the dimensions of V are small compared to the dimensions of the waveguide. With these assumptions, we can approximate the correlation function by a delta function:

$$\langle\Delta\epsilon_\omega(\mathbf{x})\Delta\epsilon_\omega(\mathbf{x}')\rangle=\langle(\delta\epsilon)^2V\rangle\delta(\mathbf{x}-\mathbf{x}'). \tag{C1}$$

The total optical field $\hat{A}_\omega(\mathbf{x})$ satisfies a wave equation

$$\left(\nabla^2+\frac{\omega^2}{c^2}[\epsilon_\omega(\mathbf{x})+\Delta\epsilon_\omega(\mathbf{x})]\right)\hat{A}_\omega(\mathbf{x})=-\frac{4\pi}{c}\hat{j}_\omega(\mathbf{x}), \tag{C2}$$

where $\hat{j}_\omega(\mathbf{x})$ is the spontaneous current density within the absorbing walls in Fig. 12. This absorption prevents the scattered light from reflecting off the boundary, thereby imitating open surroundings.

We shall approximately solve this equation as a perturbation series, where $\Delta\epsilon_\omega(\mathbf{x})$ and $\hat{j}_\omega(\mathbf{x})$ are regarded as small perturbations. We shall write the total field as the sum of three terms,

$$\hat{A}_\omega(\mathbf{x})=\hat{A}_{0\omega}(\mathbf{x})+\hat{A}_{1\omega}(\mathbf{x})+\hat{A}_{2\omega}(\mathbf{x}), \tag{C3}$$

which are zero, first, and second order in these small quantities, respectively.

The field $\hat{A}_{0\omega}(\mathbf{x})$ is that of the propagating modes in a waveguide without scattering. It satisfies the wave equation in zero order:

$$\left(\nabla^2+\frac{\omega^2}{c^2}\epsilon_\omega(\mathbf{x})\right)\hat{A}_{0\omega}(\mathbf{x})=0. \tag{C4}$$

Let us next solve the wave equation (C2) keeping terms of first order. The zero-order terms are eliminated by use of Eq. (C4), leaving

$$\left[\nabla^2+\frac{\omega^2}{c^2}\epsilon_\omega(\mathbf{x})\right]\hat{A}_{1\omega}(\mathbf{x})=-\frac{\omega^2}{c^2}\Delta\epsilon_\omega(\mathbf{x})\hat{A}_{0\omega}(\mathbf{x})-\frac{4\pi}{c}\hat{j}_\omega(\mathbf{x}). \tag{C5}$$

We regard $\hat{A}_{0\omega}$ as known and have put it on the right-hand side.

The inhomogeneous solution $A_{1\omega}(\mathbf{x})$ can be found using the Green's function of the wave equation without scattering $G_\omega(\mathbf{x},\mathbf{x}')$, which satisfies

$$\nabla^2G_\omega(\mathbf{x},\mathbf{x}')+\frac{\omega^2}{c^2}\epsilon_\omega(\mathbf{x})G_\omega(\mathbf{x},\mathbf{x}')=\delta(\mathbf{x}-\mathbf{x}'). \tag{C6}$$

The solution is

$$\hat{A}_{1\omega}(\mathbf{x})=-\int d\mathbf{x}'\,G_\omega(\mathbf{x},\mathbf{x}')\left(\frac{\omega^2}{c^2}\Delta\epsilon_\omega(\mathbf{x}')\hat{A}_{0\omega}(\mathbf{x}')+\frac{4\pi}{c}\hat{j}_\omega(\mathbf{x}')\right). \tag{C7}$$

These first-order contributions to the field are from radiation scattered out of the waveguide and vacuum fluctuations emitted by spontaneous currents in the surrounding walls.

Next, we keep all terms to second order in the wave equation (C2). The zero-order and first-order terms are eliminated by use of Eqs. (C4) and (C5):

$$\begin{aligned}\left[\nabla^2+\frac{\omega^2}{c^2}\epsilon_\omega(\mathbf{x})\right]\hat{A}_{2\omega}(\mathbf{x})&=-\frac{\omega^2}{c^2}\Delta\epsilon_\omega(\mathbf{x})\hat{A}_{1\omega}(\mathbf{x})\\&=\frac{\omega^4}{c^4}\int d\mathbf{x}'\,G_\omega(\mathbf{x},\mathbf{x}')\Delta\epsilon_\omega(\mathbf{x})\Delta\epsilon_\omega(\mathbf{x}')\hat{A}_{0\omega}(\mathbf{x}')\\&\quad+\frac{4\pi\omega^2}{c^3}\Delta\epsilon_\omega(\mathbf{x})\int d\mathbf{x}'\,G_\omega(\mathbf{x},\mathbf{x}')\hat{j}_\omega(\mathbf{x}').\end{aligned} \tag{C8}$$

The last term in Eq. (C8) is the source of noise due to the scattering of vacuum fluctuations into the modes. It can be regarded as due to an effective spontaneous current density:

$$\frac{4\pi}{c}\hat{j}_{\omega\,\mathrm{eff}}(\mathbf{x})=-\frac{4\pi\omega^2}{c^3}\Delta\epsilon_\omega(\mathbf{x})\int d\mathbf{x}'\,G_\omega(\mathbf{x},\mathbf{x}')\hat{j}_\omega(\mathbf{x}'). \tag{C9}$$

The scattering of light out of the waveguide modes results in loss described by the imaginary part of the propagation constant. Multiple scattering also alters the velocity of the modes, causing a small change in the real part of the propagation constant. To obtain the average changes in the propagation constant, we approximate $\Delta\epsilon_\omega(\mathbf{x})\Delta\epsilon_\omega(\mathbf{x}')$ by its average value [Eq. (C1)]. Averaging over a small volume was also used in deriving the Kubo formula for the susceptibility in Sec. IV.D. This averaging allows us to evaluate the first integral in Eq. (C8):

$$\int d\mathbf{x}'\,G_\omega(\mathbf{x},\mathbf{x}')\langle\Delta\epsilon_\omega(\mathbf{x})\Delta\epsilon_\omega(\mathbf{x}')\rangle\hat{A}_{0\omega}(\mathbf{x}')=G_\omega(\mathbf{x},\mathbf{x})\langle(\delta\epsilon)^2V\rangle\hat{A}_{0\omega}(\mathbf{x}). \tag{C10}$$

The coefficient of $\hat{A}_{0\omega}(\mathbf{x})$ represents a correction to the propagation constant of the guided modes.

The field $\hat{A}_{2\omega}(\mathbf{x})$ is the second-order correction to the propagating modes. There is no first-order correction. We shall write the corrected field of the modes as $\hat{A}_{\omega\ \mathrm{modes}}(\mathbf{x})=\hat{A}_{0\omega}(\mathbf{x})+\hat{A}_{2\omega}(\mathbf{x})$. An equation for this field can be found by adding Eq. (C4) for $\hat{A}_{0\omega}(\mathbf{x})$ and Eq. (C8) for $\hat{A}_{2\omega}(\mathbf{x})$ (with the averaged correlation function):

$$\nabla^2 \hat{A}_{\omega\ \mathrm{modes}}(\mathbf{x}) + \left(\frac{\omega^2}{c^2} \epsilon_\omega(\mathbf{x}) - \frac{\omega^4}{c^4} \langle (\delta\epsilon)^2 V \rangle G_\omega(\mathbf{x},\mathbf{x}) \right) \times \hat{A}_{\omega\ \mathrm{modes}}(\mathbf{x}) = -\frac{4\pi}{c} \hat{j}_{\omega\,\mathrm{eff}}(\mathbf{x}). \tag{C11}$$

We have replaced the term $\langle (\delta\epsilon)^2 V \rangle \hat{A}_{0\omega}$ by $\langle (\delta\epsilon)^2 V \rangle \hat{A}_{\omega\ \mathrm{modes}}$. The error in doing this is of fourth order and negligible.

The correlation function of the effective spontaneous current density $\hat{j}_{\omega\,\mathrm{eff}}(\mathbf{x})$ can be calculated using the fluctuation-dissipation theorem [Eq. (4.38)] with $\bar{n}_\omega = 0$ and Eq. (C1):

$$\left(\frac{4\pi}{c} \right)^2 \langle j_{\omega\,\mathrm{eff}}(\mathbf{x}) j^\dagger_{\omega'\ \mathrm{eff}}(\mathbf{x}') \rangle = 8\pi \frac{\hbar\omega^4}{c^4} \langle (\delta\epsilon)^2 V \rangle \delta(\mathbf{x}-\mathbf{x}') \times \delta(\omega-\omega') \times \int |G_\omega(\mathbf{x},\mathbf{x}'')|^2 k(\mathbf{x}'') a(\mathbf{x}'') d\mathbf{x}''. \tag{C12}$$

We show in Appendix D [Eq. (D4)] that the integral in this equation is equal to $-\mathrm{Im}[G_\omega(\mathbf{x},\mathbf{x})]$. Using this result, we find that the correlation function for $\hat{j}_{\omega\ \mathrm{eff}}(\mathbf{x})$ [Eq. (C12)] reduces to

$$\left(\frac{4\pi}{c} \right)^2 \langle \hat{j}_{\omega\ \mathrm{eff}}(\mathbf{x}) \hat{j}^\dagger_{\omega\ \mathrm{eff}}(\mathbf{x}') \rangle = -\frac{8\pi\hbar\omega^4 \langle (\delta\epsilon)^2 V \rangle}{c^4} \mathrm{Im}[G_\omega(\mathbf{x},\mathbf{x})] \delta(\mathbf{x}-\mathbf{x}') \times \delta(\omega-\omega'). \tag{C13}$$

The bracketed term in the wave equation (C11) is an effective squared propagation constant that takes scattering into account,

$$\frac{\omega^2}{c^2} \epsilon_\omega(\mathbf{x}) - \frac{\omega^4}{c^4} \langle (\delta\epsilon)^2 V \rangle \{ \mathrm{Re}[G_\omega(\mathbf{x},\mathbf{x})] + i\ \mathrm{Im}[G_\omega(\mathbf{x},\mathbf{x})] \} \equiv k_\omega{}^2(\mathbf{x}) + 2k_\omega(\mathbf{x}) \Delta k(\mathbf{x}) + i k_\omega(\mathbf{x}) a_{\omega\ \mathrm{eff}}(\mathbf{x}), \tag{C14}$$

where $a_{\omega\,\mathrm{eff}}(\mathbf{x})$ is the effective attenuation coefficient due to scattering loss. Using the imaginary part of this equation to relate $a_{\omega\,\mathrm{eff}}(\mathbf{x})$ to $\mathrm{Im}[G_\omega(\mathbf{x},\mathbf{x})]$, we find that the correlation function for $\hat{j}_{\omega\,\mathrm{eff}}(\mathbf{x})$ reduces to

$$\left(\frac{4\pi}{c} \right)^2 \langle \hat{j}_{\omega\ \mathrm{eff}}(\mathbf{x}) \hat{j}^\dagger_{\omega\,\mathrm{eff}}(\mathbf{x}') \rangle = 8\pi\hbar k_\omega(\mathbf{x}) a_{\omega\,\mathrm{eff}}(\mathbf{x}) \delta(\mathbf{x}-\mathbf{x}'). \tag{C15}$$

This is exactly the expected relation between the loss and spontaneous current density predicted by the fluctuation-dissipation theorem [Eq. (4.38)] with $\bar{n}_\omega = 0$.

APPENDIX D: GREEN'S FUNCTION INTEGRAL

Here we establish a useful integral of the product of two Green's functions and the imaginary part of the squared propagation constant. The result, derived below, is

$$\int_\Omega k_\omega(\mathbf{x}) a_\omega(\mathbf{x}) G_\omega(\mathbf{x}_1,\mathbf{x}) G_\omega(\mathbf{x}_2,\mathbf{x})^* d\mathbf{x} = -\mathrm{Im}[G_\omega(\mathbf{x}_1,\mathbf{x}_2)]. \tag{D1}$$

The integral is over a volume Ω that is enclosed by a surface where the Green's function goes to zero.

The steps leading to this integral are similar to those made in deriving the energy conservation equation (6.10). The Green's function $G_\omega(\mathbf{x},\mathbf{x}_1)$ satisfies

$$[\nabla^2 + k^2_\omega(\mathbf{x}) + i k_\omega(\mathbf{x}) a_\omega(\mathbf{x})] G_\omega(\mathbf{x},\mathbf{x}_1) = \delta(\mathbf{x}-\mathbf{x}_1). \tag{D2}$$

A similar equation is satisfied by $G_\omega(\mathbf{x},\mathbf{x}_2)^*$:

$$[\nabla^2 + k^2_\omega(\mathbf{x}) - i k_\omega(\mathbf{x}) a_\omega(\mathbf{x})] G_\omega(\mathbf{x},\mathbf{x}_2)^* = \delta(\mathbf{x}-\mathbf{x}_2). \tag{D3}$$

Let us multiply Eq. (D2) by $G_\omega(\mathbf{x},\mathbf{x}_2)^*$, multiply Eq. (D3) by $G_\omega(\mathbf{x},\mathbf{x}_1)$, and take the difference. If we integrate this difference over Ω, the terms with $\nabla^2 G_\omega(\mathbf{x},\mathbf{x}_1)$ and $\nabla^2 G_\omega(\mathbf{x},\mathbf{x}_2)^*$ can be written as the integral of a divergence, which, when converted to a surface integral, is zero. The resulting equation is (D1). In writing this equation, we have interchanged the two arguments of the Green's function. This does not alter the integral because the Green's function is symmetrical in its arguments.

An important special case of Eq. (D1) is $\mathbf{x}_1 = \mathbf{x}_2$. The integral is

$$\int_\Omega k_\omega(\mathbf{x}) a_\omega(\mathbf{x}) |G_\omega(\mathbf{x}_1,\mathbf{x})|^2 d\mathbf{x} = -\mathrm{Im}[G_\omega(\mathbf{x}_1,\mathbf{x}_1)]. \tag{D4}$$

This result resembles the "optical theorem" in the quantum theory of scattering which relates the imaginary part of the amplitude for forward scattering to the total scattering cross section (Landau and Lifshitz, 1958a).

APPENDIX E: FORMULA FOR THE AVERAGE RATE OF SPONTANEOUS EMISSION

We shall evaluate the formulas for the average rates of spontaneous emission derived in Sec. VIII.D and arrive at a general formula. We then evaluate this formula to determine the average rate of spontaneous emission into a waveguide mode and into the mode of a laser near threshold.

The expressions for $\langle \hat{\mathcal{N}}_{\mathrm{spont}} \rangle$, in Eqs. (8.30a) and (8.30b), can be immediately evaluated by writing the field and spontaneous current in terms of their frequency components,

$$\hat{A}(\mathbf{x},t) = \int \frac{d\omega}{\sqrt{2\pi}} \hat{A}_\omega(\mathbf{x}) e^{-i(\omega-\omega_S)t}, \tag{E1a}$$

$$\hat{J}(\mathbf{x},t) = \int \frac{d\omega}{\sqrt{2\pi}} \hat{J}_\omega(\mathbf{x}) e^{-i(\omega-\omega_S)t}. \tag{E1b}$$

The range integration is centered about ω_S and is small compared to the optical frequency. We can relate $\hat{A}_\omega(\mathbf{x})$ to $\hat{J}_\omega(\mathbf{x}')$ with a Green's function [Eq. (B1)]. Substituting these expansions into Eq. (8.30a) for the normally ordered spontaneous-emission rate and using the fluctuation-dissipation theorem [Eq. (4.38)], we find

$$\left\langle \frac{\partial \hat{\mathcal{N}}_{\rm spont}(\mathbf{x})}{\partial t} \right\rangle = \frac{i}{\hbar c}\langle \hat{J}^{\dagger}\hat{A} - \hat{A}^{\dagger}\hat{J}\rangle = \frac{2gkn_{\rm sp}}{\pi}\int d\omega \, {\rm Im}[G_{\omega}(\mathbf{x},\mathbf{x})], \tag{E2}$$

where ${\rm Im}[G_{\omega}(\mathbf{x},\mathbf{x})]$ is the imaginary part of the Green's function and $gkn_{\rm sp}$ is evaluated at $\mathbf{x}$.

The antinormally ordered spontaneous-emission rate, given by Eq. (8.30b), is

$$\left\langle \frac{\partial \hat{\mathcal{N}}_{\rm spont}}{\partial t} \right\rangle = -gv_g\langle \hat{\mathcal{P}}_A - \hat{\mathcal{P}}_N\rangle + \frac{i}{\hbar c}\langle \hat{A}\hat{J}^{\dagger} - \hat{J}\hat{A}^{\dagger}\rangle. \tag{E3}$$

The first term in Eq. (E3) can be evaluated using the fluctuation-dissipation theorem and Eq. (D4) with $a \rightarrow -g$ to evaluate the spatial integral over the Green's function,

$$\begin{aligned} -gv_g\langle \hat{\mathcal{P}}_A - \hat{\mathcal{P}}_N\rangle &= -\frac{gk}{2\pi\hbar}\langle A(\mathbf{x},t)A^{\dagger}(\mathbf{x},t) - A^{\dagger}(\mathbf{x},t)A(\mathbf{x},t)\rangle \\ &= \frac{2gk}{\pi}\int d\omega \int_{\Omega} k(\mathbf{x}_1)g(\mathbf{x}_1)|G_{\omega}(\mathbf{x},\mathbf{x}_1)|^2 d\mathbf{x}_1 \\ &= \frac{2gk}{\pi}\int d\omega \, {\rm Im}[G_{\omega}(\mathbf{x},\mathbf{x})]. \end{aligned} \tag{E4}$$

The second term in Eq. (E3) is the same as Eq. (E2), except that $n_{\rm sp}$ is replaced by $n_{\rm sp}-1$. Therefore the term associated with the -1 of $n_{\rm sp}-1$ cancels the contribution of Eq. (E4), leaving a rate that is identical to Eq. (E2). That is, the contribution to the rate of spontaneous emission made by the uncertainty-related field fluctuations is canceled by part of the contribution made by the spontaneous currents. The remaining contribution is proportional to $n_{\rm sp}$ and goes to zero for a cold system. This cancellation ensures that there are no spontaneous transitions when the system is in its ground state. Further interpretation of this cancellation is given in the discussion after Eq. (8.31).

The sum of the two terms in Eq. (E3) is the same as Eq. (E2). This establishes that the two expressions for the average spontaneous-emission rate, arrived at starting with the rate of carrier recombination written in normal and antinormal order, are identical. $\langle \partial \hat{\mathcal{N}}_{\rm spont}/\partial t\rangle$ is order independent.

Let us evaluate the spontaneous-emission rate in an amplifying waveguide. The Green's function is given by

$$G_{\omega}(\mathbf{x}_1,\mathbf{x}_2) = \sum_n \frac{1}{2ik_n} e^{(ik_n + g_n/2)|z_2 - z_1|}\Phi_n(x_1)\Phi_n(x_2), \tag{E5}$$

where the sum is over the transverse modes. Substituting this Green's function into Eq. (E2), keeping only the contribution of a single transverse mode 0, and replacing the integral over frequency by the interval $2\pi\Delta\nu$, we find that the spontaneous emission per unit volume into mode 0 is

$$\left\langle \frac{\partial \hat{\mathcal{N}}_{\rm spont}(\mathbf{x})}{\partial t} \right\rangle = 4gkn_{\rm sp}\Delta\nu \, {\rm Im}[G_{\omega}(\mathbf{x},\mathbf{x})] = -\frac{2gk}{k_0}n_{\rm sp}\Delta\nu\Phi_0(x)^2. \tag{E6}$$

Integrating over the transverse coordinate x, and using Eq. (7.3) to eliminate k_0, we find the spontaneous emission in both the positive and negative directions into mode 0 is $2g_0n_{\rm sp}\Delta\nu$.

The photon flux of amplified spontaneous emission can be found by multiplying the rate of spontaneous emission in one direction by the amplification and integrating over the length of the amplifier:

$$\langle Q_N\rangle = g_0 n_{\rm sp}\Delta\nu \int_0^L dz e^{g_0(L-z)} = n_{\rm sp}\Delta\nu(G-1). \tag{E7}$$

This agrees with the result established earlier in Eq. (7.37a).

The Green's function of a laser can be expanded in terms of modes. This is done in Appendix F. The Green's function $G_{\omega}(\mathbf{x},\mathbf{x})$ has a narrow resonance for each mode near threshold. The average rate of spontaneous emission into one of these modes can be found by integrating over frequency the contribution of the mode to the Green's function. For mode 0, this integral is

$$\int d\omega G_{\omega}(\mathbf{x},\mathbf{x}) = \frac{\Phi_0(\mathbf{x})^2}{\left(\frac{dk_{\omega}^2}{d\omega}\Phi_0^2\right)_{\Omega}} \times \int d\omega \frac{(\omega-\omega_0) + i\Delta G_0/2}{(\omega-\omega_0)^2 + (\Delta G_0/2)^2}. \tag{E8}$$

The first term in the integrand is odd in $\omega-\omega_0$ and integrates to zero. The modes have $\Delta G_0 < 0$ both for nonlasing modes and for the steady-state operating point of the lasing mode. For this sign of ΔG_0, the integral of the second term is $-i\pi$. Substituting the imaginary part of this equation into Eq. (E2) and replacing $2k$ by $v_g dk_{\omega}^2/d\omega$, we have

$$\left\langle \frac{\partial \hat{\mathcal{N}}_{\rm spont}(\mathbf{x})}{\partial t} \right\rangle = -g(\mathbf{x})v_g(\mathbf{x})n_{\rm sp}(\mathbf{x}) \times {\rm Re}\left[\frac{dk^2(\mathbf{x})/d\omega\Phi_0(\mathbf{x})^2}{(dk/d\omega\Phi_0^2)_{\Omega}}\right]. \tag{E9}$$

APPENDIX F: EXPANSION OF THE GREEN'S FUNCTION IN LASER MODES

We are interested in optical frequencies near ω_S. The Green's function satisfies Eq. (6.6) with the propagation constant approximated by an expansion about ω_S in Eq. (6.8):

$$\nabla^2 G_\omega(\mathbf{x},\mathbf{x}')+\left[k^2(\mathbf{x})+\frac{dk_\omega^2(\mathbf{x})}{d\omega}(\omega-\omega_S)\right.$$
$$\left.-ik(\mathbf{x})g(\mathbf{x})\right]G_\omega(\mathbf{x},\mathbf{x}')=\delta(\mathbf{x}-\mathbf{x}'), \tag{F1}$$

where $g(\mathbf{x})$ is positive for net gain and negative in regions of loss. The Green's function can be expressed as a sum of orthogonal modes satisfying the same equation, but with the right side set to zero,

$$G_\omega(\mathbf{x},\mathbf{x}')=\sum_n g_n\Phi_n(\mathbf{x}). \tag{F2}$$

The modes occupy a volume Ω and go to zero at the surface enclosing this volume; see Fig. 13. These modes have discrete frequencies $\tilde{\omega}_n\equiv\omega_n+i\Delta G_n/2$ and satisfy an orthogonality relation [Eq. (8.2)]. Substituting this expansion into Eq. (F1) and using Eq. (8.1a) reduces the equation satisfied by the Green's function to

$$\sum_n \frac{dk_\omega^2(\mathbf{x})}{d\omega}\left(\omega-\omega_n-i\frac{\Delta G_n}{2}\right)g_n\Phi_n(\mathbf{x})=\delta(\mathbf{x}-\mathbf{x}'). \tag{F3}$$

We can solve for the expansion coefficient g_m by multiplying this equation by $\Phi_m(\mathbf{x})$, integrating over Ω, and applying the mode-orthogonality relation. The resulting expression for the Green's function is

$$G_\omega(\mathbf{x},\mathbf{x}')=\sum_n \frac{\Phi_n(\mathbf{x})\Phi_n(\mathbf{x}')}{\left(\frac{dk_\omega^2}{d\omega}\Phi_n^2\right)_\Omega(\omega-\omega_n-i\Delta G_n/2)}, \tag{F4}$$

where $(\cdots)_\Omega$ indicates an integral over Ω.

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