Decay widths and total cross sections in perturbative QCD

Levan R. Surguladze

Department of Physics & Astronomy, University of Albama, Tuscaloosa, Alabama 35487; Institute of Theoretical Science, University of Oregon, Eugene, Oregon 97403; and Department of Physics, Tbilisi State University, Tbilisi 380089, Georgia

Mark A. Samuel

Department of Physics, Oklahoma State University, Stillwater, Oklahoma 74078

The current status of analytic higher-order perturbative computations of total cross sections and decay widths in Quantum Chromodynamics is reviewed. Important issues are the methodology of renormalization-group evaluations, the ambiguities of the renormalization scheme and its scale, and the technical challenge of calculating many-loop diagrams. As examples, the authors consider the quantities $\sigma_{\rm tot}(e^+e^-\to\text{hadrons})$ and $\Gamma(\tau^-\to\nu_\tau+\text{hadrons})$ up to $O(\alpha_s^3)$ as well as $\Gamma(H\to\text{hadrons})$ up to $O(\alpha_s^2)$. The evaluation of the four-loop QED beta function is also described. The problem of theoretical uncertainty estimates in perturbative calculations is briefly discussed.

CONTENTS

I. INTRODUCTION

The standard model of strong interactions, quantum chromodynamics (for a brief review, see Tavkhelidze, 1994, and references therein), is becoming a quantitative theory for treating aspects of hadrons and their interactions. The asymptotic freedom and infrared divergences of the theory have strong influences on the calculation techniques. At low energy one can resort to numerical lattice calculations, or, to some extent, to the operator product expansion techniques, but at high energy in processes involving large four-momentum transfer, the asymptotic freedom justifies a perturbative calculational scheme.

In exclusive reactions where details of the final state are measured, it is easy to enforce high-momentum transfer, and the validity of the perturbation expansion is well tested. The major applications of perturbative QCD and its phenomenology have recently been reviewed in these pages (Sterman *et al.*, 1995). In the present review, we focus on inclusive observables, namely, total cross sections and decay widths. Here the large four-momentum is provided by the initial energy of the system, if the reaction goes through a single quantum in the electroweak sector. The specific processes we consider are the total cross section for electron-positron annihilation into hadrons and the hadronic decay widths of the tau lepton and the Higgs boson.

In perturbative QCD, the rates for these processes may be expressed as a power series in the strong-coupling constant α_s. The inclusive rates start with well-known constant terms independent of α_s—the Born approximation; our objective in this review is the calculation of higher-order terms in the series. There are two important issues to be considered. First is the methodological one—how to obtain physical observables from the first principles of the theory using dispersion relations, operator product expansion, renormalization-group and other techniques. The second issue is how to interpret the results, since the running coupling constant

α_s is defined in the renormalization-group approach with a scale. Truncation of the perturbation expansion always introduces some dependence on the scale and the renormalization scheme. The question is how best to pick an optimal scheme and/or a scale in order to minimize this theoretical ambiguity.

Another issue is the technical one of calculating the higher-order graphs. A number of techniques have been applied to the problem of calculating multiloop Feynman diagrams. A recursive algorithm for analytic evaluation of one-, two-, and three-loop propagator-type diagrams has been given by Chetyrkin and Tkachov (1981) and Tkachov (1981, 1983a). This algorithm is applicable to the theory with massless particles. It allows one to expand contributions to propagators up to the three-loop level as a Laurent series in $\epsilon=(4-D)/2$, with D the (noninteger) dimension of space-time. The above algorithm has been applied to a wide class of problems up to the four-loop level. These include, for example, calculations of renormalization constants, renormalization-group functions, cross sections, and decay widths. We note once again that this algorithm deals only with massless propagator-type diagrams. Nevertheless, due to the remarkable properties of dimensional regularization ('t Hooft and Veltman, 1972) and the minimal subtraction prescription ('t Hooft, 1973), namely, that the counter-terms are polynomials in dimensional parameters within minimal subtraction (Collins, 1974; Speer, 1974; see also the textbook by Collins, 1984), a wide class of problems can be reduced to the evaluation of propagator-type diagrams (Vladimirov, 1978, 1980). At high energies, in some cases, it is possible to neglect the masses of participating particles and consider massless diagrams. The mass corrections of the type m^{2n}/s^n, where s is the center-of-mass energy squared, can also be evaluated through the calculation of massless diagrams (see, for example, Gorishny, Kataev, and Larin, 1986; Surguladze, 1989a, 1994a, 1994b, 1994c). Feynman graphs can also contain virtual heavy-particle propagators regardless of the energy scale of the particular process. If the masses of the virtual particles are much larger than the energy scale, one can neglect them, since their effects are suppressed by powers of large mass, according to the decoupling theorem (Appelquist and Carazzone, 1975). However, in some cases, such effects may not be entirely negligible (Soper and Surguladze, 1994). The prescriptions for studying asymptotic expansions of Feynman integrals in powers of m^2/s can be obtained from Chetyrkin and Tkachov (1982), Tkachov (1983b, 1983c, 1991, 1993), and Chetyrkin (1991; see also Smirnov, 1990, 1991, and references therein). An exact general expression for one-loop, N-point, massive Feynman integrals has been obtained by Davydychev (1991) and Boos and Davydychev (1992). This expression contains the generalized hypergeometric function and is complicated, except for some particular cases. An alternative method for massive Feynman integrals has been suggested by Kotikov (1991).

In practice, the perturbative calculation of physical quantities is very cumbersome and tedious beyond the one-loop level, especially in realistic quantum-field-theory models such as QCD. However, the recursive algorithms allow convenient implementation within algebraic programming systems such as REDUCE (Hearn, 1973), SCHOONSCHIP (Veltman, 1967, 1991; Strubbe, 1974), and FORM (Vermaseren, 1989). Several computer programs were written in the last decade for analytic computation of multiloop Feynman diagrams. Among them we mention the programs that fully implement the above-mentioned recursive algorithms. The program LOOPS (Surguladze and Tkachov, 1989a), written on the REDUCE system, calculates one- and two-loop massless, propagator-type Feynman diagrams for arbitrary structure in the numerator of the integrand and for an arbitrary space-time dimension. The program MINCER (Gorishny, Larin, Surguladze, and Tkachov, 1989), written on the SCHOONSCHIP system, and the program HEPLoops (Surguladze, 1992), written on the FORM system, calculate one-, two-, and three-loop massless, propagator-type diagrams. The status of the existing program packages has been discussed recently by Surguladze (1994d). The above methods, algorithms, and computer programs allow one to make significant progress on high-order analytic perturbative calculations of several important physical observables.

As we have already mentioned, there is an outstanding problem in perturbative calculations, namely, the renormalization scheme and scale ambiguities of perturbation-theory predictions. Several approaches have been suggested to deal with these ambiguities. Among them we consider the so-called fastest apparent convergence approach (Grunberg, 1980), which suggests one absorb the leading QCD corrections in the definition of the "effective" running coupling. We also consider an approach based on the *principle of minimal sensitivity* of the physical observables to nonphysical parameters (Stevenson, 1981a, 1981b), and the Brodsky, Lepage, and Mackenzie (BLM, 1983) method, which suggests that one fix the scale according to the size of quark vacuum polarization effects. The commensurate scale relations of Brodsky and Lu (1994, 1995) allow one to make scale-fixed perturbative predictions without referring to the particular renormalization prescription.

In recent work, some authors try to predict the perturbative coefficients without calculating the relevant Feynman graphs. First, we mention the method of West (1991), which is based on renormalizability, analyticity arguments, and the saddle-point technique. For comments on this work, see Barclay and Maxwell (1992a), Brown and Yaffe (1992), Surguladze and Samuel (1992a, 1992b), and Duncan *et al.* (1993). The method of Samuel *et al.*. (Samuel and Li, 1994a, 1994b, 1994c; Samuel, Li, and Steinfelds, 1994a, 1994b, 1994c), based on Padé approximants, works surprisingly well for a large number of cases considered. Recent developments have put the Padé approximant method on a much more rigorous basis, which may justify its application to perturbation series in QED, QCD, and atomic physics. This is discussed in recent papers (e.g., Samuel, Ellis, and Karliner, 1995; Ellis, Gardi, Karliner, and Samuel, 1996). An alternative method for estimating higher-order perturbative contributions can be obtained based on Stevenson's (1981a, 1981b) approach (Surguladze and Samuel, 1993; Kataev and Starshenko, 1994). The important problem of the large-order behavior of perturbation theory has been considered by Barclay and Maxwell (1992b) and Brown and Yaffe (1992). That problem has been discussed for the past 20 years. Other papers on that subject have been collected in the book edited by Le Guillou and Zinn-Justin (1990). The application of renormalon calculus in the study of the behavior of perturbative QCD series is the subject of intensive discussion in the recent literature (e.g., Mueller, 1992; Zakharov, 1992; Lovett-Turner and Maxwell, 1994; Vainshtein and Zakharov, 1994; Soper and Surguladze, 1995a, 1995b).

After some introductory comments, we turn to the discussion of the analytic high-order perturbative calculations of several physical observables, which have been completed recently with the help of the above-mentioned methods, algorithms, and computer programs. First, we consider the analytic calculation of $R(s)$ in electron-positron annihilation at the four-loop level of perturbative QCD (Surguladze and Samuel, 1991a, 1991b, see also Gorishny, Kataev, and Larin, 1991), which turned out to be the most difficult among problems of this type. This is the first and so far the only four-loop calculation of a physical quantity in QCD.[1] As a by-product, the four-loop R_τ in tau decay (Samuel and Surguladze, 1991, see also Gorishny, Kataev, and Larin, 1991) and four-loop QED β function (Surguladze, 1990; Gorishny, Kataev, and Larin, 1990) have been evaluated.[2] For earlier works, we mention, for instance, the calculation of the three-loop correction to $R(s)$ in electron-positron annihilation (Chetyrkin, Kataev, and Tkachov, 1979; Dine and Sapirstein, 1979; Celmaster and Gonsalves, 1980), the calculation of the three-loop QCD β function (Tarasov, Vladimirov, and Zharkov, 1980), and the calculation of the three-loop anomalous dimensions of quark masses (Tarasov, 1982). We should also like to list some other three- and two-loop calculations. These are the calculation of the total decay width of the neutral Higgs boson into hadrons at the three-loop level (Gorishny, Kataev, Larin, and Surguladze, 1990, 1991b; Surguladze, 1994a, 1994b), the calculation of the two- and three-loop Wilson coefficients in QCD sum rules (Surguladze and Tkachov, 1986, 1988, 1989b, 1990), and the calculation of the two-loop anomalous dimensions of the proton current (Pivovarov and Surguladze, 1991). So far only one five-loop calculation exists. This is the calculation of the five-loop renormalization-group functions in ϕ^4 theory (Kleinert *et al.*, 1991).

The paper is organized as follows. In Sec. II we introduce our notation and present some general relations. Relevant methods and tools of perturbative QCD are discussed. We briefly consider the necessary dispersion relation, the operator product expansion (OPE), the renormalization relations, and the method for evaluation of the renormalization constants. We also discuss the main ideas of the method of projectors for calculating Wilson coefficients in the OPE. In Sec. III we evaluate the quantity $\Gamma(H\to\text{hadrons})$ at the three-loop level. In Sec. IV we calculate corrections to the correlation functions due to nonvanishing quark masses. In Sec. V we describe the calculation of Wilson coefficient functions of the dim=4 operators in the OPE of the two-point correlation functions of quark currents. In Sec. VI we describe the four-loop calculation of $\sigma_{\text{tot}}(e^+e^-\to\text{hadrons})$. Sections VII and VIII are dedicated to the evaluation of $\Gamma(\tau^-\to\nu_\tau+\text{hadrons})$ and the QED β function, respectively. In Sec. IX we discuss the problem of the renormalization scheme and scale ambiguity of perturbative QCD results. As an example, we consider calculated quantities and use the known approaches to try to fix the scheme-scale parameter within the one-parameter family of MS-type schemes. We also outline the original method of scheme-invariant analysis and the optimization procedure of Stevenson (1981a, 1981b). The paper ends with concluding comments.

[1] This calculation was attempted earlier by Gorishny, Kataev, and Larin (1988), but unfortunately, errors were found.

[2] For a joint publication of the results of two independent calculations of the four-loop QED β-function, see Gorishny, Kataev, Larin, and Surguladze (1991a).

II. CALCULATIONAL METHODS

A. Notation and general relations of perturbative QCD

Throughout this paper we work within the standard model of strong interactions—QCD. For a review on QCD, see, for example, Marciano and Pagels (1978), Mueller (1981), Reya (1981), and Altarelli (1982). For a textbook, see, for example, Yndurain (1983), Quigg (1986), Muta (1987), and Ellis and Stirling (1990). For the most recent source, see, for example, Sterman *et al.* (1995). The four-loop QED calculations will be discussed in Sec. VIII.

The Lagrangian density of standard QCD is

$$L(x)=-1/4(G^a_{\mu\nu})^2-\frac{1}{2\alpha_G}(\partial^\mu A^a_\mu)^2+\sum_f \bar{q}_f(i\hat{\partial}-m_f)q_f +g\sum_f \bar{q}_f T^a\hat{A}^a q_f+\partial^\mu c^{a\dagger}(\partial_\mu\delta^{ac}+gf^{abc}A^b_\mu)c^c, \tag{2.1}$$

where $G^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu$ $(a=1,2,\ldots,8)$ are the Yang-Mills field (Yang and Mills, 1954) strengths, A^a and q_f are gluon and quark fields, m_f are the quark masses, c^a are the Faddeev-Popov ghosts, and α_G is the gauge parameter. We use the standard notation $\hat{\partial}=\gamma^\mu\partial_\mu$ and $\hat{A}^a=\gamma^\mu A^a_\mu$. The index f enumerates the quark flavors, the total number of which is N. The generators T^a of the $SU_c(N)$ gauge group, the structure constants f^{abc} and d^{abc}, obey the following relations:

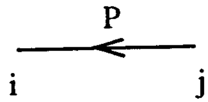
$$[T^a,T^b]=if^{abc}T^c,\quad \{T^a,T^b\}=\frac{1}{N}\delta^{ab}+d^{abc}T^c,$$
$$f^{acd}f^{bcd}=C_A\delta^{ab},\quad T^aT^a=C_F\hat{\mathbf{1}},\quad \mathrm{tr}T^aT^b=T\delta^{ab}. \tag{2.2}$$

The eigenvalues of the Casimir operators for the adjoint ($N_A=8$) and the fundamental ($N_F=3$) representations of $SU_c(3)$ are

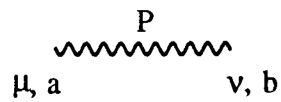
$$C_A=3,\quad C_F=4/3,\quad \text{and}\quad T=1/2,\quad d^{abc}d^{abc}=40/3. \tag{2.3}$$

We use the standard QCD Feynman rules (see, for example, Abers and Lee, 1973; Muta, 1987).

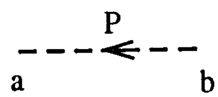
Propagators

quark 

$=\frac{1}{i}\frac{m+\hat{P}}{m^2-P^2}\delta_{ij}$

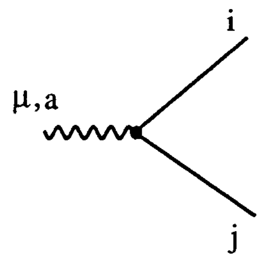
gluon 

$=\frac{1}{i}\frac{\delta_{ab}}{P^2}\left[g^{\mu\nu}-(1-\alpha_G)\frac{P_\mu P_\nu}{P^2}\right]$

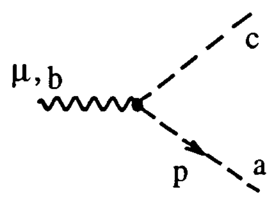
ghost 

$=\frac{1}{i}\frac{\delta_{ab}}{P^2}$

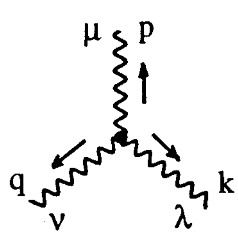
Vertices

quark-quark-gluon 

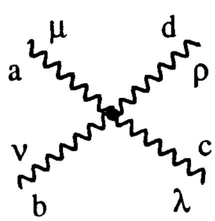
$=ig\gamma^\mu T^a_{ij}$

ghost-ghost-gluon 

$=igf^{abc}P_\mu$

3-gluon 

$=gf^{abc}[g_{\mu\nu}(q-p)_\lambda+g_{\nu\lambda}(k-q)_\mu+g_{\mu\lambda}(p-k)_\nu]$

4-gluon 

$$=ig^2[f^{abe}f^{cde}(g^{\mu\lambda}g^{\nu\rho}-g^{\mu\rho}g^{\nu\lambda})$$
$$+f^{ace}f^{bde}(g^{\mu\nu}g^{\lambda\rho}-g^{\mu\rho}g^{\nu\lambda})$$
$$+f^{ade}f^{cbe}(g^{\mu\lambda}g^{\nu\rho}-g^{\mu\nu}g^{\lambda\rho})]$$

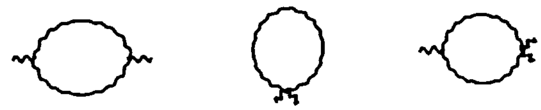
The sum of all momenta coming in each vertex of the Feynman diagram is zero (momentum conservation).

Factors

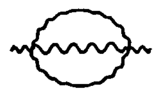
(-1) for each closed fermion or ghost loop

Statistical factors

$\frac{1}{2}$ for each graph (subgraph)



$\frac{1}{6}$ for each graph (subgraph)



etc.

Integration

Each loop corresponds to the integration $\int \frac{d^4P}{(2\pi)^4}$.

In general, the Feynman integral constructed according to the above rules is divergent. There are two kinds of divergences. One, the so-called ultraviolet (UV) divergence, is due to large integration momenta, and the other one, the so-called infrared divergence, is associated with the small integration momenta in the massless limit. The most convenient regularization of Feynman integrals is dimensional regularization (Ashmore, 1972; Bollini and Giambiagi, 1972; Cicuta and Montaldi, 1972; 't Hooft and Veltman, 1972), where the space-time dimension is analytically continued from the physical value, 4, to a complex value $D=4-2\varepsilon$. In the limit $\varepsilon\to 0$, the divergences appear as poles $1/\varepsilon$, defining the counterterms. One of the remarkable properties of dimensional regularization is that the Ward identities implied by gauge invariance are maintained for arbitrary space-time dimension D, in contrast with the old Pauli-Villars regularization (Pauli and Villars, 1949). Another useful property is a convenience in practical multiloop calculations. Thus, in dimensional regularization, we formally replace $\int d^4P/(2\pi)^4 \to \int d^DP/(2\pi)^D$. It is straightforward to extend the necessary tensor algebra into D dimensions. For example, $g^{\mu\nu}g_{\mu\nu}=D$, $\mathrm{Tr}\gamma_\mu\gamma_\nu=4g_{\mu\nu}$, etc. For the complete list of formulas see, for example, Collins (1984) and Narison (1982). Note, however, that the extension of the usual definition of the matrix γ_5

$$\gamma_5=\frac{1}{4!}\varepsilon_{\alpha\beta\mu\nu}\gamma_\alpha\gamma_\beta\gamma_\mu\gamma_\nu$$

is not straightforward. The totally antisymmetric tensor $\varepsilon_{\alpha\beta\mu\nu}$ is defined only in the four-dimensional space. In some cases the calculation of the quantities involving γ_5 is still possible within dimensional regularization. For a discussion of the problem of γ_5 in dimensional regularization, see Delbourgo and Akyeampong (1974), Trueman (1979), Bonneau (1980), Narison (1982), Collins (1984), and Larin (1993). For a calculation involving γ_5 within dimensional regularization, see, for example, Pivovarov and Surguladze (1991).

In order to get finite physical quantities, the divergences in dimensionally regularized Feynman integrals, appearing as poles in $1/\varepsilon$, need to be subtracted by adopting some specific rule. This rule is usually called a renormalization scheme. Throughout this paper we use 't Hooft's minimal subtraction (MS) scheme ('t Hooft, 1973). The subtraction of divergences is equivalent to the redefinition (renormalization) of the parameters (coupling, mass, and gauge-fixing parameter) and fields in the original "bare" Lagrangian

$$\begin{aligned}&\alpha_s^B=\mu^{2\varepsilon}Z_{\alpha_s}\alpha_s, \quad g^2/4\pi\equiv\alpha_s,\\&m^B=mZ_m,\\&\alpha_G^B=\alpha_G Z_G.\end{aligned} \tag{2.4}$$

μ is a quantity of dimension of mass which is introduced within dimensional regularization in order to make an action dimensionless. Superscript B denotes the unrenormalized quantity. We renormalize the gluon, quark, and ghost fields analogously. Within the MS scheme the N-point Green's function is renormalized in the following way,

$$\Gamma(p_1,\ldots,p_N,g,m,\alpha_G,\mu)=Z_\Gamma\Gamma^B(p_1,\ldots p_N,g,m,\alpha_G), \tag{2.5}$$

where Z_Γ is a polynomial in $1/\varepsilon$, and thus multiplying by Z_Γ, we subtract only pole parts from the divergent Γ^B. The evaluation of the renormalization constants Z will be discussed in the next subsections.

It is easy to see that the μ parameter entered through the renormalization, and hence the unrenormalized Green's function is independent of μ,

$$\mu\frac{d}{d\mu}\Gamma^B(p_1,\ldots,p_N,g,m,\alpha_G)=0.$$

Using Eq. (2.5) and expanding the full derivative, we get the renormalization-group equation in the following form,

$$\left[\mu^2\frac{\partial}{\partial\mu^2}+\beta(\alpha_s)\alpha_s\frac{\partial}{\partial\alpha_s}-\gamma_m(\alpha_s)m\frac{\partial}{\partial m}+\beta_G(\alpha_s)\frac{\partial}{\partial\alpha_G}-\gamma_\Gamma\right]\times\Gamma(p_1,\ldots,p_N,m,\alpha_s,\alpha_G,\mu)=0. \quad (2.6)$$

The QCD renormalization-group functions—the β function and the anomalous dimension functions—γ are defined in the following way,

$$\begin{aligned}\alpha_s\beta(\alpha_s)&=\mu^2\frac{d\alpha_s}{d\mu^2},\\ \beta_G(\alpha_s)&=\mu^2\frac{d\alpha_G}{d\mu^2},\\ \gamma_m(\alpha_s)&=-\frac{\mu^2}{m}\frac{dm}{d\mu^2},\\ \gamma_\Gamma(\alpha_s)&=\frac{\mu^2}{Z_\Gamma}\frac{dZ_\Gamma}{d\mu^2},\end{aligned} \quad (2.7)$$

with bare coupling and mass fixed. In the present paper we use the renormalization-group equation in the above form. The other forms are also known in the literature. The group properties of the renormalization were first discovered by Stueckelberg and Peterman (1953). The ultraviolet asymptotics of the Green's function was studied by Gell-Mann and Low (1954) in quantum electrodynamics using the group of multiplicative renormalizations. The renormalization-group formalism was further developed in the original works by Bogolyubov and Shirkov (1956). For a detailed monograph, see Bogolyubov and Shirkov (1980). The renormalization-group equation was studied by Callan (1970) and Symanzik (1970). For a recent historical review, see Shirkov (1992) and references therein.

The renormalization-group β function and anomalous dimensions of quark masses are calculated up to the three-loop level (Tarasov, Vladimirov, and Zharkov, 1980; Tarasov, 1982). The QCD β function up to and including the three-loop level in MS-type schemes is

$$\beta(\alpha_s)=-\beta_0\frac{\alpha_s}{\pi}-\beta_1\left(\frac{\alpha_s}{\pi}\right)^2-\beta_2\left(\frac{\alpha_s}{\pi}\right)^3+O(\alpha_s^4), \quad (2.8)$$

where (Tarasov, Vladimirov, and Zharkov, 1980)

$$\begin{aligned}\beta_0&=\frac{1}{4}\left(\frac{11}{3}C_A-\frac{4}{3}TN\right),\\ \beta_1&=\frac{1}{16}\left(\frac{34}{3}C_A^2-\frac{20}{3}C_ATN-4C_FTN\right),\\ \beta_2&=\frac{1}{64}\left(\frac{2857}{54}C_A^3-\frac{1415}{27}C_A^2TN+\frac{158}{27}C_AT^2N^2\right.\\ &\quad\left.-\frac{205}{9}C_AC_FTN+\frac{44}{9}C_FT^2N^2+2C_F^2TN\right).\end{aligned}$$

The quark mass anomalous dimension up to and including three-loop level is

$$\gamma_m(\alpha_s)=\gamma_0\frac{\alpha_s}{\pi}+\gamma_1\left(\frac{\alpha_s}{\pi}\right)^2+\gamma_2\left(\frac{\alpha_s}{\pi}\right)^3+O(\alpha_s^4), \quad (2.9)$$

where (Tarasov, 1982)

$$\begin{aligned}\gamma_0&=\frac{3}{4}C_F,\\ \gamma_1&=\frac{1}{16}\left(\frac{3}{2}C_F^2+\frac{97}{6}C_FC_A-\frac{10}{3}C_FTN\right),\\ \gamma_2&=\frac{1}{64}\left[\frac{129}{2}C_F^3-\frac{129}{4}C_F^2C_A+\frac{11413}{108}C_FC_A^2\right.\\ &\quad-(46-48\zeta(3))C_F^2TN-\left(\frac{556}{27}+48\zeta(3)\right)C_FC_ATN\\ &\quad\left.-\frac{140}{27}C_FT^2N^2\right].\end{aligned}$$

As shown by Caswell and Wilczek (1974) and Banyai, Marculescu, and Vescan (1974), the above renormalization-group functions are gauge independent, which greatly simplifies their evaluation. In fact, the QCD β function and the quark mass anomalous dimension have been evaluated in the Feynman gauge $\alpha_G=1$. We note that the perturbative coefficients of the renormalization-group functions are the same within the one-parameter family of the MS-type schemes. Note also the independence of these perturbative coefficients on the quark masses by their definition within the MS-type schemes.

B. Vacuum polarization function and dispersion relation

The vacuum polarization functions for various types of quark currents are crucial in the theoretical evaluation of total cross sections and decay widths. Indeed, for example, the quantity $\sigma_{\rm tot}(e^+e^-\to\text{hadrons})$, according to the well-known optical theorem (see, e.g., the textbook by Bogolyubov and Shirkov, 1980), is proportional to the imaginary part of the function $\Pi(q^2+i0)$, defined from the hadronic vacuum polarization function

$$\begin{aligned}\Pi_{\mu\nu}(q)&=i\int e^{iqx}\langle Tj_\mu(x)j_\nu(0)\rangle_0\,d^4x\\ &=(g_{\mu\nu}Q^2-Q_\mu Q_\nu)\Pi(Q^2)\frac{1}{(4\pi)^2}.\end{aligned} \quad (2.10)$$

Here, $j_\mu(x)=Q_f\bar{q}_f\gamma_\mu q_f$, Q_f is the electric charge of the quark of flavor f, and $Q^2=-q^2$ is the Euclidean momentum squared. The sum over all participating quark flavors is assumed in Π. The transverse form in the right-hand side is conditioned by the conservation of electromagnetic currents. In this paper we also consider the two-point function of quark axial-vector currents associated with the quantity $\Gamma(Z\to\text{hadrons})$ and the two-point function of quark scalar currents associated with the quantity $\Gamma(H\to\text{hadrons})$—the total decay width of the neutral standard-model Higgs boson into hadrons.

The renormalized vacuum polarization function obeys the dispersion relation

$$\Pi(Q^2)=\frac{4}{3}\int_{s_0}^{\infty}\frac{R(s)}{s+Q^2}\,ds-\text{subtractions}, \tag{2.11}$$

where

$$R(s)=\frac{\sigma_{\rm tot}(e^+e^-\to\text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)}=\frac{3}{4\pi}\text{Im}\Pi(s+i0). \tag{2.12}$$

Recall also that the muon pair-production cross section $\sigma(e^+e^-\to\mu^+\mu^-)=4\pi\alpha^2/3s$, where $\alpha=e^2/4\pi$ is the electromagnetic fine-structure constant. The above dispersion relation allows one to connect the experimentally measurable quantity $R(s)$ to the $\Pi(Q^2)$ calculable perturbatively in the deep Euclidean region (Q^2 is large compared to the typical hadron mass). For the discussion on theoretical calculability of $R(s)$, see earlier references: Adler (1974); Appelquist and Politzer (1975); de Rújula and Georgi (1976); Poggio, Quinn, and Weinberg (1976); Shankar (1977); and Barnett, Dine, and McLerran (1980). The combination of the idea of local duality in the dispersion relations (Logunov, Soloviov, and Tavkhelidze, 1967) and the operator product expansion technique (Wilson, 1969) became a basis of various versions of QCD sum rules (Shifman, Vainshtein, and Zakharov, 1979; Krasnikov, Pivovarov, and Tavkhelidze, 1983; Novikov *et al.*, 1985). For a review, see Novikov *et al.* (1978), Shifman (1992), and references therein. The methods of QCD sum rules are widely used to obtain quantitative information on the observed hadron spectrum and to extract the fundamental theoretical parameters.

In practice, sometimes it is more convenient to introduce the Adler function (Adler, 1974)

$$D(Q^2)=-\frac{3}{4}\frac{\partial}{\partial\log Q^2}\Pi(Q^2)=Q^2\int_{s_0}^{\infty}\frac{R(s)}{(s+Q^2)^2}\,ds. \tag{2.13}$$

The derivative here avoids an inconvenient extra subtraction in the right-hand side.

The leading (parton) approximation of $D(Q^2)$ in the zero quark mass limit coincides with $R(s)$,

$$D(Q^2)=3\sum_f Q_f^2, \tag{2.14}$$

where the sum runs over all participating quark charges at the given energy; 3 stands for the number of different colors. The leading "non-QCD" contribution is completely free of ultraviolet divergences, while the $\Pi(Q^2)$ needs an additive renormalization even at the leading order. At higher orders of perturbative expansion of the D function, the ultraviolet divergences appear and one should employ a procedure (usually called renormalization scheme) for their subtraction order-by-order. Because of ambiguity in the choice of subtraction scheme, the amplitude calculated within the perturbation theory depends on nonphysical parameters. Within the one-parameter family of the MS-type schemes ('t Hooft, 1973), such a parameter is usually called μ. Thus, up to power corrections, the D amplitude will be a function of $\log(\mu^2/Q^2)$ and the strong coupling α_s. On the other hand, since D is connected to the observable $R(s)$, it cannot depend on our subjective choice of the nonphysical parameter μ. This can be achieved if the strong coupling becomes a function of μ, providing independence of observables on the choice of parameter μ. Here, it is assumed that all orders of perturbation theory are summed up. Otherwise, if one considers a truncated series, the μ dependence remains. The problem of scheme-scale dependence and some possible solutions will be discussed later in this review. The set of transformations that leave observables independent of renormalization parameters has a group character and forms the renormalization group. The renormalization group in renormalizable theories (like QCD) fixes the dependence of the coupling on the μ parameter.

The function $D(Q^2)$ calculated in perturbative QCD within the MS-type schemes obeys the renormalization-group equation

$$\left(\mu^2\frac{\partial}{\partial\mu^2}+\beta(\alpha_s)\alpha_s\frac{\partial}{\partial\alpha_s}-\gamma_m(\alpha_s)m\frac{\partial}{\partial m}\right)\times D(\mu^2/Q^2,m,\alpha_s)=0. \tag{2.15}$$

Below, we consider the limit of the massless light quarks and the infinitely large top mass which decouples (Appelquist and Carazzone, 1975). The solution of Eq. (2.15) at $\mu^2=Q^2$ is

$$D(\mu^2/Q^2,\alpha_s(\mu))=D(1,\alpha_s(Q))=\sum_{i\geqslant 0}R_i(\alpha_s(Q)/\pi)^i, \tag{2.16}$$

where the $\alpha_s(\mu^2)$ is the running coupling, usually parametrized up to the three-loop level as follows,

$$\frac{\alpha_s(\mu^2)}{\pi}=\frac{1}{\beta_0 L}-\frac{\beta_1\log L}{\beta_0^3L^2}+\frac{1}{\beta_0^5L^3}(\beta_1^2\log^2L-\beta_1^2\log L+\beta_2\beta_0-\beta_1^2)+O(L^{-4}), \tag{2.17}$$

where $L=\log(\mu^2/\Lambda^2)$. Parametrization (2.17) has the same form, and the QCD β-function coefficients are the same within the MS-type schemes. The scale parameter Λ depends on the particular modification of the MS prescription. In fact, Λ is used to parametrize other versions of renormalization prescriptions as well. It is shown by Celmaster and Gonsalves (1979) that the transformation relations valid to all orders between Λ's defined by any two renormalization prescriptions can be deduced from a one-loop calculation. Comparing the bare coupling constants within different renormalization prescriptions and using the results for the one-loop renormalization constants and the property of asymptotic freedom, one obtains, for example, for momentum subtraction (MOM) and MS schemes (Celmaster and Gonsalves, 1979),

$$\Lambda_{\rm MS}=\Lambda_{\rm MOM}\exp\left[\frac{A(\alpha_G,N)}{4\beta_0}\right], \tag{2.18}$$

where

$$A(\alpha_G,N)=C_A\left[-\frac{11}{6}(\gamma_E-\ln 4\pi)+\frac{11}{3}+\frac{23}{72}I\right.$$
$$\left.+\frac{3}{8}\alpha_G(1-I)-\frac{1}{12}\alpha_G^2(3-I)+\frac{1}{24}\alpha_G^3\right]$$
$$+TN\left[\frac{2}{3}(\gamma_E-\ln 4\pi)-\frac{4}{3}-\frac{8}{9}I\right] \quad (2.19)$$

and the integral

$$I=-2\int_0^1 \frac{\ln x}{x^2-x+1}\,dx=2.3439072\ldots . \quad (2.20)$$

One note due to Stevenson (1981b, 1994) is in order. Despite its convenient form, the parametrization (2.17) produces an additional ambiguity due to the freedom with a particular definition of Λ parameter, even when the renormalization prescription is already specified. This problem was discussed by Abbot (1980), Shirkov (1980), Monsay and Rosenzweig (1981), Stevenson (1981b), and Radyushkin (1983). In fact, one can take advantage of this freedom in the choice of Λ and try to optimize the expansion in $1/L$. Indeed, as was shown by Radyushkin (1983), if one takes 0.6Λ in Eq. (2.17) instead of standard Λ (Buras, Floratos, Ross, and Sachrajda, 1977), then the $1/L^2$ and $1/L^3$ terms contribute only a few percent for a reasonably wide range of μ. On the other hand, Stevenson (1981b, 1994) has suggested that the entire problem of ambiguity in the definition of Λ can be avoided by abandoning the $1/L$ expansion and solving the renormalization-group equation (2.7) for α_s and resulting transcendental equation numerically, using the truncated β function.

According to the operator product expansion technique (Wilson, 1969), one can separate perturbative and nonperturbative contributions to the function $\Pi(Q^2)$. As shown by Shifman, Vainshtein, and Zakharov (1979), this function can be represented in the following form,

$$\Pi(Q^2)=\text{perturbation theory}+\sum_{n\geq 2}\frac{C_n(Q)\langle O_n\rangle_0}{Q^{2n}}$$
$$+\text{ instanton contributions}, \quad (2.21)$$

where $\langle O_n\rangle_0$ denote vacuum condensates parametrizing the nonperturbative contributions, and $C_n(Q)$ are their coefficient functions. The last term in the above equation describes the instanton contributions, which, in the case of electromagnetic currents, was estimated to be small (Krasnikov and Tavkhelidze, 1982; Kartvelishvili and Margvelashvili, 1995). The coefficient functions of the condensates can be calculated within perturbation theory. High-order perturbative corrections to the coefficient functions of dimension 4 and 6 power terms have been calculated in Loladze, Surguladze, and Tkachov (1984, 1985) and Surguladze and Tkachov (1989b, 1990) and also in Chetyrkin, Gorishny, and Spiridonov (1985), and those of dimension 6 power terms have been calculated in Lanin, Chetyrkin, and Spiridonov (1986) . In Sec. II.E we discuss the method for evaluating Wilson coefficient functions. Examples will be outlined in Sec. IV. Note that we consider the region of very high energies where, in fact, only perturbation-theory contributions survive in $\Pi(Q^2)$. The nonperturbative corrections could have some (small) effect in the case, for instance, of τ lepton decay (see Sec. VII). Note also that, in fact, the effects of neglected light quark masses are not entirely negligible in some phenomenological applications (see Sec. IV).

C. Renormalization relations

There are several approaches for the ultraviolet renormalization of Green's functions known in the literature. Throughout this paper we use 't Hooft's minimal subtraction method ('t Hooft, 1971, 1973). For alternative prescriptions, we refer to the works by Gell-Mann and Low (1954), Weinberg (1967), Callan (1970), Symanzik (1970), and Collins, Wilczek, and Zee (1978). For an analysis of various renormalization methods, see Collins and Macfarlane (1974). For a review, see, for example, Narison (1982) and the textbook by Collins (1984) and references therein. We focus on the renormalization relation for the two-point correlation function of quark currents relevant for the further evaluation of total cross sections and decay widths.

It is known that the vacuum polarization function is renormalized additively,

$$\Pi(\mu^2/Q^2,\alpha_s)=\Pi^B(\mu^2/Q^2,\alpha_s^B)+Z_\Pi\equiv\text{finite}. \quad (2.22)$$

The bare coupling α_s^B is related to the renormalized one by the relation (2.4). The perturbative expansion for Z_{α_s} can be found based on Eqs. (2.7) and (2.8), the MS definition of Z_{α_s}, and the renormalization-group equation

$$\mu^2\frac{d}{d\mu^2}\alpha_s^B=0. \quad (2.23)$$

We obtain

$$Z_{\alpha_s}=1-\frac{\alpha_s}{\pi}\frac{\beta_0}{\varepsilon}+\left(\frac{\alpha_s}{\pi}\right)^2\left(\frac{\beta_0^2}{\varepsilon^2}-\frac{\beta_1}{2\varepsilon}\right)$$
$$-\left(\frac{\alpha_s}{\pi}\right)^3\left(\frac{\beta_0^3}{\varepsilon^3}-\frac{7}{6}\frac{\beta_0\beta_1}{\varepsilon^2}+\frac{\beta_2}{3\varepsilon}\right)+O(\alpha_s^4). \quad (2.24)$$

In general, the polarization function depends on quark masses, and we shall need the relation between "bare" and renormalized masses up to $O(\alpha_s^2)$ (Tarasov, 1982),

$$(m_f^B)^2=m_f^2\left\{1-\left(\frac{\alpha_s}{4\pi}\right)\frac{6C_F}{\varepsilon}+\left(\frac{\alpha_s}{4\pi}\right)^2\right.$$
$$\times C_F\left[(11C_A+18C_F-4TN)\frac{1}{\varepsilon^2}\right.$$
$$\left.\left.-\left(\frac{97}{6}C_A+\frac{3}{2}C_F-\frac{10}{3}TN\right)\frac{1}{\varepsilon}\right]+O(\alpha_s^3)\right\}. \quad (2.25)$$

Within the minimal subtraction prescription ('t Hooft, 1973), the renormalization constant Z_Π can be expressed as the following double sum,

$$Z_\Pi=\sum_{\substack{-l\leqslant k<0\\ l>0}}\left(\frac{\alpha_s}{\pi}\right)^{l-1}Z_{l,k}\varepsilon^k, \qquad (2.26)$$

where Z_{lk} are numbers. Furthermore, for the "bare" vacuum polarization function, one has the following expansion in a perturbation series,

$$\Pi^B\left(\frac{\mu^2}{Q^2},\alpha_s^B\right)=\sum_{\substack{-l\leqslant k\\ l>0}}\left(\frac{\alpha_s^B}{\pi}\right)^{l-1}\left(\frac{\mu^2}{Q^2}\right)^{l\varepsilon}\Pi_{l,k}\varepsilon^k, \qquad (2.27)$$

where the first index denotes the number of loops of the corresponding Feynman diagrams at the given order of α_s. Substituting Eqs. (2.27) and (2.22) into the definition (2.13), we obtain, after the renormalization of the coupling via (2.24), at $\mu^2=Q^2$

$$D(\alpha_s)=\frac{3}{4}\Bigg\{\Pi_{1,-1}+\frac{\alpha_s}{\pi}\left[2\Pi_{2,-2}\frac{1}{\varepsilon}+2\Pi_{2,-1}\right]+\left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{1}{\varepsilon^2}(3\Pi_{3,-3}-2\beta_0\Pi_{2,-2})+\frac{1}{\varepsilon}(3\Pi_{3,-2}-2\beta_0\Pi_{2,-1})+(3\Pi_{3,-1}-2\beta_0\Pi_{2,0})\right]$$
$$+\left(\frac{\alpha_s}{\pi}\right)^3\left[\frac{1}{\varepsilon^3}(4\Pi_{4,-4}-6\beta_0\Pi_{3,-3}+2\beta_0^2\Pi_{2,-2})+\frac{1}{\varepsilon^2}(4\Pi_{4,-3}-6\beta_0\Pi_{3,-2}-\beta_1\Pi_{2,-2}+2\beta_0^2\Pi_{2,-1})\right.$$
$$\left.+\frac{1}{\varepsilon}(4\Pi_{4,-2}-6\beta_0\Pi_{3,-1}-\beta_1\Pi_{2,-1}+2\beta_0^2\Pi_{2,0})+(4\Pi_{4,-1}-6\beta_0\Pi_{3,0}-\beta_1\Pi_{2,0}+2\beta_0^2\Pi_{2,1})\right]+O(\alpha_s^4)\Bigg\}. \qquad (2.28)$$

Because of the renormalization-group invariance of $D(\mu^2/Q^2,\alpha_s)$, in the above equation we take $\mu^2=Q^2$ to avoid unnecessary logarithms. The renormalized expression for the D function must be finite in the limit $\varepsilon\to0$. Thus the coefficients of pole terms must vanish identically. This implies relations between the perturbative coefficients of Π and the QCD β function. First, we note that, prior to any renormalization, the leading poles must cancel at each order of α_s in the sum of all relevant Feynman diagrams. As shown by the actual calculation, this happens in each gauge-invariant set of diagrams.

$$\Pi_{4,-4}=\Pi_{3,-3}=\Pi_{2,-2}=0. \qquad (2.29)$$

Moreover, from the cancellation of nonleading poles, we get

$$3\Pi_{3,-2}-2\beta_0\Pi_{2,-1}=0,$$
$$4\Pi_{4,-3}-6\beta_0\Pi_{3,-2}+2\beta_0^2\Pi_{2,-1}=0, \qquad (2.30)$$
$$4\Pi_{4,-2}-6\beta_0\Pi_{3,-1}-\beta_1\Pi_{2,-1}+2\beta_0^2\Pi_{2,0}=0.$$

The above relations provide powerful tests of the calculation at its intermediate stages and are crucial.

From Eq. (2.22) we see that fully renormalized $\Pi(Q^2,\alpha_s)$ must be finite. Thus, substituting Eqs. (2.24)–(2.27) and (2.29) in Eq. (2.22), we obtain the following expression for the divergent part of $\Pi(\mu^2/Q^2,\alpha_s)$ at $\mu^2=Q^2$,

$$\mathrm{div}\Pi(\alpha_s)=\frac{1}{\varepsilon}(\Pi_{1,-1}+Z_{1,-1})+\frac{\alpha_s}{\pi}\left[\frac{1}{\varepsilon}(\Pi_{2,-1}+Z_{2,-1})\right]+\left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{1}{\varepsilon^2}(\Pi_{3,-2}-\beta_0\Pi_{2,-1}+Z_{3,-2})+\frac{1}{\varepsilon}(\Pi_{3,-1}-\beta_0\Pi_{2,0}+Z_{3,-1})\right]$$
$$+\left(\frac{\alpha_s}{\pi}\right)^3\left[\frac{1}{\varepsilon^3}(\Pi_{4,-3}-2\beta_0\Pi_{3,-2}+\beta_0^2\Pi_{2,-1}+Z_{4,-3})+\frac{1}{\varepsilon^2}(\Pi_{4,-2}-2\beta_0\Pi_{3,-1}+\beta_0^2\Pi_{2,0}-\beta_1\Pi_{2,-1}/2+Z_{4,-2})\right.$$
$$\left.+\frac{1}{\varepsilon}(\Pi_{4,-1}-2\beta_0\Pi_{3,0}+\beta_0^2\Pi_{2,1}-\beta_1\Pi_{2,0}/2+Z_{4,-1})\right]\equiv0. \qquad (2.31)$$

The leading poles in Z_Π are absent at each order of α_s $(Z_{2,-2}=Z_{3,-3}=Z_{4,-4}=0)$ except the zeroth order. Taking into account Eq. (2.30), we obtain the other set of relations between the perturbative coefficients of Π, Z and QCD β function,

$$3Z_{3,-2}+\beta_0Z_{2,-1}=0,$$
$$2Z_{4,-3}+\beta_0Z_{3,-2}=0, \qquad (2.32)$$
$$4Z_{4,-2}+2\beta_0Z_{3,-1}+\beta_1Z_{2,-1}=0.$$

$$\Pi_{1,-1}=-Z_{1,-1},$$
$$\Pi_{2,-1}=-Z_{2,-1},$$
$$\Pi_{3,-2}=-Z_{3,-2}-\beta_0Z_{2,-1},$$
$$\Pi_{3,-1}=-Z_{3,-1}+\beta_0\Pi_{2,0}, \qquad (2.33)$$
$$\Pi_{4,-1}=-Z_{4,-1}+2\beta_0\Pi_{3,0}+\beta_1\Pi_{2,0}/2-\beta_0^2\Pi_{2,1},$$
$$\Pi_{4,-2}=-Z_{4,-2}-2\beta_0Z_{3,-1}-\beta_1Z_{2,-1}/2+\beta_0^2\Pi_{2,0},$$
$$\Pi_{4,-3}=-Z_{4,-3}-2\beta_0Z_{3,-2}-\beta_0^2Z_{2,-1}.$$

In Sec. VI, the above relations will be used in the calculation of the four-loop total cross section in electron-positron annihilation.

D. Method for evaluation of renormalization constants

We now discuss the evaluation of renormalization constants within 't Hooft's MS scheme ('t Hooft, 1973), using Vladimirov's method (Vladimirov, 1978) and the so-called infrared rearrangement procedure (Vladimirov, 1980; Chetyrkin and Tkachov, 1982).

To calculate the renormalization constant Z_Γ for the one-particle-irreducible Green's function Γ, it is convenient to use the following representation (Vladimirov, 1978),

$$Z_\Gamma = 1 - \mathscr{K}R'\Gamma. \tag{2.34}$$

The operator $\mathscr{K}$ picks out all singular terms from the Laurent series in ε,

$$\mathscr{K}\sum_i c_i \varepsilon^i = \sum_{i<0} c_i \varepsilon^i.$$

R' is defined by the recursive relation

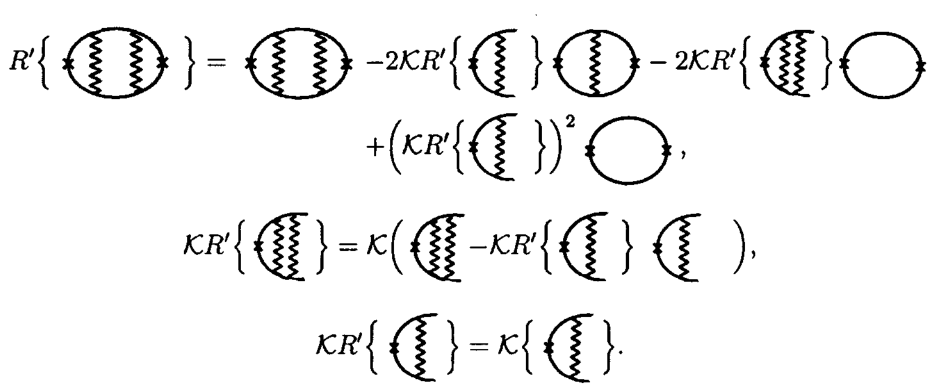
$$R'G = G - \sum_{G_i} \mathscr{K}R'G_1 \cdots \mathscr{K}R'G_n \times G_{/(G_1 \cup \cdots \cup G_n)}, \tag{2.35}$$

where the sum runs over all sets of one-particle-irreducible divergent subgraphs G_i of the diagram G. $G_{/(G_1+\cdots+G_n)}$ is the diagram G with the subgraphs $G_1, \ldots, G_n$ shrunk to a point. In fact, R' is the ordinary Bogolyubov-Parasyuk R operation (Bogolyubov and Parasyuk, 1956, 1957; for a textbook, see Bogolyubov and Shirkov, 1980) without the last (overall) subtraction. Thus, R' subtracts all "internal" divergences only and is connected to the ordinary R operation in the following way,

$$R = (1 - \mathscr{K})R'.$$

To calculate the renormalization constant Z in Eq. (2.22), one should write a diagram representation of Π and apply $\mathscr{K}R'$ to the corresponding graphs [Eq. (2.34)] or, in other words, one should evaluate the counterterms for each graph. The benefit of using relation (2.34) is based on the fact that the $\mathscr{K}R'$ for each diagram is a polynomial in dimensional parameters (Collins, 1974; Speer, 1974). This fundamental property of 't Hooft's minimal subtraction prescription is the basic idea of the various versions of the infrared rearrangement technique (Vladimirov, 1980; Chetyrkin and Tkachov, 1982).

As an example, we demonstrate the application of the $\mathscr{K}R'$ operation to the three-loop QCD diagram contributing to the $O(\alpha_s^2)$ total cross section for the process $e^+e^- \to$ hadrons.



The benefit of using the $\mathscr{K}R'$ operation, besides its convenience in actual calculations, is as follows. Using the fact that the result of the $\mathscr{K}R'$ operation is a polynomial in masses and external momenta of the diagram, one can remove the dependence on the external momenta by differentiating (usually twice is sufficient) with respect to the external momentum and then setting the external momentum to zero. However, in this case infrared divergences appear. In order to prevent this, one can introduce a new fictitious external momentum as an infrared regulator flowing along some of the lines of the diagram (Chetyrkin and Tkachov, 1982). Alternatively, one can introduce a fictitious mass in one of the lines of the diagram as an infrared regulator (Vladimirov, 1980). An appropriate choice of the fictitious momentum can drastically simplify the topology of the given diagram.

Both versions of the so-called infrared rearrangement procedure simplify the calculation and make it possible to evaluate counterterms to four- and five-loop diagrams. The main result of the application of the infrared rearrangement technique can be formulated as follows. The problem of calculating the counterterms of an arbitrary l-loop diagram with an arbitrary number of masses and external momenta within the MS prescription can be reduced to the problem of calculating some $(l-1)$-loop massless integrals to $O(\varepsilon^0)$ with only one external momentum. In the later sections, the full calculational procedure will be demonstrated for a typical four-loop diagram contributing to the photon renormalization constant.

E. Evaluation of Wilson coefficient functions in operator product expansion

In this subsection we briefly discuss the problem of the evaluation of higher-twist operator contributions to the hadronic vacuum polarization function. Those contributions are relevant in the analysis of nonperturbative contributions in some processes (e.g., hadronic decay of the τ lepton). We use the Wilson operator product expansion technique (Wilson, 1969)—the mathematical apparatus allowing a factorization of the short-distance contributions, which are calculable perturbatively, and large-distance effects, which can be parametrized with the vacuum condensates (Shifman, Vainshtein, and Zakharov, 1979; Novikov *et al.*, 1985). In the perturbative evaluation of Wilson coefficient functions, we rely on the so-called method of projectors (Gorishny, Larin, and Tkachov, 1983; Gorishny and Larin, 1987; see also Pivovarov and Tkachov, 1988, 1993 and references therein). An actual calculation for the coefficient functions of the operators of dim=4 has been performed in the work by Loladze, Surguladze, and Tkachov (1984, 1985) and Surguladze and Tkachov (1989b, 1990). The present discussion is based mainly on those works. Below, we demonstrate the above technique in the case of the coefficient functions of gluon and quark condensates.

Consider the operator product expansion of the T product of two quark currents in the deep Euclidean region, $-q^2=Q^2\to\infty$

$$\mathcal{T}(Q)=i\int d^4x\ e^{iqx}TJ(x)J(0)=\sum_i C_i(Q)O_i(0), \tag{2.36}$$

where J are quark currents. $C_i(Q)$ are c-number coefficient functions containing all dependence on Q. O_i are local operators forming, in general, a complete basis. If the currents J are gauge invariant, then, after averaging over the vacuum, only gauge-invariant operators contribute to the right-hand side of Eq. (2.36). However, the renormalization procedure mixes gauge-invariant operators with noninvariant ones, and one has to consider the complete basis of operators of the given dimension. The following set of operators of the dimension 4,

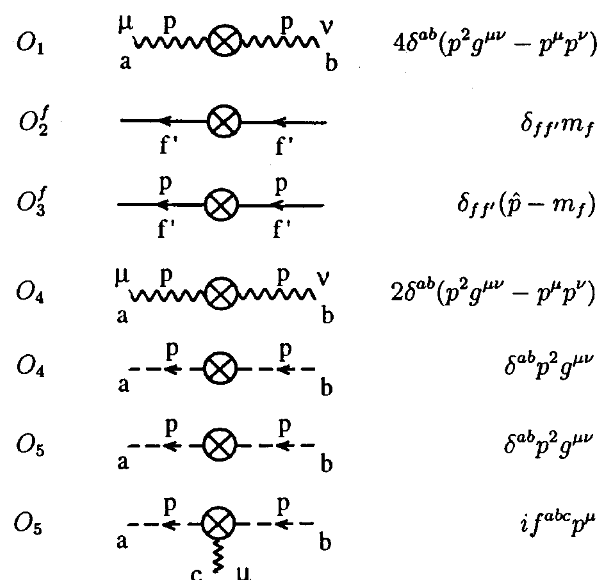
$$O_1=(G^a_{\mu\nu})^2,\quad O_2^f=m_f\bar{q}_f q_f,$$

$$O_3^f=\bar{q}_f(i\hat{\partial}-m_f+gT^a\hat{A}^a)q_f,$$

$$O_4=(\partial_\mu\bar{c}^a)(\partial_\mu c^a)+(\partial_\mu\delta^{ab}+gf^{abc}A^c_\mu)A^b_\nu G^a_{\mu\nu}$$
$$-g\sum_f \bar{q}_f T^a\hat{A}^a q_f, \tag{2.37}$$

$$O_5=\partial_\mu\bar{c}^a(\partial_\mu\delta^{ab}+gf^{abc}A^c_\mu)c^b,$$

is closed under renormalization, together with the "operator" $\sim m^4$ (Spiridonov, 1984; Loladze, Surguladze, and Tkachov, 1984, 1985). Our aim is to calculate coefficient functions of gauge-invariant operators O_1 and O_2^f. Note that $\sim m^4$ operators can be ignored because of the special structure of the renormalization matrix for the basis (2.37). The Feynman rules for the operators (2.37) are (Surguladze and Tkachov, 1990)



(2.38)

The operators of the basis (2.37) are renormalized as follows,

$$O_i=(Z_O)_{ij}O_j^B, \tag{2.39}$$

where the superscript B marks the same operators as in (2.37) but built from the "bare" fields, masses, and couplings. The structure of the renormalization matrix Z_O has been studied by Spiridonov (1984). In the MS-type schemes, Z_O has the following form (Surguladze and Tkachov, 1990),

$$Z_O = \left\| \begin{array}{c|c|c} A & B \ldots B & \ldots \\ \hline 0 & & \ldots \\ \vdots & 1_{N_f \times N_f} & \ldots \\ 0 & & \ldots \\ \hline 0 & & \ldots \\ \vdots & 0_{N_f \times N_f} & \ldots \\ 0 & & \ldots \\ \hline 0 & 0 \ldots 0 & \ldots \\ 0 & 0 \ldots 0 & \ldots \end{array} \right\| \tag{2.40}$$

where only the matrix elements A and B are relevant,

$$A = \left(1 - \frac{\beta(\alpha_s)}{\varepsilon}\right)^{-1},$$

$$B = \frac{4\gamma_m(\alpha_s)}{\varepsilon}\left(1 - \frac{\beta(\alpha_s)}{\varepsilon}\right)^{-1}. \tag{2.41}$$

Inserting Eq. (2.39) into the expansion (2.36), we get

$$\mathscr{T}(Q) = \sum_{i,j} C_i(Q) O_i^B (Z_O)_{ij}. \tag{2.42}$$

Following the method of projectors (Gorishny, Larin, and Tkachov, 1983), we define the projectors π_i satisfying the orthogonality condition and vanishing on higher spin operators,

$$\pi_i[O_j^B] = \delta_{ij}, \qquad \pi_i[\text{higher spin operators}] = 0. \tag{2.43}$$

Projectors π_i applied on the left-hand side of Eq. (2.42) separate in the right-hand side the coefficient functions in which we are interested,

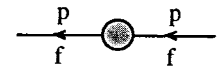
$$\pi_j[\mathscr{T}(Q)] = \sum_i C_i(Q)(Z_O)_{ij}. \tag{2.44}$$

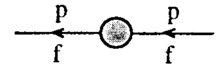
We find the coefficient functions

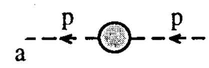
$$C_i(Q) = \sum_j \pi_j[\mathscr{T}(Q)](Z_O^{-1})_{ji}. \tag{2.45}$$

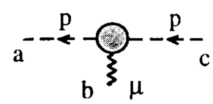
Our aim is to find the coefficient functions of gauge-invariant operators $O_1 = (G_{\mu\nu})^2$ and $O_2^f = m_f \bar{q}_f q_f$. So, we need to construct the corresponding projectors π_1 and π_2^f. Let us represent π_i as linear combinations of some "elementary" projectors $\mathscr{P}_j$ defined in the following way,

$$\mathcal{P}_1[O] = \frac{1}{N_A}\frac{\partial^2}{\partial p^2}\delta^{ab} g^{\mu\nu}\Big\{$$  $$\Big\}_{p=m_f=0}$$

$$\mathcal{P}_2^f[O] = \frac{1}{4N_F}\frac{\partial}{\partial m_f} Tr\Big\{$$  $$\Big\}_{p=m_f=0}$$

$$\mathcal{P}_3^f[O] = \frac{1}{4N_F}\frac{\partial}{\partial p^\sigma} Tr\gamma^\sigma\Big\{$$  $$\Big\}_{p=m_f=0}$$

$$\mathcal{P}_4[O] = \frac{1}{N_A}\frac{\partial^2}{\partial p^2}\delta^{ab}\Big\{$$  $$\Big\}_{p=m_f=0}$$

$$\mathcal{P}_5[O] = \frac{i f^{abc}}{g N_A C_A}\frac{\partial}{\partial p^\mu}\Big\{$$  $$\Big\}_{p=m_f=0}$$

(2.46)

where the parentheses contain the one-particle-irreducible Green's function with one operator insertion. In the case of $\mathscr{P}_2^f$ and $\mathscr{P}_3^f$, the traces are calculated over Lorentz spinor and color indices.

Acting by the projectors $\mathscr{P}_j$ on the operators (2.37), we obtain

$$\begin{aligned} &\mathscr{P}_1[O_1] = 8D(D-1), \quad \mathscr{P}_1[O_4] = 4D(D-1), \\ &\mathscr{P}_2^f[O_2^{f'}] = \delta_{ff'}, \quad \mathscr{P}_2^f[O_3^{f'}] = -\delta_{ff'}, \quad \mathscr{P}_3^f[O_3^{f'}] = D\,\delta_{ff'}, \\ &\mathscr{P}_4[O_4] = \mathscr{P}_4[O_5] = 2D, \\ &\mathscr{P}_5[O_5] = D. \end{aligned} \tag{2.47}$$

The results that are not shown in the above list are identically zero. From definition (2.43) and Eq. (2.47), we obtain the explicit form for the projectors π_1 and π_2^f,

$$\pi_1 = \frac{1}{8D(D-1)}[\mathscr{P}_1 - 2(D-1)\mathscr{P}_4 + 4(D-1)\mathscr{P}_5],$$

$$\pi_2^f = \mathscr{P}_2^f + \frac{1}{D}\mathscr{P}_3^f. \tag{2.48}$$

Combining Eqs. (2.40) and (2.41) with Eq. (2.45), we get our final expressions for the coefficient functions $C_1(Q)$ and $C_2^f(Q)$ (Surguladze and Tkachov, 1989b, 1990),

$$\begin{aligned} C_1(Q) &= \pi_1[\mathscr{T}(Q)]\left(1 - \frac{\beta(\alpha_s)}{\varepsilon}\right), \\ C_2^f(Q) &= \pi_2^f[\mathscr{T}(Q)] - \pi_1[\mathscr{T}(Q)]\frac{4\gamma_m(\alpha_s)}{\varepsilon}. \end{aligned} \tag{2.49}$$

The above expressions have a closed form and are valid at any order of perturbation theory. We note that $\mathscr{T}$ must be constructed with unrenormalized couplings and fields before one applies the projectors π_i.

The general theory of Euclidean asymptotic expansions of Feynman integrals and the methods applicable to high-order perturbative calculations have been devel-

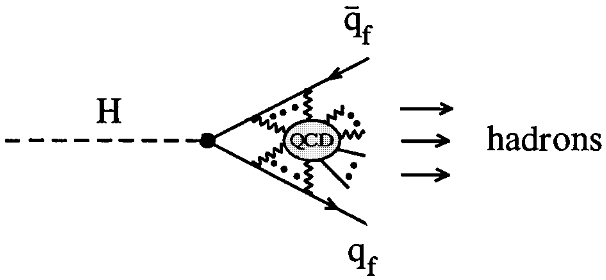


FIG. 1. The process $H\to$hadrons.

oped in the works of Tkachov (1983b, 1983c, 1991, 1993), Chetyrkin and Tkachov (1982), and Chetyrkin (1991; see also Smirnov 1990, 1991 and references therein). The technique developed in these works allows one to derive operator product expansions in the MS scheme for any Feynman integral. For more general discussion and further details, we refer to the above works and to the original calculations (Surguladze and Tkachov, 1989a, 1989b, 1990). In Sec. V we present a short description of the calculation of the coefficient functions of gluon and quark condensates up to $O(\alpha_s^2)$.

III. $\Gamma(H\to$HADRONS) TO $O(\alpha_s^2)$

A. The decay rate in terms of running parameters

In this subsection, using the above methods, we calculate the $O(\alpha_s^2)$ corrections to the total hadronic decay width of the standard-model Higgs boson in the massless quark limit (Gorishny, Kataev, Larin, and Surguladze, 1990, 1991b; Surguladze, 1994b) (see Fig. 1).

The standard SU(2)$\times$U(1) Lagrangian density of a fermion-Higgs interaction is

$$L=-g_Y\bar{q}_f q_f H=-(\sqrt{2}G_F)^{1/2}m_f\bar{q}_f q_f H =-(\sqrt{2}G_F)^{1/2}j_f H. \qquad (3.1)$$

The decay width of a scalar Higgs boson to the quark-antiquark pair is determined by the imaginary part of the two-point correlation function,

$$\Pi(Q^2=-s,m_f)=i\int e^{iqx}\langle Tj_f(x)j_f(0)\rangle_0\ d^4x, \qquad (3.2)$$

of the quark scalar currents $j_f=m_f\bar{q}_f q_f$ in the following way,

$$\Gamma_{H\to q_f\bar{q}_f}=\frac{\sqrt{2}G_F}{M_H}\,\mathrm{Im}\Pi(s+i0,m_f)\Big|_{s=M_H^2}. \qquad (3.3)$$

M_H is the Higgs mass. The total decay width will be the sum over all participating (depending on M_H) quark flavors,

$$\Gamma(H\to\text{hadrons})=\sum_{f=u,d,s,\ldots}\Gamma_{H\to q_f\bar{q}_f}. \qquad (3.4)$$

We follow the work by Gorishny, Kataev, Larin, and Surguladze (1990) and, in analogy to the vector channel, introduce the Adler function (Adler, 1974),

$$D(Q^2,m_f)=Q^2\frac{d}{dQ^2}\frac{\Pi(Q^2,m_f)}{Q^2}. \qquad (3.5)$$

The derivative avoids the additive renormalization of Π. In fact, it is possible to proceed without the introduction of the D function and deal directly with the correlation function Π (Surguladze, 1994b). Indeed, we are interested in $\mathrm{Im}\Pi(s+i0,m_f)$. Since the overall MS renormalization constant has no terms like log $(\mu^2/Q^2)^n/\varepsilon^k$, its imaginary part vanishes identically. The absence of the pole logarithms in renormalization constants is a general feature of MS-type schemes.

The D function obeys the homogeneous renormalization-group equation

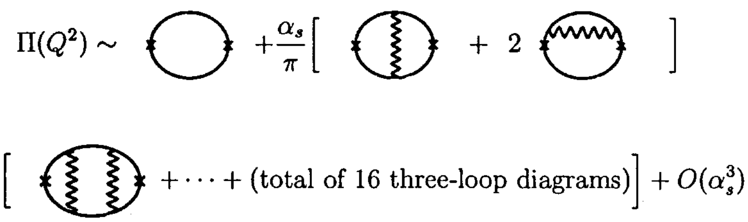
$$\left(\mu^2\frac{\partial}{\partial\mu^2}+\beta(\alpha_s)\alpha_s\frac{\partial}{\partial\alpha_s}-\gamma_m(\alpha_s)\frac{\partial}{\partial\log m_f}\right)\times D(\mu^2/Q^2,m_f,\alpha_s)=0. \qquad (3.6)$$

The QCD β function and the mass anomalous dimension γ_m are known up to the three-loop approximation and have been given in the previous section. The plan for evaluation of $\Gamma_{H\to q_f\bar{q}_f}$ is as follows. First, we write the diagram representation for $\Pi(Q^2,m_f)$ according to the standard Feynman rules up to the desired loop level. Second, we evaluate the Feynman diagrams using dimensional regularization and renormalize the coupling and quark masses within the MS renormalization prescription. Finally, to get the decay rate, we analytically continue the result for the D function obtained from Eq. (3.5) from Euclidean to Minkowski space. Following the above plan, we now demonstrate the calculation of $\Gamma_{H\to q_f\bar{q}_f}$ up to the three-loop level. First of all, note that the correlation function Π and the related D function depend on quark masses. The algorithms for evaluation of the three-loop Feynman diagrams constructed with the propagators of massive particles have not yet been developed. However, in the deep Euclidean region ($Q^2\to\infty$), it is possible to simplify the calculation using the expansion in terms of the small parameter m_f^2/Q^2,

$$\frac{1}{m_f^2Q^2}\Pi(Q^2,m_f)=\Pi(Q^2)+O\left(\frac{m_f^2}{Q^2}\right). \qquad (3.7)$$

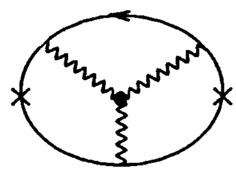
Such an expansion is legitimate, since we consider a Higgs boson much heavier than the typical hadronic mass scale. In this section we calculate the first term in the above expansion and the related decay rate. This is equivalent to the assumption that all five quarks are massless and the top quark decouples ($m_t\to\infty$).

The diagrammatic representation for Π in somewhat symbolic form looks like



(3.8)

Next, we evaluate one-, two-, and three-loop massless Feynman diagrams. By simple power counting, it is easy to find that, in general, the above diagrams are UV divergent. The unrenormalized contribution from a typical three-loop diagram in the $\overline{\rm MS}$ renormalization scheme (Bardeen, Buras, Duke, and Muta, 1978) reads

2 

$$\sim \frac{1}{(4\pi)^2}\left(\frac{\alpha_s^{\rm B}}{4\pi}\right)^2 N_F \frac{C_F C_A}{2}(m_f^{\rm B})^2 Q^2 \left(\frac{\mu_{\overline{\rm MS}}^2}{Q^2}\right)^{3\varepsilon}\left[\frac{16}{\varepsilon^3}+\frac{400}{3\varepsilon^2}+\frac{2344}{3\varepsilon}-\frac{160}{\varepsilon}\zeta(3)+\frac{11800}{3} -1312\zeta(3)-240\zeta(4)+320\zeta(5)\right],$$

where m_f^B is the f-flavor quark mass originating from the quark mass dependence of the Yukawa coupling. $\zeta(3)$, $\zeta(4)$, and $\zeta(5)$ are ordinary Riemann ζ functions. The number 2 in front of the diagram stands for the symmetry factor. The algorithms for the evaluation of propagator-type one-, two-, and three-loop massless Feynman diagrams have been given by Tkachov (1981, 1983a) and Chetyrkin and Tkachov (1981). For the description of the algorithms, see Gorishny, Larin, Surguladze, and Tkachov (1989). The results given in this section were reobtained with the help of the program HEPLoops (Surguladze, 1992), and the previous results (Gorishny, Kataev, Larin, and Surguladze, 1990, 1991a, 1991b) were independently confirmed (Surguladze, 1994b).

As one can see, each three-loop diagram may contain, in general, a pole with power $\leqslant 3$. In the vector channel, after summing the results for all diagrams with an appropriate symmetry and SU(N) group factor, the leading pole cancels. This is the consequence of the conservation of electromagnetic currents. In the scalar channel, the leading poles remain in Π. This is related to the quark mass dependence of the coupling.

Evaluating the unrenormalized correlation function (3.2) and using the definition (3.5), we obtain the unrenormalized D function in the massless limit,

$$D\left(\frac{\mu_{\overline{\rm MS}}^2}{Q^2},\alpha_s\right)=\frac{1}{(4\pi)^2}N_F(m_f^B)^2\left\{\left(\frac{\mu_{\overline{\rm MS}}^2}{Q^2}\right)^{\varepsilon}(2+4\varepsilon+8\varepsilon^2)+\left(\frac{\alpha_s^B}{4\pi}\right)\left(\frac{\mu_{\overline{\rm MS}}^2}{Q^2}\right)^{2\varepsilon}C_F\left[\frac{12}{\varepsilon}+58+\varepsilon(227-48\zeta(3))\right]\right.$$
$$+\left(\frac{\alpha_s^B}{4\pi}\right)^2\left(\frac{\mu_{\overline{\rm MS}}^2}{Q^2}\right)^{3\varepsilon}C_F\left[C_F\left(\frac{36}{\varepsilon^2}+\frac{279}{\varepsilon}+\frac{3139}{2}-360\zeta(3)\right)+C_A\left(\frac{22}{\varepsilon^2}+\frac{201}{\varepsilon}+\frac{2511}{2}-300\zeta(3)\right)\right.$$
$$\left.\left.-TN\left(\frac{8}{\varepsilon^2}+\frac{68}{\varepsilon}+414-96\zeta(3)\right)\right]+O(\alpha_s^3)\right\}. \quad (3.9)$$

The above expression requires the renormalization of the strong coupling [Eq. (2.24)] and the multiplicative renormalization [Eq. (2.25)] originating from the quark mass dependence of the Yukawa coupling.

Expanding the factors $(\mu_{\overline{\rm MS}}^2/Q^2)^{l\varepsilon}$ in terms of ε and performing the renormalizations of the coupling and the quark mass, we get a finite analytical expression for the D function in the $\overline{\mathrm{MS}}$ scheme,

$$D\left(\frac{\mu^2_{\overline{\mathrm{MS}}}}{Q^2},\alpha_s\right)=\frac{N_F}{8\pi^2}m_f^2\left\{1+\left(\frac{\alpha_s}{4\pi}\right)C_F\left[17+6\log\left(\frac{\mu^2_{\overline{\mathrm{MS}}}}{Q^2}\right)\right]+\left(\frac{\alpha_s}{4\pi}\right)^2 C_F\left[C_F\left(\frac{691}{4}-36\zeta(3)\right)+C_A\left(\frac{893}{4}-62\zeta(3)\right)\right.\right.$$
$$\left.\left.-TN(65-16\zeta(3))+\log\left(\frac{\mu^2_{\overline{\mathrm{MS}}}}{Q^2}\right)\left(105C_F+\frac{284}{3}C_A-\frac{88}{3}TN\right)+\log^2\left(\frac{\mu^2_{\overline{\mathrm{MS}}}}{Q^2}\right)(18C_F+11C_A-4TN)\right]\right\}. \tag{3.10}$$

For standard QCD with the color $SU_c(3)$-symmetry group, the analytical result for the D function reads (Surguladze, 1989d)

$$D\left(\frac{\mu^2_{\overline{\mathrm{MS}}}}{Q^2},\alpha_s\right)=\frac{3}{8\pi^2}m_f^2\left\{1+\left(\frac{\alpha_s}{\pi}\right)\left[\frac{17}{3}+2\log\left(\frac{\mu^2_{\overline{\mathrm{MS}}}}{Q^2}\right)\right]\right.$$
$$+\left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{10801}{144}-\frac{39}{2}\zeta(3)\right.$$
$$-\left(\frac{65}{24}-\frac{2}{3}\zeta(3)\right)N+\log\left(\frac{\mu^2_{\overline{\mathrm{MS}}}}{Q^2}\right)\left(\frac{106}{3}-\frac{11}{9}N\right)$$
$$\left.\left.+\log^2\left(\frac{\mu^2_{\overline{\mathrm{MS}}}}{Q^2}\right)\left(\frac{19}{4}-\frac{1}{6}N\right)\right]\right\}. \tag{3.11}$$

This completes the evaluation of the correlation function of the two scalar quark currents in the massless limit at the three-loop approximation.

There is one crucial test of this calculation based on renormalization-group constraints. The solution of the renormalization-group equation (3.6) can be conveniently rewritten as

$$D\left(\frac{\mu^2}{Q^2},m_f(\mu),\alpha_s(\mu)\right)$$
$$=\frac{3}{8\pi^2}m_f^2(\mu)\sum_{0\leqslant j\leqslant i}\left(\frac{\alpha_s(\mu)}{\pi}\right)^i a_{ij}\log^j\frac{\mu^2}{Q^2}. \tag{3.12}$$

Applying the differential operator $\mu^2 d/d\mu^2$ to both sides of Eq. (3.12), taking into account the renormalization-group invariance of the D function and Eqs. (2.8) and (2.9), we obtain to $O(\alpha_s)$

$$a_{11}=2\gamma_0 a_{00}, \tag{3.13}$$

to $O(\alpha_s^2)$

$$a_{21}=2\gamma_1 a_{00}+(\beta_0+2\gamma_0)a_{10},$$
$$a_{22}=(\beta_0+2\gamma_0)\frac{a_{11}}{2}=(\beta_0+2\gamma_0)\gamma_0 a_{00}, \tag{3.14}$$

and to $O(\alpha_s^3)$

$$a_{31}=2(\beta_0+\gamma_0)a_{20}+(\beta_1+2\gamma_1)a_{10}+2\gamma_2 a_{00},$$
$$a_{32}=(\beta_0+\gamma_0)a_{21}+(\beta_1+2\gamma_1)\frac{a_{11}}{2}$$
$$=(\beta_0+\gamma_0)[2\gamma_1 a_{00}+(\beta_0+2\gamma_0)a_{10}]$$
$$+(\beta_1+2\gamma_1)\gamma_0 a_{00}, \tag{3.15}$$
$$a_{33}=\frac{2}{3}(\beta_0+\gamma_0)a_{22}=\frac{2}{3}\gamma_0(\beta_0+\gamma_0)(\beta_0+2\gamma_0)a_{00}.$$

The relations (3.13) and (3.14) provide a powerful check of our calculation, while the relations (3.15) allow us to evaluate the log terms to $O(\alpha_s^3)$, without explicit calculations of the corresponding four-loop diagrams. With those relations, the information available at present, namely, the QCD β function, mass anomalous dimension, and the two-point correlation function up to the three-loop level, is fully exploited. In fact, similar relations can be derived for the correlation function Π. However, the renormalization-group equation for Π is not a homogeneous one, and the anomalous dimension function up to the corresponding order of α_s is necessary.

We evaluate the decay rate of the neutral Higgs boson into a quark-antiquark pair by analytical continuation of $D(\mu^2/Q^2,m_f(\mu),\alpha_s(\mu))$ from Euclidean to Minkowski space. The total decay rate can be obtained by summing up over all participating quark flavors,

$$\Gamma(H\to\text{hadrons})$$
$$=\frac{3\sqrt{2}G_F M_H}{8\pi}\sum_{f=u,d,s,\ldots}m_f^2\left\{1+\frac{\alpha_s}{\pi}\left(\frac{17}{3}+2\log\frac{\mu^2_{\overline{\mathrm{MS}}}}{M_H^2}\right)\right.$$
$$+\left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{10801}{144}-\frac{19}{2}\zeta(2)-\frac{39}{2}\zeta(3)+\frac{106}{3}\log\frac{\mu^2_{\overline{\mathrm{MS}}}}{M_H^2}\right.$$
$$+\frac{19}{4}\log^2\frac{\mu^2_{\overline{\mathrm{MS}}}}{M_H^2}-N\left(\frac{65}{24}-\frac{1}{3}\zeta(2)-\frac{2}{3}\zeta(3)\right.$$
$$\left.\left.\left.+\frac{11}{9}\log\frac{\mu^2_{\overline{\mathrm{MS}}}}{M_H^2}+\frac{1}{6}\log^2\frac{\mu^2_{\overline{\mathrm{MS}}}}{M_H^2}\right)\right]\right\}. \tag{3.16}$$

The Riemann function $\zeta(2)=\pi^2/6$ arose from the analytical continuation of the $\log^2\mu^2_{\overline{\mathrm{MS}}}/Q^2$ term and $\zeta(3)=1.202\,056\,903$. The procedure of analytical continuation and the appearance of additional invariant contributions have been discussed in several earlier works (Krasnikov and Pivovarov, 1982; Pennington and

Ross, 1982; Radyushkin, 1982; Pivovarov, 1992a). Note that in some cases those additional corrections are large and affect the result significantly. This is especially true for the total cross section in the process $e^+e^-\to$hadrons. To minimize such corrections, it was proposed, for instance, to redefine the expansion parameter (Pennington and Ross, 1982; Radyushkin, 1982).

B. The decay rate in terms of pole quark mass

For the heavy flavor decay mode of the Higgs, it is relevant to parametrize the decay rate in terms of quark pole mass (see, e.g., Kniehl, 1994a). Let us rewrite the result for $\Gamma_{H\to q_f\bar{q}_f}$ in terms of pole quark mass, assuming that heavy quark is not exactly on-shell. This subsection is based mainly on recent findings (Surguladze, 1994a, 1994b).

Solving the renormalization-group equation for the quark mass [Eq. (2.7)], we obtain the following scaling law for the running quark mass:

$$\frac{m_f(\mu_1)}{m_f(\mu_2)}=\frac{\phi(\alpha_s(\mu_1))}{\phi(\alpha_s(\mu_2))}, \tag{3.17}$$

where

$$\phi(\alpha_s(\mu))=\left(2\beta_0\frac{\alpha_s(\mu)}{\pi}\right)^{\gamma_0/\beta_0}\left\{1+\left(\frac{\gamma_1}{\beta_0}-\frac{\beta_1\gamma_0}{\beta_0^2}\right)\frac{\alpha_s(\mu)}{\pi}+\frac{1}{2}\left[\left(\frac{\gamma_1}{\beta_0}-\frac{\beta_1\gamma_0}{\beta_0^2}\right)^2+\frac{\gamma_2}{\beta_0}-\frac{\beta_1\gamma_1}{\beta_0^2}-\frac{\beta_2\gamma_0}{\beta_0^2}+\frac{\beta_1^2\gamma_0}{\beta_0^3}\right]\times\left(\frac{\alpha_s(\mu)}{\pi}\right)^2\right\}. \tag{3.18}$$

In the above equation, all appropriate quantities are evaluated for N active quark flavors. N can be determined according to the scale of M_H. At present, we usually consider $N=5$.

For the running coupling, we obtain the following evolution equation to $O(\alpha_s^3)$ (Surguladze, 1994b),

$$\frac{\alpha_s^{(n)}(\mu_1)}{\pi}=\frac{\alpha_s^{(N)}(\mu_2)}{\pi}+\left(\frac{\alpha_s^{(N)}(\mu_2)}{\pi}\right)^2\left(\beta_0^{(N)}\log\frac{\mu_2^2}{\mu_1^2}+\frac{1}{6}\sum_l\log\frac{m_l^2}{\mu_1^2}\right)+\left(\frac{\alpha_s^{(N)}(\mu_2)}{\pi}\right)^3\left[\beta_1^{(N)}\log\frac{\mu_2^2}{\mu_1^2}+\frac{19}{24}\sum_l\log\frac{m_l^2}{\mu_1^2}+\left(\beta_0^{(N)}\log\frac{\mu_2^2}{\mu_1^2}+\frac{1}{6}\sum_l\log\frac{m_l^2}{\mu_1^2}\right)^2-\frac{25}{72}(N-n)\right], \tag{3.19}$$

where the superscript N (n) indicates that the corresponding quantity is evaluated for N (n) numbers of participating quark flavors. Conventionally (see, e.g., Marciano, 1984), N (n) is specified to be the number of quark flavors with mass $\leq\mu_2$ $(\leq\mu_1)$. However, Eq. (3.19) is relevant for any $n\leq N$ and arbitrary μ_1 and μ_2, regardless of the conventional specification of the number of quark flavors. The $\log m_l/\mu_1$ terms are due to the "quark threshold" crossing effects, and the constant coefficients $1/6=\beta_0^{(k-1)}-\beta_0^{(k)}$, $19/24=\beta_1^{(k-1)}-\beta_1^{(k)}$ represent the contributions of the quark loop in the β function. The sum runs over $N-n$ quark flavors (e.g., $l=b$ if $n=4$ and $N=5$). Note that m_l is the pole mass of the quark with flavor l. For the on-shell definitions of the quark masses, Eq. (3.19) changes—the constant $-25/72$ should be replaced by $+7/72$. The above equation is derived based on Eq. (2.17), the QCD matching conditions for α_s at quark thresholds (Bernreuter and Wetzel, 1982; Marciano, 1984; Barnett, Haber, and Soper, 1988; Rodrigo and Santamaria, 1993) and the one-loop relation between on-shell and pole quark masses. Equation (3.19) is consistent with the QCD matching relation at $m_f(m_f)$ (Bernreuter and Wetzel, 1982),

$$\alpha_s^{(N_f-1)}(m_f(m_f))=\alpha_s^{(N_f)}(m_f(m_f))+[\alpha_s^{(N_f)}(m_f(m_f))]^3\times(C_A/9-17C_F/96)/\pi^2. \tag{3.20}$$

Here and below, N_f is the number of quark flavors $u,d,\ldots,f$. Note that the nonlogarithmic constant at $O(\alpha_s^3)$ in Eq. (3.19) will not contribute in further analysis.

Next, using the scaling properties of the MS running mass and Eq. (3.19), one obtains the following matching condition,

$$m_f^{(N-1)}(\mu)=m_f^{(N)}(\mu)\left\{1+\left(\frac{\alpha_s^{(N)}(\mu)}{\pi}\right)^2\times\left[\delta(m_f,m_{f'})-\frac{5}{36}\log\frac{\mu^2}{m_f^2}-\frac{1}{12}\log^2\frac{\mu^2}{m_f^2}+\frac{1}{6}\log\frac{\mu^2}{m_f^2}\log\frac{\mu^2}{m_{f'}^2}-\frac{2}{9}\log\frac{m_{f'}^2}{m_f^2}\right]\right\}, \tag{3.21}$$

where the constant terms are $1/12=\gamma_0(\beta_0^{(k-1)}-\beta_0^{(k)})/2$, $5/36=\gamma_1^{(k-1)}-\gamma_1^{(k)}$, and $2/9=C_F(\beta_0^{(k-1)}-\beta_0^{(k)})$. In general, the $\delta(m_f,m_{f'})$ is the finite contribution of the single virtual heavier quark with mass $m_{f'}$, entering when one increases the number of flavors from $N-1$ to N (one can also consider the particular case $m_{f'}=m_f$).

From the two-loop on-shell quark mass renormalization, one has (Broadhurst, Gray, and Schilcher, 1991)

$$\delta(m_f,m_{f'})=-\zeta(2)/3-71/144+(4/3)\Delta(m_{f'}/m_f), \tag{3.22}$$

where

$$\Delta(r)=\frac{1}{4}\left[\log^2 r+\zeta(2)-\left(\log r+\frac{3}{2}\right)r^2 -(1+r)(1+r^3)L_+(r)-(1-r)(1-r^3)L_-(r)\right], \tag{3.23}$$

$$L_\pm(r)=\int_0^{1/r}dx\,\frac{\log x}{x\pm 1}.$$

$L_\pm(r)$ can be evaluated for different quark mass ratios r numerically. We relate the $\overline{\rm MS}$ quark mass $m_f(m_f)$ to the pole mass m_f using the $O(\alpha_s^2)$ on-shell results of Broadhurst, Gray, and Schilcher (1991),

$$m_f^{(N_f)}(m_f)=m_f\left[1-\frac{4}{3}\frac{\alpha_s^{(N_f)}(m_f)}{\pi}+\left(\frac{16}{9}-K_f\right)\times\left(\frac{\alpha_s^{(N_f)}(m_f)}{\pi}\right)^2\right], \tag{3.24}$$

where

$$K_f=\frac{3817}{288}+\frac{2}{3}(2+\log 2)\zeta(2)-\frac{1}{6}\zeta(3)-\frac{N_f}{3}\left(\zeta(2)+\frac{71}{48}\right)+\frac{4}{3}\sum_{m_l\leqslant m_f}\Delta\left(\frac{m_l}{m_f}\right). \tag{3.25}$$

The first four terms in K_f represent the QCD contribution with N_f massless quarks, while the sum is the correction due to the N_f nonvanishing quark masses.

Combining Eqs. (3.17), (3.18), and (3.19)–(3.24), one obtains the relation between the $\overline{\rm MS}$ quark mass $m_f(M_H)$, renormalized at M_H and evaluated for the N-flavor theory, and the pole quark mass m_f (Surguladze, 1994b),

$$m_f^{(N)}(M_H)=m_f\left\{1-\frac{\alpha_s^{(N)}(M_H)}{\pi}\left(\frac{4}{3}+\gamma_0\log\frac{M_H^2}{m_f^2}\right)-\left(\frac{\alpha_s^{(N)}(M_H)}{\pi}\right)^2\left[K_f+\sum_{m_f<m_{f'}<M_H}\delta(m_f,m_{f'})-\frac{16}{9}+\left(\gamma_1^{(N)}-\frac{4}{3}\gamma_0+\frac{4}{3}\beta_0^{(N)}\right)\log\frac{M_H^2}{m_f^2}+\frac{\gamma_0}{2}(\beta_0^{(N)}-\gamma_0)\log^2\frac{M_H^2}{m_f^2}\right]\right\}. \tag{3.26}$$

Note that N is specified according to the size of M_H and has no correlation with the quark mass m_f. Thus, for instance, one can apply Eq. (3.26) to the charm mass $m_c^{(5)}(M_H)$ evaluated for five-flavor theory.

Substituting Eqs. (3.26) and (3.25) and appropriate β-function and mass anomalous dimension coefficients (see Sec. II) into Eq. (3.16), one obtains the decay rate in terms of the pole quark masses,

$$\Gamma(H\to\text{hadrons})=\frac{3\sqrt{2}G_FM_H}{8\pi}\sum_{f=u,d,s,\ldots}m_f^2\left\{1+\frac{\alpha_s^{(N)}(M_H)}{\pi}\left(3-2\log\frac{M_H^2}{m_f^2}\right)+\left(\frac{\alpha_s^{(N)}(M_H)}{\pi}\right)^2\left[\frac{697}{18}-\left(\frac{73}{6}+\frac{4}{3}\log 2\right)\zeta(2)-\frac{115}{6}\zeta(3)-N\left(\frac{31}{18}-\zeta(2)-\frac{2}{3}\zeta(3)\right)-\left(\frac{87}{4}-\frac{13}{18}N\right)\log\frac{M_H^2}{m_f^2}-\left(\frac{3}{4}-\frac{1}{6}N\right)\log^2\frac{M_H^2}{m_f^2}-\frac{8}{3}\sum_{m_l<M_H}\Delta\left(\frac{m_l}{m_f}\right)\right]\right\}. \tag{3.27}$$

Recall that at the beginning we neglected terms that were suppressed by powers m_f^2/M_H^2. Such corrections to the decay rate, in general, may not be entirely negligible and must be taken into account in precise numerical analyses. Those corrections due to the nonvanishing quark masses have also been calculated. For the explicit results, we refer to the original works (Surguladze, 1994a, 1994b; Chetyrkin and Kwiatkowski, 1995; Kniehl, 1995a). In the next section we give the results for the quark mass corrections to the correlation functions Π.

The full analytical result for the decay rate of $H\to q_f\bar{q}_f$ in terms of pole quark masses, including the leading-order (two-loop) QCD corrections, has been obtained independently by several groups: Braaten and Leveille (1980), Inami and Kubota (1981), and Drees and Hikasa (1990). In the work by Sakai (1980), the two-loop result has been obtained in the zero quark mass limit,

$$\Gamma_{H\to q_f\bar{q}_f}=\frac{3\sqrt{2}G_FM_H}{8\pi}m_f^2\left(1-\frac{4m_f^2}{M_H^2}\right)^{3/2}\times\left[1+\frac{\alpha_s(M_H)}{\pi}\delta^{(1)}\left(\frac{m_f^2}{M_H^2}\right)+O(\alpha_s^2)\right], \tag{3.28}$$

where

$$\delta^{(1)}=\frac{4}{3}\left[\frac{a(\eta)}{\eta}+\frac{3+34\eta^2-13\eta^4}{16\eta^3}\log\omega+\frac{21\eta^2-3}{8\eta^2}\right],$$

$$a(\eta)=(1+\eta^2)\left[4\mathrm{Li}_2(\omega^{-1})+2\ \mathrm{Li}_2(-\omega^{-1})-\log\omega\ \log\frac{8\eta^2}{(1+\eta)^3}\right]-\eta\ \log\frac{64\eta^4}{(1-\eta^2)^3},$$

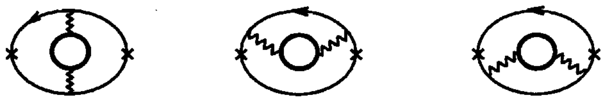


FIG. 2. $O(\alpha_s^2)$ Feynman diagrams responsible for the virtual heavy-quark contribution.

$$\omega=\frac{1+\eta}{1-\eta},\quad \eta=\left(1-\frac{4m_f^2}{M_H^2}\right)^{1/2}$$

and the Spence function is defined as usual,

$$\mathrm{Li}_2(x)=-\int_0^x dx\frac{\log(1-x)}{x}=\sum_{n=1}^{\infty}\frac{x^n}{n^2}.$$

The expansion of the right-hand side of Eq. (3.28) in a power series in terms of small m_f^2/M_H^2 has the following form,

$$\Gamma_{H\to q_f\bar{q}_f}=\frac{3\sqrt{2}G_F M_H}{8\pi}m_f^2\left\{\left(1-6\frac{m_f^2}{M_H^2}+\cdots\right)+\frac{\alpha_s(M_H)}{\pi}\left[3-2\log\frac{M_H^2}{m_f^2}-\frac{m_f^2}{M_H^2}\left(8-24\log\frac{M_H^2}{m_f^2}\right)+\cdots\right]+O(\alpha_s^2)\right\},\quad (3.29)$$

where the periods cover higher-order terms $\sim(m_f/M_H)^{2k}$, $k=2,3\ldots$. One can see that the leading terms agree with the result (3.27).

Numerically, the $\overline{\mathrm{MS}}$ high-order QCD corrections for the considered process are large and reduce the decay rates by about 40%.

IV. QUARK MASS CORRECTIONS TO THE CORRELATION FUNCTIONS

In Sec. III we neglected all quark masses in the corresponding Feynman diagrams in comparison with the momentum scale of the problem. In other words, we calculated the leading term in the expansion in terms of small m_f^2/s (for the Higgs boson decay, $s=M_H^2$) in the limit of the infinitely heavy top quark, $m_t\to\infty$. However, in the real world, quarks are massive and the leading term in the above expansion may not always give a satisfactory approximation. On the other hand, starting at $O(\alpha_s^2)$, a virtual heavy quark can also appear in certain topological types of Feynman diagrams (Figs. 2 and 3) regardless of the momentum scale of the problem.

According to the decoupling theorem (Appelquist and Carazzone, 1975), virtual quarks much heavier than the momentum scale of the problem decouple. However, in the process of Z boson decay, for instance, the effect of the top quark may not be entirely negligible, since m_t is not much greater than M_Z. A similar role could be played by the charm quark in the hadronic decay of the tau lepton. The evaluation of the virtual top quark contribution (Fig. 2) to the decay rate $Z\to$hadrons and related quantities was done by Kniehl (1990), Soper and Surguladze (1994), and Hoang, Jezabek, Kühn, and Teubner (1994) without the use of large or small mass approximations. The correction turned out to be moderate and in good agreement with the results obtained with the help of the large mass expansion technique (Chetyrkin, 1993a). The contribution of the diagrams in Fig. 2, in the presence of a virtual heavy quark, to the two-point correlation function of the electromagnetic quark currents was evaluated previously by Wetzel and Bernreuther (1981). Kniehl and Kühn (1989, 1990) calculated the $O(\alpha_s^2)$ correction to the decay rate $Z\to$hadrons due to the large mass' splitting in the top-bottom doublet (Fig. 3). This correction turned out to be large and important.

In Sec. IV we consider only the leading correction in the expansion in terms of small quark mass. For the calculations of virtual heavy quark contributions, we refer the reader to the above-mentioned original works (see also Kniehl, 1994b, 1995b). The discussion in this section is based on the works by Surguladze (1994a, 1994b, 1994c).

Let us expand the full two-point correlation function, defined by Eq. (2.10) in the vector channel and by Eq. (3.2) in the scalar and pseudoscalar channels, in powers of m_f^2/Q^2 in the "deep" Euclidean region,

$$\left(\frac{1}{m_f^2Q^2}\right)^d\Pi(Q^2,m_f,m_V)=\Pi_1(Q^2)+\frac{m_f^2}{Q^2}\Pi_{m_f^2}(Q^2)+\sum_{V=u,d,s,c,b}\frac{m_V^2}{Q^2}\Pi_{m_V^2}(Q^2)+\cdots,\quad (4.1)$$

where $d=0$ in the vector channel and $d=1$ in the scalar and pseudoscalar channels. The last term in the above expansion is due to the Feynman diagrams containing a virtual fermionic loop. Note, however, that in the vector channel the contribution from the diagrams in Fig. 3 vanishes according to Furry's theorem (Furry, 1937).

In order to evaluate the coefficient functions in the right-hand side of Eq. (4.1), it is sufficient to write the diagrammatic representation for $\Pi(Q^2,m_f^B,m_V^B)$ up to the desired level of perturbation theory and apply the appropriate projector. To $O(\alpha_s^2)$, one has

$$\Pi_{m_f^{2n}m_V^{2k}}(Q^2,\alpha_s)=\frac{1}{(2n)!(2k)!}\left(\frac{d}{dm_f^B}\right)^{2n}\left(\frac{d}{dm_V^B}\right)^{2k}\times\left\{\frac{\Pi(Q^2,m_f^B,m_V^B,\alpha_s^B)}{(m_f^B)^{2d}Q^{2(d-n-k)}}\right\}_{\substack{m_f^B=m_V^B=0\\ \alpha_s^B\to Z_\alpha\alpha_s}}\times(Z_m^2)^{(1+d)},\quad (4.2)$$

FIG. 3. $O(\alpha_s^2)$ Feynman diagrams responsible for the contribution due to the top-bottom mass splitting.

where $n,k=0,1$, $n+k\leq 1$, and superscript B denotes the "bare" quantities. The mass-renormalization constant $Z_m=m_f^B/m_f$ can be obtained from Eq. (2.25). The Feynman diagrams contributing to the $\Pi_{m_f^{2n}m_V^{2k}}$ are the same as those for the calculation of Π_1 [see Eq. (3.8)] but with massive fermion propagators. The calculations of all one-, two-, and three-loop diagrams have been performed using the program HEPLoops (Surguladze, 1992).

The obtained expressions for Π_i at each order of α_s are polynomials with respect to $1/\varepsilon$ and $\log\mu_{\overline{\rm MS}}^2/Q^2$. The poles can be removed by an additive renormalization. We note that there are no terms like $(1/\varepsilon^n)(\log\mu_{\overline{\rm MS}}^2/Q^2)^k$. They appear only at higher orders $\sim m_f^2 m_f^4/Q^4$ and represent infrared mass logarithms. The corresponding prescription similar to the Bogolyubov ultraviolet R operation has been worked out by Chetyrkin, Gorishny, and Tkachov (1982), Tkachov (1983b, 1983c), and Gorishny, Larin, and Tkachov (1983a, 1983b, 1983c; see also Tkachov, 1991, 1993 and references therein). The infrared mass singularities were studied earlier by Marciano (1975). In the present paper we consider only the terms $\sim m_f^2/Q^2$, which are sufficient for most of the phenomenologically interesting applications.

In the vector channel, we obtain the following $\overline{\rm MS}$ analytical result (Gorishny, Kataev, and Larin, 1986; Surguladze, 1994c),

$$\Pi_{m_f^2}\left(\frac{\mu_{\overline{\rm MS}}^2}{Q^2},\alpha_s\right)=\frac{N_F}{(4\pi)^2}\Bigg\{-8-\left(\frac{\alpha_s}{\pi}\right)C_F\left(16+12\log\frac{\mu_{\overline{\rm MS}}^2}{Q^2}\right)-\left(\frac{\alpha_s}{\pi}\right)^2\Bigg[C_F^2\left(\frac{1667}{24}-\frac{5}{3}\zeta(3)-\frac{70}{3}\zeta(5)+\frac{51}{2}\log\frac{\mu_{\overline{\rm MS}}^2}{Q^2}+9\log^2\frac{\mu_{\overline{\rm MS}}^2}{Q^2}\right)$$
$$+C_FC_A\left(\frac{1447}{24}+\frac{16}{3}\zeta(3)-\frac{85}{3}\zeta(5)+\frac{185}{6}\log\frac{\mu_{\overline{\rm MS}}^2}{Q^2}+\frac{11}{2}\log^2\frac{\mu_{\overline{\rm MS}}^2}{Q^2}\right)$$
$$-C_FTN\left(\frac{95}{6}+\frac{26}{3}\log\frac{\mu_{\overline{\rm MS}}^2}{Q^2}+2\log^2\frac{\mu_{\overline{\rm MS}}^2}{Q^2}\right)\Bigg]\Bigg\}, \quad (4.3)$$

$$\Pi_{m_V^2}\left(\frac{\mu_{\overline{\rm MS}}^2}{Q^2},\alpha_s\right)=\frac{N_F}{(4\pi)^2}\left(\frac{\alpha_s}{\pi}\right)^2C_FT\left[\frac{64}{3}-16\zeta(3)\right]. \quad (4.4)$$

The contribution to the physical process, in particular, to the decay rate of $Z\to$hadrons, can be obtained simply by taking the imaginary part in the right-hand side of Eqs. (4.3) and (4.4) at $Q^2=-s+i0$. We note that the $\Pi_{m_f^2}$ and $\Pi_{m_V^2}$ turned out to be finite. No overall subtraction is necessary. Moreover, one can see that the imaginary part or the contribution to the decay rate vanishes at the parton level. This can be checked by the calculation of the parton contribution in the vector channel with explicit dependence on quark mass. Indeed, calculating the trivial fermionic loop, we obtain

$$\Pi_{\rm parton}\left(\frac{\mu_{\overline{\rm MS}}^2}{-q^2},\frac{m_f^2}{-q^2}\right)=\frac{N_F}{(4\pi^2)}\left[\frac{4}{3}\frac{1}{\varepsilon}-8\int_0^1 x(1-x)\times\log\frac{m_f^2-x(1-x)q^2}{\mu_{\overline{\rm MS}}^2}dx\right]. \quad (4.5)$$

Taking the discontinuity under the integral and then evaluating the trivial integral with the Θ function, we obtain

$$\frac{1}{2\pi i}{\rm disc}\Pi_{\rm parton}\left(\frac{\mu_{\overline{\rm MS}}^2}{-q^2},\frac{m_f^2}{-q^2}\right)=\frac{N_F}{(4\pi^2)}\left(1+\frac{2m_f^2}{q^2}\right)\sqrt{1-4m_f^2/q^2}=\frac{N_F}{(4\pi^2)}O\left(\frac{m_f^4}{q^4}\right). \quad (4.6)$$

The $\sim m_f^2/Q^2$ contribution to the Adler D function can be obtained from Eqs. (4.3) and (4.4) by differentiating with respect to Q^2.

There is some confusion in the literature concerning the above results. Initially, the corrections $\sim m_f^2/Q^2$ in the vector channel were calculated by Gorishny, Kataev, and Larin (1986). Later, in similar calculations (Surguladze, 1989a), a slightly different result was obtained, which was confirmed in further publications (see, e.g., Kataev, 1990, 1991). However, in recent works (Chetyrkin and Kwiatkowski, 1993; Surguladze, 1994c), the initial result of Gorishny, Kataev, and Larin (1986) has been confirmed. Unfortunately, in the analysis of the mass corrections to the Z decay rates (Chetyrkin and Kühn, 1990), the incorrect result was used. Fortunately, the main conclusions of Chetyrkin and Kühn (1990) are not affected. Summarizing, we note that the results (4.3) and (4.4) (Gorishny, Kataev, and Larin, 1986; Chetyrkin and Kwiatkowski, 1993; Surguladze, 1994c) seem now to be reliable.

In the scalar channel, the result for the standard $\mathrm{SU}_c(3)$ gauge group reads (Surguladze, 1994b)

$$\Pi_{m_f^2}\left(\frac{\mu_{\overline{\rm MS}}^2}{Q^2},\alpha_s\right)=-\frac{1}{4\pi^2}\Bigg\{12+9\log\frac{\mu_{\overline{\rm MS}}^2}{Q^2}+\left(\frac{\alpha_s}{\pi}\right)\left(94-36\zeta(3)+60\log\frac{\mu_{\overline{\rm MS}}^2}{Q^2}+18\log^2\frac{\mu_{\overline{\rm MS}}^2}{Q^2}\right)$$
$$+\left(\frac{\alpha_s}{\pi}\right)^2\Bigg[\frac{17245}{16}-\frac{1690}{3}\zeta(3)-3\zeta(4)+\frac{385}{3}\zeta(5)+\left(\frac{7149}{8}-249\zeta(3)\right)\log\frac{\mu_{\overline{\rm MS}}^2}{Q^2}+\frac{1113}{4}\log^2\frac{\mu_{\overline{\rm MS}}^2}{Q^2}$$
$$+\frac{81}{2}\log^3\frac{\mu_{\overline{\rm MS}}^2}{Q^2}-N\left(\frac{817}{24}-6\zeta(3)+\left(\frac{313}{12}-6\zeta(3)\right)\log\frac{\mu_{\overline{\rm MS}}^2}{Q^2}+\frac{15}{2}\log^2\frac{\mu_{\overline{\rm MS}}^2}{Q^2}+\log^3\frac{\mu_{\overline{\rm MS}}^2}{Q^2}\right)\Bigg]+\text{“simple poles”}\Bigg\},\tag{4.7}$$

$$\Pi_{m_V^2}\left(\frac{\mu_{\overline{\rm MS}}^2}{Q^2},\alpha_s\right)=\frac{1}{4\pi^2}\left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{8}{3}+6\log\frac{\mu_{\overline{\rm MS}}^2}{Q^2}+\text{“simple pole”}\right],\tag{4.8}$$

where under the “simple pole” we mean number/ε^k with no dependence on $\log\mu^2/Q^2$. The simple poles have no imaginary part and consequently will not contribute to the observable quantities at the given order of α_s. Note that the $\Pi_{m_V^2}$ in Eq. (4.8) does not include the contribution from the triangle anomaly-type graphs pictured in Fig. 3. Those graphs make the following additional contribution to Π in Eq. (4.1) (Surguladze, 1994b),

$$+\sum_{f'=u,d,s,c,b}\frac{m_{f'}^2}{Q^2}\times\frac{1}{4\pi^2}\left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{118}{3}-20\zeta(3)-10\zeta(5)+12\log\frac{\mu_{\overline{\rm MS}}^2}{Q^2}+\text{“simple pole”}\right].\tag{4.9}$$

The above results are relevant for the decay rate of the standard-model Higgs boson into a quark-antiquark pair, calculated in the previous section in the massless quark limit. Corrections $\sim m_f^2/M_H^2$ can be obtained from Eqs. (4.7), (4.8), and (4.9) (Surguladze, 1994b).

In the pseudoscalar channel, we define the quark currents as $j_f=m_f\bar{q}_f i\gamma_5 q_f$. We also define the γ_5 matrix in D-dimensional space-time as an object with the following properties,

$$\{\gamma_5,\gamma_\mu\}=0,\quad \gamma_5\gamma_5=1.\tag{4.10}$$

The above definition causes no problems in dimensional regularization when there are two γ_5 matrices in a closed fermionic loop. We obtain (Surguladze, 1994a)

$$\Pi_{m_f^2}\left(\frac{\mu_{\overline{\rm MS}}^2}{Q^2},\alpha_s\right)=-\frac{1}{4\pi^2}\Bigg\{3\log\frac{\mu_{\overline{\rm MS}}^2}{Q^2}+\left(\frac{\alpha_s}{\pi}\right)\left(6-12\zeta(3)+4\log\frac{\mu_{\overline{\rm MS}}^2}{Q^2}+6\log^2\frac{\mu_{\overline{\rm MS}}^2}{Q^2}\right)$$
$$+\left(\frac{\alpha_s}{\pi}\right)^2\Bigg[-\frac{6713}{144}-116\zeta(3)-\zeta(4)+\frac{235}{3}\zeta(5)+\left(\frac{1429}{24}-83\zeta(3)\right)\log\frac{\mu_{\overline{\rm MS}}^2}{Q^2}+\frac{155}{4}\log^2\frac{\mu_{\overline{\rm MS}}^2}{Q^2}$$
$$+\frac{27}{2}\log^3\frac{\mu_{\overline{\rm MS}}^2}{Q^2}-N\left(-\frac{31}{72}-\frac{2}{3}\zeta(3)+\left(\frac{9}{4}-2\zeta(3)\right)\log\frac{\mu_{\overline{\rm MS}}^2}{Q^2}+\frac{7}{6}\log^2\frac{\mu_{\overline{\rm MS}}^2}{Q^2}+\frac{1}{3}\log^3\frac{\mu_{\overline{\rm MS}}^2}{Q^2}\right)\Bigg]$$
$$+\text{“simple poles”}\Bigg\},\tag{4.11}$$

$$\Pi_{m_V^2}\left(\frac{\mu_{\overline{\rm MS}}^2}{Q^2},\alpha_s\right)=\frac{1}{4\pi^2}\left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{8}{3}+6\log\frac{\mu_{\overline{\rm MS}}^2}{Q^2}+\text{“simple pole”}\right].\tag{4.12}$$

The result for the pseudoscalar channel is relevant, for instance, for the decay rates of the minimal supersymmetric version of the Higgs particle into a quark-antiquark pair (see Surguladze, 1994a).

Finally, we present the results of calculation of the $\sim m_f^2/Q^2$ corrections to the correlation function in the axial channel (Soper and Surguladze, 1994; Surguladze, 1994c). We use the following definition of the correlation function,

$$i\int d^4x\; e^{iqx}\langle Tj_\mu^f(x)j_\nu^f(0)\rangle_0=g_{\mu\nu}Q^2\Pi(Q,m_f)-Q_\mu Q_\nu\Pi'(Q,m_f),\tag{4.13}$$

where $j_\mu^f=\bar{q}_f\gamma_\mu\gamma_5 q_f$. Note that in the axial channel the correlation function is not transverse, in contrast to the vector channel. However, for the decay rate of the Z boson, only the $\sim g_{\mu\nu}$ part in Eq. (4.13) is relevant.

The expansions of Π and Π' in terms of small m_f^2/Q^2 have the same form as those in the vector channel [Eq. (4.1)]. The coefficient functions in this expansion can be calculated according to Eq. (4.2) in the vector channel. In the calculations of one-, two-, and three-loop Feynman diagrams, the program HEPLoops (Surguladze, 1992) was used. The final results for the $\mathrm{SU}_c(3)$ gauge group read (Soper and Surguladze, 1994; Surguladze, 1994c)

$$\Pi_{m_f^2}\left(\frac{\mu_{\overline{\mathrm{MS}}}^2}{Q^2},\alpha_s\right)=\frac{1}{4\pi^2}\Bigg\{6+6\log\frac{\mu_{\overline{\mathrm{MS}}}^2}{Q^2}+\left(\frac{\alpha_s}{\pi}\right)\left(\frac{107}{2}-24\zeta(3)+22\log\frac{\mu_{\overline{\mathrm{MS}}}^2}{Q^2}+6\log^2\frac{\mu_{\overline{\mathrm{MS}}}^2}{Q^2}\right)$$
$$+\left(\frac{\alpha_s}{\pi}\right)^2\Bigg[\frac{3241}{6}-387\zeta(3)-\frac{3}{2}\zeta(4)+165\zeta(5)+\left(\frac{8221}{24}-117\zeta(3)\right)\log\frac{\mu_{\overline{\mathrm{MS}}}^2}{Q^2}+\frac{155}{2}\log^2\frac{\mu_{\overline{\mathrm{MS}}}^2}{Q^2}+\frac{19}{2}\log^3\frac{\mu_{\overline{\mathrm{MS}}}^2}{Q^2}$$
$$-N\left(\frac{857}{36}-\frac{32}{3}\zeta(3)+\left(\frac{151}{12}-4\zeta(3)\right)\log\frac{\mu_{\overline{\mathrm{MS}}}^2}{Q^2}+\frac{8}{3}\log^2\frac{\mu_{\overline{\mathrm{MS}}}^2}{Q^2}+\frac{1}{3}\log^3\frac{\mu_{\overline{\mathrm{MS}}}^2}{Q^2}\right)\Bigg]+\text{"simple poles"}\Bigg\}, \tag{4.14}$$

$$\Pi_{m_V^2}\left(\frac{\mu_{\overline{\mathrm{MS}}}^2}{Q^2},\alpha_s\right)=\frac{1}{4\pi^2}\left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{32}{3}-8\zeta(3)\right], \tag{4.15}$$

$$\Pi'_{m_f^2}\left(\frac{\mu_{\overline{\mathrm{MS}}}^2}{Q^2},\alpha_s\right)=\frac{1}{4\pi^2}\Bigg\{-6+\left(\frac{\alpha_s}{\pi}\right)\left(-12-12\log\frac{\mu_{\overline{\mathrm{MS}}}^2}{Q^2}\right)+\left(\frac{\alpha_s}{\pi}\right)^2\Bigg[-\frac{4681}{24}-34\zeta(3)+115\zeta(5)-\frac{215}{2}\log\frac{\mu_{\overline{\mathrm{MS}}}^2}{Q^2}-\frac{57}{2}\log^2\frac{\mu_{\overline{\mathrm{MS}}}^2}{Q^2}$$
$$-N\left(-\frac{55}{12}-\frac{8}{3}\zeta(3)-\frac{11}{3}\log\frac{\mu_{\overline{\mathrm{MS}}}^2}{Q^2}-\log^2\frac{\mu_{\overline{\mathrm{MS}}}^2}{Q^2}\right)\Bigg]+\text{"simple poles"}\Bigg\} \tag{4.16}$$

$$\Pi'_{m_V^2}=\Pi_{m_V^2}. \tag{4.17}$$

The results given in this section can be tested using the renormalization group. Namely, the relations similar to Eqs. (3.13), (3.14), and (3.15) can be obtained here (Surguladze, 1994a, 1994b, 1994c). In fact, in the vector channel, one can obtain the $O(\alpha_s^3)$ logarithmic terms without actual calculation of the corresponding four-loop diagrams. On the other hand, the leading logarithmic terms in Π function form the corresponding contribution to the decay rates of, for instance, the Z boson (Chetyrkin and Kühn, 1990; Chetyrkin, Kühn, and Kwiatkowski, 1992; Surguladze, 1994c). In the axial channel the situation is more complicated. Here, because the renormalization-group equation similar to Eq. (3.6) is no longer a homogeneous one, the renormalization-group approach is restricted to $O(\alpha_s^2)$.

V. TWO-LOOP COEFFICIENT FUNCTIONS OF dim=4 POWER CORRECTIONS

In this section we outline the calculations of the two-loop coefficient functions of dim=4 power corrections. We consider the contributions that appear in the short-distance expansion of the correlation function of two flavor-diagonal vector, scalar, and pseudoscalar currents constructed from light quark fields. The methods and corresponding references are given in the earlier sections. The corrections for the vector channel have been evaluated in Loladze, Surguladze, and Tkachov (1984, 1985) and Surguladze and Tkachov (1988). In the scalar and pseudoscalar channels, the calculation has been performed in Surguladze and Tkachov (1990). The calculation for vector and axial-vector channels has been performed in Chetyrkin, Gorishny, and Spiridonov (1985), where the previous results for the vector channel have been confirmed and the calculation extended for flavor-nondiagonal currents as well. The three-loop correction to the coefficient function of gluon condensate in the scalar channel has also been computed in Surguladze and Tkachov (1989b). For the calculation of dimension 8 terms in the operator product expansion, see also Broadhurst and Generalis (1985). Here we follow the work by Surguladze and Tkachov (1990).

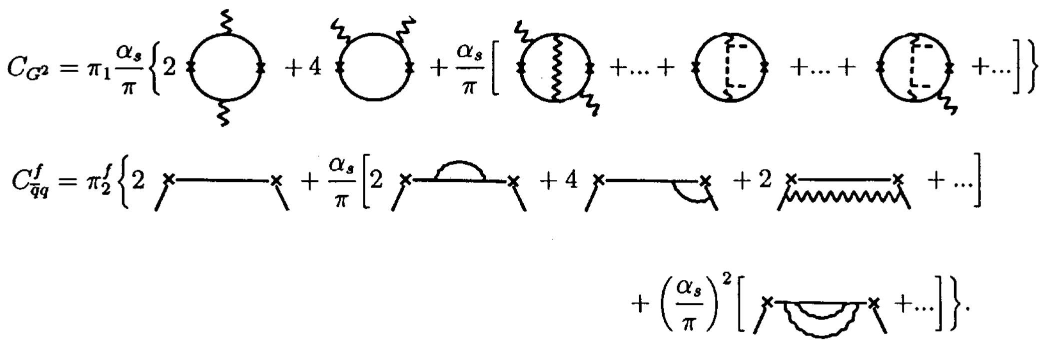
Consider first the T product of flavor-diagonal vector currents of light quarks

$$\mathscr{T}_{\mu\nu}^{f'}(Q)=i\int d^4x\, e^{iqx}TJ_\mu^{f'}(x)J_\nu^{f'}(0), \tag{5.1}$$

where $J_\mu^{f'}=\bar{q}_{f'}\gamma_\mu q_{f'}$. Taking into account the current conservation and operator product expansion technique (Wilson, 1969) for large momentum transfer ($Q^2\to\infty$), we write

$$\mathscr{T}_{\mu\nu}^{f'}(Q)=(g_{\mu\nu}Q^2-Q_\mu Q_\nu)\Bigg\{C_0+\frac{1}{Q^4}[C_{G^2}(Q^2)(G_{\mu\nu}^a)^2$$
$$+C_{\bar{q}q}^{f'}(Q^2)m_{f'}\bar{q}_{f'}q_{f'}]+\cdots\Bigg\}, \tag{5.2}$$

where C_0 is the coefficient function of the unity operator including the terms $\sim m_f^2/Q^2$ discussed in the previous section. The period covers the operators of higher twists. For the scalar and pseudoscalar channels, the transverse factor in the above equation is absent. To simplify the calculation, we contract over the Lorentz indices μ and ν. Then the expressions for C_i defined in Eq. (5.2) coincide with the ones in Eq. (2.49), if $\mathcal{T}^{f'}_{\mu_\nu}(Q)$ is replaced by $\mathcal{T}^{f'}_{\mu\mu}(Q)/(D-1)Q^2$, where $D=4-2\varepsilon$. Let us rewrite Eqs. (2.49) in a somewhat symbolic diagrammatic representation to $O(\alpha_s^2)$,



(5.3)

The total number of two-loop graphs contributing to C_{G^2} is 30 and, to $C^f_{\bar{q}q}$, 38. There is a simple rule for generating the appropriate graphs at $O(\alpha_s^n)$. One should take the graphs contributing to $O(\alpha_s^{(n+1)})$ in the unity operator and disconnect one fermion line in all possible ways for the coefficient function $C^f_{\bar{q}q}$. For the coefficient function C_{G^2}, it is necessary to write all the diagrams with one disconnected gluon line (relevant for the projector $\mathcal{P}_1$), all the diagrams with one disconnected ghost line (relevant for the projector $\mathcal{P}_4$), and all the diagrams with disconnected gluon-ghost-ghost vertex (relevant for the projector $\mathcal{P}_5$). To see this, recall Eqs. (2.46). Acting with the projectors $\mathcal{P}_j$ on the appropriate diagrams, the calculations are reduced to the evaluation of one- and two-loop propagator-type massless Feynman integrals. In the original calculation (Loladze, Surguladze, and Tkachov, 1984, 1985; Surguladze and Tkachov, 1988, 1990), all Feynman integrals were evaluated analytically using the REDUCE (Hearn, 1973) program LOOPS (Surguladze and Tkachov, 1989a).

The $\overline{\text{MS}}$ results for the projectors $\mathcal{P}_j$ in the vector channel read

$$\mathcal{P}_1[\mathcal{T}^{f'}_{\mu\mu}]=\frac{1}{Q^4}C_F\frac{N_F}{N_A}\frac{\alpha_s^B}{\pi}\left\{48-32\varepsilon+\frac{\alpha_s^B}{\pi}\left[C_F(-12)+C_A\left(\frac{18}{\varepsilon}-42+72\zeta(3)\right)\right]+O(\alpha_s^2)\right\}, \tag{5.4}$$

$$\mathcal{P}_4[\mathcal{T}^{f'}_{\mu\mu}]=\frac{1}{Q^4}C_F\frac{N_F}{N_A}C_A\left(\frac{\alpha_s^B}{\pi}\right)^2\left(\frac{3}{\varepsilon}-9+12\zeta(3)\right)+O(\alpha_s^3), \tag{5.5}$$

$$\mathcal{P}_5[\mathcal{T}^{f'}_{\mu\mu}]=0+O(\alpha_s^3), \tag{5.6}$$

$$\left(\mathcal{P}_2^{f\neq f'}+\frac{1}{D}\mathcal{P}_3^{f\neq f'}\right)[\mathcal{T}^{f'}_{\mu\mu}]=\frac{1}{Q^4}C_F T\left(\frac{\alpha_s^B}{\pi}\right)^2\left(\frac{24}{\varepsilon}-60+96\zeta(3)\right)+O(\alpha_s^3), \tag{5.7}$$

$$\left(\mathcal{P}_2^{f=f'}+\frac{1}{D}\mathcal{P}_3^{f=f'}\right)[\mathcal{T}^{f'}_{\mu\mu}]=\frac{1}{Q^4}\left\{6+\frac{\alpha_s^B}{\pi}C_F\left(\frac{3}{2}+\frac{11}{4}\varepsilon\right)+\left(\frac{\alpha_s^B}{\pi}\right)^2C_F\left[C_F\frac{387}{16}+C_A\left(\frac{11}{8\varepsilon}+\frac{7}{16}\right)+T\left(\frac{3}{4\varepsilon}-\frac{15}{4}+6\zeta(3)\right)-TN\left(\frac{1}{2\varepsilon}+\frac{7}{4}\right)\right]+O(\alpha_s^3)\right\}. \tag{5.8}$$

The vanishing of $\mathcal{P}_5[\mathcal{T}^{f'}_{\mu\mu}]$ at the two-loop level is the consequence of gauge invariance, as was shown by Spiridonov (1987). Combining Eqs. (2.48) and (2.49) with the above results and renormalizing the bare coupling via Eq. (2.24), we obtain $\overline{\text{MS}}$ $O(\alpha_s^2)$ analytical expressions for the coefficient functions in the vector channel (Surguladze and Tkachov, 1990),

FIG. 4. Two-loop diagrams forming $C_{\bar{q}q}^{f\neq f'}$.

$$C_{G^2}(Q^2)=\frac{1}{Q^4}C_F\frac{N_F}{N_A}\frac{1}{6}\frac{\alpha_s}{\pi}\left[1+\frac{\alpha_s}{\pi}\left(\frac{C_A}{2}-\frac{C_F}{4}\right)+O(\alpha_s^2)\right], \tag{5.9}$$

$$C_{\bar{q}q}^{f=f'}(Q^2)=\frac{1}{Q^4}\left\{2+\frac{\alpha_s}{\pi}\frac{C_F}{2}\left[1+\frac{\alpha_s}{\pi}\left(\frac{129}{8}C_F-\frac{25}{18}C_A-\frac{5}{9}TN+T(-3+4\zeta(3))\right)+O(\alpha_s^2)\right]\right\} \tag{5.10}$$

$$C_{\bar{q}q}^{f\neq f'}(Q^2)=\frac{1}{Q^4}\left(\frac{\alpha_s}{\pi}\right)^2 C_F T\left(-\frac{3}{2}+2\zeta(3)\right)+O(\alpha_s^3). \tag{5.11}$$

The above results are gauge invariant. This statement was checked by straightforward calculation in an arbitrary covariant gauge up to the term $\sim\varepsilon$ (Surguladze and Tkachov, 1990). The dependence on the gauge parameter canceled. Thus it is simplest to perform the calculation in the Feynman gauge. For simplicity, we have omitted the terms $\sim\log(\mu_{\overline{\rm MS}}^2/Q^2)$, taking $\mu_{\overline{\rm MS}}^2=Q^2$. The dependence on μ can be restored via the renormalization group (see below). Note that the coefficient function $C_{\bar{q}}^{f\neq f'}$ is due to the diagrams pictured in Fig. 4 with disconnected fermion lines of the virtual loop (see also Fig. 2).

Specifically for QCD with the $SU_c(3)$ symmetry group, we obtain

$$C_{G^2}(Q^2)=\frac{1}{Q^4}\frac{1}{12}\frac{\alpha_s}{\pi}\left(1+\frac{\alpha_s}{\pi}\frac{7}{6}+O(\alpha_s^2)\right), \tag{5.12}$$

$$C_{\bar{q}q}^{f=f'}(Q^2)=\frac{1}{Q^4}\left\{2+\frac{2}{3}\frac{\alpha_s}{\pi}\left[1+\frac{\alpha_s}{\pi}\left(\frac{95}{6}+2\zeta(3)-\frac{5}{18}N\right)+O(\alpha_s^2)\right]\right\}, \tag{5.13}$$

$$C_{\bar{q}q}^{f\neq f'}(Q^2)=\frac{1}{Q^4}\left(\frac{\alpha_s}{\pi}\right)^2\left(-1+\frac{4}{3}\zeta(3)\right)+O(\alpha_s^3). \tag{5.14}$$

Note the very large $O(\alpha_s^2)$ coefficient in Eq. (5.13). However, this coefficient is renormalization scheme dependent and requires special analysis (see below).

In the scalar and pseudoscalar channels, the general expression for the coefficient functions (2.45) takes the form

$$C_i(Q)=Z_m^2\sum_j \pi_j\left[\frac{\mathcal{T}(Q)}{(m_f^B)^2}\right](Z_O^{-1})_{ji}, \tag{5.15}$$

where $Z_m=m_f^B/m_f$ is the quark mass-renormalization constant [see Eq. (2.25)]. The γ^5 matrix is defined within the dimensional regularization according to Eq. (4.10). It is easy to see that in the calculations of C_{G^2}, two matrices $i\gamma^5$ can be anticommuted over the fermion propagators and can "annihilate" each other so that the results in both channels coincide. The calculational procedure is exactly the same as it was for the vector channel, except for the need for mass renormalization. The results for the coefficient functions C_{G^2} and $C_{\bar{q}q}^f$ in the $\overline{\rm MS}$ scheme are as follows (Surguladze and Tkachov, 1986, 1990). In the (pseudo)scalar channel,

$$C_{G^2}(Q^2)=\frac{1}{Q^2}C_F\frac{N_F}{N_A}\frac{1}{4}\frac{\alpha_s}{\pi}\left[1+\frac{\alpha_s}{\pi}\left(\frac{3}{2}C_A+\frac{3}{4}C_F\right)+O(\alpha_s^2)\right]. \tag{5.16}$$

In the scalar channel,

$$C_{\bar{q}q}^{f=f'}(Q^2)=\frac{1}{Q^2}\left\{3+\frac{\alpha_s}{\pi}\frac{39}{4}C_F\left(1+\frac{\alpha_s}{\pi}\left[C_F\left(\frac{447}{208}-\frac{21}{13}\zeta(3)\right)+C_A\left(\frac{389}{144}+\frac{3}{26}\zeta(3)\right)-\frac{5}{39}T-\frac{25}{36}TN\right]+O(\alpha_s^2)\right\}, \tag{5.17}$$

$$C_{\bar{q}q}^{f\neq f'}(Q^2)=\frac{1}{Q^2}\left(\frac{\alpha_s}{\pi}\right)^2 C_F T\left(-\frac{5}{4}\right)+O(\alpha_s^3), \tag{5.18}$$

and, in the pseudoscalar channel,

$$C_{\bar{q}q}^{f=f'}(Q^2)=-\frac{1}{Q^2}\left\{1+\frac{\alpha_s}{\pi}\frac{17}{4}C_F\left(1+\frac{\alpha_s}{\pi}\left[C_F\left(\frac{583}{272}-\frac{45}{17}\zeta(3)\right)+C_A\left(\frac{2443}{816}+\frac{27}{34}\zeta(3)\right)+\frac{5}{17}T-\frac{167}{204}TN\right]+O(\alpha_s^2)\right)\right\}. \tag{5.19}$$

The result for $C_{\bar{q}q}^{f\neq f'}$ coincides with the analogous one for the scalar channel.

Let us turn to the renormalization-group analysis of the above results. In this particular case it is possible to use the following trick (Surguladze and Tkachov, 1990). Note first that the vacuum average of the renormalized operators G^2 and $m\bar{q}q$ and their coefficient functions depend on the renormalization parameter μ and therefore are not convenient for further analysis. However, as was shown by Collins, Duncan, and Joglekar (1977; see also Nielsen, 1977; Tarrach, 1982; and Narison and Tarrach, 1983), the vacuum average of the trace of the energy-momentum tensor

$$\langle\Theta_{\alpha\alpha}\rangle_0=-\frac{\beta(\alpha_s)}{2\beta_0}\langle(G_{\mu\nu}^a)^2\rangle_0+\left(1-\frac{2\gamma_m(\alpha_s)}{\beta_0}\right)\times\sum_f\langle m_f\bar{q}_f q_f\rangle_0 \tag{5.20}$$

is renormalization-group invariant. On the other hand, in the MS-type schemes, the quark condensate $\langle m_f\bar{q}_f q_f\rangle_0$ is renormalization-group invariant to all orders of perturbation theory (see, e.g., Tarrach, 1982). One can introduce the renormalization-group-invariant quantity

$$\Omega=-\frac{\beta(\alpha_s)}{\beta_0}\langle (G^a_{\mu\nu})^2\rangle_0-\frac{4\gamma_m(\alpha_s)}{\beta_0}\sum_f \langle m_f\bar{q}_f q_f\rangle_0 \quad (5.21)$$

so that the new coefficient functions defined from the equation

$$C_{G^2}\left(\frac{\mu^2}{Q^2},\alpha_s\right)\langle (G^a_{\mu\nu})^2\rangle_0+C^f_{\bar{q}q}\left(\frac{\mu^2}{Q^2},\alpha_s\right)\sum_f \langle m_f\bar{q}_f q_f\rangle_0$$

$$=\bar{C}_{G^2}\left(\frac{\mu^2}{Q^2},\alpha_s\right)\Omega+\bar{C}^f_{\bar{q}q}\left(\frac{\mu^2}{Q^2},\alpha_s\right)\sum_f \langle m_f\bar{q}_f q_f\rangle_0 \quad (5.22)$$

should be the renormalization-group invariants. This is true, since the left-hand side of Eq. (5.22) is directly connected to the observables (Shifman, Vainshtein, and Zakharov, 1979) and, consequently, is invariant. From Eqs. (5.21) and (5.22), we find the invariant coefficient functions corresponding to the invariant combinations of the gluon and quark condensates,

$$\bar{C}_{G^2}\left(\frac{\mu^2}{Q^2},\alpha_s\right)=-\frac{\beta_0}{\beta(\alpha_s)}C_{G^2}\left(\frac{\mu^2}{Q^2},\alpha_s\right), \quad (5.23)$$

$$\bar{C}^f_{\bar{q}q}\left(\frac{\mu^2}{Q^2},\alpha_s\right)=C^f_{\bar{q}q}\left(\frac{\mu^2}{Q^2},\alpha_s\right)-\frac{4\gamma_m(\alpha_s)}{\beta(\alpha_s)}C_{G^2}\left(\frac{\mu^2}{Q^2},\alpha_s\right). \quad (5.24)$$

Note that, in fact, there are terms of the type $m_f^2 m_{f'}^2$ or/and m_f^4 in the right-hand side of Eq. (5.22). However, these terms obviously do not affect our equations for invariant coefficient functions. The two-loop coefficient functions for $\sim m^4$ terms have been calculated by Chetyrkin, Gorishny, and Spiridonov (1985). The contributions from such terms are negligible for phenomenological applications and will not be discussed here.

Now one can use the renormalization-group invariance of the coefficient functions and write

$$\bar{C}_i\left(\frac{\mu^2}{Q^2},\alpha_s\right)=\bar{C}_i(1,\alpha_s(Q^2)). \quad (5.25)$$

Reevaluating the coefficient functions for the u,d,s light quarks $(N=3)$, we obtain the following results in the $\overline{\mathrm{MS}}$ scheme. In the vector channel

$$\bar{C}_{G^2}(\alpha_s(Q^2))=\frac{1}{Q^4}\frac{1}{12}\left(1-\frac{\alpha_s(Q^2)}{\pi}0.6111+O(\alpha_s^2)\right), \quad (5.26)$$

$$\bar{C}^{f=f'}_{\bar{q}q}(\alpha_s(Q^2))=\frac{1}{Q^4}2\left[1+0.4074\frac{\alpha_s(Q^2)}{\pi}\times\left(1+\frac{\alpha_s(Q^2)}{\pi}14.8180+O(\alpha_s^2)\right)\right]. \quad (5.27)$$

In the scalar channel

$$\bar{C}_{G^2}(\alpha_s(Q^2))=\frac{1}{Q^2}\frac{1}{8}\left(1+\frac{\alpha_s(Q^2)}{\pi}3.7222+O(\alpha_s^2)\right), \quad (5.28)$$

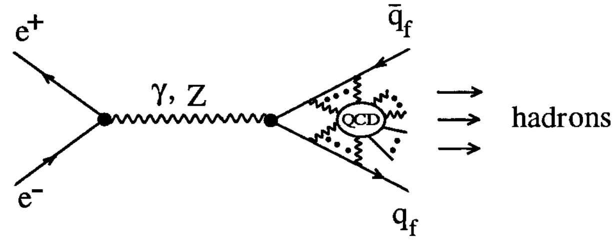


FIG. 5. The process $e^+e^-\to$hadrons. The shaded bulb includes any interactions of quarks and gluons (or ghosts) allowed in QCD. The dots cover any relevant number of gluon and quark propagators.

$$\bar{C}^{f=f'}_{\bar{q}q}(\alpha_s(Q^2))=\frac{1}{Q^2}3\left[1+4.4074\frac{\alpha_s(Q^2)}{\pi}\times\left(1+\frac{\alpha_s(Q^2)}{\pi}7.6879+O(\alpha_s^2)\right)\right]. \quad (5.29)$$

In the pseudoscalar channel

$$\bar{C}^{f=f'}_{\bar{q}q}(\alpha_s(Q^2))=-\frac{1}{Q^2}\left[1+5.4444\frac{\alpha_s(Q^2)}{\pi}\times\left(1+\frac{\alpha_s(Q^2)}{\pi}9.4559+O(\alpha_s^2)\right)\right]. \quad (5.30)$$

For all channels

$$\bar{C}^{f\neq f'}_{\bar{q}q}(\alpha_s(Q^2))=C^{f\neq f'}_{\bar{q}q}(\alpha_s(Q^2))+O(\alpha_s^3). \quad (5.31)$$

Note, again, very large $O(\alpha_s^2)$ corrections for the coefficient functions of quark condensates in the $\overline{\mathrm{MS}}$ scheme. The running coupling is evaluated at the typical hadronic mass scale. The $O(\alpha_s)$ corrections have also been calculated for the dim=6 operators (Lanin, Spiridonov, and Chetyrkin, 1986). We also mention the calculations in the case of heavy quark currents (see, e.g., Broadhurst *et al.*., 1994 and references therein).

VI. $R(s)$ IN ELECTRON-POSITRON ANNIHILATION TO $O(\alpha_s^3)$

In this section we present an outline of the evaluation of the corrections up to $O(\alpha_s^3)$ to the total cross section in the process $e^+e^-\to$hadrons (Fig. 5) in the limit of zero light quark masses and infinitely large top mass. We also mention the QCD evaluation of the hadronic decay rates of the Z boson and the relevant quark mass effects.

These calculations were first attempted by Gorishny, Kataev, and Larin (1988). However, it was shown that those results were incorrect. Indeed, about five years ago an independent calculation of the above quantity was completed (Surguladze and Samuel, 1991a, 1991b). The result is much smaller and has the opposite sign compared with the old 1988 result. This finding was confirmed shortly after that by Gorishny, Kataev, and Larin (1991).

In the process shown in Fig. 5, an electron-positron pair annihilates, producing either a photon or a Z boson, which further produces quark-antiquark pairs (in QED) plus gluons (if strong interactions are "switched on"). Finally, quarks, through hadronization, form hadronic final states with probability equal to 1 (confinement hypothesis), and the total cross section is given by

$$\sigma_{\rm tot}(e^+e^-\to \text{hadrons})=\frac{4\pi\alpha^2}{3s}3\sum_f Q_f^2(1+\delta_{\rm QCD}), \qquad (6.1)$$

where s is the total center-of-mass energy squared; Q_f is the electric charge of the quark flavor f participating at the given energy; factor 3 stands for the number of color degrees of freedom; and $\delta_{\rm QCD}$ stands for the strong-interaction contributions. The hadronic production in electron-positron annihilation is usually characterized in terms of the R ratio—the total hadronic cross section normalized by the muon pair-production cross section,

$$R(s)=\frac{\sigma_{\rm tot}(e^+e^-\to \text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)}=3\sum_f Q_f^2(1+\delta_{\rm QCD}). \qquad (6.2)$$

The above expressions are relevant at energies much less than the Z mass ($\sqrt{s}\ll M_Z$), corresponding, for instance, to the PEP/PETRA energy range. At LEP the effects of the Z boson become important. The corresponding R ratio is defined as a ratio of the hadronic and electronic widths of the Z boson,

$$R_Z=\frac{\Gamma(Z\to \text{hadrons})}{\Gamma(Z\to e^+e^-)}. \qquad (6.3)$$

Note that the total hadronic width of the Z boson in the above equation is the sum of the vector and axial current-induced decay rates. Strictly speaking, those rates get different strong-interaction contributions. In this section we calculate the QCD corrections in the vector channel—$\delta_{\rm QCD}$ in the limit of massless light quarks and the infinitely large top mass. This quantity is, in fact, relevant for the axial part as well. To get the complete axial decay rate, additional contributions are necessary. For details, see the original works: Chetyrkin and Kühn (1990); Kniehl (1990); Kniehl and Kühn (1990); Chetyrkin, Kühn, and Kwiatkowski (1992); Chetyrkin (1993a); Soper and Surguladze (1994); Surguladze (1994c); the review articles by Kniehl (1994b, 1995b); Soper and Surguladze (1995a); and Sec. IV of this paper.

A. $R(s)$ via renormalization constants

The vacuum polarization function $\Pi(Q^2)$ defined in Eq. (2.10) has a cut along the negative Q^2 axis in the massless case. The ratio $R(s)$ can be found by taking the imaginary part of $\Pi(s+i0)$, according to Eq. (2.12). Alternatively, $R(s)$ can also be found from Eq. (2.13), which in combination with Eq. (2.16) gives to $O(\alpha_s^3)$

$$R(s)=R_0+\frac{\alpha_s(s)}{\pi}R_1+\left(\frac{\alpha_s(s)}{\pi}\right)^2R_2 +\left(\frac{\alpha_s(s)}{\pi}\right)^3\left(R_3-\frac{\pi^2\beta_0^2}{3}R_1\right). \qquad (6.4)$$

The origin of the large and negative scheme-scale-independent term $R_1\pi^2\beta_0^2/3$ can be understood if one takes into account the presence of $\sim\log^3\mu^2/s$ terms at $O(\alpha_s^3)$ in the Π function and

$$\frac{1}{\pi}\,\text{Im}\,\log^3(s+i0)=-3\,\log^2 s+\pi^2.$$

The term R_1 at $O(\alpha_s^3)$ is due to the coupling renormalization. Note that the R_i in the above equation are the perturbative coefficients of the $D(Q^2)$ function defined in Eq. (2.13). For the definition of the procedure of analytical continuation and the origin of additional $\sim\pi^2$ terms, see also Krasnikov and Pivovarov (1982), Pennington and Ross (1982), Radyushkin (1982), and Pivovarov (1992a).

Substituting Eq. (2.27) with the renormalized strong coupling into Eq. (2.12) and taking into account relations (2.29), (2.30), and (2.33), we obtain

$$\begin{aligned}R(s)=-\frac{3}{4}\Bigg\{&Z_{1,-1}+\frac{\alpha_s(\mu)}{\pi}(2Z_{2,-1})+\left(\frac{\alpha_s(\mu)}{\pi}\right)^2\\ &\times\left(3Z_{3,-1}-\beta_0\Pi_{2,0}+2\beta_0Z_{2,-1}\log\frac{\mu^2}{s}\right)\\ &+\left(\frac{\alpha_s(\mu)}{\pi}\right)^3\Bigg[4Z_{4,-1}-2\beta_0\Pi_{3,0}-\beta_1\Pi_{2,0}+2\beta_0^2\Pi_{2,1}\\ &-\frac{2\pi^2\beta_0^2}{3}Z_{2,-1}+(6\beta_0Z_{3,-1}+2\beta_1Z_{2,-1}-2\beta_0^2\Pi_{2,0})\\ &\times\log\frac{\mu^2}{s}+2\beta_0^2Z_{2,-1}\log^2\frac{\mu^2}{s}\Bigg]+O(\alpha_s^4)\Bigg\}. \qquad (6.5)\end{aligned}$$

Note that the appearance of perturbative coefficients of the renormalization constant in the above equation is totally due to the relations (2.33). In fact, Z_Π has only simple poles and hence no imaginary part. The latter is the specific feature of the MS prescription. Equation (6.5) exhibits one of the main ideas of this calculation. Namely, in order to calculate the l-loop contribution to R, it suffices to calculate the l-loop counterterm Z_Π to the bare quantity Π^B, and the $(l-1)$-loop approximation to Π^B. In other words, the minimal information necessary to obtain the four-loop $R(s)$ is contained in the divergent part of the one-loop diagram, two-loop diagrams calculated up to $\sim\varepsilon$ terms, three-loop diagrams calculated up to the finite parts in the limit $\varepsilon\to 0$, and only leading $\sim 1/\varepsilon$ terms in the overall counterterms of the four-loop diagrams. In fact, as we demonstrate in Sec. VI.B, using the infrared rearrangement procedure (Vladimirov, 1980; Chetyrkin and Tkachov, 1982), one can complete the entire calculation dealing effectively with only three-loop diagrams. We mention once again that, through the procedure of infrared rearrangement, within the MS prescription, the problem of calculation of the counterterms to arbitrary l-loop diagrams with an arbitrary number of masses and external momenta can be reduced to the calculation of $(l-1)$-loop propagator-type massless integrals up to finite terms. In our case $l=4$. On the other hand, the recursive-type algorithms for multiloop Feynman integrals (Chetyrkin and Tkachov, 1981; Tkachov, 1981, 1983a, 1983b) and their computer implementation (Surguladze and Tkachov, 1989a; see also Gorishny, Larin, Surguladze, and Tkachov, 1989; Surguladze, 1989b, 1989c, 1992) allow one to calculate propagator-type Feynman diagrams to three-loop level.

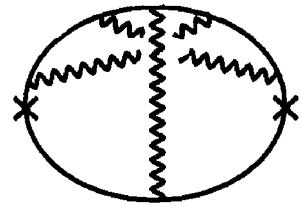


FIG. 6. A typical four-loop nonplanar-type diagram contributing to $R(s)$.

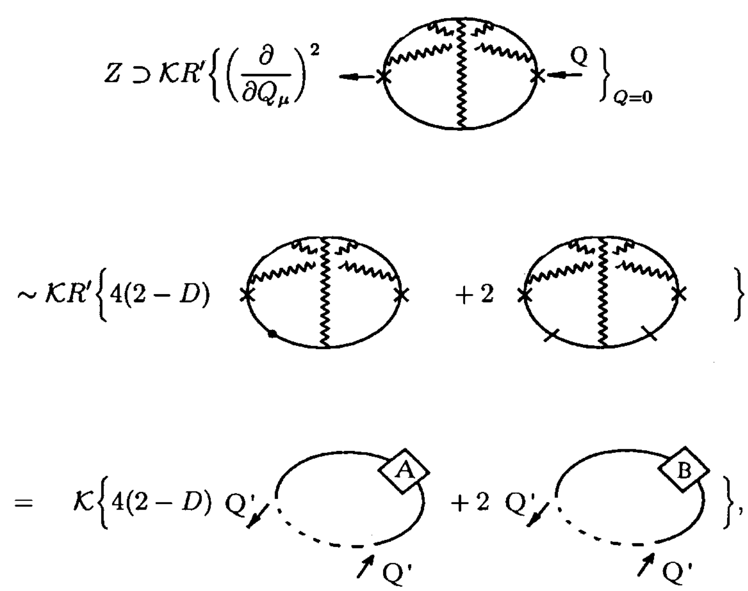
B. Full calculational procedure with a typical four-loop diagram

In this subsection we demonstrate the full calculational procedure for a typical four-loop diagram pictured in Fig. 6, which contributes to the photon renormalization constant Z_Π and hence to the R ratio. To simplify the description, in some cases we shall avoid complicated equations, substituting for them their graphical representation.

We need to evaluate the counterterm to the diagram pictured in Fig. 6. In other words, we should evaluate $-\mathcal{K}R'$ for this diagram (see Sec. II.D). A simple power counting shows that the given diagram diverges as

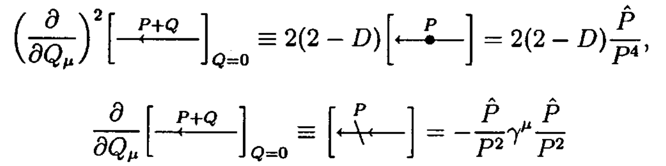
$$G \sim \lim_{Q\to\infty} Q^{4D-14},$$

and the superficial degree of divergence is 2. Using the fact that the counterterm has only a polynomial dependence on the external momenta Q within the MS prescription, one can remove such a dependence by differentiating the diagram twice with respect to Q and then setting the external momentum to zero. At the next step, since there is no dependence on the external momentum, one can introduce a new, fictitious external momentum flowing through one of the diagram lines. This line should be chosen in a way that simplifies the topology of the diagram and avoids infrared divergences. The above procedure for the diagram in Fig. 6 is displayed in the following graphical equation,



(6.6)

where the dot and dashes on the lines result from differentiating the corresponding fermion propagators

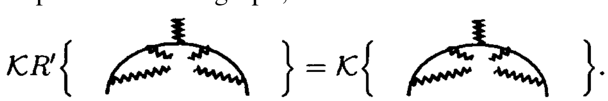


Boxes contain the corresponding three-loop propagator-type subgraphs with subtracted divergences—complete R operation (Fig. 7). The dotted lines mean that this line is temporarily "torn." After the evaluation of boxes, the parts of the torn line should be pasted and a trivial fourth-loop integration should be done, taking into account the corresponding exponents of the propagators due to the three-, two-, and one-loop "box" insertions. The above procedure gives a great simplification of the problem. Indeed, the evaluation of the four-loop counterterm is reduced to the evaluation of three-, two-, and one-loop graphs.

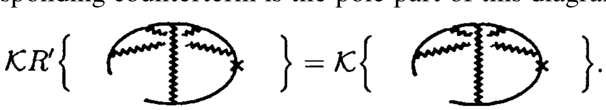
The complete R operation of the three-loop diagram insertions corresponding to the ones on the right-hand side of Eq. (6.6) is given in Fig. 7. Graphs in the brackets correspond to two- and three-loop counterterms. There is no one-loop divergent subgraph in this particular diagram. Thus

$$(G_i) \equiv \mathscr{K}R'\{G_i\},$$

where G_i is any divergent subgraph of the given diagram. It is easy to recognize that the two-loop subgraph in Fig. 7 does not have subdivergences (only an overall one), and the corresponding counterterm is simply the pole part of this subgraph,



Analogously, because of the topology, the three-loop counterterm does not have a subdivergence and the corresponding counterterm is the pole part of this diagram,



If, in general, a diagram contains divergent subgraphs, then the recursive formula (2.35) should be used.

As a result of the above manipulations, we managed to reduce the problem of calculation of the counterterm to the four-loop diagram pictured in Fig. 6 to the calculation of several three-, two-, and one-loop diagrams shown in Fig. 7. Note, however, that the "dots" and "dashes" on the diagram lines make their evaluation significantly more difficult. The computer programs for analytical programming systems capable of handling such calculations are the SCHOONSCHIP program MINCER (Gorishny, Larin, Surguladze, and Tkachov, 1989; Surguladze, 1989b, 1989c) and the FORM program HEPLoops (Surguladze, 1992). The latter is especially well suited for large-scale calculations and is much more efficient than the MINCER program.

It is important to stress that, in fact, it is sufficient to evaluate only the $\mathscr{K}R'$ for the relevant three-loop subgraphs. In other words, it is not necessary to calculate separately three-loop counterterms similar to the graph in the last brackets for the box A in Fig. 7. Indeed, a more detailed analysis gives

$$R\{G\} = R'\{G\} - (1-D/2)\mathscr{K}\left(\frac{1}{1-D/2}R'\{G\}\right), \quad (6.7)$$

where G is the corresponding three-loop subgraph. The above relation allows simple computer implementation and facilitates calculations considerably.

The complete R operation for each four-loop diagram generally has the form

$$\left(\frac{\mu^2}{Q^2}\right)^{4\varepsilon} f_4(\varepsilon) - \sum_{l=1}^{3}\left(\frac{\mu^2}{Q^2}\right)^{(4-l)\varepsilon} c_l(1/\varepsilon) f_{4-l}(\varepsilon),$$

where $f_i(\varepsilon)$ is the result of the calculation of the corresponding Feynman graphs including the last trivial loop integration, and c_l are the l-loop counterterms, polyno-

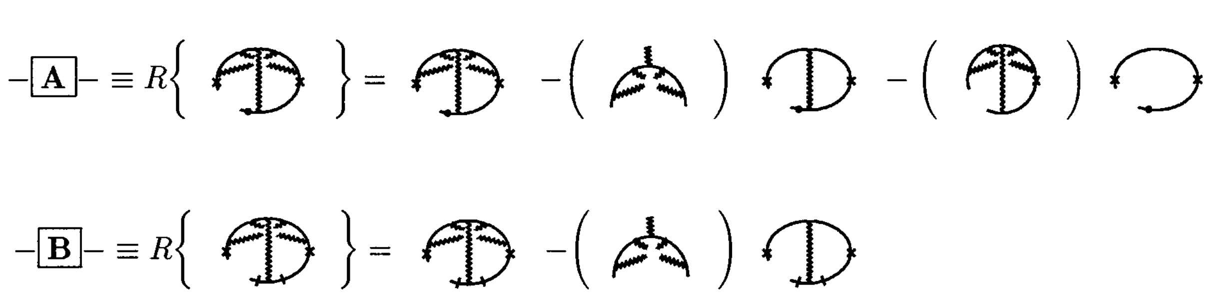


FIG. 7. Complete R operation for the three-loop subgraphs.

mials in $1/\varepsilon$. As we already mentioned, in the MS-type renormalization scheme, the counterterm for a particular diagram is a polynomial in dimensional parameters (see, e.g., the textbook by Collins, 1984 and references therein). Thus the terms of the type $(1/\varepsilon)^n \ln^m(\mu^2/Q^2)$, which appear due to the expansion of the factors $(\mu^2/Q^2)^{l\varepsilon}$ into the Laurent series in ε, must be canceled in the final answer for the particular diagram. This can be used to test the calculations at the graph-by-graph level. Recall that we calculate the counterterm $Z_{4,i}$ to the four-loop diagram.

Finally, for the contribution to the Z_Π of the diagram pictured in Fig. 6, we obtain the following result,

$$\left(\frac{\alpha_s}{4\pi}\right)^3 N_F C_F (C_F - C_A)(C_F - C_A/2)\frac{1}{3-2\varepsilon} \times\left[4\frac{1}{\varepsilon^3} - 26\frac{1}{\varepsilon^2} + \frac{65}{4}\frac{1}{\varepsilon} - 40\zeta(3)\frac{1}{\varepsilon}\right].$$

The CPU time for the above diagram on a 0.8 MFlop IBM compatible mainframe was over 6 hours. The extended version of the program MINCER for the system SCHOONSCHIP was used. Note that the above result, as well as the total result for the photon renormalization constant, does not depend on any modification of the minimal subtraction prescription.

C. Four-loop results

In this subsection, we present results and some of the details of the $O(\alpha_s^3)$ QCD evaluation of the ratio $R(s)$ in electron-positron annihilation (Surguladze and Samuel, 1991a, 1991b).

The total number of topologically distinct Feynman diagrams contributing to $Z_{1,i}$ is 1; to $Z_{2,i}$, 2; to $Z_{3,i}$, 17; and to $Z_{4,i}$, 98. However, after application of the infrared rearrangement procedure, which, as discussed earlier, involves differentiating twice with respect to the external momentum of the diagram, the number of four-loop graphs that need to be calculated increases to approximately 250. Furthermore, there are one-, two-, and three-loop diagrams, approximately 600, that need to be calculated to subtract subdivergences (evaluate R') for all four-loop diagrams.

All analytical calculations of the four-loop diagrams have been performed by using the program, which is an extended version (Surguladze, 1989c) of the program MINCER (Gorishny, Larin, Surguladze, and Tkachov, 1989; Surguladze, 1989b). This version includes new subprograms for fourth-loop integration and for ultraviolet renormalization. Evaluation of one- and two-loop counterterms has been done by using the program LOOPS (Surguladze and Tkachov, 1989a). The above programs are written on the algebraic programming systems SCHOONSCHIP (Veltman, 1967; Strubbe, 1974) and REDUCE (Hearn, 1973), respectively. The full calculation took over 700 hours of CPU time on three IBM compatible 0.8 MFlop EC-1037 mainframes with the SCHOONSCHIP system. We have also recalculated some of the difficult four-loop diagrams with HEPLoops—a new program for analytical multiloop calculations (Surguladze, 1992). The status of these and some other programs has been reviewed recently in Surguladze (1994d).

In the diagram calculations the Feynman gauge is used. The momentum integrations are performed within the $\overline{\text{MS}}$ modification (Bardeen, Buras, Duke, and Muta, 1978) of the minimal subtraction prescription ('t Hooft, 1973), which amounts to formally setting $\gamma=\zeta(2)=\log 4\pi=0$. A discussion of the scheme dependence of the results is given at the end of this section and in Sec. IX. The full graph-by-graph results will be published elsewhere.

The analytical result for the four-loop photon renormalization constant reads

$$\begin{aligned}
Z_{\rm ph} \equiv 1 + \frac{\alpha}{4\pi} Z_\Pi = 1 + N_F \frac{\alpha}{4\pi}\sum_f Q_f^2\Bigg\{ &-\frac{4}{3}\frac{1}{\varepsilon} + \frac{\alpha_s}{4\pi}\left[\frac{1}{\varepsilon}(-2C_F)\right] + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\frac{1}{\varepsilon^2}\left(\frac{22}{9}C_F C_A - \frac{8}{9}NTC_F\right)\right.\\
&\left. + \frac{1}{\varepsilon}\left(\frac{2}{3}C_F^2 - \frac{133}{27}C_F C_A + \frac{44}{27}NTC_F\right)\right] + \left(\frac{\alpha_s}{4\pi}\right)^3\left[\frac{1}{\varepsilon^3}\left(-\frac{121}{27}C_F C_A^2 + \frac{88}{27}NTC_F C_A - \frac{16}{27}N^2T^2C_F\right)\right.\\
&+ \frac{1}{\varepsilon^2}\left(-\frac{11}{9}C_F^2 C_A + \frac{2381}{162}C_F C_A^2 - \frac{14}{9}NTC_F^2 - \frac{778}{81}NTC_F C_A + \frac{88}{81}N^2T^2C_F\right)\\
&+ \frac{1}{\varepsilon}\left(\frac{23}{2}C_F^3 + \left(-\frac{430}{27} + \frac{88}{9}\zeta(3)\right)C_F^2 C_A + \left(-\frac{5815}{972} - \frac{88}{9}\zeta(3)\right)C_F C_A^2 + \left(\frac{338}{27} - \frac{176}{9}\zeta(3)\right)NTC_F^2\right.\\
&\left.\left. + \left(\frac{769}{243} + \frac{176}{9}\zeta(3)\right)NTC_F C_A + \frac{308}{243}N^2T^2C_F\right)\right] + O(\alpha_s^4)\Bigg\} + \frac{\alpha}{4\pi}\left(\frac{\alpha_s}{4\pi}\right)^3\left(\sum_f Q_f\right)^2\left(\frac{d^{abc}}{4}\right)^2\left(-\frac{176}{9} + \frac{128}{3}\zeta(3)\right)\frac{1}{\varepsilon}.
\end{aligned} \tag{6.8}$$

It should be stressed that the Riemann ζ functions, $\zeta(4)$ and $\zeta(5)$, which appear at the individual graph level, cancel in the above expression. Moreover, as we have observed, the $\zeta(4)$ has disappeared within each gauge-invariant set of diagrams. Note that $\zeta(3)$ disappears for QED ($C_F=1, C_A=0, T=1$) except the last term, which comes from the "light-by-light"-type diagrams (Fig. 8). The diagrams pictured in Fig. 8 are some of the most complicated ones, and the computation of each of them requires over 80 h of CPU time. Note, however, that the second and fourth diagrams in Fig. 8 differ correspondingly from the first and third ones only by the SU(N) group weights. So, in fact, only two of them have been calculated. The result (6.8) does not depend on the particular modification of the minimal subtraction prescription.

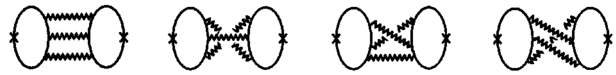


FIG. 8. "Light-by-light"-type diagrams.

In order to evaluate $R(s)$ to $O(\alpha_s^3)$, besides the four-loop Z_Π we calculate the unrenormalized hadronic vacuum polarization function $\Pi^B(Q^2)$ to the three-loop level. We get the following analytical result in the $\overline{\mathrm{MS}}$ scheme.

$$\Pi^B\left(\frac{\mu^2_{\overline{\mathrm{MS}}}}{Q^2},\alpha_s^B\right)=N_F\sum_f Q_f^2\Bigg\{\left(\frac{\mu^2_{\overline{\mathrm{MS}}}}{Q^2}\right)^{\varepsilon}\left[\frac{4}{3}\frac{1}{\varepsilon}+\frac{20}{9}+\frac{112}{27}\varepsilon+\frac{656}{81}\varepsilon^2-\frac{28}{9}\zeta(3)\varepsilon^2\right]$$
$$+\left(\frac{\alpha_s^B}{4\pi}\right)\left(\frac{\mu^2_{\overline{\mathrm{MS}}}}{Q^2}\right)^{2\varepsilon}C_F\left[2\frac{1}{\varepsilon}+\frac{55}{3}-16\zeta(3)+\varepsilon\left(\frac{1711}{18}-\frac{152}{3}\zeta(3)-24\zeta(4)\right)\right]$$
$$+\left(\frac{\alpha_s^B}{4\pi}\right)^2\left(\frac{\mu^2_{\overline{\mathrm{MS}}}}{Q^2}\right)^{3\varepsilon}\Bigg[C_F^2\left(-\frac{2}{3}\frac{1}{\varepsilon}-\frac{286}{9}-\frac{296}{3}\zeta(3)+160\zeta(5)\right)$$
$$+C_FC_A\left(\frac{44}{9}\frac{1}{\varepsilon^2}+\frac{1948}{27}\frac{1}{\varepsilon}-\frac{176}{3}\zeta(3)\frac{1}{\varepsilon}+\frac{50339}{81}-\frac{3488}{9}\zeta(3)-88\zeta(4)-\frac{80}{3}\zeta(5)\right)$$
$$+NTC_F\left(-\frac{16}{9}\frac{1}{\varepsilon^2}-\frac{704}{27}\frac{1}{\varepsilon}+\frac{64}{3}\zeta(3)\frac{1}{\varepsilon}-\frac{17668}{81}+\frac{1216}{9}\zeta(3)+32\zeta(4)\right)\Bigg]\Bigg\}. \tag{6.9}$$

The above result depends on the particular modifications of the minimal subtraction prescription, unlike the result for the renormalization constant (6.8).

Substituting the expressions for the relevant $Z_{i,j}$ and $\Pi_{i,j}$, extracted by comparing Eqs. (6.8) and (6.9) to Eqs. (2.26) and (2.27), into Eq. (6.5) and recalling the values for β_0 and β_1 from Eq. (2.8), we get the following $\overline{\mathrm{MS}}$ analytical result for $R(s)$ at the four-loop level,

$$R^{\overline{\mathrm{MS}}}(s)=N_F\sum_f Q_f^2\Bigg\{1+\left(\frac{\alpha_s(s)}{4\pi}\right)(3C_F)+\left(\frac{\alpha_s(s)}{4\pi}\right)^2\left[C_F^2\left(-\frac{3}{2}\right)+C_FC_A\left(\frac{123}{2}-44\zeta(3)\right)+NTC_F(-22+16\zeta(3))\right]$$
$$+\left(\frac{\alpha_s(s)}{4\pi}\right)^3\Bigg[C_F^3\left(-\frac{69}{2}\right)+C_F^2C_A(-127-572\zeta(3)+880\zeta(5))C_FC_A^2\left(\frac{90445}{54}-\frac{10948}{9}\zeta(3)-\frac{440}{3}\zeta(5)\right)$$
$$+NTC_F^2(-29+304\zeta(3)-320\zeta(5))+NTC_FC_A\left(-\frac{31040}{27}+\frac{7168}{9}\zeta(3)+\frac{160}{3}\zeta(5)\right)$$
$$+N^2T^2C_F\left(\frac{4832}{27}-\frac{1216}{9}\zeta(3)\right)-\pi^2C_F\left(\frac{11}{3}C_A-\frac{4}{3}NT\right)^2\Bigg]+O(\alpha_s^4)\Bigg\}$$
$$+\left(\frac{\alpha_s(s)}{4\pi}\right)^3\left(\sum_f Q_f\right)^2\left(\frac{d_{abc}}{4}\right)^2\left[\frac{176}{3}-128\zeta(3)\right]+O(\alpha_s^4). \tag{6.10}$$

The logarithmic contributions are absorbed in the running coupling by taking $\mu^2=s$. Those contributions will be presented explicitly in Sec. IX. Note that $\zeta(5)$ appears in the final result due to the contributions from $\Pi_{3,0}$. The last term $\sim(\Sigma_f Q_f)^2$ comes from the so-called light-by-light-type diagrams (Fig. 8). For standard QCD with the $SU_c(3)$ gauge group, we obtain

$$R^{\overline{\rm MS}}(s)=3\sum_f Q_f^2\Bigg\{1+\frac{\alpha_s(s)}{\pi}+\left(\frac{\alpha_s(s)}{\pi}\right)^2\left[\frac{365}{24}-11\zeta(3)-N\left(\frac{11}{12}-\frac{2}{3}\zeta(3)\right)\right]+\left(\frac{\alpha_s(s)}{\pi}\right)^3\Bigg[\frac{87029}{288}-\frac{121}{8}\zeta(2)-\frac{1103}{4}\zeta(3)$$
$$+\frac{275}{6}\zeta(5)+N\left(-\frac{7847}{216}+\frac{11}{6}\zeta(2)+\frac{262}{9}\zeta(3)-\frac{25}{9}\zeta(5)\right)+N^2\left(\frac{151}{162}-\frac{1}{18}\zeta(2)-\frac{19}{27}\zeta(3)\right)\Bigg]\Bigg\}$$
$$+\left(\sum_f Q_f\right)^2\left(\frac{\alpha_s(s)}{\pi}\right)^3\left[\frac{55}{72}-\frac{5}{3}\zeta(3)\right]+O(\alpha_s^4). \tag{6.11}$$

Finally, taking into account the values for the relevant Riemann ζ functions, $\zeta(2)=\pi^2/6$, $\zeta(3)=1.2020569\ldots$, and $\zeta(5)=1.0369278\ldots$, we obtain the numerical form

$$R^{\overline{\rm MS}}(s)=3\sum_f Q_f^2\left[1+\frac{\alpha_s(s)}{\pi}+\left(\frac{\alpha_s(s)}{\pi}\right)^2(1.9857-0.1153N)+\left(\frac{\alpha_s(s)}{\pi}\right)^3(-6.6368-1.2001N-0.0052N^2)\right]-\left(\sum_f Q_f\right)^2\left(\frac{\alpha_s(s)}{\pi}\right)^3 1.2395+O(\alpha_s^4). \tag{6.12}$$

Note that only 19 four-loop diagrams contribute to the term $\sim N$ and 2 four-loop diagrams contribute to the term $\sim N^2$. The most complicated diagrams are pictured in Fig. 9. The CPU time for each of them was over 100 h, and the intermediate expression had as many as $\sim10^5$–10^6 terms.

It is known that the perturbative coefficients for $R(s)$ are scheme dependent. The above result was obtained in the modified minimal subtraction, the so-called $\overline{\rm MS}$ scheme introduced by Bardeen, Buras, Duke, and Muta (1978). While the scheme-scale-dependence problem will be discussed in Sec. IX, here we present the results for a couple of other versions of the minimal subtraction scheme. First, we consider the so-called G scheme (Chetyrkin and Tkachov, 1979, 1981; Chetyrkin, Kataev, and Tkachov, 1980), which is convenient for practical multiloop calculations. The G scheme is defined in such a way that the trivial one-loop integral in this scheme is

$$\mu^{2\varepsilon}\int\frac{d^{4-2\varepsilon}p}{(2\pi)^{4-2\varepsilon}}\frac{1}{p^2(p-k)^2}=\frac{1}{(4\pi)^2}\left(\frac{\mu^2}{k^2}\right)^{\varepsilon}\frac{1}{\varepsilon}.$$

FIG. 9. Some of the most complicated diagrams.

The result for $R(s)$ in this scheme is

$$R^G(s)=3\sum_f Q_f^2\left[1+\frac{\alpha_s(s)}{\pi}+\left(\frac{\alpha_s(s)}{\pi}\right)^2(-3.514+0.218N)+\left(\frac{\alpha_s(s)}{\pi}\right)^3(-10.980-0.692N+0.029N^2)\right]-\left(\sum_f Q_f\right)^2\left(\frac{\alpha_s(s)}{\pi}\right)^3 1.240+O(\alpha_s^4). \tag{6.13}$$

The parametrization of the running coupling in the above equation has the same form as that in Eq. (2.17). However, the parameter Λ has to be changed to some other parameter Λ_G.

Finally, in the original MS scheme ('t Hooft, 1973), we get

$$R^{\rm MS}(s)=3\sum_f Q_f^2\left[1+\frac{\alpha_s(s)}{\pi}+\left(\frac{\alpha_s(s)}{\pi}\right)^2(7.359-0.441N)+\left(\frac{\alpha_s(s)}{\pi}\right)^3(56.026-8.778N+0.176N^2)\right]-\left(\sum_f Q_f\right)^2\left(\frac{\alpha_s(s)}{\pi}\right)^3 1.240+O(\alpha_s^4). \tag{6.14}$$

Note that the corresponding term $\sim N$ at $O(\alpha_s^2)$ given in Narison (1982) is incorrect.

As one can see, starting from $O(\alpha_s^2)$ the results heavily depend on the choice of the particular modifications of the minimal subtraction scheme. This dependence, called renormalization-group ambiguity of perturbative results, is an important problem and deserves special consideration. We shall return to this issue in Sec. IX.

Concluding this section, we mention once again that the results of the above-described calculation of the four-loop correction to the $R(s)$ have been published in Surguladze and Samuel (1991a, 1991b) and

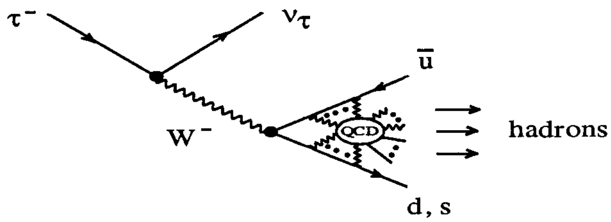


FIG. 10. Hadronic decay of the τ lepton.

independently[3] in Gorishny, Kataev, and Larin (1991) and hence, most likely, the above results are reliable. Interesting relations between the radiative corrections for different observables, found by Brodsky and Lu (1994, 1995), serve, in particular, as another confirmation of our results.

VII. $\Gamma(\tau^-\to\nu_\tau+$HADRONS) TO $O(\alpha_s^3)$

The other important inclusive process for phenomenology and testing the standard model is the hadronic decay of the τ lepton (Fig. 10). For a recent review see, for instance, Pich (1994a). For earlier references see Altarelli (1992), Marciano (1992), and Pich (1991).

In this section, using our result of four-loop calculation of the $\sigma_{\rm tot}(e^+e^-\to\text{hadrons})$ (Surguladze and Samuel, 1991a, 1991b), we evaluate the hadronic decay rate of the τ lepton to $O(\alpha_s^3)$ in perturbative QCD (Pich, 1990; Gorishny, Kataev, and Larin, 1991; Samuel and Surguladze, 1991; see also Braaten, Narison, and Pich, 1991; Diberder and Pich, 1992a, 1992b; Pich, 1992a, 1992b; Pivovarov, 1992b). We also comment on the status of the nonperturbative corrections to this quantity.

We follow the method first suggested by Tsai (1971), Shankar (1977), and Lam and Yan (1977) for theoretical evaluation of heavy lepton decay rates. This method has been further developed for the τ lepton, including the higher-order perturbative corrections and involving the operator product expansion technique (Wilson, 1969) to analyze the nonperturbative contributions (Schilcher and Tran, 1984; Braaten, 1988; Narison and Pich, 1988). As was shown in the above works, by combining the operator product expansion technique and analyticity properties of the correlation function of quark currents, the ratio

$$R_\tau=\frac{\Gamma(\tau^-\to\nu_\tau+\text{hadrons})}{\Gamma(\tau^-\to\nu_\tau e^-\bar{\nu}_e)} \tag{7.1}$$

is calculable in perturbative QCD. Strictly speaking, besides the QCD perturbative parts, the nonperturbative and weak contributions should be included to estimate R_τ. There are instanton contributions as well. However, it was shown recently by Nason and Porrati (1994; see also Kartvelishvili and Margvelashvili, 1995) that these contributions are completely negligible due to the chiral suppression factor $m_u m_d m_s/M_\tau^2$. The R_τ can be written as the following sum,

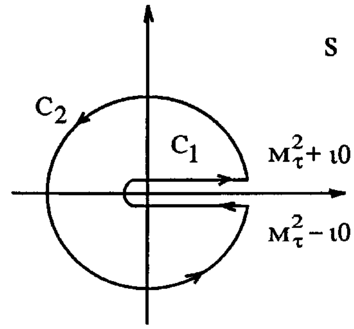


FIG. 11. Integration contour.

$$R_\tau=R_\tau^{\rm pert}+R_\tau^{\rm nonpert}+R_\tau^{\rm weak}. \tag{7.2}$$

A. Perturbative QCD contributions

The quantity $R_\tau^{\rm pert}$ can be expressed as the following integral over the invariant mass of the hadronic decay products of the τ lepton (Lam and Yan, 1977; Braaten, 1988),

$$R_\tau^{\rm pert}=\frac{3}{4\pi}\int_0^{M_\tau^2}\frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left[\left(1+2\frac{s}{M_\tau^2}\right){\rm Im}\Pi^T(s+i0)+{\rm Im}\Pi^L(s+i0)\right], \tag{7.3}$$

where M_τ is the mass of the τ lepton. The functions Π^T and Π^L are the transverse and longitudinal parts of the correlation function of weak currents of quarks coupled to a W boson. In fact, $\Pi^{T,L}$ are the appropriate combinations of vector and axial parts corresponding to the vector and axial currents of u,d,s light quarks (for details, see, for example, Pich, 1994a). The expression for $R_\tau^{\rm pert}$ in the form of (7.3) is not quite useful. The problem is that the correlation functions involved cannot be calculated at low energies because of the large nonperturbative effects that invalidate perturbative approach. However, simple analyticity properties of the correlation functions allow us to evaluate the integral in (7.3). Indeed, the function Π is analytic in the complex s plane everywhere except the positive real axis. According to the Cauchy integral theorem, an integral over s along the closed contour C_1+C_2 (Fig. 11) of the product of $\Pi(s)$ with any nonsingular function $f(s)$ is zero.

On the other hand, the imaginary part of the correlation function is proportional to its discontinuity across the positive real axis. So, the following relation holds,

[3]See, however, the discussion in the last three paragraphs of Sec. 3 in the review article by Surguladze (1994d).

$$\int_0^{M_\tau^2} ds\ f(s)\mathrm{Im}(s)=\frac{1}{2i}\int_{C_1} ds\ f(s)\Pi(s)$$
$$=-\frac{1}{2i}\int_{C_2} ds\ f(s)\Pi(s), \qquad (7.4)$$

where the C_2 is the circle of radius $|s|=M_\tau^2$ (Fig. 11). The benefit of the above relation is that, on the right-hand side, one needs to calculate the correlation function for $|s|$ at M_τ^2. It is to be hoped that M_τ is large enough to use the operator product expansion in powers of $1/M_\tau^2$, and the $\alpha_s(M_\tau)$ is small enough to use perturbative expansion in α_s. Then the perturbative method can, in principle, be used to calculate the leading term in the operator product expansion and the higher-twist terms can be estimated semiphenomenologically.

Using Eq. (7.4), one can express the perturbative part of the ratio R_τ by an integral over the invariant mass s of the final-state hadrons along the contour C_2 in the complex s plane (Fig. 11). In the chiral limit, $m_u=m_d=m_s=0$, the currents are conserved and the longitudinal part of the $\Pi(s)$ is absent. In the axial channel, $\Pi^L(s)=O(m_f^2/s)$; see Sec. IV). For the R_τ^{pert}, we get

$$R_\tau^{\mathrm{pert}}=\frac{3i}{8\pi}\int_{C_2}\frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left[\left(1+2\frac{s}{M_\tau^2}\right)\Pi^T(s)\right]. \qquad (7.5)$$

Note that the factor $(1-s/M_\tau^2)^2$ suppresses the contribution from the region near the positive real axis where the $\Pi(s)$ has a branch cut (Braaten, 1988). To simplify the description, we use the chiral limit, which is a perfect approximation for R_τ. On the other hand, the mass corrections can be included with the calculation very similar to that in Sec. IV. The actual calculations show (Chetyrkin and Kwiatkowski, 1993; see also recent analyses in Pich, 1994a) that the effects of quark mass corrections on R_τ are well below 1% and can be neglected. Note also that, in the massless quark limit, the contributions from vector and axial channels to Π coincide at any given order of perturbation theory; evidently, the results are flavor independent. So, in this case, for evaluation of $\Pi^T(s)$ in Eq. (7.5), we use our earlier results for the electromagnetic two-point correlation function that contributes to $R(s)$ in electron-positron annihilation (Sec. VI).

The function $\Pi^T(s)$ can be related to the $D(s)$ function defined in Sec. II as follows,

$$-\frac{3}{4}s\frac{d}{ds}\Pi^T(s)=\frac{\Sigma_{f=d,s}|V_{uf}|^2}{\Sigma_f Q_f^2}D(s), \qquad (7.6)$$

where V_{ud} and V_{us} are the Kobayashi-Maskawa matrix elements. $|V_{ud}|^2+|V_{us}|^2=0.998\pm0.002$ (see, for example, Pich, 1994b). The factor in the right-hand side of Eq. (7.6) is due to the replacement of the electromagnetic currents by charged weak currents in the correlation function. Note also that evidently the light-by-light-type graphs (Fig. 8) do not contribute to the decay width of the τ lepton. Thus the term $\sim(\Sigma_f Q_f)^2$ drops out in the D function. The perturbative coefficients of $D(s)$ have been given in the previous section up to the four-loop level in the vector channel [see Eqs. (2.16) and (6.11)].

Performing the contour integration in Eq. (7.5) using the relations (7.6) and (2.16), and replacing $\alpha_s(s)$ by $\alpha_s(M_\tau)$ using the evolution equation (3.19), we obtain in terms of perturbative coefficients of $R(s)$

$$R_\tau^{\mathrm{pert}}=\frac{|V_{ud}|^2+|V_{us}|^2}{\Sigma_f Q_f^2}\Bigg\{R_0+\frac{\alpha_s(M_\tau^2)}{\pi}R_1+\left(\frac{\alpha_s(M_\tau^2)}{\pi}\right)^2\left(R_2+\frac{19}{12}\beta_0R_1\right)+\left(\frac{\alpha_s(M_\tau^2)}{\pi}\right)^3\times\left[R_3+\frac{19}{6}R_2\beta_0+\frac{19}{12}R_1\beta_1+\left(\frac{265}{72}-\frac{\pi^2}{3}\right)R_1\beta_0^2\right]+O(\alpha_s^4)\Bigg\}, \qquad (7.7)$$

where, as we have already mentioned, the term $\sim(\Sigma_f Q_f)^2$ should be omitted in R_3.

Substituting the relevant expressions for R_i and β_i from the previous sections, we obtain the $O(\alpha_s^3)$ analytical result in the $\overline{\mathrm{MS}}$ scheme,

$$R_\tau^{\mathrm{pert}}(M_\tau^2)=N_F(|V_{ud}|^2+|V_{us}|^2)\Bigg\{1+\frac{\alpha_s(M_\tau^2)}{\pi}\left(\frac{3}{4}C_F\right)+\left(\frac{\alpha_s(M_\tau^2)}{\pi}\right)^2\left[C_F^2\left(-\frac{3}{32}\right)+C_FC_A\left(\frac{947}{192}-\frac{11}{4}\zeta(3)\right)+NTC_F\left(-\frac{85}{48}+\zeta(3)\right)\right]+\left(\frac{\alpha_s(M_\tau^2)}{\pi}\right)^3\left[C_F^3\left(-\frac{69}{128}\right)+C_F^2C_A\left(-\frac{1733}{768}-\frac{143}{16}\zeta(3)+\frac{55}{4}\zeta(5)\right)+C_FC_A^2\left(\frac{559715}{13824}-\frac{2591}{96}\zeta(3)-\frac{55}{24}\zeta(5)\right)+NTC_F^2\left(-\frac{125}{192}+\frac{19}{4}\zeta(3)-5\zeta(5)\right)+NTC_FC_A\left(-\frac{24359}{864}+\frac{73}{4}\zeta(3)+\frac{5}{6}\zeta(5)\right)+N^2T^2C_F\left(\frac{3935}{864}-\frac{19}{6}\zeta(3)\right)-\frac{\pi^2}{64}C_F\left(\frac{11}{3}C_A-\frac{4}{3}NT\right)^2\right]+O(\alpha_s^4)\Bigg\}. \qquad (7.8)$$

Within the standard QCD with the $SU_c(3)$ gauge group, we obtain

$$R_\tau^{\text{pert}}(M_\tau^2)=3(0.998\pm0.002)\Bigg\{1+\frac{\alpha_s(M_\tau^2)}{\pi}+\left(\frac{\alpha_s(M_\tau^2)}{\pi}\right)^2\left[\frac{313}{16}-11\zeta(3)-N\left(\frac{85}{72}-\frac{2}{3}\zeta(3)\right)\right]$$
$$+\left(\frac{\alpha_s(M_\tau^2)}{\pi}\right)^3\Bigg[\frac{544379}{1152}-\frac{121}{8}\zeta(2)-\frac{8917}{24}\zeta(3)+\frac{275}{6}\zeta(5)+N\left(-\frac{8203}{144}+\frac{11}{6}\zeta(2)+\frac{733}{18}\zeta(3)-\frac{25}{9}\zeta(5)\right)$$
$$+N^2\left(\frac{3935}{2592}-\frac{1}{18}\zeta(2)-\frac{19}{18}\zeta(3)\right)\Bigg]+O(\alpha_s^4)\Bigg\}_{N=3}, \tag{7.9}$$

and a numerical form reads

$$R_\tau^{\text{pert}}(M_\tau^2)=3(0.998\pm0.002)\Bigg[1+\frac{\alpha_s(M_\tau)}{\pi}+5.2023\left(\frac{\alpha_s(M_\tau)}{\pi}\right)^2+26.366\left(\frac{\alpha_s(M_\tau)}{\pi}\right)^3+O(\alpha_s^4)\Bigg]. \tag{7.10}$$

B. On the nonperturbative and electroweak contributions

The nonperturbative contributions to R_τ can be expressed as a power series of corrections in $1/M_\tau^2$,

$$R_\tau^{\text{nonpert}}\sim\frac{C_2(m_f^2(M_\tau),\theta_c)}{M_\tau^2}+\sum_{i\geqslant 2}\frac{C_{2i}\langle O_{2i}\rangle_0}{M_\tau^{2i}}, \tag{7.11}$$

where the m_f are u,d,s running quark masses, $\langle O_{2i}\rangle_0$ are the so-called vacuum condensates, which can be obtained phenomenologically, and the C_i are their coefficient functions describing short-distance effects. Note that in Eq. (7.11) we formally include part of the pure perturbative corrections (the first term), which is due to the nonvanishing u,d,s quark masses. These corrections for the u and d quarks are completely negligible. The contribution coming from the s quark is suppressed by $\sin^2\theta_C$ and is also below 1% (Pich, 1990). Presently, the only way to estimate the strong-interaction effects in the condensate contributions is by perturbation theory. The coefficient functions C_{2i} are asymptotic perturbative series in terms of α_s. In order to estimate the nonperturbative contributions, one needs to sum up the power series of the QCD perturbative series. In the previous section we described the calculation of the high-order perturbative QCD contributions to the coefficient functions of the dimension 4 power corrections (gluon, $\langle\alpha_s G^2\rangle_0$ and quark, $\langle m_f\bar{q}_f q_f\rangle_0$ condensates). It was shown (Loladze, Surguladze, and Tkachov, 1985; Surguladze and Tkachov, 1989b, 1990) that the high-order perturbative corrections to some of the coefficient functions are too large. For instance, for the coefficient function of the condensate $\langle m_s\bar{s}s\rangle_0$ in the vector channel [see Eq. (5.27)], $\Lambda_{\text{eff}}\approx 30\Lambda_{\overline{\text{MS}}}$. This indicates that the renormalization-group-invariant criteria to the perturbative calculability of the QCD contributions to the coefficient function are not fulfilled. The coefficient functions of the dimension 6 condensates are calculated up to $O(\alpha_s)$ (Lanin, Spiridonov, and Chetyrkin, 1986); to analyze the corresponding series, one needs at least the next-to-leading correction. The above uncertainty in coefficient functions C_{2i} allows one to estimate the condensate contributions probably not better than their order of magnitude. There is another source of theoretical uncertainties in the evaluation of condensate contributions of dimension 6 and higher, where the operator basis of expansion includes a large number of operators. Presently, there are no precise methods for estimating their matrix elements. For the matrix elements of four-quark operators (dimension 6), the vacuum saturation approximation (Shifman, Vainshtein, and Zakharov, 1979) is used to express them as the square of the two-quark matrix elements. However, the vacuum saturation approximation is not expected to be precise enough in order to use it in the analyses of the tiny nonperturbative contributions (see, for example, analysis by Altarelli, 1992; see also a brief discussion in Surguladze and Samuel, 1992b). Indeed, as found by Braaten (1988) and Pich (1990, 1992a, 1992b, 1994a), the nonperturbative corrections are below the 1% level with large theoretical error. The contributions of dim=4 condensates start at $O(\alpha_s^2)$ and thus are suppressed by two powers of α_s. The dim=6 and dim=8 corrections are suppressed by the inverse powers of M_τ (M_τ^6 and M_τ^8, respectively) and are small. On the other hand, the corrections in vector and axial channels have opposite signs, and they largely cancel each other; so the total relative error is even larger. In the works by Pumplin (1989, 1990), it was shown that the uncertainty due to threshold effects makes a significant contribution in the theoretical error for R_τ. In the works by Altarelli (1992) and Altarelli, Nason, and Ridolfi (1994), an ambiguity $\sim\Lambda^2/M_\tau^2$ is discussed. Earlier, Zakharov (1992) argued that such dim=2 terms in Eq. (7.11) can be generated by ultraviolet renormalons. For an alternative point of view on the effects of possible dim=2 terms, see Narison (1994). However, this issue is still a subject of intensive discussions and likely is far from being settled.

Summarizing, we note that the above-mentioned major sources of theoretical uncertainties in the evaluation of small power corrections place certain restrictions on the precision theoretical prediction of R_τ and consequently on $\alpha_s(M_\tau)$. Fortunately, the nonperturbative corrections are suppressed and the hadronic decay of the τ still remains as a good source to extract the low energy α_s.

Finally, we note that the electroweak contributions $R_\tau^{\rm weak}$ were calculated by Marciano and Sirlin (1988) and Braaten and Li (1990). Those corrections contain logarithms of M_τ/M_Z and are not negligible. The leading-order electroweak corrections given roughly +2% contributions to R_τ (see, for example, Pich, 1994a).

VIII. FOUR-LOOP QED RENORMALIZATION-GROUP FUNCTIONS

In this section we outline the calculation of the standard QED renormalization-group functions at the four-loop level in the minimal and momentum subtraction schemes. These quantities can be obtained as an intermediate result of the calculations of $R(s)$, described in the previous sections, by replacing the $\mathrm{SU}_c(3)$ gauge group invariants for the corresponding diagrams in a proper way. The results of two independent calculations of the four-loop QED β function by Surguladze (1990) and also by Gorishny, Kataev, and Larin (1990) have been reported in the joint publications by Gorishny, Kataev, Larin, and Surguladze (1991a, 1991c).

A. General formulas

The Lagrangian density of standard QED is

$$L_{\rm QED}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+i\sum_j \bar{\psi}_j\gamma^\mu D_\mu\psi_j-\sum_j m_j\bar{\psi}_j\psi_j -\frac{1}{2\alpha_G}\partial_\mu A^\mu\partial_\mu A^\mu, \tag{8.1}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and $D_\mu=\partial_\mu-ieA_\mu$. α_G is the gauge parameter; m_j are the fermion masses; ψ and A_μ are the fermion and photon fields; and e is the electric charge.

Renormalization constants are defined by the relations

$$\begin{aligned}&\psi_B=\mu^{-\varepsilon}\sqrt{Z_F}\psi,\\ &A_B^\mu=\mu^{-\varepsilon}\sqrt{Z_{\rm ph}}A^\mu,\\ &\alpha_B=\mu^{2\varepsilon}Z_\alpha\alpha\quad(\alpha=e^2/4\pi),\\ &\alpha_G^B=Z_G\alpha_G.\end{aligned} \tag{8.2}$$

For the fermion-fermion-photon vertex renormalization, one has

$$\mu^{-2\varepsilon}Z_{\rm vert}\,e\bar{\psi}\gamma_\mu A^\mu\psi=\mu^{-2\varepsilon}\sqrt{Z_\alpha Z_{\rm ph}}Z_F e\bar{\psi}\gamma_\mu A^\mu\psi. \tag{8.3}$$

According to the Ward identity in QED (Ward, 1950), $Z_{\rm vert}=Z_F$, which implies from Eq. (8.3) the identity

$$Z_\alpha Z_{\rm ph}=1. \tag{8.4}$$

From Eqs. (8.2) and (8.4) we get

$$\alpha_B=\mu^{2\varepsilon}Z_{\rm ph}^{-1}\alpha. \tag{8.5}$$

The gauge invariance of the QED Lagrangian implies the absence of the counterterm for the gauge-fixing term in (8.1), and thus $Z_G=Z_{\rm ph}$.

Using the relation (8.5) and the renormalization-group invariance of "bare" coupling $\mu^2 d\alpha_B/d\mu^2=0$, taking into account that $Z_{\rm ph}$ depends on μ only via α and also the standard definition of the QED MS β function

$$\beta_{\rm QED}^{\rm MS}(\alpha)=\frac{1}{4\pi}\mu^2\frac{d\alpha}{d\mu^2}\bigg|_{\alpha_B\ \rm fixed}, \tag{8.6}$$

we obtain a convenient expression for the further evaluation of the β function,

$$\beta_{\rm QED}^{\rm MS}(\alpha)=-\frac{1}{4\pi}\lim_{\varepsilon\to 0}\frac{\varepsilon\alpha}{1-\alpha\dfrac{\partial}{\partial\alpha}\log Z_{\rm ph}}. \tag{8.7}$$

B. Four-loop results

The photon field renormalization constant $Z_{\rm ph}$ can be found from the QED relation, analogous to Eq. (6.8), where only 58 QED four-loop diagrams contribute to $\Pi(\mu^2/Q^2,\alpha)$. The prescription for the evaluation of the diagram contributions to the Π_B is analogous to the one described in Sec. II. The total CPU time on the three IBM compatible mainframes was approximately 400 hours. Setting $C_F=1$, $C_A=0$, $T=1$, and $\alpha_s=\alpha$ in Eq. (6.8), we obtain the four-loop photon renormalization constant in QED, corresponding to the *minimal subtraction* prescription,

$$\begin{aligned}Z_{\rm ph}=&N-\frac{\alpha}{4\pi}\frac{4}{3\varepsilon}N-\left(\frac{\alpha}{4\pi}\right)^2\frac{2}{\varepsilon}N\\ &-\left(\frac{\alpha}{4\pi}\right)^3\left[\frac{8}{9\varepsilon^2}N-\frac{1}{\varepsilon}\left(\frac{2}{3}+\frac{44}{27}N\right)\right]N\\ &-\left(\frac{\alpha}{4\pi}\right)^4\left\{\frac{16}{27\varepsilon^3}N^2+\frac{1}{\varepsilon^2}\left(\frac{14}{9}N-\frac{88}{81}N^2\right)\right.\\ &\left.-\frac{1}{\varepsilon}\left[\frac{23}{2}-\left(\frac{190}{9}-\frac{208}{9}\zeta(3)\right)N+\frac{308}{243}N^2\right]\right\}N.\end{aligned} \tag{8.8}$$

Substituting the expression for $Z_{\rm ph}$ into Eq. (8.7), we obtain the following result for the four-loop QED β function in the MS-type schemes,

$$\begin{aligned}\beta_{\rm QED}^{\rm MS}(\alpha)=&\frac{4}{3}N\left(\frac{\alpha}{4\pi}\right)^2+4N\left(\frac{\alpha}{4\pi}\right)^3-N\left(2+\frac{44}{9}N\right)\left(\frac{\alpha}{4\pi}\right)^4\\ &-N\left[46-\left(\frac{760}{27}-\frac{832}{9}\zeta(3)\right)N+\frac{1232}{243}N^2\right]\left(\frac{\alpha}{4\pi}\right)^5.\end{aligned} \tag{8.9}$$

It is useful for further applications to present the result for the Johnson-Willey-Baker F_1 function (Johnson, Willey, and Baker, 1967; Baker and Johnson, 1971; Johnson and Baker, 1973). This function can be obtained from the result for $\beta_{\rm QED}^{\rm MS}$ by subtracting the contributions of the diagrams with fermion loop insertions into the photon lines and reducing the power in $\alpha/4\pi$ by 1 (with $N=1$). We obtain

$$F_1(\alpha)=\frac{4}{3}\left(\frac{\alpha}{4\pi}\right)+4\left(\frac{\alpha}{4\pi}\right)^2-2\left(\frac{\alpha}{4\pi}\right)^3-46\left(\frac{\alpha}{4\pi}\right)^4. \quad (8.10)$$

Note that all coefficients up to four-loop level are rational numbers. The results for most of the individual graphs do contain transcendental $\zeta(3)$, $\zeta(4)$, and $\zeta(5)$. The $\zeta(4)$ and $\zeta(5)$ cancel within each gauge-invariant set of diagrams. The three-loop results agree with the ones obtained by de Rafael and Rosner (1974). It is possible to recalculate the MS QED β function in the form of the Gell-Mann–Low $\Psi(\alpha)$ function—the QED β function in the MOM scheme. See details in Gorishny, Kataev, Larin, and Surguladze (1991a; see also Adler, 1972 and de Rafael and Rosner, 1974). We obtain the Gell-Mann–Low Ψ function at the four-loop level,

$$\Psi(\alpha)=\frac{4}{3}N\left(\frac{\alpha}{4\pi}\right)^2+4N\left(\frac{\alpha}{4\pi}\right)^3-N\left[2+\left(\frac{184}{9}-\frac{64}{3}\zeta(3)\right)N\right]\times\left(\frac{\alpha}{4\pi}\right)^4-N\left[46-\left(104+\frac{512}{3}\zeta(3)-\frac{1280}{3}\zeta(5)\right)N-\left(128-\frac{256}{3}\zeta(3)\right)N^2\right]\left(\frac{\alpha}{4\pi}\right)^5. \quad (8.11)$$

The $O(\alpha^4)$ result agrees with the one obtained by Baker and Johnson (1969) and Acharya and Nigam (1978, 1985).

Recently, Broadhurst, Kataev, and Tarasov (1993) carried out an additional calculation necessary to convert the four-loop MS QED β function to the four-loop QED on-shell β function, usually called the Callan-Symanzik function $\beta_{\rm QED}^{\rm CS}$ (Callan, 1970; Symanzik, 1970, 1971). This function is defined as

$$\beta_{\rm QED}^{\rm CS}(\alpha)=\frac{m_e}{\alpha}\frac{d\alpha}{dm_e}\bigg|_{\alpha_B\ \rm fixed}, \quad (8.12)$$

where m_e is the electron pole mass. The subtraction prescription in this case requires all subtractions to be on-shell. The three-loop $\beta_{\rm QED}^{\rm CS}$ was calculated long ago by de Rafael and Rosner (1974). The four-loop result has the following form (Broadhurst, Kataev, and Tarasov, 1993),

$$\beta_{\rm QED}^{\rm CS}(\alpha)=\frac{2}{3}N\left(\frac{\alpha}{\pi}\right)+\frac{1}{2}N\left(\frac{\alpha}{\pi}\right)^2-N\left(\frac{1}{16}+\frac{7}{9}N\right)\left(\frac{\alpha}{\pi}\right)^3-N\left[\frac{23}{64}-\left(\frac{1}{24}-\frac{5}{3}\zeta(2)+\frac{8}{3}\zeta(2)\ln 2-\frac{35}{48}\zeta(3)\right)N-\left(\frac{901}{648}-\frac{8}{9}\zeta(2)-\frac{7}{48}\zeta(3)\right)N^2\right]\left(\frac{\alpha}{\pi}\right)^4. \quad (8.13)$$

IX. RENORMALIZATION-GROUP AMBIGUITY OF PERTURBATIVE QCD PREDICTIONS

In the previous sections we have demonstrated the calculation of some of the important observables within the framework of perturbative QCD. This involves calculation of a large number of Feynman diagrams and requires a very large amount of computer and human resources. For example, to $O(\alpha_s^3)$ we have calculated 98 (effectively 250) four-loop Feynman diagrams. The next order requires calculation of approximately 600–700 five-loop diagrams. Calculations of such a scale are extremely difficult. On the other hand, perturbative QCD series are asymptotic ones, and the question of how many orders need to be calculated can be answered only from estimates of remainders (see, for example, the textbook by Collins, 1984). Moreover, perturbative coefficients beyond the two-loop level, as well as the expansion parameter, are scheme-scale dependent. The scheme-scale ambiguity, a fundamental property of the renormalization-group calculations in QCD, does not allow one to obtain reliable estimates from the first few calculated terms without involving additional criteria.

In this section we discuss the extraction of reliable estimates for observable quantities within perturbation theory. The problem of scheme-scale dependence of perturbative QCD predictions will be considered first within the MS prescription, and then we shall outline a scheme-invariant approach along the lines of Stevenson (1981a, 1981b). We apply the three known approaches for resolving the scheme-scale ambiguity. As a result, we fix the scheme-scale parameter, within the framework of MS prescription, for which all of the criteria tested are satisfied for the quantity $R(s)$ at the four-loop level (Surguladze and Samuel, 1993). On the other hand, we estimate the theoretical error by using the scheme-scale dependence as a measure of the theoretical uncertainty (Surguladze and Samuel, 1993; Surguladze, 1994b). We also mention the recent discovery of commensurate scale relations by Brodsky and Lu (1994, 1995). These relations allow one to connect several physical observables, providing important tests of QCD without scheme-scale ambiguity.

A. Perturbative QCD series: How many loops should be evaluated?

The R ratio in electron-positron annihilation is given within perturbation theory in the form

$$R(s)=r_0\left[1+r_1\left(\frac{s}{\mu^2}\right)\frac{\alpha_s(\mu)}{\pi}+r_2\left(\frac{s}{\mu^2}\right)\left(\frac{\alpha_s(\mu)}{\pi}\right)^2+r_3\left(\frac{s}{\mu^2}\right)\left(\frac{\alpha_s(\mu)}{\pi}\right)^3+\cdots\right]. \quad (9.1)$$

Our further discussion is quite general and can be applied to other observables like R_τ or Higgs decay rates. We consider high enough energies, where R is a function of a single variable—the center-of-mass energy squared. Our aim is to evaluate pure QCD effects in R, which start with the term $O(\alpha_s)$, within the minimal subtraction prescription ('t Hooft, 1973). We should stress here that the calculational methods allowing one to evaluate perturbative corrections up to the four-loop order (up to the five-loop in some cases) is essentially based on some of the unique features of the MS prescription, and our choice seems to be well justified. There is an ambiguity in the choice of renormalization scale parameter μ. Usually we set $\mu^2=s$ and absorb the large logarithms in the definition of the running coupling. On the other hand, the choice $\mu^2=\chi s$ ($\chi\equiv e^t$) for all χ gives equivalent expansions. Evidently, the sum of "all" terms in Eq. (9.1) does not depend on the choice of μ. However, in practice, we deal with truncated series, where the sum has a nontrivial dependence on the choice of renormalization parameter. Here we keep the "natural" choice $\mu^2=s$, and the ambiguity is transferred to the prescription $\int d^4p \rightarrow \int d^{4-2\varepsilon}p(\mu^2 e^{t+O(\varepsilon)})^{\varepsilon}$. By changing t, one gets different MS-type schemes. One can always reexpand (9.1) in a new scheme [with a new Λ in (2.17)] and so redistribute the values of r_i ($i>1$). All these schemes are equivalent. On the other hand, a new scheme may be "better," but one can conclude this based only on the knowledge of remainders. The problem of scheme-scale ambiguity, which, in fact, is a problem of remainders, can be formulated as follows. *How does one choose ("optimize") the scheme (or Λ) in order to make the remainder minimal in the series of the type (9.1) for the given range of energy, and what is the numerical uncertainty of the approximation (9.1)?* Here one should also distinguish the following two questions. First, what is the best accuracy to which the given quantity is calculable via perturbation theory? Second, what is the accuracy of the given approximation? A few notes are in order. It is known that perturbative QCD series are asymptotic ones. No reliable estimates of the remainders are known at present. However, it is known from the theory of asymptotic series (see, for example, Dingle, 1973) that

$$\left|\sum_{i=1}^{\mathcal{N}} r_i\alpha^i(s)-R(s)\right|=R_{\mathcal{N}}\rightarrow\Delta R_{\min},\ \text{when}\ \mathcal{N}\rightarrow\mathcal{N}_{\text{opt}}. \tag{9.2}$$

This means that the remainder $R_{\mathcal{N}}$ goes to its minimal value $\Delta R_{\min}$ when the number of orders goes to its optimal value $\mathcal{N}_{\text{opt}}$. Inclusion of the next-to-$\mathcal{N}_{\text{opt}}$ orders will lead away from the correct value. It is known (see, for example, Dingle, 1973) that for a sign-alternating asymptotic series, the remainder can be estimated by the first neglected term (or by the last included term). However, it is still unknown whether the QCD perturbative series has this character. We assume as a hypothesis that within QCD one can estimate the remainder by the first neglected or last included term. Now, the minimal possible error, which defines the best accuracy of the perturbation theory for the given quantity, has an order of $\Delta R_{\min}\sim r_{\mathcal{N}+1}\alpha^{\mathcal{N}+1}(s)$, $\mathcal{N}\rightarrow\mathcal{N}_{\text{opt}}$. Note that both the number $\mathcal{N}_{\text{opt}}$ and the value of the $\Delta R_{\min}$ depend on the range of energy for the given process. We once again emphasize that *the remainder depends on the choice of particular scheme and scale parameters, and its estimate makes sense only for the "optimized" renormalization scheme, which is unique for the given physical observable.* In fact, it was argued (Stevenson, 1984, 1994) that the "optimized" series can still converge even when the series in any fixed renormalization scheme is factorially divergent, if the "optimized" couplant shrinks in higher orders (see also Buckley, Duncan, and Jones, 1993). However, whether this applies to QCD is unknown.

B. $R(s)$ within the one-parameter family of the MS-type schemes and scale ambiguity problem

Using the results of our four-loop calculations, we obtain the analytical result for $R(s)$ with perturbative coefficients explicitly depending on the scheme-scale parameter (Surguladze and Samuel, 1993),

$$\begin{aligned}R(s,t)=R_0&+\frac{\alpha_s(s,t)}{\pi}R_1+\left(\frac{\alpha_s(s,t)}{\pi}\right)^2(R_2+\beta_0R_1t)\\&+\left(\frac{\alpha_s(s,t)}{\pi}\right)^3\Big[R_3-\frac{\pi^2}{3}\beta_0^2R_1+(2\beta_0R_2\\&+\beta_1R_1)t+\beta_0^2R_1t^2\Big].\end{aligned} \tag{9.3}$$

Recalling the values of the $\overline{\text{MS}}$ perturbative coefficients R_i from Eqs. (6.4) and (6.10) and the β_i coefficients from Eq. (2.8), we obtain numerically

$$\begin{aligned}R(s,t)=3\sum_f Q_f^2\Bigg\{&1+\frac{\alpha_s(s,t)}{\pi}+\left(\frac{\alpha_s(s,t)}{\pi}\right)^2\\&\times[(1.9857+2.75t)-N(0.1153+0.1667t)]\\&+\left(\frac{\alpha_s(s,t)}{\pi}\right)^3[(-6.6369+17.2964t+7.5625t^2)\\&-N(1.2001+2.0877t+0.9167t^2)\\&+N^2(-0.0052+0.0384t+0.0278t^2)]\Bigg\}\\&-\left(\sum_f Q_f\right)^2\left(\frac{\alpha_s(s,t)}{\pi}\right)^3 1.2395+O(\alpha_s^4),\end{aligned} \tag{9.4}$$

where $\alpha_s(s,t)$ can be parametrized in the form of (2.17) with $\mu=s$ and $\Lambda\rightarrow\Lambda_t=e^{-t/2}\Lambda_{\overline{\text{MS}}}$. Obviously, $t=0$ corresponds to the $\overline{\text{MS}}$ scheme [Eq. (6.12)]; $t=\ln 4\pi-\gamma$ will transform the result to the original MS scheme ['t Hooft, 1973; Eq. (6.14)]; $t=-2$ corresponds to the G scheme introduced by Chetyrkin and Tkachov (1979, 1981) [Eq. (6.13)]. Note that because of the one-parameter nature of the MS prescription, the t-dependent terms in Eq. (9.4) would represent also the scale dependence of the perturbative coefficients within the $\overline{\text{MS}}$ if one changes $t\rightarrow\log\mu^2/s$ and takes $\alpha_s(s,t)$ with s replaced by μ^2 and $t=0$.

Several approaches were suggested to deal with the scheme-scale-remainder problem. Among them we consider the following ones: *fastest apparent convergence*

(FAC; Grunberg, 1980, 1982, 1984), where the next-to-leading perturbative correction is absorbed in the definition of the "effective" running coupling and the scheme-scale parameter is fixed accordingly; *principle of minimal sensitivity* (PMS) of the approximant to the variation of nonphysical parameters (Stevenson, 1981a, 1981b, 1982, 1984; see also Mattingly and Stevenson, 1992, 1994); and the *Brodsky-Lepage-Mackenzie* (BLM) approach (Brodsky, Lepage, and Mackenzie, 1983), which suggests one fix the scale by the size of the quark vacuum polarization effects resulting in the independence of the next-to-leading-order perturbative correction of the number of quark flavors N. For discussions of these scheme-scale-setting methods, see Celmaster and Stevenson (1983), Brodsky and Lu (1992), and Stevenson (1992). The optimization of perturbation theory has previously been studied by Kramer and Lampe (1988) and Bethke (1989) for jet cross sections in electron-positron annihilation. The optimized perturbation theory is tested for different physical quantities in QED and QCD by Field (1993). The scale ambiguity problem has been considered by Lu and de Melo (1991) for the ϕ^3 model. The scheme-scale ambiguity problem for the quantities $R(s)$ and R_τ has been discussed by Maxwell and Nicholls (1990), Chyla, Kataev, and Larin (1991), and Grunberg and Kataev (1992). Further study of the PMS method has been done by Raczka (1995).

We apply the above methods to Eq. (9.1) and find a scale that gives good results for all criteria considered (Surguladze and Samuel, 1993). We start by noting that, in general, the renormalization scheme-scale dependence of perturbative results are parametrized by the scale parameter, say, μ and the renormalization-prescription-dependent coefficients of the β function (Stevenson, 1981a, 1981b). We should stress, however, that the β function is independent of any modification of the MS-type prescriptions; but starting from β_2, the coefficients of the β function do depend on the particular choice of subtraction prescription other than MS. In order to better visualize our discussion, we consider first the optimization procedures within the MS prescription. In other words, we fix the scheme-dependent perturbative coefficients of β function to their MS values and consider only the scale variation.

In Fig. 12 we have plotted $r_3(t)$ for different N [see Eqs. (9.1) and (9.4)]. As one can see, within the region $t\sim(-1.5,-0.5)$, r_3 has a very weak dependence on the number of flavors N as well as on the parameter t. Corresponding to the three-loop coefficient $r_2(t)$, straight lines intersect in one point for $t\approx -0.7$, which is obvious from Eq. (9.4). This value corresponds to the BLM result (Brodsky, Lepage, and Mackenzie, 1983) $\mu^2=\mu^2_{\overline{\rm MS}}e^{0.710}$, and at this scale the flavor dependence is absorbed into the definition of the coupling.

In Fig. 13 we have plotted the dependence of the partial sums

$$R_n(t)=\sum_{m=1}^{n} r_m(t)(\alpha_s/\pi)^m, \quad n=1,2,3$$

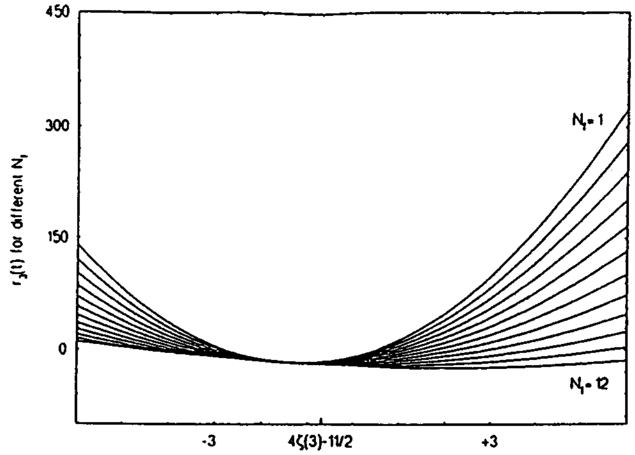


FIG. 12. $r_3(t)$ for different N.

on the parameter t. Here the parametrization (2.17) was used, $\log s/\Lambda^2_{\overline{\rm MS}}=9$ and $N=5$. The general picture does not change for other reasonable values of log and N. One can see that PMS (Stevenson, 1981a, 1981b) works perfectly for a wide range of the logarithmic scale parameter $t\sim(-1,+3)$ for the four-loop approximant and $t\sim(-2,0)$ for the three-loop approximant. A similar analysis at the three-loop level was done by Radyushkin (1983). According to the above analysis, we found that the BLM scale $t=-0.710$ is good at the four-loop level as well (Fig. 12), and this value is within the minimal sensitivity region (Fig. 13). Moreover, we found that if the t parameter is chosen in the following analytical form $t=4\zeta(3)-11/2+O(\varepsilon)$, which is equivalent to the definition of a new, say, $\widetilde{\rm MS}$ modification of the MS scheme

$$\Lambda_{\widetilde{\rm MS}}=\exp[-2\zeta(3)+11/4+O(\varepsilon)]\Lambda_{\overline{\rm MS}}, \tag{9.5}$$

then the N dependence and the $\zeta(3)$ terms cancel exactly at the three-loop level. As a result, $r_2=1/12$. Within this scheme the four-loop correction is almost independent of the number of flavors. The full result for

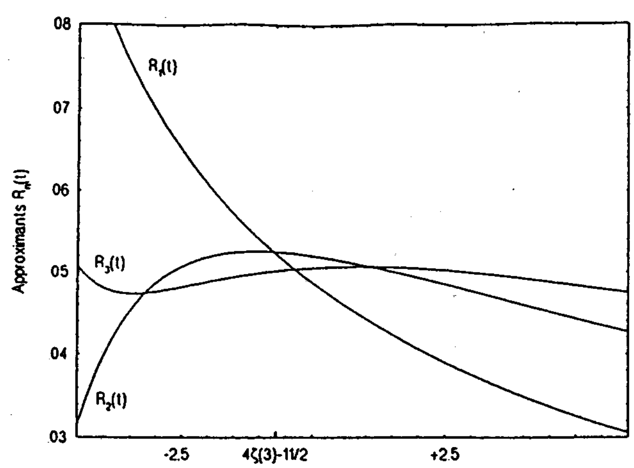


FIG. 13. The approximants R_n vs the scale parameter t.

the R ratio for the *arbitrary* number of flavors can be written in the following simple form,

$$R(s)=s\sum_f Q_f^2\left[1+\frac{\alpha_s}{\pi}+\frac{1}{12}\left(\frac{\alpha_s}{\pi}\right)^2-\left(\frac{\alpha_s}{\pi}\right)^3(16.2\pm 0.5)\right]$$
$$-\left(\sum_f Q_f\right)^2\left(\frac{\alpha_s}{\pi}\right)^3 1.2+O(\alpha_s^4), \qquad (9.6)$$

where the small uncertainty ± 0.5 stands for the remainder dependence on the number of flavors at $O(\alpha_s^3)$ for all physically reasonable N and is completely negligible for phenomenology. The last term is also very small, $\sim 0.4(\alpha_s/\pi)^3$. The running coupling can be parametrized in the standard form (2.17) with $\Lambda_{\widetilde{\rm MS}}=1.41\Lambda_{\overline{\rm MS}}$.

Using the FAC approach (Grunberg, 1980, 1982, 1984), we rewrite Eq. (9.6) as

$$R(s)=3\sum_f Q_f^2\left[1+\frac{\alpha_s^{\rm eff}}{\pi}+O(\alpha_s^3)\right], \qquad (9.7)$$

where the three-loop correction is absorbed into the definition of the effective coupling given by Eq. (2.17) with the Λ replaced by

$$\Lambda_{\rm eff}\approx\Lambda_{\widetilde{\rm MS}}\exp\left(\frac{1}{2\beta_0}\frac{r_2}{r_1}\right)\approx 1.02\Lambda_{\widetilde{\rm MS}}.$$

As one can see, the new scheme $\widetilde{\rm MS}$ almost coincides with the effective one, and the fastest convergence is guaranteed within the wide range of energy defined by the renormalization-group-invariant criteria

$$\frac{s}{\Lambda_{\rm eff}^2}\sim\frac{s}{\Lambda_{\widetilde{\rm MS}}^2}\gg 1.$$

Similar analyses can be done for the semihadronic decay rates of the τ lepton calculated to $O(\alpha_s^3)$ in Sec. VII. The result for the ratio R_τ in the $\widetilde{\rm MS}$ scheme reads

$$R^\tau=3(0.998\pm 0.002)\left[1+\frac{\alpha_s(M_\tau^2)}{\pi}+3.65\left(\frac{\alpha_s(M_\tau^2)}{\pi}\right)^2\right.$$
$$\left.+9.83\left(\frac{\alpha_s(M_\tau^2)}{\pi}\right)^3\right]+O(\alpha_s^4) \qquad (9.8)$$

and to be compared to Eq. (7.10). Note that the $\alpha_s(M_Z)$ is parametrized with the $\Lambda_{\widetilde{\rm MS}}=1.41\Lambda_{\overline{\rm MS}}$.

In Fig. 14 we plot one-, two-, and three-loop approximants to the $\Gamma_{H\to b\bar b}$ in terms of running quark mass [Eqs. (3.12)–(3.16), with $N=5$ and $m_f=m_b$] vs the scale parameter t (Surguladze, 1994b).

One can see that the higher-order corrections diminish the scale dependence from 40% to nearly 5%. The solid curve, corresponding to the three-loop result, became flat in the wide range of the logarithmic scale parameter t. Moreover, the choice $t=0$ ($\overline{\rm MS}$ scheme) satisfies Stevenson's *principle of minimal sensitivity* (Stevenson, 1981a, 1981b).

Let us now try to estimate the theoretical uncertainty in calculations of R by the last included term in the cor-

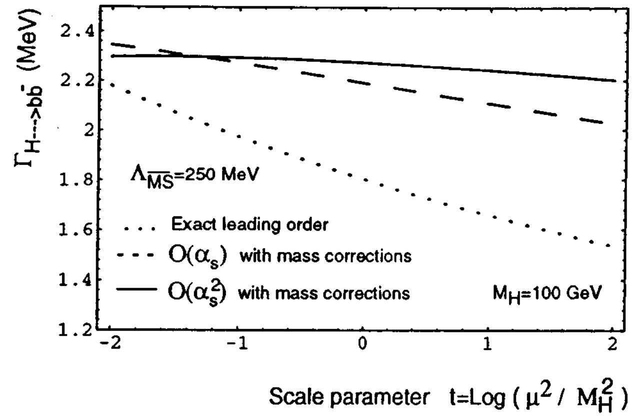


FIG. 14. The approximants of the $\Gamma_{H\to b\bar b}$ vs the scale parameter t.

responding perturbative expansion. We get for the QCD contribution within the $\widetilde{\rm MS}$ scheme the following result,

$$\delta_{\rm QCD}^{\widetilde{\rm MS}}\equiv\frac{R(s)-r_0}{r_0}=\frac{\alpha_s}{\pi}+\frac{1}{12}\left(\frac{\alpha_s}{\pi}\right)^2-(16.2\pm 0.5)\left(\frac{\alpha_s}{\pi}\right)^3$$
$$\pm(\delta_{\rm QCD}^{\rm err}=4\,\%). \qquad (9.9)$$

The analysis of Fig. 13 shows that the deviation of the four-loop approximant from the constant is also about 4% within a reasonably wide range of the t parameter. This is consistent with Stevenson's principle. One should note that the above error estimate is only for the massless quark limit. There are several different types of additional contributions, including those due to nonvanishing quark masses. This may change the above error estimate. All of the necessary information on the status of the additional corrections can be found in Kniehl (1994b, 1995b) and in Soper and Surgladze (1995a).

As we have already mentioned, recently Brodsky and Lu (1994, 1995) found the relations between the effective couplings α_A and α_B for the physical observables A and B in the following form,

$$\alpha_A(\mu_A)=\alpha_B(\mu_B)\left(1+r_{A/B}\frac{\alpha_B}{\pi}+\cdots\right). \qquad (9.10)$$

The ratio of the scales of the corresponding processes μ_A/μ_B is chosen according to the BLM scale-setting prescription so that $r_{A/B}$ is independent of the number of flavors. Thus, evolving α_A and α_B, they pass the quark thresholds at the same scale. It is shown that the relative scales satisfy the transitivity rule

$$\frac{\mu_A}{\mu_B}=\frac{\mu_A}{\mu_C}\times\frac{\mu_C}{\mu_B}.$$

So, C may correspond to any intermediate theoretical scheme such as MS, $\overline{\rm MS}$, etc., and the perturbative results can be tested without a reference to them. One of the impressive results of this method is a surprisingly simple relation between the effective couplings for the quantities R and R_τ to the next-to-next-leading order (Brodsky and Lu, 1994, 1995),

$$\frac{\alpha_\tau(M_\tau)}{\pi}=\frac{\alpha_R(\mu)}{\pi}, \quad \mu=M_\tau \exp\left[-\frac{19}{24}-\frac{169}{128}\frac{\alpha_R(M_\tau)}{\pi}\right].$$

For more details and the relations between various other observables, we refer to the original works by Brodsky and Lu (1994, 1995).

C. On scheme-invariant analyses

Let us now outline the original method of scheme-invariant analyses for the perturbation-theory results by Stevenson (1981a, 1981b, 1982, 1984). We note first that our analyses of perturbation series for $R(s)$ and R_τ have been done in Sec. IX.B within the one-parameter family of the MS-type schemes, where all β-function coefficients are the same for any modification of MS. In the PMS method, renormalization scale and scheme dependence is parametrized by the scale parameter μ/Λ and the scheme-dependent coefficients of the β function $\beta_2,\beta_3,\ldots$. Then the *principle of minimal sensitivity* is applied to the variation of the above parameters, and, to $O(\alpha_s^3)$, the "optimized" scheme corresponds to a flat two-dimensional surface. Our curve for R_3 in Fig. 13 is just a one-dimensional slice at the particular MS value of the β_2. The main points of the PMS formalism are as follows. [For the scheme-invariant analyses of $R(s)$ to $O(\alpha_s^3)$, see Mattingly and Stevenson, 1994]. To use familiar standard notation, we rewrite Eq. (2.7) for the couplant $a\equiv\alpha_s(\mu)/\pi$

$$b\frac{\partial a}{\partial\tau}=-ba^2(1+ca+c_2a^2+\cdots), \tag{9.11}$$

where

$$\tau=b\ \ln\frac{\mu}{\Lambda},\quad b=2\beta_0,\quad c=\frac{\beta_1}{\beta_0} \tag{9.12}$$

and for any modification of the minimal subtraction prescription, the scheme-dependent coefficient $c_2=\beta_2/\beta_0$. The scheme and scale can now be parametrized by the quantities $RS\equiv(\tau,c_2,c_3,\ldots)$. The *principle of minimal sensitivity* can be written as

$$\frac{dR_n}{d(\tau;c_2,c_3,\ldots)}=0. \tag{9.13}$$

The number of scheme-scale parameters in the above equation is strongly correlated with n. Indeed, it is not difficult to show that the following self-consistency condition should hold for the nth approximant,

$$\frac{\partial R_n}{\partial(RS)}=O(a^{n+1}). \tag{9.14}$$

This shows that the perturbative coefficients r_i can depend on renormalization scheme only through parameters $\tau;c_2,\ldots,c_{i-1}$. Applying the *principle of minimal sensitivity* in the form (9.13) to the approximants R_2 and R_3 and taking into account (9.14), one finds that the quantities

$$\rho_1\equiv\tau-r_2,$$

$$\rho_2\equiv r_3+c_2-\left(r_2+\frac{c}{2}\right)^2 \tag{9.15}$$

are renormalization scheme independent. Similar invariants can be constructed at each order of perturbation theory. The choice of τ as a function of the ratio μ/Λ emphasizes that the renormalization scheme dependence involves only the ratio of these quantities, and the optimization deals with τ but not μ. The "optimal" values of renormalization scheme parameters $\bar\tau$ and $\bar c_2$ are defined by the following equations. To $O(\alpha_s^2)$,

$$\left.\frac{dR_2(\tau)}{d\tau}\right|_{\tau=\bar\tau}=0. \tag{9.16}$$

To $O(\alpha_s^3)$,

$$\left.\frac{\partial R_3(\tau,c_2)}{\partial\tau}\right|_{\tau=\bar\tau}=0, \tag{9.17}$$

$$\left.\frac{\partial R_3(\tau,c_2)}{\partial c_2}\right|_{c_2=\bar c_2}=0. \tag{9.18}$$

Solving the above equations along with Eqs. (9.15) for the renormalization scheme invariants and Eq. (9.11) for the couplant with the truncated MS β function, using the $\overline{\text{MS}}$ values of r_2 and r_3, one finds the "optimized" values of $\bar\tau,\bar c_2$ and corresponding "optimized" approximants to $O(\alpha_s^3)$. The theoretical error can be estimated, as in Sec. IX.B, by the last calculated term. One obtains the following "optimized" result for the QCD contribution in R(34 GeV) in the massless quark limit (Mattingly and Stevenson, 1994; Stevenson, 1994),

$$\delta_{\text{QCD}}^{\text{PMS}}=0.051\pm0.001. \tag{9.19}$$

It is important to note that the above optimization procedure yields a negative value for the ρ_2 invariant. This results in the existence of a solution of the equation

$$\frac{7}{4}+c\bar a^*+3\rho_2(\bar a^*)^2=0 \tag{9.20}$$

with respect to $\bar a^*$—the value of the couplant for which the optimized third-order β function vanishes. This allows one, in principle, to do some analysis for $R(s)$ at the low energies $\sqrt{s}\to0$ (Mattingly and Stevenson, 1992).

Finally, we also mention that the FAC approach (Grunberg, 1980, 1982, 1984) is a special case of the PMS method (Stevenson, 1981a, 1981b, 269, 1984). Indeed, in the FAC approach all higher-order approximants are equal to the effective couplant [compare to Eqs. (9.16) and (9.18)]. From Eqs. (9.15), one gets $\rho_1=\tau$ and $\rho_2=c_2$ in the FAC approach.

X. CONCLUSIONS

We reviewed the current development of calculational methods, algorithms, and computer programs that allow one to evaluate the characteristics of the phenomenologically important physical processes to higher orders of perturbative QCD. We have considered $Z\rightarrow$hadrons, $\tau^-\rightarrow\nu_\tau+$hadrons, $H\rightarrow$hadrons. The described methods are applicable to a wide class of calculational problems of modern high-energy physics. We outlined the analytical three- and four-loop calculations for the above-mentioned processes.

The methods of analytical perturbative calculations available at present allow one, in principle, to evaluate various decay rates, cross sections, and coefficient functions in the operator product expansion, renormalization-group functions, etc., up to and including the five-loop level. This would correspond, for instance, to the decay rate in the process $Z\rightarrow$hadrons to $O(\alpha_s^4)$. It seems that such a high order will completely fit the experimental state of the problem in the observable future. Indeed, for example, the 4% estimate of the theoretical error for the decay rate of Z boson is based on the $O(\alpha_s^3)$ calculation. The present experimental error at LEP is about 5%.

The involvement of the heavier quarks in the physical processes makes it necessary to develop methods for calculation of the Feynman graphs with the propagators of massive particles. The expansion in terms of large or small masses may not always give satisfactory results.

The problem of the renormalization-group ambiguity of the perturbation-theory results and of various methods for resummation of higher-order corrections is a subject of growing interest and discussions in the literature.

The future development of analytical programming tools towards the full automation of high-order calculations would be welcome. This would greatly reduce the chance of errors in the calculations. On the other hand, the computer package with full implementation of the algorithm of high-order analytical perturbative calculations would make it realistic to step up by one more order.

We recognize that it is unavoidable that some of the relevant references have not been mentioned. We assure the reader that this is due only to our unintentional ignorance.

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