Cyclotron absorption and emission in mode conversion layers—a new paradigm

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When the analysis of absorption with mode conversion effects included began to mature in recent years, the study of the corresponding effects on emission began and has led to some surprising results. The classical expressions for cyclotron or synchrotron emission from a harmonic resonance were originally derived from models that did not include mode conversion or its attendant reflection, and classical expressions for the optical depth and opacity were obtained. When mode conversion was included, the principal surprise was that the transmission coefficient, which was understood as being due to absorption, is totally independent of absorption and due exclusively to tunneling. The other surprise from the mode conversion analysis is that the observed emission arises from two distinct sources, one direct and one from an indirect Bernstein wave source which is partially converted in the cyclotron layer to outgoing electromagnetic waves, with the net result that mode conversion cancels out for the electron case, but not for ions. The only corrections to electron cyclotron emission are then due to reflection effects, and these have been shown to be small for many laboratory plasmas, leading to the validation of the classical formula for these cases, but via an entirely new paradigm in its interpretation. This review includes a summary of the absorption process for both electron and ion cyclotron harmonics, and reviews carefully the emission physics, including both potential error estimates and a discussion of the emission source distribution in space.

CONTENTS

I. INTRODUCTION

A. Background and overview

The advent of mode conversion theory dawned when it was understood that cold plasma resonances are resolved by warm plasma effects when collisions are weak, and that when the resonance is approached in space due to an inhomogeneity, there is invariably some linear coupling between a cold plasma wave and a warm plasma wave. The original analysis (Stix, 1965) demonstrated that approaching the lower hybrid resonance in a finite-temperature plasma leads not to a pole, but to a region in the plasma where all of the incident wave energy is converted to a fundamentally different warm electrostatic wave. If there is absorption involved in such an isolated resonance, it is associated with the eventual fate of the converted wave, and almost never occurs near the conversion layer.

The connection between cyclotron harmonic resonances and mode conversion has long been established (Erokhin, 1969; Ngan and Swanson, 1977; Antonsen and Manheimer, 1978), and the role of absorption in the mode conversion process has been discussed by many [Swanson, 1980; Colestock and Kashuba, 1983; Swanson, 1985 (a review of ion cyclotron cases); Kay *et al.*, 1988; Lashmore-Davies *et al.*, 1988; Ye and Kaufman, 1988; Chow *et al.*, 1990]; but the effects of mode conversion on emission have only relatively

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recently been addressed (Swanson and Cho, 1989; Cho and Swanson, 1990a, 1990b; Swanson and Shvets, 1992). The principal focus of this review is to pull together some of the new results on emission as influenced by mode conversion, and since the absorption process is crucial to emission, the work on absorption will also be reviewed briefly. The importance of electron cyclotron emission (ECE) as a plasma diagnostic has influenced the various contributions in this area, so that most of the analysis relates to emission and absorption near the second and third harmonics of the electron cyclotron frequency, although corresponding work at ion cyclotron harmonic resonances will be included. It is not the purpose of this paper to review all of mode conversion theory, since some resonances, such as the lower hybrid and upper hybrid resonances, are virtually free of absorption, and hence of emission. Absorption and emission at the two-ion hybrid resonance could have been included, since the theory is very similar; but although this absorption process is important, the emission process seems not to be of great interest. This narrows the scope of the model equations to the class of tunneling equations, where there is propagation on both sides of the resonance or mode conversion layer in which there is finite transmission, reflection, and conversion to Bernstein modes.

As will be discussed in detail in the following sections, *absorption has no direct effect on the transmission* through one of these mode conversion layers; so all of the absorption and emission will be tied to the reflection and conversion processes, which are the *only* components affected by absorption. This flies in the face of conventional wisdom which has long estimated opacity and optical depth from a transmission coefficient calculated from integrating the imaginary part of the wave vector, which has been *assumed* to be due to absorption, across the layer. One of the remarkable results of the more nearly exact treatment which includes mode conversion effects is the exact agreement in one case with the traditional expressions for opacity, although for subtle reasons. Another case includes effects of reflection; so reflection coefficients will be carefully examined so that accurate estimates of these effects may be obtained. In many cases, these effects are small, so that classical expressions are often reliable; but the change in paradigm, or way of understanding the processes involved, is a major shift which includes a mandatory change in the language used to describe opacity, or at least a redefinition of what one means by the words. While the classical theory for electron cyclotron harmonic emission is largely validated except for our understanding of the physics involved, the ion cyclotron harmonic resonance emission is substantially different, and the classical opacity formula may fail badly except in the extreme case of very strong absorption (due to relatively large k_z). There do exist, however, cases of practical importance where electron cyclotron emission differs substantially from classical theory; so that deviations due to mode conversion theory are not merely academic. It is also found that there is a systematic shift between the peaks of the emission source distribution function and the absorption distribution function, as well as a narrowing of the source relative to the sink, due to mode conversion. This modifies our understanding of where the radiation source is located.

B. The fundamentals of mode conversion theory

The fundamental goal of mode conversion theory is an exact full-wave analysis of spatially resolved resonances. In Sec. I. B, the domain of mode conversion theory is described with minimal algebra to clarify the nature of the problems, the kinds of methods used, and the kinds of results obtainable.

1. Full-wave theory

A full-wave analysis means that a complete exact solution of a wave equation is solved as opposed to an approximate solution by eikonal methods. This does not mean that asymptotic forms are not used in the solutions, which are eikonal forms, but that the *coupling coefficients between* the asymptotically propagating wave branches are found *without any eikonal approximations*. Some of these coefficients are determined exactly, and some are evaluated numerically, but no approximations need be made. The term ''exact'' means that once the wave equation is determined, no further approximations are made. There are always some approximations in establishing the appropriate wave equation. Certain of the results are virtually independent of the approximations made in establishing the wave equation, such as the independence of the transmission coefficient from absorption, since it depends only on the analytic nature of the plasma dispersion function. This is worth noting, since this result is probably the most surprising one of all from mode conversion theory, since it is contrary to conventional wisdom which has traditionally attributed finite transmission through a resonance layer to absorption, whereas these effects are not directly related. Other scattering coefficients do depend on the approximations made in establishing the wave equation, but weakly.

2. Plasma resonances in space

The domain of mode conversion theory is restricted to plasma resonances that are approached in space due to a weak inhomogeneity in the plasma parameters, typically either the density or the magnetic field, and are resolved by the coupling to a different mode or wave type rather than being resolved solely by absorption processes. The typical cases are isolated resonances, back-to-back resonance-cutoff pairs, or a cutoff-resonance-cutoff triplet, provided that these are encountered sufficiently closely in space that each individual feature has some influence on the other. Extensions include cases in which a warm wave may change from a forward wave to a backward wave, which is effectively a mode conversion, and behaves like an isolated resonance. That such resonances lead to mode conversion is evident from the ''*Mode Conversion Theorem*'' (see Swanson, 1989, pp. 242, 243): ''In an inhomogeneous plasma, linear mode conversion is always involved to some extent in resolving every plasma resonance.'' The proof begins with a generalized dispersion relation $D(k,x)=0$, where the plasma parameters (density or magnetic field) are (slowly) spatially varying. Expanding this dispersion relation about some particular, k_c, one may write

$$D(k,x)=D[k_c(x),x]+\left.\frac{\partial D}{\partial k}\right|_{k_c}(k-k_c)+\frac{1}{2}\left.\frac{\partial^2 D}{\partial k^2}\right|_{k_c}(k-k_c)^2 + \mathcal{O}(k-k_c)^3 \ . \quad (1)$$

Choosing to expand about the point where the group velocity vanishes, or where $v_g=\partial\omega/\partial k=(\partial D/\partial k)/(\partial D/\partial\omega)=0$, one finds that the first-order term vanishes. Neglecting the higher-order terms, the result may be approximated by the expression

$$D(k,x)=P(x)+Q(x)(k-k_c)^2=0 \ , \quad (2)$$

so that, defining a shifted $k_s=k-k_c$, this may be written as

$$k_s^2=-P(x)/Q(x) \ . \quad (3)$$

Clearly, $P(x_0)=0$ is a cutoff, and $Q(x_R)=0$ is a resonance. However, any time the highest-order term in an expansion such as in Eq. (2) vanishes, one must go at least to the *next higher order* in order to adequately describe the dispersion characteristics; and from Eq. (1), this next term is of order k_s^3 (which vanishes whenever the dispersion relation is even in k, since the first-order term was chosen to vanish) or k_s^4. These higher-order terms are related to *another wave* and indicate mode conversion between at least two types of waves. The proviso in the theorem, "to some extent," notes that if there is some absorption, $Q(x)$ may never truly vanish, and in such a case it must be determined whether the fourth- (or higher-) order term exceeds the minimum of the quadratic term (mode conversion dominant) or vice versa (absorption dominant).

3. Dispersion relations

The wave equation to be analyzed is typically obtained by converting a dispersion relation (for a homogeneous plasma) into a differential equation by letting $ik\rightarrow d/dx$. A more nearly precise method is to include the weak inhomogeneity from the beginning and reduce the Vlasov-Maxwell equations to a single wave equation. The resulting wave equation is typically an ordinary differential equation, as the resonant surface is locally plane and the variation normal to the plane is one dimensional. Some implementations use a partial differential equation to include variations over the entire cross section with boundary conditions and antennas. In this review, the analysis is restricted to cyclotron harmonic resonances with slow variations in the magnetic field only, represented by $B(x)=B_0(1+x/L)$ and $\boldsymbol{B}\cdot\nabla B=0$, and L is the scale length for the magnetic-field variation ($L\sim R_0$ for a tokamak). The model equations and analysis are almost identical to those of the two-ion hybrid resonance, but that case is not considered here.

Dispersion relations, typically represented by $D(\omega,k)=0$ in a homogeneous plasma, are here represented by $D[k(x)]=0$, since at the resonance surface (at $x=0$), $\omega=n\omega_{c0}$, and the dispersion is due to the variation in $k(x)$ about $x=0$ through $\omega_c=\omega_{c0}(1+x/L)$. In addition to the cold plasma terms, which for the cases considered here describe either the fast Alfvén wave or the X-mode, there is typically a warm plasma term. For the harmonic resonances considered here, the warm plasma term is a Bernstein wave and fundamental to the analysis, since the harmonic resonances enter the dispersion relation through finite Larmor orbit effects, characterized by $\lambda_{e,i}=\frac{1}{2}k^2\rho_{L_{e,i}}$. The Bessel function expansions in $\lambda_{e,i}$ are normally truncated at the lowest order which includes the particular harmonic resonance, since $\lambda_{e,i}\ll 1$, but some integro-differential equation methods make no such approximations.

The most common dispersion relations that arise in mode conversion problems are (at least asymptotically) of the form

$$k^4+(a_2x+b_2)k^2+a_0x+b_0=0 \ , \quad (4)$$

with $a_2\neq 0$. For the isolated resonance case such as the lower hybrid resonance, $a_0=0$, while $a_0/a_2<0$ for the case with a back-to-back resonance-cutoff pair. This latter result is apparent by dropping the k^4 term (throwing away the warm wave), so that $k^2=-a_0/a_2>0$ as $|x|\rightarrow\infty$, which is the cold plasma dispersion relation away from the resonance. At least one of the b_n must be nonzero, but either one can be made to vanish by a simple translation of the origin. By changing to dimensionless variables, the two cases may be put in *standard form*, so that the isolated resonance case in standard form is

$$k^4-\lambda^2 zk^2+\beta=0 \ , \quad (5)$$

where λ^2 (*not* the same as $\lambda_{e,i}$ above) and β are real dimensionless constants and z is proportional to x. A plot of this dispersion relation for $\lambda^2>0$ may be found in Fig. 6 of my textbook (Swanson, 1989). The corresponding standard form for the resonance-cutoff pair is

$$k^4-\lambda^2 zk^2+\lambda^2 z+\gamma=0 \ , \quad (6)$$

where γ and λ^2 are real constants. Effects of localized absorption may be included by letting a_2 and a_0 be replaced by complex functions of x whose asymptotic limit is a real constant. The transmission coefficient obtained from the eikonal method uses $k(z)$ from this equation and finds the amplitude transmission coefficient to be $T=e^{-\eta}$ where η is the *tunneling factor* given by

$$\eta=\int \mathrm{Im}[k(z)]dz \ . \quad (7)$$

Depending on the value of γ, there are two distinct forms that the dispersion relation can take. For $1+\gamma>0$, which is characteristic of the ion cyclotron harmonics, a plot of Eq. (6) is given in Fig. 1. The three propagating branches are labeled such that Branch 1 represents the fast wave on the high-magnetic-field side; Branch 2 represents the fast wave on the low-magnetic-field side; and Branch 3 is the Bernstein mode. Since the plot is of k^2, each branch permits both incoming and outgoing waves, with the arrows indicating incoming waves. The complex region in the center is the tunneling region where k has an imaginary part due solely to tunneling.

For the electron case, $1+\gamma<0$, and the coupling is distinctly different, as may be seen in Fig. 2. Again the propagating branches are labeled, but here Branches 1 and 2 represent the X-mode on the high- and low-field sides, respectively. The branches are labeled so that, in either case, Branch 1 represents a (relatively) fast wave which encounters the resonance before the cutoff, while Branch 2 encounters the cutoff before the resonance. Branch 3 is always a

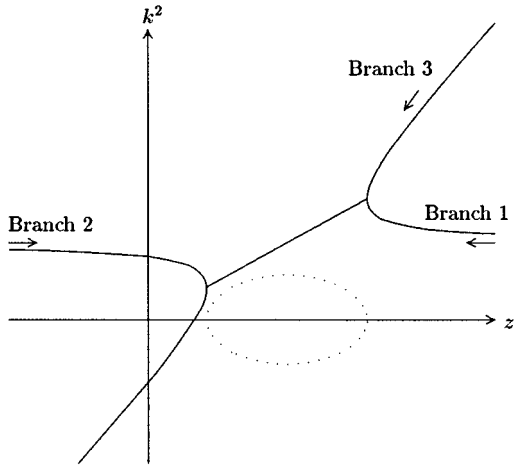


FIG. 1. Dispersion relation for tunneling equation for $n=2$ with $1+\gamma>0$. Solid lines are the $\mathrm{Re}(k^2)$, and $\mathrm{Im}(k^2)$ is dotted.

Bernstein mode. For this case, k^2 is real for all real z, and tunneling is not immediately apparent. At first glance, it would appear that a wave incident along Branch 1 would simply convert to Branch 3, while a wave incident along Branch 2 would simply reflect. This would imply no transmission at all, since transmission relates Branches 1 and 2 (fast wave on one side to a fast wave on the other side). There is transmission, but the coupling points [where the discriminant of Eq. (6) vanishes] in this case occur for complex z, such that $\lambda^2 z_\pm = 2 \pm ir$ with $r=2\sqrt{|1+\gamma|}$, and the integration path in Eq. (7) must be deformed to pass through these points to couple the fast waves on the two sides. An example of this coupling is illustrated in Fig. 3, where, with $\gamma=-2$, $r=2$, and the discriminant vanishes at $\lambda^2 z=2(1\pm i)$ where $k^2=2\pm i$. The integration path in the figure is a semicircle in the complex z plane, such that $\lambda^2 z=2(1+e^{i\theta})$ and $0\leqslant\theta\leqslant\pi$, as illustrated, or $\pi\leqslant\theta\leqslant 2\pi$ for waves traveling in the opposite direction. It is apparent from the figure that Branch 1 is now connected to Branch 2 and that k^2 is complex between $\lambda^2 z=0$ and $\lambda^2 z=4$, so that Eq. (7) yields a nonzero result. Any path in the z plane that is

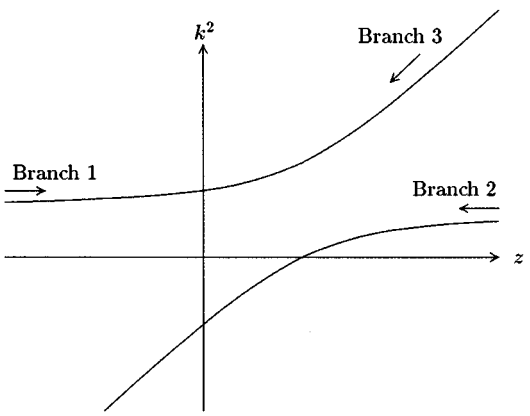


FIG. 2. Dispersion relation for tunneling equation for $n=2$ with $1+\gamma<0$.

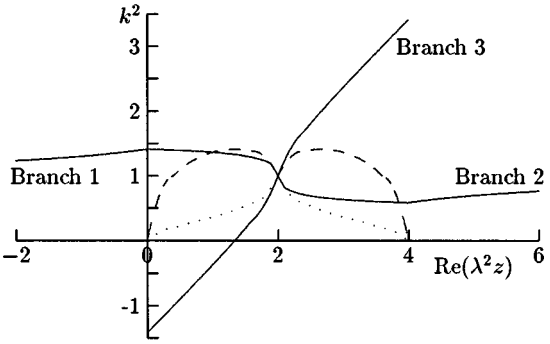


FIG. 3. Form of dispersion relation with $\gamma=-2$ and $\lambda^2 z=2(1+e^{i\theta})$ for $0\leqslant\mathrm{Re}(\lambda^2 z)\leqslant 4$. $\mathrm{Re}(k^2)$ is solid; $\mathrm{Im}(k^2)$ is dashed for the slow wave and dotted for the fast wave.

real at each end and passes through the coupling point will do, since the integral is path independent.

4. Wave equations

The wave equation is usually obtained from the appropriate dispersion relation by letting $ik\rightarrow d/dx$, or in the dimensionless form, $ik\rightarrow d/dz$. There is ambiguity in this recipe, since, in the dispersion relation, the $\lambda^2 z k^2$ term in Eq. (6) could also be written as $k^2\lambda^2 z$ or $k\lambda^2 z k$, all of which are equivalent until k becomes an operator. The proper recipe can only be deduced by going back to the Vlasov-Maxwell equations and never using the Fourier transform in the first place. Fortunately, the differences do not turn out to be large, so the simplest recipe will be used. In this case, Eq. (6) becomes the *tunneling equation*,

$$f^{iv}+\lambda^2 z f''+(\lambda^2 z+\gamma)f=0\ , \tag{8}$$

where $f(z)$ is proportional to the wave electric field, typically the y component. This equation admits an exact solution in terms of an integral via the Laplace integral method, namely,

$$f(z)=\int_\Gamma \exp\left[\left(z-\frac{1}{\lambda^2}\right)\tan u+\frac{\tan^3 u}{3\lambda^2}+\frac{1+\gamma}{\lambda^2}u\right]du\ , \tag{9}$$

where each of the four independent solutions is represented by one of the four independent contours, Γ_k, $k=1,2,3,4$, in the complex u plane. Each contour must terminate at one of the points, $u_n=(n+1/2)\pi$, approached at angles $0,\pm 2\pi/3$, so the integrand always vanishes at the end points. The scattering coefficients are obtained from the asymptotic forms of these exact solutions; and without absorption, these coefficients are exact. The asymptotic properties of the propagating solutions are determined from saddle points of the integrand, which are always determined from the original dispersion relation, regardless of whether the wave equation is obtained from the simplest recipe or from one of the more nearly exact methods.

For higher harmonics, the resonant term enters the dispersion relation through a higher-order term in the finite Larmor radius expansion, so the dispersion relation is higher order in k^2. The wave equation is correspondingly of higher order,

but exact solutions of similar type may be found for any order (Gambier and Schmitt, 1983; Gambier and Swanson, 1985).

5. Localized absorption

When localized absorption is included, the wave equation can no longer be solved exactly in terms of integrals; but several exact results have been found by separating the local and asymptotic features of the dispersion relation. To illustrate how this separation is accomplished, a representative case is considered in which $a_n x$, $n=0,2$, is replaced by $a_n x/F(x)$, and $F(x)\to 1$ as $|x|\to\infty$ but is an analytic function everywhere. For a nonrelativistic case, $F=-\zeta Z(\zeta)$, where ζ is proportional to x (and z). The appropriate dispersion relation may then be written as

$$k^4-\left(\frac{\lambda^2 z+\gamma}{F(z)}-\gamma\right)k^2+\frac{\lambda^2 z+\gamma}{F(z)}=0\ . \tag{10}$$

The eikonal method would solve this biquadratic for $k(z)$ and find η from Eq. (7).

In the separation scheme, no such approximations are made; but the terms involving $1/F$ are moved to the other side, and corresponding terms with $F=1$ are added to both sides. After letting $ik\to d/dz$, one is led to the wave equation

$$\psi^{iv}+\lambda^2 z\psi''+(\lambda^2 z+\gamma)\psi=(\lambda^2 z+\gamma)[1-1/F](\psi''+\psi)\ . \tag{11}$$

In this equation, only asymptotic terms are on the left and only local terms are on the right, since $1-1/F\to 0$ as $|z|\to\infty$. If one were to neglect absorption, setting the right-hand side of Eq. (11) to zero, one would obtain Eq. (8), whose tunneling coefficient is given exactly by

$$\eta=\frac{\pi|1+\gamma|}{2\lambda^2}\ . \tag{12}$$

For the case with $1+\gamma>0$, Eq. (7) with $F=1$ can be directly integrated with exactly the same result. In this case, there is no absorption; so whatever is not transmitted is either reflected or converted to the Bernstein wave. Numerically, with $F\neq 1$, the surprising result is that the same result is obtained. In fact, it has been shown (Ng, 1994) that, with $F=-\zeta Z(\zeta)$, this is an analytic result, depending only on $Z(\zeta)$'s being an analytic function of ζ with zeros only in the lower half ζ plane [this is also sufficient to guarantee that the right-hand side of Eq. (11) is an *absorption* function, since zeros in the upper half plane imply a source function]. The unexpected conclusion is that even the eikonal approximation finds the transmission coefficient to be independent of absorption! The reason for this is that there are *two* sources for the imaginary part of $k(z)$, one being due to absorption and one being due to tunneling. That part due to absorption cancels out because of the analytic behavior of $Z(\zeta)$, while the tunneling contribution remains. The final proof that absorption has no effect on transmission is more subtle than this, however, since these arguments are based on an approximate theory. The complete solution of Eq. (11) shows that, in general, the absorption term on the right does influence transmission; but with our choice of absorption functions, the contribution vanishes. The contribution to reflection and conversion does not vanish, however, so that conservation of energy is still guaranteed.

The solution of Eq. (11) is accomplished by first converting it to an integral equation, treating the right-hand side as a source/sink function, and finding the Green's function for the left-hand side. The advantage of this method over direct numerical integration is that the scattering parameters may then be expressed in terms of explicit integrals, some of which can be evaluated analytically and some numerically. The integral associated with the reflection coefficient on Branch 2 may be converted into a power series which admits higher accuracy than numerical integration methods.

II. ELECTRON AND ION CYCLOTRON HARMONIC RESONANCES

In this section, the fundamental wave equations from the dispersion-relation method are listed, and then the corresponding wave equations from the direct solution of the Vlasov-Maxwell equations are given. This will include higher-order harmonics and even cases in which there are five propagating branches due to the coupling of the X-mode and O-mode with finite k_z. The corresponding scattering parameters are given analytically when absorption is neglected. The effects of absorption are treated in Sec. III.

A. Tunneling equations from dispersion relations

The dispersion relations that follow are obtained from the hot plasma dispersion relation after separating out the cold plasma terms (assumed constant) in each dielectric tensor element and then adding the lowest-order term in the finite Larmor orbit expansion at which the particular resonance term first arises. In some tensor elements, the lowest-order term is of higher order than in other tensor elements, in which case the term is discarded, so that the resonance terms are all of the same order. For ions, the treatment is consistently nonrelativistic, while for electrons, the treatment is consistently weakly relativistic. For cases when stronger relativistic effects are important, the mode conversion corrections are not important.

1. Ion cyclotron harmonics

The general form of the tunneling equation for nth order ion cyclotron harmonics neglecting absorption in the finite but small Larmor orbit limit (Ngan and Swanson, 1977) is equivalent to

$$f^{(2n)}-\gamma_n f^{(2n-2)}+(-1)^n\lambda_n^{2n-2}z(f''+f)=0\ , \tag{13}$$

where $f^{(n)}$ denotes the nth derivative of f and

$$\lambda_n^{2n-2}=\left(\frac{2}{n_A n}\right)^{2n-1}\frac{n![1/(n^2-1)+p^2]c}{\omega_{pi}L\beta_i^{n-1}}\ , \tag{14}$$

$$n_A^2=\frac{[1+(n-1)p^2][1-(n+1)p^2]}{[1+(n^2-1)p^2]}\ , \tag{15}$$

$$\gamma_n=-2\left[\frac{1+(n^2-1)p^2}{1+n+(n^2-1)p^2}\right]\ , \tag{16}$$

where $p=k_zV_A/\omega$ is the parallel index of refraction (relative to V_A); L is the magnetic-field gradient scale length, where the magnetic field is taken to vary as $B(x)=B_0(1+x/L)$ so that positive x and positive $z=\omega n_A x/V_A$ represent the high-magnetic-field side of the resonance; β_i is the ratio of the ion pressure to the magnetic pressure; V_A is the Alfvén speed; and Eq. (15) is the cold plasma dispersion relation for n_A (relative to V_A). The tunneling factor is given by

$$\eta=\frac{\pi(1+\gamma_n)}{2\lambda_n^{2n-2}}\ . \tag{17}$$

It is customary to shift the origin in z for $n=2$ so that Eq. (13) is of *standard form*, or equivalent to Eq. (8).

This generalized tunneling equation arises from the hot plasma dispersion relation where the dielectric tensor components are composed of the cold plasma terms plus the harmonic resonant terms to the lowest order in $\lambda_i=\frac{1}{2}k_\perp^2\rho_{Li}^2$ where ρ_{Li} is the ion Larmor radius. Then this algebraic dispersion relation (plotted for $n=2$ in Fig. 1, where Branch 1 corresponds to the fast Alfvén wave on the high-field side and Branch 2 to the fast Alfvén wave on the low-field side) is converted to the differential equation, as in the example in the previous section. For each of the resonances represented by Eq. (13), there is no reflection from the high-field side ($R_1=0$), but there is reflection from the low-field side; and the reflected amplitude is given by $R_2=-(1-e^{-2\eta})=T^2-1$. From either side, the power neither transmitted nor reflected is converted to an ion Bernstein mode (Branch 3).

When absorption is included (which occurs whenever k_z, the wave number parallel to the magnetic field, is nonzero), the resonant terms involve the plasma dispersion function, $Z(\zeta)$, through $F(z)=-\zeta Z(\zeta)$, and the pertinent differential equations are modified so that Eq. (13) becomes Eq. (11) for $n=2$ with $\zeta=(z_0-z)/\kappa$ and $z_0=-\gamma/\lambda^2$, while for $n>2$ one finds

$$\psi^{(2n)}+\gamma_n\psi^{(2n-2)}+(-1)^n\lambda_n^{2n-2}z(\psi''+\psi)=(-1)^n\lambda_n^{2n-2}\kappa[\zeta+1/Z(\zeta)](\psi''+\psi)\ , \tag{18}$$

where $\zeta=-z/\kappa$, and $\kappa=n_Ak_zL\sqrt{\beta_i}$ for all n. It is apparent that if $k_z\to 0$, $\kappa\to 0$, so that $Z(\zeta)\to -1/\zeta$ and the right-hand sides of Eqs. (11) and (18) vanish, with the result that there is no absorption for propagation exactly perpendicular to the magnetic field. For $n=2$, it has been proved analytically that the transmission coefficient is still given by $T=e^{-\eta}$ with η unchanged except through the dispersive term p, and that there is still zero reflection from the high-field side (Swanson and Shvets, 1993a, 1993b). The proof has been generalized to $n=3$ and appears to be true for all orders (Ng and Swanson, 1994a).

In addition to the ion cyclotron harmonics, the fourth-order tunneling equation also models the two-ion hybrid resonance with appropriate choices for λ^2 and γ (Swanson, 1985).

2. Electron cyclotron harmonics

The derivation of the tunneling equations for harmonics of the electron cyclotron frequency proceeds similarly as in the ion case, except that for the case of the X-mode with $k_z=0$, one uses the weakly relativistic plasma dispersion function $F_q(\zeta)$, using textbook notation (Swanson, 1989). In this case, the dielectric tensor elements may be written as

$$K_{xx}\simeq S-\frac{Xn^2\lambda_e^{n-1}}{2^n n!}\mu F_{n+3/2}(-\mu x/L)\ , \tag{19}$$

$$iK_{xy}\simeq D-\frac{Xn^2\lambda_e^{n-1}}{2^n n!}\mu F_{n+3/2}(-\mu x/L)\ , \tag{20}$$

where $X\equiv\omega_p^2/\omega^2$, $\mu\equiv mc^2/kT_e$, and S and D are the cold plasma terms,

$$S=1-n^2X/(n^2-1)\ ,$$

$$D=-nX/(n^2-1)\ .$$

From these, the X-mode dispersion relation may be written in terms of $F(z)=\zeta F_q(\zeta-q)$, which for $n=2$ leads to Eq. (11) with $q=7/2$ and $\zeta=(z+\gamma/\lambda^2)/\kappa$, while for $n>2$ one finds

$$\psi^{(2n)}-\gamma\psi^{(2n-2)}+(-1)^n\lambda_n^{2n-2}z(\psi''+\psi)=(-1)^n\lambda_n^{2n-2}z[1-1/F](\psi''+\psi)\ , \tag{21}$$

where the variable change $z_0-z=n_\perp\omega x/c$ was used with

$$\lambda_n^{2n-2}=\frac{S2^n n!\mu^{n-1}}{Xn^{2n}n_\perp^{2n-1}}\frac{c}{\omega L}\ ,$$

$$\gamma=-2S/(S+D)\ ,$$

$$n_\perp^2=(S+D)(S-D)/S\ ,$$

$$z_0=\begin{cases}\kappa q-\gamma/\lambda_2^2, & n=2,\\ \kappa q, & n>2,\end{cases}$$

$$\zeta=z/\kappa,\quad n>2\ ,$$

$$\kappa=\omega Ln_\perp/c\mu\ ,$$

$$q=n+\frac{3}{2}\ ,$$

where $n_\perp$ is the index of refraction for the cold X-mode. As in the ion case, an additional shift of the origin for $n=2$ is required to make the tunneling equation of standard form. The tunneling factor is then given by

$$\eta=\frac{\pi|1+\gamma|}{2\lambda_n^{2n-2}}\ . \tag{22}$$

It may be noted that $1+\gamma<0$, and the dispersion relation is of the form of Fig. 2 for $n=2$.

As in the ion case, the amplitude transmission coefficient is given by $T=e^{-\eta}$, is the same from both sides, is independent of the localized absorption terms on the right of Eqs. (11) and (21), and there is no reflection on Branch 1, the high-magnetic-field side (Swanson and Shvets, 1993a, 1993b). On Branch 2, the low-field side, the reflection is nonzero and depends strongly on absorption, as do the conversion coefficients.

B. Tunneling equations from the Vlasov equations

As noted in Sec. I.B.4, there is a fundamental inconsistency in deriving a dispersion relation with a uniform plasma

assumed, then allowing for slow variations in the plasma parameters, followed by a trivial inversion back to a differential equation. Here in Sec. II.B the resolution of the ambiguities noted there will be reviewed.

Two different approaches to solving this problem were successful at about the same time. One method was to start directly with the Vlasov-Maxwell set of equations and never take a Fourier transform in the direction of the slow variation (Swanson, 1981); the other was based on a variational approach (Colestock and Kashuba, 1983). As an example of the changes from these procedures, the resonant term in the second ion harmonic case is modified so that

$$\lambda_i \frac{\omega_{pi}^2 LF}{2c^2 x} \rightarrow -\frac{d}{dx}\frac{\omega_{pi}^2 \rho_{Li}^2 LF}{4c^2 x}\frac{d}{dx}, \tag{23}$$

where $F=-\zeta Z(\zeta)$ (note that $F\rightarrow 1$ far from the resonant layer). It is clear from this change that the resonant term lies between the two derivatives rather than lying in front of them; so the differential equation will be more involved. The appropriate wave equation for the second harmonic may be written in either of two ways, such that after changing variables to $z-z_0=\omega n_A x/V_A$, where $n_A=\omega/k_\perp V_A$ with $k_\perp$ being the cold plasma wave-vector component (a constant here), one finds

$$n_A^4 K(ZE_+')''' + 2n_A^2 K\left(\frac{1}{3}-p^2\right)(ZE_+')' + n_A^2\left(\frac{1}{3}+p^2\right)E_+'' + \left(\frac{1}{3}-p^2\right)(1+p^2)E_+ = 0 , \tag{24}$$

where $K=\beta_i^{1/2}/4p$, which is an equation for the wave field $E_+=E_x+iE_y$, while

$$n_A^4 K(ZE_y''')' + 2n_A^2 K\left(\frac{1}{3}-p^2\right)(ZE_y')' + n_A^2\left(\frac{1}{3}+p^2\right)E_y'' + \left(\frac{1}{3}-p^2\right)(1+p^2)E_y = 0 \tag{25}$$

is the appropriate equation for E_y. The only difference is in the order of the derivatives in the first term. If Z [which is the abbreviated $Z(\zeta)$] were taken to commute with the derivative operator (as is assumed in the dispersion-relation formulation), then Eq. (24) would be equivalent to Eq. (25), and both would be equivalent to Eq. (11). From this it is apparent that the wave amplitude ψ in Eq. (11) could be taken to represent either E_+ or E_y.

It might appear that, since the coefficients of the odd derivative terms that result from expanding either of the equations above fall off asymptotically as $1/x$, one could obtain the same asymptotic form as Eq. (11) where the asymptotic terms are on the left and the localized terms are on the right, except that the localized terms on the right would be more complicated. Such a simple separation does not work, however, since E''' has a growing slow-wave component such that the third derivative term is not localized. The separation into asymptotic and local can be accomplished by letting $E_+=u\psi$ in Eq. (24) and then choosing $u(z)$ to eliminate the third derivative term. This results in

$$\psi^{iv}+\lambda^2(z\psi''+\alpha\psi'+z\psi)+\gamma\psi=g_0\psi+g_1\psi'+g_2\psi'' , \tag{26}$$

with $\alpha=3/2$, $u=[Z(\zeta)]^{-3/4}$, and

$$g_0=-\frac{2}{3}\lambda^2\kappa^2 v+\lambda^2\kappa(v'-v^2)/Z-\gamma\left(v'+\frac{1}{3}v^2\right)+\frac{1}{27}v^4+\frac{2}{3}v^2v'+\frac{4}{3}vv''+v'^2+v''' ,$$

$$g_1=\frac{3}{2}\lambda^2\left(1+\frac{4}{3}\kappa v/Z\right)-\frac{2}{3}\gamma v+\frac{8}{27}v^3+\frac{8}{3}(v''+vv') ,$$

$$g_2=\frac{2}{3}(v^2-\lambda^2\kappa^2 v)+2v' ,$$

where $v(z)=-u'/u=(3/4Z)(dZ/dz)$ and the prime denotes a derivative with respect to z. The corresponding equation for E_y is similar, yielding Eq. (26) again, except that $\alpha=1/2$, $u=[Z(\zeta)]^{-1/4}$, and

$$g_0=-2\lambda^2\kappa^2 v+\lambda^2\kappa(v'-v^2)/Z-\gamma(v'+3v^2)+3v^4-6v^2v'-3v'^2+v''' ,$$

$$g_1=\frac{1}{2}\lambda^2(1+4\kappa v/Z)+2\gamma v+4v''-8v^3 ,$$

$$g_2=-2\lambda^2\kappa^2 v+6(v'+v^2) ,$$

where $v(z)=-u'/u=(1/4Z)(dZ/dz)$. In both cases, the terms on the right fall off asymptotically at least as fast as $1/z$; so they represent localized effects, and the imaginary parts represent absorption.

Generally speaking, the scattering coefficients are similar for this more general case, where the transmission coefficients are still equal and independent of the absorption, and there is still no reflection from the high-magnetic-field side. Numerical studies show that R_2 differs little from the simpler case, but is not identical, and the conversion coefficients differ more but not greatly; but broad parameter range comparisons are not available.

Similar expressions follow for the electron cyclotron second harmonic, where the equations corresponding to Eqs. (24) and (25) with the variable change $z-z_0=-\omega n_X x/c$ are

$$4Xn_X^2(F_{7/2}E_-')''' + 8XL(F_{7/2}E_-')' + S(E_-''+E_-)=0 \tag{27}$$

and

$$4Xn_X^2(F_{7/2}E_y''')' + 8XL(F_{7/2}E_y')' + S(E_y''+E_y)=0 , \tag{28}$$

where $F_{7/2}$ is an abbreviation for $F_{7/2}(\zeta-7/2)$ with $\zeta=(z+\gamma/\lambda^2)/\kappa$, $z_0=7\kappa/2-\gamma/\lambda^2$, and

$$R=1-2X ,$$

$$L=1-2X/3 ,$$

$$S=1-4X/3 ,$$

$$n_X^2=RL/S .$$

Equation (28) can then be reduced to the form of Eq. (26) again, except that $\alpha=1/2$, $\gamma=-2S/R$, $u=[F_{7/2}(\zeta-7/2)]^{-1/4}$, and

$$g_0=-\frac{7}{2}\lambda^2\kappa^2(F_{9/2}/F_{7/2}-1)+\lambda^2\kappa(v'-v^2)/F_{7/2}$$
$$-\gamma(v'+3v^2)+3v^4-6v^2v'-3v'^2+v''' ,$$

$$g_1=\frac{1}{2}\lambda^2(1+4\kappa v/F_{7/2})+2\gamma v+4v''-8v^3 ,$$

$$g_2=-\frac{7}{2}\lambda^2\kappa^2(F_{9/2}/F_{7/2}-1)+6(v'+v^2) ,$$

where $v(z)=-u'/u=(1/4F_{7/2})(dF_{7/2}/dz)$.

The asymptotic solutions of Eq. (26) with the right-hand side set to zero are known (Stix and Swanson, 1983) and have the peculiar property that $|R_2|=|1-\exp(-2\eta+i\pi\alpha)|>1$ for $1-e^{2\eta}/3<2\alpha<3+e^{-2\eta}/3$. For $\alpha\neq 0$ or $\alpha\neq 2$, then, one must include the terms on the right for any physically meaningful result. When the full equations are solved, the results are close, but not equivalent, to those obtained from the tunneling equations from dispersion relations.

C. Order-reduction methods

There have been a variety of methods used to obtain numerical solutions to obtain the scattering coefficients and the wave fields with absorption included, among them being direct numerical integration (Colestock and Kashuba, 1983; Imre and Weitzner, 1985), finite element methods (Hellsten *et al.*, 1985), finite different methods (Jaeger *et al.*, 1988), and order-reduction methods. Order-reduction techniques reduce the fourth-order equation of Eq. (8) to a related second-order equation. Accomplished first without absorption (Cairns and Lashmore-Davies, 1983), later efforts with absorption included have been investigated using phase-space methods (Ye and Kaufman, 1988) and other techniques (Fuchs and Bers, 1988; Kay *et al.*, 1988; Lashmore-Davies *et al.*, 1988; Chow *et al.*, 1990). In general, all order-reduction methods make some approximations that render their results less accurate than those from the higher-order methods, but faster, and in some cases analytic results have been obtained. Since this review is primarily concerned with the connection between absorption and emission, the focus will be narrowed to the Green's-function method (Swanson, 1978, 1980), where the response to an emission source is treated by finding the Green's function for the appropriate tunneling equation with absorption, and although more complicated with higher-order equations, this technique has yielded a number of exact analytic results.

D. Five-branch tunneling equations

Most mode conversion problems are three-branch problems, where two of the branches are identified as the (relatively) fast wave on either side of the resonance layer, and the slow wave. The two cases are shown in Fig. 1 and Fig. 2, where, in each case, Branch 1 is labeled as the branch on which an incoming wave sees a resonance first, while Branch 2 sees a cutoff first. Branch 3 is the Bernstein mode in either case. At the two-ion hybrid resonance, the dispersion relation could resemble either Fig. 1 or Fig. 2, depending on the charge-to-mass ratio of the majority and minority species; but this topic is beyond the scope of this review.

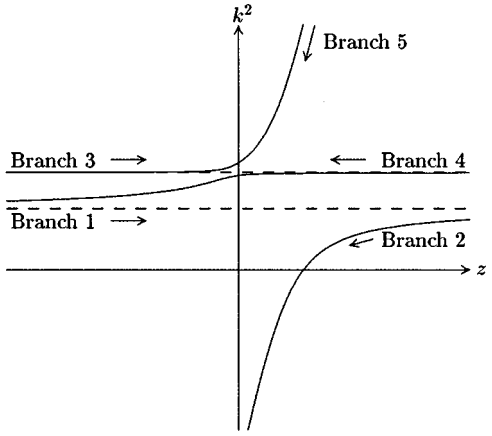


FIG. 4. Form of dispersion relation for the five-branch case with $n=2$, $X=0.3$, $n_z^2=0.1$, $\lambda^2=2$; so $k_0^2=n_O^2/n_X^2=1.59$.

When propagation is not exactly normal to the magnetic field, it is possible for an X-mode (or quasi-X-mode, since a pure X-mode has $n_z=0$) and an O-mode (or quasi-O-mode) to coexist. When the full dispersion relation is examined, it is apparent that they are coupled through the intermediate Bernstein mode. Examples of these kinds of dispersion relations are shown in Figs. 4 and 5, where Fig. 4 is characteristic of an X-mode (Branches 1 and 2) and an O-mode (Branches 3 and 4) near the second electron cyclotron harmonic for $X<1/2$, while Fig. 5 has $3/4<X<1$, the case in which the wave propagates above the upper hybrid resonance, and is not accessible from the low-density region. These problems are five-branch problems, since there are five propagating branches, including the two X-mode branches on either side, the two O-mode branches on either side, and again the Bernstein mode. When X increases beyond 1, the O-mode ceases to propagate, and one again has a three-branch problem. Due to relativistic effects, these boundaries depend somewhat on the temperature, but these

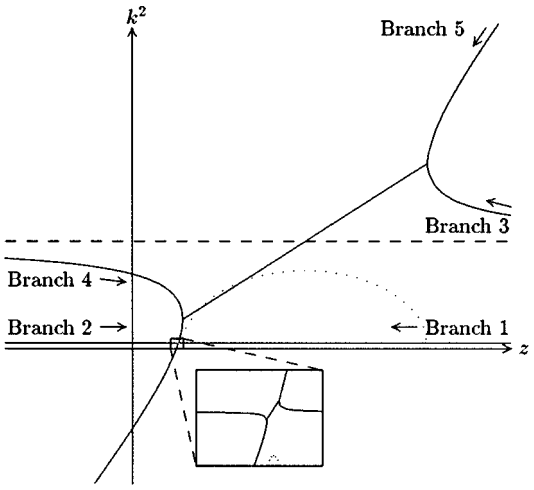


FIG. 5. Form of dispersion relation for the five-branch case with $n=2$, $X=0.91$, $n_z^2=0.12$, $\lambda^2=1.5$; so $k_0^2=0.051$. Inset is $\times 10$ magnification of coupling region.

subtleties shall be ignored, including only weakly relativistic effects.

Although higher harmonics may lead to higher-order differential equations, all mode conversion problems may be classified as either three-branch or five-branch problems, since the fast-wave branches are cold plasma waves, and there can be only two kinds of fast waves from the cold plasma dispersion relation. The addition of finite temperature adds the Bernstein wave, but there is no known coupling to any other warm plasma wave which leads to five or more propagating branches. Whenever λ_i or λ_e are not small, one may keep higher-order terms, leading to higher-order differential equations, but there are still either three or five propagating branches. Since the tunneling equations used to describe these cases are of order $2n$ or higher for a three-branch problem at the nth harmonic, and of order $2n+2$ or higher for a corresponding five-branch problem, all other solutions are nonpropagating.

1. *X*-mode–*O*-mode coupling for $n=2$

At the second electron harmonic with $n_z \neq 0$, the dominant terms in the hot plasma dispersion relation are, to lowest order in $\lambda = \frac{1}{2}k_\perp^2 \rho_{Le}^2$,

$$K_1 = S + \frac{2XLF}{\mu(x-x_0)} \frac{k_\perp^2 c^2}{\omega^2} , \tag{29}$$

$$K_2 = -iD - \frac{2iXLF}{\mu(x-x_0)} \frac{k_\perp^2 c^2}{\omega^2} , \tag{30}$$

$$K_3 = P + \mathcal{O}(\lambda^2) , \tag{31}$$

where $x_0 = qL/\mu$, $F = \zeta \mathcal{F}_q(\zeta - q, a)$, $\zeta - q = -\mu x/L$ with $q = 7/2$, and $a = \frac{1}{2}\mu n_z^2$; the cold plasma terms are $P = 1 - X$ and $D = -2X/3$. The neglected term in K_3 relates to another weak mode conversion of the *O*-mode to a different slow wave which is not a Bernstein wave and occurs at a different location and has no coupling to the *X*-mode. This process can be treated as a pair of separate three-branch problems without absorption, but the effects of absorption on the mutual coupling are not then properly described. Using these dielectric tensor components in the hot plasma dispersion relation and letting $k^2 \rightarrow -d^2/dx^2$ lead to the five-branch tunneling equation

$$\psi^{vi} + \lambda^2 z[\psi^{iv} + (1+k_0^2)\psi'' + k_0^2\psi] + \gamma_2\psi'' + \gamma_0\psi = h(z)[\psi^{iv} + (1+k_0^2)\psi'' + k_0^2\psi] , \tag{32}$$

where the variable change $z_0 - z = \omega n_X (x - x_0)/c$ has been used, and the constants are given by (Hu and Swanson, 1993a)

$$\lambda^2 = \mu S/2XLn_X^3, \tag{33}$$

$$k_0^2 = \frac{P(R-n_z^2)(L-n_z^2)}{Sn_X^4} = \frac{n_O^2}{n_X^2} , \tag{34}$$

$$\gamma_0 = \frac{(n_z^2 - P - 2L)P(R - n_z 2)(L - n_z^2)}{Sn_X^6} , \tag{35}$$

$$\gamma_2 = \{2PS(L-n_z^2) + (n_z^2 - P - 2L) \times [RL + PS - n_z^2(S+P)]\}/Sn_X^4, \tag{36}$$

$$h(z) = \lambda^2 (z - z_0)(1 - 1/F) , \tag{37}$$

$$z_0 = (P + 2L - n_z^2)/n_X^2 \lambda^2 , \tag{38}$$

$$\zeta = (z - z_0)/\kappa , \tag{39}$$

$$\kappa = n_X l/\mu , \tag{40}$$

with $l = \omega L/c$, the normalized scale length, and n_X^2 and n_O^2 are the roots of the cold plasma dispersion relation,

$$Sn_\perp^4 - [RL + PS - n_z^2(S+P)]n_\perp^2 + P(R - n_z^2)(L - n_z^2) = 0 , \tag{41}$$

which approach RL/S and P, respectively, as $n_z \rightarrow 0$.

The exact tunneling parameters for this case are

$$\eta_X = \pi |(1 - \gamma_0 + \gamma_2)/2\lambda^2(1 - k_0^2)| , \tag{42}$$

$$\eta_O = \pi |(\gamma_0 - k_0^2\gamma_2 + k_0^6)/2\lambda^2 k_0 (1 - k_0^2)| . \tag{43}$$

2. *X*-mode–*O*-mode coupling for $n=3$

Near the third electron harmonic, the resonant terms are of order λ_e^2; so, keeping resonant terms of that order and cold plasma terms, the dielectric tensor terms are

$$K_1 = S + \frac{243XLF}{16\mu^2(x-x_0)} \frac{k_\perp^4 c^4}{\omega^4} , \tag{44}$$

$$K_2 = -iD - \frac{243iXLF}{16\mu^2(x-x_0)} \frac{k_\perp^4 c^4}{\omega^4} , \tag{45}$$

$$K_3 = P + \mathcal{O}(\lambda^3) , \tag{46}$$

where, in this case, $q = 9/2$ and the cold terms are

$$R = 1 - 3X/2 = S + D ,$$

$$L = 1 - 3X/4 = S - D ,$$

$$S = 1 - 9X/8 ,$$

$$D = -3X/8 .$$

Changing variables as before, one can convert the hot plasma dispersion relation into the differential equation

$$\psi^{viii} + \psi_6\psi^{vi} - \lambda^4 z[\psi^{iv} + (1+k_0^2)\psi'' + k_0^2\psi] + \gamma_2\psi'' + \gamma_0\psi = h(z)[\psi^{iv} + (1+k_0^2)\psi'' + k_0^2\psi] , \tag{47}$$

where

$$\lambda^4 = \frac{16\mu^2 S}{243XlN_X^5} , \tag{48}$$

$$k_0^2 = \frac{P(R-n_z^2)(L-n_z^2)}{Sn_x^4} = \frac{n_O^2}{n_X^2} , \tag{49}$$

$$\gamma_0 = -\frac{2P^2(R-n_z^2)(L-n_z^2)^2}{Sn_x^8} , \tag{50}$$

$$\gamma_2 = -\frac{2P(L-n_z^2)[RL + PS - n_z^2(S+P)]}{Sn_X^6} , \tag{51}$$

$$h(z)=-\lambda^4(z-z_0)(1-1/F)\ , \tag{52}$$

$$z_0=-2P(L-n_z^2)/n_X^4\lambda^2\ , \tag{53}$$

with $\zeta=(z-z_0)/\kappa$ and $\kappa=n_X l/\mu$ as before.

The exact tunneling parameters for this case are

$$\eta_X=\pi\left|\frac{1-\gamma_0+\gamma_2-\gamma_6}{2\lambda^4(1-k_0^2)}\right|\ , \tag{54}$$

$$\eta_O=\pi\left|\frac{\gamma_0-k_0^2\gamma_2+k_0^6(k_0^2-\gamma_6)}{2\lambda^4 k_0(1-k_0^2)}\right|\ . \tag{55}$$

E. Solutions and scattering parameters

For cases without absorption, exact solutions can be found in terms of quadratures, from which a number of analytic results can be obtained. When absorption is included, some analytic results are still possible; but some of the scattering coefficients require numerical analysis, and a variety of numerical schemes have been used by various authors. In Secs. II.E.1 and II.E.2, both the exact solutions and their asymptotic forms will be given. The effects of absorption on these solutions will be included, along with some empirical results for pertinent scattering parameters which accurately summarize the numerical work and prepare the way for the emission analysis.

1. Exact analytical solutions and their classification

As an example of the exact solutions that may be found for each case without absorption, the standard tunneling equation is examined first. The differential equation is Eq. (8), and a solution in the form of a Laplace integral,

$$f(z)=\int_\Gamma e^{pz}\tilde{f}(p)dp\ , \tag{56}$$

is assumed, which, when inserted into the differential equation, results in (after some integrations by parts)

$$\int_\Gamma\{p^4\tilde{f}-\lambda^2[(p^2+1)\tilde{f}'+2p\tilde{f}]+\gamma\tilde{f}\}e^{pz}dp +\lambda^2(p^2+1)e^{pz}\tilde{f}(p)|_\Gamma=0\ , \tag{57}$$

where four independent contours, Γ, define the four solutions. This equation is satisfied if $\tilde{f}(p)$ satisfies

$$\lambda^2(p^2+1)\tilde{f}'=(p^4-2\lambda^2p+\gamma)\tilde{f}\ ,$$

and if the end points of the contours are chosen so that the final expression of Eq. (57) vanishes at the end points. This is a first-order differential equation with solutions

$$\tilde{f}(p)=(p^2+1)^{-1}\exp\left(\frac{p^3}{3\lambda^2}-\frac{p}{\lambda^2}+\frac{1+\gamma}{\lambda^2}\tan^{-1}p\right)\ .$$

The solution of the original equation is then given by

$$f(z)=\int_\Gamma\exp\left[\frac{p^3}{3\lambda^2}+\left(z-\frac{1}{\lambda^2}\right)p+\frac{1+\gamma}{\lambda^2}\tan^{-1}p\right]\frac{dp}{p^2+1}\ . \tag{58}$$

This expression is equivalent to Eq. (9) with the variable change $p=\tan u$. There is a branch cut in the p plane between $p=\pm i$, but no branch cut in the u plane. Since the emission due to a source will be treated later via the use of a Green's function, which is constructed from adjoint solutions, the solutions of the adjoint equation,

$$F^{iv}+\lambda^2(zF''+2F'+zF)+\gamma F=0\ , \tag{59}$$

are similar, given by

$$F(z)=\int_\Gamma\exp\left[\frac{p^3}{3\lambda^2}+\left(z-\frac{1}{\lambda^2}\right)p+\frac{1+\gamma}{\lambda^2}\tan^{-1}p\right]dp\ . \tag{60}$$

General techniques for solving all finite order tunneling equations whose coefficients are composed of constant and linear terms extend these results to the higher-order harmonics and to five-branch problems (Gambier and Schmitt, 1983; Gambier and Swanson, 1985).

TABLE I. Asymptotic solutions for f_n when $1+\gamma>0$.

$-\infty\leftarrow z$	f_n	$z\rightarrow\infty$
f_-	f_1	$e^{\eta}f_- -\varepsilon e^{\eta}s_-$
$-f_+ +\varepsilon(f_- +e^{\eta}\sigma_-)$	f_2	$-e^{-\eta}f_+ +\varepsilon e^{-\eta}s_-$
$e^{\eta}\sigma_- +f_-$	f_3	$e^{\eta}f_+ +e^{-\eta}s_- -e^{\eta}s_+$
$-e^{-\eta}\sigma_+ +f_-$	f_4	$e^{-\eta}s_-$

2. Asymptotics and scattering parameters

When the method of steepest descents is used on the exact solutions, each saddle point corresponds to one of the characteristic solutions. For the slow or electrostatic waves, the saddle-point method gives the asymptotic forms for large $|z|$ as

$$s_\pm(z)=\frac{\sqrt{\pi}}{\lambda^{3/2}z^{5/4}}\exp\left[\pm i\left(\frac{2}{3}\lambda z^{3/2}+\frac{\pi}{4}\right)\right],\quad z\rightarrow\infty\ ,$$

$$\sigma_+(z)=\frac{\sqrt{\pi}}{\lambda^{3/2}|z|^{5/4}}\exp\left(\frac{2}{3}\lambda|z|^{3/2}\right),\quad z\rightarrow-\infty\ ,$$

$$\sigma_-(z)=\frac{i\sqrt{\pi}}{\lambda^{3/2}|z|^{5/4}}\exp\left(-\frac{2}{3}\lambda|z|^{3/2}\right),\quad z\rightarrow-\infty\ .$$

For the fast-wave solutions, the steepest-descent method is less satisfactory, but with an expansion about the points $p=\pm i$, the Hankel integral gives

$$f_+=\frac{\pi i e^{-\eta/2}}{\Gamma(1+i\eta/\pi)}\exp\left[iz+\frac{i\eta}{\pi}\ln\left(2z-\frac{4}{\lambda^2}+\frac{\eta}{\pi}+i\right)-\frac{4i}{3\lambda^2}\right]\ , \tag{61}$$

and $f_-=f_+^*$. The adjoint solutions may be obtained from the relation $F=f''+f$, which is exact for all solutions. These, along with the slow-wave solutions, represent the independent wave solutions, but none individually satisfy the conditions for incoming and outgoing waves along with causality. By examining the appropriate contours, Γ, which must be topologically equivalent for both $z\rightarrow\infty$ and $z\rightarrow-\infty$, it is possible to establish four independent solutions for the case $1+\gamma>0$, as shown in Table I where $\varepsilon\equiv1-e^{-2\eta}$. Solution f_1 corresponds to an incident fast wave (f_- on the right) on Branch 1; f_2 corresponds to an incident fast wave (f_+ on the

TABLE II. Asymptotic solutions for f_n when $1+\gamma<0$.

$-\infty\leftarrow z$	f_n	$z\rightarrow\infty$
$e^{\eta}f_+$	f_1	$f_+ + \varepsilon e^{\eta}s_+$
$e^{-\eta}f_- - \varepsilon e^{-\eta}\sigma_-$	f_2	$f_- - \varepsilon f_+ + \varepsilon e^{-\eta}s_+$
$e^{-\eta}\sigma_- + e^{\eta}f_-$	f_3	$f_+ + e^{\eta}s_- - e^{-\eta}s_+$
$-e^{\eta}\sigma_+ + e^{\eta}f_+$	f_4	$e^{\eta}s_+$

left) on Branch 2; f_3 corresponds to an incident slow wave (s_+) on Branch 3; and f_4 corresponds to an exponentially growing solution (σ_+) for $z<0$. The expressions on the right indicate the asymptotic forms as $z\rightarrow\infty$, and the expressions on the left indicate the asymptotic forms as $z\rightarrow-\infty$.

For the case $1+\gamma<0$, the corresponding solutions are given in Table II, where, again, f_1 corresponds to an incident fast wave (f_+ on the left) on Branch 1; f_2 corresponds to an incident fast wave (f_- on the right) on Branch 2; f_3 corresponds to an incident slow wave (s_-) on Branch 3; and f_4 corresponds to an exponentially growing solution (σ_+) as $z\rightarrow-\infty$.

In each case, there is no reflection for f_1, so $R_1=0$; the amplitude reflection coefficient for a fast wave incident on Branch 2 is $R_2=-\varepsilon$; and the transmission coefficient for both cases is $T_1=T_2=e^{-\eta}$. Whatever power is neither transmitted nor reflected is converted to Branch 3, so $|C_{13}|^2=1-T_1^2=\varepsilon$ and $|C_{23}|^2=T_2^2(1-T_2^2)=\varepsilon e^{-2\eta}$. For an incident slow wave, there is no transmission, since the slow wave propagates only on one side, and the reflection coefficient is $R_3=-e^{-2\eta}$; and, from reciprocity, the power fraction converted from Branch 1 or 2 to Branch 3 is the same in either direction, so $|C_{31}|^2=|C_{13}|^2$ and $|C_{32}|^2=|C_{23}|^2$. These expressions comprise the scattering coefficients and are exact without absorption.

3. Conservation laws

Without absorption, every tunneling equation has an exact conservation law, which, for the standard fourth-order tunneling equation, Eq. (8), takes the form

$$P(f)=f'''f^{*\prime\prime}+f'''f^{*}-f''f^{*\prime}-\gamma f'f^{*}-\text{c.c.}\,, \tag{62}$$

where f is any of the solutions, and $dP/dz=0$ follows from the differential equation. From this, one finds for the asymptotic solutions that

$$\frac{P(s_\pm)}{2\pi i\lambda^2}=\pm 1, \quad \frac{P(f_\pm)}{2\pi i\lambda^2}=\mp\varepsilon\,, \tag{63}$$

so that, for example, the connection formula for f_2 from Table I leads to the relation

$$P(f_+)+R_2^2P(f_-)=T_2^2P(f_+)+C_{23}^2P(s_-)\,, \tag{64}$$

with $R_2=-\varepsilon$, $T_2=e^{-\eta}$, and $C_{23}=-e^{-\eta}\varepsilon^{1/2}$, from which it follows that everything not transmitted or reflected is converted.

For general harmonics, similar conservation laws exist, so that for Eq. (18) without absorption, the corresponding conserved quantity is (Shvets, 1992)

$$P_n=F^*f^{(2n-1)}+F'f^{*(2n-2)}+\sum_{l=0}^{n-2}(-)^l f^{(2n-l-3)}H_n^{*(l)} +(-)^nR_n-\text{c.c.}\,, \tag{65}$$

where

$$H_n=F''-\gamma_nF=f^{iv}+(1-\gamma_n)f''-\gamma_nf\,,$$
$$R_n=f^{(n)}F^{*(n-1)}+f^{*(n-1)}[f^{(n-2)}-\gamma_nf^{(n)}]\,,$$

and $F=f''+f$ is the adjoint solution.

Similar conservation laws may be found for the five-branch problems; for example, for Eq. (32), the conserved quantity is

$$P=F^*f^{v}+f^{*iv}(F'-f^{v})+[(1+k_0^2)\gamma_0-k_0^2\gamma_2]f^{*\prime}f +f'''(\gamma_2f''+\gamma_0f)-k_0^2f'''f''-\gamma_0f^{*\prime\prime}f'-\text{c.c.}\,, \tag{66}$$

where $F=f^{iv}+(1+k_0^2)f''+k_0^2f$ is the adjoint solution.

The conservation laws above illustrate the fact that wave energy is conserved without absorption. The effects of absorption on these conservation laws are discussed in Sec. III.D; their relation to emission, in Sec. IV.B.2.

III. EFFECTS OF LOCALIZED ABSORPTION

There are several ways to include absorption numerically, from a variety of numerical schemes for different integration of the governing differential equation to a variety of approximation schemes, but the method described here will be useful in both absorption and emission, an advantage not easily shared with the other methods. This method treats the localized absorption term as if it were a localized source, and then, through the use of the Green's function for the left-hand side, the differential equation is converted into an integral equation whose kernel is known analytically.

A. Physical origins of localized absorption and its impact on scattering parameters

While the mathematical descriptions of the absorption functions have been presented above, it should be noted that the basic physics of the absorption is essentially the same for both electrons and ions and independent of harmonic number. In every case, the absorption is due to an extension of Landau damping to the cyclotron harmonics, wherein an observer rotating at the wave frequency would observe the damping to be Landau damping. This means that on some time scale which is long compared to the wave period, some collisions are required, since otherwise there is no wave energy lost to the plasma, only phase mixing. The difference in the damping between the nonrelativistic ions and the weakly relativistic electrons is that, for ions, the width of the resonance region is due to the Doppler shifts due to the finite value of k_z, while, for electrons, the width is due to the actual shifts of the resonance due to the relativistic mass increase. While there is a corresponding Doppler broadening for electrons with finite k_z, described by the weakly relativistic plasma dispersion function $\mathcal{F}_q(\zeta,a)$ with $a=\mu n_z^2/2$ instead of the $F_q(\zeta)$ with $a=0$, the relativistic spread usually dominates the Doppler spread and the corresponding absorption.

The effects of this absorption are quite different on different scattering parameters. As discussed in the following sections, absorption has no effect whatsoever on certain scattering parameters, extremely weak effects on one parameter in five-branch problems, and relatively strong effects on other parameters. In particular, all of the transmission coefficients in both the three- and five-branch problems are unaffected by absorption, as are the reflection coefficients from the high-field side. In five-branch problems, one conversion coefficient from one side between the X-mode and the O-mode is zero, independent of absorption, while from the other side, the coupling is nonzero but typically weak, and nearly independent of absorption. The low-field-side reflection coefficients and all conversion coefficients between the slow and fast waves are generally strongly affected by absorption, eventually vanishing with sufficiently strong absorption. These will be described more in detail in the subsequent sections.

B. The integral equation method

Because the tunneling equations, once the localized absorption terms have been separated out and placed on the right-hand side, have only constant plus linear coefficients and admit analytic solutions, it is also straightforward to find the Green's function for the left-hand side analytically and convert the differential equation into an integral equation. As an example, writing Eq. (11) as

$$\psi^{iv}+\lambda^2 z\psi''+(\lambda^2 z+\gamma)\psi=g(z,\psi)\ , \tag{67}$$

where $g(z,\psi)\to 0$ at least as fast as $|z|^{-1}$ as $|z|\to\infty$, allows the solution to be written in terms of the Green's function so that

$$\begin{aligned}\psi&=\int_{-\infty}^{\infty}G(z,y)g[y,\psi(y)]dy\\&=\int_{-\infty}^{\infty}G(z,y)[\psi^{iv}+\lambda^2 y\psi''+(\lambda^2 y+\gamma)\psi]dy\end{aligned} \tag{68}$$

$$=\int_{-\infty}^{\infty}[G^{iv}+\lambda^2(yG''+2G'+yG)+\gamma G]\psi dy$$

$$+\text{boundary conditions}+\text{jump conditions}\ , \tag{69}$$

where the first step was by substitution and the remainder was by integration by parts after breaking $G(z,y)$ into two separate elements, one for $y>z$ and another for $y<z$. The radiating boundary condition at infinity is

$$\{\psi'''G-\psi''G'+\psi'(G''+\lambda^2 yG)-\psi[G'''+\lambda^2 yG'+\lambda^2 G]\}_{-\infty}^{\infty}=0\ . \tag{70}$$

The jump conditions at the discontinuity, where $y=z$, are

$$G(z,z_+)-G(z,z_-)=0\ , \tag{71}$$

$$G'(z,z_+)-G'(z,z_-)=0\ , \tag{72}$$

$$G''(z,z_+)-G''(z,z_-)=0\ , \tag{73}$$

$$G'''(z,z_+)-G'''(z,z_-)=1\ . \tag{74}$$

If the Green's function is then constructed from a linear combination of the adjoint solutions, defined by Eq. (59), which sets the integrand of Eq. (69) to zero, and that satisfy the jump conditions, then $\psi(z)$ is a solution of the original problem (Swanson, 1978).

The integral equation which corresponds to Eq. (67) with $1+\gamma>0$ may be written as

$$\psi_k(z)=f_k(z)+f_2(z)I_{1k}^{-}+\varepsilon f_4(z)I_{0k}^{-}+f_1(z)I_{2k}^{+}+\varepsilon f_0(z)I_{4k}^{+}, \tag{75}$$

where, with $g(z,\psi)=h(z)\Psi(z)$, $\Psi=\psi''+\psi$,

$$I_{jk}^{-}=\frac{1}{2\pi i\lambda^2\varepsilon}\int_{-\infty}^{z}F_j(y)h(y)\Psi_k(y)dy\ , \tag{76}$$

$$I_{jk}^{+}=\frac{1}{2\pi i\lambda^2\varepsilon}\int_{z}^{\infty}F_j(y)h(y)\Psi_k(y)dy\ , \tag{77}$$

where $h(z)=\lambda^2\kappa[\zeta+1/Z(\zeta)]$ for ions and $f_0=f_3-f_1$. This is valid for $k=1,2,3$, the three physically meaningful solutions; but due to a divergent integral, this must be modified for $k=4$.

The corresponding result for weakly relativistic electrons where $1+\gamma<0$ (see Table II) is

$$\psi_k(z)=f_k(z)+f_1(z)I_{2k}^{-}+\varepsilon f_4(z)I_{0k}^{-}+f_2(z)I_{1k}^{+}+\varepsilon f_0(z)I_{4k}^{+}, \tag{78}$$

where the sink term for electrons is $h(z)=\lambda^2\kappa[\zeta+1/F_{7/2}(\zeta-7/2)]$, and $f_0=e^{-2\eta}f_3-f_2$. In each case, the solution f_0 (which is not an independent solution) is exponentially decaying as σ_- as $z\to-\infty$, so that the product of an exponentially growing (f_4 or F_4) and an exponentially decaying term is bounded. Similar growing and decaying solutions occur as products in increasing numbers in all of the higher-order integral equations for the higher harmonics, but the three fundamental propagating solutions remain substantially the same.

1. Numerical solution of the integral equation

Once cast into the form of an integral equation, the method of substitution generally converges over a relatively broad range of absorption strengths, characterized by the parameter κ. The first step is to obtain accurate values of $f_k(z)$ and $F_k(z)$ over a wide enough range of z that the sink function is small at the end points and that asymptotic approximations are valid near the end points. This is accomplished by starting at either end with a contour integral evaluation of $f_k(z_m)$ and $F_k(z_m)$, $k=1,2,3,4$ and their first three derivatives. Then any numerical integrator may be used to integrate to the other limit, except that care must be taken to ensure that accuracy is maintained. This is accomplished by checking at each step the self-consistency of the solutions by the use of the identities,

$$\varepsilon F_4 f_0-\varepsilon F_0 f_4+F_2 f_1-F_1 f_2=0\ , \tag{79}$$

$$\varepsilon F_4' f_0-\varepsilon F_0' f_4+F_2' f_1-F_1' f_2=0\ , \tag{80}$$

$$\varepsilon F_4'' f_0-\varepsilon F_0'' f_4+F_2'' f_1-F_1'' f_2=0\ , \tag{81}$$

$$\varepsilon F_4''' f_0-\varepsilon F_0''' f_4+F_2''' f_1-F_1''' f_2=2\pi i\lambda^2\varepsilon\ , \tag{82}$$

which derive from the jump conditions. By use of the identity $F=f''+f$, the last two identities may be recast as

TABLE III. Amplitude scattering parameters for the ion case with $n=2$.

Transmission	Reflection	Conversion
$T_1=e^{-\eta}\left[1+\frac{I_{21}^0}{(1+C_f)}\right]$	$R_1=\frac{e^{-2\eta}I_{11}^0}{(1+C_f)}$	$C_{13}=-\frac{\varepsilon^{1/2}(1-e^{-2\eta}I_{31}^0-C_f)}{(1+\varepsilon C_s)}$
$T_2=e^{-\eta}\left[1+\frac{I_{12}^0}{(1+C_f)}\right]$	$R_2=-\frac{\varepsilon(1+C_f)+I_{22}^0}{(1-C_f)}$	$C_{23}=-\frac{e^{-\eta}\varepsilon^{1/2}(1+I_{32}^0+C_f)}{(1+\varepsilon C_s)}$
$T_3=0$	$R_3=-\frac{e^{-2\eta}(1+\varepsilon I_{33}^0-\varepsilon C_s)}{(1+\varepsilon C_s)}$	$C_{31}=C_{31}$, $C_{32}=C_{23}$

$$\varepsilon f_4' f_0-\varepsilon f_0' f_4+f_2' f_1-f_1' f_2=-\frac{2\pi i\lambda^2\varepsilon}{(1+\gamma)}\ , \tag{83}$$

$$\varepsilon F_4' F_0-\varepsilon F_0' F_4+F_2' F_1-F_1' F_2=2\pi i\lambda^2\varepsilon\ . \tag{84}$$

The corresponding identities for $1+\gamma<0$ are

$$\varepsilon F_4 f_0-\varepsilon F_0 f_4-F_2 f_1+F_1 f_2=0\ , \tag{85}$$

$$\varepsilon F_4' f_0-\varepsilon F_0' f_4-F_2' f_1+F_1' f_2=0\ , \tag{86}$$

$$\varepsilon f_4' f_0-\varepsilon f_0' f_4-f_2' f_1+f_1' f_2=-\frac{2\pi i\lambda^2\varepsilon}{(1+\gamma)}\ , \tag{87}$$

$$\varepsilon F_4' F_0-\varepsilon F_0' F_4-F_2' F_1+F_1' F_2=2\pi i\lambda^2\varepsilon\ . \tag{88}$$

Whenever serious discrepancies arise among these identities, the functions are redefined by evaluating the contour integrals at the failure point, and the integration is continued with as many restarts as necessary. Typically, restarts are necessary only on the side with the growing solutions; but in higher-order equations, there are growing solutions on both sides, so more restarts with the numerical contour integrals are required. Generally, since the topology of the contour integrals changes several times over the entire range of z, and is invariably different for z large and positive from that for z large and negative, it is wise to arrange to restart judiciously at least once so that the intermediate range will require no restart.

The integral equation then uses one of these "exact" solutions, f_k, as the trial function in the integrals for the first step to obtain the first estimate for ψ_k, and then continues by iteration with each successive ψ_k until the scattering coefficients (which are obtained from the integrals on each iteration) no longer change at the accuracy desired. If the functions were self-adjoint, convergence would be guaranteed, but if the sink distribution is too broad, the method eventually fails to converge.

2. Scattering parameters in terms of I_{jk} integrals

The asymptotic solutions for ψ may be determined directly from the integral equation by using the asymptotic forms of the exact expressions and the limiting values of the various integrals, each of which is of the form

$$I_{jk}=\frac{1}{2\pi i\lambda^2\varepsilon}\int_{-\infty}^{\infty}F_j(y)h(y)\Psi_k(y)dy\ , \tag{89}$$

where the semi-infinite integrals have been replaced by infinite integrals. For numerical accuracy, it is more efficient to break the infinite integrals into three pieces, such that

$$I_{jk}=I_{jk}^0+\Delta_{jk}^{+}+\Delta_{jk}^{-}\,, \tag{90}$$

where

$$I_{jk}^0=\frac{1}{2\pi i\lambda^2\varepsilon}\int_{-z_m}^{z_m}F_j(y)h(y)\Psi_k(y)dy\ , \tag{91}$$

and the $\Delta_{jk}^{\pm}$ are semi-infinite integrals evaluated analytically from the leading terms of the asymptotic forms. All of the leading terms in the $\Delta_{jk}^{\pm}$ reduce to a constant times one of the two integrals:

$$C_f=\frac{\lambda^2\kappa^2}{4\pi i\lambda^2\varepsilon}\int_{z_m}^{\infty}\frac{F_+F_-}{z}dz=\frac{\eta\kappa^2}{4\pi i z_m^2}\ , \tag{92}$$

$$C_s=\frac{\lambda^2\kappa^2}{4\pi i\lambda^2\varepsilon}\int_{z_m}^{\infty}\frac{S_+S_-}{z}dz=\frac{\lambda\kappa^2}{2iz_m^{1/2}}\ . \tag{93}$$

For the electron case, these correction terms must be multiplied by $2q=7$. The end points, $\pm z_m$, must be chosen to make C_s small. The scattering coefficients including these integrals which depend on absorption are given in Table III for the ion case and in Table IV for the electron case.

For a five-branch case, the corresponding scattering coef-

TABLE IV. Amplitude scattering parameters for the electron case with $n=2$.

Transmission	Reflection	Conversion
$T_1=e^{-\eta}\left[1+\frac{I_{21}^0}{(1-C_f)}\right]$	$R_1=\frac{e^{-2\eta}I_{11}^0}{(1+C_f)}$	$C_{13}=\frac{\varepsilon^{1/2}(1+e^{-2\eta}I_{31}^0-C_f)}{(1-\varepsilon C_s)}$
$T_2=e^{-\eta}\left[1+\frac{I_{21}^0}{(1+C_f)}\right]$	$R_2=-\frac{\varepsilon(1+C_f)-I_{22}^0}{(1-C_f)}$	$C_{23}=\frac{e^{-\eta}\varepsilon^{1/2}(1+I_{32}^0+C_f)}{(1-\varepsilon C_s)}$
$T_3=0$	$R_3=-\frac{e^{-2\eta}(1-\varepsilon I_{33}^0+\varepsilon C_s)}{(1-\varepsilon C_s)}$	$C_{31}=C_{13}$, $C_{32}=C_{23}$

TABLE V. Scattering parameters for a five-branch electron case with $n=2$ and $X<1/2$, where $g_X=\exp(-2\eta_X)$ and $g_O=\exp(-2\eta_O)$. For $3/4<X<1$, the X and O subscripts must be interchanged.

Transmission	Reflection	Conversion
$\|T_1\|^2=g_X\|1-g_Xg_O\varepsilon_OI_{21}\|^2$	$\|R_1\|^2=g_X^4g_O^2\varepsilon_O^2\|I_{11}\|^2$	$\|C_{13}\|^2=g_X^2g_O^3\varepsilon_X\varepsilon_O\|I_{31}\|^2$ $\|C_{14}\|^2=\varepsilon_X\varepsilon_O\|1+g_Xg_OI_{41}\|^2$ $\|C_{15}\|^2=g_O\varepsilon_X\|1-g_O\varepsilon_OI_{51}\|^2$
$\|T_2\|^2=g_X\|1-g_Xg_O\varepsilon_OI_{12}\|^2$	$\|R_2\|^2=\|\varepsilon_X-g_Xg_O\varepsilon_OI_{22}\|^2$	$\|C_{23}\|^2=g_Xg_O^3\varepsilon_X\varepsilon_O\|I_{32}\|^2$ $\|C_{24}\|^2=g_X\varepsilon_X\varepsilon_O\|1-g_OI_{42}\|^2$ $\|C_{25}\|^2=g_Xg_O\varepsilon_X\|1+\varepsilon_OI_{52}\|^2$
$\|T_3\|^2=g_O\|1-g_O\varepsilon_XI_{43}\|^2$	$\|R_3\|^2=g_O^4\varepsilon_O^2\|I_{33}\|^2$	$\|C_{31}\|^2=\|C_{13}\|^2$ $\|C_{32}\|^2=\|C_{23}\|^2$ $\|C_{35}\|^2=\varepsilon_O\|1-g_O\varepsilon_XI_{53}\|^2$
$\|T_4\|^2=g_O\|1-g_O\varepsilon_XI_{34}\|^2$	$\|R_4\|^2=\|g_X\varepsilon_O-g_O\varepsilon_XI_{44}\|^2$	$\|C_{41}\|^2=\|C_{14}\|^2$ $\|C_{42}\|^2=C_{24}\|^2$ $\|C_{45}\|^2=g_O\varepsilon_O\|g_X+\varepsilon_XI_{54}\|^2$
$\|T_5\|^2=0$	$\|R_5\|^2=\|g_Xg_O-\varepsilon_X\varepsilon_OI_{55}\|^2$	$\|C_{51}\|^2=\|C_{13}\|^2$ $\|C_{52}\|^2=\|C_{13}\|^2$ $\|C_{53}\|^2=\|C_{35}\|^2$ $\|C_{54}\|^2=\|C_{45}\|^2$

ficients in terms of the I_{jk} integrals are given in Table V for the second electron harmonic case, where in this case the integrals are defined by

$$I_{jk}=\frac{1}{2\pi i\lambda^4\varepsilon_X\varepsilon_O}\int_{-\infty}^{\infty}F_j(y)h(y)\Psi_k(y)dy\ . \tag{94}$$

3. Analytic evaluation of certain scattering parameters

In some of these cases, it is possible to find the values of the I_{jk} without solving the integral equation at all. In certain cases, it is possible to close the contour either above or below. In a few cases, the contour may be closed, and, using the analytic properties of either the $Z(\zeta)$ or the $F_q(\zeta)$ functions, one may find there are no poles in the pertinent half plane, so that $I_{jk}=0$, and absorption has no effect whatever on the pertinent scattering parameter. For the three-branch problems, this occurs for the integrals $I_{11}=I_{12}=I_{21}=0$, so that *both transmission coefficients are independent of absorption* (Swanson and Shvets, 1993a). The other result is that the reflection coefficient from the high-field side is always zero. The corresponding results for the five-branch case (Ng and Swanson, 1994b) are that $I_{11}=I_{12}=I_{21}=I_{33}=I_{34}=I_{43}=I_{31}=I_{13}=I_{32}=I_{23}=0$, so that again the transmission coefficients for both the X-mode and the O-mode are independent of absorption, and those coefficients that are zero without absorption remain zero.

The transmission coefficient does depend on the cold plasma index of refraction, $n_\perp$, which depends on the value of n_z in the ion case; otherwise, the absorption process *has no effect whatsoever on the transmission coefficient*. This result is counterintuitive, since our intuition has been based until recently on estimates of the transmission obtained from integrating the imaginary part of k across an absorption layer, so that

$$\tau=2\int \mathrm{Im}[k(x)]dx\ , \tag{95}$$

where the integral is across the entire layer. In spite of the fact that the conditions for validity of this simplistic estimate, namely, that $(1/k^2)|dk/dx|\ll 1$, are violated in the absorption layer, the answer is exact. The difficulty is in the interpretation, since, from the more nearly exact dispersion relations which include finite Larmor orbit terms, the imaginary part of $k(x)$ arises from *two* sources, namely, absorption and tunneling. Ironically, the contribution from absorption cancels identically, and only the part due to tunneling contributes. This changes the entire paradigm of resonance absorption, especially as it affects electron cyclotron emission.

In spite of the fact that the validity of the eikonal method is in grave doubt, it is remarkable that it still gives the exactly correct answer. If, for example, one takes the dispersion relation of Eq. (6) with $1>1+\gamma>0$ (typical for the ion second harmonic), then k is complex in the tunneling layer, $z_-\leqslant z\leqslant z_+$ where $z_\pm=2(1\pm\sqrt{1+\gamma})$, so that the integral is (Swanson and Shvets, 1993b)

$$\eta=\int_{z_-}^{z_+}\mathrm{Im}[k(z)]dz \tag{96}$$

$$=\frac{\pi(1+\gamma)}{2\lambda^2}\ , \tag{97}$$

and $\tau=2\eta$. This integral is exact, and the result is precisely the same as those obtained from both the tunneling equation and the usual second-order equations with absorption in-

cluded but tunneling ignored. Furthermore, if absorption is included in the dispersion relation, so that Eq. (6) is modified to

$$k^4-\left[\frac{\lambda^2 z+\gamma}{F}-\gamma\right]k^2+\frac{\lambda^2 z+\gamma}{F}=0 \ , \tag{98}$$

where now absorption is included through $F=-\zeta Z(\zeta)$, with $\zeta=(z_0-z)/\kappa$, then the result is *precisely the same for any* κ, except that now the range of the integral is infinite, since there is an imaginary part of k for any z unless $\kappa=0$ (the case without absorption). The integral is now a complicated contour integral, but, both analytically (Ng, 1994) and numerically, the result is independent of κ and hence independent of absorption.

4. Numerical evaluation of the other scattering parameters

From early numerical solutions of the integral equation for the ion case, it was conjectured that the nonzero reflection coefficient from the low-field side was equivalent to

$$R_2=R_{20}\exp(-\alpha_{Ri}\kappa^2) \ , \tag{99}$$

where $R_{20}=-\varepsilon$ is the value without absorption and $\alpha_{Ri}=1$ (Swanson, 1985). More accurate computations have shown that $\alpha_{Ri}\leqslant 1$, but the deviations from unity are very small (Ng and Swanson, 1994a). This higher accuracy was obtained when it was noted that the integral in I_{22} could be closed, but that there were an infinite number of poles. Through the use of asymptotic expressions for the $F_k(z)$, $\Psi_k(z)$, and $h(z)$, the integrals could be evaluated, leading to a power series in κ which is convergent (for the five-branch problem, the convergence of the series depend on the value of k_0). In each case the reflection coefficient R_2 can be rapidly obtained this way with high accuracy. Numerical results for the conversion coefficients are obtained from solving the integral equation iteratively, as the contours cannot be closed for these integrals. For the ion case with $n=2$, empirical formulas for the conversion coefficients and the slow-wave reflection coefficient may be represented by

$$|C_{13}|=|C_{130}|\exp(-\alpha_{C1}\kappa^2) \ , \tag{100}$$

$$|C_{23}|=|C_{230}|\exp(-\alpha_{C2}\kappa^2) \ , \tag{101}$$

$$|R_3|=|R_{30}|\exp(-\alpha_{R3}\kappa^2) \ , \tag{102}$$

where again C_{130} and C_{230} are the values without absorption, and

$$\alpha_{C1}=e^{-5\eta/4}/18.8\eta \ , \tag{103}$$

$$\alpha_{c2}=e^{-\eta/4}/1.35\sqrt{\eta} \ , \tag{104}$$

$$\alpha_{R3}=e^{-\eta/4}/1.22\eta^{0.65} \ . \tag{105}$$

These empirical formulas give the conversion coefficients within 1% over a broad range of parameters, and R_3 within 5%. These expressions indicate that, for thermal plasmas, both the reflection and the conversion coefficients are Gaussian in κ, and that the conversion coefficient for waves from the low-field side is uniformly smaller than the conversion coefficient from the high-field side, and falls faster with increasing absorption.

For the weakly relativistic electron case, the nonzero reflection coefficient may be expressed in a similar form, such that

$$R_2=R_{20}\exp[-(2n+3)\alpha_{Re}\kappa^2] \ , \tag{106}$$

where n is the harmonic number. In this case one still finds $\alpha_{Re}\leqslant 1$, but the deviations from unity are not small. The reflection coefficient falls much faster with increasing κ (where, in the electron case, κ depends mostly on temperature, while in the ion case, the strongest dependence is on k_z) due to the weakly relativistic factor $2q=2n+3$; but as α_{Re} falls significantly below unity, this effect is lessened.

An accurate empirical formula for α_{Re} for $n=2$ (Ng and Swanson, 1994a), which was obtained from the series method, may be expressed by

$$\alpha_{Re}=1-A[1-\exp(-a\kappa^2)] \ , \tag{107}$$

with

$$A=2.14\quad \exp(5.6X^{2.3})/a \ , \tag{108}$$

$$a=-19.46-6.05X+24.1/(1-2X)^{0.1} \ . \tag{109}$$

A corresponding formula for the conversion coefficient from the high-field side is given by

$$|C_{13}|^2=\varepsilon\ \exp(-14\alpha_{C1}\kappa^2) \ , \tag{110}$$

where

$$\alpha_{C1}=A_1(X)\exp[-\alpha_1(X)lT_e] + A_2(X)\exp[-\alpha_2(X)lT_e] \ , \tag{111}$$

$$A_1(X)=\frac{0.217847}{X^{0.7017146}}+\frac{0.0969301}{(1-2X)^{1.4182316}} \ , \tag{112}$$

$$A_2(X)=\frac{1.1393939}{X^{0.67166}}+\frac{0.4479536}{(1-2X)^{1.0896186}} \ , \tag{113}$$

$$\alpha_1(X)=0.0867128-0.0056594X-0.0087522X^2 + \frac{0.0898992}{(1-2X)^{0.013258}} \ , \tag{114}$$

$$\alpha_2(X)=-0.0453434-0.0204839X+0.002012X^2 + \frac{0.069961}{(1-2X)^{0.0340873}} \ , \tag{115}$$

where T_e is in KeV. (Caution: these formulas should not be extrapolated to large lT_e, since eventually they indicate a rising value of $|C_{13}|^2$, which is not physical.) The conversion coefficient $|C_{23}|^2$ is typically small compared to either $|C_{13}|^2$ or $|R_2|^2$, so that it affects absorption little and, as will be discussed later, emission not at all.

An even more accurate empirical formula for $n=3$ (Swanson and Ng, 1995) is

$$\alpha_{Re}=1-1.926[1-\exp(-a\kappa^2)]/a \ , \tag{116}$$

with

$$a(X,l)=4+[b(l)X+c(l)X^2]\exp\left[-\frac{0.01389}{(1-1.5X)}\right] \ , \tag{117}$$

$$b(l)=0.127-302.4/l \ , \tag{118}$$

$$c(l)=0.1581-472.1/l+10200/l^2\ , \tag{119}$$

where $l=\omega L/c$ is the normalized scale length. These formulas give the reflection coefficient to better than 1% accuracy over the entire range where the reflection coefficient is 1% or greater. There are no correspondingly good empirical formulas for the conversion coefficients available. These are not necessary to calculate the total emission, as will be discussed later, but they are necessary to calculate the emission profile function. The conversion coefficients are Gaussian in κ, however, and although the coefficients are not simple, it is true also for the electrons that $|C_{13}|>|C_{23}|$ uniformly.

Accurate expressions for some of the nonzero scattering coefficients in the five-branch problem have also been obtained using the power-series method (Ng and Swanson, 1994b). Some results for the limiting behavior of two of the scattering coefficients indicate that

$$|R_2|^2\simeq|R_{20}|^2\exp[-4qk_0^2\kappa^2],\quad k_0<1/3\ , \tag{120}$$

$$|C_{14}|^2\simeq|C_{140}|^2\exp[-q(1-k_0)^2\kappa^2],\quad k_0>1/3\ , \tag{121}$$

where the limits on k_0 determine the range over which the power series in κ converges, so are limits on the method, not on the expressions, which show similar tendencies for any $0<k_0<1$. The numerically observed weak dependence of C_{14} on absorption is at least in part due to the fact that most cases have k_0 close to 1, so that the exponential factor is nearly always small.

5. Power absorption fractions

The power absorbed on each branch is relatively easily given in terms of the scattering parameters, such that

$$A_1=1-T^2-|C_{13}|^2\ , \tag{122}$$

$$A_2=1-T^2-|R_2|^2-|C_{23}|^2\ , \tag{123}$$

$$A_3=1-|R_3|^2-|C_{13}|^2-|C_{23}|^2\ , \tag{124}$$

with $T=e^{-\eta}$. These relations can be derived either from the requirement that whatever is not transmitted, reflected, or converted be absorbed, or from the conservation laws of Sec. II.E.3. When absorption is included in these expressions (using ψ and Ψ instead of f and F), $dP/dz\neq 0$, which indicates that the power on the various propagating branches is not conserved. The balance is the absorbed fraction and can be obtained from the asymptotic forms of dP/dz.

For the five-branch problems, the corresponding expressions are

$$A_1=1-T_X^2-|C_{14}|^2-|C_{15}|^2\ , \tag{125}$$

$$A_2=1-T_X^2-|R_2|^2-|C_{24}|^2-|C_{25}|^2\ , \tag{126}$$

$$A_3=1-T_O^2-|C_{35}|^2\ , \tag{127}$$

$$A_4=1-T_O^2-|R_4|^2-|C_{41}|^2-|C_{45}|^2\ , \tag{128}$$

$$A_5=1-|R_5|^2-|C_{51}|^2-|C_{52}|^2-|C_{53}|^2-|C_{54}|^2\ , \tag{129}$$

where $T_X=\exp(-\eta_X)$ and $T_O=\exp(-\eta_O)$. In each case $|C_{jk}|=|C_{kj}|$ and, except for the transmission coefficients, all are Gaussian in κ with varying coefficients. The coefficient C_{41} is nonzero and extremely weakly dependent on absorption relative to the rest.

C. Reciprocity relations

Fundamental reciprocity between certain scattering parameters was first obtained by a complicated and tedious proof (Cho and Swanson, 1990b). A more straightforward proof is evident from the symmetry of the Green's function (Hu and Swanson, 1993b), where, by construction,

$$G(z,y)=G(y,z)\ , \tag{130}$$

Using this result along with the integral equation

$$\Psi_j(z)=F_j(z)+\int_{-\infty}^{\infty}G(z,y)h(y)\Psi_j(y)dy\ , \tag{131}$$

the I_{jk} integrals may be written as

$$2\pi i\lambda^2\varepsilon I_{jk}=\int_{-\infty}^{\infty}F_j(z)\Psi_k(z)h(z)dz \tag{132}$$

$$=\int_{-\infty}^{\infty}\Psi_j(z)\Psi_k(z)h(z)dz-\int_{-\infty}^{\infty}dz\Psi_k(z)h(z)\int_{-\infty}^{\infty}G(z,y)\Psi_j(y)h(y)dy \tag{133}$$

$$=\int_{-\infty}^{\infty}\Psi_j(z)\Psi_k(z)h(z)dz-\int_{-\infty}^{\infty}dy\Psi_j(y)h(y)\int_{-\infty}^{\infty}G(y,z)\Psi_k(z)h(z)dz \tag{134}$$

$$=2\pi i\lambda^2\varepsilon I_{kj}\ . \tag{135}$$

With the other relationships among the scattering coefficients, this guarantees reciprocity.

D. Conservation laws with absorption

The conservation laws with absorption included lead to explicit expressions for the net absorption. The conservation law of Eq. (62), which was exact without absorption, is now written for an absorbing solution, so that

$$P(\psi)=\psi'''\psi^{*\prime\prime}+\psi'''\psi^{*}-\psi''\psi^{*\prime}-\gamma\psi'\psi^{*}-\text{c.c.}\ , \tag{136}$$

which is no longer conserved; the loss of energy flux is given by

TABLE VI. Asymptotic form for ψ_n with $1+\gamma>0$.

$-\infty\leftarrow z$	ψ_n	$z\rightarrow\infty$
$T_1 f_-$	$e^{-\eta}\psi_1$	$f_- + R_1 f_+ + e^{\eta}\sqrt{\varepsilon}C_{13}s_-$
$-R_2 f_- - f_+$	ψ_2	$-T_2 f_+ - \sqrt{\varepsilon}C_{23}s_-$
$-C_{23}f_-/\sqrt{\varepsilon}$	$e^{-\eta}\psi_3$	$-s_+ - R_3 s_- - C_{31}f_+/\sqrt{\varepsilon}$

$$P'(\psi)=[h(z)-h^*(z)]|\Psi|^2 \ . \tag{137}$$

If the *absorption function* $w(z)$ is now defined as

$$w(z)\equiv\frac{h(z)-h^*(z)}{2\pi i\lambda^2\varepsilon} \ , \tag{138}$$

so that $w(z)$ is real, then integrating Eq. (137) over z gives

$$\frac{1}{2\pi i\lambda^2\varepsilon}\left[\lim_{z\rightarrow\infty}P(\psi_k)-\lim_{z\rightarrow-\infty}P(\psi_k)\right] = \int_{-\infty}^{\infty} w(z)|\Psi_k(z)|^2 dz \ . \tag{139}$$

Without absorption, the fact that $P'=0$ required that the limiting values of P be the same on both sides, but now it is evident that the difference between the incident side and the opposite side is the amount absorbed. The integral on the right is positive definite for all $w(z)>0$ and leads to the general expression

$$P_{\rm inc}-P_{\rm opp}=\pm P_{\rm abs}, \tag{140}$$

where $P_{\rm abs}$ is the integral in Eq. (139) and the upper sign is taken if the incident term is positive and the negative sign otherwise. The importance of $w(z)$, which is positive definite for three-branch problems and effectively so for weakly relativistic five-branch problems, is apparent in that, as a weight function in the integrals, it guarantees convergence, since it decays exponentially on either side.

For the specific case for $1+\gamma>0$, the asymptotic forms for the propagating branches are given in Table VI. Applying the power balance to the first solution leads to

$$\frac{e^{2\eta}P(f_-)}{2\pi i\lambda^2\varepsilon}+\frac{e^{2\eta}|R_1|^2P(f_+)}{2\pi i\lambda^2\varepsilon}+\frac{e^{2\eta}|C_{13}|^2P(s_-)}{2\pi i\lambda^2\varepsilon} - \frac{e^{2\eta}|T_1|^2P(f_-)}{2\pi i\lambda_2\varepsilon}=\int_{-\infty}^{\infty} w(z)|\Psi_1(z)|^2dz \ , \tag{141}$$

which, upon the use of the relations of Eq. (129), becomes

$$A_1\equiv 1-|R_1|^2-|T_1|^2-|C_{13}|^2 = e^{-2\eta}\int_{-\infty}^{\infty} w(z)|\Psi_1(z)|^2dz \ . \tag{142}$$

Proceeding in a similar manner, one finds for the second and third solutions

$$A_2\equiv 1-|R_2|^2-|T_2|^2-|C_{23}|^2=\int_{-\infty}^{\infty} w(z)|\Psi_2(z)|^2dz \ , \tag{143}$$

$$A_3\equiv 1-|R_3|^2-|C_{31}|^2-|C_{32}|^2 = e^{-2\eta}\varepsilon\int_{-\infty}^{\infty} w(z)|\Psi_3(z)|^2dz \ . \tag{144}$$

The three results can be combined into a single expression of the form

$$A_k=a_k\int_{-\infty}^{\infty} w(z)|\Psi_k(z)|^2dz \ , \tag{145}$$

with the a_k given by

$$a_1=e^{-2\eta}, \quad a_2=1, \quad a_3=e^{-2\eta}\varepsilon \ . \tag{146}$$

Although there are some differences along the way, this final result is equally valid for the case with $1+\gamma<0$.

IV. CYCLOTRON AND SYNCHROTRON EMISSION FROM NONUNIFORMLY MAGNETIZED PLASMAS

While there have been numerous papers and at least one major review of electron cyclotron emission (Bornatici *et al.*, 1983), the basis for these estimates of the radiation has generally rested on eikonal methods which relate the transmission coefficient of Eq. (95) to blackbody radiation (Bekefi, 1966), such that

$$E=(1-e^{-\tau})I_{\rm BB}, \tag{147}$$

where E is the plasma emissivity, τ is called the optical depth, $1-e^{-\tau}$ is called the opacity, and $I_{\rm BB}$ is the blackbody emissivity,

$$I_{\rm BB}=\frac{\hbar\omega^3}{8\pi^3c^2}(e^{\hbar\omega/kT}-1)^{-1}\simeq\frac{\omega^2kT}{8\pi^3c^2} \tag{148}$$

for $kT\gg\hbar\omega$. The very method ignores reflection, and the dispersion relations do not include more conversion; so with the finding that the transmission coefficient, and hence τ, is independent of absorption, it becomes urgent to find the appropriate revision of the standard theory that more accurately relates absorption to emission. Surprisingly, the classical expression is virtually exact for emission on the high-magnetic-field side of the resonance; but this occurs for subtle reasons, since it has no *direct* dependence on absorption, upon which emission is dependent.

In this section, an appropriate generalization of Eq. (147), which is called the generalized Kirchhoff's law (GKL), will be derived from thermodynamic arguments. The implications of this generalization, both in terms of nomenclature and in terms of experimental manifestations, will be discussed. The section concludes with a discussion of the source distribution function, which is associated with the distributed absorption function and is useful for estimating precisely where in the plasma the radiation originates.

A. Generalized Kirchhoff's law

In general, Kirchhoff's law is a statement that emission is related to absorption through some equilibrium relationships. From this general principle, it is obvious that no medium can emit radiation if it absorbs none. The fact that the conventional optical depth (shown to be due to tunneling only) is unrelated to absorption leaves one with an emission formula that has no connection between emission and absorption, and hence apparently *violates* Kirchhoff's law. When mode conversion effects are included, the effects of reflection and con-

version must be included; and rather than simply guessing at the modified expressions, a proof of the generalized expressions follows.

The proof of the generalized Kirchhoff's law begins with the assumption that a plasma layer of finite thickness is bounded on either side by perfectly absorbing walls. Taking, for example, the three-branch case represented by Fig. 1, one sees that the cyclotron layer will emit radiation to the right on Branches 1 and 3 of intensities E_1 and E_3 and to the left on Branch 2 of intensity E_2. The walls, which are presumed to be at the same temperature as the plasma, will radiate blackbody radiation which will be incident on the plasma layer with intensities I_1, I_2, and I_3 on the respective branches. This incident radiation will be partially transmitted, partially reflected, and partially converted, which will add to the directly emitted radiation, so that the totals on each branch are

$$E_1+|R_1|^2I_1+|T_2|^2I_2+|C_{31}|^2I_3=I_1, \tag{149}$$

$$E_2+|T_1|^2I_1+|R_2|^2I_2+|C_{32}|^2I_3=I_2, \tag{150}$$

$$E_3+|C_{13}|^2I_1+|C_{23}|^2I_2+|R_3|^2I_3=I_3, \tag{151}$$

where the equality on the right is required for thermodynamic equilibrium, since each mode is independent at the walls. Equilibrium requires that the radiation impinging on the walls from each branch exactly balance the radiation from the walls on the same branch. Since the wall is a blackbody and each branch radiates independently, equipartition requires that $I_1=I_2=I_3=I_{\rm BB}$, which leads directly to the generalized Kirchhoff's law (Swanson and Shvets, 1992),

$$E_k=A_kI_{\rm BB}, \quad k=1,2,3, \tag{152}$$

where the A_k are given in Eqs. (122) through (124) for the three-branch problem, and in Eqs. (125) through (129) for the five-branch problem (the extension of the above proof to five branches is trivial, except $k=1,2,3,4,5$). The proof is then extended to a case without walls, letting the walls be removed to infinity, in which case the emission will remain the same provided the temperature of the plasma is maintained through some external energy source via collisions.

1. Electron cyclotron emission

For the electron cyclotron harmonics, the GKL indicates that the emission from the low-magnetic-field side is different from the emission on the high-field side. In particular, the X-mode emission formulas are

$$E_1=(1-e^{-2\eta}-|C_{13}|^2)I_{\rm BB}, \tag{153}$$

$$E_2=(1-e^{-2\eta}-|R_2|^2-|C_{23}|^2)I_{\rm BB}, \tag{154}$$

since $R_1=0$. Since neither R_2 nor either of the conversion coefficients vanish, the direct emission from a cyclotron resonant layer is less than that given by the conventional formula. There is also indirect emission, however, and it must be taken into account. By indirect emission is meant the radiation from an internal thermal source of slow-wave radiation away from the resonance which is then partially converted to an X-mode. In particular, for a tokamak scenario or any device having rotational transform (assumed weak but nonvanishing), the direct emission on the Bernstein branch can be followed by ray tracing, and the slow wave turns and moves back toward the resonance, but typically at a different vertical position where it is totally absorbed. But if the absorption is total, there exists at that point a blackbody emitter radiating a slow wave of intensity $I'_{\rm BB}$, since the emitting point is at a different point in space where the temperature is T'. When this slow wave propagates back to the mode conversion layer on Branch 3, it is then partially converted to the two fast-wave branches, so that the *observed* emission is the sum of the direct and indirect emission. The net emission is then

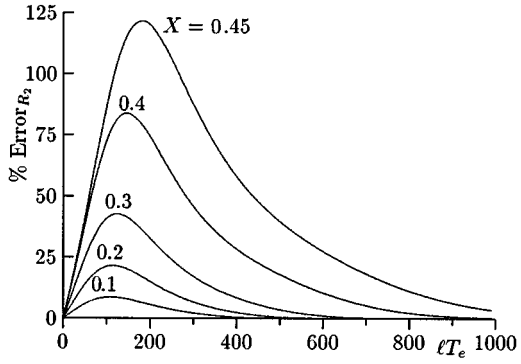


FIG. 6. Percentage error in electron cyclotron emission measurements of E_2 at the second harmonic from neglect of R_2.

$$E'_1=(1-e^{-2\eta}-|C_{13}|^2)I_{\rm BB}+|C_{31}|^2I'_{\rm BB}, \tag{155}$$

$$E'_2=(1-e^{-2\eta}=|R_2|^2-|C_{23}|^2)I_{\rm BB}+|C_{32}|^2I'_{\rm BB}. \tag{156}$$

Since, for typical plasmas, the direct source is very close to the indirect source, it is reasonable to assume $T'\sim T$. Taking them to be the same (so that $I_{\rm BB}=I'_{\rm BB}$) and using reciprocity, one finds that the converted terms cancel exactly, so that

$$E'_1=(1-e^{-2\eta})I_{\rm BB}, \tag{157}$$

$$E'_2=(1-e^{-2\eta}-|R_2|^2)I_{\rm BB}. \tag{158}$$

Remarkably, Eq. (157) is identical to the classical emission formula, so no observable differences are expected from the high-field side. The nonzero reflection coefficient on the low-field side means that the emission on the outside of a tokamak, for example, will be systematically less than the classical amount.

From the empirical formulas of Sec. III.B.4, it is possible to estimate the error that might occur if the standard formulas were used, neglecting the effects of reflection in E'_2. The percentage error for the *emission* at the second electron harmonic is given by

$$\%\ \text{Error}=\frac{100|R_2|^2}{1-e^{-2\eta}-|R_2|^2}=\frac{100}{\exp(14\alpha_R\kappa^2)/\varepsilon-1}, \tag{159}$$

with α_R obtained from Eqs. (107) and (109). This result is a function of $lT_e\propto B_0LT_e$ and X only, so the results are illustrated in Fig. 6, where T_e is in keV and X is the parameter. It is evident that the maximum error can be quite large; but for many applications, l is so large that the error is small except at extremely low temperatures. For example, if one were to consider a tokamak plasma with $B_0=5$ T and $L=3.5$ m, then

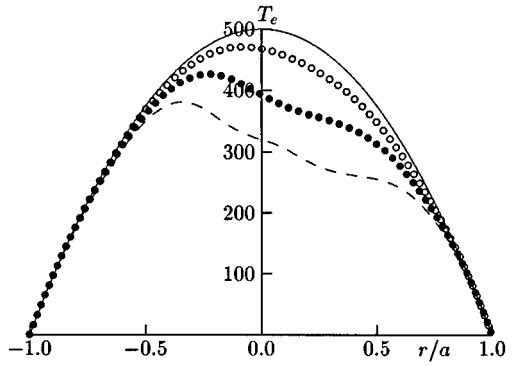


FIG. 7. Actual (solid) and inferred temperatures with reflection neglected for tokamak with $R_0=0.7$ m, $B_0=1$ T, $T_{e0}=500$ eV, $n_{e0}=1.0\times10^{19}$/ m^3 (○○○), $n_{e0}=1.35\times10^{19}$/m^3 (●●●), $n_{e0}=1.7\times10^{19}$/m^3 (− − −), $a=20$ cm.

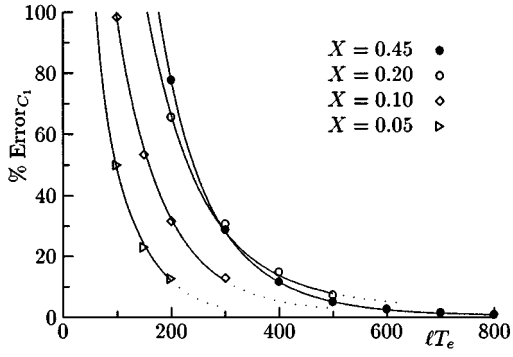


FIG. 9. Percentage error in estimating absorption, neglecting $|C_{13}|^2$ at $2\omega_{ce}$ for four values of X. Solid lines show the empirical formula result over the range fitted, and dotted lines are extrapolations. Symbols are some of the fitted points.

$l=2\times10^4$, so that the maximum error would occur for a temperature near 10 eV, and for a 1-keV temperature, the error is minimal. The error rises sharply as the wave approaches cutoff at $X=0.5$, but is still relatively unimportant for most fusion plasmas. If, on the other hand, one were to consider a smaller tokamak with $L=0.72$ m, $B_0=1$ T ($f=56$ GHz), and $T_{e0}=500$ eV, where both the temperature and density are parabolic, then the *inferred temperatures* for several peak densities would be those given in Fig. 7. Near $r/a=0.25$, the emission error is about 125% and the temperature error is 40% for the highest density shown (where $X\sim0.45$). For space or astrophysical plasmas, large errors could easily occur; and in some cases, the assumption that $I_{\rm BB}\simeq I'_{\rm BB}$ may be questionable in extended space plasmas increasing the error further.

For the corresponding third harmonic case, with $14\rightarrow18$ in Eq. (159) and α_R obtained from Eqs. (116) through (119), the error is even smaller. It is shown in Fig. 8, where again T_e is in keV and X is the parameter. The maximum error for this case is nearly two orders of magnitude smaller than for the second-harmonic case. Whereas in the second-harmonic case the maximum error continues to increase toward cutoff, the maximum error for the third harmonic is largest for $X=0.592$ and decreases as one approaches cutoff at $X=2/3$.

Even though this analysis has basically justified the classical emission formula for one case (the least likely case, representing emission on the inside of a tokamak), it has not justified the nomenclature describing its use. Calling the quantity $\tau=2\eta$ an optical depth derived from the picture that the wave energy was dissipating as it passed through the absorbing layer, but now it is established that this quantity is independent of absorption. In view of this, the term should fall into disuse for harmonic resonances and be replaced by "tunneling factor," or some such designation that connects the quantity with the physical picture. It is easy to generalize the concept of opacity by simply including the reflection and conversion coefficients in the expression. It should be noted at this point that analytic expressions for η are given along with good empirical formulas for $|R_2|$, so that accurate estimates of emission can be made without extensive numerical procedures, since the conversion coefficients are unnecessary.

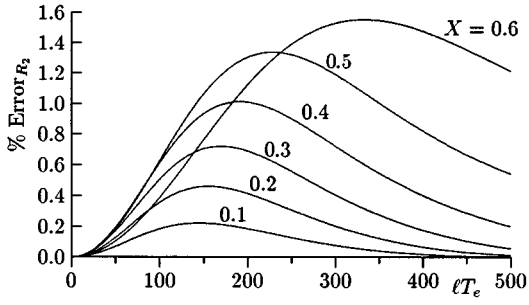


FIG. 8. Percentage error in electron cyclotron emission measurements of E_2 at the third harmonic from neglect of R_2.

To further underscore the discrepancy between the classical formula and those of the full-wave analysis, it should be noted that even though the emission from the high-field side is independent of the converted wave energy, *the absorption is not*. Just as Fig. 6 shows the error in neglecting the reflection term in estimating the emission, Fig. 9 shows the error in estimating the *absorption* without considering the converted energy. This error is obtained from

$$\%\ \text{Error}=100\frac{1-e^{-2\eta}-A_1}{A_1}=\frac{100}{\exp(\alpha_{C1}\kappa^2)-1}\ , \quad (160)$$

where $|C_{13}|^2$ is obtained from Eq. (110) through the formulas of Eqs. (111) through (115). Thus, even though the tunneling factor gives the correct emission formula, it does not describe absorption, as the differences can easily exceed a factor of 2. It is only due to the *indirect emission*, which in many cases is the dominant contribution, that the total emission happens to match the classical expression. In large tokamaks, where even the conversion is small ($lT_e>1000$), the distinction is probably unimportant; but the identification of the tunneling factor with an absorption factor must be used only with great caution, since technically they are independent.

Using the same general methods as were used to establish the GKL, one can also calculate the effects of cold reflecting walls. For this calculation, it is assumed that the slow wave is absorbed internally and that Eqs. (157) and (158) accu-

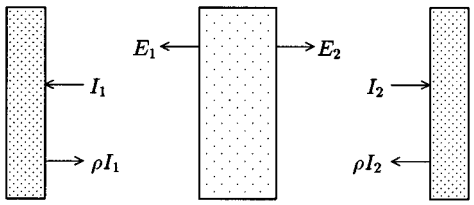


FIG. 10. Emission and reflection for a three-branch case.

rately represent the fast-wave emission with absorbing walls. If the wall reflectivity is ρ, and the radiation incident on each wall is I_1 and I_2, as in Fig. 10, then the net radiation on each wall is

$$I_1 = E_1' + \rho|T_2|^2 I_2 + \rho|R_1|^2 I_1 , \tag{161}$$

$$I_2 = E_2' + \rho|T_1|^2 I_1 + \rho|R_2|^2 I_2 . \tag{162}$$

Solving for I_1 and I_2 between these, and using the fact that $T_1 = T_2 = T = e^{-\eta}$, $R_1 = 0$, and the expressions for E_1' and E_2' from Eqs. (157) and (158), one finds

$$I_1 = \frac{(1-T^2)(1+\rho T^2) - \rho|R_2|^2}{1-\rho|R_2|^2 - (\rho T^2)^2} I_{\rm BB} , \tag{163}$$

$$I_2 = \frac{(1-T^2)(1+\rho T^2) - |R_2|^2}{1-\rho|R_2|^2 - (\rho T^2)^2} I_{\rm BB} . \tag{164}$$

When R_2 is neglected, these become equal and reduce to the usual expression (Bornatici *et al.*, 1983),

$$I = \frac{1-T^2}{1-\rho T^2} I_{\rm BB} . \tag{165}$$

2. Ion cyclotron emission

There are many similarities between the electron and ion emission cases, but one significant difference. The expressions for the direct emission are entirely parallel to Eqs. (153) and (154), as are the expressions for the sum of the direct and indirect emission given by Eqs. (155) and (156), but in this case, there is no reason to expect $I_{\rm BB} \sim I_{\rm BB}'$. Whereas in the electron case, the remote sink/source for the Bernstein wave is very close to the resonance layer, since the converted wave eventually turns back toward the resonance layer, the ion Bernstein wave does not turn back, and the location of the remote sink/source may be far from the resonance layer. The actual location must usually be determined from ray tracing, and the influence of a small vertical component of magnetic field (assumed small in the layer, since $\boldsymbol{B}\cdot\nabla B = 0$ has been assumed) makes a great deal of difference in the trajectory of the converted wave energy. The appropriate expressions for ion thermal emission are therefore given by Eqs. (155) and (156), and in this case, the empirical formulas for the conversion coefficients are important.

It may also be noted that in the ion cyclotron emission case, there are many possibilities for nonthermal emission, which may be due either to a nonthermal velocity distribution or to radiation from an instability, particularly from beam-plasma-type instabilities on the Bernstein wave branch which then mode convert to the emitting fast-wave branch. If there were a source due to an instability on the slow-wave branch that would be strong enough to dominate the thermal components, this could be ascertained from observing the ratio E_1'/E_2' which would approach the ratio $|C_{31}/C_{32}|^2$ in the limit as the thermal portion became negligible. In such a case one would interpret the $I_{\rm BB}'$ as being due to the instability source strength rather than any real temperature.

For a nonthermal velocity distribution, either for ions or electrons, it is possible to generalize the absorption term $h(z)$ to an actual velocity distribution, if known. Since the finite Larmor orbit terms and the emissivity depend only on averages, for only modest deviations from a Maxwellian, one could expect good agreement by replacing kT by $2\langle E\rangle/3$ for everything except the shape of the absorption profile or the emission profile, where the details would be more important.

B. Source distribution function and the local Kirchhoff's law

Because emission and absorption are related not only globally but locally, it is evident that knowing what the absorption profile is must somehow determine the emission profile or source distribution function. The relationship is not intuitively obvious, however, since the local emission may be partially reabsorbed by the medium before it escapes, so that the effective source will be related, but not equivalent, to the sink. The global emission is governed by the GKL and is due to some as yet undetermined combination of all three propagating branches. This emission is governed by a source distribution function, $s(z)$, which leads to a radiating solution $\phi(z)$ which is governed by (e.g., for the second-harmonic case)

$$\phi^{iv} + \lambda^2 z\phi'' + (\lambda^2 z + \gamma)\phi = h(z)\Phi + s(z) , \tag{166}$$

which can no longer be easily converted to an integral equation, since $s(z)$ is not explicitly a function of $\phi(z)$. The term involving $h(z)$ on the right could be moved to the left-hand side, in which case the basis set for the Green's function will be the $\psi_k(z)$ or the $\Psi_k(z)$, which represent the homogeneous solutions with absorption. Given the $s(z)$, one may write the solution $\phi(z)$ in terms of the Green's function and the source $s(z)$. Unfortunately, however, this is an inverse problem, since something is known about the asymptotic form of $\phi(z)$ from the GKL, but the objective is to find $s(z)$. The generalization of Eq. (166) to higher harmonics or to five-branch problems is straightforward, since one simply adds the $s(z)$ source term on the right-hand side of the governing absorption equation.

1. Integral expressions for emission

In order to derive expressions for the emission from a source in an absorbing plasma, Eq. (166) must be solved. The radiative solution is given by (Cho and Swanson, 1990c)

$$\phi(z) = S_1^+ \psi_1 + S_2^- \psi_2 + S_3^+ \psi_3 + S_4^- \psi_4 , \tag{167}$$

where the $\psi_k(z)$, $1 \leqslant k \leqslant 4$, are linearly independent solutions of Eq. (11) and where

$$S_{1,3}^+(z) = \int_z^\infty B_{1,3}(y) s(y) dy , \tag{168}$$

$$S_{2,4}^{-}(z)=\int_{-\infty}^{z}B_{2,4}(y)s(y)dy\ . \tag{169}$$

The functions $B_j(z)$ are in turn related to the ψ_k and the $\Psi_k=\psi_k''+\psi_k$ through the relations

$$\tilde{B}_1=-D_3\Psi_2+C_{23}e^{\eta}\Psi_4, \tag{170}$$

$$\tilde{B}_2=-D_3\Psi_1-D_1\Psi_3, \tag{171}$$

$$\tilde{B}_3=-D_1\Psi_2+T_1\varepsilon e^{\eta}\Psi_4, \tag{172}$$

$$\tilde{B}_4=e^{\eta}(C_{23}\Psi_1+T_1\varepsilon\Psi_3)\ , \tag{173}$$

where

$$\tilde{B}_j=2\pi i\lambda^2\varepsilon e^{\eta}(D_1C_{32}-T_1D_3)B_j\,, \tag{174}$$

$$D_1=\varepsilon I_{41},$$

$$D_3=(1+\varepsilon I_{43})\ ,$$

$$I_{4j}=\frac{1}{2\pi i\lambda^2\varepsilon}\int_{z_{\min}}^{\infty}h(y)F_4(y)\Psi_j(y)dy,\quad j=1,3\ . \tag{175}$$

The asymptotic behavior of the radiative solution, which can have only outgoing waves, may be represented in the form

$$\varphi_2 f_-(z)\leftarrow\phi(z)\rightarrow-\varphi_1 f_+(z)-\varphi_3 s_-(z)\ , \tag{176}$$

where the φ_k, $k=1,2,3$, are complex constants which are related to the energy fractions radiated on each branch, given by

$$E_{1,2}=|a\varphi_{1,2}|^2,\quad E_3=|a\varphi_3|^2/\varphi\ , \tag{177}$$

where a is some constant related to the source magnitude, which is assumed to satisfy a normalization condition of the form

$$\int_{-\infty}^{\infty}\rho(z)|s(z)|^2dz=P_0, \tag{178}$$

where $\rho(z)$ is some weight function. Since a change in the value of a just means a redefinition of P_0 for a given weight function, the value of this constant can be chosen for convenience. From Eqs. (167), (168), (169), (174), and the asymptotic behavior of the ψ_k, $1\leqslant k\leqslant 4$, along with the reciprocity relations, one obtains the relations

$$\varphi_1=T_2S_2+e^{-\eta}D_1S_4/\varepsilon\ , \tag{179}$$

$$\varphi_2=e^{\eta}(T_1S_1-C_{32}S_3)\ , \tag{180}$$

$$\varphi_3=C_{23}S_2+e^{-\eta}D_3S_4, \tag{181}$$

where the definition

$$S_j\equiv\int_{-\infty}^{\infty}B_j(y)s(y)dy,\quad j=1,2,3,4\ , \tag{182}$$

has been introduced. It is convenient to introduce the integral expressions

$$J_k=\int_{-\infty}^{\infty}s(z)\Psi_k(z)dz,\quad k=1,2,3\ , \tag{183}$$

which represents the ''projections'' of the unique source on to the kth branch. One thus expects that for any k, the radiated field ϕ_k should be determined only by the corresponding J_k for the same branch. Using Eqs. (170) through (175) and (183), one finds the integrals in Eq. (182), given by

$$\begin{pmatrix}S_1\\S_2\\S_3\\S_4\end{pmatrix}=\begin{pmatrix}-D_3J_2+C_{23}e^{\eta}J_4\\-D_3J_1-D_1J_3\\-D_1J_2+T_1\varepsilon e^{\eta}J_4\\e^{\eta}(C_{23}J_1+T_2\varepsilon J_3)\end{pmatrix}\times\frac{1}{2\pi i\lambda^2\varepsilon e^{\eta}(D_1C_{32}-T_1D_3)}\,, \tag{184}$$

so that one obtains from Eqs. (179) through (181), after some algebra, the results

$$\varphi_1=e^{-\eta}\frac{J_1}{2\pi i\lambda^2\varepsilon}\ , \tag{185}$$

$$\varphi_2=\frac{J_2}{2\pi i\lambda^2\varepsilon}\ , \tag{186}$$

$$\varphi_3=-\varepsilon e^{-\eta}\frac{J_3}{2\pi i\lambda^2\varepsilon}\ . \tag{187}$$

Since this final result is independent of the D_1 and D_3 coefficients, it is valid for any $z_{\min}$ in the definition of I_{4j}.

The final expressions are then obtained by combining Eq. (177) with Eqs. (185) through (187) and putting $|a|=2\pi\lambda^2\varepsilon$, with the result

$$E_k=a_k\left|\int_{-\infty}^{\infty}s(z)\Psi_k(z)dz\right|^2\ , \tag{188}$$

where the a_k are already given by Eq. (146).

2. Extremum theorem with the generalized Kirchhoff's law

If one models a source/sink by a single delta function, then the source automatically satisfies the GKL and the emission is in closed form. For a finite number of delta functions, the problem is already indeterminate if the number exceeds a few, since the only constraints are the GKL. For a continuous source/sink, a variational analysis is required. Since the maximum emission is constrained by the blackbody limit, the idea is to maximize E_k, which is a functional of $s(z)$ through Eq. (188), subject to the GKL of Eq. (152) and the condition that P_0 be fixed. Since the A_k are not functionals of $s(z)$, the right-hand side of Eq. (152) is fixed for each k; so the objective is to maximize the left-hand side for $k=1,2,3$ simultaneously. If the functions $s(z)$ which maximizes the emission is denoted s_m, and the E_k are related to the $J_k[s]$, the problem statement is to find s_m such that (Shvets and Swanson, 1993)

$$|J_k[s_m]|^2=\max_{s(z)}|J_k[s]|^2,\quad k=1,2,3\ . \tag{189}$$

Additionally, these maxima are related through the GKL so that

$$|J_k[s_m]|^2=\frac{A_k}{a_k}I_{\rm BB},\quad k=1,2,3\ . \tag{190}$$

The final constraint is through the boundedness of $s(z)$ through P_0, whose value for a particular weight function is to be adjusted to give the blackbody value of the GKL. This weight function will of necessity be related to the absorption function.

3. Cauchy-Schwartz estimates

Since both E_k and A_k may be represented by integral expressions, it is instructive to begin with the Cauchy-Schwartz inequality,

$$|\langle f|g\rangle_\sigma|^2 \leqslant \langle f|f\rangle_\sigma \langle g|g\rangle_\sigma , \tag{191}$$

where the scalar product of two square-integrable functions is defined by

$$\langle f|g\rangle_\sigma \equiv \int_{-\infty}^{\infty} \sigma(z) f^*(z) g(z) dz , \tag{192}$$

with a weight function $\sigma(z)$ that falls off sufficiently rapidly as $|z|\to\infty$ that the integral is bounded. The expression for absorption is then given by $A_k = a_k \langle \Psi_k|\Psi_k\rangle_w$, while the emission integral from Eq. (183) is the scalar product defined by Eq. (192) with $f(z)=s^*(z)\sigma^{-1}$ and $g(z)=\Psi_k(z)$. With the Cauchy-Schwartz inequality applied to the emission relation $E_k = a_k |J_k|^2$, the result is

$$E_k[s] \leqslant a_k \int_{-\infty}^{\infty} |s|^2 \sigma^{-1} dz \int_{-\infty}^{\infty} \sigma |\Psi_k|^2 dz \tag{193}$$

for each fixed $k=1,2,3$. In particular, if $\sigma(z)=w(z)$, then

$$E_k[s] \leqslant A_k \int_{-\infty}^{\infty} \frac{|s|^2}{w} dz , \tag{194}$$

and if, in addition, $\sigma = 1/\rho$ (so that $\rho = 1/w$), then

$$E_k[s] \leqslant A_k \int_{-\infty}^{\infty} \rho(z) |s(z)|^2 dz = A_k P_0 . \tag{195}$$

Since the inequality in Eq. (191) reduces to equality only when f and g are linearly dependent, the extremum for the emission along a particular branch k (taking them one at a time) occurs, from Eq. (193), when

$$s(z) \propto \sigma(z) \Psi_k^* . \tag{196}$$

This result indicates that for a single branch, the source must be proportional at each point to the weight function, which must be chosen from physical considerations. From the spirit of the fluctuation-dissipation theorem Kirchhoff's law, it is obvious that $\sigma(z)$ should relate to the local dissipative properties.

Following the prescription from Eq. (195), one would have a separate source for each branch; but all three waves are coupled, and the source distribution function is unique for all branches. The obvious way to treat the problem and generalize the results is to take

$$s_m(z) = \sigma(z) \sum_{k=1}^{3} \alpha_k \Psi_k^*(z) , \tag{197}$$

where the α_k are unknown complex constants to be determined from the GKL and the source normalization condition. This reasonable ansatz of Eq. (197) follows rigorously from the variational approach to the maximization of the integral of radiation.

4. Variational analysis

It is convenient to recast the variational problem in terms of the functionals

$$I_k[s] = E_k[s]/A_k , \tag{198}$$

so that the problem becomes a determination of the function $s_m(z)$ such that

$$I_1(s_m) = I_2(s_m) = I_3(s_m) = \max_s I_k[s] = I_{\rm BB} , \tag{199}$$

subject to the normalization of Eq. (178).

Using the usual prescription for a variational analysis, one introduces the functional

$$\Phi[s] = \sum_{k=1}^{3} \mu_k I_k[s] - \lambda P_0[s] , \tag{200}$$

where the Lagrange multipliers μ_k and λ correspond to the GKL and normalization conditions, respectively. From the symmetry of the GKL conditions, it follows that $\mu_1+\mu_2+\mu_3=1$. For the solution of the extremum problem, it is required that

$$\delta\Phi[s_m,\eta] = 0 , \tag{201}$$

for any smooth function $\eta(z)$.

Introducing the Hermitian matrix g_{jk} such that

$$g_{jk} = \langle \Psi_j|\Psi_k\rangle_w = g_{kj}^* , \quad k,j=1,2,3 , \tag{202}$$

one finds that the absorbed fractions are simply $A_k = a_k g_{kk}$ and $I_k[s] = |J_k[s]|^2/g_{kk}$. Taking the variation of Φ indicated in Eq. (201), one obtains

$$\delta\Phi[s,\eta] = \int_{-\infty}^{\infty} \eta^*(y) dy \left\{ -\lambda \rho(y) s(y) + \sum_{k=1}^{3} \Psi_k^* J_k[s(y)] \mu_k / g_{kk} \right\} + \text{c.c.} = 0 , \tag{203}$$

which leads to the integral eigenvalue problem,

$$\lambda \rho(y) s_m(y) - \int_{-\infty}^{\infty} K(y,z) s_m(z) dz = 0 , \tag{204}$$

with the Hermitian kernel

$$K(y,z) = \sum_{k=1,3}^{3} \mu_k \Psi_k^*(y) \Psi_k(z)/g_{kk} = K^*(z,y) . \tag{205}$$

The problem posed by Eqs. (204) and (205) has the solution

$$s_m(z) = \frac{1}{\rho(z)} \sum_{j=1}^{3} \alpha_j \Psi_j^*(z) , \tag{206}$$

where the coefficients α_j are determined from the algebraic eigenvalue problem

$$\sum_{j=1}^{3} \alpha_j \langle \Psi_j|\Psi_k\rangle|_{1/\rho} = \lambda \frac{g_{kk}}{\mu_k} \alpha_k . \tag{207}$$

Note that the expression of Eq. (206) for the source distribution function is identical to that of Eq. (197) if the weight function in the scalar product has been chosen as $\sigma(z)=1/\rho(z)$. Moreover, the physical interpretation requires one to consider both of these expressions as a local Kirchhoff's law. This physical requirement removes the ambiguity in the weight function in the normalization condition. Since the function $w(z)$ defined by Eq. (138) is the only suitable absorptive function in the physical system, one must conclude that $\rho(z)=1/w(z)$, and

$$s_m(z)=w(z)\sum_{k=1}^{3}\alpha_k\Psi_k^*(z)\ . \tag{208}$$

The problem is reduced to finding the unknown complex coefficients, α_k, that satisfy

(1) the normalization condition,

$$\sum_{k,j=1}^{3}\alpha_k^* g_{kj}^*\alpha_j=P_0; \tag{209}$$

(2) the eigenvalue problem,

$$\sum_{j=1}^{3}\alpha_j g_{jk}=\lambda\frac{g_{kk}}{\mu_k}\alpha_k,\quad k=1,2,3 \quad \text{and} \tag{210}$$

(3) the GKL conditions,

$$I_1=I_2=I_3=I_{\mathrm{BB}}, \tag{211}$$

where

$$I_k=\sum_{i,j}\alpha_i^*\frac{g_{ik}^* g_{kj}^*}{g_{kk}}\alpha_j. \tag{212}$$

It may be noted that

$$\lambda=I_{\mathrm{BB}}/P_0. \tag{213}$$

Indeed, after multiplying Eq. (204) by $s_m^*(y)$ and integrating over y, one finds, taking into account Eq. (178) and the GKL equations, $I_k[s_m]=I_{\mathrm{BB}}$, that

$$\lambda P_0=\sum_{k=1}^{3}\mu_k|J_k|^2/g_{kk}=I_{\mathrm{BB}},$$

which results in Eq. (213).

This variational problem is not of the standard form, but the evaluation of the α_k has been accomplished by a series of transformations and a minimization (Shvets and Swanson, 1993). The basic result is

$$\alpha_1=(I_{\mathrm{BB}}/g_{11})^{1/2}(\tau_1+z_3\omega_2+z_2^*\omega_3)/|Y|\ , \tag{214}$$

$$\alpha_2=(I_{\mathrm{BB}}/g_{22})^{1/2}(z_3^*+\tau_2\omega_2+z_1\omega_3)/|Y|\ , \tag{215}$$

$$\alpha_3=(I_{\mathrm{BB}}/g_{33})^{1/2}(z_2+z_1^*\omega_2+\tau_3\omega_3)/|Y|\ , \tag{216}$$

where $z_i=y_jy_k-y_i^*$, $\tau_k=1-|y_k|^2$, $|Y|=1-|y_1|^2-|y_2|^2-|y_3|^2+2\ \mathrm{Re}(y_1y_2y_3)$, $y_i=g_{jk}/(g_{jj}g_{kk})^{1/2}$, and the ω_k are given by $\omega_1=1$ and

$$\omega_2=\frac{z_1z_2-z_3x_3}{x_2x_3-q_1}\ , \tag{217}$$

$$\omega_3=\frac{z_1^*z_3^*-z_2x_2x_3}{x_2x_3-q_1}\ . \tag{218}$$

The x_3 is in turn, given by

$$x_3=c_2x_2^2+c_1x_2+c_0+\frac{c_{-1}}{x_2}+\frac{b}{x_2-d}\ , \tag{219}$$

with

$$c_2=-q_2/r\ ,$$

$$c_1=1-2s/r^2\ ,$$

$$c_0=(q_1q_2+q_1q_3+q_2q_3)/r-4q_1q_3s/r^3\ ,$$

$$c_{-1}=(q_1-q_3)/2\ ,$$

$$b=c_0d-q_3-c_{-1},$$

$$d=2q_1q_3/r\ ,$$

$$r=2\ \mathrm{Re}(z_1z_2z_3)\ ,$$

$$s=q_1q_2q_3,$$

$$q_k=|z_k|^2\ ,$$

and x_2 is one of the roots of

$$\sum_{n=0}^{8}d_nx_2^n=0\ , \tag{220}$$

where

$$d_0=q_3(c_{-1}d)^2\ ,$$

$$d_1=(2q_3^2-rd)c_{-1}d\ ,$$

$$d_2=q_3^3-2q_3c_{-1}d(c_0-c_1d)+q_1d^2(q_2-q_3)+rd(c_{-1}-q_3)\ ,$$

$$d_3=-2q_3c_{-1}d-2q_3^2(c_0-c_1d)-2q_1d(q_2-q_3)+r[d(c_0-c_1d)+q_3+d^2]\ ,$$

$$d_4=q_3(c_0-c_1d)^2-2q_3^2-2q_3c_2c_{-1}d+q_1(q_2-q_3)-r(d+c_0-c_1d)\ ,$$

$$d_5=-2q_3^2c_2+2q_3(c_0-c_1d)+dq_2,$$

$$d_6=q_3+2q_3c_2(c_0-c_1d)\ ,$$

$$d_7=2q_3c_2,$$

$$d_8=q_3c_2^2.$$

Finally, one must determine which real root for x_2 to choose by calculating

$$\sum_{k=1}^{3}(\tau_k-x_k)=P_0|Y|/I_{\mathrm{BB}}\ , \tag{221}$$

where x_1 is determined from

$$x_1x_2x_3-\sum_{k=1}^{3}q_kx_k+r=0\ . \tag{222}$$

Since it is desired to find the *smallest* (most efficient) source that will produce the blackbody radiation, the x_2 root that minimizes the sum in Eq. (221) is to be chosen. Although tedious, this produces the unique source distribution function that satisfies all of the constraints and maximizes the radiation.

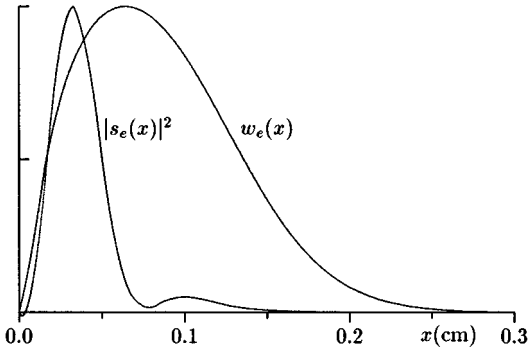


FIG. 11. Comparison of sink function $w_e(x)$ with source function $|s_e(x)|^2$ with $n=10^{20}$ m^{-3}, $B_0=3$ T, $L=0.2$ m with $T_e=400$ eV.

The procedures described above for the three-branch problems have been generalized to the five-branch problems (Ng, Shvets, and Swanson, 1995), but the minimization problem cannot be reduced to finding the roots of a polynomial. More general downhill simplex methods in multidimensions are required, but the form of the source distribution function is the same as Eq. (208) except that the sum is over $k=1,2,3,4,5$. One other difference between the three-branch and the five-branch problems is that $w(z)$ is positive definite for three-branch problems due to the behavior of $Z(\zeta)$ and $F_q(\zeta)$. For the five-branch case, the behavior of $\mathscr{F}_q(\zeta-q,a)$ leads to a narrow region with $w(z)<0$. The only pertinent emission quantity is $|s(z)|^2$, however; so the final result is positive definite.

5. Numerical results and their interpretation

The importance of the source distribution function is that it indicates both where the effective source is located and the effective width of the source. The general effect of mode conversion on the source distribution is to make a systematic shift in the peak and a narrowing of the width, relative to the corresponding peak and width of the sink or absorption function. In Fig. 11, the sink function $w_e(x)$ is plotted on the same axis with $|s_e(x)|^2$, where both the shift and the narrowing are evident. For this example, with both amplitudes normalized to unity for comparison, the source function is at

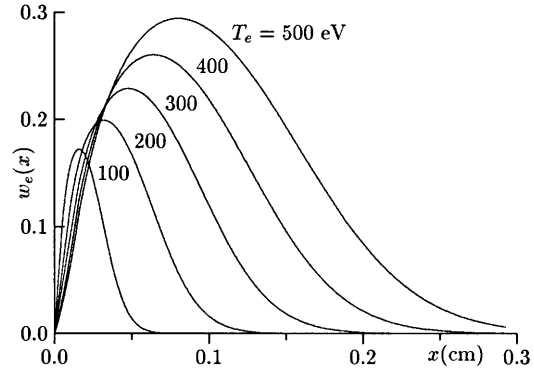
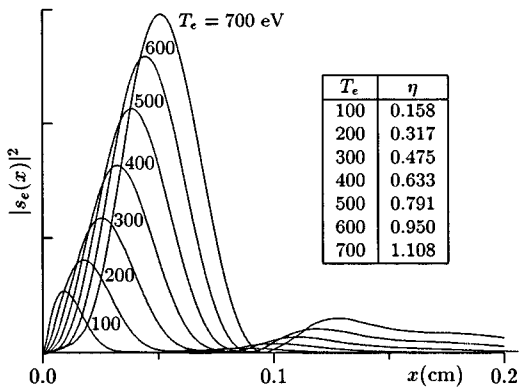


FIG. 12. Absorption function $w_e(x)$ with $n=10^{20}$ m^{-3}, $B_0=3$ T, $L=0.2$ m with $100\leqslant T_e\leqslant 500$ eV.



T_e	η
100	0.158
200	0.317
300	0.475
400	0.633
500	0.791
600	0.950
700	1.108

FIG. 13. Source function $|s_e(x)|^2$ (arb. units) with $n=10^{20}$ m^{-3}, $B_0=3$ T, $L=0.2$ m with $100\leqslant T_e\leqslant 700$ eV and corresponding values of η.

half the displacement of the sink. With a 3-m scale length, the source would be displaced 0.5 cm and the sink 1 cm from the resonance location. The narrowing is more than a factor of 2, and the bump on the tail appearing on the right is effectively a second source *on the other side of the tunneling layer*. As the layer becomes so thick that there is virtually no tunneling, the two sources become isolated from each other, and one observes only one of the two. The numerical techniques used for this analysis eventually fail for sufficiently thick layers (the integral equation fails to converge). Better numerical techniques must be developed to see the shapes and locations of these double sources, but the tendency is already clear. Figures 12 and 13 show the temperature dependence of the sink and source(s), respectively, where it is apparent that the shift of the peaks and the widths are roughly proportional to temperature (Shvets and Swanson, 1993). In each of these cases, both peaks are on the high-magnetic-field side of the resonance, since relativistic effects occur on that side.

For ions, which are nonrelativistic, the $w_i(x)$ sink function is symmetric about the resonance, but the $|s_i(x)|^2$ source function is not. In other respects, however, the shifts and narrowing of the source function are qualitatively the same. A comparison of the sink and source on the same axis for ions is shown in Fig. 14, where, again, both are normalized to unit amplitude for the comparison. For this case, the peak of the sink is, of course, unshifted, but the peak of the source has been shifted by 0.46% of L. This shift is toward the mode conversion layer (positive x is in the direction of *increasing* magnetic field) because the mode-converted branch is the strongest absorber and therefore the strongest emitter. For the ion case, k_z controls the strength of absorption and emission; so in Figs. 15 and 16 the variation of both sink and source, respectively, with k_z is shown. For the source function, again both the width and the peak increase roughly linearly with k_z, which controls the absorption/emission strength. Furthermore, the ion source distribution develops bumps on the tails on both sides of the resonance due to interference between the propagating branches, but these remain small for virtually all practical parameters.

Finally, a five-branch case is shown in Fig. 17 for weakly relativistic electrons (Ng, Shvets, and Swanson, 1995). In this case, since $k_z \neq 0$, the relativistic function $\mathscr{F}_q(z,a)$,

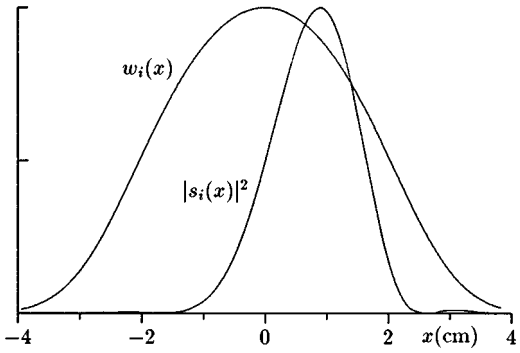


FIG. 14. Comparison of sink function $w_i(x)$ with source function $|s_i(x)|^2$ with $n=2\times10^{20}$ m^{-3}, $B_0=5$ T, $L=2$ m, $T_i=2$ keV with $k_z=8$ m^{-1}.

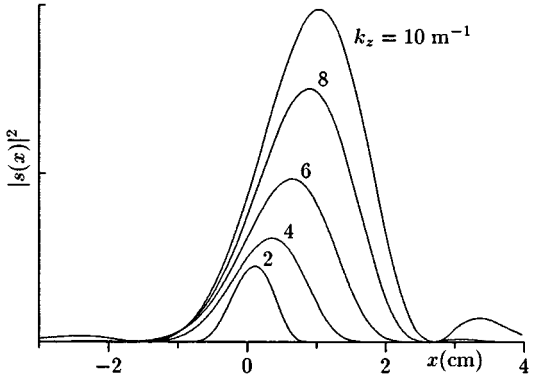


FIG. 16. Absorption function $|s_i(x)|^2$ with $n=2\times10^{20}$ m^{-3}, $B_0=5$ T, $L=2$ m, $T_i=2$ keV with $k_z=2$, 4, 6, 8, 10 m^{-1}.

which includes Doppler broadening, leads to a more complicated sink function and a correspondingly more complicated source function. Aside from the rapid transition near 0.2 cm, both the shift of the centroid and the narrowing of the source relative to the sink are evident. Note that in this example, the effective center of the source is shifted about 2.5% of L from the resonance, still about half of the shift of the sink function, if one uses the mean values instead of the peaks.

V. CONCLUSIONS

The principal conclusions are that the classical formula for electron cyclotron emission (ECE) is validated for large laboratory plasmas (but not for plasmas in general), but that a new paradigm for describing and understanding the process is necessary. The necessity for the new paradigm arises from the complete physics of the several processes in the cyclotron harmonic layer, as opposed to the previous picture which included only transmission and absorption. The complete picture shows that only reflection and mode conversion are affected by absorption and that transmission is independent of absorption. This picture has been evident from the beginning for ion cyclotron harmonics, where for small k_z the absorption is weak, but transmission, reflection, mode conversion, and absorption have all been observed simultaneously. This picture of electron cyclotron absorption has been less evident because of the much stronger absorption due to weak relativistic effects which substantially reduce or eliminate reflection and conversion. In this limit, there is again only transmission and absorption, but there is a nontrivial range of parameters where reflection may make a substantial error in the estimate of emission from the classical formulas in smaller, cooler devices. The difficulty with ignoring the new paradigm or way of understanding the processes is, first, that it describes the physics incorrectly, and, second, that nontrivial errors in the interpretation of ECE may occur.

There is also a new way to find the shape of the source distribution function. The effects of mode conversion along with the subtle effects of indirect emission have shown that the traditional formulas for emission (but not absorption) from a cyclotron harmonic layer are virtually exact in the one case from the high-field side and that the effects of reflection are small for large tokamaks and many laboratory plasmas. The fact that the tunneling factor alone, which is independent of absorption, should reproduce the result for the optical depth, which was based only on absorption, must be regarded as an accident and is another example of cases when the right answer was obtained from either wrong or insufficient analysis. For the case for ion cyclotron harmonic emission, however, the accidents of electron emission do not occur; so the classical optical depth/opacity formulas are not appropriate. The fact that the indirect emission is not neces-

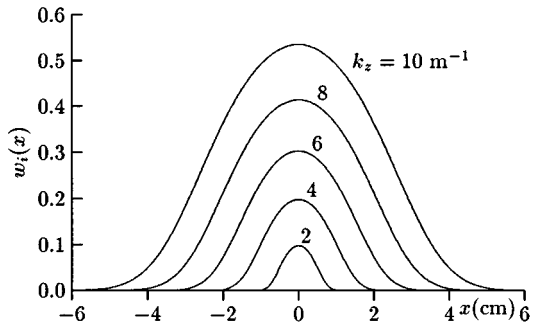


FIG. 15. Absorption function $w_i(x)$ with $n=2\times10^{20}$ m^{-3}, $B_0=5$ T, $L=2$ m, $T_i=2$ keV with $k_z=2$, 4, 6, 8, 10 m^{-1}.

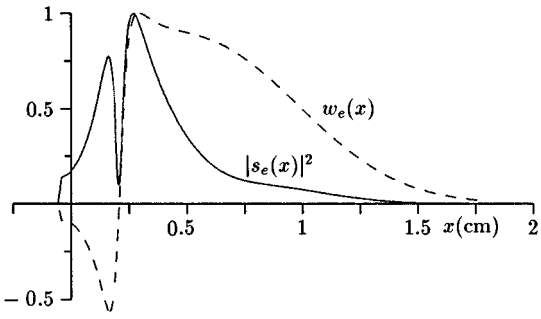


FIG. 17. Electron absorption function $w(x)$ and distributed source strength $|s(x)|^2$ with $n_e=2.7\times10^{17}$ m^{-3}, $B_0=0.1$ T, $L=0.1$ m, $T_e=5$ keV, and $n_\parallel=0.1$.

sarily near the harmonic layer requires the conversion coefficients as well as the reflection coefficient, and these are not always so small. The work could be extended to the two-ion hybrid resonances as well as the harmonic resonances; but whenever the nonthermal emission exceeds the thermal emission (which is not uncommon), the analysis of the mode conversion problem is probably fruitless. Even with pure thermal ion emission, one must know the effective temperature of the indirect emission source to be able to interpret the result, and this complicates the analysis of experimental data enormously.

The description of the emission source through the source distribution function could be useful in interpreting ECE data, since it more closely indicates the emission source location. While the shifts are not typically large, they are systematic and better than using estimates from the absorption function, which are typically off by a factor of 2 for ECE; the ion absorption function shows no shift whatsoever, while the source does.

The principal weaknesses of this analysis arise from the assumption that the finite Larmor orbit parameter, $\lambda = \frac{1}{2}k_\perp^2\rho^2$, is small and from the use of the tunneling equations derived from the dispersion relations rather than from the Vlasov equations. Both of these effects can be included, but for most cases the corrections will be predictably small. For example, including the next-higher-order terms in λ simply leads to a higher-order differential equation and a slightly modified tunneling factor. All of the analytic results remain unchanged except through the change in η, and this is sufficient for accurate estimates of emission and absorption. The more complicated tunneling equations obtained from the Vlasov equations have no effect on the tunneling factor and minimal effect on the reflection coefficient. Since the conversion coefficients cancel out in ECE, these corrections are not expected to produce any measurable error. The inherent assumption that the magnetic-field gradient is large compared to a wavelength, where, for example, $l = \omega L/c = 20$ roughly corresponds to $L \sim 3\lambda_\perp$, restricts the validity of the analysis to $l > 20$.

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