# Chaotic quantized vortices and hydrodynamic processes in superfluid helium

S. K. Nemirovskii* and W. Fiszdon†

*Max-Planck-Institut für Strömungsforschung, 37073 Göttingen, Germany*

Superfluid turbulence in He II flows appears as a stochastic tangle of quantized vortex lines. Interest in this system extends beyond the field of superfluid helium to include the study of statistical physics of extended objects and classical turbulence. A wealth of information concerning the vortex tangle has been supplied by experiments, by phenomenological models, and by numerical simulation. In the restricted case of stationary homogeneous flows, a good correlation between theory and experiment has already been demonstrated. This review therefore concentrates on nonstationary and nonhomogeneous processes with the goal of understanding the dynamical properties of the vortex tangle. The first part of this paper reviews the concept of the stochastic vortex tangle with emphasis on the work of Vinen and Schwarz. Special attention is given to justifying Vinen's equation as a basis for the study of nonstationary processes. This analysis reveals that the interpretation of experimental results requires a careful account of the mutual influence of the vortex tangle and the flow of the supefluid. The second part of this paper reviews the development of the required hyrodynamic equations of superfluid turbulence. These equations are applied to a number of examples such as linear and nonlinear sound, heat pulses, and fluctuations. The role of stochastic vortices in the phase-transition problem is discussed. Attention is focused on the relation between hydrodynamic effects and the vortex tangle properties. Finally, the main results concerning the stochastic vortex tangle are resumed and some of the important questions are pointed out.

## CONTENTS



## SYMBOLS AND ABBREVIATIONS

| | |
|---|---|
| $A(T)$ | Gorter-Mellink constant |
| $B,B'$ | Hall-Vinen coefficients |
| $b$ | coefficient of proportionality between $\mathbf{v}_L$ and $\mathbf{v}_{ns}$ |
| $E(\rho,S,\mathbf{j}_0,\mathbf{v}_s,L)$ | energy density of He II containing a VT |
| $\mathbf{F}_{ns}$ | friction force per unit of volume |
| $\mathbf{f}_D$ | friction force per unit length of vortex filament |
| HST | hydrodynamics of superfluid turbulence |
| $I_l,I_\perp,I_\parallel$ | parameters of anisotropy of the VT |
| $K$ | mutual friction coefficient for a "frozen" vortex tangle |
| $L$ | vortex line density, total quantum vortex line length per unit of volume |
| $\dot{Q}$ | heat flux density |
| $R$ | radius of the curvature of the vortex filament |
| $R_{\text{peak}}$ | most probable radius of curvature |
| $r_0$ | vortex core radius |
| ST | superfluid turbulence |
| $S,\sigma$ | entropy per unit volume and mass, correspondingly |

*Permanent address: Institute of Thermophysics, Russian Academy of Sciences, Novosibirsk, Russia.
†Permanent address: Polish Academy of Sciences and the University of Warsaw.

  

| | |
|---|---|
| $\mathbf{S}(\xi,t),\mathbf{S}(\tilde{\xi},t)$ | equation of the vortex line where $\xi$ is an arbitrary variable and $\tilde{\xi}$ is the arclength |
| $\dot{\mathbf{S}}_i$ | self-induced velocity of the vortex filament |
| $t_H$ | heat-pulse duration, time of heating |
| $t_R$ | rest time between heat pulses |
| $t_V$ | time of formation of a vortex tangle |
| $u_2$ | second-sound velocity |
| VE | Vinen's equation |
| VLD | vortex line density |
| VT | vortex tangle |
| $\mathbf{v}_L$ | drift velocity of a vortex tangle |
| $\mathbf{v}_l$ | self-induced velocity of a vortex line in the local approach |
| $\mathbf{v}_s,\mathbf{v}_n$ | superfluid and normal velocities, correspondingly |
| $\alpha,\alpha'$ | $(\alpha=\rho_n B/2\rho,\alpha'=\rho_n B'/2\rho)$ friction parameters |
| $\alpha_{\rm alt},\beta_{\rm alt}$ | coefficients in the alternative VLD evolution equation |
| $\alpha_{\rm Kh}(T)$ | coefficient of nonlinearity of second sound |
| $\alpha_s,\beta_s$ | coefficients in the Schwarz VLD evolution equation |
| $\alpha_V,\beta_V$ | coefficients in the Vinen VLD evolution equation |
| $\beta$ | coefficient in the local-approximation relation for self-induced velocity |
| $\gamma$ | coefficient of proportionality between $L$ and $v_{ns}^2$ |
| $\varepsilon_V$ | energy of a vortex line per unit of length |
| $\kappa=h/m$ | quantum of circulation |
| $\lambda$ | scaling parameter |
| $\mu$ | chemical potential |
| $\rho_s,\rho_n$ | superfluid and normal densities, correspondingly |
| $\tau_V$ | relaxation time of the VT |

## I. INTRODUCTION

This review is concerned with the very important and still open field of the theory of superfluidity—superfluid turbulence (ST). The concept of superfluid turbulence was introduced by Feynman (1955). He described superfluid turbulence as a disordered set of quantized vortex lines, called vortex tangle (VT), which appears in He II flows whenever the velocity exceeds a certain fairly small value.

Superfluid turbulence as a part of the theory of superfluidity is tightly connected with other topics of the general theory of superfluidity, e.g., the generation of vortices, the interaction between very closely spaced vortex lines and hence their reconnection, the problem of critical velocities, and the role played by quantum vortices in phase transitions. The study of ST yields often nonstandard solutions elucidating the above-mentioned problems.

The theory of ST is also important in many applied research problems of He II. Indeed, the presence of a vortex tangle (VT) strongly affects the heat flow, which can no longer be described by the simple Landau two-fluid model. The use of He II in large projects, such as the cooling of superconducting magnets or for space applications, requires adequate investigations. There is now renewed interest in the problems of interaction between classical turbulence and ST, in view of the use of helium as a test fluid in very high Reynolds number test facilities (see Donnelly, 1991b).

Besides the great importance of ST in the above-mentioned cases, the theory of the stochastic vortex tangle in He II is of great interest and importance form the standpoint of general physics. This is justified by the existence, in many physical fields, of similar systems of highly disordered sets of one-dimensional (1D) singularities, e.g., chaotic vortices in He 3 or in superconductors, where they are responsible for many of their physical properties. Polymer chains, linear defects in solids, and strings are other examples of disordered, 1D singularities. The study of the stochastical properties of similar objects (and also, e.g., chaotic sets of 2D singularities, such as membranes and triangular surfaces) belongs to the field of stochastical physics of extended objects. The vortex tangle of He II, which is formed by a stochastic complex of one-dimensional singularities conforming to the nontrivial nonlinear equation of motion with random changes of its topological structure, belongs undoubtedly to this class of objects.

It is possible that the use of the theory of stochastic vortex lines to clarify the perennial problem of classical turbulence would be of great value. Some physicists believe (see, e.g., papers of Chorin, 1982, 1991a, 1991b; Siggia, 1985; Agistein and Migdal, 1986; Hussain, 1986; Sethian, 1991b) that many features of classical turbulence can be treated using the model of interacting vortex tubes. There follows the assumption that the Kolmogorov spectrum is formed at the crossings of filaments, i.e., positions of strong singularities. It is also likely that the dissipation of the kinetic energy is not uniformly distributed in space, but occurs only at the instances of quantum vortex lines collisions and reconnections. The connection between the observed coherent structures in classical turbulence and the formation of clusters of vortex lines is also discussed. An additional asset of this model is the possibility of using effective numerical solution methods, as the dynamics of vortex filaments is described by a set of one-dimensional equations.

The theory of stochastic extended objects is far from being closed. The statement of the problem, as will be shown in Sec. II in the case of stochastic vortices in He II, is exceptionally complicated. Therefore any additional information concerning the structure and stochastic dynamics of vortex lines is very helpful and important. This is one of the main motivations of this review. The study of the VT in He II, carried out over many years, has yielded many important results leading to an understanding of the stochastic behavior of chaotic 1D singularities.

One of the aims of this paper, resulting from what we have presented above, is to review the properties of the stochastic VT in He II gained from the existing relations between the different elements appearing in the investigations of ST in He II, such as theoretical models, numerical simulations, and experimental findings.

Furthermore, a predominant part of experimental investigations of ST was obtained by means of purely hydrodynamic methods, with the exception of a few ion measurements. The simplest cases when the VLD was considered homogeneous in space and stationary in time were described in the excellent reviews of Tough (1982) and Donnelly and Swanson (1986) and in the monograph of Donnelly (1991a). The correlation between the concepts of vortex tangle and phenomenological theory and the numerical results based on a very large number of experimental observations of turbulent He II flows was there convincingly demonstrated. However, the problems of unsteady phenomena and processes were hardly touched in these publications. In the present review we would like to concentrate on nonstationary and nonhomogeneous processes. The study of nonstationary phenomena broadens greatly the possibilities for studying the dynamical properties of the vortex tangle. However, it is not only a question of adding some new observations, but also one of exploring some intrinsic properties of the macroscopic dynamics of ST which generally do not appear in homogeneous stationary problems. First of all, as we shall discuss in Sec. V, the study of such problems leads to serious questions concerning the adequacy of describing the macroscopic dynamics of a VT in terms of the single characteristic $L(t)$, even in the hydrodynamic problems. Furthermore, as opposed to the case of stationary homogeneous flows where the use of fixed flow parameters for the study of the vortex tangle is justified, in the nonstationary cases such an approach is not correct. Indeed, in such cases it is necessary to take into account the reciprocal influence of quantum turbulence on the evolution of the flow parameters, which in turn affect the dynamics of the vortex tangle. One can say that the intrinsic properties of the VT are strongly connected with pure hydrodynamic effects. Therefore, to interpret properly the experimental observations concerning the dynamics of the VT, it is necessary to use the full closed hydrodynamic description of the flow of He II in the presence of the VT, the so-called hydrodynamics of superfluid turbulence (HST). As will be shown later, in a number of cases this approach may modify some current views concerning the vortex tangle. We think that some widely known effects attributed usually to the dynamic properties of the vortex tangle (vortex anisotropy, anomalous decay, propagations of plugs, etc.) can be partially or fundamentally explained by HST.

In view of the stated aims, the review is divided into two parts. In the first part the problems concerned with the conception of the chaotic vortex tangle are reported. Section II describes the dynamics of vortex filaments. This can be regarded as the formulation of the problem of stochastic behavior of vortex lines. Section III describes Vinen's phenomenological theory of the macroscopic hydrodynamics of the VT in terms of the vortex line density $L(t)$—the total length of the VT per unit volume. Section IV exposes Schwarz's results on microscopic dynamics of disordered vortex filaments. Special attention is paid to Vinen's equation as a basis for studying nonstationary phenomena. The methods of its derivation and their justification are discussed. In spite of the large number of papers on superfluid turbulence, these questions have so far been neglected.

The second part is concerned with the study of nonstationary hydrodynamic processes affected by the presence of a superfluid vortex tangle. It begins with Sec. V, which summarizes different approaches for obtaining the equations of HST. Sections VI and VII are devoted to the methods most often for the investigation of ST with the help of linear and nonlinear sounds and intense heat pulses. Section VIII describes a number of other dynamical phenomena probably less related to the HST, but which indicate new directions of investigation of the VT or the clarification of the role played by vortices in other problems of the theory of superfluidity, in particular, in the phase-transition problem.

The closing section, IX, contains a summary of the reviewed results. This summary follows the main purpose of the review, i.e., the exploration of the relation existing between the stochastical dynamics of vortex lines and the experimentally confirmed hydrodynamic effects in superfluid He II. Special attention is paid to the relations existing between the properties of the VT and some problems of the theory of superfluidity, the theory of chaotic extended objects, and classical turbulence. In addition, trends in the research of superfluid turbulence are briefly discussed.

## II. VORTEX LINE DYNAMICS OF SUPERFLUID HELIUM

The properties and the dynamics of vortex lines in superfluid helium are described in this section. Although they may be well known, having in mind the formulation of the problem of stochastic behavior of the system, we would like to bring together the established results. This approach consists of a description of the equations of motion, including the interaction between the vortices and boundaries, if any, and their interaction with the normal component. This latter process is specific to superfluids. The considered problem also includes the reconnection process when two lines cross and reconnect, thus changing the topology of the system. Finally, the stochastical approach also requires a consideration of its origin, hence, of possible external random forces, instabilities, and so on.

The phenomenological theory of superfluidity developed by Landau (1941) was successful. This theory made it possible not only to explain a large number of unusual experimental observations, like the fountain effect, but also to predict new effects such as wavelike heat propagation. However, from its inception, this theory encountered fundamental difficulties. One difficulty was the inability to explain why, in spite of the absence of shear viscosity, the superfluid is entrained to rotate in a rotating container. The other difficulty was the explanation of the existence of a critical velocity, the process of losing its superfluid properties, and the appearance of dissipation mechanisms. Indeed, according to Landau's theory, in He II below the critical velocity of generation of rotons, about 60 m/s, the flow of the components must be frictionless. At the same time, as proved experimentally, at a much lower relative counterflow velocity $\mathbf{v}_{ns}=\mathbf{v}_n-\mathbf{v}_s$, a mutual friction between the two components appeared that resulted in a temperature gradient $\nabla T$ proportional, according to the Gorter-Mellink (1949) experiments, to $v_{ns}^3$. As we now know, both "failures" of Landau's theory are due to the existence of quantized vortices, i.e., one-dimensional topological singularities, of the superfluid component of He II.

### A. The Onsager-Feynman quantized vortices

Onsager (1949) and Feynman (1955) suggested that Landau's assumption of a rotationless flow of the superfluid component, $\omega(\mathbf{r})=\mathrm{rot}\mathbf{v}_s=0$, is violated on one-dimensional singularities $\mathbf{S}(\xi,t)$ that depend on the position parameter $\xi$ and the time $t$. This line singularity is shown in Fig. 1 where $\xi$ is the arc-length parameter. On these singularities $\mathrm{rot}\mathbf{v}_s\to\infty$, the velocity also increases to infinity so that the circulation $\kappa$ of the superfluid velocity about these lines remains constant,

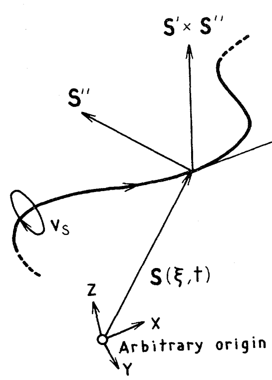


FIG. 1. Space curve representing a vortex line with its position described as $\mathbf{S}(\xi,t)$, when $\xi$ is arclength. $\mathbf{S}'=d\mathbf{S}/d\xi$ is a unit vector along the vortex line; $\mathbf{S}''=d^2\mathbf{S}/d\xi^2$ is the local curvature vector (whose magnitude is $1/R$); and $\mathbf{S}'\times\mathbf{S}''$ is binormal, which also has the magnitude $1/R$ (Donnelly, 1991, Fig. 1.14).

$\kappa=h/m_{\mathrm{He}}=9.97\times10^{-4}$ cm$^2$/s, where $h$ is Planck's constant, and $m_{\mathrm{He}}$ is the mass of the helium atom. This assertion can be written formally as

$$\omega(\mathbf{r})=\mathrm{rot}\mathbf{v}_s=\kappa\int d\mathbf{S}\delta(\mathbf{r}-\mathbf{S}(\xi,t)) , \tag{2.1}$$

where the integration is along the line singularity $\mathbf{S}(\xi,t)$. We call this line a vortex line. The explanation of this outstanding property is based on the fundamental quantum-mechanical properties of superfluid helium described in many handbooks [one of them is, of course, Feynman's (1972) handbook on statistical physics].

It follows from Eq. (2.1) that a single infinite straight line induces a velocity field, described in cylindrical coordinates by

$$\mathbf{v}_s=\left[0,\frac{\kappa}{2\pi r},0\right] , \tag{2.2}$$

having only an azimuthal component which increases rapidly on approaching the vortex axis. In particular, at $r\leq3$ Å, the velocity $v_s$ becomes greater than the velocity of roton generation, i.e., Landau's critical velocity.

Hence the relations (2.2) and (2.1) are invalid at small distances from the vortex line of the order of a few Å. At such distances the vortex line hypothesis, as expressed by (2.1), fails and must be replaced by an adequate theory. This is important, from the point of view of our review, because the small scale structure of the vortex line influences strongly its dynamics and particularly the reconnection process.

At present no exact microscopic theory of vortex lines in He II exists. The essential information about the vortex structure obtained mainly in numerous experiments can be found in the recent book of Donnelly (1991a). However, as in other statistical theories, where the stochastical characteristics are not very sensitive to details of the interaction between particles, similarly in the case of vortex lines their structure affects only slightly the average properties of the vortex tangle. In this sense the main results of interest in our case can be taken from the theory of a weakly nonideal Bose gas (see Ginzburg and Pitaevskii, 1958; Pitaevskii, 1958; Gross, 1961; and Lifshitz and Pitaevskii, 1980). Although He II cannot be considered as a weakly nonideal Bose gas, this theory allows us to understand some intrinsic properties of superfluids and to resolve some particular problems such as the nonuniform state near the wall or a free surface. In particular, this theory describes some special objects identical to vortex lines in He II. According to this theory, the state of the Bose-Einstein condensate, which in many respects is identical to the superfluid component of He II, can be described by the "macroscopic" wave function

$$\psi(\mathbf{r},t)=\sqrt{n_0(\mathbf{r},t)}\exp[i\phi(\mathbf{r},t)] , \tag{2.3}$$

where the density of the condensate $n_0(\mathbf{r},t)$ is related to the density of the superfluid component but is not identical. The superfluid velocity can be easily obtained from the quantum-mechanical relation between the current density and the wave function leading to

$$\mathbf{v}_s = \frac{\hbar}{m}\nabla\phi \ . \tag{2.4}$$

This relation shows that the motion of a Bose condensate or of the superfluid component is related to the nonuniformity of the phase $\phi(\mathbf{r})$. In particular, states with a monotonic variation of the phase around a line which corresponds to a circular motion of the Bose condensate are possible. Since the wave function is single valued, its change of phase, on returning to the starting point, is $2\pi$; hence

$$\oint \mathbf{v}_s dl = \kappa \ . \tag{2.5}$$

The integral is calculated along an arbitrary closed curve around the selected vortex line. Relation (2.5) is equivalent to (2.1) according to the vortex line hypothesis in He II. Unlike this hypothesis, the theory of a weakly interacting Bose gas predicts the microscopic structure of the vortex lines. The macroscopic wave function $\psi(\mathbf{r},t)$ satisfies the Gross-Pitaevskii equation

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar^2}{2m}\Delta\psi + U_0(|\psi|^2 - n)\psi \ , \tag{2.6}$$

where $U_0$ is the effective potential and $n$ is the particle density. In the stationary case, (2.6) describes, in particular, a straight-line vortex. The corresponding solution is

$$\psi = \sqrt{n}\exp(i\varphi) f(r/r_0) \ , \tag{2.7}$$

where $\varphi$ is the azimuthal angle about the axis and $r_0 = \hbar/\sqrt{2mnU_0}$. At $r \gg r_0$, the flow field with a constant condensate density and a velocity given by (2.2) takes place. At $r \le r_0$, the condensate density decreases, which results in suppression of the superfluid component on the vortex axis. In this model the quantity $r_0$, called the core radius, is larger than the interatomic distances. Nevertheless, it is usually assumed that the picture is similar for quantum vortices in real He II, with the difference that $r_0$ is of the order of the distance between particles. From different indirect experiments, it appeared that $r_0 = 1-2$ Å. However, close to the $\lambda$ point, $r_0$ grows rapidly and its characteristic length is of the order of the correlation length as described by Ginzburg and Sobyanin (1976).

Hills and Roberts (1977, 1978a, 1978b) succeeded in describing the structure of a vortex line without using the model of a weakly interacting Bose gas. In their phenomenological theory the superfluid density $\rho_s$ is used as an independent variable. Let us recall that in Landau's theory $\rho_s$ was considered to be an equilibrium function of the entropy $S$, density $\rho$, and relative velocity $\mathbf{v}_{ns}$. Hills and Roberts obtained an expression for the free energy as a function of all the thermodynamic variables. By minimizing the free energy, a set of differential equations for the density of the superfluid component can be obtained with parameters that can be deduced from thermodynamic data. In this way Hills and Roberts were able to obtain a solution for a single vortex line. Qualitatively, this solution is similar to the solution of the Gross-Pitaevskii equation; however, the core radius $r_0$ obtained from this theory is close to the experimentally found value.

### B. Vortex line dynamics

The notions just introduced concerning vortex lines can now be used to describe their dynamic behavior. At distances not very close to the vortex line, its equation of motion resulting from (2.1) and the condition $\nabla\mathbf{v}_s = 0$ following Biot-Savart's law is

$$\dot{\mathbf{S}}_i(\xi,t) = \frac{\kappa}{4\pi}\int \frac{[\mathbf{S}(\xi',t) - \mathbf{S}(\xi,t)]\times\mathbf{S}'_{\xi'}}{|\mathbf{S}(\xi',t) - \mathbf{S}(\xi,t)|^3} d\xi' \ , \tag{2.8}$$

where the integral should be taken along the line $\mathbf{S}(\xi',t)$ and the first argument is an arbitrary parameter not necessarily the arc length. The above integral diverges when $\mathbf{S}(\xi',t) \to \mathbf{S}(\xi,t)$, which requires additional conditions to be imposed on $\dot{\mathbf{S}}_i(\xi,t)$. This problem is closely connected with the small scale structure of the vortex line.

Let us recall how this problem is treated in the case of vortex tubes of classical fluids. As the integral (2.8) diverges logarithmically, one of the simplest, widely used approaches is to introduce a finite core radius $r_0$ and to replace the denominator in (2.8) by

$$|\mathbf{S}(\xi') - \mathbf{S}(\xi)| \to [(\mathbf{S}(\xi') - \mathbf{S}(\xi))^2 + r_0^2]^{1/2} \ .$$

The velocity distribution inside the core is calculated from either the Navier-Stokes or Euler equations, taking into account the pressure distribution inside. The resulting classical vortex tubes are assumed to have either a uniform vorticity distribution or a Gaussian distribution, or to be just a hollow tube. Frequently, for convenience, in the case of numerical modeling, such approximate distributions are assumed. A description and comparison of different methods used for smoothing the kernel in the Biot-Savart formula (2.8) for vortex tubes of classical liquids are given by Sethian (1991a). Different models of the vortex core result in different velocity fields $\dot{\mathbf{S}}_i(\xi,t)$; but they are close to each other, and the difference between them decreases logarithmically with the core radius as $1/\ln r_0$. Therefore we can use, for example, the cutoff parameter according to Hills and Roberts (1978b). This approximate procedure, which does not require the solution of the corresponding quantum-mechanical problem, is valid only for the determination of the velocities of the elements of the vortex line. However, the use of the above procedure is not justified for predicting the behavior of two vortex elements getting very close to each other. Moreover, no analogy with the case of classical vortex tubes can be used to handle this problem.

The second problem connected with the cutoff in Biot-Savart's integral (2.8) is the behavior of the core radius $r_0$

in the stretching process of the vortex lines. In most papers dealing with classical vortices, the conservation of the volume of the vortex tube is assumed. This amounts to a restriction of the relation between $r_0$ and the parameter $\xi$ to $r_0 \sim 1/|\partial \mathbf{S}/\partial \xi|$. Of course this restriction cannot be imposed on $r_0$ in the case of He II because, on quantum-mechanical grounds, the vortex core radius remains constant. On the other hand, in Agistein and Migdal's (1986) paper, it was pointed out that the introduction of a cutoff radius depending on the parameter $\xi$ violates the energy conservation law as well as the parametric invariance of the equation of motion of the vortex line. By parametric invariance, we understand the equivalence of the description of the vortex line dynamics making a transformation from the variable $\xi$ to another quantity depending monotonically on $\xi$. According to Agistein and Migdal (1986), this is too high a price to pay for a phenomenological rule of conservation of the volume of the vortex tube.

Developing $\mathbf{S}(\xi',t)$ close to $\mathbf{S}(\xi,t)$, in the Biot-Savart law, the integral in (2.8) becomes

$$\dot{\mathbf{S}}_i(\xi,t)=\frac{\kappa}{4\pi}\frac{\mathbf{S}'\times\mathbf{S}''}{|\mathbf{S}'|^3}\int\frac{d(\xi-\xi')}{(\xi-\xi')}+\text{nonlocal terms} . \tag{2.9}$$

This integral diverges logarithmically at both the upper and the lower limits of integration. As far as the low limit is concerned, the divergence can be dealt with as in the full Biot-Savart law (2.8) by introducing the radius $r_0$ of the vortex core. As for the upper limit, the radius of curvature $R$ of the vortex line at the point $\xi$ seems to be the appropriate limit. In the case of a deterministic motion, this quantity is more or less known. In the problem of a stochastic tangle, where the curvature fluctuates in a complicated way along the line, this problem is more involved and the cutoff at the upper limit is not a fixed parameter but, to the contrary, depends on the solution. In particular, it can be chosen as the average radius of curvature $\langle R \rangle$, which is of the order $L^{-1/2}$. The logarithmic dependence of the cutoff on the solution was observed in the experiments of Swanson and Donnelly (1985). Nevertheless, due to the weak logarithmic dependence of the vortex tangle structure on integration, the cutoff limits are frequently assumed to be constant. Finally,

$$\dot{\mathbf{S}}_i(\xi,t)=\beta\frac{\mathbf{S}'\times\mathbf{S}''}{|\mathbf{S}'|^3}+\text{nonlocal terms} , \tag{2.10}$$

where $\beta=(\kappa/4\pi)\ln(\langle R \rangle/r_0)$. Let us recall that $(\mathbf{S}'\times\mathbf{S}'')/|\mathbf{S}'|^3$ is directed along the binormal, and its value is equal to the curvature at the considered point (see Fig. 1). As for the nonlocal terms, according to the procedure used they are by the order of magnitude of $\ln(\langle R \rangle/r_0)$ smaller than the local term. The description of the dynamics of the vortex line using (2.10) neglecting nonlocal terms is called the local approach (LA).

As a vortex tangle in superfluid helium $\ln(\langle R \rangle/r_0)$ is typically of the order of 10, Schwarz (1988) notes that a certainty within 90% of the local approximation exists, except in the cases when the two lines are very close to each other or a line is close to a boundary. However, the problem is more involved, because the neglect of small nonlocal terms is accompanied by the discarding of the very important process of stretching the vortex lines due to nonlocal effects. We shall come back to this problem in Sec. IV.

The velocity of the vortex line, $\dot{\mathbf{S}}_i$, can vary under the influence of an external flow, $\mathbf{v}_s$. Another obvious correction must be made when the vortex line approaches a boundary. A correction due to the induced velocity $\mathbf{v}_{s,b}$, which depends on the shape of the boundary and, in particular, on its roughness, appears and must be taken into account. The vortex line velocity, taking into account these corrections, is

$$\dot{\mathbf{S}}_s=\dot{\mathbf{S}}_i+\mathbf{v}_s+\mathbf{v}_{s,b} . \tag{2.11}$$

The next factor determining the vortex line dynamics, which we shall now consider, is the mutual interaction between the quantum vortices and the normal component. This is specific for He II, and there is no analogy in the theory of vortex tubes in classical fluids. As described by Khalatnikov (1965), for example, the motion of the normal component with velocity $\mathbf{v}_n$ is equal to the drift of quasiparticles—phonons and rotons that form this component. The energy of these quasiparticles is a function of $\mathbf{v}_s$, and therefore it is a strongly varying function close to the vortex line. In other words, there exists an effective potential describing the interaction between the quasiparticles and the vortex line. Hence during the relative motion a momentum transfer results between the quasiparticles and the vortex line. Thus an interaction force, called mutual friction, appears. The corresponding theory is fully described in many reviews (see, e.g., Barenghi *et al.*, 1983; Sonin, 1983) and in Donnelly's (1991a) book. We give here only the result, important for the dynamics of vortex lines, and make some comments. The force $\mathbf{f}_D$ acting on a unit length of the vortex line is

$$\mathbf{f}_D=D_1\frac{\mathbf{S}'}{|\mathbf{S}'|}\times\left[\frac{\mathbf{S}'}{|\mathbf{S}'|}\times(\mathbf{v}_n'-\dot{\mathbf{S}})\right]+D_2\frac{\mathbf{S}'}{|\mathbf{S}'|}\times(\mathbf{v}_n'-\dot{\mathbf{S}}) , \tag{2.12}$$

where $\dot{\mathbf{S}}$ is the velocity of the vortex line, which is not identical to the previously introduced $\dot{\mathbf{S}}_s$ precisely because of the existence of the force $\mathbf{f}_D$.

Many papers have been published concerning the calculation of the coefficients $D_1$, $D_2$ (see, e.g., Barenghi *et al.*, 1983; Donnelly, 1991a). There are also many works concerning different methods of their experimental determination. These coefficients depend not only upon the thermodynamic quantities like pressure and temperature, but also on the velocities (Swanson *et al.*, 1987) and on their time derivatives, $\partial\mathbf{v}_n/\partial t$ and $\partial\mathbf{v}_s/\partial t$ (Mehl, 1974); i.e., they are nonlocal in time. This creates some uncertainty concerning (2.12), but luckily these latter effects are small and can be considered as corrections. We shall assume further that the coefficients $D_1$, $D_2$ are phenomenological constants.

The quasiparticle drift velocity close to the vortex lines is denoted $\mathbf{v}'_n$, and it must be different from $\mathbf{v}_n$, which in the average over a certain volume. Details can be found, for example, in Barenghi *et al.* (1983). Following others, we shall assume $\mathbf{v}_n = \mathbf{v}'_n$, assuming that the difference between these quantities is absorbed in the coefficients $D_1$ and $D_2$. The so-far unknown quantity $\dot{\mathbf{S}}$, which, as already stated, is not equal to $\dot{\mathbf{S}}_s$, appears also in (2.12). To find $\dot{\mathbf{S}}$ it is necessary to compare the force $\mathbf{f}_D$ with the force $\mathbf{f}_M$ due to the superfluid component acting on the vortex line when its velocity $\dot{\mathbf{S}}$ differs from $\dot{\mathbf{S}}_s$. This method was proposed by Hall and Vinen (1956a, 1956b), and the force acting on the superfluid component, called the Magnus force, is

$$\mathbf{f}_M = \rho_s \kappa \frac{\mathbf{S}'}{|\mathbf{S}'|} \times (\dot{\mathbf{S}} - \dot{\mathbf{S}}_s) \ . \tag{2.13}$$

Comparing $\mathbf{f}_M$ and $-\mathbf{f}_D$, which is justified when there are no mass effects during the vortex motion, and neglecting the velocity, $\mathbf{v}_{s,b}$, induced by the boundary effects results in

$$\dot{\mathbf{S}} = \dot{\mathbf{S}}_i + \mathbf{v}_s + \alpha \frac{\mathbf{S}'}{|\mathbf{S}'|} \times (\mathbf{v}_{ns} - \dot{\mathbf{S}}_i) - \alpha' \frac{\mathbf{S}'}{|\mathbf{S}'|} \times \frac{\mathbf{S}'}{|\mathbf{S}'|} \times (\mathbf{v}_{ns} - \dot{\mathbf{S}}_i) \ . \tag{2.14}$$

The coefficients $\alpha$ and $\alpha'$ can be expressed by the previously used coefficients $D_1$ and $D_2$ (see, e.g., Schwarz, 1978). They are also related to Hall and Vinen's $B, B'$ coefficients by the relations $\alpha = \rho_n B / 2\rho$, $\alpha' = \rho_n B' / 2\rho$. These quantities, important for future investigations, are given in the table in Sec. IV.B.

Equation (2.14) is the basic equation used to describe different problems connected with the motion of vortex lines. In particular, the problem of a stochastic vortex tangle should be investigated using this equation. The self-induced velocity $\dot{\mathbf{S}}_i$, on the right-hand side of (2.14), can be expressed in terms of the complete Biot-Savart law (2.8) or with the singled-out local part, Eq. (2.9). The alternative use of these relations will be discussed later when Schwarz's theory is described.

The process of greatest influence on the vortex tangle evolution, also to be considered, is the reconnection of vortex lines. This remarkable and scarcely studied process also appears in other extended objects such as polymers, defects in solids, etc. During their motion, the vortex lines unavoidably cross each other, and the problem of what happens when this occurs is of crucial importance for the structure and dynamics of the vortex tangle. Feynman, in his famous, fundamental (1955) paper, predicted that the crossing of two vortex lines would be accompanied by a reconnection process. This is shown schematically in Fig. 2. Lines at crossing break and reconnect with their next neighbors, thus changing the topology of the flow. A number of papers, according to Sethian's (1991a) review, deal with this problem in the case of vortex tubes of classical fluids. These results can be used partly to describe or model vortex reconnection in superfluid helium. The reconnection problem can be divided into two parts. One part describes motions of the lines in their evolution process when they approach each other up to the point where their mutual influence on the velocity of their motion becomes larger than the self-induced velocity due to the local curvature. But, at the same time, they are not close enough to influence the flow inside the vortex cores. The quantitative criterion is $r_0 < \Delta \leq R / \ln(R/r_0)$, where $\Delta$ is the distance between the lines. In these cases the evolution of the approaching segments can be followed according to Biot-Savart's law (2.8), where the cutoff radius $r_0$ either is constant (Schwarz, 1985; Tsubota *et al.*, 1992) or depends on the label $\xi$ (Siggia, 1985; Pumir and Siggia, 1987). Qualitatively, the results of these investigations are very close and can be described according to Siggia's (1985) model as follows. Due to the long-range interaction in Biot-Savart's integral, cusps may appear on the approaching and approximately antiparallel segments of two vortex lines. The curvature of these cusps may be so large that the self-induced velocity [see Eq. (2.10)] of each perturbation overcomes the repulsion from the adjoining vortex line. Further, the cusps grow and approach each other closer, thus increasing their curvature and correspondingly their self-induced velocities; this process continues faster and faster. It is important that this process grow explosively, since the distance between the two perturbed segments, $\Delta$, decreases according to the relation $\Delta \sim (t^* - t)^{1/2}$, where $t^*$ is some quantity depending on the relevant parameters and initial conditions. Thus, after a finite time, the vortex lines collapse. It is very important that the time of collapse is much smaller than the characteristic time of the motion of the other elements of the vortex line. Schwarz (1985) describes a similar behavior of vortex lines and of a vortex line and its image close to a boundary. In this way antiparallel vortex lines (a vortex line and its image), whenever they are at a distance $\Delta \leq R / \ln(R/r_0)$, suffer a rapid approach and collapse. Initially arbitrarily oriented vortex lines, as shown also by Schwarz, when they approach each other closer than some critical distance, start by reorienting their

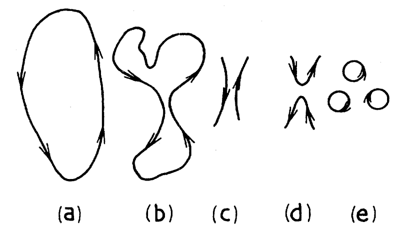


FIG. 2. Reconnection process (Feynman, 1955, Fig. 10): (a) initial stage; (b) and (c) stages of collapse; (d) reconnection stage; (e) stage of degeneration of vortex rings into thermal excitations.

close segments so as to bring them into an antiparallel position. This is followed by the collapse described above.

The second part of the process starts when the distance between the vortex lines $\Delta$ is comparable to the radius of the core $r_0$. In this case the induced velocities calculated from the Biot-Savart integral distort the flow inside the core. It then becomes necessary to solve the full Navier-Stokes or Euler equations. The full investigation of the classical vortex tube by Melander and Hussain (1989) and by Kida *et al.* (1991) indicates an extremely complicated picture of the vortex interaction. Instead of a full annihilation, some very involved structures, so-called bridgings, appear where strong dissipation effects take place. As far as helium is concerned, when atomic scales come into play it is necessary to find solutions using either the model of a weakly interacting Bose gas or Hills and Roberts's (1978b) theory. We are interested to know, first, if a full annihilation of the two antiparallel line segments occurs, as this is necessary for reconnection, and, second, how much time this process requires, if it takes place at all. The answer to the first question is positive, as shown by Nakajima *et al.* (1978) using a numerical modeling of the Gross-Pitaevskii equation (2.6): the approaching antiparallel quantized vortices are completely annihilated. Frisch *et al.* (1992) obtained a similar result for the collapse and annihilation of vortex rings. It was also shown that the duration of this process is very short compared with the characteristic time of the dynamics of the vortex tangle. This information about the short duration of the reconnection process is very useful, because it justifies the assumption used by Schwarz (1982b, 1985, 1988) and Buttke (1988) that the reconnection process is instantaneous.

Instantaneous reconnection is, of course, a great simplification of the problem. Even so, the problem remains extremely complex in spite of the simplifications made in obtaining Eq. (2.14). Indeed, this equation describing the dynamics of the vortex line motion is substantially nonlinear with several kinds of couplings; and the nonlinearity is neither polynomial nor even nonanalytical due to the existence of denominators of the type $1/|\mathbf{S}'|$. The equation also contains nonlocal terms expressed by the Biot-Savart law. The presence of mutual friction terms leads to the violation of some conservation laws, for example, the conservation of energy. Finally, the reconnection process changes the topology of the system; hence the quantity $\mathbf{S}(\xi,t)$ as a function of the parameter $\xi$ during the collision process receives and stores discontinuities which accumulate during the stochastic process of development of the vortex structure. To appreciate the complexity of the problem, we would like to point out what was shown by Hasimoto (1972): if in (2.14) all terms except the first are omitted, the local approach is used, and, finally, the local length is fixed, $|\partial\mathbf{S}/\partial\xi|=1$, Eq. (2.14) can then be reduced to the nonlinear Schrödinger equation. The stochastical behavior of the nonlinear Schrödinger equation is a very nontrivial problem which is the subject of intensive study (see, e.g., Shen and Nicholson, 1987; Lebowitz *et al.*, 1988; Dyachenko *et al.*, 1992). Nevertheless, some progress has been made in the study of the stochastic behavior of vortex lines in He II, which is described in the next two sections.

## III. THE FEYNMAN-VINEN MODEL OF SUPERFLUID TURBULENCE

Due to the great complexity of Eq. (2.14), there is practically no adequate theory concerning the stochastical dynamics of vortex lines in He II. The greatest success was gained through the phenomenological theory of superfluid turbulence. This theory, based in its original form on Feynman's (1955) qualitative considerations, was developed by Vinen (1957c). Vinen's theory has occupied during many years an important place in supplying an explanation of the processes of superfluid turbulence and was used for their quantitative interpretation. As already mentioned, this theory bridges, to some extent, the gap between the model of the chaotic vortex tangle and the experimental observations. In this section Feynman's phenomenological model and Vinen's likewise phenomenological theory are exposed and their relation with the dynamics of vortex lines discussed.

### A. Feynman's qualitative model

One of the many unusual effects encountered in the early stages of research of superfluidity was that of extremely high heat conductivity [see Keesom (1936); Keesom *et al.* (1938)]. This effect, sometimes called thermal superconductivity, refers to the observed, very high ($10^6$ times larger than that for He I) coefficient of proportionality between the temperature drop in a capillary filled with He II and the heat flux. The two-component hydrodynamic theory attributes this to the counterflow of the normal and superfluid components (Fig. 3) where heat is convected by the normal component only. This takes place at very small heat fluxes, of the order of 1 mW cm$^{-2}$, after which the linear law $Q \propto \nabla T$ is no longer valid. It is very important to note,

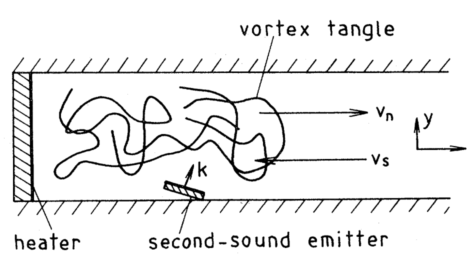


FIG. 3. Turbulent counterflow in He II. The normal component flows from the heater carrying a heat flux $Q=STv_n$; the superfluid component flows toward the heater. Total mass flux density $j=\rho_n v_n+\rho_s v_s=0$. The second-sound emitter for studying counterflow parameters is also shown (see Sec. VI).

however, that beyond this critical velocity the two-fluid model remains valid but becomes more involved. In particular, Gorter and Mellink (1949) have shown that an amplification of a temperature difference at the ends of a capillary retained counterflow, but some additional interactions between the two components occur. To overcome the resulting resistance, it is necessary to apply larger temperature drops; the heat superconductivity is greatly violated and, instead of a linear dependence between $\nabla T$ and $Q$, there follows a parabolic dependence

$$\nabla T=\frac{A(T)\rho_n}{\sigma^4\rho_s^3T^3}\dot{Q}^3 , \tag{3.1}$$

where $\dot{Q}=\rho_s S T v_{ns}/\rho$ is the heat flux density and the temperature-dependent quantity $A(T)$ is the Gorter-Mellink constant. The experimental values of this constant are shown in Fig. 19. The definitions of other symbols are given in the list of symbols.

In his famous (1955) paper, Feynman considers the experimentally observed existence of critical velocities and the appearance of mutual friction as follows. He assumes that in a helium flow—for example, a counterflow—quantized vortices will appear when a critical velocity $v_{ns}$ is exceeded, as in the case of rotating helium (the cause and processes of vortex formation were not considered). Unlike the case of rotating helium, where, due to symmetry, the alignment, of the generated vortices, or the ones coming from the wall of the container, is along the axis of rotation, the vortices in a channel flow will be arbitrarily oriented and twisted. Feynman further assumes the following scenario for the vortex structure evolution. As appears from the equation of motion of vortex lines, considered in the previous section, segments of the bent vortex line, moving with a velocity different from the local velocity of the superfluid component, are exposed to a Magnus force directed perpendicularly to the tangential vector. There follows a variation of the curvature and hence of the length of the considered segment. Depending on the orientation and curvature and on the counterflow velocity $v_{ns}$, stretching or shrinking of the segment is possible.

Feynman assumed that stretching of the lines prevails, i.e., that the length of the evolving vortex line, *on the average, grows.* While increasing their length, the lines more densely fill in the volume of the liquid, and the processes of interaction between the vortex lines become of greater importance. Feynman proposed to describe the interaction resulting from the vortex line crossing as an instantaneous reconnection. It can be seen that Feynman perspicaciously guessed the process which much later (see next section) was obtained by exact calculations. As a result of reconnection, a fusion of small vortex rings into larger ones, as well as a breakup into smaller ones, is possible (see Fig. 2). There follows again an assumption that the last property dominates; i.e., *on the average, a breakup of the vortex loops takes place.* This leads to a cascadelike process of formation of smaller and smaller loops. When the scale of the small rings becomes of the order of the interatomic distances, which is the final stage of the cascade, the vortex motion degenerates into thermal excitations. In some sense Feynman identified the thermal excitations-rotons with the microscopic small vortex rings. Onsager called rotons "ghosts of the disappearing rings." Frisch *et al.* (1992) showed in their paper, in which they studied numerically the solution of the Gross-Pitaevskii equation, that small vortex rings collapse and radiate phonons. This correction is not important from the standpoint of the macroscopic VT, as the phonon-roton equilibrium in He II is very quickly restored (see Khalatnikov, 1965). Thus reduction of the total length of the vortex lines and the transformation of the energy of the vortices, which was initially drained from the main flow, into thermal excitations take place. This decrease of the total length, at a sufficiently high density of the vortex tangle, compensates the growth process due to the mutual interaction with the normal component. Thus an equilibrium state, characterized by the total length of the vortex line, which is a function of the thermodynamic variables and the quantity $v_{ns}$, is reached. The structure of the vortex line is an intricate tangle whose chaotic dynamics is determined by the considered processes. Feynman called this intricate state superfluid turbulence.

## B. Vinen's phenomenological theory

Feynman's qualitative model was developed further in the classical works of Vinen (1957, 1958). He formulated these ideas in quantitative relations and, in particular, obtained the equation bearing his name which gives a quantitative description of the macroscopic dynamics of the VT, i.e., of the evolution of the total length of the vortex lines per unit volume $L(t)$. We shall present Vinen's theory following up how the microscopic dynamic vortex line motion laws lead to the macroscopic relations of superfluid turbulence.

Vinen considered homogeneous superfluid turbulence. Homogeneous turbulence can exist only in the case when the characteristic interline spacing, $\delta\sim L^{-1/2}$, is much smaller than the characteristic size, $D$, of the system. In his final version Vinen introduced corrections connected with the finite dimension $D$ of the container. However, we shall not consider these effects here. Following Vinen, we shall consider the case of He II counterflow due to a heat source (Fig. 3) characterized by a steady counterflow velocity $v_{ns}$. The quantity $v_{ns}$ appears as an external fixed parameter that remains constant during the evolution of the vortex structure. Vinen's aim was to obtain an evolution equation $L(t)$ for the VLD according to the processes described by Feynman. It was assumed that the time derivative $dL/dt$ is composed of two terms corresponding exactly to Feynman's qualitative model,

$$\frac{dL}{dt}=\left[\frac{\partial L}{\partial t}\right]_{\text{gen}}-\left[\frac{\partial L}{\partial t}\right]_{\text{dec}} . \tag{3.2}$$

The first term on the right-hand side of (3.2) corresponds to the growth of the VLD due to mutual friction; the second corresponds to a decay due to the breakup of the vortex rings. To find the form of the two components, Vinen used dimensional considerations, the known results concerning the dynamics of single vortex lines, and the analogy with classical turbulence. It was assumed that the term $(dL/dt)_{\rm gen}$ depends on $L$ and on the mutual interaction force between the vortex line and the normal component. Vinen made the very important assumption that $(\partial L/\partial t)_{\rm gen}$ is a function only of the instantaneous value of $L$, the friction force $\mathbf{f}$, and the circulation $\kappa$. This property was called the self-preserving state. Admitting the possibility that $(\partial L/\partial t)_{\rm gen}$ can depend on the previous history of the tangle development, Vinen nevertheless used the self-preserving principle by analogy with classical turbulence. Another assumption was that the dependence of $\mathbf{v}_{ns}$ is included in the friction force $\mathbf{f}$, which for single vortex lines must appear in the relation $\mathbf{f}/\rho_s\kappa$. Dimensional analysis leads to the equation

$$\left[\frac{\partial L}{\partial t}\right]_{\rm gen}=\kappa L^2\phi_{\rm gen}\left[\frac{\mathbf{f}}{\rho_s\kappa L^{1/2}}\right], \tag{3.3}$$

where $\phi_{\rm gen}$ is some dimensionless function of its argument. The determination of this function is one of the most delicate problems of the phenomenological theory. Vinen essentially made use in this case of the results of his and Hall's study (Hall and Vinen, 1956a, 1956b) on the dynamics of a single vortex ring oriented transversally with respect to the He II counterflow velocity. It was found in these investigations that the rate of growth of the radius, $b$, of the vortex ring and hence its length are linearly related to the force $\mathbf{f}$ according to the relation

$$\frac{db}{dt}=\frac{|\mathbf{f}|}{\rho_s\kappa}. \tag{3.4}$$

Assuming that this linear dependence of $\mathbf{f}$ remains valid in the case of a vortex tangle, Vinen concluded that

$$\left[\frac{\partial L}{\partial t}\right]_{\rm gen}=\alpha_V|\mathbf{v}_{ns}|L^{3/2}, \tag{3.5}$$

where $\alpha_V$ is proportional to the friction coefficient $\alpha$ [see Eq. (2.14)] and the coefficient of proportionality of order 1 has to be determined experimentally. Hall (1960), in his review on vortices in helium, give the following interpretation of (3.5). By assuming that the characteristic average curvature radius $R$ is of the order of the distance between the vortex lines $\delta$, which is turn is of order $L^{-1/2}$, and substituting $b\sim L^{-1/2}$ in (3.4), $(\partial L/\partial t)_{\rm gen}$ can be obtained directly, as shown in (3.5). The hypothesis that the average radius of curvature of the vortex line is of the order of the average distance between vortex lines was used by many authors in their qualitative analysis of *ST*.

The form of the $(\partial L/\partial t)_{\rm dec}$ term, responsible for the vortex decay, was determined by Vinen in analogy with classical turbulence. He assumed that Feynman's model of vortex breakup is analogous to Kolmogorov's cascade in classical turbulence. Under the theory of homogeneous isotropic turbulence, the energy dissipation connected with the cascade is given by

$$\frac{\partial u^2}{\partial t}=-\chi\frac{u^3}{l_{\rm visc}}, \tag{3.6}$$

where $u$ is a characteristic velocity of the pulsation of the length scale $l_{\rm visc}$ corresponding to the viscous limit of the inertial range (see Monin and Yaglom, 1975). Taking for $l_{\rm visc}$ in (3.6) the intervortex distance $\delta\sim L^{-1/2}$ and for $u$ the azimuthal velocity around a vortex line at a distance $\delta$ which is equal to $\kappa/2\pi\delta=(\kappa/2\pi)L^{1/2}$, Vinen obtained

$$\frac{dL}{dt}=-\chi_2\frac{\kappa}{2\pi}L^2=-\beta_VL^2, \tag{3.7}$$

where $\beta_V$ is the second parameter in Vinen's theory of the order of magnitude $\kappa/2\pi$. With the addition of the growth and decay processes, there follows immediately the relation called Vinen's equation (VE),

$$\frac{dL}{dt}=\alpha_V|\mathbf{v}_{ns}|L^{3/2}-\beta_VL^2. \tag{3.8}$$

Understanding the importance of this relation for future applications and the uncertainties concerning the strictness of the theoretical approach that led to this equation, Vinen (1957c) made a number of experiments to support his phenomenological model and to determine the coefficients $\alpha_V$ and $\beta_V$.

One of his main results is reproduced in Fig. 4, which shows $d\mathcal{L}/dt$ as a function of $\mathcal{L}^{3/2}(1-\mathcal{L}^{1/2})$. Here $\mathcal{L}=L(t)/L_\infty$ and $L_\infty=(\alpha_V/\beta_V)^2v_{ns}^2$. The experimental points shown in the figure supposedly confirm Vinen's result. However, considering the scatter and experimental errors, Vinen (1957c) noted that other relations between $dL/dt$ and $L$ and $\mathbf{v}_{ns}$ are not ruled out. In particular, Vinen admitted that the experimental results can corre-

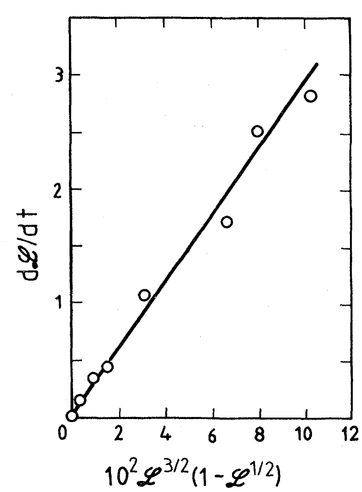


FIG. 4. Buildup of mutual friction when a heat current is suddenly switched on (Vinen, 1957c, Fig. 2).

spond, although not so well, to alternative forms of the generating term, e.g.,

$$\left[\frac{\partial L}{\partial t}\right]_{\text{gen}} = \alpha_{\text{alt}} \mathbf{v}_{\text{ns}}^2 L \ . \tag{3.9}$$

This case corresponds to a quadratic dependence of the generating function, $\phi_{\text{gen}}$ in Eq. (3.3), on its argument, $\phi_{\text{gen}}(\xi) \sim \xi^2$. Relation (3.9) is closer to the phenomenological theory of classical turbulence of Landau (see Landau and Lifshitz, 1980). Indeed, by assuming that turbulence can be characterized by a parameter, say, $L$, and that its time derivative $dL/dt$ is an analytic function of $L$, the relation (3.9) can be interpreted as the first term in the series expansion. Furthermore, as $\phi_{\text{gen}}$ in (3.3) is a scalar function of the vector argument $\mathbf{f}$, it is reasonable to assume that the series expansion of this function starts with the argument squared. If the generating term is taken in the (3.9) shape, then the alternative form of the VLD evolution equation becomes

$$\frac{dL}{dt} = \alpha_{\text{alt}} v_{ns}^2 L - \beta_{\text{alt}} L^2 \ , \tag{3.10}$$

where $\alpha_{\text{alt}}$, $\beta_{\text{alt}}$ are corresponding new parameters. A comparison of experimental results of v. Schwerdtner *et al.* (1989c) with the solutions of Vinen's equation (3.8) and the alternative form (3.10) is shown in Fig. 5. As can be seen in this figure, within the limits of experimental errors both equations fit the experimental results approximately well. It is thus seen that direct measurements of the VLD evolution can hardly allow us to select the proper form of the generating term in the VLD evolution equation. Nevertheless, in spite of the existing doubts and difficulties, the VLD evolution equation in the form (3.8) is generally used to describe nonstationary superfluid turbulence flows.

In the stationary case Vinen's equation yields the relation

$$L_\infty = \frac{\alpha_V^2}{\beta_V^2} v_{ns}^2 = \gamma^2 v_{ns}^2 \ . \tag{3.11}$$

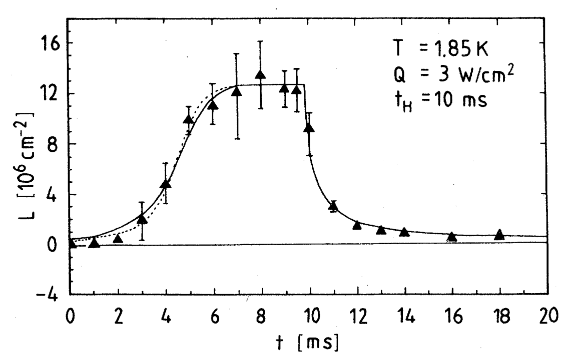


FIG. 5. Evolution of the vortex line density during the passage of a second-sound pulse. The input heat flux was 3 W/cm$^2$; the pulse duration, 10 ms; and the bath temperature, 1.85 K. The solid line corresponds to the solution of Eq. (3.8); the dotted line, to the solution of Eq. (3.10) (Nemirovskii and Schmidt, 1990, Fig. 15).

Assuming further that the tangle is isotropic and that the total friction force is proportional to the total vortex line length, we find that the Gorter-Mellink friction force acting on the superfluid component is

$$F_{ns} = (2\alpha\rho_s\kappa/3) L_\infty v_{ns} = A(T)\rho_s\rho_n v_{ns}^3 \ . \tag{3.12}$$

Introducing this relation in the superfluid hydrodynamics equation leads to the Gorter-Mellink formula (3.1).

Besides doubts concerning the generating term in Eq. (3.2), another difficulty results from the application of Vinen's equation (3.8) to simple nonstationary experiments. Vinen's data for the growth of mutual friction (Fig. 4) can be combined with steady-state values to give the parameter $\beta_V$. This parameter can also be measured in a free-decay experiment, where the input heat flux is suddenly cut off. From the integration of (3.7) it follows that

$$L^{-1}(t) = L^{-1}(0) + \beta_V t \ , \tag{3.13}$$

where $L(0)$ is the VLD at the instant the counterflow is switched off. In Vinen's experiments, the free decay followed (3.13) asymptotically for large $t$, but gave values for $\beta_V$ about 4–6 times smaller than the ones determined above. The initial decay was much more rapid and could have been consistent with a larger value of $\beta_V$; but since it appeared to depend on the channel size, Vinen dismissed it as an anomaly associated with the formation of large eddies.

To explain the discrepancy in the values of $\beta_V$, Vinen assumed that the free vortex decay and the decay of vortices in a counterflow correspond to different states of the vortex tangle, although in both cases they describe a self-preserving state of the vortex tangle. As follows from dimensional analysis in both cases, with and without counterflow, the relation (3.7) is valid, but the coefficients $\beta_V$ are different in both cases, say, $\beta_V^{\text{gr}}$ and $\beta_V^{\text{dec}}$. In other words, according to Vinen, the generating and decay terms are not simply additive but influence each other. There is an important difference in the temperature dependence of the coefficients $\beta_V^{\text{gr}}$ and $\beta_V^{\text{dec}}$. The first one grows with the temperature and the second decreases. The idea of the existence of two self-preserving states is, to some extent, intrinsically contradictory. Indeed, this terminology denotes the possibility of describing the VLD dynamics in terms of only $L(t)$, and the introduction of two self-preserving states is equivalent to admitting that there exists some additional characteristic affecting the macroscopic dynamics of the tangle. It is unknown, *a priori*, which one of the two states, or perhaps a third one, will be realized in an experiment with a harmonic heat input.

Besides the difficulties described above, another rather involved problem in Vinen's theory appears connected with the time of formation of a vortex tangle $t_V$. If, following Vinen, the quantity $t_V$ is used as the time required for the evolution of VLD up to its $L_\infty/2$ value [see (3.11)], there then follows from (3.8)

$$t_V=\int_0^{L_\infty/2}\frac{dL}{\alpha_V|\mathbf{v}_{ns}|L^{3/2}-\beta_V L^2}\ . \tag{3.14}$$

This integral diverges at its lower limit, i.e., $t_V\to\infty$. The origin of the divergence is due to the absence in the theory of any mechanism responsible for the initial appearance of vortices in (3.8).

The problem of the critical velocities and of the initial stages of formation of vortex lines is the most difficult one in the theory of quantized vortices. In the theory of homogeneous turbulence, this problem should be considered as external, and the most natural procedure is to correct the theory by introducing one more phenomenological term. From Vinen's experiments it appears that the essentially finite time $t_V$ is a function of the applied counterflow; hence Vinen's empirically found relation is

$$t_V=a_V(T)\dot{Q}^{-3/2}\ , \tag{3.15}$$

where $a_V$ is of the order of $10^{-2}$ s W$^{3/2}$ /cm$^3$. To improve the theoretical considerations, Vinen introduced a small initiating term in Eq. (3.8) which represents the mechanism of vortex line generation. This initiating term $\propto v_{ns}^{5/2}$ was chosen so as to be compatible with the phenomenological relation (3.15). However, the use of the initiating term is not widely accepted. Most often, in studies of superfluid turbulence, the existence of an initial VLD is assumed and used as a fitting parameter. Besides, there is experimental evidence (Brewer and Edwards, 1961; Childers and Tough, 1973) that this term does not exist. Such a term would give rise to an observable low level of dissipation in the steady state.

The basic ideas of the Feynman-Vinen theory have proved themselves and remain essentially unchanged up to now, although some progress has been made in the study of microscopic turbulence. It can be inferred, as will be shown in the next section, that in its contemporary version Vinen's theory is only modernized.

## IV. MODERN NOTIONS ON SUPERFLUID TURBULENCE

Vinen's phenomenological theory of superfluid turbulence (ST), described in the previous section, was derived in the late '50s. During the following two decades ST was intensely studied, mainly experimentally. A large number of observations of mostly stationary cases were made that fitted well into Vinen's theory. The fundamental concept of this theory was not questioned, and no attempts were made to substantiate it. It was only in 1978 that Schwarz published his paper on a microscopic approach to superfluid turbulence. Since then the theory on the vortex tangle model has made great progress. Thanks to the experimental and theoretical results obtained by a number of scientists, primarily by Schwarz, Donnelly, Tough, and their co-authors, the correlation between the macroscopic continuum dynamics and the microscopic motion of the vortex lines was obtained.

While preparing this section, we were faced with the dilemma of how to deal with the vortex tangle problem. As mentioned in the Introduction, the main results concerning the chaotic vortex tangle in He II were obtained, on the one hand, from measurements of different hydrodynamic characteristics. On the other hand, hydrodynamic laws also produce some effects that should be distinguished from the "pure" intrinsic properties of the vortex tangle (VT). It will be shown in Sec. V.B that these "hydrodynamic effects" are insignificant in the case of stationary homogeneous superfluid turbulence. Therefore only experiments that can be safely assumed to fit these conditions will be discussed in this section.

### A. Kinetic equations of the vortex line distribution

To obtain a quantitative description of the vortex tangle, Schwarz (1978) used Feynman's (1955) basic model of vortex lines. This was an attempt not only to obtain a qualitative description of the processes described by Feynman, but also to confirm that these processes follow from the basic principles, i.e., from the equations of motion of the vortex lines as described in Sec. II.B. Schwarz applied the main equation of motion (2.14) of the vortex line using the local approximation (2.10). This approximation was and is still widely questioned. If we also neglect the term with the coefficient $\alpha'$ of the transversal mutual friction force (which is justified over a large temperature range, $T\geq 1.4$ K), then, in the superfluid velocity reference frame, the vortex line equation of motion is reduced to

$$\dot{\mathbf{S}}=\beta\mathbf{S}'\times\mathbf{S}''+\alpha\mathbf{S}'\times(\mathbf{v}_{ns}-\beta\mathbf{S}'\times\mathbf{S}'')\ . \tag{4.1}$$

In the above relation the variable $\tilde{\xi}$ in $\mathbf{S}(\tilde{\xi},t)$ corresponds to the arclength $\xi$. This substitution allows us to dispense with terms like $1/|\mathbf{S}'|$ in Eq. (2.14). However, as the variable $\tilde{\xi}$, unlike $\xi$, is time dependent, hence generally replacing $\mathbf{S}(\xi,t)$ by $\mathbf{S}(\tilde{\xi},t)$ leads to, for the left-hand side of (4.1),

$$\dot{\mathbf{S}}=\frac{\partial\mathbf{S}}{\partial t}+\mathbf{S}'\frac{\partial\tilde{\xi}}{\partial t}\ . \tag{4.2}$$

In the numerical analysis this requires the rescaling of $\tilde{\xi}$ at each step whenever the equations of motion in the form (4.1) was used. This transformation is very important in the theoretical analysis and requires special attention.

As can be seen from (4.1) the velocity of a line element depends on the first and second derivatives of $\mathbf{S}(\tilde{\xi},t)$. From the kinematic considerations that $|\mathbf{S}'|=1$, $\mathbf{S}'\cdot\mathbf{S}''=0$, and $|\mathbf{S}''|=1/R$ ($R$ is the radius of the vortex line curvature), it follows that the velocity of a given point on the line $\dot{\mathbf{S}}(\tilde{\xi},t)$ is a function of the magnitude of the self-induced velocity $|\mathbf{v}_l|=\beta/R$ and of the angle $\theta$ between $\mathbf{v}_l$ and the counterflow velocity $\mathbf{v}_{ns}$, where $\mathbf{v}_{ns}$ is a fixed parameter, and an azimuthal symmetry is assumed.

It follows from Eq. (4.1) that during its motion the length $\Delta l$ of the vortex line element varies according to the relation

$$\frac{\partial \Delta l}{\partial t} = \frac{\alpha}{\beta}(\mathbf{v}_{ns} - \mathbf{v}_l)\mathbf{v}_l \Delta l \ ; \tag{4.3}$$

hence the variation of the length is also a function of $|\mathbf{v}_l|$ and $\theta$.

As the vortex line dynamics and the variation of its length are functions of $|\mathbf{v}_l|$ and $\theta$, it seems tempting to use the phase space $|\mathbf{v}_l|$, $\theta$ and to introduce there a distribution function of the total line length $\lambda(|\mathbf{v}_l|,\theta,t)$. The evolution equation for $\lambda$ can then be regarded as a kinetic equation in the introduced phase space

$$\frac{\partial \lambda}{\partial t} + \frac{\partial \lambda}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial \lambda}{\partial |\mathbf{v}_l|}\frac{\partial |\mathbf{v}_l|}{\partial t} = \frac{\alpha}{\beta}(\mathbf{v}_{ns} - \mathbf{v}_l)\mathbf{v}_l \lambda \ . \tag{4.4}$$

A fundamental difficulty appears at this point, because the term $\partial|\mathbf{v}_l|/\partial t$ is not only a function of $\theta$ and $|\mathbf{v}_l|$, i.e., of $\mathbf{S}'$ and $\mathbf{S}''$, but also of $\mathbf{S}'''$ and $\mathbf{S}^{\mathrm{IV}}$, i.e., the third- and fourth-order derivatives of $\mathbf{S}(\xi,t)$. Expressing it differently, Eq. (4.4) is not closed, and the introduced phase space, i.e., the variables $|\mathbf{v}_l|$, $\theta$, are not complete. Attempts to broaden the phase space up to fourth order $\mathbf{S}^{\mathrm{IV}}$ and to reformulate accordingly the distribution function to $\lambda(\mathbf{S}',\mathbf{S}'',\mathbf{S}''',\mathbf{S}^{\mathrm{IV}},t)$ will also remain unsuccessful. Indeed, it is easy to notice that the time derivatives $\partial\mathbf{S}'''/\partial t$ and $\partial S^{\mathrm{IV}}/\partial t$ will contain higher-order derivatives that will require further enlargement of the phase space and so on up to infinity. This is quite similar to the statistical-mechanics problem of finding a distribution function from Liouville's equation. Such an attempt leads to the well-known Bogoliubov's infinite chain of equations. This infinite chain can be cut if a small parameter can be found like the gaseous parameter in the kinetic theory of gases. In this last case it is possible to "cut the chain" to obtain an approximate distribution. Unfortunately, this procedure cannot be applied to the vortex tangle case because the characteristic relaxation time for all derivatives is about the same, equal to $\tau \cong \beta/v_{ns}^2$. This can be checked by differentiating (4.1) with respect to the arclength $\xi$.

Another possibility of cutting the infinite chain exists in the case of a dense gas (or in the case of long-range forces). In this case the particle trajectory is a random walk in the averaged field created by other particles. Hence the evolution of the distribution function satisfies the Fokker-Planck equation (see, e.g., Lifshitz and Pitaevskii, 1981). Indeed, Schwarz assumed that all required conditions are satisfied in deriving his stochastical dynamic model of the vortex tangle. He assumed that the nonlocal terms including the derivatives of order higher than two in $\partial\mathbf{v}_l/\partial t$ are of importance only in the intersection process of two vortex lines. The influence of the higher derivatives on the dynamics is hence reduced to a random-walk process and therefore can be described by introducing phenomenologically a diffusion term. [A slightly different approach used by Yamada *et al.* (1989) is described in Sec. V.C.] Schwarz derived the diffusion terms using a self-consistent procedure. The distribution functions $\lambda(|\mathbf{v}_l|,\theta,t)$ obtained in this way were used by Schwarz to determine many properties of superfluid turbulence. Here we describe only the results concerning the internal structure of the vortex tangle and the derivation of the equation for the vortex line density $L(t)$.

The distribution function $\tilde{\lambda}(R,\theta)R^2$ in the equilibrium state $(t \to \infty)$ is shown in Fig. 6. Unlike the case of $\lambda(|\mathbf{v}_l|,\theta)$, the first argument of the $\tilde{\lambda}$ function is the radius of curvature $R = \beta/|\mathbf{v}_l|$. As can be seen, the distribution function $\tilde{\lambda}(R,\theta)$ depends strongly on $\theta$. This means that the tangle is quite strongly anisotropic, which contradicts Vinen's model in which the tangle is assumed to be isotropic. This is one of the most important results concerning the structure of the vortex tangle.

Another important issue shown in the figure is the existence of a maximum, $R_{\mathrm{peak}}$, of the distribution function independent of $\theta$. This means that the lines with curvature $\sim R_{\mathrm{peak}}^{-1}$ prevail in the tangle. It is very important that the most probable radius of curvature be close to the average distance between the vortex lines $\delta = L^{-1/2}$ which can be expressed by $\delta = c(T)R_{\mathrm{peak}}$. The value of $c(T)$ is of order one; it depends only on the temperature and is independent of the relative velocity $v_{ns}$. Moreover, Schwarz found that the nondimensional distribution function

$$\lambda_{\mathrm{red}}\left[\frac{R}{R_{\mathrm{peak}}},\theta\right] = L^{-1}\tilde{\lambda}(R,\theta)R_{\mathrm{peak}}^3 \tag{4.5}$$

depends only weakly on $v_{ns}$ as well as on $T$.

In other words, it can be said that the dependence on the above parameters is absorbed in the values of $R_{\mathrm{peak}}$ and $L$, and the function $\lambda_{\mathrm{red}}(R,\theta,t)$ possesses scaling properties. This means that with increasing $v_{ns}$ the vortex tangle increases its density, keeping its structure, according to the new scale $R' = R/R_{\mathrm{peak}}$. In particular, the degree of anisotropy of the tangle does not depend on the relative velocity $\mathbf{v}_{ns}$. This scaling property of the vortex tangle is illustrated in Fig. 7, which shows $\lambda_{\mathrm{red}} R'^2$ at two different temperatures and velocities $\mathbf{v}_{ns}$ such that the unreduced distributions $\tilde{\lambda}R^2$ differ by four orders of magnitude.

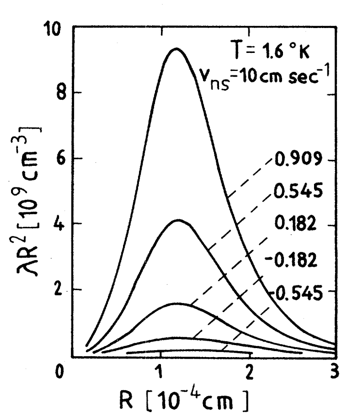


FIG. 6. Typical equilibrium distribution (Schwarz, 1978, Fig. 13). Curves are given for various values of cos$\theta$.

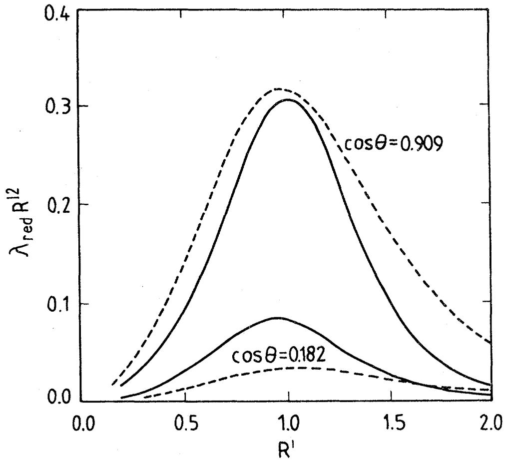


FIG. 7. Illustration of the scaling, which is a property of the steady-state λ's (Schwarz, 1978, Fig. 22). The reduced distribution functions $\lambda_{red}$ as a function of $R'=R/R_{peak}$ are plotted for the two very different cases $T=2.0$ K, $v_{ns}=20$ cm sec$^{-1}$ (dashed curves) and $T=1.2$ K, $v_{ns}=5$ cm sec$^{-1}$ (solid curves). The actual magnitudes of the unreduced distributions $\lambda R^2$ differ by a factor of order $10^4$ for these two cases.

Schwarz's (1978) next important contribution was the derivation of the VLD evolution equation from microscopic considerations. Within the limits of Schwarz's approximation, $L(t)$ is the zeroth moment of the λ distribution function; i.e.,

$$L(t)=\int \lambda(|\mathbf{v}_l|,\theta,t)d^3\mathbf{v}_l \ . \tag{4.6}$$

Hence, to obtain the evolution equation, it is necessary to integrate (4.4). The diffusion terms on the left-hand side of (4.4) containing second-order derivatives of λwith respect to $|\mathbf{v}_l|$ and θ contained in $\partial|\mathbf{v}_l|/\partial t$ lead to a redistribution of λ in the $|\mathbf{v}_l|$, θ space without changing the VLD $L(t)$. The remaining terms can be integrated. When the above-mentioned scaling properties are taken into account, the result can be expressed as

$$\dot{L}=g_1\alpha v_{ns}\frac{L}{R_{peak}}-g_2\alpha\beta\frac{L}{R^2_{peak}} \ , \tag{4.7}$$

where $g_1$and $g_2$ are temperature-dependent coefficients obtained from the integration of the scaling invariant functions $\lambda_{red}$ of $R'$ over $R'$; hence $g_1$ and $g_2$ are independent of $\mathbf{v}_{ns}$. Since $R_{peak}=\delta/c(T)$ and $\delta=L^{-1/2}$, Eq. (4.7) becomes

$$\dot{L}=\alpha_s|\mathbf{v}_{ns}|L^{3/2}-\beta_s L^2 \ , \tag{4.8}$$

in agreement with the form of Vinen's equation (3.8).

Thus Schwarz obtained the macroscopic equation for the VLD, $L(t)$, starting from first principles, i.e., from the dynamic equations of motion of a vortex line (2.14) using an approximation to (4.1). Unlike Vinen, Schwarz did not use any fitting parameters and obtained all his coefficients from his theoretical considerations. However, $\alpha_s$ and $\beta_s$ are about an order of magnitude larger than the corresponding coefficients $\alpha_V$ and $\beta_V$ in (3.8).

A second and more fundamental difference is in the physical interpretation of Eq. (4.8), which is quite different from the Feynman-Vinen conception. Unlike their model, where, we recall, the growth of the VLD is due to mutual friction and the decay is due to the cascade process of breakup of the vortex rings, Schwarz's approximation attributes the processes described by both terms to the interaction of the vortices with the normal component.

Considering the importance of the VLD evolution equation for $L(t)$ and the doubts and difficulties resulting from its phenomenological deduction, we find it worthwhile to reexamine Schwarz's main assumptions leading to Eq. (4.8). The already mentioned main doubts concern the validity of the kinetic equation approach used to obtain $\lambda(|\mathbf{v}_l|,\theta,t)$. Furthermore, as mentioned by Schwarz, the use of the solution obtained for λ assuming equilibrium processes is not valid for nonequilibrium cases. Of course, the evolution of the tangle can proceed in a quasistationary way, but this is only a hypothesis without a proof of its validity. This hypothesis is similar to Vinen's assumption of the self-preserving state of the vortex tangle, which is discussed in Sec. V.A.

However, even assuming quasiequilibrium processes, there are some open points left in Schwarz's deductions. It was important to use a scaling property of the dimensionless distribution $\lambda_{red}$ (4.5) to obtain (4.8). The use of a different expression for $\lambda_{red}$ could be equally well justified, e.g.,

$$\lambda_{red}(R\gamma c(T)v_{ns},\theta)=L^{-1}\tilde{\lambda}(R,\theta)(\gamma c(T)v_{ns})^{-3} \tag{4.9}$$

where, we recall, γ is obtained from the relation $L^{1/2}=\gamma v_{ns}$.

Indeed, since in the equilibrium case the following chain of relations exists,

$$R_{peak}=\delta/c(T)=1/c(T)L^{1/2}=1/c(T)\gamma v_{ns} \ , \tag{4.10}$$

then any one of the above three relations can be used for nondimensionalizing the argument $R$ of the function $\lambda_{red}$ without violating the scaling property. Obviously, assuming equilibrium conditions, this does not lead to any differences in the results. However, using the relation (4.9) instead of (4.5), i.e., using $R_{peak}=1/[c(T)\gamma v_{ns}]$ instead of $R_{peak}=1/[c(T)L^{1/2}]$ in (4.7) to obtain the VLD evolution equation $L(t)$, we notice that both terms in (4.7) will have the same structure and the dynamic equation will have the form

$$\dot{L}=\alpha_{alt}v^2_{ns}L \ . \tag{4.11}$$

Hence, first, as Feynman expected, the conservative interaction (although it includes both the growth and the decay of the VLD) leads, on the average, to an increase of the total length; second, there is no annihilation term in Eq. (4.11), as in Schwarz's model there is no cascade process in the space of vortex rings and a following transformation into thermal excitations; third, the generating term agrees with the alternative equations (3.9) and (3.10).

The above considerations are connected with the selection of a definite scaling parameter from the set (4.10), and each choice can be equally criticized. Indeed, we encounter here, although at a different level, the same problem Vinen had in deducing his generating term from dimensional considerations. In Schwarz's method, so as in Vinen's, the choice of the generating term based on the scaling analysis is possible with certainty only up to a certain generating function $\phi_{\text{gen}}$ of the argument $\gamma v_{ns}/L^{1/2}$. In the equilibrium case, $\gamma v_{ns}/L^{1/2}=1$, and this removes the difficult problem of a proper choice of $\phi_{\text{gen}}$; but in the nonequilibrium case, the problem remains open in both approaches.

### B. Numerical simulations of vortex tangle dynamics

Although Schwarz's kinetic equation method was the first, most important step taken in the theoretical research on microscopic dynamics of the VT, nevertheless a full clarification of the problem was not achieved. During the next decade Schwarz developed the very complicated problem of a full numerical simulation of vortex tangle dynamics (see Schwarz, 1982a, 1982b, 1985, 1988 and Schwarz and Rozen, 1991). Here we shall summarize Schwarz's results on the structure of the He II vortex tangle. One of the most important principles that allowed us to forecast some results and develop the strategy of numerical research was the scaling analysis, which was simultaneously developed by Donnelly and his group (see, e.g., Swanson and Donnelly, 1985). The scaling analysis and the numerical modeling of the VT were made according to Eq. (2.14), where the local approximation $\dot{\mathbf{S}}_i=\mathbf{S}'\times\mathbf{S}''$ to the self-induced velocity was used. The parametrization occurs along $\xi$ [see remark to Eq. (4.1)], and the time $t$ and velocities $\mathbf{v}_n$, $\mathbf{v}_s$ are replaced by $t_0=\beta t$ and $\mathbf{v}_{n,0}=\mathbf{v}_n/\beta$, $\mathbf{v}_{s,0}=\mathbf{v}_s/\beta$.

The equation of motion of the line vortex then becomes

$$\frac{\partial \mathbf{S}}{\partial t_0}=\mathbf{S}'\times\mathbf{S}''+\mathbf{v}_{s,0}+\alpha\mathbf{S}'\times(\mathbf{v}_{ns,0}-\mathbf{S}'\times\mathbf{S}'')$$
$$-\alpha'\mathbf{S}'\times[\mathbf{S}'(\mathbf{v}_{ns,0}-\mathbf{S}'\times\mathbf{S}'')] \ . \qquad (4.12)$$

This equation is invariant under the following transformations ($\lambda$ here should not be confused with the $\lambda$ of the distribution function; see Schwarz, 1982a, 1982b, 1988; Swanson and Donnelly, 1985; and Donnelly and Swanson, 1986):

$$\mathbf{S}=\lambda\mathbf{S}^*,\quad \xi=\lambda\xi^*,\quad t_0=\lambda^2 t_0^*$$
$$\mathbf{v}_{n,0}=\lambda^{-1}\mathbf{v}_{n,0}^*,\quad \mathbf{v}_{s,0}=\lambda^{-1}\mathbf{v}_{s,0}^*,\quad \mathbf{v}_{ns,0}=\lambda^{-1}\mathbf{v}_{ns,0}^* \ . \qquad (4.13)$$

The other needed variables and parameters not appearing in (4.12) must be accordingly transformed. For instance, the size of $D$ and the VLD, $L$ will be transformed as

$$D=\lambda D^*,\quad L=\lambda^{-2}L^* \ . \qquad (4.14)$$

Investigations of the VT based on the invariance of the problem as formulated in (4.12), after the introduction there of the transformation of variables according to (4.13) and (4.14), are referred to as "scaling analysis." If the number of parameters is small, then a scaling analysis can give the possibility of making some predictions. For example, with the assumption that the VLD $L$ in a homogeneous, isotropic tangle under stationary conditions is a function only of $\mathbf{v}_{ns}$, scaling will lead to the following chain of relations:

$$L(\mathbf{v}_{ns,0})=\lambda^{-2}L^*(\mathbf{v}_{ns,0}^*)=\lambda^{-2}L^*(\mathbf{v}_{ns,0}\lambda)=c_L^2 v_{ns,0}^2 \qquad (4.15)$$

or, in the usual units,

$$L(\mathbf{v}_{ns,0})=(c_L^2/\beta^2)\mathbf{v}_{ns}^2 \ . \qquad (4.16)$$

Thus, through the use of a scaling analysis, the result was the same as that obtained rom Vinen's theory, which is confirmed by a number of experiments. Another example is connected with the free decay of a vortex tangle. Assuming that in this case $\partial L/\partial t_0$ are functions only of $L$, we find

$$\dot{L}(L)=\lambda^{-4}\dot{L}^*(L^*)=\lambda^{-4}\dot{L}^*(\lambda^2 L)=-c_d L^2 \ . \qquad (4.17)$$

This result also agrees with Vinen's theory. Although the relations (4.16) and (4.17) can also be obtained from simple dimensional considerations, they have a deeper meaning, as they are connected with the initial microscopic equations of motion and thus allow us to substantiate the macroscopic laws. A number of other examples of using scaling relations to predict the dependence of the critical velocity on the characteristic dimension of the flow or of the frequency relations of the spectrum of the VLD fluctuations $\langle \delta L_\omega \delta L_{-\omega}\rangle$ are given in Donnelly's (1991a) book.

To elaborate a strategy of numerical calculations, Schwarz introduces a few quantities to characterize the vortex tangle. One of them is the VLD determined as

$$L=\frac{1}{\Omega}\int d\xi \ , \qquad (4.18)$$

where $\Omega$ is the volume of the space considered, and the integration follows the arclength of the whole vortex line in $\Omega$. In (4.18) and below, ensemble averaging of the integrand is assumed. Further, the average curvature is introduced as

$$\langle|\mathbf{S}''|\rangle=\frac{1}{\Omega L}\int|\mathbf{S}''|d\xi=c_1 L \ . \qquad (4.19)$$

The second part of (4.19) is obtained from the following considerations. According to the scaling properties of the system, the quantity $\langle|\mathbf{S}''|\rangle$ is supposed to be scale invariant and $\langle|\mathbf{S}''|\rangle=\lambda^{-1}\langle|\mathbf{S}''|\rangle^*$. As Schwarz states, selecting for convenience $(L^*/L)^{1/2}$ as a scaling parameter $\lambda$, it can immediately be concluded that $\langle|\mathbf{S}''|\rangle$ is proportional to $L$. Similarly, as will be seen later, the average value of the curvature squared $\langle|\mathbf{S}''|^2\rangle$ is an important quantity of the VT macroscopic dynamics. This quantity, using scaling analysis, can be written as

$$\langle |\mathbf{S}''|^2 \rangle = \frac{1}{\Omega L} \int |\mathbf{S}''|^2 d\tilde{\xi} = c_2^2 L \ . \tag{4.20}$$

Schwarz introduces the following parameters to describe the anisotropy of the tangle:

$$I_{\parallel} = \frac{1}{\Omega L} \int [1 - (\mathbf{S}' \hat{r}_{\parallel})^2] d\tilde{\xi} \ , \tag{4.21}$$

$$I_{\perp} = \frac{1}{\Omega L} \int [1 - (\mathbf{S}' \hat{r}_{\perp})^2] d\tilde{\xi} \ , \tag{4.22}$$

$$I_l \hat{r}_{\parallel} = \frac{1}{\Omega L^{3/2}} \int \mathbf{S}' \times \mathbf{S}'' d\tilde{\xi} \ , \tag{4.23}$$

where $\hat{r}_{\parallel}$ and $\hat{r}_{\perp}$ are, respectively, the unit vectors in the direction of $\mathbf{v}_{ns}$ and perpendicular to $\mathbf{v}_{ns}$. The parameters $I_{\parallel}$ and $I_{\perp}$ correspond to the space orientations of the vortex lines, whereas $I_l$ describes the anisotropy of the self-induced velocity $\dot{\mathbf{S}}_i = \beta \mathbf{S}' \times \mathbf{S}''$. In the case of an isotropic distribution, for instance, $I_{\parallel} = I_{\perp} = \frac{2}{3}$ and $I_l = 0$. Adding (4.21) and (4.22) and bearing in mind that $|\mathbf{S}'|^2 = 1$, we obtain

$$I_{\parallel}/2 + I_{\perp} = 1 \ . \tag{4.24}$$

From the introduced relations, many properties of the vortex tangle can be formulated. For instance, the friction force can be written as

$$\mathbf{F}_{ns} = \frac{\rho_s \kappa \alpha}{\Omega} \int \mathbf{S}' \times [\mathbf{S}' \times (\mathbf{v}_{ns} - \beta \mathbf{S}' \times \mathbf{S}'')] d\tilde{\xi} \ . \tag{4.25}$$

The term with $\alpha'$ disappears due to the azimuthal symmetry. The tangle drift velocity $\mathbf{v}_L$ in the superfluid velocity reference frame can be expressed as

$$\mathbf{v}_{Ls} = \frac{1}{\Omega L} \int \dot{\mathbf{S}} \, d\xi - \mathbf{v}_s \ . \tag{4.26}$$

The equations for the second-sound damping, temperature variation, or chemical-potential drop can be formulated analogously.

Relations like (4.25) or (4.26) are exact relations between the VT properties and the measured quantities replacing the previously assumed phenomenological relations like (3.11) or (3.12).

It is understood that, to make use of these relations, it is necessary to have some knowledge of the vortex line distribution in space or of the statistical properties of its distribution. Schwarz solved this most important and most difficult part of the problem using a direct numerical modeling method of the vortex line dynamics based on Eq. (4.12). He also used the reconnection ansatz, described in Sec. II.B, besides the evolution equation. The extreme complexity of the dynamics of the He II vortex lines can be seen in Fig. 8, which illustrates the evolution of six vortex rings.

From the obtained curves illustrating the evolution of the tangle, Schwarz concluded that the fraction of small scale vortices is relatively small. Large vortices expanded over the whole volume are prevalent. This unexpected result may contradict the homogeneity hypothesis, as the presence of a large number of large vortices must result in a strong influence of the boundaries. Another qualitative result obtained by Schwarz is the tendency of the vortex lines to align themselves transversally to the flow direction. This property contradicts experimental observations and may reflect some deficiency of the model in which an artificial mixing procedure was used. It is also worth noting that the kinkiness of the tangle increases with decreasing temperature, i.e., the average curvature $\langle R^{-1} \rangle$, as compared with the interline spacing, decreases.

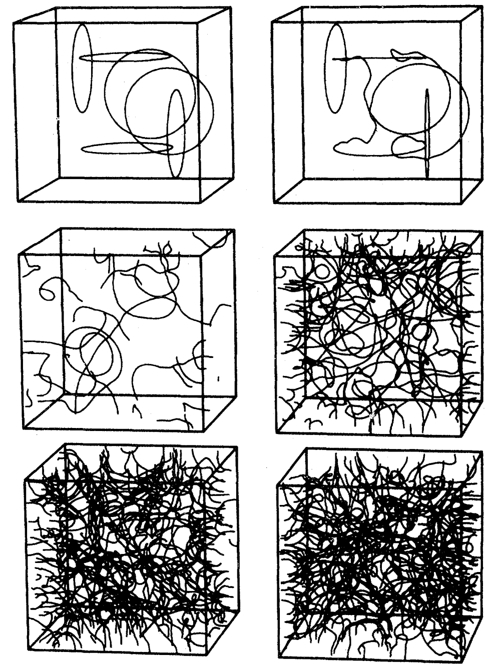

FIG. 8. Case study of the development of a vortex tangle in a real channel (Schwarz, 1988, Fig. 4). Here the temperature is about 1.6 K, and $v_{s,0} = 75$ into the front face of the channel section shown. Upper left: $t_0 = 0$, no reconnections; upper right: $t_0 = 0.0028$, three reconnections; middle left: $t_0 = 0.05$, 18 reconnections; middle right: $t_0 = 0.20$, 844 reconnections; lower left: $t_0 = 0.55$, 12 128 reconnections; lower right: $t_0 = 2.75$, 124 781 reconnections.

As far as the quantitative values are concerned, the main effort was directed at calculating the introduced variables and parameters appearing in Eqs. (4.18)–(4.23). In particular, Schwarz calculated the evolution of the VLD, $L(t)$. It was shown that $L(t)$, after a certain transition time, reaches a stationary state which is, however, accompanied by fluctuations whose nature has still to be investigated. Interestingly, the duration of the transition time is comparable with Vinen's characteristic time of development of superfluid turbulence $t_V$ [see Eq. (3.15)].

The main parameters relevant to VLD evolution and their temperature dependence have been taken from Schwarz's (1988) table and are reproduced here.

| $T$ | 1.07 | 1.26 | 1.62 | 2.01 | 2.15 |
|---|---|---|---|---|---|
| $\alpha$ | 0.0100 | 0.0300 | 0.100 | 0.300 | 1.000 |
| $\alpha'$ | 0.0050 | 0.0125 | 0.016 | 0.010 | −0.270 |
| $v_{ns,0}$ | 190.0000 | 80.0000 | 135.000 | 30.000 | 35.000 |
| $\bar{L}$ | 48.1000 | 34.0000 | 323.400 | 43.600 | 95.800 |
| $c_L$ | 0.0365 | 0.0729 | 0.133 | 0.220 | 0.280 |
| $c_1$ | 2.8100 | 1.9700 | 1.480 | 1.050 | 0.719 |
| $c_2$ | 3.3300 | 2.4200 | 1.840 | 1.430 | 1.140 |
| $\bar{I}_\parallel$ | 0.7190 | 0.7490 | 0.770 | 0.870 | 0.952 |
| $\bar{I}_l$ | 0.4280 | 0.4420 | 0.454 | 0.460 | 0.358 |

As expected from the assumed scaling properties, the above coefficients do not depend on $v_{ns}$. Using the values of his own parameters given in this table, Schwarz compares his calculated results with a number of experimental data. This is not a simple task, as most experimental results were interpreted in the frame of Vinen's theory (see Sec. III) using simple relations like (3.11) and (3.12), which must be replaced by the more rigorous ones given in this section. For example, instead of using (3.11), the relation (4.16) giving the VLD as a function of $\mathbf{v}_{ns}$ should be used. The difference between these two relations is that the latter contains the logarithmic factor $\beta=(\kappa/4\pi)\ln(1/c_2L^{1/2}r_0)$. The high-resolution data of Martin and Tough (1981) confirmed the existence of this factor. This result helped to elucidate the real meaning of the additional parameter $v_0$ used to interpret the experimental data, without the logarithmic factor $\beta$ leading to the equation $L_\infty^{1/2}=\gamma(v_{ns}-v_0)$ and hence giving a nonzero intercept in the $L$, $v_{ns}$ plane. Analogously, Schwarz had to use more involved considerations to be able to compare his results for the mutual friction force (4.25) with experimental data. The experimental data for the mutual friction force $f_{ns}$ have been obtained in a large number of experiments on stationary homogeneous superfluid turbulence. The coefficient of the $v_{ns}^3$ term of the mutual friction obtained from some representative experiments is shown in Fig. 9 and compared with Schwarz's result, which, in terms of the above parameters, is given by $(c_L^2\bar{I}_\parallel-c_L^3\bar{I}_l)^{1/3}$. The agreement with the experimental points is quite satisfactory. However, depending on the experimental conditions, the scatter is large and can be as large as 4 times the average value (see Tough, 1982). Another experimental confirmation of the theoretical results was demonstrated by Schwarz (1988), who compared the calculated and measured variation of the Gorter-Mellink coefficient with pressure.

The scarcity of experimental data does not allow a proper comparison of the anisotropic properties of the tangle. There is only one published experimental paper of Wang *et al.* (1987) concerned with anisotropy. In this paper the longitudinal and transversal damping of second sound and temperature drop were measured. The measured ratio of the anisotropy $L_T/L_A$, which in terms of the relations introduced by Schwarz is given by $I_\perp/I_\parallel$, is shown by bars in Fig. 10 as functions of the temperature $T$. The results of Schwarz's predictions are shown in the

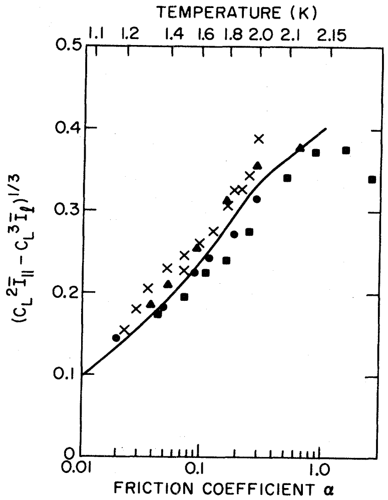


FIG. 9. Comparison of the theoretically predicted, continuous line mutual friction force coefficient with selected experiments (Schwarz, 1988, Fig. 27). Triangles are from Vinen (1957a, 1957b, 1957c); circles, from Brewer and Edwards (1961); crosses, from Opatowsky and Tough (1981); and squares, from Swanson (1985).

same figure by open circles. It can be concluded that in this case also the numerical results are within the limits of the experimental scatter. Recent experiments of Olszok (1994) confirm the observations of Wang *et al.*

There is also a reasonable agreement of the VT drift velocity $\mathbf{v}_{Ls}$ expressed in $|\mathbf{v}_{ns}|$ units in the superfluid velocity reference frame. In terms of the parameters introduced by Schwarz, the quantity $v_{Ls}/v_{ns}$ is approximately equal to $c_L\cdot I_l$. A comparison of experimental and numerical results is shown in Fig. 11, where it is seen that the vortex tangle drifts with the velocity of the superfluid component. This agrees with Vinen's original assumption, although it contradicts Ashton and Northby's (1975) ion drift measurements, which showed some slip of the tangle in the direction of the normal component velocity. Let us note that Vinen considered that the entrainment of the vortex tangle by the superfluid component is due to the assumed isotropy. The described results show a strong anisotropy but at the same time a tan-

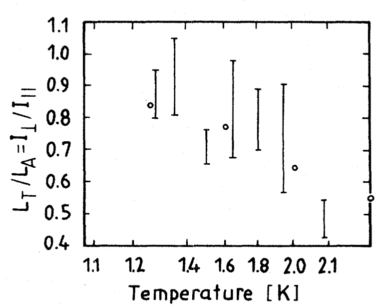


FIG. 10. Anisotropy ratio as a function of temperature: circles, computed by Schwarz (1988, Fig. 30); vertical bars, measured by Wang *et al.* (1987).

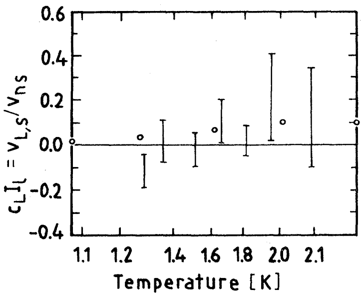


FIG. 11. Vortex drift velocity $v_{L,s}$ (in units of $v_{ns}$) as a function of temperature: circles, computed by Schwarz (1988, Fig. 31); vertical bars, measured by Wang *et al.* (1987).

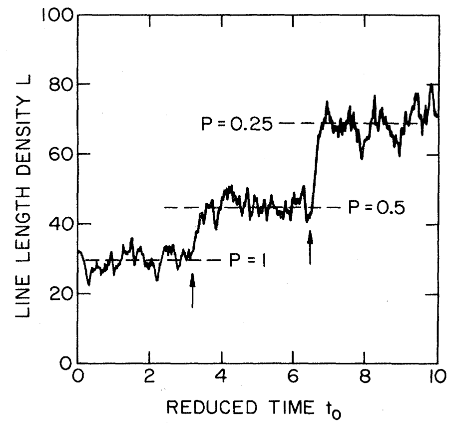


Fig. 12. Dependence of line length density on reconnection probability (Schwarz, 1988). The arrows indicate the time during the run at which the probabilities were changed. The lines have been drawn to guide the eye.

gle entrainment. So the interplay of different processes in the dynamics of the vortex line appears to be quite complex and subtle.

Schwarz's results not only confirmed the processes forecasted by Feynman and Vinen and described them quantitatively, but also give some new information about details of the vortex line dynamics. For instance, as mentioned in Sec.II.B, the description of vortex line dynamics by Biot-Savart's law leads to the possibility of collapse of two approaching antiparallel vortex line elements. However, their further fate cannot be predicted using this approach. To deal with this problem, Schwarz used a very sophisticated procedure. He introduced in his calculations a quantity characterizing the annihilation probability of colliding vortex lines, which he called the reconnection probability $P$, and studied its influence on the final result by following the VLD evolution, $L$, as the best known characteristic quantity describing superfluid turbulence. It appeared that $L$ depends strongly on the assumed reconnection probability and that the best agreement with experimental results is obtained when each vortex collision is followed by reconnection (see Fig. 12). This result agrees with the findings of Nakajima *et al.* (1978) and Frisch *et al.* (1992). From this point of view of would like to stress that the simultaneous study of the vortex line dynamics with the "hydrodynamic" measurements of the VLD made it possible to arrive at this very important conclusion about the full reconnection of quantum vortices, which otherwise would require some very involved quantum-mechanical calculations. We think that this is a good illustration of the advantages of studying vortices in He II where some theoretical difficulties can be compensated by experimental observations.

In spite of the obvious progress in understanding ST and a good agreement with experimental results, Schwarz's theory is sometimes criticized. Buttke (1988, 1991) asserts, having repeated the calculations using a different algorithm, that the main results concerning Eq. (4.16) are confirmed only for rough mesh sizes and that reducing the mesh size gave very different results. From the exchange of corresponding views between Schwarz and Buttke in *Physical Review Letters* (see Buttke, 1987 and Schwarz, 1987), it appears that Buttke's anomalous results are an artifact due to the assumption of periodic boundary conditions. Schwarz avoided the problem using the artificial mixing procedure mentioned earlier. The recent results of Aarts and de Waele (1994), which not only confirmed Schwarz's numerical findings but also obtained new results for the nonuniform, Poiseuille-like $v_{ns}$ velocity distribution, clarify somewhat this problem. We think that a full clarification of this problem is of special importance, because this controversy led to some doubts concerning the formulation of the problem, specifically, concerning the use of the local induction approximation (2.10) instead of the full Biot-Savart law (2.8). This is the second critical remark about Schwarz's results. Doubts about the use of the local induction approximation for an adequate description of the evolution of stochastic vortex lines come from scientists investigating vortex tubes of classical fluids. From their point of view, the local approximation, which in classical fluids does not lead to a change in the length of the vortex tube, i.e., does not include stretching, is not suitable for describing satisfactorily vortex dynamics (Chorin, 1979, 1982, 1991; Leonard, 1980; Siggia, 1985; Klein and Majda, 1991). Moreover, there are some statements that the omission of stretching can lead to the absence of stochastization. Hansen and Nelkin (1986) supported Schwarz's results indicating the difference between classical vortices and quantized vortices. The stretching of classical vortices is accompanied by a reduction of its core radius (Sec. II.B), whereas the core radius of quantized vortices is fixed. This difference in the vortex properties, according to Hansen and Nelkin, removes the problem of the importance of vortex stretching, due to nonlocal processes, in the vortex dynamics of He II. Agistein and Migdal (1986), on the other hand, criticized the introduction of the cutoff radius depending on the label $\xi$ (see also Sec. II.B). In their calculations, they fixed the radius of the core (which is the case of He II) and observed the stochastization of initially smooth vortex lines.

We think that to a large degree the uncertainties concerning the relevant questions are connected with the rapid development of numerical methods used in the investigation of stochastic vortices in He II and in classical fluids, which are very much ahead of the corresponding theoretical analysis. This results in the introduction of different, unverified calculating procedures, which permits the formation of different unconfirmed conclusions.

An attempt at a simultaneous theoretical and analytical investigation of the stochastic dynamics of a vortex line in the local approximation was undertaken by Nemirovskii *et al.* in (1991); they also omitted the processes of reconnection. Although this model is vary far from the physical reality, nevertheless good agreement between numerical results and analytical considerations allows us to draw some conclusions concerning the stochastical dynamics of a vortex line and even to apply them to the ST problem.

Figure 13 illustrates the evolution of a vortex ring satisfying the dynamic equation

$$\frac{\partial \mathbf{S}}{\partial t}=\beta \mathbf{S}'\times\mathbf{S}''/|\mathbf{S}'|^3+\mathbf{f}(\xi,t) \tag{4.27}$$

where $\mathbf{f}(\xi,t)$ is a random Langevin force with the correlator $\langle f_k^\alpha f_{-k}^\beta\rangle=\delta_{\alpha\beta}F_k\delta(t-t')$, where $k$ is the wave vector of a one-dimensional Fourier transformation. The Langevin force models the random long-range interaction of different elements of the line. The nonlinear interaction described by the first term on the right-hand side of (4.27) leads to the generation of higher harmonics, which in their turn generate still higher harmonics and so on. This cascade process induces the creation of segments of large curvature as shown in Fig. 13. An additional curvature generated by the force $\mathbf{f}$ stretches into the region of large $k$. Thus a flux of curvature $P$ appears in the $k$ space. At very large $|k|$, which corresponds to very kinked segments of the vortex line, a dissipation mechanism, e.g., collapse and vortex annihilation, appears. These processes lead to dissipation of the curvature of the vortex lines. Competition between the nonlinear generation of high harmonics and the dissipative mechanisms leads to a stationary picture with a spectral distribution of the vortex curvature. A vortex line in this case forms a small tangle (shown in Fig. 13). This statement of the problem is fully analogous to Kolmogorov's description of classical turbulence. The spectral distribution was obtained in the above-cited paper using field theoretical methods. It was found that the Fourier image (spectrum) of the equal-time correlator of the vortex line follows the relation

$$\langle S_k^\alpha S_{k'}^\beta\rangle/\delta(k+k')=\mathrm{const}P^{2/3}/k^3\ . \tag{4.28}$$

The spectrum obtained from an exact numerical solution of the problem (4.27) is illustrated in Fig. 14. It was found that the spectrum varies depending on the intensity of the external force, although it remains close to the theoretical predictions (4.28).

We would like to make a few remarks concerning Schwarz's results with reference to the solutions of the simple model described above. The first one is concerned with the scaling properties. When these properties were studied, it was assumed that the stochastic dynamics of the vortex tangle is described by $v_{ns}$ and $L$. Correspondingly, in the stationary case, when these quantities are connected by the relation $L^{1/2}=\gamma v_{ns}$, everything is determined by one of them (Schwarz selected $L$). However, the solution of the model problem shows that there exists one more parameter in the problem, the flux of the curvature $P$ in $k$ space. The presence of an additional dimensional parameter implies that Schwarz's scaling analysis appears, at the least, to be incomplete. Furthermore, the solution of the model problem shows the importance of random forces affecting the long-range interaction in a vortex tangle. As in Schwarz's local approximation, this long-range contribution is missing; hence the corresponding analysis is apparently not complete. If we also add here the absence of stretching as a nonlocal effect, it may be concluded that some processes remain outside Schwarz's model. Agreeing in general to this incompleteness, Schwarz (1988) states that the omitted processes have a small influence compared with the

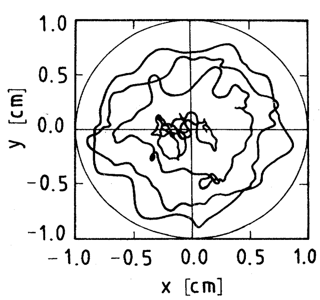


FIG. 13. Evolution of vortex ring under the influence of external random force in the local approach [Eq. (4.27)] (Nemirovskii *et al.*, 1991, Fig. 1). Projections of the vortex filament on the plane of the initial ring at different instants are shown.

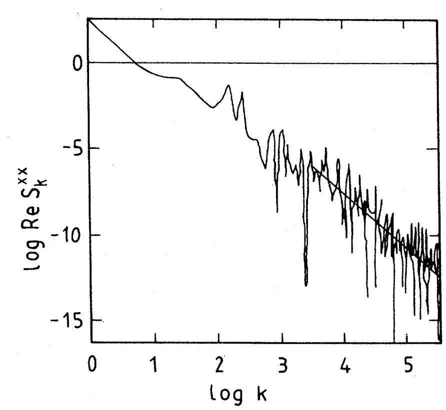


FIG. 14. Logarithm of the Fourier-transformed equal-time correlator $S_k^{xx}$ as a function of logarithm of the wave number $k$ (Nemirovskii *et al.*, 1991, Fig. 1). The straight line was calculated by linear regression; its slope is $-3.13$.

ones left in his approximation, which is confirmed by the good agreement of his calculations with experimental observations. When the absence of adjustable parameters in his calculations is considered, such an explanation is acceptable. However, one should remember that the experimental data also have a large scatter.

After Feynman's and Vinen's pioneering works, Schwarz's results were a great advance in the theory of ST. They changed the hypothesis of chaotic vortex lines into a realistic microscopic theory based on the dynamics of the vortex lines. However, when the critical remarks concerning the formulation of the problem and its solution are taken into account, it must be admitted that the theory of stochastic behavior of quantized vortices in He II is far from completion, although Schwarz's model appears to be a good working approximation.

### C. Transient behavior of superfluid turbulence

Schwarz and Rozen (1991) used the ideas and observations described in Sec. IV.B in their investigation of unsteady cases. Integrating with respect to the arclength $\xi$ and ensemble averaging of Eq. (4.3), giving the local variation of the vortex line length, result in

$$\frac{\partial L}{\partial t_0}=\frac{\alpha}{\Omega}\mathbf{v}_{ns,0}\int \mathbf{S}'\times\mathbf{S}''d\xi-\frac{\alpha}{\Omega}\int|\mathbf{S}''|^2 d\xi\ . \qquad (4.29)$$

Formally, on the right-hand side of the above equation, we find the same parameters as in the stationary case [see Eqs. (4.20) and (4.23)]. However, the use of these quantities in nonstationary cases is not obvious. But if it is assumed that the process is quasistationary, then the parameters $c_2$ and $I_l$ remain constant and (4.20) and (4.23) can be used. The relation (4.29) can then be transformed into the following Vinen-type equation (3.8)

$$\frac{\partial L}{\partial t_0}=\alpha I_l|v_{ns,0}|L^{3/2}-\alpha c_2^2 L^2\ . \qquad (4.30)$$

Thus assuming quasistationary conditions and using scaling relations leads to Vinen's equation. It is, however, necessary to add the following comment. As described in the previous section, the relations (4.20) and (4.23) used to deduce (4.30) express the scaling properties in the special case when the scaling parameter $\lambda$ is taken equal to $(L^*/L)^{1/2}$ [see Eqs. (4.13) and (4.14)]. As Schwarz stated, this was done for convenience. The same would justify the use of $(v_{ns}^*/v_{ns})$ as a scaling parameter. In view of $L=\gamma v_{ns}^2$, this would make no difference in the stationary case. However, in the nonstationary case, when $v_{ns}$ and $L$ are independent of each other, the quantities on the right-hand side of (4.29) and hence also on the right-hand side of (4.30) are determined up to a function of $L^{1/2}/\gamma v_{ns}$. In particular, by an appropriate choice, (4.29) can be transformed to the alternative form of the VLD equation (3.10). Thus we once more return to the problem of ambiguity concerning the generating term [see comments to Eqs. (3.8) and (4.8)].

To check the assumption of quasistationarity, Schwarz and Rozen (1991) investigated numerically the VLD dynamics $L(t)$ in a transient process. They examined how precisely the values of $I_l$, $c_2$, $I_\parallel$ remain constant. To avoid the question of initial vortex formation, the transition between two finite, nonzero $v_n$ values was observed. The free-decay process was also investigated.

The calculated variation in time of $L(t)/L_f$, where $L_f$ is the final equilibrium value of the VLD, is shown in Fig. 15. There is a qualitative agreement with the predictions made according to Vinen's equation, although, as stated by Schwarz and Rozen, the parameters $\alpha_V$ and $\beta_V$ must be changed. There is also a qualitative agreement with the alternative form (3.10) of the VLD equation. This problem could have been clarified if, as Vinen had done, the dependence of the dimensionless VLD, $d\mathcal{L}/dt$ on $\mathcal{L}^{3/2}(1-\mathcal{L}^{1/2})$ had been shown, which, unfortunately, it had not. The variation of the parameters of the vortex tangle structure, during the growth period illustrated in Fig. 15, is shown in Fig. 16. According to this figure, only the structure parameter $c_L^2$ remains fairly constant during the transition period. Schwarz and Rozen concluded that during the transition processes, when large changes of the heat input $Q$ occur, the equilibrium parameters of the tangle vary within 25%. Similar calculations were performed and conclusions drawn for a decaying tangle when the counterflow velocity was suddenly diminished. This numerical analysis allowed Schwarz and Rozen to infer that the Vinen VLD evolution equation (3.8) is a good approximation, at least for applied problems.

Besides numerical calculations, Schwarz and Rozen carried out an experimental investigation of the transient response of a thermal counterflow in a channel to a change in heat flux. Although these experiments fall outside the class of stationary homogeneous processes and should be discussed in the second half of this paper, the results will be considered here, since it appears that the superfluid turbulence remains quasistationary during the

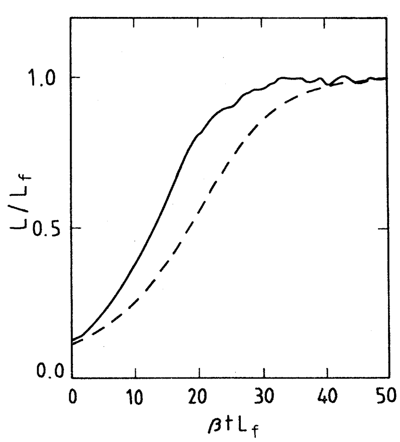


FIG. 15. Calculated growth of the line length density when $v_{ns}/\beta$ is suddenly increased from 40 to 120 (Schwarz and Rozen, 1991, Fig. 3). The dashed line is the prediction of Vinen's equation.

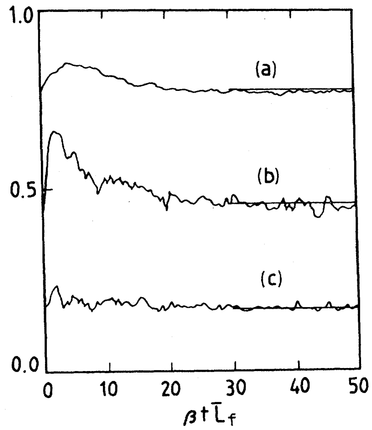


FIG. 16. Variation of the tangle-structure parameters during the growth transient of Fig. 15, showing (a) $I_{\parallel}$, (b) $I_l$, and (c) $c_2^3/20$ (Schwarz and Rozen, 1991, Fig. 4). The horizontal lines are the steady-state values given by Schwarz (1988).

transient. The results may then be compared with the calculations above and serve as a justification for the Vinen VLD evolution equation (3.8) in weakly nonstationary processes. Schwarz and Rozen measured the temperature drop $\Delta T$ between pairs of sensors placed along a uniform thermal counterflow channel. They found that the response of $\Delta T(t)$ to a small change in $Q$ could satisfactorily fit (3.8) if structure parameters that differ slightly from those obtained in the stationary case are used. These results suggest that in the nonstationary process the VT is more strongly polarized and more kinky than in the stationary homogeneous state. The final conclusion of these measurements is that VE (3.8) describes adequately VLD evolution, but the parameters $\alpha_V$ and $\beta_V$ must be 2 or 3 times larger than those obtained by Vinen.

Schwarz and Rozen also carried out a series of free-decay experiments similar to those of Vinen described above. They measured the time evolution of $\Delta T$ following the removal of a heat flux. Their results (Fig. 17) reveal exactly the same structure found by Vinen—an initial fast decay followed by a slow decay. Their interpretation of the data is quite different from Vinen's, however. They assumed that the initial fast decay is described by the Vinen equation (3.8) with the result given in (3.13). The dashed line in Fig. 17 is obtained from this equation through the use of their structure parameters obtained in the transient experiments. Schwarz and Rozen also proposed an explanation for the slow decay. However, as it is connected with the hydrodynamic model, we shall leave it to the second part of this paper, which is devoted to the corresponding considerations. Another important result was the confirmation of the existence of a small remnant VLD $L_{\mathrm{rem}} \sim 10\ \mathrm{cm}^{-2}$, which may be due to parasite heat sources. Although this question is beyond the scope of this review, it is related to the problem of initial conditions for the VE and avoids the necessity of introducing an initiating term to cancel the divergence of the integral (3.14). However, from an estimate of the integral, it follows that in this case $t_V \sim \dot{Q}^{-1}$, which contradicts (3.15). A special investigation of this problem might provide an explanation for the perennial problem in the theory of quantized vortices in He II, i.e., the initial appearance of vortex lines.

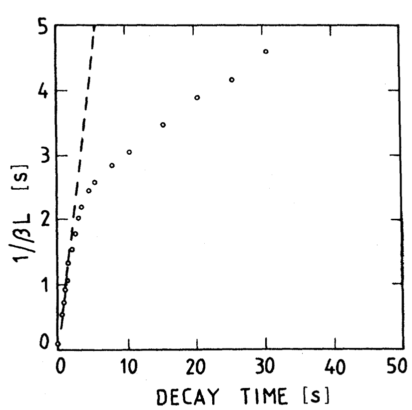


FIG. 17. Vortex-tangle decay curve (Schwarz and Rozen, 1991, Fig. 17). The dashed line is the decay behavior expected on the basis of the measurements of the growth ST and on the basis of Eq. (3.13).

Schwarz and Rozen's (1991) investigation of VLD along the channel led to one more interesting result. It appeared that the section closest to the heater responds first to the heat pulse, then the section close to the exit of the channel, and thereafter the middle section. This may be related to some boundary effects, although the authors state that this is due to the existence of two turbulence modes TI and TII (see Tough, 1982); but unlike the homogeneous TII mode, TL may exhibit some inherent spatial instability.

Besides the above-mentioned papers of Vinen and of Schwarz and Rozen, experiments by v. Schwerdtner *et al.* (1989c) were also designed to check Vinen's equation. They used the interesting method of short test pulses, "riders" superimposed on the top heat main pulse to measure the second-sound attenuation (see Fig. 18). The typical result illustrated in Fig. 5 shows a satisfactory agreement with VE with the coefficients $\alpha_V$ and $\beta_V$ close to Vinen's values. It should be noted that as the authors used the repeated pulse method, there was no initial VLD problem. It is also worth noting that no anomalous decay behavior was detected within the experimental test conditions. In connection with this experiment it can be remarked that, inside a fairly strong heat pulse, large perturbations of the temperature $T$ and counterflow velocity $v_{ns}$ may occur that should strongly distort the picture; hence some doubts about the reality of the quantitative results can exist. On the other hand, this is probably the unique, direct experimental verification of VE for heat pulses, $\dot{Q}$, very much higher than in previous experiments.

We tried to describe the works we know of devoted to a direct theoretical and experimental confirmation of the VLD evolution equation. For a complete description, the

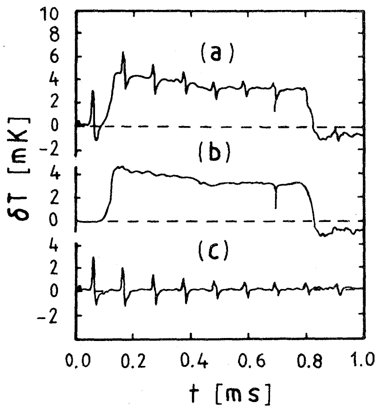


FIG. 18. Sample of damping measurements (v. Schwerdtner *et al.* 1989c, Fig. 2): (a) the primary pulse, with a superimposed testing pulse and several subsequent echoes, as registered by the sensor strip; (b) the primary pulse only; (c) the test signal, difference between (a) and (b).

papers of Sitton and Moss (1972), Ashton and Northby (1975), and Milliken *et al.* (1982), in which the VLD evolution during free decay was measured, should also be mentioned. These papers confirmed that the decay of $L$ takes place according to (3.7), although with different coefficients; however, the problem of the generating term was not considered.

Resuming this section, we can say that the progress made in studying nonstationary ST and its relation to the microscopic VT dynamics is not as advanced as in the stationary case. Therefore it is important to study the nonstationary hydrodynamic processes in He II, in the presence of the vortex tangle, as such investigations could supply essential information on the nonequilibrium structure and the macroscopic dynamics of the vortex tangle.

## V. HYDRODYNAMICS OF SUPERFLUID TURBULENCE

### A. Problem formulation

The previous sections covered the problems of stochastic behavior of vortex lines in He II, the structure of the vortex tangle, and the phenomenological description of superfluid turbulence using the VLD $L(t)$. The experimental study of superfluid turbulence is based mainly on hydrodynamic methods. The investigation of the vortex tangle by first- and second-sound probes, as well as the measurements of temperature and pressure gradients and of heat and mass flows, are just some examples of the use of such methods. On the other hand, all known methods of ST generation, for example, the use of counterflow or sound waves, are also of a hydrodynamic nature. The above examples show that there exists a reciprocal influence of the hydrodynamic quantities and the vortex tangle parameters. A change in the hydrodynamic characteristics leads to an immediate change in the VLD and vice versa. Therefore the study of superfluid turbulence and the study of hydrodynamic processes in the presence of a vortex tangle are indivisible parts of the same general problem. Hence a study of the properties of a vortex tangle, for example, in a counterflow, that assumes that the counterflow parameters are fixed ($\mathbf{v}_{ns}=\text{const}$) is not a fully justified procedure. An equally wrong procedure is an investigation of the flow properties that assumes the VLD is fixed. Although the above assertions are obvious, such approaches were widely used.

We conclude that the variations of the ST parameters and of the hydrodynamic variables are interconnected, and the aim now is to devise a system of equations describing these coordinated variations. This set of interconnected physical processes will be called the hydrodynamics of superfluid turbulence (HST). This problem appears to be quite intricate, and there exist a few approaches. We shall describe in this section the existing methods of deducing the equations of the HST and discuss the differences. Before discussing the different methods of deriving the different versions of the HST equations, it seems necessary to make a few remarks concerning the formulation of the problem. The main aim of formulating the HST is to combine Vinen's equation and the classical two-fluid Landau-Khalatnikov hydrodynamic equations (see, e.g., Khalatnikov, 1965 and Putterman, 1974). For this purpose it seems important to consider closely Vinen's assumption about the self-preservation state of the VT (see Sec. III.B), i.e., that a change of $L$ is determined only by $L$ and the hydrodynamic parameters. The more subtle characteristics of the vortex structure that differ from the VLD $L(t)$ do not participate in this macroscopic description of the process. This assumption is valid in the stationary homogeneous cases, in which only $L(t)$ and the orientation of the vortices in space determine the mutual friction force, which is responsible for most of the encountered effects. Questions concerning the possibility of describing hydrodynamic processes in the presence of a vortex tangle, in terms of only one variable $L$, when the flow cannot be considered stationary and homogeneous, arise. Indeed, in this case the quantity VLD $L(t)$ in the set of hydrodynamic equations must acquire a field property; i.e., it is necessary to impose its dependence on the coordinate. Hence $L(t)\rightarrow L(\mathbf{r},t)$. The rate of change of $L(t)$ must then obey the relation

$$\frac{dL}{dt}\rightarrow\frac{\partial L}{\partial t}+\operatorname{div}(L\mathbf{v}_L)\ , \tag{5.1}$$

where $\mathbf{v}_L$ is the drift velocity of the tangle.

Thus, already in the stationary but inhomogeneous case, a new variable connected with the fine structure of the vortex tangle appears. Obviously it is not possible to determine $\mathbf{v}_L$ from Vinen's phenomenological theory unless some other assumptions are made or experimental data acquired. We recall that Vinen assumed $\mathbf{v}_L$ to be equal to the velocity $\mathbf{v}_s$ of the superfluid component. It appears that, even in the stationary but nonhomogeneous case, it is necessary to introduce, in addition to the VLD, a new variable, which is lacking, in Vinen's theory. The nonstationary problems are much more involved.

For a better understanding of the importance of the above formulation of the problem, let us return to Schwarz's kinetic theory (Sec. IV.A). Although this theory is not perfect, it can illustrate the essence of the problem considered. This theory makes use of the distribution function $\lambda(\mathbf{v}_l,t)$. The evolution of $L$ is given by Eq. (4.4) which, integrated in the phase space $\mathbf{v}_l$, yields

$$\frac{\partial L}{\partial t}+\nabla\cdot\int \dot{\mathbf{S}}(\mathbf{v}_l)\lambda\, d^3\mathbf{v}_l=\frac{\alpha}{\beta}\int(\mathbf{v}_{ns}-\mathbf{v}_l)\mathbf{v}_l\lambda\, d^3\mathbf{v}_l \ . \tag{5.2}$$

The velocity of the elements of the vortex line $\dot{\mathbf{S}}(\mathbf{v}_l)$ should be expressed by the self-induced velocity $\mathbf{v}_l$ according to Eq. (4.1). Thus the temporal derivative of $L$, which is the zeroth moment of the distribution function $\lambda(\mathbf{v}_1,t)$, can be expressed by higher-order moments. Obviously, with the introduction of the higher-moment equations, there appear on the right-hand side still higher moments and so on. The resulting case is identical to the procedure of obtaining the gasdynamic equations using Boltzmann's kinetic theory. The difference consists in the absence of the "kinetic equation" in our case. Nevertheless, this analogy allows us to understand the essence of the problem and to point out ways to deal with it.

In the case of gasdynamics, the cutoff, or shortened description, procedure has two important features. First, due to the existence of a small parameter, the higher moments relax to equilibrium much faster than the lower ones. Second, the higher moments thus adjust themselves sooner to the lower ones and therefore can be expressed as functions of the last ones. The use of these functional relations permits us to obtain a closed set of equations. If we would like to use the same procedure in the case of ST and limit ourselves to the consideration of VLD $L(t)$ only, then it must be demonstrated that $\mathbf{v}_L=\int\dot{\mathbf{S}}\lambda\, d^3\mathbf{v}_l$, $\int \mathbf{v}_l^2\lambda\, d^3\mathbf{v}_l$ and so on relax faster than $L(t)$ and assume some functional relation, e.g., the following one,

$$\mathbf{v}_L=F(L,\mathbf{v}_n,\mathbf{v}_s,\text{ and say } p,T) \ . \tag{5.3}$$

The determination of a relation like (5.3) is a separate, difficult problem. We would like to stress that the existence of such a relation requires that the relaxation of the first moment of the distribution function $\lambda(\mathbf{v}_L,t)$ be much faster than the relaxation of the zeroth moment, i.e., $L(t)$. If this is not the case, then $\mathbf{v}_L$ must appear in the hydrodynamic theory as a separate, independent variable.

Strictly speaking, there are no theoretical grounds for assuming that the relaxation of higher moments is faster than that of the quantity $L(t)$. To respond properly to this problem, it is necessary to develop a full stochastic theory of the vortex tangle—a problem that remains open. Therefore, following Vinen, we apply his approach, remembering that it relies strongly on the assumption that $L(t)$ fully determines the macroscopic dynamics of the vortex tangle. Of course, it is understood that in using this assumption we are not interested in the special problems related to the fine structure of the vortex tangle and its effects. Some justification for using this assumption is that the resulting model describes well a multitude of nonstationary experimental observations. We assume, hence, that hydrodynamical processes can be fully described by the set of the two-fluid equations plus the VLD $L(t)$ equation according to Vinen's (3.8), which has to be slightly modified to include (5.1) and becomes

$$\frac{\partial L}{\partial t}+\operatorname{div}(L\mathbf{v}_L)=\alpha_V|\mathbf{v}_{ns}|L^{3/2}-\beta_V L^2 \ . \tag{5.4}$$

Because a relation like (5.3) is also unknown, we shall assume that $\mathbf{v}_L=b\mathbf{v}_{ns}$ where $b$ is a constant coefficient as in the case of steady flows.

### B. Phenomenological approach to the hydrodynamics of superfluid turbulence

Attempts to describe nonstationary hydrodynamic processes in He II that included ST actually began with the formulation of the Gorter-Mellink relation (see Putterman, 1974). In the simplest approach, the existence of a vortex tangle was taken into account by introducing a stationary force, given by Gorter and Mellink (1949) as $\mathbf{F}_{ns}=A(T)\rho_s\rho_n v_{ns}^2\mathbf{v}_{ns}$, on the right-hand side of the equations for the superfluid and normal velocities. The important improvement was to introduce in the Gorter-Mellink law the quantity $L$. This was achieved by making use of the asymptotic relation for $L$ leading to $\mathbf{F}_{ns}=A(T)\rho_n\rho_s(L/\gamma^2)\mathbf{v}_{ns}$, where $L$ must satisfy Viner's equation (3.8). The last mostly used approach is very effective and describes many observations. However, this approach has a limited range of applicability and, regretfully, what is very important, does not satisfy all required conservation laws.

However, had the hydrodynamic equations been considered as a set of conservation laws, one would have obtained a full closed set of equations for all the independent variables involved. This was achieved by Nemirovskii and Lebedev (1983) using the Bekarevich and Khalatnikov (1961) method.

The energy density $E$ of the He II vortex free flow is

$$E=\frac{\rho v_s^2}{2}+\mathbf{v}_s\mathbf{j}_0+E_0(\rho,S,\mathbf{j}_0) \ , \tag{5.5}$$

where $\mathbf{j}_0$ is the momentum density in the reference system relative to the superfluid component, and $E_0$ is the energy density of this system given by

$$dE_0=TdS+\mu d\rho+\mathbf{v}_{ns}d\mathbf{j}_0 \ . \tag{5.6}$$

In this classical two-fluid model the equations of motion are set up so as to obtain, using the conservation laws for $\rho$, $S$, $\mathbf{v}_s$, and $\mathbf{j}_0$, the energy conservation law. This self-consistent approach can be extended to the case of He II containing a vortex tangle.

The presence of vortices modifies the energy density $E_0$, which can be taken care of by adding the term

$$\delta E_0=\varepsilon_V dL \ , \tag{5.7}$$

where the energy of a unit length of the vortex line is

$$\varepsilon_V = \frac{\rho_s \kappa^2}{4\pi} \ln\left[\frac{1}{r_0 L^{1/2}}\right] . \tag{5.8}$$

Thus in the presence of a vortex tangle the energy of the He II flow becomes a function of $L$, which is a new independent variable in addition to the main set of variables $\rho$, $S$, $\mathbf{v}_s$, $\mathbf{j}_0$, where $L$ satisfies the relation (5.4). Besides the changes in the energy density, other hydrodynamic characteristics, such as the dissipation function, momentum density tensor, etc., must also be suitably changed. However, these changes must be such that the conservation laws for the set $\rho$, $S$, $\mathbf{v}_s$, $\mathbf{j}_0$, and $L$ lead to the conservation law for the full energy $E(\rho, S, \mathbf{v}_s, \mathbf{j}_0, L)$. This last requirement yields all corrections to the set of two-fluid equations, i.e., to a new set of equations, provided that an additional assumption has been made concerning the dissipation function. In their work the authors used the Feynman-Vinen model. In this model, two processes are responsible for entropy production. One of them is the work due to mutual friction between the vortex lines and the normal component. This work is proportional to the total length of the vortex line $L$ in unit volume and to an averaged velocity squared of the vortices relative to the normal component. Besides this mechanism of entropy production there is another one corresponding to the transformation of the energy from the breaking up of small vortex rings into thermal excitations. This last mechanism yields an entropy production equal to the product of $\varepsilon_V$ and the rate of decay of the vortex tangle $\beta_V L^2$. Thus the contribution $R'$ of the vortex tangle to the dissipation function is

$$R' = (K v_{ns}^2 L + \varepsilon_V \beta_V L^2)/T . \tag{5.9}$$

The coefficient $K$ can be determined if we assume that in the stationary homogeneous case $TR'$ corresponds to the work of the Gorter-Mellink force; hence

$$K = A(T)\rho_n \rho_s \frac{\beta_V^2}{\alpha_V^2} - \varepsilon_V \frac{\alpha_V^2}{\beta_V} . \tag{5.10}$$

The use of (5.9) allows for the unique determination of the equations of motion of He II in the presence of a vortex tangle, i.e., for the devising of a hydrodynamic theory of superfluid turbulence. The set of resulting equations is

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\mathbf{j} = 0 , \tag{5.11}$$

$$\frac{\partial j_i}{\partial t} + \frac{\partial \Pi_{ik}}{\partial x_k} = 0, \tag{5.12}$$

$$\frac{\partial S}{\partial t} + \operatorname{div}[S\mathbf{v}_n + S^L(\mathbf{v}_L - \mathbf{v}_n)] = \frac{1}{T}[K L v_{ns}^2 + \varepsilon_V \beta_V L^2] , \tag{5.13}$$

$$\rho_s \left\{ \frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \nabla)\mathbf{v}_s + \nabla\mu \right\} - b\nabla\varepsilon_V - S^L b \nabla T = K L \mathbf{v}_{ns} + \varepsilon_V \alpha_V \frac{\mathbf{v}_{ns}}{|\mathbf{v}_{ns}|} L^{3/2} , \tag{5.14}$$

where $\Pi_{ik}$ is the momentum flux tensor with an additional term, $\varepsilon_V L \delta_{ik}$, due to the presence of vortices, and $S^L$ is an additional entropy due to the vortex tangle.

Equations (5.11)–(5.14) and the modified Vinen equation (5.4) are an essential generalization of the two-fluid Landau Khalatnikov model and can be used for the study of relevant problems. The additional terms on the left-hand side of the above equations, the so-called reactive terms, are connected to the corresponding symmetry of the Hamiltonian of the system.

The dissipative terms on the right-hand side of (5.4) and (5.14) can be arranged as

$$\begin{bmatrix} \dfrac{\partial v_{si}}{\partial t} \\ \dfrac{\partial L}{\partial t} \end{bmatrix} = L \begin{bmatrix} -\dfrac{K}{\rho_s}\delta_{ik} & \dfrac{\alpha_V}{\rho_s}\dfrac{\mathbf{v}_{ns,i}}{|\mathbf{v}_{ns}|}L^{1/2} \\ -\dfrac{\alpha_V}{\rho_s}\dfrac{\mathbf{v}_{ns,k}}{|\mathbf{v}_{ns}|}L^{1/2} & -\dfrac{\beta_V}{\varepsilon_V}L \end{bmatrix} \times \begin{bmatrix} -\rho_s v_{ns,k} \\ \varepsilon_V \end{bmatrix} . \tag{5.15}$$

As $\partial E/\partial \mathbf{v}_{s,k} = -\rho \mathbf{v}_{ns,k}$ and $\partial E/\partial L = \varepsilon_V$, the relation (5.15) describes the Onsager reciprocity principle for the kinetic coefficients. The antisymmetry of the coefficients follows from the different behavior of $\mathbf{v}_s$ and $L$ under time reversal (Landau and Lifshitz, 1980). The right-hand side of (5.14), besides the customary Gorter-Mellink term $KL\mathbf{v}_{ns}$, also contains an additional term, $\varepsilon_V \alpha_V (\mathbf{v}_{ns}/|\mathbf{v}_{ns}|) L^{3/2}$. This term describes the additional damping resulting from the absorption of the kinetic energy of the flow by the VT during its growth. According to Feynman, this part of the energy is returned, during the tangle decay, to the main flow in the form of entropy. Usually, when the tangle is close to equilibrium, the Gorter-Mellink term is much larger than this additional term. However, when $L$ is very large and $\mathbf{v}_{ns}$ small, the additional term can become larger than the Gorter-Mellink term.

Note that the additional term has a very special form and does not depend on the magnitude of the velocity but only on its direction. This property recalls "dry friction" in classical mechanics and hence this additional term will be so called. The specific form of this term is directly connected to the generating term in Vinen's equation. Indeed, if we had used the alternative equation, (3.10), instead of Vinen's equation (3.8) to deduce the HST, the additional term would have been proportional to $L\mathbf{v}_{ns}$ and hence of the same form as the Gorter-Mellink term.

As in the stationary case $L \sim v_{ns}^2$, both terms on the right-hand side of (5.14) acquire the same structure; they are proportional to $v_{ns}^3$. Therefore the influence of the additional term is relevant from the standpoint of the dynamics of the VT, and its effect can be observed in nonstationary cases only (see Sec. VIII.B).

### C. A stochastic method of formulating the hydrodynamics of superfluid turbulence

Another method of deducing the set of HST equations based on the microscopic, Schwarz's "kinetic theory," was used by Yamada and Kashiwamura (1987) and Yamada *et al.* (1989). This approach was partly developed in Schwarz's (1978) original works. The problem, as formulated by Yamada *et al.*, is based on the Langevin stochastic equation and the use of the Fokker-Planck equation, allowing a deeper understanding of the essence of the stochastic approach. The following law of variation of the self-induced velocity results from the equation of motion (4.1) of the vortex line,

$$\dot{\mathbf{v}}_l = B(\mathbf{v}_l) + H(\mathbf{S}''', \mathbf{S}^{\mathrm{IV}}, \ldots) \tag{5.16}$$

where

$$B(\mathbf{v}_l) = -\frac{\alpha}{\beta}\{[(\mathbf{v}_{ns}-\mathbf{v}_l)\cdot\mathbf{v}_l]\mathbf{v}_l + \mathbf{v}_l\times(\mathbf{v}_l\times\mathbf{v}_{ns})\} \tag{5.17}$$

and $H(\mathbf{S}''', \mathbf{S}^{\mathrm{IV}}, \ldots)$ are terms with higher-order derivatives of the vortex line function $\mathbf{S}(\xi,t)$. Like Schwarz, the authors assumed that the contribution of these terms, which prevent us from closing the set of equations in the phase space $\{\mathbf{S}', \mathbf{S}''\}$, can be taken into account using the random-walk hypothesis. This allowed the authors to replace $H(\mathbf{S}''', \mathbf{S}^{\mathrm{IV}}, \ldots)$ with some quantity $\mathbf{R}(\mathbf{v}_l,t)$ which has the meaning of random actions in the $\mathbf{v}_l$ space, i.e., of a Langevin force. This force is assumed to be Gaussian with a correlator

$$\langle R_i(\mathbf{v}_l,t)R_j(\mathbf{v}_l,t)\rangle = \delta(t-t')D_{ij}(\mathbf{v}_l,\mathbf{v}_l') \ . \tag{5.18}$$

The authors introduced a distribution function $\lambda(\mathbf{v}_l,\mathbf{r},t)$ of the line length in the phase space $\mathbf{v}_l$,

$$\lambda(\mathbf{v}_l,\mathbf{r},t) = \langle \Lambda(\mathbf{r},\mathbf{v}_l,t)\rangle \ , \tag{5.19}$$

where $\langle \cdots \rangle$ denotes, as usual, ensemble averaging, and $\Lambda$ is defined as

$$\Lambda(\mathbf{r},\mathbf{v}_l,t) = \int d\xi |\mathbf{S}'(\xi,t)|\delta(\mathbf{r}-\mathbf{S}(\xi,t))\delta(\mathbf{v}_l-\mathbf{v}_l(\xi,t)) \ . \tag{5.20}$$

It should be noted, that, unlike Schwarz's approach, the distribution function $\lambda$ is dependent on $\mathbf{r}$, thus including cases of nonuniform space distributions.

The following "standard" operations consist in differentiating $\Lambda$ with respect to time and ensemble averaging. The following Fokker-Planck equation results,

$$\frac{\partial\lambda}{\partial t} + \nabla_r[\dot{\mathbf{S}}(\mathbf{v}_l)\lambda] + \frac{\partial}{\partial v_{l,i}}\left[B_i(\mathbf{v}_l)\lambda - \frac{\partial D_{ij}(\mathbf{v}_l)\lambda}{\partial v_{l,j}}\right] = \frac{\alpha}{\beta}(\mathbf{v}_{ns}-\mathbf{v}_l)\mathbf{v}_l\lambda \ . \tag{5.21}$$

This last relation is equivalent to Schwarz's equation (4.4) including spatial nonuniformity [second term on the left-hand side of (5.21)]. Unlike Schwarz, who concentrated on formulating the diffusion term and on the determination of the distribution function, the authors followed a different line. By integrating (5.21) in $d^3\mathbf{v}_l$, they eliminated the diffusion term and directly obtained the following equation for the VLD,

$$\frac{\partial L}{\partial t} + \nabla\cdot(L\mathbf{v}_L) = \frac{\alpha}{\beta}\int(\mathbf{v}_{ns}-\mathbf{v}_l)\mathbf{v}_l\lambda(\mathbf{v}_l,\mathbf{r},t)d^3\mathbf{v}_l \ . \tag{5.22}$$

The right-hand side of this equation can be calculated only after $\lambda(\mathbf{r},\mathbf{v}_l,t)$ has been determined. However, the authors, following Schwarz, considered this equation, term by term, agrees with the right-hand side of Vinen's equation (3.8).

Similarly, the authors determined the force exerted on the superfluid component by the vortex tangle. Assuming isotropy of the VT, they obtained

$$\mathbf{F}_{ns} = \rho_s\kappa\alpha\int(\mathbf{v}_{ns}-\mathbf{v}_l)\lambda(\mathbf{v}_l,\mathbf{r},t)d^3\mathbf{v}_l \ . \tag{5.23}$$

The quantity $\int \mathbf{v}_l\lambda\, d^3\mathbf{v}_l$ in (5.23), from the standpoint of its structure, coincides with the right-hand-side term of (5.22). Repeating the assumption that this expression corresponds to the generating term of Vinen's equation (3.8), the authors concluded that this term is

$$\frac{-\varepsilon_V\alpha_V}{\rho_s}\frac{\mathbf{v}_{ns}}{|\mathbf{v}_{ns}|}L^{3/2} \ . \tag{5.24}$$

Hence, besides the usual friction force proportional to $\mathbf{v}_{ns}L$, there is a supplementary term (5.24). This last term agrees with the term obtained previously in (5.14), except that the sign is reversed. Therefore the dissipative function has the structure

$$T\frac{dS}{dt} = \chi_1 L v_{ns}^2 - \varepsilon_V(\chi_2+\chi_3)L^{3/2} + \varepsilon_V^2\chi_4 L^2 \ , \tag{5.25}$$

which differs from the one obtained previously, Eq. (5.9). The quantities $\chi_i$ are related to the parameters of VE. The dissipation terms of the obtained equations lead to Onsager's reciprocal principle, but, unlike Eq. (5.15), with a symmetric coefficient.

Comparing the above with the HST equations, we note that Yamada *et al.* have not considered the reactive terms related to the energy contribution of the vortex tangle to the thermodynamic variables. However, this is of minor formal importance, since the contribution of the tangle to the energy or pressure in the usually encountered experimental conditions is negligible, although it can become noticeable at low temperatures.

Leaving out the problem of the reactive terms, we would like to discuss the remaining, very important difference between the results obtained in this and the previous section, i.e., the additional term in the mutual friction force, as we called it, the "dry friction" term. This term has an opposite sign in the work of Yamada *et al.* as compared with (5.14). From the formal point of view, this result was obtained because the authors accepted Schwarz's point of view that the generating term as well as the annihilation term of the VE is related to the action of the friction forces. Hence the decay of the vortex tangle is accompanied by the return of the energy from the vortex tangle to the kinetic energy of the main flow. (In Feynman's model the VT returned its energy to the main flow in the form of thermal excitations.) This may appear, for example, experimentally as a reduction of the effective Gorter-Mellink mutual friction.

Thus at first sight the formal difference in the interpretation of VE emerged in the HST equation. This favorable circumstance revealed the opportunity to check experimentally which mechanisms are closer to the physical reality, Feynman-Vinen's or Schwarz's. The proposed experiment is discussed later.

### D. Methods based on the variational principle

Geurst (1989, 1992) deduced the HST equations for the case of a one-dimensional flow from a variational principle. This approach, using the vortex tangle model as modifying the thermodynamic properties, is akin to the phenomenological method described in Sec. V.B. A distinctive feature of the method used is that Geurst did not consider the drift velocity $v_L$ as a variable determined from other quantities, as for example, (5.3), but used it as a new independent variable responsible for the macroscopic dynamics of the VT and obtained the differential equation for it (see the discussion in Sec. V.A). The starting premise in Geurst's approach is that besides the additional energy of the system due to the total length of the vortex tangle (5.7), the energy due to the momentum of the vortex tangle is added. Thus the differential of the internal energy is [compare with Eqs. (5.6) and (5.7)]

$$dE_0=\mu d\rho+TdS+\mu_L dL + P_n d(v_n-v)+P_L d(v_L-v) \;, \qquad (5.26)$$

where $P_n$ is the momentum density of the normal component, $P_L$ is the momentum density of the vortex tangle, and $v$ is the average mass velocity. Note that the coefficient $\mu_L$ differs from the previously introduced quantity $\varepsilon_V$. This difference is due to the inclusion of the momentum of the VT.

The formulation of the variational principle for He II turbulent flows is akin to Lin's (1963). In this method the set of variables $\rho$, $S$, $\mathbf{v}_n$, $\mathbf{v}$ is used instead of the usual one, $\rho, S, \mathbf{v}_s, \mathbf{j}$. Certainly this change of variables is possible once the set of equations is available; but it is known that in the variational method the choice of variables has an influence on the final form of the resulting equations (see Zilsel, 1950; Khalatnikov, 1952a; Lin, 1963; Ginzburg and Sobyanin, 1976). Finally, Geurst obtained six equations as compared with the five equations of Nemirovskii and Lebedev (1983) and Yamada *et al.* (1989). The additional equation is connected to the momentum of the tangle $P_L$ and is

$$\frac{\partial}{\partial t}\left[\frac{P_L}{L}\right]+\frac{\partial}{\partial x}\left[\frac{v_L P_L}{L}+\mu_L\right] = \left[F_{sL}+F_{nL}-\frac{r_L P_L}{L}\right]\Big/ L \;, \qquad (5.27)$$

where $F_{nL}$ and $F_{sL}$ are the interaction forces from the tangle acting on the normal and superfluid components, respectively. Their sum is not zero, due to the presence of the momentum of the tangle; $r_L$ is the right-hand side of the VLD evolution equation.

The additional Eq. (5.27) and the previously obtained Eqs. (5.4) and (5.11)–(5.14) are generalizations of the two-fluid model of He II to a three-fluid model. The third component is the VT.

Further, it is known that the two-velocity hydrodynamics is not closed, in the sense that, in this theoretical approach, the relation between, e.g., $\rho_s/\rho$ and $\mathbf{v}_{ns}$ cannot be determined exactly. It can be found only experimentally or result from the solution of an appropriate microscopic problem. Geurst encountered a similar difficulty. He tried to overcome this problem by using dimensional analysis. In particular, he assumed that the thermodynamic parameters of the VT can be expressed by the set of introduced variables with the help of a dimensionless number, which he called the Vinen number,

$$\mathrm{Vi}=\frac{\kappa L^{1/2}}{|v_L-v|} \;. \qquad (5.28)$$

The variational principle and the dimensional analysis allowed Geurst, using $v_L$ and $L$, to express all the introduced quantities and so to close the set of equations. In particular, the relation for the VT momentum follows:

$$P_L=\beta_g\rho_s\kappa L^{1/2}\frac{v_L-v}{|v_L-v|} \;, \qquad (5.29)$$

where $\beta_g$ is a constant undetermined in this approach.

The chemical potential of the VT is

$$\mu_L=\varepsilon_V-(v_L-v)\frac{\partial P_1}{\partial L} =\varepsilon_V-\frac{1}{2}\beta_g\rho_s\kappa L^{-1/2}|v_L-v| \;. \qquad (5.30)$$

The quantity $r_L$ corresponding to the right-hand side of the VLD evolution equation is

$$r_L=\varepsilon\alpha_g(v_n-v)L^{3/2}-\beta_g L^2 \;, \qquad (5.31)$$

where $\alpha_g$ and $\beta_g$ are parameters depending on $T$ and the quantity

$$\varepsilon=\mathrm{sgn}(v_L-v)\mathrm{sgn}(v_n-v) \;, \qquad (5.32)$$

which can be equal to $+1$ or $-1$ depending on the direction of the drifts of the normal component and of the VT relative to the average mass velocity's being, respectively, the same or opposite. Finally, the force acting, for example, on the superfluid component is

$$F_{sL}=A_g L(v_n-v)-B_g\mathrm{sgn}(v_L-v)L^{3/2} \;. \qquad (5.33)$$

Geurst determined the phenomenological coefficients in all the obtained relations by comparing his solutions in the stationary case with the VT parameters calculated by Schwarz, as described in Sec. IV.B.

In comparing Geurst's results with those described in the previous sections, we would like to make a few remarks. Concerning the reactive terms, they are close to those described in Sec. V.B because of the "similarities" between the phenomenological and variational methods. The slight differences are due to the corrections related to the VT momentum taken into account by Geurst.

These corrections are proportional to $(v_L - v)^2$, which is due to the special choice of the variational principle. If, as it is usually done, the superfluid velocity $v_s$ had been used instead of the mass velocity $v$, then $(v_L - v)$ would be replaced by $(v_L - v_s)$; but according to recent publications (see, e.g., Donnelly, 1991a), it appears that this is zero.

The additional term concerning the force acting on the superfluid component is of special interest because, as discussed previously, it has opposite signs depending on the use of the phenomenological or of the kinetic method to obtain the HST equations. This additional term, as can be seen from (5.29), has the sign of $\sin(v_L - v)$; i.e., depending on the direction of the VT drift relative to the average mass velocity, it can be positive or negative. If the variational formulation of the problem had been based on $v_s$ instead of $v$, then the sign of this term would be underdetermined as $v_L = v_s$.

The following remarks concerning Geurst's derivation of Vinen's equation from a pure macroscopic hydrodynamic analysis will not be out of place: The obtained result is rather unexpected, as usually it is accepted that the dynamics of the VT is subject to a microscopic theory. Let us note, however, that Geurst, had recourse, as had others, to dimensional analysis. He introduced the nondimensional parameter Vi (5.28) and developed some *a priori* unknown functions in terms of this parameter. As shown before, this does not lead to a unique derivation of the VLD $L(t)$ evolution equation. Note also that the quantity $r_L$ (5.31), which is the right-hand side of the VLD evolution equation, contained an unusual combination, $\varepsilon(v_n - v)$ instead of $|\mathbf{v}_{ns}|$, which in turn results from the selected formulation of the variational method.

We have shown in this section three different derivations of the HST equations. These equations are meant to describe the hydrodynamic processes in He II at velocities above the critical velocity. As the critical velocities are very small, most practical problems are in this range. The solution of these equations describes many cases of nonstationary hydrodynamic observations. They also forecast many interesting phenomena that are still to be discovered.

## VI. INTERACTION BETWEEN SECOND SOUND AND COUNTERFLOW

### A. Linear waves

Let us start by looking first at the problem of interaction between second-sound waves, heat pulses, and a counterflow including a VT of VLD $L(t)$. The VLD can depend on time, but the characteristic time of $L(t)$ must be much larger than the characteristic time of the test pulse. The study of ST using acoustic methods, primarily second sound, is the most used, widely applied experimental method.

The main idea is to take advantage of the extra attenuation of second-sound waves resulting from the friction force due to the interaction between the normal component and the VT (see Fig. 3). The corresponding relation for this damping, which is very much larger than the vicious damping, even for a very small VLD, was obtained by Vinen (1956) and is

$$\Gamma_V = B\kappa L_0/6 \;, \tag{6.1}$$

where $B$ is Hall-Vinen's coefficient given in Sec. II.A. The above relation was obtained from the solution of the set of equations for the propagation of second-sound waves where the interaction force between the normal component and the VT, proportional to $BL_0 v_n$, was added. In this approach it was assumed that the vortex tangle is isotropic (this is where the multiplier $\frac{1}{3}$ comes from) and that the value of the VLD is "frozen" equal to $L_0$. Equation (6.1), correlating the measured quantity $\Gamma_V$ with the VT, is the main relation used for the evaluation of the VLD, $L(t)$. However, a more detailed analysis of the HST equations, obtained in the previous section, shows that Vinen's approach is a greatly simplified one. One of the main simplifications is the assumption that $L_0$ is "frozen." Indeed, as the second-sound test pulse changes the value of $v_{ns}$, a corresponding change should affect $L(t)$. Hence, to the equations for $v_{ns}$ and $T$ used by Vinen, the equation for the VLD evolution must be added. Dissipative terms in addition to the Gorter-Mellink term must also be taken into account. From a consistent study of this problem by Nemirovskii and Lebedev (1983) and Kuznetsov (1991), it appears that the dispersion law of a monochromatic second-sound wave $\delta T \exp[i(\omega t - k_x x - k_y y)]$ propagating in a counterflow with a VLD $L_0$ follows the dispersion law $\omega = \omega(\mathbf{k})$ given below,

$$\omega = u_2|\mathbf{k}| + i\left\{\frac{\Gamma_y k_y^2}{k^2} + \frac{\Gamma_x k_x^2}{k^2} + A_1\frac{k_x}{k}\right\} + i\frac{A_3 k_x^2 - A_2 k k_x}{(iu_2 k\tau_V + 1)k^2} \;. \tag{6.2}$$

$K$ is given in Eqs. (5.9) and (5.10):

$$\Gamma_y = \frac{1}{2}\left[\frac{K\rho L_0}{\rho_s\rho_n} \pm \frac{\varepsilon_V \alpha_V L_0^{3/2}}{\rho_s v_s}\right] , \qquad \Gamma_x = \frac{K\rho L_0}{2\rho_s\rho_n} \tag{6.3}$$

are damping coefficients in the $x$ and $y$ directions; and

$$\tau_V = 2\beta_V/(\alpha_V^2 v_{ns}^2) \tag{6.4}$$

is the relaxation time of the VT. Other coefficients are given in the paper by Nemirovskii and Lebedev (1983). The sign $\pm$ reflects the difference between the two alternative forms of the equations discussed in the previous section.

The above relation shows that the interaction mechanisms of a second-sound wave and a counterflow are much richer than it appears only from the additional damping (6.1) introduced by Vinen. The main difference occurs when the wave propagates at an angle to the main flow; i.e., when $k_x \neq 0$. Indeed, in the case of transversal sound propagation, we have a "pure" extra attenuation with a decrement $\Gamma_y$. It should be noted that in view of (5.10), $\Gamma_y$ is strictly equal to the Vinen expression $\Gamma_V$ (6.1) only in the stationary case when $L$ and $v_{ns}$ satisfy (3.11). Numerical analysis shows that the difference between $\Gamma_y$ and $\Gamma_V$ is less than 10% smaller than in the case of a steady counterflow, but depends strongly on the relation between $L$ and $v_{ns}$ in transient processes. Formally $\Gamma_y \rightarrow \infty$ when $v_s \rightarrow 0$, i.e., when the counterflow is switched off. This, however, is an artifact, as the linearization procedure breaks down when $v_s \rightarrow 0$. Nevertheless, it can be expected that diminishing $v_s$ while keeping $L_0$ fixed or slowly decreasing leads to an increase (decrease for the minus sign) of the damping, i.e., to an apparent, but not real, increase (decrease for the minus sign) of the VLD. In principle this difference in the sign of the extra attenuation of the second sound could be used to determine which one of them is correct. Unfortunately, as this effect is very small compared with the large experimental scatter, this is hardly possible. The third term on the right-hand side of (6.2) describes both damping and dispersion of the second-sound velocity. This is explained by the modulation of the VLD by $v_{ns}$ oscillations. The equation of these $\delta L$ modulations has the character of a relaxation with a characteristic time $\tau_V$ (6.4). This appears to correspond to the classical case described by Mandelshtam and Leontovich (1937). There theory shows that the propagation of sound is subject to an additional damping and a dispersion if a relaxation process of the system exists. Let us note that there are no VLD modulations for transversal sound, as in Vinen's equation the absolute value of $v_{ns}$ appears which in the linear approximation is constant with respect to the wave amplitude. It is interesting to note that when the wave vector $\mathbf{k}$ is inclined with respect to the longitudinal axis, then $\delta \mathbf{v}_{ns}$ is not parallel with respect to $\mathbf{k}$. The following relation between them exists:

$$\frac{\delta v_{ns,y}}{\delta v_{ns,x}} = \frac{k_y}{k_x} \left[ 1 + \left[ \frac{K\rho L_0}{\rho_s \rho_n} + 2\Gamma_x + \frac{iA_3\tau_V}{iu_2 k\tau_V + 1} \right] \frac{i}{u_2 k} \right] . \tag{6.5}$$

The noncolinearity of the vectors $\mathbf{k}$ and $\delta \mathbf{v}_{ns}$ is due to the inclination of the sound wave which modulates the values of the VLD and produces a periodical variation of the Gorter-Mellink force; this in turn leads to the generation of secondary waves in the $x$ direction. The influence of the VLD modulations on the additional damping and its dependence on frequency for a purely longitudinal sound wave was considered in many papers (see, e.g., Goeje, 1986). Earlier the question of VLD modulations was discussed studying intrinsic fluctuations of ST, which are described in Sec. VIII.

Since the questions of dispersion and extra damping of second sound in the presence of vortex lines have arisen, it is necessary to recall that the Hall-Vinen coefficients $B$ and $B'$ depend on frequency (see Sec. II.B). This effect, first noticed in sound propagation in rotating bulk helium and later in the cases of supercritical counterflow (Vidal *et al.*, 1971; Vidal, 1972), was explained by Mehl (1974; see also Sec. II). It brings naturally an additional dispersion and dissipation. Studying the kinetics of the quasiparticle–vortex line interaction, Mehl (1974) assumed that the parameter $B$ has an imaginary part $iB_2$ which leads to a correction to the velocity of the second sound,

$$\Delta u_2 / u_2 = \kappa B_2 L_0 / 6\omega . \tag{6.6}$$

Thus the laws of propagation of axial and transversal sound are slightly different. In particular, the investigation of ST using transversal sound is more effective for the determination of the properties of the equilibrium VT, i.e., the Gorter-Mellink constant $A(T,p)$ and the variation of the VLD with $v_{ns}$. The measured values of $A(T)$ are shown in Fig. 19. The big differences in the data of Vinen (1957a) and Ostermeier *et al.* (1980), although transversal sound was used in both cases, are worth noting. Tough (1982) suggests that this big difference can be due to different ST states, TI and TII, in the two compared cases. The spread of the data on longitudinal sound, considering the variety of factors affecting the velocity of propagation, is not surprising.

Information on the Gorter-Mellink coefficient, which contains only the ratio $\alpha_V/\beta_V$, is gained from transversal sound, whereas from data on longitudinal soundings both coefficients of Vinen's equation can be obtained. Longitudinal sound experiments are reported in the papers of Kramers *et al.* (1960; Kramers, 1965), Ijsselstein *et al.* (1979), and de Goeje (1986). The excess attenuation $\Gamma_V$ of the longitudinal second-sound wave and of the dispersion Re$\omega$ of the second-sound velocity as functions of the counterflow velocity $v_{ns}$ is shown in Fig. 20(a) and 20(b). The kinks of the curves at $v_{ns} \sim 2-3$ cm/s are due to transition from TI-TII turbulence structure as explained

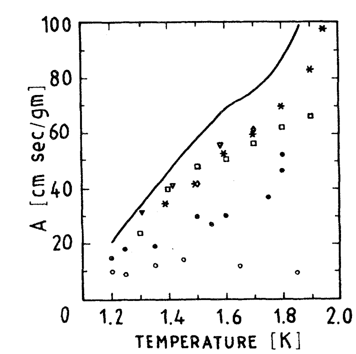


FIG. 19. The Gortert-Mellink coefficient $A(T)$ determined from second-sound attenuation measurements (Tough, 1982, Fig. 25). Triangles are from Vinen (1957a); solid circles, from Kramers *et al.* (1960); open circles, from Ostermeier *et al.* (1980); squares, from Kramers (1965); diamonds, from Ijsselstein *et al.* (1979); stars, from Fiszdon *et al.* (1991). The solid line shows $A(T)$ from Schwarz's theory (1978).

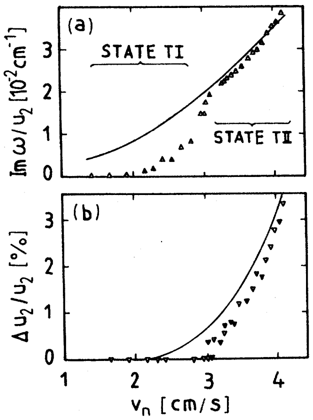


FIG. 20. (a) The excess attenuation of longitudinal second sound for a pure normal-fluid flow in a circular channel at 1.4 K (Ijsselstein *et al.*, 1979). The solid line is computed from Eq. (6.1). (b) The dispersion of second sound determined simultaneously with the attenuation data in (a) (Ijsselstein *et al.*, 1979). The solid line corresponds to Eq. (6.7) unifying the dispersion law (6.2) and Mehl's effect (6.6). (Tough, 1982, Fig. 28.)

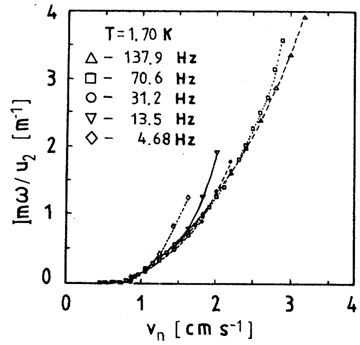


FIG. 21. All final results for the extra attenuation $\Gamma_V/u_2$ vs $v_n$ as obtained for the bath temperatures $T=1.70$ (de Goeje, 1986, Fig. 5, p. 57). These data show a significant dependence of $\Gamma_V$ on the measuring frequency.

by Tough (1982). Without contesting this explanation, we would like to point out that at least the shape of the $\Delta u_2/u_2$ curve can be explained using the relations (6.2)–(6.4) resulting from the HST theory. Combining the dispersion law (6.2) with Mehl's relation (6.6) and assuming $\omega\tau_V \gg 1$, we obtain

$$\frac{\Delta u_2}{u_2}=\frac{2\Gamma_x}{\omega^2\tau_V}-\frac{\kappa B_2 L}{6\omega} . \tag{6.7}$$

This relation is illustrated by the continuous line in Fig. 20(b). Thus the kink on the excess attenuation curves $\Gamma_V$ and $\Delta u_2/u_2$ can be explained by the combined effects as noted by Mehl and on the VLD modulation. The relation $\Delta u_2/u_2 \propto v_{ns}^4$ also agrees with Vidal's (1972) measurement.

The dependence of the damping from the sound frequency and its direction are also given by relations (6.2)–(6.4). This was confirmed by Goeje's (1986) measurements. The variation of the extra attenuation with the relative velocity is shown in Fig. 21, where it can be seen that the curves depending on the frequency branch off. Unfortunately, it is difficult to determine the $\Gamma_V$ dependence on the frequency $u_2k$ from this curve. Using similar experimental results and a theoretical analysis similar to the one described above, Goeje (1986) deduced the $\alpha_V$ and $\beta_V$ coefficients. It appeared that they are about 2 times larger than Vinen's. This agrees with Schwarz and Rozen's measurements for increasing VLD. Analogous results were obtained by Fiszdon *et al.* (1991). Unlike the Leiden group, the authors did not use monochromatic waves, but followed the dynamics of single top hat pulses which, due to dispersion and dissipation, change appreciably their shape. From the analysis of the development of the observed pulses, the value of $A(T)$ and of the coefficients $\alpha_V$, $\beta_V$ could be deduced. It can be noticed that the values of the Gorter-Mellink constant obtained from the discrete pulse measurements differ slightly from the monochromatic wave results and are closer to Schwarz's theoretical predictions. The Vinen parameters $\alpha_V$ (and, correspondingly, $\beta_V$), like Schwarz and Rozen's and de Goeje's, were found to be 1.5–2.0 larger than Vinen's.

It is interesting that the acoustic method gives the same $\alpha_V$ and $\beta_V$ values as for the growing vortex tangle. No special features can be noticed during the slow decay period. It is possible that this is an indication that the anomalously slow decay is connected to very small values of the VLD.

Thus experimental investigations of ST using second sound together with a theoretical analysis based on the HST set of equations have brought a wealth of qualitative and quantitative information concerning the structure and dynamics of the VT. Taking into account the complexity of the dispersion law (6.2)–(6.5), we can expect that the possibilities of this research approach are far from exhausted.

However, there are some acoustical experimental results that cannot be explained by the dispersion laws (6.2)–(6.4), and thus they indicate some new VT properties that do not fit into the Feynman-Vinen theory.

We have already referred to the experiments of Wang *et al.* (1987). They measured simultaneously the extra attenuation of second sound in the longitudinal and transversal directions. The temperature drop along the channel was also measured. This allowed the authors to conclude that the VT is strongly anisotropic. Analyzing the dynamics of the vortex lines, the authors determined the anisotropy characteristics of the VT and obtained an expression for the drift of the tangle. This observed anisotropy cannot be explained by the dispersion law (6.2), because at the high frequencies used by Wang *et al.* the influence of anisotropy on this effect is very small. Thus the observed phenomena are related to an additional property of the VT which is not described either by the

VE or by the HST equations. The variation of the anisotropy coefficient and on the drift velocity with temperature is shown in Figs. 10 and 11. This result agrees with Schwarz's (1988) numerical results, also shown in these figures. The anisotropy of the VT is of fundamental importance to the entire phenomenological theory of ST and would change appreciably Vinen's equation. Moreover, in this case the conception of the vortex line density $L$ must be totally modified and would very likely have a tensor structure. Wang *et al.* reported also that neither the anisotropy nor the drift velocity depended on the counterflow velocity. The independence of the degree of anisotropy on the value of the velocity $v_{ns}$ is quite unexpected. Indeed, the counterflow velocity $\mathbf{v}_{ns}$, which is here apparently an external applied field, is the only cause of anisotropy. Hence it would be appropriate to expect that, as, for example, in magnets, the degree of anisotropy should depend on the applied external field. The observed independence of the degree of anisotropy implies possibly that the isotropic VT becomes anisotropic under the influence of a small counterflow velocity $v_{ns}$; i.e., the isotropic state is unstable. Thus a small change in the direction of the counterflow velocity could produce a change in the direction of polarization of the tangle.

Nor can the acoustic measurements of the VLD made by Ostermeier *et al.* (1978) be explained by the HST theory or the Feynman-Vinen model. The authors used the second-sound burst technique and obtained extra attenuation $\Gamma_V$ which, according to (6.1), is proportional to the VLD $L$ as shown in Fig. 22. The results show some disagreement with Eq. (3.11). This result strongly contradicts the Feynman-Vinen theory. The authors also reported a strong nonuniformity of the VLD along the channel. This was refuted by Henberger and Tough (1982), who did not notice any nonuniformity of the VLD distribution in the channel. It is probably connected to the influence of the boundaries on the processes of generation of vortex lines.

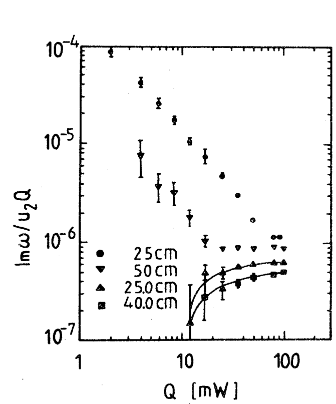


FIG. 22. $\Gamma_V/u_2Q$ vs $Q$ at 1.45 K for four positions down the channel (Ostermeier *et al.*, 1978, Fig. 2). Solid lines are obtained from corrections of VE (3.8) due to the size of the channel using a fitting procedure.

## B. Nonlinear second sound

There are only a few results on the interaction of nonlinear second-sound waves with ST, and we shall describe them briefly now. The physics of nonlinear phenomena is much richer and diverse than that of the linear ones. This broadens greatly the possibilities of its use as a probe of the VT. The properties of nonlinear sound waves in He II are described in Nemirovskii's (1990) review, in which a number of examples about the use of nonlinear effects to study superfluid properties is given. For example, Goldner *et al.* (1991) made use of such effects to measure the Kapitza resistance (see also Tsoi and Nemirovskii, 1980). As reported earlier by Khalatnikov (1965), data on nonlinear wave experiments permitted the determination of some thermodynamic quantities, in particular, their dependence on the counterflow velocity. Investigations of higher-intensity perturbations such as shock waves led Liepmann and Laguna (1984) and Cummings *et al.* (1978) to the conjecture that vortex lines can be generated by second-sound waves and shock waves of finite amplitude. One of the main advantages of this approach is that the velocity of nonlinear waves, and of shock waves, depends on their amplitude. The change of the wave amplitude, caused, for example, by extra attenuation, produces a change in the time of flight of the signals. This allows the determination of the wave amplitude variations with higher accuracy than by their direct measure.

The difficulties are thus transferred to the theoretical part, where the corresponding effects can be calculated. This will be demonstrated by solving the problem of propagation of a transversal nonlinear second-sound pulse or wave in a counterflow (Nemirovskii and Lebedev, 1983).

The nonlinear analysis of the second-order HST equations results in the following relation for the dynamics of the nonlinear wave,

$$\frac{\partial v_s}{\partial t}+[\alpha_{\mathrm{Kh}}(T)v_s+u_2]\frac{\partial v_s}{\partial y}=-\Gamma_V v_s \tag{6.8}$$

(see the notation in the previous section); $\alpha_{\mathrm{Kh}}(T)$ is the Khalatnikov (1965) nonlinearity coefficient. Equation (6.8) can be solved using the method of characteristics.

One of the results that can be used in the evaluation of experiments is the relation between the time of flight $t_{\mathrm{fl}}$ of the shock front and $\Gamma_V$ and $L_0$ from the relation (6.1).

Integrating (6.8) yields

$$\int_0^{t_{\mathrm{fl}}}[u_2+\tfrac{1}{2}\alpha_{\mathrm{Kh}}(T)\delta v_s(t)e^{-\Gamma t}]dt=y_{\mathrm{fl}}\ , \tag{6.9}$$

where $\delta v_s(t)$ is the value of the superfluid velocity at the shock front, which can also be calculated from (6.8), and $y_{\mathrm{fl}}$ is the distance between the emitter and receiver of the second-sound pulse. Thus, as expected, the experimental study of the VT using nonlinear sound (shock waves) removes the difficulty connected to measuring the pulse amplitudes.

## VII. VORTICES IN INTENSE AND MODERATE SECOND-SOUND PULSES

### A. Generation of vortex lines by second-sound pulses

Although the use of second-sound waves, described in the previous section, to study ST is widely applied, it is not very informative in studies of macroscopic VT dynamics. Indeed, as shown by the dispersion law (6.2), its dynamics is reflected only in the relaxation time $\tau_V$. Moreover, the corresponding effects appear as corrections to the additional damping (6.1) containing information only on the value of the VLD and not on its dynamics. This is due to the linearization of the HST equations, i.e., to the study of only small perturbations of the equilibrium state. The situation changes greatly when one investigates He II flows with large variations of the VLD, $\delta L \sim L$. The resulting vortex tangle affects greatly the flow that contributed to its existence, and it is necessary to account for their interaction in the first approximation. This occurs when moderate and long enough heat pulses propagate through He II.

Although ST has been observed and studied for about 30 years, the investigation of ST generated by strong heat pulses up to 100 $W/cm^2$ and microsecond duration is fairly recent. In spite of the fact that a very large number of investigations were concerned with counterflow velocities of the order of a fraction of 1 $cm\, s^{-1}$, researchers studying intense second-sound waves with counterflow velocities of the order up to 1 $m\, s^{-1}$ did not mention at all the problem of generation of quantized vortices (see, e.g., Osborne, 1951; Dessler and Fairbank, 1956; Kitabatake and Sawada, 1978; Mezhov-Deglin *et al.*, 1980; Lutset *et al.*, 1981; Atkin and Fox, 1983; see also the books of Khalatnikov, 1965; Wilks, 1967; and Putterman, 1974). Now it is obvious that the disagreements were due to the very short duration of the pulses, which were too short to generate a VT which is strong enough to influence the waves that generated them. Moreover, in the nonlinear wave theory of He II there exist many complications that overshadow the vortex-generating phenomena, considering them nonessential. In the review paper of Liepmann and Laguna (1984) concerned with nonlinear pulses and second-sound shock waves, systematic deviations from the theoretical predictions that led to the idea of breakdown of superfluidity and formation of vortices in nonstationary conditions are considered. One of the first results on the dynamics of intense second-sound waves, where a strong disagreement with the theory of nonlinear waves and generation of ST was suggested, was described in the paper of Cummings *et al.* (1978), and the variation of the second-sound velocity with the pulse intensity was obtained. Their results are reproduced in Fig. 23. The theoretical calculations of Lutset *et al.* (1981) using the two-fluid vortexless model are shown by the dashed line in this figure. It can be seen that at heat fluxes exceeding $\sim 15\ W\,cm^{-2}$, there is a strong disagreement between the measured and calculated velocities. The observed differences depend on the temperature and change sign at $T \cong 1.85$ K, i.e., at the point where the nonlinearity coefficient $\alpha_{Kh}(T)$ also changes sign (see Sec. VI.B). This can be explained by additional reduction of the wave amplitude and, consequently, to the appearance of a vortex structure.

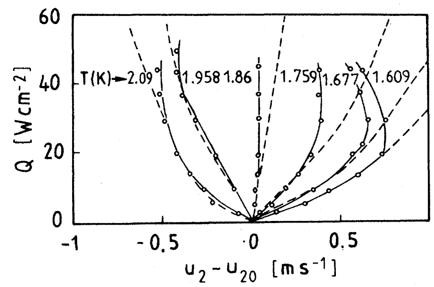


FIG. 23. Experimental measurements of the velocity of the nonlinear second-sound wave, shown by circles (Cummings *et al.*, 1978), and theoretical calculations, shown by dashed lines (Lutset *et al.*, 1981), based on vortexless equations of hydrodynamics of He II.

Turner (1979, 1983) drew similar conclusions from his observations on periodic rectangular pulses. He noticed strong distortions of the output signals, illustrated in Fig. 24, with an increase in the pulse amplitude. The distortions could not be explained by the usual nonlinear theory. Increasing further the input power, he also observed a limiting profile independent of the input (see Fig. 24). Between other explanations of his observations, Turner indicated the existence of an intrinsic critical velocity.

The described observations were an indirect indication that propagation of strong heat pulses lead to the appearance of vortices. It should be noted that the above results were interpreted as in classical gas dynamics, that the perturbations of curved shock fronts produce vortices. Nemirovskii and Tsoi (1982) undertook a direct check of these suppositions. In their experiments the authors sounded the wake of the main heat pulse by a transverse second-sound test pulse that was also nonlinear. As shown in Sec. VI.B, the time of flight of a nonlinear pulse propagating through a VT depends on the VLD $L$. The

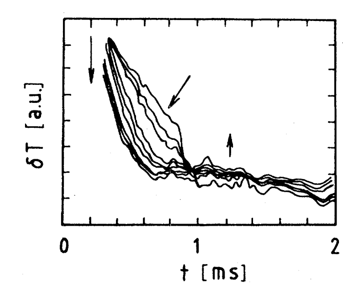


FIG. 24. Effects of heater power on the shock pulse profile [Turner, 1983, Fig. 4(d)]. Arrows indicate the change in profile with increasing heater power. There is a limiting profile which does not change with a further increase in the input power.

differences in the times of flight of test pulses propagating in the unperturbed helium and after the passage of a main heat pulse are shown in Fig. 25. The asymptotic value of about 60 $\mu$s corresponds to the difference in the time of flight of the linear and the nonlinear wave. As can be seen in the figure, the main pulses of amplitude $Q \geq 30$ W cm$^{-2}$ and duration $t > 100$ ms reduce the amplitude of the test pulse practically to zero.

It can be noticed further that, independently of the input power, very short main pulses do not affect the time of flight of the test pulse, which implies that the duration of the main pulse is as important as its amplitude for the formation of vortices. Hence the vortex tangle does not develop at the shock front but behind it during the heating time $t_H$, when the relative velocity $v_{ns}$ still exists. This agrees with Vinen's conceptions. Moreover, the order of magnitude of the time of generation of a vortex structure $t_V$ agrees with Eq. (3.15). From the same measurements, it appeared that the decay of ST also agrees roughly with Vinen's theory.

Similar experiments were conducted by Torczynski (1984b) using a longitudinal test pulse identical to the main pulse. The distortion of the test pulse that follows the main pulse depending on the rest time between pulses is shown in Fig. 26. This observation demonstrates clearly the appearance of the VT. However, in this experimental setup, it cannot be determined where, along the channel, vortices are formed. Torczynski asserts that the distortion occurs close to the heater. This assertion is based on a previous experiment on converging spherical second-sound waves (Torczynski, 1984a). Let us note that in the above-cited experiments of Nemirovskii and Tsoi (1982), the test pulse was located fairly far from the heater, which shows the possibility of vorticity generation in the volume but not necessarily close to the heater. Barenghi (1982) arrived at similar conclusions. His experiments, made using second-sound attenuation, allowed him to conclude that the vortex tangle can be created by shock waves, and the velocity at which the initiation of vorticity is propagated is the velocity of these shock waves.

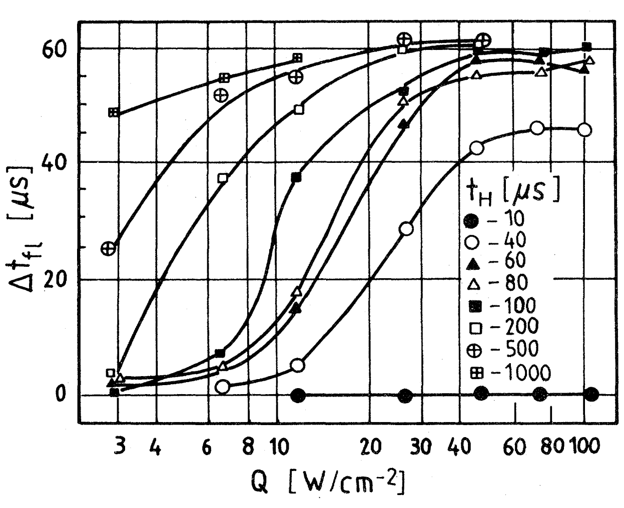


FIG. 25. Differences in the times of flight of the test signal propagating through undisturbed He II and through a wake behind the heat pulses of duration $t_H$ and amplitude $Q$ (Tsoi, 1987, Fig. 22).

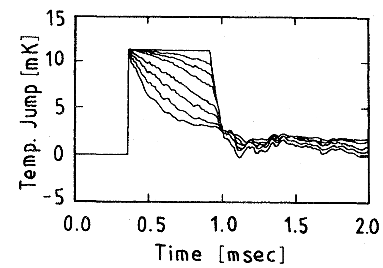


FIG. 26. Decreasing the separation time enhances degradation of the flow behind the successive shock front (Torczynski, 1984b, Fig. 3). The wave forms shown are the initial shock (highest trace) and successive shock with separation times of 20, 10, 5, 2, 1, 0.5, and 0.2 sec. The distance to the heater is 7.2 cm.

In other experiments, which also indicated the generation of a VT under the influence of short intense heat pulses, boiling of He II was observed. From the standpoint of the vortexless two-fluid model even at very strong heat pulses up to 100 W cm$^{-2}$, the temperature perturbation $\delta T$ is not higher than $\delta T \sim 0.05-0.1$ K (see Khalatnikov, 1965 and Putterman, 1974). This may turn out to be insufficient to reach even the "He II–vapor" equilibrium curve in $p$-$T$ coordinates, whereas it is known that considerable superheating is required for boiling using pulse heating (see Kraft, 1978 and Rybarcyk and Tough, 1980). Some results of experiments on nonstationary boiling of He II are shown in Fig. 27. The curve marked "VS" shows van Sciver's (1979) measurements at moderate heat fluxes, $\dot{Q} \leq 4$ W cm$^{-2}$, of top hat form. Van Sciver proposed the following relation for the time necessary for boiling to start,

$$t_B = B_{VS} \dot{Q}^{-4} , \tag{7.1}$$

where $B_{VS}$ depends on the temperature (e.g., for $T = 1.8$ K, $B_{VS} \approx 110$ W$^4$ s cm$^{-3}$). Tsoi and Lutset (1986) used higher-intensity pulses and suggested for $t_B$ the following relation (curve $TL$ in Fig. 27),

$$t_B = B_{TL} \dot{Q}^{-2} , \tag{7.2}$$

where $B_{TL} \approx 0.05$ W$^2$ s cm$^{-4}$. Vinen's (3.15) curve for the vortex formation time is also shown in Fig. 27 (curve $V$).

The relation between the boiling and vortex formation processes is clearly seen in Fig. 27, removing thus the disagreement between Eqs. (7.1) and (7.2). Indeed, in these two different experiments the states of the VT with reference to the equilibrium VLD were quite different, and this explains the difference in relations (7.1) and (7.2). He II boiling at intense heat pulses was also observed by Miklayev *et al.* (1987), Sydyganov and Kluchnikov (1988), Ametistov (1988), Ruppert *et al.* (1987), and

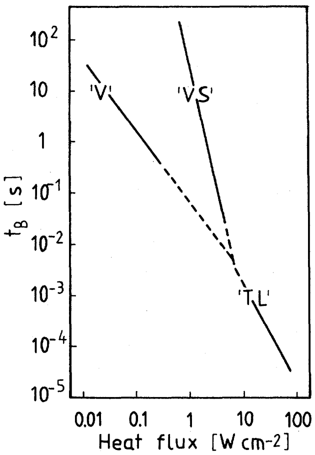


FIG. 27. Boiling time $t_B$ as a function of applied heat flux (Nemirovskii and Tsoi, 1989, Fig. 5). Curve VS, obtained in experiments of van Sciver (1979), corresponds to Eq. (7.1). Curve TL is from the experiments of Tsoi and Lutset (1986). Vinen's curve V corresponds to Eq. (3.15).

Danil'chenko and Poroshin (1983). The experiments described above proved that intense heating of He II leads very rapidly to the development of a vortex structure which changes essentially the hydrodynamic properties of He II.

Somewhat unexpected were the observations of Carey *et al.* (1978), Schwarz and Smith (1981), and Milliken *et al.* (1982) on the VT generated by high-intensity ultrasonic beams. Indeed, in the case of first sound, $v_{ns}=0$; and it may appear that there is no reason to initiate a VT. A possible explanation is that high-intense first-sound waves transform into second-sound ones due to nonlinear effects (Pokrovskii and Khalatnikov, 1976; Putterman and Garret, 1977; Nemirovskii, 1990). As shown by Kotsubo and Swift (1989), a secondary harmonic second sound leads to the initiation of a VT in the normal way due to the existence of a counterflow velocity $v_{ns}$.

### B. Propagation of moderate second-sound pulses interacting with the vortices they generated

To avoid confusion we would like to clarify our terminology concerning the intensity of the perturbations we are studying. We shall understand as "moderate" perturbations of such intensity and duration that will not lead to phase changes, i.e., evaporation of He II or transition to He I, but cannot be described by the linear approximation.

The results described in the previous section were primarily qualitative. Quantitative theoretical and experimental investigations of the dynamics of moderate heat pulses generating vortices and interacting with their "own" vortices started in the second half of the eighties. Theoretical considerations were based on the HST, with parameters obtained from corresponding experiments using a fitting procedure.

Some information concerning the interaction between the different processes involved in the dynamics of the heat pulse can be gained from the solution of the problem in the case of a low level of the VLD (Nemirovskii, 1986). In this case the problem is similar to the problem of propagation of a nonlinear heat pulse through a VT [see Sec. IV.B, Eq. (6.8)]. The main difference is in the right-hand side of Eq. (6.8), where $\Gamma_y v_s$ should be replaced by $ALv_s$, $A=K\rho/2\rho_s\rho_n$, and $L$ obeys Vinen's equation (5.4). In addition, the variation of the temperature $\delta T$ consists of the usual acoustic part $\delta T(v_s)$ and a correction $\delta T_V$ due to the presence of vortices,

$$\delta T=\delta T(v_s)+\delta T_V \ . \tag{7.3}$$

In the system of coordinates moving with the second-sound velocity $u_2$ in the $x$ direction,

$$\frac{\partial v_s}{\partial t}+\alpha_{\rm Kh}(T)v_s\frac{\partial v_s}{\partial x}=-ALv_s \ . \tag{7.4}$$

The temperature variation of the wave is hence

$$\delta T=\delta T(v_s)+\frac{u_2}{\sigma}\int_{-\infty}^{t}Av_sL\,dt' \ . \tag{7.5}$$

The temperature variation $\delta T$ is shown schematically in Fig. 28 for the case of an applied rectangular heat pulse of duration $t_H$. The acoustical part, $\delta T(v_s)$, is the familiar Burgers triangle, which is the asymptotic solution for a nonlinear traveling wave. The heat pulse can be divided into four parts corresponding to different stages. At the second stage, when the vortices are formed during the time elapsed from the start of the pulse, the quantity $Lv_s$ reaches a maximum and the right-hand side of (7.4) becomes large, leading to a break observed in many experiments described in the previous section. At the third stage, although $L$ is large but as $v_s$ is small, the left-hand side of (7.4) is small and the curve again approaches the Burgers triangle. At the fourth stage, where formally the pulse is over, the temperature perturbation $\delta T$ has not disappeared, as in the vortexless case. This temperature increase is related to two mechanisms; one is the dissipation of energy connected to the decay of the VT, and the other is more subtle. As the right-hand sides of the equations of motion are not zero,

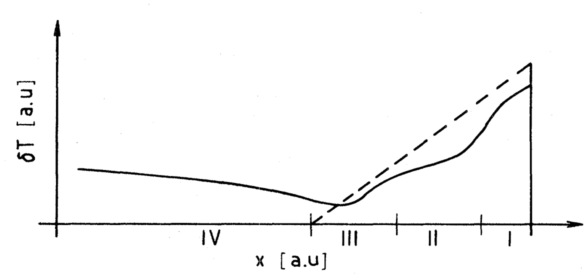


FIG. 28. Schematic of the distortion of the temperature pulse due to the interaction with is "own " vortex lines (Nemirovskii and Schmidt, 1990, Fig. 7). The dashed line represents the vortexless case when the pulse should be a "Burgers" triangle.

the wave cannot be represented by a simple Riemann invariant traveling to the right along the $x$ axis. Inevitably there appear some reflections causing the appearance of some velocity $v_s$ and temperature $\delta T$ perturbations after the heat pulse has been switched off. However, as during its propagation the pulse decays, the maximum of the temperature occurs at the heater. This leads to the typical overshoot, to be described later, observed in many experiments. This temperature increase at the heater may lead to a boiling of He II there.

At first the difference between the total $\delta T$ and the acoustical $\delta T(v_s)$ at the leading edge is surprising [see Eq. (7.5)]. Indeed, at the first instant, after switching the pulse on, the vortices do not have enough time to be initiated, and it would be expected that there should be no difference whatever from the Burgers triangle. However, Eq. (7.4) has a nonlinear term on the left-hand side, $\alpha_{\rm Kh}(T)v_s\partial v_s/\partial x$, which describes the transfer of perturbations inside the pulse; in particular, this term is responsible for the transfer of the deficiency of the velocity $v_s$ appearing in the central part of the front of the pulse. Recalling now that the velocity of the shock wave depends on the amplitude of the jump (see, e.g., Khalatnikov, 1965), we can state that a pulse generating vortices will move somewhat slower [or faster if $\alpha_{\rm Kh}(T)<0$]. Estimates based on the solution of (7.4) are in good agreement with the experiments of Cummings *et al.* (1978).

Although the above considerations confirm the vortex nature of a number of experimental effects, they are not fully satisfactory from the quantitative point of view. It is understandable that, due to the complexity of the set of the main equations, it is necessary to use numerical methods to obtain better approximations.

Results of a detailed experimental and numerical study of the temporal and spacial evolution of a moderate heat pulse in a wide channel are described in the papers of v. Schwerdtner (1988), v. Schwerdtner *et al.* (1989c), Stamm *et al.* (1989), Fiszdon and v. Schwerdtner (1989), and Fiszdon *et al.* (1990).

Examples of the evolution of an initially rectangular heat pulse depending on the perturbation parameters are shown in Fig. 29 for different heating times, $t_H$, and in Fig. 30 at different distances, $d$, from the heater. The distinguishing feature of these results is the existence of a temperature overshoot, which appears close to the heater and can be many times larger than the shock-wave amplitude. The initial stages qualitatively agree with the previously described semiqualitative analysis. The strong dependence of the overshoot on the rest time $t_R$ is related to the generated VLD, as for small $t_R$ there is not enough time for decay and hence the initial VLD, before the next pulse starts, is larger, leading to a reduction of the velocity and heat accumulation close to the heater.

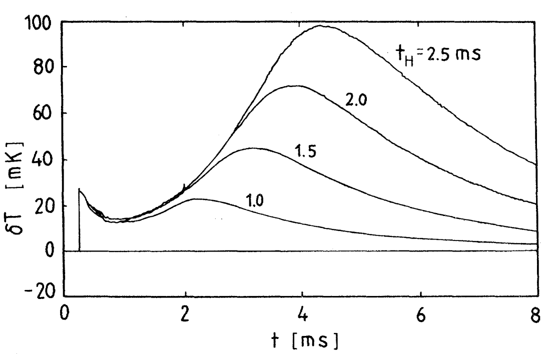


FIG. 29. Measured temperature evolution as a function of time for different heating times $t_h$ at a distance of 0.54 cm from the heater (Fiszdon *et al.*, 1990, Fig. 7).

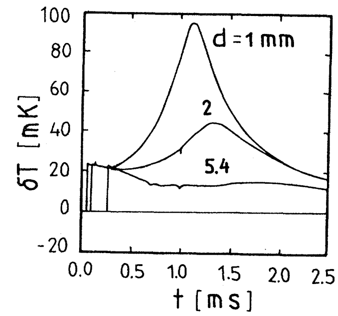


FIG. 30. Influence of distance $d$ from heater on the temperature evolution (Fiszdon *et al.*, 1990, Fig. 5a). The heat flux was 5 W/cm and pulse duration was 1 ms; $T=1.4$ K.

Numerical calculations performed using the full set of HST equations up to the second order and the Vinen coefficients are described in Fiszdon *et al.* (1988) and Noack and Fiszdon (1990). Numerical calculations were also performed by Gentile and Pakleza (1986) and Pakleza and Poppe (1989). This method allowed the investigation of the influence of many factors which otherwise could not have been done. It was found that the inclusion of the additional term proposed by Vinen as initiating the VLD evolution is not effective enough. The best results were obtained when an initial VLD $L_0$ was assumed. This also confirmed the superiority of the experimental procedure of multiple-pulse releases at constant time intervals to arrive at steady periodic test conditions. When only the initial value of the VLD was used as a fitting parameter, a very good qualitative agreement and a fair quantitative agreement between the experimental and the numerical results were obtained. A number of examples are shown in Fiszdon and v. Schwerdtner (1989) and Fiszdon *et al.* (1990). The numerical method using experimental data wherever necessary also allowed us to obtain the spatiotemporal evolution of the temperature, counterflow velocity, and VLD, as illustrated in Fig. 31, which shows the interrelated complex processes caused by a heat pulse in a one-dimensional large channel, i.e., with boundary effects neglected.

The variation of the counterflow velocity at a certain distance from the heater was measured experimentally (see v. Schwerdtner *et al.*, 1989c) using a specially devised second-sound anemometer based on the entrainment, Doppler, effect (see, e.g., Khalatnikov, 1956). An example of the measurements taken at a distance 0.54 cm

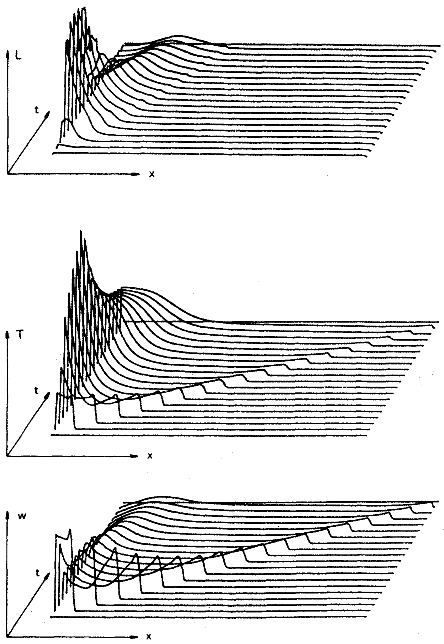


FIG. 31. Evolution of vortex line density $L$, temperature increase $T$, and counterflow velocity $v_{ns}$ in time and space, due to a second-sound pulse generated at $x=0$ and $t=0$, for a bath temperature of 1.4 K and an initial vortex line density of $3\times10^6/\mathrm{cm}^2$ (Fiszdon and v. Schwerdtner, 1989, Fig. 1).

from the heater is shown in Fig. 32. The straight line corresponds to the value $v_{ns}=\dot{Q}/\rho_s ST$. At higher heat fluxes the differences between the measured and theoretical curves are large, which corresponds to a higher level of the VLD. The differences at lower heat fluxes $\dot{Q}$ are small; but this may also be due to the short pulse duration and hence correspond to the vortexless case.

Besides the previously cited numerical calculations, Murakami and Iwashita (1990, 1991) have, unlike others, solved the full set of the HST equations without imposing counterflow conditions or limiting themselves to second-

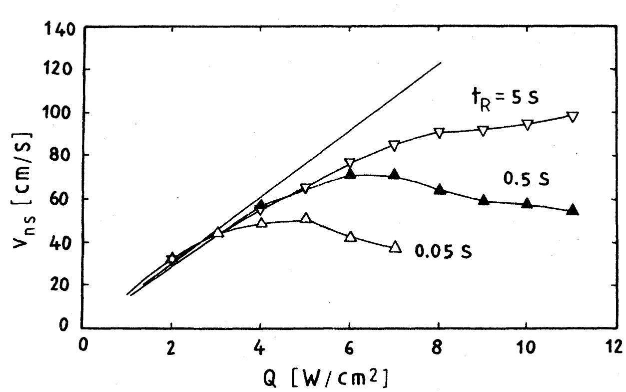


FIG. 32. Measurements of the counterflow velocity $v_{ns}$ at a distance of 5 mm from the heater for various rest times $t_R$ (v. Schwerdtner *et al.*, 1989b, Fig. 10). The pulse duration was 0.5 ms. The straight line corresponds to the usually used relation $v_{ns}=Q/\rho_s\sigma T$, $T=1.4$ K.

order deviations from the equilibrium state. This approach requires the knowledge of the dependence of the thermodynamic variables on $v_{ns}^2$, which is known up to first order only (Khalatnikov, 1965). The authors used this first-order dependence, and their results agree qualitatively with those described previously.

Dresner (1982) used a special method to study the dynamics of heat fluxes in superfluid turbulent He II. He proposed using the Gorter-Mellink law (3.1) in a similar way as the Fourier law, $Q\sim\nabla T$, in the heat transfer problem. But in this case the coefficient of heat conductivity should be considered as a function of the temperature gradient $\nabla T$. The energy conservation law leads in our case to a nonlinear heat transfer equation,

$$\frac{\partial T}{\partial t}=\lambda\nabla(\nabla T)^{1/3}\ , \tag{7.6}$$

where the constant $\lambda$ is connected to the Gorter-Mellink relation (3.1). The above equation can be obtained from the HST equations, making some substantial simplifications; in particular, it is valid close to equilibrium. It is interesting to note that for stepwise impulses below 1 $\mathrm{W\,cm^{-2}}$, in cavities of simple geometric shapes and for large enough times, the results of calculations are very close to those obtained from the solution of the full system of equations (see Nemirovskii *et al.*, 1992) and obviously can be used in applied problems.

To gain some additional insight into the process of evolution and generation of the VT, a series of experiments were conducted with converging cylindrical second-sound waves generated on the inner surface of a short cylindrical segment (Poppe *et al.*, 1992; Stamm and Fiszdon, 1992; Stamm *et al.*, 1992; Fiszdon *et al.*, 1994). In this "geometry" the attenuation of the pulses due to the induced vortex tangle is partly compensated by the increase of the amplitudes of the hydrothermodynamic parameters involved. An example of a measured temporal and spacial evolution of the VLD is shown in Fig. 33. The large increase of the VLD, at the center, of the order of $10^9$ $\mathrm{cm}^{-2}$ is remarkable. This result once more proves that the generation of the VLD takes place in the bulk of the liquid. Moreover, the ex-

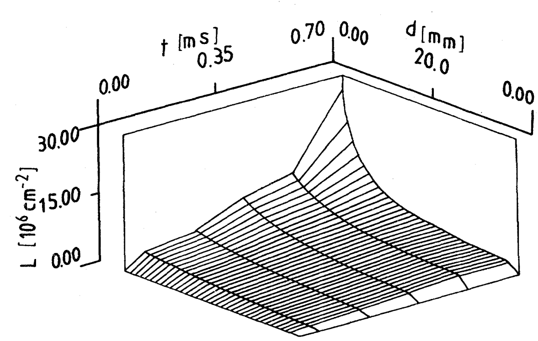


FIG. 33. Converging cylindrical wave (Stamm and Fiszdon, 1992, Fig. 4b). Measured spatial and temporal evolution of the VLD for rest times $t_R=0.5$ s. Heat flux $Q$ was 3 W/cm² and pulse duration $t_H$ was 1 ms; $T=1.4$ K.

tremely high VLD close to the axis of the cylinder, in conjunction with the zero counterflow velocity there, creates conditions of extreme nonequilibrium, which poses, in these cases, the question of applicability of the HST and VE model used.

To assess the influence of boundaries and sudden channel cross-section changes, numerical simulations of two-dimensional counterflows in simple geometries were made (Fiszdon *et al.*, 1992). They indicated that the influence of the boundary layer does affect appreciably the VLD distribution and the counterflow velocity field in the channel, but its effect on the temperature is very small. The effect of a sudden change of the cross section is much more important; it strongly affects all flow parameters and can generate their fluctuation in time and oscillations in space. However, the assumption concerning the existence of counterflow conditions except in plane, axial, or spherical cases can be questioned, as shown by Vogel and Fiszdon (1990); but as it appears from the only known numerical studies of Poppe *et al.* (1992), except very close to a geometrical singularity (e.g., a corner), the error made through the assumption of counterflow is negligible.

The cases described above of propagation of heat pulses in He II illustrate some important effects of superfluid turbulence and are valuable in the interpretation of experimental observations and possible cryogenic applications.

In Secs. III.B, IV.A, and IV.C, we described some difficulties and doubts concerning the proper form of the VLD evolution equations. It would appear that the good agreement between experiments and calculations confirms the usefulness of the evolution equations for the VLD according to Vinen's equation (3.8).

However, we would like to question this point of view, as the above-described solutions of the HST equations the specific properties of Vinen's equation (3.8) are not conspicuous. In the investigations of the dynamics of strong heat pulses, the first term of the mutual friction force $KLv_{ns}$ [see Eq. (5.14)] has the greatest influence on the dynamics of the pulse. However, this term describes the friction force of a "frozen" tangle which obviously does not depend on the form of the dynamic equation for the VLD. In particular, it remains exactly the same if, instead of Vinen's equation (3.8), we use the alternative equation (3.10). The behavior of the VLD $L(t)$, as already mentioned, is also not sensitive to the form of the equation. Hence the dynamics of the pulse can be equally well described using (3.8) or (3.10). This argumentation is strengthened if we take into account that, in the heat-pulse calculations, the fitting procedure of the parameter and coefficients is widely used. Therefore it is impossible to select the proper form of the evolution equation for the VLD $L(t)$ from the heat-pulse evolution investigations. For the same reasons, the usual experiments on heat-pulse dynamics do not allow us to select the proper sign of the dry-friction term (Secs. V.B and V.C) or to clarify which interpretation is more truthful.

Finally, the noted agreement between theory and experiment cannot be considered as a confirmation of the VLD evolution equation (3.8). Moreover, the reported results on the dynamics of heat pulses introduce additional complications in the interpretation of the experiments confirming Vinen's equation as described in Secs. III.B and IV.C. Indeed, in the evaluation of these experiments it was assumed that the velocity $v_{ns}$ is constant, during the whole process, equal to its value at the heater. But the results described in the present section show the contrary. As can be seen in Figs. 31 and 32, the variations of the counterflow velocity in time and space are very large. This results from the mutual influence of the VT on the motion that produced it. Obviously, it is inadmissible to neglect this effect and assume $v_{ns}=\text{const}$, unless this is confirmed by the solution of the full HST equations.

Finally, we would like to comment briefly on "a long-standing puzzle to which the answer remains incomplete" (McClintock, 1994), i.e., "the mechanism by which quantized vortices can be treated in He II." There are various theories that can be divided into two groups. The first group offers different mechanisms of initial generation of vortices, for example, tunneling, fluctuations, etc. Another group of the theories is based on the idea that in superfluid helium there exists permanently some background of remnant vortices. From the standpoint of the phenomenological theories, the former group supports the necessity of introducing the additional initiating term in Vinen's equation. In turn, the latter group accepts the assumption of the existence of an initial VLD whose evolution is described by Vinen's equation.

As noted previously, a better agreement between experimental data on the propagation of intense heat pulses and numerical solutions of HST was obtained with the assumption of the existence of an initial level of VLD, i.e., $L_0$, whereas the introduction of an initiating term led to an unsatisfactory correlation with experimental observations. Thus it may be surmised that this is some confirmation of the theory of remnant vortices.

This example showed that an experiment based on a phenomenological theory may allow us to draw an important conclusion concerning the subtle microscopic dynamics of quantum vortices.

## VIII. OTHER DYNAMICAL PHENOMENA

First of all we would like to say what is meant by other dynamic phenomena. From what was written in Secs. VI and VII, it can be noted that the mutual interaction between the ST and the flow of He II, and vice versa, leads to a multitude of phenomena. However, basically, these phenomena are mainly related to the existence of a mutual friction force. But there exist a number of cases when it is necessary to take into account other effects contained in the HST equations (5.4) and (5.11)–(5.14). There are also experimental research results that can be explained only partially by the solutions of the HST equations. Finally, there are research results on unsteady ST that cannot be explained at all by the HST equations. In this section we would like to discuss some relevant cases.

### A. Intrinsic fluctuations

Already in the early stages of ST research the following question arose. As we are dealing with stochastic processes, ST must be accompanied by developed fluctuations. In turn, the study of these fluctuations must yield extensive information on ST alike, e.g., the analysis of the spectrum of the velocity pulsations in classical turbulence helps to understand subtle dynamic processes.

There are many papers devoted to fluctuations in He II (Tough, 1982; Donnelly, 1991a). Regretfully, these studies have so far, not introduced much clarity in the understanding of the stochastic dynamics of the vortex lines. We think that certain contradictions in the available data and in their interpretation are connected to the larger number of degrees of freedom in ST, as compared with classical turbulence. In ST, not only can disturbances of the relative velocity $\delta \mathbf{v}_{ns}$ and of the VLD, $\delta L$ be an additional source of the observed fluctuations, but also, in some experimental setups, the resulting discreteness of the vortex lines can be a source.

Although the coexistence of VLD, $L(\mathbf{r},t)$ and relative velocity $\mathbf{v}_{ns}(\mathbf{r},t)$ introduces additional complications, in principle, as discussed below, the process can be understood in the frame of the HST theory, which allows us to connect various fluctuating quantities, e.g., $\delta L$ and $\delta v_{ns}$.

Unlike the case of averaged velocities, in experiments where fluctuations connected to the chaotic motion of the discrete vortex lines are observed, they cannot be explained by the "averaged" macroscopic equations of HST. They are more closely connected to the microscopic description of the vortex line dynamics and, from this point of view, are informative and useful. There exist a variety of methods for investigating fluctuating processes, such as mechanical probes, ion probes, second- and first-sound probes or measurements of variations of temperature and chemical potential, $\nabla T$ and $\nabla \mu$, in a counterflow and for arbitrary velocity ratios.

Allen *et al.* (1965) and Griffiths *et al.* (1964, 1966) have used mechanical probes. In a series of experiments, they measured the random displacement of a quartz fiber with a small bob at the end. They found that the mean-square displacement $(\langle x^2 \rangle)^{1/2}$ grows linearly with the heat flux. To explain their observations, Allen *et al.* made qualitative estimates. They checked two possible mechanisms leading to the deflection of the fiber, due either to the jolts by the normal fluctuating component or to the entrainment of the fiber by the quantized vortex lines. They checked a few parameters such as the average deflection, period of fluctuations, and so on. Although a comparison of the orders of magnitude showed that both mechanisms gave similar contributions to the values of both parameters, the authors concluded that the fluctuations are due primarily to the turbulence of the normal component.

Other experimental measurements consisted of measuring the response of the fiber to short ($\sim 5$ ms) heat pulses which were either superimposed on a steady heat flux or applied in a quiescent helium bath. The results were evaluated quantitatively, introducing a persistence quantity $p$ determined from a series of responses with amplitudes $\alpha_n$ by the relation

$$p = \sum \alpha_n \alpha_{n+1} \Big/ \sum \alpha_n^2 \ . \tag{8.1}$$

The quantity $p$ can be considered as a correlation function. The measurements of the variation of $p$ with the heat flux are shown in Fig. 34. The decrease of the persistence may be related to the process of pinning and depinning of vortex lines.

The author's interpretation of the results is that, without flow or in the presence of very small flow velocities, the fiber captures a single vortex line which produces a certain stiffness of the fiber–vortex line system. The heat pulses excite and deflect the fiber in a practically deterministic way which results in a "stiff" correlation, $p \approx 1$. For large heat fluxes, VLD is large; many vortex lines are captured; and they are randomly released, weakening, accordingly, the correlation. Piotrovski and Tough (1978) conducted a similar experiment using a small paddle as a mechanical probe. The highly sensitive SQUID technique (superconducting quantum interference device) was used to measure the paddle deflection with an accuracy of a few angstroms. The resonance frequency of the system, paddle, and spring support was also varied. In Fig. 35 the measured mean-square displacement of the paddle as a function of the heat flux is shown. The relation between $\langle x^2 \rangle$ and $v_{ns}$ is explained by the response of the system to a random external force with an exponential decay

$$\langle x^2 \rangle = S_0 e^{-\omega \tau_V} \ , \tag{8.2}$$

where $\tau_V$ is the relation time of the tangle (6.4). Using a fitting parameter, the authors concluded that $\beta_V$ in Vinen's equation must be about $0.04 \times 10^{-3}$, which is close to $\beta_V^{\rm dec}$ obtained by Vinen (see Sec. III.B) from free-decay experiments.

Fluctuations in ST were also measured using second-sound probes. The fluctuations in the decay of second-

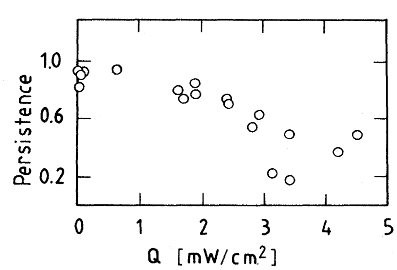


FIG. 34. Persistence $p$, Eq. (8.1), of the circulation trapped on the fiber (Tough, 1982, Fig. 34a; Allen *et al.*, 1965).

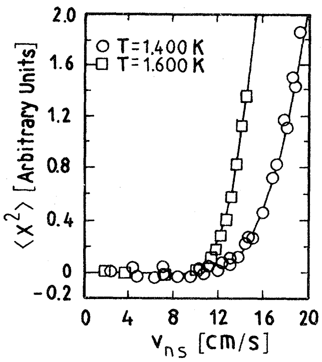


FIG. 35. Mean-square displacement of a "paddle" immersed in a turbulent counterflow as a function of the relative velocity for several temperatures (Tough, 1982, Fig. 35; Piotrovskii and Tough, 1978). The solid line is calculated from the exponential power spectral density, Eq. (8.2), with the fitting parameter $\tau_V$.

sound signals can be expressed by VLD fluctuations according to Eq. (6.1) [see Mantese *et al.* (1977) and Ostermeiser *et al.* (1980)]. In these experiments, attention was focused on the spectral density $\langle \delta L(\omega)\delta L(-\omega)\rangle$ and its dependence on the counterflow velocity and frequency. The results principally agreed with each other and can be explained using Vinen's equation (see Northby, 1978). Barenghi *et al.* (1982) finally solved this problem by comparing the observed fluctuations with the forced oscillations of the steady counterflow produced by applied second-sound random modulations. They showed that the spectral density of the VLD $S_L(\omega)$ is related to the spectral density of the noise $S_V(\omega)$ by the relation which can be derived from the dispersion law (6.2),

$$S_L(\omega)=4\frac{L_0}{v_{ns}}\frac{S_V(\omega)}{1+(\omega\tau_V)^2} . \tag{8.3}$$

A comparison of experimental results with those deduced from (8.3) is shown in Fig. 36, which illustrates the VLD spectrum $S_L(\omega)$ at $S_V(\omega)$=const, $\omega/2\pi<100$ Hz. The continuous curve corresponds to (8.3) with $\beta_V \approx 0.19$, which is closer to the value of $\beta_V^{gr}$ obtained by Vinen (see Sec. III.A) for increasing VLD. Some disagreements with (8.3) occur at large $\omega$, where, from experiments, $S_L(\omega)=\omega^{-n}$ and $n$ can reach values of up to $n=4$.

Hoch *et al.* (1975) and Sitton and Moss (1972) also

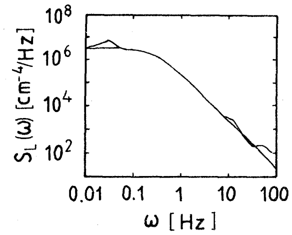


FIG. 36. Typical experimental vortex-line density frequency spectrum at 1.65 K when the counterflow is modulated by random noise (Barenghi, 1982, Fig. 1). The smooth line is a plot of Eq. (8.3) with the fitting parameter $\tau_V$.

measured fluctuations using ion currents. The general picture becomes complicated by the processes of interaction of the ions with the vortex lines, which creates difficulties in the interpretation of experimental data.

In the mid eighties Tough and co-authors made a series of experimental studies of fluctuations in ST (Lorenson *et al.*, 1985; Griswold and Tough, 1987; Griswold *et al.*, 1987; Tough, 1989). The authors measured the chemical-potential variation along the channel. The results were quite unexpected and interesting. They proposed a new approach for studying ST that can be used for the investigation of nonlinear dynamic systems. Quite unexpectedly, their results were in contradiction with the previous ones described above. In particular, and unlike the previously published results wherein an increase of the fluctuation intensity with an increase of the heat flux was observed, Tough *et al.* found that the level of fluctuations was small, of the order of 0.1%, in the whole range of the heat flux variation with the exception of the TI-TII transition range. They attributed this disagreement to the presence of small inhomogeneities in the experiments of the previous authors which could cause strong fluctuations. Another remarkable effect was the noticed peak of the characteristic response time of the system at $Q=Q_c$.

The results showing an increase of fluctuations in the transition region, as well as an increase of the system's response time, undoubtedly witness that the authors observed some kind of phase transition, however, of nonequilibrium states. An attempt at a theoretical description of the transition using the methods of nonlinear dynamical systems was made in the works of Schumaker and Horsthemke (1987; Horsthemke and Schumaker, 1989). They used the bifurcation theory to explain the experimental results.

To check these ideas, Griswold *et al.* (1987) studied the transition region TI-TII, superimposing a chaotic

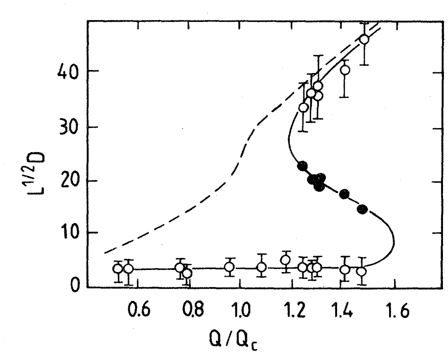


FIG. 37. "Phase diagram" for superfluid turbulence with external noise (Griswold and Tough, 1987; also Tough, 1989, Fig. 1.13). The open and solid symbols represent values of the vortex line density $L_m$ obtained from the analysis of fluctuations. The solid symbols represent unstable states of the system. The solid line is simply a guide to the eye, emphasizing the bistability induced by the noise. The dashed line is the deterministic steady state.

noise. Using a special method of examining the fluctuation distribution, they reconstructed the behavior of $L^{1/2}D$ ($D$ is the channel size) in the presence of noise. A striking proof of the appropriateness of these theories is shown in Fig. 37. It is possible these new research results will open a new chapter in the understanding of the nature of superfluid turbulence and the dynamics of chaotic vortex lines.

### B. Dry-friction effect. Validity and interpretation of the Vinen equation

The force acting on the superfluid component contains, besides the usual Gorter-Mellink force, an additional term which is called dry friction (see Sec. V). Depending on the interpretation of the physical processes leading to the Vinen equation (VE), this term can be either negative according to the Feynman-Vinen model, or positive according to the Schwarz model. The specific form of this term is directly connected to the special structure of the generating term of the VE (3.8). Therefore the study of the effects connected to this term supplies important information on the macroscopic dynamics of the VT as well as on the microscopic processes describing the stochastic behavior of the whole vortex tangle. Stamm *et al.* (1993), using second sound, investigated this problem by probing a highly nonequilibrium vortex tangle characterized by a very large VLD and a small counterflow velocity. As can be seen from Eq. (5.14), such conditions are most convenient for the investigation of dry friction. To describe the evolution of a second-sound test pulse traversing a nonequilibrium VT, let us consider, taking into account dry friction, the following equation,

$$\left[\frac{dv_{ns}}{dt}\right]_{t=x/u_2} = -A_1 v_{ns} L_0 \mp A_2 L_0^{3/2} , \tag{8.4}$$

where $A_1 = K\rho/2\rho_s\rho_n$ and $A_2 = \varepsilon_V \alpha_V / 2\rho_s$. The upper negative sign corresponds to the Feynman-Vinen model; the lower positive sign, to Schwarz's model (Sec. V). This simplified model describes the evolution of the linear test pulse along the characteristic $x - u_2 t$ crossing a nonequilibrium VT of constant VLD $L_0$. This simplification may change slightly the precise quantitative results; but, in exchange, it allows us to see clearly the expected effect.

It can be seen from (8.4) that the generated second-sound pulse of initial amplitude $v_{ns}(0)$ reaches some point $x$ with the following amplitude,

$$v_{ns}(x) = v_{ns}(0)\exp(-A_1 L_0 x/u_2) \pm (A_2/A_1) L_0^{1/2} [\exp(-A_1 L_0 x/u_2) - 1] . \tag{8.5}$$

Hence in the $v_{ns}(x)$, $v_{ns}(0)$ plane of the variation of $v_n(x)$ will be shown by a straight line with the slope $\exp(-A_1 L_0 x/u_2)$. Depending on the $\pm$ sign, these lines will cross the $v_{ns}(0)$ axis at the points $\pm v_{cr}(0)$, which are given by

$$v_{nscr}(0) = \pm (A_2/A_1) L_0^{1/2} \{\exp(-A_1 L_0 x/u_2) - 1\} . \tag{8.6}$$

The value $v_{cr}(0)$ will be called the threshold. The sign of the threshold is negative in Schwarz's model and positive in the Feynman-Vinen model.

The experiments were performed by generating a heat pulse on the inner wall of a circular ring. This configuration, as mentioned in Sec. VII.B, provides the possibility of generating very high VLD $L_0$, up to $10^8$ $cm^{-2}$ inside the ring. Preliminary calculations have shown that due to the reflection at the center, there is a short time interval when $v_{ns}$ is close to zero. Hence favorable conditions exists when the dry-friction term dominates.

The test pulse propagates parallel to the ring axis. The amplitude of the observed signal as a function of the input amplitude is shown in Fig. 38, where the existence of a negative threshold can be noted. The VLD resulting from the slope of these lines is about $L_0 \approx 5 \times 10^7$ $cm^{-2}$. From (8.6), for $T = 1.4$ K and the above VLD, the value of the threshold is of the order of 2 $W\,cm^{-2}$, which agrees fairly well with the measured value.

The analysis of the dry-friction effect and the results obtained allowed the authors to make two conclusions. First, the experimentally confirmed existence of the threshold, as illustrated in Fig. 38, proves the validity of Vinen's vortex line density evolution equation (3.8). Second, the negative value of the observed threshold corresponds to a positive value of the additional term in Eq. (8.4). This implies that He II in the presence of VT is an active medium that pumps energy into the crossing sound waves. This effect corresponds to Schwarz's model of macroscopic dynamics of the vortex tangle. Hence Schwarz's interpretation of Vinen's equation seems to be more plausible.

The special geometry of the experimental setup allowed the authors to make the following additional con-

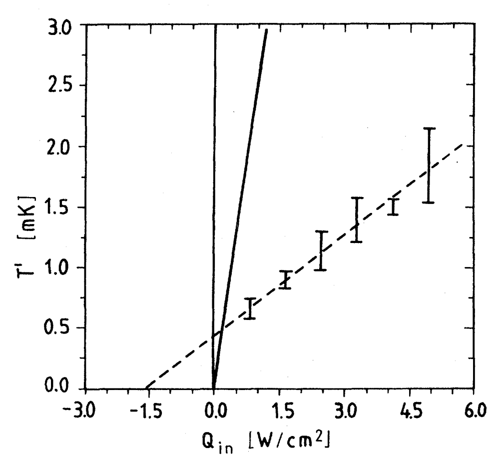


FIG. 38. Received temperature amplitude $T'$ of the test signal as a function of the input amplitude (Stamm *et al.*, 1993, Fig. 3). The solid line represents the test pulse amplitude without the main pulse.

clusion. The observed effect is connected to the anisotropy of the VT, which, in the presence of a counterflow, is polarized in the direction of the flow. But the test pulse was propagating in the transverse direction. To remove this controversy, the authors presume that, after the main flow is switched off, the VT loses its anisotropy and becomes isotropic. However, as discussed in Sec. VI.A, an isotropic vortex tangle appears to be unstable. Due to this instability, the test pulse traversing the VT causes its polarization this time in the direction of the test pulse transmitting its energy to the main flow.

### C. Slow decay of the vortex tangle

The HST methods were used to explain the problem of slow decay mentioned in Secs. III.B and IV.C. Vinen interpreted the existence of two different rates of decay of the VT by the existence of two self-preserving states, one during the growth of the VT when $v_{ns}$ exists and another one during the free decay. As this contradicts somewhat the conception of a self-preserving state, it seems appropriate to attempt to explain this process from purely HST considerations.

To explain their results shown in Fig. 17, Schwarz and Rozen suggested that, due to the entrainment of the vortex lines, a readjustment of the velocity field $v_n$ takes place and $v_{ns}$ becomes close to zero, which allows $L(t)$ to decay according to $1/L(t) \sim \beta_V t$. However, later on during the viscous decay of $v_n$, the counterflow velocity $v_{ns}$ is not equal to zero and $L(t)$ decays slower. This behavior can be described by

$$\partial v_s/\partial t = \kappa\alpha L(v_n - v_s) , \tag{8.7}$$

$$\rho_n(\partial v_n/\partial t) = -\rho_s\kappa\alpha L(v_n - v_s) - \eta v_n/d^2 , \tag{8.8}$$

and the $L(t)$ was calculated from VE (3.8). The proposed model equations are a somewhat simplified version of the HST model described in Sec. V, and, unlike this last-mentioned set, the term $\eta v_n/d^2$ representing the viscous forces is used. The quantity $d$ is a fitting parameter, and it was assumed, as it had been for the classical fluids, that $d = D/15$ where $D$ is the characteristic dimension of the channel. The calculated values shown in Fig. 39 are in qualitative agreement with experimental data.

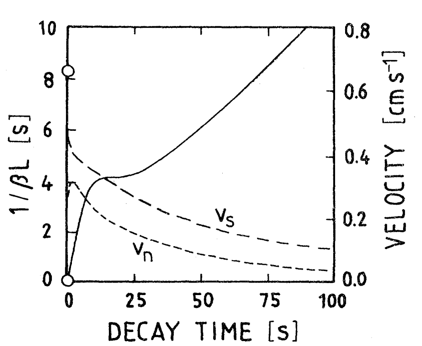


FIG. 39. Model calculation of Eqs. (8.7) and (8.8) corresponding to the experimental conditions of Fig. 17 (Schwarz and Rozen, 1991, Fig. 3). The initial values of $v_n$ and $v_s$ are indicated by the short-dashed and the long-dashed curves, respectively.

Another attempt to explain the slow decay problem based on hydrodynamic considerations was made by Geurst (1994). In his approach an additional characteristic, besides $L$, of the VT, the polarity $\mathrm{sgn}(v_L - v)$, is introduced (see Sec. V.D). This new quantity changes the form of the right-hand side of the VE [see (5.37)], retaining the assumption of self-preservation. This new form of the VE asserts that, depending on the direction of the drift of the VT with respect to the average mass velocity $v$, the generating term can either increase the VLD or decrease it. This conception is rather different from Vinen's approach, in which the generating term only increased the VLD $L(t)$. Geurst's model admits a solution in closed form that is self-preserving and exhibits three branches. On one of these the decay branches off into an initial stage of fast decay and a final stage of slow decay separated by a short transition region. This behavior is very similar to the experimentally observed one. The transition from one branch to the other is marked by a reversal of sign in the polarity of the VT.

One more way to explain the problem of slow decay by means of HST equations was developed in the work of Kondaurova (1993). The authors were seeking a direct solution of the HST equations. Unlike other authors, they did not impose the value of the counterflow velocity to be equal to zero, but solved the full problem with corresponding initial and boundary conditions. The results obtained showed that very complicated spatial and temporal variations of all the hydrodynamics variables appear. It is important that the counterflow velocity not disappear immediately after the heat flux is switched off, but that it slowly decrease. In fact, the decrease of the counterflow velocity is slower than the decrease of the VLD $L(t)$. As a result, the generating term does not disappear due to remnant counterflow velocity, and the VLD decays slower than it is required by Eq. (3.13) (see Fig. 40). As can be seen, $1/L(t)$ grows fast at first, then monotonically slows down. This behavior agrees qualitatively with the experiments of Schwarz and Rozen (1991); however, it does not explain the existence of a transition region in the behavior of the quantity $1/L(t)$. The existence of a nonzero counterflow velocity $v_{ns}$ behind the pulse leading to a slower decay than predicted by (3.13) was also reported in the numerical simulations of Fiszdon and v. Schwerdtner (1989; see also Fig. 31).

A theoretical analysis that excluded the space derivatives in HST was made by Vogel and Fiszdon (1994); the authors obtained self-similar solutions with two branches. These branches correspond qualitatively to the two decay regimes observed in the experiment. Olszok *et al.* (1994) also observed the two decay regimes over a range of temperatures, 1.5–2 K, and conjectured that the slow decay is due primarily to the normal turbulent damping and the fast decay to the mutual interaction forces. The numerical estimation of the Kolmogorov's

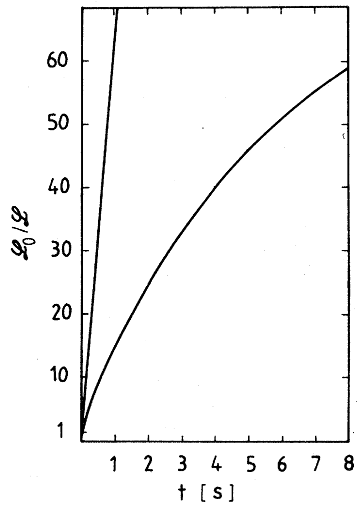


FIG. 40. The quantity $1/L(t)$ calculated from a set of HST equations (Kondaurova, 1993). The straight line corresponds to (3.13).

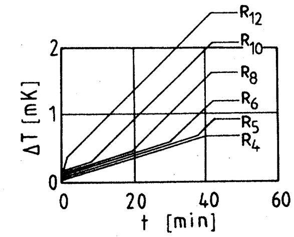


FIG. 41. Time dependence of temperatures measured along the capillary for a thermal current ($Q=4.4\times10^{-2}$ W/cm$^2$, $T=1.34$ K) (Peshkov and Tkachenko, 1961, Fig. 3).

scale agrees quite well with the average vortex line spacing at the "kink."

Summarizing it can be stated that the slow decay is of hyrodynamic nature independent of the intrinsic feature of the VT. This removes the problem of a "second self-preservation state," which is self-contradictory (see Sec. III.B).

### D. Motion of turbulent plugs and fronts

As shown in Sec. VII, the generation of the VT in heat pulses had a strong influence on the structure and dynamics of the pulses. Nevertheless, the usual wave mechanism of the hydrodynamics of He II plays a dominant role in its dynamics. Consequently, the velocity of propagation of heat pulses is close to the velocity $u_2$ of the second-sound wave, although it differs slightly from it due to the existence of a VT. In addition, the range affected by vorticity expands in helium with velocities close to $u_2$.

The nonstationary flow in He II with small heat fluxes is quite different. In this case a solitonlike evolution of the VLD $L(x,t)$, the so-called turbulent plugs and moving ST fronts, is observed. The velocities of propagation of these structures are about three orders of magnitude smaller than those of the second-sound velocity. Obviously, mechanisms (other than the above-mentioned one) connected with the structure and dynamics of the VT are involved.

Peshkov and Tkachenko (1961) and Bhagat *et al.* (1964) studied the kinetics of the formation of ST in long (up to 8 m) capillaries due to a small heat flux of the order of $10^{-3}$–$10^{-2}$ W cm$^{-2}$. Data on the evolution of ST were obtained from thermometers located along the capillaries. A set of such data from eight thermometers for a stepwise heat input at one end of the capillary is shown in Fig. 41. It appears that at the start, nearly instantaneously small, steady temperature gradients (curve marked 0) corresponding to a Poiseuille flow of the normal component appear. Thereafter, successively at regular delays, a linear increase of the temperature is registered. This behavior can be interpreted as a ST front propagation. At larger heat pulses, the picture changes. The observations show that along the channel, at different locations, regions of ST—"plugs"—appear. The boundaries of these plugs move in unperturbed helium with different velocities, which leads to their motion as a whole.

Moving fronts and plugs were investigated extensively by the Leiden group (see Slegtenhorst *et al.*, 1982a, 1982b; Marees, 1986; Marees *et al.*, 1987). Besides counterflow, they also investigated flows with arbitrary $v_n$, $v_s$ relations. In particular, the authors observed repetitive plug structures.

Van Sciver (1979) studied a similar problem, but at much larger heat fluxes, of the order of 1 W cm$^{-2}$. In these cases only monotonously varying flow parameters, no fronts, were observed. From the above observations it can be concluded that at small heat fluxes, ST is determined by some internal vortex turbulence processes which dominate the wave mechanisms.

An attempt to describe theoretically the above processes was made by van Beelen *et al.* (1988). They proposed the following modified VE,

$$\partial L/\partial t + \mathrm{div}\,\mathbf{j}_L = P_L \ , \tag{8.9}$$

where $P_L$ is the right-hand side of VE (3.8). The density flux $\mathbf{j}_L$ is assumed to be the sum of a drift term [see Eq. (5.1)] and a diffusive contribution,

$$\mathbf{j}_L = \mathbf{v}_L L + P_L \nabla L \ . \tag{8.10}$$

The addition of a diffusive contribution on the right-hand side of (8.10) seems artificial. Geurst (1989) gives some arguments supporting the existence of this term. Equations (8.9) and (8.10) have solitonlike solutions which correspond to the plugs observed in the experiments. Their dimensions and velocities depend on the flow parameters and, in particular, on the diffusion coefficient $D_L$. However, as stated by the authors, the solution obtained is related to the special model used and primarily to the artificial introduction of a diffusive contribution. Fiszdon *et al.* (1991) studied the solution of a similar problem, but without the diffusion term. In the stationary case the distribution of the VLD $L(t)$ is stepwise. Such a standing VLD wave, not confirmed experimentally, may be due to boundary effects.

Nemirovskii (1986) also obtained a running stepwise solution of the HST equations. The physical meaning of the solution is of interest. The right-hand side of (5.13), representing a dissipation function, causes an increase in temperature. In turn, a temperature difference produces a counterflow-type flow. Thus a VT that causes a counterflow decay leads by increasing the entropy to a temperature drop which in turn induces a counterflow. The interplay of these two effects leads to a propagating steplike solution. It should be noted that the approach yields qualitative solutions in good agreement with the experimental results of Peshkov *et al.* (1962).

Although these solitonlike solutions describe different physical mechanisms, they have one remarkable common feature in that they represent wavelike solutions appearing in dissipative systems. Let us recall that the study of ST from the standpoint of nonlinear dynamic systems was made while investigating fluctuations described in Sec. VIII.A. Together with the investigations reported in this section, it confirms that ST is indeed a dynamical system in which processes such as self-organized structures, dynamic phase transition, etc., may appear.

Solutions in the form of ST plugs, based on the analysis of the dynamics of vortex lines, were also obtained by Schwarz (1990). The VT plugs in his model are due to a development of an instability which results in a diffusion of vortices from the wall to the bulk flow in the form of a VT. The author states that this mechanism is sufficient to initiate and sustain the turbulent state.

### E. Chaotic quantum vortices and phase transition

The description of the phase transition is one of the important applications of the theory of chaotic quantized vortices in He II. The idea that phase transition can be related to the appearance of large vortex rings was first assumed in the classical works of Onsager (1949) and Feynman (1955), and was later frequently discussed in a number of publications as recalled in Kleinert's (1991) book.

This idea was first applied in 2D systems. Kosterlitz and Thouless (1973) studied phase transition in 2D systems and, in particular, in He II. They predicted that the appearance of large vortex-antivortex pair excitation could initiate a phase transition to the nonsuperfluid phase. The existence of this transition follows from simple energy considerations as described, for example, by Adams and Glaberson (1987; see also Donnelly, 1991a).

The entropy of a single line in the 2D case is proportional to the log of the total number of different independent positions of the vortex line which depends on the area of the system, i.e., $\sim D^2/r_0^2$. Hence the free energy,

$$F=E-TS=\frac{\rho_s d\kappa^2}{4\pi}\ln\frac{D}{r_0}-2k_B T\ln\frac{D}{r_0}\ , \tag{8.11}$$

where $k_B$ is the Boltzmann constant and $d$ the film thickness.

Thus it can be stated that at low temperatures vortices can appear only as small vortex-antivortex pairs. It follows that, when the temperature is above some critical value given by

$$T_{\mathrm{KT}}=\frac{\rho_s d\kappa^2}{8\pi k_B}\ , \tag{8.12}$$

it is energetically favorable for a single line to be excited, and hence the vortex-antivortex pairs will dissociate. The spontaneous appearance of free vortex lines implies the loss of long-range order, dissipation by mutual friction of any superflow, and hence the breakdown of superfluidity. When an external flow field is applied, the large pairs become reoriented and their net backflow reduces the external flow. This can be observed as a decrease in the superfluid density. The statistical problem of an ensemble of interacting pairs is similar to the model of a 2D gas of particles interacting via a logarithmic potential, since the energy of a pair varies logarithmically with the pair separation distance $r$:

$$E_p=2\pi\sigma_s^0\left[\frac{\hbar}{m}\right]^2\ln r/r_0+E_c\ , \tag{8.13}$$

where $\sigma_s^0\sim\rho_s^0 d$ is the bare areal superfluid density and $E_c$ is the potential energy of the smallest vortex pair. This quantity plays the role of a chemical potential necessary to create a pair. Similarly, as in the case of a plasma where other particles screen the "bare" interaction, the presence of other pairs can be taken into account by the introduction of a dielectric constant $\varepsilon(r)$. The effective interaction can be written as

$$E_p^r=2\pi\sigma_s^0\left[\frac{\hbar}{m}\right]^2\int_{r_0}^{r}\frac{dr'}{r'\varepsilon(r')}+E_c\ . \tag{8.14}$$

Continuing the analogy with plasma, it can be surmised that the dielectric constant $\varepsilon(r')$ is connected to the susceptibility by the relation

$$\varepsilon(r)=1+4\pi\chi(r)\ . \tag{8.15}$$

In turn, the susceptibility is related to the polarizability $\alpha(r)$, which is responsible for the orientation, in the external fields, of each pair and the number density $n(r)$ of the pairs of separation $r$. Hence

$$\chi(r)=\int_{r_0}^{r}\alpha(r')\exp\left\{\frac{-E_p^r(r)}{k_B T}\right\}\frac{2\pi r\,dr}{r_0^4}\ , \tag{8.16}$$

where $E_p^r(r)$ is determined by (8.14) which depends on $\chi$ via $\alpha$. The above relations can be used for a self-consistent solution of the problem.

The solution obtained by Kosterlitz and Thouless (1973) confirms the qualitative picture of the transition. It also forecasts that the appearance of the vortex-antivortex pairs of large separation leads to an abrupt change in the areal density $d\rho_s$ from a finite value to zero at the temperature $T_{\mathrm{KT}}$. The corresponding jump is given by

$$d\rho_s(T_{\mathrm{KT}})=\frac{2k_B m^2}{\pi\hbar^2}T_{\mathrm{KT}} . \tag{8.17}$$

This main theoretical result was confirmed in a number of experiments with thin He II films [see references in Donnelly's (1991a) book]. The dynamical properties of the 2D phase transition in He II were further developed by, for example, Ambegaokar *et al.* (1980).

The 2D theory of phase transitions was extended by Williams (1987, 1992, 1993a, 1993b) in his studies of the bulk $\lambda$ transition in He II. It was found that the critical temperature increases and that the jump of superfluid density is broadened, reminiscent of the behavior of $\rho_s(T)$ close to $T_\lambda$.

Williams's 3D theory is very close to the Kosterlitz-Thouless model. The role of vortex-antivortex pairs in two dimensions is played by circular vortex rings. The solution of the corresponding self-consistent problem showed that the difference in the space dimensionality of the system leads to a power-law dependence of $\rho_s(T)$ near phase transition with an exponent $\nu=0.53$, which differs from the well-known value of $\nu=0.67$. To improve this situation, Williams used the more realistic model and replaced the circular vortex rings by distorted, wiggly ones. This also overcame the deficiency of the theory, which did not take into account the variation of the core $r_0$ at temperatures close to $T_\lambda$. The calculation of the distorted vortex ring using the Flory approximation of the self-avoiding random walk was made by Shenoy (1993). This approach led to the correct value of $\nu=0.67$ of the critical exponent in the power law for $\rho_s(T)$. Later, Williams (1993a, 1993b) expanded his theory to deal with dynamical processes close to $T_\lambda$ and considered the possible applications to higher-$T_c$ superconductors.

Besides the attempts to describe the phase-transition problem using the model of interacting vortex rings, other more involved configurations were used. For instance, Owczarek (1994) considered knotted vortex lines with nonzero velocity to describe superfluid phase transition.

Although they give a clear physical picture of the phase transition, the above theories of 3D models of vortex rings are controversial. Leaving aside the critical remarks of Weichman and Fisher (1986; see also Weichman, 1988), who considered that vortex loops play no special role in the phase-transition problem, we would like to make some remarks resulting from the point of view of our review. We think that the equilibrium model of interacting vortex rings, which are governed by the Boltzmann statistics, is in contradiction with the results presented in Sec. IV. Figure 8 illustrates the evolution of six vortex rings resulting from a direct numerical calculation of the 3D equation of motion of the vortex filaments in He II. Similarly, Fig. 13 illustrates the evolution of a vortex ring in a field of stochastic forces (resulting, e.g., from the presence of other rings), calculated using the simple local approach. We assume that these examples demonstrate clearly that interacting vortex rings in He II evolve into a highly disordered set, governed by the laws of nonequilibrium statistical mechanics, thus demonstrating convincingly that the conception of a "gas" of circular vortex rings is inadequate to describe processes taking place in superfluid helium.

Therefore not rejecting the role of vortex rings in describing the bulk $\lambda$ transition, we think that a quantitative theory must use the more adequate real vortex tangle described previously instead of idealized circular vortex rings. In this respect we would like to call attention to the papers of Chorin (1991, 1992) on stochastic properties of vortex filaments in ideal fluids. Modeling filaments using the lattice vortex model, Chorin connected the $\lambda$ transition to percolation phenomena.

Summarizing the above, we can say that the vortex model of phase transition is a very promising field; however, the quantitative theory should take correctly into account the real properties of the VT. This task, however, is hindered by the problem of rather scarce information concerning the properties of the VT near $T_\lambda$, which were thoroughly studied in other temperature ranges. We think that the want of investigation of the VT near $T_\lambda$ is one of the largest gaps in the theory of ST.

Indeed, due to the divergence of the interaction between the normal component and the vortex filament (see Sec. II) and due to the growth of the core radius $r_0$, the structure of the VT is expected to be rather different from the one described before. It seems that only the Gorter-Mellink law is more or less confirmed in this region (see Ahlers, 1969; Leiderer and Pobell, 1970; and Crooks and Johnson, 1971), but other phenomena have not been studied. There are some papers dealing with this problem [see, *et al.*, Goldner *et al.* (1993)] where the unusual behavior of intense heat pulses close to $T_\lambda$ is described, or the paper of Onuki (1983), who also considered the relevant questions; but there are no systematic investigations of the ST that include unsteady phenomena near $T_\lambda$.

## IX. CONCLUSIONS

The present state of research on chaotic quantized vortices and hydrodynamic processes in superfluid helium is presented. A wide range of hydrodynamic, primarily nonstationary flow phenomena in the presence of the vortex tangle in He II are described. A large part of the observed cases is in good agreement with current notions on chaotic vortex tangles or superfluid turbulence, confirming their properties and supplying further quantitative information. The study of heat pulses essentially expanded the limits of investigations of ST. In particular, it was shown that the evolution of the VLD at moderate pulses (Sec. VII) is very much faster, by about three orders of magnitude, than at the very low intensity pulses. At the same time, the level of the VLD in heat pulses exceeded the previously observed one by four or five orders. On the other hand, the study of fluctuations of ST (Sec. VIII.A) using mechanical probes allows us to observe distinct processes of pinning and depinning of vortex filaments, confirming thus the discrete nature of ST. As described in Sec. VII.B, the vortex tangle induced by the heat pulse, which propagates with the second-sound velocity, is generated in the bulk and follows behind the shock wave. The experimental observations of propagation of heat pulses in large channels and in convergent geometries point out also that the evolution of the VLD is mainly due to the counterflow velocity $\mathbf{v}_{ns}$. As follows from the observations of heat pulses (Sec. VII) and of the threshold (see Sec. VIII.B) and also from direct observations (see Secs. III.B, IV.C), the dynamics of the VLD $L(t)$ evolution is adequately described by Vinen's equation (VE).

In the Introduction we claimed that the study of ST, with the use of macroscopic methods, can supply a lot of information on the stochastic dynamics of vortex lines. We also claimed that this information can be used in the study of some problems of the theory of superfluidity and even for the study of more general problems of stochastical physics of extended objects and classical turbulence. Following consistently these aims, we would like to summarize mainly the well-established properties of the VT in the two areas of interest.

The most important results relevant for the theory of superfluidity are the following.

Analysis of the dynamics of intense heat pulses (Sec. VIII.B) concluded that there is some remnant VLD in the bulk of He II. This was also observed in free-decay experiments (Sec. IV.C). This result seems to be important for the theory of initial nucleation of vortices in He II.

Analysis of the data on VLD and numerical results allowed Schwarz to conclude that the vortex lines must reconnect (Sec. IV.B) wherever they collapse. We would like to recall that a similar conclusion was previously obtained in a few works based on the Gross-Pitaevskii model (see Sec. II).

The corrections to the Gorter-Mellink relation and to the second-sound dispersion law, obtained from high-precision experiments, allowed the clarification of the details of the interaction between the vortex lines and the quasiparticle flow (Sec. VI.A).

Finally we would like to point out the possible role of stochastic vortices in the bulk $\lambda$ transition in He II (see Sec. VIII.E).

The following results may be of interest from both points of view, i.e., stochastic dynamics of extended objects and classical turbulence.

The self-preservation assumption (Sec. III) is fairly well confirmed by the agreement with most experimental observations. According to the discussion in Sec. V.A, this assumption points out the presence of a hierarchy of the relaxation times of different moments of the distribution function $\lambda$. Therefore there is some justification for introducing a reduced stochastical description similar to the kinetic equation or the Fokker-Planck equation.

The success of Schwarz's theory demonstrates, in particular (see, however, remarks in Sec. IV.B), the possibility of using the local self-induced approximation to calculate the dynamics of vortex lines. Another important conclusion from the comparison of the theory and experiments concerns the scaling properties of the VT. These results are very important for future analytical investigations, as they show possibilities for simplifying the basic equations of motion of the vortex lines and for an adequate formulation of the problem of the VT dynamics.

The experiments concerned with the observation of negative dry friction (Sec. VIII.B) correspond more closely to Schwarz's model of the decay process of the VLD due to mutual friction (Sec. IV.C). Schwarz's and Feynman-Vinen's conceptions of the decay process are quite different. Indeed, in the Feynman-Vinen model the decay of the vortex tangle is the consequence of the breaking up of the vortex rings transformed into thermal excitations. So there is the flux of some physical quantity, e.g., of the local curvature in the space of the sizes of vortex rings. This model is reminiscent of the Kolmogorov cascade in classical turbulence, and the stochastical behavior is essentially a nonequilibrium one. In Schwarz's interpretation only the friction force is responsible for the decrease of the vortex line length, and there is no flux in the space of sizes of the vortex rings. Thus the stochastic process is close to equilibrium. This conclusion may be of crucial importance for further theoretical developments.

In some investigations the VT displays the properties of nonlinear dynamical systems. For example, the behavior of fluctuations observed during the transition TI-TII (Sec. VIII.A) is typical for nonequilibrium phase transitions and is described with the help of the bifurcation theory. Other examples, from this point of view, are the solitonlike solutions or self-organized structures, plugs and VT fronts (cases described in Sec. VIII.D). This point of view is also confirmed by the negative dry-friction effect, which shows that ST is an active medium that can transmit energy to the traversing sound waves as described in Sec. VIII.B. With the possibility of high-resolution experiments in He II taken into account, the VT may be a convenient test model of nonlinear dynamic systems.

The existence of a vortex structure affects the flow field and leads to interesting hydrodynamic effects that must be taken into account in the evaluation of experiments and which may also be of interest in applied problems.

The unusual behavior of moderate and strong heat pulses that generate vortices and interact with them, as described in Sec. VII.B, is an important example of such effects. The temperature overshoot and the involved deformation of the flow field are particularly noteworthy.

The existence of a remnant velocity, obtained from solving the HST equations, contributes to the understanding of the slow decay problem (Sec. VIII.C).

The interaction of the second-sound waves with ST (described in Sec. VI)—which leads to a variety of processes such as extra attenuation, dispersion, anisotropy and coupling between transverse and longitudinal sound waves, and noncollinearity of velocity perturbations, $\delta\mathbf{v}_{ns}$, and of the wave vector $\mathbf{k}$—is another example of applications of HST.

We hope that we succeeded in showing that the simultaneous use of theoretical, numerical, and experimental methods to study of ST is very useful for investigations of the stochastical dynamics of vortex filaments. However, motivated by this aim and striving to present a self-consistent picture, we may have created a wrong impression that the theory is complete and closed. Actually, there remain many gaps in the evaluation of a number of investigated cases, and there remain whole regions to be studied.

We think first of all that one of the most important goals is to make an advanced theoretical study of the problem of dynamics of vortex lines. Indeed, the analytical methods developed in the theory of polymers, the recent achievements in the study of vortex filaments in ideal fluids, and the powerful modern methods of statistical physics together with the available experimental results support our belief that this task can be successfully completed. However, we realize that it cannot be completed in the near future. Therefore, realizing the complexity of a microscopic theory, we think that it is now necessary to solve the simpler problem of developing an advanced phenomenological model. In this development the more subtle characteristics of the VT structure [not only the VLD $L(t)$], its anisotropy and possibly the discrete structure of ST, should be taken into account. It can be expected that the experimental investigations based on such an advanced theory will yield new, rich information on the structure and dynamics of the VT.

As far as experimental studies are concerned, we think that, besides the particular problems already pointed out, it would be important to lay down the following "strategic" directions.

First of all, it is necessary to extend the temperature range of ST studies. As noted in Sec. IV, the coefficients $\alpha$ and $\alpha'$, which are responsible for the coupling of vortices with the normal component, exert a great influence on the evolution of the stochastic VT. Therefore it is important to see what happens at the lower temperatures where $\alpha$ and $\alpha' \to 0$ and the vortex filaments are free to evolve, not being influenced by mutual friction forces. This is of interest also because it then becomes possible to verify a number of results obtained in the case of chaotic vortex lines in ideal fluids. The region close to $T_\lambda$, where $\alpha$ and $\alpha'$ diverge and the vortex lines become frozen with respect to the normal component, is also of great interest. Therefore one can expect that the results will not only be interesting, but may also yield so-far unexpected information of importance in some applications, e.g., in the understanding of the role of stochastic vortices in the bulk phase transition (see Sec. VIII.E).

Secondly, we think that experimental studies must be expanded beyond investigations of simple flows to more involved configurations and flow conditions, e.g., the study of ST in nonuniform flows, jets, flows with $He^3$ additions, etc. Particularly, as described briefly by Tough *et al.* (1994), there appear "dramatic differences" between theory and experiment deserving further investigations. It is also important to make use of local probes and measure temperature correlations and, perhaps, velocities to obtain more information about the different stochastical flow parameters.

Finally, we think that the problem of interaction with the turbulence of the normal component should be more closely investigated, because it is possible that it is not the case of classical turbulence, but of some velocity fluctuations correlated with the motions of the chaotic quantum vortex filaments. This interesting and important problem should also be of interest and deserves the attention of theoreticians.

During the preparation of this review, two important scientific meetings on problems closely connected to the subject of this review were held: The Low Temperature XX Conference and the Workshop on Quantum Vortices and Turbulence in He II. The presentations and discussions at these meetings confirmed our conviction that the problems discussed in this review are of interest to many physicists and engineers, and some of the described problems are under investigation.

## ACKNOWLEDGMENTS

The authors would like to express their deep appreciation to Professor E.-A. Müller for his interest in and support of this study. One of the authors (W.F.) would also like to express his thanks to Professor H. Gg. Wagner for creating conditions allowing him to complete this paper. Our special thanks are due to our colleagues, Dr. G. Stamm, Dr. F. Bielert, and T. Olszok, for their advice and help in preparing this review. Our special thanks are also due to J. Krüger for her great help in editing and typing this paper, and to O. Efimova for her typing of the draft paper.

The authors would like to express also their gratitude to the Max-Planck-Gesellschaft and to the Deutsche Forschungsgemeinschaft for creating conditions that enabled us to devote our time to the task of writing this review. The participation of one of the authors (S.K.N.) in the research leading to the publication was made possible in part by Grant No. RPK000 from the International Science Foundation.

## REFERENCES

Aarts, R. G. K. M., and A. T. A. M. de Waele, 1994, in *Proceedings of the First International Workshop on Quantum*

*Vorticity and Turbulence in He II Flows,* edited by G. Stamm and W. Fiszdon (Max-Planck-Institut für Strömungsforschung, Göttingen, Report No. 2, 25).

Adams, P. W., and W. I. Glaberson, 1987, Phys. Rev. B **35**, 4633.

Agistein, M. E., and A. A. Migdal, 1986, Mod. Phys. Lett. A **1**, 221.

Ahlers, G., 1969, Phys. Rev. Lett. **22**, 54.

Allen, J. F., D. F. Griffiths, and D. V. Osborne, 1965, Proc. R. Soc. **287**, 328.

Ambegaokar, V., B. I. Halperin, D. R. Nelson, and E. D. Siggia, 1980, Phys. Rev. B **21**, 1806.

Ametistov, Ye. V., 1988, "Particularities of heat exchange with superfluid helium," Institute of Thermophysics, Siberian Branch of the Academy of Sciences, Novosibirsk, USSR, Preprint N172-88 (in Russian).

Ashton, R. A., and J. A. Northby, 1975, Phys. Rev. Lett. **35**, 1714.

Atkin, R. J., and N. Fox, 1983, J. Phys. C **16**, 1615.

Barenghi, C. F., 1982, Ph.D. thesis (University of Oregon).

Barenghi, C. F., R. J. Donnelly, and W. F. Vinen, 1983, J. Low Temp. Phys. **52**, 189.

Barenghi, C. F., C. E. Swanson, and R. J. Donnelly, 1982, Phys. Rev. Lett. **48**, 1187.

Bekarevich, I. L., and I. M. Khalatnikov, 1961, Sov. Phys. JETP **13**, 643.

Bhagat, S. M., P. R. Critchlow, and K. Mendelssohn, 1964, Cryogenics **4**, 166.

Brewer, D. F., and D. O. Edwards, 1961, Philos. Mag. **6**, 775.

Buttke, T. F., 1987, Phys. Rev. Lett. **59**, 2117.

Buttke, T. F., 1988, J. Comput. Phys. **76**, 301.

Buttke, T. F., 1991, in *Vortex Methods and Vortex Motion,* edited by K. E. Gustafson and J. A. Sethian (SIAM, Philadelphia, PA), p. 171.

Carey, R. F., J. A. Rooney, and C. W. Smith, 1978, Phys. Lett. A **65**, 311.

Childers, R. K., and J. T. Tough, 1973, Phys. Rev. Lett. **31**, 911.

Chorin, A. J., 1982, Commun. Math. Phys. **83**, 517.

Chorin, A. J., 1991a, in *Vortex Methods and Vortex Motion,* edited by K. E. Gustafson and J. A. Sethian (SIAM, Philadelphia, PA), p. 195.

Chorin, A. J., 1991b, Physica D **52**, 403.

Chorin, A. J., 1992, J. Stat. Phys. **69**, 67.

Chorin, A. J., and J. Marsden, 1979, *A Mathematical Introduction to Fluid Mechanics* (Springer, Berlin/New York).

Crooks, M. G., and D. L. Johnson, 1991, Can. J. Phys. **49**, 1035.

Cummings, J. C., D. W. Schmidt, and W. J. Wagner, 1978, Phys. Fluids **21**, 713.

Danil'chenko, V. A., and V. N. Poroshin, 1983, Cryogenics **23**, 546.

de Goeje, M. P., 1986, Ph.D. thesis (Universität Leiden).

Dessler, A., and W. Fairbank, 1956, Phys. Rev. **104**, 6.

Donnelly, R. J., 1991a, *Quantized Vortices in Helium II* (Cambridge University, Cambridge, England).

Donnelly, R. J., 1991b, *High Reynolds Number Flows Using Liquid and Gaseous Helium* (Springer, Berlin).

Donnelly, R. J., and C. E. Swanson, 1986, J. Fluid Mech. **173**, 387.

Dresner, L., 1982, Adv. Cryog. Eng. **27**, 411.

Dresner, L., 1984, Adv. Cryog. Eng. **29**, 323.

Dyachenko, S., A. C. Newell, A. Pushkarev, and V. E. Zakharov, 1992, Physica D **57**, 96.

Feynman, R. P., 1955, in *Progress in Low Temperature Physics,* edited by C. J. Gorter (North-Holland, Amsterdam), Vol. I, p. 17.

Feynman, R. P., 1972, *Statistical Physics* (Benjamin, Reading, MA).

Fiszdon, W., S. K. Nemirovskii, and M. v. Schwerdtner, 1991, Physica B **168**, 93.

Fiszdon, W., T. Olszok, G. Stamm, B. Noack, and J. Piechna, 1994, Acta Mech. [Suppl.] **4**, 313.

Fiszdon, W., J. Piechna, and W. Poppe, 1988, Max-Planck-Institut für Strömungsforschung, Göttingen, Report No. 9.

Fiszdon, W., J. Piechna, and G. Stamm, 1992, Max-Planck-Institut für Strömungsforschung, Göttingen, Report No. 23.

Fiszdon, W., and M. v. Schwerdtner, 1989, J. Low Temp. Phys. **75**, 253.

Fiszdon, W., M. v. Schwerdtner, G. Stamm, and W. Poppe, 1990, J. Fluid Mech. **212**, 663.

Frisch, T., Y. Pomeau, and S. Rica, 1992, Phys. Rev. Lett. **69**, 1644.

Gentile, D., and J. Pakleza, 1983, Numer. Heat Transfer **6**, 317.

Gentile, D., and J. Pakleza, 1986, Arch. Mech. **38**, 585.

Geurst, J. A., 1989, Physica B **154**, 327.

Geurst, J. A., 1992, Physica A **183**, 280.

Geurst, J. A., 1994, in *Proceedings of the First International Workshop on Quantum Vorticity and Turbulence in He II Flows,* edited by G. Stamm and W. Fiszdon (Max-Planck-Institut für Strömungsforschung, Göttingen, Report No. 2, 46).

Ginzburg, V. L., and L. P. Pitaevskii, 1958, Z. Eksp. Teor. Fiz. **34**, 1240 [Sov. Phys. JETP **7**, 858 (1958)].

Ginzburg, V. L., and A. A. Sobyanin, 1976, Usp. Fiz. Nauk **120**, 153 [Sov. Phys. Usp. **19**, 773 (1976)].

Goldner, L. S., G. Ahlers, and R. Mehrotra, 1991, Phys. Rev. B **43**, 12 861.

Goldner, S., N. Mulders, and G. Ahlers, 1993, J. Low Temp. Phys. **93**, 131.

Gorter, C. J., and J. H. Mellink, 1949, Physica **15**, 285.

Griffiths, D. J., D. V. Osborne, and J. F. Allen, 1964, in *Low Temperature Physics,* edited by J. F. Allen (Plenum, New York), p. 320.

Griffiths, D. J., D. V. Osborne, and J. F. Allen, 1966, in *Superfluid Helium,* edited by J. F. Allen (Academic, London/New York), p. 25.

Griswold, D., C. P. Lorenson, and J. T. Tough, 1987, Phys. Rev. B **35**, 3149.

Griswold, D., and J. T. Tough, 1987, Phys. Rev. A **36**, 1360.

Gross, E. P., 1961, Nuovo Cimento **20**, 454.

Hall, H. E., 1960, Adv. Phys. **9**, 89.

Hall, H. E., and W. F. Vinen, 1956a, Proc. R. Soc. London Ser. A **238**, 204.

Hall, H. E., and W. F. Vinen, 1956b, Proc. R. Soc. London Ser. A **238**, 215.

Hansen, A., and M. Nelkin, 1986, Phys. Rev. B **34**, 4894.

Hasimoto, H., 1972, J. Fluid Mech. **51**, 477.

Henberger, J. D., and J. T. Tough, 1982, Phys. Rev. B **25**, 3123.

Hills, R. N., and P. H. Roberts, 1977, Int. J. Eng. Sci. **15**, 305.

Hills, R. N., and P. H. Roberts, 1978a, J. Low Temp. Phys. **30**, 709.

Hills, R. N., and P. H. Roberts, 1978b, J. Phys. C **11**, 1485.

Hoch, H., L. Busse, and F. Moss, 1975, Phys. Rev. Lett. **34**, 384.

Horsthemke, W., and M. F. Schumaker, 1989, J. Stat. Phys. **54**, 1175.

Hussain, A. K. M. F., 1986, J. Fluid Mech. **173**, 303.

Ijsselstein, R. R., M. P. Goeje, and H. C. Kramers, 1979, Physica B **96**, 312.

Keesom, W. H., and A. P. Keesom, 1936, Physica **3**, 359.

Keesom, W. H., A. P. Keesom, and B. F. Saris, 1938, Physica **5**, 281.
Khalatnikov, I. M., 1952, Zh. Eksp. Teor. Fiz. **23**, 253.
Khalatnikov, I. M., 1956, Sov. Phys. JETP **3**, 649.
Khalatnikov, I. M., 1965, *An Introduction to the Theory of Superfluidity* (Benjamin, New York/Amsterdam).
Kida, S., M. Takaoka, and F. Hussain, 1991, J. Fluid Mech. **230**, 583.
Kitabatake, S., and Y. Sawada, 1978, J. Phys. Soc. Jpn. **45**, 345.
Klein, R., and A. J. Majda, 1991, Physica D **49**, 323.
Kleinert, H., 1991, *Gauge Fields in Condensed Matter Physics* (World Scientific, Singapore).
Kondaurova, L. P., 1993, unpublished.
Kosterlitz, J. M., and D. J. Thouless, 1973, J. Phys. C **6**, 1181.
Kotsubo, V. V., and G. W. Swift, 1989, Phys. Rev. Lett. **62**, 2604.
Kraft, G., 1978, J. Low Temp. Phys. **31**, 441.
Kramers, H. C., 1965, in *Superfluid Helium,* edited by J. F. Allen (Academic, London), p. 199.
Kramers, H. C., T. M. Wirada, and A. B. van Groenou, 1960, in *Proceedings of the VIIth International Conference on Low Temperature Physics,* edited by G. M. Graham and A. C. Hollis Hallet (North-Holland, Amsterdam), p. 23.
Kuznetsov, A. B., 1991, Zh. Eksp. Teor. Fiz. **100**, 1186.
Landau, L., 1941, J. Phys. **5**, 71.
Landau, L. D., and E. M. Lifshitz, 1980, *Statistical Physics,* Part 1 (Pergamon, Oxford).
Lebowitz, J. L., H. A. Rose, and E. R. Speer, 1988, J. Stat. Phys. **50**, 657.
Leiderer, P., and F. Pobell, 1970, J. Low Temp. Phys. **3**, 577.
Leonard, A., 1980, J. Comput. Phys. **37**, 289.
Liepmann, H. W., and G. A. Laguna, 1984, Annu. Rev. Fluid Mech. **16**, 139.
Lifshitz, E. M., and L. P. Pitaevskii, 1980, *Statistical Physics,* Part 2 (Pergamon, Oxford).
Lifshitz, E. M., and L. P. Pitaevskii, 1981, *Physical Kinetics* (Pergamon, Oxford).
Lin, C. C. 1963, in *Liquid Helium,* Proceedings of the Enrico Fermi School of Physics, Course 21, edited by G. Careri (Academic, New York).
Lorenson, C. P., D. Griswold, V. U. Nayak, and J. T. Tough, 1985, Phys. Rev. Lett. **55**, 1494.
Lutset, M. O., S. K. Nemirovskii, and A. N. Tsoi, 1981, Zh. Eksp. Teor. Fiz. **81**, 249 [Sov. Phys. JETP **54**, 127 (1981)].
Mandelshtam, L. I., and M. A. Leontovich, 1937, Zh. Eksp. Teor. Fiz. **17**, 438.
Mantese, J., G. Bischoff, and F. Moss, 1977, Phys. Rev. Lett. **39**, 565.
Marees, G., 1986, Ph.D. thesis (Universität Leiden).
Marees, G., R. F. Mudde, and H. van Beelen, 1987, Physica B **144**, 292.
Martin, K. P., and J. T. Tough, 1981, Phys. Rev. B **27**, 2788.
McClintock, P. V. E., 1994, in *Proceedings of the First International Workshop on Quantum Vorticity and Turbulence in He II Flows,* edited by G. Stamm and W. Fiszdon (Max-Planck-Institut für Strömungsforschung, Göttingen, Report No. 2, 9).
Mehl, J. B., 1974, Phys. Rev. A **10**, 601.
Melander, M. V., and F. Hussain, 1989, Phys. Fluids A **1**, 633.
Mezhov-Deglin, L. P., A. Yu. Iznankin, and V. P. Mineev, 1980, JETP Lett. **32**, 199.
Miklayev, V. M., I. A. Sergeev, and Yu. P. Filippov, 1987, "The peculiarities of transient heat transfer in superfluid helium," Joint Institute for Nuclear Research, Dubna, Russia, Preprint NP8-87-33 (in Russian).
Milliken, F. P., K. W. Schwarz, and C. W. Smith, 1982, Phys. Rev. Lett. **48**, 1204.
Monin, A. S., and A. M. Yaglom, 1975, *Statistical Fluid Mechanics* (MIT, Cambridge, MA).
Murakami, M., and K. Iwashita, 1990, Cryogenics **30**, 340.
Murakami, M., and K. Iwashita, 1991, Comput. Fluids **19**, 443.
Nakajima, K., Y. Sawada, and Y. Onodera, 1978, Phys. Rev. B **17**, 170.
Nemirovskii, S. K., 1986, Sov. Phys. JETP **64**, 803.
Nemirovskii, S. K., 1990, Sov. Phys. Usp. **33**, 429.
Nemirovskii, S. K., L. P. Kondaurova, and A. Ya. Baltsevich, 1992, Cryogenics **32**, 47.
Nemirovskii, S. K., and V. V. Lebedev, 1983, Zh. Eksp. Teor. Fiz. **84**, 1729 [Sov. Phys. JETP **57**, 1009 (1983)].
Nemirovskii, S. K., J. Pakleza, and W. Poppe, 1991, Notes et Documents LIMSI (Laboratoire d'Informatique pour la Mecanique et les Sciences de l'Ingenieur) No. 91-14.
Nemirovskii, S. K., and D. W. Schmidt, 1990, Max-Planck-Institut für Strömungsforschung, Göttingen, Report No. 8.
Nemirovskii, S. K., and A. N. Tsoi, 1982, Pis'ma Zh. Eksp. Teor. Fiz. **35**, 229 [JETP Lett. **35**, 286 (1982)].
Nemirovskii, S. K., and A. N. Tsoi, 1989, Cryogenics **29**, 985.
Noack, B. R., and W. Fiszdon, 1990, Max-Planck-Institut für Strömungsforschung, Göttingen, Report No. 106.
Northby, J. A., 1978, Phys. Rev. B **18**, 3214.
Olszok, T., 1994, Ph.D. thesis (Universität Göttingen).
Olszok, T., W. Fiszdon, G. Stamm, H. Vogel, and F. Bielert, 1994, in *Proceedings of the First International Workshop on Quantum Vorticity and Turbulence in He II Flows,* edited by G. Stamm and W. Fiszdon (Max-Planck-Institut für Strömungsforschung, Göttingen, Report No. 2, 64).
Onsager, L., 1949, Nuovo Cimento **6**, 249.
Onuki, A., 1983, J. Low Temp. Phys. **51**, 601.
Opatowsky, L. B., and J. T. Tough, 1981, Phys. Rev. B **24**, 5420.
Osborne, D. V., 1951, Proc. Phys. Soc. London A **64**, 114.
Ostermeier, R. M., M. W. Cromar, P. Kittel, and R. J. Donnelly, 1978, Phys. Rev. Lett. **41**, 321.
Ostermeier, R. M., M. W. Cromar, P. Kittel, and R. J. Donnelly, 1980, Phys. Lett. A **77**, 321.
Owczarek, R., 1994, in *Proceedings of the First International Workshop on Quantum Vorticity and Turbulence in He II Flows,* edited by G. Stamm and W. Fiszdon (Max-Planck-Institut für Strömungsforschung, Göttingen, Report No. 2, 32).
Pakleza, J., and W. Poppe, 1989, unpublished
Peshkov, V. P., and V. K. Tkachenko, 1961, Sov. Phys. JETP **14**, 1019.
Piotrovskii, C., and J. T. Tough, 1978, Phys. Rev. B **18**, 6066.
Pitaevskii, L. P., 1958, Zh. Eksp. Teor. Fiz. **35**, 1271 [Sov. Phys. JETP **8**, 888 (1959)].
Pokrovskii, V. L., and I. M. Khalatnikov, 1976, Sov. Phys. JETP **44**, 1036.
Poppe, W., G. Stamm, and J. Pakleza, 1992, Physica B **176**, 247.
Pumir, A., and E. D. Siggia, 1987, Phys. Fluids **30**, 1606.
Putterman, S. J., 1974, *Superfluid Hydrodynamics* (North-Holland, Amsterdam).
Putterman, S., and S. Garret, 1977, J. Low Temp. Phys. **27**, 543.
Ruppert, U., W. Z. Yang, and K. Luders, 1987, Jpn. J. Appl. Phys. **26**, Suppl. 26-3.
Rybarcyk, L. J., and J. T. Tough, 1980, J. Low Temp. Phys. **43**, 197.
Schumaker, M. F., and W. Horsthemke, 1987, Phys. Rev. A **36**, 354.
Schwarz, K. W., 1978, Phys. Rev. B **18**, 245.

Schwarz, K. W., 1982a, Phys. Rev. Lett. **48**, 1204.
Schwarz, K. W., 1982b, Phys. Rev. Lett. **49**, 283.
Schwarz, K. W., 1985, Phys. Rev. B **31**, 5782.
Schwarz, K. W., 1987, Phys. Rev. Lett. **59**, 2118.
Schwarz, K. W., 1988, Phys. Rev. B **38**, 2398.
Schwarz, K. W., 1990, Phys. Rev. Lett. **64**, 1130.
Schwarz, K. W., and J. R. Rozen, 1991, Phys. Rev. B **44**, 7563.
Schwarz, K. W., and C. W. Smith, 1981, Phys. Lett. A **82**, 251.
Sethian, J. A., 1991a, in *Vortex Methods and Vortex Motion,* edited by K. E. Gustafson and J. A. Sethian (SIAM, Philadelphia, PA), p. 1.
Sethian, J. A., 1991b, in *Vortex Methods and Vortex Motion,* edited by K. E. Gustafson and J. A. Sethian (SIAM, Philadelphia, PA), p. 59.
Shen, M. M., and D. R. Nicholson, 1987, Phys. Fluids **30**, 3150.
Shenoy, S. R., 1993, Phys. Rev. B **40**, 5056.
Siggia, E. D., 1985, Phys. Fluids **28**, 794.
Sitton, D. M., and F. Moss, 1972, Phys. Rev. Lett. **29**, 542.
Slegtenhorst, P. R., G. Marees, and H. van Beelen, 1982a, Physica B **113**, 341.
Slegtenhorst, P. R., G. Marees, and H. van Beelen, 1982b, Physica B **113**, 367.
Sonin, E. B., 1983, Pis'ma Zh. Eksp. Teor. Fiz. **37**, 82 [JETP Lett. **37**, 100 (1983)].
Stamm, G., and W. Fiszdon, 1992, Physica B **179**, 191.
Stamm, G., S. K. Nemirovskii, and W. Fiszdon, 1993, unpublished.
Stamm, G., Th. Olszok, M. v. Schwerdtner, and D. W. Schmidt, 1992, Cryogenics **32**, 598.
Stamm, G., M. v. Schwerdtner, and W. Fiszdon, 1989, Max-Planck-Institut für Strömungsforschung, Göttingen, Report No. 12.
Swanson, C. E., 1985, Ph.D. thesis (University of Oregon).
Swanson, C. E., and R. J. Donnelly, 1985, J. Low Temp. Phys. **61**, 363.
Swanson, C. E., W. T. Wagner, R. J. Donnelly, and C. F. Barenghi, 1987, J. Low Temp. Phys. **66**, 263.
Sydyganov, V. U., and I. A. Kluchnikov, 1988, in Theses of the Fifth All Union School of Young Scientists and Specialists, *Modern Problems of Thermophysics,* Novosibirsk, Russia, **1**, 71.
Torczynski, J. R., 1984a, Phys. Fluids **27**, 1138.
Torczynski, J. R., 1984b, Phys. Fluids **27**, 2636.
Tough, J. T., 1982, in *Progress in Low Temperature Physics,* edited by D. F. Brewer (North-Holland, Amsterdam), Vol. 8, p. 133.
Tough, J. T., 1989, in *Noise in Nonlinear Dynamics Systems, Vol. 3, Experiments and Simulations,* edited by F. Moss and P. V. E. McClintock (Cambridge University, Cambridge, England), p. 1.
Tough, J. T., J. F. Kafkalidis, and G. Klinich, 1994, in *Proceedings of the First International Workshop on Quantum Vorticity and Turbulence in He II Flows,* edited by G Stamm and W. Fiszdon (Max-Planck-Institut für Strömungsforschung, Göttingen, Report No. 2, 43).
Tsoi, A. N., 1987, Ph.D. thesis (Institute of Thermophysics, Soviet Academy of Sciences, Novosibirsk).
Tsoi, A. N., and M. O. Lutset, 1986, Inzh. Fiz. Zh. **51**, 5.
Tsoi, A. N., and S. K. Nemirovskii, 1980, Fiz. Nizk. Temp. **6**, 671 [Sov. J. Low. Temp. Phys. **6**, 324 (1980)].
Tsubota, M., and S. Maekawa, 1982, J. Phys. Soc. Jpn. **61**, 2007.
Turner, T. N., 1979, Ph.D. thesis (California Institute of Technology).
Turner, T. N., 1983, Phys. Fluids **26**, 3227.
van Beelen, H., W. van Joolingen, and K. Yamada, 1988, Physica B **153**, 248.
van Sciver, S. W., 1979, Cryogenics **19**, 385.
Vidal, F. C., 1972, C. R. Acad. Sci. B **275**, 609.
Vidal, F., and D. Lhuillier, 1976, Phys. Rev. B **13**, 148.
Vinen, W. F., 1956, Ph.D. thesis (Cambridge University).
Vinen, W. F., 1957a, Proc. R. Soc. London A **240**, 114.
Vinen, W. F., 1957b, Proc. R. Soc. London A **240**, 128.
Vinen, W. F., 1957c, Proc. R. Soc. London A **242**, 493.
Vinen, W. F., 1958, Proc. R. Soc. London A **243**, 400.
Vogel, H., and W. Fiszdon, 1990, Phys. Rev. B **41**, 11 585.
Vogel, H., and W. Fiszdon, 1994, in *Proceedings of the First International Workshop on Quantum Vorticity and Turbulence in He II Flows,* edited by G. Stamm and W. Fiszdon (Max-Planck-Institut für Strömungsforschung, Göttingen, Report No. 2, 70).
v. Schwerdtner, M. 1988, Ph.D. thesis (Universität Göttingen), Mitteilungen des Max-Planck-Instituts für Strömungsforschung 90/1988.
v. Schwerdtner, M. W. Poppe, and D. W. Schmidt, 1989a, Cryogenics **29**, 132.
v. Schwerdtner, M., G. Stamm, and W. Fiszdon, 1989b (Max-Planck-Institut für Strömungsforschung, Göttingen), Report No. 13.
v. Schwerdtner, M. G. Stamm, and D. W. Schmidt, 1989c, Phys. Rev. Lett. **63**, 39.
Wang, R. T., C. E. Swanson, and R. J. Donnelly, 1987, Phys. Rev. B **36**, 5240.
Weichmann, P. B., 1988, Phys. Rev. Lett. **61**, 2969.
Weichmann, P. B., and M. E. Fisher, 1986, Phys. Rev. B **34**, 7652.
Wilks, J., 1967, *The Properties of Liquid and Solid Helium* (Clarendon, Oxford).
Williams, G. A., 1987, Phys. Rev. Lett. **59**, 1926.
Williams, G. A., 1992, J. Low Temp. Phys. **89**, 91.
Williams, G. A., 1993a, Phys. Rev. Lett. **71**, 392.
Williams, G. A., 1993b, J. Low Temp. Phys. **93**, 1079.
Yamada, K., and S. Kashiwamura, 1987, Jpn. J. Appl. Phys. **26**, Suppl. **26-3**, 95.
Yamada, K., S. Kashiwamura, and K. Miyake, 1989, Physica B **154**, 318.
Zilsel, P., 1950, Phys. Rev. **79**, 309.