Primary manifestation of symmetry. Origin of quantal indeterminacy*

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Quantal physics is established as a manifestation of symmetry more far-reaching than hitherto appears to have been recognized. In this primary manifestation, the coordinate transformations of spacetime invariance are themselves the elementary variables, which define their own properties without appeal to an assumed quantal formalism. In irreducible representations, the symmetry variables are inherently indeterminate, and the probabilistic laws invoked in the interpretation of traditional quantum physics are found to originate in geometric relations between these variables. Completeness is, therefore, not an issue, and the quantum of action is not part of the theory of symmetry variables. Quantal physics thus emerges as but an implication of relativistic invariance, liberated from a substance to be quantized and a formalism to be interpreted. A symmetry variable appears in a measurement with one of its eigenvalues, but does not have a value (cannot be represented by a number) in an irreducible representation, which combines sets of eigenvalues. It is this generalized significance of a measurement that allows for correlations that cannot arise for classical variables. The observation of symmetry variables is illustrated by an interferometer experiment measuring reflection symmetry and by the equivalent coincidence experiment registering the polarization of two quanta. The measurement process becomes a matter of following the state of affairs of the symmetry variables in their unitary evolution. For the resolution of the dilemmas that quantal phenomena have been felt to pose, it appears crucial to recognize that indeterminacy, as an inherent property of a symmetry variable in a multidimensional representation, is not affected by subsequent observations. A position variable and the canonical commutator with momentum, which are basic elements of nonrelativistic quantum mechanics, emerge from spacetime symmetry, but require the link between space and time of relativistic invariance. The transition to the classical regime is analyzed in terms of a quenching of nonlocality in the state of affairs of the multidimensional symmetry variables. While the elementary variables constitute individual quanta in irreducible representations, product representations of spacetime symmetry describe systems of bosons and fermions, which form local fields with canonical properties. The discussion is focused on spacetime invariance (noninteracting quanta), but gauge invariance is itself a primary manifestation of symmetry and is as such encompassed by the theory of symmetry variables.

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*The present article has developed over a number of years. A summary of the main ideas were presented at the Symposium "Perspectives in Nuclear Structure" held in Copenhagen in June 1993 (see Bohr and Ulfbeck, 1994).

 0034-6861/95/67(1)/1(35)/$12.25

I. INTRODUCTION

A. Background

Quantum mechanics was created, in a brilliant stroke, as a generalization of classical mechanics in which conjugate variables obey canonical commutation relations, in a formalism that found its interpretation within the framework of complementarity. By the incorporation of relativistic invariance into the formalism, in terms of quantized fields with gauge symmetry, quantal physics has been successful in accounting for an ever wider range of phenomena. Nevertheless, the novel character of indeterminacy and complementarity, whose origin has remained an issue, together with the symbolic character of the formalism, which requires an interpretation with prescriptions for extracting the results of the theory, has led to a continuing debate concerning the adequacy of the foundation for quantal physics.

A clue to a more compelling basis for quantal physics is provided by the extended role of spacetime invariance in the quantal description. Classical applications of spacetime symmetry encompassed by special relativity deal with the transformation of physical quantities under a change of the coordinate system, and this role of symmetry carries over into quantal physics. However, in the quantal formalism, based on the superposition principle, spacetime symmetry acquires an extended role, since it is also carried by the states of the system, in their linear transformations.[1]

Thus quantal states with the same properties relative to different reference frames are connected by unitary transformations in Hilbert space, and the states can, therefore, be classified in terms of the irreducible representations of the group of coordinate transformations. For example, for a rotation, specified by a vector $\boldsymbol{\chi}$, the associated unitary transformation is of the form

$$\mathcal{R}(\boldsymbol{\chi})=\exp\{-i\boldsymbol{\chi}\cdot\mathbf{j}\} , \tag{1}$$

and the generator $\mathbf{j}$ represents the angular momentum vector, in units of $\hbar$. Similarly, in relativistic quantal physics, the states of a particle carry a representation of Poincaré symmetry.

B. Symmetry as basis for quantal physics

In the manner indicated, the extended role of symmetry in quantal physics is seen as a consequence of the framework of the formalism (superposition of states with observables as linear operators in Hilbert space, or equivalent formulations such as path integrals). It would appear, however, that the role of symmetry in relation to quantal physics has, so to speak, been turned upside down, and it is the purpose of the present article to show that quantal physics itself emerges, when the coordinate transformations (the elements of spacetime symmetry) are recognized as the basic variables.

As we shall attempt to demonstrate, the physics of these "symmetry variables" develops without involving a symbolic formalism and with indeterminacy and complementarity constituting inherent features of the elementary variables. The probabilistic interpretation that is invoked in quantal physics thereby finds its basis in the geometric relations between the symmetry variables, and the notion of state offers itself merely as a convenient tool for handling these variables. Thus the analysis of the measurement process, on which much debate has focused, becomes a matter of directly following the state of affairs of the symmetry variables under the specialized conditions of an experiment. Hence quantum mechanics, as the physics of these elementary variables, is liberated from the concept of quantization and the need for interpreting a formalism. It is, therefore, but an implication of spacetime invariance, with complementarity originating in the non-Abelian spacetime symmetry.[2]

C. Two manifestations of symmetry

The manifestation of symmetry that yields the basic variables will be referred to as the primary, in contrast to the secondary dealing with the transformation of physical quantities. Since symmetry variables, as members of the group of coordinate transformations, have built-in prescriptions for how they are affected by a change of reference frame, the secondary manifestation of symmetry is a corollary of the more far-reaching primary. Classical physics emerges as a limit in which symmetry is only recognized in the subordinate role of connecting quantities in different reference systems (secondary manifestation), because the origin of the variables (primary manifestation) is not apparent with the low resolution characterizing this regime.

D. Elementary substance

In its primary manifestation, symmetry, traditionally describing patterns in the configurations of matter, thus acquires an existence of its own and constitutes the elementary substance (matter, including radiation). Indeed, a quantum will be a designation for a group of symmetry variables, the full specification of which generally involves an extension of spacetime invariance to include the invariance under gauge transformations.

The changed perspective in which quantal physics is seen may perhaps be elucidated by reference to the role of the aether in the unravelling of spacetime symmetry, culminating in the establishment of special relativity. Thus Maxwell's equations were conceived as describing vibrations in the aether, but the notion of an aether was eliminated as superfluous when the equations for the electromagnetic field appeared in new perspective, as expressing the invariance of spacetime (equivalence of the reference frames of special relativity) together with gauge invariance. In the resulting development, classical relativistic physics could be seen as dealing with a substance carrying spacetime and gauge symmetry, with particle and field as the elementary degrees of freedom, and quantal physics was created in this mold by the introduction of quantization conditions, supplemented by an interpretation of the symbolic formalism. However, the notion of a substance to be quantized becomes superfluous when symmetry is recognized in its primary manifestation and

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[1]See the classic works by Wigner, 1931 and by Weyl, 1928. Perhaps Hermann Weyl (see the forewords to the editions of his book) was the most prepared in the older generation to perceive the primary manifestation of symmetry, in which case he might have issued a new edition of his classic earlier book with the new title "Raum-Zeit, als Materie."

[2]As far as we are aware, this apperception of symmetry in relation to quantal physics differs qualitatively from prevailing views presented in the literature.

itself becomes the elementary substance, with inborn complementarity.

E. Symmetry originating in aesthetics

The theory of symmetry variables may be seen as a line of evolution originating in aesthetic considerations and obtaining a quantitative (mathematical) basis in group theory. This line is distinct from, though it has strongly interacted with, the development of traditional physics, with its roots in antiquity, evolving through mechanics, thermodynamics, statistical physics of atoms and molecules, electromagnetism, and culminating in relativity theory. The attempt to pursue this classic line into quantal physics led to a formalism requiring an interpretation.

F. Synopsis

Coordinate transformations as variables that are registered with a value in a measurement are introduced in Sec. II. These variables are subject to multiplicative constraints as elements in a group, and the resulting relationship between the variables is embodied in the matrix representations of the symmetry group. In a multidimensional irreducible representation, which, for each symmetry, inextricably combines sets of eigenvalues, the variables are indeterminate in a complete description. Indeterminacy is, therefore, unaffected by subsequent observations, in contrast to uncertainties in classical probability distributions (Sec. II.C).

The complementarity of symmetry variables in the form of probability distributions are recognized as geometric relations, expressed by additive constraints between the variables in a representation. The probabilities can be seen as pairwise correlations between symmetry variables, and irreducibility implies patterns of these correlations that cannot occur for variables that have values. Hence these correlations are beyond reach of probability distributions for classical variables (Sec. II.D).

The "state of affairs" of symmetry variables will be seen to be specified by a vector in the space carrying the symmetry, and superposition of states, as vectors, is thereby part of the analysis. More general state of affairs involve quenched states (mixtures of pure states). The notions of states and probability amplitudes thus enter merely as convenient tools for handling symmetry variables (Sec. II.E).

In the manner indicated, the physics of symmetry variables develops without appeal to the framework of quantal physics, and Planck's constant is not part of the description of symmetry variables, since these involve only dimensions based on space and time. As discussed in connection with variables that have correspondence to classical physics, the quantum of action appears as a scaling factor introducing the dimension of mass, by which generators of the symmetry variables obtain dimensions of classical dynamical variables (Secs. IV.C.2 and IV.C.3).

In suitable situations, including measurements, a symmetry variable appears with one of its eigenvalues, but for variables that are inherently indeterminate, a measurement has a generalized significance. In fact, such a variable does not have a value (cannot be represented by a number), and, from an experiment yielding one of the eigenvalues for a symmetry variable, it therefore cannot be inferred that the variable, though it was present (part of reality), had this value prior to the measurement (Sec. II.C.4). This point seems to be crucial to the dilemma that has been seen as involved in quantal phenomena (see, for example, Secs. III.B and III.D.4).

The appearance of symmetry variables is illustrated by an interferometer experiment, which measures reflection and translation variables of a quantum (Sec. III.A). In the detection process, the incident quantum interacts with a large number of constituents of the detector in the production of the signal, and the measurement process, by which an indeterminate variable appears with a value, can be analyzed in terms of the evolution of the state of affairs of the symmetry variables (Sec. III.A.3).

In the experiment, the quantum is produced with one symmetry, while another symmetry is registered in the measurement, and the result of the experiment can be seen as a correlation between the two symmetry variables. The observed pairwise correlation, which expresses geometric relations between the variables of spacetime symmetry in an irreducible representation, exhibits a pattern that cannot be reproduced by classical variables (Sec. III.B), in accordance with the considerations in Sec. II.D.4.

The analysis of the experiment in terms of symmetry variables with their unambiguous terminology defines the limitations in the use of concepts originating in the development of classical physics. Thus the path of the quantum through the interferometer is a two-dimensional variable, which is indeterminate, in a complete description (Sec. III.C.3). The attribution of a value to this variable in the interferometer (or two-slit) experiment is, therefore, self-contradictory, in a mathematical sense. The correlation between variables of a single quantum can be transferred to a correlation between the variables of two quanta in an overall invariant state, as observed in the experiment measuring the correlation between the polarizations of two photons emitted in coincidence from an excited atom (Sec. III.D).

The primary manifestation of symmetry puts focus on time reversal as absent from the group of symmetry variables. Indeed, it is a basic feature of the description in space and time that the time axis has a direction. A reversal of time is, therefore, not an available coordinate transformation, and time reversal, in contrast to space reflection, is not a symmetry variable and does not appear with a value. The reversibility of motion derives from the symmetry of the representations of spacetime invariance under complex conjugation (neutrality of geometry), and time reversal connecting variables at opposite epochs thereby enters as an anti-unitary transformation (Sec. II.G).

Local symmetry variables will be seen to stem from spacetime rotations (Lorentz transformations), which single out a point in space and time that is left invariant by the transformation (Sec. IV.A). While the individual quantum is not sharply localized (Sec. IV.B), a position variable emerges when the representation of spacetime symmetry is viewed with a resolution that is low compared with its intrinsic scale (nonrelativistic regime). The canonical commutator $[p,x]=-i\hbar$ thereby arises from the geometric interplay of translations with Galilean transformations, recognized as limits of rotations in spacetime. Thus the notion of location (position at a given time) and the resulting complementarity between the conjugate variables require the link between space and time introduced by relativity (Sec. IV.C). The analysis leading to this conclusion is summarized in the self-contained Sec. IV.D.

The state of affairs of the multidimensional variables of a quantum, in the weakly relativistic regime, is fully specified by the distribution of nonlocality, constituting the mean values of products of translations and Galilean transformations. In a quenched state of affairs of the symmetry variables involving unconnected states contributing additively to the mean values (mixtures), the nonlocality is reduced, and the resolution with which the symmetry variables are observed is correspondingly lowered (as distinct from the lowering of resolution that characterizes the Galilean corner). Such a quenching of nonlocality is produced by interactions with other quanta that are subsequently irrelevant. For sufficiently strong quenching, all the visible symmetry variables commute, and classical physics emerges. This theme is touched upon in Sec. IV.C.4 and is further dealt with in the supplementary Sec. IV.E.

While irreducible representations of the symmetry give the variables of individual quanta, new dimensions of symmetry variables are associated with product representations, which describe systems of quanta in an occupation number space for bosons and fermions (Sec. V.A). Local fields thereby appear in the primary manifestation of spacetime symmetry, without any reference to an assumed notion of a field, in an underlying framework of quantal physics (Sec. V.B).

The present discussion focuses on spacetime invariance, which yields the framework for the physics of symmetry variables. However, gauge invariance, from which interactions arise, is itself a primary manifestation of symmetry and as such encompassed by the physics of symmetry variables.

The analysis of symmetry variables exploits concepts from the theory of group representation, but, in the following, an attempt is made to present the physics of symmetry variables in a form that makes the discussion accessible to readers who are not familiar with the abstract theory of group representation. More technical points are referred to in supplementary sections, footnotes, or parentheses. The discussion is intended as a self-contained presentation of the physics that derives from the primary manifestation of symmetry, but avoids elaborations that are not deemed necessary for an assessment of the main issues. The authors hope, in another context, to give a broader presentation of themes in the theory of symmetry variables.

II. PRIMARY MANIFESTATION OF SYMMETRY

The notion of spatial symmetry has roots in the aesthetic appeal of figures and patterns that can be traced to very early human expressions of art.[3] Symmetry in time has an even longer evolutionary history in the form of rhythms, tones, songs, etc. that developed into music,[4] as well as through the sense of colors that became elements of art.

Eventually, spacetime symmetry could be expressed as invariance under transformations of the coordinate system (reflections, translations, and rotations), and the idea of invariance under groups of coordinate transformations received an expanded content in the theory of representations of symmetry groups. Groups comprising noncommuting elements (non-Abelian groups) are thereby characterized by having irreducible multidimensional matrix representations. As will be seen below, this conceptual framework directly provides a basis for quantal physics, in which the irreducible representations of the symmetry are the elementary physical variables.

This framework for physics is associated with a line of development quite different from that pursued in the establishment of classical physics, from its cradle in Ancient Greece to the modern conception of classical relativistic physics, which went hand in hand with the creation of the theory of functions of continuous variables. In this classical development of physics, symmetry also became an increasingly important guidance, in particular in the characterization of the equations of motion of matter (particles and fields) by their invariance under a change of the coordinate frame. This manifestation of symmetry is referred to as the secondary, because it is subordinate to the comprehensive primary manifestation, in which the elements of the group not only perform transformations but are themselves the elementary physical variables.

[3]The development through art of the concept of spatial invariance (as distinct from representations of the symmetry), is impressively described in the classic work of Weyl, 1952, aiming at clarifying the "philosophico-mathematical significance of the idea of symmetry."

[4]The potentiality of the symmetry under time displacement and time reversal, as a skeleton around which music can develop, is exhibited by Johann Sebastian Bach in "Die Kunst der Fuge."

The basic elements in the theory of symmetry variables are developed in Secs. II.C. and II.D. As a prelude, some elementary features of symmetry and its representation that will be needed in the following sections are introduced in Secs. II.A and II.B. The relations are illustrated by the group of reflections and translations in one dimension, which exhibits the concept of irreducibility in its simplest form. Moreover, the interplay between translation symmetry, which expresses homogeneity, and symmetries such as reflections that are linked to points, is a recurring theme in the analysis of spacetime invariance (see Sec. IV).

A. Group of reflections and translations

1. Reflection and its values

A reflection $\mathcal{S}$ in the yz plane $(x \to x' = -x)$ is a transformation obeying

$$\mathcal{S}^2 = 1 , \tag{2}$$

and this constraint is satisfied by the values $s = \pm 1$. In this manner, each symmetry element is associated with a set of values that characterize its geometry.

2. Translations and their values

Translations $\mathcal{T}(a)$ along the x axis $(x \to x' = x + a)$ commute among themselves and are additive

$$\mathcal{T}(a_2)\mathcal{T}(a_1) = \mathcal{T}(a_1 + a_2) . \tag{3}$$

This constraint is satisfied by the set of values

$$\mathcal{T}(a) = \exp\{-iak\} \tag{4}$$

depending on a continuous parameters k, with the dimension of a wave number, and the set of values (4), for fixed k, is said to constitute a one-dimensional representation of the Abelian group of translations.

3. Interplay of reflections and translations

A reflection and a translation do not commute, as coordinate transformations. Thus it is part of geometry that a reflection inverts the direction of a translation

$$\mathcal{S}\mathcal{T}(a) = \mathcal{T}(-a)\mathcal{S} . \tag{5}$$

Moreover, the combination of reflections and translations leads to new symmetry elements

$$\mathcal{S}(a) \equiv \mathcal{T}(a)\mathcal{S}\mathcal{T}^{-1}(a), \quad \mathcal{S}(a=0) = \mathcal{S} \tag{6}$$

that are reflections with respect to shifted planes $x = a$. (In fact, the combination of the three transformations yields $x \to x - a \to -x + a \to -x + 2a$, which leaves $x = a$ invariant.)

The transformations $\mathcal{S}(a)$ and $\mathcal{T}(a)$ are seen to form the closed set of transformations of the x axis that leave all distances invariant. Together they constitute the elements $\mathcal{U}$ of a non-Abelian symmetry group defined by the algebraic relations (2), (3), and (5). The representations of the group are considered in Sec. II.B.

4. Equivalent transformations

The relation (6) can be seen as an equivalence between the reflections $\mathcal{S}(a)$ and $\mathcal{S}$. In fact, the transformation $\mathcal{S}' \equiv \mathcal{S}(a)$, described in the coordinate system $\mathcal{K}$, is the transformation $\mathcal{S}$, in a translated reference frame $\mathcal{K}'$ with origin at $x = a$. In this manner, for each change $\mathcal{V}$ of reference frame from $\mathcal{K}$ to $\mathcal{K}'$ (itself an element of the group), every element $\mathcal{U}$ has its equivalent $\mathcal{U}'$, which is seen as $\mathcal{U}$, in the transformed system $\mathcal{K}'$,

$$\mathcal{U}' = \mathcal{V}\mathcal{U}\mathcal{V}^{-1}, \quad (\mathcal{U})_{\mathcal{K}'} = (\mathcal{U}')_{\mathcal{K}} . \tag{7}$$

(The transformation $\mathcal{U}$ in $\mathcal{K}'$ can be carried out by first going back to $\mathcal{K}$, next performing $\mathcal{U}$ in $\mathcal{K}$, and finally returning to $\mathcal{K}'$.) Thus translations in opposite directions are equivalent under a reflection $[\mathcal{T}'(a) = \mathcal{S}\mathcal{T}(a)\mathcal{S}^{-1} = \mathcal{T}(-a)$; see Eq. (5)], and the reflections $\mathcal{S}$ and $\mathcal{S}(a)$ are equivalent under the transformation $\mathcal{T}(a)$; see Eq. (6). The mapping (7) of the group onto itself expresses the invariance under change of reference frame.

5. Classes and invariant subgroup. (Supplement)

The equivalence relations referred to above imply that each pair of translations $\mathcal{T}(a)$ and $\mathcal{T}(-a)$, the reflections taken together, and the identity, constitute the classes. Thus the translations with all values of a constitute an invariant subgroup (normal divisor).

B. Irreducible multidimensional representation

The representations of a symmetry group is the embodiment of the algebra of its elements, and, for an Abelian group, all linear representations can be expressed by assigning a value u to each element $\mathcal{U}$, such as in the relations (2) and (4). The more intimate interweaving of the elements in a non-Abelian group finds expression in multidimensional representations that are irreducible.

In a multidimensional representation of a symmetry group, the elements $\mathcal{U}$ can be expressed as matrices that satisfy the group algebra. For each element $\mathcal{U}$, the matrix combines a set of values u of this symmetry, as eigenvalues of the matrix, and the representation is irreducible when the sets of eigenvalues of the matrices cannot be divided into subsets each of which defines a representation. The discussion in the following section focuses on unitary representations in which $\mathcal{U}$ is a unitary matrix, with eigenvalues u of unit modulus. (The role played by nonunitary representations is discussed in Sec. II.F.4.)

1. Two-dimensional representation of translations and reflections

For example, the group of translations and reflections, in addition to special one-dimensional representations with $k=0$ and $s(a)=1$ (or -1) for all a, has a two-dimensional irreducible representation for each value of $k_0=|k|\neq 0$, which can be expressed by the matrices

$$\mathcal{T}(a)=\begin{Bmatrix} e^{-ik_0a} & 0 \\ 0 & e^{ik_0a} \end{Bmatrix}, \quad \mathcal{S}(a)=\begin{Bmatrix} 0 & e^{-2ik_0a} \\ e^{2ik_0a} & 0 \end{Bmatrix} \tag{8}$$

that satisfy the algebraic relations exhibited in Sec. II.A. The representation (8) combines two values for each symmetry element [$k=\pm k_0$ and $s(a)=\pm 1$], and it is seen that no combination of subsets k and $s(a)$ fulfill the relations (5) that link reflections and translations. Only together, does the pair of values represent the symmetry group, and hence the representation is irreducible.

2. Equivalent representations. Eigenvalues as intrinsic property

Multidimensional representations can be expressed in different, equivalent forms related by unitary transformations of the matrices, since such a transformation leaves the algebraic relations invariant. In the particular form (8) of the representation of translations and reflections, the matrix $\mathcal{T}(a)$ is chosen to be diagonal, and by a unitary transformation any other element can be brought to diagonal form. (The representation in which $\mathcal{S}$ is diagonal and real is exhibited in the supplementary Sec. II.B.4.) A representation refers to a coordinate system $\mathcal{K}$, and the same representation seen from $\mathcal{K}'$ is obtained by the unitary transformation (7).

The set of values u of an element $\mathcal{U}$ that are combined in a representation of a symmetry group, as the eigenvalues of the matrix, is invariant under unitary transformations and, therefore, constitutes the intrinsic property of the representation of the symmetry. (External properties of the matrix representation are considered in Sec. II.D.1.) In particular, the set of eigenvalues is independent of the coordinate system in which this symmetry is seen. For example, translations in opposite directions have the same pair of eigenvalues, and so do reflections that are translated with respect to each other.

3. Additive relations characterizing irreducible representation

An irreducible representation of the multiplicative relations between the elements of the group is characterized by additive relations between the matrices. Thus any three reflection matrices (8) are linearly dependent, as expressed, for example, by the relation

$$\mathcal{S}(a)=\cos 2k_0a\,\mathcal{S}+\sin 2k_0a\,\mathcal{S}(a_0), \quad k_0a_0=\pi/4 , \tag{9}$$

which is seen to be satisfied by the matrices (8) (and, hence, by all the equivalent forms of the representation). Clearly, relation (9) cannot be fulfilled for any combination of the eigenvalues ± 1 for the reflections, except when a is a multiple of a_0, and thus directly exhibits the irreducibility of the representation. [The relation (9) expresses the vectorial symmetry carried by the reflections under translation; see Sec. II.F.6.] More generally, equalities such as Eq. (9) are part of a comprehensive set of additive relations characterizing an irreducible representation of a group (see supplementary Sec. II.E.3).

4. Representation in which $\mathcal{S}$ is diagonal. (Supplement)

The matrices of the two-dimensional representation of translations and reflections, in which $\mathcal{S}[=\mathcal{S}(a=0)]$ is diagonal and real, are

$$\begin{aligned} \mathcal{T}(a)&=\begin{Bmatrix} \cos k_0a & -i\sin k_0a \\ -i\sin k_0a & \cos k_0a \end{Bmatrix}, \\ \mathcal{S}(a)&=\begin{Bmatrix} \cos 2k_0a & i\sin 2k_0a \\ -i\sin 2k_0a & -\cos 2k_0a \end{Bmatrix}, \\ \mathcal{T}^\dagger(a)&=\mathcal{T}(-a), \quad \mathcal{S}^\dagger(a)=\mathcal{S}(a) . \end{aligned} \tag{10}$$

The form (10) of the matrices can be obtained from Eq. (8) by the unitary transformation $\mathcal{Q}$ given by Eq. (43), which it not itself a member of the group, but is a linear combination of two symmetry matrices. The hermiticity relations are seen to be unaffected by unitary transformations.

C. Symmetry variables. Indeterminacy

1. Constraints

In the primary manifestation of symmetry, the coordinate transformations, as physical variables,[5] satisfy the geometric relations that give the result of combining successive transformations. Thus the variables belong together as elements in a symmetry group (in a mathematical sense), and the algebraic relations between the elements provide constraints among the variables that are expressed in the matrix representations of the group.

2. Quantum

As physical variables, the symmetry elements acquire an existence of their own, and the variables of a group together *constitute* an object that will be referred to as a quantum (=quantal object). A full specification of the quantum in general involves the extension of spacetime

[5]The term variable is used, as is customary in physics, to designate a quantity that can appear with different values, but only one at a time.

invariance to include variables associated with gauge transformations. The resulting interactions arising from the incorporation of local gauge invariance imply that a quantum may be a composite, such as a bound state of other quanta (or composites involving indefinite numbers of quanta). The present discussion focusses on spacetime symmetry of the quantum, which gives the framework for the physics of symmetry variables.

3. Symmetry appearing with a value

Since the intrinsic property of a symmetry is the set of (eigen)values, the recognition of a symmetry as a variable is taken to imply that, in suitable situations, the symmetry variable can *appear with* (any) one of its eigenvalues. This appearance can take place as a result of interactions between quanta, as occur ubiquitously and, under very specialized conditions, in a measurement.[6] The specification of the conditions under which a symmetry variable appears with a value in a measurement is not a requirement for the recognition of the primary manifestation of symmetry and is, therefore, only briefly dealt with in the present paper (see Sec. III.A.3).

4. Generalized significance of measurement

The distinction between *appear* with a value and *have* a value is crucial for a variable in an irreducible multidimensional representation. Indeed, the variable cannot be represented by a number and, therefore, *does not have a value,* when several eigenvalues are inextricably interwoven in the multidimensional representation. From a measurement, in which the symmetry variable appears with a value, it, therefore, cannot be inferred that the variable had (possessed) the observed value prior to the measurement, although the variable was present (part of reality).[7]

For a multidimensional variable, the significance of a measurement thus differs from that of a variable in a one-dimensional representation, which is a number, whose value prior to the measurement is established in the experiment, as in classical physics. It is this generalized significance of a measurement, rather than the process by which the measurement is accomplished, that constitutes the novel feature of a multidimensional variable, to be pursued in the following.

5. Symmetry produced

For an individual symmetry $\mathcal{U}$, it follows, from the possibility of distinguishing between the eigenvalues u in a measurement, that experiments can be designed so as to select any one of the eigenvalues u and thus to *produce* quanta that with certainty will appear with this value, in a subsequent measurement. In fact, a device that distinguishes between the eigenvalues can be used to monitor the experimental setup so as to produce quanta that all appear with a particular value of the symmetry (see the example in Sec. III.A.1). A symmetry $\mathcal{U}$ *produced* with the value u will also be referred to as *appearing* with this value. This terminology is motivated by the equivalence between production and observation, as exhibited in the basic relation (12).

6. Symmetry observed. Indeterminacy

For a group of symmetry variables in an irreducible multidimensional representation, indeterminacy is seen to be an inherent property. Indeed, one can design an experimental situation in which a variable $\mathcal{U}$ with certainty appears with a particular eigenvalue u, while the observation of other variables $\mathcal{V}$ has several different outcomes v. In fact, if each of the variables would always appear with a definite value, these values would have to obey the constraints and, hence, would constitute a one-dimensional representation. However, it is a property of an irreducible multidimensional representation that $\mathcal{U}$ and u may be so chosen that u does not occur in any one-dimensional representation of the total group (exemplified by $\mathcal{U}=\mathcal{T}(a)$ and $u=\exp[-iak_0]$). It follows that the result of an individual observation of $\mathcal{V}$ is not predictable, with certainty, and the variable is, therefore, referred to as indeterminate.

Indeterminacy is thus an immediate consequence of the multiplicative constraints for symmetry variables in an irreducible multidimensional representation. The inborn indeterminacy of these variables is also exhibited by the additive constraints that characterize an irreducible representation, as will be seen in connection with the quantitative analysis of indeterminacy in Sec. II.D.2.

7. Indeterminacy contrasted with statistical uncertainty. Completeness

An irreducible multidimensional representation is a *complete* specification of the symmetry variables in the group. Indeterminacy, therefore, has a content basically different from the uncertainty of one-dimensional (classical) variables in a statistical probability distribution. Such a distribution provides an incomplete description of the variables at a given time, in the sense that subsequent observations can lead to a more detailed specification of

[6]Words like observation, experiment, detector, etc. are not necessary in the basic theory of symmetry variables, and are only introduced in the present paper in connection with the testing of the theory by experiments.

[7]Finding a way of talking about the variables that emerge from empty (flat) space, stripped of an assumed formalism, involves the challenge of evolving a language that does not invoke associations belonging to classical variables. To *appear with* a value has much of the flavor of to *exhibit* a value, or to *show up* with a value, while to *take on* a value might convey the idea that the variable then possesses a value.

the variables at the (earlier) time considered. In contrast, indeterminacy is a property of the variables themselves in an irreducible representation, and, accordingly, *indeterminacy* of variables belonging to a given time *is not affected by subsequent observations.*

The occurrence of indeterminacy in a complete description is thus a hallmark of multidimensional variables, outside the scope of classical physics, which deals with variables that have values. The basic distinction between incompleteness in a statistical distribution of one-dimensional variables, and indeterminacy associated with the irreducibility of multidimensional variables, is seen to be linked to the different conclusions that can be drawn from a measurement of the two types of variables, as referred to above. Examples of the generalized significance of a measurement of a symmetry variable are discussed in Secs. III.C.3. and III.C.4.

D. Probabilities

Indeterminacy implies that individual observations of a symmetry variable are, in general, *fortuitous* events. Large numbers of such events occurring under identical circumstances define the conditional probability $W(u;v)$ that the symmetry $\mathcal{V}$ will appear with the eigenvalue v in a situation, where $\mathcal{U}$ exhibits the value u. This probability distribution expresses a *relationship* between the variables $\mathcal{U}$ and $\mathcal{V}$, in a representation of the symmetry group to which they belong.

1. Relative orientation of matrix variables

Relations between symmetry variables can be described in terms of the eigenvectors of the matrices seen as linear operators in a vector space carrying the representation of the symmetry. Thus a unitary matrix $\mathcal{U}$ has a set of orthogonal eigenvectors, denoted[8] by $|u\rangle$,

$$\mathcal{U}|u\rangle = u|u\rangle, \quad \mathcal{U} = \sum_u |u\rangle u\langle u| \; , \tag{11}$$

which are taken to be normalized to unit length, but which are only defined up to a phase factor, in the complex space. The discussion focuses on representations for which the eigenvectors have no degeneracy. The set of orthogonal eigenvectors for a symmetry element defines the orientation of the variable with respect to a chosen (external) basis in vector space. Correspondingly, the relative orientation of two symmetry variables $\mathcal{U}$ and $\mathcal{V}$ is specified by the scalar products $\langle v|u\rangle$ of their eigenvectors. These products are independent of the external basis in vector space (invariance with respect to unitary transformations, which include the coordinate transformations in the group) and are, therefore, like the eigenvalues of the individual matrices, intrinsic properties of spacetime, completely specified by the representation of the symmetry, apart from the undefined phase.

2. Probabilities from additive geometric constraints

The expression for the probabilities $W(u;v)$ is determined by the additive constraints that characterize the variables in a specified irreducible representation [as illustrated by Eq. (9) and discussed more generally in Sec. II.E.3]. In the first place, these constraints, like the multiplicative constraints, imply that individual observations of the variables, in successive observations in the same experimental situation, are fortuitous events. In fact, if the observations could be repeated with each variable always appearing with the same eigenvalue, these values would constitute a one-dimensional representation, as exploited in Sec. II.C.6. The orthogonality of different irreducible representations [see remarks following the relation (22)] implies, however, that no one-dimensional representation satisfies all the additive constraints characteristic of the multidimensional irreducible representation. These constraints would, therefore, be violated.

The reproducible results of the experiments are the patterns of mean values built up by large numbers of fortuitous events. Hence, in these patterns of mean values (including mean fluctuations), the constraints, as geometric relations between the variables, reveal themselves. The validity of the constraints for the mean values is fulfilled if, and only if, the conditional probabilities have the distribution

$$\begin{gathered} W(u;v) = |\langle v|u\rangle|^2 = W(v;u) \; , \\ \sum_v W(u;v) = \sum_u W(u;v) = 1 \end{gathered} \tag{12}$$

(where the sum rules express the orthonormality of the eigenvectors). It is immediately seen that the expression (12) for the probabilities guarantees that the mean values satisfy the additive constraints in a situation where $\mathcal{U}$ is produced with the value u. The mean value of the variable $\mathcal{V}$ is then, according to Eq. (12),

$$\sum_v v W(u;v) = \langle u|\mathcal{V}|u\rangle \; , \tag{13}$$

which is the diagonal element of the matrix $\mathcal{V}$ (product of the vectors $\mathcal{V}|u\rangle$ and $|u\rangle$). The linearity of Eq. (13) implies that any additive constraint between variables $\mathcal{V}, \mathcal{W}, \ldots$ is satisfied as a mean value. Conversely, the form (12) for the conditional probabilities is a necessary condition for the constraints to be valid for the mean values, as shown in the supplementary Sec. II.E.3.[9]

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[8]For compactness and flexibility, the abbreviated notation u specifies the symmetry as well as its eigenvalue and the representation.

[9]The additivity of mean values for any set of operators in Hilbert space is often invoked as a basis for the form (12) of the probabilities, in the interpretation of quantal physics. For symmetry variables, the additivity is enforced by the linear constraints, which are geometric relations to be fulfilled for any reproducible result of experiments.

3. Correlation between variables

The probability distribution (12) expresses the complementarity between the variables $\mathcal{U}$ and $\mathcal{V}$ that results from their interweaving as elements in a group.[10] As constituents of the quantum, both variables are present together, without having values. The probabilities (12) describe events in which the two variables $\mathcal{U}$ and $\mathcal{V}$ both appear with values, in an experiment that produces $\mathcal{U}$ with the value u and observes the value v of $\mathcal{V}$ or vice versa.

The symmetry of the conditional probabilities (12) under exchange of u and v implies that these complementary relations can be viewed as joint probabilities in a situation that does not single out any direction in vector space. In this invariant situation, each symmetry is produced and observed with equal probability p for every eigenvalue,

$$p(u)=d^{-1}, \quad p(v)=\sum_u p(u)W(u;v)=d^{-1} , \tag{14}$$

where d is the dimension of the representation. The joint probability is, therefore,

$$p(u,v)=d^{-1}W(u;v), \quad \sum_{u,v} p(u,v)=1 , \tag{15}$$

which contains no reference to an orientation with respect to external axes in vector space. The intrinsic relation between the symmetry variables $\mathcal{U}$ and $\mathcal{V}$, expressed as the joint distribution (15), is referred to as a pair correlation.

The pair correlation (15) can be characterized by its moments

$$\begin{aligned} \overline{\mathcal{U}^r\mathcal{V}^s} &\equiv \sum_{u,v} u^r v^s p(u,v) \\ &= \frac{1}{d}\sum_u \langle u|\mathcal{U}^r\mathcal{V}^s|u\rangle = \frac{1}{d}\mathrm{Tr}\,\mathcal{U}^r\mathcal{V}^s , \end{aligned} \tag{16}$$

which are mean values of products of powers of $\mathcal{U}$ and $\mathcal{V}$ in the probability distribution $p(u,v)$. By Eqs. (12) and (15) for $p(u,v)$, these mean values, which are generalizations of the moments (13), can be expressed as traces of the matrices and are thus independent of basis, in accordance with the invariance of the situation in which the variables occur.

The moments (16), for all values of r (and s) $=0,1,2,\ldots,d-1$, together uniquely determine the pair correlation $p(u,v)$. In fact, the set of equations (16) for $p(u,v)$ has the determinant $\prod_{i>j}(u_i-u_j)^d(v_i-v_j)^d$, which is nonvanishing, when there is no degeneracy of the eigenvalues.

[10] For symmetry variables, complementarity specifically designates the relationship that springs from their belonging together in a group with multidimensional irreducible representations.

4. Example exhibiting the generalized character of pair correlations. (Supplement)

Distinguishing features of the correlation between pairs of symmetry variables are exhibited by linear constraints [as exemplified by Eq. (9)] that cannot be fulfilled for eigenvalues. Such a constraint between three variables $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$

$$\begin{aligned} &\Delta=\alpha\mathcal{U}+\beta\mathcal{V}+\gamma\mathcal{W}=0 , \\ &\alpha u+\beta v+\gamma w\neq 0 , \end{aligned} \tag{17}$$

implies that it is not possible to derive the pairwise correlations $p(u,v)$, $p(v,w)$, and $p(w,v)$ from a more comprehensive probability distribution $p(u,v,w)$ involving all three variables. Such a distribution would have to satisfy

$$\sum_w p(u,v,w)=p(u,v) \quad \text{and cyclic} \tag{18}$$

and, hence, from Eq. (16) follows

$$\begin{aligned} &\sum_{u,v,w} |\alpha u+\beta v+\gamma w|^2 p(u,v,w) \\ &\quad =\alpha\alpha^*\overline{\mathcal{U}\mathcal{U}^\dagger}+\alpha\beta^*\overline{\mathcal{U}\mathcal{V}^\dagger}+\cdots \\ &\quad =\overline{\Delta\Delta^\dagger}=0 , \end{aligned} \tag{19}$$

which cannot be fulfilled for a positive definite distribution $p(u,v,w)$. [More generally, the conclusion is seen to hold if, for some value u (or v, or w), the constraint cannot be fulfilled for any choice of eigenvalues for the two other variables.]

The absence of more comprehensive probability distributions distinguishes multidimensional variables. Indeed, in any statistical distribution of variables that have values (such as in a classical ensemble), the pair correlations can always be expressed as a sum over probability distributions involving additional variables. Experiments testing correlations between symmetry variables that are incompatible with the notion that the variables have values are discussed in Secs. III.A and III.B. The correlations between variables of an individual quantum are also revealed in correlations between variables belonging to different quanta (see Sec. III.D).[11]

5. Summary

In summary, the primary manifestation of symmetry, which recognizes the coordinate transformations of

[11] The inability of correlations that can occur in classical ensembles to reproduce the correlations between the variables of two quanta was brought into focus by Bell, 1964. As exhibited above, and further discussed in Sec. III.B, the complementary relations that are beyond reach of classical probability distributions originate in the constraints between the variables of a single quantum (see in this connection footnote 20).

spacetime invariance as the elementary physical variables, leads in a few steps to the basic correlations governing observations of these variables. It is the point of departure that a symmetry variable, under suitable circumstances, *appears* with one of the values that are intrinsic to the symmetry, although in a multidimensional irreducible representation it *does not have* a value. It follows immediately that the variables are indeterminate, although completely specified by the representation [except for the (trivial) external orientation in vector space].

The probability distribution for the observation of the indeterminate variables follows from the additive constraints characterizing the variables in a multidimensional representation. These constraints express geometric relations between the variables and are, therefore, satisfied by the reproducible results of the observations, which are the mean values. The individual events are *fortuitous,* as a consequence of the *constraint* itself. In this manner, the probabilistic laws that are invoked in the interpretation of traditional quantum mechanics are recognized as intrinsic relations among the symmetry variables.

6. Unfolding of physics of symmetry variables. Outlook

From the spectrum of eigenvalues of the symmetries and the expression (12) for the probabilities, the physics of symmetry variables unfolds. Complementary relations between such variables are illustrated in Sec. III by the testing of the interplay between translations and reflections, as well as isomorphic relations between polarization variables. Local variables of a quantum (position at specified time) are seen in Sec. IV to emerge from the interplay of translations and spacetime rotations in the weakly relativistic regime, while field variables arise from the product representations of symmetry that describe systems of quanta in terms of bosons and fermions (Sec. V). The transition to the classical regime involves a quenching of the nonlocality in the state of affairs of the symmetry variables (as indicated in Sec. IV.C.4 and the supplementary Sec. IV.E).

The properties of symmetry variables thus follow directly from their geometrical content as embodied in the representations of the group, without appeal to quantization in an assumed symbolic formalism and without reference to Planck's constant. (Indeed, the entire argumentation could have been addressed to a 19th century physicist versed in the symmetry embodied in Maxwell's equations.)

In particular, indeterminacy and complementarity are seen to be inherent features of symmetry variables, rooted in the existence of multidimensional irreducible representations of the group of spacetime invariance. Indeed, irreducibility is the mathematical term for the impossibility of circumventing the indeterminacy of symmetry variables and, therefore, gives an expanded content to the concept of completeness. The wholeness characterizing complementary phenomena thereby appears as an expression for the interweaving of symmetry variables in a non-Abelian group.

In the traditional development of quantal physics by a metamorphosis of classical variables into quantal operators through a quantization procedure, the variables acquire properties,which are completely alien to their classical precursors, and which find their interpretation within the framework of complementarity. However, while indeterminacy is thereby incorporated into the description, its origin has remained an issue that has given rise to doubts as to the completeness of quantal physics.[12] These questions do not arise for symmetry variables, which are not representatives of physical quantities in a symbolic formalism, but are themselves the elementary variables, whose indeterminacy springs from the invariance of (flat) spacetime.

E. State vector. Superposition

1. State of affairs of quantum. Vector

Since the set of eigenvectors of a symmetry matrix form an orthogonal basis in vector space, the eigenvector $|u\rangle$ of a symmetry $\mathcal{U}$ can be expressed as a superposition of eigenvectors of any other symmetry $\mathcal{V}$,

$$|u\rangle = \sum_{v} |v\rangle\langle v|u\rangle \ . \tag{20}$$

The coefficients are the scalar products of the eigenvectors, which are amplitudes whose squares have been seen to give the probability for observing $\mathcal{V}$ with its different eigenvalues v, in a situation where $\mathcal{U}$ with certainty exhibits the value u [see Eq. (12)].

The components of the vector $|u\rangle$ in the various bases (20) thus yield all predictions that can be made concerning the quantum, in a situation in which the symmetry $\mathcal{U}$ is produced with the value u, while another symmetry $\mathcal{V}$ in the group is observed. The vector $|u\rangle$, therefore, offers itself as a means of specifying the "state of affairs" of the symmetry variables (probability distributions) in a given situation and, consequently, is referred to as a state, more specifically an eigenstate of $\mathcal{U}$. The state is thus a vector, and superposition, which is the defining property of a vector, hence carries over to states. It is seen that the notions of state and probability amplitude (wave function) are but expedient tools for describing the behavior of symmetry variables, rather than novel concepts (principles) requiring their own interpretation.

The superposition (20) is illustrated by the relation between eigenstates for translation and reflection in the representation (8)

$$\begin{aligned}|s(a)=\pm 1\rangle &= \frac{1}{\sqrt{2}}(e^{-ik_0a}|k_0\rangle \pm e^{ik_0a}|-k_0\rangle)\\ &= \cos k_0a|s=\pm 1\rangle - i\sin k_0a|s=\mp 1\rangle \ , \end{aligned} \tag{21}$$

[12] Warum der liebe Gott würfelt.

which gives the eigenstates of the reflection $\mathcal{S}(a)$ in the translation symmetric basis as well as in the basis of reflection symmetry $\mathcal{S}$ about the origin ($a=0$). The phase relations between the components in the superpositions (20) and (21), which are decisive for $\mathcal{U}[=\mathcal{S}(a)]$ to appear with a definite value, are irrelevant for predictions concerning $\mathcal{V}[=\mathcal{T}(a)$ or $\mathcal{S}]$.

The vector space that carries the symmetry in general contains vectors that are not eigenstates of a symmetry element (such as superpositions of $|k_0\rangle$ and $|-k_0\rangle$ with different intensities). However, the possible situations, allowed by the constraints between the mean values of the symmetry variables (see Sec. II.E.3.d), comprise states $|i\rangle$ associated with any vector, and the probability amplitudes are given by the expansion of the vector in the basis of the observed symmetry, as in Eq. (20).

2. General state of affairs

The most general state of affairs of a quantum involves a situation in which the mean value of a symmetry variable is the sum of contributions, each associated with a state $|i\rangle$, occurring with weight p_i (see Sec. II.E.3.d). In such a mixed state of affairs, to be referred to as a *quenched* state,[13] the nondiagonality of the symmetry variables is seen with reduced resolution. The significance of quenched states for a transition to a classical limit and for the measurement process is briefly considered in Sec. IV.C.4 (see also the supplementary Sec. IV.E) and in Sec. III.A.3, respectively.

Quenched states occur in the description of mean values of a subset of variables even though the total state for all the variables is a superposition. (Mean values are equivalent to moments of probability distributions, as discussed in Sec. II.D.3.) Examples of such substitute state of affairs for a subset of symmetry variables are discussed in the supplementary Sec. II.E.4.

In summary of Secs. II.E.1 and II.E.2, the generalized significance of a measurement of symmetry variables, associated with their indeterminacy, introduces an expanded notion of state for these variables. Only in the trivial case of a one-dimensional representation, the state is specified, as in classical physics, by the values of all the variables. In irreducible multidimensional representations, the state of affairs of the variables, with their complementary relations, is specified by a vector, as in Eq. (20) or, more generally, by a quenched state. Superposition of states thus emerges as an inherent feature of symmetry variables in their multidimensional representations, and the linearity involved in the superposition is seen to have its roots in the representation of symmetry.

[13]The notations "state" and "state of affairs" are used synonymously.

3. Uniqueness of probabilities. (Supplement)

a. Additive constraints in irreducible representation

In an irreducible representation of dimension d, the matrices satisfy the additive constraints

$$\mathcal{U}=\frac{d}{g}\sum_{\mathcal{V}}\mathcal{V}\mathrm{Tr}\mathcal{U}\mathcal{V}^{-1},\quad \mathrm{Tr}\mathcal{U}\mathcal{V}^{-1}\equiv\mathrm{Tr}(\mathcal{U}\mathcal{V}^{-1}) , \tag{22}$$

where the coefficients in the sum over all g elements in the group are traces (or characters), which are the invariants specifying the representation. For $\mathcal{U}=1$, the sum in Eq. (22) is seen to commute with all symmetry elements, since equivalent elements have the same trace, and is, therefore, a multiple of the unit matrix, expressing the orthonormality of matrix elements integrated over the group. From this result, the relation (22) follows by multiplication with $\mathcal{U}$ and a reordering of the terms. [The orthonormality extends to orthogonality for different irreducible representations, and the sum in Eq. (22), therefore, vanishes, if the matrices and characters belong to different irreducible representations.]

For example, for the two-dimensional representation k_0 of reflections and translations, the reflection variables $[\mathcal{U}=\mathcal{S}(a)]$ satisfy

$$\mathcal{S}(a)=\frac{1}{\pi}\int_0^{2\pi}dk_0b\,\cos2k_0(a-b)\mathcal{S}(b) , \tag{23}$$

since the only nonvanishing characters are $\mathrm{Tr}\mathcal{S}(a)\mathcal{S}(b)=\mathrm{Tr}\mathcal{T}[2(a-b)]=2\cos2k_0(a-b)$. For the continuous group, the sum over elements is replaced by the integral $\int_0^{2\pi}dk_0b$ over translations as well as reflections. The relation (23) describes the translation symmetry of reflections and is seen to be equivalent to Eq. (9). Correspondingly, for the group of rotations, the constraints (22) describe the rotational symmetry of rotations, which can be expressed in terms of a finite number of irreducible spherical tensors, whose components referring to coordinate systems rotated with respect to each other are linearly related, as in Eq. (9); see Sec. II.F.7.

b. Reproducible results. Mean values

In a specified situation, the observation of a variable $\mathcal{U}$ yields one of its eigenvalues u, with probability $p(u)$, obtained from a large number of events, and the resulting mean value of $\mathcal{U}$ is denoted by

$$\langle\mathcal{U}\rangle\equiv\sum_u up(u) . \tag{24}$$

While the individual events are fortuitous, the mean values (24) are the reproducible results of the experiments, which satisfy the geometric relations (22), thereby yielding the set of g linear equations

$$\langle\mathcal{U}\rangle=\frac{d}{g}\sum_{\mathcal{V}}\langle\mathcal{V}\rangle\mathrm{Tr}\mathcal{U}\mathcal{V}^{-1} \tag{25}$$

with all mean values referring to the same situation. The constraints (25) between the mean values can be expressed in the form

$$\langle \mathcal{U} \rangle = \mathrm{Tr}\,\mathcal{U}\rho \ ,$$
$$\rho \equiv \frac{d}{g} \sum_{\mathcal{V}} \langle \mathcal{V} \rangle \mathcal{V}^{-1} \left[= \frac{d}{g} \sum_{\mathcal{V}} \mathcal{V}^{-1} \mathrm{Tr}\,\mathcal{V}\rho \right] , \tag{26}$$

which gives the mean value of each symmetry variable $\mathcal{U}$ as a linear combination of its matrix elements, with coefficients that form a matrix ρ specifying the state of affairs of the quantum.

The matrix ρ is itself specified by the mean values of all the symmetries, and self-consistency is assured, for any matrix ρ, by the relation (22). In fact, the self-consistency condition, by which ρ equals the term in parenthesis in Eq. (26), is of the form (22), with $\mathcal{U}$ replaced by ρ. Since this condition is linear, it is satisfied for any linear combination of symmetry matrices in the irreducible representation and, hence, for any matrix ρ. However, the unitarity of the symmetry variables implying $\langle \mathcal{V} \rangle^* = \langle \mathcal{V}^{-1} \rangle$ requires ρ to be Hermitian. Moreover, the trace of ρ is seen to be unity, since $\langle \mathcal{U} \rangle = 1$, for $\mathcal{U} = 1$.

c. Derivation of probabilities

From the relation (26) for $\langle \mathcal{U} \rangle$, the expression (12) for the conditional probabilities $W(u;v)$ follows. Thus the matrix $\rho = |u\rangle\langle u|$ describes a situation, in which the variable $\mathcal{U}$ with certainty appears with the value u, as follows from the relations

$$\langle \mathcal{U}^r \rangle = \mathrm{Tr}\,\rho\,\mathcal{U}^r = u^r \tag{27}$$

holding for all values of $r\ (=0,1,2,\ldots,d-1)$.

For $\rho = |u\rangle\langle u|$, the mean value of powers $\mathcal{V}^s$ of another symmetry $\mathcal{V}$ can, therefore, be expressed in terms of conditional probabilities $W(u;v)$,

$$\langle \mathcal{V}^s \rangle = \sum_v v^s W(u;v) \ ,$$
$$\mathrm{Tr}\,\rho\,\mathcal{V}^s = \langle u | \mathcal{V}^s | u \rangle \ , \tag{28}$$

where the second line follows from Eq. (26). The set of equations (28) for $W(u;v)$, valid for $s=0,1,2,\ldots,d-1$, has a unique solution [see comment after Eq. (16)], and is seen to be fulfilled for the values (12).

d. Characterization of general state of affairs

An Hermitian matrix ρ can be formed from an arbitrary (normalized, but not necessarily orthogonal) set of states $|i\rangle$, with weights p_i, leading to the mean values (26) in the form

$$\langle \mathcal{U} \rangle = \sum_i p_i \langle i | \mathcal{U} | i \rangle = \mathrm{Tr}\,\rho\,\mathcal{U} \ ,$$
$$\rho = \sum_i |i\rangle p_i \langle i|, \quad \mathrm{Tr}\,\rho = \sum_i p_i = 1 \ , \tag{29}$$

by which the observations decompose into a sum of unconnected series (mixed state). For example, any ρ can be expanded in terms of its eigenvectors. (The results of unconnected observations can be freely combined, and negative weights can thereby be introduced.)

The characterization of possible situations thus introduces states $|i\rangle$ that are not eigenstates of a symmetry variable. For each state $|i\rangle$, the mean values (29) imply that the probability for observing $\mathcal{U}$ with the value u is $W(i;u) = |\langle u | i \rangle|^2$, as a generalization of Eq. (12).

The matrix ρ can be expressed as a special form of Eq. (29) in terms of its eigenvalues and eigenvectors, which form an orthonormal set. However, the states in Eq. (29) are not required to be orthogonal, and, for a given ρ (and mean values $\langle \mathcal{U} \rangle$), the decomposition (29) is, therefore, not unique. The matrix ρ is recognized as the density matrix of the quantal description.

4. Substitute state of affairs for subset of symmetry variables. (Supplement)

For a subset of all the variables, the mean values can be described by a substitute state of affairs referring only to these variables. It is a distinctive feature of multidimensional variables that, even though the total state of affairs for all the variables is a vector (pure state), the substitute state of affairs can be a mixture of states, as illustrated by the following examples.

(a) A quantum interacts with another quantum, which subsequently escapes. The remaining (first) quantum is described by a substitute state of affairs, which is a mixture of components associated with the mutually orthogonal components of the second quantum, which has escaped. The description of the remaining first quantum by the substitute state of affairs, which ignores its correlation with the second, is the more exhaustive, the more the second quantum has become inaccessible. Such a transition to a substitute state of affairs takes place in the quenching of nonlocality of a quantum through interactions with other quanta; see Sec. IV.E.5.

(b) In a measurement, the incident quantum, after its interaction with a large number of detector constituents, can be disregarded, as discussed in Sec. III.A.3. The signal can, therefore, be described by the substitute state of affairs of the detector constituents, which is a mixture, even though the total state of affairs is a superposition (see Sec. III.A.3).

(c) For a single quantum, the mean values of variables of a subgroup can be described by a substitute state of affairs which, in general, is a mixture. An example is provided by the translations, which constitute a subgroup of reflections combined with translations. Thus, for the superposition (21), the mean values for the translations can be described by a mixture of the two states $|+k_0\rangle$ and $|-k_0\rangle$, with equal weight. However, this substitute state of affairs, of course, does not describe the behavior of the quantum in situations, where reflections can occur (see Sec. III.C.3).

F. Two manifestations of symmetry

The matrices representing coordinate transformations in a symmetry group were originally introduced to characterize the transformations of physical quantities under a change of reference frame. A basic tool in such an analysis is the identification of sets of quantities that transform irreducibly among each other, the standard example being a vector. This role of the symmetry matrices is illustrated by the following situations referring to a single quantum.

1. Irreducible transformation of states. Coefficient matrix

Irreducible transformation occurs in its most elementary form for a set of eigenvectors $|u\rangle$ of a symmetry variable $\mathcal{U}$ in a specified irreducible representation of the group. Under a coordinate transformation $\mathcal{V}$ between two reference frames $\mathcal{K}$ and $\mathcal{K}'$, the symmetry $\mathcal{U}$ goes into the equivalent symmetry $\mathcal{U}'=\mathcal{V}\mathcal{U}\mathcal{V}^{-1}$ [see Eq. (7)]. Correspondingly, the eigenvector $|u\rangle$ goes into the eigenvector of $\mathcal{U}'$,

$$|u'\rangle=\mathcal{V}|u\rangle, \quad (|u'\rangle_{\mathcal{K}}=|u\rangle_{\mathcal{K}'}) , \tag{30}$$

where $|u'\rangle$ described from $\mathcal{K}$, is the same state as $|u\rangle$ seen from $\mathcal{K}'$. The two states, therefore, have the same eigenvalue for the equivalent symmetries. The relation (30) defines the phases of the states $|u'\rangle$ relative to those of $|u\rangle$, in terms of the chosen representation of $\mathcal{V}$. The transformed state $\mathcal{V}|u\rangle$ can be expanded in eigenstates of $\mathcal{U}$ to give

$$|u_1'\rangle=\sum_{u_2}|u_2\rangle\langle u_2|\mathcal{V}|u_1\rangle , \tag{31}$$

where the sum extends over all the eigenvalues of $\mathcal{U}$, labeled u_2, in order to distinguish the summation index from the eigenvalue u_1 of the state that is transformed.

The relation (31), by which the set of states $|u\rangle$ of the quantum carry symmetry, exhibits the matrix elements $\langle u_2|\mathcal{V}|u_1\rangle$ in the irreducible representation as the *coefficients* in the transformation of the states from one coordinate system to another. An example is provided by the superposition (21) by which the states $|u\rangle=|s\rangle$ are translated to $|u'\rangle=|s(a)\rangle$. [The coefficients are seen to be the matrix elements of $\mathcal{T}(a)$ in the s basis, which are given by Eq. (10)].

2. Irreducible tensors. Coefficient matrix

The transformation of symmetry variables under the changes of the reference frame is part of the algebra of the group, as given by Eq. (7). In a representation labeled u, the symmetry variables combine to form tensors T_w that transform irreducibly, according to a representation w. In analogy to Eq. (31) for states, the tensorial property is defined by the relation

$$T_{w_1'}=\mathcal{V}T_{w_1}\mathcal{V}^{-1}$$
$$=\sum_{w_2}T_{w_2}\langle w_2|\mathcal{V}|w_1\rangle, \quad (T_{w'})_{\mathcal{K}}=(T_w)_{\mathcal{K}'} , \tag{32}$$

where the variable $T_{w'}$ described from $\mathcal{K}$ is the same as T_w seen from $\mathcal{K}'$. As usual, the symbol w labels a basis as well as the representation, and the sum extends over the eigenvalues in the chosen basis. Tensorial variables are illustrated in the supplementary Secs. II.F.6 and II.F.7, for the representation k_0 of reflections and translations and for representations of rotations in three-dimensional space (spherical tensors), respectively.

In the tensorial relation (32), as in the Eq. (31) for the state, the matrix elements of the coordinate transformation $\mathcal{V}$ are *coefficients* in the linear relation between the tensor components referring to the two different reference frames. While the first relation (32) refers to symmetry variables (see examples in Secs. II.F.6 and II.F.7), the second is the same for the transformation of tensor components in classical physics. [The comparison of the two expressions for $T_{w'}$ in Eq. (32) shows the tensor as coupling the representations u and u^* to a representation w, as illustrated by the examples in Sec. II.F.6.]

3. Primary versus secondary manifestation of symmetry

The two basically different roles of a symmetry matrix, as a *variable that can appear with an eigenvalue,* discussed in Sec. II.C and II.D, and as a *set of coefficients* in a coordinate transformation, illustrated by the relations (31) and (32), are referred to as the primary and the secondary manifestation of symmetry, respectively. Indeed, the manifestation referred to as primary is the origin of the basic variables, and these variables, as elements in the group, define their own transformation properties in a secondary manifestation of the symmetry.[14]

In the primary manifestation, the vectors in the space carrying the symmetry, in the role of states, provide predictions concerning observations of symmetry variables. In the secondary manifestation, the vector components are quantities transforming irreducibly under a change of reference frame. Ultimately, any such quantity derives from the symmetry variables or from states describing these variables. The two manifestations of symmetry are compared in Table I.

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[14]As variables, symmetries are genuine, as opposed to ray, representations of the group. In a ray representation, which admits phase factors modifying the geometric relations between the symmetry elements, the eigenvalues of the matrices are changed, as, for example, by insertion of a phase factor in Eq. (5). However, symmetry variables have properties completely determined by geometry and, therefore, do not have ray representations.

TABLE I. The two manifestations of symmetry. In any linear representation of symmetry, the elements are matrices with an associated vector space, which is said to carry the symmetry, but the matrices have a fundamentally different function in the two manifestations. Thus, in the primary manifestation, a symmetry matrix is a variable that can appear with an eigenvalue, for example, in a measurement, while the vector is a state describing complementary relationships governing the observations of symmetry variables. In the secondary manifestation, in which the theory of symmetry originated, the vector is a quantity (variable or state) transforming irreducibly under a change of reference frame, and the matrix is the set of coefficients in this transformation. The primary manifestation exploits the unitary representations of the symmetry groups, characterized by orthonormal basis states, while also nonunitary matrices represent the symmetry, in its secondary manifestation.

	Primary	Secondary
Matrix representing symmetry	Variable	Transformation coefficients
Vector carrying symmetry	State	Variable State

4. Nonunitary representations

The symmetries associated with spacetime and gauge invariance have unitary representations that the elementary variables exploit, in the primary manifestation.[15] From these unitary variables, particles and fields derive (by sums and products), as shown in Secs. IV and V. In the secondary manifestation, however, the tensors may carry nonunitary representations. Thus, for the invariance in Minkowski space, all finite-dimensional representations are nonunitary, such as the symmetry carried by four-vectors [see also Sec. IV.B.4]. These nonunitary matrix representations connect variables of a quantum in different reference systems, while the variables themselves are infinite dimensional matrices in a unitary representation of Poincaré symmetry, as described in Sec. IV.A.3.

5. Low resolution. Classical physics

The variables of classical physics originate in symmetry variables (as outlined for particles and fields in Secs. IV and V). However, in the classical regime, the matrix variables are seen with such a low resolution that their indeterminacy and complementarity are hidden, and the primary manifestation of the symmetry is, therefore, not apparent. (The low resolution is associated with a quenching of nonlocality in the state of affairs of the symmetry variables, as described in Sec. IV.C.4 and in the supplementary Sec. IV.E.)

The secondary manifestation of symmetry carries over to the classical variables in terms of the linear relations between components of tensors described from different reference frames [see second line in Eq. (32)]. Indeed, it was this manifestation of symmetry, which led to the discovery of spacetime invariance, even though the origin of the variables (in spacetime symmetry) could not be seen within the framework of classical physics.

In the classical limit, the matrix transformation $\mathcal{V}T_w\mathcal{V}^{-1}$ of the *variable* T_w [see first line of Eq. (32)] can be expressed as a canonical transformation performing the change of variables associated with the shift $\mathcal{V}$ of the coordinate system. For an infinitesimal transformation, the commutator between T_w and the generator of $\mathcal{V}$ (in units of $i\hbar$), is thereby identified with the Poisson bracket between these two variables [see Eq. (97)]. In this manner, the framework of classical mechanics based on canonical variables is a remnant of the underlying complementarity (see the further discussion in Sec. IV.E.6).

The notions of space and time are based on everyday experience, and the invariance of spacetime was discovered in classical physics by means of macroscopic instruments. However, this insight gained concerns a feature of nature that is independent of observations and is recognized as the primary manifestation of symmetry. Classical physics is itself encompassed as a limit in which the primary manifestation of symmetry is no longer visible (see Sec. IV.C.4 and the supplementary Sec. IV.E).

6. Unit tensors formed by translations and reflections, in two-dimensional representation. (Supplement)

In the vector space carrying the two-dimensional representation of translations and reflections, the reflections $\mathcal{S}$ and $i\mathcal{S}(a_0)$ form components of a two-dimensional tensor

$$T_{2k_0,s}=\begin{cases}\mathcal{S}, & s=1,\\ i\mathcal{S}(a_0), & s=-1,\end{cases}\quad k_0a_0=\frac{\pi}{4}\ , \tag{33}$$

with transformation coefficients $\langle s_2|\mathcal{V}|s_1\rangle$ given by the matrix representing the symmetry $\mathcal{V}[=\mathcal{T}(a)$ or $\mathcal{S}(a)]$, in the reflection symmetric basis (10), with k_0 replaced by $2k_0$.

The tensor $T_{2k_0,s}$ thus has the transformations

$$T_{2k_0,s_1'}=\mathcal{S}T_{2k_0,s_1}\mathcal{S}^{-1}=s_1T_{2k_0,s_1}(=T^\dagger_{2k_0,s_1}),\quad s_1=\pm1\ , \tag{34}$$

under reflections, and the transformation

[15]See footnote 32 on the role of imaginary eigenvalues of the generators.

$$T_{2k_0,s_1'}=\mathcal{T}(a)T_{2k_0,s_1}\mathcal{T}^{-1}(a)=\begin{cases}\mathcal{S}(a)=\cos 2k_0a\mathcal{S}+\sin 2k_0a\mathcal{S}(a_0), & s_1=1\ ,\\ i\mathcal{S}(a+a_0)=-i\sin 2k_0a\mathcal{S}+i\cos 2k_0a\mathcal{S}(a_0), & s_1=-1\ ,\end{cases} \tag{35}$$

under translations. The translation of T_{2k_0,s_1} is seen to be identical to the linear relation (9).

Any variable built out the symmetry variables $\mathcal{T}(a)$ and $\mathcal{S}(a)$ in the representation k_0 can be expressed as linear combinations of the two-dimensional tensor $T_{2k_0,s}$ and the two one-dimensional tensors (scalars)

$$T_{0,+}=1 \tag{36}$$

and

$$T_{0,-}=\mathcal{T}(2a_0)=-ik_0^{-1}k\ . \tag{37}$$

The tensorial symmetry is seen to be the product of two representations k_0.

7. Spherical tensors. (Supplement)

The rotational variables (1) in an irreducible representation j can be decomposed into unit spherical tensors carrying the representations $\lambda=0,1,\ldots,2j$ that are contained in the product of the representation j with itself,

$$T_{\lambda\mu}=e_{\lambda\mu}(=T_w)\ ,$$
$$\langle jm_2|e_{\lambda\mu}|jm_1\rangle=\langle jm_1\lambda\mu|jm_2\rangle\ , \tag{38}$$

where the matrix element of $e_{\lambda\mu}$ is a vector-addition coefficient.

The tensors $e_{\lambda\mu}$ are polynomials in the components of the generators **j**, of order λ; for example, $e_{\lambda=1\mu}$ is proportional to the spherical components of the vector **j**. Under a rotation of the coordinate system, the tensors $e_{\lambda\mu}$ transform as in Eq. (32), with coefficients forming the rotational matrix $\langle\lambda\mu_2|\mathcal{R}(\chi)|\lambda\mu_1\rangle$ in the representation λ.

These tensorial relations express constraints between the rotational variables of the general form (22), in terms of a finite number of tensor components constituting a complete set. The relations, which are familiar from vector and spherical tensor analysis, form a set of linear constraints that cannot be fulfilled by inserting individual eigenvalues for each of the tensorial variables [in analogy to the relation (9)]. In fact, the eigenvalues for the components $e_{\lambda\mu}\pm e_{\lambda\mu}^{\dagger}$ are discrete and independent of the coordinate system to which they refer, while the constraints involve continuous functions of the rotation χ that connects the two coordinate systems, and hence cannot be fulfilled for nonvanishing eigenvalues.

G. Time reversal. Symmetry under complex conjugation

1. Directedness of time. Absence of time reversal as symmetry variable

The theory of symmetry variables, based solely on geometry, puts time reversal in a special category. Time itself is characterized by its homogeneity, which is expressed by the invariance under a time displacement $\mathcal{T}(\tau)$ of the coordinate system ($t'=t+\tau$). The symmetry variable $\mathcal{T}(\tau)$ is thus a counterpart to a spatial translation $\mathcal{T}(a)$ and, in analogy to the representation (4) of translations, the additive group of time displacements has the unitary representation

$$\mathcal{T}(\tau)=\exp\{i\omega\tau\} \tag{39}$$

characterized by a real frequency ω. [The symmetry variable $\mathcal{T}(\tau)$ as part of total spacetime symmetry is considered in Sec. IV.]

Spatial translation is inverted by a reflection of the spatial coordinate axis [see Eq. (5)]. However, a reversal of the time axis is not an available coordinate transformation, since it is an empirical fact that time has a direction. While the time axis can be shifted, its direction, therefore, does not leave a choice. Hence there is no symmetry variable that takes $\mathcal{T}(\tau)$ into $\mathcal{T}(-\tau)$, and, in the primary manifestation of symmetry, the sign of the frequency is, therefore, an invariant that may be taken to be positive. (The invariance of the time ordering of events at a specified point in space is incorporated into Minkowski metric. In this geometry, rotations in space and time can invert the wave number, but not the frequency.)

2. Symmetry under complex conjugation

Relations between symmetry variables at inverse epochs $\pm t$, referring to the same time axis, arise from the symmetry of the representations of spacetime invariance under complex conjugation. This symmetry stems from the neutrality with respect to complex conjugation of the algebraic relations for compounding spacetime transformations. Hence, to each representation $\mathcal{U}$ of a group of transformations, there corresponds a representation $\mathcal{U}^*$, in which the matrices are replaced by their complex conjugates.

The transformations of the spatial coordinate axes (translations, rotations, and reflections) have spectra of eigenvalues that are invariant under complex conjugation. Moreover, for the spatial symmetries, complex conjugate eigenvalues are always connected by a transformation in the group, as illustrated by the reflection $\mathcal{S}$ taking $\mathcal{T}(a)$ into its complex conjugate $\mathcal{T}(-a)$ [see Eq. (5)]. Hence the conjugate representations $\mathcal{U}$ and $\mathcal{U}^*$ contain the same eigenvalues and, if irreducible, are therefore identical, up to a unitary transformation. For example, translations and reflections, represented by the matrices (8), in the k basis with real $\mathcal{S}$, transform into their complex conjugates under space reflection ($\mathcal{U}^*=\mathcal{S}\mathcal{U}\mathcal{S}^{-1}$). Complex conjugation of rotation matrices is performed by a rotation of π; see supplementary Sec. II.G.5.

3. Reversibility

Complex conjugation of the time displacement (39)

$$\mathcal{F}^*(\tau)=\exp\{-i\omega\tau\}=\mathcal{F}(-\tau) \quad (40)$$

inverts the direction in which the time coordinate is shifted. [It is assumed in Eq. (40) that the variable ω is a real matrix, as in the k basis; see Eq. (59).] Thus time displacements in opposite directions, though not connected by a change of reference frame, as for spatial translations, are related by complex conjugation. This relationship carries over to variables that are functions of time (spacetime rotations; see Sec. IV.A) and thereby connects variables at opposite epochs referring to a fixed time axis. In this manner, as illustrated by the spacetime coordinate transformations in Eq. (63), reversibility of motion is part of the representation of symmetry variables, with a fixed sign of the frequency. (Microscopic reversibility carries over into classical physics, and the neutrality of geometry under complex conjugation thus provides a basis for the empirically established symmetry of the classical equations of motion under time reversal.)

4. Time reversal as anti-unitary transformation

The time-reversal invariance, which derives from the symmetry under complex conjugation, can be expressed in terms of a transformation $\mathcal{T}$, which inverts the time displacement τ,

$$\mathcal{F}(-\tau)=\mathcal{T}\mathcal{F}(\tau)\mathcal{T}^{-1} \quad (41)$$

by performing a complex conjugation (anti-unitary operation). A time-reversal operation $\mathcal{T}$ that commutes with the spatial symmetries is obtained by combining complex conjugation with the unitary transformation that reperforms complex conjugation for the spatial variables (such as space reflection $\mathcal{S}$ for the matrices (8), and a rotation of π for rotation matrices).

As an anti-unitary operation, time reversal $\mathcal{T}$ cannot be represented by a matrix, but has eigenstates in vector space (for integral spin; see supplementary Sec. II.G.5). However, time reversal, in contrast to a symmetry variable, does not appear with a value, since the eigenvalues of $\mathcal{T}$ are seen to depend on the undefined phase of the state.

5. Summary

In summary, the directedness of time implies that time reversal is not a symmetry variable. However, the neutrality of geometry under complex conjugation provides relations between symmetry variables at opposite epochs and is thus the basis for the reversibility of motion, expressed by an anti-unitary transformation.

6. Complex conjugation of rotation matrices. Rotations of 2π as square of time reversal. (Supplement)

The rotation matrices in three-dimensional space have sets of eigenvalues $\exp\{-im\chi\}$ that are invariant under complex conjugation, and complex conjugation of an irreducible representation, therefore, amounts to a unitary transformation, depending on the basis. This transformation can be expressed as a rotation of π about an axis, for which rotation is chosen to be a real matrix, and time reversal, therefore, involves a rotation of π. Hence a repetition of time reversal, which is a unitary operation, equals a rotation of 2π (about any axis) and is, therefore, not an additional variable $[\mathcal{T}^2=\mathcal{R}(2\pi)]$. For an eigenstate of $\mathcal{T}$, the eigenvalue is a phase factor, and the anti-unitarity of $\mathcal{T}$ thus implies $\mathcal{T}^2=1$ and, consequently, integral values for the angular momentum.

III. PRODUCTION AND APPEARANCE OF SYMMETRY VARIABLES. EXAMPLES

The production and appearance of a symmetry variable is illustrated by an experiment, which analyzes reflection symmetries of individual quanta by means of an interferometer.[16] The quantum is seen to manifest itself entirely in terms of its symmetry variables (translations and reflections) with complementary classical pictures (particles and waves) emerging in outline. The experiment establishes a correlation between two symmetry variables of a quantum, which is unique to variables that appear with values in a measurement, but do not have values. The same correlation occurs in coincidence experiments with two quanta.

A. Interferometer experiment

An instrument, by which reflection symmetry for individual quanta can be studied, is shown schematically in Fig. 1. The essential ingredient in the instrument is a partially reflecting mirror, which is so constructed that it transmits and reflects the individual quanta with equal probability. Such a device makes it possible to turn a translation symmetric state of a quantum into a reflection symmetric state.

1. Symmetry produced

A beam of quanta reaches the apparatus from the source S with wave number component $k_x=k_0$, which may, for example, be selected by reflections in a transla-

[16]The interferometer experiment can be compared with the two-slit experiment, which has played a prominent role in the elucidation of complementarity in quantal physics (see Bohr, 1949).

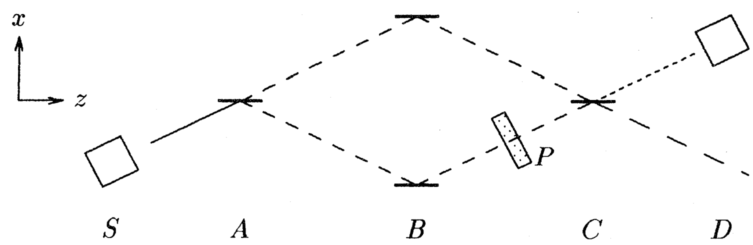

FIG. 1. Schematic illustration of interferometer. The interferometer consists of partially reflecting mirrors A and C (described in more detail below) and a set of totally reflecting mirrors B. Between A and C, the quantum is in a reflection symmetric state, and the plane of reflection symmetry can be varied by means of a phase plate P consisting of a material, in which the quantum has a wave number k_m differing from the vacuum value k (index of refraction $n=k_m/k$). The transmission through the plate thus adds a phase $(n-1)kd$ proportional to the length d of the path through the plate. This path can, for example, be varied by a small rotation of the direction of the phase plate. To simplify the analysis of the detection process, the experiment employs only a single detector intercepting one of the components of the state. Interferometers of the type illustrated were developed in classical optics in the last century, and the observed interference effect could later be ascribed to photons acting individually (incoherently). A neutron interferometer based on the same principles has been constructed from a single crystal, which performs all the operations A, B, and C with the accuracy required by the small wavelength ($k_0^{-1}\sim 10^{-8}$ cm) of the neutrons (Rauch *et al.*, 1974). The interference was studied for a beam intensity corresponding to an average distance between the neutrons large compared with the dimensions of the apparatus.

tion invariant crystal. The only variables of the quantum that are affected by the instrument are translations $\mathcal{T}(a)$ in the x direction and reflections $\mathcal{S}(a)$ in planes $x=a$ perpendicular to this direction. The incident quanta encounter the partially reflecting mirror A, which transforms the translation symmetric state $|k=k_0\rangle$ into a reflection symmetric state (see supplementary Sec. III.A.5). On account of the finite size of the reflector, the components $|k=\pm k_0\rangle$ in the reflection symmetric state separate and, after a reflection by symmetrically placed mirrors B, are brought together at C, remaining in a reflection symmetric state. (The spatial separation of the components $\pm k_0$ implies a small departure from translation symmetry, but the effect on the state at C is negligible, if the dimensions of the apparatus are large compared with k_0^{-1}.)

The relative phase of the components $|\pm k_0\rangle$ is shifted by a translation $\mathcal{T}(a)$, and this operation, which could be performed by bodily displacing parts of the apparatus, is accomplished in the experiment by a phase plate P, placed in one of the pathways (see caption to Fig. 1). The processes taking place in the apparatus can, therefore, produce any reflection symmetric state $|u\rangle=|s(a)=1\rangle$ belonging to the two-dimensional representation of translations and reflections [see Eq. (21)].

2. Symmetry observed

The state $|s(a)=1\rangle$ is analyzed by another partially reflecting mirror at C, which takes states $|s=\pm1\rangle$ with reflection symmetry $\mathcal{S}=\mathcal{S}(a=0)$ into translation symmetric states $|\pm k_0\rangle$ that separate. The quantum is registered by a detector D, by which the reflection symmetry $\mathcal{S}$ appears with the value $s=+1$ (signal) or $s=-1$ (absence of signal). The analyzer C (together with the detector), therefore, measures the reflection symmetry $\mathcal{S}$ of the individual quantum. The detection process is further characterized below, and the state of the quantum at the different stages of transmission through the interferometer is given in the supplementary Sec. III.A.5.

The interferometer is thus seen to produce the eigenstate $|u\rangle=|s(a)=1\rangle$ of the symmetry $\mathcal{U}=\mathcal{S}(a)$, which is subsequently analyzed in terms of the symmetry $\mathcal{V}=\mathcal{S}$, as in the general situation considered in Secs. II.C and II.D. The probability for the occurrence of a signal in the detector, and its absence is, therefore, given by the conditional probabilities (12), which yield

$$W[s(a)=1;s]=\begin{cases}\cos^2 k_0 a, & s=1, \quad \text{signal}\,,\\ \sin^2 k_0 a, & s=-1, \quad \text{absence of signal}\,,\end{cases} \tag{42}$$

from the products $\langle v|u\rangle=\langle s|s(a)=1\rangle$ given by Eq. (21). The pattern (42) of probabilities, which characterizes the complementarity between the variables $\mathcal{S}$ and $\mathcal{S}(a)$ in the irreducible two-dimensional representation of reflections and translations, has been observed in experiments with neutrons as well as with photons (see figure caption). The set of conditional probabilities (42) also applies to the linear polarization variables of a photon, which are reflections in planes containing the direction of propagation of the photon (see supplementary Sec. III.A.6).

The reflection variables in the two-dimensional representation satisfy additive constraints, such as Eq. (9), which are geometric relations between the variables. As discussed in Sec. II.D, the irreducibility opens the possibility for constraints of this type that cannot be fulfilled in individual observations of the variables in a given situation. Accordingly, the individual events in the interferometer experiment are observed to be fortuitous, but the constraints reveal themselves in the probabilities and the equivalent mean values, which are the reproducible results of the experiments. Thus the observed probabilities (42) yield, by translational invariance, $\langle s(a_1)=1|\mathcal{S}(a_2)|s(a_1)=1\rangle=\cos 2k_0(a_1-a_2)$, and these mean values are seen to fulfill the constraint (9) in the state $|s(a_1)=1\rangle$. [As discussed in Sec. II.D.2, the requirement that the constraints, as geometrical relations, be fulfilled for the mean values, leads uniquely to the expression (42) for the probabilities.]

The analysis of the experiment illustrated in Fig. 1 shows how the instrument observes the symmetry of the quantum by performing operations on the symmetry variables that are equivalent to the coordinate transformations themselves (reflections and translations) and further include linear combinations of such transformations (partial reflections). The instrument is thereby capable of performing any operation in the vector space carrying the symmetry.

The interferometer thus makes possible a complete analysis of the symmetry variables belonging to the two-dimensional representation of translations and reflections. The measurements described concern the reflection variables $\mathcal{S}(a)$, but can trivially be extended to include the analysis of translation variables in a reflection symmetric state, or vice versa, by the removal of one of the partially reflecting mirrors. Each set of experiments produces and observes selected symmetries, and the measured probabilities $W(u;v)$ determine the relations between the variables.

3. Detection of quantum

The detection of the quantum involves an amplification process, in which the variables of the quantum interact with a large number of constituents (symmetry variables) of the detector, thereby releasing a signal by which the quantum appears with a value of the observed symmetry. For example, a neutron detector may involve nuclei that absorb neutrons with the emission of charged particles, which in turn produce an avalanche of ions.

The functioning of the detector, like the other components of the interferometer, is fully amenable to theoretical analysis in terms of the state of affairs of the symmetry variables. The time evolution of these variables is governed by the unitary time displacement $\mathcal{T}(\tau)$, incorporating interactions between the quanta (gauge symmetry, as well as external fields), in addition to the time evolution of the free quanta (Poincaré symmetry). The crucial point in the appearance of a symmetry variable with a value, in a measurement, is the transfer of indeterminacy of the observed variable into statistical uncertainty (either/or) of a signal, as can be simply exhibited in the experiment considered.[17]

The state of affairs comprising the incident quantum and the constituents of the detector is initially a product of the state of the quantum and the state of the detector constituents. The latter may be in a mixed state with a vast number of components, but it suffices to consider a single component of this mixture.

The detection process in the experiment illustrated in Fig. 1 is described by a total state, for the quantum and the detector constituents, which is a superposition of two components deriving from the two components in the superposition $\cos k_0 a\,|k_0\rangle+i\sin k_0 a\,|-k_0\rangle$ of the quantum passing from the analyzer C to the detector D. [This form of the superposition is a consequence of Eq. (21), since the partial reflector C transforms the state $|s=\pm1\rangle$ into $\pm|\pm k_0\rangle$; see Eq. (44).]

The signal, as a variable, is only a function of the detector constituents, which is evident for a neutral incident quantum producing an electric current, and which applies with arbitrary accuracy for a charged quantum, when sufficiently many detector constituents are involved. The signal, and all the consequences that follow from it (resulting dynamical processes, registration, etc.) is, therefore, fully described by the substitute state of affairs, which comprises the detector constituents, but not the incident quantum (see supplementary Sec. II.E.4, example b).

This substitute, final state of affairs of the detector constituents with the incident quantum ignored is a mixture (quenched state) of two components resulting from either one or the other of the components $|+k_0\rangle$ or $|-k_0\rangle$ in the incident state, with probabilities $\cos^2 k_0 a$ and $\sin^2 k_0 a$, respectively (statistical uncertainty). In fact, any link (matrix element) between the two components in the superposition for the total final state must involve variables of the incident quantum that connect the region occupied by the detector to the reflected region with the detector absent. It follows that the entire detection process is independent of the phase in the incident superposition. In other words, the detector does not distinguish between a quantum arriving in a superposition and in the corre-

[17]The measurement process has received considerable attention in connection with the interpretation of the probability amplitude (collapse of the wave function). For symmetry variables, interpretative problems do not arise since, as emphasized in Sec. II.E.1, the probability amplitude is but a tool that offers itself for following the state of affairs of these variables. Thus indeterminacy is an inborn feature of the symmetry variables that shows up in the measurement (and is not introduced by the measurement process). In the following, the measurement is analyzed in terms of the effects produced by the quantum. In contrast, much of the discussion of the quantal measurement process has focused on the state of the observed object after the interaction with the detector (see references in footnote 27).

sponding mixture.[18]

In conclusion, a quantum arriving in a superposition $\cos k_0 a\,|+k_0\rangle + i\sin k_0 a\,|-k_0\rangle$ to the detector gives a signal with probability $\cos^2 k_0 a$. In the individual fortuitous event, announced by a signal, or its absence (with probability $\sin^2 k_0 a$), the reflection variable $\mathcal{S}$ appears with the value $+1$ or -1, respectively.

In the total state, the superposition between the two components continues indefinitely, but is only relevant in situations that involve not only the incident quantum, but also all the other variables that have been affected by the interaction. Such a state of affairs would, therefore, have a complexity comparable to that involved in a reversal of the detection process as well as all the subsequent processes to which the signal has given rise. The phase in the continuing superposition is, therefore, the more irrelevant, the more irreversible is the chain of events initiated by the detection process.

4. Evanescence of quantization. Fortuitousness

The experiment is seen to deal directly with the symmetries, as the variables of a quantum, without any reference to a quantization within a symbolic formalism (wave function) or to variables of classical origin (and, hence, without introduction of the quantum of action). The observed conditional probabilities establish geometric relations between translations and reflections in the two-dimensional representation of the symmetry.

The indeterminacy seen in the experiment, in the fortuitous character of the individual events, is an integral part of the properties of the symmetry variables, which is rooted in the irreducibility of their representation (in a mathematical sense). The state of affairs of these variables specifies the probabilities (equivalent to mean values), which constitute the reproducible results of the experiment, but can make no statement about the individual event, which is unpredictable in an absolute sense.

5. Partial reflection as operation in vector space. (Supplement)

The partial reflection as a superposition of transmission and reflection can be expressed as a unitary matrix

$$\mathcal{Q}=\frac{1}{\sqrt{2}}[1-i\mathcal{S}(a_0)],\quad k_0a_0=\pi/4\ ,\qquad \mathcal{Q}^\dagger\mathcal{Q}=1\ , \tag{43}$$

where the reflection is taken in a plane shifted from the origin by an eighth of a wavelength. A shift of the symmetry plane amounts to a translation and can, therefore, be accomplished by adding a phase plate to the partial reflector.

The variable $\mathcal{Q}$, derived from the symmetry elements, takes a translation symmetric state into an eigenstate of the reflection $\mathcal{S}$ [see, for example, Eqs. (8) and (21)],

$$\mathcal{Q}|\pm k_0\rangle = \pm|s=\pm 1\rangle\ . \tag{44}$$

Thus the passage of the quantum through the interferometer is described by the sequence

$$|k_0\rangle \xrightarrow{\mathcal{Q}} |s=1\rangle \xrightarrow{\mathcal{S}} |s=1\rangle \xrightarrow{\mathcal{T}(a)} |s(a)=1\rangle \tag{45}$$

leading to the state produced at C. Between A and C, the reflection symmetric state consists of components $\pm k_0$ that are separated in space (superposition of k values over a range arbitrarily small compared with k_0) with a separation that varies with z. The state $|s(a)=1\rangle$ at C is analyzed by the partial reflection $\mathcal{Q}^\dagger$ [see Eq. (21)],

$$\mathcal{Q}^\dagger|s=\pm 1\rangle = \pm|\pm k_0\rangle\ ,\qquad \mathcal{Q}^\dagger|s(a)=1\rangle = \cos k_0a|k_0\rangle + i\sin k_0a|-k_0\rangle\ , \tag{46}$$

which makes it possible to determine the reflection symmetry $\mathcal{S}$ of a quantum by transforming s into k. The registration of $k\,(=\pm k_0)$ by the detector thereby yields the probabilities $|\langle s|s(a)=1\rangle|^2$.

6. Polarization variables of a photon. (Supplement)

The linear polarization of a photon, together with its helicity, are symmetry variables that are isomorphic to the reflection and translation variables, in the two-dimensional representation. Thus linear polarization $\mathcal{S}(\phi)$ in a plane containing the wave vector of the photon and characterized by an azimuthal angle ϕ is represented by the matrices (8), with $\phi=k_0a$, in a basis in which the helicity $h=k/k_0$ is diagonal. Photons with linear polarization $s(\phi)=1$ or $s(\phi)=-1$ can be produced and observed by a polarimeter, and the correlation between two linear polarization variables can thus be analyzed by a combination of two polarimeters.

B. Impossibility of reproducing observed correlation for individual quantum in terms of classical substitute variables

1. Correlation between translated reflection variables of a quantum

The probabilities (42) measured in the interferometer experiment amount to a correlation $p(s(a_1),s(a_2))$ between pairs of two-dimensional reflection variables [see Eq. (15)], which can be characterized by its moments (16), such as

$$\overline{\mathcal{S}(a_1)\mathcal{S}(a_2)} \equiv \sum_{s(a_1),s(a_2)} s(a_1)s(a_2)p(s(a_1),s(a_2)) = \cos 2k_0(a_1-a_2)\ , \tag{47}$$

which are mean values in the distribution $p(s(a_1),s(a_2))$. The value of the moment (47) follows from the probabilities (42), in combination with transla-

[18]The "observation of a definite value" expresses a consensus within the substitute state of affairs between the observers, bookkeepers, etc., agreeing on the fortuitous value that has been registered.

tional invariance [or from the trace of $\mathcal{S}(a_1)\mathcal{S}(a_2) = \mathcal{T}(2a_1-2a_2)$; see Eqs. (8) and (16)]. For the two-dimensional representation, the second-order moment (47), together with the first-order moment $\overline{\mathcal{S}(a)}=0$, completely specify the pair correlation $p(s(a_1),s(a_2))$.

2. Incompatibility of observed correlations with the implicit assumption that the variables have values

The correlation function $p(s(a_1),s(a_2))$ is specific to two-dimensional variables and, in particular, clearly cannot describe an ensemble of classical substitute[19] variables $s_{\rm cl}(a)$ that have values $+1$ or -1. In fact, the quantity

$$\Delta_{\rm cl} \equiv s_{\rm cl}(a) - s_{\rm cl}\cos 2k_0 a - s_{\rm cl}(a_0)\sin 2k_0 a \ , \qquad s_{\rm cl} \equiv s_{\rm cl}(a=0) \tag{48}$$

would then have to vanish identically, since its square

$$\Delta_{\rm cl}^2 = 2 - 2s_{\rm cl}(a)s_{\rm cl}\cos 2k_0 a - 2s_{\rm cl}(a)s_{\rm cl}(a_0)\sin 2k_0 a + 2s_{\rm cl}s_{\rm cl}(a_0)\cos 2k_0 a \sin 2k_0 a \tag{49}$$

has the mean value

$$\overline{\Delta_{\rm cl}^2} = 2 - 2\cos 2k_0 a \sum_{s_{\rm cl}(a),s_{\rm cl}} s_{\rm cl}(a)s_{\rm cl}\, p(s_{\rm cl}(a),s_{\rm cl}) - \cdots = 2 - 2\cos^2 2k_0 a - 2\sin^2 2k_0 a = 0 \quad (\cos 2k_0 a_0 = 0) \ , \tag{50}$$

where the sum, with a change of summation indices, is the same as in Eq. (47). Thus $\overline{\Delta_{\rm cl}^2}$ vanishes for the observed pair correlations, and hence $\Delta_{\rm cl}$ would have to have the value zero for each member of the ensemble while, in fact, $\Delta_{\rm cl}^2$ is positive, since $\Delta_{\rm cl} \neq 0$, except when a is a multiple of a_0. It can, therefore, be concluded that the observed correlation (47) is incompatible with the very notion that the variables have values.[20]

[19]This use of "substitute" has no relation to "substitute state of affairs."

[20]An ensemble of classical variables yielding all mean values in a state in a two-dimensional vector space has been exhibited by Bell, 1966. However, it is a corollary of the above that the pair correlations evaluated from this ensemble differ from those of the two-dimensional symmetry variables. The difference can be exhibited by explicit evaluation of the mean value of the product of the variables $s_{\rm cl}(a_1)$ and $s_{\rm cl}(a_2)$ in the ensemble mimicking the $s=1$ state. This mean value is found to be $1-|\cos 2k_0a_1 - \cos 2k_0 a_2|$, which reduces to Eq. (47) for $a_1=0$ (or $a_2=0$), but is not a function of a_1-a_2. Hence the correlation in the ensemble violates translational invariance, which is an inborn feature of Eq. (47) for the correlation between symmetry variables. (This failure of classical ensembles to incorporate symmetry for a single quantum directly carries over to the correlations between the variables of two quanta in the invariant state; see Sec. III.D.) The experiments are seen to test the mean value (47) of the products $\mathcal{S}(a_1)\mathcal{S}(a_2)$, by producing one symmetry a _ observing another, in individual events.

This result is seen to be an example of the general conclusion arrived at in Sec. II.D.4 concerning the correlation between multidimensional variables in an irreducible representation that satisfy linear constraints of the form (17), which cannot be fulfilled for any combination of the eigenvalues, such as the relation (9),

$$\Delta = \mathcal{S}(a) - \mathcal{S}\cos 2k_0 a - \mathcal{S}(a_0)\sin 2k_0 a = 0 \tag{51}$$

for the reflection variables. The existence of such a constraint implies that the pair-wise correlations cannot be derived from a more comprehensive probability distribution of the eigenvalues $s(a)$,s, and $s(a_0)$, as would be the case for variables that have values [see Eqs. (18) and (19)]. The correlations are, therefore, outside reach of classical ensembles, in accordance with the conclusion drawn from the explicit analysis of $\Delta_{\rm cl}$ formed from the substitute variables $s_{\rm cl}(a)$.

3. Failure of classical ensemble to carry translation symmetry. (Supplement)

The constraint (51) expresses the translational symmetry carried by the reflections, as tensorial variables. [For the isomorphic polarization variables of a photon, the constraint exhibits $\mathcal{S}(\phi)$ as a component of a vector under rotations, $\mathcal{S}(\phi)=\cos\phi\,\mathcal{S}(\phi=0)+\sin\phi\,\mathcal{S}(\phi=\pi/2)$; see Sec. III.A.6.] In contrast, it is beyond the capacity of classical variables substituting for reflections to carry translational symmetry. [The substitute variables $s_{\rm cl}(\phi)$ having discrete values ± 1 clearly cannot be components of a vector.] The failure of any classical ensemble to reproduce the correlation between the reflection variables can, therefore, be traced to the inability of the ensemble to incorporate translational invariance, which is an inherent feature of the symmetry variables (see also footnote 20).

C. Connection to classical physics

The variables of the quantum in the interferometer experiment are the translation and reflection symmetries in the two-dimensional representation k_0. Already in this simple representation, the symmetry variables show correspondence to classical variables associated with waves and particles, and complementary classical pictures thereby emerge as expressions for the interplay between translation and reflection symmetry.

1. Correspondence to waves

Thus the two-dimensional representation of translations and reflections carried by the quantal state is also carried, as tensorial symmetry, by classical waves involving superpositions of traveling and standing modes, with translation symmetry $k=\pm k_0$ and reflection symmetry $s(a)=\pm 1$. An interferometer can therefore be used to analyze classical waves, in which case the operations in vector space performed by the components A, B, C, and P (see Fig. 1) carry over to the amplitude of the classical wave. An incident classical wave, with $k_x = k_0$, thus arrives in the exit channels with amplitudes that have the same relative values as the components of the quantal state. Hence the observation of intensities for a large number of quanta is in correspondence with the intensity distribution for the classical wave. However, while the individual quantum is registered in either one or the other exit, the classical amplitudes have values in both exit channels.

2. Correspondence to particles

The notion of quanta, each observed as an entity in one or the other detector, contains an element of the classical particle concept. Moreover, in the interferometer experiment, a trajectory emerges in crude outline, since the (approximate) eigenstates of translation are spatially separated, during the transmission of the quantum, as an experimental device for producing the reflection symmetric state at C. A path of the particle through the instrument, therefore, appears as a variable with two values, taken to be the sign of the wave number. However, in the irreducible two-dimensional representation, this symmetry variable does not have a value (is indeterminate) since reflection symmetry combines both values of the path (upper and lower components).

3. Terminology. Answer to tenacious question: "Did the particle pass along one of the paths?"

The links to the complementary classical pictures are part of the flavor of the phenomena, but the elementary variables are the symmetries themselves. The unambiguous terminology appropriate to the symmetry variables, therefore, defines the limitations in the use of classical notions. Thus the particle cannot be said to "pass along one of the trajectories," as would be suggested by the notion of the path as a one-dimensional (classical) variable having either one or the other of its two possible values (statistical uncertainty). However, the two dimensionality of the variable gives an unambiguous content to the path of the quantum through the interferometer.

If the translation symmetry is observed (by removing the analyzer at C; see Sec. III.A.2), the measurement gives one of the values $\pm k_0$ associated with the lower or upper path. However, the image of the particle as having been in this beam component during the passage of the apparatus, prior to the measurement, is not warranted, since, as a multidimensional variable, the path did not have a value. Indeed, in the state of the quantum between A and C, the wave number is indeterminate, and this state of affairs, which is part of a complete description, is unaffected by the subsequent observations (as discussed in Sec. II.C.7, and further illustrated in the supplementary Sec. III.C.4). Thus the quest for attributing a value (upper or lower) to the path of the particle in the interferometer experiment translates into an attempt to circumvent the irreducibility of the two-dimensional representation of translations and reflections.

4. Emission of photon from source. Complementarity of translations and rotations. (Supplement)

An additional simple example of the terminology appropriate to symmetry variables in a multidimensional representation is provided by the observation of the angular distribution of photons emitted from an excited atom, which involves the interplay between rotation and translation (specifying the direction of the photon). In this experiment, the photon is produced in a state characterized by rotational symmetry, and the complementary relation between rotation and translation gives the conditional probability for registering the photon in a given direction. However, from the observation of an individual photon by a detector at distance $\mathbf{r}$ from the source (located at the origin), by which the wave-number vector $\hat{k} = \hat{r}$ appears with a definite value, it cannot be inferred that in the decay process, the photon was emitted in this direction. In fact, as a multidimensional variable in an irreducible representation of rotations and translations, the direction of the photon did not have a value, and its indeterminacy cannot be removed by subsequent observations. The multidimensionality of the directional variable of the emitted photon before interception can be exhibited as in the interferometer, by inserting two reflectors B at distances smaller than r from the source that produce, at a location C, a superposition of states with two different directions (reflection symmetry). The excited atom thus replaces the source and the partially reflecting mirror at A in Fig. 1.

D. Correlation of two quanta in invariant state

The correlation between variables of an individual quantum, as analyzed in the interferometer experiment, also finds expression in the correlation between variables belonging to two or more quanta (in product representations of the symmetry; see Sec. V). In particular, the entire correlation pattern for the variables of a single quantum reveals itself in coincidences between the variables of two quanta that are locked together in an overall invariant state. (The product of a representation with its complex conjugate contains the invariant representation.)

1. Invariant state as superposition of products

The invariant state of two quanta has the eigenvalue unity for all symmetry transformations acting on both quanta, $\mathcal{U} = \mathcal{U}_1 \mathcal{U}_2$, and in any basis, the invariant state is a superposition of components, with equal weight, in which the quanta appear with complex conjugate eigenvalues. In the situation described by this state, the observation of the symmetry $\mathcal{U}_1$ of one quantum, with eigenvalue u_1, therefore, implies that an observation of the same symmetry $\mathcal{U}_2$ for the other quantum yields the eigenvalue $u_2=u_1^*$.

2. Coincidence experiment

It follows that, in an experiment designed to measure another symmetry $\mathcal{V}_2$ for the second quantum together with $\mathcal{U}_1$ for the first, the conditional probability $W_{12}(u_1;v_2)$ for the outcome v_2 reduces to a conditional probability $W(u;v)$ for an experiment with a single quantum,

$$W_{12}(u_1;v_2)=W(u=u_1^*;v=v_2)\ . \tag{52}$$

The correlation between two symmetries observed in coincidence, for two quanta, is thus identical to the correlation between produced and observed symmetries, for a single quantum, as a direct result of the interlocking of the symmetry variables in the invariant state of the two quanta ($\mathcal{U}_2=\mathcal{U}_1^\dagger$). Hence the coincidence experiment can be seen as an appearance of two variables of a single quantum, and the arguments in Sec. III.B, therefore, imply that the correlation between the two quanta is beyond reach of classical ensembles.

The identification (52) can be expressed in terms of the moments of the correlations. Thus, for the reflection variables, the joint moment for the two quanta is

$$\overline{\mathcal{S}_1(a_1)\mathcal{S}_2(a_2)}\equiv\sum_{s_1(a_1),s_2(a_2)} s_1(a_1)s_2(a_2)p(s_1(a_1),s_2(a_2))$$
$$=\overline{S(a_1)S(a_2)}=\cos 2k_0(a_1-a_2)\ , \tag{53}$$
$$p(u_1,v_2)=\tfrac{1}{2}W_{12}(u_1;v_2)\ ,$$

which equals the moment for the variables of a single quantum and hence has the value (47), unique to multidimensional variables.

3. Conditions for observing correlations between quanta that have separated after interaction

Correlations between symmetry variables occur in bound states, as well as in collision and decay processes. Thus the invariant state can be produced in a collision or decay, after which the two quanta separate with each pair of symmetry variables locked into definite relative orientations ($\mathcal{U}_1\mathcal{U}_2=1$, for each $\mathcal{U}$). The correlation (joint probabilities) can, therefore, be observed by measurements of the quanta at different locations. For example, an invariant state of two quanta, each in the two-dimensional representation of reflections and translations studied by the interferometer, could be produced in a colliding beam experiment, in which the collision domain is viewed at right angles to the beam by two oppositely placed sets of reflectors, as B in Fig. 1. After the reflection, the quanta could be observed in coincidence, each by a combination of a phase plate and analyzer.

4. Experimental tests verifying that the quanta do not carry values from source to detectors

Experiments testing the correlations in the invariant state of two quanta that have separated from each other have been performed for the polarization variables of two photons emitted in opposite directions in successive decays of an atomic state (Aspect *et al.*, 1981). The components $\mathcal{S}(\phi)$ of linear polarization in a plane with orientation ϕ containing the direction of propagation of the photons are reflection variables isomorphic to $\mathcal{S}(a)$ (see Sec. III.A.6), and the correlation between the polarization components is, therefore, given by Eq. (53) with ϕ replacing k_0a. This correlation between the polarization components of the two photons is a constant of the motion for freely propagating photons, and is thus, at the points of observation, identical to that produced at the source by the excited atom (considered as part of the apparatus).

The predicted correlation between the polarization of the two photons has been verified in the experiments referred to, and the observations thus confirm that the polarization components are multidimensional variables. Accordingly, the photons cannot be said to carry values of the polarization components from source to detector, as would be the case, if these components were one-dimensional variables.[21] Instead, the polarizations carried by the photons are generalized variables that, although all present in the experimental situation, do not have values and are thereby capable of correlations that are beyond reach of classical variables.

In summary, the experiments discussed in the present section III are seen as studies of symmetry variables, whose properties stem directly from spacetime invariance, without the mantle of quantization conditions or an assumed formalism of quantal physics. The observed correlations thus express geometric relations between elementary variables in a multidimensional irreducible representation. These variables are inherently indeterminate, but completely specified by the representation.

IV. POSITION OF QUANTUM

The invariance of spacetime embodied in special relativity comprises the continuous transformations that express homogeneity and isotropy. The coordinate transformations are global, but rotations in spacetime, in contrast to translations, single out sets of points (the axis of rotation) that are left invariant by the transformation. Rotational symmetries can therefore supply local variables, which include the position of a particle in nonrelativistic quantum mechanics and the relativistic invariant

[21]The notion that the photons carry values of the polarization variables from source to detector appears to be at the root of the dilemma, which the experiment has been felt to pose.

fields.[22] As a prelude, the representation of translations and rotations in spacetime is briefly outlined in Sec. IV.A, which can be seen as a counterpart to the analysis of translations and reflections in Secs. II.A and II.B.

A. Translations and rotations in spacetime (1+1 dimensions)

1. Composition of transformations

A Lorentz transformation (or boost) in the x direction with velocity u and associated rapidity ζ_u, is defined by

$$\begin{aligned} &\mathcal{L}(\zeta_u) \equiv \mathcal{L}(\zeta_u; x=t=0), \quad \tanh\zeta_u \equiv u/c \ , \\ &x' = x\cosh\zeta_u + ct\sinh\zeta_u \ , \\ &ct' = x\sinh\zeta_u + ct\cosh\zeta_u \ , \end{aligned} \tag{54}$$

which constitutes a rotation in the xt plane through the imaginary angle of magnitude ζ_u about the origin, taking the point x,t into the corresponding point x',t'. The transformations (54) are additive in the rapidity $[\mathcal{L}(\zeta_{u_1})\mathcal{L}(\zeta_{u_2}) = \mathcal{L}(\zeta_{u_1}+\zeta_{u_2})]$.

The spacetime rotation (54) takes place about the origin $x=t=0$, which is left invariant by the transformation. It is a consequence of the homogeneity of spacetime that a rotation $\mathcal{L}(\zeta_u;x,t)$ about a point x,t is obtained from $\mathcal{L}(\zeta_u)$ by the transformation

$$\mathcal{L}(\zeta_u;x,t) = \mathcal{T}(x,t)\mathcal{L}(\zeta_u)\mathcal{T}^{-1}(x,t) \ , \tag{55}$$

where $\mathcal{T}(x,t)$ is a translation in space and time (as introduced in Secs. II.A and II.G, with $x=a$ and $t=\tau$). Translations in different directions commute. In Eq. (55), the point x,t is first shifted to the origin that is invariant under $\mathcal{L}(\zeta_u)$, and then shifted back to x,t, which is, therefore, left invariant under the rotation $\mathcal{L}(\zeta_u;x,t)$. The transformation (55) exhibits rotations with the same rapidity as equivalent elements of the group of translations and rotations, in analogy to the relation (6) for reflections.[23]

The isotropy of spacetime implies that a translation, under a spacetime rotation, is carried into a translation with a rotated displacement,

$$\mathcal{T}(x',t') = \mathcal{L}(\zeta_u)\mathcal{T}(x,t)\mathcal{L}^{-1}(\zeta_u) \ , \tag{56}$$

where the coordinates x',t' are obtained from x,t by the transformation (54). By the same rotation, the locus x,t of a spacetime rotation is shifted to x',t',

$$\mathcal{L}(\zeta'_u;x',t') = \mathcal{L}(\zeta_u)\mathcal{L}(\zeta'_u;x,t)\mathcal{L}^{-1}(\zeta_u) \tag{57}$$

in accordance with Eqs. (55) and (56). [Since a translation is not linked to a spacetime point, the relation (56) is independent of the origin of the boost, in contrast to the transformation (57)].

Translations and rotations, together with their combinations, form the elements of the Poincaré group. In $1+1$ dimensions, the elements of the group constitute the displacements of the xt plane in which all relative distances are conserved. Any such transformation of the plane is characterized by a fixed point x,t and an angle of rotation ζ_u. [For an infinitely remote fixed point and a vanishingly small angle of rotation, the displacement of the plane develops into a translation.] The angle of rotation (the rapidity) is a property that is independent of the reference frame, as explicitly shown by the equivalence transformations (55) and (57).

2. Generators

The additivity of the Lorentz transformations $\mathcal{L}(\zeta_u)$ in ζ_u, as well as the translations in x and t, imply that the transformations can be expressed in the form

$$\begin{aligned} &\mathcal{L}(\zeta_u;x,t) = \exp\{i\zeta_u\xi(x,t)\} \ , \\ &\mathcal{T}(x,t) = \exp\{-ikx + i\omega t\} \end{aligned} \tag{58}$$

in terms of the generators ξ, k, and ω, which are matrices in the representation of the symmetry.

The relation (56) implies that the generators k,ω of translations transform as the components of a two-vector, whose length is, therefore, invariant under all transformations in the group,

$$\omega^2 - c^2k^2 = \omega_0^2, \quad \omega_0 = ck_0 \ . \tag{59}$$

The invariant defines a frequency ω_0 or, equivalently, a wave number k_0 [both of which are numbers (multiplying the unit matrix)]. The value of ω_0^2 is required to be positive; see below. The two-vector symmetry of k,ω is exhibited by the commutators with the generator ξ of Lorentz transformation,

$$[k,\xi(x,t)] = -i\omega c^{-1}, \quad [\omega,\xi(x,t)] = -ikc \tag{60}$$

that follow from Eq. (56). The relations (60) between the generators can be combined into

$$\begin{aligned} &[\xi(x,t),\zeta] = i \ , \\ &k \equiv k_0\sinh\zeta, \quad \omega \equiv \omega_0\cosh\zeta \end{aligned} \tag{61}$$

by expressing k and ω in terms of a rapidity variable ζ, which is an angular variable conjugate to the generator of rotations in spacetime.

[22] Sections IV and V contain a certain amount of technicality, but we have felt it necessary, in order to fully make the case, to explicitly show how local variables with correspondence to classical physics derive from the spacetime symmetries. For ourselves, an important point that had for long been an obstacle, was the realization that the position of a particle, which is a basic element of nonrelativistic quantum mechanics, requires the link between space and time of relativistic invariance.

[23] To simplify notation, the same label (x,t) is used for the fixed point in Eq. (55) and the running point in Eq. (54).

3. Representation of Poincaré symmetry

The invariant k_0 characterizes the irreducible representations of the group of spacetime translations and rotations (in $1+1$ dimensions). The (infinite) set of eigenvalues k,ω of the translations lie on one of the branches of the hyperbola (59) and, in the k basis, the symmetries are represented by the matrices

$$\langle k_2|\mathcal{T}(x,t)|k_1\rangle=\exp\{-i(k_1x-\omega_1t)\}\langle k_2|k_1\rangle\ ,$$
$$\langle k_2|\mathcal{L}(\zeta_u)|k_1\rangle=\langle k_2|k_1'\rangle,\quad \langle k_2|k_1\rangle=|\omega_1|\delta(k_2-k_1)\ , \tag{62}$$

where k_1' is the wave number resulting from k_1 by the boost ζ_u. The matrix $\mathcal{L}(\zeta_u;x,t)$ is obtained by the translation (55) of $\mathcal{L}(\zeta_u)$ and, therefore, has the phase $(k_1-k_2)x-(\omega_1-\omega_2)t$. The representation (62) of translations and rotations is thus seen to be the analog of the representation (8) of translations and reflections. For both groups, the scale of the representation is characterized by an invariant wave number k_0. For infinitesimal transformations, the matrices (62) give the representation of the generators.

The primary manifestation of spacetime symmetry is confined to positive values of ω (see Sec. II.G.1). The sign of ω is thus required to be a Lorentz invariant, as is the case for a representation with positive value of the invariant ω_0^2, in Eq. (59). Representations of Poincaré symmetry with opposite signs of the frequency are related to each other by complex conjugation and occur together in the secondary manifestation of the symmetry (see Sec. V.B.2).

4. Complex conjugation as spacetime reflection

Complex conjugation of the matrices (62) in the k basis is seen to be equivalent to a change of sign of x and t,

$$\mathcal{T}^*(x,t)=\mathcal{T}(-x,-t),\quad \mathcal{L}^*(\zeta_u;x,t)=\mathcal{L}(\zeta_u;-x,-t),\quad k\ \text{basis}\ ,$$
$$\mathcal{T}^*(x,t)=\mathcal{S}\mathcal{T}\mathcal{T}(x,t)(\mathcal{S}\mathcal{T})^{-1},\quad \mathcal{L}^*(\zeta_u;x,t)=\mathcal{S}\mathcal{T}\mathcal{L}(\zeta_u;x,t)(\mathcal{S}\mathcal{T})^{-1}\ , \tag{63}$$

which can be expressed as a spacetime reflection. In fact, reflections invert the direction of translations [see Eqs. (5) and (41)], while a rotation about the origin is invariant under reflections of both axes. The anti-unitary time reversal operation $\mathcal{T}$ thereby combines complex conjugation with a unitary operation $\mathcal{S}$, which reperforms the complex conjugation associated with the space axis; see Sec. II.G.4. [The constraints between the symmetry variables are expressed by algebraic relations between the matrices, such as $\mathcal{U}_1\mathcal{U}_2=\mathcal{U}_3$, which are invariant under complex conjugation and hence, according to Eq. (63), under the inversion of x and t. The relations between variables, therefore, do not distinguish between opposite directions in spacetime. In particular, no direction of the time displacements is singled out (reversibility of motion).]

5. States carrying Lorentz symmetry

The state $|\xi(x,t)=\xi\rangle$ carrying Lorentz symmetry about the point x,t, with eigenvalue ξ, is given in the translation symmetric basis by

$$|\xi(x,t)=\xi\rangle=\mathcal{T}(x,t)|\xi(0,0)=\xi\rangle\ ,$$
$$|\xi(0,0)=\xi\rangle=(2\pi)^{-1/2}c^{1/2}\int\frac{dk}{\omega}e^{-i\zeta\xi}|k\rangle \tag{64}$$
$$=(2\pi c)^{-1/2}\int d\zeta\exp\{-i\zeta\xi\}|k\rangle\ ,$$

since a Lorentz transformation shifts the rapidity by the amount ζ_u [see Eq. (61)] and hence multiplies the state (64) by the phase factor $\exp\{i\zeta_u\xi\}$. The states $|\xi(x,t)=\xi\rangle$, with ξ taking all real values, form a complete orthonormal basis, for each x,t.

B. Localization from spacetime rotation

1. Variables with invariant association with spacetime point

A spacetime rotation $\mathcal{L}(\zeta_u;x,t)$ singles out a spacetime point x,t that remains fixed under the rotation, and under any of the transformations of the coordinate system the rotation goes into the equivalent rotation (same ζ_u) about the corresponding point x',t'. The variable $\mathcal{L}(\zeta_u;x,t)$ and its generator $\xi(x,t)$ are, therefore, invariantly tied to a spacetime point,

$$\xi(x',t')=\mathcal{V}\xi(x,t)\mathcal{V}^{-1}\ ,$$
$$\mathcal{L}(\zeta_u;x,t)=\exp\{i\zeta_u\xi(x,t)\}\ , \tag{65}$$

where $\mathcal{V}$ is any element of the Poincaré group, and where x',t' is the point into which x,t is taken by the transformation $\mathcal{V}$.

The generators $\xi(x,t)$ at different spacetime points are constrained variables. Thus the relation (65), together with the commutators (60) between ξ and the generators of translation, yields

$$\xi(x,t)=\mathcal{T}(x,t)\xi(0,0)\mathcal{T}^{-1}(x,t)$$
$$=\xi(0,0)-\frac{1}{c}\omega x+kct\ , \tag{66}$$

connecting ξ at two different spacetime points. The additive constraint (66) between the Poincaré generators in the representation k_0 expresses the translational symmetry of the rotational variable $\xi(x,t)$ [as a tensorial relation analogous to Eq. (9); see supplementary Sec. IV.B.4].

For large values of x, an infinitesimal rotation is seen to give a time displacement, and for large t, a spatial translation. The translation from $\xi(0,0)$ to $\xi(x,t)$ is of the same form as for rotations in a spatial y,z plane,

$$j(y,z)=j(0,0)-k_z y+k_y z \ , \tag{67}$$

which gives the symmetry of rotation under the translation $\mathcal{T}(y,z)$ (from which the orbital angular momentum emerges; see supplementary Sec. V.B.7).

2. Impossibility of exact distinction between here and there, at a fixed time

The invariant association of the generator $\xi(x,t)$ of spacetime rotations with the locus x,t about which the rotation takes place is the point of departure for a localization of a quantum. However, the notion of localization further implies that the quantum, if produced here, is definitely not observed there, at the instant considered, and this feature cannot be expressed in terms of the variables $\xi(x,t)$. In fact, rotational variables referring to different x, at equal times, do not commute [see, for example, Eqs. (60) and (66)], and the eigenstate (64) for rotation about the origin is, therefore, not orthogonal to the displaced state $|\xi(x,t)=0|=\xi\rangle$ with the same eigenvalue ξ. The overlap has a range in x of order k_0^{-1}, which characterizes the scale of the representation. Thus, in an eigenstate of $\xi(0,0)$ with eigenvalue ξ, the variable $\xi(x,0)$, belonging to another point in space at the same time, is indeterminate and may appear with the same value ξ, for $|x|\le k_0^{-1}$. Hence it is not possible by a measurement of the variable $\xi(x,t)$, to single out sharply a location of the quantum.

3. Limitations to localization of reflections. (Supplement)

The distinction between localization and invariant connection to a point, which characterizes Poincaré symmetry, also applies to reflections about different planes, as considered in Sec. II.A.3. Thus a spatial reflection $\mathcal{S}(x)$ singles out a point x that remains fixed under the reflection. Moreover, under any transformation $\mathcal{V}$ (translation or reflection), $\mathcal{S}(x)$ goes into a reflection $\mathcal{S}(x')$ in the corresponding point $[\mathcal{S}(x')=\mathcal{V}\mathcal{S}(x)\mathcal{V}^{-1}$, where $\mathcal{V}$ takes x into $x']$. However, the eigenvalues ± 1 of the reflection symmetry, as a property of a quantum, does not single out a plane, since states $|s(x)=1\rangle$, with different x, have an overlap with a range k_0^{-1} [see the probability distribution (42), with $a=x$]. The width of this distribution is comparable to its periodicity, and a location is, therefore, barely definable, corresponding to the fact that the reflection only connects two wave numbers. In contrast, the Lorentz transformations bring together all wave numbers, and the degree of localization is only limited by the reduced contribution of wave numbers $k\ge k_0$, in the invariant state (64), as implied by the factor ω^{-1}.

4. Translation symmetry as tensorial relation. (Supplement)

The connection (66) between spacetime rotations about different points can be seen as a tensorial relation of the form (32) by which the Lorentz invariant $\xi[=\xi(0,0)]$ and the two-vector k,ω combine to form a three-dimensional irreducible Poincaré tensor, in a nonunitary representation, formed by a product of two infinite-dimensional complex-conjugate unitary representations k_0 [see remark in parenthesis following Eq. (32)]. The translation symmetry (66) for the rotation variables can thus be compared with the tensorial relation (9), by which reflections carry translation symmetry (see Sec. II.F.6).

C. Nonrelativistic quantum mechanics

1. Position variable

An approximate position of the quantum emerges when the representation of spacetime symmetry is viewed with a spatial resolution that is low on the scale k_0^{-1}. By such an averaging, the translation matrices (62), for $|k|\ge k_0$, are extinguished, and the representation is, therefore, confined to

$$|k|\ll k_0 \quad (\omega\approx\omega_0) \ . \tag{68}$$

In this regime, the relation (66) for the spatial translation symmetry carried by $\xi(x,t)$ reduces to

$$\xi(x,t)\approx\xi(0,t)-k_0 x \tag{69}$$

by which the averaged rotation variables at equal times are linked together, differing only by a number proportional to x. (This link reflects the merging of the rapidity ζ, which shifts the value ξ of a rotational variable at a fixed point [see Eq. (61)], with the wave number $k\approx k_0\zeta$, which translates the variable.)

It follows from the relation (69) that the eigenstate $|\xi(0,t)=\xi\rangle$, with rotational symmetry ξ about the origin, can be identified with the state $|\xi(x,t)=0\rangle$ that is invariant with respect to rotations about $x=k_0^{-1}\xi$. This identification implies that the invariant states $|\xi(x,t)=0\rangle$, belonging to different values of x at the time t, form an orthonormal basis, since the eigenstates of $\xi(0,t)$ with different eigenvalues $\xi(=k_0 x)$ are orthonormal. Hence the eigenstates of the averaged rotational variables single out points in space, at a given time, and thereby define a location of the quantum,

$$\begin{aligned} x(t)&\equiv k_0^{-1}\xi(x,t)+x=k_0^{-1}\xi(0,t) \ , \\ |x(t)=x\rangle&=k_0^{1/2}|\xi(x,t)=0\rangle \ , \end{aligned} \tag{70}$$

in terms of rotational invariance about the spacetime point x,t. The position variable $x(t)$ thus defined is seen to be equal to $\xi(0,t)$, in the scale k_0^{-1} of the representation. [As the notation indicates, $x(t)$ is a variable, and its eigenvalue is a space point x.]

The variable $x(t)$ is conjugate to the wave number k, as follows from Eq. (60), for $\omega \approx \omega_0$,

$$[k,x(t)] = -i \tag{71}$$

as an expression for the interplay between a spatial translation and a small spacetime rotation [which leaves the two-vector (k,ω) approximately in the direction of the ω axis $(\omega \approx \omega_0)$]. The variation of the matrix $x(t)$ with time follows from the time dependence (66) of $\xi(x,t)$,

$$x(t) = x(t=0) + vt \ , \quad v \equiv \frac{d}{dt}x(t) = \frac{k}{k_0}c \ , \tag{72}$$

where the velocity v is seen to be nonrelativistic, in the regime $k \ll k_0$.

2. Canonical commutator. Quantum of action

A description of the quantum with correspondence to classical physics of a particle is obtained by rescaling the two vector k,ω by the factor mc/k_0 involving a rest mass for the particle,

$$p \equiv \frac{k}{k_0}mc, \quad E \equiv \frac{\omega}{\omega_0}mc^2 \approx mc^2 + \frac{p^2}{2m} \quad (p \ll mc) \ , \tag{73}$$

which gives translational variables with dimension of momentum and energy (see Sec. IV.C.3). The commutator between momentum p and position $x\,[=x(t)]$, which follows from Eq. (71), obtains the canonical form

$$[p,x] = -i\hbar, \quad \hbar \equiv mc/k_0 \tag{74}$$

when the scaling factor is denoted by $\hbar$ (quantum of action). This action is seen to enter when the symmetry variables, defined in terms of coordinate transformations and, therefore, of dimension reciprocal length, time, or angle, are expressed as momentum, energy, and position variables,

$$p = \hbar k, \quad E = \hbar\omega, \quad x(t) = \frac{\hbar}{mc}\xi(x,t) + x \ , \tag{75}$$

which have correspondence to the classical description of a particle. (The scaling factor $\hbar$ must be taken to be a universal constant, if conservation laws for the symmetry variables k and ω of the quanta are to carry over into a description of particles with conserved momentum and energy.)

3. Dimension of mass

The need for the concept of mass in classical physics is thus seen to arise from the low resolution, which hides the wave number and frequency, of dimensions L^{-1} and T^{-1} that lie behind the conserved dynamical quantities denoted by momentum and energy. These quantities were, therefore, given dimensions MLT^{-1} and ML^2T^{-2}, in terms of a dimension M apparently not reducible to spacetime. With the discovery of the underlying quantal structure, the two scales could be identified as having the universal ratio $\hbar$. The choice of units with $\hbar = 1$ thus eliminates the need for a dimension of mass.

4. Nonlocality and its invisibility in the classical limit[24]

A translation connects two eigenvalues of the position variable $x(t)$, while a Lorentz transformation, in its weakly relativistic limit (Galilean transformation), connects eigenvalues of the momentum k (Sec. IV.E.1). The nondiagonality of these symmetry variables, which expresses properties of the variables foreign to classical physics, therefore, amounts to a nonlocality in the state of affairs of a quantum.[25] The nonlocality, defined by the mean values of the coordinate transformations, fully specifies the state of affairs (Sec. IV.E.2).

While the complementarity between the symmetry variables imply a minimum amount of nonlocality as an inherent feature of a quantal state (Sec. IV.E.3), nonlocality is reduced in a quenched state of affairs involving unconnected components (mixtures), and the resolution with which the quantum is observed is correspondingly lowered (Sec. IV.E.4).[26] Such a quenching of nonlocality is produced by interactions with other quanta that subsequently become inaccessible (Sec. IV.E.5). When the quenching is so strong that nonlocality is confined to ranges of coordinate shifts so small that all the visible symmetry variables commute, classical physics emerges (Sec. IV.E.6).[27] The analysis thus exhibits the pertinence of symmetry variables for the lowering of resolution that characterizes the transition to the classical regime.

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[24]This section is a summary of the supplementary Sec. IV.E.

[25]It seems appropriate and in line with terminology used in the literature to characterize the state of affairs of multidimensional variables in terms of a nonlocality, in contrast to the locality of a classical state of affairs. For variables, the word locality has been reserved for the singling out of a point in space and time by the spacetime rotations [the position $x(t)$ as well as the field $\phi(x,t)$ that is considered in Sec. V.B].

[26]The lowering of resolution caused by quenching towards the classical regime is distinct from the lowering of resolution associated with the restriction of the symmetry variables to the Galilean corner ($k \ll k_0$; see introduction to Sec. IV.C.1).

[27]The emergence of classical situations, as well as the analysis of measurements (see in this connection Sec. III.A.3), has been extensively discussed in recent years in the context of the decoherence resulting from interactions with the environment (see, for example, the review in *Physics Today* by Zurek, 1991). With coordinate transformations as the elementary variables, the decoherence takes the form of a quenching of the nonlocality in the state of affairs of these variables.

D. Galilean transformations as symmetry variables. Summary[28]

1. Translations and Galilean transformations in the weakly relativistic regime

Translations and Galilean transformations about the fixed point x,t are defined as coordinate transformations taking the spacetime point x'',t'' into

$$\mathcal{T}(a), \quad x''\to x''+a, \quad t''\to t'' ,$$

$$\mathcal{G}(u;x,t), \quad x''\to x''+u(t''-t), \quad t''\to t''+u(x''-x)/c^2 , \quad u \ll c , \tag{76}$$

where the Galilean transformation is recognized as a limit of a Lorentz transformation, and hence, to leading order in u, includes the departure from simultaneity, which is part of relativistic invariance. The Galilean transformation thereby acquires the character of a rotation in spacetime about the point x,t [conserving $(x''-x)^2-c^2(t''-t)^2$, to first order in u], and this link between space and time is crucial for the interplay of the symmetry variables $\mathcal{T}(a)$ and $\mathcal{G}(u;x,t)$.

In fact, without the difference in time coordination for the two reference systems, the Galilean transformation no longer singles out a space point x and thereby becomes a (time dependent) spatial translation $u(t''-t)$, rather than a spacetime rotation. The notion of a universal time completely separated from the space coordination, therefore, implies that the Galilean transformation commutes with spatial translations and hence does not yield a position variable. Indeed, a location of the quantum at a given time requires a symmetry variable that defines a point in space as well as in time, as does the transformation (76), which singles out a point x,t that is left invariant.

2. Products of translations and Galilean transformations

The product of the coordinate transformations (76) taken in opposite order are seen to differ by a time displacement $\tau=ua/c^2$,

$$\begin{aligned}\mathcal{G}(u;x,t)\mathcal{T}(a)&=\mathcal{T}(a)\mathcal{G}(u;x,t)\mathcal{T}(\tau=ua/c^2)\\&=\mathcal{T}(a)\mathcal{G}(u;x,t)\exp\left\{i\omega_0\frac{ua}{c^2}\right\},\\ \mathcal{T}(\tau)&\approx\exp\{i\omega_0\tau\} .\end{aligned} \tag{77}$$

Since translations are not linked to a spacetime point, the relation (77) is independent of the location x,t of the axis of the spacetime rotation (and is most easily evaluated for $x=t=0$). The generator ω of time displacement $\mathcal{T}(\tau)$ [see Eq. (39)], together with the generator k of spatial translations [see Eq. (4)], form a two-vector and, in the nonrelativistic regime ($ck \ll \omega$), the frequency ω is, to lowest order, approximated by the rest frequency $\omega_0=\omega(k=0)$.

[28]This section, which focuses on the canonical commutator as a constraint between the symmetry variables, is formulated (with a certain amount of repetition) so as to be accessible to readers who have not gone through the preceding parts of Sec. IV.

3. Generators of Galilean transformations

The Galilean transformations (76) are additive in the velocity u and can, therefore, be expressed in the form

$$\mathcal{G}(u;x,t)=\exp\left\{i\frac{u}{c}\xi(x,t)\right\}, \quad u\ll c \tag{78}$$

in terms of the dimensionless generator $\xi(x,t)$ of spacetime rotations about the point x,t. For infinitesimal values of u/c, the relation (77) thus yields

$$\begin{aligned}\mathcal{T}(a)\xi(x,t)\mathcal{T}^{-1}(a)&=\xi(x,t)-k_0a, \quad ck_0\equiv\omega_0\\&[=\xi(x+a,t)] ,\end{aligned} \tag{79}$$

where the effect of a translation on $\xi(x,t)$ [as expressed by the first line of Eq. (79)] is seen to derive from the link between space and time in the Galilean transformation (76). A translation can also be viewed as shifting the spacetime point about which the rotation takes place, as exhibited by the second line in Eq. (79).

4. Position variable and canonical commutator

The relation (79) shows that the generator $\xi(x,t)$ of a Galilean transformation is shifted in a translation $\mathcal{T}(a)$ by an amount proportional to a and, therefore, defines a position coordinate

$$x(t)\equiv k_0^{-1}\xi(x,t)+x=k_0^{-1}\xi(0,t) \tag{80}$$

with the transformation

$$\begin{aligned}&\mathcal{T}(a)x(t)\mathcal{T}^{-1}(a)=x(t)-a ,\\&[k,x(t)]=-i .\end{aligned} \tag{81}$$

The second form of $x(t)$ in Eq. (80) follows from the translation of $\xi(x,t)$ in the second relation (79) and implies that the different eigenstates of $x(t)$ are orthogonal, as eigenstates of $\xi(0,t)$.

The definition (80) is seen to imply that a state with $x(t)=x$ has $\xi(x,t)=0$. A quantum located at the spacetime point x,t is, therefore, characterized by rotational invariance about this point.

5. Symmetry variables in scale involving rest mass

The translational variables obtain the dimension of momentum and energy by a change of scale involving a rest mass, as described in Sec. IV.C.2 (see also Sec. IV.C.3). The commutator (81) then takes the canonical form (74). When ω_0 is expressed in terms of the rest mass ($\hbar\omega_0=mc^2$), the basic relation (77) governing the interplay of translations and Galilean transformations becomes

$$\mathcal{G}(u;x,t)\mathcal{T}(a)=\mathcal{T}(a)\mathcal{G}(u;x,t)\exp\left\{\frac{i}{\hbar}mua\right\},$$
$$\omega_0=mc^2/\hbar\,, \quad (82)$$
$$\mathcal{T}(a)=\exp\left\{-\frac{i}{\hbar}ap\right\},\quad \mathcal{G}(u;0,t)=\exp\left\{\frac{i}{\hbar}mux(t)\right\}.$$

[For a Galilean transformation at time t and $x\neq 0$, the variable $x(t)$ is replaced by $x(t)-x$; see Eq. (80).] It is seen that the departure from commutation in Eq. (82) arises from the small time shift ua/c^2 [see Eq. (77)], which is multiplied by the large frequency ω_0 proportional to the rest energy mc^2, to give a phase factor of zero order in c. The occurrence of the rest mass is a reminiscence of the origin of the phase in relativistic invariance.[29]

6. Irrelevance of the time shift in secondary manifestation of the symmetry

While the phase factor in Eq. (82) is essential to the primary manifestation of the symmetry, it is without effect for the secondary. In fact, the transformations $\mathcal{V}=\mathcal{T}(a)\mathcal{G}(u)$ and $\mathcal{W}=\mathcal{G}(u)\mathcal{T}(a)$ that only differ by a phase factor perform the same transformation $\mathcal{V}\mathcal{U}\mathcal{V}^{-1}=\mathcal{W}\mathcal{U}\mathcal{W}^{-1}$ of any variable $\mathcal{U}$. In this secondary manifestation of Galilean symmetry, therefore, the time shift in the transformation (76) can be ignored.

In summary of Sec. IV.D, the emergence of conjugate variables x and p, with the canonical commutation relation, which is at the basis of nonrelativistic quantum mechanics, is seen to hinge on the recognition of the Galilean transformation as a rotation in spacetime. Hence the complementarity of these variables, as a primary manifestation of spacetime symmetry, requires the link between space and time that special relativity introduces.

E. Quenching of nonlocality. Emergence of classical variables. (Supplement)

1. Symmetry variable $\mathcal{U}(a,q)$ at a specified time

While the translational variable $\mathcal{T}(a)$ connects positions at a distance a, the Lorentz transformation about the origin, in its weakly relativistic limit (Galilean transformation), is the translation in k space [as may, for example, be seen from Eq. (82)],

$$\mathcal{G}(q)=\exp\{iqx(t)\},\quad q=k_0\frac{u}{c}=\frac{mu}{\hbar}\,, \quad (83)$$

connecting two wave numbers separated by q. More generally, any continuous coordinate transformation in the Galilean corner of Poincaré symmetry (in 1+1 dimensions), at a fixed time, can be expressed as a product of translations along the k and x axes.

A complete set of symmetry variables is thus

$$\mathcal{U}(a,q)=\exp\{-iak+iqx(t)\}$$
$$=\mathcal{T}(a)\mathcal{G}(q)\exp\left\{i\frac{qa}{2}\right\},$$
$$\mathcal{U}^\dagger(a,q)=\mathcal{U}(-a,-q)\,, \quad (84)$$

each of which is a single translation in a skew direction in the k,x plane. The Galilean transformation (83) is a rotation about the origin ($x=0$), while a Galilean transformation about the point x involves an additional phase factor $\exp\{-iqx\}$ [see Eq. (80)]. Thus the variable $\mathcal{U}(a,q)$ can be viewed as a translation preceded by a Galilean transformation about the midpoint $x=-a/2$.

The complementarity (82) implies that the variable $\mathcal{U}(a,q)$ has the transformations

$$\mathcal{T}(a_1)\mathcal{U}(a_2,q_2)\mathcal{T}^{-1}(a_1)=\exp\{-ia_1q_2\}\mathcal{U}(a_2,q_2)\,, \quad (85)$$
$$\mathcal{G}(q_1)\mathcal{U}(a_2,q_2)\mathcal{G}^{-1}(q_1)=\exp\{+iq_1a_2\}\mathcal{U}(a_2,q_2)\,,$$

by which it carries translational and Lorentz symmetry in the Galilean corner.

2. State of affairs in terms of mean values of $\mathcal{U}(a,q)$

The mean value of the symmetry variable $\mathcal{U}(a,q)$ in any state of affairs of a quantum can be expressed by

$$\langle\mathcal{U}(a,q)\rangle=\sum_i p_i\langle i|\mathcal{U}(a,q)|i\rangle,\quad \sum_i p_i=1\,, \quad (86)$$
$$\langle\mathcal{U}(a,q)\rangle=\mathrm{Tr}(\mathcal{U}(a,q)\rho),\quad \rho=\sum_i |i\rangle p_i\langle i|,\quad \mathrm{Tr}\rho=1\,,$$

where $|i\rangle$ is an arbitrary set of unconnected states with weights p_i. The quenched (or mixed) state of affairs can be represented by a Hermitian matrix ρ (density matrix) and, as discussed in Sec. II.E.3.d, such a matrix describes the most general state of affairs of the symmetry variables. The states $|i\rangle$ in Eq. (86) need not be orthogonal, and the decomposition of a given matrix ρ is, therefore, not unique.

Any matrix, such as ρ, can be expressed as a linear combination of the complete set of matrices $\mathcal{U}(a,q)$,

[29] Galilean symmetry is usually incorporated into nonrelativistic quantum mechanics in terms of its generator, which is identified with the position variable, assumed to satisfy the canonical commutator with momentum. The resulting relation (82) is then interpreted as a representation of the inhomogeneous Galilean group up to a phase factor (ray representation). However, when the symmetries are recognized as the basic variables that determine their own properties, their representations are genuine (see footnote 14). The phase factor in Eq. (82) is, therefore, a symmetry variable (in a limiting form), which is seen to represent a time displacement [see Eq. (77)].

$$\rho=\frac{1}{2\pi}\int dadqc^*(a,q)\mathcal{U}(a,q), \quad c^*(a,q)=c(-a,-q) \ ,$$

$$c(a,q)=\mathrm{Tr}(\mathcal{U}(a,q)\rho)=\langle \mathcal{U}(a,q)\rangle, \quad c(0,0)=\mathrm{Tr}\rho=1 \ , \tag{87}$$

where use has been made of the algebraic relations

$$\mathcal{U}^\dagger(a_1,q_1)\mathcal{U}(a_2,q_2)=\mathcal{U}(a_2-a_1,q_2-q_1)\times e^{(i/2)(-q_1a_2+q_2a_1)} \ , \tag{88}$$

$$\mathrm{Tr}\mathcal{U}(a,q)=2\pi\delta(a)\delta(q)$$

that derive from Eqs. (84); see also Eq. (82). It, therefore, follows that the state of affairs is completely (and uniquely) specified by the mean values $c(a,q)$.

While $c(0,0)=1$ [see Eq. (87)], the derivatives of $c(a,q)$ at the origin are seen to give the mean values $\bar{x}\equiv\langle x\rangle$ and $\bar{k}\equiv\langle k\rangle$,

$$\frac{\partial}{\partial a}c(a,q)|_{a=q=0}=-i\bar{k}, \quad \frac{\partial}{\partial q}c(a,q)|_{a=q=0}=+i\bar{x} \tag{89}$$

of the position and momentum variables in the state of affairs of the quantum. Correspondingly, the higher derivatives give the mean values of powers of x and k.

3. Nonlocality of quantal state

The mean values $c(a,q)=\langle i|\mathcal{U}(a,q)|i\rangle$ for a quantal state give the overlap of a state with the state produced by the displacements a,q and thereby provide a measure of the nonlocality in the state of affairs of the quantum. The nonlocality distribution $c(a,q)$ expresses the extent to which the nondiagonal matrix elements of the symmetry variables are visible in the experimental situation and thus puts focus on features that are specific to multidimensional variables and hence outside the scope of classical physics.

For a pure state, the nonlocality obeys the sum rule

$$\mathrm{Tr}\rho^2=\frac{1}{2\pi}\int dadq|c(a,q)|^2=\mathrm{Tr}\rho=1 \ , \tag{90}$$

which follows from Eqs. (87) and (88), together with the relation $\rho^2=\rho$ that characterizes a quantal state. Since $|c(a,q)|\leq 1$ for the mean value of a unitary variable, it follows that $c(a,q)$ extends over an area that is of order unity, or greater. Extreme examples of the reciprocity between a and q implied by Eq. (90) are the eigenstates $|k\rangle$ and $|x\rangle$, for which $c(a,q)$ extends over the entire a or q axis, respectively, but vanishes outside the axis.

An intermediate situation is provided by a wave packet with a Gaussian distribution

$$c(a,q)=\exp\left\{-\frac{a^2}{4a_0^2}-\frac{q^2}{4q_0^2}\right\}\exp\{-ia\bar{k}+iq\bar{x}\} \ , \quad a_0q_0=1 \ , \tag{91}$$

where the phase factor involves the mean values $\bar{k}$ and $\bar{x}$ for wave number and position of the packet [see Eq. (85) or Eq. (89)]. The widths a_0^2 and q_0^2 in Eq. (91) are the mean values of a^2 and q^2 in the density distribution $(2\pi)^{-1}|c(a,q)|^2$ and are related to the mean-square deviations Δk_0 and Δx_0

$$(\Delta k_0)^2=\frac{1}{2a_0^2}, \quad (\Delta x_0)^2=\frac{1}{2q_0^2}, \quad \Delta x_0\Delta k_0=\frac{1}{2} \ , \tag{92}$$

for the variables k and x around their mean values. (An example of a distribution of nonlocality involving separate regions in a,q space is associated with the quantum in the interferometer experiment.)

4. Reduction of nonlocality in quenched state. Lowering of resolution

A reduction of nonlocality takes place by quenching, in which unconnected components in the state of affairs are combined. Such quenched states describe combinations of results from separate situations (see Sec. II.E.2), and may arise, when the correlations of the quantum with other quanta, as contained in the comprehensive state of all the quanta, are no longer accessible after the interaction has taken place [see Sec. II.E.4, example (a)].

For a quenched state (with positive weights p_i), the trace of ρ^2 is less than unity, and the integral (90), therefore, has a value less than unity. This quenching of nonlocality is associated with the fact that a phase attaches to the mean value $c(a,q)$ of a unitary variable. A reduction of $|c(a,q)|$, therefore, results when two or more components with different phases and total weight unity are added. For example, for a mixture of two distributions (91), with opposite mean values $\bar{x},\bar{k}$ and $-\bar{x},-\bar{k}$ and equal weight, the phase factor in Eq. (91) is replaced by $\cos(a\bar{k}-q\bar{x})$, which decreases the magnitude of the nonlocality.

The cumulative quenching produced by many (nonorthogonal) states is illustrated by a statistical ensemble of distributions (91) in which each element is characterized by mean values $\bar{x}$ and $\bar{k}$, with weight $p(\bar{x},\bar{k})$, giving

$$c'(a,q)=\int d\bar{x}d\bar{k}\,p(\bar{x},\bar{k})c(a,q;\bar{x},\bar{k}) \tag{93}$$

for the distribution $c'(a,q)$ in the resulting state of affairs. For a Gaussian weight factor $p(\bar{x},\bar{k})$ centered on x_1,k_1 and having widths σ_x and σ_k, the folding in Eq. (93) gives

$$c'(a,q)=\exp\left\{-\frac{a^2}{4a_1^2}-\frac{q^2}{4q_1^2}\right\}\exp\{-iak_1+iqx_1\} \ ,$$

$$\frac{1}{2a_1^2}=\frac{1}{2a_0^2}+\sigma_k^2, \quad \frac{1}{2q_1^2}=\frac{1}{2q_0^2}+\sigma_x^2, \quad a_0q_0=1 \ , \tag{94}$$

exhibiting the reduction in the range of nonlocality. The resulting widths a_1,q_1 are seen to depend partly on the indeterminacy a_0,q_0 of the quantal states and partly on the statistical uncertainty σ_x,σ_k of the ensemble. The restriction in the visibility of the symmetry variables by the quenching of nonlocality, exemplified by the distribution (94), amounts to a lowering of the resolution with which these symmetry variables are observed (see footnote 26). Nonlocality in regions of a,q space separated from the origin can be completely quenched.

5. Quenching by interactions

The reduction of nonlocality (lowering of resolution) by the transformation of a quantal state into a mixture of components takes place through interactions with a second quantum by which the variables of the two quanta become entangled. If the second quantum, after the interaction, escapes or becomes inaccessible, the state of affairs of the original quantum is a quenched substitute state of affairs (see Secs. II.E.4 and III.A.3). The quenched state can be expressed in terms of an expansion of the total state of the two quanta in sets of orthogonal components of the second quantum.

In this manner, the distribution of nonlocality can be strongly quenched, when the quantum suffers a large number of collisions with other quanta that subsequently become inaccessible (interaction with the environment; see footnote 27). The distribution (94) is obtained if the collisions produce a random (Gaussian) distribution of increments of the mean momentum and position.

While the quenching restricts the visibility of the symmetry variables, the variables themselves remain present in the quenched state of affairs, and the nonlocality can reappear under suitable conditions. For example, the quenching in the state of affairs of the quantum can be undone by Bragg reflection in a crystal.

6. Limit of small nonlocality. Classical regime

The classical regime emerges when the statistical uncertainty is large compared with the indeterminacy ($\sigma_x \gg a_0$ and $\sigma_k \gg q_0$), in which case the quenching of nonlocality satisfies the conditions

$$a_1 q_1 \ll 1, \quad \sigma_x \sigma_k \gg 1 . \tag{95}$$

In this regime, the distribution (94) is independent of the parameters a_0, q_0 of the wave packets out of which the quenched state of affairs is built, and the distribution $c(a,q;\bar{x},\bar{k})$ in Eq. (93) can be replaced by the phase factor $\exp\{-ia\bar{k}+iq\bar{x}\}$ that identifies the mean values of x and k [see Eq. (89)]. Under these circumstances $p(\bar{x},\bar{k})$ and $c'(a,q)$ are equivalent descriptions of the state of affairs, being related by a double Fourier transformation,

$$p(\bar{x},\bar{k}) = \frac{1}{4\pi^2}\int da\,dq\; c'(a,q) e^{ia\bar{k}-iq\bar{x}} ,$$
$$a_1^{-2} \approx 2\sigma_k^2, \quad q_1^{-2} \approx 2\sigma_x^2 . \tag{96}$$

In this situation, the quenched state of affairs is fully characterized by the distribution $p(\bar{x},\bar{k})$ (phase-space density).[30]

In the regime (95), in which the only visible symmetry variables $\mathcal{U}(a,q)$ have $aq \ll 1$, all the coordinate transformations within this range of parameters commute [see, for example, Eq. (88)], and the symmetry variables no longer exhibit their multidimensionality. The visible symmetry variables $\mathcal{U}(a,q)$ then behave as one-dimensional variables that can be assigned values, and the generators can be identified with the position x and momentum $p\,(=\hbar k)$ of a classical particle. Moreover, any function of x and p can be formed by a linear combination of the variables $\mathcal{U}(a,q)$. However, within the classical regime with its low resolution that hides complementarity, the origin of the variables in the primary manifestation of symmetry is not apparent. Symmetry is seen in classical physics only in its secondary manifestation ($x \to x - a, p \to p$ in a translation, and $x \to x - ut$, $p \to p - mu$ in a Galilean transformation).

While in the classical regime (strong quenching), the commutator of two symmetry variables $\mathcal{U}(a_1,q_1)$ and $\mathcal{U}(a_2,q_2)$ is zero, to lowest order, the leading nonvanishing contribution is bilinear in a and q [see Eq. (85) or (88)],

$$\begin{aligned}[\mathcal{U}(a_1,q_1),\mathcal{U}(a_2,q_2)] &= -i(a_1q_2 - a_2q_1) \\ &= -i\hbar\{\partial_p \mathcal{U}(a_1,q_1) \\ &\qquad \times \partial_x \mathcal{U}(a_2,q_2) - 1 \leftrightarrow 2\} ,\end{aligned}$$
$$\mathcal{U}(a,q) = \exp(-iak + iqx) , \tag{97}$$

where the quantities $\mathcal{U}$ in the Poisson bracket are functions of the classical variables x and p. The linearity of Eq. (97) in $\mathcal{U}(a_1,q_1)$ as well as in $\mathcal{U}(a_2,q_2)$ implies the validity of this relation between commutator and Poisson bracket for arbitrary variables that are functions of x and p.

In the transition to the classical regime, semiclassical features in individual quantal states (such as described by the WKB approximation) are distinct from phenomena (such as fluctuations, dissipation, etc.) associated with the quenching of nonlocality. The individual quantum state has $\Delta x \Delta p \sim \hbar$, while in the classical state of affairs (mixture), the phase-space density is small compared to $\hbar^{-1}$

[30]More generally, the double Fourier transformation of the symmetry variable $\mathcal{U}(a,q)$ gives the Wigner transform, which is a global symmetry $\pi^{-1}\mathcal{S}(x,k)$, where the spatial reflection $\mathcal{S}(x,k)$ is obtained from a reflection at the origin by a translation x and a Galilean transformation k (see, for example, Royer, 1977). Thus the Wigner transform has the eigenvalues $\pm\pi^{-1}$ and is, therefore, not a density, except for such strong quenching that the phase-space density is everywhere small compared to unity [$\sigma_x\sigma_k \gg 1$; see Eq. (95)]. In this classical regime, the remains of the reflection $\mathcal{S}(x,k)$ as a variable (primary manifestation of symmetry) is thus the local variable $\pi\delta[x(t)-x]\delta[k(t)-k]$.

at any point, corresponding to an uncertainty in the canonical variables with $\sigma_x \sigma_p \gg \hbar$; see Eq. (95).

V. PRODUCT REPRESENTATIONS. FIELD VARIABLES

In the primary manifestation of symmetry, the elementary variables are the irreducible representations constituting individual quanta. A new dimension is associated with product representations of the symmetry, which describe systems of quanta in an occupation number space. The number of quanta of a specified symmetry thereby becomes a physical variable, and the elementary operators in this new dimension add a quantum and thus carry the same irreducible symmetry as the states of individual quanta. For spacetime symmetry, these operators are the components of local boson and fermion fields. The steps by which field variables with canonical properties in this manner emerge from the primary manifestation of symmetry are briefly outlined below. In the framework of the present discussion, the individual quanta are independent entities (no interactions).[31]

A. Occupation number. Bosons and fermions

The representations of a symmetry group comprise products of irreducible factors giving a multitude of reducible representations of the group. Thus the direct product

$$\mathcal{U}=\mathcal{U}_1\mathcal{U}_2\cdots\mathcal{U}_N \tag{98}$$

of the matrices for individual quanta are seen to be representations of the symmetry group and to give the variables of N quanta. In Eq. (98), each factor is an irreducible representation of the same symmetry $\mathcal{U}$. When the constituent representations differ, the representation (98) describes uncorrelated quanta in product states specifying the symmetry of the individual quanta.

For identical representations, the product (98) can be classified in terms of permutation symmetry and is carried by basis vectors $|u_1,\ldots,u_N\rangle$ that are direct products of eigenvectors for the individual quanta, in a chosen basis u (for simplicity taken to be without degeneracy). The permutations act on the ordering of the eigenvalues, and the vector space can therefore be decomposed into parts that transform irreducibly under permutations. The symmetry variables (98) that are products of identical matrices only connect states with the same permutation symmetry.

The identification of a quantum with its symmetry (see Sec. II.C.2) implies a restriction in the permutation symmetry that can be carried by a system of quanta. In fact, when the symmetry of the quantum is fully specified by the label u, the ordering of the quanta, within the system is undefined, and the state is therefore invariant, up to a phase, under permutations. Hence the states are either totally symmetric, or totally antisymmetric. (The mixed permutation symmetries, which are multidimensional, involve a numbering of the quanta, which would have to refer to attributes beyond the symmetry $\mathcal{U}$.)

States with the one-dimensional permutation symmetries are completely specified by the set of eigenvalues $u_1,u_2,\ldots,u_N$ (with no ordering), or, equivalently, by the occupation numbers $n(u)$, giving the number of quanta for the different values of u. The eigenvalues of these variables are

$$n(u)=\begin{cases}0,1,2,\ldots & \text{bosons}\\ 0,1 & \text{fermions}\end{cases} \tag{99}$$

for the symmetric and antisymmetric states.

The product representations, with different values $N=0,1,2,\ldots$ for the total number of quanta (bosons or fermions), can be combined in a vector space spanned by the eigenstates of the occupation numbers $n(u)$, which thereby appear as independent variables, for the different eigenvalues u. The operators $a^\dagger(u)$ that add a quantum with symmetry u,

$$[n(u),a^\dagger(u)]=a^\dagger(u) \tag{100}$$

and the Hermitian conjugates $a(u)$ that remove a quantum offer themselves as building blocks in terms of which all the operations in the total vector space can be performed. The creation and annihilation operators obey the algebras

$$[a^\dagger(u_1),a^\dagger(u_2)]_\mp=0,\quad [a(u_1),a^\dagger(u_2)]_\mp=\delta(u_1,u_2)\ , \tag{101}$$

with commutators ($-$) and anticommutators ($+$) for bosons and fermions, respectively, expressing links between the quanta arising from their indistinguishability. The commutator between a and $a^\dagger$ in Eq. (101) involves the normalization $n(u)=a^\dagger(u)a(u)$ of these variables.

The creation operator $a^\dagger(u)$ adds the symmetry, which the quantum possesses, and the transformation (31) of the basis states $|u\rangle$ for an irreducible representation, therefore, carries over to the transformation of the variables $a^\dagger(u)$. Hence these variables are components of irreducible tensors [see Eq. (32)],

$$\begin{aligned}a^\dagger(u_1')&=\mathcal{V}a^\dagger(u_1)\mathcal{V}^{-1}\\ &=\sum_{u_2} a^\dagger(u_2)\langle u_2|\mathcal{V}|u_1\rangle\end{aligned} \tag{102}$$

with tensorial symmetry [denoted by w in Eq. (32)] equal to the symmetry u carried by the individual quantum. The variables $a^\dagger(u)$ themselves, as well as the transformation $\mathcal{V}$ in the first line of Eq. (102), are matrices in occupation number space, which carries the symmetric or antisymmetric components of the product representations for the system of quanta.

[31]Thus, for example, the possibility of quanta that obtain an arbitrary phase under exchange as a consequence of gauge interactions are not considered.

The annihilation operators $a(u)$, as Hermitian conjugates to $a^\dagger(u)$, are tensor components transforming with the complex conjugate matrices $\langle u_2|\mathcal{V}|u_1\rangle^*$. Together, the variables $a^\dagger(u)$ and $a(u)$ are the elementary tensors for the system of quanta, from which other tensorial variables are formed by products decomposed into irreducible parts [as in the one-quantum tensors discussed in Secs. II.F.6 and II.F.7, which are bilinear expressions in the products $a^\dagger(u')a(u)$].

B. Local fields. Imaginary Lorentz symmetry

The product representations, in the primary manifestation of spacetime symmetry, describe field variables with causal commutation relations, in terms of systems of quanta. The field is identified by its Lorentz symmetry and constitutes independent local degrees of freedom (that form the ingredients for expressing gauge invariance at each spacetime point). No appeal is made to a quantization of a classical field within an underlying framework of quantal physics.

1. Variables carrying Lorentz symmetry in 1+1 dimensions

In Poincaré symmetry, the tensorial variables $a^\dagger(u)$ and $a(u)$ add and remove a quantum in an irreducible representation k_0. In the translation symmetric basis, the variables $a^\dagger(k)$ transform as in Eq. (102) with the matrices $\langle k_2|\mathcal{V}|k_1\rangle$ given by Eq. (62) with positive frequencies, while the Hermitian conjugate variables $a(k)$ transform with the complex conjugate representation, with negative frequencies. Thus the description in occupation number space involves variables transforming with both signs of the frequency, in the secondary manifestation of the symmetry.

Variables that are linked to a spacetime point x,t carry symmetry with respect to Lorentz transformations (spacetime rotations) about this point, with the generator $\xi(x,t)$ (see Sec. IV.B.1). The states (64) of a quantum carry the symmetry $\xi(x,t)=\xi$, and the Lorentz symmetric variables $a^\dagger(\xi;x,t)$ that create these states, with the same symmetry ξ at each spacetime point, are

$$a^\dagger(\xi;x,t)=\left[\frac{c}{2\pi}\right]^{1/2}\int \omega^{-1}dk\, e^{-i(kx-\omega t)}e^{-i\zeta\xi}a^\dagger(k)\ ,$$
$$\mathcal{L}(\zeta_u;x,t)a^\dagger(\xi;x,t)\mathcal{L}^{-1}(\zeta_u;x,t)=e^{i\zeta_u\xi}a^\dagger(\xi;x,t) \tag{103}$$

expressed as a superposition of momentum components $a^\dagger(k)$.

While the variables $a^\dagger(k)$ for different k (and similarly the variables with different ξ, at the same point x,t) constitute independent degrees of freedom describing bosons or fermions [see Eq. (101)], the variables $a^\dagger(\xi;x,t)$ at different x, for equal time, are interrelated, as are the corresponding states of a quantum (see Sec. IV.B.2). The commutators (or anticommutators) expressing the constraints between the boson (or fermion) variables are numbers, depending on the distance between the space points. The range is of order k_0^{-1}, as given by the superposition (103).

2. Disentangling of local degrees of freedom

The disentangling of the variables into independent local degrees of freedom is achieved by a combination of the variables with negative and positive frequencies, as exhibited in a simple form by Hermitian fields. Since $a(\xi;x,t)$, as the Hermitian conjugate of the superposition (103), carries the Lorentz symmetry $-\xi^*$, a Hermitian field with Lorentz symmetry has a value for ξ that is imaginary,[32]

$$\phi(\xi;x,t)=\frac{1}{\sqrt{2}}[a^\dagger(\xi;x,t)+a(\xi;x,t)],\quad \mathrm{Re}\xi=0\ ,$$
$$\phi(\xi;x,t)=\phi^\dagger(\xi;x,t)\ . \tag{104}$$

[The Hermitian field (104) describes neutral particles, as well as particles with gauge symmetry, in a basis of eigenstates of particle-antiparticle conjugation.]

By a spacetime derivation, the Lorentz symmetry of the field is changed by an imaginary unit [see Eq. (103)],

$$(\partial_t\mp c\partial_x)\phi(\xi;x,t)=\frac{i\omega_0}{\sqrt{2}}[a^\dagger(\xi\pm i;x,t)-a(\xi\pm i;x,t)]\ ,$$
$$\omega\pm ck=\omega_0 e^{\pm\zeta}\ , \tag{105}$$

since the components $\omega\pm ck$ of the two-vector k,ω carry Lorentz symmetry $\pm i$. In the derivative fields (105), the phase between the components with positive and negative frequencies is opposite to that in the field (104).

For bosons, a field with $\xi=0$ has the commutator [see Eqs. (101) and (103)]

$$[\phi(\xi=0;0,0),\phi(\xi=0;x,t)]_-$$
$$=\frac{c}{4\pi}\int\frac{dk}{\omega}(e^{-i(kx-\omega t)}-\text{c.c.})\ , \tag{106}$$

which is seen to vanish for equal times $(t=0)$ and hence, as a consequence of Lorentz invariance, for any spacelike separation. Correspondingly, for fermions with $\xi=i/2$, the anticommutator vanishes for different x at equal t,

$$[\phi(\xi=i/2;0,0),\phi(\xi=i/2;x,0)]_+$$
$$=\frac{c}{4\pi}\int\frac{dk}{\omega}e^{-ikx}(e^{\zeta}+e^{-\zeta})=k_0^{-1}\delta(x)\ , \tag{107}$$

where, in the second term in the parenthesis, the sign of

[32] The states (64), with $\xi=i\eta$, are not normalizable, but can be expanded in components with real Lorentz symmetry ξ, with relative probabilities $W(i\eta;\xi)=|\eta|/\pi(\eta^2+\xi^2)$. Similarly, states with complex wave number and frequency (decaying states) are contained in the vector space based on translations with real generators belonging to unitary representations.

the integration variable has been inverted. [A fermion field carrying reflection symmetry involves two components with $\xi=\pm i/2$; see supplementary Sec. V.B.5.] By the derivatives (105), the causal commutators (106) and (107) are extended to fields with arbitrary integer or half-integer values of Imξ.

3. Imaginary Lorentz symmetry for bosons and fermions

Thus, for boson fields with $\xi=0$ and fermion fields with $\xi=i/2$, the degrees of freedom disentangle into local variables, even though the quanta created and annihilated by the fields are not localized. It is seen that the disentangling only occurs for bosons with integer values of Imξ and for fermions with half-integer values of Imξ. This link between imaginary Lorentz symmetry and statistics is expressed by

$$\mathcal{L}(\zeta_u=2\pi i)=e^{-2\pi\xi}=\begin{cases}1 & \text{boson}\ ,\\ -1 & \text{fermions}\ ,\end{cases} \tag{108}$$

in terms of a Lorentz transformation with rapidity $2\pi i$.[33]

4. Fields in 3+1 dimensions. Connection between spin and statistics

The extension of spacetime invariance to four dimensions brings in the spatial rotations, which have unitary representations for half-integer as well as integer values of the rotational quantum number j, distinguished by $\mathcal{R}(2\pi)=\mp 1$, for a rotation of 2π about any axis [see Eq. (1)]. The occurrence of rotation variables with $\mathcal{R}(2\pi)=-1$ reveals the double connectivity of the rotation group, as a global property of space (see also supplementary Sec. V.B.6). In four-dimensional spacetime, the finite-dimensional (nonunitary) representations of the Lorentz group, which characterize the symmetry of the field, identify $\mathcal{R}(2\pi)$ with $\mathcal{L}(2\pi i)$ (as in the identity of all rotations of 2π in an Euclidean space) and thereby, through the relation (108), establishes the connection between spin and statistics.

5. Reflection symmetry of fermions. (Supplement)

Space reflection $\mathcal{S}$ inverts ξ and is seen from Eq. (105) to take the fermion field $\phi(\xi=i/2;x,t)$ into its derivative $\omega_0^{-1}(\partial_t+c\partial_x)\phi(\xi=i/2;x,t)$, with $\xi=-i/2$, if the fermion is assigned imaginary intrinsic parity $[\mathcal{S}a^\dagger(k)\mathcal{S}^{-1}=ia^\dagger(-k)$, implying $s^2=-1$ for the fermion]. The two-component fermion field, with $\xi=\pm i/2$, connected by the first order derivatives (Dirac equation), thus carries Lorentz and reflection symmetry. The imaginary intrinsic parity applies to a space reflection that commutes with particle-antiparticle conjugation, thereby leaving the fermion fields with a minimum number of components. For a space reflection that inverts the eigenvalue of particle-antiparticle conjugation, the local fermion fields carrying reflection symmetry have real, opposite values of the intrinsic parities for particle and antiparticle ($s^2=1$).

6. Rotations of 2π. Secondary manifestation. (Supplement)

The double connectivity of the rotation group is seen in its secondary manifestation in the transformation of spinor fields. Thus, in the interference experiment (see Sec. III.A), with polarized neutrons, a rotation of 2π of one of the components can be accomplished by a magnetic field and is observed to invert the phase of this component, thereby shifting the interference pattern by $a=\pi/2k_0$ (Rauch *et al.*, 1975). In the corresponding experiment with photons, a rotation of 2π, which can, for example, be produced by an optically active material, is an invariance. In the hypothetical case that spinor waves had been discovered by their interference before the quanta had been found, these waves would have appeared as a classical field analogous to the electromagnetic field describing classical optical interference phenomena, which were later seen to represent incoherent effects of individual photons (Hepp and Jensen, 1971). Polarization experiments with such a classical spinor field would have revealed that a rotation of 2π is not the identity.

7. Four-dimensional Lorentz symmetry in nonrelativistic limit. (Supplement)

In 3+1 dimensions, the Lorentz symmetry of a quantum about a point $\mathbf{r}$, t is described by the rotational variables $\mathbf{j}(\mathbf{r},t)$ and $\boldsymbol{\xi}(\mathbf{r},t)$, in a finite-dimensional representation with eigenvalues that are real for the components of $\mathbf{j}$, and imaginary for the components of $\boldsymbol{\xi}$ (as for the field in 1+1 dimensions). The translational symmetry giving the dependence of the vectors $\mathbf{j}$ and $\boldsymbol{\xi}$ on the spacetime point is exhibited by Eqs. (67) and (66), for the x components.

In the nonrelativistic regime, where the quantum is viewed with a spatial resolution that is low on the scale of k_0^{-1} (see Sec. IV.C), the resulting latitude in a component ξ_x is large compared to unity [see Eq. (69)]. Hence, in this regime, the imaginary Lorentz symmetry, if of order unity, can be ignored, and the quantum is produced by the field in a state with $\boldsymbol{\xi}(\mathbf{r},t)\approx 0$, as in Eq. (70). In this manner, the emergence of the position variable is accompanied by the fading away of the imaginary Lorentz symmetry. [Moreover, the components of the vector $\boldsymbol{\xi}$, and thereby of $\mathbf{r}(t)$, become commutable.]

[33]A field with real Lorentz symmetry is not Hermitian, and the commutator of the field with itself, carrying the symmetry 2ξ, does not vanish at spacelike distances. The nonlocality of the field is clearly seen in the nonrelativistic limit, where a change of ξ is equivalent to a shift of the position of the quantum [see Eq. (70)]. The field ϕ at x, therefore, creates quanta at the position $x+k_0^{-1}\xi$ and annihilates quanta at $x-k_0^{-1}\xi$.

In contrast, the latitude in a component j_x resulting from the limited spatial resolution can be small compared to unity, for $|\mathbf{k}| \ll k_0$ [see Eq. (67)]. In nonrelativistic quantum mechanics, the intrinsic rotation $\mathbf{s}=\mathbf{j}(\mathbf{r}=0)-\mathbf{r}(t)\times\mathbf{k}$, therefore, clearly separates from the orbital part $\mathbf{l}=\mathbf{r}(t)\times\mathbf{k}$, where $\mathbf{r}(t)$ is the position variable defined in terms of the spacetime rotations. With the further loss of resolution that leads to the classical regime (see Sec. IV.C.4 and the supplementary Sec. IV.E), the latitude in the orbital angular momentum $\hbar\mathbf{l}$ becomes large compared with $\hbar$, and the intrinsic rotational symmetry is no longer visible.

VI. SUMMARIZING REMARKS

The role of symmetry in relation to quantal physics is turned upside down. The starting point for quantal physics is thus no longer an assumed symbolic formalism with states and operators in Hilbert space, into which symmetry is incorporated. Instead, the coordinate transformations of spacetime invariance are recognized as the elementary variables, and quantal physics itself emerges as the primary manifestation of symmetry, with no substance to be quantized.

The symmetry variables define their own relationships as irreducible matrix representations of a group and are shown to be inherently indeterminate. Individual events are, therefore, fortuitous (in an absolute sense), and completeness is not an issue.

The probabilistic laws, which are invoked in the interpretation of traditional quantum mechanics, are derived from geometric constraints among the symmetry variables. The probability distributions constitute correlations that distinguish these generalized variables.

The state vector offers itself as an expedient tool for describing the state of affairs of the symmetry matrices. Hence, state and wave function require no interpretation. Accordingly, a measurement of a symmetry variable is a unitary time evolution under specialized conditions.

It is crucial for establishing quantal physics as the primary manifestation of symmetry that space and time are linked together (by relativistic invariance). Even nonrelativistic quantum mechanics hinges on this link.

The spacetime symmetry variables comprise Poincaré invariance including spatial reflections. Time reversal is not a symmetry variable, but the neutrality of geometry under complex conjugation leads to reversibility of motion.

Quanta, as a direct implication of relativistic invariance, become the foremost manifestation of flat spacetime. It remains to be seen how the primary manifestation of symmetry and curved spacetime can coexist.

ACKNOWLEDGMENTS

We have had the benefit of numerous illuminating discussions with many colleagues, visitors, and students at the Niels Bohr Institute and Nordita over more than a decade during which the ideas presented in the paper were developed. In particular, the continuing dialogue with Ben R. Mottelson, who has followed and commented on the development of the new concepts, has been an inspiration and encouragement. We are greatly indebted to Holger Bech Nielsen, who has given so generously of his time to critically probe the main issues. His penetrating and constructive criticism has been a major challenge for us.

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