Gauge invariance and current algebra in nonrelativistic many-body theory

Jürg Fröhlich and Urban M. Studer*

Institut für Theoretische Physik, ETH-Hönggerberg, 8093 Zürich, Switzerland

The main purpose of this paper is to further our theoretical understanding of the fractional quantum Hall effect, in particular of spin effects, in two-dimensional incompressible electron fluids subject to a strong, transverse magnetic field. As a prerequisite for an analysis of the quantum Hall effect, the authors develop a general formulation of the many-body theory of spinning particles coupled to external electromagnetic fields and moving through a general, geometrically nontrivial background. Their formulation is based on a Lagrangian path-integral quantization and is valid in arbitrary coordinates, including coordinates moving according to a volume-preserving flow. It is found that nonrelativistic quantum theory exhibits a fundamental, local $U(1)\times SU(2)$ gauge invariance, and the corresponding gauge fields are identified with physical, external fields. To illustrate the usefulness of their formalism, the authors prove a general form of the quantum-mechanical Larmor theorem and discuss some well-known effects, including the Barnett-Einstein-de Haas effect and superconductivity, emphasizing the implications of $U(1)\times SU(2)$ gauge invariance. They then consider two-dimensional, incompressible quantum fluids in more detail. Exploiting $U(1)\times SU(2)$ gauge invariance, they calculate the leading terms in the effective actions of such systems as functionals of the U(1) and SU(2) gauge fields, on large-distance and low-frequency scales. Among the applications of these results are a simple proof of the Goldstone theorem for spin waves and the linear-response theory of two-dimensional, incompressible Hall fluids, including a Hall effect for spin currents and sum rules for the response coefficients. For two-dimensional, incompressible systems with broken parity and time-reversal symmetry, a particularly significant implication of $U(1)\times SU(2)$ gauge invariance is a duality between the physics inside the bulk of the system and the physics of gapless, chiral modes propagating along the boundary of the system. These modes form chiral $\hat{u}(1)$ and $\widehat{su}(2)$ current algebras. The representation theory of these current algebras, combined with natural physical constraints, permits one to derive the quantization of the response coefficients, such as the Hall conductivity. A classification of incompressible Hall fluids is outlined, and many examples, including one concerning a superfluid ^{3}He-A/B interface, are discussed.

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*Present address: Instituut voor Theoretische Fysica, Katholieke Universiteit Leuven, 3001 Leuven, Belgium.

I. INTRODUCTION

This paper has its origin in attempts to arrive at a better theoretical understanding of the integer and fractional quantum Hall effects. These truly remarkable effects, found in two-dimensional interacting electron gases subject to a strong, transverse, external magnetic field, were discovered by von Klitzing, Dorda, and Pepper (1980) and by Tsui, Stormer, and Gossard (1982), respectively. Soon thereafter, Laughlin (1981, 1983a, 1983b; see also Halperin, 1982), in his pioneering theoretical work, realized that the following two properties of such systems play a fundamental role:

First, in a fixed, external magnetic field, a two-dimensional interacting electron gas exhibits incompressibility at special values of its electron density (filling fac-

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tor). Put differently, at such values of the filling factor there are no dissipative processes in the system and hence its longitudinal resistance vanishes. (This observation has led to the term "incompressible quantum Hall fluid" for such a system.)

Second, nonrelativistic quantum mechanics exhibits a fundamental U(1) gauge invariance connected to electromagnetism, as recognized by Weyl already in 1928 (see also Weyl, 1918).

These two properties form the cornerstones of our investigation of the quantum Hall effect. In the first works of Laughlin (1981, 1983a, 1983b, 1984, 1990), Halperin (1982, 1983, 1984), Haldane (1983, 1990a), and others (Arovas, Schrieffer, and Wilczek, 1984; Trugman and Kivelson, 1985), the spins of the electrons in an incompressible quantum Hall fluid were neglected since, for example, in the lowest Landau level, the Zeeman energy is minimized by aligning all spins in the direction opposite to the external magnetic field. As early as in 1983, however, Halperin argued that, for larger electron densities, spin effects can be important. Further evidence for spin effects has been found in recent experiments (Willett *et al.*, 1987; Eisenstein *et al.*, 1988; Clark, Haynes *et al.*, 1989, 1990; Eisenstein, Willett *et al.*, 1990; Eisenstein, Stormer *et al.*, 1990a, 1990b; see also Haug *et al.*, 1987; Syphers and Furneaux, 1988a, 1988b) and in numerical studies (e.g., Chakraborty and Zhang, 1984a, 1984b; Rasolt, Perrot, and MacDonald, 1985; Yoshioka, 1986b; Maksym, 1989; and references therein). This has led to one of the questions motivating the present work: What is the most general form of gauge invariance in nonrelativistic quantum many-body systems composed of particles with spin? What are characteristic properties of such systems resulting from their spin degrees of freedom?

Our answer to the first part of this question is discussed in Secs. II and III. In Sec. II, we find a $U(1)_{\mathrm{em}}\times SU(2)_{\mathrm{spin}}$ gauge invariance of nonrelativistic, one-particle quantum mechanics based on the Pauli equation. The U(1) and SU(2) gauge transformations act on the wave function (two-component Pauli spinor) of a particle by local phase transformations and local spin rotations, respectively. Related observations have been made by Anandan (1989, 1990).

In Sec. III, extending the findings of Sec. II, we develop a general framework for describing, in arbitrary (but volume-preserving) moving coordinates, many-body systems of spinning particles coupled to external electromagnetic fields and constrained to move in some geometrically nontrivial background. This framework is based on a discussion of the geometry of the background (see Sec. III.A) and on the second-quantized Lagrangian formalism convenient for the description of many-body systems (see Sec. III.B). We show that, in arbitrary coordinates, the action functional governing such systems exhibits a fundamental $U(1)\times SU(2)$ gauge invariance. We then present a systematic identification of the associated gauge fields with physical quantities. We find that the U(1) gauge field is given in terms of the electromagnetic scalar and vector potential and the velocity field generating the moving coordinates. The SU(2) gauge field consists of terms describing spin-orbit interactions, Thomas precession, and the coupling of the spin degrees of freedom to the geometry and to the vorticity of the velocity field generating the moving coordinates. At the end of Sec. III.C, we formulate and prove a general quantum-mechanical version (including spin degrees of freedom) of the Larmor theorem.

In Sec. IV, we describe some well-known effects in quantum mechanics from the point of view of its $U(1)\times SU(2)$ gauge invariance. Examples are different realizations of the Aharonov-Bohm and the Aharonov-Casher effects, the Barnett and Einstein–de Haas effects, and the London and Landau-Ginzburg theories of superconductivity. In Sec. IV.E, we present a brief introduction to the quantum Hall effect, summarizing basic facts and some experimental data and illustrating the significance of U(1) gauge invariance. In particular, we describe an intimate connection between U(1) gauge invariance and the existence of chiral electric edge currents in two-dimensional, incompressible quantum Hall fluids, a connection that is basic for the analysis presented in Sec. VI.

In Sec. V, we propose a precise formulation of incompressibility in general two-dimensional quantum fluids in terms of clustering properties of their connected (time-ordered) current Green functions. Assuming incompressibility and exploiting the $U(1)\times SU(2)$ gauge invariance of nonrelativistic quantum mechanics in the form of Ward identities, we then calculate the "scaling limit" of the effective action (i.e., of the logarithm of the partition function) of a two-dimensional, incompressible system, as a functional of the external $U(1)\times SU(2)$ gauge fields. By "scaling limit" we mean that only those terms in the effective action are retained which are relevant for physics at large-distance scales and low frequencies. The technical details of our calculations are presented in Appendix A.

As applications of our results we find a simple proof of the Goldstone theorem for spin waves, the linear-response theory of general two-dimensional, incompressible quantum fluids (including a Hall effect for spin currents), and sum rules for linear-response coefficients such as the Hall conductivity for the electric current, or the magnetic susceptibility; see Sec. V.B. Moreover, in Sec. V.C, we discuss some aspects of the theory of chiral spin liquids and propose a mechanism for spin-singlet electron pairing in an antiferromagnetic or a resonating valence-bond background.

For two-dimensional, incompressible quantum fluids, or more generally, for two-dimensional systems exhibiting a strong form of parity and time-reversal symmetry breaking, a particularly powerful implication of $U(1)\times SU(2)$ gauge invariance is a form of "boundary-bulk duality" implying that many bulk properties of such systems are in one-to-one correspondence with properties of their boundary excitations. The tool instrumental in establishing this duality is an analysis of (gauge) anomaly cancellation.

In Sec. VI, we first derive and then apply this duality. It naturally introduces chiral $\hat{\mathrm{u}}(1)$ and $\widehat{\mathrm{su}}(2)$ current (Kac-Moody) algebra into the discussion of Hall fluids. Physically, these current algebras describe gapless, chiral electric and spin currents circulating at the edges of the systems. Combining the results of the representation theory of current algebras with some basic physical properties of Hall fluids, we derive the (integer or fractional) quantization of the linear-response coefficients found in Sec. V.B and provide a list of quantum numbers [(fractional) charges and (anyonic) statistical phases] of possible finite-energy excitations (quasiparticles) in such systems. We construct the theoretical basis for a classification of incompressible quantum fluids in terms of universality classes. We wish to emphasize that our analysis allows for an understanding of the integer and fractional quantum Hall effect on the same footing. Many examples of Hall fluids are discussed. Some mathematical details used in this section are derived in Appendix B.

Work resembling that presented in Sec. VI has also been carried out by Wen and collaborators (Wen, 1989, 1990a, 1990b, 1990c, 1991a, 1991b; Block and Wen, 1990a, 1990b; Wen and Niu, 1990), by Stone (1991a, 1991b) and others (Büttiker, 1988; Beenakker, 1990; MacDonald, 1990; Haldane, 1990b; Balatsky and Fradkin, 1991); Balatsky and Stone, 1991; Balatsky 1992), and by Fröhlich and Kerler (1991) and Fröhlich and Zee (1991).

In the last section, Sec. VII, we apply the results of Sec. VI to a detailed discussion of Hall fluids with Hall conductivities of the form $\sigma_H=[2/(4l+1)](e^2/h)$, $l=0,1,2$. We show that these Hall fluids are good candidates for observing a quantum Hall effect for spin currents. Finally, in Sec. VII.B, the same methods are applied to the study of a particular type of superfluid $^3\mathrm{He}$-A/B interface with (strongly) broken parity and time-reversal invariance. We are led to propose an even-denominator quantum Hall effect for this system.

We feel that the perspective offered in this paper, emphasizing the $U(1)\times SU(2)$ gauge invariance of quantum mechanics, is somewhat novel. Furthermore, some of the results in Secs. VI and VII are new and have not previously been published.

II. THE PAULI EQUATION AND ITS SYMMETRIES

In this section we describe nonrelativistic electrons and other nonrelativistic spinning particles in an external electromagnetic field. In a one-particle language, the wave functions of these particles satisfy the Pauli equation. We show that the Pauli equation exhibits a basic $U(1)_{\mathrm{em}}\times SU(2)_{\mathrm{spin}}$ gauge symmetry. This symmetry is a cornerstone of our subsequent analysis of incompressible quantum fluids. In order to provide a first illustration of the general usefulness of this symmetry in quantum theory we include, at the end of this section, brief discussions of the Bloch spin resonance, the Aharonov-Bohm effect, and its $SU(2)_{\mathrm{spin}}$ cousin, the Aharonov-Casher effect. Further applications to basic quantum-mechanical effects will be discussed in Sec. IV.

A. Gauge-invariant form of the Pauli equation

We first consider the Dirac equation in three space dimensions. The Dirac spinor Ψ, describing a relativistic electron/positron in an external electromagnetic field, with electromagnetic potentials Φ and $\mathbf{A}=(A_1,A_2,A_3)$, satisfies the Dirac equation

$$i\hbar\frac{\partial}{\partial t}\Psi=\left[c\boldsymbol{\alpha}\cdot\left[\frac{i}{\hbar}\nabla+\frac{e}{c}\mathbf{A}\right]+\beta m_0c^2-e\Phi\right]\Psi\ , \qquad (2.1)$$

where $-e$ and m_0 are the charge and (vacuum) mass of the electron, c is the velocity of light, $\hbar$ is Planck's quantum of action, and $\boldsymbol{\alpha}$ and β denote the usual four 4×4 Dirac matrices. Expanding the Dirac equation according to the Foldy-Wouthuysen scheme (Foldy and Wouthuysen, 1950; see also Bjorken and Drell, 1964, Hunziker, 1975, and Gesztesy, Grosse, and Thaller, 1983, 1984), we find the following equation for the 2-spinor (Pauli spinor) ψ of the 4-spinor $\Psi=\binom{\psi}{\chi}$, up to terms of fourth order in $1/m_0$:

$$i\hbar\frac{\partial}{\partial t}\psi=m_0c^2\psi-e\Phi\psi+\frac{e\hbar}{2m_0c}\mathbf{B}\cdot\boldsymbol{\sigma}\psi+\frac{1}{2m_0}\Pi^2\psi+\frac{e\hbar}{8m_0^2c^2}[\boldsymbol{\Pi}\cdot(\boldsymbol{\sigma}\times\mathbf{E})+(\boldsymbol{\sigma}\times\mathbf{E})\cdot\boldsymbol{\Pi}]\psi+\frac{e\hbar^2}{8m_0^2c^2}\mathrm{div}\mathbf{E}\psi$$
$$+\frac{e^2\hbar^2}{8m_0^3c^4}(\mathbf{E}^2-\mathbf{B}^2)\psi-\frac{1}{8m_0^3c^2}\left[\boldsymbol{\Pi}^4+\frac{e\hbar}{c}(\boldsymbol{\Pi}^2(\mathbf{B}\cdot\boldsymbol{\sigma})+(\mathbf{B}\cdot\boldsymbol{\sigma})\boldsymbol{\Pi}^2]\right]\psi+O(m_0^{-4})\psi\ , \qquad (2.2)$$

where the canonical momentum operator $\boldsymbol{\Pi}$ is defined by

$$\boldsymbol{\Pi}=\frac{\hbar}{i}\nabla+\frac{e}{c}\mathbf{A}\ . \qquad (2.3)$$

The 2-spinor ψ is the wave function of a low-energy electron. We recall the meaning of the different terms on the right-hand side (rhs) of Eq. (2.2). The first term is the rest energy of the electron, and the second term is its potential energy in the electrostatic potential Φ. The third term describes the Zeeman splitting in a magnetic field $\mathbf{B}=\mathrm{curl}\,\mathbf{A}$; $\boldsymbol{\sigma}$ is the vector formed by the three Pauli matrices. The fourth term is the kinetic energy of the electron in an electromagnetic field corresponding to the vector potential $\mathbf{A}$. The fifth term, abbreviated by $h_{\text{spin-orbit}}$, describes spin-orbit interactions in the electric field $\mathbf{E}=-\nabla\Phi-(1/c)(\partial/\partial t)\mathbf{A}$. We recall that, for static fields with a centrally symmetric potential Φ, we have that $e\mathbf{E}=\nabla\Phi=(\mathbf{r}/r)(\partial/\partial r)\Phi(r)$, and, introducing the spin operator $\mathbf{S}=(\hbar/2)\boldsymbol{\sigma}$, we find that

$$h_{\text{spin-orbit}} \propto \mathbf{S}\cdot(\mathbf{E}\times\mathbf{p}) \propto \frac{1}{r}\frac{\partial}{\partial r}\Phi(r)\mathbf{S}\cdot\mathbf{L}, \quad \text{with } \mathbf{p}=\frac{\hbar}{i}\nabla .$$

This is a more familiar form of the spin-orbit interaction in a centrally symmetric potential Φ. The orbital angular momentum operator is given by $\mathbf{L}=\mathbf{r}\times\mathbf{p}$.

The term proportional to div$\mathbf{E}$ is the so-called Darwin term, a higher relativistic correction proportional to the background charge density $\rho=\text{div}\mathbf{E}=-\Delta\Phi$. In the following we shall absorb the Darwin term in a one-body potential term. All the other terms are corrections of even higher order (we shall briefly comment on them below). Neglecting the rest-energy term (an additive constant) and all the terms of order $O(1/m_0^3)$, we find that Eq. (2.2) turns out to be the well-known Pauli equation for an electron in an electromagnetic field.

In order to describe nonrelativistic particles of arbitrary spin s, mass m, and charge q, we have to generalize Eq. (2.2). In particular, we have to find the correct Zeeman and spin-orbit terms. We recall that Bargmann, Michel, and Telegdi have found a relativistic description of the motion of a classical spin $\mathbf{S}$ in a (slowly varying) external electromagnetic field $(\mathbf{E},\mathbf{B})$. Expanding their result in powers of v/c, one obtains the equation of motion already found by Thomas (1927; see also Jackson, 1975),

$$\frac{d\mathbf{S}}{dt}=\frac{q}{mc}\mathbf{S}\times\left[\frac{g}{2}\mathbf{B}-\left[\frac{g}{2}-\frac{1}{2}\right]\frac{\mathbf{v}}{c}\times\mathbf{E}\right]+O\left[\left[\frac{v}{c}\right]^2\right], \tag{2.4}$$

where $\mathbf{v}$ is the velocity of the spinning particle (with respect to the laboratory frame) and g is its gyromagnetic ratio. We note that the spin-orbit term [second term in Eq. (2.4)] consists of two contributions: The terms proportional to $g/2$ describe the precession of the spin (or magnetic moment) in the magnetic field in the particle's rest frame. The remaining term describes the purely kinematical effect of the Thomas precession, which is a consequence of the acceleration a charged spinning particle experiences in an electric field. Recalling the Poisson bracket relations, $\{S_i,S_j\}=\varepsilon_{ijk}S_k$, for a classical spin $\mathbf{S}$, we find that Eq. (2.4) is a Hamiltonian equation of motion corresponding to the Hamilton function

$$H_{\text{cl}}=-\mathbf{S}\cdot\left[\frac{q}{mc}\frac{g}{2}\mathbf{B}-\frac{q}{mc}\left[\frac{g}{2}-\frac{1}{2}\right]\frac{\mathbf{v}}{c}\times\mathbf{E}\right]. \tag{2.5}$$

If we accept Eq. (2.4) as the appropriate nonrelativistic Heisenberg equation of motion for the (quantum-mechanical) spin operator $\mathbf{S}$ (in the spin-s representation), then the wave function $\psi^{(s)}$ of the particle, a $(2s+1)$-component complex spinor, satisfies the following Pauli equation:

$$i\hbar\frac{\partial}{\partial t}\psi^{(s)}=q\Phi\psi^{(s)}-\boldsymbol{\mu}_{\text{spin}}\cdot\mathbf{B}\psi^{(s)}+\frac{1}{2m}\boldsymbol{\Pi}^2\psi^{(s)} -\frac{1}{2mc}\left\{\boldsymbol{\Pi}\cdot\left[\left[\boldsymbol{\mu}_{\text{spin}}-\frac{q}{2mc}\mathbf{S}\right]\times\mathbf{E}\right] +\left[\left[\boldsymbol{\mu}_{\text{spin}}-\frac{q}{2mc}\mathbf{S}\right]\times\mathbf{E}\right]\cdot\boldsymbol{\Pi}\right\}\psi^{(s)}, \tag{2.6}$$

where the (intrinsic) magnetic moment of the particle is defined by

$$\boldsymbol{\mu}_{\text{spin}}=\frac{g\mu}{\hbar}\mathbf{S} \tag{2.7}$$

and, in the spin-s representation, the spin operator $\mathbf{S}$ is given by

$$\mathbf{S}=\frac{\hbar}{2}\mathbf{L}^{(s)}\equiv\frac{\hbar}{2}(L_1^{(s)},L_2^{(s)},L_3^{(s)}) . \tag{2.8}$$

Here, $(L_A^{(s)})_{A=1}^3$ are Hermitian generators of the Lie algebra su(2) in the spin-s representation, normalized such that $L_A^{(1/2)}=\sigma_A$, where σ_1, σ_2, and σ_3 are the usual Pauli matrices. Furthermore, for charged particles, $\mu=q\hbar/2mc$. In particular, for the electron, $-\mu=e\hbar/2m_0c\equiv\mu_B=5.79\times10^{-9}$ eV/G, the Bohr magneton, and Eq. (2.6) agrees with Eq. (2.2) if we set $g=2$. This is a celebrated prediction of the Dirac theory. Other examples are the neutron and the proton, where $\mu=e\hbar/2mc$ with m the corresponding mass, and the g factors are given by $g=5.59$ and $g=-3.83$, respectively.

Next we show that by "completing the square" in the Pauli equation (2.6) we obtain an equation with an astonishingly rich symmetry, namely, with a local $U(1)_{\text{em}}\times SU(2)_{\text{spin}}$ symmetry. Since the modification needed is a term of order $O(1/m^3)$, this symmetry should really be viewed as a fundamental property of nonrelativistic quantum mechanics. [This is borne out by its Galilei invariance (Piron, 1990).]

Let $x^0=ct$ and $x=(x^\mu)=(x^0,\mathbf{x})$, where $\mathbf{x}=(x^1,x^2,x^3)\in\mathbb{E}^3$ (the three-dimensional Euclidean space). We introduce the covariant derivative in the μ direction by setting

$$D_\mu=\frac{\partial}{\partial x^\mu}+ia_\mu(x)+\rho_\mu(x), \quad \mu=0,\ldots,3 , \tag{2.9}$$

where the real-valued functions a_μ are given by

$$a_0(x)=\frac{q}{\hbar c}\Phi(x) \quad \text{and} \quad a_k(x)=-\frac{q}{\hbar c}A_k(x) , \quad k=1,2,3 , \tag{2.10}$$

and where the su(2)-valued functions ρ_μ are defined by

$$\rho_\mu(x)=i\sum_{A=1}^{3}\rho_{\mu A}(x)L_A^{(s)}, \quad \mu=0,\ldots,3 . \tag{2.11}$$

The coefficients $\rho_{\mu A}$ are given by

$$\rho_{0A}(x) = -\frac{g\mu}{2\hbar c} B_A(x), \quad A = 1,2,3 , \tag{2.12}$$

and

$$\rho_{kA}(x) = \left[-\frac{g\mu}{2\hbar c} + \frac{q}{4mc^2} \right] \sum_{B=1}^{3} \varepsilon_{kAB} E_B(x) , \quad A,k = 1,2,3 , \tag{2.13}$$

where ε_{kAB} is the sign of the permutation (kAB) of (123).

With the help of the covariant derivative D_μ, we are able to write the Pauli equation (2.6) in the compact form

$$i\hbar c D_0 \psi^{(s)}(x) = -\frac{\hbar^2}{2m} \sum_{k=1}^{3} D_k D_k \psi^{(s)}(x) , \tag{2.14}$$

where a term of order $O(\rho^2)$ has been added. For an electron it can be seen to be equal to half of the term $e^2\hbar^2/(8m_0^3c^4)\mathbf{E}^2\psi$ in Eq. (2.2), which can be absorbed into a one-body potential acting on ψ.

The form (2.14) of the Pauli equation shows that nonrelativistic quantum mechanics has a basic $U(1)_{em} \times SU(2)_{spin}$ gauge symmetry. The gauge transformations are defined as follows:

$$\begin{aligned} U(1)_{em}:\ & a_\mu(x) \mapsto {}^\chi a_\mu(x) = a_\mu(x) + (\partial_\mu \chi)(x) , \\ & \psi^{(s)}(x) \mapsto {}^\chi \psi^{(s)}(x) = e^{-i\chi(x)} \psi^{(s)}(x) , \end{aligned} \tag{2.15}$$

where χ is an arbitrary, real-valued function on space-time $\mathbb{R} \times \mathbb{E}^3$, and

$$\begin{aligned} SU(2)_{spin}:\ & \rho_\mu(x) \mapsto {}^g\rho_\mu(x) = g(x)\rho_\mu(x) g^{-1}(x) \\ & \qquad + g(x)(\partial_\mu g^{-1})(x) , \\ & \psi^{(s)}(x) \mapsto {}^g\psi^{(s)}(x) = g(x)\psi^{(s)}(x) , \end{aligned} \tag{2.16}$$

where g is (the spin-s representation of) an arbitrary SU(2)-valued function on $\mathbb{R} \times \mathbb{E}^3$. Note that, for constant gauge transformations g, ρ_μ transforms according to the adjoint action of SU(2) (on its Lie algebra su(2)) which, by (2.12) and (2.14) (in an active interpretation) corresponds to global rotations of the vector fields **E** and **B** in physical space. For space-time-dependent gauge transformations, there appears an additional inhomogeneous term in (2.16). A full geometrical interpretation of the SU(2) gauge symmetry will be given in the next section. See also the work by Anandan (1989, 1990) for related observations.

We note that Eq. (2.14) can be thought of as the Euler-Lagrange equation corresponding to the following $U(1) \times SU(2)$ gauge-invariant action functional:

$$\begin{aligned} S(\psi^{(s)*}, \psi^{(s)}; a, \rho) &= \int dt\, d^3\mathbf{x} \left[i\hbar c \psi^{(s)*}(x)(D_0\psi^{(s)})(x) \right. \\ & \left. - \frac{\hbar^2}{2m} \sum_{k=1}^{3} (D_k\psi^{(s)})^*(x)(D_k\psi^{(s)})(x) \right] . \end{aligned} \tag{2.17}$$

This action functional and generalizations thereof provide a convenient starting point for a functional-integral formulation of nonrelativistic many-body theory.

We propose to illustrate the formalism described so far by reviewing some basic effects in nonrelativistic quantum mechanics from the point of view of its $U(1) \times SU(2)$ gauge symmetry.

B. Bloch spin resonance

The Bloch spin resonance is an effect caused by the Zeeman term in the Pauli equation. We consider a particle of spin s and magnetic moment $\mu_{spin} \neq 0$ [see Eq. (2.7)], in the external magnetic field

$$\mathbf{B} = \mathbf{B}(t) = (B_1 \cos\omega t, -B_1 \sin\omega t, B_0) , \tag{2.18}$$

which is a superposition of a constant background field B_0 in the z direction and a rotating radio-frequency field in the (x,y) plane, with $B_1 \ll B_0$. From Eqs. (2.12) and (2.13) we find that $\rho_{kA} = 0$ and $\rho_{0A} = -(g\mu/2\hbar c)B_A$. With the help of a purely t-dependent SU(2) gauge transformation g, we can achieve ${}^g\rho_0 = g\rho_0 g^{-1} + g\partial_0 g^{-1} \equiv 0$, with decoupled components of ${}^g\psi^{(s)} = g\psi^{(s)}$, each satisfying the *same* one-dimensional Schrödinger equation for a spinless particle in the external magnetic field (2.18). The appropriate gauge transformation g is given by

$$\begin{aligned} g^{-1}(t) &= T \exp \left[i \frac{g\mu}{2\hbar} \int_0^t d\tau (B_1 \cos\omega\tau L_1^{(s)} - B_1 \sin\omega\tau L_2^{(s)} + B_0 L_3^{(s)}) \right] \\ &= \exp\left[i L_3^{(s)} \omega \frac{t}{2} \right] \exp\left[i [L_3^{(s)}(\omega_0 - \omega) + L_1^{(s)}\omega_1] \frac{t}{2} \right] , \end{aligned}$$

where "T exp" denotes a time-ordered exponential, and we have introduced the frequencies $\omega_i = (g\mu/\hbar)B_i$, $i = 0,1$.

If we consider a spin-$\frac{1}{2}$ particle and assume its 2-spinor ${}^g\psi$ to be proportional to $\psi_{z\uparrow} = \binom{1}{0}$ at time $t = 0$, then the probability for the spin to be flipped at time t is given by

$$P_\downarrow(t) = |(\psi_{z\downarrow}, g^{-1}(t)\psi_{z\uparrow})|^2 = \frac{\omega_1^2}{2\Omega^2}(1 - \cos\Omega t) ,$$

where $\Omega = [(\omega_0 - \omega)^2 + \omega_1^2]^{1/2}$. Looking at the maximum curve of $P_\downarrow(t)$, we find the well-known resonance behavior for $\omega = \omega_0$. On resonance, the spin flips with certainty, absorbing or emitting an energy quantum $\hbar\omega_0$ from or to the radio-frequency field, periodically in time.

C. Aharonov-Bohm effect

A key effect reflecting Weyl's $U(1)_{em}$ gauge principle realized in quantum theory is the Aharonov-Bohm effect (Aharonov and Bohm, 1959): Consider the scattering of quantum-mechanical particles at a magnetic solenoid.

(The wave functions of the particles are required to vanish inside the solenoid.) Then, the diffraction pattern seen on a screen depends nontrivially on the magnetic flux Φ through the solenoid. The dependence is periodic with period hc/q, where q is the charge of the particles. This is a consequence of the fact that the vector potential **A** outside the solenoid cannot be gauged away globally, in spite of the fact that there is no electromagnetic field, thus leading to nonintegrable U(1) phases of quantum-mechanical wave functions which change interference patterns.

In formulas, we have $F_{\mu\nu}=\partial_\mu a_\nu-\partial_\nu a_\mu\equiv 0$ outside the solenoid. Thus, locally, $a_\mu=\partial_\mu\chi$, with $\chi(\mathbf{x})=(-q/\hbar c)\int_*^{\mathbf{x}}\mathbf{A}\cdot dl$, where dl denotes the line element along some path of integration from an arbitrary point $*$ to $\mathbf{x}$. The phase factors affecting the interference patterns are then given by $\exp[2\pi i(q/hc)\oint_\Gamma\mathbf{A}\cdot dl]$ $=\exp[2\pi i(q\Phi/hc)]$, where Γ is a closed path enclosing the solenoid.

We note that the Aharonov-Bohm effect explains the possibility of fractional (or θ or Abelian braid) statistics of anyons (Leinaas and Myrheim, 1977; Goldin, Menikoff, and Sharp, 1980, 1981, 1983; Wilczek, 1982a, 1982b; for a review, see Fröhlich, 1990) in two-dimensional systems. Anyons are particles carrying both electric charge q and magnetic flux Φ ($=\sigma_H^{-1}q$, where σ_H is a "Hall conductivity") and hence give rise to Aharonov-Bohm phases, which one can interpret as statistical phases; see Sec. V.C.

D. Aharonov-Casher effect

One might wonder whether there is a similar interference effect due to the $SU(2)_{\text{spin}}$ gauge symmetry of nonrelativistic quantum mechanics. The answer is yes: It is the Aharonov-Casher effect (Aharonov and Casher, 1984). Consider a system of quantum-mechanical particles with spin s, electric charge 0, but with a magnetic moment $\mu_{\text{spin}}\neq 0$, moving in a plane or in three-dimensional space. (The particles could be neutrons, or neutral atoms, etc.) Following Aharonov and Casher, we want to study the influence of a (static) external electric field on the dynamics of such particles. As a consequence of relativistic effects, the moving particles will, in their rest frame, feel a magnetic field that interacts with their magnetic moment. Up to order $O(v/c)$ this is taken into account by the spin-orbit term in the Pauli equation (2.6); see also Eq. (2.4).

In the formalism developed above, this effect should be described as follows: The SU(2) gauge potential ρ_μ is defined in Eqs. (2.11)–(2.13), and we find that

$$\rho_{0A}(\mathbf{x})=0 \quad\text{and}\quad \rho_{kA}(\mathbf{x})=-\frac{g\mu}{2\hbar c}\sum_{B=1}^{3}\varepsilon_{kAB}E_B(\mathbf{x}) , \qquad A,k=1,2,3 . \tag{2.19}$$

For general electric fields, the SU(2) curvature, defined by

$$G_{\mu\nu}^{A}=\partial_\mu\rho_{\nu A}-\partial_\nu\rho_{\mu A}-2\sum_{B,C=1}^{3}\varepsilon_{ABC}\rho_{\mu B}\rho_{\nu C} , \qquad A=1,2,3 \text{ and } \mu,\nu=0,\ldots,3 ,$$

does not vanish on full-measure sets of space, and so we are not surprised to find that the electric field **E** causes nontrivial spin-orbit interactions. However, if we consider a system of particles confined to the (x,y) plane in $\mathbb{E}^3$ which move in the electric field of a charged wire placed along the z axis, with constant charge Q per unit of length, we encounter an SU(2) version of the Aharonov-Bohm effect: Here, the electric field **E** is given by $\mathbf{E}(\mathbf{x})=(Q/2\pi r^2)(x,y,0)$, where $r=(x^2+y^2)^{1/2}$. With Eq. (2.19), we find

$$\rho(\mathbf{x})\equiv(\rho_{13}(\mathbf{x}),\rho_{23}(\mathbf{x}))=\frac{g\mu Q}{4\pi\hbar c r^2}(y,-x) , \tag{2.20}$$

and $\rho_{i1}=\rho_{i2}\equiv 0$ for $i=1,2$. Note that ρ_{3A}—which does *not* vanish for $A=1,2$—does not enter the dynamics of a system confined to the (x,y) plane. One then easily checks that, for a two-dimensional system confined to the (x,y) plane, the only component of the SU(2) curvature that does not vanish identically is given by

$$G_{12}^{3}(\mathbf{x})=-\frac{g\mu}{2\hbar c}Q\delta(\mathbf{x}) .$$

The function G_{12}^3 is supported at the origin, i.e., the SU(2) connection ρ is "flat" outside the wire. Thus, locally, it is possible to write ρ as a pure gauge, i.e., $\rho_k=g\partial_k g^{-1}$, with $g=\exp[-i\int_*^{\mathbf{x}}\rho\cdot dl\, L_3^{(s)}]$, where dl is as above. However, the scattering of the particles at the charged wire depends on its charge per unit length, Q, because, although ρ is flat except at the origin, it cannot be gauged away *globally.* Therefore ρ gives rise to "nonintegrable SU(2) phase factors" in the wave functions of the particles which affect their interference patterns. These phase factors are given by $\exp[i\oint_\Gamma\rho\cdot dl]=\exp[2\pi i(g\mu/2hc)Q]$, where Γ is a path enclosing the wire, and the patterns are periodic in Q with a period given by $2hc/g\mu$.

This effect was first described by Aharonov and Casher (1984) in a somewhat more classical language. A general discussion of this effect, much along the lines of thought sketched above, can be found in Anandan (1989, 1990).[1]

We recall that the Aharonov-Bohm effect explains why two-dimensional quantum theory can describe anyons with fractional statistics, namely, particles carrying charge and flux. It is natural to ask whether the Aharonov-Casher effect also has something to do with exotic statistics in two-dimensional quantum theory. The answer is yes! The Aharonov-Casher effect is closely related to the existence of particles in two-dimensional

[1]We thank L. Stodolsky for bringing this work to our attention.

quantum theory with non-Abelian braid statistics (Fredenhagen, Rehren, and Schroer, 1989; Fröhlich and Gabbiani, 1990; Fröhlich, Gabbiani, and Marchetti, 1990; Fröhlich and Marchetti, 1991). Such particles have topological interactions that can be described by some SU(2) Knizhnik-Zamolodchikov connection (Knizhnik and Zamolodchikov, 1984; Tsuchiya and Kanie, 1987). Consider, for example, a two-dimensional chiral spin liquid made of particles with spin $s_0 \geq 1$ (if such systems exist). An incompressible chiral spin liquid of this type will most likely exhibit excitations of arbitrary spin $s=\frac{1}{2},\ldots,s_0$. The claim is that an excitation of nonzero spin $s<s_0$ exhibits non-Abelian braid statistics, as pointed out by Zhang, Hansson, and Kivelson (1989) and Fröhlich, Kerler, and Marchetti (1991). This will be discussed further in Sec. V.C.

III. GAUGE INVARIANCE IN NONRELATIVISTIC QUANTUM MANY-PARTICLE SYSTEMS

In this section we build upon, and generalize, the formalism outlined in the preceding section. It is our aim to describe nonrelativistic quantum-mechanical systems in (one), two, and three space dimensions, composed of particles with arbitrary spin coupled to external electromagnetic fields, variable background metrics, and affine spin connections on spaces of nonvanishing curvature and torsion and "tidal" gauge fields.

Considering quantum mechanics in the presence of (strong) gravitational fields, then, in geometrical terms, the energy-mass distribution of the background gives rise to curvature (torsion is assumed to vanish in gravity). Torsion and curvature can also provide an effective description of crystalline backgrounds with dislocations and disclinations. Such a geometric description of the background is reasonable, provided the energy of the moving particles is so small that the lattice structure of the background cannot be resolved, and the background may be treated as a "smooth" manifold, i.e., provided the typical wavelength of the particles is much larger than the crystal lattice spacing. Furthermore, nontrivial background metrics can account for off-diagonal disorder in the systems and for a variable effective mass. The "tidal" gauge fields allow for a quantum-mechanical description of Coriolis forces and spin precession in moving coordinates.

We begin this section by reviewing a geometrical framework that is well suited to describe all these phenomena (for more mathematical background, see, for example, Eguchi, Gilkey, and Hanson, 1980; Bleecker, 1981; and de Rham, 1984; for a brief summary of basic notions in differential geometry, see also Sec. 2 in Alvarez-Gaumé and Ginsparg, 1985). Apart from describing possible physical effects related to curvature and torsion, the purpose of the general formalism developed here is to elucidate the geometrical meaning and origin of the $U(1)_{em}\times SU(2)_{spin}$ gauge invariance of nonrelativistic quantum mechanics.

Since we are interested in time-dependent many-body systems, it will be convenient to work in a Feynman-Berezin path-integral formalism (see, for example, Negele and Orland, 1987; Fradkin, 1991; and Feldman, Knörrer, and Trubowitz, 1992). We show that the action functionals governing such systems are $U(1)\times SU(2)$ gauge invariant. This gives rise to powerful Ward identities, which will be central in our subsequent treatment of incompressible quantum fluids in two dimensions and their generalized Hall effects. At the end of this section we present a quantum-mechanical version of Larmor's theorem, including spin degrees of freedom. Further applications of the general formalism developed in this section to basic effects in quantum many-body theory will be given in Sec. IV.

A. Differential geometry of the background

For an easy reading of this subsection and the beginning of the next one, the reader is expected to be somewhat familiar with basic notions of differential geometry. We wish to emphasize, however, that, in later sections, these notions will not be used. As stated above, the main reasons for introducing the following geometrical framework are an elucidation of the geometrical meaning of the $U(1)_{em}\times SU(2)_{spin}$ gauge invariance of nonrelativistic quantum mechanics and a preparation for treating such systems in "moving coordinates"; see Sec. III.C.

Under the condition of low energy described above, physical space is a ($d=2$ or 3)-dimensional manifold M, with a possibly time-dependent Riemannian metric, and space-time is given by $N=\mathbb{R}\times M$. The system is confined to the interior of a space-time cylinder $\Lambda\subset N$. The intersection of Λ with a fixed-time slice is denoted by Ω_t, where t is time. In local coordinates, points in M are denoted by $\mathbf{x},\mathbf{y},\ldots$ points in N by $x=(t,\mathbf{x})$, $y=(t,\mathbf{y}),\ldots$. The Riemannian metric on M is denoted by $g_{ij}(t,\mathbf{x})$ and space-time N carries the "Lorentzian" metric $\eta_{\mu\nu}(x)$, where $\eta_{00}(x)=1$, $\eta_{0i}(x)=\eta_{i0}(x)=0$, $\eta_{ij}(x)=-g_{ij}(t,\mathbf{x})$, where the indices range over $i,j=1,\ldots,d$ and $\mu,\nu=0,\ldots,d$. In the tangent space at a point $\mathbf{x}\in M$ we also have the flat Cartesian metric δ_{AB}, with $A,B=1,\ldots,d$. [Similarly, in the tangent space at a space-time point $x\in N$ we have the usual "Minkowskian" metric $\eta^0_{\alpha\beta}$, with $\alpha,\beta=0,\ldots,d$.] If the dimension of M is two, we imagine that M is a surface embedded in a three-dimensional Riemannian manifold L with metric also denoted by $g_{ij}(t,\mathbf{x})$, and the metric on M is the induced one.

Since, in nonrelativistic quantum mechanics, time is merely a parameter, we temporarily omit it from our notations and focus our attention on the description of the "spatial" geometry of M or L, respectively. In order to be able to describe particles of arbitrary spin $s=0,\frac{1}{2},1,\ldots$ moving in M, it is necessary to make use of the (co)tangent-frame or dreibein formalism. This formalism, involving local bases in the (co)tangent spaces (orthonormal frames), naturally incorporates two local symmetry groups: the group of coordinate reparametrizations of the manifold (diffeomorphisms) and the group of local frame rotations (SO(3) gauge transformations). Wave functions of particles of half-integral spin will transform under spinor representations of the frame rotation group.

In the cotangent bundle to L, $T^*(L)$, we choose (smooth) sections of 1-forms, $(e^A)_{A=1}^3$, with the property that they form an orthonormal basis (or orthonormal frame) in each cotangent space $T^*_{\mathbf{x}}(L)$, $\mathbf{x}\in L$. The components of the orthonormal frame $(e^A(\mathbf{x}))_{A=1}^3$ in the coordinate basis $(dx^j)_{j=1}^3$ of $T^*_{\mathbf{x}}(L)$ are denoted by $e_i^A(\mathbf{x})$ and are called dreibein (fields). If $\dim M=2$ we choose $(e^A(\mathbf{x}))_{A=1}^3$ such that, for $\mathbf{x}\in M\subset L$, $e^3(\mathbf{x})$ is orthogonal to $T^*_{\mathbf{x}}(M)$ in the metric of $T^*_{\mathbf{x}}(L)$. The metric on L can be expressed in terms of the dreibein as follows:[2]

$$g_{ij}(\mathbf{x})=\delta_{AB}e_i^A(\mathbf{x})e_j^B(\mathbf{x}) \ . \tag{3.1}$$

If $\dim M=2$, we choose local coordinates on L in a neighborhood of M such that the metric on M at a point $\mathbf{x}$ is given by

$$g_{ij}(\mathbf{x})=\sum_{A,B=1}^{2}\delta_{AB}e_i^A(\mathbf{x})e_j^B(\mathbf{x}), \quad i,j=1,2 \ , \tag{3.2}$$

i.e., the coordinate x^3 is transversal to M. In the following we focus on the geometry of L, thinking of the "background manifold" M as being identified with L (for $d=3$), or as being a proper submanifold of L (for $d=2$) embedded in L in the way just explained.

The inverse of the dreibein $e_i^A(\mathbf{x})$ is given by

$$\mathcal{E}^i_A(\mathbf{x})=\delta_{AB}g^{ij}(\mathbf{x})e_j^B(\mathbf{x}) \ , \tag{3.3}$$

where $(g^{ij}(\mathbf{x}))$ is the inverse matrix of $(g_{ij}(\mathbf{x}))$. Clearly,

$$\mathcal{E}^i_A(\mathbf{x})e_i^B(\mathbf{x})=\delta_A^B \quad\text{and}\quad g^{ij}(\mathbf{x})=\delta^{AB}\mathcal{E}^i_A(\mathbf{x})\mathcal{E}^j_B(\mathbf{x}) \ . \tag{3.4}$$

To summarize, the dreibein $e_i^A(\mathbf{x})$ is the matrix that transforms the coordinate basis (dx^i) of 1-forms in $T^*_{\mathbf{x}}(L)$ to an orthonormal basis of 1-forms $(e^A(\mathbf{x}))$, in $T^*_{\mathbf{x}}(L)$, i.e.,

$$e^A(\mathbf{x})=e_i^A(\mathbf{x})dx^i \quad\text{and}\quad g^{ij}(\mathbf{x})e_i^A(\mathbf{x})e_j^B(\mathbf{x})=\delta^{AB} \ . \tag{3.5}$$

Similarly, $\mathcal{E}^i_A(\mathbf{x})$ transforms the basis $(\partial/\partial x^i)$ of vector fields in $T_{\mathbf{x}}(L)$ to an orthonormal basis of vector fields $(\mathcal{E}_A(\mathbf{x}))$, in $T_{\mathbf{x}}(L)$, i.e.,

$$\mathcal{E}_A(\mathbf{x})=\mathcal{E}^i_A(\mathbf{x})\frac{\partial}{\partial x^i}\equiv\mathcal{E}^i_A(\mathbf{x})\partial_i$$

and (3.6)

$$g_{ij}(\mathbf{x})\mathcal{E}^i_A(\mathbf{x})\mathcal{E}^j_B(\mathbf{x})=\delta_{AB} \ .$$

[2]Throughout this work, if not stated explicitly, the Einstein summation convention over repeated indices is understood.

From Eq. (3.1) it follows that the dreibein e_i^A is a "square root" of the metric (g_{ij}). This "square root," however, is not unique. It is only defined up to local frame rotations. [Note that the dreibein e_i^A has nine independent components while the metric (g_{ij}) has only six. It is the group of local frame rotations which accounts for the mismatch: $\dim \mathrm{SO}(3)=3$.] Thus on every cotangent space $T^*_{\mathbf{x}}(L)$, $\mathbf{x}\in L$, we have a three-dimensional (spin-1) representation, $R(\mathbf{x})\in\mathrm{SO}(3)$, of the rotation group. The rotations $R(\mathbf{x})$ act on the dreibein $e_i^A(\mathbf{x})$ as "gauge transformations" in the following way:

$$e_i^A(\mathbf{x})\mapsto {}^R e_i^A(\mathbf{x})=R(\mathbf{x})^A{}_B e_i^B(\mathbf{x})$$

or (3.7)

$$e(\mathbf{x})\mapsto {}^R e(\mathbf{x})=R(\mathbf{x})e(\mathbf{x}) \ .$$

In order to define parallel transport on L in the (co)tangent-frame formalism, one introduces the notion of an affine spin connection $\omega^A{}_B$. This connection is an $so(3)$-valued 1-form on L [where $so(3)$ is the Lie algebra of SO(3)], and it can be expanded in the coordinate basis (dx^i) or in the orthonormal frame $(e^A(\mathbf{x}))$ of $T^*_{\mathbf{x}}(L)$:

$$\omega^A{}_B(\mathbf{x})=\omega^A{}_{Bi}(\mathbf{x})dx^i=\omega^A{}_{BC}(\mathbf{x})e_i^C(\mathbf{x})dx^i=\omega^A{}_{BC}(\mathbf{x})e^C(\mathbf{x}) . \tag{3.8}$$

Notice that, with the help of the dreibein and its inverse [see Eqs. (3.5) and (3.6)], the indices of any tensor can be changed at will from coordinate indices $i,j,\ldots$ to frame indices $A,B,\ldots$. Geometrically, the connection $\omega^A{}_B$ determines the amount by which the frame e^A rotates under a displacement by an infinitesimal vector $\boldsymbol{\xi}=(\xi^i)$,

$$e^A(\mathbf{x})\mapsto e^A(\mathbf{x}+\boldsymbol{\xi})=e^A(\mathbf{x})+\omega^A{}_B(\boldsymbol{\xi};\mathbf{x})e^B(\mathbf{x}) \ , \tag{3.9}$$

where $\omega^A{}_B(\boldsymbol{\xi};\mathbf{x})=\omega^A{}_{Bi}(\mathbf{x})\xi^i(\mathbf{x})$.

A tensor important in characterizing the affine spin connection $\omega^A{}_B$ is the torsion 2-form $\mathcal{T}^A$, associated with e^A and $\omega^A{}_B$. It is defined through Cartan's first structure equation,

$$\begin{aligned}\mathcal{T}^A(\mathbf{x})&=\mathcal{T}^A{}_{ij}(\mathbf{x})dx^i\wedge dx^j\\&=de^A(\mathbf{x})+\omega^A{}_B(\mathbf{x})\wedge e^B(\mathbf{x}) \ ,\end{aligned} \tag{3.10}$$

where d denotes exterior differentiation [given in local coordinates by $d=dx^i(\partial/\partial x^i)$] and $\wedge$ stands for the (totally antisymmetric) exterior product; see, for example, Eguchi, Gilkey, and Hanson (1980); Bleecker (1981); and de Rham (1984).

It is customary to decompose the affine spin connection into two parts:

$$\omega^A{}_B(\mathbf{x})=\lambda^A{}_B(\mathbf{x})+\kappa^A{}_B(\mathbf{x}) \ , \tag{3.11}$$

where $\lambda^A{}_B$ is the Levi-Cività connection and $\kappa^A{}_B$ is the so-called contorsion field. The Levi-Cività connection plays a prominent role in general relativity; hence it is important if we wish to study quantum mechanics in a curved space-time. On any Riemannian manifold (L,g_{ij}), it is uniquely determined by requiring that its torsion vanish, i.e., if one replaces $\omega^A{}_B$ by $\lambda^A{}_B$ in Eq. (3.10) the resulting expression has to vanish. The components $\lambda^A{}_{Bi}$ can be expressed purely in terms of the dreibein $e_i{}^A$, its derivatives, and its inverse $\mathcal{E}^i{}_A$ (Eguchi, Gilkey, and Hanson, 1980; Alvarez-Gaumé and Ginsparg, 1985):

$$\begin{aligned}\lambda^A{}_{Bi}(\mathbf{x})=\tfrac{1}{2}[\,&\mathcal{E}^k_B(\mathbf{x})(\partial_k e_i{}^A-\partial_i e_k{}^A)(\mathbf{x})\\&+\delta^{AC}\delta_{BD}\mathcal{E}^k_C(\mathbf{x})(\partial_i e^D_k-\partial_k e^D_i)(\mathbf{x})\\&+\delta^{AC}\delta_{DE}e^D_i(\mathbf{x})\mathcal{E}^k_B(\mathbf{x})\mathcal{E}^l_C(\mathbf{x})\\&\times(\partial_k e^E_l-\partial_l e^E_k)(\mathbf{x})]\;.\end{aligned} \tag{3.12}$$

The contorsion field $\kappa^A{}_B$ contains additional information about the geometry of L. This information is relevant if one considers the motion of spinning particles in L; see Sec. III.B.

As a last geometrical notion we introduce the curvature 2-form $\mathcal{R}^A{}_B(\mathbf{x})$ of the connection $\omega^A{}_B$ on L. It is defined through Cartan's second structure equation,

$$\begin{aligned}\mathcal{R}^A{}_B(\mathbf{x})&=\mathcal{R}^A{}_{Bij}(\mathbf{x})dx^i\wedge dx^j\\&=d\omega^A{}_B(\mathbf{x})+\omega^A{}_C(\mathbf{x})\wedge\omega^C{}_B(\mathbf{x})\;.\end{aligned} \tag{3.13}$$

It is easy to deduce from Eqs. (3.10) and (3.13) how ω and $\mathcal{R}$ transform under the gauge transformations (3.7) of the dreibein:

$$\begin{aligned}&\omega(\mathbf{x})\mapsto{}^R\omega(\mathbf{x})=R(\mathbf{x})\omega(\mathbf{x})R^T(\mathbf{x})+R(\mathbf{x})dR^T(\mathbf{x})\;,\\&\mathcal{R}(\mathbf{x})\mapsto{}^R\mathcal{R}(\mathbf{x})=R(\mathbf{x})\mathcal{R}(\mathbf{x})R^T(\mathbf{x})\;,\end{aligned} \tag{3.14}$$

where a superscript T denotes transposition of a matrix. Furthermore, from Eq. (3.14) and from the decomposition of ω into the parts λ and κ given in Eq. (3.11), the following transformation properties follow:

$$\begin{aligned}&\lambda(\mathbf{x})\mapsto{}^R\lambda(\mathbf{x})=R(\mathbf{x})\lambda(\mathbf{x})R^T(\mathbf{x})+R(\mathbf{x})dR^T(\mathbf{x})\;,\\&\kappa(\mathbf{x})\mapsto{}^R\kappa(\mathbf{x})=R(\mathbf{x})\kappa(\mathbf{x})R^T(\mathbf{x})\;.\end{aligned} \tag{3.15}$$

We note that the contorsion field κ transforms homogeneously under gauge transformations, i.e., according to the adjoint action of the gauge group.

We end this subsection with some remarks about the physical relevance of the geometrical notions introduced above in connection with crystalline backgrounds exhibiting dislocations and disclinations. We summarize results contained, for example, in Kleinert (1989; see also Katanaev and Volovich, 1992), where more details can be found.

Let $\mathbf{y}_n\in\mathbb{E}^3$ denote the lattice sites of a perfect crystalline background. If the crystal suffers some distortion, the original lattice sites get shifted to $\mathbf{x}_n$, where $\mathbf{x}_n=\mathbf{y}_n+\mathbf{u}(\mathbf{x}_n)$, and defects may form. In order to study these defects in the framework of differential geometry, one assumes that the crystalline background can be treated as a continuous (isotropic) medium. Then $\mathbf{u}(\mathbf{x})$ is called the total distortion field. It describes a *singular* infinitesimal transformation of Euclidean space $\mathbb{E}^3$ (with metric δ_{ij}) into a space L containing defects or, from a geometrical point of view, into a manifold with nonvanishing curvature and torsion.

Densities of different types of defects in L can be expressed in terms of (derivatives of) the distortion field $u_i(\mathbf{x})$ as follows. The density of dislocations (or translational defects) is given by

$$\alpha_{ij}(\mathbf{x})=\varepsilon_i{}^{hk}\partial_h\partial_k u_j(\mathbf{x})\;, \tag{3.16}$$

the density of disclinations (or rotational defects) by

$$\Theta_{ij}(\mathbf{x})=\tfrac{1}{2}\varepsilon_i{}^{hk}\varepsilon_j{}^{mn}\partial_h\partial_k\partial_m u_n(\mathbf{x})\;, \tag{3.17}$$

and the general defect density by

$$\eta_{ij}(\mathbf{x})=\tfrac{1}{2}\varepsilon_i{}^{hk}\varepsilon_j{}^{mn}\partial_h\partial_m(\partial_k u_n+\partial_n u_k)(\mathbf{x})\;. \tag{3.18}$$

Note that these expressions are nonvanishing, in general, because the distortion field $u_i(\mathbf{x})$ is singular! In the presence of a single defect line Γ, the dislocation and disclination densities are both proportional to a δ function along the line Γ; see Kleinert (1989).

The geometric properties of the manifold L are coded into its metric $g_{ij}(\mathbf{x})$ and contorsion field $\kappa_{jhk}(\mathbf{x})$. In terms of the distortion field $u_i(\mathbf{x})$ they are given, in linear approximation, by

$$g_{ij}(\mathbf{x})=\delta_{ij}-(\partial_i u_j+\partial_j u_i)(\mathbf{x}) \tag{3.19}$$

and

$$\begin{aligned}\kappa_{jhk}(\mathbf{x})=\tfrac{1}{2}[&\partial_k(\partial_j u_h-\partial_h u_j)(\mathbf{x})-\partial_j(\partial_h u_k+\partial_k u_h)(\mathbf{x})\\&+\partial_h(\partial_j u_k+\partial_k u_j)(\mathbf{x})]\;,\end{aligned} \tag{3.20}$$

where $\kappa_{jhk}=\delta_{AC}e^C_j e^B_h\kappa^A{}_{Bk}$; see Eqs. (3.1) and (3.11).

This allows for a comparison of the defect densities of L, given in Eqs. (3.16)–(3.18), with the expressions for torsion and curvature of the manifold L; see Eqs. (3.10)–(3.13). The following relations hold:

$$\alpha_{ij}(\mathbf{x})=\varepsilon_i{}^{hk}\kappa_{jhk}(\mathbf{x}) \tag{3.21}$$

$$\Theta_{ij}(\mathbf{x})=\mathcal{R}_{ij}(\mathbf{x})-\tfrac{1}{2}g_{ij}(\mathbf{x})\mathcal{R}(\mathbf{x})\;, \tag{3.22}$$

and

$$\eta_{ij}(\mathbf{x})=\mathcal{R}^{\rm LC}_{ij}(\mathbf{x})-\tfrac{1}{2}g_{ij}(\mathbf{x})\mathcal{R}^{\rm LC}(\mathbf{x})\;, \tag{3.23}$$

where $\mathcal{R}_{ij}=\mathcal{R}^A{}_{ijA}=\mathcal{E}^k_A e^B_i\mathcal{R}^A{}_{Bjk}$ is the Ricci tensor and $\mathcal{R}=g^{ij}\mathcal{R}_{ij}$ is the scalar curvature of the affine spin connection $\omega^A{}_{Bj}$. Similarly, $\mathcal{R}^{\rm LC}_{ij}$ and $\mathcal{R}^{\rm LC}$ denote the Ricci tensor and scalar curvature, respectively, of the Levi-Cività connection $\lambda^A{}_{Bj}$, the torsion-free part of the affine spin connection; see Eq. (3.11).

Finally, for quantum-mechanical particles moving in a crystalline background with defects, the following assumption appears to be reasonable: If the energy of the particles is so small that the lattice structure of the background cannot be resolved, then a (metrically nontrivial) Riemannian manifold provides an effective description of the background when studying the orbital motion of the particles. (Technically, the Laplacian in the

Schrödinger-Pauli equation will be replaced by the Laplace-Beltrami operator associated with the Riemannian metric on the manifold.) Moreover, the formalism presented above is well adapted to describing the orbital motion of particles confined (e.g., by some potential) to a curved surface in $\mathbb{E}^3$. The question of the motion of the spin degrees of freedom, however, is more subtle and will be addressed in the next section.

B. Systems of spinning particles coupled to external electromagnetic and geometric fields

We start this section by showing how to describe systems of spinning particles moving in a geometrically nontrivial background and coupled to an external electromagnetic field. We assume that the manifold L admits a spin structure. Then we may introduce spinor bundles over L (associated with the cotangent bundle $T^*(L)$ over L). Let $s=0,\frac{1}{2},1,\ldots$ denote the spin of the particles, i.e., $2s+1$ is the dimension of an irreducible representation of $SU(2)=\widetilde{SO}(3)$ with spin s. The fiber of the spin-s spinor bundle, $E^{(s)}(L)$, over L is isomorphic to the $(2s+1)$-dimensional Hilbert space $\mathcal{D}^{(s)}$, carrying the spin-s representation of SU(2). Sections of the spin-s spinor bundle are denoted by $\psi^{(s)}(\mathbf{x})$. From now on, we choose the gauge transformations $R(\mathbf{x})$ to be SU(2) valued. The action of these gauge transformations on the cotangent bundle $T^*(L)$ is given by their adjoint (spin-1) representation, also denoted by $R(\mathbf{x})$; see Eq. (3.7).[3] Under a gauge transformation $R(\mathbf{x})$, a section $\psi^{(s)}(\mathbf{x})$ of $E^{(s)}(L)$ transforms as follows:

$$\psi^{(s)}(\mathbf{x})\mapsto {}^R\psi^{(s)}(\mathbf{x})=U^{(s)}(R(\mathbf{x}))\psi^{(s)}(\mathbf{x}) , \tag{3.24}$$

where $U^{(s)}$ is the spin-s representation of SU(2). The transition functions of the spin-s spinor bundle $E^{(s)}(L)$—which must be specified if the topology of the base manifold L is nontrivial—are inherited from the transition functions of the cotangent bundle $T^*(L)$ by lifting them to the spin-s representation of SU(2). [Since we have assumed that L has a spin structure this is possible even if s is half-integer!]

Physically, what is meant by "spin up" or "spin down" is now a local notion, depending on the point $\mathbf{x}\in L$ at which the spin is located and determined by the local frames $(e^A(\mathbf{x}))_{A=1}^3$; see Sec. III.A.

The spaces of wave functions in nonrelativistic one-particle quantum mechanics are Hilbert spaces of sections of these spinor bundles. In nonrelativistic quantum mechanics, wave functions are complex valued. We therefore tensor the fiber space $\mathcal{D}^{(s)}$—real when s is an integer—by $\mathbb{C}$. The structure group of the resulting complexified bundle, denoted by $E_{\mathbb{C}}^{(s)}(L)$, is then $U(1)\times SU(2)$. The factor U(1) [a phase transformation of $\psi^{(s)}$] is connected to electromagnetism, as recognized by Weyl more than 60 years ago (Weyl, 1928; see also Weyl, 1918).

In order to keep our notations simple, it is advantageous to formulate quantum mechanics of many-particle systems in the language of second quantization. The sections $\psi^{(s)}(\mathbf{x})$ of $E_{\mathbb{C}}^{(s)}(L)$ over L are then interpreted as operator-valued distributions acting on Fock space and subject to canonical equal-time commutation or anticommutation relations,

$$[\psi_\alpha^{(s)}(\mathbf{x}),\psi_\beta^{(s)*}(\mathbf{y})]_\pm=\frac{1}{\sqrt{g(\mathbf{x})}}\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{y}) ,$$

and (3.25)

$$[\psi_\alpha^{(s)\natural}(\mathbf{x}),\psi_\beta^{(s)\natural}(\mathbf{y})]_\pm=0 ,$$

where $\alpha,\beta=1,\ldots,2s+1$; $[\,,]_+$ denotes the anticommutator and $[\,,]_-$ the usual commutator; $\psi^{(s)\natural}=\psi^{(s)}$ or $\psi^{(s)*}$; $\psi^{(s)*}$, the creation operator, is the adjoint (on Fock space) of $\psi^{(s)}$, the annihilation operator; and $g(\mathbf{x})$ denotes the determinant of the metric $(g_{ij}(\mathbf{x}))$ on L. The usual connection between spin and statistics is to choose anticommutators in Eq. (3.25), corresponding to Fermi statistics, when s is half-integer, and commutators, corresponding to Bose statistics, when s is integer; see, for example, Negele and Orland (1987) and Feldman, Knörrer, and Trubowitz (1992).

Our purpose is to specify some nonrelativistic dynamical laws governing the time evolution of the operators $\psi^{(s)\natural}$ in the Heisenberg picture. Let $\psi^{(s)\natural}(x)=\psi^{(x)\natural}(t,\mathbf{x})$ denote the Heisenberg picture creation and annihilation operators with initial conditions $\psi^{(s)\natural}(0,\mathbf{x})=\psi^{(x)\natural}(\mathbf{x})$. Geometrically, these operators are sections of a trivially extended spin-s spinor bundle, $E_{\mathbb{C}}^{(s)}(\mathbb{R}\times L)$, over the space-time manifold $\mathbb{R}\times L$. In order to formulate local dynamical laws for $\psi^{(s)\natural}(x)$, we need to be able to differentiate these fields in t and $\mathbf{x}$. This necessitates introducing the notion of parallel transport in $E_{\mathbb{C}}^{(s)}(\mathbb{R}\times L)$. Parallel transport in the spinor bundle $E_{\mathbb{C}}^{(s)}(\mathbb{R}\times L)$ is defined with the help of a $U(1)\times SU(2)$ connection, i.e., by a vector potential with values in $\mathbb{R}\oplus su(2)$, where su(2) is the Lie algebra of SU(2). Once such a connection is fixed, derivatives of sections $\psi^{(s)\natural}(x)$ are defined as covariant derivatives. Setting $x^0=ct$ and $x=(x^\mu)=(x^0,\mathbf{x})$, where $\mathbf{x}\in L$, we find that the covariant derivative in the μ direction is given by

$$\mathcal{D}_\mu=\frac{\partial}{\partial x^\mu}+ia_\mu(x)+w_\mu^{(s)}(x),\quad \mu=0,\ldots,3 , \tag{3.26}$$

where the real-valued 1-form $a=a_\mu(x)dx^\mu$ is the U(1) connection, and the su(2)-valued 1-form $w^{(s)}=w_\mu^{(s)}(x)dx^\mu$ is the SU(2) connection in the spin-s representation of su(2), i.e.,

$$w_\mu^{(s)}(x)=i\sum_{A=1}^{3}w_{\mu A}(x)L_A^{(s)} , \tag{3.27}$$

where we have adopted the same notation as in Eq.

[3]There is little danger of confusion.

(2.11): $(L_A^{(s)})_{A=1}^3$ are Hermitian generators of su(2) in the spin-s representation, normalized such that $L_A^{(1/2)}=\sigma_A$, where σ_1, σ_2, and σ_3 are the standard Pauli matrices.

We shall argue shortly that we can identify a with the electromagnetic vector potential (up to multiplication by physical constants). This is no surprise, given the observations in Sec. II.A [see Eqs. (2.9) and (2.10)]. What about $w^{(s)}$? First, from a geometrical point of view, it is clear that the affine spin connection $\omega^A{}_B$, introduced in Eq. (3.8), enters the definition of $w^{(s)}$, since the spinor bundle $E_{\mathbb{C}}^{(s)}(\mathbb{R}\times L)$ is associated with the cotangent bundle $T^*(\mathbb{R}\times L)$, i.e., it inherits the geometrical structure of $T^*(\mathbb{R}\times L)$. Second, based on the observations in Sec. II.A [see Eqs. (2.9) and (2.11)–(2.13)], we expect the interaction of the external electromagnetic field with the magnetic moment carried by the particles (Zeeman and spin-orbit couplings) to be described by an additional term, $\rho^{(s)}=\rho_\mu^{(s)}(x)dx^\mu$, in the SU(2) connection $w^{(s)}$. Since the sum of ω and $\rho^{(s)}$ must be an SU(2) connection, $\rho^{(s)}$ has to transform under SU(2) gauge transformations according to the adjoint action of the gauge group. We use the following notations:

$$w_\mu^{(s)}(x)=\omega_\mu^{(s)}(x)+\rho_\mu^{(s)}(x)\ , \tag{3.28}$$

where

$$\omega_\mu^{(s)}(x)=\frac{i}{2}\sum_{A,B,C=1}^{3}\varepsilon_A{}^{BC}\omega^A{}_{B\mu}(x)L_C^{(s)}\ , \tag{3.29}$$

$\varepsilon_A{}^{BC}=\varepsilon_{ABC}$ is the sign of the permutation (ABC) of (123), and

$$\rho_\mu^{(s)}(x)=i\sum_{A=1}^{3}\rho_{\mu A}(x)L_A^{(s)}\ . \tag{3.30}$$

All notions introduced in Sec. III.A—defined over space L and its cotangent bundle $T^*(L)$—can easily be extended to space-time, $\mathbb{R}\times L$, and its cotangent bundle, $T^*(\mathbb{R}\times L)\simeq\mathbb{R}\times T^*(L)$. In a nonrelativistic setting, the space-time metric $\eta_{\mu\nu}(x)$ has the property that $\eta_{0i}(x)=0=\eta_{i0}(x)$ [see the beginning of Sec. III.A], and, as a consequence, most "temporal" components of the different geometrical fields introduced in Sec. III.A vanish. In Eq. (3.29), $\omega^A{}_{Bi}(x)$, $i=1,2,3$, is given by Eqs. (3.8), (3.11), (3.12), and

$$\omega^A{}_{B0}(x)=\tfrac{1}{2}[\delta^{AC}\delta_{BD}\mathcal{E}_C^k(x)\partial_0 e_k^D(x)-\mathcal{E}_B^k(x)\partial_0 e_k^A(x)]\ . \tag{3.31}$$

Under an SU(2) gauge transformation $R(x)$, i.e., under local frame rotations in the cotangent bundle $T^*(\mathbb{R}\times L)$ [see Eq. (3.7)], the different terms of the SU(2) connection $w^{(s)}$ transform as follows:

$$\omega_\mu^{(s)}(x)\mapsto{}^R\omega_\mu^{(s)}(x)=U^{(s)}(R(x))\omega_\mu^{(s)}(x)U^{(s)}(R(x))^* + U^{(s)}(R(x))\partial_\mu U^{(s)}(R(x))^*\ , \tag{3.32}$$

which can be inferred from Eq. (3.14), and

$$\rho_\mu^{(s)}(x)\mapsto{}^R\rho_\mu^{(s)}(x) =U^{(s)}(R(x))\rho_\mu^{(s)}(x)U^{(s)}(R(x))^*\ , \tag{3.33}$$

where $*$ denotes the adjoint of a matrix.

If the metric on L is time independent, $\omega_0^{(s)}(x)$ vanishes; see Eqs. (3.29) and (3.31). After a time-dependent SU(2) gauge transformation, however, it may be different from zero. Furthermore, the $\rho^{(s)}$ part of the SU(2) connection $w^{(s)}$ will, in general, be different from zero. Geometrically, it corresponds to an additional contorsion field yielding nonvanishing torsion; see Eq. (3.10). The physical interpretation of the $\omega^{(s)}$ part of the SU(2) connection $w^{(s)}$, as well as the precise identifications with physical quantities of the $\rho^{(s)}$ part of $w^{(s)}$ and of the U(1) connection a, will be given below. (The material in Sec. II.A will serve us as a guide.)

Having introduced a U(1)×SU(2) connection and defined covariant differentiation of the sections $\psi^{(s)\natural}$, we are now in a position to formulate local dynamical laws. It is convenient to use the Lagrangian formalism, but we could also work in the Hamiltonian formalism; see Fröhlich and Kerler (1991). Let us consider a system of nonrelativistic particles of fixed spin s, and, to simplify our notations, we drop the superscript (s) from the field operators $\psi^{(s)\natural}$ and the SU(2) connection $w^{(s)}=\omega^{(s)}+\rho^{(s)}$. Our ansatz for the action of the system is an obvious generalization of the action (2.17) found in Sec. II.A. It reads [with $x=(ct,\mathbf{x})$]

$$S_\Lambda(\psi^*,\psi;g,a,w)=\int_\Lambda\sqrt{g(t,\mathbf{x})}dt\,d^3\mathbf{x}[i\hbar c\psi^*(x)(\mathcal{D}_0\psi)(x)-(\hbar^2/2m)g^{kl}(t,\mathbf{x})(\mathcal{D}_k\psi)^*(x)(\mathcal{D}_l\psi)(x)-U(\psi^*,\psi)(x)]\ , \tag{3.34}$$

where the covariant derivatives are given in Eq. (3.26), m is the effective mass of the particles (sometimes also denoted by m^*; in common situations of solid-state physics it can be considerably smaller than m_0, the mass of the particles in the vacuum), and $U(\psi^*,\psi)(x)$ is a U(1)×SU(2)-invariant functional of ψ and ψ^*, e.g. [$y=(ct,\mathbf{y})$],

$$U(\psi^*,\psi)(x)=v(x)\psi^*(x)\psi(x)+\tfrac{1}{2}\int_\Lambda\sqrt{g(t,\mathbf{y})}d^3\mathbf{y}:[\psi^*(x)\psi(x)-n]V(t,\mathbf{x},\mathbf{y})[\psi^*(y)\psi(y)-n]:\ . \tag{3.35}$$

The double colons indicate Wick ordering, v is a possibly time-dependent one-body background potential (depending on the background and on the scalar curvature $\mathcal{R}$ of M), V is some two-body potential [e.g., for charged particles a (possibly screened) Coulomb potential, or for neutral atoms or molecules a van der Waals potential], and n is approximately equal to the background density of the system (proportional to its chemical potential). We recall that $\Lambda\subset\mathbb{R}\times M$ is a cylindrical region in space-time. At fixed time t, we impose Dirichlet boundary conditions at the boundary $\partial\Omega_t$ of the region Ω_t to which the system is confined.

The field equation (or Euler-Lagrange equation) for ψ or ψ^* follows by setting the variation of S_Λ with respect to ψ^* or ψ, equal to zero. The resulting equation is a generalization of the Pauli equation (2.14) to a system of spinning particles in a geometrically nontrivial background.

In order to illustrate these matters, let us consider a simple situation: We choose space M to be given by $\mathbb{E}^2$ (the x,y plane in $L=\mathbb{E}^3$) or by $\mathbb{E}^3$; $g_{ij}(t,\mathbf{x})=\delta_{ij}$ for all times t and all $\mathbf{x}\in M$, $\Lambda=\mathbb{R}\times\Omega$, where Ω is some time-independent open set in M. The field equation for ψ, obtained by varying the action S_Λ defined in Eq. (3.34) with respect to ψ^*, then reduces to the Pauli equation (2.14). This equation coincides with the usual form given in (2.6) [up to a modification of order $O(1/mm_0^2)$; see Sec. II.A] provided we make the same identifications as in Eqs. (2.10)–(2.13): The components of the U(1) connection a [with respect to the coordinate basis (dx^μ) of the cotangent space $T_x^*(\mathbb{R}\times L)$] are given by the electromagnetic potentials Φ and $\mathbf{A}$:

$$a_0(x)=\frac{q}{\hbar c}\Phi(x) \quad\text{and}\quad a_k(x)=-\frac{q}{\hbar c}A_k(x)\ , \tag{3.36}$$

where q is the charge of the particles. Furthermore, the components of the ρ part of the SU(2) connection w are expressed in terms of the electromagnetic field $(\mathbf{E},\mathbf{B})$ as follows:

$$\rho_{0A}(x)=-\frac{g\mu}{2\hbar c}B_A(x)\ , \tag{3.37}$$

where $B_A(x)$ is the A component of the magnetic field $\mathbf{B}(x)$ in the basis $(e^A(x))$ of $T_x^*(\mathbb{R}\times L)$, and

$$\rho_{kA}(x)=\left[-\frac{g\mu}{2\hbar c}+\frac{q}{4mc^2}\right]\sum_{B=1}^{3}\varepsilon_{kAB}(x)E_B(x)\ , \tag{3.38}$$

where $E_B(x)$ is the B component of the electric field $\mathbf{E}(x)$, and the symbol $\varepsilon_{kAB}(x)$ is defined by

$$\varepsilon_{kAB}(x)=e_k^C(x)\varepsilon_{CAB}\ , \tag{3.39}$$

where ε_{CAB} is the sign of the permutation (CAB) of (123). In Eqs. (3.37) and (3.38), the magnetic moment of the particles enters via $g\mu$; see Eq. (2.7). Although the orthonormal frames $e^A(x)$ could be chosen to vary, it is simplest, in the present situation, to choose them as $e_\mu^A(x)=\delta_\mu^A$. Then the "geometrical" part ω of the SU(2) connection w clearly vanishes.

In a general situation, when the background of the system has the structure of an arbitrary Riemannian spin manifold M, the physical interpretation of the connections a and w is straightforward: The $\mathrm{U}(1)_{\mathrm{em}}$ connection a is still expressed in terms of the electromagnetic potentials, as in Eq. (3.36). The $\mathrm{SU}(2)_{\mathrm{spin}}$ connection w has been given in Eq. (3.28) with the "geometric" part ω being specified in terms of the affine spin connection $\omega^A{}_B$ on $\mathbb{R}\times L$; see Eq. (3.29). Its ρ part (describing Zeeman and spin-orbit couplings of the particles' magnetic moment to the external electromagnetic field) always contains the terms in Eqs. (3.37)–(3.39). The only difference is that, on a general Riemannian manifold it is not possible to choose the dreibein $e_\mu^A(x)$ to be constant on all of $\mathbb{R}\times L$.

Remark. Here we wish to comment on the physical status of the ω part in the SU(2) connection w. In the study of gravitational fields, the affine spin connection ω is torsion free. It is given by the Levi-Cività connection λ, which is canonically associated with the gravitational metric field g; see Eq. (3.12). Hence, if we consider a quantum-mechanical system in a (strong) external gravitational field, then ω enters into the description of the motion of the spin of the particles as a fundamental physical field.

At the end of Sec. III.A, we argued that the geometrical framework of Riemannian manifolds provides an effective description for the *orbital* motion of low-energy particles in a crystalline background with defects (or of particles confined to a curved surface in $\mathbb{E}^3$). Given the Levi-Cività connection λ, in terms of the metric (3.19) and the contorsion field κ, as specified in Eq. (3.20), we must ask whether the corresponding affine spin connection $\omega=\lambda+\kappa$ [see Eq. (3.11)] might provide an effective description of the interaction of the spin of the particles with the crystalline background through which they are moving. In general, this is not likely to be so! For example, let us consider a spinning particle with vanishing magnetic moment, which moves in a crystalline background. Then, from the point of view of basic one-body quantum mechanics (see Sec. II.A), we do not expect that the dynamics of the spin of the particle is coupled to the effective metric associated with the background. (In this situation, the spin can be viewed as an internal degree of freedom.) More generally, in one-body quantum mechanics (in the absence of gravitational fields), the dynamics of the spin of a particle (moving in some background or constrained to a surface in $\mathbb{E}^3$) is completely determined by the Zeeman effect and by spin-orbit coupling, including the kinematical effect of the Thomas precession. The ρ part of the SU(2) connection w fully accounts for these effects, and ω can be transformed to 0 in a suitable SU(2) gauge.

We wish to emphasize that the main reasons for introducing the geometrical framework have been to elucidate, from a geometrical point of view, the meaning and origin of the SU(2) gauge invariance (i.e., the introduction of an SU(2) connection and of local rotations) and to prepare for the description of quantum-mechanical systems in "moving coordinates." This will be the subject of the following section.

We finally recall that with the help of the dreibein $e_i{}^A$ and its inverse $\mathcal{E}^i{}_A$ the components of the electromagnetic field and its vector potential can easily be changed from the form they take in orthonormal frames to the form they take in local coordinates [see Eqs. (3.5) and (3.6)], e.g.,

$$B_A(x)=\mathcal{E}^k_A(x)B_k(x) \quad \text{and} \quad A_k(x)=e^C_k(x)A_C(x) \ . \quad (3.40)$$

Note that Eqs. (3.37) and (3.38) are consistent with the transformation law (3.33) of $\rho^{(s)}_\mu$ under $SU(2)_{spin}$ gauge transformations. Moreover, defining $U(1)_{em}$ gauge transformations as in Eq. (2.15) (with χ an arbitrary, real-valued function on $\mathbb{R}\times L$), we prove from the discussion above [see, in particular, Eq. (3.34) for the action S_Λ] that *nonrelativistic quantum mechanics of charged spinning particles, which move in an external electromagnetic field and in a geometrically nontrivial background, is* $U(1)_{em}\times SU(2)_{spin}$ *gauge invariant.*

C. Moving coordinates and the quantum-mechanical Larmor theorem

We now imagine that the background of the system is moving on the manifold M according to some classical flow $\phi(t,\cdot)$. Here $\phi(t,\mathbf{y})$ is the position in M of a point particle at time t starting at position $\mathbf{y}$ at time 0. Then, in the x coordinates ("laboratory coordinates") fixed to $\mathbb{R}\times M$, the one-body potential $v(x)$ and the electric and magnetic fields $\mathbf{E}(x)$ and $\mathbf{B}(x)$ created by the background are time dependent. This implies that, in the (time-independent) x coordinates on $\mathbb{R}\times M$, the Hamiltonian of the system is time dependent, which complicates the mathematical analysis of the system. In particular, it complicates the analysis of its thermal equilibrium properties. It is quite clear, physically, that approximate thermal equilibrium in such a system will be reached *locally* in regions moving with the background according to the flow $\phi(t,\cdot)$. Thus we ought to formulate quantum mechanics in "moving coordinates" (y^1,y^2,y^3), where

$$\mathbf{x}=\phi(t,\mathbf{y}) \ ,$$

that is, (3.41)

$$\mathbf{y}=\phi^{-1}(t,\mathbf{x}) \ .$$

In accordance with our nonrelativistic treatment of quantum theory, time will not be transformed, and in our calculations only terms to order $O(f/c)$ are taken into account, where f is the modulus of the velocity field of the moving background (see below). We shall see that the geometrical formalism introduced in the first part of this section allows for a natural description of the transformations to "moving coordinates," since, from the outset, it incorporates the local symmetry group of coordinate reparametrizations of the manifold, i.e., diffeomorphisms. For a different account of quantum mechanics in moving coordinates (or in noninertial reference frames), see Schmutzer and Plebański (1977).

In the new coordinates (y^1,y^2,y^3) the one-body potential $v(t,\mathbf{y})$ and the background fields $\mathbf{E}(t,\mathbf{y})$ and $\mathbf{B}(t,\mathbf{y})$ may be expected to be (approximately) time *independent*. In this situation, the Hamiltonian for spinless particles $(s=0)$ will be (approximately) time independent, and we can apply the rules of Gibbsian statistical mechanics to study thermal equilibrium.

Unfortunately, for spinning particles $(s=\frac{1}{2},1,\dots)$, the situation is not quite as neat, because, in the y coordinates, the dreibein $\tilde{e}^A_i(y)$ is time *dependent:*

$$\tilde{e}^A_k(y)=e^A_j(t,\phi(t,\mathbf{y}))\frac{\partial}{\partial y^k}\phi^j(t,\mathbf{y}) \equiv e^A_j(t,\phi(t,\mathbf{y}))\frac{\partial x^j}{\partial y^k} \ . \quad (3.42)$$

In order to eliminate as much of this undesirable time dependence as possible, we attempt to find a suitable SU(2) gauge transformation of the new dreibein $\tilde{e}^A_k(y)$; see Eq. (3.7). What is the optimal choice? The answer is, perhaps, somewhat ambiguous in general. But the following choice tends to be quite optimal. Let $(f^j(t,\mathbf{x}))$ be the velocity field generating the flow $\phi(t,\cdot)$, i.e.,

$$c\frac{\partial x^j}{\partial y^0}\equiv\frac{\partial}{\partial t}\phi^j(t,\mathbf{y})=f^j(t,\phi(t,\mathbf{y})), \quad j=1,2,3 \ . \quad (3.43)$$

Then the infinitesimal rotation of an orthonormal frame carried along by the flow $\phi(t,\cdot)$, at the point $\mathbf{x}\in M$ and at time t, is given by

$$\Omega^A{}_B(t,\mathbf{x})=\tfrac{1}{2}[(\partial_B f^A)(t,\mathbf{x})-\delta^{AC}\delta_{BD}(\partial_C f^D)(t,\mathbf{x})] \ , \quad (3.44)$$

where $\partial_A=\mathcal{E}^i_A(x)(\partial/\partial x^i)$ and $f^A(t,\mathbf{x})=e^A_j(x)f^j(t,\mathbf{x})$; see Eq. (3.6) and the remark after (3.8). The vector $\mathbf{\Omega}(t,\mathbf{x})$ dual to the antisymmetric matrix $(\Omega^A{}_B(t,\mathbf{x}))$ is called the *vorticity* or *circulation* of the vector field $\mathbf{f}(t,\mathbf{x})$ and is the local angular velocity of the rotation induced by $\phi(t,\cdot)$ of a frame at the point $\mathbf{x}$ at time t.

We define a rotation matrix $(R^A{}_B(t,\mathbf{x}))$ by setting

$$R^A{}_B(t,\mathbf{x})=T\exp\left[\gamma\int_0^t dt'\,\Omega(t',\mathbf{x})\right]^A{}_B \ , \quad (3.45)$$

where "$T\exp$" denotes a time-ordered exponential and γ is a real constant to be chosen later. (Its physical meaning will become clear at the end of this subsection; see also Sec. IV.C.) The rhs of Eq. (3.45) can be defined, for example, by a convergent Dyson series if $\mathbf{\Omega}(t,\mathbf{x})$ is uniformly bounded in t. We now define [see Eq. (3.7)]

$$\hat{e}^A_k(y)={}^R\tilde{e}^A_k(y)=R^A{}_B(t,\phi(t,\mathbf{y}))\tilde{e}^B_k(y) \ , \quad (3.46)$$

where $\tilde{e}^B_i(y)$ is given by Eq. (3.42). We also define the following transformed quantities:

$$\hat{\psi}(t,\mathbf{y})=U^{(s)}(t,\mathbf{y})\psi(t,\phi(t,\mathbf{y})) \ , \quad (3.47)$$

$$\hat{g}^{kl}(t,\mathbf{y})=\frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j}g^{ij}(t,\phi(t,\mathbf{y})) \ , \quad (3.48)$$

and

$$\hat{a}_0(t,\mathbf{y})=a_0(t,\phi(t,\mathbf{y}))+\frac{\partial x^j}{\partial y^0}a_j(t,\phi(t,\mathbf{y})) \ , \qquad \hat{a}_k(t,\mathbf{y})=\frac{\partial x^j}{\partial y^k}a_j(t,\phi(t,\mathbf{y})) \ . \quad (3.49)$$

Furthermore,

$$\hat{w}_0^{(s)}(t,\mathbf{y})=U^{(s)}(t,\mathbf{y})\left[w_0^{(s)}(t,\phi(t,\mathbf{y}))+\frac{\partial x^j}{\partial y^0}w_j^{(s)}(t,\phi(t,\mathbf{y}))\right]U^{(s)}(t,\mathbf{y})^*+U^{(s)}(t,\mathbf{y})\frac{\partial}{\partial y^0}U^{(s)}(t,\mathbf{y})^* ,$$

and (3.50)

$$\hat{w}_k^{(s)}(t,\mathbf{y})=U^{(s)}(t,\mathbf{y})\left[\frac{\partial x^j}{\partial y^k}w_j^{(s)}(t,\phi(t,\mathbf{y}))\right]U^{(s)}(t,\mathbf{y})^*+U^{(s)}(t,\mathbf{y})\frac{\partial}{\partial y^k}U^{(s)}(t,\mathbf{y})^* ,$$

where

$$U^{(s)}(t,\mathbf{y})=U^{(s)}(R(t,\phi(t,\mathbf{y}))) . \tag{3.51}$$

Our aim is to rewrite the action S_Λ introduced in Eqs. (3.34) and (3.35) in moving coordinates y, using the transformations (3.46)–(3.51). By Eqs. (3.47) and (3.51),

$$\begin{aligned}\psi(t,\mathbf{x})&=U^{(s)}(t,\phi^{-1}(t,\mathbf{x}))^*\hat{\psi}(t,\phi^{-1}(t,\mathbf{x}))\\&=U^{(s)}(R(t,\mathbf{x}))^*\hat{\psi}(t,\phi^{-1}(t,\mathbf{x})) .\end{aligned} \tag{3.52}$$

Hence [with $\partial/\partial x^0=(1/c)(\partial/\partial t)=\partial/\partial y^0$]

$$\begin{aligned}U^{(s)}(R(t,\mathbf{x}))\frac{\partial}{\partial x^0}\psi(t,\mathbf{x})\Big|_{\mathbf{x}=\phi(t,\mathbf{y})}&=\frac{\partial}{\partial y^0}\hat{\psi}(t,\mathbf{y})+U^{(s)}(t,\mathbf{y})\frac{\partial}{\partial y^0}U^{(s)}(t,\mathbf{y})^*\hat{\psi}(t,\mathbf{y})\\&\quad-\frac{1}{c}\hat{f}^k(t,\mathbf{y})\left[\frac{\partial}{\partial y^k}\hat{\psi}(t,\mathbf{y})+U^{(s)}(t,\mathbf{y})\frac{\partial}{\partial y^k}U^{(s)}(t,\mathbf{y})^*\hat{\psi}(t,\mathbf{y})\right]\\&=\frac{\partial}{\partial y^0}\hat{\psi}(t,\mathbf{y})-\frac{1}{c}\hat{f}^k(t,\mathbf{y})\frac{\partial}{\partial y^k}\hat{\psi}(t,\mathbf{y})-\frac{i\gamma}{4c}\sum_{A,B,C=1}^{3}\varepsilon_A{}^{BC}\hat{\Omega}^A{}_B(t,\mathbf{y})L_C^{(s)}\hat{\psi}(t,\mathbf{y}) ,\end{aligned} \tag{3.53}$$

where $-\hat{f}^k(t,\mathbf{y})=-(\partial y^k/\partial x^j)f^j(t,\phi(t,\mathbf{y}))$ is the kth component of the vector field generating $\phi^{-1}(t,\cdot)$ in the y coordinates, and $(\hat{\Omega}^A{}_B(t,\mathbf{y}))$ is the vorticity of the generating vector field in the y coordinates with respect to the orthonormal frame $(\hat{e}^A(t,\mathbf{y}))$, given in Eq. (3.46). By comparing Eq. (3.53) with (3.49)–(3.51) we find that

$$\begin{aligned}U^{(s)}(R(x))\left[\frac{\partial}{\partial x^0}+ia_0(x)+w_0^{(s)}(x)\right]\psi(x)\Big|_{x=(ct,\phi(t,\mathbf{y}))}&=\left[\frac{\partial}{\partial y^0}+i\hat{a}_0(y)+\hat{w}_0^{(s)}(y)\right]\hat{\psi}(y)\\&\quad-\frac{1}{c}\hat{f}^k(y)\left[\frac{\partial}{\partial y^k}+i\hat{a}_k(y)+\hat{w}_k^{(s)}(y)\right]\hat{\psi}(y) .\end{aligned} \tag{3.54}$$

We define the transformed covariant derivatives

$$\begin{aligned}\hat{\mathcal{D}}_0&=\frac{\partial}{\partial y^0}+i\hat{a}_0(y)+\hat{w}_0^{(s)}(y) ,\\\hat{\mathcal{D}}_k&=\frac{\partial}{\partial y^k}+i\hat{a}_k(y)-i\frac{m}{\hbar}\hat{f}_k(y)+\hat{w}_k^{(s)}(y) ,\end{aligned} \tag{3.55}$$

where $\hat{f}_k(y)=\hat{g}_{kl}(y)\hat{f}^l(y)$, and the transformed one-body potential

$$\hat{v}(t,\mathbf{y})=v(t,\phi(t,\mathbf{y}))-\frac{m}{2}\hat{f}_k(y)\hat{f}^k(y)-i\frac{\hbar}{2}\frac{1}{\sqrt{\hat{g}(y)}}\frac{\partial}{\partial y^k}[\sqrt{\hat{g}(y)}\hat{f}^k(y)] , \tag{3.56}$$

as well as the transformed two-body potential

$$\hat{V}(t,\mathbf{y},\mathbf{y}')=V(t,\phi(t,\mathbf{y}),\phi(t,\mathbf{y}')) . \tag{3.57}$$

After these preparations, one easily verifies the following theorem.

In moving coordinates the action of the system takes the form

$$\begin{aligned}S_\Lambda(\psi^*,\psi;g,a,w)&=\hat{S}_{\hat{\Lambda}}(\hat{\psi}^*,\hat{\psi};\hat{g},\hat{a},\hat{f},\hat{w})\\&=\int_{\hat{\Lambda}}\sqrt{\hat{g}(y)}dt\,d^3\mathbf{y}\left[i\hbar c\hat{\psi}^*(y)(\hat{\mathcal{D}}_0\hat{\psi})(y)-\frac{\hbar^2}{2m}\hat{g}^{kl}(y)(\hat{\mathcal{D}}_k\hat{\psi})^*(y)(\hat{\mathcal{D}}_l\hat{\psi})(y)-\hat{U}(\hat{\psi}^*,\hat{\psi})(y)\right] ,\end{aligned} \tag{3.58}$$

where in the definition of $\hat{U}(\hat{\psi}^*,\hat{\psi})$ *the potentials* $\hat{v}$ *and* $\hat{V}$ *of Eqs. (3.56) and (3.57) are used, and* $\hat{\Lambda}=\{(t,\mathbf{y})|(t,\mathbf{x}=\phi(t,\mathbf{y}))\in\Lambda\}$.

To prove theorem (3.58), one expands the rhs of (3.58) in powers of $\hat{f}^k$, integrates by parts, and compares the resulting expression to Eqs. (3.54), (3.34), and (3.35), using (3.55) through (3.57) and the fact that $(U^{(s)}\psi)^*(U^{(s)}\psi)=\psi^*\psi$.

Let us pause to interpret the result (3.58). By (3.55), $-(m/\hbar)\hat{f}_k(y)$ enters the action $\hat{S}_{\hat{\Lambda}}$ as a contribution to the U(1) connection. By (3.36), $m\hat{f}_k(y)$ and $(q/c)\hat{A}_k(y)$ play analogous roles, i.e. (in x coordinates),

$$m\,\mathbf{f}(x)\leftrightarrow\frac{q}{c}\mathbf{A}(x)\ . \tag{3.59}$$

The vector potential $\mathbf{A}(x)$ gives rise to the Lorentz force in the classical limit. The Lorentz force has the same form as the Coriolis force if one replaces $(q/c)\mathbf{B}(x)$ by $2m\,\mathbf{\Omega}(x)$, where $\mathbf{\Omega}(x)$ is the local angular velocity, which is precisely half the curl of the vector field $\mathbf{f}(x)$; see Eq. (3.44). Thus $\mathbf{f}(x)$ is the vector potential that gives rise to the Coriolis force in the classical limit.

By Eqs. (3.53) and (3.54), the action $\hat{S}_{\hat{\Lambda}}$ contains a term

$$\gamma\hat{\psi}^*(y)\left[\sum_{A=1}^{3}\hat{\Omega}_A(y)\frac{\hbar}{2}L_A^{(s)}\right]\hat{\psi}(y)\ , \tag{3.60}$$

where $\hat{\mathbf{\Omega}}(y)=\frac{1}{2}\,\mathrm{curl}\,\hat{\mathbf{f}}(y)$, in y coordinates. It has the same form as the Zeeman term

$$\frac{g\mu}{\hbar}\hat{\psi}^*(y)\left[\sum_{A=1}^{3}\hat{B}_A(y)\frac{\hbar}{2}L_A^{(s)}\right]\hat{\psi}(y)\ , \tag{3.61}$$

which, by (3.50), (3.37), (3.30), and (3.28), also appears in $\hat{S}_{\hat{\Lambda}}$. Recall that the magnetic moment of a particle with spin s has been defined by $\boldsymbol{\mu}_{\mathrm{spin}}=(g\mu/2)\mathbf{L}^{(s)}$; see Eq. (2.7). Thus $(g\mu/\hbar)\hat{\mathbf{B}}$ is precisely the angular velocity of *spin precession* in the magnetic field $\hat{\mathbf{B}}$.

Next, we analyze the one-body potential $\hat{v}$ in moving coordinates. By (3.56), $\hat{v}$ is complex-valued, unless

$$\mathrm{div}_{\hat{g}}\hat{\mathbf{f}}(y)\equiv\frac{1}{\sqrt{\hat{g}(y)}}\frac{\partial}{\partial y^k}[\sqrt{\hat{g}(y)}\hat{f}^k(y)]=0\ , \tag{3.62}$$

i.e., unless the vector field $\hat{\mathbf{f}}$ is divergence free. A divergence-free vector field generates a volume-preserving flow $\phi(t,\cdot)$, hence

$$\hat{g}(y)\equiv\det(\hat{g}_{kl}(t,\mathbf{y}))=g(t,\phi(t,\mathbf{y}))\ . \tag{3.63}$$

Thus, for volume-preserving (i.e., incompressible) flows, and *only* for such flows, $\hat{v}$ is again real-valued. [This is because if volume is preserved by $\phi(t,\cdot)$ then, by (3.63), the quantum-mechanical time evolution in the moving coordinate system preserves probabilities with respect to the volume element $[g(t,\phi(t,\mathbf{y}))]^{1/2}d^3\mathbf{y}$ and hence is generated by a Hermitian (self-adjoint) Hamiltonian!] But $\hat{v}$ contains an additional term, $-(m/2)\hat{f}_k\hat{f}^k$, that was not present in the original one-body potential [see Eq. (3.56)]. What does it correspond to physically? It is the potential of the centrifugal force, because $(m/2)(\partial/\partial y^l)(\hat{f}_k\hat{f}^k)$ is precisely the lth component of the centrifugal force at the point $\mathbf{y}$, at time t. [Note, incidentally, that $(m/2)\hat{f}_k\hat{f}^k$ is the classical kinetic energy of the particle in the time-independent frame, which must be subtracted in the y coordinates.]

In conclusion, we find that quantum mechanics in moving coordinates is Hamiltonian, with a Hermitian (but possibly still time-dependent) Hamilton operator, if, and only if, the flow $\phi(t,\cdot)$ defining the moving coordinate system is volume preserving, or incompressible. Henceforth this property is usually assumed. It is worthwhile recalling that in *two* space dimensions, incompressible flows are automatically symplectic (Hamiltonian) flows, because the vector fields generating them are divergence free and hence are dual to the gradient of some (scalar) Hamilton function. (This is the basis of a mathematical analysis of the two-dimensional Euler equations.)

An interesting consequence of formulating quantum mechanics in moving coordinates is a quantum-mechanical version of Larmor's theorem, in which the spin degrees of freedom of particles moving in a (variable) external magnetic field are also taken into account.

For definiteness, let us consider a system of particles with effective mass m, charge q, spin s, and magnetic moment μ_{spin} [where $\boldsymbol{\mu}_{\mathrm{spin}}=(g\mu/\hbar)\mathbf{S}$, with $\mu=q\hbar/(2m_0c)$, and m_0 is the mass of the particles in the vacuum; see Eq. (2.7)]. Furthermore, for simplicity, we choose the background manifold M to be Euclidean space $\mathbb{E}^2$ or $\mathbb{E}^3$, i.e., the metric takes the form $g_{ij}(x)=\delta_{ij}$, and all geometrical contributions to the SU(2) connection $w(x)$ are absent; see Eqs. (3.28)–(3.30). We now suppose that the system is under the influence of a (variable) external magnetic field $\mathbf{B}(x)$ and assume that there is no external electric field; $\mathbf{E}(x)=0$. As we shall see shortly, it is convenient to work in a U(1) gauge where $\mathrm{div}\,\mathbf{A}(x)=0$ (Coulomb gauge). The quantum-mechanical Larmor theorem then states the following: *For the system just described, the effect of an external magnetic field* $\mathbf{B}(x)$ *can be eliminated to first order by choosing to work in moving coordinates generated by the velocity field* $\mathbf{f}(x)=-(q/mc)\mathbf{A}(x)$ *and by performing an* SU(2) *gauge transformation* $U^{(s)}(R(x))$ *where the local frame rotation* $R(x)$ *is given by (3.45) with* $\gamma=g(m/m_0)$.

Note that the vorticity field $\mathbf{\Omega}(x)\equiv\frac{1}{2}\,\mathrm{curl}\,\mathbf{f}(x)=-q/(2mc)\mathbf{B}(x)$ of the velocity field $\mathbf{f}(x)$ is precisely the so-called Larmor angular velocity. Choosing the vector potential $\mathbf{A}(x)$ in the Coulomb gauge renders the flow generated by $\mathbf{f}(x)$ divergence free (i.e., volume preserving)!

The proof of this theorem is straightforward. We recall that $w_j^{(s)}(x)=0=a_0(x)$ by (3.28)–(3.30), and (3.36)–(3.38). Then, adopting the identifications given in the theorem, it follows from Eqs. (3.55) and (3.56), using (3.49) and (3.50), that in moving coordinates

$$\hat{\mathscr{D}}_0=\frac{\partial}{\partial y^0}+O\left[\max\left[|\hat{\mathbf{f}}|^2(y),\ \left|\frac{\partial}{\partial y^k}\hat{\mathbf{B}}\right|(y)\right]\right],$$

$$\hat{D}_k=\frac{\partial}{\partial y^k}+O\left[\left|\frac{\partial}{\partial y^k}\hat{\mathbf{B}}\right|(y)\right], \quad (3.64)$$

and

$$\hat{v}(y)=v(t,\phi(t,\mathbf{y}))+O(|\hat{\mathbf{f}}|^2(y))$$

[see also Eqs. (3.60) and (3.61)]. Note that, if the external magnetic field is constant (in the moving coordinates), $\hat{\mathbf{B}}(y)=\hat{\mathbf{B}}_0$, then the "tidal vector potential" $\hat{\mathbf{f}}(y)$ may be written as

$$\hat{\mathbf{f}}(y)=\hat{\mathbf{\Omega}}\times\mathbf{y}=\frac{q}{2mc}\mathbf{y}\times\hat{B}_0 . \quad (3.65)$$

Hence, in this situation, terms of order $O(|(\partial/\partial y^k)\hat{\mathbf{B}}|(y))$ are absent from Eq. (3.64), and $O(|\hat{\mathbf{f}}|^2(y))=O(|\hat{\mathbf{B}}_0|^2)$. Before we turn to some applications of the formalism presented in this section, we wish to emphasize, once again, that it applies equally well to (one-), two-, and three-dimensional systems.

It often happens in solid-state physics, e.g., in two-dimensional heterostructures used in measurements of the quantum Hall effect, that the system exhibits an approximate internal symmetry described by some compact group G. The spinors $\psi^{(s)\natural}$ then transform according to some nontrivial representation π of G. A breaking of G might be described as the effect of coupling $\psi^{(s)\natural}$ to an external gauge field in the representation π_* of the Lie algebra of G. Let us denote this gauge field by z. By modifying the covariant derivatives given in Eq. (3.26),

$$\mathscr{D}_\mu\mapsto\mathscr{D}'_\mu=\mathscr{D}_\mu+z_\mu(x) , \quad (3.66)$$

we may easily extend the entire formalism developed in this section to systems with gauged internal symmetries. This can be important in applications.

Note that, in this situation, the action S_Λ introduced in Eq. (3.34) is $U(1)\times SU(2)\times G$ gauge invariant, i.e., it does not change if, for an arbitrary real-valued function χ, an SU(2)-valued function R and a G-valued function g, all defined over space-time $\mathbb{R}\times M$, the following substitutions are made:

$$\psi^{(s)}(x)\mapsto e^{-i\chi(x)}U^{(s)}(R(x))\otimes\pi(g(x))\psi^{(s)}(x) ,$$

$$a_\mu(x)+\frac{m}{\hbar}f_\mu(x)\mapsto a_\mu(x)+\frac{m}{\hbar}f_\mu(x)+\partial_\mu\chi(x) , \quad (3.67)$$

$$w_\mu^{(s)}(x)\mapsto U^{(s)}(R(x))w_\mu^{(s)}(x)U^{(s)}(R(x))^* + U^{(s)}(R(x))\partial_\mu U^{(s)}(R(x))^* ,$$

and

$$z_\mu(x)\mapsto\pi(g(x))z_\mu(x)\pi(g(x))^*+\pi(g(x))\partial_\mu\pi(g(x))^* .$$

Thus, barring gauge anomalies, which actually *cannot* appear in systems of finitely many nonrelativistic particles, the nonrelativistic quantum mechanics of such systems is $U(1)\times SU(2)\times G$ gauge invariant. Ward identities expressing this gauge invariance turn out to play an important role in establishing certain universal properties of such systems; see Fröhlich and Studer (1992a–1992d) and Sec. V.

IV. SOME KEY EFFECTS RELATED TO THE U(1)×SU(2) GAUGE INVARIANCE OF NONRELATIVISTIC QUANTUM MECHANICS

Before we turn to our main topic, the analysis of two-dimensional, incompressible quantum fluids and their relation to one-dimensional chiral current algebras, we wish to describe some effects in quantum mechanics from the point of view of its $U(1)_{\text{em+tidal}}\times SU(2)_{\text{spin}}$ gauge invariance. We continue and expand the discussion started at the end of Sec. II. Most of the material reviewed here is well known, but our perspective, emphasizing gauge invariance, may be somewhat novel.

A. "Tidal" Aharonov-Bohm and "geometric" Aharonov-Casher effects

After what we have learned in Sec. III.C on the U(1) vector potential of the Coriolis force, present in moving coordinates [see Eq. (3.59)], it is clear that there must exist a "tidal" Aharonov-Bohm effect: Consider a mass-current-conducting superfluid in a large container penetrated by some straight cylindrical tube that excludes the quantum fluid. Now, set the fluid into circular motion around the axis of the tube with velocity field $\mathbf{f}$, where $|\mathbf{f}(r)|=\kappa/(2\pi r)$ at a distance r from the axis of the tube, and κ is a quantity of dimension cm^2/sec, the total vorticity or circulation: $\kappa=\oint\mathbf{f}\cdot dl$. (Note that $\kappa=2\pi\bar{L}_z/NM$, where M is the mass of the particles constituting the quantum fluid, $\bar{L}_z$ is the expectation value of the component of the total angular momentum operator parallel to the tube in the given state of the system, and N is the number of particles in the system.) Small mass currents excited in this system, scattered at the tube, will exhibit an Aharonov-Bohm effect depending periodically on κ, with period h/m, where m is the mass of the particles in the scattering currents; compare to Sec. II.C.

While this effect may be somewhat difficult to test experimentally, it is important theoretically: Consider a superfluid film with manifestly (e.g., through rotation) or spontaneously broken time-reversal and reflections-in-lines (i.e., two-dimensional parity) invariance. The formation of such films at a particular superfluid ^{3}He-A/B interface has been discussed by Salomaa and Volovik (1989). Such a two-dimensional superfluid will, in general, exhibit vortex excitations of vorticity $\kappa=n(h/M)$, $n\in\mathbb{Z}$, where M is the mass of the constituent particles in the superfluid, and fractional mass (rather than fractional charge) $\sigma_H\kappa$, where $\sigma_H=\sigma(M^2/h)$ is a "tidal Hall conductivity." Such excitations give rise to Aharonov-Bohm phases and hence are anyons if σ is not an integer, i.e., if the superfluid shows a fractional "tidal" quantum Hall effect; more details on this effect will be given in Sec.

VII.B. Such excitations may be observed experimentally by measuring fluctuations in the longitudinal resistance of superfluid current conduction. See Simmons *et al.* (1989) and Hwang *et al.* (1992) for an analogous experiment in two-dimensional electronic systems exhibiting the fractional quantum Hall effect.

A remarkable experimental observation of a "tidal" Aharonov-Bohm effect has been provided by Werner, Staudenmann, and Colella (1979). Using a neutron interferometer they have detected a quantum-mechanical interference effect due to the rotation of the Earth. For a brief theoretical comment on this interference experiment that is close to our discussion in Sec. III.C, see Sakurai (1980).

Other interesting systems where mixed "tidal" and electromagnetic effects play a role are, for example, rotating superconductors. Following an analysis by Semon (1982; see also Schmutzer and Plebański, 1977), an experiment performed by Zimmerman and Mercereau (1965) can be interpreted as the realization of a thought experiment proposed by Aharonov and Carmi (1973): Given a (uniformly) rotating sample that is not simply connected, the "tidal" forces (Coriolis and centrifugal force) felt by the particles in the moving system (all of the same charge-to-mass ratio) can be cancelled by an electromagnetic field whose vector potential does not cancel the "tidal" vector potential completely everywhere; see Eq. (3.59). This uncancelled "tidal" vector potential then leads to a quantum-mechanical interference effect.[4]

In Sec. II.D we have discussed the Aharonov-Casher effect as an SU(2) version of the Aharonov-Bohm effect. Furthermore, in Sec. III.B, systems in geometrically nontrivial backgrounds have been described by including a "geometrical" term in the SU(2) connection w; see Eqs. (3.28)–(3.31) and (3.11). Here, we combine these findings and discuss a "geometrical version" of the Aharonov-Casher effect: We consider a two-dimensional system of particles with nonzero spin on a cone with the tip at $\mathbf{x}=0$. Then, although $\rho=0$ if there are no electromagnetic fields, the SU(2) connection w determined by $\omega^{A}{}_{B}$, the affine spin connection on the cone, cannot be gauged away globally, although $\omega^{A}{}_{B}$ is flat for $\mathbf{x}\neq 0$. The SU(2) connection w has the same form as the electromagnetic part ρ given in Eq. (2.20), but Q now denotes the defect angle of the cone. Scattering of particles at the tip of the cone will yield interference patterns depending on the defect angle Q. This effect is perhaps better known than its electromagnetic cousin. It has attracted attention, for example, in connection with quantum mechanics and quantum field theory in the presence of cosmic strings (Deser and Jackiw, 1988, 't Hooft, 1988; Kay and Studer, 1991).

Do spinless particles "see" the tip of the cone, or is spin important? The answer depends on our choice of a quantum-mechanical state space: We must impose some "boundary conditions" on the wave functions, i.e., $\psi(r,\varphi+2\pi-Q)=e^{i\theta}\psi(r,\varphi)$, where φ is the polar angle, and θ is some phase to be specified, plus some boundary condition at $r=0$. But no matter how we choose θ, we can make the tip of the cone "invisible" to spinless particles by threading a magnetic flux through $\mathbf{x}=0$. If the particles have spin *and* a nonzero magnetic moment then, in addition, we would have to put a charged wire through $\mathbf{x}=0$, in order to make the tip "invisible."

Finally, we remark that there is also an analog of the Aharonov-Casher effect in which SU(2) is replaced by a gauged internal symmetry group G. This effect can, perhaps, be tested in inhomogeneous heterostructures. It is related, both physically and mathematically, to the existence of particles in two-dimensional quantum theory with topological pair interactions described by a G-Knizhnik-Zamolodchikov connection that, just as in the case of SU(2), may give rise to non-Abelian braid statistics; see Sec. V.III.

B. Flux quantization and SU(2) monopoles

A superconductor exhibits the Meissner-Ochsenfeld effect: A magnetic field cannot penetrate into the bulk of a superconducting material. However, in a type-II superconductor, thin magnetic-field tubes can thread through the bulk. They have the property that they carry a magnetic flux Φ which is an integer multiple of hc/q, where q is the charge of the particles in the condensate (e.g., $q=-2e$, for BCS pairs of electrons). These tubes are called Abrikosov vortices. The quantization of Φ is explained by requiring that, outside an Abrikosov vortex, the superconducting state of the system remain undisturbed. From what we have said about the Aharonov-Bohm effect in Sec. II.C it follows that this requirement is fulfilled precisely if Φ is an integer multiple of hc/q. Experimentally, this effect has been established first in Deaver and Fairbank (1961) and Doll and Näbauer (1961; for a review, see, for example, Chap. 6 in Tilley and Tilley, 1986).

The discussion of the "tidal" Aharonov-Bohm effect above makes it clear that the Meissner-Ochsenfeld effect and flux quantization in Abrikosov vortices have their counterparts in the theory of superfluidity: Consider a superfluid in some container. Now set the container into uniform rotation. The superfluid inside the container abhors angular velocity which could destroy its superfluidity and does *not*, therefore, follow the rotation of the container's walls. However, just as there can be Abrikosov vortices in a type-II superconductor, the superfluid can eventually be set into motion, and the motion is generated by a velocity field $\mathbf{f}$, whose circulation (or vorticity) $\mathbf{\Omega}=\frac{1}{2}\,\mathrm{curl}\,\mathbf{f}$ [see Eq. (3.44)] is localized along thin tubes. The "tidal" Aharonov-Bohm effect then predicts that the total circulation in such a tube is quantized in integer multiples of h/M, where M is the mass of the particles (e.g., ^{3}He pairs) constituting the

[4]We thank F. Jaroslav for bringing the references of this paragraph to our attention.

superfluid. (This can also be understood by appealing to the quantization of orbital angular momentum.) If, in such a fluid, one can excite mass currents of quantum-mechanical particles, "dopants" of mass $m < M$, one may be able to test the "tidal" Aharonov-Bohm effect.

Our conclusions agree with another theoretical analysis given by Pines and Nozières (1989). Experimentally, the first verification of the quantization of circulation (in superfluid He II) was given by Vinen (1961; for a review, see, for example, Chap. 6 in Tilley and Tilley, 1986). The phenomena described here may be relevant in the astrophysics of rotating neutron stars (pulsars), which appear to be superfluid (see, for example, Tsakadze and Tsakadze, 1980).

Now that we have discussed "topological" field configurations connected to the U(1) gauge invariance of quantum theory (i.e., vortices with associated quantization of flux or circulation), we may ask whether there are corresponding configurations related to the non-Abelian SU(2) gauge invariance of quantum theory.

For this purpose we recall that Abrikosov vortices can be thought of as critical configurations of a Landau-Ginzburg functional for the free energy of a type-II superconductor which have finite energy per unit length. If we restrict our attention to a plane $\mathbb{E}^2$ intersecting the bulk of a superconductor transversally to an applied magnetic field penetrating the bulk in the form of vortex tubes, then vortex configurations are characterized by the winding number of the phase of the "order parameter" ϕ, i.e., by the winding number of the map $[\mathbf{x}=(x_1,x_2)\in\mathbb{E}^2]$

$$\left.\frac{\phi(\mathbf{x})}{|\phi(\mathbf{x})|}\right|_{|\mathbf{x}|=R\to\infty} : S_R^1 \to S^1 , \tag{4.1}$$

which maps a circle S_R^1 of radius R ($\to\infty$), in the cross section $\mathbb{E}^2$ through the superconductor to the circle $S^1 \simeq \mathrm{U}(1)$. We recall that, for configurations of finite energy per unit length, the winding number—which is an integer—equals the magnetic flux through the vortex tubes threading the superconductor, in units of hc/q. This provides a "topological" explanation of the quantization of the flux through vortex lines in a superconductor. We note that, from a field-theoretic point of view, the restrictions of the above vortex configurations to a planar cross section coincide with "static, finite-energy" configurations of a U(1) Higgs model in $2+1$ dimensions (see, for example, Jaffe and Taubes, 1980).

Next, for non-Abelian Higgs models in $3+1$ dimensions, "topological" configurations are also known to exist. They are called monopoles. Let us take a closer look at these configurations in the example where the gauge group is SU(2) and where the "order parameter" (i.e., the Higgs field) ϕ takes its values in the Lie algebra su(2): $\phi=i\sum_A \phi_A\sigma_A$, with σ_A, $A=1,2,3$, the three Pauli matrices. Under gauge transformations ϕ transforms according to the adjoint representation of SU(2). Again, "static, finite-energy" configurations are classified by a topological index $[\phi]\in\pi_2(S^2)\simeq\mathbb{Z}$, the so-called monopole number, with $[\phi]$ the homotopy class of the mapping

$$\left.\frac{\phi(\mathbf{x})}{|\phi(\mathbf{x})|}\right|_{|\mathbf{x}|=R\to\infty} : S_R^2 \to S^2 \simeq \mathrm{SU}(2)/\mathrm{U}(1) , \tag{4.2}$$

where S_R^2 is a 2-sphere of radius R in physical space $\mathbb{E}^3$ (Jaffe and Taubes, 1980). If the parameters in the SU(2) Higgs model assume particular values (Bogomol'nyi limit), explicit solutions to the corresponding field equations are known with $[\phi]=\pm1$. These (static) solutions are called Prasad-Sommerfield monopoles. If "located" at the origin ($r=|\mathbf{x}|=0$) they are given by

$$w_{0A}(\mathbf{x})=0 ,$$
$$w_{kA}(\mathbf{x})=-\frac{1}{2}\left[\frac{1}{\sinh r}-\frac{1}{r}\right]\varepsilon_{kAj}\frac{x_j}{r} , \tag{4.3}$$

and

$$\phi_A(\mathbf{x})=\mp\frac{1}{2}\left[\frac{1}{\tanh r}-\frac{1}{r}\right]\frac{x_A}{r} ,$$

with the SU(2) connection 1-form given by $w=i\sum_{\mu,A} w_{\mu A}(\mathbf{x})\sigma_A\,dx^\mu$.

Physical systems where configurations of this type might arise are $(3+1)$-dimensional quantum spin liquids that are characterized by an order parameter ϕ transforming under the adjoint representation of SU(2) and that involve coupling to an SU(2) gauge field w.

C. Barnett and Einstein–de Haas effects

The Barnett and Einstein–de Haas effects (see, for example, Landau and Lifshitz, 1960) find a very natural explanation in the light of the quantum-mechanical Larmor theorem discussed at the end of Sec. III.C. Consider a cylinder of iron or some other ferromagnetic material suspended at a wire in such a way that it can freely rotate around its axis. Let us suppose that, initially, the cylinder is at rest and demagnetized. Now, imagine that the cylinder is set into rapid rotation around its axis. As explained in Sec. III.C, the quantum mechanics of the electrons in this material should now be described in a uniformly rotating coordinate system fixed to the cylindrical background. In this coordinate system the electronic Hamiltonian will be time independent, but it now contains a Zeeman term

$$\hat{\psi}^*\hat{\mathbf{\Omega}}\cdot\frac{\hbar}{2}\sigma\hat{\psi} , \tag{4.4}$$

where $\hat{\mathbf{\Omega}}=\mathrm{const}$ is the angular velocity of the rotation [see Eq. (3.60), where we have set $\gamma=1$, i.e., by Eqs. (3.44)–(3.46), the orthonormal frame in the (co)tangent space ("spin space") rotates with the same angular velocity as the rotating coordinate system]. Note that, in Eq. (4.4), $\hat{\mathbf{\Omega}}$ plays the role of the magnetic field $\mathbf{B}$ in an inertial frame. Furthermore, the Hamiltonian contains a tidal vector potential $\hat{\mathbf{f}}=\hat{\mathbf{\Omega}}\times\mathbf{y}$ in the covariant derivatives

$\hat{\mathcal{D}}_k$, and a potential $-(m/2)|\hat{\mathbf{\Omega}}\times\mathbf{y}|^2$ of the centrifugal forces; see Eqs. (3.55) and (3.65), and (3.56), respectively. All the additional terms can be combined into the term

$$\hat{\psi}^*\hat{\mathbf{\Omega}}\cdot\hat{\mathbf{J}}\hat{\psi}\ , \tag{4.5}$$

where $\hat{\mathbf{J}}=\hat{\mathbf{S}}+\hat{\mathbf{L}}$ is the total angular-momentum operator (see also Kerman and Onishi, 1981). The effect of centrifugal forces will be balanced by the chemical potential of the background. Thus the electronic Hamiltonian is essentially equivalent to the one for a cylinder at rest but in a magnetic field $\mathbf{B}=(g\mu_B/\hbar)^{-1}\mathbf{\Omega}$. The result, in both situations, is that the cylinder is magnetized, because the spins of the electrons will align with $\mathbf{\Omega}$ or $\mathbf{B}$, respectively. This is the Barnett effect.

Conversely, in the Einstein–de Haas effect, one turns on a magnetic field $\mathbf{B}$ antiparallel to the spontaneous magnetization of a magnetized piece of iron at rest, thereby increasing the free energy of the system. The system reacts to this perturbation by starting to rotate around the axis of the external magnetic field so as to offset the effect of $\mathbf{B}$ on the electrons by rotation. It thereby returns to a state corresponding to a local minimum of the free energy. By Eqs. (3.60) and (3.61), the angular velocity of this rotation, $\mathbf{\Omega}$, is given by $\mathbf{\Omega}=(g\mu_B/\hbar)\mathbf{B}$, which is precisely the angular velocity of spin precession in the magnetic field $\mathbf{B}$. A similar effect is observed when one tries to magnetize a paramagnet. It would appear interesting to test a local version of this effect in a "ferrofluid." If the magnetic field acting on a highly mobile ferrofluid, locally in thermal equilibrium, is modified locally the fluid reacts by starting to flow with a velocity field that optimally offsets the change in the magnetic field so as to restore local equilibrium. The particle and magnetic current densities induced are given by $n\,\mathbf{f}$ and $\mathbf{M}\otimes\mathbf{f}$, respectively, where $\mathbf{f}$ is the velocity field, n the particle density, and $\mathbf{M}$ the magnetization density. A somewhat analogous effect for quantum Hall fluids will be discussed in Sec. V.B.

There is another variant (Bell and Leinaas, 1983, 1987) of the Barnett effect: Consider a beam of nonrelativistic particles, e.g., heavy ions, with spin, moving in a storage ring with some mean angular velocity $\mathbf{\Omega}$. Then they experience a "tidal" Zeeman effect, given by Eq. (3.60), in addition to the usual magnetic Zeeman effect, given by Eq. (3.61). After relaxation to a steady state, the "tidal" Zeeman energy obviously affects the ratio of "spin-up" to "spin-down" ions in the beam! Similar considerations are important, e.g., in the study of electronic spectra of rotating molecules in the Born-Oppenheimer approximation (Kerman and Onishi, 1981).

D. Meissner-Ochsenfeld effect, London, and Landau-Ginzburg theories of superconductivity

Consider a superconducting condensate of charged bosons (e.g., electron pairs) of charge q and mass M, in equilibrium. Imagine that a (weak) magnetic field $\mathbf{B}_c$ is turned on inside the bulk of this system, resulting in an increase of the free energy of the system (a weak form of the Meissner-Ochsenfeld effect). Then the condensate reacts to the field $\mathbf{B}_c$ by starting to flow according to a velocity field $\mathbf{f}$ in such a way as to cancel $\mathbf{B}_c$ in the moving coordinate system. For, in this way, the free energy in regions of the system moving with the flow is minimal. Neglecting the centrifugal potential $-(m/2)\mathbf{f}\cdot\mathbf{f}$ and (in a first step) the magnetic field created by the resulting current, it follows from Eqs. (3.58), (3.59), and (3.62) that the optimal velocity field $\mathbf{f}$ is given by

$$\mathbf{f}=-\frac{q}{Mc}\mathbf{A}_c^T\ ,$$

where $\mathbf{A}_c^T$ is the vector potential of $\mathbf{B}_c$ in the Coulomb gauge (i.e., $\operatorname{div}\mathbf{A}_c^T=0$). Thus the system exhibits a supercurrent density $\mathbf{J}_s$ given in our approximation by

$$\mathbf{J}_s\simeq qn_s\mathbf{f}=-\frac{q^2n_s}{Mc}\mathbf{A}_c^T\ , \tag{4.6}$$

where n_s is the density of the condensate. Of course, this supercurrent $\mathbf{J}_s$ will given rise to an additional vector potential $\mathbf{A}_s^T$, determined by Maxwell's equation: $\Delta\,\mathbf{A}_s^T=-(1/c)\mathbf{J}_s$. Adding this potential to the rhs of Eq. (4.6), we obtain the equation

$$\mathbf{J}_s=-\frac{q^2n_s}{Mc}\mathbf{A}^T\ , \tag{4.7}$$

where $\mathbf{A}^T=\mathbf{A}_c^T+\mathbf{A}_s^T$ is the total vector potential of the external magnetic field $\mathbf{B}_c$ and the magnetic field created by the supercurrent. This is the London equation characterizing the superconducting state! Relating the external magnetic field $\mathbf{B}_c$ to an external current density $\mathbf{J}_c$ via Maxwell's equation $\Delta\,\mathbf{A}_c^T=-(1/c)\mathbf{J}_c$, and assuming that the external current $\mathbf{J}_c$ does not enter into the bulk of the superconductor, i.e., $\mathbf{J}_c=0$ inside the superconducting region, we see that the London equation (4.7) immediately implies the equation

$$\Delta\,\mathbf{A}^T=\frac{q^2n_s}{Mc^2}\mathbf{A}^T\ , \tag{4.8}$$

which shows that, in a stationary state, currents and magnetic fields in superconductors can exist only within a surface layer of thickness $\Lambda=(Mc^2/q^2n_s)^{1/2}$, the so-called London penetration depth (see, for example, de Gennes, 1966). This is the Meissner-Ochsenfeld effect. Note that by Eq. (4.7) a supercurrent $\mathbf{J}_s$ is really a sign for the presence of a vector potential $\mathbf{A}^T$ and thus can be used for experimental tests of the Aharonov-Bohm effect. However, if $\oint_\Gamma \mathbf{A}^T\cdot dl=n(hc/q)$, $n\in\mathbb{Z}$, for any closed curve Γ contained in the superconducting phase, then the Aharonov-Bohm phase factors of the charged bosons are trivial (see Sec. II.C), and no supercurrent results. This is the phenomenon of flux quantization.

The expulsion of a magnetic field from the interior of a superconductor is related to the fact that, inside a superconductor, "photons are massive," i.e., one observes the phenomenon of the Anderson-Higgs mechanism. We now show that, in a superconductor, the presence of a "mass term for the photons" can also be inferred directly from the London equation (4.7): Since the electric current density $\mathbf{J}$ and the free energy $F(\mathbf{A})$ of a system in the presence of a (static) electromagnetic field with vector potential $\mathbf{A}=(A_1, A_2, A_3)$ are related by

$$\mathbf{J}(\mathbf{x})=-c\frac{\delta F(\mathbf{A})}{\delta \mathbf{A}(\mathbf{x})} , \tag{4.9}$$

it follows from Eq. (4.7) that, in the bulk of a superconductor,

$$F(\mathbf{A})=\frac{1}{2\Lambda^2}\int d^3\mathbf{x}\ \mathbf{A}^T(\mathbf{x})\cdot \mathbf{A}^T(\mathbf{x})+\cdots , \tag{4.10}$$

where $A_i^T(\mathbf{x})=[\delta_i^j-\partial_i\Delta^{-1}\partial^j]A_j(\mathbf{x})$, with Δ the three-dimensional Laplacian, and the ellipses stand for higher-derivative terms. Notice that Eq. (4.10) is a *nonlocal* functional of $\mathbf{A}$, which is manifestly invariant under U(1) gauge transformations, $\mathbf{A}\mapsto\mathbf{A}+\nabla\chi$. It provides a "mass term for the photons" in the bulk of a superconductor.

There is an SU(2) analog of these effects in superfluid condensates of neutral bosons with magnetic moments, e.g., superfluid ^{3}He in the B phase, where the bosons are pairs of ^{3}He atoms in spin-triplet states. One then encounters an "SU(2) Anderson-Higgs mechanism."

Next, we show that the London equation (4.7) can be interpreted as the Euler-Lagrange equation corresponding to an action functional. As a first step towards finding this action, we have to look for an extension of London theory describing time-dependent phenomena. The right modification of Eq. (4.7) is given by

$$J^\mu(x)=\frac{c}{\Lambda^2}\eta^{\mu\nu}A_\nu^T(x) , \tag{4.11}$$

where $\eta^{00}=1$, $\eta^{ii}=-1$, $\eta^{\mu\nu}=0$, for $\mu\neq\nu$, and A_ν^T is defined in such a way that $\partial^\nu A_\nu^T=0$. Thus the Fourier transform $\hat{A}_\nu^T(k)$ of A_ν^T is given by

$$\hat{A}_\mu^T(k)=\frac{k^2\delta_\mu^\nu-k_\mu k^\nu}{k_0^2-\omega(\mathbf{k})^2}\hat{A}_\nu^T(k) . \tag{4.12}$$

In a superconductor with a gapless Goldstone mode we have

$$\omega(\mathbf{k})\simeq\frac{c}{n}|\mathbf{k}| , \tag{4.13}$$

for $\mathbf{k}\simeq 0$. We renormalize wave vectors in such a way that $n=1$ and neglect terms of order $|\mathbf{k}|^2$ in $\omega(\mathbf{k})$. The Fourier transform of Eq. (4.12) then reads

$$A_\mu^T(x)=([\delta_\mu^\nu-\partial_\mu\Box^{-1}\partial^\nu]A_\nu)(x) , \tag{4.14}$$

with $x=(ct,\mathbf{x})$ an arbitrary space-time point. Inserting Eq. (4.14) in (4.11) we find that J^μ satisfies the continuity equation

$$\partial_\mu J^\mu(x)=0 , \tag{4.15}$$

as it should.

In electrodynamics (in 3+1 dimensions) it is natural to interpret the electric current density as a 3-form,

$$\mathcal{J}=\frac{1}{3!}\sum_{\mu,\nu,\rho}\mathcal{J}_{\mu\nu\rho}(x)dx^\mu\wedge dx^\nu\wedge dx^\rho , \tag{4.16}$$

where

$$\mathcal{J}_{\mu\nu\rho}(x)=\varepsilon_{\sigma\mu\nu\rho}J^\sigma(x) , \tag{4.17}$$

and $\varepsilon_{\sigma\mu\nu\rho}$ is the totally antisymmetric ε tensor.

Mathematicians write this as

$$\mathcal{J}=*J , \tag{4.18}$$

where $*$ is the so-called Hodge $*$ operation, which associates to a p form a "dual" $(4-p)$ form, $p=0,1,\ldots,4$; moreover, we recall that, in 3+1 dimensions, $**=(-1)^p$ when acting on a p form. The continuity equation (4.15) now reads

$$d\mathcal{J}=\frac{1}{3!}\sum_{\mu,\nu,\rho}\partial_\alpha\mathcal{J}_{\mu\nu\rho}(x)dx^\alpha\wedge dx^\mu\wedge dx^\nu\wedge dx^\rho$$
$$=\sum_\alpha\partial_\alpha J^\alpha(x)dx^0\wedge dx^1\wedge dx^2\wedge dx^3=0 . \tag{4.19}$$

If the system is confined to a convex region in space-time, then Eq. (4.19) can be integrated. The solution is

$$\mathcal{J}=db , \tag{4.20}$$

where the "potential" $b=\frac{1}{2}\sum_{\mu\nu}b_{\mu\nu}dx^\mu\wedge dx^\nu$ is a 2-form, i.e., an antisymmetric, second-rank tensor field (just like the electromagnetic field tensor). The generalized London equation (4.11) then reads

$$*db=-\frac{c}{\Lambda^2}A^T . \tag{4.21}$$

Taking the exterior derivative of Eq. (4.21), we find that

$$d*db=-\frac{c}{\Lambda^2}F , \tag{4.22}$$

where

$$F=\tfrac{1}{2}\sum_{\mu,\nu}F_{\mu\nu}(x)dx^\mu\wedge dx^\nu ,$$

with

$$(F_{\mu\nu})=\begin{pmatrix}0 & E_1 & E_2 & E_3\\ -E_1 & 0 & -B_3 & B_2\\ -E_2 & B_3 & 0 & -B_1\\ -E_3 & -B_2 & B_1 & 0\end{pmatrix} . \tag{4.23}$$

Equation (4.22) is analogous to Maxwell's equations, but is an equation between 2-forms (rather than 1-forms). It is the Euler-Lagrange equation derived from the action

$$S(b;A)=\frac{\Lambda^2}{2c^3}\int db\wedge *db+\frac{1}{c^2}\int db\wedge A , \tag{4.24}$$

where

$$db\wedge *db=\sum_{\mu,\nu,\rho}\partial_{[\mu}b_{\nu\rho]}(x)\partial^{[\mu}b^{\nu\rho]}(x)d^4x ,$$

with the brackets [ ] indicating antisymmetrization in the corresponding indices and

$$db \wedge A = \varepsilon^{\mu\nu\rho\sigma}\partial_\mu b_{\nu\rho}(x) A_\sigma(x) \ .$$

In quantum mechanics J^μ is an operator-valued distribution. Therefore we have to quantize the 2-form potential b of the electric current density. For the time being, however, we shall treat the vector potential A as a classical, external field. Since the action $S(b;A)$ is quadratic in b, Feynman path-integral quantization is convenient. The quantum mechanics of the 2-form potential b is described, in London theory, by the functional measure

$$Z(A)^{-1}\exp\left[\frac{i}{\hbar}S(b;A)\right]\mathcal{D}b \ , \tag{4.25}$$

where the partition function $Z(A)$ is chosen such that the integral of (4.25) is unity. It is easy to see that this implies that

$$Z(A)=\exp\left[\frac{i}{\hbar}S^{\text{eff}}(A)\right] , \tag{4.26}$$

where

$$S^{\text{eff}}(A)=\frac{1}{2\Lambda^2 c}\int A^T_\mu(x) A^{T\mu}(x) d^4x \ . \tag{4.27}$$

It follows from Eqs. (4.24) and (4.25) that

$$\langle J^\mu(x)\rangle = c^2 \frac{\delta S^{\text{eff}}(A)}{\delta A_\mu(x)} \ , \tag{4.28}$$

where $\langle \ \rangle$ denotes the quantum-mechanical expectation value for a system that starts its history in its ground state; see also Sec. V. Plugging Eq. (4.27) into (4.28), we find that

$$\langle J^\mu(x)\rangle = \frac{c}{\Lambda^2} A^{T\mu}(x) \ . \tag{4.29}$$

Hence we have recovered the London equation (4.11).

Next, we should ask what kind of quantum-mechanical system is described by the functional measure (4.25), with $S(b;A)$ given by Eq. (4.24). We start by noticing that Eq. (4.20) determines the 2-form potential b only up to gauge transformations: If we define b' by setting

$$b' = b + d\chi \ , \tag{4.30}$$

that is,

$$b'_{\mu\nu}(x) = b_{\mu\nu}(x) + \partial_{[\mu}\chi_{\nu]}(x) \ ,$$

where χ is an arbitrary 1-form, then

$$db' = db \ ,$$

since $d(d\chi)\equiv 0$. The action (4.24) is obviously invariant under the gauge transformations (4.30). Furthermore, the term $(\Lambda^2/2c^3)\int db \wedge * db$ in $S(b;A)$ vanishes for every b which is a pure gauge, i.e., $b = d\chi$ and can take any positive value, for an appropriate choice of b. Thus the 2-form potential b describes gapless modes. This property and gauge invariance imply that, per wave vector $\mathbf{k}$, b describes one degree of freedom. We conclude that the quantum-mechanical current operator J^μ couples the ground state of the system to a scalar, gapless mode. This mode can be interpreted as the U(1) Goldstone boson of a type-I superconductor. Within the approximation of London theory, this Goldstone boson is apparently noninteracting.

Equilibrium states of the system at positive temperature T are described by the measure

$$dP_T(b;A) = Z_T(A)^{-1}\exp\left[-\frac{1}{\hbar}S_T(b;A)\right]\mathcal{D}b \ , \tag{4.31}$$

where

$$S_T(b;A) = \int_0^{\beta\hbar} d\tau \int d^3\mathbf{x} \left[\frac{\Lambda^2}{2c^2}(\partial_{[\mu}b_{\nu\rho]}\partial^{[\mu}b^{\nu\rho]})(\tau,\mathbf{x}) + \frac{i}{c}\varepsilon^{\mu\nu\rho\sigma}(\partial_\mu b_{\nu\rho}A_\sigma)(\tau,\mathbf{x})\right] , \tag{4.32}$$

with $\beta = 1/kT$, and periodic boundary conditions are imposed in the τ direction. In Eqs. (4.31) and (4.32), the vector potential A must be assumed to be time independent, as long as it is treated as an external field.

It is interesting to ask how the theory must be modified to account for the quantum dynamics of the electromagnetic field. The action of the electromagnetic field in the absence of matter is given by

$$S^{\text{em}}(A) = \frac{\hbar}{2e^2}\int dA \wedge * dA \ , \tag{4.33}$$

or, at positive temperature T, by

$$S^{\text{em}}_T(A) = \frac{\hbar c}{2e^2}\int_0^{\beta\hbar} d\tau \int d^3\mathbf{x}(\partial_{[\mu}A_{\nu]}\partial^{[\mu}A^{\nu]})(\tau,\mathbf{x}) \ . \tag{4.34}$$

This action will have to be added to $S_T(b;A)$ given in Eq. (4.32).

What we are after is really to understand the significance of dynamical Abrikosov vortices (typically, small vortex rings) in a type-II superconductor. Thus we must incorporate the idea of flux quantization in our formalism. We set

$$A = A_c + a \ ,$$

where A_c is the vector potential of a time-independent, external electromagnetic field and a is a dynamical vector potential, but with quantized magnetic flux. In order to be able to do explicit calculations, we must regularize the theory by introducing a short-distance cutoff. A convenient regularization is to put the theory on a space-time lattice $\Gamma_T = \mathbb{Z}_N \times \mathbb{Z}^3$, where N is proportional to $\beta = 1/kT$ and periodic boundary conditions are imposed in the τ direction. In the lattice description, we assign a real variable A_l, the lattice vector potential, to every link $l \in \Gamma_T$ and a real variable b_p, the lattice 2-form potential, to every plaquette p of the dual lattice Γ_T^*, with the convention that $A_{l^-}=-A_l$, $b_{p^-}=-b_p$, where l^- denotes the link l with reversed orientation and p^- denotes the plaquette p with reversed orientation. By c we denote a three-dimensional unit cube in Γ_T^*; moreover, if l is a link in Γ_T, $c(l)$ denotes the unit cube in Γ_T^* dual to l.

The total action for the regularized theory is then given by

$$S_T^{\text{tot}}(b;A)=\frac{\Lambda^2}{2c^2}\sum_{c\subset\Gamma_T^*}(db)_c^2+\frac{i}{c}\sum_{l\subset\Gamma_T}(db)_{c(l)}A_l+\frac{\hbar c}{2e^2}\sum_{p\subset\Gamma_T}(dA)_p^2\ , \qquad (4.35)$$

where $(db)_c=\sum_{p\subset\partial_c}b_p$, with the orientation of $p\subset\partial c$ induced by the orientation of c, and, similarly, $(dA)_p=\sum_{l\subset\partial_p}A_l$. The first two terms on the rhs of Eq. (4.35) represent the lattice approximation to the action $S_T(b;A)$ defined in Eq. (4.32), while the last term is the lattice approximation of the action $S_T^{\text{em}}(A)$ of Maxwell's theory given in Eq. (4.34).

We now recall that $A=A_c+a$, where A_c is the vector potential of a classical, external electromagnetic field of which we assume that it is time independent and has a vanishing 0 component, and a is a dynamical vector potential. The phenomenon of magnetic flux quantization described in Sec. IV.B is incorporated in our formulation by imposing the constraint

$$(da)_p=n\frac{hc}{q},\quad n\in\mathbb{Z}\ , \qquad (4.36)$$

for every plaquette p in the lattice Γ_T. The constraints (4.36) can be fulfilled by requiring that $a_1\in(hc/q)\mathbb{Z}$ (up to gauge transformations).

The model with action (4.35) and constraint (4.36) is known to be equivalent (more precisely, "dual") to a model in the universality class of the classical xy model on the space-time lattice Γ_T (see, for example, Fröhlich and Spencer, 1983). This model is a regularization of the Landau-Ginzburg theory of superconductivity. It is known to have a continuous phase transition from a superconducting phase with broken U(1) symmetry and a gapless Goldstone boson to a U(1)-symmetric high-temperature phase with rapidly decaying, connected Green functions. The same phase transition can be driven at fixed temperature $T\geq 0$ by varying the London penetration depth Λ. (The transition from the normal to the superconducting phase occurs when Λ is decreased.)

The phase transition described above could only be observed in superconducting systems of bosons that cannot disintegrate. In a realistic BCS superconductor, however, the transition from the superconducting to the normal phase is driven by the breakup of Cooper pairs.

Let us summarize the main findings of this section. We have first derived the London equation (4.7) by assuming a very weak form of the Meissner-Ochsenfeld effect. We have then shown that the generalized London equation (4.11) can be derived from an action principle, with the action given by Eq. (4.24). This enabled us to quantize the electric current density; see Eq. (4.25). The quantum theory turns out to describe the quantum mechanics of noninteracting, gapless U(1) Goldstone modes. By regularizing the theory at short distances we have been able to incorporate the effects of dynamically generated Abrikosov vortices with quantized magnetic flux. The resulting theory turns out to be in the universality class of the Landau-Ginzburg theory of superconductivity, whose phase diagram and quasiparticle spectrum (including the phenomenon of charge screening in the superconducting phase) is understood extremely well (see, for example, Fröhlich and Spencer, 1983, and references therein). For a discussion of superconductivity within the framework of algebraic statistical mechanics (also emphasizing the powerful consequences of gauge invariance), see Sewell (1992).

E. Quantum Hall effect

Just as the Aharonov-Bohm effect reflects the $U(1)_{\text{em}}$ gauge invariance of quantum theory, so does the quantum Hall effect for the electric current, as emphasized by Laughlin (1981; see also Halperin, 1982). In the same vein, the Aharonov-Casher effect and the quantum Hall effect for the spin current reflect the $SU(2)_{\text{spin}}$ gauge invariance of nonrelativistic quantum theory, as emphasized by Fröhlich and Studer (1992a, 1992c). In this section, we review some basic facts concerning the integer (von Klitzing, Dorda, and Pepper, 1980) and fractional (Tsui, Stormer, and Gossard, 1982) quantum Hall effect; for comprehensive reviews see Chakraborty and Pietiläinen (1988), Morandi (1988), Prange and Gervin (1990), and also Wilczek (1990). The purpose of this review is to set the stage for Secs. V and VI, where we attempt to unravel the universal aspects of the quantum Hall effect in two-dimensional, incompressible quantum fluids.

Experimentally, the quantum Hall effect is observed in two-dimensional systems of electrons subject to a strong (uniform) transverse magnetic field $\mathbf{B}_c$. For definiteness, we choose a Cartesian coordinate system, with the Hall system confined to a region Ω in the (x,y) plane and the magnetic field $\mathbf{B}_c=(0,0,B_c)$ along the z axis. We take Ω to be a rectangle with dimensions l_x and l_y in the x and y directions, respectively. By measuring the voltage V_x in the x direction (the difference in the chemical potentials of the electrons at the two edges in the x direction) and tuning the total electric current I_y in the y direction perpendicular to the applied voltage, one finds that the ratio

$$R_H=\frac{V_x}{I_y}\ , \qquad (4.37)$$

the so-called Hall resistance, is a constant for a fixed value of the magnetic field $\mathbf{B}_c$, a fixed density of electrons and at a constant temperature T close to absolute zero.

Two-dimensional systems of electrons (and/or holes) are realized, in the laboratory, as inversion layers. Such layers are formed at the interface between a semiconductor and an insulator or between two semiconductors, with one of them acting as an insulator (e.g., in a so-called metal-oxide-semiconductor field-effect transistor (MOSFET) or in a heterostructure made of $GaAs/Al_xGa_{1-x}As$). In the direction perpendicular to the interface an electric field is applied, which attracts electrons from the semiconductor to the interface. The electronic motion perpendicular to the interface (i.e., in the z direction) is quantized, and the energy of quantization is sufficiently large so that the electrons remain bound to the interface (i.e., in the z direction the electrons sit in a potential well). Hence, at temperatures near absolute zero, a nearly ideal, two-dimensional system of electrons is formed at the interface.

In classical physics, the connection between the electric field $\mathbf{E}=(E_x,E_y)$ in the plane of the system and the electric current density $\mathbf{J}=(J_x,J_y)$ is given by the Ohm-Hall law,

$$\mathbf{E}=\boldsymbol{\rho}\,\mathbf{J}, \quad \text{with } \boldsymbol{\rho}=\begin{bmatrix} \rho_{xx} & -\rho_H \\ \rho_H & \rho_{yy} \end{bmatrix}, \tag{4.38}$$

where the components of the resistivity tensor $\boldsymbol{\rho}$ are given as follows: $\rho_{xx}=R_L l_y/l_x$, $\rho_{yy}=R_L l_x/l_y$, and $\rho_H=R_H$, where R_L is the longitudinal resistance due to dissipative processes in the system (R_L^{-1} is proportional to the mean free time of the charge carriers) and R_H is the Hall resistance as defined in Eq. (4.37). In classical physics one easily finds that

$$R_H=\frac{B_c}{nec}\equiv\frac{h}{e^2}\nu^{-1}, \tag{4.39}$$

where B_c is the strength of the magnetic field in the z direction, n is the density of conduction electrons (minus the density of holes), e is the elementary charge, h is Planck's quantum of action, and the dimensionless quantity

$$\nu=\frac{n(hc/e)}{B_c} \tag{4.40}$$

is the filling factor. Note that ν^{-1} equals the amount of magnetic flux, in units of the flux quantum hc/e, per electron.

Since the magnetic field can be varied and the density n can be tuned by varying the electric field in the z direction (gate voltage), the law (4.39) predicted by classical physics can be tested experimentally. Experiments at very low temperatures and for pure heterostructures yield the following very surprising data:

(D1) $\sigma=(h/e^2)\sigma_H=(h/e^2)R_H^{-1}$ has plateaus at rational heights, i.e., $\sigma=p/q$, with p,q integers (see, e.g., Chakraborty and Pietiläinen, 1988; Prange and Gervin, 1990). Typically q is odd (Tao and Wu, 1985), but lately plateaus at $\sigma=\frac{5}{2}$ (Willett *et al.*, 1987; Eisenstein *et al.*, 1988; Eisenstein *et al.*, 1990) and $\sigma=\frac{1}{2}$ (Eisenstein *et al.*, 1992; Suen *et al.*, 1992) have been observed. The plateaus at integer height occur with an astronomical accuracy (measurements are precise to one part in 10^8!).

(D2) When (ν,σ) belongs to a plateau, the longitudinal resistance R_L very nearly vanishes, i.e., in plateau regions the system is dissipationless. Inverting the resistivity tensor $\boldsymbol{\rho}$ [see Eq. (4.38)] to obtain the conductivity tensor $\boldsymbol{\sigma}=\boldsymbol{\rho}^{-1}$ yields the result that the diagonal part of $\boldsymbol{\sigma}$ vanishes on a plateau.

(D3) The precession of the plateau quantization is insensitive to details of sample preparation and geometry, hence is a "universal" phenomenon.

(D4) More recently, it has been found (Clarke *et al.*, 1988; Chang and Cunningham, 1989; Simmons *et al.*, 1989; Clark *et al.*, 1990; Hwang *et al.*, 1992) that, when (ν,σ) belongs to a plateau at noninteger height, then the system exhibits fractionally charged excitations (the fractions of e being related to the value of σ).

(D5) Recent studies in "tilted magnetic fields" provide evidence that, when (ν,σ) belongs to a plateau at height $\sigma=\frac{5}{2}$ (Willett *et al.*, 1987; Eisenstein, Willett *et al.*, 1988, 1990), $\frac{4}{3}$ (Clark *et al.*, 1989; Clark *et al.*, 1990), $\frac{8}{5}$ (Eisenstein, Stormer *et al.*, 1989, 1990a), or $\frac{2}{3}$ (Clark *et al.*, 1990; Eisenstein *et al.*, 1990b), then the ground state of the system can be spin unpolarized. For certain plateaus it might be a spin-singlet state; see also the discussions in Haug *et al.* (1987), Syphers and Furneaux (1988a, 1988b), and Sec. VII.A of the present work.

Next, we propose to study what the Ohm-Hall law (4.38) tells us about a two-dimensional system of electrons in an external magnetic field when (ν,σ) belongs to a plateau. As noted in (D2), experimentally one finds that, in this situation, the longitudinal resistance R_L vanishes. This is interpreted as absence of dissipative processes. The absence of dissipative processes could be explained if one succeeded in showing that the spectrum of the many-electron Hamiltonian of the system exhibits an energy gap $\Delta>0$, above the ground-state energy (or, at least, that states of very small energy above the ground-state energy are localized). To exhibit a positive energy gap for certain values of the filling factor ν, physically interpreted as incompressibility of the system, poses difficult analytical problems. Some recent ideas about how to establish incompressibility at particular filling factors can be found in the following papers: studies of Laughlin states are given in Haldane (1983, 1990a), Halperin (1983, 1984), Laughlin (1983a, 1983b, 1984, 1990), Arovas, Schrieffer, and Wilczek (1984), and Trugman and Kivelson (1985); off-diagonal long-range order and Chern-Simons-Landau-Ginzburg theory in fractional quantum Hall fluids have been studied by Girvin and MacDonald (1987), Read (1989), Zhang, Hansson, and Kivelson (1989), Lee and Zhang (1991), Fröhlich (1992), Fröhlich, Kerler, and Marchetti (1992), and Zhang (1992); finally, for some numerical studies concerning the question of incompressibility see, for example, Fano, Ortolani, and Colombo (1986), Yoshioka (1986a), Chakraborty and Pietiläinen (1987, 1988), Rezayi (1987), and d'Ambrumenil and Morf (1989). What is easier to show is for which values of the parameter $\sigma=(h/e^2)R_H^{-1}$ a positive energy gap Δ *cannot* occur; more precisely, to prove a "gap-labeling theorem." Such a theorem is described in Secs. VI.B and VI.C [see also remark (iii) at the end of this subsection].

Thus, if the system is incompressible, in the sense that $R_L=0$, then we have the following form for the Hall law:

$$\mathbf{J}=\boldsymbol{\sigma}\mathbf{E}, \quad \text{with} \quad \boldsymbol{\sigma}=\boldsymbol{\rho}^{-1}=\begin{bmatrix} 0 & \sigma_H \\ -\sigma_H & 0 \end{bmatrix}, \tag{4.41}$$

where $\sigma_H=R_H^{-1}$. This is a phenomenological law valid at low frequencies and on large distance scales. More fundamental are the following two laws: Charge conservation,

$$\frac{1}{c}\frac{\partial}{\partial t}J^0+\nabla\cdot\mathbf{J}=0 , \tag{4.42}$$

(continuity equation), where J^0 is (c times) the electric charge density, and Faraday's induction law,

$$\frac{1}{c}\frac{\partial}{\partial t}B+\nabla\times\mathbf{E}=0 , \tag{4.43}$$

where B denotes the component of the magnetic field perpendicular to the plane of the system, and $\mathbf{E}$ is the electric field in the plane of the system. We note that the dynamics of charged, spinless particles confined to a plane depends only on the component of the magnetic field perpendicular to the plane of the system and the components of the electric field in that plane. Combining Eqs. (4.41) through (4.43), we find that

$$\frac{\partial}{\partial t}J^0=\sigma_H\frac{\partial}{\partial t}B . \tag{4.44}$$

Equation (4.44) can be integrated with respect to time t. By $J^0=J^0_{\text{tot}}-nec$ we denote the difference between the total electric charge density J^0_{tot} and the uniform background density nec of a system in a uniform background magnetic field B_c. Likewise, B denotes the difference between the total magnetic field B_{tot} and the uniform background field B_c. Then Eq. (4.44) implies that

$$J^0=\sigma_H B . \tag{4.45}$$

It is convenient to introduce the electromagnetic field tensor, which is a 2-form, F, given by

$$F=\tfrac{1}{2}\sum_{\mu,\nu}F_{\mu\nu}dx^\mu\wedge dx^\nu ,$$

with

$$(F_{\mu\nu})=\begin{bmatrix} 0 & E_x & E_y \\ -E_x & 0 & -B \\ -E_y & B & 0 \end{bmatrix}, \tag{4.46}$$

and the 2-form $\mathcal{J}$ dual to the current density $(J^0,\mathbf{J})$, i.e.,

$$\mathcal{J}=\tfrac{1}{2}\sum_{\mu,\nu}\mathcal{J}_{\mu\nu}dx^\mu\wedge dx^\nu, \quad \text{with} \quad \mathcal{J}_{\mu\nu}=\varepsilon_{\mu\nu\rho}J^\rho . \tag{4.47}$$

Then Eqs. (4.41) and (4.45) can be combined into one equation,

$$\mathcal{J}=-\sigma_H F , \tag{4.48}$$

while current conservation (4.42) is expressed as

$$d\mathcal{J}=\tfrac{1}{2}\sum_{\alpha,\mu,\nu}\partial_\alpha\mathcal{J}_{\mu\nu}dx^\alpha\wedge dx^\mu\wedge dx^\nu=0 , \tag{4.49}$$

and Faraday's induction law (4.43) becomes

$$dF=0 . \tag{4.50}$$

Equations (4.48)–(4.50) are compatible with each other if and only if σ_H is constant. If the values of σ_H along the two sides of a curve Γ differ from each other—which happens, for example, at the boundary of the system—then an additional current $\mathcal{I}$, not described by Eq. (4.41), is observed in the vicinity of Γ, in order to reconcile charge conservation with the induction law. [For time-independent fields one finds that $\nabla\cdot\mathcal{I}=(\nabla\sigma_H)\times\mathbf{E}$; see also Halperin (1982) and Sec. VI.]

Note that Eqs. (4.48)–(4.50) are generally covariant and independent of metric properties of the system. Equations (4.49) and (4.50) can be integrated by introducing the 1-forms (or "vector potentials") A and b, with

$$\mathcal{J}=db, \quad \text{that is,} \quad \mathcal{J}_{\mu\nu}=\partial_\mu b_\nu-\partial_\nu b_\mu ,$$

and (4.51)

$$F=dA .$$

Equation (4.48) then reads

$$db=-\sigma_H dA . \tag{4.52}$$

Equation (4.52) is the Euler-Lagrange equation derived from an action principle! The action $S_\Lambda(b;A)$, with $\Lambda=\mathbb{R}\times\Omega$ the space-time domain to which the system is confined, is given by

$$\begin{aligned} S_\Lambda(b;A)&=\frac{1}{2c^2\sigma_H}\int_\Lambda b\wedge db+\frac{1}{c^2}\int_\Lambda A\wedge db+\text{B.T.}(A|_{\partial\Lambda},b|_{\partial\Lambda}) \\ &=\frac{1}{2c^2\sigma_H}\int_\Lambda(d^{-1}\mathcal{J})\wedge\mathcal{J}+\frac{1}{c^2}\int_\Lambda A\wedge\mathcal{J}+\text{B.T.}(A|_{\partial\Lambda},\mathcal{J}|_{\partial\Lambda}) , \end{aligned} \tag{4.53}$$

where B.T. stands for boundary terms, which depend only on the restrictions $A|_{\partial\Lambda}$ and $b|_{\partial\Lambda}$ of A and b to the space-time boundary $\partial\Lambda$ [see the remark after Eq. (4.55) below]. Moreover, $S_\Lambda(b;A)$ shall be varied with respect to the dynamical variable, that is, with respect to b (the vector potential A of the electromagnetic field is a tunable, external field).

Why is the result (4.53) interesting? It is interesting because an equation of motion, such as (4.52), that can be derived from an action principle can be quantized easily, e.g., by using Feynman path integrals. Clearly, the current density $\mathcal{J}$ of a system of electrons must be interpreted as a quantum-mechanical operator-valued distribution. Hence (4.52) must be quantized. We note that, in the present example, Feynman path-integral quantization has a mathematically rigorous interpretation (see, for example, Negele and Orland, 1987; Feldman, Knörrer, and Trubowitz, 1992). In formal, "physical" notation, Feynman's path space measure is given by

$$d\,P_A(b)=Z_\Lambda(A)^{-1}\exp\left[\frac{i}{\hbar}S_\Lambda(b;A)\right]\mathcal{D}b\ , \tag{4.54}$$

where the partition or generating function $Z_\Lambda(A)$ is chosen such that (formally) $\int d\,P_A(b)=1$. This implies that

$$Z_\Lambda(A)=Z_0\exp\left[-\frac{i\sigma_H}{2\hbar c^2}\int_\Lambda A\wedge dA+\text{B.T.}(A|_{\partial\Lambda})\right]\ , \tag{4.55}$$

where Z_0 is a constant independent of A. [In Eq. (4.54), we have omitted a gauge-fixing term for the integration over the field b.] At this point, we wish to emphasize that a nontrivial boundary term B.T.$(A|_{\partial\Lambda})$ in Eq. (4.55) is a necessity forced upon us by the $U(1)_{em}$ gauge invariance of quantum mechanics; see Sec. III: Under a U(1) gauge transformation, $A\mapsto A+d\chi$, the Chern-Simons term in (4.55) transforms according to

$$\int_\Lambda A\wedge dA\mapsto\int_\Lambda A\wedge dA-\int_{\partial\Lambda}d\chi\wedge A\ , \tag{4.56}$$

i.e., there is a gauge anomaly localized at the boundary $\partial\Lambda$. Hence, in order for the partition function $Z_\Lambda(A)$ to be U(1) gauge invariant, the presence of a boundary term B.T.$(A|_{\partial\Lambda})$ exhibiting a gauge anomaly canceling the one in Eq. (4.56) is indispensable! We shall see that the anomalous part of B.T.$(A|_{\partial\Lambda})$ turns out to be the generating functional of the connected Green functions of chiral current operators generating a $\hat{u}(1)$ current (Kac-Moody) algebra, which physically corresponds to chiral charge-density waves circulating at the edge of the quantum Hall sample. Section VI will be devoted to an investigation of this current algebra as well as (non-Abelian) extensions thereof for general two-dimensional, incompressible Hall fluids. This analysis will lead to a complete list of the possible (quantized) values of the response coefficients in these systems, such as the Hall conductivity σ_H; see also remark (iii) at the end of this section.

Given the expression (4.55) for the partition function $Z_\Lambda(A)$, we may ask whether there is a simple way of recovering the action $S_\Lambda(b;A)$ as a functional of the vector potential b of the conserved current density $\mathcal{J}$, as given in Eq. (4.53). The answer is yes. It is provided by the following functional Fourier transform identity

$$\exp\left[\frac{i}{\hbar}S_\Lambda(b;A)\right]=\text{const}\int\mathcal{D}\alpha\exp\left[-\frac{i}{\hbar c^2}\int\alpha\wedge db\right]Z_\Lambda(A+\alpha)\ , \tag{4.57}$$

where we again omit a suitable gauge-fixing term for the integral over α. We note that, at least heuristically, one can show that Eq. (4.57) holds in general for two-dimensional quantum-mechanical systems of charged particles coupled to an external electromagnetic field with vector potential $A_{tot}=A_c+A$ (where $\nabla\times\mathbf{A}_c=B_c$ is fixed); see Fröhlich and Kerler (1991).

Let us conclude this section with a few remarks:

(i) Defining the effective action $S_\Lambda^{eff}(A)$ of a two-dimensional electronic system by

$$S_\Lambda^{eff}(A)=\frac{\hbar}{i}\ln Z_\Lambda(A) \tag{4.58}$$

[see also Sec. V], our circle of arguments can be closed from Eq. (4.55) back to the starting point of the Hall law (4.41) [and (4.45)] by noting that

$$J^i=c^2\frac{\delta S_\Lambda^{eff}(A)}{\delta A_i}=\sigma_H\,\varepsilon^{ij}E_j\ , \tag{4.59}$$

where $E_j=\partial_jA_0-\partial_0A_j$, for $j=1,2$.

Equations (4.59), (4.58), and (4.57) make it clear that one approach leading to an understanding of the quantum Hall effect is to derive from "first principles," for a quantum Hall fluid at particular values of the filling factor, the effective action $S_\Lambda^{eff}(A)$ corresponding to Eq. (4.55). In the next section, we show that gauge invariance of nonrelativistic quantum mechanics and the single assumption of incompressibility of quantum Hall fluids are sufficient to uniquely determine their effective action $S_\Lambda^{eff}(A)$ in the "scaling limit" and thereby to derive (4.55). Hence the phenomenology of quantum Hall systems at low frequencies and on large distance scales, including the quantization of the Hall conductivity σ_H [see remark (iii) below], can be derived from gauge invariance and incompressibility. This shows that a proof of incompressibility of electronic systems at particular values of the filling factor really is the essential problem in the theory of the quantum Hall effect in need of further investigation.

(ii) It can be seen directly from the Ohm-Hall law (4.38) that the incompressibility of quantum Hall fluids is a crucial property which allows for a description of these systems in terms of an effective action formalism: Only for dissipationless systems, i.e., for systems with an antisymmetric conductivity tensor σ, is it possible to functionally integrate the first relation in (4.59) in order to obtain an effective action. In other words, for electronic systems with dissipation, one *cannot* formulate the Ohm-Hall law (coming from transport theory) within an effective action formalism.

(iii) Clearly, we must require that the quantum theory with action $S_\Lambda(b;A)$ given by Eq. (4.53), defined by the Feynman path integral (4.54), describe localized, particle-like excitations with the quantum numbers of the electron or hole, i.e., with electric charge $\pm e$ and Fermi statistics. Investigating in detail the boundary term B.T.$(A|_{\partial\Lambda})$ in Eq. (4.55) and making use of the representation theory of chiral $\hat{\mathrm{u}}(1)$ current algebra in $1+1$ dimensions, we shall see in Sec. VI that the above requirement implies that, for consistency of the theory, the constant $\sigma=(h/e^2)\sigma_H$ must be a rational number. For a derivation of the quantization of σ in the simplest situation, where there is only one band of chiral edge currents, see Sec. V.C. [An alternative derivation of the existence of algebras of chiral edge currents in incompressible Hall fluids, starting from quantized Chern-Simmons theory and adopting results given in Fröhlich and Marchetti (1988), Elitzur *et al.* (1989), Fröhlich and King (1989), Moore and Seiberg (1989), and Witten (1989), can be found in Fröhlich and Kerler (1991).]

V. "SCALING LIMIT" OF THE EFFECTIVE ACTION OF A TWO-DIMENSIONAL, INCOMPRESSIBLE QUANTUM FLUID

In this section we study the partition or generating function (at $T=0$ and for real time) of a two-dimensional nonrelativistic quantum system confined to a space-time region $\Lambda=\mathbb{R}\times\Omega$ and coupled to external electromagnetic, "tidal," and possibly geometric fields,

$$Z_\Lambda(a,w)=\int \mathcal{D}\psi^*\mathcal{D}\psi \exp\left[\frac{i}{\hbar}S_\Lambda(\psi^*,\psi;a,w)\right], \qquad (5.1)$$

where the gauge potentials a and w have been introduced in Eqs. (3.26)–(3.31) and $S_\Lambda(\psi^*,\psi;a,w)$ is the action of the system given in Eqs. (3.34) and (3.35); see also Eqs. (3.55)–(3.58). The integration variables ψ^* and ψ are Grassmann variables (i.e., anticommuting c numbers) for Fermi statistics, and complex c-number fields for Bose statistics.

We have not displayed the metric g_{ij} on the background space M explicitly, since it will be kept fixed, and usually $M=\mathbb{E}^2$ with $g_{ij}=\delta_{ij}$, for simplicity. We realize that, for the study of the stress tensor, pressure and density fluctuations, and curvature and torsion effects, we would have to choose a variable external metric (or, at least, a variable conformal factor in g_{ij}). This would be important for an understanding of density waves, in particular surface density waves (which are interesting in two-dimensional quantum fluids), and of critical phenomena. We note, however, that curvature and torsion effects can be studied by analyzing the dependence of $Z_\Lambda(a,w)$ on w, which contains the affine spin connection $\omega^A{}_B$; see Eqs. (3.28) and (3.29), as well as the remarks at the end of Sec. III.A and after Eq. (3.39).

Calculating the partition function (5.1) for an arbitrary (two-dimensional) nonrelativistic quantum system is surely a major task. In the first part of this section, we show how the calculation can be carried out for incompressible systems, provided one passes to the scaling limit. Once we have an explicit expression for the partition function of a system, many of its physical properties can be derived. For incompressible systems, this is the topic of the rest of this paper. Since we are working in the scaling limit, our emphasis is on *universal* properties of such systems (i.e., properties independent of the small-scale structure of the system).

A. "Scaling limit" of the effective action

In this subsection we define the scaling limit of a system and sketch how one calculates, for incompressible systems, the "scaling limit" of the effective action [see Eq. (5.5) below] associated with the partition function (5.1). One of our main motivations for studying two-dimensional, incompressible quantum fluids comes from the phenomenology of the quantum Hall effect; see Sec. IV.E. In discussing quantum Hall fluids, it is often assumed that the magnetic fields transverse to the samples are so strong that the Zeeman energies are large enough for the systems to be totally spin polarized. Moreover, spin-orbit interaction terms are expected to be negligible in quantum Hall fluids. One might therefore ask why, when studying quantum Hall fluids, one should worry about the dependence of the partition function $Z_\Lambda(a,w)$ on the SU(2) connection w, thereby taking into account Zeeman and spin-orbit interaction effects? To answer this question, one first might argue that it is of principal, theoretical interest to know how to incorporate the spin degrees of freedom in a consistent way into the description of two-dimensional electronic systems.

Second, as pointed out first by Halperin (1983), in GaAs, for example, the g factor of the electron is $\frac{1}{4}$ of the value in the vacuum, and the effective mass m of the electron is about $\frac{7}{100}$ of the mass m_0 in the vacuum. Thus in GaAs the Zeeman energies are only approximately $\frac{1}{60}$ of the cyclotron energy (i.e., the splitting between Landau levels). Furthermore, they are of the same magnitude as the quasiparticle energies of the fractional quantum Hall states in magnetic fields of the order of 10 T. One expects therefore that, at some values of the filling factor ν, the ground state of the system will contain electrons with reversed spins. Experimental evidence that spin-unpolarized quantum Hall fluids exist has been given in the works cited in (D5) in Sec. IV.E. We emphasize that unpolarized (or partially polarized) quantum Hall fluids can arise in two fundamentally different ways: either through the presence of two (or more) independent, but oppositely polarized bands, or through the formation of spin-singlet bands; see Sec. VI.C and Fröhlich and Thiran (1993).

Third, the gauge potentials a and w (and, for that matter, any potential corresponding to a gauged internal symmetry) provide a kind of "mathematical microscope" revealing universal properties of quantum Hall fluids. We show in Sec. VI how one can infer from the form of $Z_\Lambda(a,w)$ what kind of (gapless) boundary excitations (chiral edge currents) can be observed in quantum Hall fluids. This will lead to a classification of these systems in terms of "universality classes."

Fourth, we sketch below, in Sec. V.B, how one can derive from $Z_\Lambda(a,w)$ the linear-response theory of quantum Hall fluids describing, among other effects, a quantum Hall effect for spin currents. For a discussion of possible Hall systems where this effect might be tested experimentally see Sec. VII.A.

We define the electric charge and current density, $j^0(x)$ and $\mathbf{j}(x)$, by

$$j^0(x)=\psi^*(x)\psi(x)\ ,$$
$$j^k(x)=-\frac{i\hbar}{2mc}g^{kl}(x)[(\mathcal{D}_l\psi)^*(x)\psi(x)-\psi^*(x)(\mathcal{D}_l\psi)(x)]\ , \tag{5.2}$$

and the spin and spin current densities, $s^0_A(x)$ and $\mathbf{s}_A(x)$, by

$$s^0_A(x)=\psi^*(x)L^{(s)}_A\psi(x)\ ,$$
$$s^k_A(x)=-\frac{i\hbar}{2mc}g^{kl}(x)[(\mathcal{D}_l\psi)^*(x)L^{(s)}_A\psi(x)-\psi^*(x)L^{(s)}_A(\mathcal{D}_l\psi)(x)]\ , \tag{5.3}$$

where $(L^{(s)}_1,L^{(s)}_2,L^{(s)}_3)$ are the three generators of the spin s representation of su(2) [see Eq. (3.27)], and $\mathcal{D}_l$ denotes the covariant derivative in the l direction, as specified in Eq. (3.26). Similarly, one defines currents associated with internal symmetries. The electric current is conserved [i.e., the continuity equation holds; see Eq. (4.42)], but the spin current is, in general, not conserved, because it couples to a non-Abelian gauge potential. It is, however, covariantly conserved; see Eq. (5.12) below.

It is straightforward to infer from Eqs. (5.1), (3.34), (5.2), and (5.3) that the connected, time-ordered current Green functions of the system are given, at *noncoinciding* arguments, by

$$\left\langle T\left[\prod_{i=1}^{n}j^{\mu_i}(x_i)\prod_{j=1}^{m}s^{\nu_j}_{A_j}(y_j)\right]\right\rangle^{\text{con}}_{a,w}=i^{n+m}\prod_{i=1}^{n}\frac{\delta}{\delta a_{\mu_i}(x_i)}\prod_{j=1}^{m}\frac{\delta}{\delta w_{\nu_jA_j}(y_j)}\ln Z_\Lambda(a,w)\ , \tag{5.4}$$

where $\langle\ \rangle^{\text{con}}_{a,w}$ denotes the connected expectation functional of the system in an external gauge-field configuration (a,w) (with "ground-state asymptotic conditions," as $t\to\pm\infty$, to be specified), and T indicates time ordering. At coinciding arguments, Eq. (5.4) is modified by Schwinger terms (but their precise form will not be of importance in our analysis, and therefore we do not display them).

We define the effective action of the system by

$$S^{\text{eff}}_\Lambda(a,w)=\frac{\hbar}{i}\ln Z_\Lambda(a,w)\ . \tag{5.5}$$

The idea is to try to calculate the "leading terms" in $S^{\text{eff}}_\Lambda(a,w)$ which, via Eq. (5.4), will provide us with information on the current Green functions. By "leading terms" we mean those terms which dominate at large-distance scales and low frequencies. The calculation of the leading terms in $S^{\text{eff}}_\Lambda(a,w)$ might appear to be an intractable problem. Actually, making a single assumption on the excitation spectrum of the system, incompressibility, and using the U(1)$\times$SU(2) gauge invariance of nonrelativistic quantum mechanics, we can find them explicitly.

Let χ be a real-valued function and R an SU(2)-valued function on space-time. Consider the gauge transformations in Eq. (3.67), i.e.,

$$a\mapsto{}^{\chi}a,\quad\text{with}\quad {}^{\chi}a_\mu(x)=a_\mu(x)+\partial_\mu\chi(x)\ , \tag{5.6}$$

and

$$w\mapsto{}^{R}w\ ,$$

with

$${}^{R}w_\mu(x)=U^{(s)}(R(x))w_\mu(x)U^{(s)}(R(x))^*+U^{(s)}(R(x))\partial_\mu U^{(s)}(R(x))^*\ . \tag{5.7}$$

Changing integration variables,

$$\psi(x)\mapsto{}^{\chi,R}\psi(x)=e^{-i\chi(x)}U^{(s)}(R(x))\psi(x) \tag{5.8}$$

in the functional integral (5.1), and using the gauge invariance of $S_\Lambda(\psi^*,\psi;a,w)$ under the transformations (5.6)–(5.8) and the fact that the Jacobian of (5.8) is unity, we find the Ward identity

$$S^{\text{eff}}_\Lambda({}^{\chi}a,{}^{R}w)=S^{\text{eff}}_\Lambda(a,w) \tag{5.9}$$

for all χ and R. For a system of finitely many particles in a bounded region Ω of space, Eq. (5.9) can be proven rigorously (Fröhlich and Studer, 1992b). The identity is stable under passage to limits, for χ's and $\partial_\mu R$'s of compact support.

By differentiating (5.9) in χ or R and setting $\chi=0$, $R=\mathbf{1}$, we find, using Eq. (5.4) for $n+m=1$, that

$$\frac{1}{\sqrt{g(x)}}\partial_\mu(\sqrt{g(x)}\langle j^\mu(x)\rangle_{a,w})=0\ , \tag{5.10}$$

and

$$\frac{1}{\sqrt{g(x)}}(\mathcal{D}_\mu(\sqrt{g(x)}\langle s^\mu(x)\rangle_{a,w}))_A=0,\quad A=1,2,3\ , \tag{5.11}$$

that is,

$$\frac{1}{\sqrt{g(x)}}\partial_\mu(\sqrt{g(x)}\langle s_A^\mu(x)\rangle_{a,w}) -2\varepsilon_{ABC}w_{\mu B}(x)\langle s_C^\mu(x)\rangle_{a,w}=0\ , \quad (5.12)$$

for arbitrary a and w. These infinitesimal Ward identities play an important role in determining the general form of $S_\Lambda^{\text{eff}}(a,w)$. They can be generalized, in an obvious way, to systems with internal symmetries.

We now proceed to determine the form of $S_\Lambda^{\text{eff}}(a,w)$ "in the scaling limit." We need to consider ever larger systems and ever slower variations in time. Let $1\le\theta<\infty$ be a scale parameter. We set

$$g_{ij}(x)\equiv g_{ij}^{(\theta)}(x)=\gamma_{ij}(\theta^{-1}x) \text{ and } \Lambda\equiv\Lambda^{(\theta)}=\theta\Lambda_0\ , \quad (5.13)$$

where γ_{ij} is a fixed metric on M [e.g., $\gamma_{ij}=\delta_{ij}$] and $\Lambda_0\subset N=\mathbb{R}\times M$ is a fixed space-time cylinder;

$$x=(x^0=ct,\mathbf{x})=\theta(\xi^0,\xi)=\theta\xi,\ \ \xi\in\Lambda_0\ ; \quad (5.14)$$

hence

$$\frac{\partial}{\partial x^\mu}=\theta^{-1}\frac{\partial}{\partial\xi^\mu}\ . \quad (5.15)$$

We propose to study the reaction of the system to a small change in the external gauge potentials a and w. We choose fixed background potentials a_c and w_c defined on all of space-time N and set

$$a_\mu^{(\theta)}(x)=a_{c,\mu}(x)+\theta^{-1}\tilde{a}_\mu(\theta^{-1}x) \quad (5.16)$$

and

$$w_\mu^{(\theta)}(x)=w_{c,\mu}(x)+\theta^{-1}\tilde{w}_\mu(\theta^{-1}x)\ , \quad (5.17)$$

where $\tilde{a}_\mu(\xi)$ and $\tilde{w}_\mu(\xi)$ are fixed functions defined on Λ_0. [From now on we drop the superscript (s) from the components of the SU(2) connection w; cf. Eq. (3.27).] If m is the effective mass of the particles and μ is the quantity determining their magnetic moment [see Eq. (2.7)] in physical $(t,\mathbf{x})$ coordinates, then the mass $m^{(\theta)}$ and the strength of the magnetic moment $\mu^{(\theta)}$ in rescaled coordinates $(\tau=(\xi^0/c),\xi)$ are given by

$$m^{(\theta)}=\theta m \text{ and } \mu^{(\theta)}=\theta^{-1}\mu\ , \quad (5.18)$$

as follows from Eqs. (3.34), (3.37), and (3.38). (That is, in the rescaled systems, the particles are heavy and have small magnetic moment. Moreover, the range of the two-body potential in the rescaled systems becomes shorter and shorter, as the scale parameter θ becomes large.)

One basic assumption underlying our analysis is that $S_{\theta\Lambda_0}^{\text{eff}}(a^{(\theta)},w^{(\theta)})$ is four times continuously (Fréchet) differentiable in $\tilde{a}_\mu^{(\theta)}(x)\equiv\theta^{-1}\tilde{a}_\mu(\theta^{-1}x)$ and $\tilde{w}_\mu^{(\theta)}(x)\equiv\theta^{-1}\tilde{w}_\mu(\theta^{-1}x)$ at $\tilde{a}_\mu^{(\theta)}(x)=0-\tilde{w}_\mu^{(\theta)}(x)$, for a suitable choice of background potentials $a_{c,\mu}(x)$ and $w_{c,\mu}(x)=\omega_{c,\mu}(x)+\rho_{c,\mu}(x)$, and for $\tilde{a}_\mu(\xi)$ and $\tilde{w}_\mu(\xi)=\tilde{\omega}_\mu(\xi)+\tilde{\rho}_\mu(\xi)$ constrained to suitable function spaces $\mathcal{A}$ and $\mathcal{W}$, of perturbation potentials, to be specified later. We may then (functionally) expand $S_{\theta\Lambda_0}^{\text{eff}}(a^{(\theta)},w^{(\theta)})$ to third order in $\tilde{a}^{(\theta)}(x)$ and $\tilde{w}^{(\theta)}(x)$, with a fourth-order remainder term. Among the terms thus generated we shall retain only the leading terms in θ, namely, those scaling with a non-negative power of θ, which are commonly called relevant and marginal terms. The sum of these terms will be denoted by $S_{\Lambda_0}^*(\tilde{a},\tilde{w})$, a functional that we call the *"scaling limit" of the effective action.*

Using identity (5.4) and (5.5) to find the Taylor coefficients of $S_{\theta\Lambda_0}^{\text{eff}}(a^{(\theta)},w^{(\theta)})$, plugging Eqs. (5.16) and (5.17) into the resulting expression, and finally passing to (ξ^0,ξ) coordinates [cf. Eq. (A1) in Appendix A], we find that the coefficient of the term of nth order in $\tilde{a}(\xi)$ and of mth order in $\tilde{w}(\xi)$ in $S_{\theta\Lambda_0}^{\text{eff}}(a^{(\theta)},w^{(\theta)})$ is given by a distribution

$$\varphi^{(\theta)\mu_1,\ldots,\mu_n,\nu_1,\ldots,\nu_m}_{A_1,\ldots,A_m}(a_c,w_c;\xi_1,\ldots,\xi_n,\ \eta_1,\ldots,\eta_m) \equiv\varphi^{(\theta)\underline{\mu},\underline{\nu}}_{\underline{A}}(a_c,w_c;\underline{\xi},\underline{\eta})\ , \quad (5.19)$$

which, at noncoinciding arguments, is given by

$$\frac{(-i)^{n+m+1}\hbar}{n!m!}\left\langle T\left[\prod_{i=1}^{n}[\theta^2 j^{\mu_i}(\theta\xi_i)]\prod_{j=1}^{m}[\theta^2 s^{\nu_j}_{A_j}(\theta\eta_j)]\right]\right\rangle^{\text{con}}_{a_c,w_c} \quad (5.20)$$

in accordance with the circumstance that, in $2+1$ space-time dimensions, the scaling dimension of currents is 2! [Note, for example, that $\int_{t=\text{const}}j^0(ct,\mathbf{x})d^2\mathbf{x}$ is a dimensionless conserved charge.]

We may now formulate our basic *assumption of incompressibility:* We assume that, for certain choices of the background potentials a_c and w_c, the excitation spectrum of the system above its ground-state energy is such that (in the bulk) the connected Green functions of its currents have "good" cluster properties (better than in a system with Goldstone bosons). More precisely, we assume that the distributions given in (5.19) exhibit the following behavior, for $\theta\to\infty$:

$$\varphi^{(\theta)\underline{\mu},\underline{\nu}}_{\underline{A}}(a_c,w_c;\underline{\xi},\underline{\eta})=\sum_{\Delta=0}^{N}\theta^{-D_\Delta}\varphi^{\underline{\mu},\underline{\nu}}_{\Delta\underline{A}}(a_c,w_c;\underline{\xi},\underline{\eta}) +\text{B.T.}^{\underline{\mu},\underline{\nu}}_{\underline{A}}(a_c,w_c;\underline{\xi},\underline{\eta})+o(\theta^{-D_N})\ , \quad (5.21)$$

where $\varphi^{\underline{\mu},\underline{\nu}}_{\Delta\underline{A}}(a_c,w_c;\underline{\xi},\underline{\eta})$ are *local* distributions, i.e., sums of products of derivatives of δ functions, and the scaling dimensions D_Δ are given by

$$D_\Delta=-2(n+m)+3(n+m-1)+\Delta =n+m-3+\Delta\in\mathbb{Z}\ , \quad (5.22)$$

with Δ the number of derivatives present in the corresponding local distribution. The upper limit N in the sum on the rhs of Eq. (5.21) is chosen such that $D_N\ge0$. Finally, $\text{B.T.}^{\underline{\mu},\underline{\nu}}_{\underline{A}}(a_c,w_c;\underline{\xi},\underline{\eta})$ are distributions (*not* necessarily local ones) that are completely localized on the space-time boundary $\partial\Lambda_0$ of the rescaled system in Λ_0.

These terms will not be discussed in this section; they form the subject matter of Sec. VI. [For a different way of formulating the incompressibility of a system, see the discussion in Sec. IV.E on the quantum Hall effect.]

This incompressibility assumption is by no means a mild or minor assumption. It tends to be a hard analytical problem of many-body theory to show that, for a concrete system, it is satisfied. [For some recent ideas about how to establish it for quantum Hall fluids at certain filling factors, see the references given after (D5) in Sec. IV.E.] What we propose to do here is to use it to calculate the general form of $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$, the "scaling limit" of the effective action of the system, thereby elucidating the universal properties of two-dimensional, incompressible systems. We only sketch some ideas; for the details see Appendix A and Fröhlich and Studer (1992b).

The calculation is based on the following four principles.

(P1) *Incompressibility:* For all n and m, with $2\le n+m\le 4$, the distributions $\varphi^{(\theta)\underline{\mu},\underline{\nu}}_{\underline{A}}(a_c,w_c;\underline{\xi},\underline{\eta})$ "converge" (in the bulk), for $\theta\to\infty$, to local distributions, as specified in Eq. (5.21).

(P2) *U(1)×SU(2) gauge invariance:* Ward identities (5.9)–(5.12).

(P3) *Only relevant and marginal terms are kept in* $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$.

(P4) *Extra symmetries of the system,* e.g., for $w_{c,\mu A}(x)=\delta_{A3}w_{c,\mu 3}(x)$ [and hence, by Eq. (3.38), for $a_{c,\mu}(x)$ such that $E_{c,3}(x)=0$], global rotations around the 3-axis in spin space are a continuous, global symmetry of the system with an associated conserved Noether current $s^\mu_3(x)$; or translation invariance in the scaling limit $(\theta\to\infty)$; . . . , *are exploited to reduce the number of terms.*

From (P1) and Eqs. (5.16) and (5.17) it immediately follows that all terms contributing to $S^{\text{eff}}_{\theta\Lambda_0}(a^{(\theta)},w^{(\theta)})$ of order 4 or higher in $\tilde{a}(\xi)$ and $\tilde{w}(\xi)$ are irrelevant, i.e., they scale like θ^{-D}, with $D>0$. In particular, a fourth-order remainder term does *not* contribute to $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$ [principle (P3)]. We now present the final result in the special case of a system that is incompressible for a choice of $w_c(x)$ satisfying

$$w_{c,\mu A}(x)=\delta_{A3}w_{c,\mu 3}(x)\ , \tag{5.23}$$

or, in view of Eqs. (3.28)–(3.30), (3.37), and (3.38), for a background electromagnetic field $(\mathbf{E}_c(x),\mathbf{B}_c(x))$ with

$$\begin{aligned}\mathbf{E}_c(x)&=(E_{c,1}(x),E_{c,2}(x),0)\ ,\\ \mathbf{B}_c(x)&=(0,0,B_c(x))\ ,\end{aligned} \tag{5.24}$$

and possibly for some affine spin connection ω of the following form [see Eqs. (3.29), (3.11), (3.12), and (3.31), as well as the remarks about the physical relevance of ω at the end of Sec. III.A and after Eq. (3.39)]:

$$(\omega^A{}_{B\mu}(x))=\begin{pmatrix}0 & \omega_\mu(x) & 0\\ -\omega_\mu(x) & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{5.25}$$

relative to some orthonormal frames $(e^1(x),e^2(x),e^3(x))$. [It is natural to work in an SU(2) gauge, which respects our convention of choosing $e^3(x)$ perpendicular to the cotangent plane $T^*_{\mathbf{x}}(M)$ at $\mathbf{x}\in M$, for all times t; see the discussion preceding Eq. (3.1). E.g., if M were the (x,y) plane in $\mathbb{E}^3$, we would choose $e^3(x)$ to coincide with dz. Hence the choice of the electromagnetic background field specified by Eq. (5.24) corresponds to an electric field $\mathbf{E}_c$ that is tangential to the sample and to a magnetic field $\mathbf{B}_c$, which is perpendicular to it.] In this situation, the "scaling limit" of the effective action is given by

$$\begin{aligned}-\frac{1}{\hbar}S^*_{\Lambda_0}(\tilde{a},\tilde{w})=&\int_{\Lambda_0}j^\mu_c\tilde{a}_\mu dv+\int_{\Lambda_0}m^\mu_3\tilde{w}_{\mu 3}dv\\ &+\sum_{A=1}^{2}\int_{\Lambda_0}\tau^{\mu\nu}_1\tilde{w}_{\mu A}\tilde{w}_{\nu A}dv+\sum_{A,B=1}^{2}\int_{\Lambda_0}\tau^{\mu\nu}_2\,\varepsilon_{AB}\,\tilde{w}_{\mu A}\tilde{w}_{\nu B}dv+\frac{k}{4\pi}\int_{\Lambda_0}\mathrm{tr}(w\wedge dw+\tfrac{2}{3}w\wedge w\wedge w)\\ &+\frac{\sigma}{4\pi}\int_{\Lambda_0}\tilde{a}\wedge d\tilde{a}+\frac{\chi_s}{2\pi}\int_{\Lambda_0}\tilde{a}\wedge d\tilde{w}_3+\frac{\sigma_s}{4\pi}\int_{\Lambda_0}\tilde{w}_3\wedge d\tilde{w}_3\\ &+\sum_{A,B,C=1}^{3}\int_{\Lambda_0}\eta^{\mu\nu\rho}_{ABC}\tilde{w}_{\mu A}\tilde{w}_{\nu B}\tilde{w}_{\rho C}dv+\mathrm{B.T.}(a|_{\partial\Lambda_0},w|_{\partial\Lambda_0})\ ,\end{aligned} \tag{5.26}$$

where $j^\mu_c(\xi)$ is an electric and $m^\mu_3(\xi)$ a magnetic (super-) current circulating in the system when $\tilde{a}(\xi)=0=\tilde{w}(\xi)$; $\tau^{\mu\nu}_1(\xi)$ is a function symmetric in μ and ν, while $\tau^{\mu\nu}_2(\xi)$ is antisymmetric in μ and ν; the function $\eta^{\mu\nu\rho}_{ABC}(\xi)$ is symmetric under interchanges of (μA), (νB), and (ρC) and vanishes if two or more of the indices A,B,C are equal to 3; $dv=[\gamma(\xi)]^{1/2}d^3\xi$, where $\gamma(\xi)=\det[\gamma_{ij}(\xi)]$ is the volume element on the space-time cylinder Λ_0 [see Eq. (5.13)]; σ, χ_s, σ_s, and k are real constants; $w(\xi)$ is the total SU(2) connection, given by

$$w(\xi)=w^{(\theta)}_c(\xi)+\tilde{w}(\xi),\quad\text{with}\quad w^{(\theta)}_c(\xi)=\theta w_c(\theta\xi) \tag{5.27}$$

[see Eq. (5.17)]; and $\mathrm{B.T.}(a|_{\partial\Lambda_0},w|_{\partial\Lambda_0})$ denotes boundary terms depending only on the gauge potentials $a|_{\partial\Lambda_0},w|_{\partial\Lambda_0}$, restricted to the boundary $\partial\Lambda_0$ of the space-time cylinder

Λ_0. They will be studied in Sec. VI. Moreover, on the rhs of Eq. (5.26) we are using the notation

$$\tilde{a}=\sum_{\mu=0}^{2}\tilde{a}_\mu(\xi)d\xi^\mu,\quad d\tilde{a}=\sum_{\mu,\nu=0}^{2}(\partial_\mu\tilde{a}_\nu)(\xi)d\xi^\mu\wedge d\xi^\nu\,,$$
$$\tilde{w}_3=\sum_{\mu=0}^{2}\tilde{w}_{\mu 3}(\xi)d\xi^\mu,\quad \tilde{w}=i\sum_{\mu=0}^{2}\sum_{A=1}^{3}\tilde{w}_{\mu A}(\xi)L_A^{(s)}d\xi^\mu\,, \tag{5.28}$$

where $\partial_\mu=\partial/\partial\xi^\mu$ whenever we are working in rescaled ξ coordinates. Finally, we note that the terms in Eq. (5.26) are ordered according to their scaling dimensions, which are implicit in their coefficients; see Appendix A. In Sec. VI, we shall use results on chiral $\hat{\mathrm{u}}(1)$ and $\widehat{\mathrm{su}}(2)$ current algebras to determine the possible values of the constants σ, χ_s, σ_s, and k.

Here, we wish to point out that the functions j_c^μ, m_3^μ, $\tau_1^{\mu\nu}$, $\tau_2^{\mu\nu}$, and $\eta_{ABC}^{\mu\nu\rho}$ are *not* all independent, but are constrained by the infinitesimal Ward identities (5.10) and (5.12). According to Eqs. (5.4) and (5.5)

$$\langle j^\mu(\xi)\rangle_{a^{(\theta)},w^{(\theta)}}=-\frac{1}{\hbar}\frac{\delta S^*_{\Lambda_0}(\tilde{a},\tilde{w})}{\delta\tilde{a}_\mu(\xi)}+\cdots \tag{5.29}$$

and

$$\langle s_A^\mu(\xi)\rangle_{a^{(\theta)},w^{(\theta)}}=-\frac{1}{\hbar}\frac{\delta S^*_{\Lambda_0}(\tilde{a},\tilde{w})}{\delta\tilde{w}_{\mu A}(\xi)}+\cdots\,. \tag{5.30}$$

The ellipses stand for contributions from irrelevant terms in the effective action. We calculate the rhs of these equations by using Eq. (5.26) and plug the result into Eqs. (5.10) and (5.12). As a result we obtain the following constraints (Fröhlich and Studer, 1992b):

$$\text{(i)}\quad \frac{1}{\sqrt{\gamma(\xi)}}\partial_\mu(\sqrt{\gamma}j_c^\mu)(\xi)=0\,.$$
$$\text{(ii)}\quad \frac{1}{\sqrt{\gamma(\xi)}}\partial_\mu(\sqrt{\gamma}m_3^\mu)(\xi)=0\,.$$
$$\text{(iii)}\quad \sum_{B=1}^{2}\varepsilon_{AB}\,[m_3^\mu(\xi)-2\tau_1^{0\mu}(\xi)w_{c,03}^{(\theta)}(\xi)]\tilde{w}_{\mu B}(\xi)+2w_{c,03}^{(\theta)}(\xi)\sum_{j=1}^{2}\tau_2^{0j}(\xi)\tilde{w}_{jA}(\xi)=0,\quad A=1,2\,.$$
$$\text{(iv)}\quad \frac{1}{\sqrt{\gamma(\xi)}}\partial_\mu\left[\sqrt{\gamma}\tau_1^{\mu\nu}\tilde{w}_{\nu A}+\sqrt{\gamma}\sum_{B=1}^{2}\varepsilon_{AB}\tau_2^{\mu\nu}\tilde{w}_{\nu B}\right](\xi)=-2\left[\sum_{B=1}^{2}\varepsilon_{AB}\,\tau_1^{\mu\nu}(\xi)\tilde{w}_{\nu B}(\xi)-\tau_2^{\mu\nu}(\xi)\tilde{w}_{\nu A}(\xi)\right]\tilde{w}_{\mu 3}(\xi)$$
$$-3\sum_{B=1}^{2}\varepsilon_{AB}\sum_{C,D=1}^{3}\eta_{BCD}^{0\nu\rho}(\xi)w_{c,03}^{(\theta)}(\xi)\tilde{w}_{\nu C}(\xi)\tilde{w}_{\rho D}(\xi),\quad A=1,2\,. \tag{5.31}$$

Constraints (i) and (ii) just express the conservation of the (super-) currents j_c^μ and m_3^μ when $\tilde{a}=0=\tilde{w}$; see also Eqs. (A9) and (A12) in Appendix A.

If we impose constraints (iii) and (iv), for *arbitrary* smooth perturbation potentials $\tilde{w}$, then it follows that

$$m_3^\mu(\xi)=\tau_1^{\mu\nu}(\xi)=\tau_2^{\mu\nu}(\xi)=0\quad\text{for all }\mu,\nu=0,1,2\,, \tag{5.32}$$

in particular, the system *cannot be magnetized* ($m_3^0=0$) and cannot support persistent spin currents. This may seem rather strange, because we would expect that if $\rho_{c,03}=-(g\mu/2\hbar c)B_c$ [see Eq. (3.37)], for some large background magnetic field $\mathbf{B}_c=(0,0,B_c)$, then the system would be magnetized in the 3 direction. What has gone wrong? The point is that the assumed properties—that $S^{\text{eff}}_{\theta\Lambda_0}(a^{(\theta)},w^{(\theta)})$ is four times continuously differentiable in $\tilde{a}$ and $\tilde{w}$ and that the system remains incompressible in an *arbitrary* function-space neighborhood of (a_c,w_c) of sufficiently small diameter—must fail for magnetized systems! The reason is that an arbitrarily small perturbation field $\tilde{w}$ which oscillates rapidly in time can destroy the incompressibility of the system, and hence our estimate of the fourth-order remainder term in the Taylor expansion of $S^{\text{eff}}_{\theta\Lambda_0}(a^{(\theta)},w^{(\theta)})$ breaks down!

We thus assume, for example, that, for a time-independent background field w_c satisfying Eq. (5.23), the system remains incompressible and $S^{\text{eff}}_{\theta\Lambda_0}(a^{(\theta)},w^{(\theta)})$ is four times continuously differentiable in $(\tilde{a},\tilde{w})$ on the set of function spaces

$$\mathcal{A}=\{\tilde{a}\,|\,\tilde{a}_\mu(\tau,\xi)=\eta(\tau)f_\mu(\xi),\ f_\mu(\xi)\in\mathcal{S}\}\,,$$
$$\mathcal{W}=\{\tilde{w}\,|\,\tilde{w}_{\mu A}(\tau,\xi)=\eta(\tau)g_{\mu A}(\xi),\ g_{\mu A}(\xi)\in\mathcal{S}\}\,, \tag{5.33}$$

where $\eta(\tau)$ describes an adiabatic process of turning on and off the perturbations: $\eta(\tau)=1$ for $\tau\in[\tau_1,\tau_2]$, some finite interval in (rescaled) time, and $\eta(\tau)=0$ for $\tau\ll\tau_1$ or $\tau\gg\tau_2$, while smoothly interpolating in between; and $\mathcal{S}$ in some Schwartz space neighborhood of 0. Then constraints (iii) and (iv) in Eq. (5.31) imply that

$$\tau_1^{00}(\xi)=\frac{m_3^0(\xi)}{2w_{c,03}^{(\theta)}(\xi)},\quad \tau_1^{0i}(\xi)=\tau_1^{ij}(\xi)=0\,,$$
$$\tau_2^{\mu\nu}(\xi)=0 \tag{5.34}$$

and

$$\eta_{AA3}^{000}(\xi)=-\frac{\tau_1^{00}(\xi)}{3w_{c,03}^{(\theta)}(\xi)}\quad\text{for }A=1,2\,, \tag{5.35}$$

all other $\eta^{\mu\nu\rho}_{ABC}(\xi)$ vanish. Hence $(m_3^\mu)=(m_3^0,\mathbf{0})$. Under somewhat more restrictive assumptions on $\mathcal{W}$, for example, imposing relations of the form (3.37) and (3.38) on $\varpi=\tilde{\omega}+\tilde{\rho}$ which couple ϖ and $\tilde{a}$, a nonzero spin current $\mathbf{m}_3=(m_3^1,m_3^2)$ is possible, too. For a more detailed discussion, see Fröhlich and Studer (1992b).

A corollary of our derivation of $S^*_{\Lambda_0}(\tilde{a},\varpi)$, using gauge invariance and incompressibility, is the Goldstone theorem (Goldstone, 1961; Goldstone, Salam, and Weinberg, 1962). Recalling that $w_{c,03}=-(g\mu/2\hbar c)B_c$, if $\omega_0=0$ [see Eqs. (3.37) and (5.25)], and denoting by $\mathcal{M}=(g\mu/2)m_3^0$ the magnetization in the background field B_c [see Eq. (5.52) below], one finds, by Eq. (5.34), that, for an incompressible system, the following identity must hold:

$$\mathcal{M}(\xi)=-\frac{2\hbar c}{g^2\mu^2}\tau_1^{00}(\xi)B_c(\xi)\ .$$

Hence, with $|\tau_1^{00}|<\infty$, one finds that, if $\mathcal{M}$ does *not* tend to 0, as the background magnetic field B_c tends to 0, then the system *cannot* be incompressible at $B_c=0$, i.e., there are gapless extended modes, the Goldstone bosons, coupled to the ground state by the spin current (Fröhlich and Studer, 1992b). We note that our proof also works for systems with continuous non-Abelian internal symmetries.

Remark. In Eq. (5.26) for the "scaling limit" of the effective action $S^*_{\Lambda_0}(\tilde{a},\varpi)$ of a two-dimensional, incompressible quantum fluid we have, as explained above, retained only relevant and marginal terms, i.e., terms scaling as θ^{-D} for $\theta\to\infty$, with $D\le -1$ and $D=0$, respectively. Although, in this paper, we are mainly interested in the physics corresponding to the relevant and marginal terms in the effective action $S^{\mathrm{eff}}_{\theta\Lambda_0}(a^{(\theta)},w^{(\theta)})$, we display here the most important subleading order terms. These are the unique terms that are of second order in the perturbation potentials $\tilde{a}$ and ϖ and of scaling dimension $D=1$, the so-called Maxwell terms. Added to the rhs of Eq. (5.26) they take the form

$$-\tfrac{1}{4}\left[\sum_{i=1}^{2} g_\parallel^{(i)}\int_{\Lambda_0}\tilde{f}_{0i}^2 dv+g_\perp\int_{\Lambda_0}\tilde{f}_{12}^2 dv+\sum_{i=1}^{2} l_\parallel^{(i)}\int_{\Lambda_0}\mathrm{tr}[h_{0i}^2]dv+l_\perp\int_{\Lambda_0}\mathrm{tr}[h_{12}^2]dv\right]\ , \qquad (5.36)$$

where $\tilde{f}_{\mu\nu}=\partial_\mu\tilde{a}_\nu-\partial_\nu\tilde{a}_\mu$ is the U(1) curvature (or field strength), and likewise $h_{\mu\nu}=\partial_\mu w_\nu-\partial_\nu w_\mu+[w_\mu,w_\nu]$, where w, given by Eq. (5.27), is the SU(2) curvature. Moreover, $g_\parallel^{(i)}$, $g_\perp$, $l_\parallel^{(i)}$, and $l_\perp$ are constants of dimension *cm*. Rotation invariance in the scaling limit would imply that $g_\parallel^{(1)}=g_\parallel^{(2)}\equiv g_\parallel$ and $l_\parallel^{(1)}=l_\parallel^{(2)}\equiv l_\parallel$. A brief discussion of the consequences of the U(1) curvature terms can be found in Fröhlich and Studer (1992b). For an application of the SU(2) curvature terms to a spin-pairing mechanism, see the end of Sec. V.C.

B. Linear-response theory and current sum rules

We briefly discuss the linear-response equations (5.29) and (5.30) that follow from our (universal) expression (5.26) for the "scaling limit" $S^*_{\Lambda_0}(\tilde{a},\varpi)$ of the effective action of systems characterized by the conditions (5.23) and (5.33)–(5.35). It is a simple exercise to verify that

$$\sqrt{\gamma(\xi)}\langle j^\mu(\xi)\rangle_{a,w}=\sqrt{\gamma(\xi)}j_c^\mu(\xi)+\frac{\sigma}{2\pi}\varepsilon^{\mu\nu\rho}(\partial_\mu\tilde{a}_\rho)(\xi)+\frac{\chi_s}{2\pi}\varepsilon^{\mu\nu\rho}(\partial_\nu\varpi_{\rho3})(\xi)+\cdots \qquad (5.37)$$

and

$$\sqrt{\gamma(\xi)}\langle s_A^\mu(\xi)\rangle_{a,w}=\sqrt{\gamma(\xi)}\delta_{A3}\delta_0^\mu m_3^0(\xi)+\delta_{A3}\frac{\chi_s}{2\pi}\varepsilon^{\mu\nu\rho}(\partial_\nu\tilde{a}_\rho)(\xi)+\delta_{A3}\frac{\sigma_s}{2\pi}\varepsilon^{\mu\nu\rho}(\partial_\nu\varpi_{\rho3})(\xi)$$
$$-\frac{k}{\pi}\varepsilon^{\mu\nu\rho}[(\partial_\nu w_{\rho A})(\xi)-\varepsilon_{ABC}w_{\nu B}(\xi)w_{\rho C}(\xi)]+\sqrt{\gamma(\xi)}2(1-\delta_{A3})\delta_0^\mu\tau_1^{00}(\xi)\varpi_{0A}(\xi)+\cdots\ , \qquad (5.38)$$

where the ellipses stand for terms coming from irrelevant terms in the effective action or from terms of second order in ϖ (e.g., a term proportional to η^{000}_{ABC}), which are of little interest in linear-response theory. Furthermore, we recall that $w_{\mu A}=w^{(\theta)}_{c,\mu A}+\varpi_{\mu A}$; see Eq. (5.27).

In order to understand the physical contents of these equations, we should recall the physical meaning of the connections a and w elucidated in Sec. III.B. From Eqs. (3.36), (3.55), and (3.59) we know that

$$a_j(x)=-\frac{q}{\hbar c}A_j(x)-\frac{m}{\hbar}f_j(x)\ , \qquad (5.39)$$

where $\mathbf{A}$ is the electromagnetic vector potential, q is the charge, m is the effective mass of the particles in the quantum fluid (for electrons we have $q=-e$), and $\mathbf{f}$ is a divergence-free velocity field generating some incompressible superfluid flow. Furthermore, by Eq. (3.36),

$$a_0(x)=\frac{q}{\hbar c}\Phi(x)\ , \qquad (5.40)$$

where Φ is the electrostatic potential.

We recall that, for our study of two-dimensional, incompressible quantum fluids on a surface M embedded in $\mathbb{E}^3$, it is natural to choose an SU(2) gauge with the property that $e^3(t,\mathbf{x})$ is orthogonal to the cotangent space M at $\mathbf{x}$, for all times t; see Eqs. (5.23)–(5.25) and the discussion at the beginning of Sec. III.A. Then, by Eq. (5.25), a possible affine SU(2) connection $\omega_\mu^{(s)}$ has the form

$$\omega_\mu^{(s)}(x)=i\omega_\mu(x)L_3^{(s)} . \tag{5.41}$$

It then follows from Eqs. (3.28)–(3.30), (3.37), (3.55), and (3.60) that

$$w_{0A}(x)=-\frac{g\mu}{2\hbar c}B_A(x)+\delta_{A3}[\Omega(x)+\omega_0(x)] , \tag{5.42}$$

where, by Eq. (3.44), $\mathbf{\Omega}(x)=(0,0,\Omega(x))=\frac{1}{2}\operatorname{curl}\mathbf{f}(x)$ with respect to the orthonormal frame $(e^A(x))_{A=1}^3$ at x, and the magnetic moment of the particles is determined by μ [for electrons we have $\mu=-\mu_B$; see Eq. (2.7)]. Moreover, by Eqs. (3.28)–(3.30), (3.38), (3.50), and (5.41),

$$w_{jA}(x)=\left[-\frac{g\mu}{2\hbar c}+\frac{q}{4mc^2}\right]\sum_{B=1}^{3}\varepsilon_{kAB}(x)E_B(x)+\delta_{A3}[\omega_j(x)+\cdots] , \tag{5.43}$$

where the ellipses correspond to terms proportional to derivatives of $\Omega(t',\mathbf{x})$, $t'\le t$ [and are generated by the SU(2) gauge transformation $U^{(s)}(R)$ with R defined in Eq. (3.45)].

Finally, we define, in physical units, the charge-density (operator) by

$$\rho(\xi)=q\sqrt{\gamma(\xi)}j^0(\xi) , \tag{5.44}$$

the electric current density by

$$\mathcal{J}^i(\xi)=qc\sqrt{\gamma(\xi)}j^i(\xi) , \tag{5.45}$$

the spin density by

$$\mathcal{S}^0_A(\xi)=\frac{\hbar}{2}\sqrt{\gamma(\xi)}s_A^0(\xi) , \tag{5.46}$$

and the spin current density by

$$\mathcal{S}^i_A(\xi)=\frac{\hbar c}{2}\sqrt{\gamma(\xi)}s_A^i(\xi) . \tag{5.47}$$

Then Eq. (5.37) for the ($\mu=0$) component reads

$$\langle\rho(\xi)\rangle_{a,w}=\rho_c(\xi)-\frac{\sigma_H}{c}\tilde{B}_3(\xi)-\sigma\frac{2qm}{h}\tilde{\Omega}(\xi)-\chi_s\left[\frac{qg\mu}{4hc}\nabla\cdot\tilde{\mathbf{E}}(\xi)-\frac{q}{2\pi}\mathcal{R}(\xi)\right]+\cdots , \tag{5.48}$$

where the Hall conductivity (for the electric current) is given by

$$\sigma_H=\frac{q^2}{h}\sigma , \tag{5.49}$$

$\tilde{\mathbf{E}}(\xi)=(E_1(\xi),E_2(\xi))$, and the operator $\nabla=(\partial_1,\partial_2)\equiv(\partial/\partial\xi^1,\partial/\partial\xi^2)$, $\mathcal{R}(\xi)=\operatorname{curl}\boldsymbol{\omega}(\xi)$ is the scalar curvature of M at ξ, and the ellipses stand for contributions from irrelevant terms. It will turn out that

$$\chi_\perp=-\frac{qg\mu}{2hc}\chi_s \tag{5.50}$$

is the magnetic susceptibility of the system in the 3-direction normal to the surface. In Eq. (5.48) and the following formulas the tildes indicate contributions from the perturbation potentials $\tilde{a}$ and $\tilde{w}$ (we have absorbed the affine spin connection ω into $\tilde{w}$, but without decorating it with a tilde). Next, one verifies that

$$\langle\mathcal{J}^i(\xi)\rangle_{a,w}=\mathcal{J}^i_c(\xi)-\sigma_H\,\varepsilon^{ij}\tilde{E}_j(\xi)+\sigma\frac{qm}{h}\varepsilon^{ij}\frac{\partial}{\partial\tau}\tilde{f}_j(\xi)-\chi_s\left[\frac{qg\mu}{2h}\varepsilon^{ij}\partial_j\tilde{B}_3(\xi)-\frac{qc}{2\pi}\varepsilon^{ij}\partial_j\tilde{\Omega}(\xi)\right]+\chi_s\left[\frac{qg\mu}{4hc}\frac{\partial}{\partial\tau}\tilde{E}^i(\xi)-\frac{q}{2\pi}\varepsilon^{ij}\frac{\partial}{\partial\tau}\lambda_j(\xi)\right]+\cdots , \tag{5.51}$$

where $\tau=\xi^0/c$ is the rescaled time variable.

From Eq. (5.38) we find, for example, that for $\mu=0$ and $A=3$ [i.e., for the spin density along the 3-direction in the spin or (co)tangent space]

$$\frac{g\mu}{\hbar}\langle\mathcal{S}^0_3(\xi)\rangle_{a,w}=\mathcal{M}_c(\xi)+\sigma_H^{\text{spin}}\left[\frac{g\mu}{2\hbar c}\nabla\cdot\tilde{\mathbf{E}}(\xi)-2\mathcal{R}(\xi)\right]+k\frac{g^2\mu^2}{4hc}\nabla\cdot\tilde{\mathbf{E}}_c(\xi)+\chi_\perp\left[\tilde{B}_3(\xi)-\frac{hc}{g\mu\pi}\tilde{\Omega}(\xi)\right]+\cdots , \tag{5.52}$$

where $\mathcal{M}_c$ is the magnetization of the system in the background field (a_c,w_c) given by Eqs. (5.23)–(5.25), $\chi_\perp$ is the magnetic susceptibility at (a_c,w_c) given in Eq. (5.50), and

$$\sigma_H^{\text{spin}}=\frac{g\mu}{4\pi}k-\frac{g\mu}{8\pi}\sigma_s \tag{5.53}$$

is the Hall conductivity for the spin current. As Eqs. (5.52) and (5.26) show, σ_H^{spin} is a pseudoscalar. Next, for $\mu=i=1,2$ and $A=3$ [i.e., for the spin current density in the i-direction in the surface M and polarized along the 3-

direction in the spin or (co)tangent space],

$$\langle \mathcal{S}_3^i(\xi)\rangle_{a,w} = \sigma_H^{\text{spin}}\left[\varepsilon^{ij}\partial_j\tilde{B}_3(\xi) - \frac{hc}{g\mu\pi}\varepsilon^{ij}\partial_j\tilde{\Omega}(\xi) - \frac{1}{2c}\frac{\partial}{\partial\tau}\tilde{E}^i(\xi) + \frac{h}{g\mu\pi}\frac{\partial}{\partial\tau}\lambda^i(\xi)\right] + k\frac{g\mu}{4\pi}\varepsilon^{ij}\partial_j B_c(\xi) + \chi_\perp\left[\frac{\hbar c}{g\mu}\varepsilon^{ij}\tilde{E}_j(\xi) - \frac{m\hbar c}{qg\mu}\frac{\partial}{\partial\tau}\tilde{f}^i(\xi)\right] + \cdots , \tag{5.54}$$

where the ellipses stand for terms proportional to $\omega_0(\xi)$ and further irrelevant and higher-order terms. Similar equations hold for the remaining su(2) components of $\langle \mathcal{S}_A^\mu(\xi)\rangle_{a,w}$, but we refrain from displaying them explicitly and refer the reader to the discussion in Fröhlich and Studer (1992b).

We encourage the reader to notice how neatly our formulas summarize the laws of the Hall effect, including effects due to tidal forces coming from (superfluid) flow and effects due to curvature. (We believe that the tidal terms might be relevant in the study of the transition from one plateau of σ_H to the next in very pure samples.)

Let us briefly comment on the relation of our definition of the Hall conductivity $\sigma_H = (e^2/h)\sigma$ as the coefficient of a Chern-Simons term $(\sigma/4\pi)\int_{\Lambda_0}\tilde{a}\wedge d\tilde{a}$ in the effective action $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$ [see Eq. (5.26)], of an incompressible quantum Hall fluid to the more conventional definition via the Kubo formula (see, for example, Fradkin, 1991). It follows easily from Eqs. (5.4), (5.5), and (5.26) that σ appears in the following current sum rules: For every choice of a permutation $(\mu\nu\rho)$ of (012),

$$\sigma = 2\pi i\,\text{sgn}(\mu\nu\rho)\int (y-x)^\mu \langle T[j^\nu(x)j^\rho(y)]\rangle^{\text{con}}_{a,w} d^3y . \tag{5.55}$$

These are three equations for one and the same quantity σ. The equation for $(\mu\nu\rho)=(012)$ is

$$\sigma = 2\pi i\int (s-t)\langle T[j^1(t,\mathbf{x})j^2(s,\mathbf{y})]\rangle^{\text{con}}_{a,w} ds\, d^2\mathbf{y} , \tag{5.56}$$

which is just the Kubo formula (in "mathematical" units); compare, for example, Fradkin (1991). The other two equations are an automatic consequence of U(1) gauge invariance.

Thouless and co-workers (Thouless *et al.*, 1982; Niu and Thouless, 1984, 1987; Kohmoto, 1985), and followers (Avron, Seiler, and Simon, 1983; Avron and Seiler, 1985; Dana, Avron, and Zak, 1985; Avron, Seiler, and Yaffe, 1987; Kunz, 1987), have derived from the Kubo formula that

$$\sigma = \frac{1}{n_0}c_1 , \tag{5.57}$$

where n_0 is the ground-state degeneracy and c_1 is the first Chern number of a vector bundle over a 2-torus of magnetic fluxes (Φ_1,Φ_2). Thus c_1 is an integer. It can be identified with the number of electrons N created when one turns on a local magnetic field of total flux n_0; see Eq. (5.65) below. Does our formulation "know" that n_0 is the degeneracy of the ground state? Yes, it does! This follows, for example, from the material in Sec. VI and has been noted by Wen (1989, 1990a) and Wen and Niu (1990).

Bellissard (1988a, 1988b) and Avron, Seiler, and Simon (1990, 1992)) have also given a definition of σ as an index. Their definition is equivalent to ours, too, and the proof follows from the material in Sec. VI; see also Sec. 6 in Fröhlich and Kerler (1991).

Finally, we note that from Eqs. (5.4), (5.5), and (5.26) it also follows that $\sigma_H^{\text{spin}} = (g\mu_B/8\pi)\sigma_s$ (for $k=0$, i.e., fully spin-polarized quantum Hall fluids) is given by a Kubo formula involving spin currents,

$$\sigma_s = 2\pi i\,\text{sgn}(\mu\nu\rho)\int (y-x)^\mu\langle T[s_3^\nu(x)s_3^\rho(y)]\rangle^{\text{con}}_{a,w} d^3y , \tag{5.58}$$

and it can then be shown to be proportional to a first Chern number of a vector bundle over a two-dimensional torus of electric charges per unit length (Q_1,Q_2). We refer the reader to Fröhlich and Studer (1992b) for a more systematic study of current sum rules and "proofs." Here, we merely give a last example by expressing the magnetic susceptibility $\chi_\perp = -(qg\mu/2hc)\chi_s$ in the form of a (mixed) sum rule,

$$\chi_s = 2\pi i\,\text{sgn}(\mu\nu\rho)\int (y-x)^\mu\langle T[j^\nu(x)s_3^\rho(y)]\rangle^{\text{con}}_{a,w} d^3y . \tag{5.59}$$

C. Quasiparticle excitations and a spin-singlet electron pairing mechanism

In this subsection we present a first analysis of quasiparticle excitations above the ground state in a two-dimensional, incompressible quantum fluid, whose "scaling limit" of the effective action is given by the action $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$ presented in Eq. (5.26). A systematic, general analysis of quasiparticle excitations in electronic quantum Hall fluids is given in Sec. VI based on an analysis of the so-far mysterious boundary terms "B.T." in Eq. (5.26). At the end of this section we describe a natural mechanism for spin-singlet pairing of electrons that are moving in some two- or three-dimensional, antiferromagnetic background.

1. "Laughlin vortices" and fractional statistics

For simplicity, we begin our analysis by considering a flat, two-dimensional system of charged fermions with vanishing magnetic moment ($\mu=0$) so that the SU(2) connection w vanishes identically in an appropriate SU(2) gauge [the local frames $e^1(x)$, $e^2(x)$, and $e^3(x)$ are chosen to be time independent, so that there is no tidal Zeeman term; see Sec. III]. We suppose that, in a small neighborhood of a suitably chosen background potential a_c (typically $a_{c,0}=0$, $b_c=\partial_1 a_{c,2}-\partial_2 a_{c,1}=\text{const}$ and of suitable magnitude), the system is incompressible. Then the "scaling limit" of the action is given by

$$-\frac{1}{\hbar}S^*_{\Lambda_0}(\tilde{a})=\int_{\Lambda_0} j_c^\mu \tilde{a}_\mu dv+\frac{\sigma}{4\pi}\int_{\Lambda_0}\tilde{a}\wedge d\tilde{a} , \qquad (5.60)$$

up to boundary terms. The first term on the rhs is unimportant in the following discussion, and we set $j_c^\mu=0$.

Let us produce a "Laughlin vortex" (Laughlin, 1983a, 1983b, 1990) in this system by turning on a (perturbing) magnetic field $\tilde{b}=\partial_1\tilde{a}_2-\partial_2\tilde{a}_1$ in a small disk. (Actually, $\tilde{b}$ could be a vorticity field of a superfluid flow if, instead of an electronic quantum Hall fluid, we consider a superfluid film; see Sec. VII.B. We shall, however, use "magnetic language" in the following discussion.) From our discussion of the Aharonov-Bohm effect in Sec. II.C we know that this excitation only disturbs the system locally, and thus may have a finite energy difference from the ground-state energy, if

$$\frac{1}{2\pi}\int \tilde{b}(t,\xi)d^2\xi=n, \quad \text{with } n\in\mathbb{Z} . \qquad (5.61)$$

By Eq. (5.37) for $\mu=0$ [see also Eq. (4.45)], we have

$$\langle j^0(\xi)\rangle_{a,w}=\frac{\sigma}{2\pi}\tilde{b}(\xi) , \qquad (5.62)$$

and hence the charge of this excitation (in units of e and with the background charge normalized to 0) is given by

$$Q=\int \langle j^0(t,\xi)\rangle_{a,w}d^2\xi=\sigma n . \qquad (5.63)$$

If σ is not an integer, then Q will be *fractional,* in general. Now consider two such excitations localized in two disjoint small disks and interchange them (adiabatically) along some path oriented anticlockwise. According to Sec. II.C, the Aharonov-Bohm phase picked up in this process is given by

$$e^{i\pi\theta}\equiv e^{i\pi Qn}=e^{i\pi\sigma n^2} , \qquad (5.64)$$

where we have normalized the statistical phase θ such that $\theta=1$ (mod 2) corresponds to Fermi statistics, $\theta=0$ (mod 2) corresponds to bosons, and $\theta\neq 0,1$ (mod 2) corresponds to anyons (Leinaas and Myrheim, 1977; Goldin, Menikoff, and Sharp, 1980, 1981, 1983; Wilczek, 1982a, 1982b; for a review see Fröhlich, 1990). Thus Laughlin vortices are anyons, unless σn^2 is an integer.

Among the excitations that one can produce in this fashion there should be the particles constituting the system, namely, electrons (or holes). Let us suppose that the state of the system is fully spin polarized (as is the case for filling factors $\nu=\frac{1}{3},\frac{1}{5}$ (say) in electronic quantum Hall fluids). Supposing that a magnetic flux n_0 (in units of the elementary flux quantum hc/e) produces a state of N electrons, we infer, from Eq. (5.63), that

$$\sigma=\frac{N}{n_0} . \qquad (5.65)$$

If N is odd, this state is composed of N fermions and hence describes a fermion, so that, by Eq. (5.64),

$$e^{i\pi N n_0}=-1 . \qquad (5.66)$$

Thus n_0 must be odd too. In fact, one may show that if N and n_0 have no common divisor then n_0 is odd. In particular, for $N=1$, we conclude that

$$\sigma=\frac{1}{n_0}, \quad \text{with } n_0 \text{ odd} . \qquad (5.67)$$

This is the famous odd-denominator rule (see, for example, Tao and Wu, 1985). [See also Eq. (6.90) in Sec. VI.B.] An excitation associated with a magnetic flux (or vorticity) 1 (in units of hc/e) then has fractional charge $Q=1/n_0$ (in units of e) and is an anyon for $n_0>1$.

Note that the vector potential $\tilde{a}$, created by a pointlike excitation of charge Q located at $\xi=\mathbf{0}$, is given by

$$\tilde{a}_i(t,\xi)=-\frac{Q}{\sigma}\sum_{j=1}^{2}\varepsilon_{ij}\frac{\xi^j}{|\xi|^2}, \quad i=1,2 , \qquad (5.68)$$

as follows from Eqs. (5.61) and (5.63), for $\langle j^0(t,\xi)\rangle_{a,w}=Q\delta(\xi)$. This is the "U(1) Knizhnik-Zamolodchikov connection."

2. Spinon quantum mechanics

Next, we consider another "in vitro" system, namely, a "chiral spin liquid." (It is not entirely clear that such systems exist in nature, but they might appear as subsystems of superfluid ^{3}He layers.) A chiral spin liquid is a system of neutral particles of spin $s_0>0$ with nonzero magnetic moment (i.e., $\mu\neq 0$), having a spin-singlet ground state for some constant magnetic field $\mathbf{B}_c$. It is assumed here to be incompressible and to exhibit breaking of parity (reflections in lines) and time reversal, but no spontaneous magnetization. In our formalism, the "scaling limit" of the effective action of such a system is given by

$$-\frac{1}{\hbar}S^*_{\Lambda_0}(w)=\frac{k}{4\pi}\int_{\Lambda_0}\mathrm{tr}(w\wedge dw+\tfrac{2}{3}w\wedge w\wedge w) , \qquad (5.69)$$

up to boundary terms. Under reflections in lines, w_i transforms as a vector, w_0 as a pseudoscalar, and k as a pseudoscalar. Let us consider an excitation created by turning on an SU(2) gauge field w with curvature (or field strength) h, given by

$$h(\xi)=dw(\xi)+w(\xi)\wedge w(\xi)$$
$$=\frac{i}{2}\sum_{\mu,\nu=0}^{2}\sum_{A=1}^{3}h_{\mu\nu A}(\xi)L_A^{(s_0)}d\xi^\mu\wedge d\xi^\nu , \quad (5.70)$$

where $h_{\mu\nu A}=\partial_\mu w_{\nu A}-\partial_\nu w_{\mu A}-2\varepsilon_{ABC}w_{\mu B}w_{\nu C}$ [see also the discussion following Eq. (5.36)]. For example, we may choose h to be given by

$$\mathbf{h}_{0i}(\xi)\equiv(h_{0i1}(\xi),h_{0i2}(\xi),h_{0i3}(\xi))=\mathbf{0}$$

and (5.71)

$$\mathbf{h}_{12}(\xi)=-\mathbf{n}h_0(\xi) ,$$

where $\mathbf{n}$ is some unit vector in $\mathbb{R}^3$ and h_0 is a time-independent function. By Eq. (5.38), the spin density of this excitation is given by

$$\langle \mathbf{s}^0(\xi)\rangle_w\equiv\langle (s_1^0(\xi),s_2^0(\xi),s_3^0(\xi))\rangle_w$$
$$=-\frac{k}{\pi}\mathbf{h}_{12}(\xi)=\mathbf{n}\frac{k}{\pi}h_0(\xi) , \quad (5.72)$$

so that the expectation value of its total spin operator, $\mathbf{S}=(\hbar/2)\mathbf{L}^{(s)}$ [in the spin-s representation; see Eq. (2.8)], is given by

$$\frac{\hbar}{2}\langle \mathbf{L}^{(s)}\rangle_w=\langle \mathbf{S}\rangle_w=\mathbf{n}\frac{k\hbar}{2\pi}\int h_0(\xi)d^2\xi . \quad (5.73)$$

Such an excitation is commonly called a "spinon." Quantum-mechanically, spin is quantized: $\mathbf{S}\cdot\mathbf{S}=s(s+1)\hbar^2$, with $s\in\frac{1}{2}\mathbb{Z}$. Consider a spinon of spin s localized at the point $\xi=\xi_1$. Then Eq. (5.72) says that $\mathbf{h}_{12}$ has to solve the equation

$$\langle \mathbf{L}^{(s)}\rangle_w\delta(\xi-\xi_1)=-\frac{k}{\pi}\mathbf{h}_{12}(\xi) . \quad (5.74)$$

An SU(2) connection $w=i\sum_\mu \mathbf{w}_\mu\cdot\mathbf{L}^{(s)}d\xi^\mu$ for the field strength h satisfying Eq. (5.74), with $\mathbf{h}_{0i}=0$, is given by

$$\mathbf{w}_0(\xi)=\mathbf{0}$$

and (5.75)

$$\mathbf{w}_j(\xi)=\frac{1}{2k}\langle \mathbf{L}^{(s)}\rangle_w\sum_{l=1}^{2}\varepsilon_{jl}\frac{\xi^l-\xi_1^l}{|\xi-\xi_1|^2}, \quad j=1,2 .$$

Suppose we now create a second spinon of spin s' moving in the background gauge field w excited by the first spinon. Its dynamics is coupled to w through the covariant derivatives [see Eqs. (3.26) and (3.27)]

$$\mathcal{D}_\mu=\partial_\mu+i\mathbf{w}_\mu(\xi)\cdot\mathbf{L}^{(s')} , \quad (5.76)$$

with $\mathbf{w}$ as in Eq. (5.75). Let us imagine that it makes sense to do "two-spinon quantum mechanics" on a Hilbert space $\mathcal{H}^{(s)}\otimes\mathcal{H}^{(s')}$, with

$$\mathcal{H}^{(s)}=\mathcal{D}^{(s)}\otimes L^2(M,dv) ,$$

where $\mathcal{D}^{(s)}$ carries the spin-s representation of SU(2). By Eqs. (5.75) and (5.76), the covariant derivatives on $\mathcal{H}^{(s)}\otimes\mathcal{H}^{(s')}$ are then given by

$$\mathcal{D}_0^1=\frac{\partial}{\partial\xi_1^0} ,$$
$$\mathcal{D}_j^1=\frac{\partial}{\partial\xi_1^j}+\frac{i}{2k}\sum_{l=1}^{2}\varepsilon_{jl}\frac{\xi_1^l-\xi_2^l}{|\xi_1-\xi_2|^2}\sum_{A=1}^{3}L_A^{(s)}\otimes L_A^{(s')} , \quad (5.77)$$

and

$$\mathcal{D}_0^2=\frac{\partial}{\partial\xi_2^0} ,$$
$$\mathcal{D}_j^2=\frac{\partial}{\partial\xi_2^j}+\frac{i}{2k}\sum_{l=1}^{2}\varepsilon_{jl}\frac{\xi_2^l-\xi_1^l}{|\xi_1-\xi_2|^2}\sum_{A=1}^{3}L_A^{(s)}\otimes L_A^{(s')} . \quad (5.78)$$

These are the covariant derivatives associated with the celebrated Knizhnik-Zamolodchikov connection (Knizhnik and Zamolodchikov, 1984; Tsuchiya and Kanie, 1987). For "two-spinon quantum mechanics" with parallel transport given by Eq. (5.78) to be consistent with unitarity, it is necessary that

$$k=\pm(\kappa+2), \quad \kappa=1,2,\ldots . \quad (5.79)$$

This follows from results in Belavin, Polyakov, and Zamolodchikov (1984), Knizhnik and Zamolodchikov (1984), Kohno (1987, 1988), and Tsuchiya and Kanie (1987). Recalling what we have said in Sec. II.D about the Aharonov-Casher effect, we observe that the "phase factor" arising in the parallel transport of a quantum-mechanical spinon in the field excited by a classical spinon with spin orthogonal to the plane of the system is an "Aharonov-Casher phase factor."

Consider an exchange of the positions of two quantum-mechanical, pointlike spinons along some anticlockwise-oriented path. Then the "Aharonov-Casher phase factor" multiplying the wave function is given by a matrix

$$R_{ss'}^{(\kappa)}:\ \mathcal{D}^{(s)}\otimes\mathcal{D}^{(s')}\to\mathcal{D}^{(s')}\otimes\mathcal{D}^{(s)} ,$$

which is the braid matrix for exchanging a chiral vertex of spin s with a chiral vertex of spin s' in the chiral Wess-Zumino-Novikov-Witten model (Belavin, Polyakov, and Zamolodchikov, 1984; Knizhnik and Zamolodchikov, 1984; Tsuchiya and Kanie, 1987; see also Gawedzki, 1990) at level κ. It is given by

$$R_{ss'}^{(\kappa)}=T\pi_s\otimes\pi_{s'}(\mathcal{R}^{(\kappa)}) , \quad (5.80)$$

where $\mathcal{R}^{(\kappa)}$ is the universal R matrix of the quantum group $U_q(sl_2)$, with $q=\exp[i\pi/(\kappa+2)]$, and T is the flip (transposition of factors). All this can be extended to "n-spinon quantum mechanics." The matrices $(R_{ss'}^{(\kappa)})$ determine an exotic quantum statistics described by non-Abelian (for $\kappa>1$ and $s,s'<\kappa/2$) representations of the braid groups (more precisely, the groupoids of colored braids) which is commonly called non-Abelian braid statistics (Fredenhagen, Rehren, and Schroer, 1989; Fröhlich and Gabbiani, 1990; Fröhlich, Gabbiani, and Marchetti, 1990; Fröhlich and Marchetti, 1991). We wish to note that s and s' are forced to be $\leq\kappa/2$, i.e., there are no spinons of spin $>\kappa/2$. One might call this phenomenon "spin screening." If the particles of spin s_0 constituting the chiral spin liquid appear as spinon excitations above the ground state and s_0 is half-integer, then

$$\kappa \geq 2s_0 \ ,$$

since these particles carry spin s_0. One can argue that the statistics of these particles must be Abelian braid statistics, i.e., they are anyons. In fact, it then follows that they are semions ($\theta=\frac{1}{2}$). Now, for a given level κ, the matrices $(R_{ss}^{(\kappa)})$ define an Abelian representation of the braid groups if and only if $2s=\kappa$. Thus it follows that, for a chiral spin liquid made of particles of spin s_0,

$$\kappa=2s_0 \ , \tag{5.81}$$

any spinon excitation of spin $s<s_0$ exhibits non-Abelian braid statistics!

The reader may feel that our "derivation" of spinon quantum mechanics from the effective action $S^*_{\Lambda_0}(w)$ given in Eq. (5.69) is based on idealizations—see Eq. (5.74)—and jumps in the logics—reasoning between (5.76) and (5.78)—that might make it appear to be quite problematic. Actually, it turns out that our conclusions concerning spinon statistics, in particular Eqs. (5.80) and (5.81), are perfectly correct. This follows from an analysis of the boundary terms "B.T." in the effective action (5.26); see Sec. VI.

In order to understand electronic quantum Hall fluids with spin-singlet ground state, one must glue the Laughlin vortices described in Eqs. (5.61)–(5.66) to the spinons discussed above. One checks that for $\sigma=2/n_0$, n_0 odd, and $\kappa=2s_0=1$, a Laughlin vortex of vorticity $n=-n_0/2$ (!) glued to a spinon of spin $s=\frac{1}{2}$ is an excitation of charge $Q=-1$, spin $\frac{1}{2}$ and Fermi statistics; see Fröhlich and Kerler (1991) and the discussion in Sec. VI. These are the properties of an electron. In an electronic quantum Hall fluid (without any very exotic internal symmetries) one does *not* find any excitations with non-Abelian braid statistics. However, if one could manufacture a quantum Hall fluid made of charge carriers of spin $s_0=\frac{3}{2},\frac{5}{2},\ldots$, with a spin-singlet ground state, it would display excitations with non-Abelian braid statistics (Zhang, Hansson, and Kivelson, 1989; Fröhlich, Kerler, and Marchetti, 1992). It may appear difficult to build such a system in practice. But, perhaps, one can think of incompressible superfluid films of particles of higher spin, with broken parity (reflections-in-lines) and time reversal, which would also exhibit excitations with non-Abelian braid statistics. The analysis sketched above extends, in a straightforward way, to systems with continuous internal symmetries and corresponding gauge fields; see the discussion in Sec. VI.C.

It may be worthwhile emphasizing that in quantum Hall fluids with nonvanishing magnetic susceptibility (spin-polarized Hall fluids) the fractional statistics of Laughlin vortices always appears as a consequence of a combination of the Aharonov-Bohm *and* the Aharonov-Casher effect (but notice that, for spin-polarized quantum Hall fluids, the Aharonov-Casher phase factors are automatically Abelian). This is a consequence of the fact that electrons have a nonvanishing magnetic moment and follows from Eq. (5.26).

3. Spin-singlet electron pairing mechanism

In this subsection we describe a mechanism for spin-singlet pairing of dopant electrons (or holes) moving in an antiferromagnetic or a resonating valence-band (RVB) background. Our mechanism may be related to the "spin-bag mechanism" (Schrieffer, Wen, and Zhang, 1988). For definiteness, we consider two-dimensional systems, but our arguments can easily be extended to systems in three dimensions.

The magnetic properties of the system are described by an order parameter ϕ transforming under the adjoint representation of $SU(2)_{spin}$. We assume that, after coupling ϕ to an external SU(2) gauge field w, the system is an SU(2) diamagnet. Integrating out the order parameter ϕ, at a fixed temperature $T>0$, we obtain the free energy $F_T(w)=-kT\ln Z_T(w)$ as a functional of the gauge field w.

For an antiferromagnet or a system with a resonating valence-band ground state, described, for example, by a Landau-Ginzburg-type Lagrangian in which ϕ is coupled minimally to w, the free energy $F_T(w)$ is expected to be smooth in w in a neighborhood of $w=0$. This is in contrast to the behavior of $F_T(w)$ in a system with ferromagnetic ordering. In a ferromagnetic system, the order parameter is the spin density, which is the time component of the spin current density. The spin current density is the variable conjugate to the SU(2) gauge field w; see Eq. (5.4). Thus if the expectation value of the total spin operator in an equilibrium state of the system at some temperature T is nonzero, then the free energy $F_T(w)$ must have a cusplike singularity at $w=0$. But in systems with an antiferromagnetic or RVB magnetic structure, the order parameter ϕ is *not* given by a component of some current density. Viewing $SU(2)_{spin}$ as an internal symmetry group of the system, we see that ϕ turns out to be a scalar field transforming under the adjoint representation of the internal symmetry group. In this situation it is consistent to assume that $F_T(w)$ is quadratic in w at $w=0$.

We shall assume that, in the scaling limit, the system does not break parity or time-reversal invariance and is rotation and translation invariant. It then follows from the assumed smoothness of $F_T(w)$ near $w=0$, from SU(2) diamagnetism, and from the invariance of $F_T(w)$ under time-independent SU(2) gauge transformations, that $F_T(w)$ is given by

$$F_T(w)=-\frac{1}{4}\left[\frac{1}{l(T)}\int \mathrm{tr}[w_0^2(\xi)]d^2\xi + \frac{1}{l'(T)}\sum_{i=1}^{2}\int \mathrm{tr}[(w_i^T)^2(\xi)]d^2\xi\right] -\frac{1}{4}\left[l_\parallel(T)\sum_{i=1}^{2}\int \mathrm{tr}[h_{0i}^2(\xi)]d^2\xi + l_\perp(T)\int \mathrm{tr}[h_{12}^2(\xi)]d^2\xi\right]+\cdots . \quad (5.82)$$

Here w is a time-independent, external SU(2) gauge field, and our conventions have been chosen such that the traces in Eq. (5.82) are negative; see Eq. (3.27). The third and the fourth term on the rhs of Eq. (5.82) are the "Maxwell terms" discussed in Eq. (5.36) [note that, because of rotation and translation invariance, we have $l_\parallel^{(1)}=l_\parallel^{(2)}=l_\parallel$ and $g_{ij}(\xi)=\delta_{ij}$ in Eq. (3.27)]. In the first and second term on the rhs of Eq. (5.82), l and l' are constants of dimension *cm*, and the "transversal" gauge field w_i^T, $i=1,2$, is defined by $w_i^T=[\delta_i^j-\mathcal{D}_i\Delta_{\mathrm{cov}}^{-1}\mathcal{D}^j]w_j$, with $\Delta_{\mathrm{cov}}=\mathcal{D}_j\mathcal{D}^j$, where $\mathcal{D}_j$ has been defined in Eqs. (5.11) and (5.12). Note that the first term is invariant under time-independent SU(2) gauge transformations (5.7), and the second term respects the SU(2) Ward identity (5.11) [i.e., it respects infinitesimal (time-independent) SU(2) gauge transformations], but it is *nonlocal*. In fact, this term is the non-Abelian analog of the term we have encountered in the free energy (4.10) of a superconductor. Its presence mirrors the fact that we do *not* require the system to be incompressible. SU(2) diamagnetism implies that $l_\parallel$, $l_\perp$, l, and l' are non-negative. (For a system with a RVB or VBS ground state, the nonlocal term is expected to be absent.) Finally, the dots in Eq. (5.82) stand for terms of higher order in w or involving higher derivatives acting on w.

Given the free energy (5.82), we can study how the system responds to turning on an (external) SU(2) gauge field w. Choosing some w with the property that $\mathbf{w}_j=0$, $j=1,2$, we find, similarly to Eqs. (5.38) and (5.72), that

$$\langle \mathbf{s}^0(\xi)\rangle_w=\frac{\delta F_T(w)}{\delta \mathbf{w}_0(\xi)}=-l_\parallel\Delta\mathbf{w}_0(\xi)+\frac{1}{l}\mathbf{w}_0(\xi)+\cdots , \quad (5.83)$$

where Δ is the two-dimensional Laplacian.

In order to create a spin-$\frac{1}{2}$ excitation (a dopant electron) localized at $\xi=\xi_1$, the three SU(2) gauge-field components of $\mathbf{w}_0$ have to solve Eq. (5.83) for a lhs of the form

$$\langle \mathbf{s}^0(\xi)\rangle_w=\frac{2}{\hbar}\langle \mathbf{S}\rangle_w\delta(\xi-\xi_1)\equiv\mathbf{\Sigma}_1\delta(\xi-\xi_1) . \quad (5.84)$$

If $l_\parallel$ and l are positive constants, the solution of Eqs. (5.83) and (5.84) is given by

$$\mathbf{w}_0(\xi)=\frac{\mathbf{\Sigma}_1}{l_\parallel}K\left[\frac{1}{l^*}|\xi-\xi_1|\right] \text{ and } \mathbf{w}_j(\xi)=\mathbf{0},\ j=1,2 , \quad (5.85)$$

where $l^*=(l_\parallel l)^{1/2}$, and K is a function with the following asymptotic behavior:

$$K(x)=\sqrt{1/x}\,e^{-x}[\alpha+O(1/x)], \text{ if } x\gg 1 , \quad (5.86)$$

with α a positive constant.

We now consider a second dopant electron moving in the background gauge field w excited by the first one; see Eq. (5.85). Recalling the form of the coupling in (5.76), we expect the motion of the second electron to be subject to a force resulting from a "Zeeman term" given by $2c\mathbf{w}_0\cdot\mathbf{S}_2$, where $\mathbf{S}_2$ denotes the spin operator of the second electron. Classically, we find a contribution to the energy of the two-electron system of the form

$$E_{12}=\hbar c\frac{\mathbf{\Sigma}_1\cdot\mathbf{\Sigma}_2}{l_\parallel}K\left[\frac{1}{l^*}|\xi_2-\xi_1|\right] , \quad (5.87)$$

where $\mathbf{\Sigma}_2$ is the expectation value of the spin operator $\mathbf{S}_2$ of the second electron, which we assume to be localized at $\xi=\xi_2$. A similar expression to (5.87) is obtained in a "more symmetric" treatment: One solves Eq. (5.83) for several dopant electrons localized at points $\xi_1,\ldots,\xi_n$ and considers the interaction term in the free energy $F_T(w)$ corresponding to the resulting SU(2) gauge field w.

The form of the energy E_{12} in (5.87) suggests that a term of the form

$$J(\xi_1-\xi_2)\mathbf{S}_{\xi_1}\cdot\mathbf{S}_{\xi_2}$$

must be included in the Hamiltonian of the dopant system, where $J(\xi)$ is some positive function with $J(\xi)\sim e^{-|\xi|/l^*}$ for $|\xi|\to\infty$.

To summarize, when we consider two dopant electrons moving in an antiferromagnetic background characterized by the free energy (5.82), then Eq. (5.87) implies that, as a result of the collective response of the background, the two electrons experience a mutual attraction if their spins are "antiparallel" and a mutual repulsion if their spins are "parallel." This interaction could yield binding of dopant-electron pairs in spin-singlet states (i.e., with "antiparallel spin orientations") and hence could result in a superconducting state for the dopant electrons.

VI. ANOMALY CANCELLATION AND ALGEBRAS OF CHIRAL EDGE CURRENTS IN TWO-DIMENSIONAL, INCOMPRESSIBLE QUANTUM FLUIDS

The purpose of this section is to discuss the origin of the quantization of the constants σ, χ_s, σ_s, and k, which appear as the coefficients of the Chern-Simons terms in the "scaling limit" of the effective action of two-dimensional, incompressible quantum fluids; see Eq. (5.26). From the linear-response equations displayed in (5.48) through (5.54) we recall that these constants completely determine the response (on large-distance scales and at low frequencies) of such quantum fluids when perturbed by small external electromagnetic "tidal" and geometric fields. In particular, they specify the Hall effect for the electric and for the spin current. We show how the rationality of the constants σ, χ_s, σ_s, and k follows from a consistency analysis of the theory presented hitherto. This analysis rests on a more thorough inspection of the so-far mysterious boundary terms "B.T." on the rhs of Eq. (5.26). By the requirement of anomaly cancellation we find, among these boundary terms, gauge-anomalous contributions which turn out to be the generating functionals of the connected, time-ordered Green functions of chiral current operators which generate $\hat{\mathfrak{u}}(1)$ and $\widehat{\mathfrak{su}}(2)$ current (Kac-Moody) algebras. Some basic, physical requirements on the spectrum of (finite-energy) excitations in incompressible quantum fluids, together with some elements of the representation theory of current algebras, enter this consistency analysis. Our analysis naturally leads to a complete list of possible quantum numbers [such as (fractional) charges and (fractional) statistical phases] of physical excitations that one expects to find in such quantum fluids.

For the sake of concreteness, we restrict our attention to two-dimensional, incompressible quantum fluids composed of electrons (or holes). For a different example of a physical system in which we can apply similar ideas, see the discussion of superfluid ^{3}He-A/B interfaces with broken parity and time-reversal invariance given in Sec. VII.B.

The plan of this section is as follows. First, we review the physics at the boundary of incompressible quantum Hall fluids by following some basic ideas of Halperin (1982). Extending these ideas by making use of some facts concerning chiral $\hat{\mathfrak{u}}(1)$ current algebra and introducing the idea of anomaly cancellation, we present a very natural explanation of the integer quantum Hall effect. The purpose of this first part is to illuminate the physical basis and provide a concrete illustration of the basic ideas underlying the more technical material in Secs. VI.B and VI.C where the general implications of $\hat{\mathfrak{u}}(1)$ and $\widehat{\mathfrak{su}}(2)$ current algebras describing the boundary excitations of a Hall sample are investigated. In these subsections we sketch a classification of (electronic) quantum Hall fluids in terms of universality classes. We discuss many examples, including the recently discovered two-layer systems with Hall plateaux at $\sigma=\frac{1}{2}$. More examples can be found in Sec. VII.A, where, based on the results in Sec. VI.C, we present a detailed analysis of Hall fluids with $\sigma=2/(4l+1)$, $l=0,1,2$, which turn out to be good candidates for an observation of the Hall effect for spin currents.

Independent work on current algebras in incompressible quantum Hall fluids that resembles ours, as presented in this section and in Fröhlich and Kerler (1991), Fröhlich and Zee (1991), and Fröhlich and Studer (1992b, 1992c), has been carried out by Wen and collaborators (Wen, 1989, 1990a–1990c, 1991a, 1991b; Block and Wen, 1990a, 1990b; Wen and Niu, 1990). Additional work vaguely or closely related to Wen's and ours can be found in Büttiker (1988), Beenakker (1990), Haldane (1990b), MacDonald (1990), Balatsky and Fradkin (1991), Balatsky and Stone (1991), and Stone (1991a, 1991b).

A. Integer quantum Hall effect and edge currents

Let us consider an electronic system confined to a two-dimensional domain Ω in the (x,y) plane. We choose Ω to be an annulus and denote by $\partial\Omega$ the boundary of Ω. In our example, $\partial\Omega$ consists of two connected components C_1 and C_2, which are circles of radii R_1 and R_2, respectively. We imagine that there is a (uniform) external magnetic field $\mathbf{B}_c=(0,0,B_c)$ (with vector potential $\mathbf{A}_c$, i.e., $B_c=\operatorname{curl}\mathbf{A}_c$), perpendicular to the plane of the sample. Note that the magnetic field $\mathbf{B}_c$ breaks time-reversal and parity (reflections-in-lines) invariance.

In Sec. IV.E, we mentioned that if the Hall conductivity of the system is on a plateau the longitudinal resistance R_L vanishes and the system is dissipationless, or incompressible. In the preceding section we showed that the bulk physics of such a system then exhibits universal features. In this subsection we intend to study the physics at the boundary of the sample in a similar manner. (If R_L is nonvanishing, the physics of the electronic system is complicated, and simple concepts of universality fail to capture the basic properties of the system.)

Classically, in the absence of an external electric field, there are no currents in the system. Quantum mechanically, however, the picture is different, as has been emphasized by Halperin (1982). In the absence of an external electric field, currents supported by the system are localized within approximately one cyclotron radius of the boundary $\partial\Omega$, and they are expected to persist even in the presence of a moderate amount of random disorder in the sample. Because of the presence of the external magnetic field $\mathbf{B}_c$ the edge currents are chiral, i.e., the electrons drifting in the field $\mathbf{B}_c$ can move in only one direction along the boundary components C_1 and C_2 of $\partial\Omega$. We can choose the orientation of the annulus Ω, and therefore of C_1 and C_2, such that the chirality of the edge current localized near C_i is given by the orientation of C_i, $i=1,2$.

In order to be more explicit, we temporarily assume that there is no disorder in the system, and the electrons are moving in a confining one-body potential V, which is constant in the bulk and rises steeply at the boundaries of the sample, i.e., at $|\mathbf{x}|=R_1,R_2$. Furthermore, we assume that electron-electron interactions are turned off (independent electron approximation), so that the many-electron states of the system can be constructed by filling up one-electron states $\psi_{\uparrow/\downarrow}$, in accordance with the Pauli principle. The one-particle wave function $\psi_{\uparrow/\downarrow}$ describes a two-dimensional, charged scalar fermion [i.e., a fermion with a fixed spin polarization "up" ($\uparrow$) or "down" ($\downarrow$)]. It satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi_{\uparrow/\downarrow}(t,\mathbf{x}) = \left[-\frac{\hbar^2}{2m^*}\left[\nabla+i\frac{e\hbar}{c}\mathbf{A}_c(\mathbf{x})\right]^2+V_{\uparrow/\downarrow}(\mathbf{x})\right]\psi_{\uparrow/\downarrow}(t,\mathbf{x}) , \tag{6.1}$$

where

$$V_{\uparrow/\downarrow}(\mathbf{x})=V(\mathbf{x})\pm\frac{g\mu_B}{2}B_c \ .$$

The second term in $V_{\uparrow/\downarrow}$ is the Zeeman energy ($\mu_B>0$), whose sign depends on whether the spin of the electron is parallel ($\uparrow$) or antiparallel ($\downarrow$) to the magnetic field $\mathbf{B}_c$. We assume that the effective electron mass m^* is less then the vacuum mass m_0, so that all Landau bands in the common spectrum of the two Hamiltonians in Eq. (6.1) have different energies. Hence each Landau band of one-electron states is fully spin polarized.

Because of the rotational symmetry of the annular domain Ω, the eigenvalues $m\in\mathbb{Z}$ of the z component L_z of the orbital angular-momentum operator are good quantum numbers for the one-electron states. For a given Landau band, a one-electron state with magnetic quantum number m is localized, in the radial direction, within about one cyclotron radius $r_c=(\hbar c/eB_c)^{1/2}$, from some mean radius r_m, with $r_m=r_c(m/\pi)^{1/2}$, provided that $R_1<r_m<R_2$ and $|R_i-r_m|\gg r_c$ for $i=1,2$. In the presence of a confining one-body potential $V(|\mathbf{x}|)$, the energies $\mathcal{E}$ (in units of $\hbar\omega_c$, where $\omega_c=eB_c/m^*c$ is the cyclotron frequency) of one-electron states as a function of their angular momentum m (i.e., of the square of their mean radius r_m) qualitatively look as indicated in Fig. 1 (Halperin, 1982).

In Fig. 1 the quantities m_i are determined by setting $r_{m_i}\simeq R_i$, $i=1,2$. The magnetic quantum numbers $m_{i,\nu}^F$, $i=1,2$ are determined by requiring that all one-particle levels up to the Fermi energy E^F be filled in the Landau band indexed by $\nu=(n,s)$, where $n\in\mathbb{N}_0$, and $s=\uparrow$ or $\downarrow$. [Note that the assumption of the incompressibility for the system is reflected by the requirement that, in the bulk region (i.e., for $m_1\ll m\ll m_2$), the Fermi energy E^F lies between two Landau bands.] States corresponding to values of m well between m_1 and m_2 carry no net electric current. However, states with m below, but close to, m_1 or above m_2 carry *gapless,* chiral currents localized within approximately one cyclotron radius of the boundaries C_1 and C_2, respectively (Halperin, 1982).

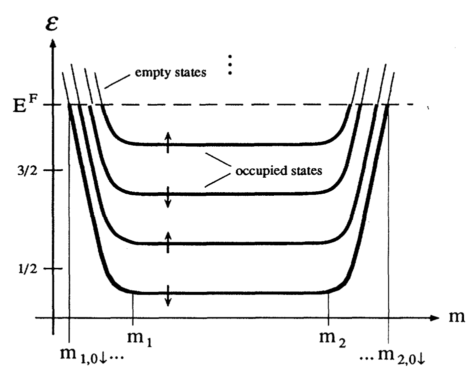

FIG. 1. One-electron energy levels $\mathcal{E}$ (in units of $\hbar\omega_c$) in an annular sample as a function of angular momentum m (neglecting disorder and electron-electron interactions, but for a g factor <2).

To summarize, we find that the gapless electronic excitations near the Fermi surface of a given filled Landau band are charged, chiral, scalar fermions propagating along the boundary of the sample. In order to describe the dynamics of these chiral fermions (chiral Luttinger model) in more detail, we introduce the one-dimensional momenta

$$p_{i,\nu}=\frac{m-m_{i,\nu}^F}{R_i}\hbar,\quad i=1,2$$

and (6.2)

$$\nu=(n,s)\in\mathbb{N}_0\times\{\uparrow,\downarrow\}\ ,$$

and we redefine the energy of the one-electron states relative to the Fermi surface, i.e., we write $E=\mathcal{E}-E^F$. Let us set $R_i=\theta r_i$, with r_i fixed, $i=1,2$, $0<\theta<\infty$. We scale space and time coordinates as in Eq. (5.14), i.e., $(t,\mathbf{x})=\theta(\tau,\xi)$, where (τ,ξ) belongs to a fixed space-time domain $\Lambda_0=\mathbb{R}\times\Omega_0$. Then, as θ grows, we are interested in that part of the spectrum of the edge excitations, $E=E(p_{i,\nu})$, which belongs to an ever smaller interval around $p_{i,\nu}=0$; see Fig. 1. Note that, by Eq. (6.2), $p_{i,\nu}$ scales with θ^{-1}. Thus, for the rescaled systems in Ω_0, the spectra of the edge excitations associated with a given Landau band converge towards the linear energy-momentum dispersion law of a massless, chiral "relativistic" Fermi field propagating along a circle of radius r_i, $i=1,2$; see Fig. 2. More details on this point and about the (chiral) Luttinger model can be found in Heidenreich, Seiler, and Uhlenbrock (1980, and the references therein).

Before we turn to a description of relativistic Fermi fields, we note that, in order to observe the quantum Hall effect experimentally, it is necessary to perturb the electronic system. This can be achieved, for example, by applying a low voltage between the inner and outer edges of the annular sample, thereby changing the chemical potentials of the electrons at the two edges. More generally, we shall couple the electronic system to an additional, external electromagnetic vector potential A, where $A=A_{\rm tot}-A_c$ is a small perturbation and A_c is the vec-

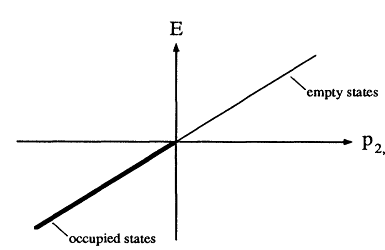

FIG. 2. Energy-momentum dispersion law in the scaling limit ($\theta\to\infty$) of chiral edge excitations circulating at r_2 and associated with some filled Landau band indexed by ν.

tor potential corresponding to $\mathbf{B}_c$. Our next step is thus to recall the description of a massless, chiral relativistic Fermi field circulating along one component C_0 of the boundary of the (rescaled) system and coupled to $A|_{\Gamma_0}$ (the restriction of A to $\Gamma_0=\mathbb{R}\times C_0\subset\partial\Lambda_0$).

We introduce some notation: It is convenient to use "light-cone" coordinates on the (1+1)-dimensional boundary space-time Γ_0. We set

$$u_\pm=\frac{1}{\sqrt{2}}(\zeta^0\pm\zeta^1)\equiv\frac{1}{\sqrt{2}}\left[v\tau\pm\frac{L}{2\pi}\eta\right], \tag{6.3}$$

where $\eta\in[0,2\pi)$ is an angular variable along C_0 (whose length is given by L), τ is (rescaled) time, and the constant v physically corresponds to the propagation speed of charge-density waves at the edge of the sample. (The value of the velocity v does not matter in the following. The calculation of this physically interesting quantity is the subject of a more complete microscopic analysis.) We write

$$A|_{\Gamma_0}=A_+(u)du_++A_-(u)du_-\ , \tag{6.4}$$

where

$$A_\pm(u)=\frac{1}{\sqrt{2}}(A_0|_{\Gamma_0}(\zeta)\pm A_1|_{\Gamma_0}(\zeta))|_{\zeta=\zeta(u)}\ ,$$

with $u=(u_+,u_-)$ and $\zeta=(\zeta^0,\zeta^1)$. The (1+1)-dimensional d'Alembertian, $\Box=\partial_0^2-\partial_1^2\equiv(1/v^2)(\partial^2/\partial\tau^2)-(2\pi/L)^2(\partial^2/\partial\eta^2)$, is given in "light-cone" coordinates by

$$\Box=2\,\partial_+\partial_-\ , \tag{6.5}$$

where $\partial_\pm=\partial/\partial v_\pm$.

In 1+1 dimensions, a relativistic Fermi field (fermion, for short) is described by a two-component Dirac (i.e., complex) spinor ψ. We choose the chiral representation of Dirac matrices

$$\gamma^0\equiv\sigma_1,\quad \gamma^1\equiv-i\sigma_2,\quad\text{and}\quad \gamma_5=\gamma^0\gamma^1\equiv\sigma_3\ , \tag{6.6}$$

where $\sigma_1,\sigma_2,\sigma_3$ are the standard Pauli matrices. Moreover, we define by $\bar\psi=\psi^*\gamma^0$ the conjugate of the Dirac spinor ψ and identify its left-/right-handed component with

$$\psi_{L/R}(\zeta)=\tfrac{1}{2}(1\mp\gamma_5)\psi(\zeta)\ . \tag{6.7}$$

Our aim is to write down an action for a massless, chiral fermion $\psi_{L/R}$ coupled to the vector potential $A|_{\Gamma_0}$, and to calculate an effective gauge-field action, $\Gamma_{L/R}(A|_{\Gamma_0})$, by integrating out the chiral fermion degrees of freedom located at the edge of the sample. Naively, one might try to start with an action of the form

$$\text{“}S_{L/R}(\bar\psi_{L/R},\psi_{L/R};A|_{\Gamma_0})=i\int_{\Gamma_0}\bar\psi_{L/R}(\zeta)D(A|_{\Gamma_0})\psi_{L/R}(\zeta)d^2\zeta,\text{”} \tag{6.8}$$

where the Dirac operator $D(A|_{\Gamma_0})$ is defined by

$$D(A|_{\Gamma_0})=\gamma^\mu\left[\partial_\mu+i\frac{e}{\hbar c}A_\mu|_{\Gamma_0}(\zeta)\right]\equiv\partial\!\!\!/+i\frac{e}{\hbar c}A\!\!\!/|_{\Gamma_0}(\zeta)\ . \tag{6.9}$$

However, it is *not* possible to compute the effective action of a massless, chiral fermion coupled to $A|_{\Gamma_0}$ by a fermionic (Berezin) path integral based on the action (6.8). Put differently, it is not possible to calculate the determinant of the Dirac operator $D(A|_{\Gamma_0})$ restricted to the subspace of either only left- or only right-handed field modes. This is because of the simple fact that the Dirac operator $D(A|_{\Gamma_0})$ maps left-handed to right-handed modes and vice versa, i.e., the chirality subspaces are not invariant under the action of $D(A|_{\Gamma_0})$; see, for example, Alvarez-Gaumé and Ginsparg (1984). Using Eqs. (6.4) and (6.7) we can rewrite the standard action of a massless, two-component Dirac field ψ on Γ_0 in terms of its components ψ_L and ψ_R. We find that

$$i\int_{\Gamma_0}\bar\psi(\zeta)D(A|_{\Gamma_0})\psi(\zeta)d^2\zeta=i\sqrt{2}\int_{\Gamma_0}\left[\psi_L^*\left[\partial_-+i\frac{e}{\hbar c}A_-\right]\psi_L+\psi_R^*\left[\partial_++i\frac{e}{\hbar c}A_+\right]\psi_R\right](\zeta)d^2\zeta\ . \tag{6.10}$$

(The equation of motion for $\psi_{L/R}$ following from Eq. (6.10) reads $[\partial_\mp+i(e/\hbar c)A_\mp]\psi_{L/R}=0$. Hence, in 1+1 dimensions the left-/right-handed modes are actually left-/right-moving excitations, provided $A_\mp=0$.) By inspecting the coupling structure of $A_\mp$ to $\psi_{L/R}$ in Eq. (6.10), we are led to the following expression for the effective gauge-field action of a massless, chiral (left-/right-moving), relativistic Fermi field coupled to the external vector potential $A|_{\Gamma_0}$:

$$\begin{aligned}\frac{i}{\hbar}\Gamma_{L/R}(A|_{\Gamma_0})&=[\ln\det D(A|_{\Gamma_0})]_{A_\pm=0}+\frac{a}{4\pi}\left[\frac{e}{\hbar c}\right]^2\int_{\Gamma_0}A_+(u)A_-(u)d^2u\\&\equiv\ln\det\left[\partial\!\!\!/+i\frac{e}{\hbar c}A\!\!\!/|_{\Gamma_0}\tfrac{1}{2}(1\mp\gamma_5)\right]\ ;\end{aligned} \tag{6.11}$$

see Jackiw (1985) and Jackiw and Rajaraman (1985). In Eq. (6.11), the determinant of the Dirac operator $D(A|_{\Gamma_0})$ is calculated on the full mode space of a (1+1)-dimensional, two-component Dirac field, and its evaluation goes back to

Schwinger (1962). In the second term a is an *arbitrary* real constant. This term stands for a finite renormalization ambiguity and mirrors the fact that one cannot invoke U(1) gauge invariance as a guiding principle in the calculation of chiral effective actions (Jackiw and Rajaraman, 1985; Leutwyler, 1986 and references therein). Chiral effective actions are anomalous, a fact that we shall exploit shortly. We set $a=1$, which is a particularly convenient choice for the subsequent discussion (see also Jackiw, 1985), and find that

$$\frac{1}{\hbar}\Gamma_{L/R}(A|_{\Gamma_0})=\frac{1}{4\pi}\left[\frac{e}{\hbar c}\right]^2\int_{\Gamma_0}\left[A_+(u)A_-(u)-2A_\mp(u)\frac{\partial_\pm^2}{\Box}A_\mp(u)\right]d^2u \ . \tag{6.12}$$

Let us define the left-/right-handed current (operator) $j^\mu_{L/R}$ by

$$j^\mu_{L/R}(\xi)=:\bar{\psi}(\xi)\gamma^\mu\tfrac{1}{2}(1\mp\gamma_5)\psi(\xi): \ , \tag{6.13}$$

where :: indicates normal ordering. Then, using Eqs. (6.10) and (6.11), we observe that the effective action $\Gamma_{L/R}(A|_{\Gamma_0})|_{A_\pm=0}$ is the generating functional for time-ordered, connected Green functions of $j^\mu_{L/R}$. Both currents j^μ_L and j^μ_R independently generate a chiral $\hat{\mathrm{u}}(1)$ current algebra. This will be discussed in more detail in the next section. We note that the choice of the left- or right-handed current, j^μ_L or j^μ_R, depends on the physics of a given system, namely, on the sign of the external magnetic field and on whether the physical edge currents are carried by electrons or holes.

Next, we exploit the fact that effective actions for chiral fermions are breaking U(1) gauge invariance, i.e., that they are *anomalous.* When we perform a U(1) gauge transformation, $A\mapsto A+d\chi$, the explicit expression for the chiral effective action $\Gamma_{L/R}(A|_{\Gamma_0})$ given by Eq. (6.12) implies that

$$\frac{1}{\hbar}\Gamma_{L/R}([A+d\chi]|_{\Gamma_0})=\frac{1}{\hbar}\Gamma_{L/R}(A|_{\Gamma_0})\pm\frac{1}{4\pi}\left[\frac{e}{\hbar c}\right]^2\int_{\Gamma_0}[A_+(u)\partial_-\chi(u)-A_-(u)\partial_+\chi(u)]d^2u \ . \tag{6.14}$$

Thus the chiral anomaly of the effective action produced by the quantum-mechanical degrees of freedom located at the edge of the sample takes the form

$$\pm\frac{1}{4\pi}\left[\frac{e}{\hbar c}\right]^2\int_{\Gamma_0}[A_+(u)\partial_-\chi(u)-A_-(u)\partial_+\chi(u)]d^2u=\pm\frac{1}{4\pi}\left[\frac{e}{\hbar c}\right]^2\int_{\Gamma_0}d\chi\wedge A \ . \tag{6.15}$$

Remember that a Landau band gives rise to an algebra of chiral edge currents at *each* connected component C_0 of the boundary $\partial\Omega_0$ of the sample; see Fig. 1. Hence the total chiral effective boundary action resulting from a given Landau band is obtained by simply adding up the contributions of the form (6.12) for the different connected components of $\partial\Omega_0$. We recall our assumption that, in the bulk, the Fermi energy E^F lies well between two Landau bands (incompressibility) and that there are $N=0,1,2,\ldots$ filled Landau bands below the Fermi energy, each of fixed spin polarization (i.e., all the electrons can be treated as scalar fermions). Then, in the scaling limit, all the quantum-mechanical degrees of freedom localized near the boundary $\partial\Omega_0$ of the sample together give rise to the boundary contribution $\zeta^{L/R}_{\partial\Lambda_0}(A|_{\partial\Lambda_0})$ to the total partition function $Z_{\Lambda_0}(A)$ of the electronic system:

$$\zeta^{L/R}_{\partial\Lambda_0}(A|_{\partial\Lambda_0})=\prod_{j=0}^{N}\prod_{\Gamma_0\subset\partial\Lambda_0}\exp\left[\frac{i}{\hbar}\Gamma_{L/R}(A|_{\Gamma_0})\right] . \tag{6.16}$$

From Eq. (6.15) it follows that the total chiral anomaly of the boundary contribution (6.16) is given by

$$\pm\frac{N}{4\pi}\left[\frac{e}{\hbar c}\right]^2\int_{\partial\Lambda_0}d\chi\wedge A \ . \tag{6.17}$$

We are now in a position to describe the basic idea of *anomaly cancellation:* In Secs. II and III we have seen that nonrelativistic quantum mechanics is U(1) gauge invariant, which means that the total partition function $Z_{\Lambda_0}(A)$ of the electronic system *is* U(1) gauge invariant! In other words, the total chiral anomaly (6.17) due to the degrees of freedom localized at the boundary has to be cancelled by an anomalous term in the total effective action associated with the degrees of freedom in the bulk of the system. The term with the required anomaly under U(1) gauge transformations is the (by now familiar Abelian) Chern-Simons term,

$$\mp\frac{N}{4\pi}\left[\frac{e}{\hbar c}\right]^2\int_{\Lambda_0}A\wedge dA \ . \tag{6.18}$$

Thus, to leading order in the scale parameter θ, the total partition function $Z_{\Lambda_0}(A)$ of a (disorder-free, noninteracting) quantum Hall fluid with $N=0,1,2,\ldots$ filled Landau bands coupled to a small perturbing vector potential A takes the form

$$Z_{\Lambda_0}(A)=\zeta^{L/R}_{\partial\Lambda_0}(A|_{\partial\Lambda_0})\times\exp\left[\mp i\frac{N}{4\pi}\left[\frac{e}{\hbar c}\right]^2\int_{\Lambda_0}A\wedge dA+\text{G.I.}\right] , \tag{6.19}$$

where "G.I." stands for possible U(1) gauge-invariant bulk terms. As we have seen in Eqs. (4.55) and (4.59), it is the Chern-Simons term (6.18) in the total (bulk plus boundary) effective action which reproduces the basic response equations (4.41) and (4.45) of the Hall effect. In particular, comparing Eqs. (4.55) and (6.19), we identify the coefficient of the Chern-Simons term (6.18) with the Hall conductivity σ_H of the system, i.e.,

$$\sigma_H = \pm N \frac{e^2}{h}, \quad N = 0, 1, 2, \ldots \tag{6.20}$$

These considerations yield a natural description of the physics underlying the integer quantum Hall effect, provided the system is free of disorder and consists of noninteracting electrons.

Let us briefly comment on these two assumptions: One can argue that the picture of chiral edge excitations given above, and hence the form of the anomaly, still hold when the system is perturbed by a small amount of disorder. A chiral Luttinger liquid perturbed by a weak random potential will not exhibit Anderson localization, because there is no interference between left- and right-moving waves. This is in contrast to what happens in the bulk, where a moderate amount of disorder leads to plenty of localized states (we then require that the Fermi energy E^F lie in a region of localized states, in order to conclude incompressibility of the fluid). In fact, we recall that Anderson localization in the bulk is crucial in order for the (integer) quantum Hall effect to be observable experimentally (Halperin, 1983; Morandi, 1988; Prange, 1990; and the references therein). More precisely, the width of a Hall plateau depends on the amount of disorder in the system—which determines the density of localized states— and, in the thermodynamic limit, would tend to 0 as the strength of the disorder tended to 0.

Finally, taking electron-electron interactions into account, the *form* of the anomaly will not change, because it is universal. However, the value of the Hall conductivity σ_H can and will change.

In the following sections, we discuss two-dimensional, incompressible electronic systems that do not necessarily have integral filling factors ν due to electron-electron interactions; see Eq. (4.40). In our discussion we also include the spin degrees of freedom of the electrons.

In contrast to the logic of this subsection, we start from the universal form $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$ given in Eq. (5.26), which the effective action for the bulk degrees of freedom takes in the scaling limit. We recall that Eq. (5.26) of $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$ takes the spin degrees of freedom into account and that it does not exclude any effects of electron-electron interactions that respect U(1)×SU(2) gauge invariance and are compatible with incompressibility of the electron fluid! We identify those terms in $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$ which exhibit anomalous behavior under U(1) and SU(2) gauge transformations *not* vanishing at the *boundary* $\partial\Lambda_0$. The idea of anomaly cancellation then leads to the study of the dynamics of degrees of freedom at the boundary of the system compensating the gauge noninvariance of some of the bulk terms in $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$. As above, we find bands of charge *and* spin carrying chiral edge currents forming $\hat{\mathfrak{u}}(1)$ and $\widehat{\mathfrak{su}}(2)$ current (Kac-Moody) algebras, respectively. We show how the representation theory of these current algebras nicely captures the universal features of systems exhibiting a fractional quantum Hall effect for the electric and for the spin current. In Sec. VI.B, we discuss the properties of "chiral boundary systems" [$\hat{\mathfrak{u}}(1)$ current algebra] formed in fully spin-polarized quantum Hall fluids, and in Sec. VI.C we study the additional features of systems when spin [$\widehat{\mathfrak{su}}(2)$ current algebra] and possibly internal symmetries [$\hat{g}$ current algebra associated with some compact Lie group G] must be taken into account.

B. Edge excitations in polarized quantum Hall fluids

In this section we consider interacting, spin-polarized, two-dimensional, incompressible quantum Hall fluids. Here "spin-polarized" means that the spin degrees of freedom are "frozen." Moreover, as a first step, we neglect the magnetic moments of the electrons. (For a treatment of spin-polarized Hall fluids including the effects of the magnetic moments of the electrons see the general discussion in Sec. VI.C.) For such systems, the universal form of the "scaling limit" $S^*_{\Lambda_0}(\tilde{a})$ of the effective action is thus obtained by discarding all terms depending on the SU(2) gauge fields w or $\tilde{w}$ in expression (5.26). We find that

$$-\frac{1}{\hbar} S^*_{\Lambda_0}(\tilde{a}) = \int_{\Lambda_0} j^\mu_c(\xi)\tilde{a}_\mu(\xi)\, d^3\xi + \frac{\sigma}{4\pi}\int_{\Lambda_0} \tilde{a}\wedge d\tilde{a} + \mathrm{B.T.}(\tilde{a}|_{\partial\Lambda_0}) \,, \tag{6.21}$$

where $\mathrm{B.T.}(\tilde{a}|_{\partial\Lambda_0})$ stands for boundary terms depending only on the restriction of the perturbing, external vector potential $\tilde{a}$ to the boundary of the system. [Note that we have returned to working in "mathematical units"; see, for example, Eq. (3.36). Moreover, we consider a Euclidean background metric $\gamma_{ij} = \delta_{ij}$; see Eq. (5.13).] So far, the coefficient σ of the Chern-Simons term on the rhs of Eq. (6.21) can be an arbitrary real constant.

The particular situation in which σ takes integral values was identified in the preceding section as basic in describing the integer quantum Hall effect. The main purpose of this section is to understand the more general, fractional quantization of the values of the constant σ that arise in systems where electron-electron interactions cannot be neglected. We recall that, by Eq. (5.49), σ determines the value of the Hall conductivity $\sigma_H = \sigma e^2/h$, which completely specifies the (linear) response properties of incompressible quantum Hall fluids, provided we neglect the spin degrees of freedom; see Eqs. (5.48) and (5.51). Hence, in this section, we develop a complete picture of the universal aspects of the fractional quantum Hall effect in spin-polarized, two-dimensional electronic systems.

We make use of the ideas elaborated in Sec. VI.A: As a first step, we wish to implement the idea of anomaly cancellation in the direction opposite to that explained in Sec. VI.A, i.e., we start from the bulk terms. As explained in Fröhlich and Studer (1992b, 1992d), $S^*_{\Lambda_0}(\tilde{a})$ must be invariant under U(1) gauge transformations

$$\tilde{a} \mapsto {}^{\chi}\tilde{a} = \tilde{a} + d\chi \ , \tag{6.22}$$

in spite of the fact that $\tilde{a}$ is the vector potential of a perturbation of the electromagnetic field (i.e., $\tilde{a} = a - a_c$ is the *difference* of two connection 1-forms)! We recall that, for a trivial U(1) bundle, relevant for our choice of space-time domains $\mathbb{R}\times\Omega_0$, the associated space of connections (vector potentials) is a real *vector space.* In particular, any gauge transformation of a sum of connections can be rewritten as the result of gauge-transforming each summand in the sum separately. Note that this is in contrast to the situation encountered for non-Abelian gauge fields. Performing a gauge transformation (6.22) on $\tilde{a}$, with χ not vanishing at $\partial\Lambda_0$, we find from Eq. (6.21) that

$$\begin{aligned}\frac{1}{\hbar}S^*_{\Lambda_0}(\tilde{a}+d\chi) = \frac{1}{\hbar}S^*_{\Lambda_0}(\tilde{a}) + \int_{\partial\Lambda_0}(*j_c)\chi \\ + \frac{\sigma}{4\pi}\int_{\partial\Lambda_0} d\chi\wedge\tilde{a} - \mathrm{B.T.}([\tilde{a}+d\chi]|_{\partial\Lambda_0}) \\ + \mathrm{B.T.}(\tilde{a}|_{\partial\Lambda_0}) \ ,\end{aligned} \tag{6.23}$$

where $*j_c = \frac{1}{2}\sum_{\mu,\nu}\varepsilon_{\mu\nu}{}^{\sigma} j_{c,\sigma}(\xi)d\xi^{\mu}\wedge d\xi^{\nu}$ is the (Hodge) dual of the persistent current $j_c = \sum_{\mu} j_{c,\mu}(\xi)d\xi^{\mu}$.

We note that $S^*_{\Lambda_0}(\tilde{a}+d\chi)$ would be equal to $S^*_{\Lambda_0}(\tilde{a})$, and hence we could set $\mathrm{B.T.}(\tilde{a}|_{\partial\Lambda_0}) \equiv 0$ (up to gauge-invariant terms at $\partial\Lambda_0$), if

$$*j_c|_{\partial\Lambda_0} = \frac{\sigma}{4\pi} d\tilde{a}|_{\partial\Lambda_0} \ , \tag{6.24}$$

since $\int_{\partial\Lambda_0} d\chi\wedge\tilde{a} = -\int_{\partial\Lambda_0}\chi d\tilde{a}$. However, j_c is a persistent current supported by the Hall fluid when $a = a_c$ (i.e., $\tilde{a}=0$), and $\tilde{a}$ is the potential of a small but arbitrary external perturbation. Therefore Eq.(6.24) cannot be imposed.

Experimentally, for a Hall fluid in a heterostructure or MOSFET, the boundary $\partial\Lambda_0$ of the sample is such that there is no leakage of electric charge through $\partial\Lambda_0$, which means that the normal component of j_c^{μ} at $\partial\Lambda_0$ has to vanish, or equivalently,

$$*j_c|_{\partial\Lambda_0} = 0 \ . \tag{6.25}$$

In this case, the second term on the rhs of Eq. (6.23) vanishes, and, requiring the total effective action $S^*_{\Lambda_0}(\tilde{a})$ to be U(1) gauge invariant, one obtains the following functional equation for the boundary terms:

$$\mathrm{B.T.}([\tilde{a}+d\chi]|_{\partial\Lambda_0}) - \mathrm{B.T.}(\tilde{a}|_{\partial\Lambda_0}) = \frac{\sigma}{4\pi}\int_{\partial\Lambda_0} d\chi\wedge\tilde{a} \ , \tag{6.26}$$

which has to hold for arbitrary $\tilde{a}$ and χ.

From the discussion in the previous subsection we can immediately infer the solution to Eq. (6.26). Introducing "light-cone" coordinates on each connected component Γ_0 of the (1+1)-dimensional boundary space-time $\partial\Lambda_0$ [see Eq. (6.3)], we have, as in Eq. (6.4),

$$\begin{aligned}\tilde{a}|_{\Gamma_0} &= \frac{e}{\hbar c}A_+(u)du_+ + \frac{e}{\hbar c}A_-(u)du_- \\ &\equiv \alpha_+(u)du_+ + \alpha_-(u)du_- \ .\end{aligned} \tag{6.27}$$

From Eqs. (6.14) and (6.15) it follows that the solution to Eq. (6.26) is given by

$$\begin{aligned}\mathrm{B.T.}(\tilde{a}|_{\partial\Lambda_0}) = \sigma_L \sum_{\Gamma_0\subset\partial\Lambda_0}\frac{1}{\hbar}\Gamma_L(A|_{\Gamma_0}) \\ + \sigma_R \sum_{\Gamma_0\subset\partial\Lambda_0}\frac{1}{\hbar}\Gamma_R(A|_{\Gamma_0}) + \mathrm{G.I.}(\tilde{a}|_{\partial\Lambda_0}) \ ,\end{aligned} \tag{6.28}$$

with

$$\sigma = \sigma_L - \sigma_R \ , \tag{6.29}$$

where $\Gamma_{L/R}(A|_{\Gamma_0})$ is as in Eq. (6.12), σ_L and σ_R are non-negative constants, and "G.I." stands for manifestly gauge-invariant terms supported at the boundary space-time $\partial\Lambda_0$. We recall from Eq. (6.14) that the contributions to the total anomaly of the two terms $\Gamma_L(A|_{\Gamma_0})$ and $\Gamma_R(A|_{\Gamma_0})$ are of opposite sign, which explains Eq. (6.29). We do not need to discuss the terms in G.I. any further and hence omit them in the following discussion.

In Sec. VI.A, we saw that, for $\sigma_{L/R} = N$ a (positive) integer, there is a straightforward interpretation of the boundary terms in Eq. (6.28) in terms of N bands of noninteracting, chiral (left-/right-moving) fermions propagating along the different connected components of the sample boundary $\partial\Omega_0$. In order to understand the more general, fractional quantization of the values of the constant σ, we have to generalize this physical picture to coupled bands of boundary excitations. For this purpose, we use bosonization techniques always available in two space-time dimensions. First, we derive an expression for $\Gamma_{L/R}(A|_{\Gamma_0})$ [see Eq. (6.11)] in terms of one chiral Bose field. Then this bosonic expression is generalized to one describing several coupled bands of excitations at the boundary.

In the following we assume, for simplicity, that the topology of the sample Ω_0 is that of a disk, i.e., the boundary $\partial\Omega_0$ consists of but *one* connected component C_0, the general situation being given by simply adding up the contributions from the different connected components of $\partial\Omega_0$.

Let us first suppose that the external gauge field $\alpha \equiv \tilde{a}|_{\partial\Lambda_0}$ is set to 0. In Eq. (6.13) we have introduced the currents

$$j^{\mu}_{L/R}(\zeta)=\tfrac{1}{2}[j^{\mu}(\zeta)\mp j^{\mu}_5(\zeta)] \tag{6.30}$$

of a massless Dirac spinor ψ. Recalling the equation of motion satisfied by the left-/right-handed component of ψ [see Eqs. (6.7), (6.6), and the remark after (6.10)], we see that j^{μ} and j^{μ}_5 are conserved currents, i.e.,

$$\partial_{\mu}j^{\mu}(\zeta)=0=\partial_{\mu}j^{\mu}_5(\zeta) \ . \tag{6.31}$$

The general solution to the equations in (6.31) is

$$j^{\mu}(\zeta)=\sqrt{2}\varepsilon^{\mu\nu}\partial_{\nu}\phi(\zeta)$$

and (6.32)

$$j^{\mu}_5(\zeta)=\sqrt{2}\varepsilon^{\mu\nu}\partial_{\nu}\phi_5(\zeta) \ ,$$

where ϕ and ϕ_5 are scalar fields and $\varepsilon^{01}=-\varepsilon^{10}=1$. However, in $1+1$ dimensions, it follows from Eqs. (6.13) and (6.6) that $j^0_5=j^1$ and $j^1_5=j^0$. Thus $j^{\mu}_5=-\sqrt{2}\partial^{\mu}\phi$, and Eq. (6.31) implies that

$$\partial_{\mu}\partial^{\mu}\phi(\zeta)\equiv\Box\phi(\zeta)=0 \ , \tag{6.33}$$

i.e., ϕ is a free, massless, relativistic Bose field. Any solution to Eq. (6.33) has the form [see Eqs. (6.3) and (6.5)]

$$\phi(\zeta(u))=\phi_L(u_+)+\phi_R(u_-) \ . \tag{6.34}$$

Moreover, since j^{μ} and j^{μ}_5 are currents propagating along the boundary $\partial\Omega_0$, ϕ has to satisfy the periodicity conditions

$$\partial_{\pm}\phi(\zeta^0,\zeta^1+L)=\partial_{\pm}\phi(\zeta^0,\zeta^1) \ . \tag{6.35}$$

In terms of the chiral components ϕ_L and ϕ_R of the Bose field ϕ we can define the chiral currents J_L and J_R,

$$\frac{2\pi}{l}J_L(u_+)\equiv\partial_+\phi_L(u_+)=j^0_L(\zeta(u)) \ ,$$

and (6.36)

$$\frac{2\pi}{l}J_R(u_-)\equiv-\partial_-\phi_R(u_-)=j^0_R(\zeta(u)) \ ,$$

where $l=L/\sqrt{2}$, with L the length of the boundary $\partial\Omega_0$ [see Eq. (6.3)]. Clearly $\partial_{\mp}J_{L/R}=0$. We emphasize that Eqs. (6.30)–(6.36) hold at the level of quantized fields. They are at the origin of Abelian bosonization in two space-time dimensions.

The currents J_L and J_R both generate a chiral $\hat{\mathrm{u}}(1)$ current algebra as follows: We can decompose the current J_L (and similarly J_R) into its Fourier modes:

$$J_L(u_+)=\frac{l}{2\pi}\partial_+\phi_L(u_+) =\frac{1}{\kappa}p+\frac{1}{\sqrt{\kappa}}\sum_{n\neq0}\alpha_n e^{-2\pi i n u_+/l} \ , \tag{6.37}$$

where κ is a positive normalization constant whose meaning will become apparent shortly. Integrating Eq. (6.37) we find that

$$\phi_L(u_+)=q+\frac{2\pi}{l\kappa}pu_+ +\frac{i}{\sqrt{\kappa}}\sum_{n\neq0}\frac{1}{n}\alpha_n e^{-2\pi i n u_+/l} \ , \tag{6.38}$$

with q some real integration constant. Since ϕ_L is a real quantum field, the operator α_{-n} is the adjoint of α_n, i.e., $\alpha_{-n}=(\alpha_n)^{\dagger}$, $n\in\mathbb{Z}\setminus\{0\}$. Moreover, the Fourier coefficients in Eq. (6.38) are subject to the commutation relations

$$[q,p]=i \quad\text{and}\quad [\alpha_m,\alpha_n]=m\delta_{m,-n}, \quad m,n\in\mathbb{Z}\setminus\{0\} \ . \tag{6.39}$$

Equation (6.38) and the relations (6.39) define the celebrated $\hat{\mathrm{u}}(1)$ current (Kac-Moody) algebra (at level κ); see, for example, Goddard and Olive (1986), Buchholz, Mack, and Todorov (1990), Ginsparg (1990), and also Frenkel (1981) and Kac (1983). The vacuum (charge-0) sector of this quantum field theory is the Fock space built from the vacuum state, $|0\rangle$, which is defined by

$$\alpha_n|0\rangle=0, \quad n>0$$

and (6.40)

$$p|0\rangle=0$$

by applying polynomials in the creation operators α_{-n}, $n=1,2,\ldots$, to $|0\rangle$.

These constructions are well known in string and conformal field theory. We describe more details below. First, however, we derive a general expression, in terms of Bose fields, for the boundary term B.T., given in Eq. (6.28).

In the path-integral formulation, a free, massless Bose field ϕ, propagating along the boundary $\partial\Omega_0$ of the sample, is described by a Gaussian action

$$\frac{1}{\hbar}S(\phi)=\frac{\kappa}{4\pi}\int_{\partial\Lambda_0}\partial_-\phi(u)\partial_+\phi(u)d^2u \ , \tag{6.41}$$

where $d^2u=du_-\wedge du_+$ is the (oriented) space-time measure on $\partial\Lambda_0$ and κ is the same normalization constant that appeared in Eqs. (6.37) and (6.38) (on the question of normalizations, see Appendix B). If we consider but one chiral component of the field ϕ we have to supplement the action (6.41) by a chirality constraint

$$\partial_-\phi(u)=0 \quad\text{or}\quad \partial_+\phi(u)=0 \ . \tag{6.42}$$

Next, we wish to couple ϕ to an external gauge potential $\alpha\equiv\tilde{a}|_{\partial\Lambda_0}$. We present a formal argument allowing us to identify the correct way of coupling ϕ to α and providing a path-integral derivation of the fermion-boson equivalence in $1+1$ dimensions. The state sum (a divergent constant) for a free, chiral (left-moving), massless Bose field reads

$$Z_L=\int\mathcal{D}\phi\exp\left[\frac{i}{\hbar}S(\phi)\right]\delta(\partial_-\phi) \ , \tag{6.43}$$

where $S(\phi)$ is given by Eq. (6.41). Since the field ϕ is an angle variable, we may shift it according to $\phi \mapsto {}^{\chi}\phi = \phi + (1/\kappa)\chi|_{\partial\Lambda_0}$. Using the fact that the integration measure is gauge invariant, i.e., $\mathcal{D}^{\chi}\phi = \mathcal{D}\phi$, we find, after partial integration, that

$$Z_L = \exp\left[\frac{i}{\hbar\kappa^2}S(\chi)\right]\int \mathcal{D}\phi \exp\left[\frac{i}{\hbar}S(\phi) + \frac{i}{2\pi\hbar}\int_{\partial\Lambda_0}\partial_+\phi(u)\partial_-\chi(u)d^2u\right]\delta\left[\partial_-\phi + \frac{1}{\kappa}\partial_-\chi|_{\partial\Lambda_0}\right] . \tag{6.44}$$

Let us choose χ such that $\partial_-\chi|_{\partial\Lambda_0} = -Q\alpha_-$, with Q a real constant. Then Eq. (6.44) and the expression for the action functional $\Gamma_L(\alpha)$ given in Eq. (6.12) imply that the following identity holds:

$$\exp\left[\frac{iQ^2}{\hbar\kappa}\Gamma_L(\alpha)\right] = \frac{1}{Z_L}\int \mathcal{D}\phi \exp\left[\frac{i}{\hbar}S_{\mathrm{WZNW}}(\phi;\alpha)\right]\delta\left[\partial_-\phi - \frac{Q}{\kappa}\alpha_-\right] , \tag{6.45}$$

where

$$\frac{1}{\hbar}S_{\mathrm{WZNW}}(\phi;\alpha) = \frac{\kappa}{4\pi}\int_{\partial\Lambda_0}\partial_-\phi(u)\partial_+\phi(u)d^2u$$
$$-\frac{Q}{2\pi}\int_{\partial\Lambda_0}\left[\partial_+\phi(u)\alpha_-(u) - \left[\partial_-\phi - \frac{Q}{\kappa}\alpha_-\right](u)\alpha_+(u)\right]d^2u + \frac{Q^2}{4\pi\kappa}\int_{\partial\Lambda_0}\alpha_-(u)\alpha_+(u)d^2u , \tag{6.46}$$

with α_+ specified in Eq. (6.27). We note that the α_+ term in the second line of Eq. (6.46) vanishes because of the constraint in (6.45). It has been added for symmetry reasons which will be discussed below. Moreover, the third term on the rhs of Eq. (6.46) is independent of ϕ. It has been added in order for the lhs of (6.45) to coincide with the expression given in (6.12).

The field theory with action (6.46) is known as the gauged, Abelian Wess-Zumino-Novikov-Witten (WZNW) model; see, for example, Gawedzki (1990) and for related considerations also Floreanini and Jackiw (1987), Sonnenschein (1988), and Harada (1990a). We emphasize that the chirality constraint in Eq. (6.45),

$$\partial_-\phi(u) - \frac{Q}{\kappa}\alpha_-(u) = 0 , \tag{6.47}$$

is invariant under gauge transformations

$$\phi(u) \mapsto {}^{\chi}\phi(u) = \phi(u) + \frac{Q}{\kappa}\chi|_{\partial\Lambda_0}(u) ,$$
$$\alpha \mapsto {}^{\chi}\alpha = \alpha + d\chi|_{\partial\Lambda_0} . \tag{6.48}$$

By Eqs. (6.14) and (6.45), the theory specified by the rhs of (6.45), with an action as given in Eq. (6.46), gives rise, under a gauge transformation (6.48), to the anomaly

$$\frac{\sigma}{4\pi}\int_{\partial\Lambda_0} d\chi \wedge \tilde{a}, \quad \text{with } \sigma = \frac{Q^2}{\kappa} . \tag{6.49}$$

It is clear from Eq. (6.46) that, physically, Q specifies the charge of the left-moving component of the Bose field ϕ in units of the electronic charge $-e$; see Eqs. (6.27) and (6.36). If $Q^2/\kappa = 1$, then Eq. (6.45) can be used to prove the equivalence of the theory of one chiral fermion, given in (6.11), to the theory of one chiral boson, given in (6.46).

Notice that replacing L by R on the lhs of Eq. (6.45) corresponds to interchanging $+$ and $-$ and replacing Q by $-Q$, and κ by $-\kappa$ on the rhs of (6.45). [Interchanging $+$ and $-$, we find that the measure d^2u goes over into $-d^2u$. In order for this symmetry property to become evident, we have included, on the rhs of Eq. (6.46), an α_+ term that vanishes upon imposing the constraint (6.47).] Clearly, the resulting anomaly then has the opposite sign to that given in Eq. (6.49). Physically, this symmetry property of (6.45) corresponds to replacing electrons (L) by holes (R) as the elementary charge carriers for the edge currents along the boundary of the sample. It mirrors the fact that, for a given external magnetic field $\mathbf{B}_c$, electrons and holes will circulate in opposite directions. From this it follows that we can continue our discussion by considering only the left-moving excitations along the boundary $\partial\Omega_0$, the corresponding equations for the right-moving ones following by applying this symmetry.

Before turning to a generalization of the action (6.46) we wish to mention yet another way of deriving expression (6.46): We start from the Chern-Simons term $\exp[-(i\sigma/4\pi)\int_{\Lambda_0}\tilde{a}\wedge d\tilde{a}]$ and perform a gauge transformation, $\tilde{a} \mapsto {}^{\varphi}\tilde{a} = \tilde{a} + d\varphi$. Then, integrating the gauge-transformed Chern-Simons term $\exp[-(i\sigma/4\pi)\int_{\Lambda_0}{}^{\varphi}\tilde{a}\wedge d\,{}^{\varphi}\tilde{a}]$ over all gauge transformations φ satisfying the boundary condition $\partial_-\varphi - \alpha_- = 0$, where $\alpha \equiv \tilde{a}|_{\partial\Lambda_0}$, we reproduce the same expressions as in Eqs. (6.45) and (6.46) by setting $\varphi = (\kappa/Q)\phi$. This procedure can also be applied in the non-Abelian situation considered in the next section.

In order to generalize Eqs. (6.41)–(6.49) to a situation in which we have $N(\geq 1)$ "coupled bands (or subbands)" of excitations at the boundary of the sample, we use the fact that any sum of free fields is still a free field. Thus

$$\phi(u) = \kappa_1\phi_1(u) + \cdots + \kappa_N\phi_N(u) , \tag{6.50}$$

where $\phi_1, \ldots, \phi_N$ are distinct, free, massless Bose fields and $\kappa_1, \ldots, \kappa_N$ are arbitrary, real normalization constants, is a free field. We set

$$\hat{\phi}(u)=\begin{bmatrix}\phi_1(u)\\ \vdots \\ \phi_N(u)\end{bmatrix}, \tag{6.51}$$

and the generalization of the Gaussian action (6.41) is given by

$$S(\hat{\phi})=\frac{1}{4\pi}\int_{\partial\Lambda_0}\partial_-\hat{\phi}(u)\cdot K\partial_+\hat{\phi}(u)d^2u \ , \tag{6.52}$$

where K is some positive $N\times N$ matrix and $\hat{a}\cdot\hat{b}=\sum_{i=1}^{N}a_ib_i$. We call K the "band-coupling matrix." Of course, we could diagonalize K (since K is positive and hence symmetric) and study N independent, free Bose fields. However, we shall see below that it pays to keep a general K.

The generalization of the remaining terms of the action given in Eq. (6.46) is obvious. Let us introduce the vectors

$$\hat{\alpha}=\begin{bmatrix}Q_1\alpha_1\\ \vdots \\ Q_N\alpha_N\end{bmatrix} \quad\text{and}\quad \hat{Q}=\begin{bmatrix}Q_1\\ \vdots \\ Q_N\end{bmatrix}, \tag{6.53}$$

where $Q_1, \ldots, Q_N$ are real constants and $\alpha_1, \ldots, \alpha_N$ are the restrictions of gauge potentials to the boundary space-time $\partial\Lambda_0$; cf. Eq. (6.27). The generalized, gauged, Abelian WZNW model then reads

$$\frac{1}{\hbar}S_{\mathrm{WZNW}}(\hat{\phi};\hat{\alpha})=\frac{1}{4\pi}\int_{\partial\Lambda_0}\partial_-\hat{\phi}(u)\cdot K\partial_+\hat{\phi}(u)d^2u-\frac{1}{2\pi}\int_{\partial\Lambda_0}[\partial_+\hat{\phi}(u)\cdot\hat{\alpha}_-(u)-(\partial_-\hat{\phi}-K^{-1}\hat{\alpha}_-)(u)\cdot\hat{\alpha}_+(u)]d^2u$$
$$+\frac{1}{4\pi}\int_{\partial\Lambda_0}\hat{\alpha}_-(u)\cdot K^{-1}\hat{\alpha}_+(u)d^2u \ . \tag{6.54}$$

The chirality constraint generalizing Eq. (6.47) can be inferred from the second term in (6.54). It is given by

$$\partial_-\hat{\phi}(u)-K^{-1}\hat{\alpha}_-(u)=0 \ . \tag{6.55}$$

Clearly, Eq. (6.55) is invariant under the generalized gauge transformations

$$\begin{aligned}&\hat{\phi}(u)\mapsto{}^{\hat{\chi}}\hat{\phi}(u)=\hat{\phi}(u)+K^{-1}\hat{\chi}|_{\partial\Lambda_0}(u) \ ,\\ &\hat{\alpha}\mapsto{}^{\hat{\chi}}\hat{\alpha}=\hat{\alpha}+d\hat{\chi}|_{\partial\Lambda_0} \ ,\end{aligned} \tag{6.56}$$

where the vector $\hat{\chi}$ is given by

$$\hat{\chi}(\xi)=\begin{bmatrix}Q_1\chi_1(\xi)\\ \vdots \\ Q_N\chi_N(\xi)\end{bmatrix}, \tag{6.57}$$

with $Q_1, \ldots, Q_N$ the same constants as in Eq. (6.53) and $\chi_1, \ldots, \chi_N$ arbitrary real-valued functions on Λ_0.

Physically, the quantity Q_i is the charge (in units of $-e$) of the left-moving component of the Bose field ϕ_i, $i=1, \ldots, N$; see also the remark after Eq. (6.49). Hence the vector $\hat{Q}$ is called the "charge vector" of the system. A system of Bose fields $\phi_1, \ldots, \phi_N$ which describes N bands of edge excitations at $\partial\Lambda_0$ and whose dynamics is specified by the action (6.54) and the chirality constraint (6.55) is called a "chiral boundary system." A chiral boundary system is characterized by the two quantities K and $\hat{Q}$, and its partition function is given by

$$\zeta^L_{\partial\Lambda_0}(\hat{\alpha})=\int\mathcal{D}\hat{\phi}\exp\left[\frac{i}{\hbar}S_{\mathrm{WZNW}}(\hat{\phi};\hat{\alpha})\right]\delta(\partial_-\hat{\phi}-K^{-1}\hat{\alpha}_-) \ . \tag{6.58}$$

By Eqs. (6.21) and (6.58), the "scaling limit" $Z_{\Lambda_0}(\tilde{a})$ of the total partition function of a two-dimensional, incompressible quantum Hall fluid [whose elementary charge carriers are electrons (L)], confined to a space-time domain Λ_0 and coupled to an external vector potential $a=a_c+\tilde{a}$, is given by

$$Z_{\Lambda_0}(\tilde{a})=\exp\left[i\int_{\Lambda_0}j^{\mu}_c(\xi)\tilde{a}_{\mu}(\xi)d^3\xi+\frac{i\sigma}{4\pi}\int_{\Lambda_0}\tilde{a}\wedge d\tilde{a}\right]$$
$$\times\zeta^L_{\partial\Lambda_0}(\hat{\alpha}=\hat{Q}\tilde{a}|_{\partial\Lambda_0}) \ . \tag{6.59}$$

Performing a U(1) gauge transformation (6.22) which, at the boundary $\partial\Lambda_0$, corresponds to a gauge transformation of the form (6.56), with $\hat{\chi}=\hat{Q}\chi|_{\partial\Lambda_0}$, and using the gauge invariance of the integration measure in (6.58), i.e., $\mathcal{D}^{\hat{\chi}}\hat{\phi}=\mathcal{D}\hat{\phi}$, one verifies that the total partition function $Z_{\Lambda_0}(\tilde{a})$ is U(1) gauge invariant if and only if

$$\sigma=\hat{Q}\cdot K^{-1}\hat{Q} \ . \tag{6.60}$$

This equation generalizes the relationship found in (6.49).

We note that if we choose $Q=(1, \ldots, 1)^T\in\mathbb{R}^N$ (where T denotes transposition) and if we set K equal to the identity matrix in N dimensions, Eqs. (6.58)–(6.60) provide a description of an *integer quantum Hall fluid* with $\sigma=N$ in the bosonized language.

Again we note that replacing L (electrons) by R (holes) in Eq. (6.59) corresponds to interchanging $+$ and $-$ and replacing $\hat{Q}$ by $-\hat{Q}$ and K by $-K$ on the rhs of Eq. (6.58). One easily verifies that the sign of σ changes, which is consistent with Eq. (6.60). A unified description of Hall fluids in which both types of elementary charge carriers are present, electrons and holes, will be given at the end of this subsection.

So far, the coupling matrix K is an arbitrary real, positive $N\times N$ matrix and $\hat{Q}$ is an arbitrary real N-component vector. Next, we show that there are natural constraints on these two quantities coming from the representation theory of $\hat{\mathrm{u}}(1)$ current algebra and some basic requirements on the spectrum of physical excitations in a Hall fluid. As a consequence of these constraints we shall find that σ has to take *rational* values.

The basic objects in the representation theory of chiral $\hat{\mathrm{u}}(1)$ current algebra are the chiral vertex (Weyl) operators. We recall some basic properties of these operators. (They play the role of Clebsch-Gordan operators.) It is convenient [see (6.37) and (6.38)] to introduce the coordinates

$$z=e^{2\pi i u_+/l},\quad \bar{z}=e^{2\pi i u_-/l}. \tag{6.61}$$

In terms of the left-moving Bose field ϕ_L given in Eq. (6.38), we define the chiral vertex operators

$$V_n(z)=:e^{in\phi_L(z)}:,\quad \text{with}\quad n\in\mathbb{R}\ , \tag{6.62}$$

where $::$ denotes normal ordering of moving all α_n with $n>0$ to the right of α_m with $m<0$, and p to the right of q. Applying the commutation relations (6.39), one obtains the basic exchange (or Weyl) relations of chiral vertex operators,

$$V_n(z)V_m(w)=e^{\pm i\pi(nm/\kappa)}V_m(w)V_n(z),\quad z\neq w\ , \tag{6.63}$$

where the sign $+$ or $-$ depends on the sign of $(\arg z-\arg w)$ relative to a fixed choice of a "point at ∞"; see Appendix B.

In the presence of an external gauge field α the charge operator $\mathbb{Q}$ (in units of $-e$) is defined by

$$\mathbb{Q}=\oint_{|z|=\text{const}}\left[J_L(z)-\frac{Q}{\kappa}\alpha_z(z)\right]\frac{dz}{2\pi i z}\ , \tag{6.64}$$

where $\alpha_z=\alpha_+\partial u_+/\partial z$. Note that this operator is manifestly gauge invariant. Recalling Eqs. (6.37) and (6.39), we find that

$$[\mathbb{Q},V_n(z)]=\frac{n}{\kappa}V_n(z)\ . \tag{6.65}$$

Thus the chiral vertex operator $V_n(z)$ creates a left-moving excitation of charge $q=n/\kappa$ (in units of $-e$). It follows from Eq. (6.64) that, in order for the charge of the system to be changed by an amount n/κ, the magnetic flux penetrating the system has to be changed by an amount n (in units of $-hc/e$).

Equations (6.62)–(6.65) are easily generalized to the chiral boundary system (6.58) composed of N chiral Bose fields $\phi_1,\dots,\phi_N$ and characterized by the band-coupling matrix K and the charge vector $\hat{Q}$. We introduce the chiral vertex operators

$$V_{\hat{n}}(z)=:e^{i\hat{n}\cdot\hat{\phi}_L(z)}:,\quad \text{with}\quad \hat{n}=\begin{bmatrix}n_1\\ \vdots\\ n_N\end{bmatrix}\in\mathbb{R}^N\ . \tag{6.66}$$

Then the exchange relations (6.63) generalize to

$$V_{\hat{n}}(z)V_{\hat{m}}(w)=e^{\pm i\pi\hat{n}\cdot K^{-1}\hat{m}}V_{\hat{m}}(w)V_{\hat{n}}(z),\quad z\neq w\ ; \tag{6.67}$$

see Appendix B. From the obvious generalization of Eq. (6.65) to N bands we conclude that a chiral vertex operator $V_{\hat{n}}(z)$ changes the charge in the ith band by an amount $q_i=\sum_{j=1}^N(K^{-1})_{ij}n_j$ (in units of $-e$) for $i=1,\dots,N$. We write

$$\hat{q}=K^{-1}\hat{n} \tag{6.68}$$

and note that, in order for the charge in the ith band to be changed by an amount $q_i=\sum_{j=1}^N(K^{-1})_{ij}n_j$ for $i=1,\dots,N$, the magnetic flux penetrating the system has to be changed by an amount n_j (in units of $-hc/e$) in the jth band for $j=1,\dots,N$. Hence states generated by chiral vertex operators applied to the ground state are characterized by the "magnetic-flux (or vorticity) vectors" $\hat{n}$, or, equivalently, by the "electric charge vectors" $\hat{q}=K^{-1}\hat{n}$.

Next, we require that the states generated by the chiral vertex operators (6.66) applied to the ground state exhibit properties consistent with two basic assumptions, (A1) and (A2) below, concerning the physics of two-dimensional, incompressible quantum Hall fluids whose elementary charge carriers are (spin-polarized) electrons.

(A1) *Each of the N bands admits excitations with the quantum numbers of a (scalar) electron.* That is, for $i=1,\dots,N$, there are states of the system corresponding to a charge vector of the form $\hat{q}_{\text{el}}^{(i)}=(0,\dots,0,1,0,\dots,0)^T$, where 1 stands in the ith place, and obeying Fermi statistics.

The choice of the electric charge vectors for the electrons, $\hat{q}_{\text{el}}^{(i)}$, $i=1,\dots,N$, given in assumption (A1) is consistent with a charge vector $\hat{Q}$ given by

$$\hat{Q}=(\underbrace{1,\dots,1}_{N})^T\ ; \tag{6.69}$$

see Eq. (6.53). The electric charge $Q(\hat{n}_{\text{el}}^{(i)})$ (in units of $-e$) of an electron specified by the flux vector $\hat{n}_{\text{el}}^{(i)}=K\hat{q}_{\text{el}}^{(i)}$ can be written as

$$1=Q(\hat{n}_{\text{el}}^{(i)})=\hat{Q}\cdot K^{-1}\hat{n}_{\text{el}}^{(i)}=\hat{Q}\cdot\hat{q}_{\text{el}}^{(i)}\ . \tag{6.70}$$

Remark. The choice of $\hat{q}_{\text{el}}^{(i)}$ in assumption (A1) and of $\hat{Q}$ in Eq. (6.69) corresponds to a particular choice of a basis in $\mathbb{R}^N$. An invariant way of formulating (A1) would be merely to require the existence of N linearly independent vectors, $\hat{q}_1,\dots,\hat{q}_N$, and a charge vector $\hat{Q}$ such that the excitations corresponding to $\hat{q}_1,\dots,\hat{q}_N$ are fermions of charge 1 [see Eq. (6.70)]. At this point, however, it is convenient to work in the particular basis corresponding to assumption (A1) and Eq. (6.69); see also Eq. (6.84) below and Fröhlich and Thiran (1993).

By Eq. (6.67), the chiral vertex operators $V_{\hat{n}_{\text{el}}^{(i)}}$, with $\hat{n}_{\text{el}}^{(i)}=K\hat{q}_{\text{el}}^{(i)}$, creating electrons anticommute if, and only if,

$$\begin{aligned}K_{ii}&=\hat{q}_{\text{el}}^{(i)}\cdot K\hat{q}_{\text{el}}^{(i)}\\&=\hat{n}_{\text{el}}^{(i)}\cdot K^{-1}\hat{n}_{\text{el}}^{(i)}\in 2\mathbb{N}_0+1\quad\text{for}\quad i=1,\dots,N\ .\end{aligned} \tag{6.71}$$

Moreover, electrons are excitations that are relatively local to each other (meaning that microscopic electronic wave functions are single valued). Hence a vertex operator creating an electron in the ith band must commute or anticommute with a vertex operator creating an electron in the jth band, for all pairs of $i,j \in \{1,\ldots,N\}$. By Eq. (6.67) this will be the case if, and only if,

$$\begin{aligned} K_{ij} &= \hat{q}_{\rm el}^{(i)} \cdot K \hat{q}_{\rm el}^{(j)} \\ &= \hat{n}_{\rm el}^{(i)} \cdot K^{-1} \hat{n}_{\rm el}^{(j)} \in \mathbb{Z} \quad \text{for all } i,j=1,\ldots,N \ . \end{aligned} \tag{6.72}$$

We note that if we assume that the vertex operators creating electrons in the ith and jth band commute then

$$K_{ij} \in 2\mathbb{Z} \quad \text{for} \quad i \neq j, \ i,j=1,\ldots,N \ . \tag{6.73}$$

In order not to violate the Pauli principle, electrons in different bands must then be distinguishable (e.g., by their spins, "up" or "down," or by some other quantum numbers).

Next, we wish to find the spectrum of "finite-energy excitations" in an incompressible quantum Hall fluid that has a band-coupling matrix K satisfying Eqs. (6.71) and (6.72) [or possibly (6.73)] and whose charge vector is given by (6.69). If an electron in the ith band is transported around a dynamical excitation of the system of finite energy (above the ground-state energy), it should not pick up a nontrivial statistical (or Aharonov-Bohm) phase factor, because electronic wave functions are single valued. Thus we expect that

(A2) *Every finite-energy excitation of the system is relatively local to the electrons in all N bands.*

At the boundary of the system, a finite-energy excitation can be described by applying some chiral vertex operator $V_{\hat{n}}(z)$, $\hat{n} \in \mathbb{R}^N$, to the ground state. By Eq. (6.67), it follows that, in order for assumption (A2) to hold, the corresponding flux vector $\hat{n}$ has to satisfy

$$\hat{n}_{\rm el}^{(i)} \cdot K^{-1} \hat{n} = \hat{q}_{\rm el}^{(i)} \cdot \hat{n} \in \mathbb{Z} \quad \text{for } i=1,\ldots,N \ . \tag{6.74}$$

Recalling the form of $\hat{q}_{\rm el}^{(i)}$ [see (A1)], we infer that the flux vectors corresponding to finite-energy excitations are given by

$$\hat{n} \in \mathbb{Z}^N \ . \tag{6.75}$$

Finally, we recall the symmetry properties of chiral boundary systems under the replacement of electrons by holes (i.e., of L by R, z by $\bar{z}$, Q by $-Q$, and K by $-K$) that we mentioned after Eq. (6.60). If holes are the elementary charge carriers, then relations similar to Eqs. (6.66)–(6.75) hold, up to changes of signs. By forming K matrices of block-diagonal form (with positive and negative blocks along the diagonal; see below) we can describe, in a unified way, systems with bands of electrons *and* holes. We find the following general characterization of two-dimensional, incompressible quantum Hall fluids with spin-polarized edge current bands.

(i) The large-distance and low-frequency physics of such systems can be discussed completely in terms of chiral boundary systems of electrons and/or holes [see Eqs. (6.58)–(6.60)]. Such chiral boundary systems are characterized by a band-coupling matrix K and a charge vector $\hat{Q}$. Describing the elementary excitations, electrons and/or holes, as in (A1), we find that the charge vector $\hat{Q}$ can be chosen to take the form

$$\hat{Q} = \hat{Q}_L \oplus \hat{Q}_R = (\underbrace{1,\ldots,1}_{l}, -\underbrace{1,\ldots,-1}_{r})^T \ , \tag{6.76}$$

corresponding to l bands of electrons and r $(=N-l)$ bands of holes. The band-coupling matrix K is a regular, symmetric $N \times N$ matrix with coefficients satisfying

$$K_{ii} \in 2\mathbb{Z}+1 \quad \text{for } i=1,\ldots,N \tag{6.77}$$

and

$$K_{ij} \in \mathbb{Z} \quad \text{for } i \neq j, \ i,j=1,\ldots,N \ ; \tag{6.78}$$

see Eqs. (6.71)–(6.73). Actually, in the basis of chiral Bose fields where the vector $\hat{Q}$ takes the form (6.76), the matrix K assumes the following block-diagonal form: $K = K_L \oplus K_R$, with a positive $l \times l$ submatrix K_L and a negative $r \times r$ submatrix K_R, where $r = N - l$. The general expression for the Hall constant σ is given by

$$\sigma = \hat{Q} \cdot K^{-1} \hat{Q} = \hat{Q}_L \cdot K_L^{-1} \hat{Q}_L + \hat{Q}_R \cdot K_R^{-1} \hat{Q}_R \ ; \tag{6.79}$$

see Eqs. (6.60) and (6.29). From this equation and Eqs. (6.76)–(6.78), it clearly follows that σ is a rational number. Hence the Hall conductivity $\sigma_H = \sigma e^2/h$ *is a rational multiple of* e^2/h. In particular, if we consider a situation in which we have only electrons *or* only holes as elementary excitations, we find that

$$\sigma = \sum_{i,j=1}^{N} (K^{-1})_{ij} \ , \tag{6.80}$$

where σ is positive for systems of electrons and negative for systems of holes (for a fixed sign of $\mathbf{B}_c$).

(ii) The chiral vertex operators $V_{\hat{n}}(z,\bar{z}) = V_{\hat{n}_L}(z) \otimes V_{\hat{n}_R}(\bar{z})$, creating left- and right-moving finite-energy excitations at the boundary of the system, are specified by flux vectors $\hat{n} = \hat{n}_L \oplus \hat{n}_R$, which form the sites of the "flux lattice" $\Phi = \mathbb{Z}^N = \mathbb{Z}^l \oplus \mathbb{Z}^r$; see Eqs. (6.66) and (6.75). The electric charge vectors $\hat{q}$ specifying the charges created by these vertex operators in the N bands of the system are given by $\hat{q} = K^{-1}\hat{n}$; see Eq. (6.68). They form the sites of the "charge lattice" $\Gamma = K^{-1}\Phi$. Note that, in accordance with our convention for the charge vector $\hat{Q}$ in Eq. (6.76), the charges q_i created in the ith band, $i=1,\ldots,N$, are given in units of $-e$ (the charge of an electron) for bands of electrons and in units of e (the charge of a hole) for bands of holes. Hence the total charge $Q(\hat{n})$ of a finite-energy excitation created by the vertex operator $V_{\hat{n}}(z,\bar{z})$, with $\hat{n} = \hat{n}_L \oplus \hat{n}_R \in \Phi$, applied to the ground state is given by

$$Q(\hat{n}) = \hat{Q} \cdot K^{-1} \hat{n} = \hat{Q} \cdot \hat{q} \ . \tag{6.81}$$

In order to determine the possible charges of finite-energy excitations (so-called *quasiparticles),* we note that the charge lattice Γ contains the sublattice $\Gamma_{\rm int} = \mathbb{Z}^N$, corresponding to excitations with integer charges (i.e., to multielectron and multihole excitations). The information about the possible *fractional charges* of quasiparticles is encoded into the quotient space $\Gamma/\Gamma_{\rm int}$. More explicitly, it is contained in the set

$$\mathcal{F}=\{Q(\hat{n})|\hat{n}\in\Phi\}/\mathbb{Z}\ ; \tag{6.82}$$

see also Fröhlich and Zee (1991). It follows from this equation and Eqs. (6.81) and (6.75) that the possible fractional charges of quasiparticles in an incompressible quantum Hall fluid are fully determined by the pair $(K,\hat{Q})$ of the associated chiral boundary system. The question of uniqueness of the pair $(K,\hat{Q})$ will be addressed below.

By generalizing Eq. (6.67) to the case of bands of electrons and holes, we finally see that the *statistics phase* $\pi\theta(\hat{n})$ of a finite-energy excitation specified by the flux vector $\hat{n}\in\Phi$ is given by

$$\theta(\hat{n})=\hat{n}\cdot K^{-1}\hat{n}\ \ (\mathrm{mod}\ 2)\ . \tag{6.83}$$

The results in Eq. (6.76)–(6.79), can be interpreted as a "gap-labeling theorem": Assuming incompressibility of a two-dimensional quantum Hall fluid (whose spin degrees of freedom are "frozen out"), we have proven that its Hall conductivity σ_H has to be a rational multiple of e^2/h. Conversely, if σ_H is *not* a rational multiple of e^2/h, the corresponding two-dimensional electronic system *cannot* be incompressible, i.e., there *cannot* be a positive energy gap above the ground-state energy in the spectrum of the many-body Hamiltonian of this system.

Next, we ask whether, for a given rational value of $\sigma=\sigma_H h/e^2$, there exists a *unique* chiral boundary system, characterized by a pair $(K,\hat{Q})$ of a band-coupling matrix K and a charge vector $\hat{Q}$, that explains this value of σ via the formulae (6.79) and (6.76)–(6.78). The answer is clearly *no.* A given rational value of σ corresponding to a plateau of the Hall conductivity can, in general, be reproduced by infinitely many different chiral boundary systems specified by distinct K matrices and $\hat{Q}$ vectors. This might be viewed as an intrinsic weakness of our general approach. In order to find out which chiral boundary system is the most likely candidate corresponding to a given plateau of the Hall conductivity, one must invoke additional information on the quantum Hall fluid. In particular, one might investigate stability properties of the system against small perturbations, something that, in general, would require more analytical or numerical work, or one might study symmetry properties of the system. Below, and in Sec. VI.C, we illustrate these ideas by some examples.

As a first step towards reducing the plethora of possible pairs $(K,\hat{Q})$ explaining a given plateau of the Hall conductivity σ_H, we propose to study what kind of *invariant* information is coded into a pair $(K,\hat{Q})$. From the scalar-product form of Eqs. (6.79), (6.81), and (6.83) for the physically interesting quantities σ, $Q(\hat{n})$, and $\theta(\hat{n})$, it clearly follows that all these quantities can be reproduced by a whole "orbit" of pairs $(K,\hat{Q})$.

More specifically, if S is some integral $N\times N$ matrix of determinant ±1, i.e., $S\in\mathrm{GL}(N;\mathbb{Z})$, then S, S^{-1}, S^T, and $(S^T)^{-1}$ map the flux lattice $\Phi=\mathbb{Z}^N$ and the sublattice $\Gamma_{\rm int}=\mathbb{Z}^N\subset\Gamma$ of charges of multielectron and multihole excitations onto themselves. Defining $\hat{n}'=S^T\hat{n}$ and $\hat{q}'=S^{-1}\hat{q}$, it follows that two chiral boundary systems specified by the pairs $(K,\hat{Q})$ and $(K',\hat{Q}')$, with

$$K'=S^TKS\quad\text{and}\quad\hat{Q}'=S^T\hat{Q}\ , \tag{6.84}$$

describe equivalent incompressible quantum Hall fluids. Here "equivalent" means that the systems exhibit the same Hall conductivity σ_H [see Eq. (6.79)], the same set of charges $Q(\hat{n})$ and statistics phases $\theta(\hat{n})$ for finite-energy excitations [see Eqs. (6.81) and (6.83)], and the same set $\mathcal{F}$ of fractional charges for quasiparticles [see Eq. (6.82)]. The only difference lies in the assignment of electric charges to the fields $\phi_1,\ldots,\phi_N$ forming their respective chiral boundary systems [see Eqs. (6.63), (6.54), and (6.59)], and a different choice of basis in the flux lattice.

These observations pose the problem of finding and characterizing equivalence classes of chiral boundary systems subject to the equivalence relation (6.84). The solution of this problem turns out to be mathematically involved. It is discussed in detail by Fröhlich and Thiran (1993). So far, we have been considering spin-polarized, incompressible quantum Hall fluids. We wish to mention, however, that the classification of Hall fluids in which the spin degrees of freedom are taken into account turns out to proceed along the same lines of reasoning. The difference is that the pairs $(K,\hat{Q})$ have to satisfy some additional symmetry properties. This will be discussed by means of examples in the following section. The general discussion is given in Fröhlich and Thiran (1993).

We now turn to the question of *symmetries* in incompressible quantum Hall fluids. We ask what kind of additional constraints on the K matrices can be inferred from symmetry requirements. For simplicity, we restrict our attention to Hall fluids whose elementary excitations are electrons; see Eqs. (6.71)–(6.73). [For systems with holes as elementary excitations the discussion is the same, up to changes of signs, and for systems with bands of electrons and holes the discussion is easily generalized; see Eqs. (6.76)–(6.79).]

(A3) A natural symmetry one might expect of an "elementary" Hall fluid is *invariance under arbitrary permutations* of the bands formed by $\phi_1,\ldots,\phi_N$ [provided we are describing the system in a "symmetric" basis, where the charge vector $\hat{Q}$ is of the form $\hat{Q}=(1,\ldots,1)^T$; see Eq. (6.76)].

This symmetry implies that

$$K_{ij}=K_{\pi(i)\pi(j)},\quad\text{for } i,j=1,\ldots,N\ , \tag{6.85}$$

where π denotes an arbitrary permutation of $\{1,\ldots,N\}$. Together with the conditions (6.77) and (6.78), this implies that

$$K_{ii}=2l+1 \quad \text{for } i=1,\ldots,N\ , \tag{6.86}$$

and

$$K_{ij}=p \quad \text{for } i\neq j,\ i,j=1,\ldots,N\ , \tag{6.87}$$

for some $l\in\mathbb{N}_0$ and some $p\in\mathbb{Z}$, both independent of i and j (and such that K is positive). Thus we may write

$$K=\begin{bmatrix} 2l+1 & p & \cdots & p \\ p & 2l+1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & p \\ p & \cdots & p & 2l+1 \end{bmatrix}$$

$$=(2l+1-p)\mathbf{1}_N+pN\mathbf{P}_N\ , \tag{6.88}$$

where $\mathbf{1}_N$ is the unit matrix in N dimensions and $\mathbf{P}_N$ is the orthogonal projector onto the diagonal in $\mathbb{R}^N$, i.e., $\mathbf{P}_N$ is the $N\times N$ matrix all of whose components are given by $1/N$. Using the fact that $\mathbf{P}_N$ is an orthogonal projector, one easily verifies that

$$K^{-1}=\frac{1}{2l+1-p}\left[\mathbf{1}_N-\frac{pN}{2l+1+p(N-1)}\mathbf{P}_N\right]. \tag{6.89}$$

Recalling identity (6.80) for the Hall constant σ, this equation yields

$$\sigma\equiv\sigma_K=\frac{N}{2l+1+p(N-1)}\ . \tag{6.90}$$

If we want to impose the constraint (6.73) we must assume that p is an even integer. This reproduces the odd-denominator rule (see, for example, Tao and Wu, 1985). In general, the odd-denominator rule holds only for an odd number of bands.

We may define "second-generation states" of an incompressible quantum Hall fluid as states of a system with a charge vector $\hat{Q}=(1,\ldots,1)^T\in\mathbb{R}^{nN}$ and an $(nN)\times(nN)$ band-coupling matrix K_h given by

$$K_h=\begin{bmatrix} K & & 0 \\ & \ddots & \\ 0 & & K \end{bmatrix}+r(nN)\mathbf{P}_{nN}\ , \tag{6.91}$$

where the first matrix on the rhs is block diagonal and built from n copies of an $N\times N$ matrix K of the form (6.88), and r is an even integer. For the Hall constant σ_{K_h} of this system we find

$$\sigma_{K_h}=\frac{1}{r+(1/n\sigma_K)}=\frac{n\sigma_K}{rn\sigma_K+1}\ . \tag{6.92}$$

We can go on in this vein and define "third-generation states" etc. We can also consider K_h matrices of the form

$$K_h=\begin{bmatrix} K_1 & & & 0 \\ & K_2 & & \\ & & \ddots & \\ 0 & & & K_n \end{bmatrix}, \tag{6.93}$$

with different blocks $K_1,\ldots,K_n$ of the form (6.91) along the diagonal (Fröhlich and Zee, 1991; see also Wen and Zee, 1992). We then find that the Hall constant is given by

$$\sigma_{K_h}=\sigma_{K_1}+\cdots+\sigma_{K_n}\ . \tag{6.94}$$

Requirements on the pair $(K,\hat{Q})$ characterizing a chiral boundary system with a unitary symmetry are discussed in the next section. We conclude this section by considering some examples of fractional quantum Hall fluids covered by our theory.

The simplest examples correspond to $N=1$, $Q=\pm1$, and $K=\pm(2l+1)$, with $l\in\mathbb{N}_0$. For $l=0$, this is an integer quantum Hall fluid with $\sigma=\pm1$. This is the simplest example of the situation discussed in Sec. VI.A. For $l=1$ we find Laughlin's fluid (Laughlin, 1983a, 1983b) with $\sigma=\pm\frac{1}{3}$. Experimentally, a fractional quantum Hall fluid corresponding to $l=2$, i.e., to $\sigma=\pm\frac{1}{5}$, has been observed only recently (Jiang *et al.*, 1990). Experimental evidence for fractional quantum Hall fluids corresponding to $l=3$ and 4 ($\sigma=\pm\frac{1}{7}$ and $\sigma=\pm\frac{1}{9}$) is only partial (e.g., Williams, 1992 and the references indicated therein). There are no known quantum Hall fluids corresponding to $l\geq5$.

Actually, one anticipates that, for two-dimensional electronic systems with a filling factor $\nu\lesssim\frac{1}{5}$ [see Eq. (4.40)], the formation of a triangular Wigner crystal is favored over that of an incompressible quantum Hall fluid. By forming an electron crystal the system loses its incompressibility. Moreover, one expects that random impurities pin the electron crystal to the background and, as a result, such two-dimensional electronic systems become insulators. For a more detailed discussion of these issues and experimental results, see Jiang *et al.* (1991).

The charges of finite-energy excitations in a quantum Hall fluid with $\kappa\equiv K=\pm(2l+1)$ are determined by the (one-dimensional) charge lattice $\Gamma=[1/(2l+1)]\mathbb{Z}$. Thus the fluid is expected to exhibit quasiparticle excitations with fractional charges (in units of $\mp e$) given by

$$\frac{n}{2l+1} \quad \text{for } n=1,\ldots,2l\ . \tag{6.95}$$

For experimental signatures of fractional charges in quantum Hall fluids see Clark *et al.* (1988), Chang and Cunningham (1989), Simmons *et al.* (1989), Clark *et al.* (1990), and Hwang *et al.* (1992). By Eq. (6.65), the magnetic fluxes (in units of $\mp hc/e$) associated with these excitations are determined by an integer n mod $2l+1$. Equations (6.95) and (6.63) then tell us that these quasiparticles are *anyons.*

At the end of Sec. V.B, we mentioned that, in the conventional approach to the quantum Hall effect (starting from the Kubo formula), the denominator $n_0=\pm(2l+1)$ of the Hall constant σ is interpreted as the degeneracy of the ground state of the fractional quantum Hall fluid; see Eq. (5.57) and the references given there. In our approach this has a straightforward explanation: The algebra of the chiral edge current [see Eqs. (6.37)–(6.39)] of a quantum Hall fluid with $\sigma=\pm 1/(2l+1)$ [$N=1$, $Q=\pm 1$, $\kappa \equiv K=\pm(2l+1)$] has $2l+1$ inequivalent representations labeled by magnetic fluxes $n=1,\ldots,2l+1$, corresponding to electric charges $q=\pm 1/(2l+1)$, $\pm 2/(2l+1),\ldots,\pm 1$ [see Eqs. (6.62)–(6.65)]. In the thermodynamic limit, approached when the scale parameter θ tends to ∞, every one of these representations corresponds to a ground state of the fractional quantum Hall fluid with a one-component boundary. In this limit, these $2l+1$ distinct ground states have the same energy per electron and hence are degenerate.

Finally, we note that if a vortex of strength $n=2l+1$ is created in the bulk of a quantum Hall fluid with $\sigma=\pm 1/(2l+1)$ and a one-component boundary, then, in the thermodynamic limit ($\theta\to\infty$), the total charge of the fluid changes by $K^{-1}(2l+1)=\sigma(2l+1)=\pm 1$, as follows from Eq. (6.64). More precisely, a charge of ± 1 is transferred from the place where the vortex is created to the boundary of the system; see Sec. 6 in Fröhlich and Kerler (1991) for more details. This result relates our definition of the Hall conductivity $\sigma_H=\sigma e^2/h$ to one in which σ_H is defined as an index (Bellissard, 1988a, 1988b; Avron, Seiler, and Simon, 1990, 1992).

The results reviewed above for the simple fractional quantum Hall fluids with $N=1$, $Q=\pm 1$, and $K=\pm(2l+1)$, where $l\in\mathbb{N}_0$, have straightforward extensions to fluids corresponding to more general chiral boundary systems consisting of N (>1) bands (or subbands) and characterized by a general pair $(K,\hat{Q})$, as discussed above; see also Fröhlich and Thiran (1993). We illustrate the general situation by an example: We consider two-dimensional electronic systems exhibiting a fractional quantum Hall effect with $\sigma=\frac{1}{2}$, as discovered recently by Eisenstein *et al.* (1992) and Suen *et al.* (1992). Since, experimentally, these systems consist of two layers, we expect the following chiral boundary system $(K,\hat{Q})$ to provide a natural explanation for this effect: We set $N=2$, corresponding to two bands or layers. Working in a basis in which the charge vector is given by $\hat{Q}=(1,1)^T$ and assuming invariance under permutations of the two bands, as discussed in assumption (A3) above and Eqs. (6.85)–(6.88), we propose a K matrix of the form

$$K=\begin{pmatrix} 2l+1 & p \\ p & 2l+1 \end{pmatrix},$$

for some $l\in\mathbb{N}_0$ and some $p\in\mathbb{Z}$. The simplest realizations of such a matrix is given by

$$K=\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix};$$

see also Wen and Zee (1992). For this system we predict the existence of quasiparticles with fractional charge $Q=\frac{1}{4},\frac{1}{2},\frac{3}{4}$, as follows from Eq. (6.82)! [Actually, within the framework given above, one can prove that *any* explanation of an even-denominator quantum Hall fluid ($\sigma=n/d$, d even) has the property that the minimal fractional charge Q^* exhibited by quasi-particles is a *fraction* of $1/d$ ($Q^*=1/\lambda d$, with $\lambda\geq 2$); see Fröhlich and Thiran (1993).] We note that other explanations, e.g., with $(l,p)=(2,-1)$, $(3,-3)$, $(4,-5),\ldots,$ predict the same fractional charges, but the number of independent, fractionally charged, anyonic excitations is larger in the these systems. [This number is actually given by $\det K$.] One feature to which we draw the reader's attention is that for all these explanations p is odd. Recalling Eq. (6.73), this is consistent with the assumption that the electrons in the two bands or layers are indistinguishable. In particular, they must have the same spin polarization. Physically, this indistinguishability suggests that, in addition to interlayer Coulomb interactions, tunneling of electrons between the two layers is important in producing a plateau at $\sigma=\frac{1}{2}$; see Eisenstein *et al.* (1992) and Suen *et al.* (1992).

In the following subsection we generalize our analysis to incompressible quantum Hall fluids in which the dynamics of the spins of the particles (and possibly of (gauged) internal degrees of freedom) are taken into account. These generalizations are important for an understanding of quantum Hall fluids with unpolarized ground states, e.g., for $\sigma=\frac{8}{5}$ (Eisenstein, Stormer, *et al.*, 1989, 1990a) or the unpolarized, even-denominator Hall fluid with $\sigma=\frac{5}{2}$ (Willett *et al.*, 1987; Eisenstein, Willett, *et al.*, 1988, 1990).

C. Edge excitations in unpolarized quantum Hall fluids

The purpose of this section is to extend the discussion presented in the two preceding sections to two-dimensional, incompressible quantum Hall fluids where the dynamics of the spin degrees of freedom is taken into account. For convenience, we recall from Eq. (5.26) the universal form of the "scaling limit" $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$ of the effective gauge-field action for such systems:

$$\begin{aligned}-\frac{1}{h}S^*_{\Lambda_0}(\tilde{a},\tilde{w})=&\int_{\Lambda_0}(*j_c)\wedge\tilde{a}+\int_{\Lambda_0}(*m_3)\wedge\tilde{w}_3+\sum_{A=1}^{2}\int_{\Lambda_0}\tau_1^{\mu\nu}\tilde{w}_{\mu A}\tilde{w}_{\nu A}\,dv+\sum_{A,B=1}^{2}\int_{\Lambda_0}\tau_2^{\mu\nu}\varepsilon_{AB}\tilde{w}_{\mu A}\tilde{w}_{\nu B}\,dv\\&+\frac{k}{4\pi}\int_{\Lambda_0}\mathrm{tr}(w\wedge dw+\tfrac{2}{3}w\wedge w\wedge w)+\frac{\sigma}{4\pi}\int_{\Lambda_0}\tilde{a}\wedge d\tilde{a}+\frac{\chi_s}{4\pi}\int_{\Lambda_0}(\tilde{a}\wedge d\tilde{w}_3+d\tilde{a}\wedge\tilde{w}_3)+\frac{\sigma_s}{4\pi}\int_{\Lambda_0}\tilde{w}_3\wedge d\tilde{w}_3\\&+\sum_{A,B,C=1}^{3}\int_{\Lambda_0}\eta^{\mu\nu\rho}_{ABC}\tilde{w}_{\mu A}\tilde{w}_{\nu B}\tilde{w}_{\rho C}\,dv+\mathrm{B.T.}(a|_{\partial\Lambda_0},w|_{\partial\Lambda_0}),\end{aligned}\qquad(6.96)$$

where $a=a_c+\tilde{a}=\sum_\mu a_\mu(\xi)d\xi^\mu$ is the vector potential of the total, external electromagnetic field $(\mathbf{E},\mathbf{B})$ acting on the system [see Eq. (3.36)], $w=w_c+\tilde{w}=i\sum_{\mu,A}w_{\mu A}(\xi)\sigma_A\,d\xi^\mu$ is the total SU(2) connection [see Eqs. (5.27) and (3.28)], with σ_1, σ_2, and σ_3 the three Pauli matrices. (Recall that we are considering electronic systems constituted by electrons and/or holes, i.e., systems of spin-$\frac{1}{2}$ particles). The gauge field w determines the motion of the spins (magnetic moments) of the particles in the electromagnetic field $(\mathbf{E},\mathbf{B})$ [see Eqs. (3.30), (3.37), and (3.38)] and, possibly, in some geometrical field specified by Eq. (3.29). We recall that $\tilde{a}$ and $\tilde{w}$ describe small perturbations of the external electromagnetic field around some background field $(\mathbf{E}_c,\mathbf{B}_c)$, specified (in a suitable SU(2) gauge) in Eqs. (5.23)–(5.25). The quantities j_c, m_3, $\tau_\alpha^{\mu\nu}$, $\alpha=1,2$, and $\eta_{ABC}^{\mu\nu\rho}$ have been discussed after Eq. (5.26). Finally, σ, χ_s, σ_s, and k are real constants, and $\mathrm{B.T.}(a|_{\partial\Lambda_0},w|_{\partial\Lambda_0})$ denotes boundary terms depending only on the restriction of the gauge potentials to the boundary $\partial\Lambda_0$ of the system.

In this section, we choose time-independent coordinates and an SU(2) gauge such that there are no "tidal" terms contributing to a and w; see Sec. III.C. An example in which "tidal" terms are relevant is provided in Sec. VII.B, where we discuss the physics of a superfluid ^{3}He-A/B interface with broken symmetries.

The aim of this section is to explain the quantization of the values of all the constants σ, χ_s, σ_s, and k for incompressible quantum Hall fluids. Again, we make extensive use of gauge invariance and the idea of anomaly cancellation.

For this purpose, we recall the action of U(1)×SU(2) gauge transformations on the Abelian and non-Abelian gauge potentials. The action of U(1) gauge transformations on the perturbation potential $\tilde{a}$ was explained in Eq. (6.22); see also the discussion following that equation. Let g denote an SU(2)-valued function on Λ_0 (with dg not necessarily vanishing at $\partial\Lambda_0$). Then the total SU(2) connection w transforms according to

$$w\mapsto {}^g w=gwg^{-1}+g\,dg^{-1}\ , \tag{6.97}$$

whereas the perturbation potential $\tilde{w}=w-w_c$, which is the difference of two SU(2) connections w and w_c, transforms homogeneously (i.e., under the adjoint representation of the gauge group),

$$\tilde{w}\mapsto {}^g\tilde{w}=g\tilde{w}g^{-1}\ ; \tag{6.98}$$

see also the remark after Eq. (6.22).

The "scaling limit" $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$ of the effective action is U(1)×SU(2) gauge invariant under gauge transformations for which (χ,dg) has support in the interior of the space-time region Λ_0 and g is in the component of the identity; see Sec. V.A and Appendix A. [We infer from Appendix A or Fröhlich and Studer (1992b) that the quantities m_3, $\tau_\alpha^{\mu\nu}$, $\alpha=1,2$, $\eta_{ABC}^{\mu\nu\rho}$, χ_s, and σ_s are, in fact (components of) vectors or tensors transforming under SU(2) gauge transformations according to the adjoint representation or some tensor product thereof. This guarantees the aforementioned gauge invariance. The specific form (6.96) of the effective action refers to an SU(2) gauge in which the background field w_c is given by Eqs. (5.23)–(5.25).]

For the SU(2) gauge invariance to hold, the constant k has to be an *integer*. This can be inferred by the following well-known "winding number argument" (Deser, Jackiw, and Templeton, 1982a, 1982b; Witten, 1984; see also Novikov, 1982): Let g be an SU(2) gauge transformation with the property that

$$g(\xi^0,\xi)\to\mathbf{1}\ \text{continuously}$$
$$\text{as }(\xi^0,\xi)\to\partial\Lambda_0\ \text{or}\ \xi^0\to\pm\infty\ . \tag{6.99}$$

Then, by SU(2) gauge invariance of nonrelativistic quantum mechanics, the generating function

$$Z_{\theta\Lambda_0}(a^{(\theta)},w^{(\theta)})\equiv\exp\left[\frac{i}{\hbar}S^{\mathrm{eff}}_{\theta\Lambda_0}(a^{(\theta)},w^{(\theta)})\right]$$
$$\simeq\exp\left[\frac{i}{\hbar}S^*_{\Lambda_0}(\tilde{a},\tilde{w})\right] \tag{6.100}$$

(see Sec. V.A) must be invariant under gauge transformations (6.97) satisfying (6.99). Asymptotically, as $\theta\to\infty$, the *only* gauge variance of $Z_{\theta\Lambda_0}(a^{(\theta)},w^{(\theta)})$ comes from the SU(2) Chern-Simons term in Eq. (6.96). In order to see this, let us consider the factor

$$z_k(w)=\exp\left[-\frac{ik}{4\pi}\int_{S^3}\mathrm{tr}(w\wedge dw+\tfrac{2}{3}w\wedge w\wedge w)\right]\ , \tag{6.101}$$

contributing to the partition function $Z_{\theta\Lambda_0}(a^{(\theta)},w^{(\theta)})$. In Eq. (6.101) we are integrating over S^3 because, for Λ_0 a cylinder, the topological nature of the Chern-Simons term (i.e., its independence from metric properties of Λ_0) and the choice in Eq. (6.99) allow for an identification of Λ_0 with S^3. Since, topologically, SU(2) is the 3-sphere, as well, there exist SU(2) gauge transformations g [satisfying Eq. (6.99)] with nontrivial winding number

$$n(g)=\frac{1}{24\pi^2}\int_{S^3}\mathrm{tr}(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg)\in\mathbb{Z} \tag{6.102}$$

[recall that $\pi_3(\mathrm{SU}(2))=\mathbb{Z}$]. A straightforward calculation then shows that

$$z_k({}^g w)=z_k(w)\exp[-2\pi ik\,n(g)]\ . \tag{6.103}$$

Hence, by Eqs. (6.102) and (6.103) and the fact that there is no other term in $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$ cancelling the factor $\exp[-2\pi ik\,n(g)]$ in (6.103), we infer the famous constraint

$$k\in\mathbb{Z}\ . \tag{6.104}$$

We recall that, by (5.53), k enters the Hall conductivity σ_H^{spin} for the spin current. Below, we shall discuss the Hall effect for spin currents in systems with vanishing and with nonvanishing values of k. We shall see that k counts the number of spin-singlet (edge current) bands in the corresponding Hall fluid.

Next, we study $U(1)\times SU(2)$ gauge transformations which are nontrivial at the boundary $\partial\Lambda_0$. What can be inferred from them about the boundary terms, B.T.$([a_c+\tilde{a}]|_{\partial\Lambda_0},[w_c+\tilde{w}]|_{\partial\Lambda_0})$ in Eq. (6.96)? Or, more specifically, what can be inferred about the constants σ, χ_s, and σ_s? In order to answer this question, we elaborate on the ideas of Sec. VI.B, where those terms in Eq. (6.96) depending only on $\tilde{a}$ have already been discussed.

(i) Concerning the terms involving m_3, $\tau_\alpha^{\mu\nu}$, $\alpha=1,2$, and $\eta_{ABC}^{\mu\nu\rho}$, we note that full SU(2) gauge invariance of the partition function (6.100) implies boundary constraints on these quantities of a similar nature to the constraint on j_c discussed after Eq. (6.23). We do not discuss these terms further, since they do not constrain the values of σ, χ_s, and σ_s.

(ii) We now turn to the Abelian Chern-Simons terms on the rhs of Eq. (6.96), which we collectively denote by

$$-\frac{1}{\hbar}S_{\Lambda_0}^{\text{CS}}(\tilde{a},\tilde{w}_3)=\frac{\sigma}{4\pi}\int_{\Lambda_0}\tilde{a}\wedge d\tilde{a}+\frac{\chi_s}{4\pi}\int_{\Lambda_0}(\tilde{a}\wedge d\tilde{w}_3+d\tilde{a}\wedge\tilde{w}_3)+\frac{\sigma_s}{4\pi}\int_{\Lambda_0}\tilde{w}_3\wedge d\tilde{w}_3\ . \tag{6.105}$$

These terms are anomalous under $U(1)\times SU(2)$ gauge transformations that are nontrivial at the boundary $\partial\Lambda_0$. The anomaly of the first term on the rhs of Eq. (6.105) under U(1) gauge transformations χ, defined in Eq. (6.22), has been displayed in Eq. (6.23). In the following, we present an analysis of $S_{\Lambda_0}^{\text{CS}}(\tilde{a},\tilde{w}_3)$ as a whole.

As mentioned above, $S_{\Lambda_0}^{*}(\tilde{a},\tilde{w})$ assumes the form (6.96) in an SU(2) gauge where the background field w_c takes the form specified in Eqs. (5.23)–(5.25); and under SU(2) gauge transformations g, the quantities χ_s and σ_s are components of quantities transforming as an su(2) vector and as a second-rank su(2) tensor, respectively. However, in order to determine the possible values of the constants χ_s and σ_s, it suffices to study those SU(2) gauge transformations which leave the form of Eq. (6.96) or, for that matter, of (6.105) invariant. This is the case for SU(2) gauge transformations corresponding to local rotations around the 3-axis in spin (or tangent) space, i.e., to transformations of the form $g(\xi)=\cos\lambda(\xi)\mathbf{1}-i\sin\lambda(\xi)\sigma_3$. These transformations form an Abelian subgroup of $U(1)_{\text{spin}}$ gauge transformations. By the remark following Eq. (6.22), we must therefore study the transformation properties of $S_{\Lambda_0}^{\text{CS}}(\tilde{a},\tilde{w}_3)$ under gauge transformations of the form

$$\tilde{w}_3\mapsto{}^{\lambda}\tilde{w}_3=\tilde{w}_3+d\lambda\ , \tag{6.106}$$

where $\tilde{w}_3$ is the su(2) 3-component (a real 1-form) of the perturbation part $\tilde{w}$ of the total SU(2) connection $w=w_c+\tilde{w}$.

Hence, denoting by χ a U(1) gauge transformation as in Eq. (6.22) and by λ a $U(1)_{\text{spin}}$ gauge transformation as in Eq. (6.106), we find the following anomalous gauge behavior of the Abelian Chern-Simons terms in Eq. (6.96):

$$\frac{1}{\hbar}S_{\Lambda_0}^{\text{CS}}(\tilde{a}+d\chi,\tilde{w}_3+d\lambda)=\frac{1}{\hbar}S_{\Lambda_0}^{\text{CS}}(\tilde{a},\tilde{w}_3)+\frac{\sigma}{4\pi}\int_{\partial\Lambda_0}d\chi\wedge\tilde{a}+\frac{\chi_s}{4\pi}\int_{\partial\Lambda_0}(d\chi\wedge\tilde{w}_3+d\lambda\wedge\tilde{a})+\frac{\sigma_s}{4\pi}\int_{\partial\Lambda_0}d\lambda\wedge\tilde{w}_3\ . \tag{6.107}$$

From the discussion in Sec. VI.B we can immediately infer the form of boundary terms in Eq. (6.96) which cancel the total anomaly in (6.107). Let $\hat{Q}=\hat{Q}_L\oplus\hat{Q}_R\in\mathbb{R}^l\oplus\mathbb{R}^r$ be the charge vector given in Eq. (6.76), and let $K=K_L\oplus K_R$ be a real, symmetric $(l+r)\times(l+r)$ matrix with a positive and negative block along the diagonal, as explained after Eq. (6.78). We denote by $\hat{\delta}=\hat{\delta}_L\oplus\hat{\delta}_R$ an arbitrary vector in $\mathbb{R}^l\oplus\mathbb{R}^r$. The physical meaning of this vector is discussed below. Finally, we denote by $\zeta_{\partial\Lambda_0}^{L}(K_L;\hat{\alpha})$ the partition function of a chiral boundary system [of left-moving excitations coupled to arbitrary U(1) gauge potentials $\alpha_1,\dots,\alpha_l$], as given in Eqs. (6.58) and (6.54), with positive $K=K_L$; and we denote by $\zeta_{\partial\Lambda_0}^{R}(K_R;\hat{\alpha})$ the partition function of a chiral boundary system [of right-moving excitations coupled to arbitrary U(1) gauge potentials $\alpha_1,\dots,\alpha_r$], determined by Eqs. (6.58) and (6.54), with $K=K_R$ negative and + and − interchanged. Then it is a straightforward calculation to show that the factor

$$Z_{\Lambda_0}^{\text{CS}}(\tilde{a},\tilde{w}_3)=\exp\left[\frac{i}{\hbar}S_{\Lambda_0}^{\text{CS}}(\tilde{a},\tilde{w}_3)\right]\zeta_{\partial\Lambda_0}^{L}(K_L;\hat{\alpha}=\hat{Q}_L\tilde{a}|_{\partial\Lambda_0}+\hat{\delta}_L\tilde{w}_3|_{\partial\Lambda_0})\,\zeta_{\partial\Lambda_0}^{R}(K_R;\hat{\alpha}=\hat{Q}_R\tilde{a}|_{\partial\Lambda_0}+\hat{\delta}_R\tilde{w}_3|_{\partial\Lambda_0}) \tag{6.108}$$

of the total partition function (6.100) is $U(1)\times U(1)_{\text{spin}}$ gauge invariant [see Eqs. (6.22) and (6.106)], provided that

$$\sigma=\hat{Q}\cdot K^{-1}\hat{Q}=\hat{Q}_L\cdot K_L^{-1}\hat{Q}_L+\hat{Q}_R\cdot K_R^{-1}\hat{Q}_R\ , \tag{6.109}$$

$$\chi_s=\hat{\delta}\cdot K^{-1}\hat{Q}=\hat{\delta}_L\cdot K_L^{-1}\hat{Q}_L+\hat{\delta}_R\cdot K_R^{-1}\hat{Q}_R\ , \tag{6.110}$$

and

$$\sigma_s=\hat{\delta}\cdot K^{-1}\hat{\delta}=\hat{\delta}_L\cdot K_L^{-1}\hat{\delta}_L+\hat{\delta}_R\cdot K_R^{-1}\hat{\delta}_R\ . \tag{6.111}$$

[In the calculation one uses the fact that the measure in the path integral for $\xi^{L/R}_{\partial\Lambda_0}(K_{L/R};\hat{\alpha})$ is invariant under $U(1)\times U(1)_{spin}$ gauge transformations (χ,λ), i.e., $\mathcal{D}^{(\hat{\chi},\hat{\lambda})}\hat{\phi}=\mathcal{D}\hat{\phi}$, where ${}^{(\hat{\chi},\hat{\lambda})}\hat{\phi}=\hat{\phi}+K^{-1}_{L/R}(\hat{\chi}|_{\partial\Lambda_0}+\hat{\lambda}|_{\partial\Lambda_0})$, with $\hat{\chi}=\hat{Q}_{L/R}\chi$ and $\hat{\lambda}=\hat{\delta}_{L/R}\lambda$.] Equations (6.109)–(6.111) generalize Eq. (6.79). However, we emphasize that, so far, we have not argued for any constraints on K similar to Eqs. (6.77) and (6.78).

We note that if $k=0$, i.e., if there is *no* non-Abelian anomaly, then the result in Eq. (6.108) is the complete solution to the problem of anomaly cancellation in $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$. Moreover, in this situation, the discussion of Sec. VI.B applies, and the constraints (6.77) and (6.78) must be satisfied by $(K,\hat{Q})$. A consistent choice of $\hat{\delta}$ as an integral vector will be discussed shortly. Hence [modulo the equivalence discussed in Eq. (6.84)] the constraints (6.77) and (6.78) and the equations (6.109)–(6.111) and (6.81)–(6.83) provide a complete characterization of the universal properties of "Abelian" quantum Hall fluids. In particular, σ, χ_s, and σ_s take *rational* values.

We still need to provide a physical interpretation of the "polarization vector" $\hat{\delta}$. To this end, we recall the linear-response equation (5.51) and specialize it to a static system in the absence of a persistent current $\mathcal{J}_c$ and of "tidal" fields $\tilde{f}$ and $\tilde{\Omega}$. We find that

$$\langle \mathcal{J}^i(\xi)\rangle_{a,w}=-\sigma_H\varepsilon^{ij}\tilde{E}_j(\xi)+\chi_\perp c\,\varepsilon^{ij}\partial_j\tilde{B}_3(\xi)\quad\text{for}\quad i=1,2\ , \tag{6.112}$$

with

$$\sigma_H=\frac{q^2}{h}\sigma\quad\text{and}\quad\chi_\perp=-\frac{qg\mu}{2hc}\chi_s\ , \tag{6.113}$$

where q is the charge of the particles constituting the system and $g\mu$ specifies their magnetic moments; see Eqs. (2.7), (5.49), and (5.50).

For definiteness, we consider a rectangular sample in the (1,2) plane of Euclidean space $\mathbb{E}^3$ of dimensions l_1 and l_2 in the 1- and 2-directions, respectively, and with the background magnetic field $\mathbf{B}_c$ along the 3-axis, as described in Sec. IV.E. Moreover, we temporarily assume that the system exhibits one edge current band of charged particles *with* spin (electrons or holes). Introducing the electrostatic potential $\tilde{\Phi}$, corresponding to the electric field $\tilde{\mathbf{E}}$ (i.e., $\tilde{E}_j=-\partial_j\tilde{\Phi}$, $j=1,2$), we can integrate Eq. (6.112), e.g., for $i=2$, along the 1-axis from $\xi_1=0$ to $\xi_1=l_1$. We find that

$$I_2=\sigma_H\left[\left[\tilde{\Phi}(\xi_1=l_1)+\frac{\chi_\perp c}{\sigma_H}\tilde{B}_3(\xi_1=l_1)\right]-\left[\tilde{\Phi}(\xi_1=0)+\frac{\chi_\perp c}{\sigma_H}\tilde{B}_3(\xi_1=0)\right]\right]\ , \tag{6.114}$$

where I_2 is the total electric current flowing in the 2-direction in response to an external perturbation by an electric field $\tilde{E}_1$ in the 1-direction and a magnetic field $\tilde{B}_3$ in the 3-direction.

Let us confront Eq. (6.114) with the experimental procedure of measuring the Hall conductivity, denoted here by $\sigma_{H,\text{exp}}$. Experimentally, one tunes the total electric current I_2 in the 2-direction. Then one measures the voltage in the 1-direction. One really measures a difference qV_1 between the chemical potential μ_{chem} at $\xi_1=0$ and the one at $\xi_1=l_1$. Finally, one defines

$$\sigma_{H,\text{exp}}=\frac{I_2}{V_1} \tag{6.115}$$

(see also Sec. IV.E). Denoting by μ_3 the "mean" 3-component of the magnetic moments of the particles forming the edge current band of the system (actually, $\mu_3=\partial\mathcal{M}_3/\partial N_p$, where $\mathcal{M}_3$ is the 3-component of the total magnetization of the band and N_p is the total number of particles in the band), we obtain the chemical potential μ_{chem} associated with this band,

$$\mu_{\text{chem}}(\xi)=q\tilde{\Phi}(\xi)-\mu_3\tilde{B}_3(\xi)$$

and (6.116)

$$qV_1=\mu_{\text{chem}}(\xi_1=l_1)-\mu_{\text{chem}}(\xi_1=0)\ .$$

Comparing Eqs. (6.114)–(6.116), we find that

$$\sigma_H=\sigma_{H,\text{exp}}\quad\text{and}\quad\mu_3=-\frac{\chi_\perp cq}{\sigma_H}\ . \tag{6.117}$$

Recalling the definitions in Eq. (6.113), as well as Eqs. (6.109) and (6.110), we infer that, for $Q=\pm1$,

$$\chi_s=\pm\delta\sigma\quad\text{with}\quad\delta\equiv\mu_3\left[\frac{g\mu}{2}\right]^{-1}\ . \tag{6.118}$$

Hence δ specifies the "mean" polarization of the particles constituting the band of the system.

This picture is easily generalized to the situation of an incompressible quantum Hall fluid with N (edge current) bands. In a basis where the charge vector $\hat{Q}$ is given by Eq. (6.76) we assign to fully spin-polarized bands (or subbands) the values $\delta=\pm1$, where the sign depends on the orientation of the spins (parallel or antiparallel to the magnetic field $\mathbf{B}_c$). Examples will be provided in Sec. VII.

(iii) Finally, we have to investigate the properties of the non-Abelian Chern-Simons term on the rhs of Eq. (6.96),

$$-\frac{1}{\hbar}S^{\text{CS}}_{\Lambda_0}(w)=\frac{k}{4\pi}\int_{\Lambda_0}\text{tr}(w\wedge dw+\tfrac{2}{3}w\wedge w\wedge w)\ , \tag{6.119}$$

under arbitrary SU(2) gauge transformations g which are nontrivial at the boundary $\partial\Lambda_0$. Similarly to (6.103), one calculates the full anomaly of $S^{\text{CS}}_{\Lambda_0}(w)$ to be given by

$$\frac{1}{\hbar}S^{\mathrm{CS}}_{\Lambda_0}({}^g w)=\frac{1}{\hbar}S^{\mathrm{CS}}_{\Lambda_0}(w)-\frac{k}{4\pi}\int_{\partial\Lambda_0}\mathrm{tr}(g^{-1}dg\wedge w)-2\pi k\, n(g;\Lambda_0)$$
$$=\frac{1}{\hbar}S^{\mathrm{CS}}_{\Lambda_0}(w)-\frac{k}{4\pi}\int_{\partial\Lambda_0}\mathrm{tr}[(g^{-1}\partial_- g)(u)w_+(u)-(g^{-1}\partial_+ g)(u)w_-(u)]d^2u-2\pi k\, n(g;\Lambda_0)\ , \tag{6.120}$$

where ${}^g w$ has been defined in Eq. (6.97) and $n(g;\Lambda_0)$ denotes the term in Eq. (6.102) with the difference that, here, we integrate over the space-time Λ_0 instead of the 3-sphere S^3 [we do not require (6.99) to hold]. Moreover, as in Eq. (6.27), $w_\pm$ is defined by

$$w|_{\partial\Lambda_0}=w_+(u)du_+ + w_-(u)du_-\ . \tag{6.121}$$

Next, we determine an appropriate boundary term B.T.$(w|_{\partial\Lambda_0})$, whose SU(2) gauge anomaly cancels that in Eq. (6.120). In a first step, we outline a general construction, which is a straightforward generalization of the discussion in Sec. VI.B [see Eqs. (6.45)–(6.49)] to the non-Abelian situation. In a second step, we show that there is a unified way of treating the problem of Abelian *and* non-Abelian anomaly cancellation in terms of one chiral boundary system whose K matrix satisfies some additional constraints, besides the one discussed in Sec. VI.B [Eqs. (6.77) and (6.78)].

First, let h be an SU(2)-valued function supported on the boundary $\partial\Lambda_0$, and let $\tilde{h}$ be a smooth extension of h to the entire space-time Λ_0. Then the gauged SU(2) Wess-Zumino-Novikov-Witten model at level k (see, for example, Gawedzki, 1990) is defined by the action

$$\frac{1}{\hbar}S_{\mathrm{WZNW}}(h;w|_{\partial\Lambda_0})=-\frac{k}{4\pi}\int_{\partial\Lambda_0}\mathrm{tr}[(h^{-1}\partial_- h)(u)(h^{-1}\partial_+ h)(u)]d^2u-2\pi k\, n(\tilde{h};\Lambda_0)$$
$$+\frac{k}{2\pi}\int_{\partial\Lambda_0}\mathrm{tr}(w_-(u)[(h^{-1}\partial_+ h)(u)-w_+(u)]+w_+(u)[(h\partial_- h^{-1})(u)+(hw_-h^{-1})(u)])d^2u$$
$$+\frac{k}{4\pi}\int_{\partial\Lambda_0}\mathrm{tr}[w_-(u)w_+(u)]d^2u\ , \tag{6.122}$$

with notations as above. [Note that, for different extensions $\tilde{h}$ of a given h, the terms $n(\tilde{h};\Lambda_0)$ differ at most by an integer corresponding to a winding number difference [see Eq. (6.102)] of the two extensions (Witten, 1984; see also Novikov, 1982). This observation actually leads to another proof of the integral quantization of k.]

Similarly to the Abelian situation in Sec. VI.B, we have to consider only the left-chirality or the right-chirality sector of the theory specified in Eq. (6.122) if we wish to cancel the anomaly in Eq. (6.120). What is the correct non-Abelian chirality constraint generalizing the one in (6.47)? By the form of the action (6.122), we are led to the constraint

$$h(u)\partial_- h^{-1}(u)+h(u)w_-(u)h^{-1}(u)=0\ . \tag{6.123}$$

We note that Eq. (6.123) is invariant under SU(2) gauge transformations

$$\tilde{h}\mapsto {}^g\tilde{h}=\tilde{h}g^{-1}\ ,$$
$$w\mapsto {}^g w=gwg^{-1}+g\,dg^{-1}\ , \tag{6.124}$$

where $h=\tilde{h}|_{\partial\Lambda_0}$. The partition function $\zeta^L_{\partial\Lambda_0}(k_L;w|_{\partial\Lambda_0})$ of the (left) chiral, gauged SU(2) WZNW model at level k is given by

$$\zeta^L_{\partial\Lambda_0}(k;w|_{\partial\Lambda_0})=\int \mathcal{D}h\,\exp\left[\frac{i}{\hbar}S_{\mathrm{WZNW}}(h;w|_{\partial\Lambda_0})\right]$$
$$\times\delta(h\partial_- h^{-1}+hw_-h^{-1})\ . \tag{6.125}$$

We note that the chiral current $J_+(w_+=0)$ $=(l/2\pi)h^{-1}\partial_+ h$ of the theory (6.125) generates a chiral $\widehat{\mathrm{su}}(2)$ current (Kac-Moody) algebra at level k denoted by $\widehat{\mathrm{su}}(2)_k$; see Knizhnik and Zamolodchikov (1984), Witten (1984), Gepner and Witten (1986), Felder, Gawedzki, and Kupiainen (1988), Salomonson and Skagerstam (1989), and Harada (1990b).

Using the gauge invariance of the Haar measure in Eq. (6.125), i.e., $\mathcal{D}^g h=\mathcal{D}h$, one verifies that, under SU(2) gauge transformations (6.124), the chiral effective action $(\hbar/i)\ln\zeta^L_{\partial\Lambda_0}(k;w|_{\partial\Lambda_0})$ associated with Eq. (6.125) exhibits an anomaly of exactly the same form as in (6.120) but of opposite sign. In the proof one makes use of the Polyakov-Wiegmann identity (Polyakov and Wiegmann, 1984; see also Gawedzki, 1990):

$$\Gamma(gh)=\Gamma(g)+\Gamma(h)$$
$$-\frac{1}{2\pi}\int_{\partial\Lambda_0}\mathrm{tr}[(g^{-1}\partial_+ g)(u)(h\partial_- h^{-1})(u)]d^2u\ , \tag{6.126}$$

where

$$\Gamma(h)=\frac{1}{4\pi}\int_{\partial\Lambda_0}\mathrm{tr}[(h^{-1}\partial_- h)(u)(h^{-1}\partial_+ h)(u)]d^2u$$
$$+2\pi\, n(\tilde{h};\Lambda_0)\ . \tag{6.127}$$

Finally, as for the Abelian situation in Sec. VI.B, there is a symmetry between the left- and right-moving degrees of freedom in the SU(2) WZNW model (6.122). Hence replacing k by $-k$ and $+$ by $-$ on the rhs of Eq. (6.125), we find the partition function $\zeta^R_{\partial\Lambda_0}(k;w|_{\partial\Lambda_0})$ of the (right) chiral, gauged SU(2) WZNW model at level k. This time the anomaly is of the same sign as that in Eq. (6.120).

Thus, for two integers $k_L, k_R \geq 0$, we infer from the discussion above that the factor

$$Z^{\rm CS}_{\Lambda_0}(w)=\exp\left[\frac{i}{\hbar}S^{\rm CS}_{\Lambda_0}(w)\right]\zeta^L_{\partial\Lambda_0}(k_L;w|_{\partial\Lambda_0})\,\zeta^R_{\partial\Lambda_0}(k_R;w|_{\partial\Lambda_0}) \tag{6.128}$$

in the total partition function (6.100) is SU(2) gauge invariant, provided that

$$k=k_L-k_R \quad (\in\mathbb{Z})\ . \tag{6.129}$$

In the remaining part of this section we concentrate on the left-moving degrees of freedom with dynamics specified by Eq. (6.125); the discussion for the right-moving ones is implied by the symmetry above.

From the representation theory of (chiral) current (Kac-Moody) algebra we recall that, for $k>1$, there are representations of $\widehat{\mathrm{su}}(2)_k$ [see Eq. (6.125)] which exhibit excitations obeying non-Abelian braid statistics (Fredenhagen, Rehren, and Schroer, 1989; Fröhlich and Gabbiani, 1990; Fröhlich, Gabbiani, and Marchetti, 1990; Fröhlich and Marchetti, 1991; see also the remarks about spinon quantum mechanics in Sec. V.C). This raises the question whether such representations can be realized in (electronic) quantum Hall fluids. A consistency analysis of this question, which also accounts for the possibility of internal symmetries (see the remark at the end of Sec. III.C), suggests that, in quantum Hall fluids, only representations generated by U(1) currents, i.e., given in terms of the vertex operator construction, are realized (Fröhlich and Thiran, 1993).

We recall that, by the vertex operator construction (Frenkel, 1981; Goddard and Olive, 1986), it is possible to give an explicit realization of a (chiral) $\widehat{\mathrm{su}}(N)$ current algebra at level 1 in terms of $N-1$ free, massless chiral Bose fields, $\chi_1,\dots,\chi_{N-1}$.

Given this fact, we may ask whether there is a unified way of treating the problem of Abelian [see Eqs. (6.108)–(6.111)] *and* non-Abelian [see Eqs. (6.128) and (6.129)] anomaly cancellation in terms of one chiral boundary system of the form given in (6.108). We proceed in two steps: First, we establish the form that the K matrix of a (left) chiral boundary system has to have in order for the system to exhibit an $\widehat{\mathrm{su}}(N)$ current algebra at level 1. Second, for the particular situation of $N=2$, we give an explicit description of the coupling structure of the corresponding chiral boundary system to the external SU(2) gauge field $w|_{\partial\Lambda_0}$ for it to generate the same non-Abelian anomaly as the model system presented in Eq. (6.125) (in addition, of course, to the Abelian anomaly arising from the coupling of the system to the Abelian gauge field $\hat{\alpha}=\hat{Q}\tilde{\alpha}|_{\partial\Lambda_0}+\hat{\delta}\tilde{w}_3|_{\partial\Lambda_0}$).

A discussion of the first step can be found in Fröhlich and Zee (1991). It amounts to studying chiral boundary systems exhibiting a full unitary group U(N) of symmetries permuting N bands of edge currents such that the algebra of edge currents contains a current subalgebra $\widehat{\mathrm{su}}(N)$ at level 1. Clearly, the unitary symmetry we are requiring here is much larger than the permutation symmetry we considered in assumption (A3) [see Eqs. (6.85)–(6.88)]. Correspondingly, the K matrices of chiral boundary systems compatible with this larger symmetry are more constrained. In a symmetric basis where $\hat{Q}=(1,\dots,1)^T$, they are of the form

$$K=\mathbf{1}_N+2lN\mathbf{P}_N \tag{6.130}$$

for some $l\in\mathbb{N}_0$; see Eq. (6.88) for notations. By Eq. (6.90), the corresponding Hall constant is given by

$$\sigma_K=\frac{N}{2lN+1}\ . \tag{6.131}$$

In order to prove that the presence of an $\widehat{\mathrm{su}}(N)_1$ current algebra implies Eq. (6.130), it is useful first to show that there is a matrix $S\in\mathrm{GL}(N;\mathbb{Z})$ (S has components 1 along the diagonal, -1 along the first upper off-diagonal, and 0 elsewhere) such that

$$K'=S^TKS=\begin{bmatrix}2l+1 & -1 & 0 & \cdots & 0\\ -1 & 2 & -1 & \ddots & \vdots\\ 0 & -1 & 2 & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & -1\\ 0 & \cdots & 0 & -1 & 2\end{bmatrix}$$

and (6.132)

$$\hat{Q}'=S^T\hat{Q}=\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix},$$

where $(K'_{ij})^N_{i,j=2}$ is the Cartan matrix of su(N), i.e., $K'_{i+1,j+1}=\hat{\beta}^{(i)}\cdot\hat{\beta}^{(j)}$, $i,j=1,\dots,N-1$, where the $\hat{\beta}^{(j)}$'s are the simple roots of su(N). Second, one determines $N-1$ linear combinations χ_m, $m=1,\dots,N-1$, of the N chiral Bose fields $\phi'_{L,1},\dots,\phi'_{L,N}$ [forming the basis in which (6.132) holds] by solving

$$\hat{\beta}^{(j)}\cdot\hat{\chi}(u_+)=\sum_{m=1}^{N-1}\beta^{(j)}_m\chi_m(u_+)=(K'\hat{\phi}'_L)_{j+1}(u_+)$$
$$\equiv\hat{n}^{(j)\prime}\cdot\hat{\phi}'_L(u_+)\ . \tag{6.133}$$

Then one verifies that the Fourier(-Laurent) coefficients of the currents $(l/2\pi)\partial_+\chi_m$, $m=1,\dots,N-1$ [see Eqs. (6.37) and (6.38)], generate the Cartan subalgebra of $\widehat{\mathrm{su}}(N)_1$; moreover, one verifies that the chiral vertex operators $V_{\pm\hat{n}(j)'}\equiv:e^{\pm i\hat{\beta}^{(j)}\cdot\hat{\chi}}:=:e^{\pm i\hat{n}^{(j)\prime}\hat{\phi}'_L}:$, $j=1,\dots,N-1$, create finite-energy excitations and that, up to some constant "cocycles," their Fourier(-Laurent) coefficients provide the remaining step operators in the current algebra $\widehat{\mathrm{su}}(N)_1$ (Fröhlich and Zee, 1991; see also Goddard and Olive, 1986). We emphasize that the chiral vertex operators $V_{\hat{n}^{(j)\prime}}$, $j=1,\ldots,N-1$, do not generate charge, i.e., $Q(\hat{n}^{(j)\prime})=0$; see Eq. (6.81). This reflects the fact that the chiral Bose fields χ_m, $m=1,\ldots,N-1$, do not couple to the U(1) gauge field $\tilde{a}|_{\partial\Lambda_0}$; see, for example, Eq. (6.135) below.

The construction above may be summarized as follows: Given the integral flux lattice $\Phi=\mathbb{Z}^N$, equipped with the quadratic form $(\hat{n}_1,\hat{n}_2)\mapsto\hat{n}_1\cdot K^{-1}\hat{n}_2$, where K is of the form (6.130), we have shown that there exists a neutral sublattice $\Sigma\subset\Phi$, i.e., $Q(\hat{n})=0$, for all $\hat{n}\in\Sigma$, generated by $\hat{n}^{(j)}=(S^T)^{-1}\hat{n}^{(j)\prime}$, $j=1,\ldots,N-1$, which forms the root lattice of su(N), i.e., $\hat{n}^{(i)}\cdot K^{-1}\hat{n}^{(j)}=\vec{\beta}^{(i)}\cdot\vec{\beta}^{(j)}$, $i,j=1,\ldots,N-1$. This point of view provides a natural starting point for a systematic discussion of symmetry properties of chiral boundary systems. For a detailed analysis along these lines, see Fröhlich and Thiran (1993).

We note that, in connection with quantum Hall fluids, the matrix K' in (6.132) first appeared in Read (1990). Furthermore, quantum Hall fluids with K matrices of the form given in (6.130) correspond to Jain's states (Jain, 1989a, 1989b).

Next, we turn to the question of how to couple chiral boundary systems to Abelian *and* non-Abelian external gauge fields. For definiteness, we treat a two-band (left) chiral boundary system characterized by $(K',\hat{Q}',\hat{\delta}')$, which we couple to the external U(1) gauge field $\tilde{a}|_{\partial\Lambda_0}$ and to the full SU(2) gauge field $w|_{\partial\Lambda_0}$. According to the discussion above we choose

$$K'=\begin{bmatrix}2l+1 & -1\\ -1 & 2\end{bmatrix},\quad \hat{Q}'=\begin{bmatrix}1\\0\end{bmatrix},$$

and (6.134)

$$\hat{\delta}'=\begin{bmatrix}-1\\2\end{bmatrix}.$$

That the polarization vector $\hat{\delta}'$ $[=S^T\hat{\delta}$, with

$$S=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix};$$

see Eq. (6.118)] is chosen correctly can be seen by calculating the j_3 value of electronic excitations in the first and second bands of the system, $\hat{q}_{\rm el}^{(1)\prime}=(1,0)$ $(=S^{-1}\hat{q}_{\rm el}^{(1)})$, and $\hat{q}_{\rm el}^{(2)\prime}=(1,1)$ $(=S^{-1}\hat{q}_{\rm el}^{(2)})$, respectively; see Eq. (6.84) and assumption (A1) in Sec. VI.B. The analysis above shows that the chiral current algebra $\widehat{\rm su}(2)_1$ exhibited by the system is generated by $J_+^3=(1/\sqrt{2})(l/2\pi)\partial_+\chi$ and $J_+^{\pm}=J_+^1\pm iJ_+^2=:e^{\pm i\sqrt{2}\chi}:$ $=:e^{\pm i\hat{n}^{(1)\prime}\cdot\vec{\phi}'_L}:=V_{\pm\hat{n}^{(1)\prime}}$, with $\hat{n}^{(1)\prime}=(-1,2)^T$. Then the j_3 value of an excitation created by the chiral vertex operator $V_{\pm\hat{n}'}$, with $\hat{n}'\in\Phi'=\mathbb{Z}^N$, is given by $2j_3=\hat{n}^{(1)\prime}\cdot K'^{-1}\hat{n}'=\hat{n}^{(1)\prime}\cdot\hat{q}'$. Hence $\delta_1=2j_3(\hat{q}_{\rm el}^{(1)\prime})=-1$ and $\delta_2=2j_3(\hat{q}_{\rm el}^{(2)\prime})=1$.

Now we define the partition function, $\zeta^L_{\partial\Lambda_0}(K';\tilde{a}|_{\partial\Lambda_0},w|_{\partial\Lambda_0})$, of a (left) chiral boundary system coupled to $\tilde{a}|_{\partial\Lambda_0}$ *and* to $w|_{\partial\Lambda_0}$:

$$\begin{aligned}\zeta^L_{\partial\Lambda_0}(K';\tilde{a}|_{\partial\Lambda_0},w|_{\partial\Lambda_0})=&\int\mathcal{D}\phi'_1\exp\Bigg\{\frac{i}{4\pi}(K'_{11}-\tfrac{1}{2})\int_{\partial\Lambda_0}\partial_-\phi'_1(u)\partial_+\phi'_1(u)d^2u\\&-\frac{i}{2\pi}\int_{\partial\Lambda_0}\partial_+\phi'_1(u)\tilde{a}_-(u)d^2u+\frac{i}{4\pi}(K'^{-1})_{11}\int_{\partial\Lambda_0}\tilde{a}_-(u)\tilde{a}_+(u)d^2u\Bigg\}\\&\times\delta(\partial_-\phi'_1-(K'^{-1})_{11}\tilde{a}_-)\\&\times\int\mathcal{D}\chi\exp\Bigg\{\frac{i}{4\pi}\int_{\partial\Lambda_0}\partial_-\chi(u)\partial_+\chi(u)d^2u\\&-\frac{i}{2\pi}\int_{\partial\Lambda_0}\Bigg[\frac{2\pi}{l}\{w_-^+(u):e^{-i\sqrt{2}\chi}:(u)+w_-^-(u):e^{i\sqrt{2}\chi}:(u)\}\\&+\sqrt{2}w_-^3(u)\partial_+\chi(u)\Bigg]d^2u\\&-\frac{i}{4\pi}\int_{\partial\Lambda_0}\mathrm{tr}(w_-(u)w_+(u))d^2u\Bigg\}\delta(\partial_-\chi-\sqrt{2}w_-^3)\\\equiv&\exp\left[\frac{i}{\hbar}S^{\rm eff}_{\partial\Lambda_0}(\tilde{a}|_{\partial\Lambda_0})\right]\exp\left[\frac{i}{\hbar}S^{\rm eff}_{\partial\Lambda_0}(w|_{\partial\Lambda_0})\right],\end{aligned}\tag{6.135}$$

where w_-^A, $A=1,2,3$, are the su(2) components of w_- [see Eq. (6.121)], and $w_-^{\pm}=w_-^1\pm iw_-^2$.

Recalling that Eq. (6.135) is expressed in a basis ϕ'_1,ϕ'_2 where Eq. (6.134) holds, and that $\sqrt{2}\chi=-\phi'_1+2\phi'_2$, we notice that, for $w_-^{\pm}\equiv 0$, the partition function in (6.135) reduces to that considered in Eqs. (6.58) or (6.108). Hence, under the U(1) gauge transformations defined in Eqs. (6.22) and (6.106), the theory in (6.135) reproduces the required Abelian anomaly canceling that in Eq. (6.107), and the identities (6.109)–(6.111) (restricted to the left-chirality sector) hold. [Note that under $U(1)_{spin}$ gauge transformations we have $\chi \mapsto \chi + \sqrt{2}\lambda$, $w_-^3 \mapsto w_-^3 + \partial_- \lambda$, and $w_-^{\pm} \mapsto e^{\pm i\lambda} w_-^{\pm}$.]

In the following we show that the second factor on the rhs of Eq. (6.135), $\exp[(i/\hbar) S^{\text{eff}}_{\partial\Lambda_0}(w|_{\partial\Lambda_0})]$, reproduces the correct non-Abelian anomaly under SU(2) gauge transformations (6.97), canceling that in Eq. (6.120) (with coefficient $k=1$). For this purpose, we show that the effective action $S^{\text{eff}}_{\partial\Lambda_0}(w|_{\partial\Lambda_0})$ equals that determined by the theory defined in (6.125), i.e., we show that

$$\frac{\hbar}{i}\ln \zeta^{L}_{\partial\Lambda_0}(k=1; w|_{\partial\Lambda_0}) = S^{\text{eff}}_{\partial\Lambda_0}(w|_{\partial\Lambda_0}) \ . \tag{6.136}$$

By inspection of expressions (6.125) and (6.135), we find that, upon adding the "finite renormalization" term $\text{F.R.}(w|_{\partial\Lambda_0}) \equiv (\hbar/4\pi)\int_{\partial\Lambda_0} \text{tr}(w_- w_+) d^2u$ to both sides of Eq. (6.136), the two sides in (6.136) depend *only* on w_-. In particular, we note that on the lhs of Eq. (6.136) the coupling of the chiral SU(2) WZNW field h to w_- is given by

$$\frac{1}{2\pi}\int_{\partial\Lambda_0} \text{tr}[w_-(u)(h^{-1}\partial_+ h)(u)] d^2u = -\frac{1}{l}\int_{\partial\Lambda_0} [w_-^{+}(u) J_+^{-}(u) + w_-^{-}(u) J_+^{+}(u) + 2 w_-^3(u) J_+^3(u)] d^2u \ , \tag{6.137}$$

where $J_+^{\pm} = J_+^1 \pm i J_+^2$ and J_+^A, $A=1,2,3$, are the su(2) components of the chiral current $J_+(w_+=0) = (l/2\pi) h^{-1}\partial_+ h$ [see Eq. (6.36)].

Next, we expand the lhs of Eq. (6.136) [without the term $\text{F.R.}(w|_{\partial\Lambda_0})$] in a functional Taylor series in $w_-^{\pm}$ and w_-^3 around $w_-=0$. The coefficient of the term of lth order in w_-^{+}, mth order in w_-^{-}, and nth order in w_-^3 is given by the following connected, time-ordered Green function of chiral su(2) currents:

$$\left\langle T\left[\prod_{i=1}^{l} J_+^{-}(u_i) \prod_{j=1}^{m} J_+^{+}(u_j) \prod_{k=1}^{n} J_+^3(u_k)\right]\right\rangle^{\text{SU(2)}}_{w_-=0} \ ; \tag{6.138}$$

compare this to Sec. V.A and also Appendix A.

Similarly, expanding the rhs of Eq. (6.136) [without the term $\text{F.R.}(w|_{\partial\Lambda_0})$] in a (functional) Taylor series in $w_-^{\pm}$ and w_-^3 around $w_-=0$, we find that the corresponding coefficients are given by the Green functions

$$\left\langle T\left[\prod_{i=1}^{l} :e^{-i\sqrt{2}\chi}:(u_i) \prod_{j=1}^{m} :e^{i\sqrt{2}\chi}:(u_j) \prod_{k=1}^{n} \frac{1}{\sqrt{2}}\frac{l}{2\pi}\partial_+\chi(u_k)\right]\right\rangle^{\text{Gauss}}_{w_-=0} \ . \tag{6.139}$$

Finally, we note that, by the vertex operator construction (Frenkel, 1981; Goddard and Olive, 1986), the Green functions in (6.138), calculated in the chiral SU(2) WZNW model at level $k=1$, are identical to those in Eq. (6.139), calculated in the Gaussian model of a *free*, massless chiral Bose field χ, for $w_-=0$. This proves Eq. (6.136).

Applications of the general constructions presented in this section are given in the next section. We discuss electronic quantum Hall fluids that are good candidates for observing a (fractional and integer) quantum Hall effect for spin currents, and we propose a (fractional) "tidal" quantum Hall effect in a superfluid ^{3}He-A/B interface with broken parity and time-reversal invariance.

VII. NEW GENERALIZED QUANTUM HALL EFFECTS

In this last section, we illustrate the general constructions given in Sec. VI by discussing some new effects that might be observed in two-dimensional quantum fluids. We study a quantum Hall effect for spin currents in electronic quantum Hall fluids, e.g., with $\sigma=\frac{2}{5}$ or $\frac{2}{9}$, exhibiting both spin-unpolarized (spin-singlet) *and* spin-polarized ground states depending on the strength of the applied, external magnetic field. Second, we discuss an even-denominator quantum Hall effect in a recently discovered superfluid ^{3}He-A/B interface with broken parity and time-reversal invariance.

A. Quantum Hall effect for spin currents in polarized and unpolarized Hall fluids

In this section we consider two-dimensional, incompressible quantum Hall fluids exhibiting a plateau in their Hall conductivity, $\sigma_H = \sigma(e^2/h)$, for

$$\sigma = \frac{2}{4l+1}, \quad \text{with } l = 0,1,2,(3,\dots) \ . \tag{7.1}$$

One motivation for studying these Hall fluids stems from the interesting experimental observations made by Eisenstein *et al.* (1989, 1990a). In their experiments evidence was found for a phase transition at $\sigma=\frac{8}{5}$, from a fractional quantum Hall fluid whose ground state is spin-unpolarized to one with a fully spin-polarized ground state. This phase transition is driven by tilting the two-dimensional Hall sample relative to the external magnetic field, keeping the filling factor (or the Hall conductivity) the same, but increasing the Zeeman energy for the constituting particles. By the electron-hole symmetry discussed in Eqs. (6.108)–(6.111), (6.128), and (6.129), we may discuss the Hall effect at $\sigma=\frac{8}{5}=2-\frac{2}{5}$ and the observed phase transition by considering spin-unpolarized and spin-polarized quantum Hall fluids at $\sigma=\frac{2}{5}$, a value contained in the list (7.1).

Below, we present different realizations of quantum Hall fluids with $\sigma=2/(4l+1)$ consistent with spin-unpolarized or fully spin-polarized ground states. We show that these different realizations display quantitatively distinct quantum Hall effects for their spin currents. For different points of view on such quantum Hall fluids showing spin-unpolarized ground states, we refer the reader to Halperin (1983), Chakraborty and Zhang (1984a, 1984b), Rasolt, Perrot, and MacDonald (1985), Yoshioka (1986b), and Maksym (1989).

First, we present a realization of a Hall fluid with $\sigma=2/(4l+1)$ consistent with a *spin-singlet* ground state. According to the discussion above, the simplest (left) chiral boundary system allowing for spin-singlet excitations is given in Eq. (6.135), with K', $\hat{Q}'$, and $\hat{\delta}'$ as in Eq. (6.134), i.e., in a "symmetric" basis, we consider a system with band-coupling matrix

$$K=\begin{bmatrix} 2l+1 & 2l \\ 2l & 2l+1 \end{bmatrix},$$

charge vector $\hat{Q}=(1,1)^T$, and polarization vector $\hat{\delta}=(-1,1)^T$. By Eq. (6.136), a characteristic of the Hall fluid described here is the presence of an SU(2) Chern-Simons term with coefficient $k=1$ in its effective action (6.96).

Second, with the same choice of K and $\hat{Q}$ as above, but with $\hat{\delta}=(1,1)^T$ and $k=0$, we find a description of a quantum Hall fluid with $\sigma=2/(4l+1)$ and fully *spin-polarized* ground state. This fluid exhibits a $\widehat{su}(2)_1$ current algebra due to an *internal* SU(2) symmetry (not connected to spin).

Third, another description of a Hall fluid with $\sigma=2/(4l+1)$ and fully *spin-polarized* ground state is as follows: The system exhibits two independent, polarized edge current bands, i.e.,

$$K=\begin{bmatrix} 4l+1 & 0 \\ 0 & 4l+1 \end{bmatrix},$$

in a basis where $\hat{Q}=(1,1)^T$ and $\hat{\delta}=(1,1)^T$.

Fourth, we may also consider another realization of a *spin-unpolarized* Hall fluid with $\sigma=2/(4l+1)$ that is very similar to the one just given for the polarized situation: a two-band system with the same K matrix as above, consisting, however, of two oppositely polarized bands, i.e., the polarization vector is given by $\hat{\delta}=(-1,1)^T$. In this realization, the ground state happens to be unpolarized because the 3-component of the total spin of all the "spin-up" electrons in one band is compensated by that of all the "spin-down" electrons in the other band of the system; i.e., the ground state is unpolarized due to an "occupation-number" symmetry.

Clearly, from a dynamical point of view, the two realizations of a polarized/unpolarized ground state at $\sigma=2/(4l+1)$ discussed above differ in essential ways. This brings us to the question of whether we can find measurable quantities that allow for an experimental distinction between the two fluids. To answer this question, we recall the linear-response equations (5.48)–(5.54) described in Sec. V.B and Eqs. (6.109)–(6.111) and (6.136) for the constants σ, χ_s, σ_s, and k determining the response of a quantum Hall fluid quantitatively. In Table I, the predictions for these quantities are presented for the four inequivalent realizations of a quantum Hall fluid with $\sigma=2/(4l+1)$ discussed above.

For all four realizations, the spectra of possible fractional charges of quasiparticles are given by

$$\mathcal{J}=\{n/(4l+1)\,|\,n=1,\ldots,4l\}\ ; \tag{7.2}$$

see Eq. (6.82).

Next, we sketch some physical implications of the results in Table I. From Eqs. (5.50) and (5.52), we recall that χ_s determines the magnetic susceptibility of the

TABLE I. Linear-response parameters for different realizations of incompressible quantum Hall fluids with $\sigma_H=[2/(4l+1)](e^2/h)$.

Ground state	K	$\hat{Q}$	$\hat{\delta}$	σ	χ_s	σ_s	k
Unpolarized (spin singlet)	$\begin{bmatrix} 2l+1 & 2l \\ 2l & 2l+1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$	$\frac{2}{4l+1}$	0	2	1
Unpolarized	$\begin{bmatrix} 4l+1 & 0 \\ 0 & 4l+1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$	$\frac{2}{4l+1}$	0	2	0
Polarized (internal symmetry)	$\begin{bmatrix} 2l+1 & 2l \\ 2l & 2l+1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\frac{2}{4l+1}$	$\frac{2}{4l+1}$	$\frac{2}{4l+1}$	0
Polarized	$\begin{bmatrix} 4l+1 & 0 \\ 0 & 4l+1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\frac{2}{4l+1}$	$\frac{2}{4l+1}$	$\frac{2}{4l+1}$	0

ground state of the system (in the direction normal to the plane of the sample). Hence the vanishing of χ_s, for a spin-singlet ground state or an unpolarized ground state with an occupation-number symmetry, is expected. For both polarized Hall fluids, however, our discussion predicts a *rationally quantized* magnetic susceptibility.

A more interesting prediction can be inferred from the linear-response equation (5.54). For definiteness, let us consider a two-dimensional Hall system confined to the (1,2) plane in $\mathbb{E}^3$ that is incompressible in some uniform background field $\mathbf{B}_c$ perpendicular to the sample (3-direction) and that shows a plateau in the Hall conductivity at height $\sigma_H=[2/(4l+1)](e^2/h)$. If we perturb this system by an inhomogeneous magnetic field $\widetilde{\mathbf{B}}=(\widetilde{B}_1,\widetilde{B}_2,\widetilde{B}_3)$, then Eqs. (5.54) and (5.53) and their analogs for the su(2) components $A=1,2$ (Fröhlich and Studer, 1992b) predict the following quantum Hall effect for spin currents:

$$\langle \mathcal{S}^i_A(\xi)\rangle_{\mathbf{B}_c+\widetilde{\mathbf{B}}}=k\frac{g\mu_B}{4\pi}\sum_{j=1}^{2}\varepsilon^{ij}\partial_j\widetilde{B}_A(\xi)\quad \text{for } i,A=1,2\ , \tag{7.3}$$

$$\langle \mathcal{S}^i_3(\xi)\rangle_{\mathbf{B}_c+\widetilde{\mathbf{B}}}=\sigma_H^{\text{spin}}\sum_{j=1}^{2}\varepsilon^{ij}\partial_j\widetilde{B}_3(\xi)\ ,\quad \text{for } i=1,2\ , \tag{7.4}$$

where

$$\sigma_H^{\text{spin}}=(2k-\sigma_s)\frac{g\mu_B}{8\pi}\ . \tag{7.5}$$

For example, if $\sigma_H^{\text{spin}}\neq 0$ then, by Eq. (7.4), there exists, in regions of the (1,2) plane where the 3-component of the perturbing field $\widetilde{B}_3$ varies, a nonvanishing spin current (density) polarized along the 3-direction and flowing in the direction perpendicular to the (two-dimensional) gradient of $\widetilde{B}_3$. [For a straightforward, semiclassical derivation of Eq. (7.4) see Fröhlich and Studer (1992b).]

Equations (7.3)–(7.5) and Table I imply the following results: For an incompressible quantum Hall fluids with $\sigma=2/(4l+1)$ and spin-singlet ground state, the quantum Hall effect for the 3-component of the spin current (7.4) is absent ($k=1$, $\sigma_s=2$), while Eq. (7.3) predicts an *integer* ($k=1$) quantum Hall effect for the ($A=1,2$) components of the spin current. This is in contrast to the behavior of the Hall fluid exhibiting a spin-unpolarized ground state with an occupation-number symmetry. Here, we expect an *integer* quantum Hall effect for the 3-component of the spin current, while there is no effect for the other components of the spin current ($k=0$, $\sigma_s=2$). Finally, in the two incompressible quantum Hall fluids with $\sigma=2/(4l+1)$ realizing a fully spin-polarized ground state, Eqs. (7.4) and (7.5) predict a *fractional* quantum Hall effect for the 3-component of the spin currents, while there is again no similar effect for the other su(2) components of the spin currents ($k=0$, $\sigma_s=2/(4l+1)$). It would prove very interesting if an experimental observation of these predictions became possible!

In conclusion, we note that one straightforward generalization of the results above is obtained by combining them according to the hierarchy construction given in Eqs. (6.93) and (6.94). We refer the reader to Fröhlich and Thiran (1993) for discussions of many more examples.

B. Even-denominator quantum Hall effect in a ^{3}He-A/B interface with broken symmetries

In this section we study a superfluid ^{3}He-A/B interface realizing a superfluid film in which (two-dimensional) parity (P) and time-reversal symmetry (T) are broken. We argue that such a superfluid film may represent another example of a system (besides electronic quantum Hall fluids) in which the techniques of Secs. VI.B and VI.C yield results. In particular, we describe a fractional "tidal" quantum Hall effect for the mass current in this system. An integer quantum Hall effect for the mass and spin current in a superfluid ^{3}He-A film of different origin has been discussed by Volovik (1988) and by Volovik and Yakovenko (1989).

A recent analysis by Salomaa and Volovik (1989) of some NMR experiments performed by Hakonen and Nummila (1987) provides strong evidence for the following phenomenon: Consider a rotating cylindrical vessel filled with superfluid ^{3}He-B at a particular pressure and temperature and subject to a magnetic field $\mathbf{B}_c$ directed along the axis of the cylinder. One finds that, if the rotation is performed in such a manner that the bulk of the superfluid remains free of vortices, an interesting surface structure forms at the wall of the vessel. Namely, between the wall of the cylinder and the ^{3}He-B bulk, a thin film of superfluid ^{3}He-A with broken P and T invariance forms.

In order to elucidate some properties of such a ^{3}He-A/B interface (surface structure), we describe this (essentially two-dimensional) system in moving coordinates, as explained in Sec. III.C. For simplicity, we consider a sample of such a superfluid film to be confined to a rectangular region Z in the (1,2) plane of Euclidean space $\mathbb{E}^3$. We denote by $\mathbf{v}_s=(v_s^1,v_s^2)$ the velocity field describing the flow of the superfluid film relative to the fixed coordinates (x^1,x^2) of the (1,2) plane. Then the transformation to coordinates adapted to the flow of the superfluid film is generated by the (two-dimensional) vector field $\mathbf{f}=\mathbf{v}_s$; see Eq. (3.43).

According to the discussion in Sec. III.C [see, in particular, Eq. (3.55)], the two-dimensional superfluid system is then coupled to an effective U(1) gauge field a given by

$$a_\mu(x)=-\frac{M_p}{\hbar}(u(x),v_{s,1}(x),v_{s,2}(x))\ , \tag{7.6}$$

where u denotes the chemical potential of the system and M_p is the mass of the (quasi-)particles constituting the superfluid. Since the superfluid state of ^{3}He is formed, at very low temperatures ($\lesssim 3$ mK), by Bose-Einstein condensation of Cooper pairs of ^{3}He atoms, we set M_p equal to the mass of a ^{3}He *pair.* [Note that since the constituent ^{3}He pairs are neutral there are no electromagnetic terms in Eq. (7.6).]

Furthermore, we recall that, in superfluid ^{3}He, the spins of two ^{3}He atoms forming a Cooper pair arrange in a spin-triplet state. For some general references on superfluid ^{3}He, see Leggett (1975), Vollhardt and Wölfle (1990), and Volovik (1991). Given the fact that the constituent Cooper pairs of superfluid ^{3}He are bosons with nonzero spin, we may also consider the superfluid condensate to be coupled to some external SU(2) gauge field w. This gauge field is a function of the superfluid flow $\mathbf{v}_s$ and of the magnetic field $\mathbf{B}_c$ coupling to the magnetic moments of the ^{3}He pairs. The explicit expressions for the components of w depend on the orientation of $\mathbf{B}_c$ relative to the sample and on the particular choice of basis (orthonormal frames) in spin (or tangent) space. Because we do not need these expressions, we do not display them explicitly. Actually, we emphasize that the mere possibility of coupling the system to an SU(2) gauge field is sufficient for the following discussion.

Attempting to describe such a ^{3}He-A/B interface by specifying its effective action as discussed in the previous sections, we make one basic *assumption:* Denoting by $S^{\text{eff}}_{PT}(a,w)$ the P- and T-breaking terms in the effective action of the system, we assume that the bulk contributions to $S^{\text{eff}}_{PT}(a,w)$ are given by Chern-Simons terms in a and w, i.e., we assume that

$$S^{\text{eff}}_{PT}(a,w)=\frac{\sigma}{4\pi}\int_\Lambda a\wedge da +\frac{k}{4\pi}\int_\Lambda \operatorname{tr}(w\wedge dw+\tfrac{2}{3}w\wedge w\wedge w) +\text{B.T.}(a|_{\partial\Lambda},w|_{\partial\Lambda}) , \tag{7.7}$$

where Λ denotes the space-time domain $\mathbb{R}\times Z$, and $\text{B.T.}(a|_{\partial\Lambda},w|_{\partial\Lambda})$ stands for the by now familiar boundary terms localized at the edge of the rectangle Z (required by anomaly cancellation). These terms describe mass and spin currents circulating at the edge; see Sec. VI and the discussion below. [It would be interesting to investigate whether, under the conditions satisfied in the experiments mentioned above, such ^{3}He-A/B surface structures exhibit incompressibility in the sense specified in Sec. V.A. This would be a first step towards justifying our ansatz for the P- and T-breaking terms given in Eq. (7.7).]

Defining the mass current density $(\rho_m,\mathcal{J}_m)$ as in Eqs. (5.44) and (5.45), replacing the charge q by the pair mass M_p, we find, similarly to Eqs. (5.48) and (5.51), that the following linear-response equations hold ("tidal" Hall effect):

$$\langle \rho_m(x)\rangle_{a,w}=\sigma\frac{M_p^2}{h}\operatorname{curl}\mathbf{v}_s(x) , \tag{7.8}$$

and

$$\langle \mathcal{J}^i_m(x)\rangle_{a,w}=\sigma\frac{M_p^2}{h}\varepsilon^{ij}\partial_j u(x) . \tag{7.9}$$

By integrating Eq. (7.8) over the rectangle Z, we infer that through some (localized) perturbation of the superfluid flow, $\mathbf{v}_s\mapsto\mathbf{v}_s+\tilde{\mathbf{v}}_s$, one may produce an excitation in the system of mass $\tilde{m}$ given by

$$\tilde{m}=\sigma\frac{\tilde{\phi}}{\phi_0} , \tag{7.10}$$

where $\tilde{\phi}=\int_Z \operatorname{curl}\tilde{\mathbf{v}}_s d^2x$, and $\phi_0=h/M$ is the circulation quantum in our system; see Sec. IV.B. Similarly, integrating Eq. (7.9), for $i=1$, along the 2-axis, one finds that

$$I^1_m=\sigma\frac{M_p^2}{h}\Delta_2 u , \tag{7.11}$$

where I^1_m is a mass current flowing in the 1-direction, as a result of a difference $\Delta_2 u$ between the chemical potentials at the rectangle's lower and upper edges in the 2-direction.

Our next task is to find the possible (quantized) values of the Hall constant σ and of the constant k. For this purpose, we present a consistency analysis of the system described by the action (7.7) that parallels the one for electronic quantum Hall fluids presented in Sec. VI.

The basic physical requirement on the model system corresponding to Eq. (7.7) is that it exhibit an excitation spectrum which contains (i) bosonic excitations of mass M_p describing the constituent ^{3}He Cooper pairs and (ii) spin-$\frac{1}{2}$ excitations of mass $M_p/2$ that may be identified with ^{3}He atoms. As we have shown in Sec. VI, the possible excitation spectrum of the model system (7.7) can be found by studying chiral boundary systems that give rise to boundary terms $\text{B.T.}(a|_{\partial\Lambda},w|_{\partial\Lambda})$ which cancel the gauge anomalies of the Abelian and non-Abelian Chern-Simons term in Eq. (7.7).

Here, we discuss the simplest chiral boundary systems fulfilling requirements (i) and (ii) above. We note that, since we wish to describe both bosonic [$s=0$ (mod 1)] and fermionic [$s=\frac{1}{2}$ (mod 1)] excitations, we need to consider chiral boundary systems that exhibit an SU(2) symmetry [or, equivalently, we require $k\neq 0$ in the effective action (7.7)]. The simplest system realizing this is a two-band system whose 2×2 coupling matrix K and "mass vector" $\hat{M}$ [instead of the charge vector $\hat{Q}$; compare Eqs. (3.36) and (7.6)] are of such a form that the algebra of edge currents contains an $\widehat{su}(2)$ current algebra at level $k=1$. We now determine the form of the pair $(K,\hat{M})$ by following, for two-dimensional, P- and T-breaking superfluid systems, the discussion given after Eq. (6.133) for two-dimensional, incompressible electronic systems.

We start with a positive 2×2 matrix K. Then the *evenness* of the diagonal elements K_{ii}, $i=1,2$, is a consequence of the fact that the constituent particles in the superfluid film (^{3}He pairs) are *bosons.* This is in contrast to the situation in Secs. VI.B and VI.C, where we considered fermions (electrons/holes) as the elementary excitations in the system which led to odd K_{ii}. The integrality of K_{12} follows as in Eq. (6.72). Next, we denote by Φ the "circulation lattice" of finite-energy excitations, i.e., of excitations that are relatively local to the constituent ^{3}He-pair excitations in the two edge current bands of the system. Similarly to Eqs. (6.74) and (6.75), one finds that $\Phi=\mathbb{Z}^2$. We recall that characterizing a finite-energy excitation by a "circulation vector" $\hat{n}\in\Phi$ [see the discussion after Eq. (6.68) and compare with (7.10)], its mass (in units of the ^{3}He-pair mass M_p) is given by

$$M(\hat{n})=\hat{M}\cdot K^{-1}\hat{n}\equiv\hat{M}\cdot\hat{m} \tag{7.12}$$

[compare with Eq. (6.81)], and its statistics phase is given by

$$\theta(\hat{n})=\hat{n}\cdot K^{-1}\hat{n}\equiv\hat{m}\cdot K\hat{m} \pmod 2 ; \tag{7.13}$$

see Eq. (6.83).

In the remaining discussion, it is convenient to work in a basis (of fields forming the chiral boundary system) where the mass vector $\hat{M}$ takes the form $\hat{M}=(1,0)^T$. [Notice that the evenness of the diagonal elements of K is invariant under GL(2;$\mathbb{Z}$) equivalence transformations (6.84).]

Setting $K_{22}=2$, one allows for the existence of a "massless" (one-dimensional) sublattice $\Sigma\subset\Phi$, which forms the root lattice of su(2); i.e., for $\hat{n}\in\Sigma$, we have $M(\hat{n})=0$. Moreover, Σ is generated by the circulation vector $\hat{\alpha}\equiv\hat{n}_\alpha=(K_{12},2)^T$, with $\hat{\alpha}\cdot K^{-1}\hat{\alpha}=2$; see the discussion after Eq. (6.134).

Finally, acting on K with GL(2;$\mathbb{Z}$) equivalence transformations S that leave the above mass vector $\hat{M}$ invariant [see Eq. (6.84), with

$$S=\begin{bmatrix}1 & 0\\ z & \pm 1\end{bmatrix},$$

where $z\in\mathbb{Z}$], one infers that K_{12} can be restricted to the values 0 or 1. In order to determine K_{12}, we may proceed as follows: Let us choose, similarly to the discussion after Eq. (6.134), the "mass vectors" ($\hat{m}=K^{-1}\hat{n}$) of ^{3}He-pair excitations in the first and second edge current bands to be given by $\hat{m}_p^{(1)}=(1,0)^T$ and $\hat{m}_p^{(2)}=(1,1)^T$, respectively. Then recalling that the j_3 value of an excitation with $\hat{n}\in\Phi$ is given by

$$2j_3(\hat{n})=\hat{\alpha}\cdot K^{-1}\hat{n}\equiv\hat{\alpha}\cdot\hat{m} , \tag{7.14}$$

and requiring consistency with the fact that ^{3}He pairs are bosons [$j_3=0 \pmod 1$], we are forced to set $K_{12}=0$. Hence, in a basis where $\hat{M}=(1,0)^T$, the band-coupling matrix K can be chosen to take the form

$$K=\begin{bmatrix}2p & 0\\ 0 & 2\end{bmatrix},$$

for some positive integer p.

The last consistency constraint on K originates from the requirement that the system exhibit excitations that may naturally be identified with ^{3}He atoms. Denoting the circulation vector corresponding to a ^{3}He atom excitation by $\hat{n}_a$, one infers from Eqs. (7.12)–(7.14) that

$$M(\hat{n}_a)=\frac{n_{a,1}}{2p}=\tfrac{1}{2} ,$$
$$\theta(\hat{n}_a)=\frac{1}{2p}(n_{a,1}^2+pn_{a,2}^2)=1 \pmod 2 , \tag{7.15}$$

and

$$2j_3(\hat{n}_a)=n_{a,2}=1 \pmod 2 .$$

These equations have no solutions unless $p=4l+1$, for some $l\in\mathbb{N}_0$. Now, for $p=4l+1$, the circulation vectors characterizing ^{3}He atom excitations in the first and second edge currents are given by $\hat{n}_a^{(1)}=(p,-1)^T$ and $\hat{n}_a^{(2)}=\hat{n}_a^{(1)}+\hat{\alpha}=(p,1)^T$, respectively; see Fig. 3.

We note that, at level $k=1$, the current (Kac-Moody) algebra $\widehat{\mathrm{su}}(2)$ has *two* (integrable, irreducible highest weight) representations, which are characterized by their spins $s=0$ and $s=\frac{1}{2}$, respectively (see, for example, Felder, Gawedzki, and Kupiainen, 1988 and Gepner and Witten, 1986). By the identifications given above, ^{3}He-pair excitations belong to the spin-0 and ^{3}He atom excitations belong to the spin-$\frac{1}{2}$ representation of $\widehat{\mathrm{su}}(2)_1$.

To summarize, we have found chiral boundary systems that are consistent with basic physical properties of a superfluid film, i.e., our model systems exhibit excitations that can naturally be identified with ^{3}He pairs and atoms, respectively. The systems are characterized by a mass vector $\hat{M}$ and a band-coupling matrix K of the form

$$\hat{M}=\begin{bmatrix}1\\0\end{bmatrix} \quad\text{and}\quad K=\begin{bmatrix}8l+2 & 0\\ 0 & 2\end{bmatrix}, \tag{7.16}$$

where l is a non-negative integer. Similarly to Eq. (6.79), this implies, for the Hall constant σ, a *fractionally quantized* value given by

$$\sigma=\hat{M}\cdot K^{-1}\hat{M}=\frac{1}{8l+2},\quad l=0,1,2,\ldots . \tag{7.17}$$

Recalling Eq. (7.10), we see that the total circulation of

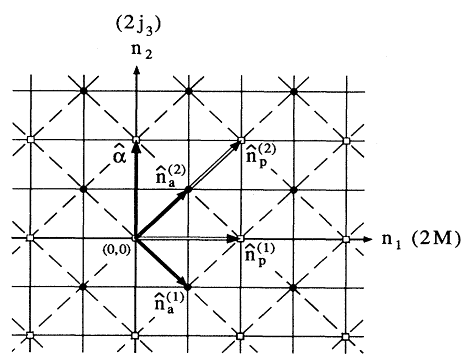

FIG. 3. The circulation lattice $\Phi=\mathbb{Z}^2$ (solid lines) and its sublattice Φ_c of "spin-mass-confined" excitations (dashed lines) for a superfluid film with $\sigma=\frac{1}{2}$ ($l=0$). $\hat{n}_p^{(i)}$ describes a ^{3}He pair and $\hat{n}_a^{(i)}$ a ^{3}He atom in the ith band, $i=1,2$; $\hat{\alpha}$ is the root vector of su(2); □, bosonic ($s=0$) excitations; ●, fermionic ($s=\frac{1}{2}$) excitations.

a perturbation of the superfluid flow, corresponding to an excitation of mass $M(\hat{n})$, is given by

$$\tilde{\phi}(\hat{n})=\frac{M(\hat{n})}{\sigma}\phi_0 \ . \tag{7.18}$$

In particular, it follows from the identifications given above [see the remarks preceding Eq. (7.14) and following Eq. (7.15)] that, in all systems determined by (7.16), the total circulation associated with a ^{3}He pair equals twice that associated with a ^{3}He atom: $\phi(\hat{n}_p^{(i)})=2\phi(\hat{n}_a^{(i)})=(8l+2)\phi_0$, $i=1,2$.

We conclude our discussion by observing that if a finite-energy excitation specified by some $\hat{n}\in\Phi$ is required to be relatively local not only to the "elementary" ^{3}He pair but also to the ^{3}He atom, one finds that $\hat{n}$ has to satisfy the constraints

$$n_1-n_2\in 2\mathbb{Z} \quad \text{and} \quad n_1+n_2\in 2\mathbb{Z} \ . \tag{7.19}$$

Then, denoting by Φ_c the subset of circulation vectors obeying (7.19), we note that there are no massless spin-$\frac{1}{2}$ excitations in Φ_c [i.e., there is no $\hat{n}\in\Phi_c$ such that $M(\hat{n})=0$ and $2j_3(\hat{n})=1$ (mod 2)]. This implies that if there is no spin-mass separation in the superfluid film then one must consider the sublattice $\Phi_c\subset\Phi$ as the one describing the physical excitations of the system; see Fig. 3.

It would be interesting to test the quantum Hall effect for a ^{3}He-A/B interface discussed in this section experimentally.

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APPENDIX A

In this appendix we summarize the basic steps in the calculation of $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$, the "scaling limit" of the effective action of a two-dimensional, incompressible quantum fluid, whose form we have displayed in Eq. (5.26). For more details the reader is referred to Fröhlich and Studer (1992b).

Adopting the notations given in Eqs. (5.13)–(5.20), we expand the effective action $S^{\text{eff}}_{\theta\Lambda_0}(a^{(\theta)},w^{(\theta)})$ in a Taylor series, around a_c and w_c, to third order in $\tilde{a}^{(\theta)}$ and $\tilde{w}^{(\theta)}$, with a fourth-order remainder term $R(a_c+\alpha,w_c+\omega;\tilde{a}^{(\theta)},\tilde{w}^{(\theta)})$, where $\alpha=\lambda_\alpha\tilde{a}^{(\theta)}$ and $\omega=\lambda_\omega\tilde{w}^{(\theta)}$, with $0<\lambda_\alpha,\lambda_\omega<1$:

$$\begin{aligned}
S^{\text{eff}}_{\theta\Lambda_0}(a^{(\theta)},w^{(\theta)})&=S^{\text{eff}}_{\theta\Lambda_0}(a_c,w_c)\\
&\quad+\sum_{n,m=1}^{3}\frac{1}{n!m!}\int_{(\theta\Lambda_0)^{n+m}}\frac{\delta^{(n+m)}S^{\text{eff}}_{\theta\Lambda_0}}{\delta a_{\mu_1}(x_1)\cdots\delta a_{\mu_n}(x_n)\delta w_{\nu_1,A_1}(y_1)\cdots\delta w_{\nu_m,A_m}(y_m)}(a_c,w_c)\\
&\qquad\times\tilde{a}^{(\theta)}_{\mu_1}(x_1)\cdots\tilde{a}^{(\theta)}_{\mu_n}(x_n)\tilde{w}^{(\theta)}_{\nu_1,A_1}(y_1)\cdots\tilde{w}^{(\theta)}_{\nu_m,A_m}(y_m)dv_g(x_1)\cdots dv_g(y_m)\\
&\quad+R(a_c+\alpha,w_c+\omega;\tilde{a}^{(\theta)},\tilde{w}^{(\theta)})\\
&=S^{\text{eff}}_{\theta\Lambda_0}(a_c,w_c)+\sum_{n,m=1}^{3}\int_{\Lambda_0^{n+m}}\varphi^{(\theta)\mu_1\ldots\mu_n,\nu_1\ldots\nu_m}_{\qquad A_1\ldots A_m}(a_c,w_c;\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_m)\\
&\qquad\times\tilde{a}_{\mu_1}(\xi_1)\cdots\tilde{a}_{\mu_n}(\xi_n)\tilde{w}_{\nu_1,A_1}(\eta_1)\cdots\tilde{w}_{\nu_m,A_m}(\eta_m)dv(\xi_1)\cdots dv(\eta_m)\\
&\quad+R^{(\theta)}(a_c+\alpha,w_c+\omega;\tilde{a},\tilde{w}) \ ,
\end{aligned} \tag{A1}$$

where $dv_g(x)=\sqrt{g(x)}d^3x$ and $dv(\xi)\equiv dv_\gamma(\xi)=\sqrt{\gamma(\xi)}d^3\xi$ denote the volume elements on $\Lambda=\theta\Lambda_0$ and Λ_0, respectively. Note that, by Eqs. (5.13) and (5.14), $dv_g(\theta\xi)=\theta^3 dv_\gamma(\xi)$, and recall that $\tilde{a}^{(\theta)}_\mu(\theta\xi)\equiv\theta^{-1}\tilde{a}_\mu(\xi)$ and $\tilde{w}^{(\theta)}_\mu(\theta\eta)\equiv\theta^{-1}\tilde{w}_\mu(\eta)$, where the scale parameter θ satisfies $1\le\theta<\infty$. Moreover, the distributions $\varphi^{(\theta)\underline{\mu},\underline{\nu}}_{\underline{A}}(a_c,w_c;\underline{\xi},\underline{\eta})$ have been identified in Eqs. (5.19) and (5.20).

In the remaining part of this appendix, we determine the leading-order terms (scaling like θ^{-D} with $D\le 0$, for $\theta\to\infty$) of the distributions $\varphi^{(\theta)\underline{\mu},\underline{\nu}}_{\underline{A}}(a_c,w_c;\underline{\xi},\underline{\eta})$. We exploit the four principles (P1)–(P4) given in Sec. V.A.

By principle (P1), we have to calculate the *local* distributions

$$\varphi^{(\theta)\underline{\mu},\underline{\nu}}_{\text{loc}\,\underline{A}}(a_c,w_c;\underline{\xi},\underline{\eta})\equiv\sum_{\Delta=0}^{N}\theta^{-D_\Delta}\varphi^{\underline{\mu},\underline{\nu}}_{\Delta\underline{A}}(a_c,w_c;\underline{\xi},\underline{\eta}) \ , \tag{A2}$$

where

$$\operatorname{supp}\varphi^{(\theta)\underline{\mu},\underline{\nu}}_{\text{loc}\,\underline{A}}(a_c,w_c;\underline{\xi},\underline{\eta})=\{\xi_1,\ldots,\eta_m\,|\,\xi_1=\cdots=\eta_m\} \ .$$

By principle (P3), we consider only terms with $D_\Delta\le 0$ (relevant and marginal terms) on the rhs of Eq. (A2). In particular, by Eq. (5.22), this implies that we have to determine only the local distributions $\varphi^{(\theta)\underline{\mu},\underline{\nu}}_{\text{loc}\,\underline{A}}(a_c,w_c;\underline{\xi},\underline{\eta})$ with $n+m\le 3$.

Next, principles (P2) and (P4) provide enough constraints on these local distributions that they can be determined explicitly. First, U(1) gauge invariance and the fact that the space of connections (vector potentials) on a trivial U(1) bundle is a real vector space imply that

$$\nabla_{\mu_i}\varphi_{\rm loc}^{(\theta)\mu_1\cdots\mu_i\cdots\mu_n,\underline{\nu}}{}_{\underline{A}}(a_c,w_c;\xi_1,\ldots,\xi_i,\ldots,\xi_n,\underline{\eta})=0\ , \tag{A3}$$

for all $i=1,\ldots,n$, in the sense of distributions on Λ_0^{n+m}; see the remark following Eq. (6.22) and Fröhlich and Studer (1992b). In Eq. (A3) the divergence operator is defined by $\nabla_{\mu_i}=[1/\sqrt{\gamma(\xi_i)}](\partial/\partial\xi_i^{\mu_i})[\gamma(\xi_i)\cdot\ ]^{1/2}$.

Second, we investigate the constraints due to SU(2) and $U(1)_{\rm spin}$ gauge invariance: It is important to note that the local distributions $\varphi_{\rm loc}^{(\theta)\underline{\mu},\underline{\nu}}{}_{\underline{A}}(a_c,w_c;\underline{\xi},\underline{\eta})$ are not SU(2) gauge invariant. The forms they take depend on the choice of gauge in which we describe $w=w_c+\tilde{w}$; see Eqs. (3.7), (3.32), and (3.33).

Explicitly, let g be an SU(2) gauge transformation. Then ${}^g w_c=gw_cg^{-1}+g\,dg^{-1}$ and similarly for the total gauge field w. Moreover, the perturbing field $\tilde{w}$ transforms homogeneously according to $\tilde{w}\mapsto{}^g\tilde{w}=g\tilde{w}g^{-1}$, which may also be written as

$$\tilde{w}_A\mapsto({}^g\tilde{w})_A=\sum_{B=1}^{3}R_{AB}(g)\tilde{w}_B \quad \text{for } A=1,2,3\ , \tag{A4}$$

where $\tilde{w}_A$ denotes the su(2) components of $\tilde{w}$, i.e., $\tilde{w}=i\sum_{A=1}^{3}\tilde{w}_AL_A^{(s)}$ (and similarly for ${}^g\tilde{w}$), and $R(g)$ is the SO(3) rotation representing the element g of SU(2) in the adjoint representation.

Turning to the local distributions (A2), one verifies by a change of variables in the path integral (5.1) ($\psi\mapsto g^{-1}\psi$) that their behavior under an SU(2) gauge transformation g is given by

$$\varphi_{\rm loc}^{(\theta)\underline{\mu},\nu_1\cdots\nu_m}{}_{A_1\cdots A_m}(a_c,{}^gw_c;\underline{\xi},\eta_1,\ldots,\eta_m) =\sum_{B_1,\ldots,B_m=1}^{3}R_{A_1B_1}(g(\eta_1))\ldots R_{A_mB_m}(g(\eta_m))\varphi_{\rm loc}^{(\theta)\underline{\mu},\nu_1\cdots\nu_m}{}_{B_1\cdots B_m}(a_c,w_c;\underline{\xi},\eta_1,\ldots,\eta_m)\ , \tag{A5}$$

where the matrix $R(g)$ has been defined in (A4).

Provided the external electromagnetic, "tidal" (and possibly geometric) background fields are of a form that allows for an SU(2) gauge where

$$w_{c,\mu A}(y)=\delta_{A3}w_{c,\mu 3}(y)\ , \tag{A6}$$

then the spin current s_3^μ is conserved; see Eq. (5.12). An explicit realization of such background fields, corresponding to a standard experimental situation in two-dimensional condensed-matter physics [$\mathbf{B}_c$ perpendicular to the sample and pointing along the 3-axis in spin or (co)tangent space, and $\mathbf{E}_c$ tangential to the sample], has been given in Eqs. (5.24) and (5.25). Note that, for consistency with Eq. (A6), the electromagnetic vector potential a_c has to be of such a form that $E_{c,3}=0$; see Eq. (3.38). In the following we assume Eq. (A6) to hold [principle (P4)]. The SU(2) gauge transformations leaving the form (A6) invariant are the local rotations around the 3-axis in spin space. They form a $U(1)_{\rm spin}$ gauge (sub)group. As in the U(1) situation, the vector-space nature of the space of connections on a trivial U(1) bundle [see the remark following Eq. (6.22)] naturally leads to an ("inhomogeneous") action of $U(1)_{\rm spin}$ gauge transformations on the su(2) 3-component of the perturbing field $\tilde{w}$. It is given by

$$\tilde{w}_3\mapsto{}^\lambda\tilde{w}_3=\tilde{w}_3+d\lambda\ . \tag{A7}$$

Similarly to Eq. (A3), the corresponding $U(1)_{\rm spin}$ gauge invariance implies that

$$\nabla_{\nu_j}\varphi_{\rm loc}^{(\theta)\underline{\mu},\nu_1\cdots\nu_j\cdots\nu_m}{}_{3\cdots3\cdots3}(a_c,w_c;\underline{\xi},\eta_1,\ldots,\eta_j,\ldots,\eta_m)=0\ , \tag{A8}$$

for all $j=1,\ldots,m$, and again in the sense of distributions on Λ_0^{n+m}.

We are now ready to display all the leading-order terms possibly contributing to $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$, the "scaling limit" of the effective action $S^{\rm eff}_{\theta\Lambda_0}(a^{(\theta)},w^{(\theta)})$. We present the result in an SU(2) gauge and for background fields such that condition (A6) holds. For physical interpretations of the different terms, we refer the reader to Secs. V and VI.

Since, in all our considerations, the constant term $S^{\rm eff}_{\theta\Lambda_0}(a_c,w_c)$ plays no role, we omit it from $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$.

(a) Terms involving only $\tilde{a}$:

(a1) $\varphi^{(\theta)\mu}$: By Eqs. (5.19) and (5.19) we have

$$\varphi^{(\theta)\mu}(a_c,w_c;\xi)=\theta^2\langle j^\mu(\theta\xi)\rangle_{a_c,w_c}\equiv j_c^\mu(a_c,w_c;\xi)\ ,$$

where by Eq. (A3)

$$\nabla_\mu j_c^\mu(a_c,w_c;\xi)=0\ . \tag{A9}$$

Hence there is a relevant ($D=-2$) term contributing to $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$,

$$\int_{\Lambda_0}j_c^\mu(a_c,w_c;\xi)\tilde{a}_\mu(\xi)dv(\xi)=\int_{\Lambda_0}(*j_c)\wedge\tilde{a}\ . \tag{A10}$$

(a2) $\varphi_{\rm loc}^{(\theta)\mu_1,\mu_2}$: Exploiting Eqs. (A2) and (A3) one finds that

$$\varphi_{\rm loc}^{(\theta)\mu_1,\mu_2}(a_c,w_c;\xi_1,\xi_2) =\frac{\alpha(a_c,w_c)}{\sqrt{\gamma(\xi_1)}\sqrt{\gamma(\xi_2)}}\varepsilon^{\mu_1\mu_2\rho}\frac{\partial}{\partial\xi_1^\rho}\delta^{(3)}(\xi_1-\xi_2)\ ,$$

where $\alpha(a_c,w_c)$ is a constant depending on the background fields. This leads to the marginal ($D=0$) Chern-

Simons term

$$\alpha(a_c,w_c)\int_{\Lambda_0}\varepsilon^{\mu_1\rho\mu_2}\tilde{a}_{\mu_1}(\xi)\partial_\rho\tilde{a}_{\mu_2}(\xi)d^3\xi \equiv \frac{\sigma(a_c,w_c)}{4\pi}\int_{\Lambda_0}\tilde{a}\wedge d\tilde{a}\ . \quad (\mathrm{A11})$$

(a3) $\varphi_{\mathrm{loc}}^{(\theta)\mu_1,\mu_2,\mu_3}$: There are no local distributions (in 1+2 dimensions) that are compatible with (A3) and that have non-negative scaling dimensions D. Thus there is no term of third order in $\tilde{a}$ present in $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$.

(w) Terms involving only $\tilde{w}$ and w:

(w1) $\varphi_{\mathrm{loc}\,A}^{(\theta)\nu}$: As in (a1) we have

$$\varphi_{\mathrm{loc}\,A}^{(\theta)\nu}(a_c,w_c;\eta)=\theta^2\langle s_A^\nu(\theta\eta)\rangle_{a_c,w_c}\equiv m_A^\nu(a_c,w_c;\eta)\ .$$

If w_c corresponds to a magnetic field in the 3-direction [in spin or (co)tangent space], as is assumed in Eq. (A6), then we expect that

$$m_A^\nu(a_c,w_c;\eta)=\delta_{A3}m_3^\nu(a_c,w_c;\eta)\ ,$$

and by $\mathrm{U}(1)_{\mathrm{spin}}$ gauge invariance, i.e., by Eq. (A3), $m_3^\nu(a_c,w_c;\eta)$ satisfies

$$\nabla_\nu m_3^\nu(a_c,w_c;\eta)=0\ . \quad (\mathrm{A12})$$

Furthermore, we notice that, by Eq. (A5),

$$m_A^\nu(a_c,{}^g w_c;\eta)=R_{A3}(g(\eta))m_3^\nu(a_c,w_c;\eta)\ .$$

This makes the relevant ($D=-2$) contribution

$$\int_{\Lambda_0}m_3^\nu(a_c,w_c;\eta)\tilde{w}_{\nu 3}(\eta)dv(\eta)=\int_{\Lambda_0}(*m_3)\wedge\tilde{w}_3 \quad (\mathrm{A13})$$

compatible with SU(2) gauge invariance; see also Eq. (A4).

(w2) $\varphi_{\mathrm{loc}\,A_1,A_2}^{(\theta)\nu_1,\nu_2}$ and $\varphi_{\mathrm{loc}\,A_1,A_2,A_3}^{(\theta)\nu_1,\nu_2,\nu_3}$: The procedure for determining these distributions comprises the following steps: First, we temporarily restrict our attention to perturbing SU(2) gauge fields $\tilde{w}$ of the same form as w_c given in Eq. (A6), i.e.,

$$\tilde{w}_A=\delta_{A3}\tilde{w}_3\ . \quad (\mathrm{A14})$$

Then, according to what we have seen above in Eqs. (A6)–(A8), the theory is a $\mathrm{U}(1)_{\mathrm{spin}}$ gauge theory, and the calculation of the second- and third-order terms in $\tilde{w}_3$ contributing to $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$ proceeds along the very same lines of reasoning as presented in (a2) and (a3) above. The result is given by the second-order term

$$\int_{\Lambda_0}\tilde{w}_3\wedge d\tilde{w}_3\ , \quad (\mathrm{A15})$$

and there is no third-order term in $\tilde{w}_3$. Second, we remember that, as a functional of the total SU(2) gauge field w, $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$ must be SU(2) gauge invariant [principle (P3)], where we recall that, in rescaled variables, the total gauge field w is of the form

$$w_{\mu A}(\eta)=w_{c,\mu A}^{(\theta)}(\eta)+\tilde{w}_{\mu A}(\eta)\equiv\theta w_{c,\mu A}(\theta\eta)+\tilde{w}_{\mu A}(\eta)\ ;$$

see Eqs. (5.27) and (5.17). Now, the idea is that if we restrict the gauge field w in $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$ to be of the form $w=(w_{c,3}^{(\theta)}+\tilde{w}_3)L_3^{(s)}$ [see Eqs. (A3) and (A14)] then the second-order term in $\tilde{w}_3$ has to reduce to one proportional to Eq. (A15). The relevant and marginal ($D=-1$ and 0) terms in w and $\tilde{w}$ having the required properties are the two Chern-Simons terms,

$$\frac{k(a_c,w_c)}{4\pi}\int_{\Lambda_0}\mathrm{tr}(w\wedge dw+\tfrac{2}{3}w\wedge w\wedge w) \quad (\mathrm{A16})$$

for some constant $k(a_c,w_c)$ and

$$\int_{\Lambda_0}(\beta_3\tilde{w}_3)\wedge d(\beta_3\tilde{w}_3)\equiv\frac{\sigma_s(a_c,w_c)}{4\pi}\int_{\Lambda_0}\tilde{w}_3\wedge d\tilde{w}_3\ . \quad (\mathrm{A17})$$

In an SU(2) gauge where (A6) holds, $\beta_3=\beta_3(a_c,w_c)$ is a constant. Moreover, under SU(2) gauge transformations, $\beta_3(a_c,w_c)$ transforms according to

$$\delta_{A3}\beta_3(a_c,w_c)\mapsto\beta_A(a_c,{}^g w_c;\eta)=R_{A3}(g(\eta))\beta_3(a_c,w_c)\ , \quad (\mathrm{A18})$$

which ensures the SU(2) gauge invariance of the term in (A17); see Eq. (A5). Finally, a winding number argument, which shows that, by SU(2) gauge invariance, the coefficient $k(a_c,w_c)$ of (A16) has to be an integer, is given in the text; see Eqs. (6.99)–(6.104).

Next, we show that, in addition to the terms in (A16) and (A17), there can be further relevant and marginal terms in $\tilde{w}$ that contribute to $S^*_{\Lambda_0}(\tilde{a},\tilde{w})$. The reason for this is that we have the transformation properties (A4) and (A5) and the fact that the constraint (A8) has to hold *only* for the su(2) 3-components of the local distributions we are considering. Hence, applying this and exploiting, in particular, invariance under global rotations around the 3-axis in spin space [principle (P4) in the situation of (A6)], one finds that for $\varphi_{\mathrm{loc}\,A_1,A_2}^{(\theta)\nu_1,\nu_2}(a_c,w_c;\eta_1,\eta_2)$ the most general expression reads

$$\begin{aligned}\varphi_{\mathrm{loc}\,A_1,A_2}^{(\theta)\nu_1,\nu_2}(a_c,w_c;\eta_1,\eta_2)=&\frac{k(a_c,w_c)}{2\pi\sqrt{\gamma(\eta_1)}\sqrt{\gamma(\eta_2)}}\varepsilon^{\nu_1\nu_2\rho}\left[\delta_{A_1A_2}\frac{\partial}{\partial\eta_1^\rho}-2\theta\varepsilon_{A_1A_23}w_{c,\rho3}(\theta\eta_1)\right]\delta^{(3)}(\eta_1-\eta_2)\\&+\frac{\sigma_s(a_c,w_c)}{4\pi\sqrt{\gamma(\eta_1)}\sqrt{\gamma(\eta_2)}}\varepsilon^{\nu_1\nu_2\rho}\delta_{A_13}\delta_{A_23}\frac{\partial}{\partial\eta_1^\rho}\delta^{(3)}(\eta_1-\eta_2)\\&+\theta[(1-\delta_{A_13})\delta_{A_1A_2}\tau_1^{\nu_1\nu_2}(a_c,w_c;\eta_1)+\varepsilon_{A_1A_23}\tau_2^{\nu_1\nu_2}(a_c,w_c;\eta_1)]\delta^{(3)}(\eta_1-\eta_2)\ ,\end{aligned} \quad (\mathrm{A19})$$

where $\tau_1^{\nu_1\nu_2}(a_c,w_c;\eta_1)$ is symmetric and $\tau_2^{\nu_1\nu_2}(a_c,w_c;\eta_1)$ is antisymmetric in ν_1 and ν_2. In the expression (A19), the first two [marginal and relevant ($D=-1$)] terms correspond to (A16), the third (marginal) term to (A17), and the last two terms give rise to an additional [relevant ($D=-1$)] contribution of the form

$$\sum_{A=1}^{2}\int_{\Lambda_0}\tau_1^{\nu_1\nu_2}(a_c,w_c;\eta)\tilde{\varpi}_{\nu_1 A}(\eta)\tilde{\varpi}_{\nu_2 A}(\eta)dv(\eta)+\sum_{A_1,A_2=1}^{2}\int_{\Lambda_0}\varepsilon_{A_1A_2 3}\tau_2^{\nu_1\nu_2}(a_c,w_c;\eta)\tilde{\varpi}_{\nu_1 A_1}(\eta)\tilde{\varpi}_{\nu_2 A_2}(\eta)dv(\eta)\ , \tag{A20}$$

where we have absorbed the scale factor θ into $\tau_1^{\nu_1\nu_2}$ and $\tau_2^{\nu_1\nu_2}$. For further constraints on $\tau_1^{\nu_1\nu_2}$ and $\tau_2^{\nu_1\nu_2}$, if the system displays, for example, rotation invariance in the scaling limit ($\theta\to\infty$) [principle (P4)], see Fröhlich and Studer (1992b). Finally, all possible third-order terms in $\tilde{\varpi}$ are summarized by

$$\varphi_{\mathrm{loc}\,A_1,A_2,A_3}^{(\theta)\nu_1,\nu_2,\nu_3}(a_c,w_c;\eta_1,\eta_2,\eta_3)=\frac{k(a_c,w_c)}{3\pi\sqrt{\gamma(\eta_1)}\sqrt{\gamma(\eta_2)}\sqrt{\gamma(\eta_3)}}\varepsilon^{\nu_1\nu_2\nu_3}\varepsilon_{A_1A_2A_3}\delta^{(3)}(\eta_1-\eta_2)\delta^{(3)}(\eta_1-\eta_3)$$
$$+\eta_{A_1,A_2,A_3}^{\nu_1,\nu_2,\nu_3}(a_c,w_c;\eta_1)\delta^{(3)}(\eta_1-\eta_2)\delta^{(3)}(\eta_1-\eta_3)\ , \tag{A21}$$

where $\eta_{A_1,A_2,A_3}^{\nu_1,\nu_2,\nu_3}(a_c,w_c;\eta_1)$ is symmetric under permutations of (ν_1A_1), (ν_2A_2), and (ν_3A_3). Moreover, it vanishes if two or three of the indices A_1,A_2,A_3 take the value 3 simultaneously. The first term on the rhs of Eq. (A21) gives a contribution that is contained in (A16). Again the restoration of rotation or translation symmetry in the scaling limit can be used to imply further constraints on the resulting marginal contribution to $S^*_{\Lambda_0}(\tilde{a},\tilde{\varpi})$,

$$\sum_{A_1,A_2,A_3=1}^{3}\int_{\Lambda_0}\eta_{A_1,A_2,A_3}^{\nu_1,\nu_2,\nu_3}(a_c,w_c;\eta)\tilde{\varpi}_{\nu_1A_1}(\eta)\tilde{\varpi}_{\nu_2A_2}(\eta)\tilde{\varpi}_{\nu_3A_3}(\eta)dv(\eta)\ . \tag{A22}$$

(aw) Mixed terms in $\tilde{a}$ and $\tilde{\varpi}$:

(aw1) $\varphi_{\mathrm{loc}}^{(\theta)\mu,\ \nu}{}_{A}$: By invariance under global rotations around the 3-axis in spin space [principle (P4)] $\varphi_{\mathrm{loc}}^{(\theta)\mu,\ \nu}{}_{A}(a_c,w_c;\xi,\eta)$ has to vanish unless $A=3$. In addition, exploiting U(1) and $\mathrm{U(1)_{spin}}$ gauge invariance, i.e., imposing Eqs. (A3) and (A8), one concludes that

$$\varphi_{\mathrm{loc}}^{(\theta)\mu,\ \nu}{}_{A}(a_c,w_c;\xi,\eta)=\gamma_3(a_c,w_c)\delta_{A3}\varepsilon^{\mu\nu\rho}\frac{\partial}{\partial\xi^\rho}\delta^{(3)}(\xi-\eta)\ ,$$

where $\gamma_3(a_c,w_c)$ is a constant in an SU(2) gauge where (A6) holds. Under SU(2) gauge transformations $\gamma_3(a_c,w_c)$ transforms like $\beta_3(a_c,w_c)$; see Eq. (A18). Hence there is a mixed marginal term contributing to $S^*_{\Lambda_0}(\tilde{a},\tilde{\varpi})$ that takes the form

$$\gamma_3(a_c,w_c)\int_{\Lambda_0}d\tilde{a}\wedge\tilde{\varpi}_3\equiv\frac{\chi_s(a_c,w_c)}{2\pi}\int_{\Lambda_0}d\tilde{a}\wedge\tilde{\varpi}_3\ . \tag{A23}$$

(aw2) $\varphi_{\mathrm{loc}\ \ A_1,A_2}^{(\theta)\mu,\nu_1,\nu_2}$ and $\varphi_{\mathrm{loc}\ \ A_1,A_2,A_3}^{(\theta)\mu,\nu_1,\nu_2,\nu_3}$: It follows from the constraint (A3) that these local distributions must have scaling dimensions $D\geq 1$, and hence they give rise to irrelevant terms which we discard in $S^*_{\Lambda_0}(\tilde{a},\tilde{\varpi})$.

This completes the calculation (in the bulk) of the most general (universal) form of the "scaling limit" $S^*_{\Lambda_0}(\tilde{a},\tilde{\varpi})$ that is consistent with principles (P1)–(P4) given in Sec. V.A. For a compilation of the terms discussed in (a1)–(aw2) above, see Eq. (5.26).

APPENDIX B

The purpose of this appendix is to collect a few basic facts about Gaussian models in two dimensions. Thereby we fix our normalizations and verify the identities (6.63) and (6.67), which play a crucial role in the discussion of Sec. VI.

We consider the Euclidean path-integral formulation of the theory of a free Bose field in two dimensions and recall the form of the corresponding two-point function:

$$\langle\phi(z,\bar z)\phi(w,\bar w)\rangle=\frac{\delta}{\delta J(z,\bar z)}\frac{\delta}{\delta J(w,\bar w)}\times\ln\int\mathcal{D}\phi\exp[-S_E(\phi)+\langle J,\phi\rangle]\ , \tag{B1}$$

where

$$\langle J,\phi\rangle=\int J(z,\bar z)\phi(z,\bar z)d\bar z\wedge dz\ , \tag{B2}$$

and

$$S_E(\phi)=-\frac{i\kappa}{4\pi}\int\phi(z,\bar z)\partial_{\bar z}\partial_z\phi(z,\bar z)d\bar z\wedge dz\ , \tag{B3}$$

with κ *positive*. $S_E(\phi)$ is the Euclidean form of the action (6.41), and z and $\bar z$ are the analytic continuations of the variables (on the unit circle) defined in Eq. (6.61); note that $\bar z=z^*$ in the Euclidean domain. Evaluating the Gaussian integral in (B1) by recalling the fundamental solution of the Laplacian $\Delta_z=\partial_{\bar z}\partial_z$, one finds that

$$\langle\phi(z,\bar z)\phi(w,\bar w)\rangle=-\frac{2}{\kappa}\ln|z-w|\ . \tag{B4}$$

Writing $\phi(z,\bar z)=\phi_L(z)+\phi_R(\bar z)$, similarly to Eq. (6.34), one obtains

$$\langle\phi_L(z)\phi_L(w)\rangle=-\frac{1}{\kappa}\ln(z-w)\ , \tag{B5}$$

and

$$\langle\phi_R(\bar z)\phi_R(\bar w)\rangle=-\frac{1}{\kappa}\ln(\bar z-\bar w)\ . \tag{B6}$$

Given two operators A and B that are *linear* in harmonic-oscillator creation and annihilation operators, one easily verifies the general identity (see, for example, the Appendix of Chap. 7 in Green, Schwarz, and Witten, 1987)

$$\langle :e^{A}::e^{B}:\rangle = e^{\langle AB\rangle} . \tag{B7}$$

Hence, by Eqs. (B5) and (B7), we infer that the vacuum expectation value of two (left) chiral vertex operators as in Eq. (6.62) is given by

$$\langle :e^{in\phi_L(z)}::e^{im\phi_L(w)}:\rangle = (z-w)^{nm/\kappa} . \tag{B8}$$

It is straightforward to generalize these results to the situation of N Bose fields, $\phi_1, \ldots, \phi_N$, coupled by a *positive* matrix K; see Eq. (6.52). The generalization of Eq. (B5), for example, is given by

$$\langle \phi_{L,i}(z)\phi_{L,j}(w)\rangle = -(K^{-1})_{ij}\ln(z-w), \quad i,j=1,\ldots,N , \tag{B9}$$

which leads to an obvious generalization of Eq. (B8).

Now, starting from the mode expansion of a free chiral Bose field [see Eqs. (6.38) and (6.39)], one finds that

$$\begin{aligned} V_n(z)V_m(w) &\equiv :e^{in\phi_L(z)}::e^{im\phi_L(w)}: \\ &= (z-w)^{nm/\kappa}:e^{i[n\phi_L(z)+m\phi_L(w)]}: \quad \text{if } |z|>|w| ; \end{aligned} \tag{B10}$$

see, for example, Goddard and Olive (1986). Comparing Eqs. (B10) and (B8), one verifies that the normalization in Eq. (6.41) or (B1) and correspondingly in (6.46) is consistent with that in Eq. (6.37)–(6.39).

Next, if $|z|<|w|$, we define the lhs of (B10) as the analytic continuation of the product $V_n(w)V_m(z)$ along a path such that z circumvents w anticlockwise or clockwise. This then leads to the identity (6.63) of Sec. VI when restricting the variables to the unite circle (see the Appendix of Chap. 7 in Green, Schwarz, and Witten, 1987). Identity (6.67) for the general chiral vertex operators (6.66) is established along the same lines; see, for example, Eq. (B9).

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