Dynamic symmetries and supersymmetries in nuclear physics

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The role played by dynamic symmetries and supersymmetries in nuclear physics is described and interpreted with experimental examples. Implications for other fields of physics are reviewed briefly.

CONTENTS

I. INTRODUCTION

The structure of nuclei, especially medium-mass and heavy nuclei, is rather complex. An important tool in its study is provided by symmetry, a wide-reaching concept used in physics in a variety of ways. The symmetries that are particularly useful in nuclear physics are those called *dynamic* symmetries. This article reviews the concept of dynamic symmetry briefly and discusses its use in the study of nuclear structure, based on the "interacting boson model." An account of this model is given by Iachello and Arima (1987).

In the last 15 years, the concept of symmetry has been further enlarged with the introduction of supersymmetries (or graded symmetries). The use of supersymmetry for the study of odd-even nuclei through the "interacting boson-fermion model" will also be described briefly. An account of this model is given by Iachello and van Isacker (1991). The type of supersymmetry found in nuclei is the only one that has been experimentally verified so far. Since the supersymmetric partners are nucleons (fundamental fermions) and correlated pairs of nucleons (bosons), we shall speculate briefly on whether or not this type is the only one that can be realized in physics.

II. DYNAMIC SYMMETRIES

The word "symmetry," from the Greek $\sigma\upsilon\mu\mu\epsilon\tau\rho o\nu$, meaning well-ordered, well-proportioned, is used in physics in many ways. Originally it applied to the geometric structure of physical systems, such as molecules, or crystals, as illustrated in Fig. 1. The geometric structure of the molecule in this figure is invariant under the group C_3 (rotations of 120° around the horizontal axis). Later, the concept was used to describe other situations, as those occurring in kinematic (or fundamental) symmetries. Examples of these symmetries are Poincaré invariance in relativistic quantum field theories and rotational invariance in nonrelativistic quantum mechanics. In addition, there are permutational symmetries, gauge symmetries, etc. Here, I review only one type of symmetry, dynamic symmetry. This is a situation represented by the following.

(a) The Hamiltonian describing the system can be constructed in terms of the elements of a Lie algebra, $\mathcal{G}$ (called the spectrum-generating algebra).

A Lie algebra is a set of operators $X_i \in \mathcal{G}$ satisfying the commutation relations

$$[X_i, X_j] = \sum_k c_{ij}^k X_k \ , \tag{2.1}$$

together with the Jacobi identities. (For a review of Lie algebras and Lie groups, see Wybourne, 1974.) Quite often the Hamiltonian H is a polynomial in the elements X_i,

$$H = E_0 + \sum_i \epsilon_i X_i + \sum_{i \le j} v_{ij} X_i X_j + \cdots \ . \tag{2.2}$$

The expansion of H into the elements of an algebra is called algebraic theory.

(b) The Hamiltonian H does not contain all elements of $\mathcal{G}$ but only special combinations of them, called "invariant" (or "Casimir") operators of a chain of algebras, $\mathcal{G} \supset \mathcal{G}' \supset \mathcal{G}'' \supset \cdots$. An invariant operator $\mathcal{C}$ of an algebra $\mathcal{G}$ is an operator that commutes with all the elements of $\mathcal{G}$,

$$[\mathcal{C}, X_i] = 0 \quad \text{for any } i \ . \tag{2.3}$$

If the Hamiltonian H is a polynomial in the elements X_i,

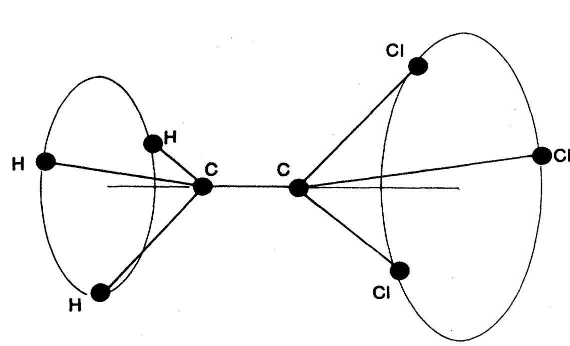

FIG. 1. H_3-C-C-Cl_3. The geometric structure of this molecule is invariant under rotations of 120° around the horizontal axis.

dynamic symmetry is the situation in which H is a polynomial in the Casimir operators

$$H=\alpha\mathcal{C}(\mathcal{G})+\alpha'\mathcal{C}(\mathcal{G}')+\alpha''\mathcal{C}(\mathcal{G}'')+\cdots \ . \qquad (2.4)$$

Functions more complicated than polynomials can be (and have been) constructed, as it will be mentioned in the following section in the case of the hydrogen atom. Since the Casimir operators are linear combinations of the elements of the algebra, X_i, and their products, dynamic symmetry is a special case of algebraic theory.

The main advantage of dynamic symmetries is that, whenever one such symmetry occurs, all properties of the system can be given in closed form in terms of quantum numbers. These closed forms are very useful in analyzing experimental data, especially in complex situations such as those encountered in the spectroscopy of nuclei. Since the Casimir operators are diagonal in the basis provided by the representations of $\mathcal{G}\supset\mathcal{G}'\supset\mathcal{G}''\cdots$, the eigenvalues of H of Eq. (2.4) are given by

$$E=\alpha\langle\mathcal{C}(\mathcal{G})\rangle+\alpha'\langle\mathcal{C}(\mathcal{G}')\rangle+\alpha''\langle G(\mathcal{G}'')\rangle+\cdots \ , \qquad (2.5)$$

where $\langle\mathcal{C}(\mathcal{G})\rangle$ denotes the expectation value of $\mathcal{C}(\mathcal{G})$ in the appropriate representation of $\mathcal{G}$. [For explicit expressions of the eigenvalues of the Casimir operators $\mathcal{C}$, see Wybourne (1974), Chap. 15.] The energy formula (2.5) implies a splitting, but not mixing, of the representations of $\mathcal{G}$.

III. DYNAMIC SYMMETRIES IN ATOMIC AND PARTICLE PHYSICS

The oldest example of a dynamic symmetry is provided by the spectrum of the nonrelativistic hydrogen atom. Pauli (1926), Fock (1935), and Bargmann (1936) showed that the Schrödinger equation for the hydrogen atom has a larger symmetry than rotational invariance. The Hamiltonian H can be written in terms of one of the Casimir operators of an algebra, O(4), whose elements are the three components of the angular momentum **J** and the three components of the Runge-Lenz vector **A**, as

$$H=-\frac{me^4/\hbar^2}{2(\mathcal{C}_2+1)} \ . \qquad (3.1)$$

Here $\mathcal{C}_2$ is the first quadratic Casimir invariant of O(4). (In this article, following common practice, capital letters will be used to denote both algebras and groups.) An account of the O(4) symmetry of the hydrogen atom is given in Chap. 21 of Wybourne (1974). By noting that the eigenvalues of $\mathcal{C}_2+1$ are n^2 ($n=0,1,\ldots$), one obtains the Bohr formula

$$E(n)=-\frac{me^4/\hbar^2}{2n^2} \ , \quad n=1,2,\ldots \ . \qquad (3.2)$$

As one can see from Eq. (3.1), in this case it is actually $1/H$ and not H that is linear in the invariants. This is due to the special nature of the Coulomb interaction. The presence of a dynamic symmetry manifests itself in (a) the closed form (3.2) of the energy eigenvalues which produces a regular pattern of energy levels, as shown in Fig. 2, and (b) the occurrence of degeneracies in addition to those due to rotational invariance.

Although known for at least 60 years, dynamic symmetries did not receive considerable attention in physics until the '60s, when they were introduced in particle physics. Gell-Mann (1962) and Ne'eman (1961) suggested that the internal degrees of freedom of hadrons be described in terms of the Lie algebra SU(3). This algebra is now called "flavor" SU(3) in order to distinguish it from other SU(3) algebras that play a role in hadronic physics (such as the "color" algebra). Furthermore, it was suggested that a dynamic symmetry exists corresponding to the chain of algebras,

$$SU(3)\supset SU_I(2)\oplus U_Y(1)\supset O_I(2)\oplus U_Y(1) \ , \qquad (3.3)$$

where $SU_I(2)$ is the isospin algebra and $U_Y(1)$ the algebra of hypercharge. By writing the mass operator M (which replaces for relativistic situations the Hamiltonian operator) describing the mass of the hadron as

$$M=M_0+a\mathcal{C}_1(U_Y(1))+b[\mathcal{C}_2(SU_1(2))-\tfrac{1}{4}\mathcal{C}_1^2(U_Y(1))] \qquad (3.4)$$

and evaluating the expectation value of (3.4) in the representation described by the diagram

$$\left|\begin{array}{ccccccccc} SU(3) & \supset & SU_I(2) & \otimes & U_Y(1) & \supset & O_I(2) & \otimes & U_Y(1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ (\lambda,\mu) & & I & & Y & & I_3 & & \end{array}\right\rangle \qquad (3.5)$$

where λ, μ, I, Y, I_3 are the quantum numbers that label the representations, one obtains the Gell-Mann–Okubo mass formula (Okubo, 1962)

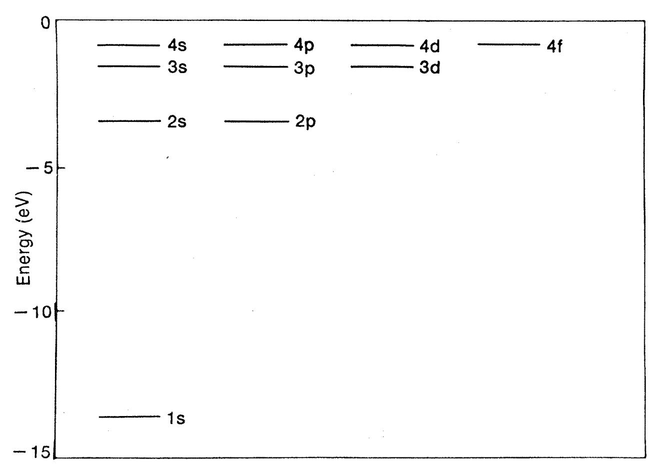

FIG. 2. A portion of the level diagram of the nonrelativistic hydrogen atom showing the regular pattern of energy levels and the occurrence of additional degeneracies.

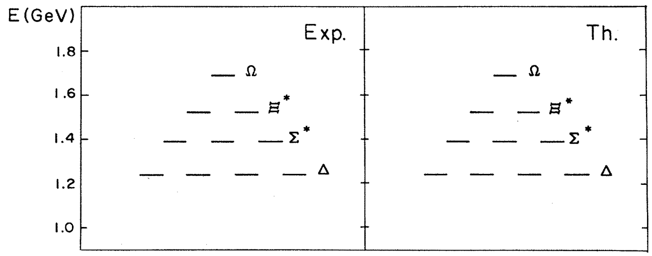

FIG. 3. Level diagram of the baryon decuplet. The energy levels are computed using the mass formula (3.6).

$$M(I,I_3,Y)=M_0+aY+b[I(I+1)-Y^2/4] \; . \quad (3.6)$$

In the diagram (3.5) the direct product sign $\otimes$ is used instead of the direct sum sign $\oplus$ of Eq. (3.3), since the wave functions are products rather than sums. Equation (3.6) provides an excellent description of hadronic spectra, as shown, for example, in Fig. 3. In this example the mass (or Hamiltonian) operator is linear and quadratic in the Casimir operators.

IV. DYNAMIC SYMMETRIES IN NUCLEAR PHYSICS

The best and most complete example of dynamic symmetries to date is provided by nuclear physics. Dynamic symmetries here are based on the interacting boson model (Arima and Iachello, 1976, 1978, 1979), which describes nuclei with an even number of neutrons and protons as aggregates of N bosons with angular momenta $J^P=0^+$ or $J^P=2^+$. The bosons represent correlated pairs of neutrons and protons. In order to construct the Hamiltonian and other physical operators of the interacting boson model, it is convenient to introduce creation and annihilation operators for s ($J=0$) and d ($J=2$) bosons, $s^\dagger$, $d^\dagger_\mu$, s, d_μ ($\mu=0,\pm1,\pm2$), denoted generically $b^\dagger_\alpha$, b_α ($\alpha=1,\ldots6$). There are six of these operators, since the s boson has one single component, while the d boson has five components corresponding to the five projections of its angular momentum along an axis. The bilinear products of the boson operators

$$\mathcal{G}: \; G_{\alpha\beta}=b^\dagger_\alpha b_\beta \; , \quad \alpha,\beta=1,\ldots,6 \; , \quad (4.1)$$

are the elements of the Lie algebra U(6). All operators are constructed from the operators $G_{\alpha\beta}$ (algebraic theory). For example, the Hamiltonian H is written as

$$H=E_0+\sum_{\alpha\beta}\epsilon_{\alpha\beta}G_{\alpha\beta}+\frac{1}{2}\sum_{\alpha\beta\gamma\delta}u_{\alpha\beta\gamma\delta}G_{\alpha\beta}G_{\gamma\delta} \; . \quad (4.2)$$

The algebra $\mathcal{G}\equiv$U(6) is thus the spectrum-generating algebra of this problem. The knowledge of the spectrum-generating algebra allows one to study in a straightforward way its dynamic symmetries. These are obtained by a mathematical procedure, known as the branching rule, which studies all possible breakings of an algebra $\mathcal{G}$ in its subalgebras, $\mathcal{G}\supset\mathcal{G}'\supset\mathcal{G}''\supset\cdots$. Since we want states to be characterized by good values of the angular momentum, the angular momentum algebra O(3) must always be contained in the chain of subalgebras of $\mathcal{G}$. There are three, and only three, such chains:

$$\mathrm{U}(6)\begin{cases}\nearrow \mathrm{U}(5)\supset\mathrm{O}(5)\supset\mathrm{O}(3)\supset\mathrm{O}(2) \quad (\mathrm{I}) \; , \\ - \mathrm{SU}(3)\supset\mathrm{O}(3)\supset\mathrm{O}(2) \quad (\mathrm{II}) \; , \\ \searrow \mathrm{O}(6)\supset\mathrm{O}(5)\supset\mathrm{O}(3)\supset\mathrm{O}(2) \quad (\mathrm{III}) \; .\end{cases} \quad (4.3)$$

In general, each nucleus (i.e., all its low-lying states) is described by a set of parameters $\epsilon_{\alpha\beta}$, $u_{\alpha\beta\gamma\delta}$, and the problem of finding the eigenvalues of H must be solved numerically. However, for some nuclei, the values of the parameters are such that the Hamiltonian H can be written in terms only of the Casimir operators of a single chain in (4.3). For example, for chain III,

$$H^{(\mathrm{III})}=E_0+A\,\mathcal{C}_2(\mathrm{O}(6))+B\,\mathcal{C}_2(\mathrm{O}(5))+C\,\mathcal{C}_2(\mathrm{O}(3)) \; , \quad (4.4)$$

where $\mathcal{C}_2(\mathcal{G})$ denotes the quadratic invariant of the algebra $\mathcal{G}$, and A, B, and C are arbitrary coefficients (parameters) not determined by symmetry. By evaluating the expectation values of H in the appropriate representation, one can then obtain the eigenvalues of H in terms of quantum numbers labeling the representations. In the case of chain III, the representations are labeled by the diagram

$$\left|\begin{array}{ccccccccc}\mathrm{U}(6) & \supset & \mathrm{O}(6) & \supset & \mathrm{O}(5) & \supset & \mathrm{O}(3) & \supset & \mathrm{O}(2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ N & & \sigma & & \tau(\nu_\Delta) & & L & & M_L\end{array}\right\rangle \quad (4.5)$$

and one obtains

$$E^{(\mathrm{III})}=E_0+A\sigma(\sigma+4)+B\tau(\tau+3)+CL(L+1) \; . \quad (4.6)$$

In the diagram (4.5), N, σ, τ, ν_Δ, L, and M_L denote the quantum numbers that characterize uniquely the states. These are obtained by studying the representations of U(6) and its branchings into representations of the subalgebras [see Wybourne (1974) for a description of branching rules]. Repeating the same procedure for other chains, one can obtain the full set of dynamic symmetry formulas,

$$\begin{aligned}&E^{(\mathrm{I})}(N,n_d,v,n_\Delta,L,M_L)\\&\quad=E_0+\epsilon n_d+\alpha n_d(n_d+4)+\beta v(v+3)+\gamma L(L+1) \; ,\\&E^{(\mathrm{II})}(N,\lambda,\mu,K,L,M_L)\\&\quad=E_0+\kappa(\lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu)+\kappa' L(L+1) \; , \quad (4.7)\\&E^{(\mathrm{III})}(N,\sigma,\tau,\nu_\Delta,L,M_L)\\&\quad=E_0+A\sigma(\sigma+4)+B\tau(\tau+3)+CL(L+1) \; .\end{aligned}$$

These formulas describe in many cases the experimental data very well (Figs. 4, 5, and 6). For example, for sym-

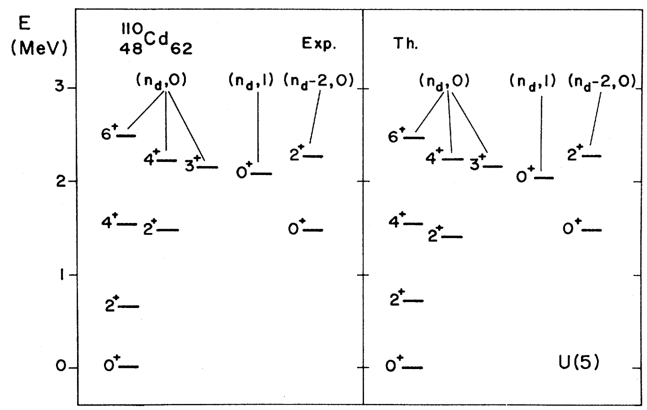

FIG. 4. U(5) dynamic symmetry (chain I) in nuclei: $^{110}_{48}Cd_{92}$. On the left is the experimental spectrum, on the right the spectrum predicted by the energy formula (4.7), I.

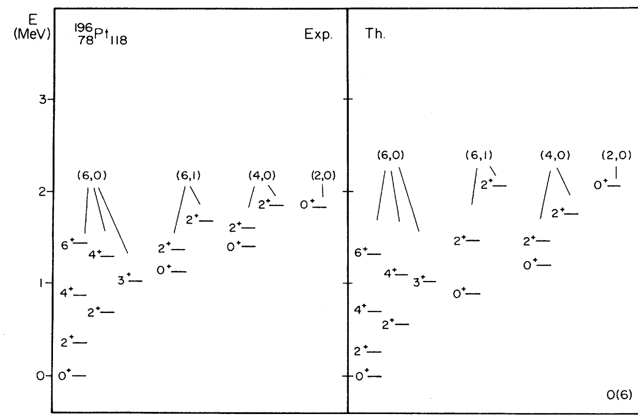

FIG. 6. O(6) dynamic symmetry (chain III) in nuclei: $^{196}_{78}Pt_{118}$. On the left is the experimental spectrum, on the right the spectrum predicted by the energy formula (4.7), III.

metry (III), there are two free parameters κ and κ'. With these parameters one can correlate the energies of 20 levels, as shown in Fig. 5, with an accuracy of a few percent.

It must be said that the interacting boson model does much more than just describe some nuclei with dynamic symmetry (the aspect described in this article). If, instead of using special values of the parameters $\epsilon, u, \ldots$, such that the Hamiltonian is a sum of Casimir operators, one uses the full set of parameters, then one is able to describe *all* collective low-lying quadrupole states of nuclei. There are six parameters that describe this most general situation. This can be seen from Eq. (4.7) by counting the number of independent terms for all chains. Since the rotational term $L(L+1)$ is common to all three chains, there are just six independent terms in (4.7).

Since its introduction in 1974, the interacting boson model has been extended to include proton-neutron degrees of freedom, through the algebra $U_\pi(6)\oplus U_\nu(6)$, called the "interacting boson model 2" (Arima *et al.*, 1977; Otsuka *et al.*, 1988). In a further generalization, necessary when protons and neutrons occupy the same shell, proton-neutron pairs both with isospin one and zero have been introduced (interacting boson models 3 and 4; Elliott and White, 1980; Elliott and Evans, 1981). The dynamic symmetries of these more complex models have also been analyzed and some experimental examples found.

It should also be remarked briefly that, in addition to the particle interpretation of the bosons given above (correlated pairs), it is possible to give another (classical or geometric) interpretation. This is done through the the introduction of a coherent (or intrinsic) state

$$|N,\alpha_\mu\rangle = \left[s^\dagger + \sum_\mu \alpha_\mu d_\mu^\dagger\right]^N|0\rangle \ , \quad \mu=0,\pm1,\pm2 \ , \tag{4.8}$$

where the α_μ's are classical variables (c numbers). The variables α_μ can then be associated with the shape of an object described by the surface

$$R=R_0\left[1+\sum_\mu \alpha_\mu Y_{2\mu}(\theta,\phi)\right] \ , \tag{4.9}$$

leading to a description of collective states in terms of

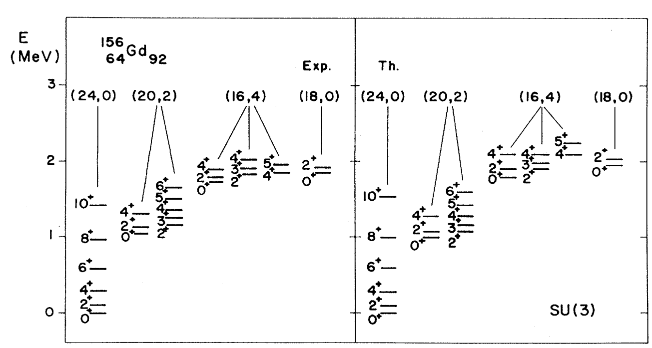

FIG. 5. SU(3) dynamic symmetry (chain II) in nuclei: $^{156}_{64}Gd_{92}$. On the left is the experimental spectrum, on the right the spectrum predicted by the energy formula (4.7), II.

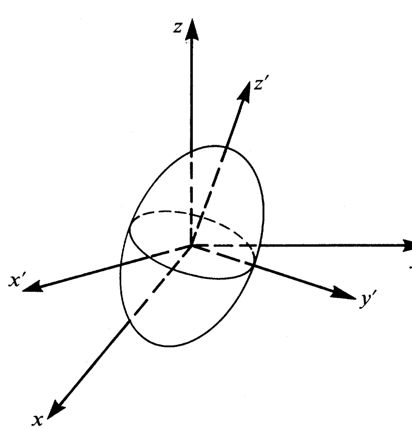

FIG. 7. Nuclear shape corresponding to symmetry II.

shape variables (Bohr and Mottelson, 1975). The three dynamic symmetries, Eq. (4.3), provide then a classification of shapes of nuclei in terms of symmetry groups: (I) spherical shape, (II) axially symmetric deformed shape, (III) nonaxially symmetric deformed shape (γ-unstable shape). Figure 7 shows the shape corresponding to symmetry II. The classification of shapes of nuclei in terms of symmetry groups is similar to the classification of shapes of crystals in terms of point groups.

V. DYNAMIC SUPERSYMMETRIES

In recent years the concept of symmetry has been enlarged even further with the introduction of supersymmetries or graded symmetries. Supersymmetries were originally introduced in elementary-particle physics (Miyazawa, 1966; Ramond, 1971) in an attempt to provide a unification of particle properties. In contrast with normal symmetries, where the symmetry operations transform bosons into bosons or fermions into fermions, in supersymmetry there are also operations that transform bosons into fermions and vice versa. As in the case of normal symmetries, there are several types of supersymmetries (for example "kinematic" supersymmetries; Wess and Zumino, 1974). Here only "dynamic" supersymmetries will be discussed.

Dynamic supersymmetries can be characterized in the same way as "normal" dynamic symmetries:

(a) The Hamiltonian describing the system is constructed in terms of the elements of a Lie superalgebra $\mathcal{G}^*$ (called the spectrum-generating superalgebra).

(b) The Hamiltonian can be written in terms only of invariant (Casimir) operators of a chain of superalgebras, $\mathcal{G}^* \supset \mathcal{G}'^* \supset \mathcal{G}''^* \supset \cdots$, i.e.,

$$H=\alpha^*\mathcal{C}(\mathcal{G}^*)+\alpha'^*\mathcal{C}(\mathcal{G}'^*)+\alpha''^*\mathcal{C}(\mathcal{G}''^*)+\cdots , \tag{5.1}$$

where $\mathcal{C}(\mathcal{G}^*)$ denotes a Casimir operator of the superalgebra $\mathcal{G}^*$.

This definition of a dynamic supersymmetry is identical to that of a dynamic symmetry given in Sec. II, except for the appearance of superalgebras. The main difference between a normal and a super Lie algebra is that in the superalgebras there appear two sets of operators, bosonic G_α ($\alpha=1,\ldots,n$), and fermionic F_i ($i=1,\ldots,m$), satisfying the relations

$$\begin{aligned}
[G_\alpha,G_\beta]&=\sum_\gamma c^\gamma_{\alpha\beta}G_\gamma ,\\
[G_\alpha,F_i]&=\sum_j c^j_{\alpha i}F_j ,\\
\{F_i,F_j\}&=\sum_\alpha c^\alpha_{ij}G_\alpha ,
\end{aligned} \tag{5.2}$$

together with the Jacobi identities. In Eq. (5.2) the square brackets denote commutators, while the curly brackets denote anticommutators,

$$\begin{aligned}
[A,B]&=AB-BA ,\\
\{A,B\}&=AB+BA .
\end{aligned} \tag{5.3}$$

The presence in (5.2) of anticommutators makes the algebra a superalgebra. [A normal Lie algebra is defined only in terms of commutators, Eq. (2.1).] The importance of dynamic supersymmtries is that the eigenvalue problem for the Hamiltonian of a mixed system of bosons and fermions can be written explicitly in terms of quantum numbers, thus providing a simple classification scheme that can be checked easily by experiment. The eigenvalues of (5.1) are in fact obviously given by

$$E=\alpha^*\langle\mathcal{C}(\mathcal{G}^*)\rangle+\alpha'^*\langle\mathcal{C}(\mathcal{G}'^*)\rangle+\alpha''^*\langle\mathcal{C}(\mathcal{G}''^*)\rangle+\cdots , \tag{5.4}$$

where $\langle\mathcal{C}(\mathcal{G}^*)\rangle$ denotes the expectation value of $\mathcal{C}(\mathcal{G}^*)$ in the appropriate representation of $\mathcal{G}^*$.

VI. DYNAMIC SUPERSYMMETRIES IN NUCLEAR PHYSICS

The use of dynamic supersymmetries in nuclear physics is based on the interacting boson-fermion model (Iachello and Scholten, 1979; Iachello and Kuyucak, 1981). This model is a generalization of the interacting boson model and introduces, in addition to bosons describing the correlated pairs, fermions describing unpaired particles. The simplest example of nuclei where one must consider situations of this type is provided by nuclei with an odd number of protons and an even number of neutrons, or vice versa. In these nuclei, at least one particle must be unpaired. The fundamental ingredients in constructing operators that describe these situations are a set of fermion creation and annihilation operators, $a_i^\dagger$, a_i ($i=1,\ldots,m$). The elements of the Lie superalgebra can be written explicitly in terms of these operators as

$$\begin{aligned}
G_{\alpha\beta}&=b^\dagger_\alpha b_\beta ,\\
G_{ij}&=a_i^\dagger a_j ,\\
F^\dagger_{\alpha i}&=b^\dagger_\alpha a_i ,\\
F_{i\alpha}&=a_i^\dagger b_\alpha ,
\end{aligned} \tag{6.1}$$

where $G_{\alpha\beta}$ and G_{ij} are bosonic operators and $F^\dagger_{\alpha i}$ and $F_{i\alpha}$ are fermionic operators.

The Hamiltonian of the interacting boson-fermion model can be written as

$$H=H_B+H_F+V_{BF} , \tag{6.2}$$

where H_B describes the bosons, H_F the fermions, and V_{BF} their interaction. Each individual piece can be written in terms of the elements (6.1) of the superalgebra as

$$\begin{aligned}
H_B&=E_{0B}+\sum_{\alpha\beta}\epsilon_{\alpha\beta}G_{\alpha\beta}+\tfrac{1}{2}\sum_{\alpha\beta\gamma\delta}u_{\alpha\beta\gamma\delta}G_{\alpha\beta}G_{\gamma\delta} ,\\
H_F&=E_{0F}+\sum_{ij}\epsilon_{ij}G_{ij}+\tfrac{1}{2}\sum_{ijkl}u_{ijkl}G_{ij}G_{kl} ,\\
V_{\rm BF}&=\sum_{\alpha\beta ij}w_{\alpha\beta ij}G_{\alpha\beta}G_{ij}+\sum_{\alpha\beta ij}w'_{\alpha\beta ij}F^\dagger_{\alpha i}F_{j\beta} .
\end{aligned} \tag{6.3}$$

If one includes only s and d bosons and single particles in a shell with degeneracy Ω, the spectrum-generating superalgebra of (6.3) is the unitary superalgebra $U(6/\Omega)$.

Since there are three possible subalgebras of U(6) and many subalgebras of $U(\Omega)$, depending on the actual degeneracy of the shell, a vast number of possibilities can occur, several of which have been investigated in detail. Here only the case in which the single particle occupies a shell with $j=3/2$, $\Omega=2j+1=4$, will be discussed (Balantekin, Bars, and Iachello, 1981a, 1981b). The corresponding supersymmetry is U(6/4). By breaking the superalgebra into its maximal Lie subalgebra,

$$U(6/4) \supset U(6) \oplus U(4) \supset O(6) \oplus SU(4) \supset Spin(6) \supset Spin(5) \supset Spin(3) \supset Spin(2) , \tag{6.4}$$

one can then construct a dynamic supersymmetry. The eigenvalues of the Hamiltonian constructed in terms of invariants of (6.4) in the representation

$$\left| \begin{array}{ccccccccccccccc} U(6/4) & \supset & U(6) & \otimes & U(4) & \supset & O(6) & \otimes & SU(4) & \otimes & Spin(6) & \supset & Spin(5) & \supset & Spin(3) & \supset & Spin(2) \\ \downarrow & & \downarrow & & \downarrow & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{N} & & N & & M & & & \Sigma & & & \sigma_1,\sigma_2,\sigma_3 & & \tau_1,\tau_2,(\nu_\Delta) & & J & & M_J \end{array} \right\rangle \tag{6.5}$$

are given by

$$E(\mathcal{N},N,M,\Sigma,\sigma_1,\sigma_2,\sigma_3,\tau_1,\tau_2,\nu_\Delta,J,M_J)$$
$$=E_0+E_1N+E_2N^2+A\Sigma(\Sigma+4)+A'[\sigma_1(\sigma_1+4)+\sigma_2(\sigma_2+2)+\sigma_3^2]+B[\tau_1(\tau_1+3)+\tau_2(\tau_2+1)]+CJ(J+1) . \tag{6.6}$$

Here Σ, σ_1, σ_2, σ_3, ... are the quantum numbers that characterize uniquely the states of this complicated system of bosons and fermions and A, A', B, and C are parameters not determined by supersymmetry. The formula (6.6) describes several nuclei (a supermultiplet), characterized by a given number of bosons plus fermions, $\mathcal{N}$. To each supermultiplet there belong five nuclei with $N=\mathcal{N}$, $M=0$; $N=\mathcal{N}-1$, $M=1$; $N=\mathcal{N}-2$, $M=2$; $N=\mathcal{N}-3$, $M=3$; $N=\mathcal{N}-4$, $M=4$. The quantum number M does not appear explicitly in Eq. (6.6), since it can be eliminated using the condition $N+M=\mathcal{N}$ and thus absorbed into E_0, E_1, and E_2. Typical supermultiplets are shown in Fig. 8. The predicted spectra of a pair of nuclei (even-even and even-odd) are shown in Fig. 9(a). They can be compared with the experimental spectra shown in Fig. 9(b). Several other examples of supersymmetry have been found in the same region and in other regions of the periodic table.

Again, it must be said that the interacting boson-fermion model does much more than describe spectra of odd-even nuclei with supersymmetry (the aspect discussed briefly here). The model provides a complete description of these spectra even in situations when there is no supersymmetry; i.e., the coefficients ϵ, u, ... in the

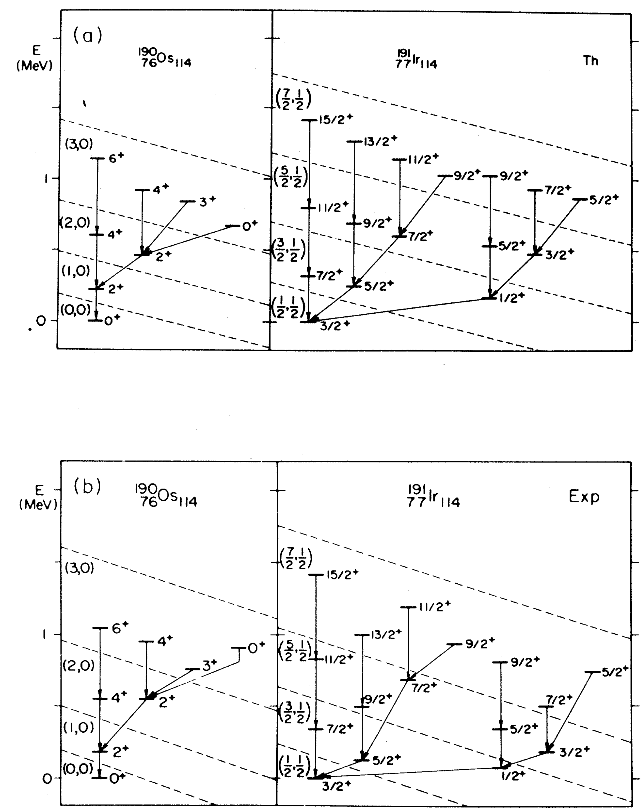

FIG. 9. U(6/4) supersymmetry in nuclei: (a) excitation spectra predicted by Eq. (6.6); (b) experimental spectra of the pair of nuclei ^{190}Os-^{191}Ir.

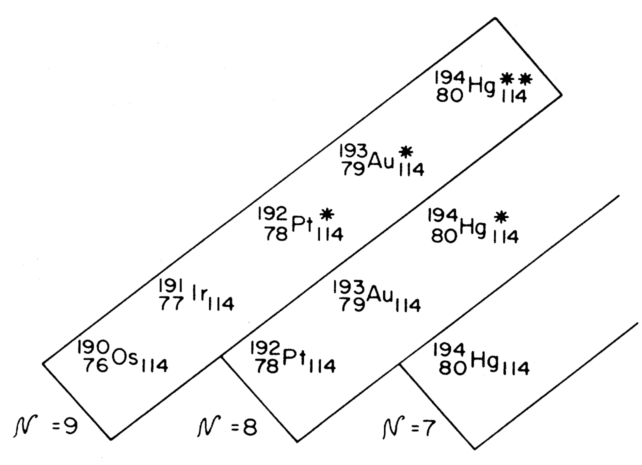

FIG. 8. U(6/4) supermultiplets in the Os-Ir region.

Hamiltonian (6.2) are not such that H can be written purely in terms of Casimir operators. For these cases, the algebra U(6/Ω), or U(6) when there are no unpaired particles, plays the same role as Racah algebra plays for angular momentum states. (For a review of Racah algebra, see Fano and Racah, 1959.) The strikingly successful implementation of this approach in nuclear spectroscopy can be viewed as an extension of Racah's methods to a larger set of operators.

VII. DYNAMIC SUPERSYMMETRIES IN PARTICLE PHYSICS

Dynamic supersymmetries of the type described above have also been considered in elementary-particle physics. Indeed, the original suggestion of Miyazawa (1966) was for a supersymmetry of this type. Recently, Catto and Gürsey (1985) have suggested that the internal degrees of freedom of hadrons be described in terms of the superalgebra U(6/21). In this case, 6 is the dimension of the fermionic sector and 21 that of the bosonic sector. The supersymmetric partners are quarks and their associated pairs (diquarks). The fundamental supersymmetric multiplet is

$$\psi = \begin{bmatrix} q \\ \bar{D} \end{bmatrix} , \tag{7.1}$$

where q denotes a quark and $\bar{D}$ an antidiquark. By constructing the mass operator in terms of Casimir operators of the appropriate algebras, one can obtain mass formulas that describe simultaneously baryons and mesons. These mass formulas account for the observed spectra quite well. One of the consequences of the supersymmetric mass formula is that the slopes of the Regge trajectories for mesons and baryons are expected to be equal, as experimentally observed.

Both supersymmetries described above are based on fundamental fermions and their associated pairs. Nambu (1985) has suggested that these form a general class of supersymmetries that could eventually be applied in other situations, such as those encountered in the electron gas or liquid ^{3}He. One can further speculate whether or not this type of supersymmetry is the only one that can be realized in physics. The experimental situation at the present time is in support of this speculation, since no example has been found so far of fundamental supersymmetries.

VIII. CONCLUSIONS

I have briefly reviewed here applications of the concept of dynamic symmetry (and supersymmetry) to several areas of physics, with particular emphasis on applications to nuclear physics. The applications to nuclear physics are based on the interacting boson (Iachello and Arima, 1987) and boson-fermion (Iachello and van Isacker, 1991) models. These models provide a unified framework within which properties of nuclei can be described. In this article, only the symmetry aspects of the models have been reviewed briefly. Their relationship with the microscopic structure of nuclei (in particular the nuclear shell model) has been discussed in a previous review article (Iachello and Talmi, 1987). In view of its relation with the collective model (Bohr and Mottelson, 1975), mentioned briefly at the end of Sec. IV, and to the nuclear shell model, the interacting boson model combines both collective and single-particle aspects into a single theoretical framework. It is for this reason that it has proven to be very useful in the analysis of the complex situations encountered in nuclei.

The methods of spectrum-generating algebras and dynamic symmetry have recently been applied to the study of molecules (Iachello, 1981; Iachello and Levine, 1982), where they have proven also to be useful, especially in the study of polyatomic molecules (Iachello and Oss, 1992). From all these applications, it appears that algebraic methods provide a general framework for attacking many quantum-mechanical problems in physics and chemistry.

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