Instabilities in a sandpile

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Bak, Tang, and Wiesenfeld proposed the idea of self-organized criticality in order to gain a general understanding of the behavior of extended dynamical systems driven in a nonequilibrium state. In particular this idea was intended to explain the ubiquitous scaling behavior and fractal structures that are observed in many different phenomena occurring spontaneously in nature. Recent experiments on the dynamics of a pile of sand, which had been expected to show self-organized criticality, are reviewed and it is shown that sand behaves in a manner more reminiscent of a first-order transition than of a second order (or critical) one.

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I. INTRODUCTION

Geological faults and piles of granular material are examples of driven dynamical systems that often demonstrate large and catastrophic events in the form of earthquakes and avalanches. Are these events similar in any way, if so, what kind of behavior can one expect from this *class* of systems? Is there anything we can say about these systems from first principles *without knowing about the "microscopic" details of the problem?* Renewed interest in this type of phenomenon was generated a few years ago by a series of papers by Bak, Tang, and Wiesenfeld (Bak, Tang, and Wiesenfeld, 1987; Tang and Bak, 1988), in which they introduced the notion of self-organized criticality to describe the behavior of spatially extended, driven dynamical systems in a steady state. Their idea was to make an analogy between such systems and the better understood phenomena that occur in an equilibrium system near a critical point. Following this analogy they proposed a possible underlying cause for the variety of fractal structures that occur ubiquitously in nature. The ideas were initially expressed in terms of a model "pile of sand," and I believe that this is still the simplest way of gaining an intuitive grasp of the concept. We shall see later that real sandpiles actually do not behave in this manner.

In this Colloquium, I shall talk about the dynamics of granular materials, in particular about the behavior of avalanches that occur when the slope of a pile of sand becomes too great. I shall first give an introduction to the ideas of self-organized criticality and then describe some experiments that have been performed to see if these ideas relate to the dynamics of real avalanches. As we shall see, the dynamics of sand is much more complicated than is envisioned in the model of self-organized criticality; I shall describe some of the more salient features of real granular materials and a simple theory of friction to account for some of this behavior. The experiments were performed with Heinrich Jaeger and Chu-heng Liu, and the model of friction was developed in collaboration with them and Thomas Witten.

II. INTRODUCTION TO SELF-ORGANIZED CRITICALITY

If we take a flat, horizontal platform and place one grain of "sand" on it, we expect that it will be stable and not move. If we continue to place grains at random places on the platform, eventually one particle will land on top of one already there, in which case it may be unstable and fall off to one side. This is a microscopic avalanche. As we continue to add particles at random positions, the slope of the pile increases and it becomes more likely that a new particle added at any position will be unstable. When an added particle is unstable, it will again fall to its neighbor, which is also likely to be unstable, so that the avalanche will continue to propagate.

Thus, as the slope of the pile increases, one expects that avalanches should become larger in extent. This should persist until the steady-state angle is reached where, for every particle that is added to the pile, *on average* one particle will fall off the edge of the platform. From our childhood experience with sandpiles we remember that the sand grains do not leave the pile one at a time, but rather do so in larger avalanches which may fluctuate in size. Thus we would expect that as the slope of the pile increases the rearrangements of the particles (i.e., the avalanches) will also increase in size until they span the entire system at the steady-state angle. If we prepare the pile to be above the steady-state angle, we expect that a small perturbation will initiate a massive rearrangement of particles, which will bring the pile back to the steady-state conditions.

This picture of what happens in a pile of sand bears a remarkable resemblance to what occurs near a second-order, or critical, phase transition. For example, in a paramagnet at high temperatures, far above the critical temperature T_c, the spins will have a very small correlation length. The spins will only be aligned over a very small region of space. However, as the temperature is lowered, the correlation length grows until at T_c it becomes infinite and spans the entire system. Bak, Tang, and Wiesenfeld made an analogy between the angle of the sandpile and the temperature of the magnetic system. If the angle is very small, far below the steady-state angle, the avalanches, or fluctuations in the sandpile, will be microscopic. This is analogous to the small fluctuations in the paramagnet at high temperature. As the angle is increased, the avalanches are expected to increase in size, similarly to the increase in the correlation length as the temperature is lowered in the spin system. The steady-state angle, where the avalanches are supposed to be global in extent, is analogous to the critical temperature, where the correlation length has diverged. Thus this model suggests that there is a strong analogy between a dynamical system, such as a pile of sand, at its steady-state condition and an equilibrium system sitting at its critical point.

There is, however, one important difference between the two situations. In order to observe critical behavior in a magnet, one must carefully tune the temperature to be at exactly the critical temperature. In the pile of sand, however, the system apparently tunes itself. If we simply add particles very slowly, the model sandpile will increase its slope until the critical point is reached. If the angle were ever to get too large, a global avalanche would occur to bring the slope back to its critical value. In this sense the sandpile is self-organized critical, since the experimentalist did not have to tune any of the parameters by hand. [Although this seems to be very different from the behavior of a magnet, one could certainly imagine designing a feedback loop that automatically adjusted the temperature to keep the system at the critical point. In such a case it is less clear that there is a distinction between the two systems as regards the self-tuning. One must also ask (Hwa, 1991) at this point whether the condition of adding the grains of sand indefinitely slowly is equivalent to adjusting a parameter such as the temperature in the spin system.]

I have so far followed the original authors, Bak, Tang, and Wiesenfeld, in describing the idea of self-organized criticality in terms of the behavior of a sandpile. It is clear, as they pointed out, that the same ideas could also be used to describe many different kinds of dynamic problems. Indeed self-organized criticality has been suggested as a possible explanation for the power-law behavior seen in many systems. Thus $1/f$ noise in metal wires, the power laws seen in turbulent systems, the power-law distribution of earthquake sizes, the fractal nature of coastlines, the power-law distribution of mass in the universe, the distributions observed in the cellular automaton game of life, have all been ascribed to some self-organized critical behavior. Clearly, if even a small portion of these systems turns out to be self-organized critical, the model is important to understand. For this reason we have tried to test its validity in the simplest of systems to which it was applied—the dynamics of avalanches in a sandpile, the system for which the ideas were first enunciated.

III. EXPERIMENTS ON SAND

The experiment we designed is very simple in concept (Jaeger, Liu, and Nagel, 1989). We start with a three-sided box. As shown in Fig. 1(a), we add sand from above in such a way that it is spread evenly over the surface of the container. As sand is added it forms a wedge-shaped pile starting at the open edge of the box. Once the pile has reached the steady-state condition, as sand is added from above the avalanches will remove the same average amount from this open edge. In order to measure the size distributions of the avalanches as well as their temporal occurrence, we have placed a pair of capacitor plates immediately below the front edge of the box, so that all the sand that falls off the system must flow through the capacitor. The size of each avalanche is detected by measuring the change of the capacitance as a function of time. The capacitor is sensitive enough to measure even the presence of a single grain of sand flowing through it. The time dependence of the capacitance can be measured to determine whether there is a power-law behavior in the avalanche distribution.

There are two possible problems with this experiment, which concern the mechanism for adding particles: any fluctuation in the *rate* of addition of particles or any non-randomness in the *positions* of where the sand is placed might influence the rate of avalanche occurrence. A second experiment, shown in Fig. 1(b), can eliminate these problems. In this case we fill a semicylindrical drum with sand and rotate it, at a constant angular velocity, about its axis. The lower half of the drum is closed off, so that as it is rotated the sand will initially increase to its steady-state slope and thereafter will maintain that slope through avalanches across the top lip of the parti-

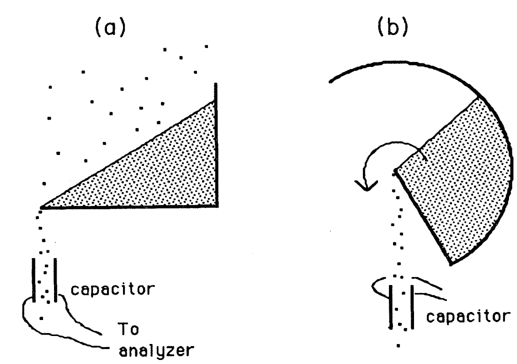

FIG. 1. A schematic diagram of the side view of two experimental configurations: (a) an open box with sand added from the top and (b) a rotating semicylindrical drum. In both cases the sand falls through a pair of capacitor plates. The signal from the capacitor is sent to a spectrum analyzer.

tion. Again the sand that flows out of the system is monitored with a pair of capacitor plates as in the first version of the experiment described above. In this experiment, since the angular rotation velocity is kept constant, there is no fluctuation in the rate of increase of the upper surface of the pile, and since the entire surface is tilted uniformly there is no randomness in the positions of where the sand is placed. In what follows, I shall show results for the second of these experiments, although the same qualitative behavior is found in both.

If the ideas of self-organized criticality were applicable to the behavior of avalanches, we would expect that the distribution of avalanche sizes would show power-law behavior. What we find is actually quite different. In Fig. 2(a), I show the change in capacitance ΔC as a function of time, as the drum is rotated. Since the capacitance changes due to the amount of sand flowing through the plates, ΔC is simply proportional to the current of flowing sand. Instead of finding avalanches of all sizes obeying a power-law distribution, we find only avalanches of one typical size. All the avalanches repeatedly correspond to global ones which span the entire surface of the sandpile. The spikes in the figure occur quite regularly in an almost periodic pattern.

This leads us to the realization that the sandpile does not act in the manner hypothesized above but rather has two important angles between which the slope oscillates. The pile is stable and does not collapse until the slope reaches an upper value θ_m. When the slope becomes greater than this maximum angle of stability the pile is unstable and a global avalanche occurs. This avalanche brings the slope to a smaller value, called "the angle of repose." As more sand is added to the system, the slope of the pile again increases until θ_m is reached and another avalanche is generated. The regularity of the spikes in Fig. 2(a) indicates that the difference between the two angles, $\delta \equiv \theta_m - \theta_r$, is well defined. If we take the power spectrum of the data shown in Fig. 2(a), we get the results shown in Fig. 2(b). There is a peak at low frequencies corresponding to the average interval between spikes. At intermediate frequencies, the spectrum is flat, and at high frequencies it falls off rapidly. The high-frequency rolloff is due to the shape of each individual spike and occurs at a frequency corresponding to the time it takes an avalanche to sweep through the system.

This behavior is more reminiscent of a first-order transition than it is of second-order behavior. In a first-order transition there is a barrier in the free-energy surface that separates the two phases. See Fig. 3. If the system is initially in the high-temperature phase, then as the temperature is lowered past the transition temperature it will remain in the high-temperature state until a fluctuation occurs that takes it over the barrier into the low-temperature state. Thus a first-order transition occurs by a process of nucleation and growth, where a fluctuation spontaneously creates a nucleus of the equilibrium phase, which then grows to envelope the entire system. In a first-order transition it is therefore possible to supercool one phase past its equilibrium phase transition. This leads naturally to hysteresis.

In our experiments there is hysteresis in the behavior similar to the ability to supercool a liquid below its equilibrium first-order freezing temperature. Thus the slope can increase past the angle of repose and exist in a metastable situation with $\theta_r < \theta < \theta_m$. One might even go so far as to associate the upper maximum angle of stability, θ_m, with a spinodal point. Above this point the pile is no longer metastable but is totally unstable.

A recent set of experiments by Held and co-workers (Held, Solina, Keane, Haag, Horn, and Grinstein, 1990) indicated that a different behavior could be observed if the pile were made sufficiently small. These experimenters added single grains of sand to a pile confined to a very small platform and measured the distribution of particles falling off the edge after each grain was added. For

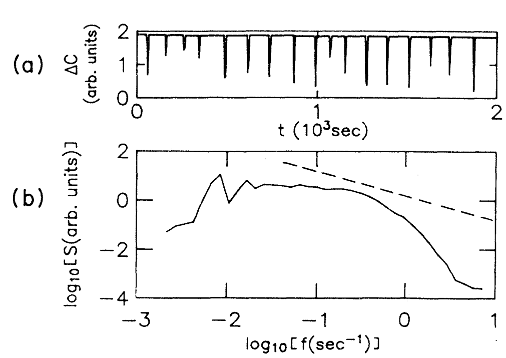

FIG. 2. Avalanches in the rotating drum experiment: (a) The change in capacitance ΔC as a function of time, as the drum in Fig. 1(b) is rotated. The sand consists of spherical glass beads of diameter 0.5 mm. (b) The power spectrum $S(f)$ for the same trace as shown in (a). For comparison, a $1/f$ power spectrum is shown by the dashed line.

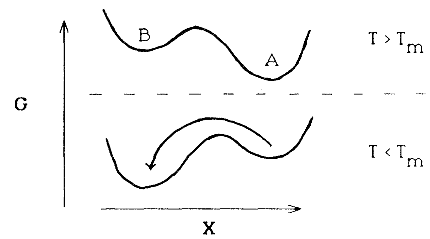

FIG. 3. A schematic illustration of the free-energy surface at a first-order phase transition. G represents the free energy and X represents a coordinate of the system such as the density. At high temperature, above the phase-transition temperature T_m,the lowest free-energy minimum occurs at A, whereas at low temperature it occurs at B. As the temperature is lowered, the system remains in the minimum at A until a nucleation event occurs. Nucleation is caused by a fluctuation that takes a small part of the sample over the energy barrier from the metastable minimum at A to the stable minimum at B. If the nucleus is sufficiently large, it will grow to envelop the entire sample.

platforms with very small diameter they found that the distributions of avalanches could be superimposed on top of one another after being suitably scaled. This result has been taken as evidence (Bak and Chen, 1991) that sandpiles do, after all, demonstrate self-organized criticality.

We have proposed an argument showing how finite-size effects can affect the behavior of avalanches and obscure the first-order nature of the transition when the pile is made sufficiently small (Liu, Jaeger, and Nagel, 1991). The difference between the two angles $\delta \equiv \theta_m - \theta_r$ is approximately two degrees. If the base of the pile is small, it may not be possible to add even a single grain, of diameter d, to the top of the pile without bringing the slope from below θ_r to above θ_m. As shown in Fig. 4, the base of the pile, L, must be greater than d/δ in order for an entire grain to fit between the two angles. For our case of $\delta \approx 2°$ this implies that $L > 30d$. For smaller piles of sand, the pile is so small that the slope is not defined with a resolution that can distinguish between θ_m and θ_r. In this case the first-order behavior of the pile must be cut off. The experiments of Held *et al.* indicate that the transition to the small-size regime occurs almost precisely at this value of the platform radius. Thus the sandpile must be very small indeed before the first-order nature of the transition is blurred by the finite-size effects. Although for platform sizes smaller than $L \approx 30d$ it is possible to superimpose the different distribution curves on top of one another, this should not be taken as evidence of true critical behavior. First, with the largest length scale of only 30 bead diameters, there is not sufficient dynamic range to tell whether any significant scaling does indeed occur. Second, there is no region where asymptotic behavior can be observed. (One would normally think that the scaling should occur in the limit of large distances, whereas here it is precisely at large distances that the scaling results disappear. One cannot look at the asymptotically small piles, since piles cannot be made with diameters less than a single grain.)

Self-organized criticality was not observed in our large sandpiles primarily because the slope of the pile oscillated between two distinct angles. It was plausible that if we could get rid of one of these angles the self-organized critical behavior would become manifest. In order to do this we introduced vibrations to the system, so that the system would be made unstable as soon as the slope increased above the angle of repose and the upper angle θ_m would no longer be encountered. In Fig. 5 we show the results of the power spectrum for the avalanches when different amounts of vibration were used. No new scaling behavior was observed as the vibration intensity was increased. Instead the high-frequency rolloff became progressively more rounded. Thus this experiment, as well, did not indicate that avalanches in a sandpile behave in a self-organized critical manner.

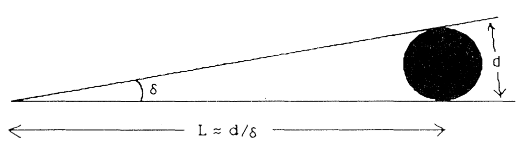

FIG. 4. Finite-size effects should become important if the radius of the pile is smaller than $L \approx d/\delta$. For L less than this value the pile is too small to be able to distinguish between θ_r and θ_m.

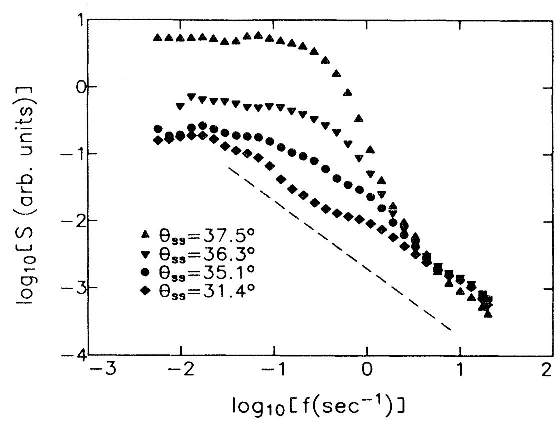

FIG. 5. The power spectra for avalanches occurring in a rotating drum in the presence of vibrations. The intensity of the vibration is parametrized by θ_{ss}, which is the steady-state angle of the pile when the drum is rotated with a given angular velocity Ω (in this case $\Omega = 1.3°/\text{min}$). As in Fig. 2, a $1/f$ power spectrum is shown by the dashed line for comparison.

Clearly sand does not behave in a manner predicted by the theory of self-organized criticality. Instead it has many intriguing properties of its own which we do not fully understand. For example, sand behaves like a solid in that its upper surface remains stable at a nonhorizontal angle even under the influence of gravity. Yet when the angle grows too large, the sand begins to flow like a liquid. The flowing sand, however, is quite different from liquids with which we are well acquainted. The region where the flow takes place is confined to a narrow layer near the surface of the pile. The bulk does not flow at all. There is hysteresis in the sandpile as well. When the slope lies between θ_m and θ_r the pile can have two states, depending on its initial conditions: it can either be stationary or it can be flowing.

IV. FRICTION IN SAND

In order to explain some of these phenomena we have proposed (Jaeger, Liu, Nagel, and Witten, 1990) a simple friction law for the motion of a grain of sand on the surface of a pile with slope θ. We start with Newton's laws:

$$ma = mg \sin\theta - F , \tag{1}$$

where F is the friction force retarding the acceleration down the pile. In a steady state with no acceleration, the friction force just balances the driving force: $F = mg \sin\theta$. We can calculate the friction from

$$F=\frac{dE}{dx}\ , \tag{2}$$

where E is the energy of the particle and x is the distance traveled in the forward direction. There will be two contributions to E. The first will be the kinetic energy $E_k=\frac{1}{2}mv^2$ and the second will be the potential energy $E_p=mgh$ where h is the height. Every time the particle has moved a distance d corresponding to its own diameter, it will lose much of its kinetic energy, due to collisions in the closed-packed system, and a portion of its potential energy due to falling into depressions between particles in the layer on which it is moving. From these two sources of dissipation we can write down a friction law, which has the form

$$F=\frac{amg}{1+b\left[\frac{v^2}{gd}\right]}+cmg\left[\frac{v^2}{gd}\right]\ , \tag{3}$$

where a, b, and c are constants.

As can be seen in Fig. 6, the friction starts out at zero velocity with a finite value, then decreases as the velocity increases, and then starts to increase rapidly after reaching its minimum value. It is easy to see that the value of F at $v=0$ corresponds to $mg\ \sin\theta_m$, its value just before the pile becomes unstable. The value of the friction at its minimum value near $v\simeq(gd)^{1/2}$ is $mg\ \sin\theta_r$. That is, the only steady-state solution for the pile is to be stationary below the angle of repose. In between θ_m and θ_r there are three steady-state solutions. The middle one, which occurs in the region where the slope of F versus v is negative, is an unstable solution. The other two solutions, at $v=0$ and at large velocity, are both stable. In this region there is hysteresis since, depending on the initial conditions of the system, the pile may be in one or the other of the two stable solutions. Finally it is possible to show that, because the minimum in the $F(v)$ curve occurs at a finite velocity, this form of friction law can account for the existence of a boundary layer for the flowing material.

In this article I have tried to review some of the ideas of self-organized criticality. Although the theory was originally formulated in terms of avalanches in granular systems, a series of experiments on real sandpiles indicates that these systems behave in a much different and more complicated manner than that predicted by the model. Our demonstration that granular materials do not show self-organized critical behavior should not be taken as evidence that the self-organized critical systems do not exist. There may be many systems that display

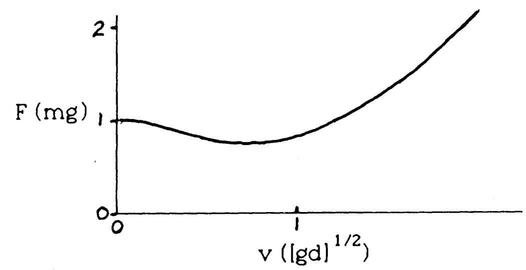

FIG. 6. The friction as a function of the velocity, as given by Eq. (3) for a grain of sand moving on the top surface of a sandpile. The parameters used are $a=b=1$ and $c=0.5$.

this behavior. What our results do show is that the ideas of self-organization must be extended to include first-order as well as critical behavior. Indeed, first-order behavior may be the generic result and self-organized critical behavior might be the exceptional situation. What is also apparent is that we do not yet know what conditions would be necessary and sufficient for critical behavior to exist. After all, granular materials were originally proposed as the prototypical example of self-organized critical behavior. The fact that they do not behave in the predicted way should give us pause.

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