Quantum mechanics and field theory with fractional spin and statistics

Stefano Forte*

Service de Physique Théorique,† C.E. Saclay, F-91191 Gif-sur-Yvette CEDEX, France

Planar systems admit quantum states that are neither bosons nor fermions, i.e., whose angular momentum is neither integer nor half-integer. After a discussion of some examples of familiar models in which fractional spin may arise, the relevant (nonrelativistic) quantum mechanics is developed from first principles. The appropriate generalization of statistics is also discussed. Some physical effects of fractional spin and statistics are worked out explicitly. The group theory underlying relativistic models with fractional spin and statistics is then introduced and applied to relativistic particle mechanics and field theory. Field-theoretical models in 2+1 dimensions are presented which admit solitons that carry fractional statistics, and are discussed in a semiclassical approach, in the functional integral approach, and in the canonical approach. Finally, fundamental field theories whose Fock states carry fractional spin and statistics are discussed.

CONTENTS

I. INTRODUCTION

Planar physical systems, i.e., systems in two space and one time dimensions, display a variety of peculiar quantum-mechanical phenomena, ranging from massive gauge fields to soluble quantum gravity (see Jackiw, 1991). These are linked to the peculiar structure of the rotation, Lorentz, and Poincaré groups in 2+1 dimensions and to the possibility (characteristic of odd-dimensional space-time) of introducing local, Lorentz-invariant interaction Lagrangians with unusual transformation properties under CP and general coordinate transformations, the Chern-Simons and Hopf topological interactions. These features combine in giving rise to the physical effect that is the subject of the present paper, the existence of quantum states that carry angular momentum which is not quantized in half-integer units and whose statistics is neither bosonic nor fermionic.

It is *a priori* clear that for a planar system there is no reason why angular momentum should be quantized: the (spatial) rotation group is Abelian, SO(2), and admits a continuum of representations, characterized by the eigenvalue j of its only generator J, the angular momentum operator. Physically, the angular momentum vector has fixed direction (orthogonal to the plane where the system is confined) and arbitrary length. Upon rotation by an angle θ the wave function ψ_j of a system with angular momentum j acquires a phase

$$e^{i\theta J}\psi_j = e^{i\theta j}\psi_j \tag{1.1}$$

and, in general, is not invariant upon rotation by 2π.

It is natural to expect that it should be possible to generalize accordingly the usual definition of statistics, as well as the spin-statistics theorem. That is, one would expect that the wave function for an n-particle system $\psi(q_1,\ldots,q_n)$ (where q_i denotes the set of all quantum numbers characterizing the ith particle) may be chosen to satisfy

$$\psi(\ldots,q_i,\ldots,q_j,\ldots)=e^{2\pi i\sigma}\psi(\ldots,q_j,\ldots,q_i,\ldots) \tag{1.2}$$

with σ, the statistics parameter, an arbitrary real number (modulo integer, by definition). The cases of bosons and fermions correspond, respectively, to $\sigma=0$ and $\sigma=\frac{1}{2}$, and one would expect there to be a generalized spin-statistics relation, of the form

$$\sigma=\pm j\ . \tag{1.3}$$

Note that the sign here is immaterial in the case of bo-

*On leave from I.N.F.N., Sezione di Torino, via P. Giuria 1, I-10125, Torino, Italy (present address).

†Laboratoire de la Direction des Sciences de la Matière du Commissariat à l'Energie Atomique.

sons or fermions, while it is significant in the case of generic statistics.

This is indeed the case, although the justification for the existence of arbitrary statistics and a generalized spin-statistics theorem is not entirely trivial: as a matter of fact, Eq. (1.3) turns out to be correct only in special cases (cf. Secs. V.C and VI.C below). Also, it is not obvious how quantum-mechanical theories that display such physical properties should look. In this paper we shall review the actual implementation of these ideas in quantum models, within the realms of both particle mechanics and field theory, nonrelativistic and relativistic.[1]

First, we shall deal with nonrelativistic quantum mechanics (Sec. II). We shall exhibit several simple systems in which fractional spin and statistics can be made to appear rather naturally. These will serve as reference working examples in the subsequent, more formal treatment. Then, in Sec. III, we shall discuss the appearance of fractional spin and statistics from first principles in the path-integral approach. In Sec. IV we shall discuss some physical applications to simple systems (harmonic oscillator, ideal gas), and sketch some theoretical ideas that suggest the relevance of fractional statistics to the theory of the fractional quantum Hall effect, and, more generally, to condensed matter physics.

Next, in Sec. V, we shall see how the previous group-theoretical considerations can be made relativistic, by looking at the $2+1$-dimensional Lorentz group, and we shall outline the characteristic features of fractional spin and statistics in relativistic theories. The group-theoretical approach will be applied to a detailed discussion of relativistic particle mechanics with fractional spin and statistics. In Sec. VI we shall consider relativistic field theory, and in particular we shall discuss the construction of solitons with fractional spin and statistics in the O(3) and CP^1 model, first semiclassically, then from a path-integral and canonical point of view. Finally, we shall discuss how fundamental theories whose states carry fractional spin and statistics may be constructed, and we shall outline their features.

The main emphasis of our treatment will be on the construction of the various models from first principles, and on explicit calculations: we shall give a self-contained rerun of the usual quantum mechanics and field theory for systems with fractional spin and statistics, and stress its peculiarities. By contrast, we shall treat only in a cursory fashion, or not at all, several further related lines of development, most notably, the algebraic (axiomatic) approach to fractional spin sectors; the discussion of the remarkable Chern-Simons theories, which (both quantum-mechanically and field-theoretically) display besides fractional statistics numerous startling effects, such as the appearance of novel soliton classical solutions and unexpected links to two-dimensional conformal field theory, knot theory, and integrable models; and the applications of fractional spin and statistics to the theory of the fractional quantum Hall effect and high-T_c superconductivity, which are at the origin of a large part of the recent interest in the subject. For these topics, we refer the reader to some ample and clear reviews that have appeared in the literature: Fröhlich, Gabbiani, and Marchetti (1989) on the axiomatic approach, Jackiw (1991) and de Sousa Gerbert (1991) on Chern-Simons theory, and Wilczek (1990a, 1990b) on the applications to condensed-matter physics. Previous reviews that cover some parts of the material discussed here are those of Mackenzie and Wilczek (1988) and Arovas (1989).

We shall see that quantum mechanics and field theory with fractional spin and statistics reveal a score of new nontrivial quantal phenomena that are intrinsically interesting and justify a treatment for their own sake.

II. THE ORIGIN OF FRACTIONAL SPIN AND STATISTICS

A. Towards fractional spin: the solenoid

We shall discuss first a very simple system (Wilczek, 1982a; Jackiw and Redlich, 1983): a point electric charge moving on a plane in the (external) magnetic field of an infinitely thin solenoid perpendicular to the plane. Although this system *does not* display fractional spin, it provides a convenient stage from which to study the appearance of the effect.

Our system is described by the Lagrangian[2]

$$L=\tfrac{1}{2}m\dot{\mathbf{x}}^2+e\dot{\mathbf{x}}\cdot\mathbf{A}(\mathbf{x})\ , \tag{2.1}$$

where e and m, are respectively, the charge and mass of the particle, located at $\mathbf{x}$ and interacting with the poten-

[1]Excitations with fractional statistics are often referred to as *anyons*. They should not be confused with *parafermions*, which have canonical (integer or half-integer) spin but satisfy a modified exclusion principle. The objects of our study, instead, have fractional spin and obey (see Sec. III.A, below) the usual exclusion principle. In this paper we shall refrain from using unconventional nomenclature. Integer or half-integer spin and statistics will be referred to as canonical, while spin and statistics that are not quantized in half-integer units will be called fractional. This is somewhat of a misnomer, as spin and statistics will be allowed to take any real value and not only rational values, as the term fractional would seem to imply.

[2]Our notational conventions will be as follows: latin indices take the values 1,2, while greek indices run from 0 to 2; x^1,x^2 are space coordinates and $x^0\equiv t$ is the time coordinate; the three-dimensional metric is $(+,-,-)$; the vector notation always denotes the (two) spatial components of vectors; ρ,ϕ are polar coordinates on the space plane; repeated indices are summed over; ϵ^{ab} and $\epsilon^{\mu\nu\rho}$ are, respectively, the two- and three-dimensional completely antisymmetric tensors, with the convention $\epsilon^{12}=\epsilon^{012}=1$; the exterior product of two vectors is defined as $\mathbf{v}\times\mathbf{w}=\epsilon^{ab}v^a w^b$ (notice that it is a scalar).

tial $\mathbf{A}$. The momentum canonically conjugate to $\mathbf{x}$ is

$$\mathbf{p}=m\dot{\mathbf{x}}+e\,\mathbf{A} \tag{2.2}$$

and leads to the Hamiltonian

$$H=\frac{1}{2m}(\mathbf{p}-e\,\mathbf{A})^2 . \tag{2.3}$$

The potential $\mathbf{A}$ is chosen to be that of a solenoid field,

$$A^a=-\frac{\epsilon^{ab}}{2\pi}\frac{\hat{x}^b}{\rho}\Phi(\rho) , \tag{2.4}$$

where $\Phi(\rho)$ is a function of the radial coordinate ρ which satisfies $\Phi(\rho)=\Phi_0=\text{const}$ when $\rho>\rho_0$ (outside the solenoid), leading to the magnetic field[3]

$$B=\frac{1}{2\pi}\frac{1}{\rho}\frac{d}{d\rho}\Phi(\rho) , \tag{2.5}$$

whose flux through any surface containing the solenoid is constant and equal to Φ_0. The solenoid can be made infinitely thin by letting $r_0\to 0$.

Suppose now that the A field is switched on adiabatically; at time t_0 we take $\Phi(t_0)=0$ for all ρ, that is, the A field vanishes everywhere, and then A is slowly turned on until, at time t_1, $\Phi(t_1)=\Phi_0$. According to Faraday's law an electric field

$$E^a=\frac{1}{2\pi\rho}\epsilon^{ab}\hat{x}^b\frac{d}{dt}\Phi(t) \tag{2.6}$$

is generated. This exerts a force on the particle, whose mechanical angular momentum

$$J_m\equiv\mathbf{x}\times m\dot{\mathbf{x}} \tag{2.7}$$

grows accordingly:

$$\frac{dJ_m}{dt}=\mathbf{x}\times e\mathbf{E}=-\frac{e}{2\pi}\dot{\Phi} . \tag{2.8}$$

Thus if the spectrum of J_m in the absence of background (at initial time t_0) is given by the integers $J_m=\ell$, $\ell\in\mathbb{Z}$, with the background field (2.4) (at time t_1) it is given by

$$\begin{aligned} &J_m=\ell-\frac{\Phi}{2\pi};\quad \ell\in\mathbb{Z} ,\\ &\Phi=e\Phi_0 . \end{aligned} \tag{2.9}$$

An equivalent way of seeing this is to assume that the A field has the form of Eq. (2.4) from the onset and to notice that, since the potential outside the solenoid is constant, we may set it to zero by a gauge transformation:

$$\mathbf{A}'=\mathbf{A}-\nabla\Omega=0 , \tag{2.10}$$

$$\Omega=\frac{\phi}{2\pi}\Phi . \tag{2.11}$$

The wave function is transformed to

$$\psi'=e^{-i\Omega}\psi . \tag{2.12}$$

If ψ satisfies periodic boundary conditions, $\psi(\rho,\phi+2\pi)=\psi(\rho,\phi)$, then ψ' satisfies

$$\psi'(\phi+2\pi)=e^{-i\Phi}\psi'(\phi) . \tag{2.13}$$

Noting that the mechanical angular momentum is the operator $J_m=-i(\partial/\partial\phi)$, we see again that its spectrum is shifted according to Eq. (2.9) (Wilczek, 1982a).

However, of course, nothing remarkable has happened (Jackiw and Redlich, 1983): the mechanical angular momentum J_m, defined in Eq. (2.7), is not in general equal to the total canonical angular momentum operator $J=\mathbf{x}\times\mathbf{p}$. Rather, with $\mathbf{p}$ given by Eq. (2.2),

$$J=\mathbf{x}\times(m\dot{\mathbf{x}}+e\,\mathbf{A})=J_m+J_f . \tag{2.14}$$

According to the correspondence principle, it is J, not J_m, that generates rotations of the wave function, i.e., it is $\mathbf{p}$, and not $\dot{\mathbf{x}}$, which becomes the operator $-i\nabla$; therefore the spectrum of J remains canonical. This is also clear if one observes that angular momentum is conserved in the model (2.1), thus $\dot{J}=0$, and necessarily $\dot{J}_f=-\dot{J}_m$; if we switch on the field adiabatically $J_f=-J_m$.

Equivalently, if we gauge-transform the potentials A [Eq. (2.4)] by Eqs. (2.10) and (2.11), we should recall that, in general, an operator $\mathcal{O}$ also transforms according to $\mathcal{O}\to e^{-i\Omega}\mathcal{O}e^{i\Omega}$. Because the gauge function Ω (2.11) is singular in the origin, it is not surprising that it may change both the boundary conditions on the admissible wave functions and the form of the operator itself. Indeed, it is easy to verify that

$$e^{-i\Omega}(\mathbf{x}\times\mathbf{p})e^{i\Omega}=\mathbf{x}\times\mathbf{p}+\frac{\Phi}{2\pi} . \tag{2.15}$$

Acting on the wave functions ψ' [Eq. (2.13)], this operator has the usual, canonical spectrum.

We conclude that the unusual spectrum of J_m is not the sign of any new physics in the model (2.1) (Jackiw and Redlich, 1983). However, this simple exercise will teach us how to construct a variant of the model (2.1) in which the angular momentum reduces to J_m [Eq. (2.7)] and the spectrum of angular momentum gets shifted. We shall do this in the next section.

B. The virtual flux tubes

In order to understand better the origin of the various contributions to the angular momentum J [Eq. (2.14)], let us look at the total angular momentum of the particle-field system, i.e., let us consider now the electromagnetic field as a dynamical, rather than external, field (Jackiw and Redlich, 1983; see also Paranjape, 1987, for a detailed discussion). The gauge-invariant angular momentum stored in the field is

$$\begin{aligned} J_{em}&=\int d^2y\;\mathbf{y}\times\mathbf{P}(\mathbf{y})\\ &=-\int d^2y\;\mathbf{y}\cdot\mathbf{E}(\mathbf{y})B(\mathbf{y}) , \end{aligned} \tag{2.16}$$

[3]Notice that in 2+1 dimensions the electric field is a two-vector and the magnetic field is a (pseudo)scalar.

where $\mathbf{P}(\mathbf{y})$ is the field's momentum density.

The total gauge-invariant angular momentum is the sum

$$J_t = J_m + J_{em} \ . \tag{2.17}$$

With the potential (2.4) in the limit $\rho_0 \to 0$ the magnetic field is concentrated in the origin, thus $J_{em}=0$, and it would seem that the angular momentum reduces to J_m. There is, however, a caveat: the gauge-invariant total angular momentum J_t [Eq. (2.17)] does not coincide with the canonical, conserved angular momentum J_c, obtained from the application of Noether's theorem. Rather, J_t and J_c differ by a surface term—the integral of the total derivative (see, for example, Jackiw, 1985):

$$\begin{aligned} J_c &= J_t + J_s \ , \\ J_s &= \int d^2y \, \partial_i [E^i(\mathbf{y}) \mathbf{y} \times \mathbf{A}(\mathbf{y})] \ . \end{aligned} \tag{2.18}$$

The canonical angular momentum J_c is not gauge invariant. However, if the fields are sufficiently long ranged, J_s may be nonvanishing and J_t may not be conserved, whereas J_c always is. Indeed, it is easy to see that, with the potential (2.4),

$$\dot{J}_t = -\dot{J}_s = -\frac{e}{2\pi}\dot{\Phi} \ . \tag{2.19}$$

It is the conserved quantity, J_c, that becomes the angular momentum operator in the quantum theory. With a potential of the form (2.4), J_c coincides with the operator J [Eq. (2.14)]: it is now clear that its two components J_m and J_f are to be interpreted, respectively, as the angular momentum associated with the particle itself and that due to the electromagnetic field, which is present because of the long-range nature of the field, as demonstrated by Eq. (2.19).

The angular momentum should reduce to J_m if we eliminate the electromagnetic field and the particle-field interaction in Eq. (2.1) in favor of an effective particle-particle interaction. We define, therefore, the following *new* theory

$$\begin{aligned} L &= \tfrac{1}{2} m \dot{\mathbf{x}}^2 + L_i[x] \ , \\ L_i[x] &= \frac{\Phi}{2\pi} \epsilon^{ab} \dot{x}^a \frac{\hat{x}^b}{r} \ . \end{aligned} \tag{2.20}$$

Here there is no electromagnetic field; rather, $L_i[x]$ is directly a functional of the particle's position $\mathbf{x}$. In order to avoid problems due to the fact that the interaction in Eq. (2.20) is manifestly ill defined at $\mathbf{x}=0$, we exclude the origin from the configuration space, i.e., we assume $\mathbf{x} \neq 0$. Since there is no field, now $J_c = J_t = J_m$, that is, the angular momentum has the fractional spectrum (2.9).

It is easy to verify by direct computation that the spectrum of the canonical angular momentum defined by the Lagrangian (2.20) is indeed fractional. The interaction term may be written as a total derivative,

$$L_i = \frac{\Phi}{2\pi} \frac{d}{dt} \Theta[\mathbf{x}] \ , \tag{2.21}$$

where Θ is the polar angle of $\mathbf{x}$:

$$\Theta[\mathbf{x}] = \tan^{-1}\left[\frac{x^2}{x^1}\right] \ . \tag{2.22}$$

Addition of a total derivative shifts the constants of motion by a term that is equal to (minus) the variation of the argument of the total derivative under the symmetry associated with the constant of motion (see for example, Jackiw, 1985). In the case at hand, the (canonical Noether) angular momentum J_c is augmented by the variation of $-(\Phi/2\pi)\Theta$ under rotations, i.e., it is shifted according to Eq. (2.9).

Although the theory (2.20) displays a spectrum of the total angular momentum that is shifted with respect to the integers, there is a problem in interpreting the fractional contribution to the angular momentum as a fractional spin, i.e., as an intrinsic angular momentum. Indeed, the term (2.21) transforms in such a way as to shift the angular momentum according to Eq. (2.9) only upon rotations about the origin. An intrinsic contribution to the angular momentum should instead be independent of the choice of center of rotation. This is of course a consequence of the fact that the origin is unnaturally singled out by the interaction (2.21).

We can obviate this problem by considering the many-particle generalization of the above construction. We attach to each particle an infinitely thin solenoid (Wilczek, 1982a), which interacts with the other particles, then we dispose of the electromagnetic field and we keep just an effective particle-particle interaction. We are thus led to the Lagrangian (Wu, 1984b; Arovas *et al.*, 1985; see also Jackiw, 1991)

$$L = \sum_i \tfrac{1}{2} m \dot{\mathbf{x}}_i^2 + L_i \ , \tag{2.23}$$

$$L_i = \sum_{i \neq j} \frac{\Phi}{2\pi} \frac{d}{dt} \Theta(\mathbf{x}_i(t) - \mathbf{x}_j(t)) \ , \tag{2.24}$$

where Θ is as in Eq. (2.22). The sums run over particles, and the points $\mathbf{x}_i = \mathbf{x}_j$ are excluded from configuration space. A rerun of the previous argument for the Lagrangian (2.23) shows that the spectrum of angular momentum for a system of n particles is given by

$$J_t = \ell - \frac{\Phi}{2\pi} n(n-1); \quad \ell \in \mathbb{Z} \ . \tag{2.25}$$

Since the function $\Theta(\mathbf{x}_i - \mathbf{x}_j)$ [Eq. (2.24)] depends on the relative positions of pairs of particles, the fractional contribution to the angular momentum (2.25) does not depend on the center of rotation.

Finally, it is instructive to look at the theory (2.23) in the Hamiltonian formalism. The Hamiltonian is

$$\begin{aligned} H &= \sum_i \frac{1}{2m} [\mathbf{p}_i - \mathcal{A}(\mathbf{x}_i)]^2 \ , \\ \mathcal{A}(\mathbf{x}_i) &= \frac{\Phi}{2\pi} \sum_{j \neq i} \nabla_i \Theta(\mathbf{x}_i(t) - \mathbf{x}_j(t)) \ . \end{aligned} \tag{2.26}$$

The interaction $\mathcal{A}$ may be eliminated by gauge transformation (which now is not singular, since $\mathbf{x}_i \neq \mathbf{x}_j$ by assumption). The transformed wave function is

$$\psi'(\mathbf{x}_1,\ldots,\mathbf{x}_n)=\prod_i\left[\exp\left[-i\Phi/2\pi\sum_{i\neq j}\Theta(\mathbf{x}_i-\mathbf{x}_j)\right]\right]\psi(\mathbf{x}_1,..,\mathbf{x}_n)$$
$$=\prod_{i<j}\exp\left[-i\frac{\Phi}{\pi}\Theta(\mathbf{x}_i-\mathbf{x}_j)\right]\psi(\mathbf{x}_1,..,\mathbf{x}_n)\ . \tag{2.27}$$

If ψ satisfies the Schrödinger equation associated with the Hamiltonian (2.26), then ψ' [Eq. (2.27)] satisfies the free Schrödinger equation, but if ψ is subject to ordinary boundary conditions, ψ' satisfies "twisted" boundary conditions. For example, in the two-particle case, if ϕ is the relative polar angle of the two particles, ψ' satisfies

$$\psi'(\phi+2\pi)=e^{-2i\Phi}\psi'(\phi)\ . \tag{2.28}$$

It follows that (Wu, 1984b) an equivalent way of formulating the theory (2.20)–(2.26) is to solve the free Schrödinger equation while requiring the wave function to satisfy twisted boundary conditions of the form (2.28) in all couples of variables. It is interesting to observe (Thouless and Wu, 1985) that upon interchange of two of its arguments the wave function ψ' satisfies a modified statistics relation of the form (1.2) with

$$\sigma=-\frac{\Phi}{2\pi}\ . \tag{2.29}$$

It is, however, premature to draw any conclusion about a spin-statistics relations, since we have so far no control over internal quantum numbers associated with the particles; we shall come back to this problem in Secs. III.C and VI.C.

In summary, we have constructed a many-particle theory that displays a spectrum of angular momentum that is shifted with respect to the integers, Eq. (2.25). The infinite-range interaction (2.24) that changes the particles' spin may be regarded as the relic of "virtual" flux tubes (i.e., solenoids of infinitesimal radius) attached to the particles, after eliminating the electromagnetic interaction. It is important to remember, however, that this heuristic picture should not be taken literally. These are not physical solenoids, like those of the system discussed in the previous section, since the latter does not display fractional spin. The interaction that contributes the fractional part of the angular momentum may be eliminated provided the wave function is subject to peculiar boundary conditions. Notice that we nowhere assumed that the theory, apart from the "fractional spin" interaction (2.21), is free: all of the above holds true for generic n-body systems interacting with each other through an arbitrary potential.

C. The Hopf term

The particle-particle interaction (2.21) has the shortcoming of being nonlocal; on the other hand we cannot view it as generated by an ordinary Maxwell field because then the angular momentum stored in the field restores the conventional integer angular momentum spectrum, as discussed in Sec. II.A. In this section we shall see that the interaction (2.21) may be obtained by coupling the particle current to an Abelian gauge field whose dynamics is not governed by the usual Maxwell Lagrangian, but rather by a topological Lagrangian.

Consider indeed the Lagrangian obtained by adding to the particle Lagrangian L_0 a coupling L_c to an Abelian gauge field whose dynamics is provided by L_f (Hagen, 1984):

$$L=L_0+L_c+L_f\ , \tag{2.30}$$
$$L_i=e\sum_i(\dot{\mathbf{x}}_i\cdot\mathbf{A}-A^0)\ , \tag{2.31}$$
$$L_f=\frac{\pi}{\Phi}\int d^2y(\mathbf{A}(\mathbf{y})\times\dot{\mathbf{A}}(\mathbf{y})+A^0(\mathbf{y})B(\mathbf{y}))\ . \tag{2.32}$$

In order to describe the particle-field coupling it is convenient to introduce a covariant charged-particle current defined (in the general n-particle case) as

$$j^\mu(x^\alpha)=\sum_{i=1}^n\int ds\,\delta^{(3)}(x-x_i)\frac{dx^\mu}{ds}\ , \tag{2.33}$$

where s parametrizes the space-time trajectories of the particles, whose locations are $(t(s),\mathbf{x}_i(s))$. We shall always assume that the particle Lagrangian L_0 is such that the particle number is conserved,

$$\partial_\mu j^\mu=0\ . \tag{2.34}$$

The action I associated with the Lagrangian (2.30)–(2.32) may now be written in covariant notation as

$$I=I_0+I_c+I_f\ , \tag{2.35}$$
$$I_c=\int d^3x\,j^\mu(x)A_\mu(x)\ , \tag{2.36}$$
$$I_f=\frac{\pi}{\Phi}\int d^3x\,\epsilon^{\mu\nu\rho}A_\mu(x)\partial_\nu A_\rho(x)\ . \tag{2.37}$$

The field action I_f (2.36) is the Abelian version of the celebrated Chern-Simons action (see de Sousa Gerbert, 1991; Jackiw, 1991). Its peculiar properties are due to the fact that the field is coupled through the $\epsilon^{\mu\nu\rho}$ tensor, which is a generally covariant object. It is often referred to as a topological action because of its sensitivity to the global features of the gauge potential A. For our purposes, however, it is enough to observe that the action I_f is quadratic in the field A: we can therefore compute the path integral over the A field exactly (Polyakov, 1988) and work out the effective current-current interaction induced by the particle-field coupling. We get, up to an irrelevant (albeit infinite) additive constant,

$$I_{\rm eff}[j]=-i\ln\int \mathcal{D}A^{\mu}e^{i(I_c+I_f)}$$
$$=I_H\equiv-\frac{\Phi}{2}\int d^{3x}d^{3}y\, j^{\mu}(x)K_{\mu\nu}(x,y)j^{\nu}(y)\ , \tag{2.38}$$

where the bilocal kernel

$$K_{\mu\nu}(x,y)=-\frac{1}{2\pi}\epsilon_{\mu\rho\nu}\frac{(x-y)^{\rho}}{|x-y|^{3}} \tag{2.39}$$

is the inverse of the operator $\epsilon^{\mu\nu\rho}\partial_{\nu}$ when acting on the current j^{ν}, i.e., it satisfies

$$\epsilon_{\mu\nu\rho}\partial_{\nu}K^{\rho\sigma}(x,y)=\delta_{\mu}{}^{\sigma}\delta^{(3)}(x-y) \tag{2.40}$$

up to the addition of terms proportional to ∂_{σ}, which are irrelevant because j^{σ} is conserved. The bilocal functional of the currents I_H (2.38) is known as the Hopf term, because it is related (in a way we shall discuss in Sec. VI.A) to the Hopf invariant of differential geometry.

We proceed now to show that the current-current interaction (2.38) leads to an effective particle interaction of the form (2.24) (Arovas *et al.*, 1984). The fact that the result turns out to be a total derivative demonstrates the topological nature of the interaction. By substituting the current (2.33) in the action $I_{\rm eff}$ (2.38), we may reduce the volume integrals to line integrals:

$$I_{\rm eff}=\frac{\Phi}{4\pi}\sum_{i,j}\int ds\, ds'\epsilon_{\mu\nu\rho}\frac{dx_i^{\mu}(s)}{ds}$$
$$\times\frac{(x_i(s)-x_j(s'))^{\nu}}{|x_i(s)-x_j(s')|^{3}}\frac{dx_j^{\rho}(s')}{ds'}\ . \tag{2.41}$$

The diagonal terms in the sum over particles are somewhat problematic because of the divergences in the integrand when both $i=j$ and $s=s'$. Indeed, these terms are ill defined in a nonrelativistic treatment; therefore we shall defer their discussion to Secs. V.B and V.C, where we deal with the relativistic theory, and we shall ignore them for the time being. Indeed, in a nonrelativistic theory, these terms may be set to zero by a suitable regularization (Jackiw, 1991).

Let us concentrate on the off-diagonal terms in the sum (2.41). If the curves $x_i^{\mu}(s)$ traversed by the particles were closed curves in Euclidean 3-space, the double integral (2.41) for each value of the indices i,j would be an expression for Gauss's linking number of the curves x_i, x_j

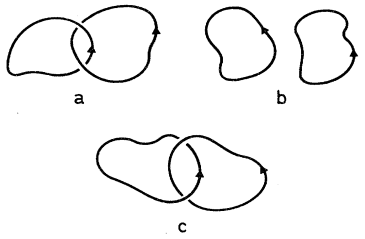

FIG. 1. Linking number of two curves: (a) $l=-1$; (b) $l=0$; (c) $l=1$.

(see Dubrovin *et al.*, 1984); i.e., the integral, evaluated along the two curves, would be proportional to the number of times the two curves link with each other (see Fig. 1). Even though here $x_i(s)$ traverse open curves in Minkowski space, $I_{\rm eff}$ is still related to a linking number, as we show shortly.

Consider a single term in the sum (2.41), I_{ij}, with $i\neq j$, which we rewrite in the form

$$I_{ij}=\int dx_i^{\mu}dx_j^{\nu}\epsilon_{\mu\rho\nu}\frac{(x_i-x_j)^{\rho}}{|x_i-x_j|^{3}}\ . \tag{2.42}$$

In this form, the invariance of I_{ij} upon changes of parametrization of the curves is manifest. Next, observe (Forte and Jolicoeur, 1991) that the integrand in Eq. (2.42) may be written as the curl of a "potential" $\tilde{A}^{\mu}$:

$$\frac{x^{\mu}}{|x|^{3}}=\epsilon^{\mu\alpha\beta}\partial_{\alpha}\tilde{A}_{\beta}(x)\ . \tag{2.43}$$

The function $\tilde{A}$ must be singular, because the left-hand side (lhs) of Eq. (2.43) may be written as divergence. As a matter of fact, the lhs of Eq. (2.43) is the field of a Dirac magnetic monopole (see, for example, Balachandran *et al.*, 1983), and Eq. (2.43) defines $\tilde{A}$ as its potential, which notoriously has a string of singularities (that can be put anywhere by a choice of gauge). Anyway, for our purposes it is enough to pick a particular form of $\tilde{A}$ that satisfies Eq. (2.43); a convenient one is

$$\tilde{A}_0(t,\mathbf{x})=0;\quad \tilde{A}_a(t,\mathbf{x})=-\frac{\epsilon_{ab}x^{b}}{r(t-r)}\ , \tag{2.44}$$

where $r^2=|x|^2=t^2-x_1^2-x_2^2$. We may now use Eq. (2.43) to rewrite I_{ij} [Eq. (2.42)]; we parametrize paths with time:

$$I_{ij}=\int_0^T dt\int_0^T dt'\frac{dx_i^{\mu}(t)}{dt}[\partial_{\mu}\tilde{A}_{\nu}(x_i-x_j)-\partial_{\nu}\tilde{A}_{\mu}(x_i-x_j)]\frac{dx_j^{\nu}(t')}{dt'}$$
$$=\left[\int_0^T dt'\int_0^{t'}dt\frac{dx_i(t)^{\mu}}{dt}\frac{\partial}{\partial x_i^{\mu}}\left[\tilde{A}_{\rho}(x_i-x_j)\frac{dx_j(t')^{\rho}}{dt'}\right]+\int_0^T dt\int_t^T dt'\frac{dx_j(t')^{\mu}}{dt'}\frac{\partial}{\partial x_j^{\mu}}\left[\tilde{A}_{\rho}(x_i-x_j)\frac{dx_i(t)^{\rho}}{dt}\right]\right]+x_i\leftrightarrow x_j$$
$$=-2\int_0^T dt\,\epsilon^{ab}\left[\frac{dx_i^{a}}{dt}-\frac{dx_j^{a}}{dt}\right]\frac{[x_i(t)-x_j(t)]^{b}}{|x_i(t)-x_j(t)|^{2}}+I_g\ , \tag{2.45}$$

where in the last step we used the explicit form of $\tilde{A}$ (2.44) and

$$I_g=\int_0^T dt\left[\tilde{A}_\mu(x_i(t)-x_j(T))\frac{dx_i^\mu}{dt}-\tilde{A}_\mu(x_i(0)-x_j(t))\frac{dx_j^\mu}{dt}\right]+x_i\leftrightarrow x_j\ . \tag{2.46}$$

Now, we may use the fact that

$$\frac{\partial}{\partial y^a}\Theta(\mathbf{x}-\mathbf{y})=\epsilon_{ab}\frac{(x-y)^b}{|\mathbf{x}-\mathbf{y}|^2} \tag{2.47}$$

where $\Theta(\mathbf{x}-\mathbf{y})$ is as in Eq. (2.22) and (2.24). It is important to notice that Eq. (2.47) is satisfied only if in Eq. (2.22) we take the $\tan^{-1}$ to be multivalued, i.e., such that if the vector $(\mathbf{x}-\mathbf{y})$ is rotated by 2π, then the value of Θ is also shifted by 2π, rather than remaining the same. If we took a single-valued definition, the function Θ would necessarily have a branch cut, and there would be an additional contribution on the rhs of Eq. (2.47), concentrated along the cut (see Sec. V for a more detailed discussion on this point). Using Eq. (2.47) we get finally

$$I_{ij}=2\int dt\frac{d}{dt}\Theta(\mathbf{x}_i-\mathbf{x}_j)+I_g\ . \tag{2.48}$$

Equation (2.48) shows that, if we can forget about the term I_g, then I_{ij} has a simple geometrical meaning: it is equal to twice the total angle of rotation of the trajectory of particle i around that of particle J in the course of their evolution. The total angle of rotation, in turn, is equal to (2π times) the number of times the two paths link, i.e., if we close the paths by joining their endpoints to a point at infinity along a fixed direction and in a fixed order (see Fig. 2), the total angle of rotation is equal to (2π) times their linking number. The term I_g vanishes for closed paths; for open paths it is associated with a contribution to the Lagrangian which does not modify angular momentum and statistics (as can be explicitly verified by checking that it is rotationally invariant) and need not concern us here.

Adding up the contributions (2.48) from all pairs of particles, we find that (up to the addition of I_g) $I_{\rm eff}$ is the action associated with the Lagrangian L_i [Eq. (2.23)],

$$\begin{aligned}I_{\rm eff}&=\int dt\,L_i+\frac{\Phi}{4\pi}I_g\\&=l\Phi+\frac{\Phi}{4\pi}I_g\ ,\end{aligned} \tag{2.49}$$

where in the last step l indicates the total linking number (integer for a closed path). We have succeeded in reproducing the nonlocal interaction (2.24) through the coupling to an external field. Note that it is crucial that this field be considered as a genuine dynamical field, rather than an external one, otherwise we would be led back to the action (2.1)–(2.4). Indeed, one may verify explicitly (Jackiw, 1991) that the total angular momentum for the full theory (2.30)–(2.32) has the fractional spectrum (2.25), whereas if one considered the gauge field as external (thus not contributing to the angular momentum) one would get the canonical spectrum of the theory discussed in Sec. II.A. Henceforth we shall always work with the formulation of the theory (2.24) or (2.49), which does not require the introduction of gauge fields and is, in a sense that will be made more precise in the next section, more fundamental. The angular momentum operator for this theory, as discussed in Sec. II.B, in the n-particle case is (Thouless and Wu, 1985)

$$J=J_0-\frac{\Phi}{2\pi}n(n-1)\ , \tag{2.50}$$

$$J_0=\sum_{i=1}^n \mathbf{x}_i\times(-i)\nabla_i\ . \tag{2.51}$$

We are now equipped with the knowledge of a class of models with fractional angular momentum. Starting with any system of particles with conventional angular momentum we can obtain fractional angular momentum in several equivalent ways: we can add to the Lagrangian a nonlocal interaction (2.24), or we can let the particles interact through an Abelian gauge field with "topological" dynamics (2.30), or we can couple the particle's currents through the Hopf term [Eqs. (2.38) and (2.39)] or, finally, we can require the wave function of the system to satisfy twisted boundary conditions of the form (2.28). The angular momentum operator in the first three cases has the form (2.51) and has the spectrum (2.25) when acting on single-valued wave functions; in the last case the wave functions are multivalued, according to Eq. (2.28), in the relative polar angles of all pairs of particles, and the angular momentum, with spectrum (2.25), is found by acting with the operator J_0 (2.51).

It is important to observe that while the interaction (2.24) is equivalent to the requirement of twisted boundary conditions (2.28), and the Hopf dynamics (2.38) and gauge dynamics (2.30) are strictly the same, the latter two theories are not precisely the same as the former two (only the spin-changing terms are) due to the presence of additional terms (such as I_g) which we neglected for the

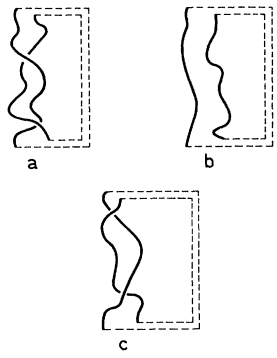

FIG. 2. Two particle trajectories (solid line) define a linking number when their endpoints are joined to infinity (dashed line) along a fixed direction. (a) $l=-1$; (b) $l=0$; (c) $l=1$.

moment, since they have no effect on spin and statistics, but may have other physical effects (see Jackiw and Pi, 1990).

III. NONRELATIVISTIC QUANTUM MECHANICS

A. The path integral on homotopically nontrivial spaces

In the previous section we have seen that fractional angular momentum may be constructed by imposing "twisted" boundary conditions (2.28) on the wave function of a system that, if quantized in the usual way, would have the canonical angular momentum spectrum. It is clear that the possibility of consistently imposing such boundary conditions is due to the peculiar structure of the two-dimensional configuration space: for example, suppose we tried to impose a boundary condition of the form (2.28) on a two-body wave function in three spatial dimensions, by using spherical coordinates and assuming ϕ to be the azimuthal angle. Along the z axis we would get $\psi'=e^{-2i\Phi}\psi'$, which is a contradiction unless $2\Phi=2\pi$ modulo integer, which implies [take Eq. (2.25) with $n=2$] that the spectrum of angular momentum is quantized in the usual way.

In fact, we shall now show that both the possibility of fractional spin and statistics, and the general form of the Lagrangian which embodies it, are consequences of the structure of the configuration space of indistinguishable particles in two (space) dimensions. This result can be obtained in several ways; historically, it was first found in a Hamiltonian approach by looking at the most general Schrödinger equation in two dimensions (Leinaas and Myrheim, 1977). An equivalent way of getting at it is to derive all possible representation of the algebra of observables (Goldin *et al.*, 1981). Yet another distinct possibility is to proceed in a Lagrangian framework and derive the result by looking at the most general path integral (Wu, 1984a). This last option, which is the simplest, is the one we shall follow; its equivalence with the Hamiltonian approach (Wu, 1984b) will be discussed in Sec. III.C; in Sec. V.B (in the relativistic case) we shall provide a derivation based on the representation theory of the Lorentz group (de Sousa Gerbert, 1990; Forte, 1991c).

Let us first see what is special about the configuration space in two dimensions (Leinaas and Myrheim, 1977). The configuration space for n particles in d dimensions is given by the set of d-component vectors. If we wish to allow all kinds of statistics, we ought to assume that the vectors are distinct, i.e., that no two particles may occupy the same point in space, because otherwise the statistics would be necessarily bosonic. For example, in the two-particle case, if

$$\psi(\mathbf{x}_1,\mathbf{x}_2)=e^{2\pi i\sigma}\psi(\mathbf{x}_2,\mathbf{x}_1)\ , \tag{3.1}$$

then if $\mathbf{x}_1=\mathbf{x}_2$ we get a contradiction unless σ is integer. According to Eqs. (2.29) and (2.25) this would imply that the angular momentum is canonical.

Furthermore, since particles are indistinguishable, configurations that differ by the interchange of two particles (i.e., two two-vectors) should be identified. Therefore, if $(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ is an element of $\mathbb{R}^{dn}$, if we call $\mathcal{D}\subset\mathbb{R}^{dn}$ the set of points where $\mathbf{x}_i=\mathbf{x}_j$ for some i, j, and if S_n is the group of permutations of n objects, then the configuration space is the coset

$$\begin{aligned}\mathcal{C}&=\frac{\overline{\mathcal{C}}}{S_n}\ ,\\ \overline{\mathcal{C}}&=\mathbb{R}^{dn}-\mathcal{D}\ ,\end{aligned} \tag{3.2}$$

where S_n acts on the n-tuples of vectors $(\mathbf{x}_i,\ldots,\mathbf{x}_j)$. The main difference between $d=2$ and $d>2$ is that in the former case $\overline{\mathcal{C}}$ is multiply connected, i.e., only in the former case do there exist pairs of paths in $\overline{\mathcal{C}}$ that share the same endpoints and that cannot be deformed continuously into each other (homotopically disconnected paths, henceforth). This in turn implies that the connectedness of $\mathcal{C}$ or, more precisely, the fundamental group (see Dubrovin *et al.*, 1984) of $\mathcal{C}$, is special when $d=2$.

Therefore let us first discuss path integration on multiply connected spaces (Laidlaw and Morette de Witt, 1971). We shall see that this is the natural framework in which to introduce fractional spin and statistics. The path integral on a multiply connected space differs from that on a simply connected space in that, in general, we are free to assign different weights χ to homotopically disconnected paths. The quantum transition amplitude from $q\in\mathcal{C}$ at time t to q' at time t' is

$$\begin{aligned}K(q',t';q,t)&=\langle q',t'|q,t\rangle\\ &=\sum_{\alpha\in\pi_1(\mathcal{C})}\chi(\alpha)K^{\alpha}(q',t';q,t)\ .\end{aligned} \tag{3.3}$$

Here α labels equivalence classes of homotopically connected paths (homotopy classes, henceforth); these form a group, the fundamental group $\pi_1(\mathcal{C})$ (see, for example, Dubrovin *et al.*, 1984). K^{α} is the usual path integral for a theory with Lagrangian L

$$K^{\alpha}(q',t';q,t)=\int_{q(t)\in\alpha}Dq(t)\exp\left[i\int_t^{t'}dt_0L(q(t_0))\right] \tag{3.4}$$

with the integration restricted to paths belonging to the αth homotopy class.

In order to compute the weights χ we must assign paths from point a to point b $(a,b,\in\mathcal{C})$ to homotopy classes. This can be done by choosing a "homotopy mesh," i.e., by picking arbitrarily a reference point $x\in\mathcal{C}$ and defining a (continuous and smooth) path $C(a)$ from x to each point $a\in\mathcal{C}$ (this is possible because $\mathcal{C}$ is connected) (Fig. 3). We assume that $C(a)$ is oriented (i.e., it is traversed in a fixed direction), and we call $C^{-1}(a)$ the path with the opposite orientation. Any path $p(ab)$ from a to b is associated with the closed path

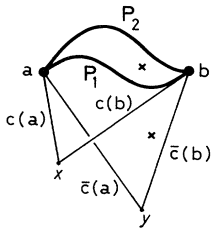

FIG. 3. A path ab is assigned to a homotopy class by closing it through the mesh C to point x and counting the number of points that do not belong to configuration space (indicated by $\times$) which are encircled by the closed path. Paths P_1 and P_2 are not homotopic. Their class assignment always differs by one unit, regardless of the choice of mesh. However, the class assignment of each path with mesh $\bar{C}$ based on y differs by one unit from that computed with mesh C.

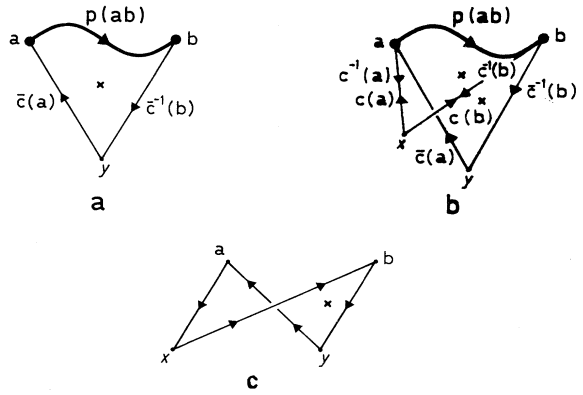

FIG. 4. Computation of the homotopy class through different choices of mesh: (a) computation of the homotopy class of ab through mesh $\bar{C}$ based at y; (b) relationship between computations of the class with mesh C and mesh $\bar{C}$ based at y. The arrows indicate path $\bar{C}(a)C^{-1}(a)C(a)p(ab)C^{-1}(b)C(b)\bar{C}^{-1}(b)$ [Eq. (3.9)]; $\times$ indicates points excluded from configuration space. (c) The path $\lambda\mu$.

$P(ab)\equiv C(a)p(ab)C^{-1}(b)$. Since closed paths on a multiply connected space fall into equivalence classes—the elements of $\pi_1(\mathcal{C})$—this construction naturally assigns $p(ab)$ to an element of $\pi_1(\mathcal{C})$, that to which $P(ab)$ belongs (Fig. 3). Of course, changing the mesh may change the class to which a path belongs; however, it is easy to see that a change of mesh cannot change the "differential" assignment, i.e., if two paths with the same endpoints belong to different classes they cannot be brought into the same class by a change of mesh (see Fig. 3).

Now, there are some physical requirements that the weighted sum (3.3) must fulfill, namely, (a) physical observables must not depend on the mesh. This implies that a change of mesh may at most change all propagation amplitudes $K(q',t';q,t)$ by a universal multiplicative phase factor; (b) the weighted sum (3.3) must satisfy the convolutive property

$$K(q'',t'',q,t)=\int dq'\,\langle q'',t''|q',t'\rangle\langle q't'|q,t\rangle = \int dq'\,K(q'',t'';q',t')K(q',t';q,t)\ . \tag{3.5}$$

This follows from the assumption that $|q,t\rangle$ be a complete set of states, and is satisfied by each of the K^α (3.4) separately by definition of path integral.

These conditions in turn imply a set of constraints on the weights χ. Indeed, for the above requirements to be satisfied, it is necessary and sufficient that (Laidlaw and Morette de Witt, 1971)

$$|\chi(\alpha)|=1\ , \tag{3.6}$$

$$\chi(\alpha)\chi(\beta)=\chi(\alpha\circ\beta)\ . \tag{3.7}$$

In Eq. (3.7) α and β are the homotopy classes of two paths with a common endpoint, say $p(ab)$ and $p(bc)$, and $\alpha\circ\beta$ is the homotopy class of the path $p(ac)$ obtained by joining the two.

We shall now show that Eqs. (3.6) and (3.7) are sufficient for requirements (a) and (b); the proof that they are also necessary is somewhat technical and we shall skip it. That Eq. (3.6) is sufficient for (b) is straightforwardly seen by rewriting Eq. (3.5) as

$$\sum_\gamma \chi(\gamma)K^\gamma(q'',t'';q,t)=\sum_\alpha\sum_\beta \chi(\alpha)\chi(\beta)\int dq'\,K^\alpha(q'',t'';q',t')K^\beta(q',t';q,t)\ . \tag{3.8}$$

In order to show that (a) follows from Eq. (3.6), consider the computation of the class of $p(ab)$ using two different meshes, C and $\bar{C}$. For greatest generality, we may take the mesh $\bar{C}$ to be based on a different point y. The homotopy class assignment of $p(ab)$ according to the new mesh $\bar{C}$ is that of the closed path $\bar{P}(ab)\equiv\bar{C}(ab)p(ab)\bar{C}^{-1}(b)$ [Fig. 4(a)]. The two paths $P(ab)$ and $\bar{P}(ab)$ are related by [Fig. 4(b)]

$$\begin{aligned}\bar{P}(ab)&=\bar{C}(a)p(ab)\bar{C}^{-1}(b)\\ &=\bar{C}(a)C^{-1}(a)C(a)p(ab)C^{-1}(b)C(b)\bar{C}^{-1}(b)\\ &=\lambda P(ab)\mu\ ,\end{aligned} \tag{3.9}$$

where

$$\lambda=\bar{C}(a)C^{-1}(a);\quad \mu=C(b)\bar{C}^{-1}(b)\ . \tag{3.10}$$

The difference in assignment of $P(ab)$ and $\bar{P}(ab)$ is thus given by the class of $\lambda\mu$ [Fig. 4(c)].

It follows that the weights χ and $\bar{\chi}$, computed, respectively, with mesh C and mesh $\bar{C}$, are related by a universal factor, $\chi(\lambda\mu)$, and, because of Eq. (3.7), so are the respective amplitudes:

$$\sum_\gamma \bar{\chi}(\gamma)K^\gamma(q',t';q,t)=\chi(\lambda\mu)\sum_\gamma \chi(\gamma)K^\gamma(q',t';q,t)\ . \tag{3.11}$$

This proves requirement (a): amplitudes computed through two different meshes differ at most by the universal phase factor $\chi(\lambda\mu)$.

Now, the two conditions (3.6) and (3.7) just mean that the weights χ are phases, which provide a one-dimensional (Abelian) unitary representation of the fundamental group $\pi_1(\mathcal{C})$. As we shall see shortly, their most general form is determined by the relevant representation theory. This will also make the previous statements about the peculiarity of the two-dimensional case mathematically precise.

B. The braid group

In order to construct the weights χ that enter in the path integral (3.3), we are interested in studying the group $\pi_1(\mathcal{C})$, with $\mathcal{C}$ given by Eq. (3.2) (Laidlaw and Morette de Witt, 1971; Leinaas and Myrheim, 1977; Wu, 1984a). When $d>2$, $\bar{\mathcal{C}}$ is simply connected [that is, $\pi_1(\bar{\mathcal{C}})=0$]. The permutation group S_n always acts effectively on $\bar{\mathcal{C}}$, i.e., there is no element of S_n except the identity that maps every point of $\bar{\mathcal{C}}$ onto itself. These two properties imply that $\pi_1(\mathcal{C})=S_n$. To understand this, consider a path in $\bar{\mathcal{C}}$ and the path in $\mathcal{C}$ to which it corresponds by identifying points under S_n. A closed path in $\mathcal{C}$ can be obtained either from a closed path on $\bar{\mathcal{C}}$ or from an open one. If the path in $\bar{\mathcal{C}}$ is closed, then it is always homotopically trivial, i.e., it can be smoothly shrunk to a point (because $\bar{\mathcal{C}}$ is simply connected). But then so is the path in $\mathcal{C}$, because to a family of paths smoothly interpolating between a closed path and a point in $\bar{\mathcal{C}}$ there corresponds a family of paths with the same properties in $\mathcal{C}$. Thus a nontrivial path on $\mathcal{C}$ must be obtained by projecting an open path in $\bar{\mathcal{C}}$. Furthermore, the sets of points on $\bar{C}$ that project to the same point on $\mathcal{C}$ are in one-to-one correspondence with elements of S_n, because the latter acts effectively. All this means that $\bar{\mathcal{C}}$ is the universal cover of $\mathcal{C}$, and that equivalence classes of paths on $\mathcal{C}$ are in one-to-one correspondence with elements of S_n.

The results of the previous section therefore imply that, in more than two space dimensions, the weights χ must provide a one-dimensional unitary representation of the permutation group. There are only two such representations, the trivial one, which associates $\chi=1$ with any element of S_n, and the alternating one, which associates $\chi=1$ with an even permutation and $\chi=-1$ with an odd permutation. Clearly, the two cases correspond, respectively, to bosons and fermions.

In $d=2$, instead, $\bar{\mathcal{C}}$ is multiply connected, and $\pi_1(\mathcal{C})$ is an infinite non-Abelian group, known as the braid group (see, for example, Guadagnini, 1991, and refs. therein). Because of Eq. (3.6), however, we are interested only in the Abelian representations of this group, which are easy to work out (see Wu, 1984a). We can represent pictorially an element of the group—an equivalence class of closed paths—by drawing a braid related to a representative of the class: for a given value of n (number of particles) this is given by a set of n curves in three dimensions (two space and one corresponding to a parameter along the curves, say time) which never intersect and such that the final positions of the n particles are a permutation of the initial ones (which corresponds to the same point in $\mathcal{C}$ because S_n is divided out) [Fig. 5(a)]. Equivalent braids (corresponding to a single element of the group) are those which can be deformed into each other without moving the endpoints and without letting a curve pass through another one [Fig. 5(b), 5(c)].

Let us now project the braid on a plane spanned by the time and one of the space coordinates. Of course the projected curves will generally intersect (Fig. 6); in order to keep track of the way the curves in three dimensions wind it is clearly enough to indicate (Fig. 6) in the two-dimensional projection at each intersection which of the two intersecting curves is on top. Each two-dimensional braid may now be characterized by the set of its intersections, which we may denote by the action of an operator σ_i that performs the interchange of two neighboring curves i and $i+1$, with the left curve on top. The exchange with the right curve on top is clearly the inverse operation (i.e., the combination of the two interchanges is equivalent to no interchange), thus it is performed by the operator σ_1^{-1} (Fig. 7).

A braid, i.e., an element of the group, is uniquely

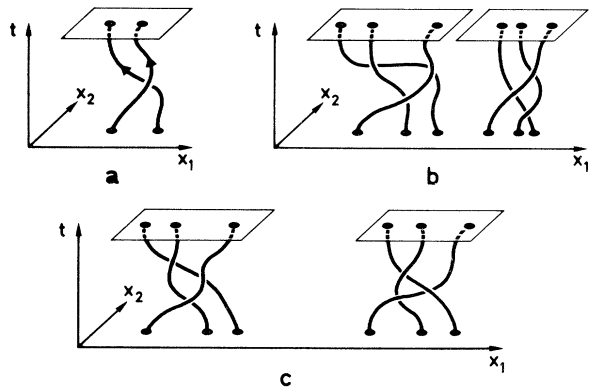

FIG. 5. Examples of braids defined by particle trajectories: (a) the braid corresponding to a two-particle trajectory; the parameter along the curve (time) flows in the vertical direction; (b) two equivalent three-particle braids; (c) two inequivalent three-particle braids.

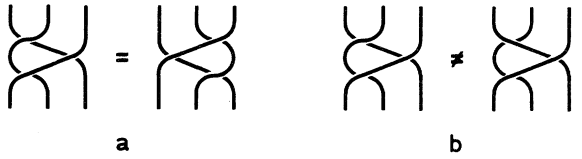

FIG. 6. Two-dimensional projection of the braids of Figs. 5(b) and 5(c). (a) two equivalent braids; (b) two inequivalent braids.

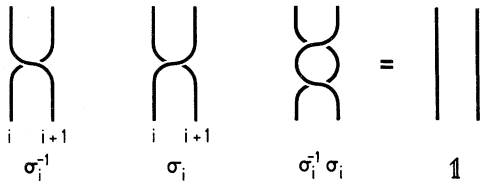

FIG. 7. The exchange operator σ_i and its inverse.

characterized by an ordered product of σ's. There are, however, different sequences of σ's that correspond to the same element of the group (i.e., to homotopic braids). Two equivalent braids are displayed in Fig. 6(a), and two more in Fig. 8. These lead to the relations

$$\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1} \tag{3.12}$$

from Fig. 6(a) and

$$\sigma_i\sigma_j=\sigma_j\sigma_i \quad \text{if } |i-j|>1 \tag{3.13}$$

from Fig. 8. It can actually be proven that there are no more independent relations among σ's.

Now, let us concentrate on the case we are interested in: that is, let us represent each element of the group, i.e., each sequence of σ's, with a phase χ. Equation (3.13) implies

$$\chi(\sigma_i\sigma_j)=\chi(\sigma_i)\chi(\sigma_j) \quad \text{if } |i-j|>1\ , \tag{3.14}$$

i.e., phases corresponding to different interchanges factorize, while Eq. (3.12) implies that

$$\chi(\sigma_i)=\chi(\sigma_j) \quad \text{for all } i,j\ , \tag{3.15}$$

i.e., the interchange phase is universal. Therefore, a unitary Abelian representation of the braid group is uniquely specified by the assignment of a universal phase $\chi=e^{i\Phi}$ which is associated with the operator σ_i. Consequently, the phase $\chi=e^{-i\Phi}$ is assigned to the inverse operator σ_i^{-1}. A generic element of the braid group $\sigma_{i_1}\cdots\sigma_{i_n}$, where σ_{i_k} stands generically for σ or σ^{-1}, is represented by the phase

$$\chi(\sigma_{i_1}\cdots\sigma_{i_n})=\chi(\sigma_{i_1})\cdots\chi(\sigma_{i_n})$$
$$=\exp\left[i\Phi\sum_{k=1}^{n}\epsilon_k\right]\ , \tag{3.16}$$

where ϵ_k is the signature of the kth element, i.e., $\epsilon_k=1$ if the kth element is σ and $\epsilon_k=-1$ if it is σ^{-1}. These are the weights that we shall assign to paths that contribute to the propagator (3.3).

C. Path integral, wave function, and Schrödinger equation

We are now ready to work out explicitly (Wu, 1984a) the most general propagator on the multiply connected configuration space $\mathcal{C}$ [Eq. (3.2)], by substituting the weights (3.16) in the path integral [Eqs. (3.3) and (3.4)]. The representation of the braid group which enters in the construction is fixed by the choice of the parameter Φ in Eq. (3.16). It is easy to provide an analytic expression of the phases (3.16) in terms of the paths to which they correspond. We note that the signature ϵ in Eq. (3.16) may be expressed as $\epsilon_i=\Delta\Theta_i/\pi$ in terms of the variation $\Delta\Theta_i$ of the relative polar angle Θ_i of the two particles that are interchanged. The phase χ (3.16) which weights the n-particle path in $\mathcal{C}$ given (as a function of time) by $(\mathbf{x}_i(t),\ldots,\mathbf{x}_n(t))$ is then

$$\chi=\exp\left[i\frac{\Phi}{\pi}\sum_{i<j}\Delta\Theta_{ij}\right]$$
$$=\exp\left[i\frac{\Phi}{\pi}\sum_{i<j}\int dt\frac{d}{dt}\Theta(\mathbf{x}_i(t)-\mathbf{x}_j(t))\right] \tag{3.17}$$

where Θ_{ij} is the relative polar angle of particles i and j and the function $\Theta(\mathbf{x})$ was defined in Eq. (2.22). Notice that although the phase χ was constructed in Sec. III.A for closed paths in $\mathcal{C}$, Eq. (3.17) provides the generalization to arbitrary open paths. Using this form of the phases χ in the path integral [Eqs. (3.3) and (3.4)], we get finally

$$K(q',t';q,t)=\int_{q(t)=q;q(t')=q'}Dq(t_0)\exp\left[\left[i\int_t^{t'}dt_0(L(q(t_0))+\frac{\Phi}{2\pi}\sum_{i\neq j}\frac{d}{dt_0}\Theta(\mathbf{x}_i-\mathbf{x}_j))\right]\right]\ , \tag{3.18}$$

where $q(t)$ stands for the point in configuration space $(\mathbf{x}_1,\ldots,\mathbf{x}_n)$.

The effect of the weights χ is just to supplement the Lagrangian that appears in the path integral with a particle-particle interaction of the form (2.24). It is now apparent that the one-parameter family of Lagrangians (2.23) and (2.24) describes the most general quantum dy-

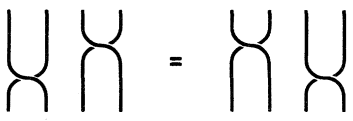

FIG. 8. The two equivalent braids of Eq. (3.13).

namics in two dimensions; the parameter Φ characterizes both the spectrum of angular momentum [given by Eq. (2.25)] and the boundary conditions on the wave functions, or, equivalently, the symmetry properties of the wave function of the system upon interchange of two particles, as expressed by Eqs. (1.2) and (2.29).

We may now come full circle, and rederive the spin and statistics properties of the system directly from the path integral (3.18) (Wu, 1984a). The wave function of a system that evolves according to Eq. (3.18) can be expressed as

$$\psi(q,t)=\langle q,t|\psi\rangle = \int dq_0\,\langle q,t|q_0,t_0\rangle\langle q_0,t_0|\psi\rangle = \int dq_0\,K(q,t;q_0,t_0)\psi(q_0,t_0) \tag{3.19}$$

in terms of some initial condition described by the wave function $\psi(q_0,t_0)$. Also, the path integral (3.3) can be related to the path integral of the purely bosonic theory $K_0(q',t';q,t)$, which is obtained by setting $\Phi=0$ in Eq. (3.18), or equivalently setting $\chi(\alpha)=1$ for all α in Eq. (3.3). Use of Eq. (3.17) in reverse obtains

$$K(q',t';q,t)=\sum_{n_{ij}\ (i\neq j)=-\infty}^{\infty}\exp\left[i\frac{\Phi}{2\pi}\sum_{i\neq j}\Theta_{ij}(t')+2\pi n_{ij})\right]K_0^{(n)}(q',t';q,t)\exp\left[-i\frac{\Phi}{2\pi}\sum_{i\neq j}\Theta_{ij}(t)\right]. \tag{3.20}$$

The sums over n_{ij} appear because of the multivaluedness of the function Θ [Eq. (2.22)]. They correspond to contributions to the path integral from paths that wind n_{ij} times on the configuration space. For each set of values of n_{ij} only paths with the corresponding winding numbers are included in the computation of $K^{(n)}$. This provides an explicit representation of the sum over homotopy classes of paths α in the definition (3.3) of the propagator.

The wave function ψ (3.19) is single-valued. Because n_{ij} are summed over, we can replace $\Theta_{ij}(t)$ and $\Theta_{ij}(t')$ by their determination modulo 2π. However, we may define a new wave function

$$\psi_0(q,t)=\exp\left[-i\frac{\Phi}{2\pi}\sum_{i\neq j}\Theta_{ij}(t)\right]\psi(q,t)\ . \tag{3.21}$$

Here $\Theta(\mathbf{x})$ is defined as a multivalued function on the punctured plane $\mathbb{R}^2_P\equiv\mathbb{R}^2-\{0\}$; it is single valued on its universal covering $\widetilde{\mathbb{R}}^2_P$, which is the Riemann surface of the complex logarithm. The sum of Θ_{ij} for all i,j which appears as a phase in Eq. (3.21) is a multivalued function over $\mathcal{C}$ [Eq. (3.2)], single valued (by definition) over its universal cover $\widetilde{\mathcal{C}}$. We can fix the choice of branch for the phase in Eq. (3.21) by defining

$$\Theta(q)=\int_{q_0}^{q}dq'\frac{d}{dq'}\Theta(q')\ , \tag{3.22}$$

where $q\in\mathbb{R}^2_P$ is a point in the punctured plane spanned by $\mathbf{x}_i-\mathbf{x}_j$ for all i,j and where the integration runs along a path that joins a fiducial reference point $q_0\in\mathbb{R}^2_P$ to the point q at which Θ is evaluated. The set of paths from q_0 to all points $q\in\mathbb{R}^2_P$ forms a mesh, as discussed in Sec. III.A. The prescription described in Sec. II.C to compute the linking number of paths i,j by attaching their endpoints to infinity (Fig. 2) is a particular case of the prescription (3.22), corresponding to the choice of q_0 as the point at infinity of the compactified manifold $\mathbb{R}^2_P+\{\infty\}$.

Because of the definition (3.22), the wave function ψ_0 at point q carries a path joining q_0 to q; this allows it to "remember" along its evolution the sheet of the Riemann surface on which it should be evaluated. Otherwise stated, ψ_0 is to be considered a wave function on the universal cover $\widetilde{\mathcal{C}}$; when viewed as a function on $\mathcal{C}$, it is a function that depends not only on the instantaneous configuration $q(t)$, but also on a path that connects the reference point q_0 to $q(t)$.[4]

Therefore both the sum over paths in the propagator (2.20) and the phases at initial and final times can be simply absorbed into the redefinition (3.21) of the wave function: the multivalued wave function ψ_0 is propagated by the "bosonic" propagator $K_0(q',t';q,t)$. Because ψ_0 [Eq. (3.21)] is multivalued on $\mathcal{C}$, it satisfies twisted boundary conditions of the form (2.13), or equivalently it has fractional statistics (2.29). The canonical angular momentum operator (2.51) acting on it produces the (fractional) spectrum (2.25).

It is now straightforward to make contact with the Hamiltonian formalism (Wu, 1984b): because the wave function (3.21) is propagated by K_0, it satisfies the ordinary Schrödinger equation (without spin-changing interaction). The transformation (3.21) is the same as that of Eq. (2.27) which allowed us to eliminate the long-range interaction in the Hamiltonian approach of Sec. II.B. We may therefore revert to a single-valued wave function, which then satisfies the Schrödinger equation with the Hamiltonian (2.26), that contains an additional long-range interaction.

Again, we find that a theory with fractional spin and statistics may be viewed in two alternative, equivalent ways (Leinaas and Myrheim, 1977; Wu, 1984a): either as a theory with the nonlocal interaction (2.24) and the propagator K of Eq. (3.18) or a theory with the usual interactions and the propagator K_0, but whose wave functions are subject to twisted boundary conditions. In the former case physical observables are modified because the added interaction may contribute to conserved quan-

[4]The theory of wave functions localized on paths has been developed by Zaccaria *et al.* (1983) and Balachandran, Gomm, and Sorkin (1987). For an axiomatic (algebraic) approach see also Fröhlich, Gabbiani, and Marchetti (1989).

tities, while in the latter case the operators associated with observables are as in the free theory, but their spectrum is modified by the boundary conditions. The angular momentum operator, in particular, is given in the former case by J [Eq. (2.50)] and acts on single-valued functions, whereas in the latter case it is given by J_0 [Eq. (2.51)] but it acts on multivalued functions. In both instances its spectrum is the fractional one (2.25). The Chern-Simons coupling (2.35)–(2.37) is a way of producing the required nonlocal interaction; it does, however, produce extra terms, as well, that might further affect the dynamics.

Whatever description we choose, the dynamics of fractional spin is characterized by the fact that nonhomotopic paths are weighted with different phases in the path integral. This provides a unique unambiguous characterization of a theory with fractional spin and statistics and allows us to avoid the ambiguities that are inevitably present in other approaches, as seen in Secs. II.A and II.B, and that are due to the fact that some observables, like the angular momentum, may be redefined by the addition of a constant.

The factors (3.17) and wave functions (3.21) may be presented in an alternate form that is useful for some applications (Wu, 1984a). Points in two-space may be parametrized with complex coordinates $z=x^1+ix^2$. The phase factors (3.17) may be expressed in this parametrization as

$$\chi=\prod_{i<j}\left[\frac{(z_i-z_j)}{|z_i-z_j|}\right]^{i\Phi/\pi} . \tag{3.23}$$

Any wave function of the form (3.21), satisfying twisted boundary conditions, may thus be written as

$$\begin{aligned}\psi(q,t)&=\psi(z_1,\ldots,z_n;t)\\&=\prod_{i<j}(z_i-z_j)^{-\Phi/\pi}f(z_1,\ldots,z_n) ,\end{aligned} \tag{3.24}$$

where $f(z_1,\ldots,z_n)$ is a single-valued function of z and z^*.

The phases χ [Eq. (3.17)] [or, equivalently, the spin-changing interaction in Eq. (3.18)] break the symmetry of the theory under time reversal if Φ/π is not an integer, i.e., if the spin is neither bosonic nor fermionic. The simplest way of seeing this is to observe that under time reversal the spin s changes sign (see, for example, Schiff, 1968); therefore, if $s\notin\mathbb{Z}/2$, the spectrum of angular momentum (2.28) is not invariant. Equivalently, under time reversal $\chi\to\chi^*$; if we require $\chi=\chi^*$ for a closed path (which is a necessary condition for time-reversal invariance to hold), we get $\chi=\pm1$ which, according to Eqs. (3.17) and (2.25), implies that the spin is integer or half-integer.

Time-reversal invariance is, of course, restored (Semenoff and Weiss, 1990) in a theory that contains both particles of spin s and $-s$. The derivation of the path integral given in this section, however, shows that the phase χ must be universal: this means that if particles of different spin are present, the total propagator must be constructed as a sum over superselection sectors, i.e., particles with different spins do not interfere with each other (Fröhlich, Gabbiani, and Marchetti, 1989, and references therein). The fact that it is necessary either to give up time-reversal invariance or to introduce superselection sectors (or both) if we want to consider wave functions that are more than double valued was already shown by Schrödinger (1938) to follow from the general principles of quantum mechanics.

In conclusion, it is interesting to observe that the close connection between fractional spin and statistics and the topology of the configuration space makes it nontrivial to generalize the theory to the case in which the particles are located on a generic compact surface with arbitrary topology, rather than on a plane, as we have discussed so far. The simplest case is that of a particle on a torus $\mathcal{T}$ (Einarsson, 1990). Then, although $\pi_1(\mathcal{C})$ is still a braid group, it is a larger group than that discussed in Sec. III.B. This is due to the fact that now even the one-particle configuration space is multiply connected; $\pi_1(\mathcal{C})$ is generated by the σ_i [Eqs. (3.12) and (3.13)], supplemented (in the n-particle case) by $2n$ generators τ_i, ρ_i, $i=1,\ldots,n$, of $\pi_1(\mathcal{T})$. These are defined as the operators that take each particle around the two inequivalent noncontractible loops on the torus.

Although the group generated by the σ_i is a subgroup of this group, fractional statistics may be obtained only from higher-dimensional (i.e., more than one-dimensional) representations of the extended group. Moreover, the allowed values of the representation parameter (3.17), which fixes the value of spin and statistics, are, for a system of n particles, the rational fractions (Lee, 1989; Einarsson, 1990)

$$\frac{\Phi}{2\pi}=\frac{k}{2n},\quad k\in\mathbb{Z} . \tag{3.25}$$

This restriction on the allowed values of the spin parameter may also be derived (Lee, 1989) by using arguments analogous to those that lead to the Dirac monopole quantization. In this guise, it may be shown that the restriction (3.25) holds on a sphere as well. These results are not only of academic interest, since the geometry of the torus is relevant for the description of planar systems with periodic boundary conditions, which, in turn, may have physical relevance in the physics of the quantum Hall effect (see Sec. IV.C; Einarsson, 1990; Iengo and Lechner, 1990).

IV. SOME PHYSICAL EFFECTS

As we have seen in the previous section, any system can be endowed with fractional spin and statistics either by adding the nonlocal interaction (2.24) or by imposing twisted boundary conditions (2.13) on the wave function. Both approaches are in practice rather difficult to pursue, and even simple problems like the calculation of the par-

tition function for the free-particle gas are highly nontrivial and as yet unsolved. There is nevertheless a small class of models in which the physical effects of fractional statistics may be seen explicitly at work.

A. Two-particle systems: configuration space and partition function

The simplest case in which we may see the effects of fractional spin and statistics explicitly is that of a two-particle system. Indeed, the structure of configuration space which underlies the possibility of fractional statistics is particularly simple in the two-particle case (Leinaas and Myrheim, 1977). Let us use relative coordinates $\mathbf{R}=(\mathbf{x}_1+\mathbf{x}_2)/2$ and $\mathbf{r}=(\mathbf{x}_1-\mathbf{x}_2)$. The configuration space $\mathcal{C}$ [Eq. (3.2)] when $n=2$ is given by the Cartesian product of the $\mathbb{R}^2$ spanned by the center-of-mass coordinate $\mathbf{R}$ times the set of points spanned by $\mathbf{r}$ with $\mathbf{r}\neq 0$ and the points $\mathbf{r}$ and $-\mathbf{r}$ identified. The latter space can also be viewed as a Cartesian product, by representing $\mathbf{r}$ in polar coordinates (ρ,ϕ), namely, as the product of the $\mathbb{R}$ spanned by ρ and the real projective space spanned by ϕ, that is, the circle with points ϕ and $\phi+\pi$ identified, which is again a circle. Therefore, when $n=2$, the configuration space is $\mathcal{C}=\mathbb{R}^2\times\mathbb{R}\times S^1$.

The advantage of this way of looking at $\mathcal{C}$ is that all the nontrivial effects of the statistics are contained in the relative coordinate $\mathbf{r}$, in the sense that the topological interaction (2.24) depends only on $\mathbf{r}$, or, equivalently, the twisted boundary condition (2.13) is imposed on the $\mathbf{r}$ dependence of the wave function. It is clear why the two-particle case is special: for $n>2$ we can still separate the center-of-mass coordinate, which does not play any role as far as spin and statistics are concerned, but then the space of relative positions of the particles no longer has a simple form.

We can now see how fractional spin enters the Schrödinger equation. A system of two particles of mass m and spin s is found by solving for the eigenfunctions of the Hamiltonian (Leinaas and Myrheim, 1977; Arovas *et al.*, 1985)

$$H=H_R+H_r+V(\mathbf{r},\mathbf{R}) , \tag{4.1}$$

where

$$\begin{aligned} H_R&=-\frac{1}{4m}\nabla_R^2 , \\ H_r&=-\frac{1}{m}\nabla_r^2=-\frac{1}{m}\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right] \end{aligned} \tag{4.2}$$

with the boundary condition

$$\psi(\mathbf{R},\rho,\phi+\pi)=e^{i\pi 2s}\psi(\mathbf{R},\rho,\phi) . \tag{4.3}$$

Equivalently, the same physics is described by the eigenstates of the Hamiltonian

$$\begin{aligned} H_s&=H_R+H_r^s+V(\mathbf{R},\mathbf{r}) , \\ H_r^s&=-\frac{1}{m}\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\left[\frac{\partial}{\partial\phi}+2is\right]^2\right] \end{aligned} \tag{4.4}$$

with the (usual) boundary condition

$$\psi(\mathbf{R},\rho,\phi+\pi)=\psi(\mathbf{R},\rho,\phi) . \tag{4.5}$$

Of course, given an eigenstate ψ of H [Eq. (4.2)], an eigenstate of H_s [Eq. (4.4)] is

$$\psi_s=e^{-i2s\phi}\psi . \tag{4.6}$$

Let us now consider the simplest possible case, namely that of free particles, by setting the potential $V(\mathbf{r},\mathbf{R})=0$. Eigenfunctions of Eq. (4.6) then have the factorized form

$$\psi(\mathbf{R},\mathbf{r})=\chi(\mathbf{R})\xi(\mathbf{r}) , \tag{4.7}$$

where $\chi(\mathbf{R})$ is a solution of the free Schrödinger equation while $\xi(\mathbf{r})$ is an eigenstate of H_r, which is best chosen as an eigenstate of angular momentum [Eq. (2.50)]:

$$\xi(\mathbf{r})=e^{i2q\phi}R_{k,q}(\rho) . \tag{4.8}$$

Here q is an integer because of the boundary condition (4.5); $R_{k,q}$ is an eigenfunction of the radial Schrödinger equation, which is the Bessel equation of order $\nu=2|q+s|$. The (un-normalized) generic eigenstate of Eq. (4.4) is thus

$$\psi_{\mathbf{K};k,q}(\mathbf{R},\mathbf{r})=e^{i\mathbf{K}\cdot\mathbf{R}}e^{i2q\phi}J_{2|q+s|}(k\rho) , \tag{4.9}$$

where $J_{2|q+s|}$ is the Bessel function of the first kind, of order $2|q+s|$.

The effects of statistics on the energy spectrum are seen, for example, by putting the system in a cylindrical box of radius ρ_0, that is, by imposing the boundary condition $\xi(\phi,\rho_0)=0$, which implies $J_{2|q+s|}(k\rho_0)=0$. This leads to the energy spectrum

$$E_{\mathbf{K};n,q}=\frac{\mathbf{K}^2}{4m}+\frac{x_{n,q}^2}{m\rho_0^2} , \tag{4.10}$$

where $x_{n,q}$ is the location of the nth zero of $J_{2|q+s|}$. The spectrum (4.10) interpolates continuously between the bosonic case (s integer) and the fermionic one (s half-integer) and is periodic in s with period one. The spectrum of angular momentum is given by Eq. (2.25) with $n=2$ and $-\phi/2\pi=s$ because of the additive constant in Eq. (2.50). Notice that indeed all integer values of ℓ are allowed in Eq. (2.25) because the relative angular momentum $2q$ (which is even) must be composed with the center-of-mass angular momentum (which can take any integer value) carried by $\chi(\mathbf{R})$ [Eq. (4.7)] in order to get the total angular momentum (2.25).

The spectrum (4.10) may be used to compute the partition function for the two-particle gas in an expansion around Bose or Fermi statistics (Arovas *et al.*, 1985; see also Arovas, 1989, for a detailed review). The partition

function may alternatively be calculated exactly in a path-integral approach. The path integral [Eqs. (3.3) and (3.4)] with the phases (3.17) in the two-particle case may be written as

$$K_2(\mathbf{R}',\mathbf{r}',t';\mathbf{R},\mathbf{r},t)=Z_{\mathbf{R}}\sum_{n=-\infty}^{\infty}e^{2ins}K_1^n(\mathbf{r}',t';\mathbf{r},t)\ , \quad (4.11)$$

where $Z_{\mathbf{R}}$ is the free-particle path integral that describes the center-of-mass motion, while $K_1^n(\mathbf{r}',t';\mathbf{r},t)$ is the propagator for free one-particle paths from $\mathbf{r}$ to $\mathbf{r}'$ that belong to the nth homotopy class, i.e., that wind n times about the origin (which is excluded from configuration space). Explicitly,

$$K_1^n(\mathbf{r}',t';\mathbf{r},t)=\int_{\mathbf{r}_0(t)=\mathbf{r};\mathbf{r}_0(t')=\mathbf{r}'}D\mathbf{r}_0(t_0)\exp\left[i\int_t^{t'}dt_0L(q(t_0))\right]\delta[\phi(r_0(t'))-\phi(r_0(t))-2n\pi-\Theta] \quad (4.12)$$

where $\Theta\equiv\phi(\mathbf{r}')-\phi(\mathbf{r})$, L is the Lagrangian of a free particle with mass $m/2$, and the δ enforces the constraint that only paths with winding number n may contribute. A *tour de force* computation (Inomata and Singh, 1978; Gerry and Singh, 1979) allows the explicit evaluation of K_1^n [Eq. (4.12)] in closed form. The result when $\mathbf{r}=\mathbf{r}'$ is

$$K_1^n(\mathbf{r},t';\mathbf{r},t)=\frac{m}{4\pi i(t'-t)}\exp\left[\frac{im}{2(t'-t)}\rho^2\right]\int_{-\infty}^{\infty}d\lambda\exp[i\lambda(\Theta+2\pi n)]I_{|\lambda|}\left[\frac{m\rho^2}{2i(t'-t)}\right]\ , \quad (4.13)$$

where $I_\lambda(z)$ is the modified Bessel function. The partition function is found (Arovas *et al.*, 1985) by summing over all n and integrating over all $\mathbf{r}$. This result, which displays the nontriviality of the statistics interaction even in the simple case at hand, may be used to calculate the second virial coefficient of an ideal gas with s spin and statistics (Arovas *et al.*, 1985). It is worth noting that the result, although a smooth periodic function of the parameters s, is not differentiable (it has a cusp) when s is an integer, i.e., when the statistics is bosonic.

B. Central potentials, statistics, and spectra

The effects of fractional spin on the spectrum of the two-particle Hamiltonian (4.4) may be explicitly determined whenever the angular momentum operator (2.50) commutes with the Hamiltonian, i.e., for central potentials $V=V(\rho)$. In this case, the wave function still has the form of Eqs. (4.7) and (4.8), and $R_{k,q}(\rho)$ is a solution to the radial equation

$$-\frac{1}{m}\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\left[V(\rho)+\frac{4(q+s)^2}{\rho^2}\right]\right]R_{k,q}=E_kR_{k,q}(\rho)\ . \quad (4.14)$$

As expected, spin has the effect of shifting the radial quantum number $q\to q+s$. For example, suppose we take a harmonic-oscillator potential $V(\rho)=\frac{1}{4}m\omega^2\rho^2$. The spectrum of energy eigenvalues is simply obtained by performing the above replacement in the usual spectrum, with the result (Leinaas and Myrheim, 1977; Wilczek, 1984b)

$$E_n^{(s)}=\omega(2(n+\tfrac{1}{2})+|q+s|)\ , \quad (4.15)$$

to which of course must be added the energy of the center-of-mass motion. When $s=0$ (mod$\mathbb{Z}$), the spectrum is the usual one (Fig. 9), whereas when $s=\frac{1}{2}$ (mod$\mathbb{Z}$) the spectrum is that of a fermionic oscillator, which may be obtained from the bosonic case by using the exclusion principle. For example, in the ground state the two particles cannot both be in the $E_0=\omega/2$ state, so the ground-state energy is $E_0^{(1/2)}=(\frac{1}{2}+\frac{3}{2})\omega$ rather than the bosonic value $E_0^{(0)}=\omega$, and so forth. The parameter s provides a smooth interpolation between these two cases (The case $s=\frac{1}{4}$ is displayed in Fig. 9); notice that although there are no level crossings (i.e., the ordering of energy levels does not change with s) the pattern of degeneracies depends on the value of s (Fig. 9).

With a harmonic-oscillator potential it is possible to go a few steps further and discuss the many-particle case as well. When $n>2$ it is actually convenient to use the complex parametrization described in Sec. III.C. The motion of a system of n particles in a harmonic potential well centered in the origin is given by eigenstates of the form (3.24) of the Hamiltonian

$$H=\sum_{i=1}^{n}\left[-\frac{2}{m}\frac{\partial^2}{\partial z_i\partial z_i^*}+\tfrac{1}{2}m\omega^2|z_i|^2\right]\ . \quad (4.16)$$

When $n=2$, the dynamics of Eq. (4.16) basically reduces to that discussed in the previous section, in that the rela-

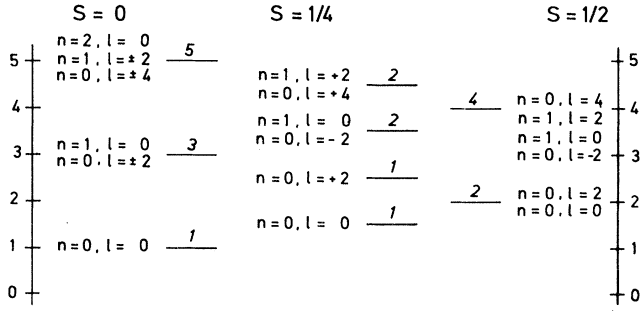

FIG. 9. The energy spectrum (4.15) of a two-particle system interacting through a harmonic-oscillator potential (4.14) in units of ω. The cases $s=0$ (bosons), $s=\frac{1}{2}$ (fermions), and $s=\frac{1}{4}$ (intermediate spin) are shown. The values of the principal quantum number n and radial quantum number l as well as the degeneracies of the levels are shown.

tive motion is described as before by the Hamiltonian H_r [Eq. (4.2)], although now the center-of-mass motion is also subject to a harmonic potential rather than being free. For $n>2$, however, this dynamics is more easily manageable than the other natural generalization of the two-particle case, namely, that of $\frac{1}{2}n(n-1)$ interparticle harmonic forces.

The Hamiltonian (4.16) is of considerable physical interest in that if we let $\omega=(eB)/(2m)$ in Eq. (4.16) then

$$H_B=H-\frac{eB}{2m}\sum_{i=1}^{n}\frac{\partial}{\partial\phi_i} \tag{4.17}$$

is the Hamiltonian that describes n particles with fractional spin and charge e on a plane with an external magnetic field B perpendicular to the plane (Dunne, Lerda, and Trugenberger, 1991). The diagonalization of H_B [Eq. (4.17)] when $n=2$ reduces trivially to Eq. (4.14) and may also be acccomplished in the presence of an additional Coulomb repulsive potential $V_c=e^2(|\mathbf{r}_1-\mathbf{r}_2|)$ (Verçin, 1991).

When $n=3$ a set of eigenfunctions of (4.16) may be obtained (Wu, 1984b) by positing a wave function of the form (3.24) with $\Phi/\pi=-2s$ and the ansatz

$$f(z_1,z_2,z_3)=e^{-\frac{1}{2}\omega r^2}P(z_1,z_2,z_3)\ ,\qquad r^2=\sum_{i=1}^{3}|z_i|^2\ , \tag{4.18}$$

which is suggested by the fact that the asymptotic behavior of the eigenfunctions of (4.16) for large r is $\phi\sim e^{\pm[(\omega r^2)/2]}r^k$. The center-of-mass motion is again given by harmonic-oscillator wave functions (Laguerre polynomials) L_N^L, and if we set

$$P(z_1,z_2,z_3)=(z_1+z_2+z_3)^L L_N^L(\tfrac{1}{3}\omega R^2)(z_1-z_2)^l(2z_1-z_2-z_3)^m L_n^{l+3m+6s-5}(\tfrac{1}{3}\omega\tau^2)$$
$$+\text{symmetrization w.r., to } z_1,z_2,z_3 \tag{4.19}$$

where the center-of-mass radial coordinate is $\frac{1}{3}R=\frac{1}{3}|z_1+z_2+z_3|$, while $\tau=(|2z_1-z_2-z_3|^2+\text{cyclic permutations})^{1/2}$, then the functions f [Eq. (4.18)] are eigenstates of the Hamiltonian (4.16) with energy

$$E=(2N+2n+L+l+m+6s+3)\omega\ , \tag{4.20}$$

where N,n,L,l,m are non-negative integers and l,m are further constrained by the requirement that P not vanish after symmetrization.

The eigenstates (4.20) do not form a complete set: for example, when $s=\frac{1}{2}$ the lowest energy state (4.20) is $E_0=(6s+3)\omega=6\omega$. The energy of the ground state of the three-fermion system, instead, is found by using the exclusion principle, and it is obtained by putting one particle in the lowest state and two in the first excited state (which is doubly degenerate); since the zero-point energy is 3ω, the result is $E_0'=5\omega$. Therefore the ground state is missing from the set (4.19) when $s=\frac{1}{2}$. When s is close to or equal to zero, instead, the lowest energy state (4.20) does coincide with the ground state. This means that in this case the complete energy spectrum must display level crossings as a function of the spin parameter s, that is, not only the degeneracies but also the ordering of the energy levels depends on s. Indeed, it is clear that for small enough s the ground-state energy of the Hamiltonian (4.16) with arbitrary n is given by the sum of the zero-point energy $E_0=n\omega$ and the energy due to the interaction $\mathcal{A}$ (2.26) between all pairs of particles, i.e., it is $E_0=(n+sn(n-1))\omega$. The energy due to the spin grows like a two-body interaction energy and determines level crossings as s grows if $n>2$ (Wu, 1984b).

The eigenstates (4.19) of the Hamiltonian (4.16) may also be generalized to the generic n-particle case (Chou, 1991a). Wider classes of solutions have been found by Polychronakos (1991), who discussed the most general rotationally invariant Hamiltonian quadratic in coordinates and momenta, while a general classification of the available eigenfunctions of the Hamiltonian (4.17) is given by Dunne, Lerda, Sciuto, and Trugenberger (1991). In all cases the fermionic ground state and the energy levels which should display level crossings are missing. The full spectrum has been determined numerically when $n=3$ by Sporre *et al.* (1991), and displays an intricate pattern of level crossings; it is as yet unknown when $n>3$. The complete spectrum may also be found perturbatively in the n-particle case (Chou, 1991b) for statistics infinitesimally close to the bosonic and fermionic statistics, by retaining the spin interaction $\mathcal{A}$ [Eq. (2.26)] in the Hamiltonian and treating the statistics parameter Φ as a perturbative expansion parameter. The three-body spectrum, which was first calculated in the absence of magnetic field (McCabe and Ouvry, 1991) may then be used to calculate the third virial coefficient of the free-particle gas (McCabe and Ouvry, 1991).

The perturbative expansion around Bose statistics is technically intricate due to the fact that the spin interaction must have a singular behavior when the spin parameter vanishes. Heuristically, this is a manifestation of the fact that the exclusion principle, which holds for any nonzero value of the spin parameter, however small, ceases to hold when Bose statistics are exact and is displayed by the nonanalytic behavior of the second virial coefficient discussed above (McCabe and Ouvry, 1991). Due to the singular nature of the spin interaction (which diverges as $1/r_{ij}^2$ when the separation r_{ij} of two particles tends to zero), one may actually compute the first-order perturbative correction to all the n cluster coefficients, as well as the grand-canonical equation of state, without explicitly solving the Schrödinger equation (Comtet *et al.*,

1991). The result is

$$P\beta = P_0\beta + |s|\frac{\rho^2}{\lambda^2} + O(|s|^2) , \tag{4.21}$$

where $P_0\beta$ is given by the equation of state of an ideal Bose gas, $\rho = n/V$ is the particle density, $\beta \equiv 1/kT$, and the thermal wavelength is $\lambda = (2\pi\beta/m)^{1/2}$.

Finally, the exact ground state for arbitrary values of the statistics parameter in the n-free-particle case may be determined on the torus (Iengo and Lechner, 1990) thanks to the translational symmetry of wave functions on this manifold.

C. Adiabatic evolution and the Laughlin wave function

The recent renewal of interest in fractional spin is in large part due to its possible relevance to the theory of the quantum Hall effect and high-T_c superconductivity. Although the discussion of these developments is outside the scope of our treatment, we would like to show at least formally how fractional spin surfaces in this context. The offspring of all these developments is a variational wave function proposed by Laughlin (1983) as the ground state of the Hamiltonian for a planar system of electrons interacting electrostatically with each other, and with an external magnetic field orthogonal to their plane of motion. As we shall show shortly, the localized particlelike excitations of this wave function behave under adiabatic transport as one would expect from a wave function that describes excitations with fractional spin (Arovas *et al.*, 1985).

This leads us to a discussion of the adiabatic approximation for particles with fractional spin and its relation to Berry's phase (Forte, 1991b). Let us consider a quantum system containing particlelike excitations with fractional spin, with dynamics given by a Hamiltonian $H(t)$, whose time dependence is due to the fact that the particles are allowed to move. This kind of system may be treated in the adiabatic approximation (see, for example, Schiff, 1968) if the time scale Δ_T of variation of $H(t)$ is much slower than the time scale Δ_t associated with the spectrum of $H(t_0)$ at fixed $t = t_0$. The latter is of the order of the inverse spacing ΔE of energy levels of $H(t_0)$: $\Delta_t \sim 1/\Delta E$. The adiabatic approximation consists of diagonalizing the Hamiltonian at fixed time (i.e., for fixed positions of the "slow" particles, which in our case have fractional spin):

$$H(t)|\psi_n(t)\rangle = E_n(t)|\psi_n(t)\rangle \tag{4.22}$$

and of assuming that if the system is prepared in a state $|\psi_n\rangle$ it remains in the same instantaneous eigenstate as time evolves. The propagator in this approximation is therefore given simply by

$$K^a(t';t) = \sum_n |\psi_n(t')\rangle \exp\left[-i\int_t^{t'} E_n(t_0)dt_0\right] \exp\left[-\int_t^{t'} \left\langle n,t_0 \left| \frac{d}{dt_0} \right| n,t_0 \right\rangle dt_0\right] \langle \psi_n(t)| . \tag{4.23}$$

Notice that H (and E_n) depend on time through the positions of the slow excitations, which are treated as parameters.

The phase factors (Berry's phases) $\int_t^{t'} dt_0 \langle n,t_0|(d/dt_0)|n,t_0\rangle$ (which are purely imaginary because the states ψ_n are normalized $\langle \psi_n|\psi_n\rangle = 1$) in the standard treatments of the adiabatic approximation are said to be unobservable (see, for example, Schiff, 1968), and are therefore neglected. It has been pointed out relatively recently (Berry, 1984), however, that this is true only if the phases are single valued; otherwise they have observable physical consequences. By writing the time dependence of $|\psi_n\rangle$ through its dependence on the slow coordinates (i.e., the parameters) q as

$$\left\langle n,t \left| \frac{d}{dt} \right| n,t \right\rangle = \dot{q}\cdot\mathcal{A}(q) , \tag{4.24}$$

$$\mathcal{A}(q) = \langle n,t|\nabla_q|n,t\rangle , \tag{4.25}$$

one sees immediately that the phases are single valued if and only if $\mathcal{A}(q)$ has vanishing curl, i.e., if its line integral along any closed loop vanishes.[5]

Let us now come back to the case in which q are positions of (slow) excitations with fractional spin and statistics. We may for instance assume that some external device generates a deep and narrow potential well centered at points q which vary slowly with time. These excitations can be described in two ways: either by a multivalued wave function that satisfies twisted boundary conditions (2.13), or by a single-valued wave function with a long-range, velocity-dependent interaction (2.26). Let us first consider the former picture: the states $|n\rangle$ are multivalued, and they are related to single-valued states $|n^s\rangle$ by Eq. (3.21) (with the replacement $\psi_0 \to |n\rangle$). This implies

[5]Notice that q is a point in a generally k-dimensional parameter space, i.e., a k-component vector, and so is $\mathcal{A}$. By curl of $\mathcal{A}$ is meant the appropriate k-dimensional generalization, which comes from the use of Stokes's theorem in calculating the line integral of $\mathcal{A}$ along a closed curve in the k-dimensional parameter space. This generalization reduces to the ordinary curl when $k=3$.

$$\left\langle n,t\left|\frac{d}{dt}\right|n,t\right\rangle=-\frac{i\Phi}{2\pi}\frac{d}{dt}\sum_{i\neq j}\Theta_{ij}(t)+\left\langle n^s,t\left|\frac{d}{dt}\right|n^s,t\right\rangle . \tag{4.26}$$

When integrated along a path, the second term on the r.h.s. of Eq. (4.26) is single valued, whereas the first term is multivalued: indeed, it can be written in the form (4.24) with $\mathcal{A}$ given by Eq. (2.26). Recalling Eqs. (2.44) and (2.47), we see that if we regard the path (in parameter space) followed by the particles as a path in the space spanned by $q_{ij}=(t,\mathbf{x}_i-\mathbf{x}_j)$, then $\mathcal{A}$ is the Dirac monopole potential (2.44).[6] This has of course nonvanishing curl, and indeed it is the prototypical example of a Berry phase (Berry, 1984). On the other hand, it is immediately obvious that these phases, when used in the propagator (4.23), exactly reproduce the homotopy weights χ of Eq. (3.17) which characterize the propagator for systems with fractional spin (Forte, 1991b). Therefore the statistics phases Θ coincide with a Berry phase for the system at hand. The fact that a Berry phase is related to a modification of the rotational transformation properties of the theory [as we know to be the case for the χ phases (3.17)] is generally true and may be verified explicitly (Jackiw, 1988).

If we now turn to the other picture, i.e., if we eliminate the multivaluedness of the states at the expense of introducing a velocity-dependent potential, there no longer is any multivalued contribution to the propagator (4.23). However, the adiabatic approximation ceases to be valid, and the form of the propagator (4.23) itself is not correct: the velocity operator has an unbounded spectrum, hence the off-diagonal matrix elements of the velocity-dependent potential cannot be neglected, contrary to what assumed in deriving Eq. (4.23) (Forte, 1991b).

Even without knowledge of the Hamiltonian, therefore, given the wave function of an n-particle system we can establish that it describes fractional spin by computing the Berry phase (4.26) and verifying that it is multivalued, provided we know that the adiabatic approximation holds. Of course, the latter is dynamical information which requires some knowledge of the Hamiltonian.

We can now finally come to the Laughlin wave function, which describes (Laughlin, 1983; see also Arovas, 1989 for a review) the ground state of an electron gas in the plane in a strong transverse magnetic field B_0 as

$$\psi_m=N_0\prod_{i<j}(z_i-z_j)^m\exp\left[-\tfrac{1}{4}\sum_k|z_k|^2\right] , \tag{4.27}$$

where the sums and products run over the electron locations z_i, N_0 is a normalization factor, and we have used the complex parametrization (3.24). The electron density with the wave function (4.27) is constant and equal to (Laughlin, 1983)

$$\rho_0=\frac{eB_0}{2\pi m} . \tag{4.28}$$

Elementary localized excitations are found by acting on the wave function (4.27) with operators $A_{\pm z_0}$:

$$\psi_m^{\pm z_0}=N^{\pm}A_{\pm z_0}\prod_{i<j}(z_i-z_j)^m\exp\left[-\tfrac{1}{4}\sum_k|z_k|^2\right] , \tag{4.29}$$

where

$$A_{+z_0}=\prod_i(z_i-z_0) \tag{4.30}$$

creates a quasihole and

$$A_{-z_0}=\prod_i\left[\frac{\partial}{\partial z_i}-eB_0z_0\right] \tag{4.31}$$

creates a quasiparticle localized at z_0; $N^{\pm}$ are normalization factors. Higher excitations are obtained by repeatedly acting with $A_{\pm z}$.

We may now compute (Arovas *et al.*, 1985) the Berry phase (4.26) due to adiabatic displacement of a pair of excitations and show that it is multivalued. We do this in two steps. First, we compute the Berry phase for a single excitation (4.29), say a quasihole. The time dependence is entirely due to the adiabatic motion of the quasihole $z_0=z_0(t)$, hence

$$\frac{d\psi_m^{+z_0}}{dt}=\psi_m^{+z_0}\sum_i\frac{d}{dt}\ln[z_i-z_0(t)] , \tag{4.32}$$

and the Berry phase is

$$\left\langle\psi_m^{z_0}\left|\frac{d}{dt}\right|\psi_m^{z_0}\right\rangle=\left\langle\psi_m^{z_0}\left|\frac{d}{dt}\sum_i\ln[z_i-z_0(t)]\right|\psi_m^{z_0}\right\rangle . \tag{4.33}$$

Introducing the electron density in presence of the quasihole

$$\rho_e(z)=\left\langle\psi_m^{z_0}\left|\sum_i\delta(z-z_i)\right|\psi_m^{z_0}\right\rangle \tag{4.34}$$

we can rewrite the phase in a more transparent form:

$$\left\langle\psi_m^{z_0}\left|\frac{d}{dt}\right|\psi_m^{z_0}\right\rangle=\int dx\,dy\,\rho_e(z)\frac{d}{dt}\ln[z-z_0(t)] \tag{4.35}$$

where $z=x+iy$.

The Berry phase due to Eq. (4.35) is multivalued if its integral along a closed loop is nonzero. The integral along a circle of radius R_0 of Eq. (4.35) is

$$\oint_{R_0}\left\langle\psi_m^{z_0}\left|\frac{d}{dt}\right|\psi_m^{z_0}\right\rangle dt=2\pi i\int_0^{R_0}d\rho\int_0^{2\pi}d\phi\,\rho_e(\rho,\theta)$$
$$=2\pi i\langle n\rangle_{R_0} , \tag{4.36}$$

where (ρ,ϕ) are polar components of the vector (x,y) and $\langle n\rangle_{R_0}$ is the average number of electrons in a circle of

[6] In the n-particle case the parameter space may be considered as a tensor product of $\frac{1}{2}n(n-1)$ three-dimensional spaces spanned by q_{ij} for all distinct i,j, and $\mathcal{A}$ is a sum of monopoles living in the subspaces spanned by each q_{ij}.

radius R_0. Equation (4.36) allows us to establish the charge of the quasihole, if we recall that the wave function of a particle with charge q that traverses a closed path in the presence of an external electromagnetic potential $\mathbf{A}$ picks up a phase

$$\gamma = iq \oint d\mathbf{l} \cdot \mathbf{A} \ . \tag{4.37}$$

If we set $\mathbf{A}$ equal to the potential associated with the magnetic field B_0, then the line integral is equal to the flux $\Phi(B)$ through the portion of the surface enclosed by the path and

$$\gamma = iq\Phi(B) = i\pi R_0^2 B_0 q \ . \tag{4.38}$$

It can be shown (see Arovas, 1989) that the electron density is not modified by the presence of the quasihole; therefore ρ_e is still equal to ρ_0 [Eq. (4.28)]. Then, $\langle n \rangle_{R_0} = \pi R_0^2 \rho_0$, and equating the Berry phase (4.36) with γ (4.38) we determine the charge of the quasihole:

$$q = \frac{e}{m} \ . \tag{4.39}$$

This shows that the quasihole charge is not quantized in units of the electron charge.

We are now ready for the final step in the determination of the spin and statistics of the quasiholes. We take a two-quasihole state

$$\psi_m^{z_1,z_2} = N^{+2} A_{+z_1} A_{+z_2} \psi_m \ , \tag{4.40}$$

normalized by the factor N^{+2}, and we consider a motion in which z_1 is fixed whereas z_2, as before, describes a circle of radius R_0. The above calculation of the Berry phase goes through unchanged, and we get again Eq. (4.36), where the density ρ_e is now to be evaluated by averaging according to Eq. (4.34) in the two-quasiparticle state (4.40). If $|z_2| < |z_1|$ the density inside the circle is the same as above (neglecting again finite-size effects). If, instead, z_1 lies inside the circle traversed by z_2, the number of encircled electrons $\langle n \rangle_{R_0}$ is diminished by the fraction of electrons needed in the average to build the quasihole, thus in this case

$$\langle n \rangle_{R_0} = \pi \rho_0 R_0^2 - \frac{1}{m} \ . \tag{4.41}$$

Now, recalling the discussion in Sec. II, we see that the two paths with particle z_1 respectively within and outside the circle traversed by z_2 belong to different homotopy classes. If there are no other excitations present, the difference in homotopy class is one unit. Because the Berry phase (4.26) provides the spin-changing weights, the exponential of the difference in phase between the two cases gives us the weight χ [Eq. (3.17)]. Comparing the adiabatic propagator, Eqs. (4.23) and (4.26), with the path integral (3.18) we find immediately that the quasiholes have weights χ [Eq. (3.17)] with

$$\Phi = -\frac{2\pi}{m} \tag{4.42}$$

and therefore have spin and statistics $1/m$. Through similar arguments, quasiparticles can be shown to have charge, spin and statistics with the same magnitude and the opposite sign.

It has been suggested (Halperin, 1984) that these features of the quasiparticle excitations, rather than being just a haphazard feature of the phenomenological wave function (4.27), are at the origin of the fractional quantum Hall effect—i.e., the observation of dissipationless current flow for fractionally quantized values of the Hall conductivity (von Klitzing *et al.*, 1980; Tsui *et al.*, 1982; see also Prange and Girvin, 1987). This claim has been substantiated by the construction of effective field theories (Girvin, 1986; Zhang *et al.*, 1989) that support fractionally charged vortices which may be identified with the quasiparticles discussed above and which display superfluid behavior. Indeed, these theories have been shown (Read, 1989; Zhang *et al.*, 1989) to reproduce the physics of the Laughlin wave function (4.27) and the phenomenological features of the quantum Hall effect.

It was later conjectured (Kalmeyer and Laughlin, 1987; Laughlin, 1988a, 1988b) that the superfluid behavior is a universal property of a gas of particles with fractional statistics, and moreover that the same mechanism—based on fractional statistics—is at the origin of the fractional quantum Hall effect and high-T_c superconductivity. The first part of this claim has been supported by mean-field computations (Chen *et al.*, 1989; Fetter *et al.*, 1989; Hanna *et al.*, 1989), in which a gas of particles with fractional statistics is described by the Hamiltonian (2.26), and the many-body statistics-changing interaction $\mathcal{A}$ [Eq. (2.26)] is approximated by an average mean field.

The application of these ideas to the high-T_c copper-oxide superconductors is still rather speculative and is currently the subject of very intense theoretical work. Recent reviews and more extensive lists of references are given by Wilczek (1990a, 1990b). Although the phenomenological import of these investigations is still hard to assess, input from condensed-matter physics has greatly contributed to the development of the theory of fractional spin and statistics.

V. RELATIVISTIC THEORIES

The construction of quantum-mechanical models with fractional spin in the previous sections has been intrinsically nonrelativistic, in that we have repeatedly made use of the peculiar structure of the spatial rotation group (for spin) and of the fundamental group of the spatial configuration space (for statistics), that is, we have tacitly singled out the spatial coordinates. It turns out, however, that this assumption is not crucial, and relativistic models with fractional spin and statistics may be constructed as well. In this section we shall first discuss the general features of relativistic particle mechanics from a general, group-theoretical viewpoint, then study the gen-

eralization of the path-integral approach of Secs. II.C and III to the relativistic case.

A. The Lorentz and Poincaré groups in 2+1 dimensions

Fractional spin in two spatial dimensions is possible because the rotation group, SO(2), is infinitely connected: the group manifold is the circle S^1, and $\pi_1(S^1)=\mathbb{Z}$. In general, a wave function may provide a multivalued representation of rotations, provided the multivaluedness is contained in a phase (recall the discussion of Sec. III). A wave function that carries fractional spin provides a multivalued representation of the group, in that a rotation of 2π does not leave the wave function invariant, but rather, it multiplies it by the phase $e^{2i\pi j}$, according to Eq. (1.1). This is possible only if the group is multiply connected. Then the wave function provides a multivalued representation of the group or, equivalently, a true[7] representation of its universal cover.

In a relativistic theory, rotations are a subgroup of the Lorentz group, which in 2+1 dimensions is SO(2,1). The wave function must provide a representation of SO(2,1) up to a phase, i.e., either a multivalued representation of SO(2,1) with the multivaluedness contained in a phase, or a true representation of its universal cover $\widetilde{\mathrm{SO}}(2,1)$. A necessary condition for fractional spin is that the manifold of the Lorentz group be infinitely connected, which is sufficient to insure that the group admits multivalued representations. Furthermore, it is necessary that the restriction to the rotation subgroup of a multivalued representation be multivalued, too, i.e., that the corresponding submanifold be also infinitely connected.

Let us now take a closer look at SO(2,1) (see, for example, Wybourne, 1974). The generators in the fundamental representation are the 3×3 matrices

$$L^{(\mu\nu)\alpha}{}_{\beta}=-i(g^{\mu\alpha}g^{\nu}{}_{\beta}-g^{\nu\alpha}g^{\mu}{}_{\beta}) . \tag{5.1}$$

The operator $\frac{1}{2}(L^{(12)}-L^{(21)})\equiv R$ generates the compact rotation subgroup, while the operators $\frac{1}{2}(L^{(0a)}-L^{(a0)})\equiv B^a$ generate the noncompact boosts. The Lie algebra is

$$\begin{aligned}[B^a,R]&=-i\epsilon^{ab}B^b ,\\ [B^a,B^b]&=i\epsilon^{ab}R ,\end{aligned} \tag{5.2a}$$

or, in covariant notation,

$$\begin{aligned}[L^{(\mu\nu)},L^{(\rho\sigma)}]=i(&g^{\mu\sigma}L^{(\nu\rho)}+g^{\nu\rho}L^{(\mu\sigma)}\\ &-g^{\mu\rho}L^{(\nu\sigma)}-g^{\nu\sigma}L^{(\mu\rho)}) .\end{aligned} \tag{5.2b}$$

This is the same as the Lie algebra of SL(2,$\mathbb{R}$), hence the two groups admit the same universal cover.

An SL(2,$\mathbb{R}$) matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} , \tag{5.3}$$

where a,b,c,d are real numbers that satisfy $ad-bc=1$. This condition on the determinant of Eq. (5.3) can also be written

$$\left[\frac{a+d}{2}\right]^2+\left[\frac{b-c}{2}\right]^2-\left[\frac{a-d}{2}\right]^2-\left[\frac{b+c}{2}\right]^2=1 , \tag{5.4}$$

which is the equation of a three-dimensional one-sheeted hyperboloid. Its section by the plane $a-d=b+c=0$ is a real circle, which in SO(2,1) language corresponds to the SO(2) subgroup generated by R, the rotation subgroup. It follows that there indeed exist noncontractible paths on the group manifold, namely, those that wind around this circle: $\pi_1(\mathrm{SL}(2,\mathbb{R}))=\pi_1(\mathrm{SO}(2))=\mathbb{Z}$.

We conclude that the group SO(2,1) is infinitely connected, and its multivalued representations correspond to multivalued representations of rotations. Notice that if, instead, we considered theories defined in Euclidean space-time, the Lorentz group would be SO(3). The group manifold of SO(3) is notoriously doubly connected [the universal cover is SU(2)], $\pi_1(\mathrm{SO}(3))=\mathbb{Z}_2$. This implies that representations of SO(3) can only be single valued or double valued, that is, that spin may be either integer or half-integer: the Minkowski signature of the metric is essential if we wish to consider fractional spin.

Irreducible representations (irreps) of $\widetilde{\mathrm{SO}}(2,1)$ may be obtained by fixing the eigenvalue of the Casimir operator $\mathcal{Q}=(B^1)^2+(B^2)^2-R^2$ and diagonalizing, say, the rotation generator R. By means of the algebra (5.2) it can be shown that the spectrum of R is always given by a ladder of states ξ_m that satisfy

$$R\xi_m=m\xi_m , \tag{5.5}$$

$$B^+\xi_m=\sqrt{(d+m)(-d+m+1)}\,\xi_{m+1} , \tag{5.6}$$

$$B^-\xi_m=\sqrt{(-d+m)(d+m+1)}\,\xi_{m-1} , \tag{5.7}$$

where the raising and lowering operators are defined in terms of the boost generators as $B^{\pm}=B^1\pm iB^2$, and the eigenvalue of $\mathcal{Q}$ associated with a given irrep is $d(d-1)$.

Representations may be classified by looking at the behavior of the above ladders of states for all possible values of d. Clearly, if $d=m \bmod(\mathbb{Z})$ the ladder (5.6), (5.7) terminates, and there exist two sets of infinite-dimensional irreps, according to whether the ladder is unbounded from above (if all $m>0$) or from below (if all $m<0$); these are the so-called discrete series of representations (unitary if $d>0$). If d is integer or half-integer, there also exist representations that are bounded both from above and from below, found by setting $d=m \bmod(\mathbb{Z})$, but none is unitary. Finally, for $d\neq m \bmod(\mathbb{Z})$ there are irreps unbounded both from above and from below, which are unitary either when $d=\frac{1}{2}+i\alpha$ (for all real $\alpha\neq 0$) (principal series) or when

[7] A true representation is one in which only the unity of the group is mapped onto the identity transformation.

$\frac{1}{2}-|j-\frac{1}{2}|<d<\frac{1}{2}+|j-\frac{1}{2}|$, where $j\equiv m \bmod(\mathbb{Z})$ (supplementary series).

All of the above are true representations of SO(2,1) (rather than its cover) when m is chosen to be integer. We conclude that SO(2,1) has no finite-dimensional unitary representations. Moreover, even if we are willing to give up unitarity (after all, the usual spinor representation of the Lorentz group is not unitary), it can be shown that there exist no true finite-dimensional matrix representations of the universal cover $\widetilde{\mathrm{SO}}(2,1)$ (see Dubrovin *et al.*, 1984). Indeed, the above enumeration shows that the finite-dimensional representations are at most double valued.

Therefore, if we insist that spin be fractional, we must necessarily work with an infinite-dimensional representation of the Lorentz group, in general, or the rotation group, in particular. Suppose we consider a system with definite spin. Then the wave function carries an irrep of $\widetilde{\mathrm{SO}}(2,1)$. In order to write the wave function explicitly we have two alternatives: either we define an infinite-component wave function or we stick to a wave function that gives the trivial representation of SO(2,1) (a scalar) and we supplement it with a phase that has multivalued transformation properties upon the action of SO(2,1). In the first case the wave function provides a true irrep of $\widetilde{\mathrm{SO}}(2,1)$; in the latter case it provides a multivalued irrep of SO(2,1).

If $g: q\to q^g$ is the action of the group on a point q that belongs to the configuration space $\mathcal{C}$, then the action of the group on wave functions is given by $U(g)\,\psi(q)=\psi(q^g)$, which provides a single-valued representation if the action of g on $\mathcal{C}$ is single valued.[8] A multivalued representation may be obtained by introducing a phase (see, for example, Jackiw, 1985)

$$U(g)\,\psi(q)=e^{i\omega_1(q;g)}\psi(q^g)\ . \tag{5.8}$$

The group composition law for the operators $U: U(g_1)U(g_2)=U(g_{12})$ implies that the phase ω_1 must be a one-cocycle phase, i.e., it must satisfy the constraint

$$\Delta\omega_1\equiv\omega_1(q^{g_1};g_2)-\omega_1(q;g_1g_2)+\omega_1(q;g_1)=0 \bmod(2\pi\times\text{integer})\ . \tag{5.9}$$

In fact, we are only interested in a particular subclass of cocycles $\omega_1(q;g)$: the phase in Eq. (5.8), which must be multivalued upon rotations, should shift the spectrum of the generator of rotations (the angular momentum) without spoiling its conservation law. Therefore it is sufficient that the phase (5.8) be due to the presence of a multivalued prefactor in the wave function, as in Eq. (3.21). In this case, the wave function can be written [cf. Eq. (3.21)]

$$\psi(q)=e^{i\alpha_0(q)}\tilde{\psi}(q) \tag{5.10}$$

in terms of a wave function $\tilde{\psi}$ that carries a single-valued irrep of SO(2,1), and

$$\omega_1(q;g)=\alpha_0(q^g)-\alpha_0(q)\equiv\Delta^g\alpha_0\ , \tag{5.11}$$

which automatically satisfies condition (5.9). A cocycle that admits the expression (5.11) is said to be trivial. In our case, the cocycle must be only *locally* trivial, that is, if $g=g_0$ is the identity of SO(2,1) but not the identity of $\widetilde{\mathrm{SO}}(2,1)$, then

$$\omega_1(q;g_0)\neq 0 \tag{5.12}$$

although $q^{g_0}=q$. This is of course just a restatement of the multivaluedness of the function α_0. If point q is brought back to the same location after traversing a homotopically nontrivial path, the function α_0 does not get back to its original value.

The function Θ [Eq. (2.22)] as used in Sec. III.C is an explicit example of the function α_0 [Eq. (5.11)] for the rotation group SO(2), and the wave function ψ_0 [Eq. (3.21)] carries a multivalued representation of SO(2) or equivalently a true representation of its universal cover $\widetilde{\mathrm{SO}}(2)$. In that case, if g is a rotation of angle β, then $\Delta^g\alpha_0=\beta$. We can construct an explicit expression of $\Delta^g\alpha_0$ in the SO(2,1) case, too (see Forte and Jolicœur, 1991), by associating each element $g\in\widetilde{\mathrm{SO}}(2,1)$ with a path on the group manifold from a reference element (say, the unity of the group) to the element $\Lambda(g)\in\mathrm{SO}(2,1)$. Because of the infinite-valuedness of the universal cover there is an infinity of g's that correspond to the same Λ; however, to each g corresponds a distinct path from unity to Λ, or rather, an equivalence class of homotopic paths. Paths corresponding to different g's associated with the same Λ are instead nonhomotopic.

Once we associate with Λ a path P, an expression of the cocycle (5.11) is given by any local expression of the winding number density along this path. If p is an integer that labels homotopy classes of paths on the group manifold, a local expression of the winding number density is a function $w(t)$ along the path such that

$$p=\oint_P dt\, w(t)\ . \tag{5.13}$$

An expression for the cocycle is then

$$\omega_1(q;g)=s\int_{t_0}^{t_1}dt'\,w(t)\ , \tag{5.14}$$

where, in terms of some reference configuration q_0, $\Lambda(t_0)q_0=q$ and $\Lambda(t_1)q_0=q^g$. This definition satisfies all the requirements on the cocycle (5.8): it is a multivalued function, since the same point may be associated with an infinity of different values of the integral (5.14) (differing by s times an arbitrary integer according to how many times the path winds) and, in agreement with Eq. (5.11), it is associative along the path, i.e.,

[8] Notice that in the relativistic case the configuration space $\mathcal{C}$ is given by Eq. (3.2) with $d=3$ and with Minkowski metric; thus, for example, when $n=1$ in the nonrelativistic case, $\mathcal{C}$ is the plane, whereas in the relativistic case it is $2+1$ dimensional space-time.

$$\Delta^{g_2 g_1}\alpha=\Delta^{g_2}(\Delta^{g_1}\alpha)\ . \tag{5.15}$$

An explicit expression of the function $w(t)$ [Eq. (5.13)] for a path originating from the identity of the group is

$$w(t)=\mathrm{tr}\frac{1}{2i}[\Lambda^{-1}\dot{\Lambda}(t)R]\ , \tag{5.16}$$

where R is the generator of the rotation subgroup of SO(2,1). Clearly, for a rotation of $2\pi p$, use of this expression for w in Eq. (5.13) gives the winding number p. For an arbitrary closed path P this is still true. One way of seeing this is to decompose the instantaneous transformation $\Lambda(t)$ into its rotation and boost components. The winding number is then found by adding up the infinitesimal rotational components along the path, and the trace in Eq. (5.19) has precisely the effect of projecting out the rotation component of the infinitesimal transformations that build up Λ (see Forte and Jolicœur, 1991).

If we are interested in constructing relativistic one-particle states, the Poincaré, rather than the Lorentz, group is relevant: indeed, general quantum-mechanical principles tell us that a physical (on-shell) state must provide a unitary irreducible representation of the universal covering of the Poincaré group (Wigner, 1939; see also Balachandran *et al.*, 1983). The Poincaré group (see, for example, Barut and Raczka, 1986) is the semidirect product of the Lorentz group and the translation group; in $2+1$ dimensions it is the group $\mathrm{ISO}(2,1)=\mathbb{R}^3\overset{s}{\otimes}\mathrm{SO}(2,1)$. Its universal cover is $\widetilde{\mathrm{ISO}}(2,1)=\mathbb{R}^3\overset{s}{\otimes}\widetilde{\mathrm{SO}}(2,1)$. The Lie algebra of ISO(2,1) is generated by the three generators $L^{(\mu\nu)}$ of the Lorentz group and the three generators P^μ of translations, and it is given by extending the Lorentz algebra (5.2) by the further relations

$$[P^\mu,P^\nu]=0,\quad [L^{(\mu\nu)},P^\rho]=i(P^\mu g^{\nu\rho}-P^\nu g^{\mu\rho})\ . \tag{5.17}$$

The unitary irreps of $\widetilde{\mathrm{ISO}}(2,1)$ may be easily classified and constructed (Binegar, 1982) through Wigner's method of induced representations (see Barut and Raczka, 1986). A complete classification is given by the set of distinct orbits of a point $p^\mu\in\mathbb{R}^3$ under the action of $\widetilde{\mathrm{SO}}(2,1)$, times all irreps of the subgroup of $\widetilde{\mathrm{SO}}(2,1)$ that leaves p^μ invariant (little group). General quantum-mechanical principles (Wigner, 1939) lead us to identify p^μ, which belongs to the space spanned by group transformations generated by the operators P^μ, with momentum. The orbits of p^μ under Lorentz transformation are classified by the value of $p^2=m^2$. The little group of p^μ is the group of spatial rotations about p^μ, i.e., $\widetilde{\mathrm{SO}}(2)$, whose representations are classified by the eigenvalue s of its only generator. Physically, m is the mass and s the spin of the particle.

The infinitesimal transformation laws for the states $u(p)$ of an irrep with mass m and spin s are determined again by Wigner's method (Binegar, 1982). The mass and spin are given by the eigenvalues of the Casimir operators,

$$P_\mu P^\mu u(p)=m^2u(p)\ , \tag{5.18a}$$

$$\epsilon_{\mu\nu\rho}P^\mu\tfrac{1}{2}M^{\nu\rho}u(p)=msu(p)\ . \tag{5.18b}$$

The transformation of $u(p)$ upon translation along $a^\mu(a^\mu a_\mu=1)$ is

$$e^{i\epsilon P\cdot a}u(p)=e^{i\epsilon p\cdot a}u(p)\ ; \tag{5.19}$$

its transformation under infinitesimal rotation is

$$e^{i\epsilon R}u(p)=e^{is\epsilon}u(e^{-i\epsilon R}p) \tag{5.20}$$

and its transformation under an infinitesimal boost along $\theta^a(\theta^a\theta_a=1)$ is

$$e^{i\epsilon\theta_a B^a}u(p)=\exp is\epsilon\left\{\frac{\epsilon^{ab}\theta_a p_b}{E+m}\right\}u(e^{-i\epsilon\theta_a B^a}p)\ , \tag{5.21}$$

where we have denoted by E the time component of p_μ (the energy). In the next section we shall work out explicit examples of physical systems that provide representations of the Lorentz and Poincaré groups and see that the cocycle representations of the Lorentz group [Eqs. (5.8)–(5.16)] automatically reproduce the irreps of the Poincaré group (5.19)–(5.21) when the states $\psi(q)$ [Eq. (5.10)] are momentum eigenstates.

B. Relativistic point particles

Among the different approaches to fractional spin considered so far there is one that lends itself naturally to a relativistic treatment, namely, adding to the Lagrangian of an n-particle system the Hopf interaction [Eqs. (2.38)–(2.41)], discussed in Sec. II. Indeed, it is clear that the n-particle current j^μ [Eq. (2.33)] is a Lorentz vector, and both the Chern-Simons interaction [Eqs. (2.36) and (2.37)] and the Hopf Lagrangian (2.48) are Lorentz scalars.[9] In the treatment of Sec. II.C the covariance of the theory was broken by the choice of boundary conditions, and led to a noncovariant expression of the spin-changing interaction, Eq. (2.48), which endowed the wave function of the system with a multivalued representation of the rotation group. Let us now see how a fully relativistic treatment leads to the multivalued representations of the Lorentz and Poincaré groups discussed in the

[9] As a matter of fact, because of the form (2.39) of the kernel $K^{\mu\nu}$, all indices are contracted with the tensor density $\epsilon_{\mu\nu\rho}$, and it can be shown (see Forte, 1991a) that the Hopf Lagrangian is actually invariant under general coordinate transformations (Einstein scalar).

previous section.[10]

Let us first consider the case of a one-particle system; the extension to the many-particle case and its implication for the spin-statistics relation will be discussed in the next section. We look at a theory of particles interacting through the Hopf Lagrangian (2.38). In the one-particle case, this reduces to

$$I_H=\frac{\Phi}{4\pi}\int ds\,dt\,\epsilon_{\mu\nu\rho}\frac{dx^\mu(s)}{ds}\frac{[x(s)-x(t)]^\nu}{|x(s)-x(t)|^3}\times\frac{dx^\rho(t)}{dt}\ , \tag{5.22}$$

where the two integrations run along the same curve $x(s)$ traversed by the particle, and s,t are invariant parameters along the curve, for example the arc length $ds^2=dx^\mu dx^\nu g_{\mu\nu}$. Although the integrand in Eq. (5.22) looks singular as $s\to t$, if we expand $x(s)$ in a Taylor series in the vicinity of $s=t$ we get (Calugareanu, 1959)

$$\epsilon_{\mu\nu\rho}\dot{x}^\mu(s)\dot{x}^\rho(t)\frac{[x(s)-x(t)]^\nu}{|x(s)-x(t)|^3}=-\frac{1}{6}|s-t|\epsilon_{\mu\nu\rho}\frac{\dot{x}^\mu(s)\ddot{x}^\nu(s)\dddot{x}^\rho(s)}{|\dot{x}(s)|^3}+O(|s-t|^2)\ , \tag{5.23}$$

where the dot denotes differentiation with respect to s. This expression is $O(|s-t|)$ as $s\to t$, implying that the integrand is not only regular but actually vanishing when $s\to t$.

The integral (5.22) may be computed explicitly [(Calugareanu, 1959; Pohl, 1968); a "physicist's" discussion is given by Frank-Kamenetskiĭ and Vologodskiĭ (1981; see also Tze, 1988)] by introducing a framing of the curve $x(s)$, i.e., by defining a new curve

$$x^\mu_\epsilon(s)=x^\mu(s)+\epsilon n^\mu(s)\ , \tag{5.24}$$

where n^μ satisfies $n\cdot n=1$, $n\cdot dx/ds=0$, and $\epsilon\to 0$. In this approach I_H [Eq. (5.22)] is written as

$$I_H=\frac{\Phi}{4\pi}\int ds\,dt\lim_{\epsilon\to 0}\epsilon_{\mu\nu\rho}\frac{dx^\mu_\epsilon(s)}{ds}\frac{[x_\epsilon(s)-x(t)]^\nu}{|x_\epsilon(s)-x(t)|^3}\frac{dx^\rho(t)}{dt}\ . \tag{5.25}$$

If we take the limit out of the integral sign in Eq. (5.25) we recover Gauss's integral (5.22), evaluated for the two curves x and x_ϵ, which, as shown in Sec. II.C, is proportional to their linking number l; let us call this I_ϵ. According to Eq. (2.49)

$$I_\epsilon=l\Phi\ , \tag{5.26}$$

where l is an integer for closed paths (recall Fig. 2). For ϵ sufficiently small, l does not depend on ϵ (see Fig. 10), and we can take $\epsilon\to 0$ in Eq. (5.26). Of course, l depends on the choice of framing, i.e., on the choice of n [Eq. (5.24)]. If n is the principal normal, defined in terms of the tangent e as

$$n^\mu_p=\frac{\dot{e}^\mu}{|\dot{e}|};\quad e^\mu=\frac{\dot{x}}{|\dot{x}|}\ , \tag{5.27}$$

then l is called the *self-linking* number of the curve (Pohl, 1968).[11] Clearly, this cannot be equal to I_H [Eq. (5.25)], that is, $\lim_{\epsilon\to 0}I_\epsilon\neq I_H$, because l [Eq. (5.26)] is manifestly framing dependent, while I_H must be a framing-independent quantity.

This means that the integral and the limit in Eq. (5.25) do not commute. The noncommutativity is given by (Calugareanu, 1959)

$$I_H-l\Phi=-\Phi\int\frac{ds}{2\pi}e_{\mu\nu\rho}e^\mu n^\nu\frac{dn^\rho}{ds}\ , \tag{5.28}$$

where e is as in Eq. (5.27). Rewriting Eq. (5.28) as

$$I_H=\Phi(l-\tau)\ , \tag{5.29}$$

$$\tau=\int\frac{ds}{2\pi}b\cdot\dot{n}\ , \tag{5.30}$$

$$b^\mu=\epsilon^{\mu\nu\rho}e_\nu n_\rho \tag{5.31}$$

FIG. 10. Framing x_ϵ of a curve x.

[10]The possibility of formulating the propagator for relativistic spinning particles in terms of a bosonic path integral with a Chern-Simons coupling was first proven by Polyakov (1988, see also 1989), who laid the bases of the theory, in the spin-$\frac{1}{2}$ case. The quantization of relativistic particles with arbitrary fractional statistics is discussed by de Sousa Gerbert (1990) from a canonical point of view, using the general formalism of Balachandran *et al.* (1983). The path integral with arbitrary statistics in the n-particle case is worked out by Forte (1991c). The formal theory of spin and statistics superselection sectors is developed by Fröhlich and Marchetti (1989) and Fröhlich, Gabbiani, and Marchetti (1989).

[11]Strictly speaking this terminology applies to the case of an Euclidean metric, i.e., to the case of curves in three-dimensional space, rather than 2+1-dimensional space-time. There is, however, no obstacle to defining linking, self-linking, etc. for a Minkowski metric, either by performing the computations in Euclidean space and Wick-rotating the result by supplementing appropriate factors of i, or by performing the computation in Minkowski space directly. It is, for instance, straightforward to verify that the computation of the linking number (2.45)–(2.48) goes through regardless of the signature of the metric; the same applies to the results discussed here. In order to facilitate the geometrical understanding of the results we shall stick to the usual Euclidean terminology.

shows that if $n=n_p$ [Eq. (5.27)], then b [Eq. (5.31)] is the binormal vector and τ [Eq. (5.30)] is the torsion of the curve.

Upon changes of framing, the variation of the total torsion τ is exactly equal to that of the self-linking l, thus leading to a framing-independent value of I_H. The difference of linking number and torsion in Eq. (5.29) is called the *cotorsion* or *writhing number* of the curve. Equations (5.29)–(5.31) imply that the writhing number is a functional of the time evolution of the moving frame[12] of vectors $e(s)$, $n(s)$, $b(s)$. Because of its framing independence, however, the writhing number may be expressed as a function of the tangent vector only. If we set

$$e^{\mu}(s)=\begin{bmatrix}\cosh\theta\\ \sinh\theta\sin\phi\\ \sinh\theta\cos\phi\end{bmatrix} \tag{5.32}$$

then it can be shown (Forte, 1991c) that

$$I_H=-\Phi\left[\int\frac{ds}{2\pi}\dot{\phi}\cosh\theta-\frac{1}{4\pi}I_g[n]\right]. \tag{5.33}$$

Here $I_g[n]$ are functions of the form (2.46), evaluated for the pair of curves formed by x and its framing, $x_i=x$, $x_j=x_\epsilon$, and also depend on the framing vector n. For a closed path the terms I_g vanish; for an open path they depend on the boundary conditions imposed on the frame at the endpoints of the path and may be set to zero by a choice of boundary conditions (Forte, 1991c). These terms are present because for an open path there is a certain ambiguity in defining the tangent to the path at its endpoints. They do not affect the Lorentz transformation properties of the theory and will be neglected henceforth.

Equation (5.33) shows that (because of the factor of $\cosh\theta$), even for a closed path, the writhing number need not be an integer, while the linking number necessarily is. Clearly, I_H is a kind of relativistic generalization of the Θ term (3.18): in the nonrelativistic limit the tangent vector is $e^{\mu}\approx(1,0,0)$, that is, $\theta\approx 0$ and I_H reduces to a polar angle in the space plane. However, the nonrelativistic Θ was defined as the polar angle of a two-particle relative coordinate, whereas Eq. (5.33) holds for a one-particle system. Indeed, as a function of e, I_H is singular when $e=(1,0,0)$, as is apparent if we write

$$I_H=-\frac{\Phi}{2\pi}\int ds\frac{de}{ds}\cdot\tilde{A}'[e], \tag{5.34}$$

where

$$\tilde{A}'[e]=\left[0,-\frac{\epsilon_{ab}e^0e^b}{((e^0)^2-1)}\right] \tag{5.35}$$

is the potential of a Dirac monopole in e space, with two Dirac strings going through the north and south poles (see Balachandran *et al.*, 1983).

This singularity has a geometrical meaning (Pohl, 1968; in this context see Forte, 1991c). Suppose we compute I_H with the canonical Frenet framing (5.27). Then the writhing number is the difference of the torsion and the self-linking of the curve. Now, the torsion is a smooth function of the shape of the curve: a small deformation of the curve results in a small variation of the torsion. The self-linking, instead, takes only integer values and may jump one unit upon arbitrarily small deformations of the curve; thus the same must be true for the writhing number, independently of the framing we use to compute it. When written as a functional of $e(s)$, the writhing number is a function of the path traversed by e. Because e is timelike, it spans the upper sheet of a two-sheeted hyperboloid (which we may parametrize by θ and ϕ). The fundamental group of this manifold is trivial, i.e., all paths on it are homotopic. Thus I_H as a function of this path must have singularities if it is to vary discontinuously upon small deformations of a smooth path. Notice that there need be no singularity if the writing is expressed, instead, as a function of the path traversed by a *spacelike* vector (like n): this spans a one-sheeted hyperboloid, which is homotopically nontrivial. Therefore the writhing can be written as a regular function as soon as we introduce an n vector along the path, i.e., a framing.

This argument also shows that, although I_H has singularities as a function of the path traversed by e [Eq. (5.34)], it is regular and well defined as a function of the path $x(s)$ itself and its derivatives, and as such it may be used as an action in a path integral. Choosing for definiteness the canonical Frenet framing, we have the path-integral expression for the propagator

$$K(q',s';q,s)=\int_{q(s)=q;q(s')=q'}\sum_{n=-\infty}^{\infty}Dq^{(n)}(s_0)\times e^{i\frac{\Phi}{2\pi}(\psi(s')+2\pi n)}\left[\exp\left[i\int_s^{s'}ds_0\{L_0[q(s_0)]-\Phi\tau[e]\}\right]\right]e^{-i\frac{\Phi}{2\pi}\psi(s)}, \tag{5.36}$$

where τ is the torsion of the path, L_0 is the Lagrangian of the theory without Hopf term, and ψ is the multivalued polar angle of the projection of the principal normal on the plane orthogonal to e. The path integration runs over all possible paths from q to q', including those which go backwards in time.[13] This means that the tangent vector e [Eq. (5.32)] may be spacelike; hence θ might be imaginary.

[12] If $n=n_p$ this is the usual Frenet frame along the path.

[13] The rigorous derivation of a path integral of the form (5.36)

Equation (5.36) shows that the addition of the Hopf interaction (5.22) to the one-particle Lagrangian $L_0(q)$ has a twofold effect; not only are the paths weighted with a multivalued topological phase which depends only on the endpoints (as in the nonrelativistic case), but they also receive a weight related to their torsion τ [Eq. (5.30)]. The sum over the integer n in the path integral (5.36) is not related to a classification of paths into homotopy classes: indeed, in the one-particle sector, the configuration space is just $\mathcal{C}=\mathbb{R}^3$ (with Minkowski metric), which is homotopically trivial (all paths are homotopically connected). Rather, paths are classified according to their self-linking number and assigned weights χ [Eq. (3.17)] which reproduce the sum over n in Eq. (5.36) (as in the nonrelativistic case) according to their self-linking class.[14] The measure of integration over paths of the nth self-linking class $Dq^{(n)}$ may in practice be rather complicated; for practical purposes different parametrizations and classifications of paths (corresponding to different choices of framing) may be more manageable (see Forte, 1991c).

We are now ready to show that the Hopf interaction endows the particle with spin $s=-\Phi/2\pi$, as one would expect comparing Eq. (5.36) to the nonrelativistic propagator (3.20). It is clear that the nonrelativistic construction cannot be reproduced literally, since the effect of the Hopf term is not merely to endow the path integral with the integral of a total derivative, as would necessarily be the case if the action (5.33) were due to homotopy weights [cf. Eq. (3.17)]. Rather, the writhing (5.33) may be viewed as a Wess-Zumino term (see, for example, Jackiw, 1985), i.e, as a total derivative in one dimension more. For a closed path, use of Stokes's theorem leads to

$$\oint_P ds \frac{de}{ds}\cdot \tilde{A}'[e]=\int dS^{\mu}\epsilon_{\mu}{}^{\nu\rho}\partial_{\nu}\tilde{A}'_{\rho}[e]\ . \tag{5.37}$$

Nevertheless, given the wave function $\psi(q)$ propagated by the path integral (5.36), we may define a new wave function ψ_0 which depends not only on the point $q\in\mathcal{C}$, but also on a path that joins a reference point q_0 to q (Forte, 1991c):

$$\psi_0(q)=e^{i\frac{\Phi}{2\pi}\Theta_{P_0}(q)}\psi(q)\ , \tag{5.38}$$

$$\Theta_{P_0}(q)=\int_{q_0\,P_0}^{q} dq'\frac{de}{dq'}\cdot\tilde{A}'[e]\ . \tag{5.39}$$

Since the wave function must describe a state of a physical system, the requirement that boundary conditions be imposed on a spacelike surface, as dictated by causality, implies that the path P_0 must be contained in a spacelike surface. Furthermore, if we wish that the choice of path P_0 be immaterial, the curve must be planar, since in that case the difference in phase for two different paths P_0, P_0' vanishes. Indeed, thanks to Eq. (5.37), if S_0 is a surface bound by P_0 and P_0',

$$\Theta_{P_0}(q)-\Theta_{P_0'}(q)=\int_{S_0} d\cosh\theta\, d\phi\ , \tag{5.40}$$

which vanishes because for a planar curve $d\cosh\theta=0$. Without loss of generality, we may therefore take the path P_0 to be a straight line joining spatial infinity to q along a spacelike plane.[15]

Patterning after the nonrelativistic case, we should now like to interpret the set of paths $P_0(q)$ as a mesh over space-time (cf. Sec. III.A). For this purpose, we must connect the paths P_0 and P at q in such a way that the tangent at q is smooth, because the writhing number depends on the tangent, rather than on the path itself. We can do this by positing that, given the tangent $e_q(\theta_q,\phi_q)$ to the path at q, (a) P_0 is the line from infinity to q on some spacelike surface along $\phi=\phi_q$; (b) the tangent to P_0 at q is joined to e_q by keeping $\phi=\phi_q$ and varying θ [see Fig. 11(a)]. The corresponding path of tangent vectors e [shown in Fig. 11(b)] may be realized by deforming the end of the path P at q.

With this prescription the torsion (which is a function of e only) vanishes along P_0 (see Fig. 11). This allows one to express the writhing number of an arbitrary open path $P(q,q')$ in terms of the writhing number of the closed path $P_0(q)P(q,q')P_0^{-1}(q')$ (using the notation introduced in Sec. III.A). Furthermore, the wave function ψ_0 [Eq. (5.38)] is propagated (Forte, 1991c) by the path integral K_0 of the theory without Hopf term, in that the propagation of the phase Θ_{P_0} localized on a path automatically produces the topological contribution to the action in the path integral (5.36). Therefore, as usual, the effects of the topological interaction may be viewed in a dual way, either as a modification of the propagator or as a modification of the wave function.

It should be observed that the phase (5.39) is a function

---

from the scaling limit of a sum over random walks, as well as a discussion of the scaling properties of random paths for a spinning particle, is given (in the spin-$\frac{1}{2}$ case) by Ambjørn, Durhuus, and Jonsson (1989) (in four dimensions) and Jaroszewicz and Kurzepa (1991) (in arbitrary dimension).

[14]The proof that paths can be classified according to their self-linking number, as well as the topological interpretation of the latter, is nontrivial and relies on somewhat more advanced arguments in homology theory. Roughly speaking, the self-linking is related to the number of times the path intersects the surface formed by the envelope of its tangents; as such, it is related to the "knottedness" of the path. A detailed discussion is given by Pohl (1968).

[15]Quite in general it may be shown (Zaccaria *et al.*, 1983; Balachandran *et al.*, 1987) that the effects of the addition of a Wess-Zumino term to the action may be reproduced by a wave function localized on a path. The fact that wave functions with fractional spin must be localized on a spacelike line was first proven by Fröhlich and Marchetti (1989; see also Fröhlich, Gabbiani, and Marchetti, 1989).

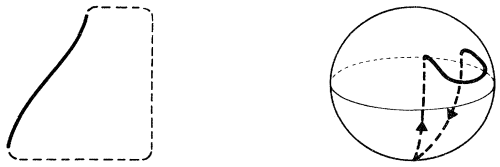

FIG. 11. Calculation of the writhing number of an open curve: (a) an open curve in space-time (solid line) is closed through a torsionless mesh (dashed line); (b) path of the tangent vector corresponding to the curve (solid line) and the mesh (dashed line) of Fig. (a). The Euclidean (rather than Minkowski) tangent is shown for simplicity.

of the tangent to the path. At the endpoints of the path itself the tangent is uniquely defined by the boundary conditions only if the initial and final states are momentum eigenstates: if ψ is an eigenstate of momentum p, then the tangent to the path at q is fixed and parallel to p. If, instead, ψ is a position eigenstate, or generally a superposition of momentum eigenstates, it should be written as a superposition of momentum eigenstates each of which carries a phase [Eqs. (5.38) and (5.39)].

Because the wave function ψ_0 has the "bosonic" dynamics defined by K_0, the fractionization of its angular momentum is due to the boundary conditions, that is, ψ_0 provides a multivalued representation of SO(2,1) (Forte, 1991c). By construction, the wave function ψ in Eq. (5.38) is single valued, and the function Θ_{P_0} plays the role of the multivalued phase α_0 [Eq. (5.10)]. This means that upon Lorentz transformation Θ_{P_0} produces the cocycle ω_1 [Eq. (5.14)]. Indeed, upon Lorentz transformation by $\Lambda \in \widetilde{\mathrm{SO}}(2,1)$ we have

$$\Theta_{P_0}(\Lambda q)=\int_{q_0}^{\Lambda_q} dq' \frac{de}{dq'}\cdot \tilde{A}'[e]=\Theta_{P_0}(q)+\omega_1(q,g) , \tag{5.41}$$

$$\omega_1(q,g)=\int_{q}^{\Lambda_q} dq' \frac{de}{dq'}\cdot \tilde{A}'[e] . \tag{5.42}$$

The integration (5.42) runs over a path of tangent vectors obtained by acting on $e(q)$ with the path of matrices $\Lambda(s)$ which corresponds to the given element of $\widetilde{\mathrm{SO}}(2,1)$ [see the general discussion of the cocycle (5.14)]. The integral (5.42) is the writhing number of a path whose tangent vector is $e(s)=\Lambda(s)e(q)$, and it can be written, as usual, as the difference of torsion and self-linking of this path. Now, the torsion is a smooth, single-valued function of the path; therefore it transforms covariantly. Although its presence may affect the dynamics described by the wave function ψ_0, it does not modify its Lorentz transformation properties. Let us concentrate, therefore, on the self-linking. This is the integral along the path of the polar component of the projection of n on the normal plane to e. For the path that we are considering this is the instantaneous rate of rotation of n around the comoving e axis; but this is, by definition, the contribution to the winding number from the path we are considering:

$$\omega_1(q,g)=\frac{1}{2i}\int_{B_e}^{\Lambda B_e} ds\ \mathrm{tr}(\Lambda'^{-1}\dot{\Lambda}'R) , \tag{5.43}$$

where B_e is the boost that takes the unit vector $\hat{t}\equiv(1,0,0)$ into e: $B_e\hat{t}=e$.

This proves that the wave function ψ_0 [Eq. (5.38)] carries a multivalued representation of SO(2,1) (see Forte, 1991c, for a more explicit proof), and, because the angular momentum operator acting on it is the canonical one, it carries fractional angular momentum. If we consider in particular a state with definite momentum p, the tangent vector e is parallel to p, and the transformations laws take the simple form of Eq. (5.20) and (5.21), as expected on group-theoretical grounds. The transformation law under rotations (5.20) is trivial to verify, because if R^β is a rotation by angle β, then $(R^\beta)^{-1}\dot{R}^\beta=2i\beta R$. The transformation upon a boost B is easily found by noting that (Forte and Jolicœur, 1991; Forte, 1991c)

$$\mathrm{tr}(\Lambda'^{-1}\dot{\Lambda}'R)\Big|_{B_e}^{BB_e}=\dot{\phi}\cosh\theta\Big|_{B_e}^{BB_e}$$
$$=(Be_q-e_q)^a\left[\frac{\epsilon_{ab}e^b}{e^0+1}\right] . \tag{5.44}$$

The form (5.21) of the phase follows immediately, since $e_q=p/m$ for momentum eigenstates. This shows that the phases (5.20), (5.21) are reproduced by the cocycle (5.16) when the latter is evaluated for a momentum eigenstate, as claimed at the end of Sec. V.A.

To summarize, if the theory is formulated in terms of the wave function ψ_0 [Eq. (5.38)], fractionization of spin follows from the Lorentz transformation properties of the multivalued phase Θ_{P_0} [Eq. (5.39)]. If, instead, the theory is formulated in terms of the single-valued wave function ψ and the path integral (5.36), fractional spin must follow from a modification of the canonical angular momentum due to the topological interaction terms in Eq (5.36). Whereas it is easy to verify that (as in the nonrelativistic case) the topological interaction shifts the canonical angular momentum J_0 by an additive constant, as in Eq. (2.50),

$$J=J_0-\frac{\Phi}{2\pi} , \tag{5.45}$$

it is clear that this cannot be done without modifying the rest of the Poincaré generators, if relativistic covariance is to be preserved.

Indeed, in the nonrelativistic case the multivalued phase Θ [Eq. (3.21)] can be absorbed in a redefinition of the angular momentum according to

$$e^{i\theta J_0}\psi_0=e^{i\theta J_0}(e^{-i(\Phi/2\pi)\theta}\psi)$$
$$=e^{i\theta J}\psi ; \tag{5.46}$$

$$J=J_0-\frac{\Phi}{2\pi} . \tag{5.47}$$

As shown by Eq. (5.20), Eq. (5.46) holds for the relativis-

tic wave function (5.38), (5.41) and cocycle Θ_{P_0} as well. However, Θ_{P_0} is noninvariant upon boosts, because of Eqs. (5.44) and (5.21). Yet, the noninvariance may be expressed fully in terms of Poincaré generators, and this makes an analogous redefinition possible (Forte, 1991c):

$$e^{i\theta\hat{\theta}_a B^a_0}\psi_0 = e^{i\theta\hat{\theta}_a B^a_0}\exp\left[-i\frac{\Phi}{2\pi}\theta\left[\frac{\epsilon^{ab}\hat{\theta}_a p_b}{E+m}\right]\right]\psi$$

$$= e^{i(\Phi/2\pi)\theta\hat{\theta}_a B^a}\psi \ ; \tag{5.48}$$

$$B^a = B^a_0 - \frac{\Phi}{2\pi}\frac{\epsilon^{ab}P^b}{P^0+m} \ . \tag{5.49}$$

It can be verified explicitly that the topological interaction modifies the canonical currents $M_0^{(\mu\nu)}$ according to Eqs. (5.47) and (5.49), while leaving P^μ invariant, i.e., in covariant notation,

$$P^\mu = P^\mu_0; \quad M^{(\mu\nu)} = M_0^{(\mu\nu)} - \frac{\Phi}{2\pi}\epsilon^{\mu\nu\rho}\frac{P^\rho + t^\rho m}{P\cdot t + m} \ , \tag{5.50}$$

where $t=(1,0,0)$. The covariance of both theories entails the somewhat unexpected result that there exists a redefinition of the Poincaré generators, given by Eq. (5.50), that preserves the algebra while shifting the spectrum of the angular momentum. This is actually a known fact (Jackiw and Nair, 1991a, 1991b), although not an obvious one.

C. Many-particle states, spin, and statistics

In the previous section we have seen that the addition of a Hopf term to a theory of relativistic particles endows the one-particle sector of the theory with fractional spin. We can now look at the many-particle case by simply letting j^μ be an n-particle current, according to Eq. (2.33). Substituting this expression of the current in the Hopf action when $n>1$, we obtain

$$I_H = \sum_{i=1}^{n} I_H^1[x_i] + 2\frac{\Phi}{4\pi}\sum_{i=1}^{n}\sum_{j=1}^{i-1} I_{ij} \ , \tag{5.51}$$

which contains two different kind of terms (Fröhlich and Marchetti, 1989): n one-particle (diagonal) terms I_H^1, given by Eq. (5.22), and $n(n-1)$ two-particle (off-diagonal) terms I_{ij} of the form (2.42).

The diagonal terms have the properties we discussed extensively in the previous section in the one-particle case. The off-diagonal terms were discussed in the nonrelativistic case. The treatment of these terms in Sec. II.C is actually Lorentz covariant. The explicit invariance is broken by the choice of boundary conditions when computing the Gauss integral (2.42). However, Lorentz covariance is preserved. If, for example, we impose on the path integral boundary conditions at fixed time by requiring the *in* and *out* states to be $\langle \mathbf{x};t|\psi_{i,o}\rangle = \psi_{i,o}(\mathbf{x};t)$, then, upon Lorentz transformation by Λ, the *in* and *out* states become $\langle \Lambda\mathbf{x};\Lambda t|\psi_{i,o}\rangle = \psi_{i,o}(\Lambda\mathbf{x};\Lambda t)$. Furthermore, the only effect of the linking number terms is to endow the path integral with the multivalued phases Θ_{ij} that depend only on the endpoints of the path (see Sec. III.C). In a relativistic treatment, these phases are defined as polar angles on the arbitrary spacelike plane on which boundary conditions at initial and final times are imposed. Without further ado, we can give to all the results derived in the nonrelativistic case in Sec. III.C a Lorentz-covariant interpretation.

Explicitly, the n-particle propagator from $\psi_i(\mathbf{x}_1,\ldots,\mathbf{x}_n;t_i)$ to $\psi_f(\mathbf{x}'_1,\ldots,\mathbf{x}'_n;t_f)$ has the form (Forte, 1991c)

$$K(\mathbf{x}'_1,\ldots,\mathbf{x}'_n;t_f;\mathbf{x}_1,\ldots,\mathbf{x}_n;t_i) = \sum_{n_{ij}\,(i\neq j)=-\infty}^{\infty}\exp\left[-i\sigma\left[\sum_{i\neq j}\Theta_{ij}(t_f)+2\pi n_{ij}\right]\right]$$
$$\times\tilde{K}(\mathbf{x}'_1,\ldots,\mathbf{x}'_n;t_f;\mathbf{x}_1,\ldots,\mathbf{x}_n;t_i)\exp\left[i\sigma\sum_{i\neq j}\Theta_{ij}(t_i)\right] \ ; \tag{5.52}$$

$$\tilde{K}(\mathbf{x}'_1,\ldots,\mathbf{x}'_n;t_f;\mathbf{x}_1,\ldots,\mathbf{x}_n;t_i) = \sum_{n_1=-\infty}^{\infty}\cdots\sum_{n_n=-\infty}^{\infty}\int\left[\prod_{i=1}^{n}Dx_i(t_0)\right]\exp\left[-is\left[\sum_{i=1}^{n}\psi_i(t_f)+2\pi n_i\right]\right]$$
$$\times\exp\left[i\int_{t_i}^{t_f}dt_0\left[L_0(x_1(t_0),\ldots,x_1(t_0))+2\pi s\sum_{i=1}^{n}\tau_i[e_i]\right]\right]$$
$$\times\exp\left[is\left[\sum_{i=1}^{n}\psi_i(t_i)\right]\right] \ . \tag{5.53}$$

Here $\sigma = s = -\Phi/2\pi$; the two parameters have been distinguished for reasons that will become clear shortly. The integration is performed over all n-particle paths with the boundary conditions specified above. To simplify the notation we have left implicit that for each value of n_{ij} and n_i only paths with the corresponding linking and self-linking are to be integrated over. The diagonal terms contribute to the sum over paths the weights proportional to the torsion τ and the self-linking phases ψ. These terms are both inside the path integral, if boundary conditions corresponding to position eigenstates are imposed, since in this case all (timelike) values of the

tangent to the path at the endpoints contribute to the integral. The nondiagonal terms contribute the additional phases Θ_{ij}, which are (with position-space boundary conditions) outside the integral (i.e., fixed by the boundary conditions). The functions Θ_{ij} are defined as polar angles (2.22) on the spacelike plane on which boundary conditions are imposed (a space plane at fixed time, in the example at hand). Notice that the coefficients of the diagonal phases ψ_i and the off-diagonal phases Θ_{ij} are the same, and both are equal to the coefficient of the torsion.

The various interaction terms induced by the Hopf term in the path integral (5.53) may be, as usual, absorbed in a redefinition of the wave function of the form (Forte, 1991c)

$$\psi_0(\mathbf{x}_1,\ldots,\mathbf{x}_n;t)=\exp\left[2i\sigma\sum_{i=1}^{n}\sum_{j=1}^{i-1}\Theta_{ij}(t)\right]\times\int\frac{m}{E_1}d^2k_1,\ldots,\frac{m}{E_n}d^2k_n\times\exp\left[-is\sum_{i=1}^{n}\Theta_{P_0}(k_i)\right]\times\langle k_1,\ldots,k_n|\psi(\mathbf{x}_1,\ldots,\mathbf{x}_n;t)\rangle\ , \tag{5.54}$$

where the phases Θ and Θ_{P_0} are given by Eqs. (3.22) and (5.39), respectively. The phase (3.22) is defined in position space and is evaluated for each pair of particles along the paths P_0 associated with the two particles. The phase Θ_{P_0} is defined in momentum space (the space of tangent vectors to the path), and for a position eigenstate must be defined by expanding ψ in momentum eigenstates, as indicated in Eq. (5.54). This procedure parallels that of constructing one-particle position-space solutions to the Dirac equation: the general solution is found by expanding in plane waves and weighting each plane wave of fixed momentum with the spinor that solves the pertinent Dirac equation in momentum space.

Upon Lorentz transformation each phase Θ_{P_0} transforms with the cocycle (5.43). Furthermore, it is clear that the same argument used to derive the transformation properties of Θ_{P_0} implies that the phase Θ_{ij} defined on the spacelike surface orthogonal to a timelike vector n transforms with the cocycle (5.43), with the replacement $e\rightarrow n$: in fact, these two phases are due to one and the same object, the linking number density (2.45), computed, respectively, for a pair of paths i,j or for the ith path and its canonical Frenet framing. To summarize, upon Lorentz transformation the wave function acquires n copies of the Fourier transform of the cocycle (5.43), evaluated for all $e=p/m$, and $n(n-1)$ copies of the cocycle (5.43), evaluated for $e=n$. Upon spatial rotation, in particular, the wave function acquires a phase (5.46) (the same for all the contributions to the cocycle) which shifts the angular momentum according to (Thouless and Wu, 1985; Fröhlich and Marchetti, 1989)

$$J=J_0+[n+n(n-1)]s=J_0+n^2s\ . \tag{5.55}$$

The somewhat haphazard-looking path integral (5.53) and wave function (5.54) actually have a very simple physical interpretation (Polyakov, 1988; Forte, 1991c), which is perhaps best understood by comparison to the usual spin-$\frac{1}{2}$ case. First, let us look at the diagonal terms, which are related to the one-particle dynamics. The multivalued phases ψ have the role of generating the cocycle (5.43) upon Lorentz transformation of on-shell momentum eigenstates; they play the same role as the definite-momentum spinors $u(p)$, in that an on-shell one-particle state is a Fourier superposition of plane waves multiplied by the appropriate phase, or spinor, respectively. The pertinent contribution to the angular momentum spectrum has the form of an intrinsic angular momentum, since it is independent of the choice of the origin. The torsion terms in the path integral provide the kinetic term that describes the spin dynamics. Indeed, it has been shown explicitly by Nielsen and Rohrlich (1988) and Johnson (1989) that the action (5.34) used in a path integral, with the choice $s=\frac{1}{2}$, leads to the correct canonical quantization of $e/2$, interpreted as a spin degree of freedom.[16]

The nondiagonal terms, instead, appear at the level of the n-particle position-space wave function. Upon Lorentz transformation they produce a cocycle that depends on the choice of the (spacelike) plane of quantization, i.e., on the form of the wave function. In particular, the cocycle determines a shift in the angular momentum spectrum due to the relative orbital angular momentum of each pair of particles. Indeed, suppose we let particles i and j rotate about their center of mass while keeping the other particles fixed. This rotation may be realized by acting on the wave function with the operator

$$L_{x_ix_j}=-i\epsilon_{ab}\left[x_i^a\frac{\partial}{\partial x_i^b}+x_j^b\frac{\partial}{\partial x_j^a}\right]\ ,$$

and the corresponding cocycle produces a phase proportional to the angle of rotation β:

$$e^{i\beta L_{x_ix_j}}\psi_0(\ldots,\mathbf{x}_i,\ldots,\mathbf{x}_j,\ldots)=e^{2is\beta}\psi_0(\ldots,R^\beta\mathbf{x}_i,\ldots,R^\beta\mathbf{x}_j,\ldots)\ , \tag{5.56}$$

[16]In the construction of a spin path integral from first principles the inclusion of the multivalued "self-linking" phase appears to be inevitable if the correct S-matrix elements are to be obtained. The need for a multivalued phase may be understood as the consequence of the fact that the coherence effects that yield the desired quantization rules are effective only if one path-integrates over a noncompact phase space (I thank K. Johnson for pointing this out to me). This dynamical result complements the purely group-theoretical (kinematical) one according to which the multivaluedness is needed in order to obtain multivalued representations of SO(2,1).

where R^{β} is the rotation matrix of angle β. Since the generator of rotations of x_i and x_j, $L_{x_i x_j}$ is the orbital angular momentum operator for the pair of particles x_i, x_j, the phase in Eq. (5.56) corresponds to a contribution of $2s\beta$ to the total orbital angular momentum (Forte and Jolicœur, 1991).

In the customary spin-$\frac{1}{2}$ case, the phases Θ_{ij} represent just the antisymmetrization of the wave function with respect to all pairs of particles: if we interchange particles i and j by means of a rotation of π, Eq. (5.56) shows that when $s=\frac{1}{2}$ the wave function is antisymmetrized upon interchange. For arbitrary fractional spin the wave function is "symmetrized" with phases, rather than just factors of ± 1, and upon interchange acquire a phase according to Eq (5.56), i.e., it carries fractional statistics. Notice that if s is not quantized in half-integer units the statistics phase depends on whether the interchange is realized through a clockwise or counterclockwise rotation. Of course, this just means that the wave function on a spacelike plane carries a representation of the braid group, rather than the permutation group, as discussed in Sec III.C.

All this implies that there is a natural spin-statistics relation in the theory we are discussing. In a $2+1$-dimensional quantum-mechanical theory there are generally two distinct relations between statistics and angular momentum (Forte and Jolicœur, 1991). The first is expressed by Eq. (5.56), and it is always true—it follows from the identification of the orbital angular momentum operator $L_{x_i x_j}$ as the generator of rotations of particles i,j about each other. It states that the statistics σ [Eq. (1.2)] is equal to (one-half) the spectrum of the orbital angular momentum operator $L_{x_i x_j}$ modulo integer. The second is the relation between the statistics phase [Eqs. (1.2) and (5.56)] and the phase acquired by a one-particle state upon rotation, modulo integer. The latter is the coefficient of the one-particle cocycle (5.42), and, as discussed above, it coincides with the particle's spin s in Eqs. (5.53) and (5.54). This last relation is what is usually referred to, and what we shall call, the spin-statistics relation: it is, in the general n-particle case, a relationship between the coefficients σ and s in the path integral [Eqs. (5.52) and (5.53)] and wave function (5.54). The spin-statistics theorem states that, for boson and fermion fields, $\sigma = s \bmod(\mathbb{Z})$ if the underlying relativistic field theory is to be local; in a first-quantized theory (quantum mechanics), however, it may or may not be verified. For a generic theory of particles with spin s and statistics σ the orbital angular momentum L, spin S, and total angular momentum J of an n-particle state are thus

$$L = n(n-1)\sigma + \ell, \quad \ell \in \mathbb{Z}, \qquad S = ns, \quad J = L + S. \tag{5.57}$$

Comparing Eq. (5.55) with Eq. (5.57), we see that in our theory the spin-statistics theorem $\sigma = s$ is automatically satisfied (Fröhlich and Marchetti, 1989).

To summarize, the relativistic treatment of the Hopf interaction (5.22) leads naturally to the dynamics associated with spin degrees of freedom in $2+1$ dimensions. The action (5.34) that determines this dynamics is intrinsically relativistic, in that it requires that the particle's paths be viewed as paths in $2+1$-dimensional Minkowski space-time. The spin degrees of freedom are coupled to the translational degrees of freedom, as one would expect in a relativistic theory, because the spin action is a function of the tangent vector to the particle paths. Since semiclassically the tangent to the path is identified with the particle momentum, the semiclassical spin vector along the path is always parallel to the momentum (this can be shown to follow directly from the semiclassical limit of the Dirac propagator; see Coste and Lüscher, 1989).

The "diagonal" terms I_H^1 [Eq. (5.51)] that determine the spin dynamics cannot be disposed of in a relativistic theory (contrary to the nonrelativistic case; cf. Sec. II). Because paths contributing to the path integral can go both forward and backward in time, intermediate states corresponding to propagation of an n-particle state may contain an arbitrary number of particles $k \geq n$ (Fig. 12). This implies that there is no universal way of defining the two-particle braiding phases Θ_{ij} [Eq. (3.17)] used in the nonrelativistic construction: both the braiding and the self-linking phases must be included at once. Indeed, one would expect the self-linking phases to be responsible for the nontrivial relativistic spin dynamics, as is the case for fermions. For instance, this term leads to the Pauli coupling of spin to the magnetic field once the particle is coupled to an electromagnetic field (Cortés *et al.*, 1991).

This concludes our discussion of the theory of relativistic particles with fractional statistics obtained by coupling the particle's current through a Hopf interaction. No proof is as yet available that the path integral [Eqs. (5.52) and (5.53)] (even with $\sigma \neq s$) is the most general one for relativistic particles with fractional spin and statistics, contrary to the nonrelativistic case (3.18), although this seems plausible.

In the spin-$\frac{1}{2}$ case the present approach to the theory of relativistic spinning particles yields the same results as the usual Dirac theory. For example, the propagator (5.36) can be shown to coincide with the usual Dirac propagator (Polyakov, 1988, 1989; Ambjørn, Durhuus,

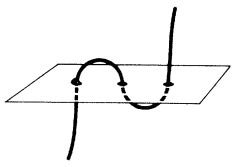

FIG. 12. A three-particle state obtained as a section of a single space-time path by a spacelike plane.

and Jonsson, 1989; Orland, 1989)—it is actually in this particular case that the theory was first developed. The possibility of trading the multivaluedness of the wave function for a long-range interaction, once rephrased in terms of spinor fields, allows one to establish an isomorphism between a bosonic theory and a fermionic one, i.e., a bosonization in $2+1$ dimensions (Lüscher, 1989; Ambjørn and Semenoff, 1989). The equivalence with the Dirac formalism, as well as the numerous developments of the quantum theory that this equivalence makes possible, have spawned an extended literature (see Jaroszewicz and Kurzepa, 1991, and references therein).

Of course, a quite different approach to relativistic theory is possible: rather than starting with the path integral, one can start with the Schrödinger equation, or rather, the Dirac equation, adapted to the case of fractional statistics (Jackiw and Nair, 1991a, 1991b). The advantage of this approach is that it disposes with the cumbersome cocycle formalism. One straightforward way of writing this equation is to postulate, according to Wigner (1939, see also Balachandran *et al.*, 1983), that the one-particle states provide irreps of the Poincaré group. In this case the equations of motion reduce to just the two equations (5.18)–the mass-shell condition (5.18a) and the spin condition (5.18b). There is an important ambiguity in this procedure, namely, that of choosing the representation space (necessarily infinite-dimensional) in which the wave functions live. Moreover, a purely group-theoretical approach does not give us any clue as to the construction of the covariant field theory. Patterning after the familiar integer or half-integer spin cases, the covariant fields should provide a linear representation of the Lorentz group and a generally reducible representation of the Poincaré group, whereas the equation of motion should project out the Poincaré irreps. Although, in principle, the path integral of the theory should contain all the relevant dynamical information, it is in practice rather unmanageable: compare, for example, the task of computing a fermion-fermion scattering amplitude from the path integral [Eqs. (5.52) and (5.53)] rather than from the $2+1$-dimensional Dirac equation.

A dynamical equation that satisfies these requirements has been proposed recently (Jackiw and Nair, 1991a, 1991b). The wave functions are chosen to be of the spinor-vector form: $F_n^\mu(x)$, where μ is a vector index and $0\le n\le\infty$ runs over an infinite-dimensional irrep bounded either below or above of $\widetilde{\mathrm{SO}}(2,1)$ (cf. Sec II.A). Then the spin condition (Pauli-Lubanski equation), Eq. (5.18b), is imposed on F_n^μ. Setting $P_\mu = i\partial_\mu$, the generator of space translations, and $M^{(\mu\nu)}$ the Lorentz generators in the tensor product of the spaces spanned by the indices μ and n in Eq. (5.18b), one obtains

$$\epsilon_{\mu\nu\rho}P^\mu(M^{(\nu\rho)}_{\alpha\beta}\delta_{nn'}+M^{(\nu\rho)}_{nn'}\delta_{\alpha\beta})F_n^\beta(x)=msF_{n'}^\alpha(x)\ . \tag{5.58}$$

The value of the spin s in Eq. (5.58) is fixed in terms of the value of d that characterizes the infinite-dimensional irrep [cf. Eqs. (5.5)–(5.7)] as $s=1-d$.

In order to project out the physical degrees of freedom, two additional subsidiary conditions have to be imposed, namely (Jackiw and Nair, 1991a, 1991b),

$$\begin{aligned} &P_\mu M^{(\mu\nu)}F_n^\nu(x)=0\ , \\ &\epsilon^\mu{}_{\nu\rho}P_\alpha M^{(\alpha\nu)}F_n^\rho(x)=0\ . \end{aligned} \tag{5.59}$$

By expanding the solutions to the spin equation in plane waves it is then easy to show that the solutions that satisfy the subsidiary conditions (5.59) also satisfy the transversality condition (Jackiw and Nair, 1991a)

$$P_\mu F_n^\mu(x)=0\ . \tag{5.60}$$

The general solution to these equations can be constructed explicitly. Only one component of F_n is nonvanishing, the highest-weight one $n=0$, and this satisfies the mass-shell conditions automatically by iteration. The explicit form of the solution in the particle's rest frame is (Jackiw and Nair, 1991b)

$$\begin{aligned} f_0^\mu(p_0)&=N^\mu{}_\nu\begin{pmatrix}0\\0\\\Psi(p_0)\end{pmatrix}\ , \\ N^\mu{}_\nu&=\begin{pmatrix}\sqrt{2}&0&0\\0&1&1\\0&-i&i\end{pmatrix}\ , \end{aligned} \tag{5.61}$$

where $P_0=(m,0,0)$ and $\psi(p_0)$ is a function that provides a one-dimensional positive-energy Poincaré irrep with zero spin and the given momentum. The solution for arbitrary p is of course obtained by boosting Eq. (5.61), and a negative-energy solution can be constructed as well.

The explicit relation between these solutions and the wave functions that may be obtained from a path integral of the form (5.52) and (5.53) is as yet unexplored. We shall come back to this problem when discussing fractional spin and statistics in field theory in Sec. VI.C.

VI. FIELD-THEORETICAL MODELS

Field theory with fractional spin and statistics is still an open subject: although it is reasonable to expect that various effects of fractional spin, such as the spin-statistics theorem, should be understood only in a full-fledged relativistic field theory, relatively little is known in this respect.

The simplest approach to a field theory of excitations with fractional spin and statistics consists of looking at field theories that support topological solitons. Indeed, it is clear that, insofar as topological solitons may be described as localized particle-like excitations, the treatment of the soliton sector of such a theory may be reduced to that of point-particle mechanics. In fact as we shall see, the soliton approach is in a way more natural than the particle approach. A different option consists of trying to construct a field theory in which the fundamental field quanta themselves (rather than some soliton states) carry fractional spin. We shall see that the first

results obtained in this approach hint at new physics not seen in the soliton approach.

A. Solitons in the O(3) and CP^1 models: topology and the adiabatic limit

The simplest example of a theory that supports static topological solitons in 2+1 dimensions is the O(3) nonlinear sigma model (see Rajaraman, 1982, for a review). In this theory the fields are three-vectors $n^a(x)$,[17] normalized to $n^a n^a = 1$, that is, they take values on the sphere S^2. The Lagrangian density is

$$\mathcal{L}_0 = \frac{1}{2f}\partial_\mu n^a \partial^\mu n^a \ , \tag{6.1}$$

where f has the dimensions of [length].

If we compactify the fixed-time surfaces to a sphere by requiring the fields to tend to a constant value at spatial infinity, the classical field configurations at fixed time are maps $S^2 \to S^2$ and thus fall into disconnected homotopy classes (see, for example, Dubrovin *et al.*, 1984). A representative of the kth homotopy class is

$$n^a(\mathbf{x}) = \begin{bmatrix} \hat{r}\sin kf(r) \\ \cos kf(r) \end{bmatrix} ; \qquad f(r) = \begin{cases} \pi, & \text{if } r = 0 \\ 0, & \text{if } r = \infty \end{cases} , \tag{6.2}$$

where r is the radial coordinate and $\hat{r}$ the radial unit vector on the space plane. Fields in different homotopy classes cannot be deformed continuously into each other and are classified by a two-dimensional winding number

$$Q = \frac{1}{8\pi}\int d^2x \, \epsilon_{abc}\epsilon^{ij} n^a \partial_i n^b \partial_j n^c \ , \tag{6.3}$$

which counts how many times n^a spans S^2 as x runs over all space, i.e., for n^a given by Eq. (6.2) $Q = k$.

Classical solutions to the equations of motion are obtained for a wide class of functions $f(r)$ [Eq. (6.2)]. These solutions, extended yet localized, are referred to as solitons of the classical theory. Time-dependent solitons may be obtained by letting the location of the soliton depend on time, i.e, by letting $\mathbf{x} \to \mathbf{x} - \mathbf{x}_0(t)$ in Eq. (6.2). The current that carries the soliton excitations is then

$$j^\mu(x) = \frac{1}{8\pi}\epsilon^{\mu\nu\rho}\epsilon_{abc} n^a \partial_\nu n^b \partial_\rho n^c \ . \tag{6.4}$$

This current is identically conserved, $\partial_\mu j^\mu = \det(\partial_\mu n^\nu) = 0$ (as a consequence of the normalization of n), independently of the dynamics. The charge density from Eq. (6.4) coincides with the winding number density that appears in Eq. (6.3). The soliton is a generally extended object, localized in the region where most of the charge density is concentrated; the size of the latter, in turn, is related to the range of variation of the function $f(r)$ [Eq. (6.2)] (measured on the scale set by f): for example, if $F(r) = \pi[\lambda/(\lambda + r)]$, the soliton size is of order λ.

A classically equivalent field theory is the CP^1 model (see Rajaraman, 1982), whose fundamental field takes values in the space of unimodular two-component complex vectors

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad z^\dagger z = 1 \ ,$$

with vectors differing by an overall phase identified (CP^1 space). This space is isomorphic to the sphere (on which the fields n^a live); the isomorphism is given by

$$n^a = z^\dagger \sigma^a z \ , \tag{6.5}$$

where σ^a are the usual Pauli matrices. The relation between z fields and n fields is the usual one between spinors and vectors made from spinor bilinears. The Lagrangian (6.1) and current (6.4) can easily be written in terms of the z field by substituting Eq. (6.5). The current (6.4), in particular, takes the simple form

$$j^\mu = \frac{i}{2\pi}\epsilon^{\mu\nu\rho}(\partial_\nu z^\dagger \partial_\rho z) \ . \tag{6.6}$$

As we shall see shortly, the CP^1 formulation may be more convenient for some applications.

The soliton sector of these theories may be quantized by canonical or path-integral methods, although the nonrenormalizability[18] of the theory (6.1) makes its quantum treatment arduous. At the very least, the model can be treated as a low-energy effective one, and this is enough to be able to discuss its long-range symmetry properties—like spin and statistics. It is rather natural to expect that fractional spin and statistics may be induced in the soliton sector of the theory (6.1) by coupling the current (6.4) to itself through the Hopf Lagrangian (2.38), or equivalently, by coupling it to a Chern-Simons term (2.35)–(2.37) (Wilczek and Zee, 1983). Indeed, this is trivially true in the limit in which the current (6.4) is concentrated in one point z_0, so that it may be approximated by a point-particle current (2.33).[19] In this case, we are led back to the treatment of a relativistic point particle coupled via the Hopf interaction (Wu, 1984b; Wu and Zee, 1984), discussed in the previous section.

---

[17] In this section latin indices from the beginning of the alphabet take the values $a = 1,2,3$. These indices are always to be summed with an Euclidean metric.

[18] It is interesting to observe that the O(N) model, which generalizes the above model to the case in which n^a takes values in the $N-1$ dimensional sphere, although perturbatively nonrenormalizable, admits a renormalizable $1/N$ expansion (Parisi, 1975). This feature has been used to discuss the equivalence of the phase structure of the O(3) model with Hopf term, with that of a fermonic theory (Kovner, 1990).

[19] Notice that if $j^0(\mathbf{x} - \mathbf{x}_0) = \delta^{(2)}(\mathbf{x} - \mathbf{x}_0(t))$ then it automatically follows from Eq. (6.4) that the full current has the form of Eq. (2.33).

A simple topological argument (Wilczek and Zee, 1983; see also Wu and Zee, 1984) shows that extended solitons (6.2) can carry fractional angular momentum, irrespective of the pointlike approximation. This is because the configuration space of the O(3) model fields at fixed time is infinitely connected. The configuration space is the space $\mathcal{C}$ of all maps $n:S^2\to S^2$, and $\pi_1(\mathcal{C})=\pi_3(S^2)$, but $\pi_3(S^2)=\mathbb{Z}$ (see Dubrovin *et al.*, 1984). Otherwise stated, all possible one-parameter families of field configuration over space (*not* space-time) $n^a(\mathbf{x})$ fall into disconnected classes. This is good news, because each of these one-parameter families may be thought of as a time evolution (i.e., a motion) of a given static soliton, and then we may weight the contribution to the path integral of paths in each equivalence class with a phase, as discussed in Sec. III.C (Wu and Zee, 1984). It remains to establish that these equivalence classes correspond to solitons that perform different numbers of space rotations, so that the multivalued phases in the path integral may be interpreted as contributions to the angular momentum, and to relate these phases to the presence of a topological term in the action.

Rather than constructing the phases from first principles and then deriving the action, let us show that the Hopf action, evaluated for a rotating soliton, does the job (Wilczek and Zee, 1983). That is, let us assume that the Hopf interaction is added to the action, and consider the path integral in the one-soliton sector of the theory, i.e., the sum over paths of one-soliton configurations, weighted by their action. We consider in particular the contribution to the path integral from an evolution of the soliton (6.2) (with $k=1$), in which the soliton remains at a fixed location, does not change shape, and rotates about itself:

$$n^a(\mathbf{x},t)=n^a(R(t)\mathbf{x})=\begin{bmatrix}[R(t)\hat{r}]\sin f(r)\\ \cos f(r)\end{bmatrix}=R^a{}_b(t)n^b(\mathbf{x}) , \tag{6.7}$$

where $R(t)$ is a matrix that performs a spatial rotation by 2π in a period T. If we let $T\to\infty$, ie., if we assume that the rotation is infinitely slow, the kinetic term (6.1), which is of order $1/T$, gives a vanishingly small contribution to the action. The Hopf term, instead, is of order $T^0=1$ and survives even in this limit. We show now that for a rotation of $2\pi n$ the Hopf term evaluated for this field configuration takes the value n, and that this implies that the angular momentum spectrum is shifted by an amount proportional to the coefficient of the Hopf term.

The Hopf action for the rotating soliton (6.7) is easy to compute (Wilczek and Zee, 1983); however, the result can be obtained directly by inspection, thanks to the topological meaning of the Hopf term (Wilczek and Zee, 1983). As mentioned above, all maps from S^3 to S^2 fall into disconnected homotopy classes. A field $n^a(x,t)$ may be viewed as one such map, if we impose boundary conditions that allow compactifying space-time like S^3. There exists an invariant (called Hopf invariant) that classifies these maps, just as the winding number (6.2) classifies maps from S^2 to S^2 (see Dubrovin *et al.*, 1984). The Hopf term (2.38), evaluated for the current (6.4), provides precisely an expression of this invariant. Note that it is nonlocal. Moreover, this invariant has a simple geometrical interpretation: if f is a map $S^3\to S^2$, we can define the inverse map f^{-1}. The inverse image of a point $p\in S^2$, given by $f^{-1}(p)$, will be in general a curve in S^3, which can be shown to be closed if the map f is sufficiently regular. Now, two closed curves always define a linking number (Fig. 1), and we can consider the linking number of the two inverse images $f^{-1}(p)$ and $f^{-1}(q)$ of two distinct points $p,q\in S^2$. It turns out that this linking number is a universal invariant for the map f

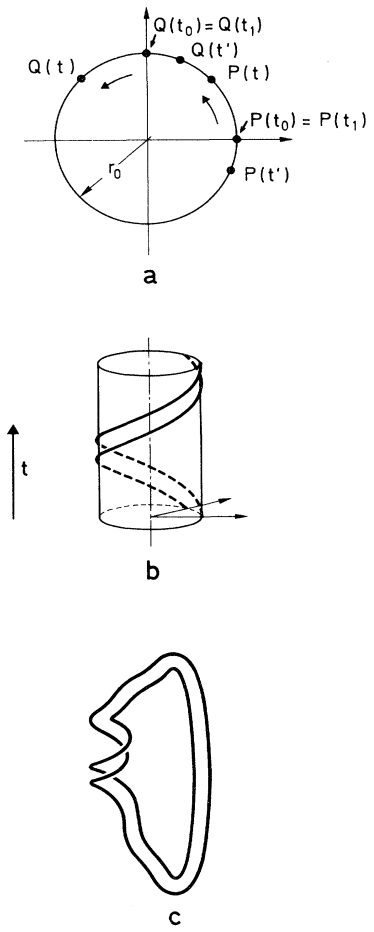

FIG. 13. Determination of the Hopf invariant of a rotating soliton: (a) location on the space plane of the points p and q where the soliton field takes the values (1,0,0) and (0,1,0), called $P(t)$ and $Q(t)$, respectively (inverse images of points p,q). The locations at initial and final times t_0 and t_1, as well as at intermediate times $t_0\le t\le t'\le T_1$ are shown. (b) Space-time trajectory of points $P(t)$ and $Q(t)$: two linked helices wound on a cylinder of radius r_0. (c) Space-time trajectory of the inverse images of p and q for a soliton-antisoliton pair in which the soliton rotates. In the vicinity of the rotating solitons the trajectory coincides with that shown in (b).

(it is the same for any pair of points p,q)and that the Hopf invariant is proportional to it.

It is easy to see that this produces the desired result. Consider a rotating soliton (6.7) and choose as points p,q two points on S^2, say the points $n^a=(1,0,0)$ and $n^a=(0,1,0)$. At an initial time t_0, when $R=\mathbb{1}$, the soliton has the form (6.2), and the inverse images of p and q are, respectively [see Fig. 13(a)], the points $P\equiv(t_0,r_0,0)$ and $Q\equiv(t_0,0,r_0)$, where r_0 is a point such that $f(r_0)=\pi/2$ (there is always one such point for a map of the first homotopy class: we assume for simplicity and with no loss of generality that there is only one). As the soliton rotates, the points $P(t)$, $Q(t)$ that are mapped, respectively, into p and q, describe a helix; for a rotation of 2π the inverse images of p and q are the two helicoidal paths shown in Fig. 13(b).[20] These are open paths, but of course a soliton (6.7) does not define a map $S^3\to S^2$: in order for the fields to be compactified on S^3 they must tend to the same constant at both space and time infinity, whereas any one-soliton configuration tends to (1,0,0) at spatial infinity, while it has the space-dependent form (6.2) at time infinity.[21]

However, we can consider (Wilczek and Zee, 1983) a process in which a soliton-antisoliton pair is created at $t_i<t_0$ [that is, a pair of fields (6.2) with $k=1$ and $k=-1$]. Then the two solitons are taken apart, and at $t=t_0$ only one of the two solitons starts to rotate and performs a rotation by 2π until $t=t_1$. Finally, the solitons are brought to annihilation at $t=t_f$. In this case, for fixed time $t<t_i$ and $t>t_f$ the n field is $n=(0,0,1)$, that is, no point is mapped into p and q. For times $t_i<t<t_f$ there are two pairs of space points where the n field takes the values p and q, associated with the soliton and antisoliton, respectively, and when the soliton rotates Fig. 13(b) represents the trajectories of the points associated with the soliton only. In sum, the inverse image of p and q for this configuration is given by the pair of linked paths of Fig. 13(c), and it should be clear that this pair-production picture just reproduces the usual prescription (Fig. 2) for calculating the linking number of open space-time curves.

We conclude that the Hopf invariant, evaluated for any rotating soliton configuration, just measures the linking number of the space-time trajectories followed by any two points arbitrarily picked on the soliton shape.[22] This proves that in the limit of an adiabatic rotation the soliton's action is proportional to the coefficient of the Hopf term times the total rotation angle. But this also implies that the soliton's path integral contains a weight of the form (3.17)–(3.18), which shifts the angular momentum spectrum, thus concluding the argument.

Finally, a similar argument (Wilczek and Zee, 1983) may be used to show that if two solitons are interchanged, the path integral acquires a phase equal to that acquired when rotating one soliton by 2π, thus establishing the spin-statistics connection that we know (from Sec. V.C) to hold in the point-particle limit. For this it is enough to evaluate the linking number for a process in which two soliton-antisoliton pairs are created, then the solitons are interchanged (Fig. 14).

In a way, if we regard the soliton as an assembly of particles, the Hopf term simply measures the linking of the space-time trajectories of the particles and endows the soliton with fractional angular momentum by the arguments of Sec. III.C. In this sense, the discussion of fractional spin is more natural for a single soliton than for a single particle, in that a soliton, being an extended object, can be viewed as a many-particle system and can be endowed with fractional spin by the simple homotopy arguments of Sec. III, without the need to face the subtle problems related to the definition of the self-linking of a one-particle trajectory.

A precise evaluation of the coefficient of proportionality between the fractional contribution to the angular momentum and the coefficient of the Hopf term requires us to look more closely at the quantization of the theory. We shall do this in the next section.

B. Soliton dynamics and quantization

The topological interpretation of the Hopf term suggests that its role in the field theory of solitons with fractional spin may be as simple as it was in the case of non-relativistic particles: namely, that the Hopf term in the one-soliton sector may be written as a total derivative—or rather, being a field-theoretical object, a total divergence. This turns out to be the case (Din and Zakrzewski, 1984; Wu and Zee, 1984) and makes the discussion of its effect on the quantization of the theory

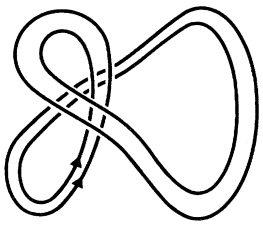

FIG. 14. The inverse images of p and q for a process in which two soliton-antisoliton pairs are created, the solitons are interchanged, and then the pairs are brought to annihilation. The picture is homotopically equivalent to the two linked loops shown in Fig. 1(a).

[20]Notice that a counterclockwise rotation of the inverse images (as shown in Fig. 13) corresponds to a clockwise rotation [Eq. (6.7)] of the soliton profile.

[21]Indeed, both the equality $\pi_1(\mathcal{C})=\pi_3(S^2)$ and the identification of the Hopf invariant as a linking number hold only for maps that on fixed-time surfaces are homotopically trivial, i.e., that are homotopic to the trivial map obtained putting $k=0$ in Eq. (6.2).

[22]This somewhat formal argument has been checked by explicit calculation by Jaroszewicz (1985).

rather simple.

The expression of the Hopf term in terms of the O(3) and CP^1 fields is found by substituting the current (6.4) or (6.6), respectively, in its definition, Eq. (2.38). It is convenient, to this purpose, to rewrite the Hopf term as

$$I_H=-\frac{\Phi}{2}\int d^3x\, j^\mu(x)A'_\mu(x)\ , \tag{6.8}$$

$$A'_\mu(x)=\int d^3y\, K_{\mu\nu}(x,y)j^\nu(y)\ . \tag{6.9}$$

The expression of A'_μ in terms of j^μ in Eq. (6.9) coincides (up to the normalization) with the classical equation of motion of the Chern-Simons field A_μ [Eqs. (2.36)–(2.37)] (Wilczek and Zee, 1983). Equations (6.8) and (6.9) express the Hopf term as a local function of the current j^μ and a composite (i.e., nonlocal) gauge field A'_μ.

It turns out that if we express the current in CP^1 form the composite gauge field becomes local, and thus so does the Hopf term (Wu and Zee, 1984). Indeed, Eq. (6.9) can be written equivalently as

$$j^\mu(x)=\epsilon^{\mu\nu\rho}\partial_\nu A'_\rho(x)\ , \tag{6.10}$$

which, for j^μ given in CP^1 form by Eq. (6.6), is solved by

$$A'_\mu=\frac{i}{2\pi}z^\dagger\partial_\mu z\ , \tag{6.11}$$

leading to

$$I_H=-\frac{\Phi}{2}\int d^3x\left[-\frac{1}{4\pi^2}\right]e^{\mu\nu\rho}(z^\dagger\partial_\mu z)(\partial_\nu z^\dagger\partial_\rho z)\ . \tag{6.12}$$

This is not fortuitous: the field z actually takes values on S^3, rather than on S^2 as the n field does, as may be seen by observing that each spinor z uniquely specifies an SU(2) matrix (which takes a fixed spinor, say $\binom{1}{0}$, into the given one), and that the group manifold of SU(2) is S^3. The CP^1 field is obtained by identifying all spinors that differ by a phase, i.e., by identifying $CP^1=S^3/U(1)$. As a function of space-time the z field is a map $S^3\to S^3$, rather than $S^3\to S^2$ as the n field is. This is called a lift of the original map. Now, it may be shown (see Pak and Percacci, 1991) that the lift of the Hopf invariant (i.e., the Hopf invariant calculated for the lifted map) just coincides with the winding number of the lifted map, which is obviously local.

This discussion also tells us immediately that the Hopf term is (locally) a total derivative, since the winding number is (see Dubrovin *et al.*, 1984; Jackiw, 1985). An explicit expression is found by parametrizing the components of z with Cartesian or polar coordinates on the complex plane:

$$z=\begin{bmatrix}x_1+iy_1\\ x_2+iy_2\end{bmatrix}=\begin{bmatrix}\rho_1 e^{i(\phi_1/2)}\\ \rho_2 e^{i(\phi_2/2)}\end{bmatrix}\ . \tag{6.13}$$

Straightforward though cumbersome manipulations lead to (Wu and Zee, 1984)

$$I_H=-\frac{\Phi}{2}\int d^3x\left[-\frac{1}{\pi^2}\right]e^{\mu\nu\rho}\partial_\mu(\phi_2\partial_\nu x_1\partial_\rho y_1 - \phi_1\partial_\nu x_2\partial_\rho y_2)\ , \tag{6.14}$$

which is the sought-for total-divergence form of the Hopf term. Equation (6.14) may be rewritten in a more transport form by choosing arbitrarily the overall phase of z in such a way that $\phi_1=-\phi_2\equiv\phi_0$. Then

$$I_H=\frac{\Phi}{2\pi}\int d^3x\,\partial_\mu(\phi_0(x)j^\mu(x))\ . \tag{6.15}$$

It is now easy to see how the Hopf term induces fractional spin of the solitons when the theory is quantized (Wu and Zee, 1984): the argument closely parallels the construction of the path integral for particles on a multiply connected space, given in Sec. II.[23] In a field theory, the points in configuration space are functions, and the state vectors are functionals. If we quantize the theory *à la* Schrödinger, time is singled out, the fields $n^a(x)$ are quantized canonically, and the state vectors are $\langle q,t|\Psi\rangle=\langle n^a(\mathbf{x}),t|\Psi\rangle=\Psi[n^a(\mathbf{x});t]$. Equation (3.19) then holds for these state functionals, with

$$\begin{aligned}K(q',t';q,t)&=K(n'^a(\mathbf{x}),t';n^a(\mathbf{x}),t)\\ &=\int\mathcal{D}n^a(\mathbf{x},t)e^{iI[n]}\ ,\\ I[n]&=\int d^3x\,\mathcal{L}_0+I_H\ ,\end{aligned} \tag{6.16}$$

where the boundary conditions are the field configurations $n^a(\mathbf{x})$ at initial time t and $n'^a(\mathbf{x})$ at final time t'. Now, the topological action I_H is a total derivative; let us moreover suppose that space is compactified to S^2 by requiring that fields fall off at infinity. Each field configuration that contributes to the path integral (6.16) is defined on a cylinder $S^2\times I$, where I is the time interval $t\le t_0\le t'$ and vanishes at space infinity, while at initial and final times it satisfies the given boundary conditions. It follows that the topological action reduces to surface terms at initial and final times:

$$\begin{aligned}I_H&=\int d\mathbf{x}\,dt_0\,\partial_\mu\Omega^\mu[\phi(\mathbf{x},t_0)]=H(t')-H(t)\ ,\\ H(t)&=\int d\mathbf{x}\,\Omega_0(\mathbf{x},t)\ .\end{aligned} \tag{6.17}$$

Upon Lorentz boost the surface on which the boundary conditions are defined is transformed into a generic spacelike plane. These surface terms are the exact analogue of the phases Θ [Eq. (3.17)], in that, as shown in the previous section, they measure the linking of the soliton profile, considered as a path (parametrized by time) on the multiply connected space of static one-soliton configurations (Wu and Zee, 1984).

At this point, we may proceed exactly as we did in Sec. III.C for nonrelativistic particles. From Eq. (6.15) the

[23]The theory of path integration on multiply connected space by Laidlaw and Morette de Witt (1971), described in Sec. III.A, has been generalized to field theory by Dowker (1972).

surface terms are

$$H(t)=\frac{\Phi}{2\pi}\int d^3x\,\phi_0(\mathbf{x})j^0(\mathbf{x})\ . \tag{6.18}$$

This quantity is multivalued in configuration space: if we write the unit three-vector n in spherical coordinates θ_n, ϕ_n [as in Eq. (5.32) with cosh→cos], then Eq. (6.5) implies that $\phi_n=(\phi_1-\phi_2)$; therefore if we let $\phi_n\to\phi_n+\alpha$ we have $\phi_0\to\phi_0+\alpha$. Therefore the surface terms evaluated for the field $n(x)$ and its rotation by α, $R^\alpha n(x)$, are related by

$$H[R^\alpha n]=H[j^0(R^\alpha n)]+\frac{\Phi}{2\pi}\alpha Q\ , \tag{6.19}$$

where Q is given by Eq. (6.3). Because Q is identified with the soliton number, the multivaluedness is present only in the soliton sectors of the theory.

It follows that the propagator is given by

$$K(q',t';q,t)=\sum_{n=-\infty}^{\infty}e^{i[H(t')+(\phi/2\pi)nQ]}\times K_0(q',t';q,t)e^{-i\frac{\Phi}{2\pi}H(t)}\ , \tag{6.20}$$

where $K_0(q',t';q,t)$ is the path integral (6.15) in the absence of a topological term I_H, and the sum over n corresponds to the sum over all possible determinations of $H(t')$. We may at this point absorb the topological term by redefining the state vectors by a phase obtained by integrating $H(t)$ from a certain reference field configuration to the given one, etc. Let us just verify that the redefined functionals carry a spin $-\Phi/2\pi$ representation of rotations.[24] Space-time transformations are related to transformations in field space by the soliton ansatz (6.2). In particular, with $n=1$, Eq. (6.7) shows that a space rotation is the same as a rotation of the soliton field n. Thus H transforms upon space rotations as do the phases (3.22) and (5.38), and it endows the one-soliton states with spin $-\Phi/2\pi$.

To evaluate the spin of a k-soliton state, as well as its statistics, along the same lines, is rather difficult, because the explicit form of the n field plays a crucial role in the above derivation, while the k-soliton field has a rather contrived form. It is possible, nevertheless, to verify explicitly (Din, 1990) that in the k-soliton sector the spin-statistics relation we discussed in the point-particle case, Sec. V.C, also holds. A determination of the complete Lorentz transformation properties of the n-soliton states is, however, lacking.

The total angular momentum carried by a k-soliton state can also be determined directly by quantizing the theory canonically (Bowick *et al.*, 1986; see also Voruganti, 1989, for the quantization of the CP^1 theory in the BRST formalism, and Pak and Percacci, 1991, for a thorough discussion of the topological aspects). The canonical momenta conjugate to the fields $n^a(x)$ are

$$\pi^a(x)=\frac{1}{f}\partial_0 n^a(x)-\mathcal{A}^a(x)\ ,\qquad \mathcal{A}^a(x)=\frac{\Phi}{4\pi}\epsilon_{abc}\epsilon^{ij}A_i'(x)(n^b(x)\partial_j n^c(x))\ , \tag{6.21}$$

[where A' is given by Eq. (6.9)] and satisfy the canonical Dirac brackets (the use of Dirac brackets is necessary because of the constraint $n^a n^a=1$):

$$\{\pi^a(\mathbf{x}),n^b(\mathbf{y})\}=-(\delta^{ab}-n^a(\mathbf{x})n^b(\mathbf{y}))\delta^{(2)}(\mathbf{x}-\mathbf{y})\ ,$$
$$\{\pi^a(\mathbf{x}),\pi^b(\mathbf{y})\}=-(\pi^a(\mathbf{x})n^b(\mathbf{y})-n^a(\mathbf{x})\pi^b(\mathbf{y}))\delta^{(2)}(\mathbf{x}-\mathbf{y})\ , \tag{6.22}$$
$$\{n^a(\mathbf{x}),n^b(\mathbf{y})\}=0\ .$$

Note that the Hopf term (being a total derivative) has no effect on either the Dirac brackets or the classical equations of motion [although it modifies the canonical momenta (6.21) by the term $\mathcal{A}$].

The canonical energy-momentum tensor $T^{\mu\nu}(x)$ can be determined by use of Noether's theorem, as well as the angular momentum tensor

$$J^{\mu\nu}=\int d^2x\,x^\mu T^{0\nu}(x)\ , \tag{6.23}$$

and it may be verified explicitly that they satisfy the Poincaré algebra. In particular, the angular momentum operator turns out to be

$$J\equiv\frac{\epsilon_{ij}}{2}T^{ij}=\left[\int d^2x\,\epsilon^{ij}x_j\pi^a(x)\partial_j n^a(x)\right]-\frac{\Phi}{2\pi}Q^2 =J_0-\frac{\Phi}{2\pi}Q^2\ , \tag{6.24}$$

where J_0 is the angular momentum operator in the absence of a Hopf term, and Q is the soliton charge (6.3). This shows explicitly that the total angular momentum is fractional and that it scales with the soliton number (as measured by the total soliton charge) just as in the point-particle case (5.55). Because we would expect the spin carried by the n soliton state to be given by n times the one-soliton spin, this suggests that Eq. (5.55) is still true, term by term, and solitons satisfy the same spin-statistics relation as relativistic point particles.

A completely different approach to the soliton theory, hampered by its lack of renormalizability, consists of trying to define soliton creation operators (Semenoff and Sodano, 1989; Karabali, 1991). These are formally constructed as follows: given the one-soliton field (6.2), there exists a space-dependent SO(3) matrix $R(\mathbf{x})$ such that $n^a(\mathbf{x})=R(\mathbf{x})t$, where $t=(1,0,0)$; thus a classical soliton field located at $\mathbf{x}_0$ is $n(\mathbf{x}-\mathbf{x}_0)=[R(\mathbf{x}-\mathbf{x}_0)]t$. R may be expressed in terms of the SO(3) generators J_a as $R=\exp(iJ_a\theta^a(\mathbf{x}-\mathbf{x}_0))$. Because the field theory is O(3) invariant, there exist three Noether currents $I_a^\mu(x)$ associated with the transformations generated by J_a. In the quantum theory the Noether charges I_a^0 generate the O(3)

[24] Gamboa (1990) has shown directly that any propagator of the form (6.20) that contains multivalued surface terms has a pole corresponding to a physical excitation with fractional spin.

transformations on the fields upon commutation. Therefore we may identify

$$U(x_0)=\exp\left[\int d^2x\,\theta^a(x-x_0)I_a^0(x)\right] \tag{6.25}$$

as the operator that—acting on the vacuum functional Ψ_0—creates a one-soliton excitation, provided the charges I^0 are interpreted as operators in the quantum theory.

If the theory is quantized canonically, the charges I_a^0 are expressed in terms of the canonical fields $n^a(x)$ and their conjugate momenta (6.21) as

$$I_a^0=\frac{1}{f}\epsilon_{abc}\partial_0 n^b(x)n^c(x)=\epsilon_{abc}(\pi^b(x)+\mathcal{A}^b(x))n^c(x)\ . \tag{6.26}$$

It is then sufficient to use this expression for the currents in Eq. (6.25), while replacing the canonical momenta (6.22) with functional derivatives; $\pi_a\to -i\delta/\delta n^a$. The operators U thus obtained may be verified to carry bosonic statistics, in that $U(\mathbf{x})U(\mathbf{y})=U(\mathbf{y})U(\mathbf{x})$. Their transformation properties under rotations may be found by commuting with the canonical generator of rotations (6.24). While the standard operator J_0 has the effect of rotating the fields, the Q-dependent term produces an additional contribution. Because by construction $QU(x)=U(x)(Q+1)$ [that is, $U(x)$ creates one unit of soliton charge], it follows that

$$e^{i\alpha J}U(\mathbf{x})e^{-i\alpha J}=e^{-i\alpha(\Phi/2\pi)(2Q-1)}U(R^\alpha\mathbf{x})\ , \tag{6.27}$$

i.e., the soliton operators, although single-valued, carry fractional angular momentum because of the modified form of J [Eq. (6.24)].

The solitons created by U [Eq. (6.25)] experience a long-range interaction due to the Hopf term. In view of previous experience, it is natural to define new operators

$$U_0(x)=e^{-i\Lambda(x)}U(x) \tag{6.28}$$

such that the canonical transformation from U to U_0 eliminates the Hopf interaction [i.e., if H is the Hamiltonian with Hopf term, then $H_0=e^{-i\Lambda(x)}He^{i\Lambda(x)}$ does not contain the Hopf term], while the redefined states are multivalued and carry fractional statistics. However, no entirely satisfactory form of the functional $\Lambda(\mathbf{x})$ has been found. One possibility is to define (Semenoff and Sodano, 1989)

$$\begin{aligned}\Lambda(x)&=\frac{\Phi}{2\pi}S(\mathbf{x})\ ,\\ S(\mathbf{x})&=\int d^2y\,\Theta(\mathbf{x}-\mathbf{x}_0)j^0(\mathbf{x})\ ,\end{aligned} \tag{6.29}$$

where Θ is the multivalued function (2.22) and j^0 is the soliton charge density. Although it is easy to verify that the operators U_0 have fractional statistics (we shall discuss this in the next section), it has not been possible to prove that the redefinition (6.25) eliminates the Hopf interaction potentials. Because of the integration over y in Eq. (6.25), there is also a technical problem in establishing whether the function $S(\mathbf{x})$ is multivalued, as one would expect and as the function Θ [Eq. (2.22)] is (Hagen, 1989; Semenoff, 1989); we shall come back to this problem in the next section.

An alternative proposal (Karabali, 1991) consists of defining

$$\Lambda(x)=I_H[\tilde{n}]\ , \tag{6.30}$$

where $I_H[\tilde{n}]$ is the Hopf action (6.8) evaluated, with time $t_i\le t\le t_f$, for a field configuration $\tilde{n}(t,\mathbf{x})$ that at initial time reduces to the vacuum field configuration while at final time it coincides with the one-soliton field (6.2):

$$\begin{aligned}\tilde{n}^a(t=t_i,\mathbf{x})&=(1,0,0)\ ,\\ \tilde{n}^a(t=t_f,\mathbf{x})&=n^a(\mathbf{x})\ .\end{aligned} \tag{6.31}$$

It is straightforward to verify that this redefinition of the fields eliminates the Hopf interaction. Moreover, the operators U_0 have the same multivalued transformation law (6.27) as the fields U, provided the operator J_0 (6.24) is used to generate the rotations. It is unfortunately impossible to determine the statistics displayed by the operators U_0 with this form of Λ and to ascertain whether the two definitions of $S(\mathbf{x})$, (6.29) and (6.30), are equivalent. In the next section we shall see that they are equivalent in a second-quantized theory, in that, up to terms that do not have any effect on the Lorentz transformation properties of the state functionals, Λ [Eq. (6.29)] can be interpreted as the surface term that results from integration of Eq. (6.30).

In conclusion, we have recovered in the path-integral approach the results of the one-point-particle treatment. Because the soliton contains internal degrees of freedom associated with its shape, while the soliton ansatz (6.2) automatically treats space and time on a different footing, even the single-particle integral has the simple form of the nonrelativistic path integral, with which it shares the simple topological interpretation as a path integral on a multiply connected configuration space. The somewhat loose remark at the end of the previous section, to the effect that the soliton treatment allows one to regard the self-linking of each particle's trajectory as a linking of the soliton profile, is made precise by Eqs. (6.13)–(6.15), which express the Hopf term as the linking number (2.48) computed for the soliton's profile polar coordinate ϕ_0 [Eq. (6.13)]. The role of the Hopf term in describing spin also appears to be related to the fact that the phases that provide fractional spin and statistics have the form of the holonomy of a Dirac monopole potential (2.44), (5.35), and the latter can always be written (Ryder, 1980) as the holonomy of a U(1) connection naturally induced by the Hopf map. The O(3) solitons produce this map in the most economic way. The price to pay for these niceties is the loss of covariance of the theory restricted to its soliton sector.

C. Fundamental field theories with fractional spin and statistics

The construction of a quantum field theory whose fields carry multivalued representations of SO(2,1) is the logical conclusion of the line of argument that we have pursued so far. The treatment of fractional spin in soliton sectors of a field theory, discussed in the previous two sections, further suggests that, while fractional spin seems to be compatible with field theory, some of the problems in the soliton approach might disappear if fractional spin were carried by the fundamental fields of the theory. In particular, the models discussed in Secs. V.A and V.B are plagued by nonrenormalizability, and their investigation is made difficult by the lack of knowledge of the (nonlinear) soliton dynamics; finally, the soliton profile breaks Lorentz invariance explicitly, and indeed most of the results found in the previous sections are not manifestly Lorentz covariant.

As we mentioned already in Sec. V.C, the fundamental fields of a relativistic theory ought to provide a linear representation of the Lorentz group, with a generally reducible Poincaré representation content. If the fundamental fields are to carry fractional spin they must carry a multivalued irrep of SO(2,1). As discussed in Sec. V.A, this means that either we define infinite-component fields or we work with multivalued fields, which transform with a cocycle (5.8) upon Lorentz action. The construction of the theory can thus be approached from two opposite points of view: either we construct the one-particle dynamics by starting with infinite-component fields and imposing equations of motion that satisfy the requirement that one-particle states provide multivalued Poincaré irreps (5.18)–(5.21), and then derive an action that reproduces these equations of motion, or we proceed as we did in all the instances we have considered so far, i.e., we add a topological term to the action that endows the path integral with multivalued phases, and then we seek a transformation to a basis of one-particle states that eliminates the topological interaction while endowing the wave function with a multivalued Lorentz representation.

In the former case, the only available set of equations of motion with fractional spin is given by Eqs. (5.58) and (5.59). The infinite-component wave function $F_n^\mu(x)$ introduced in Sec. V.C may be promoted to a relativistic field (Jackiw and Nair, 1991a, 1991b). However, it may be shown that these equations cannot be derived from an action principle as they stand (they do not satisfy the integrability condition that would follow if they were the variation of something). It is possible, though, to introduce a nonlocal set of equations containing a further auxiliary (infinite-component) field whose classical solutions coincide with Eq. (5.61) and can be obtained by varying a nonlocal action (Jackiw and Nair, 1991b). The quantization of the ensuing theory is an open problem, as well as its physical interpretation.

On the other hand, it seems plausible that the addition of the Hopf current-current interaction (6.8) to the action of any field theory that admits a U(1) conserved current j^μ should have an effect on the spin and statistics of the theory (Semenoff, 1988; Semenoff and Sodano, 1989; Forte and Jolicœur, 1991). It turns out that indeed the theory can be developed from this point of view. Henceforth we shall assume that the fundamental field of the theory is a complex scalar field $\phi(x)$ that satisfies canonical commutation relations and for which the one-particle states are found by acting with the creation operator $\phi^\dagger(x)$ on the vacuum state $|0\rangle$ (as in the usual Klein-Gordon theory).[25] The latter assumption can be formalized as the requirement that the commutator

$$[j^0(\mathbf{y}),\phi^\dagger(\mathbf{x})]=\delta^{(2)}(\mathbf{x}-\mathbf{y})\phi^\dagger(\mathbf{x}) \tag{6.32}$$

hold. We consider a theory with action

$$I=I_0+I_H\ , \tag{6.33}$$

where I_0 is a matter action for the fields ϕ that admits a $U(1)$ charge symmetry, whose Noether current j^μ satisfies Eq. (6.32), and I_H is the Hopf action (2.38).

It is immediately clear that if we are given fundamental canonical fields $\phi(x)$, $\pi(x)$ that carry bosonic statistics we can define new field operators that carry arbitrary statistics by proceeding in analogy to Eqs. (6.28) and (6.29) (Semenoff, 1988; see also Matsuyama, 1989), i.e., by defining

$$\tilde{\phi}^\dagger(\mathbf{x})\equiv e^{-2i\sigma S(\mathbf{x})}\phi^\dagger(\mathbf{x})\ . \tag{6.34}$$

The Baker-Campbell-Hausdorff formula, together with the commutator (6.32) then implies that

$$e^{-2i\sigma S(\mathbf{x})}\phi^\dagger(\mathbf{y})e^{2i\sigma S(\mathbf{x})}=e^{-2i\sigma\Theta(\mathbf{x}-\mathbf{y})}\phi^\dagger(\mathbf{y})\ , \tag{6.35}$$

whence

$$\tilde{\phi}^\dagger(\mathbf{x})\tilde{\phi}^\dagger(\mathbf{y})=e^{\pm 2\pi i\sigma}\tilde{\phi}^\dagger(\mathbf{y})\tilde{\phi}^\dagger(\mathbf{x}) \tag{6.36}$$

because $\Theta(\mathbf{x}-\mathbf{y})=\Theta(\mathbf{y}-\mathbf{x})\pm\pi$. This shows that if the fields ϕ commute,then the fields $\tilde{\phi}^\dagger$ obey graded commutation relations. For example, if σ is half-integer they anticommute (Semenoff, 1988; Matsuyama, 1989). The sign ambiguity in the phase is due to the fact that the function Θ is defined only mod(2π). If $2\sigma\notin\mathbb{Z}$, when $\phi^\dagger(\mathbf{x})$ and $\phi^\dagger(\mathbf{y})$ are interchanged the result depends on whether the interchange is performed by means of a clockwise or a counterclockwise rotation. If the rotation is clockwise, the sign in Eq. (6.36) is minus, whereas if it is counterclockwise the sign will be plus. In particular, a state obtained by applying two (or more) operators $\tilde{\phi}^\dagger$ [Eq. (6.34)] to the vacuum has statistics σ [according to the definition of statistics (1.2)].

The possibility of constructing fields that satisfy arbi-

[25]The field must be complex if we want to allow for fractional statistics. Because antiparticles, which are generated by the complex-conjugate operator, have spin and statistics equal in magnitude and opposite in sign (recall Sec. III.C), a real field is necessarily either bosonic or fermionic.

trary commutation relations can be given a physical meaning if we are able to show that n-particle states are created by acting on the vacuum with the operators $\tilde{\phi}^\dagger$. A first indications that this might be the case is the observation (Semenoff, 1988; Semenoff and Sodano, 1989; see also Matsuyama, 1989) that formally the transformation (6.35) eliminates the effects of the Hopf interaction at the classical level. To prove this, consider the Hopf interaction as the effect of a Chern-Simons coupling (2.35)–(2.37). The A field satisfies the classical equation of motion (6.10). The latter is actually a constraint, and as such persists in the quantum theory. This is most easily seen by working in the $A_0=0$ gauge, because then the nonvanishing components A^i are determined by Eq. (6.10) with $\mu=0$, which is a secondary constraint, obtained by demanding that at all times the canonical variable π_0 conjugate to A_0 vanish, $\pi_0\equiv 0$.

Now, Eq. (6.10) with $\mu=0$ is solved by

$$A^i(\mathbf{x},t)=-\epsilon^{ij}\partial_j\int d^2y\, G(\mathbf{x}-\mathbf{y})j^0(\mathbf{y},t)\ , \tag{6.37}$$

where $G(\mathbf{x}-\mathbf{y})$ is the Green's function of the two-dimensional Laplacian $\partial_i\partial_i G(\mathbf{x})=\delta^{(2)}(\mathbf{x})$. Using the explicit form of the Green's function $G(\mathbf{x})=(1/2\pi)\ln|\mathbf{x}|$ in Eq. (6.37), we obtain

$$\begin{aligned}A^i(\mathbf{x},t)&=-\frac{1}{2\pi}\int d^2y\,\epsilon^{ij}\frac{(x-y)_j}{|\mathbf{x}-\mathbf{y}|^2}j^0(\mathbf{y},t)\\&=\frac{1}{2\pi}\int d^2y\,\frac{\partial}{\partial x^i}\Theta(\mathbf{x}-\mathbf{y})j^0(\mathbf{y},t)\ ,\end{aligned} \tag{6.38}$$

where in the last step we used Eq. (2.47). This shows that, *if* we can interchange integral and derivative in the last step, then, in terms of $S(\mathbf{x})$ [Eq. (6.29)],

$$A_i(x)=\frac{1}{2\pi}\partial_i S(\mathbf{x})\ , \tag{6.39}$$

i.e., the gauge potential is a pure gauge and may be removed by a gauge transformation. Because we assumed the ϕ fields to be charged, this transformation will have precisely the form (6.34), with the value of s determined by the details of the gauge-matter coupling.

Unfortunately, the formal manipulation leading from Eq. (6.38) to Eq. (6.39) is incorrect (Jackiw and Pi, 1990): the interchange of integral and derivative required in order to go from (6.37) to (6.38) or, equivalently, the local use of Eq. (2.47) to express A (6.38) in terms of Θ, are not allowed. The simplest way of seeing this is to observe that the integral (6.37) has support on a space plane, where the function Θ (2.47) must have a discontinuity of 2π along a straight line that originates at $\mathbf{x}$ and goes to infinity. The derivative across the discontinuity contains an extra δ-like term: if, for definiteness, we suppose that the discontinuity is along the positive axis of abscissae, then Eq. (2.47) is modified into (Forte and Jolicœur, 1991)

$$\frac{\partial}{\partial x^a}\Theta(\mathbf{x})=-\epsilon_{ab}\frac{x^a}{|\mathbf{x}|^2}-2\pi H(x^1)\delta(x^2)\begin{bmatrix}0\\1\end{bmatrix}_a\ , \tag{6.40}$$

where H is the Heaviside function. This term modifies Eq. (6.38) and prevents the identification of A as a pure gauge (Jackiw and Pi, 1990). Equivalently, it is possible to change coordinates and domain of integration in Eq. (6.38) in such a way that Eq. (2.47) holds everywhere in the interior of the integration domain, but at the expense of introducing a noncommutativity of derivative and integral in Eq. (6.39), which eventually yields the same correction (Jackiw, 1990; Jackiw and Pi, 1990).

Apart from this problem, even if Eq. (6.39) were true, it would still have to be shown that the redefined operators (6.34) create one-particle states. Indeed, in a second-quantized theory, the appearance of fractional spin and statistics is inevitably a consequence of an anomaly, i.e., of the fact that a symmetry of the classical theory is broken when represented on the quantum state vectors (see, for example, Jackiw, 1985). Fractional spin and statistics in field theory appear when the Lorentz covariance of the state functionals is spoilt by a cocycle ω_1 [Eq. (5.8)]: the state $\Psi=\Psi[\phi(\mathbf{x});t]$ transforms as

$$U(\Lambda)\Psi[\phi(\mathbf{x};t)]=e^{i\omega_1(\phi(\mathbf{x},t);\Lambda)}\Psi[\phi(\Lambda\mathbf{x};\Lambda t)]\ . \tag{6.41}$$

The cocycle must depend on the particle content of the state Ψ, because the transformation law of an n-particle state depends generally on n. In particular, the vacuum of the theory should be Lorentz invariant; this, however, is possible only if the classical Lagrangian is strictly Lorentz invariant (rather than invariant up to a total derivative), which in turn implies that a cocycle of the form discussed in Sec. V.A must vanish at the classical level. The rotational symmetry is classically exact, and the modification of the angular momentum spectrum produced by the cocycle appears only when the theory is quantized (Forte and Jolicœur, 1991).

The fact that an anomaly may modify the quantum numbers of a theory and, in particular, produce quantum numbers that are not integer multiples of those carried by the fundamental fields of the theory (quantum number fractionization) is well known when internal symmetries are involved (see Niemi and Semenoff, 1986, for a review). The possibility of fractionization of angular momentum—a space-time quantum number—was first envisaged by Paranjape (1985).

It turns out that rotational anomalies do appear in the theory (6.33). It is actually possible to follow the approach that we described for point particles and for solitons, namely, to work out the effect of the Hopf term on the path integral explicitly (Forte and Jolicœur, 1991, see also Forte, 1991d). A few surprises are, however, in store.

The key observation is that in a quantized field theory it is actually possible to write the Hopf action as a surface term (6.17) without invoking any specific form of the currents j^μ (in particular, without having to use any soliton ansatz), up to terms with trivial Lorentz transformation properties. Indeed, if one replaces the point-particle currents by the currents of a field theory, the computation of the Hopf Lagrangian (2.42)–(2.45) goes through

unchanged. One gets

$$I_H=\frac{\Phi}{\pi}\left[\int_0^T dt_y \int d^2x\, d^2y\, j^0(\mathbf{x},t_y)j^a(\mathbf{y},t_y) \times\epsilon_{ab}\frac{(x-y)^b}{|\mathbf{x}-\mathbf{y}|^2}+\tfrac{1}{2}I_g\right], \tag{6.42}$$

where I_g is the field-theoretic generalization of Eq. (2.46), with which it shares the trivial Lorentz transformation properties.

We may now use Eq. (2.47). Integrating by parts and using current conservation we get

$$I_H=\frac{\Phi}{\pi}\left[\int_0^T dt_y \int d^2x\, d^2y\, \Theta(\mathbf{x}-\mathbf{y})j^0(\mathbf{x},t_y)\frac{\partial}{\partial t_y} \times j^0(\mathbf{y},t_y)+\tfrac{1}{2}\tilde{I}_g\right]. \tag{6.43}$$

When integrating by parts, we find that surface terms are produced that are also covariant upon Lorentz transformation (Forte and Jolicœur, 1991) and that have been lumped into $\tilde{I}_g$. Integrating by parts with respect to time once more leads to the desired result:

$$I_H=\frac{\Phi}{\pi}[(H(T)-H(0))+\tfrac{1}{2}\tilde{I}_g], \tag{6.44}$$

$$H(t)=\tfrac{1}{2}\int d^2x\, d^2y\, j^0(\mathbf{x};t)j^0(\mathbf{y};t). \tag{6.45}$$

The terms contained in $\tilde{I}_g$ have no effect on the space-time symmetries of the state functionals and need not concern us; note, however, that they do prevent the elimination of the Hopf term by gauge transformation (6.39), and in particular they might have dynamical effects (as nonrelativistic arguments seem to suggest; Jackiw and Pi, 1990).

The fact that the function Θ is ill defined in the origin is of no concern, since in the quantized field theory the currents j are operators, and the product of currents in Eqs. (6.42) and (6.45) diverges as $\mathbf{x}\to\mathbf{y}$. This point is thus effectively excluded from the domain of the double space integral (6.42), which becomes the multiply connected space $\mathcal{C}$ given by Eq. (3.2) with $d=2$ and $n=2$. The regularization of this singularity is responsible for the anomalous representation of the Lorentz symmetry. If the divergence is subtracted in such a way that the vacuum is invariant, the n-particle states are not. This may be checked by performing an operator-product expansion of the current operators in Eq. (6.46) (Forte and Jolicœur, 1990, see Forte, 1991d).[26] As expected, second quantization provides both particles' indistinguishability and the exclusion principle automatically.

When going from Eq. (6.42) to (6.43), we find that the double space integration is lifted to the universal cover of the space $\mathcal{C}$ (cf. Sec. III.A), i.e., spelling out the extremes of integration, Eq. (6.45) reads

$$H(t)=\tfrac{1}{2}\int_0^\infty \rho_x\, d\rho_x \int_\alpha^{\alpha+2\pi} d\theta_x \int_0^\infty \rho\, d\rho \times\int_{\theta_x}^{\theta_x+2\pi} d\theta\, \theta[j^0(\mathbf{x},t)j^0(\mathbf{x}+\mathbf{r},t)], \tag{6.46}$$

where (ρ_x,θ_x) and (ρ,θ) are polar components of the vectors $\mathbf{x},\mathbf{r}$, respectively, and α is an arbitrary (multivalued) reference angle, which may be chosen, as usual, by defining $H(t)$ as the integral of its time derivative from a reference field configuration to the given one.

We may now proceed as in Secs. II.C and VI.B (Forte and Jolicœur, 1991) and eliminate the Hopf term from the propagator by redefining the state functionals according to

$$\Psi_0[\phi(\mathbf{x}),t]=e^{-i\frac{\Phi}{\pi}H(t)}\Psi[\phi(\mathbf{x}),t]. \tag{6.47}$$

As usual, the topological interactions may be eliminated by a phase redefinition of the state vectors. In the field-theoretic case, however, the phase is operator valued. The phase factor $e^{-i(\Phi/\pi)H(t)}$ turns out to provide the cocycles that modify the representation of the Lorentz and Poincaré groups given by the n-particle states, and it will be referred to as an operator cocycle.

The spin, statistics, and space-time symmetry properties of n-particle states Ψ_0 can now be obtained by using repeatedly the commutator (6.32). These properties are nontrivial and depend on the number of particles n, as they ought to, because the operators $H(t)$ and $\phi^\dagger(t)$ do not commute:[27]

$$[H(t),\phi^\dagger(\mathbf{z},t)]=S(\mathbf{z},t)\phi^\dagger(\mathbf{z},t), \qquad S(\mathbf{x},t)=\int d^2y\,\Theta(\mathbf{x}-\mathbf{y})j^0(\mathbf{y},t), \tag{6.48}$$

where the integration over $\mathbf{y}$ runs over the same range as the integration over r, θ in Eq. (6.46).

The commutation relation (6.48) implies, again by the Baker-Campbell-Hausdorff formula, that the operator H, when acting on an n-particle state $|\Psi^n\rangle$, effectively modifies the field operators according to Eq. (6.34):

[26] In a rigorous treatment these divergences are treated by splitting in a position-dependent way the times at which the operators $j(\mathbf{x})$ and $j(\mathbf{y})$ act. Whereas the nontrivial transformation properties of $H(t)$ are due to a contribution to the double integral (6.45), which is localized on a spacelike line (Forte and Jolicœur, 1991), if this prescription is used the nontrivial contribution becomes localized on a spacelike cone whose axis is this line (Fröhlich and Marchetti, 1989; see also Fröhlich, Gabbiani, and Marchetti, 1989).

[27] Commutation with $\phi(\mathbf{x},t)$ leads to the same result but with the opposite sign.

$$|\Psi_0^n\rangle \equiv e^{-2i\sigma H}|\Psi^n\rangle = e^{-2i\sigma S(\mathbf{x}_1)}\phi^\dagger(\mathbf{x}_1)e^{-2i\sigma H}\left[\prod_{i=2}^{n}\phi^\dagger(\mathbf{x}_i)\right]|0\rangle$$
$$= e^{-2i\sigma S(\mathbf{x}_1)}\phi^\dagger(\mathbf{x}_1)e^{-2i\sigma S(\mathbf{x}_2)}\phi^\dagger(\mathbf{x}_2)e^{-2i\sigma H}\left[\prod_{i=3}^{n}\phi^\dagger(\mathbf{x}_i)\right]|0\rangle$$
$$= \left[\prod_{i=1}^{n}\tilde{\phi}^\dagger(\mathbf{x}_i)\right]|\tilde{0}\rangle \ , \tag{6.49}$$

where we have set

$$|\tilde{0}\rangle = e^{-2i\sigma H}|0\rangle \ , \tag{6.50}$$

$$\sigma = \frac{\phi}{2\pi} \ . \tag{6.51}$$

Because of Eqs. (6.35) and (6.36), the statistics of an n-particle state is given by σ [Eq. (6.51)], which has the same magnitude as the point-particle result (5.52) and (5.53) but the opposite sign. In order to determine the contribution of the phases in Eq. (6.49) to the spin and orbital angular momentum, it is sufficient to observe that the phase $S(\mathbf{x})$ (6.48) is not invariant upon rotations:

$$R^\beta S[j^0(\mathbf{x})]R^{\beta^{-1}} = \beta Q + S[j^0(R^\beta\cdot\mathbf{x})] \ , \tag{6.52}$$

where Q is the charge operator and $S[j^0(R^\beta\cdot\mathbf{x})]$ denotes the covariant transform of $S[j^0(\mathbf{x})]$ upon rotation, obtained by transforming the argument of the field operators on which S depends. It follows that each phase factor $e^{2i\sigma S(\mathbf{x})}$ is multivalued upon rotation of 2π and contributes a phase $e^{2i\sigma\beta Q}$ to the transformation of the state functionals $|\Psi_0\rangle$ [Eqs. (6.49) and (6.50)]. Accordingly, the total angular momentum becomes fractional and equal to

$$J = 2\sigma Q = -\frac{\Phi}{\pi}Q \ \mathrm{mod}(\mathbb{Z}) \ . \tag{6.53}$$

The transformation law of the vacuum under the Lorentz transformation $T(\Lambda)$ is left unchanged by the phase redefinition, Eq. (2.33). Because of the Poincaré invariance of the vacuum,

$$T(\Lambda)|\tilde{0}\rangle = T(\Lambda)e^{2i\sigma H}T(\Lambda^{-1})|0\rangle \ . \tag{6.54}$$

Equivalently, the vacuum expectation value of $e^{2i\sigma H}$ may be subtracted by normal ordering. A one-particle state, instead, acquires a phase

$$R^\beta|\Psi_0^1[\phi(\mathbf{x})]\rangle = R^\beta e^{-i(\Phi/\pi)}\phi^\dagger(\mathbf{x})|\tilde{0}\rangle$$
$$= e^{-i\beta(\Phi/\pi)}|\Psi_0^1[\phi(R^\beta\mathbf{x})]\rangle \tag{6.55}$$

and carries spin

$$s = -\frac{\Phi}{\pi} \ . \tag{6.56}$$

An n-particle state is endowed by the operator cocycle with a fractional total angular momentum equal to

$$J = -\frac{\Phi}{2\pi}n(n+1) \ \mathrm{mod}(\mathbb{Z}) \ . \tag{6.57}$$

Comparing the values of the spin and statistics (6.51), (6.56), (6.57) with each other and with the point-particle result (5.55), it is apparent that (i) the spin-statistics theorem is not verified (for example, fermions, which correspond to $s=\frac{1}{2}$, do not anticommute, but rather, commute with a factor of i); and (ii) the point-particle result is not reproduced.

To disentangle the various contributions to spin and orbital angular momentum, it is convenient to rewrite the n-particle functional (6.49) as

$$|\Phi_0^n\rangle = \exp\left[2i\sigma\sum_{j=1}^{n}\sum_{i=1}^{j-1}\Theta(\mathbf{x}_i-\mathbf{x}_j)\right]$$
$$\times\left[\exp\left[is\sum_{i=1}^{n}S(\mathbf{x}_i)\right]\prod_{i=1}^{n}\phi^\dagger(\mathbf{x}_i)|0\rangle\right] \ , \tag{6.58}$$

where σ and s are given by Eqs. (6.51) and (6.56), respectively. Because $[S(\mathbf{x}),S(\mathbf{y})]=0$, the statistics is entirely given by the Θ-dependent prefactor, and the different contributions to the induced phase may be interpreted as was done for the point-particle wave function (5.54) in Sec. V.C. Comparison with Eq. (5.54) shows that the statistics carried by Eq. (6.58) is σ, while its orbital angular momentum L, spin S, and total angular momentum J are (Forte and Jolicœur, 1991)

$$L \equiv \sigma n(n-1)+\ell = \frac{\Phi}{2\pi}n(n-1)+\ell, \quad \ell\in\mathbb{Z} \ ,$$
$$S = n^2 s = -\frac{\Phi}{2\pi}2n^2, \quad J = L+S \ . \tag{6.59}$$

It may be analogously shown that the operators ϕ generate the corresponding antiparticles, i.e., states with opposite spin, angular momentum, and statistics.

Comparison with the point-particle values of the same quantities, Eq. (5.57), shows that (i) the dependence of the orbital angular momentum (and statistics) on the number of particles is the same, while the dependence on the coefficient of the Hopf term Φ has the opposite sign; (ii) the (one) particle spin has the same sign but twice the magnitude it used to; (iii) the dependence of the spin on the particle number is quadratic rather than linear. It follows in particular that the spin-statistics theorem is never satisfied by the states (6.58). Notice that all this holds true for any state obtained by applying a string of operators $\tilde{\phi}^\dagger$ to the vacuum, regardless of the details of the interaction.

This somewhat surprising result indicates that the point-particle limit and the quantization of the theory do

not commute. In particular, the different dependence of the spin term on n can be traced back to the fact that fractional spin is due to the current self-interaction in the Hopf action (2.38) [i.e., the diagonal terms (5.51) in the point-particle case], and the interaction kernel $K^{\mu\nu}(\mathbf{x},\mathbf{y})$ (2.39) is singular as $\mathbf{x}\to\mathbf{y}$. In the point-particle case the singularity is regulated geometrically by evaluating the interaction along the classical particle's trajectories, then the theory is quantized by path integration. In the field-theoretical case the theory is second-quantized first, and the singularity turns out to be taken care of by the second quantization of the theory. This may be made manifest by computing the phase induced on states (6.49) directly through the expansion of the operator product of the Hopf action and the field operators, $e^{iI_H}\phi^\dagger(\mathbf{x})$ (Forte and Jolicœur, 1990; see Forte, 1991c).

The symmetry of the physical states, however, is not determined by the action or the path integral, i.e., the states (6.58) need not be physical. Rather, we are free to impose any symmetry on the state functionals that we propagate through the path integral of the theory. This is just a choice of boundary conditions, which we may enforce by symmetrizing the states $|\Psi\rangle$, i.e., by setting

$$|\Psi^n_{\sigma_0}\rangle=\exp\left[i\sigma_0\sum_{j=i}^{n}\sum_{i=1}^{n}\Theta(\mathbf{x}_i-\mathbf{x}_j)\right]|\Psi_0\rangle\ . \tag{6.60}$$

When acted upon by the operator cocycle, the states $|\Psi_{\sigma_0}\rangle$ acquire extra spin and statistics phases and may be written in the form (6.58), with s given by Eq. (6.56), and statistics

$$\sigma'=\sigma_0+\sigma=\sigma_0-\tfrac{1}{2}s\ . \tag{6.61}$$

In general, a state with spin s and statistics σ' has a total angular momentum spectrum given by (Forte and Jolicœur, 1991)

$$J=n^2s+n(n-1)\sigma'+\ell,\quad \ell\in\mathbb{Z}\ . \tag{6.62}$$

It is clear that no value of σ_0 reproduces the point-particle result (5.57); however, we may satisfy the requirement that the spin-statistics relation hold for fermions by tuning σ_0 so that $s=\pm\sigma$. Because the operators $\tilde\phi^\dagger$ [Eq. (6.34)] can be identified with creation operators for physical states only if $\sigma_0=0$, it follows that this identification is incompatible with the spin-statistics theorem (even in the case of fermions). This means that neither the point-particle wave functions (5.54) nor their nonrelativistic limit (3.21), nor the wave functions associated with the state functionals (6.60) can be obtained as the vacuum expectation value of a string of $\tilde\phi$ and $\tilde\phi^\dagger$ operators. There exist, however, operators that do have this property with respect to the nonrelativistic wave function (3.21) (Fubini, 1991; Fubini and Lütken, 1991). These operators are closely related to the so-called FV operators of string theory (Fubini and Veneziano, 1970) and satisfy

$$\langle A|U_{\alpha_1(z_1)}\cdots U_{\alpha_n(z_n)}|0\rangle=\prod_{i\neq j}(z_i-z_j)^{\alpha_i\cdot\alpha_j}\ , \tag{6.63}$$

where $|A\rangle$ is a suitably redefined vacuum state, U_{α_i} (the FV operator) is an operator-valued function of z_i, and α_i is an arbitrary two-vector. The r.h.s. of Eq. (6.63) can be recognized as the generalization of the complex parametrization (3.23) and (3.24) of the nonrelativistic wave function (3.21). Since this is the most general wave function with braid statistics, any wave function with generic statistics may be constructed in this manner [for example, Dunne, Lerda, and Trugenberger (1991) construct in this fashion eigenfunctions of Eq. (4.17)]. It is then trivial to endow the FV operators with spin, too, by supplementing them by prefactors (6.34), and thus to construct factorized creation operators for states with arbitrary spin and statistics. Whether these operators might be related to a local Lagrangian field theory is still an open problem.

If we now require that the physical in and out states describe a local, noninteracting system, then the angular momentum J [Eq. (6.62)] must be linear in the number of particles. This entails the spin-statistics relation (Forte and Jolicœur, 1991)

$$\sigma=-s\ . \tag{6.64}$$

Notice that, in the case of fermions ($s=\frac{1}{2}$), Eq. (6.64) reduces to the spin-statistics relation found in the point-particle case; moreover, in this case the difference in sign in σ [Eq. (6.51)] and the dependence of the spin (6.59) on n^2 are unobservable.

The multivalued transformation properties of the state functionals $|\Psi_0\rangle$ [Eqs. (6.47) and (6.58)] upon rotations are fully compatible with the Lorentz covariance of the theory: it is possible to show that upon Lorentz transformation the phases induced by the operator cocycle transform with a cocycle (5.16), with coefficient equal to the fractional part of J [Eq. (6.59)]. The proof proceeds along the lines of that given in the relativistic point-particle case, Eqs. (5.41)–(5.43). Upon a combination of Lorentz rotations and boosts the phases $S(\mathbf{x})$ and $\Theta(\mathbf{x})$ transform covariantly, up to the addition of a term equal to the winding number of the path, which builds up the cocycle (Forte and Jolicœur, 1991).

Finally, if we look in particular at one-particle states with definite momentum, i.e., with current

$$J^\mu[f_{k^\mu}]=k^\mu\frac{\rho}{\omega} \tag{6.65}$$

(where $\rho=1/V$ is a normalized constant charge density), it is easy to verify that the state functionals (6.47) satisfy Eqs. (5.18) and (5.19), i.e., they provide a (multivalued) irreducible representation of the Poincaré group, as they ought to (Forte and Jolicœur, 1991). Although we can formally identify the phase induced by the operator cocycle on one-particle states $e^{isS(\mathbf{x})}$ with a functional representation of the infinite-component wave function F^μ_n [Eqs. (5.58)–(5.60)], a direct link with the field theory and equations of motion described by the latter is missing.

The appearance of fractional spin and statistics in some field theories of the class considered here, such as

the Klein-Gordon theory with Hopf term, can be seen also in a canonical approach, where one may show that the canonical fields develop graded commutators, and an anomalous transformation law upon rotations (Foerster and Girotti, 1990); or, equivalently, one computes directly the canonical angular momentum operator and shows that it is supplemented by an anomalous addition, which turns out to have the form (6.24) (Semenoff and Sodano, 1989). This seems to be a universal feature of field theories whose charged current is coupled to a Chern-Simons term through Eqs. (2.35)–(2.37). When pursued in a fundamental field theory (rather than the soliton sector of a theory), these approaches are plagued by the difficulty of identifying the physical degrees of freedom; accordingly, doubts have been cast on the results thus obtained (Boyanovsky, 1989).

However, one may take the shift of the angular momentum spectrum (6.24) as an indication that fractional statistics is a general feature of a field theory coupled to a Chern-Simons term, and try to tackle the problems of a field-theoretic description of fractional spin and statistics by quantizing the Chern-Simons theory on the lattice (a review is given by Semenoff, 1991). Indeed, because in the lattice theory the charge density is necessarily localized in points, many difficulties are resolved, in that field-theoretic objects coincide with point-particle quantities. For example, the manipulations leading from Eq. (6.37) to Eq. (6.39), which are illegitimate in field theory, are allowed if j^μ is a point-particle current, as in Eq. (2.33), because if j^0 is a sum of Dirac's δ the integration in (6.38) can be done immediately and there are no problems in interchanging derivation and integration in Eq. (6.38) (Jackiw and Pi, 1990). Thus the Chern-Simons interaction may be eliminated by a (singular) gauge transformation, which of course coincides with that displayed in Eqs. (2.26)–(2.28). This carries over to the lattice theory, where j^μ is a sum of an infinite number of point charges, localized on all lattice sites. Then the gauge transformation that removes the Chern-Simons field may be viewed as a three-dimensional generalization of the Jordan-Wigner transformation (Fradkin, 1989; Ambjørn and Semenoff, 1989), in that it is a nonlocal transformation (depending on all pairs of lattice points), which relates fields (or state functionals) that carry different spin.

An understanding of the spin and statistics of the physical states, however, requires the lattice quantization of the full theory coupled to a Chern-Simons term. The point-particle lattice theory has been quantized and shown to reproduce the usual continuum results by Fröhlich and Marchetti (1988), while the lattice field theory is discussed by Lüscher (1989), albeit only for those values of the Chern-Simons coefficient that lead to a fermionic angular momentum spectrum; the generalization to the case of generic spin and statistics has been presented by Müller (1990). In these references, however, the gauge-field action is not just the Chern-Simons action (2.37), but rather, it contains a Maxwell term as well. Although fractional spin and statistics do seem to appear, the theory must contain extra dynamical gauge degrees of freedom. A lattice quantization of the pure Chern-Simons theory (Eliezer *et al.*, 1990) requires either a gauge-noninvariant lattice Chern-Simons term (Fradkin, 1989) or a lattice non-nearest-neighbor generalization of the Chern-Simons interaction (Eliezer *et al.*, 1990). Within the latter approach (Eliezer *et al.*, 1990), a theory of lattice fermions interacting through a Chern-Simons term has been mapped exactly onto a theory of free particles with fractional spin and statistics. The construction of the continuum limit and of the interpolating operators for the asymptotic states is still an open problem in all of these models.

VII. CONCLUSIONS

We have tried to present the quantum mechanics and field theory of particles with fractional spin and statistics as a self-contained and consistent theory, which should eventually be available for the development of models that may be of interest for physical applications, just as the usual theories of bosons and fermions are. The discussion and development of these applications, which is the ultimate goal of any physical theory, is already a flourishing subject, but lies outside the scope of the present review. We would like to conclude by summarizing the status of current knowledge and by pointing out where our attempts at a consistent and complete presentation have inevitably failed due to lack of knowledge.

As far as nonrelativistic quantum mechanics is concerned, the situation is entirely satisfactory from the formal point of view: the theory is well understood in the path-integral as well as in the Schrödinger equation approach; its topological underpinnings and even its axiomatic formulation are known in detail. The situation is much less satisfactory from the point of view of applications: very little is known beyond the very simplest models. Even the partition function for an ideal gas has been determined only in the simplest possible case of a two-particle system, and explicit solutions are available only for a handful of models. On the other hand, due to possible applications to superconductivity, the mean-field theory has been vigorously pursued in recent times and hints at several promising developments (see Wilczek, 1990a, 1990b).

The theory of relativistic particles is more or less understood, at least from the viewpoint of kinematics and symmetry properties; however, the nontrivial spin dynamics appearing in a relativistic treatment have not been disentangled from the rather formal path-integral formulation—except, of course, in the spin-$\frac{1}{2}$ case. This amounts to saying that the dynamical consequences of fractional spin for relativistic particles are virtually unexplored. This is an aspect of the theory of considerable potential interest, as a wide range of problems (particle scattering, bound-state problems, coupling to electrodynamics, . . .) is open to investigation and perhaps will

offer surprising effects.

Finally, the field-theoretical knowledge of fractional statistics is full of gaps even from a purely formal viewpoint. Different formulations of the same theory lead to contradictory results. The results found in the point-particle limit do not seem to be reproduced by a fundamental field theory, while in theories of solitons it is difficult to establish reliable results due to nonrenormalizability. Although progress in these directions may turn out to be hard, the consistent formulation of a field theory holds the promise of adding new items to the very limited list of known renormalizable field-theoretic models.

The intricacies of the theory even in the simplest cases, as well as the numerous surprising connections with topology and differential geometry that it displays, suggest that answering these questions may be of considerable theoretical interest. Perhaps the results may prove important and eventful even in their phenomenological applications.

ACKNOWLEDGMENTS

This paper is the outgrowth of a series of seminars I gave in Saclay in 1989–1990. I thank J. B. Zuber for prompting me to this endeavor and for his interest, and several people who attended the seminars for critical remarks and comments, in particular D. Altschüler, D. Bernard, C. Itzykson, and M. Rho. During the course of this work I also benefited from discussions with M. Bauer, F. David, K. Johnson, D. Karabali, A. Lerda, P. A. Marchetti, M. Monastirsky, V. P. Nair, M. Nowak, R. Percacci, P. Sodano, and especially M. Bos and R. Jackiw, whom I also thank for communicating to me unpublished notes. Most of my understanding of this subject was shaped in discussions with T. Jolicœur, whom I also thank for a critical reading of the manuscript. Part of this work was done during a visit to N. Khuri's laboratory at the Rockefeller University, whose hospitality and support I gratefully acknowledge.

REFERENCES

Ambjørn, J., B. Durhuus, and T. Jonsson, 1989, Nucl. Phys. B **330**, 509.

Ambjørn, J., and G. W. Semenoff, 1989, Phys. Lett. B **226**, 107.

Arovas, D. P., 1989, in *Geometric Phases in Physics,* edited by A. Shapere and F. Wilczek (World Scientific, Singapore), p. 284.

Arovas, D. P., R. Schrieffer, and F. Wilczek, 1984, Phys. Rev. Lett. **53**, 722.

Arovas, D. P., R. Schrieffer, F. Wilczek, and A. Zee, 1985, Nucl. Phys. B **251**, 117.

Balachandran, A. P., H. Gomm, and R. D. Sorkin, 1987, Nucl. Phys. B **281**, 573.

Balachandran, A. P., G. Marmo, B.-S. Skagerstam, and A. Stern, 1983, *Gauge Theories and Fiber Bundles* (Springer, Berlin).

Barut, A. O., and R. Raczka, 1986, *Theory of Group Representations and Applications* (World Scientific, Singapore).

Berry, M. V., 1984, Proc. R. Soc. London Ser. A **392**, 45.

Binegar, B., 1982, J. Math. Phys. **23**, 1511.

Bowick, M. J., D. Karabali, and L. C. R. Wijewardhana, 1986, Nucl. Phys. B **271**, 417.

Boyanovsky, D., 1989, "Chern-Simons with Matter Fields: Schrödinger Picture Quantization," University of Pittsburgh Preprint No. PITT-89-12.

Calugareanu, G., 1959, Rev. Roum. Math. Pures Appl. (Bucharest) **4**, 5.

Chen, Y. H., F. Wilczek, E. Witten, and B. I. Halperin, 1989, Int. J. Mod. Phys. B **3**, 1001.

Chou, C., 1991a, Phys. Lett. A **155**, 245.

Chou, C., 1991b, "The Multi-Anyon Spectra and Wave Functions," MIT Preprint No. MIT-CTP-1973.

Comtet, A., J. McCabe, and S. Ouvry, 1991, Phys. Lett. B **260**, 372.

Cortés, J., J. Gamboa, and L. Velazquez, 1991, "Electromagnetic Interaction of Anyons," University of Zaragoza Preprint No. DFTUZ 91/16.

Coste, A., and M. Lüscher, 1989, Nucl. Phys. B **323**, 631.

de Sousa Gerbert, P., 1990, Nucl. Phys. B **346**, 440.

de Sousa Gerbert, P., 1991, Int. J. Mod. Phys. A **6**, 173.

Din, A. M., 1990, Nucl. Phys. B **330**, 757.

Din, A. M., and W. J. Zakrzewski, 1984, Phys. Lett. B **146**, 341.

Dowker, J. S., 1972, J. Phys. A **5**, 936.

Dubrovin, B., S. Novikov, and A. Fomenko, 1984, *Modern Geometry* (Springer, Berlin).

Dunne, G. V., A. Lerda, and C. Trugenberger, 1991, Mod. Phys. Lett. A **6**, 2819.

Dunne, G. V., A. Lerda, S. Sciuto, and C. Trugenberger, 1991, Nucl. Phys. B, in press.

Einarsson, T., 1990, Phys. Rev. Lett. **64**, 1995.

Eliezer, D., G. W. Semenoff, and S. S. C. Wu, 1990, "Anyonization and Superfluidity of Lattice Chern-Simons Theory," University of British Columbia preprint.

Fetter, A. L., C. B. Hanna, and R. B. Laughlin, 1989, Phys. Rev. B **39**, 9679.

Foerster, A., and H. O. Girotti, 1990, Nucl. Phys. B **342**, 680.

Forte, S., 1991a, Phys. Lett. B **253**, 403.

Forte, S., 1991b, Mod. Phys. Lett. A **6**, 3153.

Forte, S., 1991c, Int. J. Mod. Phys., in press.

Forte, S., 1991d, Acta Phys. Pol., in press.

Forte, S., and T. Jolicœur, 1990, unpublished.

Forte, S., and T. Jolicœur, 1991, Nucl Phys. B **350**, 589.

Fradkin, E., 1989, Phys. Rev. Lett. **63**, 322.

Frank-Kamenetskiĭ, M. D., and A. V. Vologodskiĭ, 1981, Usp. Fiz. Nauk **134**, 641 [Sov. Phys. Usp. **24**, 679 (1981)].

Fröhlich, J., F. Gabbiani, and P. A. Marchetti, 1989, in *Knots, Topology and Quantum Field Theories,* Johns Hopkins Workshop Proceedings No. 13 (Florence, 1989), edited by L. Lusanna (World Scientific, Singapore), p. 335.

Fröhlich, J., and P. A. Marchetti, 1989, Commun. Math Phys. **121**, 177.

Fubini, S., 1991, Mod. Phys. Lett. A **6**, 347.

Fubini S., and C. A. Lütken, 1991, Mod. Phys. Lett. A **6**, 487.

Fubini, S., and G. Veneziano, 1970, Nuovo Cimento A **67**, 29.

Gamboa, J., 1990, Phys. Lett. B **251**, 548.

Gerry, C. C. G., and V. A. Singh, 1979, Phys. Rev. D **20**, 2550.

Girvin, S. M., 1986, in *The Quantum Hall Effect,* edited by R. E. Prange and S. M. Girvin (Springer, Berlin), Chap. 10.

Goldin, G. A., R. Menikoff, and D. H. Sharp, 1981, J. Math. Phys. **22**, 1664.

Guadagnini, E., 1991, in *Nonperturbative Methods in Low Dimensional Quantum Field Theories,* Johns Hopkins Workshop Proceedings No. 14 (Debrecen, 1990), edited by G. Domokos, Z. Horváth, and S. Kovesi-Domokos (World Scientific, Singapore), p. 1.
Hagen, C. R., 1984, Ann. Phys. (N.Y.) **157**, 342.
Hagen, C. R., 1989, Phys. Rev. Lett. **63**, 1025.
Halperin, B. I., 1984, Phys. Rev. Lett. **52**, 1583.
Hanna, C. B., R. B. Laughlin, and A. L. Fetter, 1989, Phys. Rev. B **40**, 8745.
Iengo, R., and K. Lechner, 1990, Nucl. Phys. B **346**, 551.
Inomata, A., and V. A. Singh, 1978, J. Math. Phys. **19**, 2318.
Jackiw, R., 1985, in S. Treiman, R. Jackiw, C. Zumino, and E. Witten, *Current Algebra and Anomalies* (World Scientific, Singapore), p. 81 and p. 211.
Jackiw, R., 1988, Int. J. Mod. Phys. A **3**, 285.
Jackiw, R., 1990, unpublished.
Jackiw, R., 1991, Nucl. Phys. B (Proc. Suppl.) **18A**, 107.
Jackiw, R., and V. P. Nair, 1991a, in *Quarks, Symmetries and Strings,* Proceedings of the Sakita-Fest (New York, 1990), edited by M. Kaku, A. Jevicki, and K. Kikkawa (World Scientific, Singapore), p. 361.
Jackiw, R., and V. R. Nair, 1991b, Phys. Rev. D **43**, 1933.
Jackiw, R., and S.-Y. Pi, 1990, Phys. Rev. D **42**, 3500.
Jackiw, R., and A. N. Redlich, 1983, Phys. Rev. Lett. **50**, 555.
Jaroszewicz, T., 1985, Phys. Lett. B **159**, 299.
Jaroszewicz, T., and P. S. Kurzepa, 1991, Ann. Phys. (N.Y.) **210**, 255.
Johnson, K., 1989, Ann. Phys. (N.Y.) **192**, 104.
Kalmeyer, V., and R. B. Laughlin, 1987, Phys. Rev. Lett. **59**, 2095.
Karabali, D., 1991, Int. J. Mod. Phys. A **6**, 1369.
Kovner, A., 1990, Int. J. Mod. Phys. A **5**, 3999.
Laidlaw, M. G. G., and C. Morette de Witt, 1971, Phys. Rev. D **6**, 1375.
Laughlin, R. B., 1983, Phys. Rev. Lett. **50**, 1395.
Laughlin, R. B., 1988a, Science **242**, 525.
Laughlin, R. B., 1988b, Phys. Rev. Lett. **60**, 2677.
Lee, K., 1989, "Anyons on Spheres and Tori," Boston University Preprint No. BUHEP-89-28.
Leinaas, J. M., and J. Myrheim, 1977, Nuovo Cimento B **37**, 1.
Lüscher, M., 1989, Nucl. Phys. B **336**, 557.
Mackenzie, R., and F. Wilczek, 1988, Int. J. Mod. Phys. A **3**, 2827.
Matsuyama, T., 1989, Phys. Lett. B **228**, 99.
McCabe, J., and S. Ouvry, 1991, Phys. Lett. B **260**, 113.
Müller, V., 1990, Z. Phys. C **47**, 301.
Nielsen, H. B., and D. Rohrlich, 1988, Nucl. Phys. B **299**, 471.
Niemi, A., and G. W. Semenoff, 1986, Phys. Rep. **135**, 99.
Orland, P., 1989, Int. J. Mod. Phys. A **4**, 3615.
Pak, N., and R. Percacci, 1991, Phys. Rev. D **43**, 1375.
Paranjape, M., 1985, Phys. Rev. Lett. **55**, 2390.
Paranjape, M., 1987, Phys. Rev. D **36**, 3766.
Parisi, G., 1975, Nucl. Phys. B **100**, 368.
Pohl, W. F., 1968, J. Math. Mech. **17**, 975.
Polyakov, A. M., 1988, Mod. Phys. Lett. A **3**, 325.
Polyakov, A. M., 1989, in *Fields, Strings and Critical Phenomena,* Les Houches summer school proceedings, Session XLIX, edited by E. Brézin and J. Zinn-Justin (North-Holland, Amsterdam), p. 305.
Polychronakos, A. P., 1991, Phys. Lett. B **264**, 362.
Prange, R. E., and S. M. Girvin, 1987, Eds., *The Quantum Hall Effect,* second edition (Springer, Berlin).
Rajaraman, R., 1982, *Solitons and Instantons* (North-Holland, Amsterdam).
Read, N., 1989, Phys. Rev. Lett. **62**, 86.
Ryder, L. H., 1980, J. Phys. A **13**, 437.
Schiff, L. I., 1968, *Quantum Mechanics, 3rd ed.* (McGraw-Hill, New York).
Schrödinger, E., 1938, Ann. Phys. (Leipzig) **32**, 49.
Semenoff, G. W., 1988, Phys. Rev. Lett. **61**, 517.
Semenoff, G. W., 1989, Phys. Rev. Lett. **63**, 1026.
Semenoff, G. W., 1991, "Anyons and Chern Simons Theory," University of British Columbia preprint.
Semenoff, G. W., and P. Sodano, 1989, Nucl. Phys. B **328**, 753.
Semenoff, G. W., and N. Weiss, 1990, Phys. Lett. B **250**, 117.
Sporre, M., J. J. M. Verbaarschot, and I. Zahed, 1991, "Solution of the Three Anyon Problem," Stony Brook Preprint No. SUNY-NTG-91/22.
Thouless, D., and Y.-S. Wu, 1985, Phys. Rev. B **31**, 1191.
Tsui, D. C., H. L. Stormer, and A. C. Gossard, 1982, Phys. Rev. Lett. **48**, 1559.
Tze, C. H., 1988, Int. J. Mod. Phys. A **3**, 1959.
Verçin, A., 1991, Phys. Lett. B **260**, 120.
von Klitzing, K., G. Dorda, and M. Pepper, 1980, Phys. Rev. Lett. **45**, 494.
Voruganti, P., 1989, Phys. Rev. D **39**, 1179.
Wigner, E. P., 1939, Ann. Math. **40**, 149.
Wilczek, F., 1982a, Phys. Rev. Lett. **48**, 1144.
Wilczek, F., 1982b, Phys. Rev. Lett. **49**, 957.
Wilczek, F,. 1990a, in *Anomalies, Phases, Defects,* . . . , Proceedings of the 1989 Ferrara school, edited by M. Bregola, G. Marmo, and G. Morandi (Bibliopolis, Naples), p. 329.
Wilczek, F., 1990b, Ed., *Fractional Statistics and Anyon Superconductivity* (World Scientific, Singapore).
Wilczek, F., and A. Zee, 1983, Phys. Rev. Lett. **51**, 2250.
Wu, Y.-S., 1984a, Phys. Rev. Lett. **52**, 2103.
Wu, Y.-S., 1984b, Phys. Rev. Lett. **53**, 111.
Wu, Y.-S., and A. Zee, 1984, Phys. Lett. B **147**, 325.
Wybourne, B. G., 1974, *Classical Groups for Physicists* (Wiley, New York).
Zaccaria, F., E. C. G. Sudarshan, J. S. Nilsson, N. Mukunda, G. Marmo, and A. P. Balachandran, 1983, Phys. Rev. D **27**, 2327.
Zhang, S. C., T. H. Hansson, and S. Kivelson, 1989, Phys. Rev. Lett. **62**, 82.