Boson realizations of Lie algebras with applications to nuclear physics

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The concept of boson realization (or mapping) of Lie algebras appeared first in nuclear physics in 1962 as the idea of expanding bilinear forms in fermion creation and annihilation operators in Taylor series of boson operators, with the object of converting the study of nuclear vibrational motion into a problem of coupled oscillators. The physical situations of interest are quite diverse, depending, for instance, on whether excitations for fixed- or variable-particle number are being studied, on how total angular momentum is decomposed into orbital and spin parts, and on whether isotopic spin and other intrinsic degrees of freedom enter. As a consequence, all of the semisimple algebras other than the exceptional ones have proved to be of interest at one time or another, and all are studied in this review. Though the salient historical facts are presented in the introduction, in the body of the review the progression is (generally) from the simplest algebras to the more complex ones. With a sufficiently broad view of the physics requirements, the mathematical problem is the realization of an arbitrary representation of a Lie algebra in a subspace of a suitably chosen Hilbert space of bosons (Heisenberg-Weyl algebra). Indeed, if one includes the study of odd nuclei, one is forced to consider the mappings to spaces that are direct-product spaces of bosons and (quasi)fermions. Though all the methods that have been used for these problems are reviewed, emphasis is placed on a relatively new algebraic method that has emerged over the past decade. Many of the classic results are rederived, and some new results are obtained for odd systems. The major application of these ideas is to the derivation, starting from the shell model, of the phenomenological models of nuclear collective motion, in particular, the geometric model of Bohr and Mottelson and the more recently developed interacting boson model of Arima and Iachello. A critical discussion of those applications is interwoven with the theoretical developments on which they are based; many other applications are included, some of practical interest, some simply to illustrate the concepts, and some to suggest new lines of inquiry.

CONTENTS

I. INTRODUCTION

A. Early history and general outline

In this paper we review the methods of boson mappings or boson expansions as applied to the nuclear shell

model. These methods may also be applicable to other many-fermion systems. It is well known in many-body theory that under suitable circumstances pairs of fermions can exhibit bosonlike behavior, especially when the density of such pairs is sufficiently low that they do not get in the way of each other, thereby minimizing the effects of the Pauli principle. Under these circumstances the low-lying excitations of such systems can often be identified as a collection of a relatively small number of weakly interacting, nearly harmonic oscillators. But even in situations where interactions may be strong and where large deviations from independent harmonic motion occur for these degrees of freedom, the introduction of a Hilbert space of small dimensionality, which can be considered to be decoupled from the remaining degrees of freedom, can confer both conceptual and technical advantages. The overriding aim of the present review is to provide substantiation for this assertion. Though the emphasis will be on the *representation* of the decoupled degrees of freedom, the idea that the decoupling must occur is a uniform, though not always sufficiently explicit, feature of all applications.

The boson methods are intended to exploit this kind of situation by replacing the degrees of freedom of fermion pairs *directly* with *exact* boson degrees of freedom. Contact with the actual fermion problem is maintained by performing a mapping from the original many-fermion Hilbert space into another Hilbert space called the *ideal space.* In the case of a system with an even number of fermions, the ideal space is generated by boson creation and annihilation operators, with one-boson degree of freedom corresponding to each fermion pair of interest. For an odd number of fermions, in one approach, the ideal space is usually taken to be the tensor product of the boson space with the space of a fermionlike object called the "ideal odd particle" that represents the odd fermion. In another approach currently much in vogue, which relates either to even or to odd systems, only a subset of all the fermion-pair degrees of freedom is mapped into bosons, the remaining degrees of freedom being represented as kinematically independent "quasifermions." What one gains through either metamorphosis is the potential for new kinds of many-body approximations for describing collective motion that would be difficult to implement in the usual fermion formulation, for example, approximations that allow for small violations of the Pauli principle. Moreover, since boson degrees of freedom have their counterparts in classical canonical variables, they have more of an intuitive appeal than fermion operators and are more readily related to semiclassical approximations. In addition, boson mappings provide a direct link between microscopic models of nuclear structure and phenomenological collective models.

Indeed the Copenhagen collective model (Bohr 1952; Bohr and Mottelson, 1953), which evolved from a quantized description of the oscillations of a liquid drop, provided an early precedent for regarding collective excitations as bosons. This idea was reinforced by the introduction of the random-phase approximation (RPA) into nuclear physics (Arvieu and Veneroni, 1960; Baranger, 1960; Marumori, 1960). The RPA, which is a microscopic approach, had first been consistently derived in a boson framework by Sawada (1957) for the electron-gas problem and was often referred to as the "quasiboson approximation" by nuclear physicists. The derivation of the RPA as the small-oscillation limit of the time-dependent Hartree-Fock (TDHF) equations (Thouless and Valatin, 1962) made the bosonic interpretation of the excitations particularly natural. In this review, we shall encounter the same approximation several times within the framework of a boson space. In this context, the equivalence of this approximation to the standard problem of small oscillations in classical mechanics becomes obvious.

It was Belyaev and Zelevinsky (1962) who actually took the first major step in the development of boson expansions in nuclear physics, although similar concepts had been invoked in isolated instances in solid-state physics (Blatt and Matsubara, 1958; Usui, 1960). Now, it was already well known that the RPA gives corrections to the nuclear mean-field approximations of the order of $1/\Omega$, where Ω measures the average shell capacity. Belyaev and Zelevinsky proposed to carry this expansion to higher orders by literally expanding fermion-pair operators as infinite series in boson polynomials, with coefficients chosen so that the commutation rules of fermion-pair operators [now well known to be equivalent to the algebra of SO($2n$) for a space of n single-particle levels] would be satisfied *order by order* in the smallness parameter. This means that their expansion is a true Taylor series in this parameter. Belyaev and Zelevinsky perturbatively calculated the next higher correction to the RPA, equivalent to including cubic and quartic anharmonic corrections to the harmonic RPA.

The seminal paper of Belyaev and Zelevinsky left unanswered the essential question of the formal justification for replacing fermion-pair operators with infinite boson expansions. Is the mere fulfillment of the commutator algebra equivalent to SO($2n$) sufficient to guarantee the equivalence of the boson and fermion systems? Of course, we now know that the answer to this question is in general negative. It is also necessary to demonstrate a correspondence of fermion to boson states that preserves the matrix elements of the generators, i.e., that the boson space actually carries a full spinor representation of SO($2n$). The next major step was carried out by Marumori and co-workers (Marumori, Yamamura, and Tokunaga, 1964; Yamamura, 1965), who, taking a hint from the earlier work of Usui (1960), attacked the problem from the viewpoint of mapping state vectors, relegating commutation rules to a secondary role. The idea is extremely simple and can be described as follows. Let $\mathcal{H}_F$ be the finite-dimensional fermion Fock space generated by creation and annihilation operators $c_i^\dagger, c_i$ corresponding to n single-particle levels, and let $\{|p_i\rangle\}$

be any complete orthonormal basis for $\mathcal{H}_F$ (in reality, any well-defined subspace of $\mathcal{H}_F$, such as a collective subspace, could also be mapped). Also, let $\mathcal{H}_B$ be the boson Fock space generated by creation and annihilation operators $B_{ij}^\dagger, B_{ij}$, with one antisymmetric boson $B_{ij}^\dagger = -B_{ji}^\dagger$ for each fermion pair $c_i^\dagger c_j^\dagger$, and let $\{|p_i)\}$ be an orthonormal basis for the *physical subspace* of $\mathcal{H}_B$, defined as a subspace chosen to be in one-to-one correspondence with the original shell-model space. That is, $\mathcal{H}_F$ is mapped injectively into the physical subspace with the correspondence

$$|p_i\rangle \rightarrow |p_i) \ . \tag{1.1}$$

Then for an arbitrary fermion operator A_F,

$$A_F = \sum_{ij} A_{ij} |p_i\rangle\langle p_j| \ , \tag{1.2}$$

there is a corresponding boson operator A_B given by

$$A_B = \sum_{ij} A_{ij} |p_i)(p_j| \ , \tag{1.3}$$

with the property

$$(p_i| A_B |p_j) = \langle p_i| A_F |p_j\rangle = A_{ij} \ . \tag{1.4}$$

Hence the mapping of state vectors (1.1) together with the mapping of operators $A_F \rightarrow A_B$ *preserves all matrix elements* in the physical subspace. On the other hand, if $|u)$ is any *unphysical* vector, i.e., a vector lying entirely in the orthogonal complement of the physical subspace, it immediately follows that

$$A_B |u) = 0 \ . \tag{1.5}$$

Thus, in the formalism of Usui and Marumori *et al.*, the many-fermion problem can be neatly tucked away in the finite-dimensional physical subspace of the ideal (boson) space while the unphysical orthogonal complement is annihilated. Moreover, it is clear that the physical subspace can be chosen arbitrarily as long as it has the same dimension as the fermion space. (In fact, from the above, it is clear that almost any system, not necessarily a fermion one, can be mapped into a boson space.) The trick, of course, is to choose a physically useful mapping. Both Usui and Marumori *et al.* elected to map an independent fermion basis, with Usui choosing a corresponding boson basis of simple boson product states with an ordering convention for the indices, and the Marumori group choosing a basis of antisymmetric boson states, which has proven easier to work with. In both cases, the fermion vacuum is mapped into the boson vacuum. Of course, if one has prior insight into the correlations in the system, it might be more profitable to map correlated bases. Yamamura also extended the formalism to odd-particle systems by appending the *ideal odd-particle* (IOP) degree of freedom mentioned above. The main characteristic of the IOP is that the physical vectors can have at most one of them, since any pair of fermions would be preempted by a boson. Yamamura chose the IOP to obey fermion anticommutation rules, and vectors having more than one of this species were relegated to the unphysical subspace. However, there is another possible way to handle this degree of freedom, discussed later in Sec. X, that makes use of IOP operators that do not obey fermion anticommutation rules but automatically satisfy the condition that no more than one can exist.

Now the operator image (1.3) implicitly contains the boson vacuum projector $P_0 = |0)(0|$ in every term of the sum. By using the well-known expression for this projector, namely,

$$P_0 = :\exp\left[\tfrac{1}{2}\sum_{ij} B_{ij}^\dagger B_{ij}\right]:$$

(:: is the normal-ordering symbol), one can develop the operator image (1.3) as an infinite normal-ordered expansion, called the *Marumori expansion* [in the case of odd-particle systems, P_0 should be multiplied by $(1-\hat{n})$, where $\hat{n}$ is the IOP number operator]. A reader new to this subject might expect at this point the pronouncement that the expansions of Belyaev and Zelevinsky and those of Marumori *et al.* are one and the same. In fact, the expansions as we have defined them are *different,* but, as we shall show later, there is an intimate connection. At the time, however, this difference led to considerable confusion. Although superficially similar, the expansion of Belyaev and Zelevinsky has only linked intermediate indices, while that of Marumori *et al.* also has unlinked terms. The Marumori expansion has the property that, for any finite truncation, there is always some subspace of the physical subspace in which the exact fermion matrix elements are preserved. In other words, with the restriction to even fermion numbers for definiteness, as the series is extended, this subspace enlarges to include states with 1, 2, 3, etc. antisymmetric bosons, which reproduce the matrix elements of the corresponding states with 2, 4, 6, etc. fermions. The truncated expansion of Belyaev and Zelevinsky did not seem to have this property at all, prompting Marumori *et al.* to assert that it violated the Pauli principle and was thus inferior to their expansion.

Other works on boson expansions began to appear within a few years. For example, Banville and Simard (Simard, 1967, 1969a, 1969b; Banville and Simard, 1970a, 1970b, 1973; Simard and Banville, 1971, 1972) developed an expansion similar to that of Marumori *et al.*, but utilizing a different formal approach. They adopted the antisymmetric physical subspace of Marumori and Yamamura, including the IOP, introduced an ansatz for the expansion of the image of a single-fermion creation or annihilation operator in terms of boson and IOP's, and determined certain of the coefficients by sequentially operating on the physical basis vectors starting with the vacuum and requiring that all fermion matrix elements be exactly preserved up to a certain number of fermions (five was the maximum in practice). Undetermined coefficients were set equal to zero afterwards for the sake of simplicity, which accounts for the differences between the Marumori and Banville-Simard expansions. To uniquely determine all coefficients, it is necessary to specify the action of the operators in the whole ideal

space, not just the physical subspace. This is done in the Marumori expansion by requiring the strong condition (1.5) that all unphysical vectors be annihilated. In the Banville-Simard approach, the situation is reversed; operator coefficients not determined by the action in the physical subspace are arbitrarily set equal to zero, which effectively determines the action in the unphysical subspace. One does not really care about the latter as long as the physical subspace is invariant under the operator images, which is certainly the case for both the Marumori and the Banville-Simard expansions. Banville and Simard utilized their expansions to show that particle and quasiparticle shell-model calculations could be reproduced, which is not surprising in view of the construction. This could just as well have been done using the Marumori expansion, or, for that matter, without boson mappings at all, although Banville and Simard seemed to imply some technical advantages using the boson approach.

At about the same time, an ambitious, calculationally oriented program utilizing boson expansions was initiated by Sorensen (1966a, 1966b, 1967, 1968a, 1968b, 1969, 1970a, 1970b, 1971, 1973; Broglia and Sorensen, 1968). Sorensen's aim was not to reproduce shell-model calculations, for which bosons were not really needed, but to calculate the properties of collective states by diagonalizing the Hamiltonian in a sizable multiboson basis. He expressed the view that the rigorous order-by order Pauli-principle constraints satisfied by the Marumori expansion made its practical convergence too slow. On the other hand, the Belyaev-Zelevinsky expansion was designed to calculate small anharmonic corrections to the RPA, whereas Sorensen was interested in soft nuclei with large anharmonicities. For these reasons, he introduced a new expansion that was designed, like that of Belyaev and Zelevinsky, to satisfy the SO($2n$) algebra of fermion-pair operators, with no explicit consideration being given to the underlying mapping of state vectors. The chief innovation in Sorensen's mapping compared to that of Belyaev and Zelevinsky was the addition of a constant term to the boson images of fermion-scattering operators $c_i^\dagger c_j$. This constant, called y_0, which was taken to be an adjustable parameter, was alleged to perform various useful functions, including speeding up convergence, introducing ground-state correlations from the outset, and correcting for particle-number fluctuations introduced by the Bardeen-Cooper-Schrieffer (BCS) quasiparticle representation (Bardeen, Cooper, and Schrieffer, 1957). However, as shown many years later, the introduction of y_0, which is zero in the Belyaev-Zelevinsky expansion, together with certain other differences, may have the unfortunate consequence that, in spite of fulfillment of the algebra up to a certain order, no physical subspace exists because there is no analog of a fermion vacuum state in the ideal space (Marshalek, 1980b). While a rigorous proof of such a catastrophe is difficult to give because of ambiguities in Sorensen's definition of his expansion, it can be demonstrated for certain reasonable choices of those coefficients left undefined by the commutation rules. Even if there were an underlying physical subspace, no attempt was made in the diagonalization procedure to insure reasonable confinement of the wave functions to the physical subspace. Thus, although Sorensen introduced some interesting ideas, there is reason for doubting the validity of his calculations.

The Era of Confusion with its diversity of disparate boson expansions that followed the pioneering papers of Belyaev and Zelevinsky and of Marumori *et al.* gave way to the Era of Enlightenment (relatively speaking) and unification in 1971. First, Marshalek (1971a) effectively summed the Belyaev-Zelevinsky expansion by brute force for the algebra of particle-hole operators of a system with fixed-particle number, equivalent to U(n) for n single-particle levels, and showed that a physical subspace indeed existed and was just that of Marumori *et al.* In addition, it was shown that the Marumori operator (1.3) is just the projection onto the physical subspace of the corresponding infinite Belyaev-Zelevinsky expansion. The latter is well defined only in the physical subspace, but multiplication by the projector defines the resulting Marumori operator in the unphysical subspace as well, with the property (1.5). It was also remarked that the summed form of the Belyaev-Zelevinsky expansion is analogous to the well-known Holstein-Primakoff representation of the SU(2) algebra (Holstein and Primakoff, 1940). The summation of the Belyaev-Zelevinsky expansion into the Holstein-Primakoff form for the SU(2) case had, in fact, been done earlier (Pang, Klein, and Dreizler, 1968). This paper also introduced both the Holstein-Primakoff mapping and method into nuclear physics.[1] It failed to distinguish, however, between the Marumori mapping and a normal-ordered form of the Belyaev-Zelevinsky mapping.

All these points and more were made independently in the paper of Janssen *et al.* (Janssen, Dönau, Frauendorf, and Jolos, 1971). In this *tour de force,* the authors started with the mapping of Marumori *et al.* for the general SO($2n$) case and showed that the image of a two-fermion creation operator, for example, could be decomposed as the product of an operator involving a formal square root and the projector to the physical subspace. The former operator was properly identified as the summation of the series initiated by Belyaev and Zelevinsky and a direct generalization of the Holstein-Primakoff expansion. In addition, Janssen *et al.* derived a finite boson mapping that, unlike the Marumori mapping, did not preserve Hermitian conjugation. They identified this as the

[1]The authors were saved the embarrassment of claiming a fresh discovery twenty-eight years after the event by the good fortune of communicating the results to their colleague, Henry Primakoff, prior to publication. It is somewhat surprising that this mapping was completely overlooked by the earlier workers in the field.

SO($2n$) generalization of the Dyson-Maleev mapping associated with SU(2) (Dyson, 1956; Maleev, 1958) used in the theory of ferromagnetism. The authors showed that this mapping is equivalent to the Marumori mapping, the equivalence transformation being one that normalizes the basis vectors for the physical subspace. Many other interesting results were obtained as well, including the relation between the boson mapping method and the generator-coordinate approach to collective motion.

From this point onward, the theory and application of boson expansions developed more quickly and rationally. The achieved unification was extended to systems with odd particle number (Marshalek, 1974a, 1974b; Okubo, 1974b, 1974c) including a new form of the expansion with an independent odd particle. It was also proven (Marshalek and Weneser, 1970; Marshalek and Holzwarth, 1972) that in the classical (c-number) limit boson expansion theory reduces to the Hartree-Bogoliubov approximation (Valatin, 1961), or, put another way, the Hartree-Bogoliubov approximation can be regarded as a classical field theory, whose canonical quantization leads back to the exact many-body problem but in the boson form. This connection was exploited to provide a heuristic justification of the self-consistent cranking model (Marshalek and Weneser, 1970; Marshalek, 1971b). Another advance occurring during this time was the adaptation of the Belyaev-Zelevinsky expansion for a perturbative treatment of deformed (broken-symmetry) systems (Marshalek and Weneser, 1969, 1970). We also remark here that the expansion initiated by Belyaev and Zelevinsky was first rechristened the generalized Holstein-Primakoff (GHP) expansion and lately has been often referred to as the Belyaev-Zelevinsky-Marshalek or BZM method, unjustly omitting credit due Janssen *et al.*, but seven-letter acronyms *are* awkward.

Many applications of boson mappings have been developed in the past eighteen years. In addition to applications to instructive toy models there have been many more serious applications. Most of these fall into one or two broad categories. The first consists of perturbative calculations using the BZM expansion, while the second, which is more appropriate to soft anharmonic systems, involves diagonalizations in a multiboson basis. The generalized Dyson-Maleev mappings have been especially popular in recent years because the mapped operators are finite expansions. There has been an increasing awareness of the need to satisfy the Pauli principle. Very recently, boson mean-field approximations have been developed that are different from the familiar fermion ones, such as the Hartree-Bogoliubov approximation, and that incorporate more correlations. There is also a growing interest in thermal boson expansions, but that is still a fledgling field. All of these developments will be described in the review.

The successes of the phenomenological interacting boson model (Iachello and Arima, 1987) in the past fifteen years have greatly sparked the interest in boson mappings. While several attempts have been made in the past ten years to derive the interacting boson model microscopically from boson mappings, the problem is far from settled. This is not surprising to the old-timers who recall the attempts to derive the old Copenhagen phenomenology from a microscopic basis. In any case, many of these developments are covered in our review.

The general interest in the theory of Lie groups sparked by the interacting boson model has probably been at least partially responsible for stimulating a more group-theoretical approach to the derivation of boson mappings in the past decade. This is quite natural in our context, in which we are deriving boson realizations of Lie algebras of operators, in whose enveloping algebra are contained the Hamiltonian and transition operators of interest. For the most part, we have adopted this more modern Lie-algebraic viewpoint in our derivations of boson mappings, but we believe we have done so in a manner that requires only a modicum of technical expertise. There has been a mutual feedback between practical boson mappings and the formal theory of induced group representations, and these developments are also briefly discussed.

B. Guide for the reader

Since we have not followed historical sequence in our exposition, some remarks addressed more specifically to organization are in order before finally allowing the review to speak for itself. Our aim has been to provide a complete work of reference, and therefore it is hardly necessary that it be read linearly. In a general sense this paper consist of three large subdivisions: the first part, Secs. II–VIII, studies the simplest algebras, and it is here that the reader not versed in the field may discover the basic ideas and encounter illustrative applications, not all devoid of physical interest; the second part, comprising Secs. IX–XVIII, addresses the core subject, the realistic description of collective motion in nuclei within the framework of the nuclear shell model, and contains derivations of all the relevant mappings and accounts of all the major programs of application; in the third subdivision, Secs. XIX–XXII, one finds a variety of relatively recent and only partially developed ideas, representing areas that call for further theoretical and practical exploration. We now turn to a somewhat more detailed exposition of the contents of these parts. A detailed summary of the results of this paper, organized somewhat differently, can also be found in Sec. XXIII.

For pedagogical reasons, we have chosen to describe the basic mapping ideas with the help of the simplest algebras for which useful closed-form mappings are known. This material, presented in Secs. II–VIII, runs the gamut from the early work of Pang *et al.* (Pang, Klein, and Dreizler, 1968), which first called attention to the possible interest of studying such simplified models, to a recent study of the quantized Bogoliubov-Valatin transformation (Hahne and Klein, 1989), now understood

as an alternative (see below) to the boson-fermion mappings approach that was developed historically and described above. We see in these sections that semirealistic and realistic applications have been possible with these algebras, though application to the one fundamental problem that originally stimulated interest in boson mappings in nuclear physics, the microscopic description of nuclear vibrations and rotations using the full shell model, is not included. This problem is first attacked in Sec. IX.

Before turning to this material, we interpose some remarks about the quantized Bogoliubov-Valatin transformation. In this approach, as now understood, one bosonizes only a fraction of the pair degrees of freedom, for example, those associated with a subalgebra of the full shell-model algebra. The remaining degrees of freedom are mapped as quasifermions, kinematically independent of the bosons, whose modified algebraic properties have to be established as part of the procedure. The original fermions, which do not commute with the bosons, are mapped in terms both of bosons and of quasifermions. [For the case of SU(2), where the concept was first introduced (Suzuki and Matsuyanagi, 1976), this mapping appears in the guise of an operator-valued Bogoliubov-Valatin transformation, thus the origin of the name.] Altogether, the extension of the idea of the quantized Bogoliubov-Valatin transformation is one of the main new themes emphasized in this review (Klein and Marshalek, 1988, 1989; Hahne and Klein, 1989; Klein and Walet, 1990), of immediate interest mathematically in the theory of group representations, and of ultimate interest physically in providing a basis for the study of the coupling between collective and individual degrees of freedom.

Beginning with Sec. IX and running though Sec. XIII, one finds an account of the main contributions of one of the co-authors of this paper (E.R.M.) and of related work, including derivation of the BZM mapping for even and for odd nuclei, classical limit, perturbative application to vibrations, and the most recent work, formal completion of the Marshalek-Weneser program. The next three sections, XIV–XVI, are concerned with programs designed to study collective motion nonperturbatively with the aid of boson mappings. In Sec. XIV we describe such work as exists for doing dynamics exclusively in the boson space. We then focus on the program of Tamura, Kishimoto, and their associates in Sec. XV and follow that by a discussion in Sec. XVI of the more recently launched program for utilizing the generalized Dyson mapping, by Takada and his collaborators. (See the sections named for the many publications of these authors.) One other section, XVIII, is fully devoted to a critical review of the attempts at a microscopic derivation of the interacting boson models.

The remaining sections are primarily but not exclusively devoted to interests developed during the past decade by the other author of this review (A.K.) and his collaborators. Thus Sec. XVII contains an account of symmetry-conserving mappings of possible interest in connection with derivations of the interacting boson model. In Sec. XIX we discuss the concept of the vector coherent state and its applications, not so much on its own merits, but rather to emphasize that the algebraic techniques used throughout the review to present modern derivations of even classical results are equivalent in power to what can be achieved through this technique. A second goal of this section has already been emphasized above in the discussion of the quantized Bogoliubov-Valatin transformation.

The final three sections are accounts of programs in an early stage of development: Sec. XX on the idea of quantum coherent states as the basis for a fully quantum theory of collective motion, Sec. XXI on boson mappings and large-amplitude collective motion, and Sec. XXII on thermal boson mappings.

C. Apologia

The authors set out to write an exhaustive scholarly account of a subfield of theoretical nuclear physics that is more than a quarter of a century old. We have touched many bases, but we have also omitted some topics that other authors might have included. In making such difficult choices, we have been guided by the thought that our subject was the theory and application of boson mappings of Lie algebras that arise in the nuclear shell model. We are aware of at least five principal omissions made in conformity with this criterion. There is first of all the extensive literature on the development of the phenomenology associated with the Bohr-Mottelson program. [See, for example, Bohr and Mottelson (1975) and Eisenberg and Greiner (1970). Also belonging to this category is a phenomenological boson coherent-state model such as that developed most extensively by Rădută and his associates, reviewed by Rădută (1987).] Second, we have omitted the purely phenomenological aspects of the interacting boson model (Iachello and Arima, 1987; Bonatsos, 1988). Third, we have not included the work of Kumar and Baranger (1967, 1968) and Kumar (1983, 1984). Though bosons enters this microscopic theory of collective motion in a way related to the considerations developed in Sec. XXI, our reason for omitting this subject is that it has been well reviewed in the standard texts and in the cited work of Kumar. Fourth, we have not discussed the Sp(6,R) collective model, which involves a noncompact Lie algebra in an essential way; though this work stimulated the discoveries concerned with the vector coherent-state method, until recently, bosons mappings have not played an essential role in the working out of the physics. Furthermore a good review exists (Rowe, 1985). Fifth, we have chosen not to give an account of the ingenious mapping method of Wu and Feng because it is *sui generis* and does not fit naturally into our account. Again these authors are the best source for review (Wu and Feng, 1981, 1982).

Another limit set on this work, made with considerable regret, was not to venture beyond the borders of our mutual expertise in nuclear physics, and thus to omit work on boson mapping stimulated by other many-body problems.

A word about style and notation. We have tried to be reasonably consistent in our notation for fermion operators, fermion pair operators, generators of an algebra, bosons, bras, and kets. Nevertheless, it was too difficult to completely homogenize the notation of material taken from so many diverse sources and styles. In extreme (but rare) instances, we have even changed notation within a given major section.

II. MAPPING METHODS FOR THE ALGEBRA SU(2)

A. Holstein-Primakoff mapping

In this section, we study realizations of the Lie algebra SU(2) in terms of boson operators, i.e., operators obeying the Heisenberg-Weyl algebra. An irreducible representation (irrep) of the corresponding group is then carried by a subspace of the boson Fock space called the *physical subspace.* Subspaces orthogonal to the physical subspace that do not carry irreps are called *unphysical.*

The reason for choosing to study elementary algebras first and, in particular, SU(2) is that it allows us to exhibit in a relatively short span almost all of the mathematical techniques that will be used or alluded to throughout this review. It will also permit us to illustrate in succeeding sections a wide variety of applications, though admittedly some are only illustrative of the concepts entering into realistic physical applications.

Thus we consider the usual SU(2) generators $J_+=(J_-)^\dagger$ and J_z satisfying the commutation relations

$$[J_+,J_-]=2J_0 \ , \tag{2.1}$$

$$[J_0,J_\pm]=\pm J_\pm \ . \tag{2.2}$$

The progenitor of boson mappings is the Holstein-Primakoff mapping (Holstein and Primakoff, 1940; Pang, Klein, and Dreizler, 1968). Here one examines the nonvanishing representation matrices in the space of states $|j,m\rangle, -j\le m\le j$,

$$\langle j,m|J_0|j,m\rangle=m \ , \tag{2.3}$$

$$\langle j,m+1|J_+|j,m\rangle=[(j-m)(j+m+1)]^{1/2} \ , \tag{2.4}$$

and notices, first, that one can map the sequence of integers or half integers m onto a set of non-negative integers $n, 0\le n\le 2j$, by the displacement

$$m=-j+n \ . \tag{2.5}$$

Thus Eqs. (2.3) and (2.4) become, in an obvious notation that suppresses the eigenvalue j,

$$\langle n|J_0|n\rangle=-j+n \ , \tag{2.6}$$

$$\langle n+1|J_+|n\rangle=[(n+1)(2j-n)]^{1/2} \ . \tag{2.7}$$

One recognizes immediately that Eqs. (2.6) and (2.7) are the matrix elements of the boson operators $b,b^\dagger([b,b^\dagger]=1)$,

$$(J_0)_B=-j+b^\dagger b \ , \tag{2.8}$$

$$(J_+)_B=(J_-)_B^\dagger=b^\dagger(2j-b^\dagger b)^{1/2} \ , \tag{2.9}$$

acting on a subspace of the infinite boson Fock space with basis

$$|n)=(n!)^{-1/2}(b^\dagger)^n|0) \ . \tag{2.10}$$

The form of Eq. (2.9) reminds us that it is defined only in the subspace $n\le 2j$, the physical subspace as previously defined. With this understanding, Eqs. (2.8)–(2.10) provide us with all the irreps of SU(2). Let us indeed remark that within the allowed subspace Eqs. (2.8) and (2.9) can be inverted. The formula

$$b^\dagger=(j+1-J_0)^{-1/2}J_+ \tag{2.11}$$

converts Eq. (2.9) to the standard form of the basis vectors for the representation j, including the correct normalization factors. In Eq. (2.11) $b^\dagger$ is to be understood as an operator that obeys boson commutation relations within the limited vector space of an irrep of SU(2). Within this space it is a well-defined operator. We shall later exhibit similar operators for some more complicated algebras.

The formulas (2.8)–(2.10) were first applied to a nuclear physics model by Pang, Klein, and Dreizler (1968). Marshalek (1971a) suggested that a refinement was necessary to avoid errors when carrying out further manipulations with Eqs. (2.8) and (2.9): Let $\mu=\pm$ or 0, let P_j be the projection operator onto the physical subspace, and let I be the unit operator. Then

$$P_j=\sum_{n=0}^{2j}|n)(n|=I-\sum_{n=2j+1}^{\infty}|n)(n| \ . \tag{2.12}$$

It follows that the operators

$$(J_\mu)'_B=P_j(J_\mu)_BP_j \tag{2.13}$$

have the same matrix elements as $(J_\mu)_B$ in the physical subspace and that they annihilate the unphysical subspace. [Actually the projection operator on the left may be removed, for a very fundamental reason: The $(J_\mu)_B$ leave the physical subspace invariant.]

The introduction of Eqs. (2.12) and (2.13) may appear pedantic as long as we work directly with Eqs. (2.8)–(2.10) where the separation between physical and unphysical is obvious. On the other hand, under one of at least two circumstances the caution entailed in the introduction of Eqs. (2.12) and (2.13) is fully warranted. The first of these, which will be encountered imminently, occurs when we try to expand Eq. (2.9) in powers of $(b^\dagger b/j)$ and thus lose sight of the branch point of the operators $(J_\pm)_B$. The second, to be encountered later,

occurs when we mix the physical and unphysical subspaces by a unitary transformation dictated by the physics of the particular Hamiltonian under consideration.

We shall indicate the mapping involved in Eqs. (2.8)–(2.10) by the notation

$$|n\rangle \rightarrow |n) \ , \tag{2.14}$$

where the angular bracket refers to the basis constructed directly from the raising operators J_+ acting on the state with $J_0=-j$. In the literature of nuclear physics, Eq. (2.14) is usually replaced by the more explicit

$$|n\rangle_F \rightarrow |n)_B \ , \tag{2.15}$$

where F refers to the fermion and B to the boson space.

B. Marumori, Yamamura, and Tokunaga mapping

Before attention was called to the Holstein-Primakoff mapping, two different methods already had currency in nuclear physics. We shall discuss them in reverse historical order. In the method of Marumori, Yamamura, and Tokunaga (1964; Marumori, Yamamura, Tokunaga, and Takada, 1964; Yamamura, 1965), as applied to SU(2), the formula for the generator or any function G of the generators,

$$G=\sum_{n,n'}^{2j} \langle n|G|n'\rangle |n\rangle\langle n'| \ , \tag{2.16}$$

is replaced under the mapping (2.14) by its boson image G_B,

$$G_B=\sum_{n,n'}^{2j} \langle n|G|n'\rangle |n)(n'| \ . \tag{2.17}$$

It is clear that

$$\langle n|G|n'\rangle = (n|G_B|n') \tag{2.18}$$

and that all matrix elements of G_B that couple to unphysical states vanish. Formally, Eq. (2.17) may be written as

$$G_B=UGU^\dagger \ , \tag{2.19}$$

where U is the operator

$$U=\sum_{n=0}^{2j} |n)\langle n| \ . \tag{2.20}$$

We shall henceforth refer to Eq. (2.20) as the Marumori operator. It is a unitary version of an operator previously studied by Usui (1960). Given the basis (2.10), U can be constructed explicitly, provided one utilizes the easily verified expression for the boson projector $|0)(0|$,

$$|0)(0| = :\exp(-b^\dagger b): \ , \tag{2.21}$$

where the customary notation for normal ordering has been utilized. In the present case when Eq. (2.21) is expanded, the generators can be shown to take the forms

$$(J_+)_B = b^\dagger \sum_{k=0}^{\infty} c_k (b^\dagger)^k (b)^k \ , \tag{2.22}$$

where

$$c_k=\sum_{p=0}^{p_k} (-1)^{k-p}(2j+p)^{1/2}/p!(k-p)! \ , \tag{2.23}$$

with $p_k=k$ for $k\le 2j$, $p_k=2j$ for $k\ge 2j$, and

$$\begin{aligned} (J_0)_B &= \sum_{k=0}^{\infty} d_k (b^\dagger)^k b^k \ , \\ d_k &= \sum_{p=0}^{p_k} (-1)^{k-p}(p-j)/p!(k-p)! \ . \end{aligned} \tag{2.24}$$

In the early literature (Pang, Klein, and Dreizler, 1968; Kleber, 1969) one encounters the normal-ordered Holstein-Primakoff mapping misidentified as the Marumori-Yamamura-Tokunaga mapping. In the former, Eqs. (2.22) and (2.23) remain valid except that now the coefficients c_k are defined by the condition $p_k=k$ for all k, and Eq. (2.24) is replaced by Eq. (2.8). With this new definition of the coefficients in Eq. (2.23), one observes that they become imaginary for $k>2j$. Such terms contribute, however, only if spurious basis states intervene in a calculation. For the examples quoted in the next two sections, the admixture of unphysical states is sufficiently small that for all intents and purposes the results for the two mappings are almost the same. In any event, in any approximation, the Marumori-Yamamura-Tokunaga mapping also strictly loses its salient property of fully excluding the spurious subspace.

The difference between the two mappings suggests another possibility that has not been investigated in any systematic manner. Let $|\mu\rangle$ be a member state of the unphysical space. Then if we replace Eq. (2.17)

$$\begin{aligned} \tilde{G}_B = G_B &+ \sum_{\mu n} \{c_{\mu n}|\mu\rangle\langle n| + c_{n\mu}|n\rangle\langle\mu|\} \\ &+ \sum_{\mu\mu'} c_{\mu\mu'}|\mu\rangle\langle\mu'| \ , \end{aligned} \tag{2.25}$$

$\tilde{G}_B$ is completely equivalent to G_B in its actions on the physical subspace, but is no longer so if approximations are made. Can one invent criteria for choosing the coefficients in Eq. (2.25) that will improve the approximation to a prescribed order?

C. Method of Belyaev and Zelevinsky

The introduction of bosons into nuclear physics (as a microscopic concept) dates back to Belyaev and Zelevinsky (1962). Though it is simplest to describe their method within the context of the SU(2) algebra, we shall attempt to do so in a manner that lends itself to easy generalization for more complicated algebras. We should also remark that Belyaev and Zelevinsky (BZ) did not present their method in the spirit of group theory, nor did they make any attempt to sum the full expansions as will be done later. Let us then consider an irrep of a Lie algebra. Generally speaking, the generators can be divid-

ed into three sets: The raising operators [J_+ for SU(2)] that are used to construct the basis by operating on a reference state ("state of maximum weight"); the lowering operators (J_-) that construct the dual basis; and the remaining generators (J_0) that contain as a subset (J_0) diagonal operators whose eigenvalues are, up to a displacement, the non-negative integers (Gilmore, 1974; Wybourne, 1974). (A more precise separation into sets will be found in Sec. XIX in connection with the concept of the vector coherent state.) For each raising operator, we introduce a boson creation operator $b^\dagger$, which in the simplest case has the same selection rules with respect to the chosen basis as the raising operator. Under these conditions the diagonal operators, such as J_0, are linear in the boson number operators and can be written down by inspection. Here we regard Eq. (2.8) as a part of the mapping that is given to us, or rather forced on us by the consideration above. Turning to the form of J_+, one sees that it can only be a function of boson operators with the same selection rule as $b^\dagger$, which restricts us to operators of the form $b^\dagger$ multiplied by any function of $b^\dagger b$. In the original BZ method, J_+ is taken in the form

$$(J_+)_B=\sum_{n=0}^{\infty} c_n b^\dagger (b^\dagger b)^n \ , \tag{2.26}$$

$$(J_-)_B=(J_+)_B^\dagger \ , \tag{2.27}$$

which satisfies our requirement in series form. The unknown coefficients c_n are to be determined from the commutation relations (2.1) and (2.2) "order by order." Let us examine what this means.

First, the choices already made for J_0 and $J_\pm$ satisfy Eq. (2.2), so that only Eq. (2.1) need be studied. Without evaluating the commutator in detail, we see that the result of commuting J_+ with J_- is a power series in $(b^\dagger b)$, whose coefficients are equated power by power with the corresponding coefficients of $2J_0$. While in the present example each power of $(b^\dagger b)$ is divided by j, the determination of the coefficients c_n is not formally dependent on the presence of an obvious small parameter, though the domain of convergence of the resulting series surely is. In any event, the coefficients discovered are precisely those which encourage the summation of the series to the form of Eq. (2.9). Thus the BZ method provides the expanded form of the Holstein-Primakoff mapping.

This suggests the development of a modified version of the BZ argument that will lead directly to the summed form, an argument that will furthermore prove its value for more elaborate algebras. Our previous reasoning assured us that $(J_+)_B$ can be chosen in the form

$$(J_+)_B=(J_-)_B^\dagger=b^\dagger f(b^\dagger b)\equiv b^\dagger f(\hat{n}) \ . \tag{2.28}$$

With the help of the relations

$$b^\dagger f(\hat{n})=f(\hat{n}-1)b^\dagger \ , \tag{2.29}$$

$$f(\hat{n})b=bf(\hat{n}-1) \ , \tag{2.30}$$

the commutation relation (2.1) can be evaluated in closed form, yielding the difference equation

$$\hat{n}f^2(\hat{n}-1)-(\hat{n}+1)f^2(\hat{n})=2(\hat{n}-j) \ , \tag{2.31}$$

with the expected solution

$$f(\hat{n})=(2j-\hat{n})^{1/2} \ . \tag{2.32}$$

We have thus regained the results of the original Holstein-Primakoff mapping and all that is thereby implied, including the mapping of states (2.14) or (2.15).

Let us nevertheless examine this last point with some care. The mapping of the original states

$$|n\rangle=|j,m\rangle=\phi_m(J_+)|j,-j\rangle=\phi_m(J_+)|0\rangle \ , \tag{2.33}$$

where

$$\phi_m(J_+)=\left[\frac{(j-m)!}{2j!(j+m)!}\right]^{1/2}(J_+)^{j+m} \ , \tag{2.34}$$

may be summarized in two steps: (i) vacuum maps to vacuum, $|0\rangle\rightarrow|0)$ and (ii) $\phi_m(J_+)\rightarrow\phi_m((J_+)_B)$. The result of these steps in the present example is that the resulting boson basis is Eq. (2.10). However, the two steps just given and the corresponding steps for the dual space, where these are distinct, always apply when we determine the mapping by the algebraic requirement that the commutation relations be preserved. This will prove a useful formulation when we turn below to nonunitary mappings, where the precise forms of the mapped states is not so obvious.

D. Dyson-Maleev mapping

So far we have dealt with mappings that preserve the orthonormality of the bases and also thereby preserve the formal relation $(J_+)^\dagger=J_-$. We consider next a mapping, the Dyson-Maleev mapping (Dyson, 1956; Maleev, 1958), that does not preserve the orthonormality. In accordance with custom we shall call it the Dyson mapping or, when extended to other algebras, the generalized Dyson mapping. The motivation for seeking such a mapping was to avoid the convergence questions associated with the expansion of square-root operators. This motivation also applies to nuclear problems, and therefore the Dyson mapping has attracted increasing attention here as well. In Sec. XVI we shall describe what has been accomplished in a practical sense by the use of this mapping. Here we shall be concerned only with the formal aspects of the mapping—with its direct derivation and with its status as an intermediate step in obtaining unitary mappings.

There are essentially two ways of deriving the Dyson mapping, either algebraically or as a realization of the algebra in the space of coherent states.

Let us consider the algebraic approach first. The argument is very much the same as for the BZ method. We introduce the bosons b and $b^\dagger$ and choose J_0 diagonal as in Eq. (2.8),

$$(J_0)_D = -j + b^\dagger b \ . \tag{2.35}$$

(Diagonal operators remain diagonal under the mapping.) We also insist that J_+ and J_- have the selection rules as previously understood, but we no longer insist that they be formal Hermitian conjugates in the mapped space. Thus we generalize Eq. (2.28) to

$$(J_+)_D = b^\dagger f_+(\hat{n}) \ , \tag{2.36}$$

$$(J_-)_D = f_-(\hat{n}) b \ , \tag{2.37}$$

forms that are guaranteed to satisfy Eq. (2.2) for arbitrary $f_\pm$. (We return below to a discussion of the meaning of such a step.) The commutation relation (2.1) then replaces Eq. (2.32) by the condition

$$f_+(\hat{n}) f_-(\hat{n}) = 2j - \hat{n} \tag{2.38}$$

and shows that we have some freedom in the choice of the functions in Eqs. (2.36) and (2.37). The Dyson mapping corresponds to choosing either f_+ or f_- equal to unity. We make the conventional choice $f_- = 1$. Thus

$$f_+(n) = (2j - b^\dagger b) \ . \tag{2.39}$$

It follows from the state mapping rules reviewed after Eqs. (2.33) and (2.34) that the basis of right vectors

$$|n)_D = \phi_n((J_+)_D)|0) \tag{2.40}$$

consists still of orthogonal states, which are, however, no longer normalized, but rather have unit overlap with a corresponding member of the left or dual bases (distinguished by a bar)

$${}_D(\bar{n}| = (0|\phi_n((J_-)_D) \ . \tag{2.41}$$

We now face more squarely the question of whether we have the right to give up the formal relation $J_+ = (J)^\dagger_-$. Isn't this relation a part of the definition of the algebra? The answer is that it is, as long as we insist on a unitary representation, but it is lost if we go over to a nonunitary representation, which we can certainly do without losing the property of having a representation. The loss of unitarity is related, of course, to the lack of normalization of the basis vectors Eq. (2.40). We do maintain the Hermitian conjugation property for matrix elements in the sense

$${}_D(\bar{n}'|(J_+)_D|n)_D = {}_D(\bar{n}|(J_-)_D|n')^*_D \ , \tag{2.42}$$

which is the mapped version of the corresponding statement in a unitary basis. This elementary point will play an essential role when we consider applications in Sec. XVI.

We turn next to a very brief exposition of the use of generalized coherent states. To obtain equivalent results from the viewpoint of coherent states, let $|F\rangle$ be an arbitrary state vector defined on the space of the irrep of SU(2) with angular momentum j, i.e., it is a linear combination of the states $|j,m\rangle$. The formula

$$f(z) = \langle 0|e^{zJ_-}|F\rangle \equiv \langle z|F\rangle \tag{2.43}$$

contains the dual of the un-normalized coherent state of SU(2) (Gilmore, 1972; Perelomov, 1972, 1977),

$$|z\rangle \equiv e^{z^* J_+}|0\rangle \ , \tag{2.44}$$

and defines a mapping from the space of states $|F\rangle$ to a space $f(z)$ of holomorphic functions of the complex variable z. The state $\langle 0|$ satisfies

$$\langle 0|J_0 = -j\langle 0| \ , \tag{2.45}$$

$$\langle 0|J_+ = 0 \ . \tag{2.46}$$

The mappings

$$\langle z|J_-|F\rangle = (d/dz) f(z) \ , \tag{2.47}$$

$$\langle z|J_0|F\rangle = (-j + z(d/dz)) f(z) \ , \tag{2.48}$$

$$\langle z|J_+|F\rangle = z(2j - z(d/dz)) f(z) \tag{2.49}$$

are special cases of the formula, valid for a general operator $\mathcal{O}$,

$$\langle z|\mathcal{O}|F\rangle = \langle 0|e^{zJ_-}\mathcal{O}e^{-zJ_-}e^{zJ_-}|F\rangle \ , \\ = \langle 0|\{\mathcal{O} + z[J_-,\mathcal{O}] + \cdots\}e^{zJ_-}|F\rangle \ , \tag{2.50}$$

that simplifies by persistent application of the commutation relations and of Eqs. (2.45) and (2.46). But these equations are immediately identified with the set (2.35)–(2.39) when we notice that the mapping

$$b^\dagger \to z \ , \tag{2.51a}$$

$$b \to (d/dz) \tag{2.51b}$$

is a valid realization of the boson commutation relations.

The use of coherent states to obtain the Dyson mapping has a long history in nuclear physics, though the language and viewpoint utilized has often been that of generator coordinates (Jancovici and Schiff, 1964; Brink and Weiguny, 1968; Ui and Biedenharn, 1968; Hage-Hassan and Lambert, 1972; Holzwarth, 1972; Dasso and Klein, 1973).

E. Unitarization of the Dyson mapping

For the present we consider the Dyson mapping as an intermediate stage in the problem of producing a unitary mapping. We thus seek to unitarize the mapping. This can be done with the help of a similarity transformation S, with the properties

$$S(J_0)_D S^{-1} = (J_0)_B \ . \tag{2.52}$$

$$S(J_\pm)_D S^{-1} = (J_\pm)_B \ , \tag{2.53}$$

where we require (since it is already Hermitian)

$$(J_0)_B = (J_0)_D \tag{2.54}$$

and

$$(J_+)_B=(J_-)_B^\dagger \ . \tag{2.55}$$

From Eqs. (2.53) and (2.55), we derive

$$V^{-1}(J_-)_D^\dagger V=(J_+)_D \ , \tag{2.56}$$

where

$$V=S^\dagger S \ . \tag{2.57}$$

From Eqs. (2.52) and (2.53), it follows that V is the metric tensor for the right vectors $|n)_D$,

$$_D(n|V|n')_D=\delta_{nn'} \ . \tag{2.58}$$

Since V is Hermitian and positive definite, there is no loss of generality in identifying S with the square root of V, i.e., we may take S to be real and symmetric.

To find S, observe that Eqs. (2.52) and (2.54) imply that S commutes with J_0 and is thus at most a function of $\hat{n}=b^\dagger b$. Thus let us take the matrix element of Eq. (2.56) between the (unitary) basis states $(n|$ and $|n+1)$. Substituting Eqs. (2.36)–(2.38) and noting that the matrix element of $b^\dagger$ cancels from both sides, we derive

$$V(n+1)=V(n)(2j-n)^{-1} \tag{2.59}$$

or

$$S(n+1)=S(n)(2j-n)^{-1/2} \ . \tag{2.60}$$

When we apply this result to the appropriate nonvanishing matrix element of Eq. (2.53) and repeat the argument leading to Eq. (2.59), it is a simple exercise to regain the Holstein-Primakoff formulas for $(J_\pm)_B$. In effect, Eq. (2.60) reestablishes the normalization lost in the Dyson mapping. This method will also find extensive use for more general algebras.

To summarize thus far, we have presented at least four distinct methods of boson mapping, all designed to yield the same boson realization of a given irrep of SU(2):

(i) The Holstein-Primakoff method works directly from the matrices representing the generators—in the exceptional case that these are available.

(ii) The BZ or commutator method determines the mapping from restricted but essential information about the basis, coupled with the application of the commutation relations. When it can be carried to completion, it produces the same result as (i); in other cases the result is an "approximation" to (i).

(iii) In a modified Holstein-Primakoff method, one first obtains the nonunitary Dyson mapping and subsequently unitarizes it. Again the result duplicates (i). The common feature of (i)–(iii) is that the mapping yields correct results only when applied to the physical subspace of boson states that are in one-to-one correspondence with the finite-dimensional starting basis. When applied in the unphysical space, the result is nonsense. This defect can be avoided by surrounding the mapped operators with suitable projection operators onto the physical subspace. This modified Holstein-Primakoff result is then exactly equivalent to the result of the next method.

(iv) The Marumori-Yamamura-Tokunaga method by its very nature produces a mapping that annihilates the unphysical subspace if carried to completion. It is equivalent to the modified Holstein-Primakoff result, Eq. (2.13).

F. Schwinger mapping

Two additional types of mappings of SU(2) will prove of practical and theoretical interest. In one, a boson-fermion mapping to be discussed in Sec. III.E, we utilize the notion of SU(2) as a subalgebra of a shell-model algebra and shall be interested as well in the mapping of single-fermion operators that decompose into simple tensor sets under the subalgebra. In the other, the famous Schwinger mapping (Schwinger, 1965), we remain strictly within the confines of SU(2) [or U(2)]; we complete this section with a discussion of this mapping.

The original derivation can be paraphrased as follows: Consider two bosons $b_\pm^\dagger, b_\pm$ such that $b_+^\dagger$ increases J_z by one-half and $b_-^\dagger$ decreases it by one-half (in units of $\hbar$). It follows that possible allowed values of J_z can be expressed as a difference of number operators

$$J_z=\tfrac{1}{2}(b_+^\dagger b_+ - b_-^\dagger b_-)=\tfrac{1}{2}(\hat{n}_+ - \hat{n}_-) \ . \tag{2.61}$$

Clearly also

$$J_+=b_+^\dagger b_- , \quad J_-=b_-^\dagger b_+ \ , \tag{2.62}$$

are unit raising and lowering operators. Furthermore it is easily verified that Eqs. (2.61) and (2.62) satisfy the SU(2) algebra.

But how do we select an irreducible representation? Calculate $\vec{J}^2$ and find

$$\vec{J}^2=\tfrac{1}{2}\hat{N}(\tfrac{1}{2}\hat{N}+1) \ , \tag{2.63}$$

where

$$\hat{N}=\hat{n}_+ + \hat{n}_- \ . \tag{2.64}$$

With n the eigenvalues of the operator (2.64), we therefore have

$$j=\tfrac{1}{2}n \ . \tag{2.65}$$

Equation (2.64) is an operator constraint in the space of each irreducible representation. Finally the basic states are

$$\begin{aligned}|n_+,n_-\rangle &= |n,m\rangle \\ &= ((b_+^\dagger)^{n_+}(b_-^\dagger)^{n_-}/\sqrt{n_+!}\sqrt{n_-!})|0) \\ &= ((b_+^\dagger)^{j+m}(b_-^\dagger)^{j-m}/\sqrt{(j+m)!}\sqrt{(j-m)!})|0) \ .\end{aligned} \tag{2.66}$$

The Schwinger mapping has so far found its main application to nuclear physics in phenomenological models (see Secs. V and VIII).

III. APPLICATION TO THE LIPKIN MODEL. A BOSON-FERMION MAPPING

A. Vibrations in the LMG model

In this section and in the succeeding two, we shall describe a number of illustrative applications and extensions of the mappings derived in the previous section. These applications involve mostly oversimplified shell models, though applications to phenomenology will also prove illuminating.

We start with the well-studied LMG or Lipkin model (Lipkin, Meshkov, and Glick, 1965). As seen below, in this model the SU(2) algebra is realized by bilinear forms in single-particle fermion operators: Consider two levels, each of degeneracy Ω, separated by energy ϵ; in the absence of interaction, Ω particles fully occupy the lower level. This state, designated $|0\rangle$, will be our reference, and excitations will be measured relative to it. We introduce fermion operators $\beta_m^\dagger$, β_m, $m=1,\ldots,\Omega$ that create and annihilate holes in the lower level and $\alpha_m^\dagger$, α_m that create and annihilate particles in the upper level. Thus the coherent sums

$$J_0=-\tfrac{1}{2}\Omega+\tfrac{1}{2}\sum_{m=1}^{\Omega}(\alpha_m^\dagger\alpha_m+\beta_m^\dagger\beta_m) \tag{3.1}$$

$$J_+=\sum_{i=1}^{\Omega}\alpha_m^\dagger\beta_m^\dagger,\quad J_-=J_+^\dagger=\sum_{i=1}^{\Omega}\beta_m\alpha_m \tag{3.2}$$

satisfy the SU(2) algebra [Eqs. (2.1) and (2.2)] if the following anticommutators are satisfied

$$\{\alpha_m,\alpha_{m'}^\dagger\}=\{\beta_m,\beta_{m'}^\dagger\}=\delta_{mm'}\ , \tag{3.3}$$

other anticommutators vanishing. The reference state $|0\rangle$ is defined by

$$\alpha_m|0\rangle=\beta_m|0\rangle=0\ . \tag{3.4}$$

We study the Hamiltonian

$$H=\epsilon J_0+\tfrac{1}{2}V(J_+^2+J_-^2)\ , \tag{3.5}$$

which belongs to the enveloping algebra of SU(2). This means that we are describing a simplified shell model with a high degree of symmetry, since the spectrum decomposes into irreps of SU(2); our description above implies, as is obvious from Eqs. (3.1) and (3.4), that the ground-state "band" belongs to $j=\frac{1}{2}\Omega$.

Indeed, for $V=0$, introducing the Holstein-Primakoff mapping Eqs. (2.8) and (2.9) and choosing $\epsilon=1$ (or in units of ϵ), one obtains

$$H=-\tfrac{1}{2}\Omega+b^\dagger b \tag{3.6}$$

and thus a perfect harmonic spectrum of $\Omega+1$ states (within the physical subspace). The first question to ask is whether this b degree of freedom remains the natural choice for describing the system as one turns up the value of the interaction. We describe briefly an early investigation of Pang *et al.* devoted to this subject (Pang, Klein, and Dreizler, 1968). This work exploited the normal-ordered Holstein-Primakoff mapping [see the discussion following Eq. (2.24)], which to fourth-order terms replaces Eq. (3.5) by the expression

$$H_B\cong-\tfrac{1}{2}\Omega+b^\dagger b+\tfrac{1}{2}\delta[1-\Omega^{-1}]^{1/2}((b^\dagger)^2+b^2)+\tfrac{1}{2}\delta[1-\Omega^{-1}]^{1/2}\{[1-2\Omega^{-1}]^{1/2}-1\}((b^\dagger)^3b+b^\dagger b^3)\ , \tag{3.7}$$

where $\delta\equiv(\Omega V/\epsilon)$ measures the size of the interaction. In addition, there is the "parameter" $(b^\dagger b)/\Omega$ measuring the ratio of anharmonic to harmonic interaction terms. In the work cited, it was shown that, if one carried out a linear Bogoliubov transformation (Bogoliubov, 1958) to a new boson $B^\dagger$ according to the equations

$$b^\dagger=c_1B^\dagger+c_2B \tag{3.8}$$

and chose c_1,c_2 from the twin conditions of canonicity and that the coefficient of the "dangerous" terms $(B^2+B^\dagger)^2$ should vanish (equivalent to the random-phase approximation), the resulting Hamiltonian, when diagonalized, correctly described the low-lying eigenstates for $\delta<0.9$. In this regard, full fourth-order results were markedly superior to the quadratic approximation (RPA in the boson space) or even to approximate quartic results obtained by further expansion in Ω^{-1}. Numerical results will not be illustrated here, but comparisons of various approximations with exact results will be shown for a related calculation in Sec. IV.A.

Our immediate interest resides mainly in the observation that the transformation (3.8) necessarily introduces spurious components of the boson space into the treatment, even if we restrict the size of the original vector space to the physical size. The numerical results indicate that these components have little effect for the range of parameters considered. The treatment breaks down as $\delta\to1$ for the completely different reason that the model exhibits a phase transition: for larger values of δ the approximation (3.7) develops an instability that no longer reflects the behavior of the full Hamiltonian. This point can be established by the substitution, sensible for large δ and justified below,

$$\begin{aligned} b&=b^\dagger=x/\sqrt{2},\quad \text{if } \delta<0\ ,\\ b&=b^\dagger=ix/\sqrt{2},\quad \text{if } \delta>0\ , \end{aligned} \tag{3.9}$$

where x can be considered as the collective coordinate of the system. We then find that Eq. (3.7) becomes a potential-energy function dominated by an unstable quartic interaction.

B. Phase transition

We turn next to the notion that the system under discussion undergoes a phase transition as a function of the

parameter δ. Though the idea of phase transitions in nuclear systems (for instance shape transitions) is widely understood in qualitative and approximately quantitative terms, it has been rendered mathematically precise only for certain simplified models, the simplest of which is the one under discussion. In such models there is a degeneracy parameter D, which counts the number of single-particle orbits and a total occupation number N of these levels. By studying the limit N and $D\to\infty$, N/D remaining finite, as a function of the parameters that occur in the Hamiltonian, we can derive rigorous conclusions concerning the possible phase transitions in such models.

We now describe a method (Klein 1980a; Klein and Vallières, 1980; Klein, Li, and Vallières, 1981), based on the results of the previous section, that can be utilized, not only to discuss phase transitions, but also to treat the dynamics in each of the phases. We start by substituting the Holstein-Primakoff mapping (2.8) and (2.9) into Eq. (3.5), with the result (subscript B omitted)

$$H=\epsilon(-\tfrac{1}{2}\Omega+b^\dagger b)+\tfrac{1}{2}V\{[b^\dagger(\Omega-b^\dagger b)^{1/2}]^2+\text{H.c.}\}\ , \tag{3.10}$$

where H.c. stands for Hermitian conjugate. H now appears to be an explicit function of Ω, the degeneracy parameter, and our task would seem to be straightforward, but the matter is more subtle than that. To get at the physics, let us introduce canonical variables x and p, satisfying $[x,p]=i$, according to the standard definitions,

$$b^\dagger=2^{-1/2}(\Omega^{1/2}x-i\Omega^{-1/2}p)\quad\text{(and H.c. Eq.)}\ . \tag{3.11}$$

Here the scale used has no absolute significance but is simply convenient. The argument of the square root in Eq. (3.10) becomes

$$\Omega-\tfrac{1}{2}\Omega x^2-\tfrac{1}{2}p^2/\Omega+\tfrac{1}{2}\ . \tag{3.12}$$

Now in the small-vibrations domain, previously studied, one has $b^\dagger b\sim 1$ and equipartition between the x^2 and p^2 contributions. This dictates that $x\sim\Omega^{-1/2}$, $p\sim\Omega^{1/2}$, which implies that the entire square root can be expanded in powers of $(b^\dagger b/\Omega)$. We thereby obtain (for δ small enough) a stable polynomial Hamiltonian describing small oscillations about $x=0$, as previously studied.

As δ increases, the frequency of this oscillation tends to zero and the Hamiltonian passes to a new domain where $x\sim 1$ and $p\sim 1$, which means that we can expand only in powers of the last two terms of (3.12) rather than the last three, as in the vibrational domain. Because the leading term of this expansion yields a potential-energy function that can "change shape" as a function of the parameter δ, it determines different phases. By defining

$$v(x)=\lim(b^\dagger=b\to 2^{-1/2}\Omega^{1/2}x)H/\Omega\epsilon\ , \tag{3.13}$$

one finds the simple function

$$v(x)=-\tfrac{1}{2}+\tfrac{1}{2}x^2-\tfrac{1}{2}\delta x^2+\tfrac{1}{4}\delta x^4\ , \tag{3.14}$$

which always has a critical point at $x=0$ and which for $\delta>1$ has critical points at

$$x=\pm(1-\delta^{-1})^{1/2}\ . \tag{3.15}$$

In the latter case these points are equivalent minima separated by a maximum at $x=0$, giving a symmetric double well. This is the domain of large-amplitude collective motion for this model. In the limit $\Omega\to\infty$, the system would have to reside in one or the other of these minima, i.e., one would have calculated the transition to a phase in which the reflection symmetry of the underlying Hamiltonian [cf. Eq. (3.14)] has been broken.

The behavior just noted is one of the simplest rigorous examples of a "nuclear" system exhibiting a phase transition. The most famous of nuclear shape transitions, from spherical to quadrupole shape, retains reflection symmetry, but tentative evidence for octupole shape transitions in the actinides (Żylicz, 1986) may be an example of a shape transition breaking reflection symmetry.

For the finite system, one can add the previously discarded kinetic-energy (and zero-point energy) terms and proceed to study the dynamics of each "phase."

C. Transition-operator boson

A study, particularly of the "deformed" phase ($\delta>1$), was carried out previously for the LMG model using a method related to that described, but obtained by rather different reasoning (Klein, 1972; Dreizler and Klein, 1973). Though the detailed results for the model are of no current interest, this work contains a proposal of a boson mapping which appears distinct from anything presented in Sec. II. A rigorous connection with the Holstein-Primakoff mapping will follow, but the "physical" argument leading to the result will be presented first.

For an attractive interaction $V=-|V|$, one can write Eq. (3.5) as

$$H=\epsilon J_0-|V|(J_x^2-J_y^2)\ . \tag{3.16}$$

Thus for large $|V|$, the structure of Eq. (3.16) implies that the energy can be minimized by maximizing $\langle J_x^2\rangle$. This suggests further that one look for a boson mapping in which $J_x\sim X$, the "coordinate" with a large amplitude. It is easily seen that the tentative choice

$$J_x=\mathcal{N}X,\quad J_y=-P,\quad J_0=-\mathcal{N}\ , \tag{3.17}$$

where $\mathcal{N}$ is a constant, will satisfy the commutation relation

$$[J_x,J_y]=iJ_0\ , \tag{3.18}$$

but, to fully satisfy the commutation relations, this choice must be complicated by adding to the operators in Eq. (3.17) powers of P^2 and X^2. One is soon led to the ansatz

$$J_x=\tfrac{1}{2}\mathcal{N}\{\sin X,\psi(P^2)\}\ , \tag{3.19}$$

$$J_y=-P\ , \tag{3.20}$$

$$J_z = -\tfrac{1}{2}\mathcal{N}\{\cos X, \psi(P^2)\} \ , \tag{3.21}$$

which is certainly not unique, though the simple form of Eq. (3.20) is suggested by the decreasing importance of J_y and P at large $|V|$. The assumptions (3.19)–(3.21) satisfy exactly all the commutation relations except that between J_0 and J_x, which must be used to determine $\psi(P^2)$. This leads to the difference equation

$$P = \tfrac{1}{2}\mathcal{N}^2(\psi_{-1}-\psi_1)(\psi_{-1}+2\psi+\psi_1) \ , \tag{3.22}$$

where $\psi_\lambda = \psi((P+\lambda)^2)$. Equation (3.22) determines ψ up to the scale $\mathcal{N}$; the latter is fixed by the one remaining condition not yet imposed in the present development, namely the value of the Casimir invariant. This gives the equation

$$\tfrac{1}{2}N(\tfrac{1}{2}N+1) = P^2 + \tfrac{1}{2}\mathcal{N}^2[2\psi^2 + 2\psi(\psi_1+\psi_{-1}) + \psi_1^2+\psi_{-1}^2] \ . \tag{3.23}$$

For reasons that have been explained, the quantity X appearing in Eqs. (3.19)–(3.21) was termed the transition-operator boson. An equivalent final form of the mapping is quoted below in Eq. (3.29) and therefore an explicit form for the operator ψ will not be given here.

In their account, Dreizler and Klein failed to emphasize the obvious fact that X and P above must be angle-action variables rather than a Cartesian canonical pair. These authors were interested in the physics of the strong-coupling limit, as had been explained in a previous communication (Klein, 1972). In this domain a semiclassical approximation, in which X and P are treated as a canonical pair, is valid, and the lack of mathematical rigor *in the application of the mapping* had no practical consequences. The oversight was gleefully corrected by Okubo and Marshalek (Okubo, 1974a; Marshalek, 1976b). In the following new account, we show how to transform the Holstein-Primakoff mapping into a mapping equivalent to Eqs. (3.19)–(3.21)

To transform the Holstein-Primakoff result, we define an operator $\mathcal{U}$ by the formula

$$\mathcal{U} = b^\dagger/\sqrt{\hat{n}+1} \ . \tag{3.24}$$

For any function $f(\hat{n})$ of the number operator, we have, clearly,

$$\mathcal{U} f(\hat{n}) = f(\hat{n}-1)\mathcal{U} \ . \tag{3.25}$$

If I is the unit operator, it follows from Eqs. (3.24) and (3.25) that

$$\mathcal{U}^\dagger\mathcal{U} = I \ , \tag{3.26a}$$

$$\mathcal{U}\mathcal{U}^\dagger = 1 - P_0 \ , \tag{3.26b}$$

where P_0 is the vacuum projector. Thus the operator $\mathcal{U}$ is "not quite" unitary, and herein lies the need for some care.

It is convenient at this junction to reintroduce the conventional magnetic quantum number by means of the formula [cf. Eq. (2.5)]

$$\hat{m} = -\tfrac{1}{2}\Omega + \hat{n} \tag{3.27a}$$

and to change the definition of the vacuum state,

$$|0)_m = (\mathcal{U})^{(1/2)\Omega}|0) \ . \tag{3.27b}$$

It follows that

$$|n) \equiv |m) = (\mathcal{U})^m|0)_m \ . \tag{3.28}$$

If we substitute Eqs. (3.24) and (3.27a) into Eqs. (2.8) and (2.9), with $j = \frac{1}{2}\Omega$, the results are

$$J_0 = \hat{m} \ , \tag{3.29a}$$

$$J_+ = \mathcal{U}[(\tfrac{1}{2}\Omega+\hat{m}+1)(\tfrac{1}{2}\Omega-\hat{m})]^{1/2} \ . \tag{3.29b}$$

Equations (3.29a) and (3.29b), acting on the basis (3.28), constitute a simple, rigorous transformation of the Holstein-Primakoff mapping. To implement the formulations and to make contact with previous work, we go to a representation first utilized by Nodvik (1969) and studied further by Marshalek (1976b): we map the physical subspace to the $2j+1$ functions $(\phi|m)$, $-\pi \le \phi \le \pi$,

$$(\phi|m) = \frac{1}{\sqrt{2\pi}}\exp(im\phi) \ . \tag{3.30}$$

It follows from the given formulas [except Eq. (3.26b), see below], that on this space of states the operators $\mathcal{U}$ and $\hat{m}$ are realized by the mappings

$$\mathcal{U} \to \exp(i\hat{\phi}) \ , \tag{3.31a}$$

$$\hat{m} \to -id/d\phi \ . \tag{3.31b}$$

Thus when Eq. (3.31) is substituted into (3.29), we obtain operators that, acting on the periodic functions (3.30), have the following desirable properties: they reproduce the well-known angular momentum matrix elements with which we began this review; they yield an equivalent of the Klein-Dreizler form (3.19)–(3.21) (after a cyclic permutation of axes); they can be transformed to the Nodvik form for J_+, namely,

$$J_+ = \exp(\tfrac{1}{2}i\phi)[\tfrac{1}{2}(\Omega+1)^2 - \hat{m}^2]^{1/2}\exp(\tfrac{1}{2}i\phi) \ . \tag{3.29c}$$

Though all previous results have thus been found, the mapping (3.31a) has at least one fault, in that it does not satisfy Eq. (3.26b). However, this blemish can be removed by replacing Eq. (3.31a) by the mapping

$$\mathcal{U} \to (I - P_0)\exp(i\hat{\phi}) \ . \tag{3.31c}$$

At first sight this would seem to lead to a violation of Eq. (3.26a), but as long as $\exp(i\hat{\phi})$ acts first on any state in the physical subspace, including the vacuum, the projector may be dropped since the result is orthogonal to the vacuum, and thus (3.26a) is in fact satisfied. For the same reason, the projector may be dropped from Eqs. (3.29b) and (3.29c), and we regain the old results.

We refer the reader to the literature for further discussion of the use of action-angle variables in quantum

mechanics (Carruthers and Nieto, 1968; Moshinsky and Seligman, 1978, 1979; Deenen, Moshinsky, and Seligman, 1980; Newton, 1980; Oliveira and Malta, 1984; Leacock and Padget, 1987). In the material immediately following we illustrate a semiclassical approximation without the use of action-angle variables.

D. Semiclassical approximation

We have thus far indicated how to discuss the LMG model in both the weak-coupling and the strong-coupling limits with the help of boson mappings. There is one further limit where, surprisingly, progress can be made. This is the semiclassical limit (Klein and Li, 1981). Referring to Eqs. (3.10) and (3.11), the semiclassical limit corresponds to the domain of large x and large p. In terms of the quantities defined in Eq. (3.11), we now have $x \sim 1$ and $p \sim \Omega^{1/2}$. This means that in Eq. (3.12) only the zero-point energy can be expanded compared to everything else and the mathematical situation appears to be hopeless. The special structure of the Hamiltonian (3.10) together with Eqs. (2.29) and (2.30) allows us, however, to approximate (3.10) by the following expression, which correctly includes terms of order Ω and unity (i.e., sufficient to yield WKB accuracy) and is straightforwardly obtained:

$$H=\epsilon(-\tfrac{1}{2}\Omega+b^\dagger b)+\tfrac{1}{2}V[b^\dagger b^\dagger(\Omega-b^\dagger b)+\text{H.c.}] -\tfrac{1}{2}V(b^\dagger b^\dagger+bb)+O(1/\Omega)\ . \tag{3.32}$$

The result, Eq. (3.22), leads under the transformation (3.11) to a Schrödinger equation, albeit of fourth order in the momenta as well as in the coordinates. This is no hindrance, however, to treating it by the WKB approximation using Keller's method (Keller, 1958). Notice that the error of the Hamiltonian itself is beyond the WKB order, since the first two orders in Ω^{-1} are correctly included (Ω^{-1} here plays the role of $\hbar$). The impressive results found for $N=30$ are reproduced in Table I. Here $2J=N$, and δ was defined after Eq. (3.7). The levels are symmetrical with respect to zero energy and therefore only half are listed. For $\delta=5$, we are well into the strong-coupling region and thus the doublet structure of the energy levels of a symmetric double well are clearly in evidence.

E. Boson-quasifermion mapping

The model and the applications of it described above are far too restricted to accord with reality. The Hamiltonian (3.5) describes a set of uncoupled bands. The applications described thus far have all referred to the ground-state band of the system with Ω particles, i.e., exactly half the total single-particle degeneracy. this is the prototype of a close-shell nucleus. We shall now turn to the consideration of an extended description which subsumes at once all the bands implied in the LMG model. Moreover, this will enable us to describe the cases in which several particles or holes are added to the closed shell of the LMG model. The idea about to be described was first proposed within the context of the theory of pairing correlations (see the next section) by Suzuki and Matsuyanagi, 1976; see also Geyer and Hahne, 1980a, 1983; Marshalek, 1981; Matsuyanagi, 1982) and termed the "quantized Bogoliubov transformation." (The reason for this name will be explained in Sec. IV.E.) The development that follows is isomorphic in physical content to that given by Suzuki and Matsuyanagi. The derivation will be distinct, however, in that they applied the Marumori-Yamamura-Tokunaga method (to eventually

TABLE I. Negative energy eigenvalues (in units of ϵ) of the Lipkin-Meshkov-Glick Hamiltonian for $\delta=0.6$, 1, and 5 for $J=15$ compared with the results of the WKB quantization. From Klein and Li, 1981.

	$\delta=0.6$		$\delta=1$		$\delta=5$	
n	WKB	Exact	WKB	Exact	WKB	Exact
0	−15.100	−15.094	−15.359	15.314	−38.025	−38.049
1	−14.263	−14.258	−14.808	−14.800	−38.025	−28.049
2	−13.382	−13.378	−14.094	−14.087	−31.410	−31.436
3	−12.464	−12.461	−13.275	−13.269	−31.410	−31.436
4	−11.517	−11.514	−12.375	−12.370	−25.388	−25.422
5	−10.543	−10.540	−11.411	−11.407	−25.388	−25.419
6	−9.546	−9.544	−10.396	−10.392	−20.047	−20.142
7	−8.531	−8.529	−9.337	−9.334	−20.047	−20.053
8	−7.498	−7.497	−8.241	−8.239	−15.704	−16.135
9	−6.452	−6.451	−7.116	−7.114	−15.704	−15.244
10	−5.394	−5.393	−5.966	−5.964	−12.805	−12.633
11	−4.3264	−4.3256	−4.796	−4.795	−10.457	−10.466
12	−3.2512	−3.2507	−3.610	−3.609	−7.955	−7.933
13	−2.1705	−2.1702	−2.4131	−2.4125	−5.354	−5.348
14	−1.0862	−1.0860	−1.2084	−1.2081	−2.691	−2.688
15	0.0000	0.0000	0.0000	0.0000	0.000	0.000

extract a Holstein-Primakoff representation), whereas we shall apply an algebraic technique based on satisfying the commutation relations and passing through the intermediary of the Dyson mapping, a method we favor frequently in this review.

Let us begin by generalizing the boson description of the ground state given in Eqs. (2.8)–(2.11), where $j=\frac{1}{2}\Omega$. The simplest class of excited states for $N=\Omega$ are those built on one hole in the lower level and one particle in the upper one. In the fermion space such states have the form

$$|m;m'\rangle=\alpha_m^\dagger\beta_{m'}^\dagger|0\rangle \ . \tag{3.33}$$

We should like to carry out a mapping

$$|m;m'\rangle\rightarrow|m;m') \ , \tag{3.34}$$

analogous to Eq. (2.14), for each m and m' and then construct an irrep of SU(2) with Eq. (3.34) as a reference state,

$$\begin{aligned}|\nu;m;m')&=\frac{1}{\sqrt{\nu!}}(b^\dagger)^\nu|m;m')\\&\equiv|j=\tfrac{1}{2}(\Omega-2);J_0\\&=-\tfrac{1}{2}(\Omega-2)+\nu;m,m') \ ,\end{aligned} \tag{3.35}$$

i.e., the irrep has $j=\frac{1}{2}(\Omega-2)$ because the two orbits m and m' ($m\neq m'$ or $m=m'$) have been rendered inert and $(J_0)_{\max}=\frac{1}{2}(\Omega-2)$.

There are two related conceptual problems that must be solved before the program just described can be carried out: (i) how to take into account our inability to define an excited-state band for every state (3.33), since according to Eq. (3.2), setting $m=m'$ and summing generates a state in the ground-state band and (ii) how to realize the mapping (3.34). In this connection, recall that the fermion operators, α_m, etc., do not commute with the SU(2) generators. Indeed, one has

$$[\alpha_m,J_0]=\tfrac{1}{2}\alpha_m, \quad [\beta_m,J_0]=\tfrac{1}{2}\beta_m \quad \text{(and H.c. Eqs.)} \ , \tag{3.36}$$

$$[\alpha_m,J_+]=\beta_m^\dagger, \quad [\beta_m,J_+]=-\alpha_m^\dagger \quad \text{(and H.c. Eqs.)} \ . \tag{3.37}$$

It also follows that these fermion operators do not commute with the bosons.

As we shall show, the above requirements can be satisfied by replacing the given fermions by a set of ideal (quasi)fermions, $\alpha_m\rightarrow a_m$, $\beta_m\rightarrow b_m$, whose properties must include the following: (i) The mapping (3.34) is realized by the equations

$$|m;m')=a_m^\dagger b_{m'}^\dagger|0) \ , \tag{3.38}$$

$$a_m|0)=b_m|0)=0 \ . \tag{3.39}$$

(ii) The states (3.38) are all orthogonal to the ground-state band. This is guaranteed by requiring that the quasifermions commute with the bosons. (iii) the states (3.38) generate the correct number of linearly independent two-quasiparticle bands and at the same time no linear combination of these states belongs to the ground-state band. This can be satisfied by the operator condition

$$\sum_m a_m^\dagger b_m^\dagger=\sum_m b_m a_m=0 \ . \tag{3.40}$$

Operators satisfying such a constraint cannot be garden-variety fermions but must satisfy a new algebra. Nevertheless, it is an interesting consequence of our approach that the discovery of this algebra can be postponed until the end of the considerations. The reason for this is the following: we introduce a formal quasifermion number operator $\hat{n}$ (the boson number in Sec. II will henceforth be written $\hat{N}_B$),

$$\hat{n}=\sum_m(a_m^\dagger a_m+b_m^\dagger b_m) \ , \tag{3.41}$$

and assume that whatever the algebra of the quasifermions, the very convenient relations

$$[a_m,\hat{n}]=a_m, \quad [b_m,\hat{n}]=b_m \quad \text{(and H.c. Eqs.)} \tag{3.42}$$

continue to hold, the same as for normal fermions. This assumption will turn out to be a consistent one and is all that one needs for most of the computations.

The most important further consequence of this assumption is that the boson mappings of Sec. II are easily generalized to boson-fermion mappings applicable not only to the set of irreps associated with Eq. (3.38) but with representations associated with any allowed set of inert particles. Of the various mapping methods utilized in Sec. II, we shall here follow the one of first obtaining the generalized Dyson form and then unitarizing it. The SU(2) mapping can be written

$$(J_0)_D=-\hat{J}+B^\dagger B \ , \tag{3.43}$$

$$(J_-)_D=\Omega^{1/2}B \ , \tag{3.44}$$

$$(J_+)_D=\Omega^{-1/2}B^\dagger[2\hat{J}-B^\dagger B] \ , \tag{3.45}$$

where the only difference compared to Sec. II, other than the capitalization of the boson operators, is that the value j of the angular momentum has been replaced by a Hermitian operator

$$\hat{J}\equiv\tfrac{1}{2}(\Omega-\hat{n}) \ , \tag{3.46}$$

which commutes with the bosons and becomes a c-number with the correct value of the angular momentum in any irrep in which the states are formed as has been described. From Eq. (3.42) it follows that

$$[\hat{J},a_m^\dagger]=-\tfrac{1}{2}a_m^\dagger, \quad [\hat{J},b_m^\dagger]=-\tfrac{1}{2}b_m^\dagger \ , \quad \text{(and H.c. Eqs.)} \ . \tag{3.47}$$

To describe interband transitions, we must also map operators other than the SU(2) generators and, in particular, the single-fermion operators. Here we must recognize a property of the quasifermions that distinguishes

them from the fermions. The latter can change the angular momentum by $\pm\frac{1}{2}$ unit. However, according to Eq. (3.47), $a_m^\dagger$ and $b_m^\dagger$ can only decrease the angular momentum by $\frac{1}{2}$ unit each, while a_m, b_m accomplish the corresponding increase. Thus even if we dictate, by analogy with Eq. (3.44), that α_m and β_m carry the simplest possible operator structure, the minimum number of terms is two, each of which must separately satisfy Eq. (3.36). We write

$$(\alpha_m)_D = a_m + \sqrt{\Omega} f(\hat{n}) b_m^\dagger B \ , \tag{3.48}$$

$$(\beta_m)_D = b_m - \sqrt{\Omega} g(\hat{n}) a_m^\dagger B \ . \tag{3.49}$$

Here an overall scale factor has been chosen by requiring the fermion and quasifermion operators to coincide in the absence of bosons. That f and g must be independent of the boson number operator follows from Eqs. (3.44), (3.36), and (3.37). The commutation relations (3.37) then provide the following representations:

$$(\alpha_m^\dagger)_D = (2\hat{J}+1-\hat{N}_B) g a_m^\dagger + \Omega^{-1/2} b_m B^\dagger \ , \tag{3.50}$$

$$(\beta_m^\dagger)_D = (2\hat{J}+1-\hat{N}_B) f b_m^\dagger - \Omega^{-1/2} a_m B^\dagger \ . \tag{3.51}$$

We now assert that

$$f(\hat{n}) = g(\hat{n}) = (\Omega - \hat{n} + 1)^{-1} = (2\hat{J}+1)^{-1} \ . \tag{3.52}$$

This follows from the requirements

$$(\beta_m^\dagger \beta_m)_0 = (b_m^\dagger b_m)_0 \ , \tag{3.53}$$

$$(\alpha_m^\dagger \alpha_m)_0 = (a_m^\dagger a_m)_0 \ , \tag{3.54}$$

where the notation designates the subspace with no bosons present. These requirements follow from the structure of the basis states, of which Eq. (3.38) is an example, plus the commutation relations (3.42). It is then straightforward to verify that the fermion anticommutation relations are satisfied provided the quasifermion operators satisfy the "anomalous" anticommutators

$$\{a_m, b_{m'}^\dagger\} = f(\hat{n}) b_m^\dagger a_{m'} \ , \tag{3.55}$$

$$\{a_m, a_{m'}^\dagger\} = \delta_{mm'} - f(\hat{n}) b_m^\dagger b_{m'} \ , \tag{3.56}$$

$$\{b_m, b_{m'}^\dagger\} = \delta_{mm'} - f(\hat{n}) a_m^\dagger a_{m'} \ . \tag{3.57}$$

The remaining anticommutators, except for the Hermitian conjugate of Eq. (3.55), all vanish.

We can summarize the results found for the "singles" by the mapping equations

$$(\alpha_m)_D = a_m + \Omega^{1/2} B (2\hat{J}+1)^{-1} b_m^\dagger \ , \tag{3.58}$$

$$(\beta_m)_D = b_m - \Omega^{1/2} B (2\hat{J}+1)^{-1} a_m^\dagger \ , \tag{3.59}$$

$$(\alpha_m^\dagger)_D = (2\hat{J}+1-\hat{N}_B)(2\hat{J}+1)^{-1} a_m^\dagger + \Omega^{-1/2} B^\dagger b_m \ , \tag{3.60}$$

$$(\beta_m)^{\dagger_D} = (2\hat{J}+1-\hat{N}_B)(2\hat{J}+1)^{-1} b_m^\dagger - \Omega^{-1/2} B^\dagger a_m \ . \tag{3.61}$$

Surprisingly, the derivation has been carried through without allusion to Eq. (3.40). The latter does play a role, however, in the verification of the consistency requirement that when we form the SU(2) generators from Eqs. (3.58)–(3.61) we thereby recover the mapping (3.43)–(3.45).

The last step in the technical development is the unitarization of the mapping (3.43)–(3.45) and (3.58)–(3.61). The algebraic technique, to be exhibited in sufficient detail below, will again prove a model for much of our later work. Let us first consider Eqs. (3.43)–(3.45). We utilize the technique of the S operator introduced in Eq. (2.52) and sequel. This first of all requires

$$S(J_0)_D S^{-1} = (J_0)_B = (J_0)_D \ , \tag{3.62}$$

so that S commutes with $(J_0)_D$ and is thus a function of $\hat{N}_B$ and $\hat{n}$ only. Next come the same steps as carried out in Eqs. (2.56)–(2.59), which need not be repeated. The result is

$$(J_+)_B = (J_-)_B^\dagger = B^\dagger \sqrt{2\hat{J} - \hat{N}_B} \ . \tag{3.63}$$

Turning to Eqs. (3.58)–(3.61), the analog of Eq. (2.56) is

$$V^{-1} (\alpha_m)_D^\dagger V = (\alpha_m^\dagger)_D \ , \tag{3.64}$$

which leads directly to the equations

$$S^{-1}(\hat{N}_B, n+1) S(\hat{N}_B, n) = [(2\hat{J} - \hat{N}_B)/2\hat{J}]^{1/2} \ , \tag{3.65}$$

$$S^{-1}(\hat{N}_B + 1, n-1) S(\hat{N}_B, n) = [(2\hat{J}+1)/\Omega]^{1/2} \ . \tag{3.66}$$

To use these relations, we write the direct mapping conditions, for example,

$$S(\alpha_m)_D S^{-1} = (\alpha_m)_B \ , \tag{3.67}$$

$$S(\alpha_m^\dagger)_D S^{-1} = (\alpha_m^\dagger)_B = (\alpha_m)_B^\dagger \ . \tag{3.68}$$

These equations imply both Eq. (3.64), already exploited, and the determining relation

$$S^{-1}(\alpha_m)_D^\dagger S = (\alpha_m^\dagger)_B = \phi_1 a_m^\dagger + B^\dagger b_m \phi_2 \ , \tag{3.69}$$

where the assumption of the form of Eq. (3.69) reflects again the fact that there are only two independent tensors that contribute, or equivalently the two independent matrix elements (3.65) and (3.66). Utilizing these latter conditions, we determine

$$\phi_1 = [(2\hat{J}+1-\hat{N}_B)/(2\hat{J}+1]^{1/2} \ , \tag{3.70}$$

$$\phi_2 = (2\hat{J}+1)^{-1/2} \ , \tag{3.71}$$

and

$$(\alpha_m^\dagger)_B = [(2\hat{J}+1-\hat{N}_B/(2\hat{J}+1)]^{1/2} a_m^\dagger + B^\dagger b_m (2\hat{J}+1)^{-1/2} \ , \tag{3.72}$$

$$(\beta_m^\dagger)_B = [(2\hat{J}+1-\hat{N}_B)/(2\hat{J}+1)]^{1/2} b_m^\dagger - B^\dagger a_m (2\hat{J}+1)^{-1/2} \ . \tag{3.73}$$

Applications of this formalism will be given in the next section.

IV. APPLICATION OF SU(2) MAPPINGS TO PAIRING PROBLEM

A. Pairing vibrations

The Lipkin model discussed in the previous section may be considered to be a (not very realistic) model of a monopole excitation or other giant resonance at closed shells. Somewhat more realistic applications of the SU(2) mappings may be made by consideration of the pairing interaction. The most important application, to nuclear superconductivity, will be taken up in Sec. IV.C and sequels. The most direct application is to what are dubbed pairing vibrations, which we shall consider first. For our purposes it is convenient to follow the schematic account of Kleber (1969). For a more exhaustive treatment of the theory and comparison with experiment, see Broglia and Sorensen (1968); Jolos (1969, 1971); Sorensen (1969); Jolos, Dönau, Kartavenko, and Janssen (1973).

For simplicity, consider a two-level model (Högaasen-Feldman, 1961) with pairing force interaction

$$H=e_1N_1+e_2N_2-G\sum_{i,j=1}^{2}A_i^\dagger A_j\ , \tag{4.1}$$

where $G\geq 0$, $e_1<e_2$, and both levels are assumed to have the same degeneracy $2\Omega=2j+1$. The operators N_i and $A_i^\dagger$ are defined in terms of the individual fermion operators, $\alpha_{im}^\dagger$,

$$\begin{aligned} N_i&=\sum_m \alpha_{im}^\dagger\alpha_{im}\ ,\\ A_i^\dagger&=(A_i)^\dagger=\sum_{m>0}(-1)^{j-m}\alpha_{im}^\dagger\alpha_{i,-m}^\dagger\ . \end{aligned} \tag{4.2}$$

The phase factor $(-1)^{j-m}$ is proportional to the Clebsch-Gordan coefficient for coupling the two-fermion creation operators to angular momentum zero. The product $(-1)^{j-m}\alpha_{i,-m}^\dagger$ also creates the time-reversed orbit to $\alpha_{i,m}^\dagger$. (The association of operators with time-reversed orbits continues to be the appropriate method for describing pairing for nonspherical systems.)

For each value of i, we have an SU(2) algebra with the identification $J_{i+}=A_i^\dagger$, $J_{i0}=-\frac{1}{2}(\Omega_i-N_i)$. We confine ourselves to a particle number $N=2\Omega$ simulating a closed-shell nucleus like ^{208}Pb. The ground state in the absence of interaction would be

$$|0\rangle_F=(\Omega!)^{-1/2}(A_1^\dagger)^\Omega|\text{vac}\rangle_F\ , \tag{4.3}$$

i.e., a state with the first level fully occupied. The processes of interest to us are those in which we add a pair of particles coupled to angular momentum zero to the ground state described by Eq. (4.3) (two-particle stripping) or remove such a pair of particles (two-particle pickup). We ask to what extent one can use the boson picture to describe these processes. To answer this question, Kleber carried out a normal-ordered Holstein-Primakoff mapping for each SU(2) [see the discussion following Eq. (2.24) for a definition of this mapping and its limitations]. It is thus unnecessary to repeat the details of this mapping except for one important change. For index 2, the mapping is as before, i.e., $J_{2+}\rightarrow b_2^\dagger$, and

$$|p_2\rangle_F=\mathcal{N}_2(A_2^\dagger)^p|0\rangle_F\rightarrow|p_2)_B=(p!)^{-1/2}(b_2^\dagger)^p|0)_B\ , \tag{4.4}$$

$\mathcal{N}_2=[(\Omega-p)!/\Omega!p!]^{1/2}$, but in view of the structure of Eq. (4.3), the mapping for index 1 must be made with $A_1\rightarrow\overline{A}_1^\dagger$, i.e., one interchanges particles and holes. With that interchange the formulas described in Sec. II carry over. They can be summarized as follows:

$$\begin{aligned} (\overline{A}_i^\dagger)_B&=\sum_{k=0}^{\infty}c_k(b_i^\dagger)^{k+1}b_i^k\ ,\\ c_k&=\sum_{n=0}^{k}(-1)^n(\Omega-k+n)^{1/2}/n!(k-n)!\ , \end{aligned} \tag{4.5}$$

$$(N_1)_B=2(\Omega-b_1^\dagger b_1),\quad (N_2)_B=2b_2^\dagger b_2\ . \tag{4.6}$$

The series (4.5) should converge rapidly as long as $(\langle b_i^\dagger b_i\rangle/\Omega)$, $i=1,2$ are small, i.e., as long as we study nuclei only a few pairs away from the closed shell. It is then sufficient to carry the expansion of the Hamiltonian to fourth order only. We thereby find

$$\begin{aligned} H_B\cong H_B^{(2)}&+G(b_1^{\dagger 2}b_1^2+b_2^{\dagger 2}b_2^2)\\ &+G[\Omega-(\Omega(\Omega-1))^{1/2}][(b_2^\dagger b_2+b_1^\dagger b_1)b_1b_2+\text{H.c.}]\ , \end{aligned} \tag{4.7}$$

$$\begin{aligned} H_B^{(2)}=\langle 0|H|0\rangle_F&+2Gb_1^\dagger b_1-\Omega G(b_1^\dagger b_2+b_2^\dagger b_1)\\ &-(2e_1+\Omega G)b_1^\dagger b_1+(2e_2-G\Omega)b_2^\dagger b_2\ . \end{aligned} \tag{4.8}$$

Diagonalization of this Hamiltonian was carried out for $N=6$ and $N=40$. The fourth-order terms are essential for precise agreement with the exact eigenvalues (as we also found for the Lipkin model). This implies that the addition and subtraction eigenbosons are (slightly) nonlinear transforms of the kinematic bosons. Nevertheless a linear transformation of the type described by Eq. (3.8) yields qualitatively correct results for these pair-transfer degrees of freedom.

These conclusions are illustrated in Fig. 1, which contains two sets of plots, for $\Omega=3$ and $\Omega=20$, respectively, of the energy of the first excited state as a function of the pairing strength. Five values are given in each case: The letter A designates the exact result, B the value obtained by diagonalizing Eqs. (4.7) and (4.8), and C and D further approximations to Eqs. (4.7) and (4.8), whose exact definition need not concern us here, except that they ignore some terms of higher order in Ω^{-1}. These approximations should be justified for Ω large enough, as is evi-

dent from the graphs in Fig. 1. Also shown is the RPA, which breaks down at the superconducting transition. By contrast, the value of the boson mapping, at least for the pairing interaction, was clearly established.

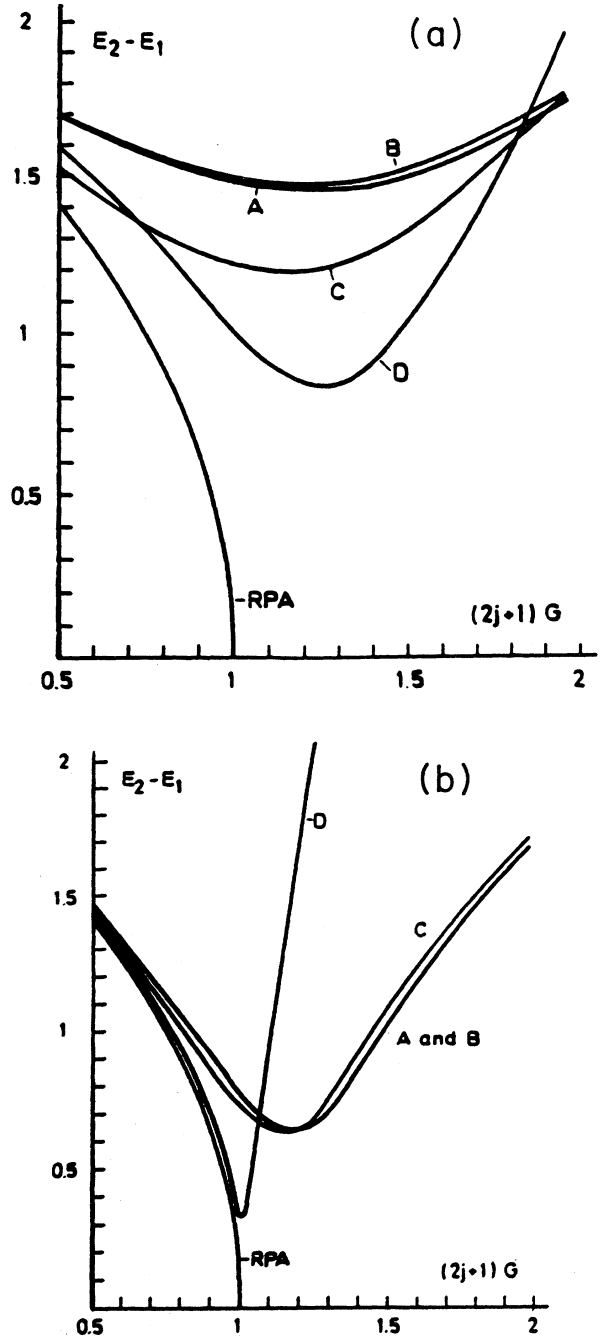

FIG. 1. Lowest excitation energy for the symmetric two-level model ($\Omega_1=\Omega_2=\Omega$) with pairing force interaction. (a) $N=2\Omega=6$. All values are in units of the single-particle excitation $\epsilon_2-\epsilon_1$: A, exact solution; B, fourth-order expansion; RPA, nonsuperconducting branch of the random-phase approximation; C and D are approximations of no interest for the present discussion. (b) Same as (a) but $N=40$. From Kleber, 1969.

B. Variational method for choosing collective variables

We turn next to an application that illustrates the problem of choosing collective degrees of freedom from a larger space of fermion-pair variables (Klein and Vallières, 1981). We again take the Hamiltonian (4.1) but now extended over three nondegenerate levels e_i, $i=1,2,3$, of different pair multiplicity Ω_i. Once more we carry out a normal-ordered Holstein-Primakoff mapping, restricted, however, to the vacuum state as reference state, so that only Eq. (4.6) needs to be modified to

$$(N_i)_B=2b_i^\dagger b_i \ . \tag{4.9}$$

We compute the Hamiltonian again to fourth-order terms.

Now let us look for new bosons $S_\lambda^\dagger$, $\lambda=1,2,3$, related to the $b_i^\dagger$ by a linear transformation with coefficients $\alpha_{i\lambda}$,

$$b_i^\dagger=\sum_\lambda \alpha_{i\lambda}S_\lambda^\dagger \ , \tag{4.10}$$

and assume that the lowest-lying states can be constructed largely from the basis states generated by two of these new bosons, $S=S_1$ and $T=S_2$ (Cooper pair and "pairing vibration," respectively). We thus truncate the sum (4.10), writing $\alpha_{i1}=\alpha_i$, $\alpha_{i2}=\beta_i$ and imposing orthonormalization conditions

$$\sum_i \alpha_i^\dagger\alpha_i=\sum_i \beta_i^\dagger\beta_i=1 \ , \tag{4.11}$$

$$\sum_i \alpha_i^\dagger\beta_i=\sum_i \beta_i^\dagger\alpha_i=0 \ . \tag{4.12}$$

How shall one determine the transformation (4.10)? Klein and Vallières applied a variational condition requiring that in an average sense the states constructed from $S^\dagger$, $T^\dagger$, i.e., from the basis

$$\frac{(S^\dagger)^{N-n}}{[(N-n)!]^{1/2}}\frac{(T^\dagger)^n}{(n!)^{1/2}}|0) \ , \tag{4.13}$$

lie lower in energy than any other eigenstates. (Notice that, in this example, the choice of a number-conserving Hamiltonian is merely a convenience, whereas in the treatment of pairing vibrations it was a natural choice dictated by the physics.) Since the average energy of a set of states is proportional to the trace of the Hamiltonian over the subspace considered, a necessary condition for determining this subspace is

$$\delta(\mathrm{Tr}H)=0 \ . \tag{4.14}$$

The trace is over the eigenstates, which are linear combinations of (4.13), but by the invariance of the trace can be taken over the set (4.13) itself. This simplifies the calculation, as only terms of H diagonal in the basis (4.13) need be retained. The variational principle subject to the constraints (4.11) and (4.12) yield nonlinear algebraic equations to determine α_i, β_i. This procedure determines a definite approximate collective Hamiltonian under the

transformation (4.10), which then can be further diagonalized in the basis (4.13). From the results found in this and other examples (Cohen and Klein, 1981), the method appears to be promising. To achieve its full potential, however, requires a generalization of Eq. (4.10) to nonlinear transformations. First efforts in this direction (Pannert and Ring, 1987) will be described in Sec. XVI.

It is apparent that the application of the trace variational principle is not restricted to the boson space. It was first conceived for application in this space for the purely technical reason that the traces are more easily evaluated for the Heisenberg-Weyl algebra than for the Lie algebras relevant to shell-model calculations with fermions.

C. Derivation of the BCS theory from a number-conserving boson Hamiltonian

We next show how to utilize a Holstein-Primakoff-mapped Hamiltonian in order to derive the number-conserving version of the BCS theory of superconductivity. What is suggested here is a way of exploiting the square-root operators that intervene in this mapping. Though deriving the BCS approximation involves only the crudest of approximations for these square roots, we shall briefly consider afterwards how the approximations can be improved.

Again consider the Hamiltonian (4.1) extended to an arbitrary number of levels. We apply a Holstein-Primakoff mapping for each SU(2) in the form ($j_a=\frac{1}{2}\Omega_a$)

$$N_a=2b_a^\dagger b_a,\quad A_a^\dagger=b_a^\dagger\sqrt{\Omega_a-b_a^\dagger b_a}\ . \tag{4.15}$$

We then study the problem by the equation of motion method:

$$\begin{aligned}[b_a,H]=&2e_a b_a-\sqrt{\Omega_a-b_a^\dagger b_a}\,\hat{\Delta}\\&-b_\alpha^\dagger\{\sqrt{\Omega_a-b_a^\dagger b_a-1}-\sqrt{\Omega_a-b_a^\dagger b_a}\}b_a\hat{\Delta}\\&-\hat{\Delta}^\dagger\{\sqrt{\Omega_a-b_a^\dagger b_a-1}-\sqrt{\Omega_a-b_a^\dagger b_a}\}b_a b_a\ ,\end{aligned} \tag{4.16}$$

where

$$\hat{\Delta}\equiv G\sum_a\sqrt{\Omega_a-b_a^\dagger b_a}\,b_a\ , \tag{4.17}$$

and evaluate Eq. (4.16) between states $|N-2\rangle$ and $|N\rangle$. Defining

$$\langle b_a\rangle\equiv\langle N-2|b_a|N\rangle\ , \tag{4.18a}$$

$$\Delta\equiv\langle N-2|\hat{\Delta}|N\rangle\equiv\langle\hat{\Delta}\rangle\ , \tag{4.18b}$$

where the states involved are the ground states of neighboring even nuclei, with energies $W(N-2)$ and $W(N)$ and with λ, the chemical potential, given by

$$2\lambda=W(N)-W(N-2)\ , \tag{4.19}$$

we assume that $\langle b_a^\dagger b_a\rangle=|\langle b_a\rangle|^2\sim N\gg 1$. We can then be rather cavalier in evaluating the right-hand side of Eq. (4.16), keeping the difference of the square roots to first order only and replacing $\Delta\langle b_a\rangle$ by $\langle b_a^\dagger\rangle\Delta$. The major approximation, however, is the replacement of $b_a^\dagger b_a$ by $\langle b_a^\dagger b_a\rangle$ under the square root. (This approximation is evaluated at the end of the discussion.)

From Eq. (4.16), with the help of the above approximations, we thus obtain

$$\begin{aligned}2\lambda\langle b_a\rangle=&2e_a\langle b_a\rangle-\sqrt{\Omega_a-\langle b_a^\dagger b_a\rangle}\,\Delta\\&+[\Omega_a-\langle b_a^\dagger b_a\rangle]^{-1/2}\langle b_a^\dagger b_a\rangle\Delta\ .\end{aligned} \tag{4.20}$$

This equation can be further rewritten ($\epsilon_a=e_a-\lambda$)

$$2\epsilon_a\kappa_a=(\Omega_a-\rho_a)\Delta\ , \tag{4.21}$$

where

$$\kappa_a=\langle b_a^\dagger\rangle\sqrt{\Omega_a-\langle b_a^\dagger b_a\rangle}\ , \tag{4.22}$$

$$\rho_a=2\langle b_a^\dagger b_a\rangle\ , \tag{4.23}$$

$$\Delta=G\sum_a\kappa_a\ . \tag{4.24}$$

Equation (4.21) is of standard BCS form. From Eqs. (4.22) and (4.23) one then derives the relation

$$\Omega_a-\rho_a=\sqrt{\Omega_a^2-4\kappa_a^2}\ . \tag{4.25}$$

From Eqs. (4.21) and (4.24) one finds

$$\kappa_a=\frac{\Omega_a\Delta}{2E_a}\ , \tag{4.25a}$$

$$E_a=\sqrt{\epsilon_a^2+\Delta^2}\ , \tag{4.25b}$$

and

$$\rho_a=\Omega_a(1-\epsilon_a/E_a)\ . \tag{4.26}$$

By combining Eqs. (4.24) and (4.26), one obtains

$$G=\sum_a\frac{\Omega_a}{2E_a}\ , \tag{4.27}$$

and from Eq. (4.26) one finds

$$N=\sum_a\Omega_a(1-\epsilon_a/E_a)\ . \tag{4.28}$$

We recognize in Eqs. (4.27) and (4.28) the standard equations of the BCS theory, determining the parameters Δ and λ.

How can we go beyond the crude approximations presented above? The trick is to write

$$b_a^\dagger b_a=\langle b_a^\dagger b_a\rangle+\delta(b_a^\dagger b_a) \tag{4.29}$$

and to expand the square root operators in powers of $\delta(b_a^\dagger b_a)$. The actual form of $\delta(b_a^\dagger b_a)$, useful within the context of the equations of motion method, is obtained by writing

$$b_a^\dagger b_a=\sum_{I,I'}\langle I|b_a^\dagger b_a|I'\rangle|I\rangle\langle I'|\ , \tag{4.30}$$

$$\langle b_a^\dagger b_a \rangle = \langle 0|b_a^\dagger b_a|0\rangle \sum_I |I\rangle\langle I| \ , \tag{4.31}$$

where $|I\rangle$ refers to a complete set of states of given N and angular momentum zero (more precisely, "seniority" zero), including the ground state $|0\rangle$. A program of this type combining the equation of motion method with the "quantized Bogoliubov transformation," introduced in the previous section, has been carried through (Hasegawa and Kanesaki, 1980a, 1980b, 1980c).

D. Coherent-state calculation

As the next application of this section, we shall give another derivation of the BCS theory with the help of the fermion coherent state and the associated Dyson realization of SU(2). The un-normalized BCS state has, in fact, the form

$$|\underline{z}\rangle = \exp\left[\sum_a z_a A_a^\dagger\right]|\text{vac}\rangle \ , \tag{4.32}$$

where the z_a are complex parameters. This is a fermion coherent state for SU(2)×SU(2)×$\cdots$. Now we obtain, by reasoning similar to that utilized in Eqs. (2.47)–(2.49),

$$A_a^\dagger|\underline{z}\rangle = (\partial/\partial z_a)|\underline{z}\rangle \equiv d_a|\underline{z}\rangle \ , \tag{4.33}$$

$$N_a|\underline{z}\rangle = 2z_a d_a|\underline{z}\rangle \ , \tag{4.34}$$

$$A_a|\underline{z}\rangle = z_a(\Omega_a - z_a d_a)|\underline{z}\rangle \ . \tag{4.35}$$

We summarize these formulas by the notation

$$\vec{D}(A_a^\dagger) = d_a \ , \tag{4.36a}$$

$$\vec{D}(N_a) = 2z_a d_a \ , \tag{4.36b}$$

$$\vec{D}(A_a) = z_a(\Omega_a - z_a d_a) \ . \tag{4.36c}$$

Similarly, when acting on $\langle\underline{z}| = \langle \text{vac}|\exp(\sum_a z_a^* A_a)$, we have the representation

$$\overleftarrow{D}(A_a) = d_a^* \ , \tag{4.37a}$$

$$\overleftarrow{D}(N_a) = 2z_a^* d_a^* \ , \tag{4.37b}$$

$$\overleftarrow{D}(A_a^\dagger) = z_a^*(\Omega_a - z_a^* d_a^*) \ . \tag{4.37c}$$

The average of any operator Θ is given by the equivalent formulas

$$\langle\Theta\rangle = \langle\underline{z}|\Theta|\underline{z}\rangle / \langle\underline{z}|\underline{z}\rangle = \vec{D}(\Theta)\mathcal{S}/\mathcal{S} = \mathcal{S}\overleftarrow{D}(\Theta)/\mathcal{S} \ , \tag{4.38}$$

where

$$\mathcal{S} \equiv \langle\underline{z}|\underline{z}\rangle \ . \tag{4.39}$$

It follows that the entire theory can be constructed from the scalar product $\mathcal{S}$.

We calculate $\mathcal{S}$ by applying Eq. (4.38) to the operator A_a with the help of Eqs. (4.36a) and (4.37a). First we evaluate directly

$$\mathcal{S}(\underline{z}) = \langle \text{vac}|\exp\left\{\sum_a [z_a^* z_a(\Omega_a - z_a d_a)]\right\}|\text{vac}\rangle \ . \tag{4.40}$$

The further evaluation of Eq. (4.40) is simplified by noting that from the definition (4.39) it follows that $\mathcal{S}(\underline{z})$ can depend only on the variables $|z_a|^2$. Thus we write, with $t_a = |z_a|^2$, as a definition of the function $g_a(t_a)$,

$$\mathcal{S} = \prod_a e^{g_a(t_a)} \ . \tag{4.41}$$

We then obtain a first-order differential equation for the function $g_a(t_a)$ by equating from Eq. (4.38) the two definitions of $\langle A_a\rangle$ (with prime meaning derivative),

$$z_a\Omega_a - z_a^2 z_a^* g_a'(t_a) = z_a g_a'(t_a) \ , \tag{4.42}$$

which easily yields the solution

$$\mathcal{S} = \prod_a (1+t_a)^{\Omega_a} \ . \tag{4.43}$$

From Eq. (4.43), we calculate, for example,

$$\langle N_a\rangle \equiv \rho_a = 2t_a\Omega_a/(1+t_a) \tag{4.44}$$

and

$$\langle A_a\rangle \equiv \kappa_a = z_a\Omega_a/(1+t_a) \ . \tag{4.45}$$

A short calculation shows that Eqs. (4.44) and (4.45) are related by Eq. (4.24) of BCS theory. Similarly, all the other results of BCS theory can be reproduced from the generating function $\mathcal{S}(\underline{z})$. In the above derivation, we have followed the reasoning of Dasso and Klein (1973), but similar methods, which have become very popular lately (Blaizot and Ripka, 1986), appeared quite early in nuclear physics, as we have already remarked following Eqs. (2.50) and (2.51).

E. Quantized Bogoliubov transformation

We turn now to some simple applications of the results of Sec. III.E, restricting ourselves to the same pairing Hamiltonian as has been discussed throughout Sec. III. The applications in this and in the following subsection are again meant to be partial and illustrative. Treatment of the full problem, with coupling of all modes, will be carried out in Secs. XI–XIII by an alternative, but theoretically equivalent, method. Here we follow the discussion of Suzuki and Matsuyanagi (1976).

We must first transcribe the results of Sec. III.E. Toward this end, we collapse the two levels of the LMG model into a single j level with $2\Omega = 2j+1$ and

$$\begin{aligned} &\alpha_m^\dagger \to \alpha_m^\dagger \ , \\ &\beta_m^\dagger \to (-1)^{j-m}\alpha_{-m}^\dagger = \tilde{\alpha}_m^\dagger \ . \end{aligned} \tag{4.46}$$

We replace $J_{\pm,0} \to S_{\pm,0}$ and $\hat{J} \to \hat{S}$, which is a change of notation compared to earlier parts of this section. Thus we have the following (Hermitian) mappings for the SU(2) generators:

$$(S_0)_B = -\hat{S} + b^\dagger b \ , \tag{4.47}$$

$$\hat{S} = \tfrac{1}{2}(\Omega - \hat{n}) \ , \tag{4.48}$$

$$\hat{n} = \sum_{m=-\Omega}^{\Omega} a_m^\dagger a_m \ , \tag{4.49}$$

$$(S_+)_B = (S_-)_B^\dagger = b^\dagger \sqrt{2\hat{S} - b^\dagger b} \ . \tag{4.50}$$

For the fermions, we have

$$\begin{aligned} \alpha_m^\dagger &= \sqrt{1 - b^\dagger b/(2\hat{S}+1)}\, a_m^\dagger + b^\dagger \tilde{a}_m \frac{1}{\sqrt{2\hat{S}+1}} \\ &\equiv \hat{u} a_m^\dagger + \tilde{a}_m \hat{v}^\dagger \ . \end{aligned} \tag{4.51}$$

Notice that the operators $\hat{u}$ and $\hat{v}$ satisfy

$$\hat{u}^\dagger \hat{u} + \hat{v}^\dagger \hat{v} = 1 \ . \tag{4.52}$$

The form of the operator identity (4.52) immediately suggests the name quantized Bogoliubov transformation for (4.51). As we shall consider below, this name is not completely apt.

Finally the quasiparticles satisfy the anticommutation relations

$$\{a_m, a_{m'}^\dagger\} = \delta_{mm'} - \tilde{a}^\dagger{}_m \tilde{a}_{m'} \frac{1}{2\hat{S}+1} \ . \tag{4.53}$$

The BCS limit of the pairing theory, as we shall verify below, is reached by the replacement (compare also Sec. IV.C)

$$b^\dagger \to \langle b^\dagger \rangle = \sqrt{2\hat{S}}\, v = \langle b \rangle \ . \tag{4.54}$$

Then Eq. (4.51) becomes

$$\begin{aligned} \alpha_m^\dagger &= \sqrt{1-v^2}\, a_m^\dagger + v\tilde{a}_m \ , \\ \tilde{\alpha}_m &= -v a_m^\dagger + \sqrt{1-v^2}\, \tilde{a}_m \ . \end{aligned} \tag{4.55}$$

This has the BCS form, but it must be remembered that the $a_m^\dagger$ cannot form $j=0$ pairs, whereas the Bogoliubov-Valatin quasiparticles do. In the present theory the $j=0$ pairs are constructed from the bosons.

To see this point, let us treat the Hamiltonian $\tilde{H} = H - \lambda N$, where H is given by Eq. (4.1) and we have an arbitrary number of single-particle levels. Thus

$$\begin{aligned} \tilde{H} &= \sum_j (e_j - \lambda) \sum_m \alpha_{jm}^\dagger \alpha_{jm} - G \sum_{jj'} \hat{S}_{j+} \hat{S}_{j'-} \\ &= \sum_j (e_j - \lambda)[\hat{n}_j + 2b_j^\dagger b_j] \\ &\quad - G \sum_{jj'} b_j^\dagger \sqrt{2\hat{S}_j - b_j^\dagger b_j} \sqrt{2\hat{S}_{j'} - b_{j'}^\dagger b_{j'}}\, b_{j'} \ , \end{aligned} \tag{4.56}$$

where we use the same symbol $\tilde{H}$ for the mapped operator. To study the superconducting phase transition, we write [cf. Eq. (4.54)]

$$b_j^\dagger = \sqrt{\Omega_a} v_a + \mathcal{B}_j^\dagger \quad \text{(and H.c. Eq.)} \ , \tag{4.57}$$

where the first term represents the mean field defined in Eq. (4.54) and the second the fluctuating field. With this assumption, we expand the square root up to first-order terms; the expansion parameter is $\Omega^{-1/2}$. We then obtain

$$\tilde{H} = W + H_{\text{intr}} + H_{\text{pair}} \ , \tag{4.58}$$

$$W = \sum_j 2(e_j - \lambda)\Omega_j v_j^2 - \Delta^2/G \ , \tag{4.59}$$

$$H_{\text{intr}} = \sum_j (e_j - \lambda + \frac{v_j}{u_j}\Delta)\hat{n}_j \ , \tag{4.60}$$

$$H_{\text{pair}} = H_{\text{pair}}^{(0)} + H_{\text{pair}}^{(1)} + H_{\text{pair}}^{(2)} \ , \tag{4.61}$$

$$\begin{aligned} H_{\text{pair}}^{(0)} = \sum_j \Big\{ & 2(e_j - \lambda)\mathcal{B}_j^\dagger \mathcal{B}_j + \frac{u_j v_j}{4u_j^2}\Delta[2(4u_j^2 + v_j^2)\mathcal{B}_j^\dagger \mathcal{B}_j \\ & + (2u_j^2 + v_j^2)(\mathcal{B}_j^\dagger \mathcal{B}_j^\dagger + \mathcal{B}_j \mathcal{B}_j)] \Big\} \ , \end{aligned} \tag{4.62}$$

$$H_{\text{pair}}^{(1)} = \sum_j \frac{\sqrt{\Omega_j}}{u_j}[2u_j v_j (e_j - \lambda) - (u_j^2 - v_j^2)\Delta](\mathcal{B}_j^\dagger + \mathcal{B}_j) \ , \tag{4.63}$$

$$\begin{aligned} H_{\text{pair}}^{(2)} = -G \sum_{jj'} & \frac{\sqrt{\Omega_j}}{2u_j} \frac{\sqrt{\Omega_{j'}}}{2u_{j'}} [(2u_j^2 - v_j^2)\mathcal{B}_j^\dagger - v_j^2 \mathcal{B}_j] \\ & \times [(2u_{j'}^2 - v_{j'}^2)\mathcal{B}_{j'} - v_{j'}^2 \mathcal{B}_{j'}^\dagger] \ , \end{aligned} \tag{4.64}$$

with $\Delta = G\sum_j \Omega_j u_j v_j$ and $u_j = \sqrt{1 - v_j^2}$.

At this stage the parameters u_j are determined either by minimizing W or equivalently by requiring the vanishing of the "dangerous diagrams" (4.63). By setting the square bracket in Eq. (4.63) to zero, we obtain the usual BCS conditions. Under these conditions the term H_{intr} reduces to

$$H_{\text{intr}} = \sum_j E_j \hat{n}_j, \quad E_j = \sqrt{(e_j - \lambda)^2 + \Delta^2} \ , \tag{4.65}$$

which implies that the energy of an ideal quasiparticle is the same as that of the Bogoliubov quasiparticle. Next we diagonalize $H_{\text{pair}}^{(0)}$ by the unitary transformation

$$\begin{aligned} B_j^\dagger &= \xi_j \mathcal{B}_j^\dagger + \eta_j \mathcal{B}_j \ , \\ \xi_j &= \tfrac{1}{2}(u_j^{-1} + u_j), \quad \eta_j = \tfrac{1}{2}(u_j^{-1} - u_j) \ . \end{aligned} \tag{4.66}$$

Then $H_{\text{pair}}^{(0)}$ becomes

$$H_{\text{pair}}^{(0)} = -\sum_j \frac{v_j^4}{2u_j^2} E_j + \sum_j 2E_j B_j^\dagger B_j \ , \tag{4.67}$$

and $H_{\text{pair}}^{(2)}$ reduces to

$$\begin{aligned} H_{\text{pair}}^{(2)} = -G \sum_{jj'} & \sqrt{\Omega_j}\sqrt{\Omega_{j'}}[u_j^2 B_j^\dagger - v_j^2 B_j] \\ & \times [u_{j'}^2 B_{j'} - v_{j'}^2 B_{j'}^\dagger] \ . \end{aligned} \tag{4.68}$$

The operator part of Eqs. (4.67) and (4.68) provides the harmonic approximation to the pairing vibrational modes, including a spurious zero-energy solution if $\Delta \neq 0$. The next order in the expansion of the square roots in

Eq. (4.56) will provide the anharmonic correction. Although a full and systematic treatment of the pairing interaction can be laid out along the lines so far indicated, we halt the general development at this point because of the study promised in Secs. XI–XIII. A special line of inquiry is developed below.

F. Pairing rotation

We consider the formalism of the previous section in more detail for the special case of a single level. In that case it is, of course, trivial to find the exact eigenvalues, and the procedure starting with Eq. (4.57) and culminating in Eqs. (4.66)–(4.68) is superfluous, for we then have

$$H=-G\hat{S}_+\hat{S}_- \\ =-G[\vec{S}^2-S_0^2+S_0]\ , \tag{4.69}$$

and

$$E(N,n)=-\tfrac{1}{4}G(N-n)(2\Omega-N-n+2)\ , \tag{4.70}$$

where N is the total number of particles and n the number of unpaired quasiparticles.

Nevertheless, we study the approximate method as an introduction to another treatment of the pairing theory embodied in the Hamiltonian (4.58). In that event, taking note of the fact that the single-particle energy $E=\frac{1}{2}G\Omega$, we have two terms of the order of $G\Omega$,

$$\tilde{H}=W+E\hat{n}+G[\sqrt{\Omega}uv(\mathcal{B}^\dagger+\mathcal{B})]^2 \\ =W+E\hat{n}+(\tilde{N}^{(1)})^2/2\mathcal{I}\ , \tag{4.71}$$

where $\mathcal{I}=2G^{-1}$ and

$$\tilde{N}^{(1)}=\sqrt{\Omega}uv(\mathcal{B}^\dagger+\mathcal{B})\ . \tag{4.72}$$

If we also introduce the operator

$$\phi^{(1)}=\frac{1}{2i\sqrt{\Omega}uv}(\mathcal{B}^\dagger-\mathcal{B})\ , \tag{4.73}$$

with the motivation that $\phi^{(1)}$ is canonically conjugate to $\tilde{N}^{(1)}$, then

$$[\phi^{(1)},\tilde{N}^{(1)}]=i\ , \tag{4.74}$$

and also

$$[\tilde{H},\tilde{N}^{(1)}]=0\ , \tag{4.75}$$

$$[\tilde{H},i\phi^{(1)}]=\tilde{N}^{(1)}/\mathcal{I}\ . \tag{4.76}$$

Thus $\tilde{N}^{(1)}$ resembles a two-dimensional angular momentum operator and the last term of Eq. (4.71) appears as a (pairing) rotational energy uncoupled from the quasiparticle energy. As we shall see below, $\tilde{N}^{(1)}$ also has the interpretation of number-fluctuation operator; since the average number has been fixed by the BCS calculation, one is tempted to try to define an improved ground state by the condition

$$\tilde{N}^{(1)}|\Psi_0^{(0)})=0\ . \tag{4.77}$$

As is well known (Marshalek and Weneser, 1969), however, Eq. (4.77) does not lead to a normalizable ground state. In the present case, this problem can be solved exactly, as we now show (Suzuki, Fuyuki, and Matsuyanagi, 1979a, 1979b).

Recalling the definition Eq. (4.57) of the script boson operators, we see that the BCS state in the boson space is a coherent state $|\mathrm{coh}\rangle$ defined by

$$\mathcal{B}|\mathrm{coh})=(b-\sqrt{\Omega}v)|\mathrm{coh})=0 \tag{4.78}$$

or

$$|\mathrm{coh})=\exp[(b^\dagger-b)\sqrt{\Omega}v]|0)\ . \tag{4.79}$$

In this case $\tilde{H}$ can be written (exactly)

$$\tilde{H}=W+E\tilde{n}+G(\delta N)^2+G\hat{n}\delta N\ , \tag{4.80}$$

where

$$\delta N=b^\dagger b-\Omega v^2=\sqrt{\Omega}v(\mathcal{B}^\dagger+\mathcal{B})+\mathcal{B}^\dagger\mathcal{B} \tag{4.81}$$

is the exact number-fluctuation operator. Now in terms of the operator $\Pi\equiv\delta N+\frac{1}{2}\hat{n}$, which differs only by a constant from half the total number of particles, $\tilde{H}$ may be rewritten as

$$\tilde{H}=W+E\hat{n}+(\Pi^2/2\mathcal{I})-\tfrac{1}{4}G\hat{n}^2\ . \tag{4.82}$$

Thus, in the exact case, as we expect, the coupling of the quasiparticles to the pairing rotation disappears when the latter is suitably defined.

One may be tempted to carry the discussion further by seeking an operator $\hat{\phi}$ canonically conjugate to II. Such an enterprise is subject to the same limitations mentioned in connection with the study of the transition-operator boson in Sec. III and will not be pursued here. One must also introduce the equivalent of a pairing rotation in the case of many levels. A general treatment will be found in Sec. XIII.

V. APPLICATION OF SU(2) MAPPINGS TO PHENOMENOLOGY

A. Equivalent descriptions of a breathing mode

For didactic purposes we shall invent nuclear properties to illustrate ideas which will have some application to the real nucleus, requiring only a change of mathematical details (Klein, Li, and Vallières, 1982a). Thus imagine that in some region of the periodic table even-even nuclei exhibit a spectrum of relatively low-lying 0^+ states, which we label $n=0,1,2,3,\ldots$, $n=0$ representing the ground state. These bands are observed to have the following properties: (i) The spacing is nearly harmonic. Thus the spacing E_n relative to the ground state is represented by the formula

$$E_n=a_1n+a_2n(n-1)+a_3n(n-1)(n-2)+\cdots\ , \tag{5.1}$$

where $a_1 \gg a_2 \gg a_3 \gg \cdots$. (ii) Monopole transitions between neighboring states are observed to be enhanced compared to single-particle values and depend on n in a characteristic way prescribed below. The monopole operator is defined by the expression

$$M = \sum_i (r^i)^2 \ . \tag{5.2}$$

The matrix element of M between neighboring states is given by the expansion

$$\langle n+1|M|n\rangle = M_0\sqrt{n+1}[1+q_1 n + q_2 n(n-1)+\cdots] \ , \tag{5.3}$$

where $1 \gg q_1 \gg q_2 \gg \cdots$ and $M_0 \gg \langle r^2\rangle$, with $\langle r^2\rangle$ a typical single-particle matrix element. (iii) Crossover intraband transitions are suppressed compared to the direct transitions described in (ii), i.e.,

$$|\langle n+1|M|n\rangle| \gg |\langle n+2|M|n\rangle| \ . \tag{5.4}$$

(iv) The relative change of the average mean-square radius in going from state to state in the band is small. (v) Interband monopole transitions are not much enhanced, if at all, over single-particle values.

Given the above "experimental data," it would not take us long to conclude that the systems under discussion could sustain a well-developed "breathing mode" of oscillation. The most obvious representation of the data is by means of a one-dimensional nearly harmonic oscillator, with variables $b, b^\dagger$,

$$[b, b^\dagger] = 1 \ , \tag{5.5}$$

such that the *exact eigenstates* are represented as uncoupled harmonic-oscillator states,

$$|n\rangle \rightarrow |n) = (n!)^{-1/2}(b^\dagger)^n|0)_b \ , \tag{5.6}$$

$$b|0)_b = 0 \ . \tag{5.7}$$

In terms of these variables, the Hamiltonian that yields the eigenvalues (5.1) is

$$H_{\rm coll} = a_1 b^\dagger b + a_2 (b^\dagger)^2 b^2 + a_3 (b^\dagger)^3 (b)^3 + \cdots \ . \tag{5.8}$$

The transition data, of which Eq. (5.3) is a sample, is reproduced by the operator

$$\begin{aligned} M = {} & M_1 b^\dagger [1 + q_1 b^\dagger b + q_2 (b^\dagger)^2 b^2 + \cdots] + \text{H.c.} \\ & + q_{11} b^\dagger b + \cdots + q_{20}[(b^\dagger)^2 + b^2] + \cdots \ , \end{aligned} \tag{5.9}$$

where the additional parameters $q_{11}\ldots$ have been included to describe a possible change in mean-square radius with excitation; the additional parameters $q_{20}\ldots$ represent crossover transitions when n changes by two or more units. Other, less "natural" choices of boson variables are possible, but will not be pursued here. The method of phenomenological description given above has actually been applied to the vibrational regime of the Bohr-Mottelson model (Ferreira, Alcarás Castilho, and Agiulera Navarro, 1964; Brink, de Toledo Piza, and Kerman, 1965; Das, Dreizler, and Klein, 1970; Bohr and Mottelson, 1975).

If the nuclei imagined above exhibited this breathing mode for a sequence of neighboring nuclei, we might expect the various phenomenological parameters to vary slowly and regularly from nucleus to nucleus. If, in addition to the breathing mode, the sequence of nuclei were also superconducting, we might also end up inventing a monopole version of the interacting boson model (Arima and Iachello, 1976a, 1976b, 1978, 1979). For this purpose we introduce two monopole bosons $s^\dagger$ and $t^\dagger$, where $s^\dagger$ creates the superconducting pair and $t^\dagger$ the monopole vibration. The latter differs from $b^\dagger$, which does not change the number of fermions, whereas $t^\dagger$ and $s^\dagger$ are each assumed to add two particles to the nucleus. Thus the sum

$$N = s^\dagger s + t^\dagger t \ , \tag{5.10}$$

which represents half the number of nucleons added to some reference nucleus, is conserved. We then postulate the simplest possible phenomenological Hamiltonian ($\epsilon_1 < \epsilon_2$) that commutes with N, namely,

$$\begin{aligned} H = {} & \epsilon_1 s^\dagger s + \epsilon_2 t^\dagger t + v_1 s^\dagger s^\dagger ss + v_{12} s^\dagger t^\dagger ts + v_2 t^\dagger t^\dagger tt \\ & + u(s^\dagger s^\dagger tt + t^\dagger t^\dagger ss) + w_1(s^\dagger t^\dagger ss + s^\dagger s^\dagger ts) \\ & + w_2(s^\dagger t^\dagger tt + t^\dagger t^\dagger ts) \ . \end{aligned} \tag{5.11}$$

Our interest in this model stems from the fact that H is a polynomial of degree two in the generators of the Lie algebra U(2), namely, $s^\dagger s$, $t^\dagger t$, $s^\dagger t$ and $t^\dagger s$. Under the given condition (5.10), the remaining linearly independent combinations may be identified with the generators of SU(2),

$$J_+ = (J_-)^\dagger = s^\dagger t \ , \tag{5.12a}$$

$$J_0 = \tfrac{1}{2}(s^\dagger s - t^\dagger t) \ , \tag{5.12b}$$

and if we use Eq. (5.10), H may be rewritten in terms of these. In Eq. (5.12) we recognize the Schwinger realization of SU(2). The best known example of a Schwinger realization of SU(n) occurs for $n=6$ within the context of the interacting boson model.

If we focus now on the Lie-algebraic structure of the model defined by Eq. (5.11), we would expect the leading terms of the monopole operator to have the form

$$M = \alpha(s^\dagger t + t^\dagger s) + \beta t^\dagger t \ , \tag{5.13}$$

namely a linear combination of generators. The final point in the definition of the model is that the eigenstates are obtained by diagonalizing the Hamiltonian (5.11) in the space specified by the basis

$$|n_2, n_1) = (n_2! n_1!)^{-1/2}(t^\dagger)^{n_2}(s^\dagger)^{n_1}|0) \ , \tag{5.14}$$

where $(n_1 + n_2) = N$ and $|0)$ is the "vacuum," the ground state of the reference nucleus, assumed to remain unexcited by the processes under discussion.

We come now to the main reason for including the

present discussions: We show that there is a one-to-one correspondence between the monopole interacting boson model we have been discussing and a suitably defined version of the model for a breathing mode introduced at the beginning of this section. This is established by mapping the Schwinger realization of SU(2) onto the Holstein-Primakoff realization according to the formulas [actually a mapping of U(2)]

$$t^\dagger t = b^\dagger b \ , \tag{5.15a}$$

$$t^\dagger s = b^\dagger (N - b^\dagger b)^{1/2} = (s^\dagger t)^\dagger \ , \tag{5.15b}$$

$$s^\dagger s = N - b^\dagger b \ , \tag{5.15c}$$

where N fixes the representation and is to be identified with the operator in Eq. (5.10) (Blaizot and Marshalek, 1978b; Klein, Li, and Vallières, 1982a).

The proof actually consists in showing that under the mapping (5.15) the basis (5.14) is converted into the basis (5.6). We define the basis vectors

$$|n_2, N) \equiv (n_2! N!)^{-1/2} (b^\dagger)^{n_2} (s^\dagger)^N |0)$$
$$\equiv (n_2!)^{-1/2} (b^\dagger)^{n_2} |0, N\rangle \ , \tag{5.16}$$

which are a set of states (5.6) (corresponding to a fixed N). Notice next that Eq. (5.15) may be inverted to yield a formula for $b^\dagger$,

$$b^\dagger = [N + 1 - t^\dagger t]^{-1/2} t^\dagger s \ . \tag{5.17}$$

If we substitute Eq. (5.17) into (5.16), using the commutation relations and the condition $s|0) = 0$, we are led immediately to the states (5.14). The transformation can also be performed in the reverse sense. The equivalence is predicted on the assumption that the breathing mode is described in a finite vector space whose size is that of the appropriate irrep of SU(2). There is thus, strictly, a mathematical distinction between the concept of the original Bohr-Mottelson model, which worked in the full boson Hilbert space, and the algebraic limitations of the interacting boson model. This distinction is physically interesting only for small enough N.

B. Triaxial rotor at high spin: Approximate treatment of the spectrum

We next describe another application of the HP mapping that appears to have some relevance to high-spin states of nuclei, if one admits the approximation that such motion can be described in terms of the quantum triaxial rotor (Tanabe and Sugawara-Tanabe, 1971, 1976); Sugawara-Tanabe and Tanabe, 1973; Tanabe, 1973; Bohr and Mottelson, 1975; Marshalek, 1975b). Consider the Hamiltonian

$$H = \sum_{i=1,2,3} I_i'^2 / 2\mathcal{I}_i$$
$$= \vec{I}^{\,2} / (2\mathcal{I}_3) - \tfrac{1}{2}(1/\mathcal{I}_3 - 1/\mathcal{I}_1) I_1'^2$$
$$- \tfrac{1}{2}(1/\mathcal{I}_3 - 1/\mathcal{I}_2) I_2'^2 \ , \tag{5.18}$$

where the I_i' are intrinsic components of the angular momentum along the principal axes satisfying the commutation relations with the famous sign change,

$$[I_i', I_j'] = -i\epsilon_{ijk} I_k' \ . \tag{5.19}$$

In the classical theory of the asymmetric top, the motion reduces for high spin to a simple rotation without precession of the axes, if the angular momentum is along the axis corresponding to the largest or smallest moment of inertia (which our reexpression of Eq. (5.18) has already taken to be the third axis). Correspondingly, in the quantum theory, the states of smallest (or largest) angular momentum I acquire a simple structure, which can be exploited with the aid of the Holstein-Primakoff mapping [trivially modified for the sign change in Eq. (5.19)]. We write

$$I_+' = I_1' + iI_2' = (I_-')^\dagger = (2I)^{1/2} b^\dagger [1 - b^\dagger b / 2I]^{1/2} \ , \tag{5.20a}$$

$$I_3' = I - b^\dagger b \ . \tag{5.20b}$$

Then states that are nearly equal in energy to the simple states and correspond classically to the precessional (wobbling) motion of the angular momentum vector with respect to the axes can be studied by expanding $I_\pm'$ in powers of $(b^\dagger b / 2I)$. The quadratic part of the Hamiltonian has the general form seen, for example, in Eq. (3.6), and can be diagonalized by a linear transformation

$$b = \eta_+ B + \eta_-^* B^\dagger \ , \tag{5.21a}$$

$$b^\dagger = \eta_- B + \eta_+^* B^\dagger \ , \tag{5.21b}$$

with

$$|\eta_+|^2 - |\eta_-|^2 = 1 \ . \tag{5.22}$$

Standard calculations yield for the diagonal part of the Hamiltonian

$$H = I(I+1)/2\mathcal{I}_3 - 2I[\beta - (\alpha - \beta)/2I](B^\dagger B + \tfrac{1}{2})$$
$$+ \alpha (B^\dagger B)^2 \ , \tag{5.23}$$

where

$$\alpha = \tfrac{1}{4}(2/\mathcal{I}_3 - 1/\mathcal{I}_1 - 1/\mathcal{I}_2) \ , \tag{5.24a}$$

$$\beta = \pm \tfrac{1}{2}[(1/\mathcal{I}_3 - 1/\mathcal{I}_1)(1/\mathcal{I}_3 - 1/\mathcal{I}_2)]^{1/2} \ . \tag{5.24b}$$

Here the upper sign corresponds to the prolate-like shape ($\mathcal{I}_3$ minimum) and the lower sign to the oblate-like shape ($\mathcal{I}_3$ maximum). The last term in Eq. (5.23) has been included because of stability requirements. Sugawara-Tanabe and Tanabe (1973) have shown that, with this term, the theory connects smoothly with the symmetric-rotor limit.

C. Triaxial rotor at high spin: Electromagnetic transitions

The treatment of the high-spin asymmetric rotor by Tanabe and Sugawara-Tanabe did not handle elec-

tromagnetic transitions in a manner consistent with that of the angular momentum and Hamiltonian. The remedy can be found in the work of Marshalek (1975b), who provided a Holstein-Primakoff boson mapping of all the elementary operators associated with the quantized rotor. In this treatment, a separate Holstein-Primakoff mapping is provided for the laboratory components I_k ($k=x,y,z$) of the angular momentum and for the body-fixed components I'_α ($\alpha=1,2,3$). Now, an electromagnetic transition operator T^λ_μ is a spherical tensor of rank λ and thus can be written in the form (Bohr and Mottelson, 1975)

$$T^\lambda_\mu=\sum_\nu T'^\lambda_\nu D^\lambda_{\mu\nu} , \tag{5.25}$$

where the operators T'^λ_μ depend on intrinsic degrees of freedom (including possibly the I'_α) and where the action of the operators $D^\lambda_{\mu\nu}$ on the rotor eigenstates $|IMK\rangle$ is given by the well-known decomposition

$$D^\lambda_{\mu\nu}|IMK\rangle=\sum_{I'=|I-\lambda|}^{I+\lambda}\left[\frac{2I+1}{2I'+1}\right]^{1/2}\langle I\lambda M\mu|I'M+\mu\rangle \times\langle I\lambda K\nu|I'K+\nu\rangle|I'M+\mu K+\nu\rangle . \tag{5.26}$$

The set of angular momentum components together with the operators $D^\lambda_{\mu\nu}$ satisfy the following algebra:

$$\begin{aligned}
&[I_+,I_-]=2I_z,\quad [I_z,I_\pm]=\pm I_\pm ,\\
&[I'_+,I'_-]=-2I'_1,\quad [I'_1,I'_\pm]=\mp I'_\pm ,\\
&[I'_\alpha,I_k]=0 , \qquad (5.27)\\
&[D^{\lambda_1}_{\mu_1\nu_1},D^{\lambda_2}_{\mu_2\nu_2}]=0 , \qquad (5.28)\\
&[I_\pm,D^\lambda_{\mu\nu}]=[(\lambda\mp\mu)(\lambda\pm\mu+1)]^{1/2}D^\lambda_{\mu\pm1\,\nu} ,\\
&[I_z,D^\lambda_{\mu\nu}]=\mu D^\lambda_{\mu\nu} ,\\
&[I'_\pm,D^\lambda_{\mu\nu}]=[(\lambda\pm\nu)(\lambda\mp\nu+1)]^{1/2}D^\lambda_{\mu\,\nu\pm1} ,\\
&[I'_1,D^\lambda_{\mu\nu}]=\nu D^\lambda_{\mu\nu} . \qquad (5.29)
\end{aligned}$$

The subalgebra (5.27) is readily recognized as that corresponding to SU(2)×SU(2), and since the $D^\lambda_{\mu\nu}$ generate an Abelian group, as indicated by Eq. (5.28), the addition of (5.29) shows that the full algebra corresponds to the semidirect product of SU(2)×SU(2) with this Abelian group. In Eq. (5.27), $I_\pm\equiv I_x\pm iI_y$, while $I'_\pm\equiv I'_2\pm iI'_3$. In the case of the body-fixed components, a cyclic permutation was performed so that the first axis, which is the conventional axis of rotation in the cranking model, would also be the axis of quantization. (From the mathematical viewpoint, we could just as well choose $I'_\pm=I'_1\pm iI'_2$ as we have done in the previous subsection.)

Consider now the Holstein-Primakoff boson representation of the rotor basis states and the angular momentum components. The action of the latter on these states is given by

$$\begin{aligned}
&I_z|IMK\rangle=M|IMK\rangle ,\\
&I_\pm|IMK\rangle=[(I\mp M)(I\pm M+1)]^{1/2}|IM\pm1K\rangle ,\\
&I_1|IMK\rangle=K|IMK\rangle ,\\
&I'_\pm|IMK\rangle=[(I\pm K)(I\mp K+1)]^{1/2}|IMK\mp1\rangle .
\end{aligned}\tag{5.30}$$

It is clear from Eq. (5.30) that the matrix elements for a given value of the angular momentum I can be realized by taking the direct product of two Holstein-Primakoff representations, therefore requiring the introduction of at least two independent bosons. However, looking ahead to the inclusion of the $D^\lambda_{\mu\nu}$, one sees that these operators change the quantum number I, so that the physical subspace must contain all the irreducible representations and thus a third boson to carry the quantum number I. By "inspection," it is not hard to see that the Holstein-Primakoff representation of the angular momentum components is given by

$$\begin{aligned}
&I'_1=\hat I-\hat N_b ,\\
&I'_+=b^\dagger S_{ab},\quad I'_-=S_{ab}b ,\\
&S_{ab}\equiv(2\hat I-\hat N_b)^{1/2},\quad \hat I\equiv\tfrac12\hat N_a=\tfrac12 a^\dagger a ,\\
&\hat N_b\equiv b^\dagger b ,
\end{aligned}\tag{5.31}$$

$$\begin{aligned}
&I_z=\hat I-\hat N_c ,\\
&I_+=S_{ac}c,\quad I_-=c^\dagger S_{ab} ,\\
&S_{ac}=(2\hat I-\hat N_c)^{1/2},\quad \hat N_c=c^\dagger c ,
\end{aligned}\tag{5.32}$$

acting on the boson states

$$|IMK\rangle=\frac{(a^\dagger)^{2I}(b^\dagger)^{I-K}(c^\dagger)^{I-M}}{[(2I)!(I-K)!(I-M)!]^{1/2}}|0\rangle , \qquad 0\le I<\infty,\quad -I\le K,M\le I . \tag{5.33}$$

In Eq. (5.33) one has $[a,a^\dagger]=1$, $[b,b^\dagger]=1$, $[c,c^\dagger]=1$, with all other combinations commuting, and $|0\rangle$ is the vacuum state satisfying $a|0\rangle=b|0\rangle=c|0\rangle=0$. Note that the convention chosen here is $b|III\rangle=c|III\rangle=0$, where $|III\rangle=a^{\dagger 2I}/[(2I)!]^{1/2}|0\rangle$ is the fully aligned state of maximal weight. The extra a boson is introduced to label basis vectors with different values of I, which are eigenstates of the operator $\hat I=\frac12\hat N_a$. It may be noted that the representation (5.31) and (5.32) satisfies the requirement that the laboratory and body-fixed components commute. In addition, the condition that the sum of the squares of the laboratory and of the body-fixed components must equal the square of the length of the angular momentum vector is satisfied, i.e.,

$$\tfrac12\{I_+,I_-\}+I_z^2=\tfrac12\{I'_+,I'_-\}+I_1^2=\hat I(\hat I+1) , \tag{5.34}$$

as may be directly verified. A similar situation, in which one introduces a boson that changes the value of the total angular momentum, will be encountered in somewhat more detail in the next section, in the discussion following Eq. (6.10).

Next, the $D^{\lambda}_{\mu\nu}$ can be determined in principle from the commutators (5.29), together with the well-known requirement that they form a unitary matrix:

$$\sum_{\mu} D^{\lambda\dagger}_{\mu\nu} D^{\lambda}_{\mu\nu'} = \delta_{\nu\nu'} ,$$
$$\sum_{\nu} D^{\lambda\dagger}_{\mu\nu} D^{\lambda}_{\mu'\nu} = \delta_{\mu\mu'} . \tag{5.35}$$

In practice, it is easier to invoke the action of these operators on the basis vectors (5.33) directly, using Eq. (5.26). An especially convenient procedure is to begin with the case $\lambda = \frac{1}{2}$, since these operators are relatively simple, and then synthesize the operators for higher values of λ using the composition rule

$$D^{\lambda}_{\mu\nu} = \sum_{\mu_1\mu_2} \sum_{\nu_1\nu_2} \langle \lambda_1\lambda_2\mu_1\mu_2|\lambda\mu\rangle \times \langle \lambda_1\lambda_2\nu_1\nu_2|\lambda\nu\rangle D^{\lambda_1}_{\mu_1\nu_1} D^{\lambda_2}_{\mu_2\nu_2} , \tag{5.36}$$

which can be derived from Eq. (5.26). Thus, for example, application of Eq. (5.26) to the case $\lambda = \frac{1}{2}$, with $|IMK\rangle$ denoted by $|N_a, N_b, N_c\rangle$, where $N_a = 2I$, $N_b = I - K$, $N_c = I - M$, gives

$$D^{1/2}_{1/2\,1/2}|N_a N_b N_c\rangle = \left[\frac{(2I+1-N_b)(2I+1-N_c)}{(2I+1)(2I+2)}\right]^{1/2} |N_a+1 N_b N_c\rangle + \left[\frac{N_b N_c}{2I(2I+1)}\right]^{1/2} |N_a-1 N_b-1 N_c-1\rangle , \tag{5.37}$$

with a similar expression for the action of $D_{1/2\,-1/2}$. It is then straightforward to obtain the boson representation of the $D^{1/2}_{\mu\nu}$ by replacing the numbers I, N_b, N_c with the corresponding operators and taking into account the changes in the boson numbers of the basis vectors. One is then led to the following compactly written expressions:

$$D^{1/2}_{1/2\,1/2} = S_{ab} S_{ac} A^{\dagger} + Abc ,$$
$$D^{1/2}_{1/2\,-1/2} = b^{\dagger} S_{ac} A^{\dagger} - A S_{ab} c , \tag{5.38}$$

where the operators A and $A^{\dagger}$ are defined by

$$A \equiv (2\hat{I}+1)^{-1/2} E_{-} (2\hat{I}+1)^{-1/2} ,$$
$$A^{\dagger} = (2\hat{I}+1)^{-1/2} E_{+} (2\hat{I}+1)^{-1/2} , \tag{5.39}$$

and where $E_{\pm}$ are defined as

$$E_{+} = a^{\dagger}(2\hat{I}+1)^{-1/2}, \quad E_{-} = E^{\dagger}_{+} = (2\hat{I}+1)^{-1/2} a . \tag{5.40}$$

The operators (5.40) change N_a by one unit (and I by a half-unit), while maintaining the normalization of the basis vectors. The operator E_{+}, which satisfies

$$E_{-}E_{+} = E^{\dagger}_{+}E_{+} = 1, \quad E_{+}E_{-} = 1 - |0\rangle\langle 0| , \tag{5.41}$$

is isometric, rather than precisely unitary, since $E_{+}E^{\dagger}_{+}$ annihilates the vacuum (or, for that matter, any boson vector with no $a^{\dagger}$ bosons).

All other $D^{1/2}_{\mu\nu}$ can be obtained from the relation $D^{\lambda\dagger}_{\mu\nu} = (-1)^{\nu-\mu} D^{\lambda}_{-\mu-\nu}$. As previously mentioned, additional $D^{\lambda}_{\mu\nu}$ with higher values of λ can be synthesized with the aid of Eq. (5.36). Marshalek (1975b) lists all the operators with $\lambda = 1$. For many purposes it is sufficient to work with the $D^{\lambda}_{\mu\nu}$ defined in the subspace with $M = I$; this is especially true when calculating electromagnetic transitions of lower multipole order between high-spin states (Marshalek, 1975b). In that case, in place of the $D^{\lambda}_{\mu\nu}$ one needs only the restriction of these operators to the $M = I$ subspace, i.e., the operators

$$\hat{D}^{\lambda}_{\mu\nu} \equiv P D^{\lambda}_{\mu\nu} P , \tag{5.42}$$

where $P = \sum_{IK} |IIK\rangle\langle IIK|$ is the projector to the $M = I$ subspace. The resulting expressions are considerably simpler, since they involve only the bosons a, b, with $N_c = 0$ in this subspace. Of particular interest for high-spin states are the $\hat{D}^{2}_{\mu\nu}$ associated with electric quadrupole ($E2$) transitions. For example, one obtains

$$\hat{D}^{2}_{22} = (S_{ab}\alpha^{\dagger})^4, \quad \hat{D}^{2}_{21} = 2(S_{ab}\alpha^{\dagger})^2 \alpha^{\dagger} I'_{+} \alpha^{\dagger} ,$$
$$\hat{D}^{2}_{20} = \sqrt{6}(\alpha^{\dagger} I'_{+} \alpha^{\dagger})^2, \quad \hat{D}^{2}_{2-1} = 2(\alpha^{\dagger})^2 b^{\dagger} I'_{+} b^{\dagger} (\alpha^{\dagger})^2 ,$$
$$\hat{D}^{2}_{2-2} = (b^{\dagger}\alpha^{\dagger})^4 , \tag{5.43}$$

where

$$\alpha^{\dagger} \equiv (2\hat{I}+1)^{-1/2} E_{+} = (2\hat{I}+1)^{-1/2} a^{\dagger} (2\hat{I}+1)^{-1/2} . \tag{5.44}$$

[The remaining $\hat{D}^{2}_{\mu\nu}$ are given in Marshalek (1975b)]. When operating on a high-spin state with a large value of I, one may replace $\hat{I}$ by its eigenvalue I in Eq. (5.43). One then obtains the following approximations to Eq. (5.43), correct through order $I^{-1/2}$:

$$\hat{D}^{2}_{22} \approx E^4_{+}, \quad \hat{D}^{2}_{21} \approx (2/I)^{1/2} b^{\dagger} E^4_{+} ,$$
$$\hat{D}^{2}_{2\nu} \approx 0 \quad \text{for } \nu \le 0 . \tag{5.45}$$

From Eqs. (5.25) and (5.43), the $E2$ transition operator T_{22} is then approximated by

$$T_{22} \approx [T'_{22} + (2/I)^{1/2} b^{\dagger} T'_{21}] E^4_{+} , \tag{5.46}$$

where E^4_{+} changes the quantum number I by 2 units. The intrinsic $E2$ moments $T'_{2\mu}$ are constants in the simple asymmetric-rotor model discussed earlier. In the same

way, we can show that other components of the $E2$ operator are approximated by

$$T_{21}=[-(2/I)^{1/2}bT'_{22}+T'_{21}+(3/I)^{1/2}b^{\dagger}T'_{20}]E^2_+ , \tag{5.47}$$

and

$$T_{20}=-(3/I)^{1/2}bT'_{21}+T'_{20}+(3/I)^{1/2}b^{\dagger}T'_{2-1} . \tag{5.48}$$

In Eq. (5.47), the operator E^2_+ changes I by one unit, while Eq. (5.48) involves $\Delta I=0$ transitions.

Formulas equivalent to Eqs. (5.46)–(5.48) have been derived by Bohr and Mottelson (1975) for the asymmetric rotor at high spin by using a large-I approximation for Clebsch-Gordan coefficients. The boson operators b and $b^{\dagger}$ may be expressed in terms of the normal-mode bosons B, $B^{\dagger}$ introduced earlier [Eq. (5.21)] to obtain the reduced transition probabilities. For our purposes, it is sufficient to observe that the $\Delta I=2$ term $T'_{22}E^4_+$ in T_{22} is the largest, all others being of order $(n/I)^{1/2}$. It is this large term that is responsible for the "parallel tracks" of the yrast cascade, while the other terms, which give rise to transitions between the tracks, lead to transition probabilities that are smaller by factors of order n/I.

The extended Holstein-Primakoff representation discussed in this subsection has also proven useful in elucidating the structure of the self-consistent cranking+RPA approximation, which describes small oscillations (including the wobbling motion) about a state of steady rotation (Marshalek, 1977). It has also been used to obtain approximate solutions for the model of a particle in a large j shell coupled to a rotor (Marshalek, 1982b).

D. Triaxial rotor at high spin: Schwinger mapping and other results

From the above work, Yamamura, Suzuki, and Ichihashi (1978) and Ichihashi and Yamamura (1978) set out to develop a Schwinger representation in place of the Holstein-Primakoff one for the quantized rotor. Since a Schwinger representation does not lend itself *a priori* to an I^{-1} expansion, these authors used a variant of the cranking model (with effective "Routhian" $H_{\omega}=H-\hbar\omega\hat{I}$) in the framework of oscillator coherent states. Now, a four-boson Schwinger representation obtained in a straightforward extension from SU(2) to SU(2)×SU(2) does not automatically fulfill the condition (5.34), although one could arrange for Eq. (5.34) to hold within some appropriately defined physical subspace. In order to preserve Eq. (5.34) identically over the whole boson space, the authors chose a representation of the Schwinger type for the body-fixed components of the angular momentum, i.e., one that is bilinear in the bosons. However, the representation of the laboratory components is not of the Schwinger type, but involves a complicated square-root function. Fortunately, only the body-fixed components are needed for the Hamiltonian. However, the representation of the $D^{\lambda}_{\mu\nu}$ then becomes quite complex. Because of the complexity of the formulas and the fact that no new physical results emerged, the interested reader is referred to the original papers.

Another closely related work is that of D. Janssen (1977), who used SU(2) coherent states to derive a functional representation for the quantized rotor, equivalent to a Dyson representation. By averaging with respect to the coherent state, he derived the classical Euler equations for the asymmetric top. He also sketched the application of the averaging to the cranking model at high angular momentum.

As a further contribution to the present topic, we mention the work of Badea and Raduta (1979), who applied the Holstein-Primakoff and also the Dyson mappings to the asymmetric rotor at high spin. Aside from a discussion of the connection between the two mappings, the main purpose of the work was to obtain improved approximate eigenvalues. The authors studied two methods: a variational Bogoliubov transformation that takes into account higher orders in the I^{-1} expansion, and, second, a boson coherent state as a variational wave function. Both methods when appropriately employed give excellent approximations to the true eigenvalues.

A true Schwinger representation for the quantal rotor that also satisfies Eq. (5.34) was first found by Gulshani (1979a, 1979b), who provided generalizations as well. In the simplest version, Gulshani began by parametrizing the system in terms of the Cayley-Klein parameters a, b, c, and d, related to the Euler angles by

$$\begin{aligned} a&=D^{1/2}_{1/2\,1/2}(\varphi,\theta,\psi)=e^{i(\varphi+\psi)/2}\cos\frac{\theta}{2} ,\\ b&=D^{1/2}_{1/2\,-1/2}(\varphi,\theta,\psi)=-e^{i(\varphi-\psi)/2}\sin\frac{\theta}{2} ,\\ c&=D^{1/2}_{-1/2\,1/2}(\varphi,\theta,\psi)=-b^* ,\\ d&=D^{1/2}_{-1/2\,-1/2}(\varphi,\theta,\psi)=a^* , \end{aligned} \tag{5.49}$$

where the conventional zyz definition of the Euler angles is used. The angular momentum components represented by differential operators in the Euler angles are then transformed into the space of Cayley-Klein parameters, using the relations

$$\frac{\partial}{\partial\varphi}=\frac{\partial a}{\partial\varphi}\frac{\partial}{\partial a}+\frac{\partial b}{\partial\varphi}\frac{\partial}{\partial b}+\frac{\partial c}{\partial\varphi}\frac{\partial}{\partial c}+\frac{\partial d}{\partial\varphi}\frac{\partial}{\partial d} , \tag{5.50}$$

and similar equations for

$$\frac{\partial}{\partial\theta}, \quad \frac{\partial}{\partial\psi} .$$

After a straightforward but tedious calculation, one arrives at the following expressions for the angular momentum components:

$$I_+=a\frac{\partial}{\partial c}+b\frac{\partial}{\partial d}, \quad I_-=c\frac{\partial}{\partial a}+d\frac{\partial}{\partial b} ,$$

$$I_z=\frac{1}{2}\left[a\frac{\partial}{\partial a}+b\frac{\partial}{\partial b}-c\frac{\partial}{\partial c}-d\frac{\partial}{\partial d}\right]$$

(lab. components), (5.51a)

$$I'_+=b\frac{\partial}{\partial a}+d\frac{\partial}{\partial c},\quad I'_-=a\frac{\partial}{\partial b}+c\frac{\partial}{\partial d},$$

$$I'_3=\frac{1}{2}\left[a\frac{\partial}{\partial a}-b\frac{\partial}{\partial b}+c\frac{\partial}{\partial c}-d\frac{\partial}{\partial d}\right]$$

(principal-axis components). (5.51b)

Next, the angular momentum components (5.51) are expressed in terms of four kinds of bosons $A^\dagger$, A, $B^\dagger$, B, $C^\dagger$, C, and $D^\dagger$, D defined by

$$\begin{aligned}
A^\dagger&\equiv\frac{1}{\sqrt{2}}\left[a-\frac{\partial}{\partial d}\right],\quad A\equiv\frac{1}{\sqrt{2}}\left[d+\frac{\partial}{\partial a}\right],\\
B^\dagger&\equiv\frac{1}{\sqrt{2}}\left[b+\frac{\partial}{\partial c}\right],\quad B\equiv\frac{1}{\sqrt{2}}\left[-c+\frac{\partial}{\partial b}\right],\\
C^\dagger&\equiv\frac{1}{\sqrt{2}}\left[c+\frac{\partial}{\partial b}\right],\quad C\equiv\frac{1}{\sqrt{2}}\left[-b+\frac{\partial}{\partial c}\right],\\
D^\dagger&\equiv\frac{1}{\sqrt{2}}\left[d-\frac{\partial}{\partial a}\right],\quad D\equiv\frac{1}{\sqrt{2}}\left[a+\frac{\partial}{\partial d}\right],
\end{aligned}\tag{5.52}$$

which obey the commutation rules

$$[A,A^\dagger]=[B,B^\dagger]=[C,C^\dagger]=[D,D^\dagger]=1\ , \tag{5.53}$$

all other pairs commutating. Substitution of the inverse of Eqs. (5.52) into (5.51) then gives the final Schwinger boson image of the angular momentum components:

$$I_+=A^\dagger C+B^\dagger D,\quad I_-=I_+^\dagger\ ,$$

$$I_z=\tfrac{1}{2}(A^\dagger A+B^\dagger B-C^\dagger C-D^\dagger D)$$

(lab. components), (5.54a)

$$I'_+=B^\dagger A+D^\dagger C,\quad I'_-=I'^\dagger_+\ ,$$

$$I'_3=\tfrac{1}{2}(A^\dagger A-B^\dagger B+C^\dagger C-D^\dagger D)$$

(intrinsic components). (5.54b)

It is readily checked that the commutation rules [Eq. (5.27)] are satisfied, as well as Eq. (5.34). Unlike the "Schwinger" mappings of Yamamura *et al.*, Eq. (5.50) represents both the laboratory and principal-axis components by simple quadratic forms in the bosons. In this representation, the total angular momentum operator $\hat{I}^2$ is given by

$$\hat{I}^2=\tfrac{1}{2}\hat{N}(\tfrac{1}{2}\hat{N}+1)-(A^\dagger D^\dagger-B^\dagger C^\dagger)(AD-BC)\ , \tag{5.55}$$

where $\hat{N}$ is the total boson number operator.

The physical subspace of angular momentum eigenvectors $|IMK\rangle$ can be generated by first constructing a "vacuum" state $|I-I-I\rangle$ satisfying the conditions

$$\begin{aligned}
&I_-|I-I-I\rangle=0,\quad I'_+|I-I-I\rangle=0\ ,\\
&\hat{I}^2|I-I-I\rangle=I(I+1)|I-I-I\rangle\ .
\end{aligned}\tag{5.56}$$

By inspection, using Eqs. (5.54a) and (5.54b), one finds that the normalized eigenvector satisfying (56) is given by

$$|I-I-I\rangle=\frac{1}{\sqrt{(2I)!}}(D^\dagger)^{2I}|0\rangle\ , \tag{5.57}$$

where $|0\rangle$ is the true boson vacuum. We note that the condensate form of Eq. (5.57) is characteristic of a true Schwinger representation as well as the quadratic form of the generators. The general eigenvectors $|IMK\rangle$ can then be generated as usual from the formula

$$|IMK\rangle=\frac{[(I-M)!(I-K)!]^{1/2}}{(2I)!}I'^{I+K}_-I^{I+M}_+|I-I-I\rangle\ . \tag{5.58}$$

For a closed form of Eq. (5.58) written in terms of the four boson creation operators, the reader is referred to the paper of Gulshani (1979b). Since the generators conserve $\hat{N}$, the states (5.58) all have the same number of bosons, $n=2I$, as follows from Eqs. (5.55) and (5.56).

From Eqs. (5.49) and the inverse of (5.52), the Cayley-Klein variables, which are just the $D^{1/2}_{\mu\nu}$, can be expressed as linear combinations of bosons as follows:

$$\begin{aligned}
a&=D^{1/2}_{1/2\,1/2}=\frac{1}{\sqrt{2}}(A^\dagger+D)\ ,\\
b&=D^{1/2}_{1/2\,-1/2}=\frac{1}{\sqrt{2}}(B^\dagger-C)\ ,\\
c&=D^{1/2}_{-1/2\,1/2}=-b^\dagger,\quad d=D^{1/2}_{-1/2\,-1/2}=a^\dagger\ .
\end{aligned}\tag{5.59}$$

Other $D^\lambda_{\mu\nu}$ can then be synthesized with the help of Eq. (5.36) and thus must always be expressable as finite polynomials in the bosons.

Finally, Gulshani pointed out that the Schwinger representation can be generalized by choosing as coordinates the rotation matrix elements $D^j_{\mu\nu}$ for any j, not just $j=\frac{1}{2}$. Moreover, the coordinates need not be limited to the $D^j_{\mu\nu}$, but can be replaced by any set of abstract double tensors under SU(2). By going through a procedure analogous to the one just described, Gulshani obtained the generalized Schwinger representation, in which

$$\begin{aligned}
I_M&=\sum_{\mu\nu\mu'\nu'=-j}^{j}\langle j\nu'\mu'|I_M|j\nu\mu\rangle A^{(j)\dagger}_{\mu'\nu'}A^{(j)}_{\mu\nu}\ ,\\
I'_K&=\sum_{\mu\nu\mu'\nu'=-j}^{j}\langle j\nu'\mu'|I'_K|j\nu\mu\rangle A^{(j)\dagger}_{\mu'\nu'}A^{(j)}_{\mu\nu}\ ,
\end{aligned}\tag{5.60}$$

where the $A^{(j)\dagger}_{\mu\nu}$ are creation operators for independent bosons.

VI. MAPPINGS OF SO(4) MODELS AND APPLICATIONS

A. Definition of models studied

We now consider a model that is only slightly more complicated than SU(2) [and is in fact isomorphic to the

algebra of SU(2)×SU(2)], which can be used to illustrate a number of different physical ideas. (Piepenbring, Silvestre-Brac, and Szymanski, 1980, 1982; Klein, 1982; Marshalek, 1982a; Matsuyanagi, 1982). We begin with the two-level pairing model with equal pair degeneracy Ω for each level, as described by Eqs. (4.1) and (4.2). The linear combinations of the quasispins,

$$\vec{J}=\vec{J}_1+\vec{J}_2, \quad \vec{K}=\vec{J}_1-\vec{J}_2 , \tag{6.1}$$

generate the algebra SO(4) with the commutators

$$[J_+,J_-]=[K_+,K_-]=2J_0 , \tag{6.2a}$$

$$[J_+,K_-]=[K_+,J_-]=2K_0 , \tag{6.2b}$$

$$[K_+,J_0]=[J_+,K_0]=-K_+ \quad \text{(and H.c. Eqs.)} , \tag{6.2c}$$

$$[J_+,J_0]=[K_+,K_0]=-J_+ \quad \text{(and H.c. Eqs.)} . \tag{6.2d}$$

We shall, for example, study a model in which we ignore the single-particle splitting of the two levels and choose a Hamiltonian of the form

$$H=-GJ_+J_- -FK_+K_- . \tag{6.3}$$

For $F=0$, Eq. (6.3) describes the strong-coupling limit of superconductivity with SU(2) symmetry associated with the total quasispin $\vec{J}$. The remaining term is a monopole-monopole interaction that breaks this symmetry in analogy with a quadrupole-quadrupole interaction.

A completely different physics for the same mathematics has been achieved by Piepenbring *et al.* (1980, 1982) by the following mapping of single-particle indices [refer to the definitions associated with Eqs. (4.1) and (4.2)]:

$$\begin{aligned} &2,|m|\rightarrow\tfrac{3}{2},p=(1,\ldots,\Omega); \quad 2,-|m|\rightarrow-\tfrac{3}{2},\bar{p} ; \\ &1,|m|\rightarrow\tfrac{1}{2},p; \quad 1,-|m|\rightarrow-\tfrac{1}{2},\bar{p} . \end{aligned} \tag{6.4}$$

If we interpret $k(=\pm\frac{3}{2},\pm\frac{1}{2})$ as a projection of angular momentum along a symmetry axis, then in the language of nuclear physics the model is one of Nilsson levels (Nilsson, 1955) with axial symmetry, where $k=\frac{3}{2}$ and $k=\frac{1}{2}$ are split, but each is p-fold degenerate. By using the notation $(2,p)$, $(2,\bar{p})$, $(1,p)$, $(1,\bar{p})\rightarrow(i,p)$, $(i,\bar{p})$ for the sets described in Eq. (6.4), these authors consider a Hamiltonian of the form

$$H=\sum_i e_iN_i-GJ_+J_- -\tfrac{1}{2}\chi q^2(N_2-N_1)^2 , \tag{6.5}$$

where

$$N_i=\sum_p[\alpha^\dagger_{ip}\alpha_{ip}+\alpha^\dagger_{i\bar{p}}\alpha_{i\bar{p}}] \tag{6.6}$$

is a number operator for each level. Thus we have a single-particle splitting, a pairing interaction, and a monopole-monopole interaction. In the literature under discussion the latter is often referred to as a quadrupole-quadrupole interaction, by analogy with more realistic models that truly include angular momentum.

This model, in a version that drops the single-particle splitting, has been discussed within the context of the "mode-mode" coupling theory by Matsuyanagi (1982) as an example of what can be done with the "quantized Bogoliubov transformation." We shall take this matter up in Sec. VI.D and indeed generalize Matsuyanagi's considerations. More recently (Jammari, Piepenbring, and Silvestre-Brac, 1983), the model of Piepenbring *et al.* (1980, 1982) described above has been extended to include octupole coupling. This was done by exactly doubling the degeneracy, adding a set of odd-parity levels in a model that is isomorphic to SU(2)×SU(2)×SU(2)×SU(2). The main object of this and their previous work was to establish, as they succeed in doing, the accuracy of the multiphonon method, a modified version of the Marumori-Yamamura-Tokunaga mapping that we shall discuss in Sec. XIV.

A third physical model associated with the SO(4) algebra has been studied in considerable detail by boson methods (Marshalek, 1982a) and therefore will be of interest to us. Consider a system of nucleons confined to a single major shell, N, of a two-dimensional harmonic oscillator. The orbitals are labeled for fixed N by $m\pm$, $\pm$ for spin up or down, and $m=N,N-2,\ldots,-N$. The single-particle operators (mass$=\omega_0=1$ for the oscillator)

$$\begin{aligned} t_x&=\tfrac{1}{4}[(x^2-y^2)+(p_x^2-p_y^2)] , \\ t_y&=\tfrac{1}{2}[xy+p_xp_y] , \\ t_z&=\tfrac{1}{2}l_z=\tfrac{1}{2}(xp_y-yp_x) \end{aligned} \tag{6.7}$$

are the components of a vector under SU(2). It follows that, for $\mu=\pm1,0$ (spherical components), if $\alpha^\dagger_{m\pm}$, $\alpha_{m\pm}$ are the single-fermion creation and annihilation operators, the operators

$$T_\mu(\pm)=\sum_{mm'}\langle Nm|t_\mu|Nm'\rangle\alpha^\dagger_{m\pm}\alpha_{m'\pm} \tag{6.8}$$

are generators of two commuting SU(2) algebras. Let $\vec{T}=\vec{T}(+)+\vec{T}(-)$, as in the first of Eqs. (6.1). For this model, the physical total angular momentum is $2T_0$. We shall then study the Hamiltonian

$$\begin{aligned} H&=-\tfrac{1}{4}G^2(T_+T_-+T_-T_+)-C(T_0(+)-T_0(-)) \\ &=-\tfrac{1}{2}G^2(\vec{T}^2-T_0^2)-C(T_0(+)-T_0(-)) . \end{aligned} \tag{6.9}$$

For $C=0$, this is the two-dimensional version of the Elliot model (1958a, 1958b). The term proportional to C is a two-dimensional spin-orbit coupling that breaks the SU(2) symmetry (Moszkowski, 1958).

B. Two SO(4) mappings of type SO(4)⊃SU(2)⊃SU(1)

For detailed study, we turn first to the model of SO(4) embodied in the Hamiltonian (6.3). This model suggests the mappings that can be used in subsequent treatments. For the problem at hand we are interested in the irrep of the algebra containing the vacuum state. We shall apply

the method described in Eqs. (2.28)–(2.32), that we henceforth call the commutator method, which generates the Holstein-Primakoff mapping without our having first to calculate all the matrices of the generators. These matrices, in fact, are a result of the calculation. The irrep we wish to map is that obtained by orthonormalizing the set of vectors

$$(J_+)^{n_b}(K_+)^{n_c}|0) , \tag{6.10}$$

under the restriction [see Eq. (6.14) below]

$$0 \le n_b + 2n_c \le 2\Omega . \tag{6.11}$$

The essential remark is that the only information necessary to proceed is contained in Eq. (6.10), which suggests by analogy with the previously studied SU(2) case that we look for a mapping onto a boson basis

$$|n_b, n_c) = (n_b! n_c!)^{-1/2} (b^\dagger)^{n_b} (c^\dagger)^{n_c} |0) . \tag{6.12}$$

Equations (6.10) or (6.12) are understood to represent a basis described by the group chain $SO(4) \supset SU(2) \supset U(1)$ where Ω is an SO(4) label (we are dealing with an especially simple class of irreps), n_c defines the pseudospin of the SU(2) associated with $\vec{J}$, and n_b is the corresponding magnetic quantum number. The latter point is obvious from Eq. (6.10). The meaning of $c^\dagger$ will emerge in rapid course, but it is clear from the structure of Eq. (6.10) and the analogy with the well-known Lenz vector for the bound-state spectrum of hydrogen that K_+ must change the eigenvalue J of the quasispin. Actually, the introduction of a boson that changes the value of the angular momentum has already been encountered in the previous section in connection with the study of the triaxial rotor.

An essential aspect of this method is that we build on solutions previously established. Thus we write down by inspection the appropriate mapping for the SU(2) algebra associated with the quasispin $\vec{J}$, namely,

$$J_0 = -\Omega + \hat{n}_c + \hat{n}_b , \tag{6.13}$$

$$J_+ = b^\dagger [2\Omega - 2\hat{n}_c - \hat{n}_b]^{1/2} \equiv b^\dagger r(\hat{n}_b + 2\hat{n}_c) , \tag{6.14}$$

where $\hat{n}_b$ and $\hat{n}_c$ are the boson number operators. Since $|0\rangle$ is, in a more informative labeling, the state $|\Omega, -\Omega)$, these formulas already show that the boson $c^\dagger$ decreases the total quasispin by unity and increases its zero component by the same amount. To obtain formulas for the remaining operators, we recognize that $\vec{K}$ is a vector operator under the transformations generated by $\vec{J}$. Its selection rules are thus well known. The general form of K_+, for example, is

$$K_+ = c^\dagger \Phi_1(\hat{n}_b, \hat{n}_c) + b^\dagger b^\dagger c \Phi_2(\hat{n}_b, \hat{n}_c) , \tag{6.15}$$

which states that K_+ must increase J_0 by one unit and may increase or decrease J by a unit. (The absence of a term in which J does not change is a simplification applicable only to the special representation considered here, as will be shown later in this section.) By studying the commutations relations (6.2) in conjunction with the form (6.13) and (6.14), we are led to difference equations that are easily solved and yield the results (Klein, Rafelski and Rafelski, 1981; Klein, 1982; Matsuyanagi, 1982)

$$K_+ = K_-^\dagger = c^\dagger r(\hat{n}_b + 2\hat{n}_c) r(\hat{n}_b + 2\hat{n}_c - 1) \phi(\hat{n}_c) - b^\dagger b^\dagger c \phi(\hat{n}_c - 1) , \tag{6.16}$$

$$K_0 = c^\dagger b r(\hat{n}_b + 2\hat{n}_c) \phi(\hat{n}_c) + \phi(\hat{n}_c) r(\hat{n}_b + 2\hat{n}_c) b^\dagger c , \tag{6.17}$$

where

$$\phi(n) = \left[\frac{2\Omega - n + 1}{(2\Omega - 2n - 1)(2\Omega - 2n + 1)} \right]^{1/2} . \tag{6.18}$$

We shall generalize these results in Sec. VI.D, where some details of the derivation will be given.

For application to the analysis of the physical content of the Hamiltonian (6.3), we need a second mapping that can be written down by inspection. Toward this end, note that $K_\pm, J_0$ also constitute an SU(2) algebra and, consequently, by examination of the commutation relations (6.2), we see that by interchanging $K_\pm \leftrightarrow J_\pm$ and $b \leftrightarrow c$ in the relations (6.13)–(6.18), a distinct mapping is obtained. In fact, there are no further chains of subalgebras available for the classification of mappings.

We now study the Hamiltonian (6.3). The concept of a phase transition first discussed in Sec. III in connection with the LMG model is also applicable here. We shall, however, consider only well-defined limiting regimes, corresponding to definite phases, where one term or the other of Eq. (6.3) dominates. For $F=0$, H has the eigenvalues

$$H(F=0) = -Gn_b(2\Omega - 2n_c - n_b + 1) = -G(n - n_c)(2\Omega - n - n_c + 1) , \tag{6.19}$$

where $n = n_b + n_c$ is half the number of fermions. For fixed n, we may interpret n_c as a vibrational quantum number. For low-lying states, $(n_c/\Omega) \ll 1$. We expect this condition to hold even if $F \neq 0$ as long as we are in the phase dominated by the first term of Eq. (6.3). Thus we expect to be able to expand all formulas in powers of (n_c/Ω). The mapping under study may thus be called the seniority mapping by analogy with the seniority limit of the shell model.

By the same token, in the limit $G \to 0$, using the alternative mapping, one obtains

$$H(G=0) = -F(n - n_b)(2\Omega - n - n_b - 1) , \tag{6.20}$$

which interchanges the roles of n_b and n_c, as previously stated. Here, the low-lying states have $(n_b/\Omega) \ll 1$. Approximations incorporating this condition would then be valid for the low-lying states in the deformed phase $G \ll F$.

C. Application to the Moszkowski model

Turning next to the Moszkowski model (Moszkowski, 1958), Marshalek has proposed a boson mapping ap-

propriate to the study of the deformed limit of this model (Marshalek, 1982a). The derivation given in this reference and the application for which it was intended will be described below. First we shall show that Marshalek's result can be obtained by transformation of the mapping (6.13)–(6.18). The reason for expecting such a transformation to be possible is that physically one is dealing with the same basis $SO(4)\supset SU(2)\supset U(1)$. The first point, then, is to transform the basis (6.12) to the form utilized by Marshalek. Generalizing the method described for SU(2) in Sec. III.C, one introduces the "phase operator" $\mathcal{U}$ and, in addition, a boson $B^\dagger$, defined by the equations

$$b^\dagger=\mathcal{U}\sqrt{\hat{n}_b+1}\ , \tag{6.21}$$

which duplicates the previous definition, and

$$c^\dagger=\mathcal{U}B^\dagger=B^\dagger\mathcal{U}\ . \tag{6.22}$$

We conclude that the phase operator raises J_0 by unity and that $B^\dagger$ therefore lowers J by a unit without changing J_0. Next, we define a new vacuum state $|0)_B$,

$$|0)_B=\mathcal{U}^\Omega|0)\ . \tag{6.23}$$

With the further definition

$$L_z\equiv 2J_0=-2\Omega+2\hat{n}_b+2\hat{n}_c\equiv\hat{M}\ , \tag{6.24}$$

whose eigenvalues are designated by M, we find from Eq. (6.12), after setting $J=\Omega-n_c$,

$$|n_b,n_c)\equiv|J,\tfrac{1}{2}M)=\mathcal{U}^{M/2}\frac{(B^\dagger)^{\Omega-J}}{\sqrt{(\Omega-J)!}}|0)_B\ . \tag{6.25}$$

By introducing Eqs. (6.21) and (6.22) into (6.14), (6.16), and (6.17) [Eq.(6.13) has already been rewritten], and by utilizing the mapping

$$\hat{M}\rightarrow(-id/d\theta)\ , \tag{6.26a}$$

$$\mathcal{U}\rightarrow(I-P_0)\exp(2i\hat{\theta}) \tag{6.26b}$$

and the fact that $\hat{n}_b$ and $\hat{n}_c$ can be eliminated, the latter from the relation $\hat{J}=\Omega-\hat{n}_c$ and the former from the relation $n_b=\hat{J}+\frac{1}{2}\hat{M}$, we obtain the equations

$$J_+=\tfrac{1}{2}e^{i\hat{\theta}}[(2\hat{J}+1)^2-\hat{M}^2]^{1/2}e^{i\hat{\theta}}\ , \tag{6.27}$$

$$K_+=\frac{1}{2}e^{i\hat{\theta}}\left[\frac{(2\hat{J}+1-\hat{M})}{2\hat{J}+1}\right]^{1/2}B^\dagger\left[\frac{(2\hat{J}+1-\hat{M})(\hat{J}+\Omega+1)}{2\hat{J}+1}\right]^{1/2}$$
$$-\frac{1}{2}e^{i\hat{\theta}}\left\{\frac{(\hat{J}+\Omega+1)[(2\hat{J}+1)^2-\hat{M}^2]}{2\hat{J}+1}\right\}B\frac{e^{i\hat{\theta}}}{(2\hat{J}+1)^{1/2}}\ . \tag{6.28}$$

We have suppressed the projection operator for the reason explained in Sec. III.E. The factor of 2 in Eq. (6.26b) can be understood by reference to Eq. (6.24).

These results agree with those found by Marshalek with the transcription $B^\dagger\rightarrow b^\dagger$, $\Omega\rightarrow T_0$, $\hat{J}\rightarrow\hat{T}$, $\hat{M}\rightarrow L_z$. The derivation given in the original paper is based on the calculation of a set of matrix elements of $\vec{J}_1$ and $\vec{J}_2$ in the coupled basis $|\Omega/2,\Omega/2,\Omega,\frac{1}{2}M\rangle$ by standard methods of the theory of angular momentum, followed by the utilization of the original Holstein-Primakoff method to transcribe these to a boson basis in which the mapping of the above coupled states is to the basis (6.25).

Let us describe, briefly, the treatment of the Hamiltonian (6.9) in terms of the bosons just introduced. In units of G^2 and with Marshalek's notation, we have ($\hat{n}=b^\dagger b$)

$$H=-\tfrac{1}{2}T_0(T_0+1)+T_0\hat{n}-\tfrac{1}{2}\hat{n}(\hat{n}-1)+(L_z^2/8)$$
$$-\left[\frac{T_0}{2x}(2T_0+1-2\hat{N})^{1/2}b^\dagger\left\{\frac{(2T_0+1-\hat{n})[4(T_0-\hat{n})^2-L_z^2]}{2T_0+1-2\hat{n}}\right\}^{1/2}+\text{H.c.}\right]\ . \tag{6.29}$$

Here $x=G^2T_0/C\gg 1$ in the domain of interest to us, that is, the deformation energy dominates the spin-orbit coupling. The steps taken to investigate Eq. (6.29) are as follows.

(i) One replaces the boson operators by c numbers,

$$b\rightarrow\beta,\quad b^\dagger\rightarrow\beta^*\ , \tag{6.30}$$

and the classical minimum $\beta=\beta_0$ is found. The substitution

$$b=\beta_0+B,\quad b^\dagger=\beta_0^*+B^\dagger\ ,\quad [B,B^\dagger]=1 \tag{6.31}$$

is then made and an expansion in powers of B and $B^\dagger$ is carried out. The linear terms disappear when β_0 has the value determined by the minimization. In this approximation the ground state is the coherent state

$$|0)_B=\exp[\beta_0(b^\dagger-b)]|0)_b\ , \tag{6.32}$$

and

$$\beta_0 = T_0^{1/2}(1-\alpha_0)^{1/2} ,$$
$$\alpha_0 = [1-(1/x^2)]^{1/2} . \tag{6.33}$$

(ii) The transformed Hamiltonian is quadratic and higher in $B^\dagger, B$. The quadratic terms are diagonalized by a linear transformation (RPA), and residual cubic and quartic terms are treated by perturbation theory, which mixes bands. As is well known, the second-order effect of cubic terms is comparable to the first-order effect of quartic terms. In practice, the cubic terms are eliminated by a unitary transformation to new bosons such that the Hamiltonian depends only on their number operator, similar to the so-called physical monopole boson described at the beginning of Sec. V. Energy values are computed as well as transition matrix elements of a suitably defined quadrupole operator.

The results just described provide a paradigm for the treatment of deformed systems when an exact boson mapping is at hand. In this respect, it should be compared with corresponding treatments of the pairing interaction described in Sec. IV. The main purpose of the exercise, however, was to compare the results thus obtained, which are guaranteed to be valid, with the results of another boson method (Marshalek and Weneser, 1969, 1970; Marshalek 1987a, 1987b) that can be applied to fully realistic shell models where no exact mappings are available for the description of the deformed regime. In this way the correctness of the Marshalek-Weneser method can be substantiated. This part of the exercise will be described in Sec. XIII as an introduction to the general case, also treated there.

D. Generalization of the quantized Bogoliubov transformation

In Sec. III.E we described a boson-fermion mapping that was referred to as the quantized Bogoliubov transformation. In objective terms, we are given a Lie algebra, in practice associated with a shell model, together with a subalgebra (the quasispin) associated with collectivity in a limiting physical situation (pairing). The collective pairs are replaced by bosons under a mapping. We wish first to generalize the mapping of the subalgebra so that it is defined in the full shell-model space of the algebra. This requires an extension of the mapping so that it contains all representations of the subalgebra realized in the full space. But it also means that we must map the individual fermion operators to quasifermions with two salient properties. One is that, in contrast with the original fermions, they must commute with the bosons. The second, related property is that, in order to avoid a redundant description, there must be no quasifermion pairs corresponding to the collective bosons.

It is very natural then to ask how we may extend this idea when the collectivity is not exhausted by the quasispin. Here two cases should be distinguished. In the first, one deals with a larger subalgebra of the shell model. A simple example is the algebra treated in this section. The second case involves additional collectivity but no precisely defined subalgebra. As an example, in the usual j-j coupled shell model, the interpretation of the observated quadrupole collectivity cannot be associated with a subalgebra, at least in a straightforward way (see below, however). Nevertheless, it would be of interest to bosonize the $J=2$ pairs as well as the $J=0$ pairs and have modified quasifermions that describe the remaining "strength."

Subalgebras subsuming $J=0$ and $J=2$ pairs can be realized by recoupling of angular momenta utilizing the related concepts of pseudospin and pseudo-orbital angular momenta first applied to the pseudo-SU(3) scheme (Arima, Harvey, and Shimizu, 1969; Arvieu, 1969; Hecht and Adler, 1969; Ratna Raju, Draayer, and Hecht, 1973). Two significant developments have built on this recoupling in quite distinct ways only tangentially related to boson mappings. In the work of Draayer and his associates (Draayer and Weeks, 1984; Draayer and Rosensteel, 1985; Leschber and Draayer, 1986; Castaños, Draayer, and Leschber, 1988), the pseudo-SU(3) model has provided a new algebraic basis for the description of deformed nuclei. Boson mappings for the underlying symplectic shell-model algebras, described at the end of Sec. XVII, have not yet been applied in a systematic way to this problem. The second implementation, based on several models of Ginocchio (1980), is the fermion dynamical symmetry model (Wu, Feng, Chen, Chen, and Guidry, 1986, 1987). This scheme has produced impressive phenomenological results, but it is not yet understood how this truncation can emerge as a dynamical approximation to the conventional shell model. This model is, however, a prime candidate for the application of the ideas that follow.

We turn then to the specific task of studying an extended mapping for SO(4). Here again previous work (Matsuyanagi, 1982; Kaup and Ring, 1987) makes a sharp distinction between the pairing degree of freedom and a possible second collective degree of freedom or, in terms of the operators appearing earlier in this section, between the vector $\vec{J}$ and the vector $\vec{K}$. In both of these works one carries out a two-step process. First one performs the quantized Bogoliubov transformation, introducing quasifermions as in Sec. III.E. One then seeks further to bosonize the remaining quasifermion pairs. This procedure was carried out fully by Kaup and Ring, requiring the introduction of modified quasifermions. Their work, which will be examined in the next section, has the additional feature that it utilizes a Schwinger-type boson mapping, but this latter point is not germane to the current investigation.

In the work following below (Hahne and Klein, 1989) we shall show how the quantized Bogoliubov transformation can be derived in a straightforward way by generalizing Eqs. (6.13)–(6.18). A further difference in this case, even compared to Sec. III.E, is that we shall produce a

unitary mapping directly, i.e., without the intermediary of a generalized Dyson mapping.

To fix clearly the particular model to be employed we write

$$J_+ = (J_-)^\dagger = \sum_{m=1}^{\Omega} \sum_{\sigma=\pm 1} \alpha^\dagger_{m\sigma} \beta^\dagger_{m\sigma} , \tag{6.34a}$$

$$J_0 = -\Omega + \tfrac{1}{2} \sum_{m,\sigma} (\alpha^\dagger_{m\sigma} \alpha_{m\sigma} + \beta^\dagger_{m\sigma} \beta_{m\sigma}) , \tag{6.34b}$$

$$K_+ = (K_-)^\dagger = \sum_{m,\sigma} \sigma \alpha^\dagger_{m\sigma} \beta^\dagger_{m\sigma} , \tag{6.35a}$$

$$K_0 = \sum_{m,\sigma} \sigma (\alpha^\dagger_{m\sigma} \alpha_{m\sigma} + \beta^\dagger_{m,\sigma} \beta_{m,\sigma}) . \tag{6.35b}$$

The picture here is of a Lipkin model with each level possessing an additional two-valued variable $\sigma = \pm$. The physics is trivially converted to two-level pairing, as previously described.

In addition to the algebra given by Eq. (6.2), one has now to consider the commutation relations between the fermion operators and the generators of SO(4), of which we record only one-half, the remaining set following by Hermitian conjugation or interchange of $\alpha_{m,\sigma}$ and $\beta_{m,\sigma}$:

$$[\alpha_{m\sigma}, J_-] = [\alpha_{m\sigma}, K_-] \tag{6.36a}$$

$$= 0 , \tag{6.36b}$$

$$[\alpha_{m\sigma}, J_0] = \tfrac{1}{2} \alpha_{m\sigma} , \tag{6.37a}$$

$$[\alpha_{m\sigma}, K_0] = \tfrac{1}{2} \sigma \alpha_{m\sigma} , \tag{6.37b}$$

$$[\alpha_{m\sigma}, J_+] = \beta^\dagger_{m\sigma} , \tag{6.38a}$$

$$[\beta_{m\sigma}, J_+] = -\alpha^\dagger_{m\sigma} , \tag{6.38b}$$

$$[\alpha_{m\sigma}, K_+] = \sigma \beta^\dagger_{m\sigma} , \tag{6.39a}$$

$$[\beta_{m\sigma}, K_+] = -\sigma \alpha^\dagger_{m\sigma} . \tag{6.39b}$$

Finally there are the anticommutation relations

$$\{\alpha_{m\sigma}, \alpha^\dagger_{m'\sigma'}\} = \{\beta_{m\sigma}, \beta^\dagger_{m'\sigma'}\} = \delta_{\sigma\sigma'} \delta_{mm'} , \tag{6.40}$$

all other anticommutators vanishing.

The method to be followed is first to generalize the boson mapping of Sec. VI.B and then to study the mapping of the fermion operators. Following the reasoning of Sec. IV.B, we introduce the bosons B and C with the same properties assigned there (but notice the shift to capital letters), and in addition quasifermions $a_{m\sigma}$, $b_{m\sigma}$ corresponding to the fermions $\alpha_{m\sigma}$, $\beta_{m\sigma}$. These must commute with the bosons and satisfy

$$\sum_m a_{m\sigma} b_{m\sigma} = \sum_m a^\dagger_{m\sigma} b^\dagger_{m\sigma} = 0 . \tag{6.41}$$

It also turns out to be consistent that all creation or annihilation operators mutually commute. Selection rules for the quasifermions are as specified in Sec. III.E. We thus have four number operators, n_B, n_C, $n_\pm$, which is precisely what is needed to specify the basis of an arbitrary irrep of SO(4) or SU(2)×SU(2). We work in a basis in which $\vec{J}^2$, J_0, and $\vec{J}^2_\pm$ are diagonal, the number operators being equivalent to these four operators (see below for details).

We first generalize Eqs. (6.13) and (6.14) to

$$J_0 = -\Omega + \hat{n}_C + \tfrac{1}{2}\hat{n} + \hat{n}_B , \tag{6.42}$$

$$J_+ = B^\dagger \sqrt{2\hat{J} - \hat{n}_B} , \tag{6.43}$$

where

$$\hat{n} = \sum_{m\sigma} (a^\dagger_{m\sigma} a_{m\sigma} + b^\dagger_{m\sigma} b_{m\sigma}) = \hat{n}_+ + \hat{n}_- , \tag{6.44}$$

and

$$\hat{J} \equiv \Omega - \hat{n}_C - \tfrac{1}{2}\hat{n} \tag{6.45}$$

satisfies

$$\vec{J}^2 = \hat{J}(\hat{J}+1) . \tag{6.46}$$

Thus the eigenvalues of $\hat{J}$ specify the total angular momentum. It is also convenient to define

$$\mathcal{N} = 2\hat{n}_C + \hat{n}_B + \hat{n} \tag{6.47}$$

and to write

$$J_+ \equiv B^\dagger r(\mathcal{N}) . \tag{6.48}$$

The generalizations of Eqs. (6.16) and (6.17) are

$$\begin{aligned} K_+ &= K^\dagger_- \\ &= C^\dagger r(\hat{\mathcal{N}}) r(\hat{\mathcal{N}}+1) \phi(\hat{n}_C, \hat{n}_\sigma) \\ &\quad - B^\dagger B^\dagger C \phi(\hat{n}_C - 1, \hat{n}_\sigma) + B^\dagger r(\hat{\mathcal{N}}) \chi(\hat{n}_C, \hat{n}_\sigma) , \end{aligned} \tag{6.49}$$

$$K_0 = C^\dagger B r \phi + \phi r B^\dagger C + (-\hat{J} + \hat{n}_B)\chi . \tag{6.50}$$

We note the occurrence of an additional term involving the operator χ, which will naturally turn out to be proportional to $(\hat{n}_+ - \hat{n}_-)$, thus explaining its previous absence. In Eq. (6.50), we have also suppressed the arguments of r, ϕ, and χ. In the following these will be shown only when they are shifted up or down by some integer.

In writing Eqs. (6.49) and (6.50) we have already satisfied the Wigner-Eckart theorem as well as Eq. (6.2b). From the second part of Eq. (6.2d), we then find the difference equations (also dropping the hats, which distinguish operators from their eigenvalues)

$$\begin{aligned} -1 &= n_C \phi^2(n_C - 1)(2J+3) \\ &\quad - (n_c + 1)\phi^2(n_c)(2J-1) - \chi^2 , \end{aligned} \tag{6.51}$$

$$\chi(n_C - 1)(J+2) - \chi(n_C) J = 0 . \tag{6.52}$$

Equation (6.52) has the solution

$$\chi(n_C, n_\sigma) = A(n_\sigma)/J(J+1) . \tag{6.53}$$

With the substitution

$$\phi^2 = \psi^2/(n_C + 1) , \tag{6.54}$$

Eq. (6.51) is reduced to the form

$$1-[A^2/J^2(J+1)^2] =(2J-1)\psi^2(J)-(2J+3)\psi^2(J+1)\ . \quad (6.55)$$

The solution of this difference equation is a sum of the general solution of the homogeneous equation and a special solution of the inhomogeneous equation,

$$\psi^2(J)=[C_0^2/(2J+1)]-\tfrac{1}{4}+[A^2/J^2]\ , \quad (6.56)$$

where C_0^2 and A^2 are both functions of $n_\pm$.

The last task in this initial part of the work is to determine the functions C_0 and A. This can be done by evaluating suitably chosen algebraic constraints in the subspace containing no bosons. From the equations

$$\vec{J}_\pm^{\,2}=\tfrac{1}{4}\vec{J}^{\,2}+\tfrac{1}{4}\vec{K}^{\,2}\pm\tfrac{1}{4}(\vec{K}\cdot\vec{J}+\vec{J}\cdot\vec{K}) =(\tfrac{1}{2}\Omega-\tfrac{1}{2}n_\pm)(\tfrac{1}{2}\Omega-\tfrac{1}{2}n_\pm+1)\ , \quad (6.57)$$

we deduce

$$\vec{J}_+^{\,2}-\vec{J}_-^{\,2}=\tfrac{1}{2}(\vec{K}\cdot\vec{J}+\vec{J}\cdot\vec{K}) =-\tfrac{1}{2}(\hat{J}_n+1)(\hat{n}_+-\hat{n}_-)\ , \quad (6.58)$$

where

$$\hat{J}_n\equiv\Omega-\tfrac{1}{2}\hat{n}\ . \quad (6.59)$$

Equating the expectation values of the two forms of Eq. (6.58) in a state with arbitrary allowed $n_\pm$, but with $n_C=n_B=0$, we easily find

$$A(n_+,n_-)=\tfrac{1}{2}(n_--n_+)(J_n+1)\ . \quad (6.60)$$

To evaluate C_0^2, we than calculate the average of the second commutator in Eq. (6.2a) for the same class of states and find

$$C_0^2=(J_n+\tfrac{1}{2})(J_n+\tfrac{3}{2})[1-(n_--n_+)^2]\ . \quad (6.61)$$

The generalization of Eq. (6.18) that results from Eqs. (6.54), (6.56), (6.60), and (6.61) has the form

$$\phi(J,n_+,n_-)=\left[\frac{J_+J_-(J_n+1+J)}{J^2(2J-1)(2J+1)}\right]^{1/2}\ , \quad (6.62)$$

where (in a notation not to be confused with a previous designation for angular momentum operators)

$$J_\sigma=\Omega-n_C-n_\sigma\equiv J_C-n_\sigma\ . \quad (6.63)$$

We turn then to the task of mapping the fermion operators. We outline the slightly tedious calculation. From the commutators of $\alpha_{m\sigma}$ and $\beta_{m\sigma}$ with the components of $\vec{J}$, we determine the forms

$$\alpha_{m\sigma}=F_{1\sigma}(n_C,n_\sigma)ra_{m\sigma}+F_{2\sigma}b_{m\sigma}^\dagger B +F_{3\sigma}rb_{m\sigma}^\dagger C+F_{4\sigma}a_{m\sigma}C^\dagger B\ , \quad (6.64a)$$

$$\beta_{m\sigma}=F_{2\sigma}(n_\sigma+1)rb_{m\sigma}-F_{1\sigma}(n_\sigma-1)a_{m\sigma}^\dagger B +F_{4\sigma}(n_C+1,n_\sigma-1)ra_{m\sigma}^\dagger C -F_{3\sigma}(n_C-1,n_\sigma+1)b_{m\sigma}C^\dagger B\ . \quad (6.64b)$$

From Eq. (6.36b), we obtain eight homogeneous conditions on the $F_{i\sigma}, i=1,\ldots 4$ which yield the solutions

$$F_{2\sigma}(n_\sigma+1)=F_{1\sigma}\ , \quad (6.65a)$$

$$F_{4\sigma}(n_C+1,n_\sigma-1)=-F_{3\sigma}\ , \quad (6.65b)$$

$$F_{1\sigma}=f_{1\sigma}(n_\sigma)\left[\frac{J_\sigma(J_n+1+J)}{J(2J+1)}\right]^{1/2}\ , \quad (6.66a)$$

$$F_{3\sigma}=f_{3\sigma}(n\sigma)[\bar{J}_\sigma/J(2J+1)]^{1/2}\ , \quad (6.66b)$$

where $\bar{J}_\pm=J_\mp$, and

$$f_{3\sigma}=f_{1\sigma}(n_\sigma-1)\ . \quad (6.67)$$

These relations completely exhaust the content of Eqs. (6.36)–(6.39).

The analog of the normalization argument associated with Eqs. (3.53) and (3.54) provides values for the functions $f_{1\sigma}$. We find that

$$f_{1\sigma}\equiv f_\sigma(n_\sigma)=[2(\Omega-n_\sigma)]^{-1/2}\ . \quad (6.68)$$

The anticommutation relations of the fermions then yield the following nonvanishing anticommutators among the quasifermions:

$$\{a_{m\sigma},a_{m'\sigma}^\dagger\}=\delta_{mm'}-b_{m\sigma}^\dagger(\Omega-n_\sigma)^{-1}b_{m'\sigma}\ , \quad (6.69)$$

$$\{b_{m\sigma},b_{m'\sigma}^\dagger\}=\delta_{mm'}-a_{m\sigma}^\dagger(\Omega-n_\sigma)^{-1}a_{m'\sigma}\ , \quad (6.70)$$

$$\{a_{m\sigma},b_{m'\sigma}^\dagger\}=b_{m\sigma}^\dagger(\Omega-n_\sigma)^{-1}a_{m'\sigma}\ . \quad (6.71)$$

An interesting, largely open problem is the extension of these results to other exact subalgebras. [See Sec. XIX for the case of U(3).]

If the Hamiltonian of a system of interest belongs to the enveloping algebra of the subalgebra, it is clear from the concepts already discussed that there will be simple numerical relations between observables of a "nucleus" and its neighbors, an observation that carries the essence of the idea of supersymmetry. [For an introductory discussion, see Bonatsos (1988).]

E. Schwinger mapping

As remarked previously, Kaup and Ring (1987) have pursued ideas related to those presented in the previous subsection, their prior work differing from ours in three respects: (i) They sought to reach the final result in two steps, the first of which is the usual quantized Bogoliubov transformation. (ii) For the bosonization they proposed a Schwinger-type mapping. (iii) They utilized the Marumori-Yamamura-Tokunaga method. Without attempting to reproduce their reasoning in detail, we shall

now apply the method of the previous subsection in order to present the essence of their results. We consider only the simplified situation without quasifermions, analogous to Eqs. (6.13)–(6.18), since exactly the same methods as we have developed above can then be applied to include the latter.

As described in Sec. II.F, we replace the mapping (6.13), (6.14) by the representation

$$J_+ = v^\dagger u, \quad J_- = u^\dagger v, \quad J_0 = \tfrac{1}{2}(\hat{n}_v - \hat{n}_u) \ . \tag{6.72}$$

where

$$n_v = v^\dagger v, \quad n_u = u^\dagger u \ , \tag{6.73}$$

in the notation of Ring and Kaup. In this representation we also have

$$\hat{J} = \tfrac{1}{2}(\hat{n}_v + \hat{n}_u) \tag{6.74}$$

and

$$|\Omega, -\Omega\rangle \equiv |\text{vac}\rangle = (2\Omega!)^{-1/2}(u^\dagger)^{2\Omega}|0\rangle \ . \tag{6.75}$$

When we turn to the operator $\vec{K}$, which may change the values of $\hat{J}$ (consider, for example, K_+), notice the availability of the two operators $u^\dagger u^\dagger$, $v^\dagger v^\dagger$, both of which increase $\hat{J}$ by unity but also alter J_0 by ∓ 1, respectively. Since these operators can also be used to keep track of fermion number N [defined as the number of particles plus number of holes for the realization (6.34) and (6.35) of the algebra], according to the formula

$$N = 2\Omega - n_u + n_v \ , \tag{6.76}$$

these are a sufficient set to complete the mapping, but we shall not pursue this remark, since it is not the path chosen by Kaup and Ring.

They choose to introduce two additional bosons s and d and (in our transcription) write

$$K_+ = d^\dagger suu\Phi_1 + \Phi_2 v^\dagger v^\dagger s^\dagger d \ , \tag{6.77}$$

where Φ_i are, as usual, only functions of the number operators. What can such an ansatz mean? Since the products uu and $v^\dagger v^\dagger$ already carry the needed selection rules, the operator products $d^\dagger s$ and $s^\dagger d$ must effect no change in quantum numbers. The simplest way to achieve this result is to choose these bosons to carry no quantum numbers, but to perform a bookkeeping function only. From Eq. (6.76) it is clear that the vacuum state must now be occupied by a sufficient number of s bosons so that K_+ can act a requisite number of times in order to generate the correct number of states of the irrep. To achieve this result, we postulate a representation of the vacuum of the form

$$|\Omega, -\Omega\rangle = (2\Omega!)^{-1/2}(u^\dagger)^{2\Omega}[(2\Omega + k)!]^{-1/2} \times (s^\dagger)^{2\Omega+k}|0\rangle \ , \tag{6.78}$$

k to be determined. The only way this can make sense is if one imposes the constraints

$$\hat{J} = \tfrac{1}{2}(n_v + n_u) = n_s - \Omega - k = \Omega - n_d \ , \tag{6.79}$$

or

$$n_s + n_d = 2\Omega + k \ , \tag{6.80a}$$

$$n_s - n_d - k = n_u + n_v \ . \tag{6.80b}$$

By combining Eqs. (6.76) and (6.80), we can derive various formulas for N, for example,

$$N = 2n_d + 2n_v \ . \tag{6.81}$$

Kaup and Ring choose $k = 1$ (their Ω is our 2Ω), but it seems to us that any value of $k \geq -\Omega$ will serve. Below the choice $k = 1$ is made.

If we now apply the algebraic method of the previous section, the result

$$\Phi_1 = -\Phi_2 = \Phi_2(n_u + n_v, n_s, n_d) \tag{6.82}$$

satisfies a difference equation that with proper normalization [determination of the constant corresponding to the C_0^2 of Eq. (6.56)] can be written as

$$\Phi = [n_s(n_d+1)]^{-1/2}\left[\frac{(\Omega+1-J)(\Omega+1+J)}{(2J+1)(2J-1)}\right]^{1/2} = [(n_s-1)/n_s]^{1/2}[(n_s-n_d-1)(n_s-n_d-3)]^{-1/2} \ , \tag{6.83}$$

where Eqs. (6.78)–(6.80) have been utilized. For the sake of comparison, the second square root may be identified in Eq. (4.5a) of Kaup and Ring. As previously remarked, the calculation can be extended to include the quasifermion mapping.

VII. SO(5)=Sp(4) MAPPINGS AND APPLICATIONS

A. Physical models associated with algebra

In this section, we shall study some properties of the algebra SO(5) [isomorphic to Sp(4)], which has served as a basis both for approximate realistic physics and for toy models (Hecht, 1965a, 1965b, 1967; Agassi, 1968; Goshen and Lipkin, 1968; Schütte and Bleuler, 1968; Ichimura, 1969; Evans and Kraus, 1971; Krumlinde and Syzmański, 1971; Chattopadhyay, Krejs, and Klein, 1972; Eichler and Yamamura, 1972; Dasso and Klein, 1973, 1974; Dasso, Krejs, Klein, and Chattopadhyay, 1973; Krumlinde and Marshalek, 1973; Krumlinde and Syzmański, 1973; Vassanji and Klein, 1978; Klein, Rafelski and Rafelski, 1981). Given a set of single-particle fermion states labeled by m, σ where

$$-j \leq m \leq j, \quad \sigma = \pm 1, \quad 2\Omega = 2j + 1 \ , \tag{7.1}$$

one may define the ten operators

$$A^{\dagger}_{\sigma\sigma'}=[2\Omega(1+\delta_{\sigma\sigma'})]^{-1/2}\sum_m(-1)^{j-m}\alpha^{\dagger}_{m\sigma}\alpha^{\dagger}_{-m\sigma'}$$
$$=A^{\dagger}_{\sigma'\sigma}=(A_{\sigma\sigma'})^{\dagger}\ \text{(6 operators)}\ , \tag{7.2a}$$
$$B_{\sigma\sigma'}=(2\Omega)^{-1/2}\sum_m\alpha^{\dagger}_{m\sigma}\alpha_{m\sigma'}$$
$$\equiv(2\Omega)^{-1/2}N_{\sigma\sigma'}\ \text{(4 operators)}\ . \tag{7.2b}$$

Notice that the scaling of the pair operators is different from that previously employed in this work.

In the history of nuclear physics, the algebra of the set (2) has been utilized in at least three distinct ways, according to the physical interpretation given the quantum numbers, m,σ. In the earliest use, the model is that of a single j level and $\sigma=\pm1$ distinguishes neutrons from protons which, neglecting the Coulomb interaction, are assumed to occupy a degenerate single-particle orbit. The problem of interest is that of charge-independent pairing, since in this interpretation the ten operators (2) carry angular momentum zero. Here N_{+-}, N_{-+}, and $\frac{1}{2}(N_{++}-N_{--})$ are the generators of an SU(2) subalgebra describing the isospin and $A^{\dagger}_{++}$, $A^{\dagger}_{-+}$, $A^{\dagger}_{--}$ form a vector under this SU(2). The basic requirement of this model is that the Hamiltonian H be a scalar under the same subalgebra.

In the second interpretation of the algebra, j is again an angular momentum coupled to a resultant of zero, but $\sigma=\pm1$ distinguishes two separated single-particle levels of equal degeneracy. In this form a scalar Hamiltonian need only conserve fermion number. This model is often referred to as the monopole-plus-pairing model because of the usual form chosen for H, which makes the model a generalization both of the Lipkim model and of the two-level pairing model.

In the third interpretation, we again (as in the first instance) choose a completely degenerate orbit, but now interpret $\sigma=\pm1$ as a unit of angular momentum along or opposed to a fixed (spatial) axis. For example, the operator $A^{\dagger}_{++}$ creates two particles and two units of angular momentum; $A^{\dagger}_{+-}$ creates two particles and no angular momentum; $B^{\dagger}_{+-}$ conserves number but adds two units of angular momentum, etc. The physical angular momentum for this model is

$$J_0=(N_{++}-N_{--})\ , \tag{7.3}$$

and an admissible Hamiltonian is one that commutes with

$$N=N_{++}+N_{--} \tag{7.4}$$

and with J_0. Hamiltonian operators in this set can describe rotations and vibrations in two dimensions and thus can provide tractable analogs for the collective behavior found in real nuclear systems. Furthermore, by considering more than one level, we can study models with the symmetry $SO(5)\times SO(5)\times\cdots$, still relatively tractable. We can thus refer to this version of SO(5) as the vibration-rotation model.

B. Characterization of the mappings to be studied

In the following, we shall derive and study three different boson mappings for the irrep of SO(5) that contains the vacuum state for fermions (Klein, Cohen, and Li, 1982). Before describing how to derive these mappings and what we use them for, it is appropriate to explain why three is a natural number to consider [just as we found two mappings for SO(4) for the corresponding reason]. We start with the fact that a basis for the representation in question can be obtained by orthonormalization of a linearly independent subset of the vectors

$$(A^{\dagger}_{++})^{n_+}(A^{\dagger}_{--})^{n_-}(A^{\dagger}_{+-})^{n_0}|0\rangle\ . \tag{7.5}$$

For a physically interesting choice, the three quantum numbers needed to specify basis states should be eigenvalues of operators associated with closed subalgebras of SO(5). These basis vectors will also be eigenstates of limiting forms of a suitably chosen model Hamiltonian. There are precisely three subalgebras of SO(5) that can be used for this purpose:

(i) A_{++}, $A^{\dagger}_{++}$, and $(N_{++}-\Omega)$ form an SU(2) that commutes with the corresponding SU(2) formed from A_{--}, $A^{\dagger}_{--}$, and $(N_{--}-\Omega)$. This $SU(2)\times SU(2)\equiv SO(4)$ is a natural choice of subalgebra for the monopole-plus-pairing model in the pairing limit. In the rotation-vibration model, it plays a very special role that will be described later. In the basis under discussion, one can compute the eigenvalue of each Casimir operator,

$$C_\sigma\equiv\tfrac{1}{2}\Omega\{A^{\dagger}_{\sigma\sigma},A_{\sigma\sigma}\}+\tfrac{1}{4}(N_{\sigma\sigma}-\Omega)^2$$
$$=\tfrac{1}{2}(\Omega-(n_0)_{\max})\tfrac{1}{2}(\Omega-(n_0)_{\max}+2)\ , \tag{7.6}$$

where $(n_0)_{\max}$, half the usual seniority quantum number, is the maximum value of n_0 that occurs in any component of the form of Eq. (7.5) contributing to a basis vector. Thus the two quasispins are equal and yield just one quantum number. The other two quantum numbers are provided by the operators

$$N_{++}=2n_++n_0\ , \tag{7.7a}$$
$$N_{--}=2n_-+n_0\ , \tag{7.7b}$$

both values allowing mixing of components (7.5) with n_0 changing by two and n_+, n_- changing by one each in the opposite sense. By applying the commutator method, we find that the information provided above suffices for a determination of the mapping.

(ii) $A^{\dagger}_{+-}$, A_{+-}, and $(2\Omega-N_{++}-N_{--})$ generate an SU(2). In the vibration-rotation model, this is the natural SU(2) associated with the pairing limit of the model. In this case the seniority is $2(n_++n_-)_{\max}$, with a second quantum number provided by

$$N=2(n_++n_-+n_0)\ . \tag{7.8}$$

This SU(2) supplies only two quantum numbers, but a third is provided by the angular momentum operator

$$J_0=2(n_+-n_-)\ . \tag{7.9}$$

(iii) The operators $\sqrt{2\Omega}B_{+-}$, $\sqrt{2\Omega}B_{-+}$, and $\frac{1}{2}J_0$ generate an SO(3) subalgebra. For a mapping based on this subalgebra, the pair of quantum numbers l, m, using standard angular momentum notation, is supplemented by Eq. (7.8). This SO(3) is useful in all three interpretations of SO(5)—as the total isospin for charge-independent pairing, for the monopole limit of the monopole-plus-pairing model, and for the rotational limit of the vibration-rotation model. This form of the mapping is actually a special case of mappings developed for realistic shell-model algebras, where they are commonly referred to as Belyaev-Zelevinsky-Marshalek (BZM) mappings. These will be studied in the general case, starting with Sec. IX.

We shall describe mappings from each of the orthonormal bases characterized above to a basis of orthonormal states in a space of three bosons. We begin with the boson bases, which are very simple. (It will be unnecessary to specify the fermion bases in any more detail than we have done). For example, it is almost immediately evident from our discussions under (i) and (ii) above that for mappings associated with some seniority quantum number, the integers n_+, n_-, n_0, introduced in Eq. (7.5), are natural choices for boson occupation numbers. Thus we introduce boson creation (annihilation) operators $a^\dagger_\lambda(a_\lambda)$, $\lambda=\pm$, 0 with the identifications

$$N_{++}=2n_++n_0\equiv 2a^\dagger_+a_++a^\dagger_0a_0\ , \tag{7.10a}$$

$$N_{--}=2n_-+n_0\equiv 2a^\dagger_-a_-+a^\dagger_0a_0\ . \tag{7.10b}$$

These operators are diagonal in the direct product basis

$$|n_+,n_-,n_0)=\frac{(a^\dagger_+)^{n_+}}{\sqrt{n_+!}}\frac{(a^\dagger_-)^{n_-}}{\sqrt{n_-!}}\frac{(a^\dagger_0)^{n_0}}{\sqrt{n_0!}}|0)\ . \tag{7.11}$$

We define the mappings (i) and (ii) to be from the distinct fermion bases described under these subheadings to the single basis (7.11).

The boson basis needed for the mapping described under (iii) is characterized by the states $|n,l,m)$, where n is the total boson number,

$$n=n_++n_-+n_0=\tfrac{1}{2}(N_{++}+N_{--})\ , \tag{7.12}$$

l and m are the eigenvalues, respectively, of

$$\vec{J}^{\,2}=\tfrac{1}{2}\{J_+,J_-\}+J_z^2=l(l+1)\ , \tag{7.13}$$

$$J_z=\tfrac{1}{2}J_0=(n_+-n_-)\ , \tag{7.14}$$

and

$$J_+=\sqrt{2\Omega}B_{+-}=\sqrt{2\Omega}(B_{-+})^\dagger=\sqrt{2}(a^\dagger_+a_0+a^\dagger_0a_-)\ . \tag{7.15}$$

Note that this is a Schwinger realization of SU(2).

From the Bose statistics, for given l and m, n takes on the values

$$n=l,l+2,l+4,\ldots\ . \tag{7.16}$$

Without providing any details, we record the basis in the form ($n=l+2\nu$)

$$|n,l,m)=\left[\frac{(2l+1)!!}{2^\nu\nu!(2l+2\nu+1)!!}\right](C^\dagger)^\nu|l,l,m\rangle\ , \tag{7.17}$$

where the scalar operator

$$C^\dagger=2a^\dagger_+a^\dagger_--(a^\dagger_0)^2 \tag{7.18}$$

commutes with $J_\pm$, J_0 and implies the selection rule (7.16). It is easy to verify that

$$C_2\equiv C^\dagger C=\hat{n}(\hat{n}+1)-\vec{J}^{\,2}\ . \tag{7.19}$$

Further, the states $|l,l,m)$ have the form

$$|l,l,m)=\left[\frac{(l-m)!}{(l+m)!(2l)!}\right]^{1/2}(J_+)^{l+m}|l,l,-l) \tag{7.20}$$

and

$$|l,l,-l)=\frac{(a^\dagger_-)^l}{\sqrt{l!}}|0)\ . \tag{7.21}$$

Finally we shall require reduced matrix elements $a_{n,l}(\sigma)$ of the boson operators, defined by the formula

$$(nlm|a^\dagger_\lambda|n-1,l+\sigma,m-\lambda)\equiv(-1)^{l-m}\begin{pmatrix} l+\sigma & l & 1\\ m-\lambda & -m & \lambda\end{pmatrix}a_{nl}(\sigma)\ , \tag{7.22}$$

in terms of the Wigner $3j$ symbol. Only the values $\sigma=\pm1$ yield nonvanishing elements according to the selection rules established. The matrix elements required can be calculated easily from sum rules for number and angular momentum. We find

$$a_{nl}(1)=[(n-l)(l+1)]^{1/2}\ , \tag{7.23a}$$

$$a_{nl}(-1)=[l(n+l+1)]^{1/2}\ . \tag{7.23b}$$

C. Derivation of seniority mappings

We are finally ready to derive the mappings. In fact, two derivations of each mapping were given in the paper under review (Klein, Cohen, and Li, 1982). It turns out that the commutator method is most convenient for mappings (i) and (ii), which we study first. Consider, for example, mapping (ii) appropriate to the pairing limit of the vibration-rotation model. It is convenient to change notation for the generators:

$$N_{\pm\pm}\to N_\pm,\quad B_{\pm,\mp}\to B_\pm\ ,$$
$$A_{\pm,\pm}\to A_\pm,\quad A_{+-}\to A_0\ . \tag{7.24}$$

Now in the mapping to the basis (ii), we interpret (n_++n_-) as the seniority. We start with a Holstein-Primakoff mapping of the associated SU(2) defined in

connection with Eq. (7.11), i.e., with a knowledge of four operators, $N_\pm$ given by Eq. (7.10), and $A_0^\dagger$, A_0 given by

$$A_0^\dagger = a_0^\dagger[1-(2n+n_0)/2\Omega]^{1/2} \equiv a_0^\dagger r(\mathcal{N}) \ . \tag{7.25}$$

Of the remaining six operators, four are determined from the prototype form

$$A_+^\dagger = a_+^\dagger F_1(n_+,n_-,n_0) + (a_0^\dagger)^2 a_- F_2(n_+,n_-,n_0) \tag{7.26}$$

and the remaining two from the commutation relations

$$[A_0, A_\pm^\dagger] = -\Omega^{-1/2} B_\pm \ . \tag{7.27}$$

Let us consider the justification of the form (7.26), which is a paradigm of the arguments that led to the result for SO(4) in the preceding section. The operator $A_+^\dagger$ increases or decreases the seniority by two units and increases particle number by two. (It can also leave the seniority unchanged, but such terms do not occur for the irrep under study.) Now the operator $a_+^\dagger$ only increases seniority. Thus two tensors $a_+^\dagger$ and $(a_0^\dagger)^2 a_-$ (up to powers of the number operators), can contribute to $A_+^\dagger$. Each does part of the job that $A_+^\dagger$ can do by itself. The simplicity of the results to be obtained thus arises from the relatively small number of independent tensors obeying the required selection rules that can be constructed from the given bosons. The form of the functions F_1 and F_2 can be obtained from a suitable set of commutation relations that yield easily solvable difference equations. The techniques are the same as described for SO(4). The result is

$$A_+^\dagger = a_+^\dagger r(\mathcal{N}) r(\mathcal{N}+\tfrac{1}{2}) f(n_+,n_-) - (2\Omega)^{-1}(a_0^\dagger)^2 a_- f(n_- -1, n_+) \ , \tag{7.28a}$$

$$\sqrt{\Omega} B_+ = a_+^\dagger a_0 r(\mathcal{N}) f(n_+,n_-) + r(\mathcal{N}) f(n_-,n_+) a_0^\dagger a_- \ , \tag{7.28b}$$

where ($n_d = n_+ + n_-$)

$$f(n_+,n_-) = r(n_+ - \tfrac{1}{2})/r(n_d - \tfrac{1}{2}) r(n_d + \tfrac{1}{2}) \ . \tag{7.29}$$

The mapping corresponding to basis (i) in the fermion space and appropriate to the monopole-plus-pairing model can be derived by the same method. It is actually contained as a special case of a more general class of mappings first obtained by Evans and Kraus (1971). See also Klein, Rafelski and Rafelski (1981). Here an appropriate notation refers to levels 1 and 2 (lower and upper), so that we write

$$N_+ \to N_2, \quad B_+ \to B_{21}, \quad A_+^\dagger \to A_2^\dagger, \quad A_0^\dagger \to A_{21}^\dagger, \quad \text{etc.} \ , \tag{7.30}$$

$$a_+^\dagger \to a_2^\dagger, \quad a_-^\dagger \to a_1^\dagger, \quad a_0^\dagger \to a_{21}^\dagger \ , \tag{7.31}$$

and

$$n_i = a_i^\dagger a_i, \quad n_{21} = a_{21}^\dagger a_{21} \ . \tag{7.32}$$

The mapping is specified by the equations

$$N_i = 2n_i + n_{21} \quad (i=1,2) \ , \tag{7.33a}$$

$$A_i^\dagger = a_i^\dagger r_i \ , \tag{7.33b}$$

$$r_i = [1-\Omega^{-1}(n_i+n_{21})]^{1/2} \ . \tag{7.34}$$

From the algebra, we then derive the representations

$$A_{21}^\dagger = a_{21}^\dagger r_1 r_2 \phi(n_{21}) - \Omega^{-1}\phi(n_{21}) a_2^\dagger a_1^\dagger a_{21} \ , \tag{7.35a}$$

$$\sqrt{\Omega} B_{21} = r_1 \phi(n_{21}) a_2^\dagger a_{21} + a_{21}^\dagger a_1 \phi(n_{21}) r_2 \ , \tag{7.35b}$$

where

$$\phi(n) = \left[\frac{\Omega(\Omega+1-\frac{1}{2}n)}{(\Omega-n+1)(\Omega-n)}\right]^{1/2} \ . \tag{7.36}$$

Before going on to derive the remaining mapping, we discuss just once more the observation made in connection with Eq. (2.11). It is that, when properly viewed, the boson basis (7.11) within the physical subspace can be understood to represent an orthonormal basis in the fermion space. This can be seen, for example, by formally inverting the mapping (ii), specified by Eqs. (7.25)–(7.29). We derive [see Eq. (7.25) for the definition of $\mathcal{N}$]

$$(a_0^\dagger)_F = A_0^\dagger [r(\mathcal{N})]^{-1} = [r(\mathcal{N}-\tfrac{1}{2})]^{-1} A_0^\dagger \ , \tag{7.37}$$

$$(a_+^\dagger)_F = \{A_+^\dagger r(\mathcal{N}-\tfrac{1}{2}) r(\mathcal{N}-1) + [2\Omega r(\mathcal{N}-\tfrac{1}{2}) r(\mathcal{N}-1)]^{-1} (A_0^\dagger)^2 A_-\} \times \{f(n_+,n_-)[r(\mathcal{N}+\tfrac{1}{2}) r(\mathcal{N}) r(\mathcal{N}-\tfrac{1}{2}) r(\mathcal{N}-1)] - (2\Omega)^{-2} n_0(n_0-1)\}^{-1} \ , \tag{7.38}$$

with a corresponding equation for $(a_-^\dagger)_F$. The notation is chosen to emphasize that these are fermion operators that behave like bosons within the finite-dimensional vector space of the irrep. The specification of the right-hand sides of Eqs. (7.37) and (7.38) as a fermion operator is completed by the replacements

$$v = 2(n_+ + n_-) \tag{7.39}$$

and Eqs. (7.8) and (7.9), namely,

$$n_\pm = \tfrac{1}{4}(v \pm J) \ , \tag{7.40a}$$

$$n_0 = \tfrac{1}{2}(N-v) \ . \tag{7.40b}$$

Thus, as a result of the mapping, we can now give the orthonormal basis in the fermion space.

D. Derivation of the Belyaev-Zelevinsky-Marshalek (BZM) mapping

We turn finally to mapping (iii), the BZM mapping. Although the result can be derived by a suitable adaptation of the commutator method, in this case we have found that it is more systematic to use the method already illustrated for SU(2) in which the Dyson mapping is used as an intermediary.

We begin with the appropriate SO(3) mapping as well as the mapping for the number operator,

$$(N)_D=2(a_0^\dagger a_0+a_+^\dagger a_+ +a_-^\dagger a_-)\ , \tag{7.41a}$$

$$(J_0)_D=2(a_+^\dagger a_+ -a_-^\dagger a_-)\ , \tag{7.41b}$$

$$\sqrt{\Omega}(B_+)_D=\sqrt{\Omega}(B_-)_D^\dagger=(a_+^\dagger a_0+a_0^\dagger a_-)\ . \tag{7.41c}$$

[As remarked below Eq. (7.9), the usual angular momentum operators are half of Eq. (7.41b) and the root two times Eq. (7.41c), respectively.] The remaining six operators are given by an easily verified Dyson form

$$(A_\lambda)_D=a_\lambda\ , \tag{7.42a}$$

$$\begin{aligned}(A_\lambda^\dagger)_D&=a_\lambda^\dagger(l-\Omega^{-1}n)+(2\Omega)^{-1}C^\dagger\tilde{a}_\lambda\\&=a_\lambda^\dagger(1-\Omega^{-1}n)+(4\Omega)^{-1}[C_2,a_\lambda^\dagger]\ ,\end{aligned} \tag{7.42b}$$

where C_2 is the Casimir operator defined by Eq. (7.19), and

$$\tilde{a}_\lambda=(-1)^{1-\lambda}a_{-\lambda}\ . \tag{7.43}$$

Next we seek a similarity transformation S, which first of all leaves Eq. (7.41) unaffected

$$SNS^{-1}=N\ , \tag{7.44a}$$

$$SJ_\lambda S^{-1}=J_\lambda\ . \tag{7.44b}$$

These conditions tell us that S commutes with N and J_λ. Consequently [we use $\hat{l}$ instead of $\vec{l}^{\,2}\equiv\hat{l}(\hat{l}+1)$],

$$S=S(\hat{n},\hat{l})\ . \tag{7.45}$$

Given Eq. (7.45), we shall determine S from the conditions

$$S(A_\lambda^\dagger)_D S^{-1}=A_\lambda^\dagger\ , \tag{7.46a}$$

$$S(A_\lambda)_D S^{-1}=A_\lambda\ , \tag{7.46b}$$

and

$$A_\lambda^\dagger=(A_\lambda)^\dagger\ . \tag{7.47}$$

From Eqs. (7.46) and (7.47) we deduce

$$V^{-1}(A_\lambda)_D^\dagger V=(A_\lambda^\dagger)_D\ , \tag{7.48}$$

where $V=S^\dagger S$ or, substituting Eq. (7.42) into (7.48), we have

$$V^{-1}a_\lambda^\dagger V=a_\lambda^\dagger[1-\Omega^{-1}n]+(4\Omega)^{-1}[C_2,a_\lambda^\dagger]\ . \tag{7.49}$$

Equation (7.49) can be studied by taking nonvanishing matrix elements in the orthonormal basis $|n,l,m\rangle$ in which V is diagonal. In this case the reduced matrix elements of $a_\lambda^\dagger$ simply cancel out, and we obtain the equations

$$V^{-1}(n+1,l+1)V(n,l)=1-(2\Omega)^{-1}(n+l)\ , \tag{7.50a}$$

$$V^{-1}(n+1,l-1)V(n,l)=1-(2\Omega)^{-1}(n-l-1)\ . \tag{7.50b}$$

Assuming (without loss of generality) that $S^\dagger=S=$real or $V=S^2$, we can take the square root of Eq. (7.50) in the form

$$S^{-1}(n+1,l+1)S(n,l)=[1-(2\Omega)^{-1}(n+l)]^{1/2}\ , \tag{7.51a}$$

$$S^{-1}(n+1,l-1)S(n,l)=[1-(2\Omega)^{-1}(n-l-1)]^{1/2}\ . \tag{7.51b}$$

With these matrix elements, we can return to Eq. (7.46b), for instance. To utilize this condition, we need a general form of A_λ,

$$\begin{aligned}A_\lambda&=f(\hat{n},\hat{l})a_\lambda+g(\hat{n},\hat{l})\tilde{a}_\lambda^\dagger C\\&=f(\hat{n},\hat{l})a_\lambda+g(\hat{n},\hat{l})\tfrac{1}{2}[a_\lambda,C_2]\ .\end{aligned} \tag{7.52}$$

Taking matrix elements of both sides of Eq. (7.46b) and, proceeding as before, we derive

$$\begin{aligned}(2l+1)f(n,l)=&(l+n+1)[1-(2\Omega)^{-1}(n+l)]^{1/2}\\&-(n-l)[1-(2\Omega)^{-1}(n-l-1)]^{1/2}\ ,\end{aligned} \tag{7.53a}$$

$$\begin{aligned}(2l+1)g(n,l)=&[1-(2\Omega)^{-1}(n-l-1)]^{1/2}\\&-[1-(2\Omega)^{-1}(n+l)]^{1/2}\ .\end{aligned} \tag{7.53b}$$

These results can also be used to define an orthonormal fermion basis.

E. Applications

We next apply the results found above to the vibration-rotation model. Consider first a single SO(5) for which the most general admissible quadratic Hamiltonian may be written

$$\begin{aligned}H_{V\text{-}R}=&-GA_0^\dagger A_0-G_2(A_+^\dagger A_+ +A_-^\dagger A_-)\\&-\tfrac{1}{2}F\{B_+,B_-\}+DJ_0^2+CN^2\ .\end{aligned} \tag{7.54}$$

For the study of excitation spectra, we can set $C=0$. Furthermore, since the quadratic Casimir invariant is

$$\begin{aligned}\Lambda=\Omega+3=&\{A_0^\dagger,A_0\}+\{B_+,B_-\}+\{A_+^\dagger,A_+\}\\&+\{A_-^\dagger,A_-\}+(4\Omega)^{-1}[J_0^2+(2\Omega-N)^2]\ ,\end{aligned} \tag{7.55}$$

where $\Omega+3$ is the value for the irrep under study [obtained by calculating the vacuum expectation value of Eq. (7.55)], we can drop one of the remaining terms in Eq. (7.54). This we do by setting $G_2=0$. Finally, since J_0^2 will always be diagonal, we can add its effect at any time.

Thus we are left with the two-term Hamiltonian

$$H=-GA_0^\dagger A_0-\tfrac{1}{2}F\{B_+,B_-\}\ , \tag{7.56}$$

the sum of a pairing plus quadrupole-quadrupole interaction, which was first studied numerically by Chattopadhyay (Chattopadhyay, Klein, and Krejs, 1972). For $F=0$ or $G=0$, we have well-known limiting cases corresponding for $F=0$ to single-level pairing with an almost harmonic excitation spectrum, and for $G=0$ to the two-dimensional Elliott model discussed in the previous section.

In Fig. 2, a series of spectra that demonstrate this limiting behavior is shown, calculated for different values of the ratio $x=(F/4G\Omega)$. These calculations exhibit one surprising feature. As F increases from zero, one expects to see the originally degenerate multiplets of fixed seniority but different angular momentum split apart increasingly with F (anharmonic splitting of phonon multiplets). This fails to occur because of a special symmetry associated with the value $F=G$. For this value, using Eq. (7.55), we may rewrite Eq. (7.56)

$$(H/G)=[1-(2\Omega)^{-1}N]-\tfrac{1}{2}\Lambda+\Omega^{-1}(C_++C_-)\ , \tag{7.57}$$

where ($\sigma=\pm$) [cf. Eq. (7.6)]

$$C_\sigma=\tfrac{1}{2}\Omega\{A_\sigma^\dagger,A_\sigma\}+\tfrac{1}{4}(N_\sigma-\Omega)^2\ . \tag{7.58}$$

The eigenvalues of C_σ can be evaluated in terms of boson variables from the mapping results (7.33)–(7.36), and we find

$$\begin{aligned}C_+=C_-&=\tfrac{1}{4}(\Omega-n_0)(\Omega-n_0+2)\\&=\tfrac{1}{4}(\Omega-n+\tfrac{1}{2}v)(\Omega-n+2+\tfrac{1}{2}v)\ .\end{aligned} \tag{7.59}$$

Thus the *excitation* spectrum for a fixed n depends only on v, which means that phonon splitting will be largely absent for this model in the immediate neighborhood of the phase transition from spherical to deformed shape. This is in sharp contrast to the usual three-dimensional situation.

As a second application of the mappings derived above, we consider a system with two levels, each described by the SO(5) algebra. We take the Hamiltonian

$$\begin{aligned}H=&e_1N_1+e_2N_2\\&+\sum_{\lambda,\lambda'}[-GA_{0\lambda}^\dagger A_{0\lambda'}-\tfrac{1}{2}F\{B_{+\lambda},B_{-\lambda'}\}]\ ,\end{aligned} \tag{7.60}$$

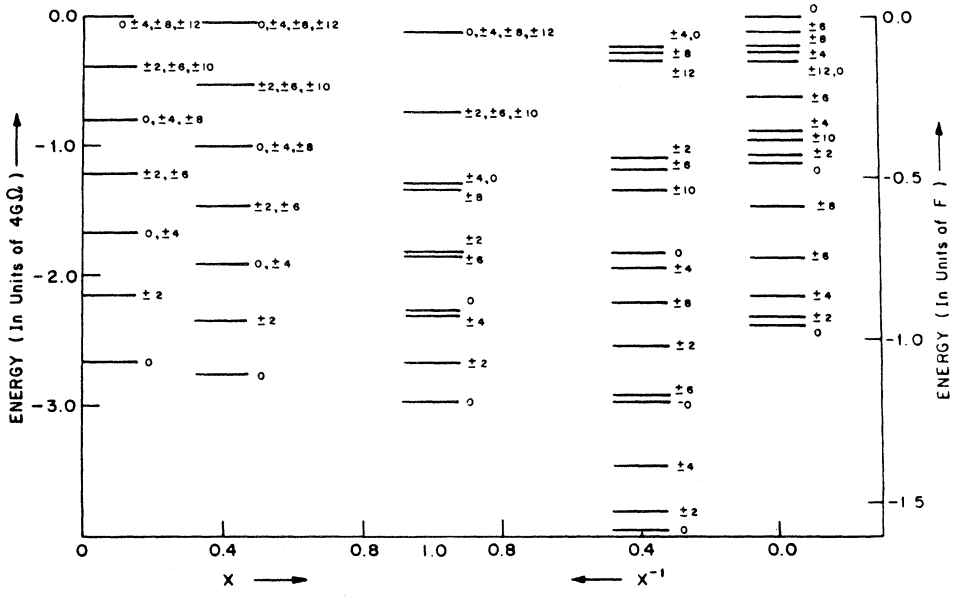

FIG. 2. Energy levels of the Hamiltonian (7.56) for various values of the ratio $(x/4G\Omega)$. Notice that at $x=1$ the scale of the abcissa has been changed to x^{-1}. For $x\leq 1$, the energy scale on the left applies, and for $x>1$ the energy scale on the right is appropriate. The levels correspond to the system $\Omega=22$, $N=12$. From Chattopadhyay, Klein, and Krejs, 1972.

where $\lambda=1,2$ distinguishes the two levels. We now have a system with two $J=0$ degrees of freedom and two $J=2$ degrees of freedom. In the spirit of the interacting boson model, we are interested in finding a single collective degree of freedom of each kind to describe a set of low-lying collective states. Toward this end we shall apply the trace variational principle described in Sec. IV.

For illustrative purposes we utilize the seniority-adapted mapping called (i) above, with Eq. (7.60) as an operator on the space of six bosons with basis

$$|n_{+1},n_{-1},n_{01},n_{+2},n_{-2},n_{02}\rangle=\mathcal{N}(a^{\dagger}_{+1})^{n_{+1}}(a^{\dagger}_{+2})^{n_{+2}}(a^{\dagger}_{-1})^{n_{-1}}(a^{\dagger}_{-2})^{n_{-2}}(a^{\dagger}_{01})^{n_{01}}(a^{\dagger}_{02})^{n_{02}}|0\rangle ,$$
$$\mathcal{N}=(n_{+1}!n_{+2}!n_{-1}!n_{-2}!n_{01}!n_{02}!)^{-1/2} . \tag{7.61}$$

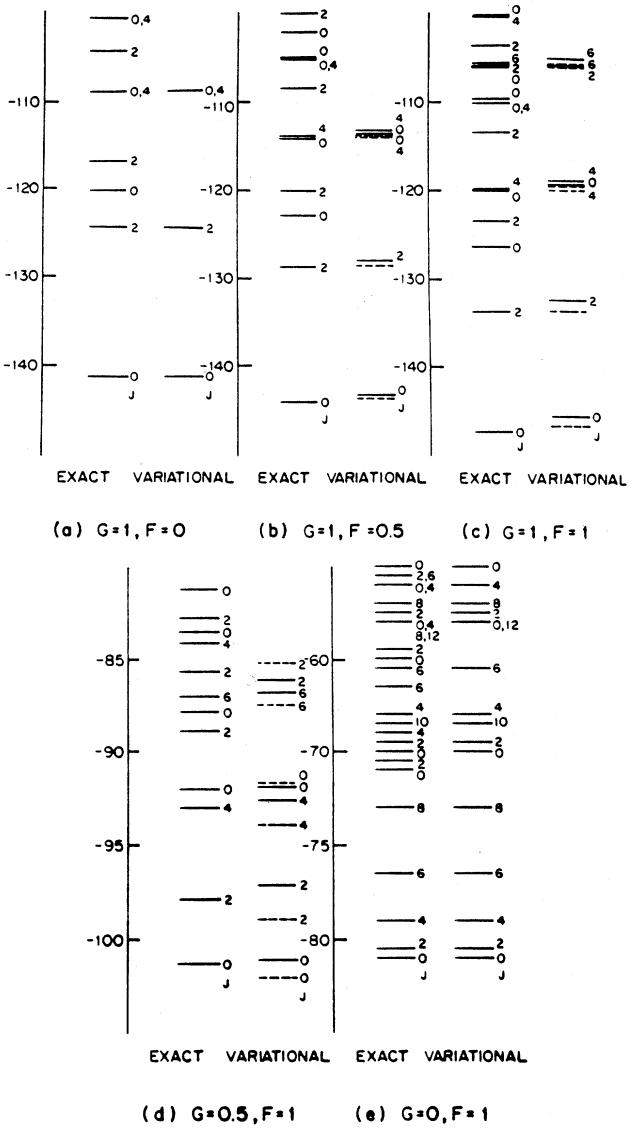

FIG. 3. Energy-level diagrams for the SO(5)×SO(5) Hamiltonian: (a) $\Omega_1=\Omega_2=10$, $\epsilon_1=-10$, $\epsilon_2=0$, $N=12$, $G=1$, $F=0$; (b) $G=1$, $F=0.5$; (d) $G=1$, $F=1$; (e) $G=0$, $F=1$. The calculations are done exactly, by the trace variational method and by a truncation of the trace variational method (the interacting boson model analog). The IBM analog is given on the same plot as the trace variational calculation. Where they differ, the IBM analog states are represented by dashed lines. From Cohen and Klein, 1982.

We introduce new bosons $c^{\dagger}_{\sigma i}$, which are linear in the old,

$$c_{\sigma i}=\sum_{l=1,2}\alpha^{*}_{\sigma il}a^{\dagger}_{\sigma l} \quad (\sigma=\pm,0,\quad i=1,2) , \tag{7.62}$$

conserving angular momentum in the transformation. The $\alpha^{*}_{\sigma il}$ define a unitary (ultimately orthogonal) transformation. In the method in question, in the inverse transformation,

$$a^{\dagger}_{\sigma l}=\sum_{i}\alpha_{\sigma li}c^{\dagger}_{\sigma i} , \tag{7.63}$$

we suppose $c^{\dagger}_{\sigma 1}\equiv c^{\dagger}_{\sigma}$ to be the collective boson and drop the dependence on $c^{\dagger}_{\sigma 2}$,

$$a^{\dagger}_{\sigma l}\cong\alpha_{\sigma l}c^{\dagger}_{\sigma} . \tag{7.64}$$

When Eq. (7.64) is substituted into Eq. (7.60), one obtains an operator function of the $c^{\dagger}_{\sigma}$ and c_{σ} depending also on the unknown coefficients $\alpha_{+l}=\alpha_{-l}$ and α_{0l}. This can be determined from the variational principle $\delta\,\mathrm{Tr}H=0$, where the trace is taken in the space of states

$$|n_{+}n_{-}n_{0}\rangle=(n_{+}!n_{-}!n_{0}!)^{-1/2}(c^{\dagger}_{+})^{n_{+}}(c^{\dagger}_{-})^{n_{-}}(c^{\dagger}_{0})^{n_{0}}|0\rangle , \tag{7.65}$$

from which the collective states are supposed to be constructed. The results for the spectrum are quite satisfactory (Cohen and Klein, 1982), as illustrated in Fig. 3. Here we see five spectra calculated for various values of the pairing and quadrupole interaction strengths. The trace variational principle yields only a subset of the energy levels, of course. The reader should ignore the dashed lines, which represent an approximation of no interest to the present discussion. The quadrupole matrix element connecting the ground and first excited state was also computed and excellent agreement found.

For the many other applications of the SO(5) model to schematic models, the reader is referred to the references quoted in the first paragraph of this section.

F. Boson mappings for a general representation

The developments described in this section have so far been confined to the so-called vacuum representation, the one which contains the state with no particles. We have emphasized that the states in this representation are specified by three quantum numbers. Of course there is a

fourth quantum number Ω, which is equivalent to the specification of the (two) Casimir invariants. That only one quantum number is thus provided signals that the vacuum representation is degenerate in the sense that the two Casimir operators of SO(5) are not algebraically independent. The states of a general irrep of SO(5) must be specified by six quantum numbers (one half the sum of the number of generators and the number of Casimir operators). By generalizing the ideas associated with boson-fermion mappings of the type studied so far for SU(2) and SO(4), one can understand how to assign the additional two quantum numbers and what their physical significance is.

Thus for a boson-fermion mapping we introduce in addition to the three bosons utilized so far a set of quasifermions $a^\dagger_{m\sigma}$, $a_{m\sigma}$, $\sigma=\pm$, which are subject to the constraints that they cannot form the pairs that define the pair generators of SO(5). We can, however, form a U(2) algebra, generated by

$$\mathcal{N}_{\sigma\sigma'}=\sum_m a^\dagger_{m\sigma}a_{m\sigma'}\ , \tag{7.66}$$

or, removing the total number,

$$\hat{n}=\sum_{m,\sigma} a^\dagger_{m\sigma}a_{m\sigma}\ , \tag{7.67}$$

from the algebra, an SU(2). It can be shown that, despite the altered algebra of the quasifermions, the algebra (7.66) of the multipole operators will be the same as that of the $N_{\sigma\sigma'}$, Eq. (7.2b). This algebra is conventionally identified with the concept of *reduced* isospin t, because of the historical connection with the problem of charge-independent pairing (Hecht, 1965a, 1965b).

Thus we introduce a state of maximum weight in the quasifermion space,

$$|m_1,\ldots,m_K\rangle=\prod_{i=1}^{K} a^\dagger_{m_i-}|0\rangle\ . \tag{7.68}$$

This state carries $t=\frac{1}{2}K$. In additional to the raising operators $A^\dagger_{\sigma\sigma'}$, introduced in Eq. (7.5), the $\mathcal{N}_{+-}$ will now be raising operators acting on Eq. (7.68), but they cannot change the value of t, which is fixed for the representation and provides a fifth quantum number. The operator

$$t_z=\tfrac{1}{2}(\mathcal{N}_{++}-\mathcal{N}_{--}) \tag{7.69}$$

defines the sixth quantum number. None of the work on boson mapping of general SO(5) reported in the literature (Evans and Kraus, 1971; Geyer and Hahne, 1980b; Klein and Zhang, 1986) realizes the reduced isospin in this way, which is the natural road to take to generalize the quantized Bogoliubov transformation (Hahne and Klein, 1989).

Turning to the results available in the literature, we find that they are of three kinds. The first represents the generalization of the mapping that utilizes the group chain $SO(5)\supset SU(2)\times SU(2)$ and either realizes the reduced isospin in terms of an extra boson (Evans and Kraus, 1971) or leaves it in abstract form (Klein and Zhang, 1986). We refer to the literature for details of this work. The second (Hecht and Elliott, 1985) is a derivation of the general irrep in the BZM representation carried out by the new method based on the concept of the generalized coherent state (Deenen and Quesne, 1984, 1985; Rowe, 1984; Rowe, Rosensteel, and Gilmore, 1985; Quesne, 1986; Rowe, Le Blanc, and Hecht 1988). This method is closely allied to those mapping techniques utilized throughout this review to build a final unitary mapping through the intermediary of a Dyson mapping; it will be discussed in Sec. XIX.

We dwell somewhat upon the third result in the literature, the work of Geyer and Hahne (1980b), because it is informed by a viewpoint that differs from the one described by us so far in this section. We have emphasized the importance of identifying the quantum numbers of the fermion basis, from which one maps, and of the boson basis to which one maps. Geyer and Hahne only worry about the second of these sets. In addition, they stress the simplicity of the generalized Dyson mapping. They succeed in imposing a simple structure on this mapping without having to consider which operators are diagonal in the fermion basis. We quote their results: In their notation (only the creation operators given)

$$\begin{aligned} S_+&=(2\Omega)^{1/2}A^\dagger_{+-}\ ,\\ L_+&=(4\Omega)^{1/2}A^\dagger_{++}\ ,\\ l_+&=(4\Omega)^{1/2}A^\dagger_{--}\ , \end{aligned} \tag{7.70}$$

[cf. Eq. (7.2)]

$$\begin{aligned} S_z&=\tfrac{1}{2}(N_{++}+N_{--})-\Omega\ ,\\ L_z&=\tfrac{1}{2}(N_{++}-\Omega)\ ,\\ l_z&=\tfrac{1}{2}(N_{--}-\Omega)\ . \end{aligned} \tag{7.71}$$

They insist that (in our notation for bosons)

$$S_-=a_0,\quad L_-=a_+,\quad l_-=a_-\ . \tag{7.72}$$

The commutation relations are then satisfied by the equations

$$\begin{aligned} S_+=&a^\dagger_0(2\Omega-v-n_0-2n_+-2n_-)\\ &-a^\dagger_+a^\dagger_-a_0-a^\dagger_+t_--a^\dagger_-t_+\ , \end{aligned} \tag{7.73a}$$

$$\begin{aligned} L_+=&a^\dagger_+(\Omega-\tfrac{1}{2}v-t_z-n_+-n_0)\\ &-(a^\dagger_0)^2a_--a^\dagger_0t_+\ , \end{aligned} \tag{7.73b}$$

$$\begin{aligned} l_+=&a^\dagger_-(\Omega-\tfrac{1}{2}v+t_z-n_--n_0)\\ &-(a^\dagger_0)^2a_+-a^\dagger_0t_-\ , \end{aligned} \tag{7.73c}$$

$$L_z=n_++\tfrac{1}{2}n_0+\tfrac{1}{2}(\tfrac{1}{2}v+t_z-\Omega)\ , \tag{7.74a}$$

$$l_z=n_-+\tfrac{1}{2}n_0+\tfrac{1}{2}(\tfrac{1}{2}v-t_z-\Omega)\ , \tag{7.74b}$$

$$N_{+-}\equiv T_+=2a^\dagger_0a_-+a^\dagger_+a_0+t_+=(T_-)^\dagger\ . \tag{7.74c}$$

These are remarkably simple formulas. The implication, which remains to be justified, is that these formulas will be simple to apply. For this point we must defer to the discussion in Sec. XVI. Finally, we refer the reader to the original work of Geyer and Hahne for ways of realizing $\vec{t}$ in terms of additional bosons. As remarked earlier in this section, for nuclear applications, it seems more natural to realize the reduced spin in terms of quasifermions.

Finally, Geyer (1986b), following a suggestion of Kim and Vincent (1987), has shown how to construct (for the case $t=0$) a similarity transformation connecting the results just quoted with one of the sets of results discussed earlier in this section.

VIII. SU(n) SYMMETRIC REPRESENTATION

A. Introduction

Before we turn to the general shell-model problem, we consider a set of examples associated with the algebra SU(n) (Li, Klein, and Dreizler, 1970; Louck and Biedenharn, 1970; Meshkov, 1971; Okubo, 1975; Klein, 1980b; Klein and Vallières, 1980; Marshalek, 1980a; Klein, Li, and Vallières, 1982a, 1982b). We begin with some general remarks.

The Lie algebra of the n-dimensional unitary group U(n) is specified by the commutation relations

$$[A^{\mu}_{\nu}, A^{\alpha}_{\beta}]=\delta^{\mu}_{\beta}A^{\alpha}_{\nu}-\delta^{\alpha}_{\nu}A^{\mu}_{\beta} , \tag{8.1}$$

in terms of n^2 generators A^{μ}_{ν}, where all Greek indices μ,ν,α,β assume n values $1,\ldots,n$. For the SU(n) subgroup, we have only to replace A^{μ}_{ν} by a traceless tensor

$$B^{\mu}_{\nu}=A^{\mu}_{\nu}-\frac{1}{n}\delta^{\mu}_{\nu}\sum_{\lambda=1}^{n}A^{\lambda}_{\lambda} . \tag{8.2}$$

An operator T^{μ}_{ν} is called a vector operator under U(n) if it satisfies the commutation relations

$$[A^{\mu}_{\nu}, T^{\alpha}_{\beta}]=\delta^{\mu}_{\beta}T^{\alpha}_{\nu}-\delta^{\alpha}_{\nu}T^{\mu}_{\beta} . \tag{8.3}$$

We see from Eq. (8.1) that A^{α}_{β} itself is a vector operator. It follows from Eq. (8.3) that the trace of a vector operator

$$\mathrm{Tr}T\equiv\sum_{\lambda=1}^{n}T^{\lambda}_{\lambda} \tag{8.4}$$

commutes with all the generators A^{μ}_{ν}.

If we have two vector operators S^{μ}_{ν} and T^{μ}_{ν}, we can define a product vector R^{μ}_{ν} by

$$R^{\mu}_{\nu}=(ST)^{\mu}_{\nu}=\sum_{\lambda=1}^{\mu}S^{\lambda}_{\nu}T^{\mu}_{\lambda} . \tag{8.5}$$

It is easy to verify that R^{μ}_{ν} satisfies the definition (8.3) of a vector operator. Combining this observation with the observation associated with Eq. (8.4), we conclude that the n operators

$$M_1=\mathrm{Tr}A,\quad M_2=\mathrm{Tr}(AA),\quad M_3=\mathrm{Tr}(AAA),\ \ldots ,$$
$$M_n=\mathrm{Tr}\underbrace{(AA\cdots A)}_{n} \tag{8.6}$$

commute with all the U(n) generators A^{μ}_{ν} and hence with each other. Here the product $(AAA)^{\mu}_{\nu}$, for example, is defined by

$$(AAA)^{\mu}_{\nu}=\sum_{\lambda=1}^{n}A^{\lambda}_{\nu}(AA)^{\mu}_{\lambda}=\sum_{\lambda,\rho=1}^{n}A^{\lambda}_{\nu}A^{\rho}_{\lambda}A^{\mu}_{\rho} . \tag{8.7}$$

These n quantities $M_j (j=1,2,\ldots,n)$ are the algebraically independent Casimir operators of the U(n) group and specify its irreducible representations (Biedenharn, 1962; Klein, 1963; Umezawa, 1963, 1964a, 1964b.) Irreducible representations of U(n) may be characterized by n integers

$$f_1\geq\cdots\geq f_n . \tag{8.8}$$

Then M_j is a polynomial of degree j in $f_1,\ldots,f_n$. The explicit form for M_1 is simply

$$M_1=f_1+f_2+\cdots+f_n . \tag{8.9}$$

Convenient formulas for evaluating the M_j are known (Perelomov and Popov, 1966). Above and in the following discussion we use the same symbols for operators and for their eigenvalues, depending on the context to distinguish them.

The simplest class of representations are the completely symmetric ones for which

$$f_1=f,\quad f_2=f_3=\cdots=f_n=0 . \tag{8.10}$$

For this set of representations all the M_j are algebraically dependent. This can be seen by means of an example. Consider n bosons $(a_{\mu},a^{\dagger}_{\mu})$, $\mu=1,\ldots,n$, satisfying

$$[a_{\mu},a^{\dagger}_{\nu}]=\delta_{\mu,\nu},\ \text{etc} . \tag{8.11}$$

It is easily checked that the operators

$$A^{\mu}_{\nu}=a^{\dagger}_{\nu}a_{\mu} \tag{8.12}$$

satisfy the commutation relations (we have already studied the case $n=2$). Further, we have

$$(AA)^{\mu}_{\nu}=(n-1+N)A^{\mu}_{\nu} \tag{8.13}$$

and

$$M_1=N=\sum_{\lambda=1}^{n}a^{\dagger}_{\lambda}a_{\lambda} . \tag{8.14}$$

It follows from Eqs. (8.13) and (8.14) that by specifying a value of $N=f$ we have completely specified the irrep, and all the M_j are given polynomials in f (or N). The basis for this irrep is the subspace with $N=f$ of the n-dimensional harmonic oscillator. For U(2), this is the Schwinger realization (Schwinger, 1965). [For SU(2) and SU(2) alone, as we have shown, we can obtain all the representations this way.]

B. Relation of interacting boson model to Bohr-Mottelson model

The most important application of the algebra (8.12) to nuclear physics is the interacting boson model known as IBM-1 (Arima and Iachello, 1976a, 1976b, 1978, 1979; Iachello and Arima, 1987; Bonatsos, 1988), which works with U(6) and consists of a monopole boson $s^\dagger(s)$ and a tensor of rank 2, $d_\mu^\dagger(d_\mu)$, $(\mu=-2,-1,\ldots,2)$. Thus the operators $s^\dagger s$, $s^\dagger d_\mu$, $d_\mu^\dagger s$, and $d_\mu^\dagger d_\nu$ generate U(6). The physics of a given nucleus is confined to subspaces that are defined by a fixed value of the number operator

$$N=s^\dagger s+\sum_\mu d_\mu^\dagger d_\mu \equiv s^\dagger s+d^\dagger\cdot d=\hat{n}_s+\hat{n}_d \ . \tag{8.15}$$

Thus each nucleus is described by a given irrep of SU(6). The simplest way of characterizing this space is in terms of the states

$$|[n_\mu],n_s)=\prod_{\mu=-2}^{2}(n_\mu!)^{-1/2}(d_\mu^\dagger)^{n_\mu}(n_s!)^{-1/2}(s^\dagger)^{n_s}|0) \ , \tag{8.16}$$

subject to the restriction

$$N=\sum_\mu n_\mu+n_s=n_d+n_s=\text{const} \ . \tag{8.17}$$

For present purposes, we define IBM-1 as follows: For the nucleus characterized by N, there is a class of eigenstates that can be modeled by suitable linear combinations of the states (8.16), carrying, in particular, a definite value of the total angular momentum. With this definition the Hamiltonian can be any well-behaved rotationally invariant function of the generators and is not restricted to a simple polynomial in the generators.

In Sec. V, we "invented" a corresponding SU(2) model in order to illustrate the idea of two mathematically equivalent versions of the model connected by a mapping from the Schwinger realization of the algebra to the Holstein-Primakoff version. Here we shall do the same, since, particularly for the symmetric representation, a similar mapping exists and may be said to establish the link between IBM-1 and the conventional Bohr-Mottelson model (Bohr and Mottelson, 1975). We consider a definition of the Bohr-Mottelson model in its simplest terms, in which we work with boson operators $(b_\mu^\dagger,b_\mu)$, $(\mu=-2,-1,\ldots,2)$, the creation operator constituting a tensor of rank two under three-dimensional rotations. Otherwise (i) $b_\mu^\dagger$ (in contrast to $d_\mu^\dagger$) conserves the number of nucleons; (ii) the Hamiltonian $H_{\rm BM}$ of this model may be any well-behaved, rotationally invariant function of the $b_\mu^\dagger,b_\mu$ and the transition operators may be any well-behaved tensors of appropriate rank; (iii) the vector space is the Hilbert space of the five-dimensional oscillator, i.e., the set of states

$$|[n])=\prod_{\mu=-2}^{2}(n_\mu!)^{-1/2}(b_\mu^\dagger)^{n_\mu}|0)_N \ , \tag{8.18}$$

where the designation $|0)_N$ implies that the Hilbert space is constructed independently for each nucleus.

We would like to restrict the set (8.18) and connect N to the eigenvalues of (8.17) so as to establish a relationship between IBM-1 and the Bohr-Mottelson model. In analogy with the considerations of Sec. V, we can do this by postulating the relation

$$|0)_N=(N!)^{-1/2}(s^\dagger)^N|0) \tag{8.19}$$

and by restricting n_b, the number of b bosons to satisfy the equality $n_b=n_d$, where then

$$0\le n_b\le N \ . \tag{8.20}$$

We are thus led to consider a modified Bohr-Mottelson basis

$$|[n],N)=\prod_{\mu=-2}^{2}(n_\mu!)^{-1/2}(b_\mu^\dagger)^{n_\mu}(N!)^{-1/2}(s^\dagger)^N|0) \ . \tag{8.21}$$

The proof of the equivalence of the bases (8.16) and (8.21) is made via a mapping that generalizes the Holstein-Primakoff mapping for SU(2) to the symmetric irrep of SU(n). For our case, $n=6$, it reads

$$d_\mu^\dagger d_\nu=b_\mu^\dagger b_\nu \ , \tag{8.22a}$$

$$d_\mu^\dagger s=(s^\dagger d_\mu)^\dagger=b_\mu^\dagger[N-\sum_\lambda b_\lambda^\dagger b_\lambda]^{1/2} \ , \tag{8.22b}$$

$$s^\dagger s=N-\sum_\mu d_\mu^\dagger d_\mu=N-\sum_\mu b_\mu^\dagger b_\mu \ . \tag{8.22c}$$

It is an elementary exercise to show that under (8.22) the bases (8.16) and (8.21) map into each other, one to one.

If we start with a Hamiltonian in IBM form, it maps under (8.22) to a Bohr-Mottelson form. The mapping also gives a natural method for discussing phase transitions in terms of geometric shape, as described in Sec. III for the Lipkin model. Toward this end, one changes to canonical coordinates,

$$b_\mu^\dagger=2^{-1/2}(x_\mu-ip_\mu^\dagger) \ , \tag{8.23a}$$

$$\tilde{b}_\mu=(-1)^\mu b_{-\mu}=2^{-1/2}(x_\mu+ip_\mu^\dagger) \ , \tag{8.23b}$$

$$\tilde{x}_\mu=x_\mu^\dagger, \quad \tilde{p}_\mu=p_\mu^\dagger \ , \tag{8.24}$$

$$[x_\mu,p_\nu]=[x_\mu^\dagger,p_\nu^\dagger]=i\delta_{\mu\nu} \ . \tag{8.25}$$

Setting $p_\mu=0$ defines a potential-energy surface whose geometrical structure determines the "shape" of the system [Klein and Vallières, 1981; Klein, Li, and Vallières, 1982b).

The problem of geometrical shape in the interacting boson model was first studied using a coherent-state method (Dieperink, Scholten, and Iachello, 1980; Ginocchio and Kirson, 1980a, 1980b). In contrast to the mapping method described above, which provides a full "geometrical" Hamiltonian, this method is limited to computing the potential energy. To obtain a full Hamiltonian by this route requires the introduction of the

method of generator coordinates. By editorial decision this topic has been excluded from the present review. However, a brief survey of the relevant literature can be found at the beginning of Sec. XIX.

C. Generalized Lipkin model

We study next a distinctly different application of the symmetric representation of SU(n), namely, as it occurs in the n level generalization of the Lipkin model (Okubo, 1975; Klein, 1980b). This generalization was first considered in some detail for SU(3) (Li, Klein, and Dreizler, 1970; Meshkov, 1971). We consider n nondegenerate levels, each with sublevel degeneracy N. Let $a_{ir}^{\dagger}$ be the creation operator for a fermion in level r and its ith degenerate sublevel, a_{ir} the corresponding destruction operator. The bilinear operators

$$A_r^s=(A_s^r)^{\dagger}=\sum_{i=1}^{N} a_{ir}^{\dagger}a_{is},\quad r,s=1,\ldots,n\ , \tag{8.26}$$

are a set of generators of the Lie algebra U(n). We select a Hamiltonian

$$H=H_0+H_1\ , \tag{8.27}$$

where H_0 is the Hamiltonian of the individual levels with energies η_r,

$$H_0=\sum_{r=1}^{n}\eta_r A_r^r\ , \tag{8.28}$$

and H_1 is an interaction that we choose, for illustration only, in the form

$$H_1=-G\sum_{r=2}^{n}[(A_r^1)^2+(A_1^r)^2]\ , \tag{8.29}$$

"anchored" at the first level, taken to be the lowest one.

The number of parameters can be reduced by one and the algebra reduced to SU(n) by choosing a zero of energy according to the relation

$$\sum_{r=1}^{n}\eta_r=0 \tag{8.30}$$

and by defining new parameters and operators,

$$\epsilon_r\equiv 2\eta_{r+1}\ , \tag{8.31}$$

$$J_0^{(r)}\equiv\tfrac{1}{2}(A_{r+1}^{r+1}-A_1^1)\ , \tag{8.32a}$$

$$J_+^{(r)}=(J_-^{(r)})^{\dagger}\equiv A_{r+1}^1,\quad r=1,\ldots,n-1\ . \tag{8.32b}$$

Thus H_0 and H_1 become

$$H_0=\sum_{r=1}^{n-1}\epsilon_r J_0^{(r)}\ , \tag{8.33a}$$

$$H_1=-G\sum_{r}[(J_+^{(r)})^2+(J_-^{(r)})^2]\ . \tag{8.33b}$$

For $n=2$, this is the standard LMG model and for $n=3$ has been studied before, as previously asserted.

We now consider the simplest problem, in which there are N particles, exactly the degeneracy of each level. In effect, this is a "closed-shell" nucleus in which, in the absence of interactions, all particles are in the lowest level. Such a state is one of minimum weight of SU(n), for each $J_0^{(r)}$ [from Eq. (8.32a)] has the same eigenvalue $=-\frac{1}{2}N, r=1,\ldots,n-1$. The equality of these eigenvalues implies that one is dealing with the symmetric representation, which can be mapped by the generalized Holstein-Primakoff mapping (Janssen, Jolos, and Dönau, 1974; Okubo, 1975)

$$A_{r+1}^{s+1}=b_r^{\dagger}b_s,\quad r,s=1,\ldots,n-1\ , \tag{8.34a}$$

$$A_{r+1}^1=(A_1^{r+1})^{\dagger}=b_r^{\dagger}\theta(N)\ , \tag{8.34b}$$

$$A_1^1=(\theta(N))^2\ , \tag{8.34c}$$

$$\theta(N)=\left[N-\sum_{r=1}^{n-1}b_r^{\dagger}b_r\right]^{1/2}\equiv[N-\mathcal{N}]^{1/2}\ . \tag{8.35}$$

With the help of this mapping, we obtain a Hamiltonian describing $n-1$ interacting bosons. It is most straightforward to discuss the vibrational domain (small interaction), where an expansion in powers of $(\mathcal{N}/N)$ can be carried out. The possibility of phase transitions has been discussed by the method already described several times, namely, by making a transformation to canonical pairs,

$$b_r^{\dagger}=2^{-1/2}[N^{1/2}x_r-iN^{-1/2}p_r]\ , \tag{8.36}$$

and then setting $p_r=0$ (Klein, 1980b).

Of these models, the SU(3) model, in particular, is rapidly overtaking the original MGL model as a favorite for testing many-body approximation schemes. Two recent kinds of applications have some connection with the subject of this review—the study of semiclassical approximation schemes (Williams and Koonin, 1982) and of large amplitude collective motion (Holzwarth and Yukawa, 1974; Villars, 1977; Klein, Marumori, and Une, 1982, 1984; Umar and Klein, 1986). The latter subject will be described in Sec. XXI.

D. Relation of the interacting boson model to the geometrical model: Method of Hahne and Scholtz

Although the material presented at the beginning of Sec. VIII.B provides a complete method for the the study of the relationships, mathematical and physical, between the IBM and the Bohr-Mottelson picture, it has been applied only sparsely and then only to IBM-1. Recently an alternative method has been introduced, which is most useful for the study of the deformed limit. Detailed investigations have been described for a U(3) model (Scholtz and Hahne, 1987), an IBM-1 model (Scholtz, 1987), a U(3)×U(3) model (Hahne and Scholtz, 1988a), and a U(6)×U(6) model (IBM-2; Hahne and Scholtz, 1988b). The major results not covered by previous work are contained in this last work.

We consider the underlying physical idea within the framework of IBM-1. The objective is to make the transition from the IBM (Schwinger) description of the bosons to the Holstein-Primakoff version appropriate to the study of the geometrical limit. This involves first of all the mapping (8.22). Next, one introduces shape-positional operators according to the equation

$$\hat{\alpha}_\mu=\frac{1}{\sqrt{2}}(b_\mu^\dagger+\tilde{b}_\mu) , \tag{8.37}$$

which are precisely the x_μ operators of Eqs. (8.23)–(8.25), although it is convenient here to change notation because of Eqs. (8.40) and (8.41) below. Reference to "sharp position" will mean definite orientation and definite quadrupole shape, as becomes clear from any standard discussion of the meaning of the variables α_μ (Bohr and Mottelson, 1975).

The essential new point is the introduction of eigenstates of the positional operators. It is shown that the states

$$|\alpha\rangle\equiv\frac{2}{\pi^{5/4}}\exp(\tfrac{1}{2}\alpha\cdot\alpha)\exp(-\tfrac{1}{2}b^\dagger\cdot b^\dagger+\sqrt{2}\alpha\cdot b^\dagger)|0\rangle , \tag{8.38}$$

where the complex numbers α_μ satisfy $\alpha_\mu^*=(-1)^\mu\alpha_{-\mu}$, are simultaneous eigenstates of the $\hat{\alpha}_\mu$ with eigenvalues α_μ. Together with the dual space of states

$$\langle\alpha|=\langle 0|\exp(-\tfrac{1}{2}\tilde{b}\cdot\tilde{b}+\sqrt{2}\alpha\cdot\tilde{b})\exp(-\tfrac{1}{2}\alpha\cdot\alpha)\frac{2}{\pi^{5/4}} , \tag{8.39}$$

they provide a resolution of unity

$$\int(d\alpha)|\alpha\rangle\langle\alpha|=1 , \tag{8.40a}$$

where the integration is over the entire five-dimensional space,

$$(d\alpha)\equiv\prod_{\mu>0}(dx_\mu dy_\mu)d\alpha_0 , \tag{8.40b}$$

and

$$\alpha_\mu=x_\mu+iy_{-\mu},\quad \mu>0 ,$$
$$\alpha_\mu=(-1)^\mu(x_{-\mu}-iy_{-\mu}),\quad \mu<0 . \tag{8.40c}$$

We also have an orthogonality relation

$$\langle\alpha|\alpha'\rangle=\prod_{\mu>0}[\delta(x_\mu-x'_\mu)\delta(y_\mu-y'_\mu)]\delta(\alpha_0-\alpha'_0) . \tag{8.41}$$

[The concept of strictly localized states in the space of collective coordinates can be found in the early work of Kerman and Klein (Kerman and Klein, 1962; Klein, Dreizler, and Johnson, 1968; Klein *et al.*, 1982). In these investigations no effort was made to find explicit forms for the states, but rather the work was based on a fundamental notion of collectivity: If Q is a one-body operator and the state $|\alpha\rangle$ is a sharp positional state in the sense defined above, then the matrix element $\langle\alpha|Q|\alpha'\rangle$ must be strongly peaked in the differences $\alpha_\mu-\alpha'_\mu$. Consequently the matrix element can be expanded in terms of delta functions of these differences and of their derivatives. It follows that an expansion in powers of the collective momenta is valid.]

Returning to the work of Scholtz and Hahne, they study the transition matrix element

$$\langle JM\Gamma|Q|J'M'\Gamma'\rangle=\int(d\alpha)(d\alpha')\langle JM\Gamma|\alpha\rangle\langle\alpha|Q|\alpha'\rangle\langle\alpha'|J'M'\Gamma'\rangle . \tag{8.42}$$

Here it is assumed that the observed collective states $|JM\Gamma\rangle$ span the same space as the $|\alpha\rangle$, Γ specifying the additional quantum numbers necessary for this purpose. In the deformed case, the states $|\alpha\rangle$ are conveniently transformed to the intrinsic frame. It then follows from Eq. (8.42) and the related discussion that one can derive the full phenomenology of the geometrical model.

Hahne and Scholtz have another aim, however. Since they are interested in the extent to which the geometrical model can be reached starting from a single irrep of SU(6) [or in the case of IBM-2, SU(6)×SU(6)], they must decompose the resolution of unity (8.40a) into a direct sum (of irreps) of SU(6) with the help of the projection operators

$$P_N=\sum_{n_\mu\le N}|n_2,n_1,n_0,n_{-1},n_{-2}\rangle\times\langle n_2n_1n_0n_{-1}n_{-2}|=1_N . \tag{8.43}$$

Then Eq. (8.40a) is replaced by

$$\int(d\alpha)P_N|\alpha\rangle\langle\alpha|=1_N . \tag{8.44}$$

The essence of their work is then the evaluation of a formula like (8.42) when all quantities in the latter are restricted to an irrep of SU(6). This involves transforming the states (8.38) to an intrinsic system, assuming there is a large static deformation, and expanding the resulting expression in powers of the ratio of the quantum-mechanical fluctuations to the static deformation, which turns out also to be an expansion in reciprocal powers of the number of bosons. We refer the reader to the published work for details of the procedure.

IX. BZM MAPPINGS FOR SHELL-MODEL ALGEBRAS. EVEN NUCLEI

A. Introduction

The problem of obtaining formally exact mappings of realistic shell-model algebras for both even and odd nuclei has been solved by a number of authors (Janssen, Dönau, Frauendorf, and Jolos, 1971; Marshalek, 1971a; Marshalek and Holzwarth, 1972; Donau and Janssen, 1973; Marshalek, 1973, 1974a 1974b, 1975a, 1980a, 1983;

Okubo, 1974b, 1974c). In this section we shall discuss only even systems. The extension to odd systems is made in the next section. Though we shall borrow from all these authors, we begin our account with ideas from more recent studies of the appropriate algebra (Bonatsos, Klein and Zhang, 1986b; Rowe and Carvalho, 1986, Klein and Marshalek, 1988). The basic method to be followed will be first to obtain the non-Hermitian Dyson mapping and subsequently to unitarize it.

We shall illustrate the approach for a class of algebras that will include, of course, the one of major interest, because it provides the basis for the nuclear shell model, namely, the *j-j* coupled shell-model algebra. But we shall also include a larger class of shell-model algebras, obtained by the well-known device of contraction with respect to a suitably chosen subset of the indices that label the generators, a device that in physical terms is described as the decomposition of each individual j into a sum of pseudo-orbital and pseudospin angular momenta, where the latter may have any mathematically compatible half-integral value (Hecht and Adler, 1969; Ginocchio, 1980). Thus consider a set of fermion creation and annihilation operators, $\alpha^{\dagger}_{m,\mu}$, $\alpha_{m,\mu}$, $-l\le\mu\le l$, $2l+1=2\Lambda$, l integral, $-j\le m\le j$, $2j+1=2\Omega$, j half-integral. The pair and multipole operators

$$A^{\dagger}_{mm'}=(A_{mm'})^{\dagger}=-A^{\dagger}_{m'm}$$
$$=(2\Lambda)^{-1/2}\sum_{\mu}(-1)^{l-\mu}\alpha^{\dagger}_{m,\mu}\alpha^{\dagger}_{m',-\mu}\ , \tag{9.1}$$

$$N_{mm'}=\sum_{\mu}\alpha^{\dagger}_{m',\mu}\alpha_{m,\mu} \tag{9.2}$$

satisfy the algebra

$$[N_{m_1m_2},N_{m_3m_4}]=\delta_{m_1m_4}N_{m_3m_2}-\delta_{m_2,m_3}N_{m_1m_4}\ , \tag{9.3}$$

$$[A^{\dagger}_{m_1m_2},N_{m_3m_4}]=\delta_{m_1m_3}A^{\dagger}_{m_2m_4}-\delta_{m_2m_3}A^{\dagger}_{m_1m_4}\ , \tag{9.4}$$

$$[A_{m_lm_2},A^{\dagger}_{m_3m_4}]=\delta_{m_1m_3}\delta_{m_2m_4}-\delta_{m_1m_4}\delta_{m_2m_3}$$
$$+(2\Lambda)^{-1}(\delta_{m_2,m_3}N_{m_1m_4}-\delta_{m_1m_3}N_{m_2m_4}-\delta_{m_2m_4}N_{m_1m_3}+\delta_{m_1m_4}N_{m_2m_3})\ . \tag{9.5}$$

The algebra represented [which can be put in standard form by the replacement $N_{m_1m_2}\to N_{m_1m_2}+(1/4\Lambda)\delta_{m_1m_2}$] is SO(4Ω). In this connection, although the requirement that l be integral is essential, the index m need not be associated with a given value of angular momentum. A number of j values could be involved or indeed the index could have a different significance altogether. The commutation relations (9.3) imply that $N_{mm'}$ span the unitary subalgebra U(2Ω), and Eq. (9.4) informs us that the $A^{\dagger}_{m_1m_2}$ transforms as a tensor under this subalgebra. The same is true of the operators $A_{m_1m_2}$. We use the notation SO(4Ω|Λ) to distinguish different values of Λ.

In the present discussion, we consider only the vacuum representations of the algebra SO(4Ω|Λ), i.e., those which, in the language of shell-model theory, contain the vacuum state, satisfying

$$A_{mm'}|0\rangle=0\ , \tag{9.6a}$$

$$N_{mm'}|0\rangle=0\ . \tag{9.6b}$$

A (generally) nonorthogonal basis for such an irrep is provided by the vectors

$$[|n_{mm'}]\rangle=\prod_{m<m'}(A^{\dagger}_{mm'})^{n_{mm'}}|0\rangle\ , \tag{9.7}$$

where the maximum allowed value of $n_{mm'}$ is determined by the value of Λ. We shall study mappings characterized by the subgroup chain SO(4Ω)⊃U(2Ω) (BZM mapping). As is well known (and was reviewed briefly in Sec. VIII), the irreps of U(2Ω) are characterized by a sequence of nonincreasing integers, $[f_1,f_2,\ldots,f_{2\Omega}]$, that constitute a partition of N, the number of particles in the state. In the usual shell-model case ($l=0$, $\Lambda=\frac{1}{2}$), $n_{mm'}=0,1$ and for each N, one has an irreducible representation of U(2Ω) consisting of the set of all Slater determinants that can be formed from N particles in 2Ω orbitals. This is the representation $[1,\ldots,1_N,0,\ldots,0_{2\Omega}]$, where the notation indicates N ones and $(2\Omega-N)$ zeros, often represented by a column of N boxes (Young diagram). Thus the entire representation of SO(4Ω) decomposes into a direct sum of irreducible representations of U(2Ω), each corresponding to a definite number of particles.

For general Λ the story is a little more complicated. The space of N particles contains a number of irreps of U(2Ω) that can be deduced from several well-known results (Wybourne, 1974): (i) The tensor $A^{\dagger}_{mm'}$ transforms as $[1,1,0,\ldots]\equiv[1^2]$ (two-particle state). (ii) A state such as $(A^{\dagger}_{mm'})^2|0\rangle$ belongs to an irrep $[2,2,0,\ldots]\equiv[2^2]$. Building on this, one can conclude that the irreps for U(2Ω|Λ) consists of partitions with $f_1=f_2,f_3=f_4,f_5=f_6\ldots$ and $f_1\le 2\Lambda$. An equivalent expression of this result is that, when one applies $A^{\dagger}_{mm'}$ to such a state, it corresponds to the reduction of the direct product according to the Clebsch-Gordan series (branching rule)

$$[1]^2\otimes[f_1,f_1,f_2,f_2,\ldots,f_{\Omega},f_{\Omega}]$$
$$=\sum_i[f_1,f_1,\ldots,f_i+1,f_i+1,\ldots,f_{\Omega},f_{\Omega}]\ , \tag{9.8}$$

if the term is allowed, i.e., if $f_i<f_{i-1}$.

B. Generalized Dyson mapping

We are now ready to consider the mapping problem. As usual, one introduces one boson creation operator $b^\dagger_{mm'}=-b^\dagger_{m'm}$ for each "raising" operator $A^\dagger_{mm'}$ thereby raising the fermion number by two. This boson operator $b^\dagger_{mm'}$ and its associated destruction operator, $b_{mm'}$, satisfy the commutation relations

$$[b_{mm'},b^\dagger_{m''m'''}]=\delta_{mm''}\delta_{m'm'''}-\delta_{mm'''}\delta_{m'm''} . \tag{9.9}$$

A central task is to find the mapping of the operators belonging to the subgroup U(2Ω), namely the operators $N_{mm'}$. Here we recall that the diagonal operators N_{mm}, 2Ω in number, mutually commute, and that distinct sets of eigenvalues of stretched states (states of maximum weight) are precisely the sets of integers that define the partition $[f_1\ldots f_{2\Omega}]$ that specifies the representation. The solution

$$(N_{mm'})_D=\sum_{m''} b^\dagger_{m'm''}b_{mm''}\equiv b^\dagger_{m'm''}b_{mm''} \tag{9.10}$$

is often written down as an *ansatz*, but it can be argued on the basis of the above observations and the requirement, *which we impose as part of the definition of the bosons,* that $b^\dagger_{mm'}$ transforms like $A^\dagger_{mm'}$ under U(2Ω). It follows from the commutation relation (9.4) that $N_{mm'}$ must be an operator of structure $(b^\dagger b)$ (multipole operator), the exact form being verified by the commutation relation (9.3).

We turn next to the mapping of $A^\dagger_{mm'}$ and $A_{mm'}$. Here the essential point is that, in addition to $b^\dagger_{mm'}$, there are an infinite number of $[1^2]$ tensors, namely (summation convention),

$$t^{(1)}_{mm'}=b^\dagger_{mm'},\quad t^{(2)}_{mm'}=b^\dagger_{mm_1}b^\dagger_{m'm_2}b_{m_1m_2} ,$$
$$t^{(3)}_{mm'}=b^\dagger_{mm_1}b^\dagger_{m'm_2}b^\dagger_{m_3m_4}b_{m_3m_1}b_{m_4m_2},\ldots . \tag{9.11}$$

In order to satisfy both the commutation relation (9.5) and the Hermiticity requirement stated in (9.1), we need to form $A^\dagger_{mm'}$ from a linear combination of all the possibilities in (9.11). The result to which one is led by this means will be discussed below, at least for the case $\Lambda=\frac{1}{2}$, the usual shell model. If we give up the Hermiticity requirement on the mapping, then we can utilize one of the Dyson solutions, e.g.,

$$(A_{mm'})_D=b_{mm'} , \tag{9.12}$$

$$(A^\dagger_{mm'})_D=b^\dagger_{mm'}-(2\Lambda)^{-1}b^\dagger_{mm_1}b^\dagger_{m'm_2}b_{m_1m_2}$$
$$\equiv b^\dagger_{mm'}-(2\Lambda)^{-1}[\mathbf{b}^\dagger(\mathbf{b}^\dagger\mathbf{b})^T]_{m'm} , \tag{9.13}$$

where boldface indicates matrix and T signifies transposition with respect to the matrix indices only. The form of Eqs. (9.12) and (9.13) means that we have mapped to a nonunitary basis, as discussed in detail later in this section. For the remainder of the present discussion, we shall be concerned with unitarizing this basis, as reflected by the reestablishment of the mutual Hermiticity of the A and $A^\dagger$ sets, and with establishing the detailed structure of the basis. Below, an alternative form of Eq. (9.13) will be required, namely,

$$(A^\dagger_{mm'})_D=b^\dagger_{mm'}+(1/8\Lambda)[b^\dagger_{mm'},\bar{C}_2]$$
$$\equiv\mathcal{A}^{(2\Lambda)\dagger}_{mm'} , \tag{9.14}$$

where

$$\bar{C}_2=b^\dagger_{m_1m_2}b^\dagger_{m_3m_4}b_{m_2m_3}b_{m_4m_1} \tag{9.15}$$

is the two-particle part of the quadratic Casimir operator for U(2Ω); it has the eigenvalues (Perelomov and Popov, 1966)

$$\bar{C}_2=\sum_{i=1}^{2\Omega} f_i(f_i+2-2i) . \tag{9.16}$$

C. Unitarization procedure. General method

As regards the unitarization process, in this subsection we study the general case. We subject the mapping (9.10), (9.12), and (9.13) to a similarity transformation S, intended to effect the unitarization (specified by the subscript B)

$$S(N_{mm'})_DS^{-1}=(N_{mm'})_B\equiv(N_{mm'})_D , \tag{9.17}$$

$$S(A^\dagger_{mm'})_DS^{-1}=(A^\dagger_{mm'})_B , \tag{9.18}$$

$$S(A_{mm'})_DS^{-1}=(A_{mm'})_B=(A^\dagger_{mm'})^\dagger_B . \tag{9.19}$$

Since, according to Eq. (9.17), S commutes with all operators $N_{mm'}$, it can depend only on the Casimir operators of U(2Ω), or equivalently, it is a function of the partition "operators" $f_1,\ldots,f_{2\Omega}$. Without loss of generality we can choose S to be a real operator, since, as pointed out in Sec. II, the operator

$$V=S^\dagger S=S^2 \tag{9.20}$$

is a metric tensor for a real vector space. By combining Eqs. (9.14), (9.18), and (9.19), we can easily deduce

$$V^{-1}b^\dagger_{mm'}V=(A^\dagger_{mm'})_D$$
$$=b^\dagger_{mm'}+(1/8\Lambda)[b^\dagger_{mm'},\bar{C}_2] . \tag{9.21}$$

The consequences of Eq. (9.21) may be obtained by evaluating it in the mapped basis, which is also taken to be composed of irreducible representations of U(2Ω). In this basis, V and $\bar{C}_2$ are diagonal and the nonvanishing matrix elements of $b^\dagger_{mm'}$ are specified by the branching rule or decomposition (9.8). It thus follows from Eqs. (9.16) and (9.21) that

$$V^{-1}([f_1,f_1,\ldots,f_i+1,f_i+1,\ldots,f_\Omega,f_\Omega])V([f_1,f_1,\ldots,f_\Omega,f_\Omega])=1-(2\Lambda)^{-1}(f_i+2-2i)\ , \tag{9.22}$$

the matrix element of $b^\dagger_{mm'}$ canceling from both sides. In practice we need the square root of Eq. (9.22),

$$S^{-1}([f_1,f_1,\ldots,f_i+1,f_i+1,\ldots,f_\Omega,f_\Omega])S([f_1,f_1,\ldots,f_\Omega,f_\Omega])=[1-(2\Lambda)^{-1}(f_i+2-2i)]^{1/2}\ . \tag{9.23}$$

The last step that can be carried out in full generality is to derive from Eqs. (9.18) and (9.21) the formula

$$(A^\dagger_{mm'})_B=S^{-1}b^\dagger_{mm'}S\ . \tag{9.24}$$

In order to apply this formula, we need the essential result [of which the transition from (9.13) to (9.14) is a special case]

$$S^{-1}b^\dagger_{mm'}S=[b^\dagger_{mm'},K]\ , \tag{9.25}$$

where K is an operator function of the Casimir operators (or of the partition). The existence of K follows from the fact that every tensor in Eq. (9.11) is a special case of (9.25), and the right-hand side of Eq. (9.25) must be a linear combination of such tensors.

D. Completion of mapping for shell-model algebra

To make further progress, we must choose a value of Λ (or l), since the detailed form of K depends on this choice, K growing in complexity for increasing Λ. Further reasoning will be restricted to two special cases.

Consider first the familiar case $l=0$. Because of the Pauli principle only completely antisymmetric irreps of U(2Ω) occur, namely the irreps $[1^N]$, where $N\le 2\Omega$. As a special case of Eq. (9.8) we write

$$A^\dagger_{mm'}|[1^N]\rangle=\alpha|[2,2,1^{N-1}]\rangle+\beta|[1^{N+2}]\rangle\ . \tag{9.26}$$

Note that this equation is in part symbolic because we have suppressed all quantum numbers other than the partition numbers. In Eq. (9.26), however, we must require $\alpha=0$ and $\beta\neq 0$ (if $N<2\Omega$), the former being a consequence of the Pauli principle. We then write, as a variant of (9.25) applicable to the special case under study,

$$\begin{aligned}(A^\dagger_{mm'})_B&=S^{-1}b^\dagger_{mm'}S\\&=b^\dagger_{mm'}f(N)+\tfrac{1}{4}[b^\dagger_{mm'},\bar{C}_2]g(N)\ .\end{aligned} \tag{9.27}$$

The condition $\alpha=\langle[2,2,1^{N-2}]|A^\dagger_{mm'}|[1^N]\rangle=0$, yields by the same calculation as led to Eq. (9.22),

$$0=[f(N)-g(N)]\langle b^\dagger_{mm'}\rangle\ , \tag{9.28}$$

or $f(N)=g(N)$. Evaluating β by the same means, we obtain $f(N)=(1+N)^{-1/2}$, or altogether [cf. (9.14)]

$$(A^\dagger_{mm'})_B=\mathcal{A}^{(1)\dagger}_{mm'}(1+N)^{-1/2}\ , \tag{9.29}$$

which differs from the Dyson form only by the square-root operators. We emphasize again that this result has been constructed to satisfy the Pauli principle constraint. It produces antisymmetric states when acting on properly antisymmetric states in the physical subspace. Another way of saying this is that Eq. (9.29), which is understood to be correct only within the physical subspace, can be replaced by the expression

$$(A^\dagger_{mm'})_B=\mathcal{A}^{(1)\dagger}_{mm'}(1+N)^{-1/2}P\ , \tag{9.30}$$

where P is the projector onto the physical subspace of the total boson space. Equation (9.29) or (9.30) is just the tool required to construct the basis in the boson space, as we shall emphasize below.

Next, we remark that an even simpler form than Eq. (9.30) can be given, but with a different significance. Instead of Eq. (9.27), we write

$$(A^\dagger_{mm'})_B=S^{-1}b^\dagger_{mm'}S=b^\dagger_{mm'}F(N) \tag{9.31}$$

and evaluate (9.31) by studying the overlap $\langle[1]^{N+2}|A^\dagger_{mm'}|[1^N]\rangle$ with the help of Eq. (9.23). We thus obtain

$$F(N)=(1+N)^{1/2}\ . \tag{9.32}$$

It should be emphasized once more that Eqs. (9.31) and (9.32) are correct only if the formula is surrounded by projection operators, i.e., strictly

$$(A^\dagger_{mm'})_B=Pb^\dagger_{mm'}(1+N)^{1/2}P\ . \tag{9.33}$$

Here only the right-hand projector may be removed, and this may limit the value of this formula in practice, as we shall argue when we consider applications in Sec. XIV.

E. Ginocchio model

Before completing the discussion of the shell-model case, we wish to include at least one additional example of the application of Eq. (9.25), namely, to the case $j=\frac{3}{2}$, l arbitrary. For all intents and purposes, this is the SO(8) model of Ginocchio (1980). Here the simplifying feature is the value $\Omega=2$, so that we are dealing with the U(4) subalgebra. Recalling the discussion associated with Eq. (9.8), we encounter the U(4) irreps

$$[f_1,f_1,n-f_1,n-f_1,0,\ldots,0]=[(f_1)^2(n-f_1)^2]\ ,$$

where $n=f_1+f_2$ is the number of bosons. Provided $(n-f_1)<f_1$, there are two nonvanishing contributions to Eq. (9.23):

$$\begin{aligned}S^{-1}([(f_1+1)^2(n-f_1)^2])S([f_1^2(n-f_1)^2])\\=[1-(2\Lambda)^{-1}f_1]^{1/2}\ ,\end{aligned} \tag{9.34}$$

$$S^{-1}([f_1^2(n-f_1+1)^2])S([f_1^2(n-f_1)^2]) = \{1-(2\Lambda)^{-1}[(n-f_1+2)]\}^{1/2} . \quad (9.35)$$

Thus one can satisfy Eqs. (9.24) and (9.25) for this case by writing

$$A^\dagger_{m,m'} = S^{-1}b^\dagger_{mm'}S = b^\dagger_{mm'}\phi_1(n,f_1) + (1/8\Lambda)[b^\dagger_{mm'},\bar{C}_2]\phi_2(n,f_1) , \quad (9.36)$$

and find with the help of Eqs. (9.34) and (9.35)

$$\phi_2(n,f_1) = 2\Lambda[n-2f_1-2]^{-1}\{[1-(2\Lambda)^{-1}f_1]^{1/2} - [1-(2\Lambda)^{-1}(n-f_1+2)]^{1/2}\} , \quad (9.37)$$

$$\phi_1(n,f_1) = [l-(2\Lambda)^{-1}f_1]^{1/2} + f_1(2\Lambda)^{-1}\phi_2 . \quad (9.38)$$

This result is equivalent to one given by Arima *et al.* (Arima, Yoshida, and Ginocchio, 1981).

We add a few remarks before returning to our primary study of the shell model. First, the result (9.36)–(9.38) has the same status as Eq. (9.33), and we would have to work harder to derive the analog of (9.30). For larger values of Λ and/or larger values of Ω, the procedure described for the two examples studied can be generalized, at the expense of increasing algebraic complexity, which can, however, be handled by a suitable computer program.

An extension of the Ginnochio model to include isospin, utilizing the methods of this section, has been carried out recently by Peres Menezes (Peres Menezes and Bonatsos, 1989; see also Peres Menezes, Brink, and Bonatsos, 1989).

F. Further discussion of shell-model case

Let us return now to the shell model to complete our discussion. So far, we have derived two forms of the mapping in Eqs. (9.30) and (9.33). A different mapping was given by several authors (Marshalek, 1971a, 1973, 1974a, 1980a, Janssen, Dönau, Frauendorf, and Jolos, 1971) in the form

$$A^\dagger_{mm'} = P\{\mathbf{b}^\dagger[\mathbf{I}-(\mathbf{b}^\dagger\mathbf{b})^T]^{1/2}\}_{m'm}P , \quad (9.39)$$

where again the right-hand P may be dropped in practice. This apparently more complex form has been more widely applied so far than the other forms, and it therefore behooves us to establish its equivalence to Eqs. (9.29) and (9.33), in particular the latter. The proof follows from the identity

$$Pb^\dagger_{mm'}\hat{N} = -P[\mathbf{b}^\dagger(\mathbf{b}^\dagger\mathbf{b})^T]_{m'm} . \quad (9.40)$$

This relation is a consequence of the equation

$$-P[\mathbf{b}^\dagger(\mathbf{b}^\dagger\mathbf{b})^T]_{m'm} = P\tfrac{1}{4}[b^\dagger_{m'm},\bar{C}_2] , \quad (9.41)$$

used in deriving Eq. (9.14). Equation (9.40) can be derived from (9.41) by noting that in addition to the form (9.15), $\bar{C}_2$ may also be written in the form

$$\bar{C}_2 = -\hat{N}(\hat{N}-2) , \quad (9.42)$$

which can be derived from Eq. (9.16) by evaluating the latter for the antisymmetric representation. [The relation between (9.41) and (9.42) was actually the main result needed to derive (9.29) and (9.33)]. Equation (9.39) then follows from (9.33) by repeated application of (9.41) as a formal expansion. The problem of convergence in the present setting was discussed by Okubo (1974c). The introduction of a small parameter by angular momentum coupling will be discussed below.

We found off the present discussion with remarks about the mapping of states, which has been implicit up to now. For the shell-model problem, the fermion states ($n=\frac{1}{2}N$)

$$|n\rangle = \prod_{k=1}^{n} A^\dagger_{i_k j_k}|0\rangle = a^\dagger_{i_1}a^\dagger_{j_1},\ldots,a^\dagger_{i_n}a^\dagger_{j_n}|0\rangle \quad (9.43)$$

constitute an orthonormal set. Equation (9.29) implies that the mapping is to the boson states

$$|n) = \prod[\mathcal{A}^{(1)\dagger}_{i_k j_k}(1+N)^{-1/2}]|0) = [(2n-1)!!]^{-1/2} \times \sum(-1)^P b^\dagger_{i_1 j_1},\ldots,b^\dagger_{i_n j_n}|0) . \quad (9.44)$$

In this equation P is the parity of the permutation of all $2n$ indices from a standard order. The last line of Eq. (9.44) is most easily derived by induction, establishing the case $n=2$ directly and then showing that $\mathcal{A}^{(1)\dagger}_{mm'}$ annihilates the state if m or m' already appear in the list of indices to the right and otherwise multiplies by $b^\dagger_{mm'}$ and antisymmetrizes among m,m' and all the other indices (Okubo, 1974c).

G. Early work

Concerning the seminal efforts in this area, we first comment on the work of Janssen *et al.* These authors obtained all the results in this section pertaining to the final unitary mapping by applying the Marumori operator

$$U_M = \sum_m |m)\langle m| \quad (9.45)$$

in the manner explained in detail for SU(2) in Sec. II. They derived the Dyson results independently by using an analogous operator, first introduced by Usui (1960),

$$U_S = \sum_m |\tilde{m})\langle m| , \quad (9.46)$$

where

$$|\tilde{m}\rangle=\prod \mathcal{A}^{(1)\dagger}_{i_k j_k}|0\rangle \tag{9.47}$$

are the non-normalized states corresponding to the mapping. An independent derivation of the mappings, using the generator-coordinate method, can also be found in their work, as well as suggestions for applications. Though the same remarkable paper also contains a derivation of Eqs. (9.29) and (9.33), there is, perhaps surprisingly, no discussion of their possible utility (Marshalek, 1989).

We turn then to the contributions of Marshalek (1971a, 1974a), which will be more heavily in evidence in succeeding sections as we seek to describe the possible applications of these results. The derivation of Marshalek can be summarized as follows. Recognizing that there is an implicit underlying mapping of states, he nevertheless emphasizes determination of the operator mapping from a study of the commutation relations (9.3)–(9.5), which are to be satisfied order by order, as explained below. After the mapping of the U(2Ω) subalgebra by Eq. (9.10) is imposed, and it is recognized that $A^{\dagger}_{mm'}$ must be chosen as a linear combination of the tensors $t^{(k)}$, $k=1,2,\ldots,$ of Eq. (9.11), then only Eq. (9.5) of the algebra remains to be satisfied. If a multipole operator $\sim(b^{\dagger}b)$ is said to be of order one and the kth power of order k, as long as there are k b and k $b^{\dagger}$ operators in some order, then from the fact that $[t^{(p)\dagger},t^{(q)}]=0((b^{\dagger}b)^{p+q-1})$, it follows that equating left- and right-hand sides of Eq. (9.5) order by order uniquely determines all expansion coefficients, once Eq. (9.10) has been specified. This procedure, when carried out fully, was recognized to be the formal expansion of Eq. (9.39)

Though the rigor of this procedure may be questioned because of the absence of a visible small parameter, the final result is correct and, as we shall demonstrate in the next sections, directly useful. However, even the expansion is useful as an intermediate step in the transition to a different result, essential for the major applications. The reference here is to the form taken by the mapping when the generators are combined so as to carry a definite angular momentum, i.e., when the generators are assembled into irreducible tensors under the rotation group. It turns out to be much easier to obtain the series or order-by-order solution in this case by recoupling the uncoupled solution than by doing the algebra freshly with the coupled operators.

We shall now record this expansion, which is needed for later application, and then comment on its salient property. We start with the spherical tensors

$$\mathcal{A}^{\dagger}_{JM}(\nu\nu')=\sum_{mm'}(jj'mm'|JM)a^{\dagger}_{\nu m}a^{\dagger}_{\nu' m'}\ , \tag{9.48a}$$

$$\mathcal{B}_{JM}(\nu\nu')=\sum_{mm'}(jj'm-m'|JM)(-1)^{j'-m'}a^{\dagger}_{\nu m}a_{\nu' m'}\ , \tag{9.48b}$$

where $(jj'mm'|JM)$ is a Clebsch-Gordan coefficient. These satisfy the algebra

$$[\mathcal{B}_1,\mathcal{B}_2^{\dagger}]=\sum_3(1-\hat{p}_1\hat{p}_2\hat{p}_3)Y(1,3,2)\mathcal{B}_3\ , \tag{9.49a}$$

$$[\mathcal{B}_1,\mathcal{A}_2^{\dagger}]=2\sum_3\tfrac{1}{2}(1+\hat{p}_2)\tfrac{1}{2}(1+\hat{p}_3)Y(2,3,1)\mathcal{A}_3^{\dagger}\ , \tag{9.49b}$$

$$[\mathcal{A}_1,\mathcal{A}_2]=0\ , \tag{9.49c}$$

$$[\mathcal{A}_1,\mathcal{A}_2^{\dagger}]=\delta_{12}(-)-4\sum_3\tfrac{1}{2}(1+\hat{p}_1)\tfrac{1}{2}(1+\hat{p}_2)Y(1,2,3)\mathcal{B}_3\ , \tag{9.49d}$$

derived from Eqs. (9.3)–(9.5). The compact notation means, for example, $1=(J_1M_1;\nu_1\nu_1')$ as in $\mathcal{A}_1=\mathcal{A}_{J_1M_1}(\nu_1\nu_1')$. The symbol $\hat{p}_1$ changes the order of coupling as follows:

$$\hat{p}_1\phi(1)=\hat{p}_1\phi(J_1M_1;\nu_1\nu_1')=-(-1)^{j_1+j_1'+J_1}\phi(J_1M_1;\nu_1'\nu_1)\ . \tag{9.50}$$

It acts on the quantities $Y(1,2,3)$ and generalizations of these denoted by $Y(1,2,3,\ldots,n)$ to be defined below. The symbol $\delta_{12}(-)$ means

$$\delta_{12}(-)=\delta_{J_1J_2}\delta_{M_1M_2}(\delta_{\nu_1\nu_2}\delta_{\nu_1'\nu_2'}-(-1)^{j_1+j_1'+J_1}\delta_{\nu_1'\nu_2}\delta_{\nu_1\nu_2'})\ . \tag{9.51}$$

Finally, the important symbol $Y(1,2,3)$, introduced by Belyaev and Zelevinsky (1962), is defined by

$$Y(1,2,3)=\delta_{\nu_1\nu_3}\delta_{\nu_1'\nu_2'}\delta_{\nu_2\nu_3'}\sqrt{(2J_2+1)(2J_3+1)}\times W(j_3j_1'J_3J_2;J_1j_2)\times(J_2J_3M_2M_3|J_1M_1)\ , \tag{9.52}$$

where W is a Racah coefficient.

Introducing the average shell size of the single-particle space 2Ω:

$$2\Omega\equiv\frac{1}{n}\sum_{\nu=1}^{n}(2j_\nu+1)\ , \tag{9.53}$$

Belyaev and Zelevinsky noted that in the asymptotic limit of large j values, there is the proportionality

$$Y(1,2,3)\propto[(2j_1'+1)(2j_2+1)(2j_3+1)]^{-1/6}\sim(2\Omega)^{-1/2}\ . \tag{9.54}$$

It will turn out that this provides the small expansion parameter we have been seeking. In carrying out the algebra, we encounter linked sums over products of the Y's for even numbers of indices,

$$Y(1,2,3,4)=\sum_{5} Y(1,2,5)Y(3,4,5)\ , \tag{9.55a}$$

$$\begin{aligned} Y(1,2,3,4,\ldots,2n+2)=&\sum_{2n+3,\ldots,4n+1} Y(1,2,2n+3) \\ &\times\prod_{k=0}^{n-2}[\,Y(2n+4+2k,4+2k,2n+3+2k) \\ &\times Y(2n+4+2k,3+2k,2n+5+2k)] \\ &\times Y(2n+1,2n+2,4n+1),\quad n\geq 2\ . \end{aligned} \tag{9.55b}$$

It is convenient also to record the Y with odd numbers of indices, although these are first useful for the angular momentum recoupling of the mappings involving an odd number of fermions, described in the next section:

$$Y(1,2,3,4,5)=\sum_{6,7} Y(1,2,6)Y(6,4,7)Y(5,3,7)\ ,$$

$$Y(1,2,3,\ldots,2n+1)=\sum_{2n+2,\ldots,4n-1} Y(1,2,2n+2) \tag{9.56a}$$

$$\begin{aligned} &\times\prod_{k=0}^{n-3} Y(2n+2+2k,4+2k,2n+3+2k)Y(2n+4+2k,3+2k,2n+3+2k) \\ &\times Y(4n-2,2n,4n-1)Y(2n+1,2n-1,4n-1),\quad n\geq 3\ . \end{aligned} \tag{9.56b}$$

It is important to note that asymptotically the orders of the coefficients are given by

$$Y(1,2,3,\ldots,2n+1)\cong(2\Omega)^{1/2-n}\ , \tag{9.57a}$$

$$Y(1,2,\ldots,2n+2)\cong(2\Omega)^{-n}\ . \tag{9.57b}$$

Finally, the expansions of the operators can be written as follows in terms of the coupled bosons:

$$B^{\dagger}_{JM}(\nu,\nu')=\sum_{mm'}(jj'mm'|JM)b^{\dagger}_{\nu m,\nu' m'}\ , \tag{9.58}$$

$$\mathcal{B}_1=\sum_{23} Y(3,2,1)B^{\dagger}_2B_3\ , \tag{9.59}$$

$$\mathcal{A}^{\dagger}_1=B^{\dagger}_1-\sum_{n=1}^{\infty}\frac{(2n-3)!!}{2^n n!}\sum_{2,3,\ldots,2n+2} Y(1,2,\ldots,2n+2)B^{\dagger}_2\prod_{k=1}^{n}B^{\dagger}_{2k+1}B_{2k+2}\ . \tag{9.60}$$

It follows from the analysis of the Y's that all terms proportional to a coefficients $(2n-3)!!/(2^n n!)$ should be regarded as being of order $(2\Omega)^{-n}$, so that the series contains at least a kinematic expansion parameter. This does not settle the question of convergence completely, as we saw already in Sec. III for the case of the Lipkin model, where there was a dynamic parameter determining the occurrence of a phase transition that strictly limited the convergence. Nevertheless, the appearance even of a kinematic convergence factor where none was apparent before recoupling must be accounted an encouraging development, implying that the recoupling can be viewed roughly as an analytic continuation of the original series. At the same time the detailed structure of the basis vectors underlying the mapping has been modified.

X. BZM MAPPING FOR SHELL-MODEL ALGEBRA. BOSE-FERMI MAPPINGS

A. Introduction and mathematical preliminaries

Before describing some applications, it is convenient to extend the arguments of the previous section to include the important case of a single odd particle. The algebra of even nuclei, if there are 2Ω orbits is SO(4Ω), as previously noted. When one brings in the single-fermion operators, the algebra becomes SO($4\Omega+1$), and the commutation relations (9.3)–(9.5) are augmented by the set

$$[\alpha_i,N_{jk}]=\delta_{ik}\alpha_{ij}\ \text{ (and H.c. Eq.)}\ , \tag{10.1}$$

$$[\alpha_i,A_{jk}]=0\ \text{ (and H.c. Eq.)}\ , \tag{10.2a}$$

$$[\alpha_i,A^{\dagger}_{jk}]=\delta_{ij}\alpha^{\dagger}_k-\delta_{ik}\alpha^{\dagger}_j\ \text{ (and H.c. Eq.)}\ , \tag{10.2b}$$

$$[\alpha_i,\alpha^{\dagger}_j]=\delta_{ij}-2N_{ij}\ , \tag{10.3a}$$

$$[\alpha^{\dagger}_i,\alpha^{\dagger}_j]=2A^{\dagger}_{ij}\ . \tag{10.3b}$$

It may be noted that (10.3a) and (10.3b) are a rearrangement of the fermion anticommutation rules. A boson or boson-fermion realization of this algebra requires not only the mapping of pair transfer and multipole operators, which are bilinear in the fermions, but also the mapping of single-fermion operators. Such a mapping was first given in the Holstein-Primakoff form by Marshalek (1974a, 1974b, 1980a) and in the generalized Dyson form

by Okubo (1974). The presentation that follows reflects the recent work of Klein and Marshalek (1988).

In the present case the physical subspace will be the direct sum of the space for even systems as mapped by Eq. (9.44) and the subspace for odd nuclei. For the expression of the latter, one introduces an *ideal* quasiparticle $a_k^\dagger$ and adjoins to Eq. (9.44) the mapping,

$$\alpha_{i_1}^\dagger \alpha_{j_1}^\dagger \cdots \alpha_{i_n}^\dagger \alpha_{j_n}^\dagger \alpha_k^\dagger |0\rangle \rightarrow \frac{1}{\sqrt{(2n+1)!!}} \sum (-1)^P b_{i_1 j_1}^\dagger \cdots b_{i_n j_n}^\dagger a_k^\dagger |0\rangle , \quad (10.4)$$

where

$$b_{ij}|0\rangle = a_k|0\rangle = 0 . \quad (10.5)$$

The mapping to be described is appropriate for problems in which an odd particle can make transitions from one of its substates k to another, but the total number is restricted to zero or one, since all other particles are paired as bosons. It is intuitively clear that the Hamiltonian can only depend on the ideal fermion multipole operator $(a_i^\dagger a_j)$. Any state containing more than one ideal fermion belongs to the unphysical subspace, by definition. This implies that ideal fermon pair operators $(a_i a_j)$ or $(a_i^\dagger a_j^\dagger)$ must annihilate the physical subspace. This restriction imposes algebraic constraints upon these operators that alter their algebraic properties compared to "normal" fermions, as will be seen. (We have already encountered this phenomenon in Secs. III and VI.)

Let us assemble first the needed properties of the ideal quasiparticles. In addition to Eq. (10.5), we require the correspondence

$$\alpha_i^\dagger|0\rangle \rightarrow a_i^\dagger|0\rangle , \quad (10.6)$$

and therefore the ideal odd-particle (IOP) operators transform under U(2Ω) in the same way as the original fermion operators. To accommodate this, we may postulate the following algebraic properties for the ideal odd-particle operators:

$$[a_i^\dagger a_j, a_k^\dagger a_l] = \delta_{jk} a_i^\dagger a_l - \delta_{il} a_k^\dagger a_j , \quad (10.7)$$

and

$$[a_i^\dagger a_j, a_k^\dagger] = \delta_{jk} a_i^\dagger \quad \text{(and H.c. Eq.)} . \quad (10.8)$$

Equations (10.7) and (10.8), which are the only algebraic properties of the ideal particles entering into the derivation of the Dyson mapping for bilinear operators, are certainly satisfied if the ideal particles are chosen to be fermions. However, Eqs. (10.7) and (10.8) are equally compatible with the choice of the ideal particles as bosons. In fact, other choices that correspond to neither fermions nor bosons, such as the one utilized here (Marshalek, 1974a, 1974b, 1980a; Okubo, 1974c) are also compatible. This ambiguity is the natural consequence of the restriction that the representation map contain no more than one ideal odd particle, imposed since the degrees of freedom of pairs is preempted by the bosons $b_{ij}^\dagger$. The resolution of this ambiguity in the current application, where the correct algebra for the IOP operators is that of neither fermions nor bosons, is described in Klein and Marshalek (1988). The required rules, which are intuitively clear, are listed:

$$a_j a_i = a_i^\dagger a_j^\dagger = 0 , \quad (10.9a)$$

$$a_i a_j^\dagger = \delta_{ij} P_0 , \quad (10.9b)$$

$$P_0 a_i^\dagger = a_i P_0 = 0 , \quad (10.9c)$$

$$P_0^2 = P_0 , \quad (10.9d)$$

where P_0, the projector to the purely bosonic sector, may be represented by

$$P_0 = 1 - \hat{n} , \quad (10.10)$$

with

$$\hat{n} \equiv a_i^\dagger a_i \quad (10.11)$$

(recall the summation convention!) being the IOP number operator, satisfying

$$\hat{n}^2 = \hat{n} . \quad (10.12)$$

Equation (10.12) expresses the property that the ideal space is the direct sum of the even subspace, in which $\hat{n}$ has the eigenvalue zero, and the odd subspace, in which $\hat{n}$ has the eigenvalue unity. The algebra (10.9) is easily seen to imply Eqs. (10.7) and (10.8), but the additional properties are essential for the mapping of single-fermion operators given below.

In order not to interrupt the main flow of the argument, we assemble next some mathematical preliminaries that rewrite and extend some of the results utilized in Sec. IX. These refer to the unitary subalgebra U(2Ω) of SO(4Ω+1). Again one maps an irrep of the latter, which is a direct sum (one for each particle number) of irreducible representations of this unitary subalgebra. The generators of U(2Ω),

$$N_{ik} = \alpha_k^\dagger \alpha_i , \quad (10.13)$$

are mapped uniquely by a generalization of Eq. (9.10),

$$(N_{ik})_D = b_{kl}^\dagger b_{il} + a_k^\dagger a_i . \quad (10.14)$$

Consistent with our previous practice, we utilize parentheses with a suitable subscript to indicate mapped operators, omitting the subscript on occasion if the various mappings coincide. In turn, this yields for the number operator $\hat{N}$ the image

$$(\hat{N}) = (\hat{N})_D = (\hat{N})_B = 2\hat{N}_b + \hat{n} , \quad (10.15)$$

where

$$\hat{N}_b \equiv \sum_{i>j} b_{ij}^\dagger b_{ij} \quad (10.16)$$

is the boson number operator and $\hat{n}$ is the IOP number operator, Eq. (10.11).

The importance of the Casimir invariants of U(2Ω) has already been established in Sec. IX. We consider them in more detail here. The mapped invariants have the form

$$C_1=(\hat{N})\ , \tag{10.17a}$$

$$C_2=(N_{ij})(N_{ji})\ , \tag{10.17b}$$

$$C_k=(N_{i_1i_2})(N_{i_2i_3})\cdots(N_{i_{k-1}i_k})(N_{i_ki_1})\ . \tag{10.17c}$$

Note that the images of the Casimir operators under the mapping (10.14) depend on both the boson and the ideal odd-particle operators. However, we may also consider the purely bosonic parts of the Casimir operators associated with the boson sector of the representation. Of particular importance is the quadratic Casimir operator, which, after normal ordering, is given by

$$C_2=2\Omega(2\hat{N}_b+\hat{n})-2\hat{N}_b+b^\dagger_{ij}b^\dagger_{kl}b_{kj}b_{il} +a^\dagger_ia_jb^\dagger_{jk}b_{ik}+a^\dagger_ja_ib^\dagger_{ik}b_{jk}\ , \tag{10.18}$$

where Eq. (10.9) was invoked to obtain the simplification $a^\dagger_ja_ia^\dagger_ia_j=2\Omega\hat{n}$. It will also be convenient to define the two-body part of Eq. (10.18), $\bar{C}_2$:

$$\bar{C}_2\equiv C_2-2\Omega(2\hat{N}_b+\hat{n})+2\hat{N}_b \tag{10.19}$$

and its purely bosonic part

$$\bar{C}_{2b}\equiv b^\dagger_{ij}b^\dagger_{kl}b_{kj}b_{il}\ . \tag{10.20}$$

The relative simplicity of the derivations to follow is based on the use of a small number of identities associated with the quadratic Casimir operators for U(2Ω) and $U_b(2\Omega)$. According to Perelomov and Popov (1966), the eigenvalues of the Casimir operators C_k for the antisymmetric representations $[1^N]$, which are the ones under consideration, are given by the simple formula

$$C_k=N(2\Omega-N+1)^{k-1}\ , \tag{10.21}$$

using the convention of denoting the Casimir operator and its eigenvalue by the same symbol. Thus, within the physical subspace only, which is the direct sum of antisymmetric representations, the Casimir operators (10.17) may be replaced by a function of the number operators obtained by replacing $N\to\hat{N}$ in Eq. (10.21). In particular, for the quadratic Casimir operator this gives

$$\begin{aligned}C_2&=(\hat{N})[2\Omega-(\hat{N})+1]\\&=(2\hat{N}_b+\hat{n})[2\Omega-(2\hat{N}_b+\hat{n})+1]\\&=2\Omega(2\hat{N}_b+\hat{n})-2\hat{N}_b(2\hat{N}_b-1)-4\hat{N}_b\hat{n}\ ,\end{aligned} \tag{10.22}$$

where, in the last step, the property $\hat{n}-\hat{n}^2=0$ [Eq. (10.12)] was used.

From Eq. (10.19) one immediately obtains for the two-body part of (10.22)

$$\bar{C}_2=-2\hat{N}_b(2\hat{N}_b-2)--4\hat{N}_b\hat{n}\ , \tag{10.23}$$

whose boson part is given by

$$\bar{C}_{2b}=-2\hat{N}_b(2\hat{N}_b-2)\ , \tag{10.24}$$

and the difference by

$$\bar{C}_2-\bar{C}_{2b}=-4\hat{N}_b\hat{n}\ . \tag{10.25}$$

Next, we may calculate the commutators of $b^\dagger_{ij}$ with Eqs. (10.24) and (10.25), as well as the commutator of $a^\dagger_i$ with (10.25). We may also calculate commutators with the generic forms (10.18)–(10.20) and thus altogether arrive at the following identities:

$$[b^\dagger_{ij},\tfrac{1}{4}\bar{C}_{2b}]=2b_{ij}\hat{N}_b=-[\mathbf{b}^\dagger(\mathbf{b}^\dagger\mathbf{b})^T]_{ji}\ , \tag{10.26}$$

$$[b^\dagger_{ij},\tfrac{1}{2}(\bar{C}_2-\bar{C}_{2b})]=2b^\dagger_{ij}\hat{n}=b^\dagger_{jk}a^\dagger_ia_k-b^\dagger_{ik}a^\dagger_ja_k\ , \tag{10.27}$$

$$[a^\dagger_i,\tfrac{1}{2}(\bar{C}_2-\bar{C}_{2b})]=2a^\dagger_i\hat{N}_b=-a^\dagger_j(\mathbf{b}^\dagger\mathbf{b})^T_{ji}\ . \tag{10.28}$$

We emphasize again that the identities (10.26)– (10.28) hold only in the physical subspace, so that some care is required in further manipulations. Postmultiplication of these equations by operators that leave the physical subspace invariant, such as group generators, however, is always safe. Thus, if Eq. (10.27) is postmultiplied by the group generator b_{ij} [cf. (10.30a) below] and summed over repeated indices, the result is the identity

$$2\hat{N}_b\hat{n}=-(\mathbf{b}^\dagger\mathbf{b})^T_{ij}a^\dagger_ia_j\ , \tag{10.29}$$

which will also be needed later. This identity can also be obtained by postmultiplying Eq. (10.28) by the SO(4Ω+1) generator $(\alpha_i)_D$ [cf. Eq. (10.31a) below] and summing over repeated indices, taking account of Eq. (10.9a). A simpler procedure yet is to postmultiply (10.28) by a_i, but since this is not a group generator, further justification would have to be provided.

B. Dyson mapping

We can now proceed to the Dyson mapping, beginning with the bilinear operators. The mapping $N_{ij}\to(N_{ij})$ given by Eq. (10.14), as we have already seen, satisfies the unitary subalgebra (9.3). This is the crucial first step that guarantees that, as long as $(A^\dagger_{ij})_D$ transforms like $b^\dagger_{ij}$ and $(A_{ij})_D$ like b_{ij} under U(2Ω), then the commutators (9.4) will be satisfied. It is the commutators (9.5) that provide a stronger restriction. In order to find the mappings that satisfy these commutators, one must in principle construct the totality of operators that transform like $b^\dagger_{ij}$ and b_{ij}, i.e., one must generalize the set (9.4). A systematic procedure for generating these is to construct the commutators of $b^\dagger_{ij}$ and b_{ij} with the hierarchy of generalized Casimir invariants. For the Dyson mapping, only a subset of such relations, already expressed by Eqs. (10.26) and (10.27), is required.

Having discussed preliminaries at considerable length, it suffices to record the mapped pair operators. We choose

$$(A_{ij})_D=b_{ij}\ , \tag{10.30a}$$

which then necessitates that

$$(A_{ij}^{\dagger})_D = b_{ij}^{\dagger} + [b_{ij}^{\dagger}, \tfrac{1}{4}\bar{C}_{2b}] + [b_{ij}^{\dagger}, \tfrac{1}{2}(\bar{C}_2 - \bar{C}_{2b}]$$
$$= b_{ij}^{\dagger} - b_{ik}^{\dagger} b_{jl}^{\dagger} b_{kl} + b_{ki}^{\dagger} a_j^{\dagger} a_k - b_{kj}^{\dagger} a_i^{\dagger} a_k \ . \quad (10.30b)$$

We also remark that all commutators of bilinears are preserved if the ansatz (10.30a) is rescaled by an arbitrary constant κ, provided that the right-hand side of Eq. (30.b) is also rescaled by κ^{-1}. The choice $\kappa=1$ corresponds to normalizing the state $(A_{ij}^{\dagger})_D|0\rangle = b_{ij}^{\dagger}|0\rangle$.

Next, we consider the Dyson mapping of single-fermion operators, first stating the results and then outlining the derivation. The Dyson images of the singles are then given by

$$(\alpha_i)_D = a_i + b_{ij} a_j^{\dagger} \ , \quad (10.31a)$$

$$(\alpha_i^{\dagger})_D = a_i^{\dagger} + b_{ij}^{\dagger} a_j - (\mathbf{b}^{\dagger}\mathbf{b})_{ji}^{T} a_j^{\dagger}$$
$$= a_i^{\dagger} + b_{ij}^{\dagger} a_j + [a_i^{\dagger}, \tfrac{1}{2}(\bar{C}_2 - \bar{C}_{2b})] \ , \quad (10.31b)$$

where the identity (10.28) was used.

Equations (10.31) can be obtained from the mapping of the commutators (10.1)–(10.3) of the single-fermion operators with the bilinear products. Thus inserting the Dyson images (10.30) provides the following commutator equations to be solved for the singles:

$$[(\alpha_i)_D, (N_{kj})] = \delta_{ij}(\alpha_k)_D \ ,$$
$$[(\alpha_i^{\dagger})_D, (N_{jk})] = -\delta_{ij}(\alpha_k^{\dagger})_D \ , \quad (10.32)$$

$$[(\alpha_i^{\dagger})_D, (A_{jk}^{\dagger})_D] = 0 \ , \quad (10.33a)$$

$$[(\alpha_i)_D, b_{jk}] = 0 \ , \quad (10.33b)$$

$$[(\alpha_i^{\dagger})_D, b_{jk}] = \delta_{ik}(\alpha_j)_D - \delta_{ij}(\alpha_k)_D \ , \quad (10.34a)$$

$$[(\alpha_i)_D, (A_{jk}^{\dagger})_D] = \delta_{ij}(\alpha_k^{\dagger})_D - \delta_{ik}(\alpha_j^{\dagger})_D \ . \quad (10.34b)$$

Equations (10.32) imply, of course, that $(\alpha_i)_D$ and $(\alpha_i^{\dagger})_D$ must transform like a_i and $a_i^{\dagger}$, respectively. Furthermore, it is easily seen that $(\alpha_i)_D$ and $(\alpha_i^{\dagger})_D$ can only depend linearly on $a_j, a_j^{\dagger}$ as follows:

$$(\alpha_i)_D = a_i + f_{ij}(\mathbf{b}, \mathbf{b}^{\dagger}) a_j + g_{ij}(\mathbf{b}, \mathbf{b}^{\dagger}) a_j^{\dagger} \ ,$$
$$(\alpha_i^{\dagger})_D = a_i^{\dagger} + \phi_{ji}(\mathbf{b}, \mathbf{b}^{\dagger}) a_j^{\dagger} + \gamma_{ij}(\mathbf{b}, \mathbf{b}^{\dagger}) a_j \ . \quad (10.35)$$

Substitution of Eq. (10.35) into (10.32) shows that these commutators are satisfied as long as f_{ij} and ϕ_{ij} are vector operators under $U_b(2\Omega)$, the boson unitary subgroup, i.e., they transform like the generators $(\mathbf{b}^{\dagger}\mathbf{b})_{ij}^{T}$, while g_{ij} transforms like b_{ij} and γ_{ij} like $b_{ij}^{\dagger}$. The same is true of the commutators (10.33a). The commutator (10.33b), however, places a very stringent restriction on $(\alpha_i)_D$, limiting it to the form

$$(\alpha_i)_D = \lambda a_i + \mu b_{ij} a_j^{\dagger} \ , \quad (10.36)$$

where λ and μ are arbitrary constants. Likewise, the commutator (10.34a) places a very stringent restriction on $(\alpha_i^{\dagger})_D$, limiting it to the form

$$(\alpha_i^{\dagger})_D = \eta a_i^{\dagger} - \mu(\mathbf{b}^{\dagger}\mathbf{b})_{ji}^{T} a_j^{\dagger} + \lambda b_{ij}^{\dagger} a_j \ , \quad (10.37)$$

where η is another arbitrary constant. The last commutator, (10.34b), is then satisfied by Eqs. (10.36) and (10.37), providing only that

$$\eta = \mu \ . \quad (10.38)$$

This still leaves two free parameters, λ and μ. These, in turn, can be fixed by invoking Eq. (10.6), i.e., the condition $(\alpha_i^{\dagger})_D|0\rangle = a_i^{\dagger}|0\rangle$, and also requiring $(\alpha_j)_D(\alpha_i^{\dagger})_D|0\rangle = \delta_{ij}|0\rangle$. This yields $\lambda=\mu=1$ and the final result (10.31). Another means of fixing the scales is to consider the fermion anticommutation rules. Finally, we remark that the calculation of the commutators (10.32)–(10.34) requires the full algebra (10.9), i.e., (10.7) and (10.8) are not sufficient.

C. Derivation of the unitary mapping of Okubo

We now come to the central problem of transforming the nonunitary Dyson representation into a unitary one. We seek a real positive-definite similarity transformation S such that for any operator O under consideration

$$S(O)_D S^{-1} = (O)_U \ , \quad (10.39)$$

$$S(O^{\dagger})_D S^{-1} = (O^{\dagger})_U = (O)_U^{\dagger} \ , \quad (10.40)$$

where $(\ )_U$ denotes the image under the unitarized mapping. Equations (10.39) and (10.40) imply that

$$(O)_D^{\dagger} V = V(O^{\dagger})_D \ , \quad (10.41)$$

where

$$V \equiv S^{\dagger} S = S^2 \ . \quad (10.42)$$

From Eq. (10.41) it follows that V commutes with any operator that satisfies $(O^{\dagger})_D = (O)_D^{\dagger}$, which is certainly the case for $O = N_{ij}$, the generators of $U(2\Omega)$, and functions thereof, such as the number operator. This, in turn, implies that V, and therefore S is a function of the Casimir invariants and hence, in the present case, of the particle numbers $2\hat{N}_b$ and $\hat{n}$. Because $\hat{n}$ is a projector, it must occur linearly in V or S. Thus, for example, S can be written in the form

$$S(2\hat{N}_b, \hat{n}) = S_e(2\hat{N}_b)(1-\hat{n}) + S_o(2\hat{N}_b)\hat{n}$$
$$= S_e(2\hat{N}_b) + [S_o(2\hat{N}_b) - S_e(2\hat{N}_b)]\hat{n} \ , \quad (10.43)$$

where $S_e(2\hat{N}_b)$ acts in the subspace of even particle numbers and $S_o(2\hat{N}_b)$ acts in the subspace of odd particle numbers.

We now proceed as we have done on previous occasions. Equation (10.41), written in the form

$$V^{-1}(O)_D^{\dagger} V = (O^{\dagger})_D \ , \quad (10.44)$$

is applied to the lowering operators A_{ij} and α_i, these being the ladder operators with the simplest Dyson images, in order to determine the essential properties of $S = V^{1/2}$.

The unitarized images of the corresponding raising operators is then obtained from

$$(O^\dagger)_U = S^{-1}(O)^\dagger_D S \ . \tag{10.45}$$

For the case $O = A_{ij}$, we obtain from Eqs. (10.44), (10.30), (10.26), and (10.27)

$$\begin{aligned} &V^{-1}(2\hat{N}_b,\hat{n})b^\dagger_{ij}V(2\hat{N}_b,\hat{n}) \\ &\quad = b^\dagger_{ij} + [b^\dagger_{ij}, \tfrac{1}{4}\bar{C}_{2b}] + [b^\dagger_{ij}, \tfrac{1}{2}(\bar{C}_2 - \bar{C}_{2b})] \\ &\quad = b^\dagger_{ij}(1+2\hat{N}_b+2\hat{n}) \ . \end{aligned} \tag{10.46}$$

With the help of the identity

$$f(2\hat{N}_b,\hat{n})b^\dagger_{ij} = b^\dagger_{ij} f(2\hat{N}_b+2,\hat{n}) \tag{10.47}$$

for an arbitrary function $f(2\hat{N}_b,\hat{n})$, one may move $b^\dagger_{ij}$ to the far left on the left-hand side and extract the result

$$b^\dagger_{ij}V^{-1}(2\hat{N}_b+2,\hat{n})V(2\hat{N}_b,\hat{n}) = b^\dagger_{ij}(1+2\hat{N}_b+2\hat{n}) \ . \tag{10.48}$$

This implies for $S = V^{1/2}$ the formula

$$\begin{aligned} &b^\dagger_{ij}S^{-1}(2\hat{N}_b+2,\hat{n})S(2\hat{N}_b,\hat{n}) \\ &\quad = b^\dagger_{ij}(1+2\hat{N}_b+2\hat{n})^{1/2} \\ &\quad = b^\dagger_{ij}[(1+2\hat{N}_b)^{1/2}(1-\hat{n}) + (2\hat{N}_b+3)^{1/2}\hat{n}] \ . \end{aligned} \tag{10.49}$$

In the last step of Eq. (10.49), the square root was expanded in powers of $2\hat{n}(1+2\hat{N}_b)^{-1}$, and, using the idempotency of $\hat{n}$, the result then was resummed.

Next, we apply Eq. (10.44) to $O = \alpha_i$, making use of Eq. (10.31) and (10.28) to obtain

$$\begin{aligned} &V^{-1}(2\hat{N}_b,\hat{n})(a^\dagger_i + b^\dagger_{ij}a_j)V(2\hat{N}_b,\hat{n}) \\ &\quad = a^\dagger_i + [a^\dagger_i, \tfrac{1}{2}(\bar{C}_2 - \bar{C}_{2b})] + b^\dagger_{ij}a_j \\ &\quad = a^\dagger_i(1+2\hat{N}_b) + b^\dagger_{ij}a_j \ . \end{aligned} \tag{10.50}$$

From the identity (10.47), as well as the additional identities

$$\begin{aligned} f(2\hat{N}_b,\hat{n})a^\dagger_i &= a^\dagger_i f(2\hat{N}_b,1) \ , \\ f(2\hat{N}_b,\hat{n})a_i &= a_i f(2\hat{N}_b,0) \ , \end{aligned} \tag{10.51}$$

one obtains from Eq. (10.50) the results

$$\begin{aligned} a^\dagger_i S^{-1}(2\hat{N}_b,1)S(2\hat{N}_b,\hat{n}) &= a^\dagger_i(1+2\hat{N}_b)^{1/2} \ , \\ b^\dagger_{ij}a_j S^{-1}(2\hat{N}_b+2,0)S(2\hat{N}_b,\hat{n}) &= b^\dagger_{ij}a_j \ . \end{aligned} \tag{10.52}$$

The unitarized mappings then follow from Eqs. (10.49) and (10.52). Thus, applying (10.45) to $O^\dagger = A^\dagger_{ij}$, one obtains from Eqs. (10.47) and (10.49)

$$\begin{aligned} (A^\dagger_{ij})_U = (A_{ij})^\dagger_U &= S^{-1}(2\hat{N}_b,\hat{n})b^\dagger_{ij}S(2\hat{N}_b,\hat{n}) \\ &= b^\dagger_{ij}S^{-1}(2\hat{N}_b+2,\hat{n})S(2\hat{N}_b,\hat{n}) \\ &= b^\dagger_{ij}(1+2\hat{N}_b+2\hat{n})^{1/2} \\ &= b^\dagger_{ij}(1+2\hat{N}_b)^{1/2}(1-\hat{n}) + (1+2\hat{N}_b)^{1/2}b^\dagger_{ij}\hat{n} \\ &= b^\dagger_{ij}(1+2\hat{N}_b)^{1/2} + [(1+2\hat{N}_b)^{1/2}, b^\dagger_{ij}]\hat{n} \ . \end{aligned} \tag{10.53}$$

Finally, applying Eq. (10.45) to $O^\dagger = \alpha^\dagger_i$, one obtains from Eqs. (10.51) and (10.52)

$$\begin{aligned} (\alpha^\dagger_i)_U = (\alpha_i)^\dagger_U &= S^{-1}(2\hat{N}_b,\hat{n})(a^\dagger_i + b^\dagger_{ij}a_j)S(2\hat{N}_b,\hat{n}) \\ &= a^\dagger_i S^{-1}(2\hat{N}_b,1)S(2\hat{N}_b,\hat{n}) + b^\dagger_{ij}a_j S^{-1}(2\hat{N}_b+2,0)S(2\hat{N}_b,\hat{n}) \\ &= a^\dagger_i(1+2\hat{N}_b)^{1/2} + b^\dagger_{ij}a_j \ . \end{aligned} \tag{10.54}$$

We now summarize the unitary mapping obtained in this section:

$$(N_{ij})_U = (N_{ij})_D = b^\dagger_{jk}b_{ik} + a^\dagger_j a_i \ , \tag{10.55a}$$

$$(A^\dagger_{ij})_U = b^\dagger_{ij}(1+2\hat{N}_b)^{1/2} + [(1+2\hat{N}_b)^{1/2}, b^\dagger_{ij}]\hat{n} \ ,$$

$$(A_{ij})_U = (A^\dagger_{ij})^\dagger \ , \tag{10.55b}$$

$$(\alpha^\dagger_i)_U = a^\dagger_i(1+2\hat{N}_b)^{1/2} + b^\dagger_{ij}a_j \ ,$$

$$(\alpha_i)_U = (\alpha^\dagger_i)^\dagger_U \ . \tag{10.55c}$$

Note that, for an even number of particles, $\hat{n} = 0$ and Eq. (10.55b) reduces to Eq. (9.33), apart from notational differences. This unitary mapping, which was first derived *in toto* by Okubo, is however, formally different from that of Marshalek, which is the one usually referred to as an example of a generalized Holstein-Primakoff mapping.

D. Unitary mapping of Marshalek

We now proceed to transform the unitary mapping of Okubo (1974c) into that of Marshalek (1974a) with the aid of previously derived identities. By iterated postmultiplication of Eq. (10.26) with the $U_b(2\Omega)$ generators $(b^\dagger b)^T_{ij}$ and recursive use of Eq. (10.26) one readily ob-

tains the identity

$$b_{ij}^{\dagger}(2\hat{N}_b)^p=\{\mathbf{b}^{\dagger}[-(\mathbf{b}^{\dagger}\mathbf{b})^T]^p\}_{ji} , \tag{10.56}$$

for arbitrary powers p. Application to the formal series expansion of the square-root operator then gives the identity

$$b_{ij}^{\dagger}(1+2\hat{N}_b)^{1/2}=\{\mathbf{b}^{\dagger}[\mathbf{I}-(\mathbf{b}^{\dagger}\mathbf{b})^T]^{1/2}\}_{ji} , \tag{10.57}$$

where $\mathbf{I}$ denotes the identity matrix. Next, by repeated exponentiation of both sides of Eq. (10.29), using $\hat{n}^2=\hat{n}$ as well as Eq. (10.9b), one easily arrives at the identity

$$(2\hat{N}_b)^p\hat{n}=\{[-(\mathbf{b}^{\dagger}\mathbf{b})^T]^p\}_{ij}a_i^{\dagger}a_j , \tag{10.58}$$

which implies that

$$(1+2\hat{N}_b)^{1/2}\hat{n}=\{[\mathbf{I}-(\mathbf{b}^{\dagger}\mathbf{b})^T]^{1/2}\}_{ij}a_i^{\dagger}a_j . \tag{10.59}$$

Application of Eqs. (10.57) and (10.59) to (10.55b) gives

$$\begin{aligned}(A_{ij}^{\dagger})_{\mathrm{HP}}&=(A_{ij}^{\dagger})_U\\&=\{\mathbf{b}^{\dagger}[\mathbf{I}-(\mathbf{b}^{\dagger}\mathbf{b})^T]^{1/2}\}_{ji}\\&\quad+[\{(\mathbf{I}-(\mathbf{b}^{\dagger}\mathbf{b})^T)^{1/2}\}_{kl},b_{ij}^{\dagger}]a_k^{\dagger}a_l ,\\(A_{ij})_{\mathrm{HP}}&=(A_{ij}^{\dagger})_{\mathrm{HP}}^{\dagger} ,\end{aligned} \tag{10.60}$$

where the notation $(\ )_{\mathrm{HP}}$ was used to denote the generalized Holstein-Primakoff form, and a commutator occurs in the second term of (10.60).

Finally, iterated use of Eq. (10.28) yields

$$a_i^{\dagger}(2\hat{N}_b)^p=a_j^{\dagger}\{[-(\mathbf{b}^{\dagger}\mathbf{b})^T]^p\}_{ji} , \tag{10.61}$$

leading to the identity

$$a_i^{\dagger}(1+2\hat{N}_b)^{1/2}=a_j^{\dagger}\{[\mathbf{I}-(\mathbf{b}^{\dagger}\mathbf{b})^T]^{1/2}\}_{ji} . \tag{10.62}$$

Consequently, the single-fermion operators (10.55c) are transformed into

$$\begin{aligned}(\alpha_i^{\dagger})_{\mathrm{HP}}&=(\alpha_i^{\dagger})_U\\&=a_j^{\dagger}\{[\mathbf{I}-(\mathbf{b}^{\dagger}\mathbf{b})^T]^{1/2}\}_{ji}+b_{ij}^{\dagger}a_j ,\\(\alpha_i)_{\mathrm{HP}}&=(\alpha_i^{\dagger})_{\mathrm{HP}}^{\dagger} .\end{aligned} \tag{10.63}$$

The U(2Ω) generators are, as always, unchanged in the generalized Holstein-Primakoff mapping.

E. Concluding remarks

We remind the reader that the equivalence between the two unitary forms of the mapping holds only in the physical subspace, as do the identities used in deriving them. Nevertheless, the structure of the two mappings is different, and thus the possible range of applications also differs. As already remarked, the generalized Holstein-Primakoff form has a perturbative structure, satisfying commutation rules order by order. It is therefore most suitable for perturbative applications, as in the regions near closed shells and also in strongly deformed nuclei. It can also reproduce results of both nuclear field theory and the nuclear version of Fermi-liquid theory. These applications and others will be reviewed in Secs. XI–XIII. Although the generalized Holstein-Primakoff expansions appear to be nonconvergent at first sight, because of the absence of an explicit expansion parameter, the corresponding expansions of operators coupled to good angular momentum (as reviewed in Sec. IX) or expansions in random-phase approximation bosons introduce an explicit smallness parameter.

On the other hand, the Okubo form of the unitary mapping (see also Janssen *et al.*, 1971) seems most suitable for nonperturbative applications. As already noted by Okubo, the mapping of a Hamiltonian that conserves the number of fermions takes a simple, finite, and Hermitian form. Therefore, this mapping may have a role to play in understanding the microscopic foundations of the interacting boson model, although this has not yet been studied. Such applications as have used this form are reviewed in Sec. XIV.

XI. APPLICATION OF BZM MAPPING: DERIVATION OF HARTREE-BOGOLIUBOV THEORY FROM THE MAPPING AND THE INVERSE PROBLEM

A. Hartree-Bogoliubov theory as classical limit of a boson-mapped theory

The Belyaev-Zelevinsky-Marshalek (BZM) mappings have thus far proved most useful when applied to the quasiparticle representation of fermion algebras. This includes closed-shell nuclei, if one uses the language of particles and holes. The main requirement is that the definition of quasiparticle be such that occupation probabilities are small near an equilibrium configuration; the latter can be spherical or deformed, normal or superconducting. In this section, the boson images of quasiparticle pairs $\alpha_i^{\dagger}\alpha_j^{\dagger}$ and multipoles $\alpha_i^{\dagger}\alpha_j$ are therefore given by the expressions

$$(\alpha_i^{\dagger}\alpha_j^{\dagger})_B=[\mathbf{b}^{\dagger}\sqrt{\mathbf{I}-(\mathbf{b}^{\dagger}\mathbf{b})^T}]_{ji}=(\alpha_j\alpha_i)_B^{\dagger} , \tag{11.1a}$$

$$(\alpha_i^{\dagger}\alpha_j)_B=(\mathbf{b}^{\dagger}\mathbf{b})_{ij} , \tag{11.1b}$$

where

$$(\mathbf{b}^{\dagger}\mathbf{b})_{ij}^T=b_{jl}^{\dagger}b_{il} , \tag{11.2}$$

and $b, b^{\dagger}$ are the bosons defined in Eq. (9.9). In the applications of this section, the bosons will be replaced by c numbers, i.e., we go to the classical limit. Of course, the ordering of the boson operators prior to the c-number replacement affects the results obtained. We choose an ordering that replaces commutators of mapped operators by the corresponding classical Poisson brackets, which has been dubbed "c ordering" (Marshalek and Weneser, 1970; see also Johnson, Klein, and Dreizler, 1968); Marshalek and Holzwarth, 1972; Yamamura and Nishiyama, 1977; Yamamura, 1980). This turns out to be ex-

actly the ordering obtained by freezing the operators in Eqs. (11.1) and (11.2) in the positions shown.

Below we shall also need the BZM representation for particle creation and annihilation operators $(a_i^\dagger, a_i)$, which are easily obtained from Eq. (11.1). The two sets of fermion operators are related by a general Bogoliubov transformation

$$\begin{bmatrix} \boldsymbol{\alpha}^\dagger \\ \boldsymbol{\alpha}^T \end{bmatrix} = \mathcal{U} \begin{bmatrix} \mathbf{a}^\dagger \\ \mathbf{a}^T \end{bmatrix} , \tag{11.3}$$

$$\mathcal{U} = \begin{bmatrix} \mathbf{U} & \mathbf{V} \\ \mathbf{V}^* & \mathbf{U}^* \end{bmatrix} . \tag{11.4}$$

The boson images of the pair operators are consequently related by

$$\begin{bmatrix} (\mathbf{a}^\dagger \otimes \mathbf{a})_B & (\mathbf{a}^\dagger \otimes (\mathbf{a}^T)^T)_B \\ (\mathbf{a}^T \otimes \mathbf{a})_B & (\mathbf{a}^T \otimes (\mathbf{a}^\dagger)^T)_B \end{bmatrix} = \mathcal{U}^\dagger \begin{bmatrix} (\boldsymbol{\alpha}^\dagger \otimes \boldsymbol{\alpha})_B & (\boldsymbol{\alpha}^\dagger \otimes (\boldsymbol{\alpha}^\dagger)^T)_B \\ (\boldsymbol{\alpha}^T \otimes \boldsymbol{\alpha})_B & (\boldsymbol{\alpha}^T \otimes (\boldsymbol{\alpha}^\dagger)^T)_B \end{bmatrix} \mathcal{U} , \tag{11.5}$$

where $\otimes$ denotes the direct product.

Suppose now that the boson operators in Eq. (11.1) are replaced by c numbers:

$$b_{ij} \to c_{ij}, \quad b_{ij}^\dagger \to c_{ij}^* . \tag{11.6}$$

Then we have

$$(\alpha_i \alpha_j)_B \to \boldsymbol{\kappa}_{ji}(\mathbf{c}^*, \mathbf{c}) ,$$

$$(\alpha_i^\dagger \alpha_j^\dagger)_B \to \boldsymbol{\kappa}_{ij}^*(\mathbf{c}^*, \mathbf{c}) , \tag{11.7a}$$

$$(\alpha_i^\dagger \alpha_i)_B \to \mathbf{r}_{ji}(\mathbf{c}^*, \mathbf{c}) , \tag{11.7b}$$

where $\boldsymbol{\kappa}$ is an antisymmetric matrix and $\mathbf{r}$ a Hermitian matrix given by

$$\boldsymbol{\kappa} = \sqrt{\mathbf{I} + \mathbf{c}\mathbf{c}^*}\,\mathbf{c} = \mathbf{c}\sqrt{\mathbf{I} + \mathbf{c}^*\mathbf{c}} , \tag{11.8a}$$

$$\mathbf{r} = -\mathbf{c}\mathbf{c}^* . \tag{11.8b}$$

These matrices, as one checks, identically satisfy the equations

$$\mathbf{r}^2 - \mathbf{r} - \boldsymbol{\kappa}\boldsymbol{\kappa}^* = 0 , \tag{11.9a}$$

$$\mathbf{r}\boldsymbol{\kappa} - \boldsymbol{\kappa}\mathbf{r}^* = 0 . \tag{11.9b}$$

Assembling a matrix $\mathbf{K}$,

$$\mathbf{K} = \begin{bmatrix} \mathbf{r} & \boldsymbol{\kappa} \\ -\boldsymbol{\kappa}^* & \mathbf{I} - \mathbf{r}^* \end{bmatrix} , \tag{11.10}$$

we can write Eq. (11.9) as

$$\mathbf{K}^2 = \mathbf{K} . \tag{11.11}$$

Under the replacements (11.6) and (11.7), the boson images of pairs in the particle representation become

$$(a_i a_j)_B \to \boldsymbol{\chi}_{ji}(\mathbf{c}^*, \mathbf{c}) ,$$

$$(a_i^\dagger a_j^\dagger)_B \to \boldsymbol{\chi}_{ij}^*(\mathbf{c}^*, \mathbf{c}) , \tag{11.12a}$$

$$(a_i^\dagger a_j)_B \to \boldsymbol{\rho}_{ji}(\mathbf{c}^*, \mathbf{c}) , \tag{11.12b}$$

where $\boldsymbol{\chi}$ is antisymmetric and $\boldsymbol{\rho}$ is Hermitian. Let us define the matrix

$$\mathcal{K} = \begin{bmatrix} \boldsymbol{\rho} & \boldsymbol{\chi} \\ -\boldsymbol{\chi}^* & \mathbf{I} - \boldsymbol{\rho}^* \end{bmatrix} . \tag{11.13}$$

It follows from Eq. (11.5) that

$$\mathcal{K} = \mathcal{U}^{*\dagger} \mathbf{K} \mathcal{U}^* . \tag{11.14}$$

Because $\mathcal{U}$ is unitary, Eq. (11.11) implies that

$$\mathcal{K}^2 = \mathcal{K} . \tag{11.15}$$

It is therefore established that if the boson operators are replaced by c numbers, then the images of pair operators can be interpreted as elements of the generalized density matrix $\mathcal{K}$ of Hartree-Bogoliubov (HB) theory (Valatin, 1961). In particular, $\boldsymbol{\rho}$ may be identified with the one-particle density matrix and $\boldsymbol{\chi}$ with the pairing tensor. The reason for making the identification with HB rather than with Hartree-Fock-Bogoliubov (HFB) theory will be clarified later. It should perhaps be emphasized that the parametrization of an HB or HFB density matrix by the substitution (11.6) works only for the BZM mapping and not for the (normal-ordered) Marumori-Yamamura-Tokunaga mapping (see below, however).

The substitution (11.6) can be considered as a passage to the classical limit by means of the Wigner transform (Carruthers and Zachariasen, 1983), in which a product of operators is replaced by the same product of average values, i.e., a function of the classical canonical variables. In this case (c_{ij}, ic_{ij}^*) play the role of (complex) canonical variables. Commutators go over into (i times) the classical Poisson bracket,

$$[A,B] \to \tfrac{1}{2} \sum_{ij} \left(\frac{\partial A}{\partial c_{ij}} \frac{\partial B}{\partial c_{ij}^*} - \frac{\partial A}{\partial c_{ij}^*} \frac{\partial B}{\partial c_{ij}} \right) \equiv [A,B]_P . \tag{11.16}$$

It is therefore guaranteed that the algebra of the pair operators is replaced by the Poisson bracket relations,

$$\begin{aligned} &[r_{ji}, r_{lk}]_P = \delta_{jk} r_{li} - \delta_{li} r_{jk} , \\ &[r_{lk}, \kappa_{ji}]_P = \delta_{jk} \kappa_{il} - \delta_{ki} \kappa_{jl} , \\ &[\kappa_{ji}, \kappa_{lk}]_P = 0 , \\ &[\kappa_{ji}, \kappa_{kl}^*]_P = \delta_{jk}\delta_{li} - \delta_{ki}\delta_{lj} + \delta_{ki} r_{jl} \\ &\qquad + \delta_{jl} r_{ik} - \delta_{jk} r_{il} - \delta_{li} r_{jk} , \end{aligned} \tag{11.17}$$

with similar relations involving matrix elements of $\boldsymbol{\rho}$ and $\boldsymbol{\chi}$.

Another way of "deriving" the substitution (11.6) is to consider average values in a boson coherent state. But in such a case, the substitution is only strictly justified for the normal-ordered operator. There is no contradiction because the commutators arising from normal ordering

are quantum fluctuations and should be husbanded for use in the study of quantum corrections. The result is then the same as for the Wigner transform method. We turn then to the application of these results.

B. Application to general schematic Hamiltonian

The BZM expansion stemming from Eq. (11.1) has proved its usefulness when employed in conjunction with a Hamiltonian of the form

$$H=\sum_{ij}(\epsilon_{ij}-\lambda\delta_{ij})a_i^\dagger a_j+\tfrac{1}{2}\sum_{ijkl}F_{ij,kl}a_i^\dagger a_k a_j^\dagger a_l +\tfrac{1}{4}\sum_{ij,kl}G_{ij,kl}a_i^\dagger a_j^\dagger a_l a_k \ , \tag{11.18}$$

where the one-body operator includes the Lagrange-multiplier term λN, the second term is an effective particle-hole force, and the last term is an effective particle-particle force, with antisymmetrized matrix elements $G_{ij,kl}=-G_{ji,kl}=-G_{ij,lk}$. Included in such schemes are the schematic models such as the quadrupole-quadrupole plus pairing force model and also the density-dependent interaction of Migdal's Fermi-liquid theory (Sorensen, 1971).

We have two choices for proceeding with the study of Eq. (11.18). The first is to make a transformation to quasiparticles before boson mapping, a method we shall use in succeeding sections. Here we shall pursue a second approach, which is to introduce particle bosons immediately according to the transcription

$$H_B=\sum_{ij}(\epsilon_{ij}-\delta_{ij}\lambda)(a_i^\dagger a_j)_B +\tfrac{1}{2}\sum_{ijkl}F_{ij,kl}(a_i^\dagger a_k)_B(a_j^\dagger a_i)_B +\tfrac{1}{4}\sum_{ijkl}G_{ij,kl}(a_i^\dagger a_j^\dagger)_B(a_l a_k)_B \ . \tag{11.19}$$

Next we set

$$b_\mu=c_\mu^0+\tilde{b}_\mu \ , \qquad b_\mu^\dagger=c_\mu^{(0)*}+\tilde{b}_\mu^\dagger, \quad \mu=\{ij\} \ . \tag{11.20}$$

Setting $\tilde{b}_\mu,\tilde{b}_\mu^\dagger=0$ initially, H_B becomes a c number,

$$H_B\rightarrow\mathcal{E}_0(\mathbf{c}^*,\mathbf{c})=\sum_{ij}(\epsilon_{ij}-\delta_{ij}\lambda)\rho_{ji}(\mathbf{c}^*,\mathbf{c})+\tfrac{1}{2}\sum_{ijkl}F_{ij,kl}\rho_{ki}(\mathbf{c}^*,\mathbf{c})\rho_{lj}(\mathbf{c}^*,\mathbf{c})+\tfrac{1}{4}\sum_{ij,kl}G_{ij,kl}\chi_{ij}^*(\mathbf{c}^*,\mathbf{c})\chi_{kl}(\mathbf{c}^*,\mathbf{c}) \ , \tag{11.21}$$

which is nothing other than the HB energy functional associated with the original Hamiltonian. In this case the elements of the generalized density matrix have already been parametrized (in terms of the c_{ij}^*,c_{ij}) so that the (Pauli principle) constraint $\mathcal{H}^2=\mathcal{H}$ is automatically satisfied. It follows that the equations

$$\partial\mathcal{E}_0(\mathbf{c}^*,\mathbf{c})/\partial c_\mu=0, \quad \mu=\{ij\} \tag{11.22}$$

and complex conjugates are equivalent to the HB equations and therefore determine the equilibrium configuration specified by the set $c_\mu^{(0)},c_\mu^{(0)*}$. The corresponding value of Eq. (11.21) is the equilibrium HB energy. The vacuum of the shifted bosons $(\tilde{b}_\mu,\tilde{b}_\mu^\dagger)$ is given by the boson coherent state

$$|\mathbf{c}(0))=\exp[iS(\mathbf{c}^0)]|0) \ , \tag{11.23a}$$

$$S(\mathbf{c}^{(0)})=-i\sum_\mu(c_\mu^{(0)}b_\mu^\dagger-c_\mu^{(0)*}b_\mu) \ . \tag{11.23b}$$

The solution of the many-body problem is continued by expanding H_B about $\mathcal{E}_0(\mathbf{c}^{(0)*},\mathbf{c}^{(0)})$. It is guaranteed that there are no linear boson terms, and the quadratic terms yield the standard RPA (without exchange). Exchange terms are taken care of in higher order. Higher RPA effects will be studied in the next section.

Suppose one starts with a general nuclear Hamiltonian, rather than a schematic one. If this is put into normal form with respect to a chosen quasiparticle vacuum, one certainly obtains the HFB energy from the constant term, rather than the HB result. The problem is to decide how to map the remaining operator part of the Hamiltonian. This problem is of considerable interest and importance for future applications. We postpone a full discussion of the possibilities until Sec. XIV.

C. Time-dependent Hartree-Bogoliubov theory

Next we go a step further and show that, in the c-number limit, the equations of motion in the BZM theory reduce to the time-dependent HB equations. The time dependence of the boson operators is given, of course, by the equations

$$i\hbar\frac{db_\mu}{dt}=[b_\mu,H_B] \ . \tag{11.24}$$

Since all the commutators are preserved, the equations of motion for fermion pair images are the exact equations

$$i\hbar\frac{d(a_i^\dagger a_j)_B}{dt}=\tfrac{1}{2}\{\hat{h}_{jk},(a_i^\dagger a_k)_B\}-\tfrac{1}{2}\{(a_k^\dagger,a_j)_B,\hat{h}_{ki}\} +\sum_k[\hat{\Delta}_{ki}^\dagger(a_k a_j)_B-\hat{\Delta}_{jk}(a_k^\dagger a_i^\dagger)_B] \ , \tag{11.25a}$$

$$i\hbar\frac{d(a_i a_j)_B}{dt}=\frac{1}{2}\sum_k\{\hat{h}_{jk},(a_i a_k)_B\}+\frac{1}{2}\sum_k\{(a_k a_j)_B,\hat{h}_{ki}^\dagger\} -\sum_k[(a_k^\dagger a_j)_B\hat{\Delta}_{ki}+(a_k^\dagger a_i)_B\hat{\Delta}_{jk}] \ , \tag{11.25b}$$

where

$$\hat{h}_{ki}=\epsilon_{ki}-\lambda\delta_{ki}+\sum_{lm}F_{kl,im}(a_l^\dagger a_m)_B\ ,$$
$$\hat{\Delta}_{ki}=\tfrac{1}{2}\sum_{lm}G_{ki,lm}(a_m a_l)_B\ . \tag{11.26}$$

In the c-number limit Eqs. (11.25) become

$$i\hbar\frac{dc_\mu}{dt}=\frac{\partial\mathcal{E}_0(\mathbf{c}^*,\mathbf{c})}{\partial c_\mu^*}\ , \tag{11.27}$$

$$i\hbar\frac{d\rho}{dt}=[\mathbf{h},\rho]+\chi\Delta^*-\Delta\chi^*\ ,$$
$$i\hbar\frac{d\chi}{dt}=\mathbf{h}\chi+\chi\mathbf{h}^*-\rho\Delta-\Delta\rho^*\ , \tag{11.28}$$

where $\mathbf{h}$ and Δ are the self-consistent field and pairing potentials, respectively:

$$h_{ki}=\epsilon_{ki}-\lambda\delta_{ki}+\sum_{lm}F_{kl,im}\rho_{ml}(\mathbf{c}^*,\mathbf{c})\ ,$$
$$\Delta_{kl}=\tfrac{1}{2}\sum_{lm}G_{ki,lm}\chi_{lm}(\mathbf{c}^*,\mathbf{c})\ , \tag{11.29}$$

and where the condition $\mathcal{H}^2=\mathcal{H}$ holds at each instant. With the aid of the Poisson bracket notation (11.16), one may write

$$i\hbar\frac{d\rho}{dt}=[\rho,\mathcal{E}_0]_P\ ,$$
$$i\hbar\frac{d\chi}{dt}=[\chi,\mathcal{E}_0]_P\ , \tag{11.30}$$

with $\mathcal{E}_0$, defined by Eq. (11.21), playing the role of Hamiltonian.

We have thus recovered the time-dependent HB equations, which differ from the exact equations only in lacking quantization. Put in another way, the boson representation seems to be the most rigorous way to quantize the time-dependent HB equations. This is an elegant "theorem," which has been utilized extensively by Kuriyama and Yamamura (1981a, 1981b, 1981c, 1981d, 1981e; Iwasaki, 1981; Yamamura, 1983; Raduta *et al.*, 1984; Yamamura and Kuriyama, 1987b, 1987c) to discover or rediscover boson mappings starting from Poisson bracket relations, i.e., classical-mechanical realizations of Lie algebras. We turn next to this inverse problem, which has ramifications much wider than the rediscovery of known boson mappings.

D. The inverse problem: Quantization of time-dependent Hartree-Fock theory. Canonical formulation of TDHF

For the discussion of this problem we shall omit the pairing correlations and confine our attention to time-dependent Hartree-Fock theory (TDHF). It is well to recognize three aspects of the problem: (i) First one establishes that the TDHF equations are a disguised form of Hamilton's classical equations of motion. The simplest way is to find at least one mapping of the equations to the Hamiltonian form. This, in essence, is what has been accomplished in Eq. (11.27), where the average values of the boson operators are revealed as complex canonical variables. The more complex method, which will be exemplified in the present discussion, is to provide an existence theorem for the canonical form. To be useful the existence theorem must be constructive, providing a means for choosing a specific set of canonical variables. (ii) The second and most important part is to discover *interesting* mappings. We shall explore this aspect in more detail below. (iii) Finally we quantize the classical system. This procedure also requires elaboration.

Interesting mappings are of at least three kinds. The first leads to canonical variables that yield the classical limits of the exact BZM (and extended BZM) mappings derived in Secs. IX and X. If this has been done independently of knowledge of the exact boson mappings, then upon quantization we have, up to operator ordering, a new derivation of these mappings. This approach is of considerable formal interest, especially in the case of the boson-fermion mapping of Sec. X, where we have constraints. Here one may call upon Dirac's theory of quantization of systems with constraints (Dirac, 1950; Ruggeri, 1988). Below, for illustrative purposes, we shall describe the method as applied to the boson mapping alone, since we have already established methods of obtaining the appropriate operator algebras when there are constraints.

A second class of problems arises when one is near the vibrational or rotational limit. Here one starts with the uncorrelated canonical variables furnished by the classical version of the BZM model and introduces linear transformations to RPA variables. Further canonical transformations are then required to diagonalize the Hamiltonian to higher order. For this type of problem, initial passage to the classical limit does not appear to offer great advantages. In fact, the following two sections are devoted to a relatively detailed study of these problems entirely within the quantum framework established at the start of this section.

A third class of problems may provide the natural hunting ground for the classical dynamical analysis, namely, the problem of large-amplitude collective motion. Here one must develop techniques for decoupling collective modes—where they exist but cannot be treated by small-amplitude approximations—from a system with many degrees of freedom. This problem is much too extensive (and complicated) to be wedged into the end of this section. Several aspects of this problem will be discussed in Sec. XXI.

Concerning quantization in realistic problems, one cannot, in dynamically interesting cases, obtain other than an approximately valid requantization of a system that has been taken to the classical limit. Even if we begin with a system described by Cartesian coordinates, a nonlinear transformation will take us to a curved space, leading to ambiguity in the requantization of the kinetic energy. As will be discussed, if the classical limit can be justified in the first place, then numerical errors resulting

from the ambiguities will be small.

We devote the remainder of this section to an illustration of the work of Kuriyama and Marumori, following the account of Yamamura and Kuriyama (1987b). If $|0\rangle$ is a reference Slater determinant that is the solution of a static HF calculation, we utilize below the particle-hole formalism, introducing particle and hole creation operators, $a_\alpha^\dagger$ and $b_\alpha^\dagger$. Thus if the set α is divided into subsets h and p, $\alpha=\{h,p\}$, occupied and unoccupied, respectively, in $|0\rangle$, then

$$b_h|0\rangle=a_p|0\rangle=0 \ . \tag{11.31}$$

The Slater determinant $|\gamma\rangle$, which is a solution of the TDHF equations, will be taken in the form

$$|\gamma\rangle=\hat{U}|0\rangle, \hat{U}^\dagger\hat{U}=\hat{U}\hat{U}^\dagger=1 \ , \tag{11.32}$$

where

$$\hat{U}=\exp\left[\sum_{ph}(a_p^\dagger b_h^\dagger\Gamma_{hp}-b_h a_p\Gamma_{hp}^*)\right] \ . \tag{11.33}$$

The Γ_{hp} and Γ_{hp}^* are time-dependent parameters, totaling $2NM\equiv 2f$ in all, where N is the number of hole and M the number of particle states, respectively. In place of the set Γ we wish to substitute canonical variables Q_r and P_r. We next describe a general method for doing this proposed by Marumori, Maskawa, Sakata, and Kuriyama (1980).

The TDHF theory can be based on the variational principle

$$\begin{aligned}&\delta\int_{t_1}^{t_2}L\,dt=0 \ ,\\ &L=\left\langle\gamma\left|\left[i\frac{\partial}{\partial t}-\hat{H}\right]\right|\gamma\right\rangle \ ,\end{aligned} \tag{11.34}$$

where Q_r and P_r are chosen as variational parameters. These can be determined in a manner that leaves open the possibility of a canonical transformation, by means of the equations

$$\begin{aligned}&\left\langle\gamma\left|i\frac{\partial}{\partial Q_r}\right|\gamma\right\rangle\equiv P_r+\frac{\partial S}{\partial Q_r} \ ,\\ &\left\langle\gamma\left|i\frac{\partial}{\partial P_r}\right|\gamma\right\rangle=\frac{\partial S}{\partial P_r} \ .\end{aligned} \tag{11.35}$$

It follows from Eq. (11.35) that L can be expressed as

$$\begin{aligned}&L=\sum_r P_r\dot{Q}_r-H+\dot{S} \ ,\\ &H=\langle\gamma|\hat{H}|\gamma\rangle \ .\end{aligned} \tag{11.36}$$

It follows from Eqs. (11.34) and (11.36) that the TDHF equations take the Hamiltonian form

$$\dot{Q}_r=\frac{\partial H}{\partial P_r}, \quad \dot{P}_r=-\frac{\partial H}{\partial Q_r} \ . \tag{11.37}$$

Equations (11.35) are called the canonicity conditions. A definite choice of canonical variables emerges from a specification of the generating function $S(Q_r,P_r)$.

To apply this procedure we utilize the fact already established in Secs. XI.A and XI.B, namely, that the expectation value of any operator in the state $|\gamma\rangle$ may be regarded as a classical image of that operator. For example, H in Eq. (11.36) (called $\mathcal{E}_0$ earlier in this section) is a classical image of $\hat{H}$. In this sense, TDHF theory gives us a classical image of the original many-fermion system. By requantization of some choice of canonical variables, one can regain the quantum domain from which one started, at least in some special cases.

E. Boson expansion derived from canonical quantization through the Poisson bracket

To illustrate the technique, we quote the results for the classical limits derived by Yamamura and Kuriyama (1987b). These authors show that

$$\langle\gamma|b_h a_p|\gamma\rangle=[(1-A)^{1/2}C]_{hp}=[(1-A^T)^{1/2}C^T]_{ph} \ , \tag{11.38}$$

$$\begin{aligned}&\langle\gamma|a_p^\dagger a_{p'}|\gamma\rangle=(C^\dagger C)_{pp'} \ ,\\ &\langle\gamma|b_h^\dagger b_{h'}|\gamma\rangle=(C^*C^T)_{hh'} \ ,\end{aligned} \tag{11.39}$$

$$(A)_{pp'}\equiv(C^\dagger C)_{pp'}, \quad (A)_{hh'}\equiv(C^*C^T)_{h'h} \ . \tag{11.40}$$

Here C is a matrix whose hp element has the value

$$\begin{aligned}(C)_{hp}&=[\Gamma(\Gamma^\dagger\Gamma)^{-1/2}\sin(\Gamma^\dagger\Gamma)^{1/2}]_{hp}\\ &=[\sin(\Gamma\Gamma^\dagger)^{1/2}(\Gamma\Gamma^\dagger)^{-1/2}\Gamma]_{hp} \ .\end{aligned} \tag{11.41}$$

It is convenient to use complex canonical coordinates,

$$X_r=\frac{1}{\sqrt{2}}(Q_r+iP_r), \quad X_r^*=\frac{1}{\sqrt{2}}(Q_r-iP_r) \ , \tag{11.42}$$

$$\begin{aligned}&i[X_r,X_s]_P=i[X_r^*,X_s^*]_P=0 \ ,\\ &i[X_r,X_s^*]_P=\delta_{rs} \ .\end{aligned} \tag{11.43}$$

The essential role played by the canonicity condition (11.35) is illustrated by the choice

$$S=-\tfrac{1}{2}\sum_r Q_r P_r \tag{11.44}$$

and by the evaluation

$$\begin{aligned}&\left\langle\gamma\left|\frac{\partial}{\partial X_r}\right|\gamma\right\rangle=\frac{1}{2}\mathrm{Tr}\left[C^\dagger\frac{\partial C}{\partial X_r}-\frac{\partial C^\dagger}{\partial X_r}C\right]=\frac{1}{2}X_r^* \ ,\\ &\left\langle\gamma\left|\frac{\partial}{\partial X_r^*}\right|\gamma\right\rangle=\frac{1}{2}\mathrm{Tr}\left[C^\dagger\frac{\partial C}{\partial X_r^*}-\frac{\partial C^\dagger}{\partial X_r^*}C\right]=-\frac{1}{2}X_r \ .\end{aligned} \tag{11.45}$$

A possible solution of Eq. (11.45) is

$$(C)_{hp}=X_r, \quad (C^*)_{hp}=X_r^*, \quad r=(hp) \ . \tag{11.46}$$

Substitution of Eq. (11.46) into Eqs. (11.38)–(11.40) yields the classical limit of the BZM mapping. Maintaining the order *as written*, and interpreting X_{hp} as a boson

operator satisfying the commutation relation

$$[X_{hp},X_{h',p'}]=[X^{\dagger}_{hp},X^{\dagger}_{h'p'}]=0\ , \tag{11.47}$$

$$[X_{hp},X^{\dagger}_{h'p'}]=\delta_{hh'}\delta_{pp'}\ , \tag{11.48}$$

one obtains the mapping

$$\begin{aligned}(b_h a_p)_B&=[(1-X^{\dagger}X)^{1/2}X]_{hp}\\&=[(1-(X^{\dagger}X)^{T})^{1/2}X^{T}]_{ph}\ ,\end{aligned} \tag{11.49}$$

$$\begin{aligned}(a^{\dagger}_p a_{p'})_B&=(X^{\dagger}X)_{pp'}\ ,\\(b^{\dagger}_h b_{h'})_B&=(X^{\dagger}X^{T})_{hh'}\ ,\end{aligned} \tag{11.50}$$

which agrees with the results of Sec. IX, except for notation.

The extension of these results to the boson-fermion case is rather more complex, and, in any event, obtaining the exact quantum system requires "malice aforethought," which can only succeed for the purely kinematical mappings considered here.

Let us remark in closing that results equivalent to those described in the last part of this section, i.e., the derivation of the BZM mapping by requantization, are also to be found in the work of Blaizot and Marshalek (1978a, 1978b). A primary aim of these papers, however, was to obtain finite boson mappings of the Schwinger form.

XII. PERTURBATIVE APPLICATION OF BZM MAPPINGS: BEYOND RPA AND CONNECTION WITH FINITE FERMI-SYSTEM THEORY AND NUCLEAR FIELD THEORY

A. A boson-fermion Hamiltonian for the study of vibrational nuclei

In this section, we present some examples (Marshalek, 1974b, 1975a) of the perturbative application of the generalized Holstein-Primakoff (BZM) boson expansion to spherical nuclei; deformed nuclei are discussed in an ensuing section. The basic assumption of the approach is that the mean-field plus random-phase approximation provides an adequate basis for treating higher-order anharmonic corrections by perturbation theory. The Hamiltonian is assumed to have the form given by Eq. (11.18). However, instead of mapping particle pairs onto boson operators and then performing the boson-shift transformation to eliminate linear boson terms, we now map quasiparticle pairs, where the quasiparticles are defined by the general Bogliubov transformation (11.5). In accord with most applications, only $l=0$ pairing will be assumed, in which case the simpler Bogoliubov-Valatin transformation is sufficient, provided that the so-called canonical single-particle basis is used (Valatin, 1961). The quasiparticle creation operators $\alpha^{\dagger}_i$ are then given by

$$\alpha^{\dagger}_i=U_i a^{\dagger}_i-\theta_i V_i a_{-i}\ , \tag{12.1}$$

where the coefficients satisfy $U_i^2+V_i^2=1$, thereby guaranteeing that the transformation is unitary and the quasiparticles obey fermion commutation rules. The notation $a^{\dagger}_{-i}=\theta_i T a^{\dagger}_i T^{-1}$ is used, where T is the time-reversal operator. Since the index i may refer to either time-positive or time-negative orbitals, the phase factor θ_i, defined by $\theta_i=1$ for $i>0$ and $\theta_i=-1$ for $i<0$, is introduced in order to satisfy $T^2=-1$. It should also be noted that in the case of closed-shell systems, $V_i=1$ for orbitals below the Fermi sea and $V_i=0$ for those above, so that the transformation (12.1) just introduces the particle and hole operators for which the Fermi sea is the vacuum.

In order to take into account the possible presence of an odd nucleon, we map into the full ideal space (Sec. X). The mapping is given by

$$\begin{aligned}a^{\dagger}_i a_j&\rightarrow(a^{\dagger}_i a_j)_I\\&=V_i^2\delta_{ij}+U_iU_j(\alpha^{\dagger}_i\alpha_j)_I-\theta_i\theta_jV_iV_j(\alpha^{\dagger}_{-j}\alpha_{-i})_I\\&\quad+U_iV_j\theta_j(\alpha^{\dagger}_i\alpha^{\dagger}_{-j})_I+U_jV_j(\alpha_{-i}\alpha_j)_I\ ,\\a^{\dagger}_i a^{\dagger}_{-j}&\rightarrow(a^{\dagger}_i a^{\dagger}_{-j})_I\\&=U_iV_i\theta_i\delta_{ij}+U_iU_j(\alpha^{\dagger}_i\alpha^{\dagger}_{-j})_I\\&\quad-\theta_i\theta_jV_iV_j(\alpha_{-i}\alpha_j)_I-U_iV_j\theta_j(\alpha^{\dagger}_i\alpha_j)_I\\&\quad-U_jV_j\theta_i(\alpha^{\dagger}_{-j}\alpha_{-i})_I\ \ \text{(and H.c. Eq)}\ .\end{aligned} \tag{12.2}$$

where $(\ )_I$ denotes the Holstein-Primakoff image derived in Sec. X.D. The corresponding image of the Hamiltonian (11.18), denoted by $(H)_I$, takes the form

$$\begin{aligned}(H)_I=&\ W_0+(H_{11})_I+(H_{22})_I+(H_{31}+\text{H.c.})_I\\&+(H_{40}+\text{H.c.})_I+(H'_{22})_I\ ,\end{aligned} \tag{12.3}$$

where W_0 turns out to be the Hartree-Bogoliubov ground-state energy, H_{11} is the image of the independent quasiparticle Hamiltonian, and the remaining terms represent residual interactions among the quasiparticles. Specifically,

$$W_0=\sum_i\epsilon_iV_i^2-\tfrac{1}{2}\sum_{ij}F_{ij,ij}V_i^2V_j^2-\tfrac{1}{2}\sum_i\Delta_iU_iV_i\ , \tag{12.4a}$$

$$(H_{11})_I=\sum_i E_i(\alpha^{\dagger}_i\alpha_i)_I\ , \tag{12.4b}$$

$$(H_{22})_I=\tfrac{1}{8}\sum_{ijkl}F_{il,jk}f_{ij}^{(+)}f_{kl}^{(+)}\{(\alpha_i^\dagger\alpha_{-j}^\dagger)_I,(\alpha_{-l}\alpha_k)_I\}-\tfrac{1}{4}\sum_{ijkl}G_{i-j,k-l}[\,U_iU_jU_kU_l(\alpha_i^\dagger\alpha_{-j}^\dagger)_I(\alpha_{-l}\alpha_k)_I$$
$$-V_iV_jV_kV_l(\alpha_{-l}\alpha_k)_I(\alpha_i^\dagger\alpha_{-j}^\dagger)_I]\ , \tag{12.4c}$$

$$H_{31}=\tfrac{1}{4}\sum_{ijkl}F_{ik,jl}f_{ij}^{(+)}g_{kl}^{(-)}\{(\alpha_i^\dagger\alpha_{-j}^\dagger)_I,(\alpha_k^\dagger\alpha_l)_I\}+\tfrac{1}{2}\sum_{ijkl}G_{i-j,l-k}\theta_k[U_kV_lV_iV_j(\alpha_k^\dagger\alpha_l)_I(\alpha_i^\dagger\alpha_{-j}^\dagger)_I$$
$$-U_lV_kU_iU_j(\alpha_i^\dagger-\alpha_{-j}^\dagger)_I(\alpha_k^\dagger\alpha_l)_I]\ , \tag{12.4d}$$

$$H_{40}=\tfrac{1}{8}\sum_{ijkl}[F_{ik,jl}f_{ij}^{(+)}f_{kl}^{(+)}-\tfrac{1}{2}G_{i-j,l-k}\theta_k\theta_l(g_{ij}^{(+)}g_{kl}^{(+)}-g_{ij}^{(-)}g_{kl}^{(-)})](\alpha_i^\dagger\alpha_{-j}^\dagger)_I(\alpha_k^\dagger\alpha_{-l}^\dagger)_I\ , \tag{12.4e}$$

$$H'_{22}=\sum_{ijkl}[\tfrac{1}{2}F_{ik,jl}g_{ij}^{(-)}g_{kl}^{(-)}+G_{i-j,l-k}U_iV_jU_lV_k\theta_j\theta_k](\alpha_i^\dagger\alpha_j)_I(\alpha_k^\dagger\alpha_l)_I\ , \tag{12.4f}$$

where $f_{ij}^{(\pm)}$ and $g_{ij}^{(\pm)}$ are defined by

$$f_{ij}^{(\pm)}\equiv(U_iV_j\pm U_jV_i)\theta_j\ ,$$
$$g_{ij}^{(\pm)}\equiv U_iU_j\pm V_iV_j\ . \tag{12.5}$$

The quasiparticle energies E_i in Eq. (12.4b) are given by the BCS formula

$$E_i=[(\mathcal{E}_i-\lambda)^2+\Delta_i^2]^{1/2}\ ,$$

where the $\mathcal{E}_i$ are self-consistent single-particle energies and the Δ_i are the gap parameters obtained from the simultaneous diagonalization of the matrices $\mathcal{E}_{ij}$ and Δ_{i-j} as follows:

$$\mathcal{E}_{ij}\equiv\epsilon_{ij}+\sum_k F_{ik,jk}V_k^2=\mathcal{E}_i\delta_{ij}\ ,$$
$$\Delta_{i-j}\equiv\sum_{k>0}G_{i-j,k-k}U_kV_k=-\delta_{ij}\theta_i\Delta_i\ . \tag{12.6}$$

A possible term $(H_{20})_i+\text{H.c.}$ in the Hamiltonian (12.3), given by

$$(H_{20})_I=\sum_{ij}[\,(\mathcal{E}_{ij}-\lambda\delta_{ij})U_iV_j\theta_j$$
$$+\tfrac{1}{2}\Delta_{i-j}(U_iU_j-V_iV_j)](\alpha_i^\dagger\alpha_{-j}^\dagger)_I \tag{12.7}$$

is eliminated, i.e., the condition $(H_{20})_I=0$ is achieved, first of all, by satisfying Eq. (12.6) and, second, by determining U_i, V_i so that

$$(\epsilon_i-\lambda)U_iV_i-\tfrac{1}{2}\Delta_i(U_i^2-V_i^2)=0\ , \tag{12.8}$$

which, together with the condition $U_i^2+V_i^2=1$, yields the BCS occupation probabilities

$$V_i^2=\tfrac{1}{2}[1-(\mathcal{E}_i-\lambda)/E_i]\ ,$$
$$U_i^2=\tfrac{1}{2}[1+(\mathcal{E}_i-\lambda)/E_i]\ . \tag{12.9}$$

Equations (12.6), (12.8), and (12.9) are equivalent to the Hartree-Bogoliubov approximation. We have seen in the last section that this approximation can be achieved by mapping the Hamiltonian onto particle-based bosons in an arbitrary single-particle basis and then performing the boson-shift transformation. Equivalently, one may first introduce the so-called canonical single-particle basis in which Eqs. (12.6) are satisfied and then map onto quasiparticle-based bosons. In both cases, the linear boson terms are removed to lowest order in the smallness parameter and all final physical results are the same. Note, however, that the shifted bosons of the previous section have an oscillator coherent state as the vacuum, while the bosons discussed in this section have as vacuum the boson image of the HB state, which is not the same.

To proceed further, we introduce the generalized Holstein-Primakoff images of the quasiparticle pairs in the ideal space, given by

$$(\alpha_i^\dagger\alpha_j)_I=\sum_k B_{ij}^\dagger B_{jk}+c_i^\dagger c_j=(\mathbf{B}^\dagger\mathbf{B})_{ji}^T+c_i^\dagger c_j\ , \tag{12.10a}$$

$$(\alpha_j\alpha_i)_I=\{[\mathbf{I}-(\mathbf{B}^\dagger\mathbf{B})^T]^{1/2}\mathbf{B}\}_{ij}$$
$$-\sum_{kl}[(\mathbf{I}-(\mathbf{B}^\dagger\mathbf{B})^T)_{kl}^{1/2},B_{ij}]c_k^\dagger c_l\ , \tag{12.10b}$$

$$(\alpha_i^\dagger\alpha_j^\dagger)_I=(\alpha_j\alpha_i)_I^\dagger\ , \tag{12.10c}$$

where the $B_{ij}^\dagger$ are the boson creation operators, while $c_k^\dagger$ creates an ideal odd particle. These equations reproduce Eqs. (10.14) and (10.60), except that we have changed symbols both for the bosons and for the quasifermions.

In order to lighten the notation we have not introduced angular momentum coupling, and thus an explicit smallness parameter is absent. Nevertheless, in expanding Eq. (12.10) and then substituting in the Hamiltonian defined by Eqs. (12.3) and (12.4), one should keep in mind the implicit relative orders of the terms determined by the kinematic expansion parameter $(2\Omega)^{-1/2}$, as discussed in Sec. IX. For example, to obtain the leading-order corrections to the RPA, one must retain the following terms in the expansion of $(\alpha_i^\dagger\alpha_j^\dagger)_I$:

$$(\alpha_i^\dagger\alpha_j^\dagger)_I\approx B_{ij}^\dagger-\tfrac{1}{2}\sum_{kl}B_{ik}^\dagger B_{lj}^\dagger B_{lk}$$
$$-\tfrac{1}{2}\sum_k(c_i^\dagger c_k B_{kj}^\dagger-c_j^\dagger c_k B_{ki}^\dagger)\ . \tag{12.11}$$

Compared to the first term $B_{ij}^\dagger$ on the right-hand side, the other terms should be regarded as $O((2\Omega)^{-1})$, while $(\alpha_i^\dagger\alpha_j)_I$, given by Eq. (12.10a), is $O((2\Omega)^{-1/2})$. Then, for example, the following terms contribute to $(H_{40})_I$:

$$(\alpha_i^\dagger\alpha_{-j}^\dagger)_i(\alpha_k^\dagger\alpha_{-l}^\dagger)_I \approx B_{i-j}^\dagger B_{k-l}^\dagger - \tfrac{1}{2}\sum_{mn}(B_{i-j}^\dagger B_{m-l}^\dagger B_{kn}B_{mn} + B_{m-j}^\dagger B_{in}^\dagger B_{mn}B_{k-l}^\dagger)$$
$$-\tfrac{1}{2}\sum_m B_{i-j}^\dagger B_{m-l}^\dagger c_k^\dagger c_m + \tfrac{1}{2}\sum_m B_{i-j}^\dagger B_{mk}^\dagger c_{-l}^\dagger c_n - \tfrac{1}{2}\sum_m B_{m-j}^\dagger B_{k-l}^\dagger c_i^\dagger c_m + \tfrac{1}{2}\sum_m B_{mi}^\dagger B_{k-l}^\dagger c_{-j}^\dagger c_m \ , \tag{12.12}$$

where the $B^\dagger B^\dagger$ term in the right-hand side provides the backward-going graphs of the RPA. The quartic boson terms provide anharmonic corrections to the RPA of relative order $(2\Omega)^{-1}$, while the $B^\dagger B^\dagger c^\dagger c$ terms are also of order $(2\Omega)^{-1}$ compared with the ideal odd-particle term $\sum_i E_i c_i^\dagger c_i$ arising from $(H_{11})_I$. As another example, we note that the leading-order contributions to $(H_{31})_I$ arise from

$$(\alpha_i^\dagger\alpha_{-j}^\dagger)_I(\alpha_k^\dagger\alpha_l)_I \approx \sum_m B_{i-j}^\dagger B_{km}^\dagger B_{lm} + B_{i-j}^\dagger c_k^\dagger c_l \ , \tag{12.13}$$

which should be considered as $O((2\Omega)^{-1/2})$ compared to the RPA and ideal odd-particle terms. Since $(H_{31})_I$ has no diagonal matrix elements, it first contributes in second-order perturbation theory terms of relative order $(2\Omega)^{-1}$, which are comparable to the diagonal contributions arising from quartic terms.

For the purpose of calculating the leading-order corrections to the RPA, it is only necessary to expand through quartic terms as in the above examples. Indeed, we know of no perturbation applications that have been carried out to higher orders. To the order of interest then

$$(H)_I \approx W_0 + \sum_i E_i c_i^\dagger c_i + H_B^{(2)} + H_B^{(3)} + H_{BF}^{(1)} + H_B^{(4)} + H_{BF}^{(2)} \ , \tag{12.14}$$

where

$$H_B^{(2)} \equiv H_{\rm RPA} = \tfrac{1}{2}\sum_{ij}(\dot{E}_i + E_j)B_{ij}^\dagger B_{ij} + \tfrac{1}{8}\sum_{ijkl} F_{il,jk} f_{ij}^{(+)} f_{kl}^{(+)}\{B_{ij}^\dagger, B_{k-l}\}$$
$$-\tfrac{1}{4}\sum_{ijkl} G_{i-j,k-l}(U_iU_jU_kU_lB_{i-j}^\dagger B_{k-l} - V_iV_jV_kV_lB_{k-l}B_{i-j}^\dagger)$$
$$+\tfrac{1}{8}\sum_{ijkl}\{[F_{ik,jl}f_{ij}^{(+)}f_{kl}^{(+)} - \tfrac{1}{2}G_{i-j,l-k}\theta_k\theta_l(g_{ij}^{(+)}g_{kl}^{(+)} - g_{ij}^{(-)}g_{kl}^{(-)}]B_{i-j}^\dagger B_{k-k}^\dagger + \text{H.c.}\} \tag{12.15a}$$

is the RPA Hamiltonian arising from $(H_{11})_I$, $(H_{22})_I$, and $(H_{40})_I + \text{H.c.}$, and where

$$H_B^{(3)} \equiv \tfrac{1}{4}\sum_{ijklm} F_{ik,jl}f_{ij}^{(+)}g_{kl}^{(-)}\{B_{i-j}^\dagger, B_{km}^\dagger B_{lm}\}$$
$$+\tfrac{1}{2}\sum_{ijklm}[G_{i-j,l-k}\theta_k(U_kV_lV_iV_jB_{km}^\dagger B_{lm}B_{i-j}^\dagger - U_lV_kU_iU_jB_{i-k}^\dagger B_{km}^\dagger B_{lm}) + \text{H.c.}] \tag{12.15b}$$

and

$$H_{BF}^{(1)} = \tfrac{1}{2}\sum_{ijkl}[F_{ik,jl}f_{ij}^{(+)}g_{kl}^{(-)} - \tfrac{1}{2}G_{i-j,l-k}(f_{lk}^{(+)}g_{ij}^{(-)} + f_{lk}^{(-)}g_{ij}^{(+)})]B_{i-j}^\dagger c_k^\dagger c_l + \text{H.c.} \tag{12.15c}$$

with both arising from $(H_{31})_I + \text{H.c.}$ Furthermore,

$$H_B^{(4)} \equiv -\tfrac{1}{16}\sum_{ijklmn} F_{il,jk}f_{ij}^{(+)}f_{kl}^{(+)}\{B_{i-j}^\dagger, B_{mn}^\dagger B_{kn}B_{m-l}\}$$
$$+\tfrac{1}{8}\sum_{ijklmn} G_{i-jk-l}(U_iU_jU_kU_lB_{i-j}^\dagger B_{mn}^\dagger B_{kn}B_{m-l} - V_iV_jV_kV_lB_{mn}^\dagger B_{kn}B_{m-l}B_{i-j}^\dagger)$$
$$+\tfrac{1}{4}\sum_{ijklmn}(F_{ik,jl}g_{ij}^{(-)}g_{kl}^{(-)} + 2G_{i-j,l-k}\theta_j\theta_kU_iV_jU_lV_k)B_{im}^\dagger B_{jm}B_{kn}^\dagger B_{ln}$$
$$-\tfrac{1}{16}\sum_{ijklmn}(F_{ik,jl}f_{ij}^{(+)}f_{kl}^{(+)} - 2G_{i-j,l-k}\theta_k\theta_lU_iU_jV_kV_l)B_{i-j}^\dagger B_{m-l}^\dagger B_{kn}B_{mn}$$
$$-\tfrac{1}{16}\sum_{ijklmn}(F_{ik,jl}f_{ij}^{(+)}f_{kl}^{(+)} - 2G_{i-j,l-k}\theta_k\theta_lU_kU_lV_iV_j)B_{m-l}^\dagger B_{kn}^\dagger B_{mn}B_{i-j}^\dagger + \text{H.c. of all terms} \ , \tag{12.15d}$$

and, finally,

$$H_{BF}^{(2)} \equiv -\tfrac{1}{8}\sum_{ijklm} F_{il,jk} f_{ij}^{(+)} f_{kl}^{(+)} \{B_{i-j}^{\dagger}, B_{m-l}\} c_m^{\dagger} c_k + \tfrac{1}{4}\sum_{ijklm} G_{i-j,k-l}(U_i U_j U_k U_l B_{i-j}^{\dagger} B_{m-l} - V_i V_j V_k V_l B_{k-l} B_{i-j}^{\dagger}) C_m^{\dagger} C_k$$
$$+\tfrac{1}{2}\sum_{ijklm} [F_{il,jk} g_{ij}^{(-)} g_{kl}^{(-)} + \tfrac{1}{2} G_{i-j,k-l}(f_{ij}^{(+)} f_{kl}^{(+)} + f_{ij}^{(-)} f_{kl}^{(-)})] B_{im}^{\dagger} B_{jm} c_l^{\dagger} c_k$$
$$-\tfrac{1}{4}\sum_{ijklm} [F_{ik,jl} f_{ij}^{(+)} f_{kl}^{(+)} - \tfrac{1}{2} G_{i-j,l-k} \theta_k \theta_l (g_{ij}^{(+)} g_{kl}^{(+)} - g_{ij}^{(-)} g_{kl}^{(-)})] B_{i-j}^{\dagger} B_{m-l}^{\dagger} c_k^{\dagger} c_m$$
$$+\tfrac{1}{4}\sum_{ijk} (F_{ij,jk} g_{ij}^{(-)} g_{kj}^{(-)} + 2 G_{i-j,k-j} U_i V_j U_k V_j) c_i^{\dagger} c_k + \text{H.c. of all terms} . \tag{12.15e}$$

The contributions (12.15d) and (12.15e) arise from $(H_{22})_I$, $(H'_{22})_I$, and $(H_{40})_I$ + H.c.

In the Hamiltonian given by Eqs. (12.14) and (12.15), the Hartree-Bogoliubov energy is of the order of (2Ω), the RPA Hamiltonian (12.15a) is of the order of unity, $H_B^{(3)}$ and $H_{BF}^{(1)}$ are of order $(2\Omega)^{-1/2}$, while $H_B^{(4)}$ and $H_{BF}^{(3)}$ are of order $(2\Omega)^{-1}$. Since $H_B^{(3)}$ and $H_{BF}^{(1)}$ have vanishing diagonal matrix elements in an RPA basis, they first contribute to the energy in second-order perturbation theory and are thus of the same order as the diagonal contributions of $H_B^{(4)}$ and $H_{BF}^{(2)}$.

B. Perturbative diagonalization. The random-phase approximation and beyond

We have carefully avoided contracting operators up to this point. The reason, of course, is to avoid mixing up terms of different orders, which is easy to do in the absence of an explicit expansion parameter. However, once all of the terms of a given order have been accounted for, there is certainly no harm in normal ordering. Thus, for example, $H_B^{(4)}$, when rearranged into normal order, gives rise to quadratic terms, but these are smaller by a factor of the order of $(2\Omega)^{-1}$ than those included in the RPA. Likewise, the linear boson terms arising from normal ordering of $H_B^{(3)}$ are smaller by a factor of the order of $(2\Omega)^{-1}$ than those of $(H_{20})_B$ removed by the choice of the HB basis.

The zero-order basis vectors for the perturbation treatment are provided by the Hamiltonian $\sum_i E_i c_i^{\dagger} c_i + H_B^{(2)}$. The RPA Hamiltonian $H_B^{(2)}$ can be brought into the general diagonal form

$$H_B^{(2)} = W_{\text{RPA}} + \sum_{\sigma} \hbar\omega_{\sigma} \mathcal{O}_{\sigma}^{\dagger} \mathcal{O}_{\sigma} + \tfrac{1}{2}\sum_r \mathcal{P}_r^2 / \mathcal{I}_r , \tag{12.16}$$

W_{RPA} being the correction to the ground-state energy. The $\mathcal{O}_{\sigma}, \mathcal{O}_{\sigma}^{\dagger}$ are normal-mode bosons ("phonons") corresponding to nonzero eigenfrequencies ω_{σ} and related to the $B_{ij}, B_{ij}^{\dagger}$ by a Bogoliubov transformation

$$\mathcal{O}_{\sigma}^{\dagger} = \tfrac{1}{2}\sum_{ij} [X_{ij}(\sigma) B_{ij}^{\dagger} - Y_{ij}(\sigma) B_{ij}] \tag{12.17}$$

whose coefficients are the eigenvectors of the eigenvalue equation (Rowe, 1970; Ring and Schuck, 1980),

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B}^* & -\mathcal{A}^* \end{bmatrix} \begin{bmatrix} \mathbf{X}(\sigma) \\ \mathbf{Y}(\sigma) \end{bmatrix} = \hbar\omega_{\sigma} \begin{bmatrix} \mathbf{X}\sigma \\ \mathbf{Y}_{\sigma} \end{bmatrix} , \tag{12.18}$$

where the submatrices $\mathcal{A}$ and $\mathcal{B}$ are obtained by writing Eq. (12.15a) in normal order:

$$H_B^{(2)} = \text{const} + \sum_{i<j,k<l} \mathcal{A}_{ij,kl} B_{ij}^{\dagger} B_{kl} + \tfrac{1}{2}\sum_{i<j,k<l} (\mathcal{B}_{ij,kl} B_{ij}^{\dagger} B_{kl}^{\dagger} + \text{H.c.}) . \tag{12.19}$$

Specifically,

$$\mathcal{A}_{i-j,k-l} = (\delta_{ki}\delta_{lj} - \delta_{k-j}\delta_{l-i})(E_i + E_j) + F_{il,jk} f_{ij}^{(+)} f_{kl}^{(+)} - \tfrac{1}{2} G_{i-j,k-l}(g_{ij}^{(+)} g_{kl}^{(-)} + g_{ij}^{(-)} g_{kl}^{(+)}) \theta_k \theta_l ,$$
$$\mathcal{B}_{i-j,k-l} = F_{ik,jl} f_{ij}^{(+)} f_{kl}^{(+)} - \tfrac{1}{2} G_{i-j,l-k} \theta_k \theta_l (g_{ij}^{(+)} g_{kl}^{(+)} - g_{ij}^{(-)} g_{kl}^{(-)}) . \tag{12.20}$$

In addition, the $\mathcal{P}_r$ in Eq. (12.16) (the $\mathcal{I}_r$ are inertial parameters), which commute with the $\mathcal{O}_{\sigma}, \mathcal{O}_{\sigma}^{\dagger}$ and can be chosen to commute among themselves, represent the linear boson approximations to generators of broken symmetries, i.e., those constants of motion not conserved by the HB approximation. If the system is superfluid, i.e., for some range of single-particle levels $\Delta_i \neq 0$, then the total particle number of the paired nucleons is certainly among these constants of motion, with other possibilities including linear momentum, isospin, and, in the case of deformed nuclei, the total angular momentum. Such degrees of freedom, together with canonically conjugate coordinates, are often referred to (misleadingly) as "spurious modes," since they are associated with vanishing RPA frequencies. In the present section, zero-frequency modes will be disregarded, but as shown in the next section, the expansion of $(H)_I$ contains all the information necessary to correctly reconstruct the dependence on such constants of motion.

We now are prepared to discuss the perturbative diagonalization of $(H)_I$ using the (nonzero-frequency) RPA states plus those of the odd particle as a basis. A convenient technique consists of applying successive unitary transformations to diagonalize the Hamiltonian to the order of interest. Since $H_B^{(3)}$ and $H_{BF}^{(1)}$ have no diagonal matrix elements, it is clear that to proceed one order

beyond the RPA, the unitary transformation must entirely remove these terms. This is conveniently accomplished by means of two successive unitary transformations. The first, given by

$$(H)_I \rightarrow (H_I)' = \exp(-iS_B^{(3)})(H)_I \exp(iS_B^{(3)}) = (H)_I + [(H)_I, iS_B^{(3)}] - \tfrac{1}{2}[[(H)_I, S_B^{(3)}], S_B^{(3)}] + \cdots , \quad (12.21)$$

may be chosen to remove $H_B^{(3)}$ through the requirement

$$[H_B^{(2)}, iS_B^{(3)}] = -H_B^{(3)} . \quad (12.22)$$

In the absence of an odd particle, $H_{BF}^{(1)}$, of course, can be neglected. Otherwise, a second transformation

$$(H_I)' \rightarrow (H_I)'' = \exp(-iS_{BF}^{(1)})(H_I)' \exp(iS_{BF}^{(1)}) , \quad (12.23)$$

with $S_{BF}^{(1)}$ chosen to satisfy

$$\left[\sum_i E_i c_i^\dagger c_i + H_B^{(2)}, iS_{BF}^{(1)} \right] = -H_{BF}^{(1)} , \quad (12.24)$$

eliminates $H_{BF}^{(1)}$ to lowest order.

Subsequent to the two transformations, the Hamiltonian takes the form

$$(H_I)'' = W_0 + \sum_i E_i c_i^\dagger c_i + H_B^{(2)} + \tfrac{1}{2}[H_B^{(3)}, iS_B^{(3)}] + H_B^{(4)} + [H_{BF}^{(1)}, iS_B^{(3)}] , + \tfrac{1}{2}[H_{BF}^{(1)}, iS_{BF}^{(1)}] + H_{BF}^{(2)} + (\text{higher orders}) . \quad (12.25)$$

The corrections to the RPA are now all of relative order $(2\Omega)^{-1}$, with those of relative order $(2\Omega)^{-1/2}$ having been removed. Thus, to obtain corrections to the RPA energies to this order, it is only necessary to calculate expectation values of Eq. (12.25). Of course, one could perform an additional unitary transformation to remove *off-diagonal* terms of relative order $(2\Omega)^{-1}$, but that would only be necessary if the energies through relative order $(2\Omega)^{-2}$ were desired [in which case the off-diagonal terms of order $(2\Omega)^{-3/2}$ would also have to be removed] or if one wanted the effects of these off-diagonal terms on transition matrix elements. In principle, the method of successive unitary transformations could be used to transform the Hamiltonian into the diagonal form of a polynomial in the boson and odd-particle number operators $\mathcal{O}_i^\dagger \mathcal{O}_i$ and $c_i^\dagger c_i$, respectively, with coefficients given as expansions in a smallness parameter of the order of $(2\Omega)^{-1}$. [This is the quantum analog of the classical Birkhoff-Gustavson method (Birkhoff, 1966; Gustavson, 1966). Since the classical method is known not to converge, the same is expected for the quantum expansion. Nevertheless, it may still provide a useful asymptotic expansion.] The physical expansion parameter is of the order of the product of the kinematic parameter and the RPA zero-point amplitudes. Since the latter blow up in the neighborhood of a phase transition, such as the transition from spherical to deformed shapes, perturbation theory necessarily breaks down in such a situation.

C. Anharmonic corrections to observables

In the viewpoint taken here, operators get transformed while state vectors remain unchanged and equal to the zero-order vectors. To calculate matrix elements of transition operators, we must take into account the transformation of the pair and density operators, which, to lowest order, are given by

$$(\alpha_j \alpha_i)_I'' \approx B_{ij} + [B_{ij}, iS_B^{(3)}] + [B_{ij}, iS_{BF}^{(1)}] , \quad (\alpha_i^\dagger \alpha_j)_I'' \approx (\alpha_i^\dagger \alpha_j)_I = \sum_k B_{ik}^\dagger B_{jk} + c_i^\dagger c_j , \quad (12.26)$$

The quasiparticle transition density matrix is given by

$$\kappa_{ij}(ab) \equiv (a|(\alpha_j \alpha_i)_I''|b) , \quad r_{ji}(ab) \equiv (a|(\alpha_i^\dagger \alpha_j)_I''|b) , \quad (12.27)$$

where $|a)$ and $|b)$ denote any two eigenstates of $\sum_i E_i c_i^\dagger c_i + H_B^{(2)}$. With a knowledge of the matrix elements (12.27), we may calculate all particle-transition density-matrix elements and thereby all moments and transition matrix elements.

In order to compare these matrix elements with results of the theory of finite Fermi systems, it is convenient to introduce the quantities

$$\delta\kappa_{ij}(ab) \equiv \kappa_{ij}(ab) - \delta_{ab}(0|(\alpha_j \alpha_i)_I''|0) , \quad \delta r_{ij}(ab) \equiv r_{ij}(ab) - \delta_{ab}(0|(\alpha_i^\dagger \alpha_j)_I''|0) , \quad (12.28)$$

where $|0)$ is the RPA vacuum satisfying $\mathcal{O}_\sigma|0) = c_i|0) = 0$. The commutators (12.22) and (12.24) can be converted to linear inhomogeneous equations for the quantities (12.28) by first commuting them with a B_{ij} and using the Jacobi identity, to obtain

$$[H_B^{(2)}, [B_{ij}, iS_B^{(3)}]] + \tfrac{1}{2} \sum_{kl} \{\mathcal{A}_{ij,kl}[B_{kl}, iS_B^{(3)}] + \mathcal{B}_{ij,kl}[B_{kl}^\dagger, iS_B^{(3)}]\} = [H_B^{(3)}, B_{ij}] \quad (12.29)$$

and

$$\left[\sum_i E_i c_i^\dagger c_i + H_B^{(2)}, [B_{ij}, iS_{BF}^{(1)}] \right] + \tfrac{1}{2} \sum_{kl} \{\mathcal{A}_{ij,kl}[B_{kl}, iS_{BF}^{(1)}] + \mathcal{B}_{ij,kl}[B_{kl}^\dagger, iS_{BF}^{(1)}]\} = [H_{BF}^{(1)}, B_{ij}] . \quad (12.30)$$

By taking matrix elements of both sides of Eqs. (12.29)

and (12.30), one obtains integral equations for the corrections to the quasiparticle transition density matrix. We consider first an even-even nucleus with arbitrary eigenstates $|e\rangle$ and $|e'\rangle$ of $H_B^{(2)}$. Then $\delta\kappa(ee')\cong\delta\kappa_{i-j}^{(1)}(ee')+\delta\kappa_{i-j}^{(2)}(ee')$, where, according to Eq. (12.26), $\delta\kappa_{i-j}^{(1)}(ee')=(e|B_{i-j}|e')$, an RPA amplitude. From Eq. (12.29) one readily finds the inhomogeneous equation for $\delta\kappa_{i-j}^{(2)}(ee')$:

$$(e-e')\delta\kappa_{i-j}^{(2)}(ee')+\tfrac{1}{2}\sum_{kl}[\mathcal{A}_{i-j,k-l}\delta\kappa_{k-l}^{(2)}(ee')+\mathcal{B}_{i-j,k-l}\delta\kappa_{k-l}^{(2)*}(e'e)]$$
$$=(e|[H_B^{(2)},B_{i-j}]|e')-\delta_{e,e'}\langle 0|[H_b^{(3)},B_{i-j}]|0\rangle\ , \quad (12.31)$$

where

$$\delta\kappa_{i-j}^{(2)}(ee')=\langle 0|[B_{i-j},iS_B^{(3)}]|0\rangle-\delta_{ee'}\langle 0|[B_{i-j},iS_B^{(3)}]|0\rangle\ . \quad (12.32)$$

To the present order of approximation, δr_{ij} is determined entirely by the RPA, as can be seen from the second of Eqs. (12.26).

As a more specific application of Eq. (12.31), we consider the case when both states are of the one-phonon type, so that $e=\hbar\omega_\sigma$ and $|e\rangle=|\sigma\rangle=\mathcal{O}_\sigma^\dagger|0\rangle$. Then Eq. (12.31) becomes

$$\hbar(\omega_\sigma-\omega_{\sigma'})\delta\kappa_{i-j}^{(2)}(\sigma\sigma')+\tfrac{1}{2}\sum_{kl}[\mathcal{A}_{i-j,k-l}\delta\kappa_{k-l}^{(2)}(\sigma\sigma')+\mathcal{B}_{i-j,k-l}\delta\kappa_{k-l}^{(2)*}(\sigma'\sigma)]$$
$$=\sum_{klm}[-\mathcal{V}_{ik,jl}(X_{km}^*(\sigma)X_{lm}(\sigma')+Y_{km}(\sigma')Y_{lm}^*(\sigma))-\tfrac{1}{2}\mathcal{V}_{ki,lm}(X_{k-l}^*(\sigma)X_{m-j}(\sigma')+Y_{k-l}(\sigma')Y_{m-j}^*(\sigma))$$
$$-\tfrac{1}{2}\mathcal{V}_{km,lj}\theta_j\theta_m(X_{k-l}^*(\sigma)X_{i-m}(\sigma')+Y_{k-l}(\sigma')Y_{i-m}^*(\sigma))$$
$$+\tfrac{1}{2}\mathcal{V}_{li,km}\theta_k\theta_l(Y_{k-l}^*(\sigma)X_{m-j}(\sigma')+X_{k-l}(\sigma')Y_{m-j}^*(\sigma))$$
$$+\tfrac{1}{2}\mathcal{V}_{lm,kj}\theta_j\theta_k\theta_l\theta_m(Y_{k-l}^*(\sigma)X_{i-m}(\sigma')+X_{k-l}(\sigma')Y_{i-m}^*(\sigma))]\ , \quad (12.33)$$

where

$$\mathcal{V}_{ik,jl}\equiv F_{ik,jl}f_{ij}^{(+)}g_{kl}^{(-)}-\tfrac{1}{2}G_{i-j,l-k}(g_{ij}^{(-)}f_{lk}^{(+)}+g_{ij}^{(+)}f_{lk}^{(-)})\ . \quad (12.34)$$

Since $\delta\kappa_{i-j}^{(1)}(ee')=0$ in this case, we have $\delta\kappa_{i-j}(ee')\approx\delta\kappa_{i-j}^{(2)}(ee')$.

Equations having exactly the form of Eqs. (12.31)–(12.34) have also been derived within the framework of the theory of finite Fermi systems by using Green's functions (Birbrair, 1976, Speth, Werner, and Wild, 1977). The latter approach makes use of a specific form of the residual interaction, namely, a density-dependent delta interaction due to Migdal (1967).

From the solution of Eq. (12.33) for the diagonal case $\sigma=\sigma'$ one may, for example, calculate the electric quadrupole moments of the one-phonon 2+ states of spherical nuclei. A relatively simple closed form emerges for the popular interaction consisting of the quadrupole-quadrupole particle-hole interaction given by

$$F_{ik,jl}=-\chi\sum_{M=-2}^{2}\langle i|r^2Y_{2M}|j\rangle\langle l|r^2Y_{2M}|k\rangle^* \quad (12.35a)$$

plus the particle-particle pairing interaction with constant matrix elements given by

$$G_{ij,kl}=-G\delta_{j,-i}\delta_{l,-k}\theta_i\theta_k\ . \quad (12.35b)$$

Since the interaction is a sum of separable forces, the integral equation (12.33) has a degenerate kernel, thus allowing a simple closed solution. With allowance for different strengths of the interaction (12.35a) for *nn*, *pp*, and *np* pairs, denoted by χ_n, χ_p, and χ_{np}, respectively, it can be shown that the $E2$ moments are given by

$$Q_{2+}=-\left[\frac{16\pi}{5}\right]^{1/2}\left[e_n(\mathrm{pol})\frac{\partial\hbar\omega}{\partial\alpha_n}+e_p(\mathrm{pol})\frac{\partial\hbar\omega}{\partial\alpha_p}\right]\ , \quad (12.36)$$

where the quantities $\partial\hbar\omega/\partial\alpha_n$ and $\partial\hbar\omega/\partial\alpha_p$, which can be shown to represent the changes of the lowest RPA frequency with no respect to neutron and proton deformations of the average field, are given by the microscopic sums

$$\frac{\partial\hbar\omega}{\partial\alpha_n}=Z_n\begin{bmatrix}2&2&2\\2&0&-2\end{bmatrix}\sum_{123(\mathrm{neutrons})}\begin{Bmatrix}2&2&2\\j_1&j_2&j_3\end{Bmatrix}\frac{q_{12}q_{23}q_{31}f_{12}^{(+)}g_{23}^{(-)}f_{13}^{(+)}}{E_{13}^2-\hbar^2\omega^2}\left[\frac{2E_{13}}{E_{12}}+\frac{E_{12}E_{13}+\hbar^2\omega^2}{E_{12}^2-\hbar^2\omega^2}\right]\ , \quad (12.37)$$

where the sum runs over neutron single-particle levels, and an analogous sum for $\partial\hbar\omega/\partial\alpha_p$ runs over proton levels. The reduced matrix elements in (12.37) are given by

$$q_{12}\equiv\langle 1\|r^2Y_2\|2\rangle\ , \tag{12.38}$$

$E_{12}\equiv E_1+E_2$, a two-quasiparticle excitation energy; $\hbar\omega$ is the RPA frequency of the 2+ level; the symbol ( ) is a 3-j symbol; and { } is a 6-j symbol. The factors Z_n and Z_p are given by

$$\begin{aligned} Z_n&=\frac{\Sigma_p(\omega)(1-2\chi_p\,\Sigma_p(\omega))}{N^2}\ ,\\ Z_p&=\frac{\Sigma_n(\omega)(1-2\chi_n\,\Sigma_n(\omega))}{N^2}\ , \end{aligned} \tag{12.39}$$

where $\Sigma_n(\omega)$, $\Sigma_p(\omega)$ are the neutron and proton contributions, respectively, to the sums

$$\Sigma(\omega)\equiv\tfrac{1}{10}\sum_{12}\frac{|q_{12}|^2 f_{12}^{(+)2}E_{12}}{E_{12}^2-\hbar^2\omega^2}\ , \tag{12.40}$$

which occur in the RPA dispersion equation

$$[1-2\chi_n\Sigma_n(\omega)][1-2\chi_p\Sigma_p(\omega)]-4\chi_{np}^2\Sigma_n(\omega)\Sigma_p(\omega)=0\ , \tag{12.41}$$

and where N is the RPA normalization factor

$$\begin{aligned} N^{-2}=\omega[&\,\Sigma_p(\omega)(1-2\chi_p\Sigma_p(\omega))B_n\\ &+\Sigma_n(\omega)(1-2\chi_n\Sigma_n(\omega))B_p]\ , \end{aligned} \tag{12.42}$$

B_n and B_p being the neutron and proton contributions, respectively, to the sum

$$B=\tfrac{1}{5}\sum_{12}\frac{|q_{12}|^2 f_{12}^{(+)2}E_{12}^2}{(E_{12}^2-\hbar^2\omega^2)^2}\ , \tag{12.43}$$

which is proportional to an RPA mass parameter.

The quantities e_n(pol) and e_p(pol) in Eq. (12.36) are the neutron and proton polarization charges, where e_n(pol), for example, is given by

$$e_n(\text{pol})=\frac{\chi_n[e_n\chi_pC_p+e_p\chi_{np}(1-C_p)]}{\chi_n\chi_pC_nC_p-\chi_{np}^2(1-C_n)(1-C_p)}\ , \tag{12.44}$$

with a similar expression for e_p(pol) obtained by interchanging "n" and "p" subscripts. In Eq. (12.44), e_n, e_p are the effective neutron and proton charges, and C_n, C_p are given by

$$C_n=1-2\chi_n\Sigma_n(0),\quad C_p=1-2\chi_p\Sigma_p(0)\ . \tag{12.45}$$

Equation (12.36) with $\chi_n=\chi_p=\chi_{np}$ has been applied to the Sn isotopes, with the calculated quadrupole moments coming out generally larger than experimental values by factors of two. Almost exactly the same equation had been derived within the framework of nuclear field theory (Broglia, Liotta, and Paar, 1972), with one difference. It can be shown (Marshalek and da Providência, 1973b) that to a good approximation

$$e_n(\text{pol})\propto\sum_\sigma\frac{B(E2,0_{g^+}\to 2_{\sigma^+})}{\hbar\omega_\sigma}\ , \tag{12.46}$$

i.e., it is proportional to a sum over all $B(E2)$'s from the ground state to the various RPA one-phonon 2+ modes, inversely weighted by the excitation energy.

In the work of Broglia *et al.*, it was assumed that this sum is saturated already by just the one contribution for the first excited state, which can be obtained empirically. This leads to a considerably smaller value than the purely microscopic expression, especially for the single-closed-shell nuclei which were calculated, since their 2+ states are not very collective. The result was that the calculations of Broglia *et al.* agreed much better with experiment, but that agreement must be considered as an artifact of the drastic approximation. The quadrupole-quadrupole interaction probably provides excessive collectivity. In this connection, the work of Meyer-ter-Vehn (1979) should be mentioned, in which an equivalent theory was derived entirely within the framework of the classical time-dependent self-consistent field method. This is not surprising since, in fact, Eq. (12.36) was first derived by the self-consistent cranking method (Marshalek and Sabato, 1971), taking advantage of the connection between the c-number limit of the boson expansion theory and the self-consistent field, as discussed in Sec. XI. The main point here, however, is that Meyer-ter-Vehn showed that the addition of quadrupole pairing (particle-particle channel) considerably tempered the excessive collectivity from the quadrupole-quadrupole force, thereby reducing the quadrupole moments. As far as we know, the theoretical status of the quadrupole moments of vibrational nuclei is still an open, if generally forgotten, question.

Next, let us briefly turn to odd nuclei, where we consider states of the form $|m\rangle=c_m^\dagger|0\rangle$, i.e., an odd particle added to the RPA vacuum. Taking matrix elements in Eq. (12.30) between two such states, we obtain the equation

$$\begin{aligned} &(E_m-E_n)\delta\kappa_{i-j}^{(2)}(mn)\\ &\quad+\tfrac{1}{2}\sum_{kl}[\mathcal{A}_{i-j,k-l}\delta\kappa_{k-l}^{(2)}(mn)\\ &\qquad+\mathcal{B}_{i-j,k-l}\delta\kappa_{k-l}^{(2)*}(nm)]=-\mathcal{V}_{im,jn}\ , \end{aligned} \tag{12.47}$$

where

$$\delta\kappa_{ij}\approx\delta\kappa_{i-j}^{(2)}(mn)=(m|[B_{i-j},iS_{BF}^{(1)}]|n)\ , \tag{12.48}$$

with the interaction matrix element on the right defined by Eq. (12.34). To the given order, $\delta r_{ij}(mn)\approx\delta_{in}\delta_{jm}$. Solution of Eq. (12.47) permits the calculation of transitions between single-particle states, the renormalization of the single-particle states by the vibration-particle coupling $H_{BF}^{(1)}$, and determination of the changes in the static moments of the ground state due to the addition of the odd nuclei. Again, the integral equation, (12.47) is for-

mally identical to results of finite Fermi-system theory and nuclear field theory (Bortignon *et al.*, 1977).

One can go beyond Eqs. (12.29) and (12.30) to include the effects of $H_{BF}^{(2)}$. In particular, in order to calculate the splittings of the multiplet formed by the coupling of the odd nucleon to the one-phonon vibrational state, it is essential to take into account the diagonal contribution of $H_{BF}^{(2)}$ [Eq. (12.15e)] along with the off-diagonal contribution of $H_{BF}^{(1)}$. The latter is just the conventional particle-vibration coupling taken into account, for example, by Kisslinger and Sorensen (1963), while $H_{BF}^{(2)}$ is the Pauli principle correction for the antisymmetrization between the bosonized "core" and the odd particle. The effects of $H_{BF}^{(2)}$ are in complete agreement with nuclear field theory (Bortignon *et al.*, 1977).

In summary, we can say that the results of perturbative boson expansions for spherical nuclei formally agree with the corresponding results, derived from both the theory of finite Fermi systems and nuclear field theory, although the formalisms of the three methods appear to be quite different. This is not really so surprising, since all three methods share the common assumption that the self-consistent field plus random-phase approximation constitutes a good starting point for treating higher-order corrections as perturbations. Since all three methods involve a perturbation expansion in powers of $(2\Omega)^{-1/2}$, the concordance suggests that they are all equivalent and, hopefully, correct. However, outside of the results of specific applications, we are not aware of any general proof of equivalence between the boson expansion approach and either the theory of finite Fermi systems or nuclear field theory. However, derivations of the latter starting from the method of Green's functions have been given, as well as demonstrations that nuclear field theory sums all Feynman graphs to all orders for some simple models (Reinhardt, 1975, 1977; Kleinert, 1977). Since boson expansions, when carried to infinite order, are equivalent to the exact many-fermion problem, they too must sum all Feynman graphs. On this basis, one expects order-by-order agreement between boson expansions and nuclear field theory, but that hardly constitutes a rigorous proof.

XIII. PERTURBATIVE TREATMENT OF DEFORMED SYSTEMS

A. The problem of Goldstone modes

The success of the perturbative boson expansion in spherical nuclei hinges primarily on the smallness of the RPA zero-point amplitudes, although some resonance denominators can provide an additional source of difficulty (Marshalek and Sabato, 1972). In the case of deformed systems, the occurrence of Goldstone modes in the RPA presents yet another problem, but, as we shall show, a problem that can be completely overcome. Thus perturbative boson expansions are at least as viable in strongly deformed nuclei as in spherical ones, perhaps even more so, since the zero-point amplitudes of the vibrational modes tend, generally speaking, to be smaller in deformed nuclei. We remark, parenthetically, that there may be some evidence for larger-amplitude "γ vibrations" in some deformed nuclei (Bohr and Mottelson, 1982). Nevertheless, a theoretical method essentially equivalent to one described in Sec. XII has been applied to these phenomena by Matsuo and Matsuyanagi (1985, 1986, 1987).

The mean field of a finite nucleus can (and normally does) violate one or more continuous symmetries obeyed by the full Hamiltonian. Thus the localization of the center of mass by the mean field violates momentum conservation; if the mean field is spatially deformed, the localization of its orientation violates angular momentum conservation and, if the system is superfluid, particle-number conservation and possibly isospin conservation as well. These broken-symmetry solutions are always part of a degenerate continuum, for example, the set of states corresponding to different orientations of a spatially deformed field. Since real nuclei are finite systems, the solutions with broken symmetry cannot be real eigenstates, which must fully reflect the symmetries of the Hamiltonian; thus the continuous degeneracy is spurious.

We shall first describe the problem of Goldstone modes and later present the solution. For simplicity, the discussion will be limited to even-even systems, which can be treated by means of expansions in pure bosons. Just as in the case of spherical nuclei, the Hamiltonian H_B is expanded in the form

$$H_B = W_0 + H^{(2)} + H^{(3)} + H^{(4)} + \cdots , \tag{13.1}$$

where W_0 is the mean-field energy and $H^{(n)}$ is an nth-order polynomial in the bosons. Now, the RPA Hamiltonian $H^{(2)}$ can be diagonalized in the following form (Marshalek and Weneser, 1969):

$$\begin{aligned} H^{(2)} &= \sum_{ij} \{ A_{ij} b_i^\dagger b_j + \tfrac{1}{2} B_{ij} [b_i^\dagger b_j^\dagger + b_j b_i] \} \\ &= W^{(2)} + \sum_\lambda \tfrac{1}{2} [P_\lambda^2 + \omega_\lambda^2 Q_\lambda^2] , \end{aligned} \tag{13.2}$$

where the $b_i^\dagger$ represent the two-quasiparticle bosons $b_{\alpha\beta}^\dagger$ while P_λ, Q_λ are Hermitian normal-mode canonical momentum-coordinate pairs corresponding to the frequency ω_λ. The quantity $W^{(2)}$ is a constant. The generator of each broken symmetry, denoted generically by J_q, is necessarily nondiagonal in the deformed basis and therefore has a linear term, $J_q^{(1)}$, in its boson expansion. Since H_B must preserve the symmetry, i.e.,

$$[H_B, J_q] = 0 , \tag{13.3}$$

and since the boson expansion satisfies commutation rules order by order, it follows that

$$[H^{(2)}, J_q^{(1)}] = 0 . \tag{13.4}$$

This implies that one of the normal-mode momenta P_q is

proportional to $J_q^{(1)}$ and the corresponding frequency $\omega_q=0$. Such a mode with vanishing frequency is called a Goldstone mode. Although J_q has a discrete spectrum in all cases except when it is a component of the total momentum, $J_q^{(1)}$, being a linear combination of $b_i^\dagger$ and b_i, necessarily has a continuous spectrum. As a consequence, the eigenstates of $H^{(2)}$ have infinite norms, with an infinite-norm factor arising from each Goldstone mode (Marshalek and Weneser, 1969). Of course, the physical reason for this problem is that the coordinate corresponding to a Goldstone mode is really a cyclic variable that ought not be treated as a small displacement. For the nonzero-frequency modes, one may introduce boson annihilation and creation operators $\mathcal{B}_\mu$ and $\mathcal{B}_\mu^\dagger$ in the usual way:

$$\begin{aligned}\mathcal{B}_\mu&=(2\hbar\omega_\mu)^{1/2}(P_\mu-i\hbar\omega_\mu Q_\mu)\ ,\\ \mathcal{B}_\mu^\dagger&=(2\hbar\omega_\mu)^{-1/2}(P_\mu+i\hbar\omega_\mu Q_\mu)\ .\end{aligned}\tag{13.5}$$

Then, the RPA Hamiltonian can be written as

$$H^{(2)}=W_{\rm RPA}+\sum_\mu \hbar\omega_\mu\mathcal{B}_\mu^\dagger\mathcal{B}_\mu+\sum_q(\hbar J_q^{(1)})^2/(2\mathcal{J}_q)\ ,\tag{13.6}$$

where the constant $W_{\rm RPA}$ is essentially the RPA correlation energy, and $\mathcal{J}_q$ is an inertial parameter associated with the Goldstone mode. In the case when J_q is a component of the angular momentum, it can be shown that $\mathcal{J}_q$ is identical to the self-consistent cranking-model moment of inertia (Marshalek and Weneser, 1969). In spite of the correct identification of the moment of inertia, the eigenstates of $H^{(2)}$ obviously cannot be used as the zero-order basis for a perturbative treatment of the higher-order terms $H^{(3)}$, $H^{(4)}$, etc.

Because of this difficulty, it has been widely assumed that perturbative boson expansions are unsuitable for deformed systems. However, this is not the case. As first shown by Marshalek and Weneser (1969, 1970), although the expansion (13.1), which is a *local* Taylor expansion, cannot be used directly for a perturbative treatment, it is possible to reconstruct from the local expansions a *global* expansion of the Hamiltonian and other operators (i.e., one not restricted to small displacements of the variables conjugate to the symmetry operators); it is the global expansions that can be treated directly by perturbation theory. The Marshalek and Weneser reconstruction method will first be discussed and illustrated for the case of two-dimensional rotation, which was the scope of the original 1970 method. However, the method has recently been generalized to the non-Abelian case of three-dimensional rotation (Marshalek, 1987a, 1987b), discussed later in this section.

B. Two-dimensional rotation

We consider first an intrinsically deformed many-body system that can only rotate about a single axis. As shown by Villars (1965) and Klein, Dreizler, and Johnson (1968), the rotationally invariant Hamiltonian can be expanded in powers of the angular momentum J as follows:

$$H=\sum_{n=0}^{\infty}H_nJ^n/(n!)\ ,\tag{13.7}$$

where the H_n are *intrinsic* operators; i.e., they commute with both J and any canonically conjugate Hermitian angle variable ϕ, satisfying $[\phi,J]=i$. Moreover, the H_n can be expressed in terms of multiple commutators of ϕ with H, as follows:

$$H_n=\sum_{k=0}^{\infty}(-1)^kJ^kC_k/(k!)\ ,\tag{13.8}$$

where

$$C_0\equiv H, C_k\equiv[[\cdots[[H,\underbrace{i\phi}],\underbrace{i\phi}],\cdots],\underbrace{i\phi}]\quad(k>0)\ .\tag{13.9}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad k\ i\phi's$$

Equations (13.7)–(13.9), which constitute what is called the Villars expansion, emphasize the central role of the angle variable ϕ. This angle variable, which is not unique, determines the convergence rate of the angular momentum expansion (13.7). For a certain choice of ϕ, H can even be made diagonal. In practice, we require an expansion of the intrinsic operators H_n in a small parameter of the order of $\Omega^{-1/2}$. Equations (13.8) and (13.9) suggest that a Taylor expansion of H and ϕ may be used to compute the desired expansion of the H_n. Of course, the H_n must be expressed afterwards in terms of intrinsic variables, such as the creation and annihilation operators for the nonzero-frequency vibrational modes. This will provide the global expansion alluded to above. We proceed to outline a practical program for achieving this aim.

We assume that the mean field of the system breaks SO(2) invariance. The Hamiltonian then has the Taylor expansion (1), and the RPA Hamiltonian corresponding to Eq. (13.6) can be written in the form

$$H^{(2)}=W_{\rm RPA}+\sum_\mu\hbar\omega_\mu\mathcal{B}_\mu^{(1)\dagger}\mathcal{B}_\mu^{(1)}+(\hbar J^{(1)})^2/(2\mathcal{J})\ ,\tag{13.10}$$

where $J^{(1)}$ is the first term of the Taylor expansion of J:

$$J=J^{(1)}+J^{(2)}+J^{(3)}+\cdots\ .\tag{13.11}$$

The notation $\mathcal{B}_\mu^{(1)\dagger}$ and $\mathcal{B}_\mu^{(1)}$ for the creation and annihilation operators of the true RPA vibrational modes emphasizes that these operators are the leading-order approximations in a Taylor expansion that is yet to be found. Within the RPA, one can find a real angular variable $\phi^{(1)}$ canonically conjugate to $J^{(1)}$,

$$[\phi^{(1)},J^{(1)}]=i\ ,\tag{13.12}$$

which, like $J^{(1)}$, commutes with all the $\mathcal{B}_\mu^{(1)\dagger},\mathcal{B}_\mu^{(1)}$. Hence, $\phi^{(1)}$ must be the solution of the inhomogeneous linear equation [cf. Eq. (4.76)]

$$[H^{(2)},\phi^{(1)}]=-i\hbar^2 J^{(1)}/\mathcal{I} \ . \tag{13.13}$$

By using the first form of $H^{(2)}$ given in Eq. (13.2), and writing $J^{(1)}$ and $\phi^{(1)}$ as linear combinations of the elementary bosons $b_i^\dagger$ and b_i, one sees that Eq. (13.13) together with the normalization (13.12) is just the Thouless-Valatin equation of the self-consistent cranking model (Thouless and Valatin, 1962), whose solution determines the moment of inertia

$$\hbar^2/\mathcal{I}=[\phi^{(1)},[H^{(2)},\phi^{(1)}]] \ . \tag{13.14}$$

The complete set of RPA variables $(J^{(1)},\phi^{(1)},\mathcal{B}_\mu^{(1)\dagger},\mathcal{B}_\mu^{(1)})$ can be used as a starting point for generating a new set $(J,\phi,\mathcal{B}_\mu^\dagger,\mathcal{B}_\mu)$ with the same mutual commutation relations. One way to proceed is to seek first an expansion of the angle variable ϕ that commences with the RPA value:

$$\phi=\phi^{(1)}+\phi^{(2)}+\phi^{(3)}+\cdots \ . \tag{13.15}$$

The corrections are determined from the requirement that

$$[\phi,J]=i \tag{13.16}$$

be satisfied order by order as follows:

$$\begin{aligned}&[\phi^{(1)},J^{(1)}]=i \ ,\\ &[\phi^{(2)},J^{(1)}]+[\phi^{(1)},J^{(2)}]=0 \ ,\\ &\sum_{k=1}^{n}[\phi^{(n-k+1)},J^{(k)}]=0 \quad (n\geq 1) \ .\end{aligned} \tag{13.17}$$

Since the expansion of J is known *a priori,* the hierarchy of Eqs. (13.17) can readily be solved order by order to determine the expansion of ϕ. With a knowledge of the expansions of both J and ϕ, one may obtain the expansions of the boson creation and annihilation operators $\mathcal{B}_\mu^\dagger$ and $\mathcal{B}_\mu$ for the true vibrational modes,

$$\mathcal{B}_\mu=\mathcal{B}_\mu^{(1)}+\mathcal{B}_\mu^{(2)}+\mathcal{B}_\mu^{(3)}+\cdots \text{ (and H.c. Eq.)} \ , \tag{13.18}$$

from the requirements

$$[\mathcal{B}_\mu,\mathcal{B}_\nu^\dagger]=\delta_{\mu\nu}, \quad [\mathcal{B}_\mu,\mathcal{B}_\mu]=0 \text{ (and H.c. Eq.)} \ , \tag{13.19a}$$

$$[\mathcal{B}_\mu,J]=0, \quad [\mathcal{B}_\mu,\phi]=0 \text{ (and H.c. Eqs.)} \ , \tag{13.19b}$$

which are to be satisfied order by order.

An equivalent but more convenient procedure is to construct a formal unitary transformation $\exp(iS)$ defined by

$$\exp(iS)J^{(1)}\exp(-iS)=J^{(1)}+J^{(2)}+J^{(3)}+\cdots \ , \tag{13.20}$$

where S is given by an expansion beginning with cubic terms:

$$S=S^{(3)}+S^{(4)}+\cdots \ , \tag{13.21}$$

so that upon separation of orders, Eq. (13.20) is equivalent to

$$\begin{aligned}&[S^{(3)},J^{(1)}]=-iJ^{(2)} \ ,\\ &[S^{(4)},J^{(1)}]=-iJ^{(3)}-\tfrac{1}{2}[S^{(3)},J^{(2)}] \ ,\\ &\cdots \ .\end{aligned} \tag{13.22}$$

After solving the hierarchy of equations successively for the $S^{(n)}$ up to the order of interest, one can then generate the expansions of ϕ, $\mathcal{B}_\mu^\dagger$, and $\mathcal{B}_\mu$ from the RPA variables as follows:

$$\begin{aligned}&\phi=\exp(iS)\phi^{(1)}\exp(-iS) \ ,\\ &\mathcal{B}_\mu=\exp(iS)\mathcal{B}_\mu^{(1)}\exp(-iS) \text{ (and H.c. Eq.)} \ .\end{aligned} \tag{13.23}$$

Since the transformation is unitary, it is clear that Eq. (13.23) must satisfy the commutation relations (13.16) and (13.19).

At this point, two remarks are in order. First, it would appear that Eq. (13.20) involves the unitary transformation of $J^{(1)}$, an operator with a continuous spectrum, into J, an operator with a discrete spectrum. Of course, no such unitary transformation exists. Equation (13.20) should be viewed only as a formal trick for generating the *expansion* of J. Because of the improper treatment of the rotation as a small oscillation, both the expansions of J and $\exp(iS)$ contain divergent parts associated with $\phi^{(1)}$. The final results, however, are independent of these divergent parts. Moreover, no difficulties occur with the solutions of the hierarchy of Eqs. (13.22).

The second remark is that the solutions of Eqs. (13.17) and (13.19), or, equivalently, (13.22) are not unique. It is obvious that to any solution $S^{(n)}$ one may add an arbitrary nth-order polynomial function of $J^{(1)}$, $\mathcal{B}_\mu^{(1)\dagger}$, and $\mathcal{B}_\mu^{(1)}$, which all commute with $J^{(1)}$, and still have a solution. This arbitrariness provides the freedom to shape the final forms of the intrinsic operators H_n, including the possibility of diagonalizing them. The arbitrary functions can always be chosen as zero if desired, in which case additional unitary transformations can be used to diagonalize H afterwards.

Having obtained the expansions of H, J, ϕ, $\mathcal{B}_\mu^\dagger$, and $\mathcal{B}_\mu$, one may proceed in one of two ways. The first is to make use of the Villars expansion (13.7). Leaving the powers of J intact, one can obtain the expansions of the intrinsic coefficients H_n by evaluating the multiple commutators (13.9) up to the order of interest and then expressing them as functions of the bosons $\mathcal{B}_\mu^\dagger$ and $\mathcal{B}_\mu$. The second and equivalent way is to transform the local expansion (13.1) directly from the RPA variables $(J^{(1)},\phi^{(1)},\mathcal{B}_\mu^{(1)\dagger},\mathcal{B}_\mu^{(1)})$ to the variables $(J,\phi,\mathcal{B}_\mu^\dagger,\mathcal{B}_\mu)$. In practice, this turns out to be the more convenient approach. Following this approach, we define for any function of the RPA variables $F(J^{(1)},\phi^{(1)},\mathcal{B}_\mu^{(1)\dagger},\mathcal{B}_\mu^{(1)})$ the transform $\tilde{F}$, given by

$$\begin{aligned}\tilde{F}&\equiv\exp(iS)F(J^{(1)},\phi^{(1)},\mathcal{B}_\mu^{(1)\dagger},\mathcal{B}_\mu^{(1)})\exp(-iS)\\ &=F(J,\phi,\mathcal{B}_\mu^\dagger,\mathcal{B}_\mu) \ .\end{aligned} \tag{13.24}$$

By inversion, the original operator F may be expressed in

terms of the final variables as follows:

$$F=\exp(-i\tilde{S})\tilde{F}\exp(i\tilde{S})$$
$$=F(J,\phi,\mathcal{B}_\mu^\dagger,\mathcal{B}_\mu)-[i\tilde{S}^{(3)},F(J,\phi,\mathcal{B}_\mu^\dagger,\mathcal{B}_\mu)]+\cdots . \tag{13.25}$$

Of course, $\tilde{S}=\tilde{S}^{(3)}+\tilde{S}^{(4)}+\cdots=S=S^{(3)}+S^{(4)}+\cdots$. Thus for H one obtains through quartic terms in the new variables

$$H=W_0+H'^{(2)}+H'^{(3)}+H'^{(4)}+\cdots , \tag{13.26}$$

where

$$H'^{(2)}=\tilde{H}^{(2)} , \tag{13.27a}$$

$$H'^{(3)}=\tilde{H}^{(3)}-[i\tilde{S}^{(3)},\tilde{H}^{(2)}] , \tag{13.27b}$$

$$H'^{(4)}=\tilde{H}^{(4)}-[i\tilde{S}^{(4)},\tilde{H}^{(2)}]-[i\tilde{S}^{(3)},\tilde{H}^{(3)}]$$
$$+\tfrac{1}{2}[i\tilde{S}^{(3)},[i\tilde{S}^{(3)},\tilde{H}^{(2)}]] . \tag{13.27c}$$

In particular, the quadratic contribution $H'^{(2)}$ is explicitly given by

$$H'^{(2)}=W_{\rm RPA}+\sum_\mu \hbar\omega_\mu\mathcal{B}_\mu^\dagger\mathcal{B}_\mu+\hbar^2J^2/(2\mathcal{I}) , \tag{13.28}$$

which differs from the RPA Hamiltonian (13.10) mainly by the replacement of $J^{(1)}$ by J. The final result (13.26) must be independent of ϕ because of the commutation of H and J, which is satisfied order by order.

The method described is not limited to the Hamiltonian but can be extended to any transition operator. Thus, if T_M is a two-dimensional analog of a spherical tensor, it obeys the commutation rule

$$[J,T_M]=MT_M , \tag{13.29}$$

where M is an integer. It follows that T_M can be written in the form

$$T_M=\tfrac{1}{2}\{e^{iM\phi},\hat{T}_M\} , \tag{13.30}$$

where $\hat{T}_M$ is a scalar and therefore has a Villars expansion like the Hamiltonian. The local expansion of T_M can be transformed to the variables $J,\phi,\mathcal{B}_\mu^\dagger$, and $\mathcal{B}_\mu$ in accord with Eq. (13.25). The resulting expression will contain powers of ϕ arising from the expansion of the exponential of Eq. (13.30). The operator $\hat{T}_m$ can then be obtained from

$$\hat{T}_M=\lim_{\phi\to0}T_M , \tag{13.31}$$

and used in the global expression (13.30), where the full ϕ dependence is resummed.

The Moszkowski SO(4) model discussed in Sec. IV.C provides an illustration and a test of the method for two dimensions (Marshalek, 1982). On the one hand, the Hamiltonian (as well as transition operators) may be written in the "action-angle" variable representation, as is done in Eqs. (6.27)–(6.29). Following the procedure outlined in Eqs. (6.30)–(6.32), one obtains a polynomial expansion in normal-mode bosons $\mathcal{B}^\dagger,\mathcal{B}$ and powers of the angular momentum $J\equiv L_z$; this is an expansion in powers of the small parameter $T_0^{-1/2}$. Since this procedure does not involve small-angle expansions at any stage, the result may be regarded as the "exact solution" against which the outcome of the reconstruction method should be compared. On the other hand, the model Hamiltonian, given by Eq. (6.9),

$$H=-C[T_z(+)+T_z(-)]-\tfrac{1}{4}G^2(T_+T_-+T_-T_+) , \tag{13.32}$$

(T_z is the same as T_0) may be expanded about a deformed Hartree extremum so as to generate a Goldstone mode. This procedure may be carried out in several equivalent ways. One way, for example, would be to express Eq. (13.32) in terms of two kinds of bosons via an SU(2)×SU(2) Holstein-Primakoff mapping to spherical (circular?) bosons, followed by a boson-shift transformation to deformed bosons that eliminates linear bosons terms of lowest order. We follow instead the method used by Marshalek, which is first to express Eq. (13.32) in terms of the deformed Hartree basis, followed by an SU(2)×SU(2) Holstein-Primakoff mapping to deformed bosons. In this way, the linear boson terms never appear. It is easily shown that the transformation to the deformed Hartree basis is given by the pseudospin rotation

$$T_x(\pm)=\cos\gamma T'_x(\pm)\pm\sin\gamma T'_z(\pm) ,$$
$$T_y(\pm)=T'_y(\pm) , \tag{13.33}$$
$$T_z(\pm)=\mp\sin\gamma T'_x(\pm)+\cos\gamma T'_z(\pm) ,$$

where

$$\sin\gamma=\left[1-\frac{1}{x^2}\right]\equiv\alpha_0 , \tag{13.34}$$

and, as defined earlier, $x=G^2T_0/C$. The deformed solution exists, provided that $x>1$, with the perfect rotor limit (two-dimensional Elliott model) corresponding to $x=\infty$, so that the deformation parameter $0<\alpha_0\le1$. The Hartree energy for the deformed solution is

$$W=-G^2T_0^2(1-\tfrac{1}{2}\alpha_0^2) . \tag{13.35}$$

In the Hartree basis, the angular momentum is given by

$$J=L_z=2T_z=2(T_z(+)+T_z(-))$$
$$=2\{\cos\gamma T'_z-\sin\gamma[T'_x(+)-T'_x(-)]\} . \tag{13.36}$$

As shown by Marshalek, the correct Holstein-Primakoff transformation is given by

$$T'_\mp(\pm)=b_\pm^\dagger(T_0-\hat{n}_\pm)^{1/2} ,$$
$$T'_\pm(\pm)=(T_0-\hat{n}_\pm)^{1/2}b_\pm , \tag{13.37}$$
$$T'_z(\pm)=\pm(\tfrac{1}{2}T_0-\hat{n}_\pm),\quad \hat{n}_\pm\equiv b_\pm^\dagger b_\pm .$$

After expanding these expressions in powers of T_0^{-1} and

introducing the RPA normal modes one obtains, the Hamiltonian in the form of expansion (13.1), where the RPA normal-mode operators referred to in Eqs. (13.10) and (13.12) are given by

$$\begin{aligned}
\mathcal{B}^{(1)\dagger}&=\tfrac{1}{2}(2\alpha_0)^{-1/2}[\,(1+\alpha_0)(b_+^\dagger+b_-^\dagger)\\
&\quad-(1-\alpha_0)(b_++b_-)]\,,\\
\mathcal{B}^{(1)}&=-\tfrac{1}{2}(2\alpha_0)^{-1/2}[\,(1-\alpha_0)(b_+^\dagger+b_-^\dagger)\\
&\quad-(1+\alpha_0)(b_++b_-)]\,,\\
J^{(1)}&=L_z^{(1)}=-T_0^{1/2}\alpha_0(b_+^\dagger-b_-^\dagger+b_+-b_-)\,,\\
\phi^{(1)}&=\frac{i}{4}(\alpha_0^2T_0)^{-1/2}(b_+^\dagger-b_-^\dagger-b_++b_-)\,,
\end{aligned}\tag{13.38}$$

with the vibrational frequency

$$\hbar\omega=G^2T_0\alpha_0\,,\tag{13.39}$$

and rotational parameter

$$\frac{\hbar^2}{2\mathcal{I}}=\frac{G^2}{8\alpha_0^2}\,.\tag{13.40}$$

Since there is only one vibrational mode, the index μ has been dropped.

The higher-order terms of the expansion (13.1) expressed in terms of the RPA operators in normal order, are given by

$$\begin{aligned}
H^{(3)}=&\frac{1}{4}G^2\left[\frac{T_0(1-\alpha_0)^2}{2\alpha_0}\right]^{1/2}[\,(1-\alpha_0^2)(\mathcal{B}^{(1)\dagger 3}+\mathcal{B}^{(1)3})+(3+\alpha_0^2)(\mathcal{B}^{(1)\dagger 2}\mathcal{B}^{(1)}+\text{H.c.})\\
&+(1-\alpha_0)(3-\alpha_0)(\mathcal{B}^{(1)\dagger}+\mathcal{B}^{(1)})]\\
&+\frac{1}{8}G^2\left[\frac{1-\alpha_0^2}{2T_0\alpha_0^3}\right]^{1/2}(\mathcal{B}^{(1)\dagger}+\mathcal{B}^{(1)})J^{(1)2}+G^2[2T_0^3\alpha_0^5(1-\alpha_0^2)]^{1/2}(\mathcal{B}^{(1)\dagger}+\mathcal{B}^{(1)})\phi^{(1)2}\,,
\end{aligned}\tag{13.41a}$$

$$\begin{aligned}
H^{(4)}=&\frac{G^2}{32\alpha_0^2}[2(3-5\alpha_0^2-3\alpha_0^4-3\alpha_0^6)\mathcal{B}^{(1)\dagger 2}\mathcal{B}^{(1)2}+4(1-\alpha_0^2)(3+2\alpha_0-4\alpha_0^2-2\alpha_0^3-3\alpha_0^4)\mathcal{B}^{(1)\dagger}\mathcal{B}^{(1)}\\
&+3-8\alpha_0-3\alpha_0^2+16\alpha_0^3-15\alpha_0^4+8\alpha_0^5-3\alpha_0^6)]-\frac{G^2}{128T_0^2\alpha_0^2}J^{(1)4}\\
&+\frac{G^2}{64T_0\alpha_0^3}[2(3+\alpha_0^2)(1-2\alpha_0^2)\mathcal{B}^{(1)\dagger}\mathcal{B}^{(1)}+3-5\alpha_0^2+8\alpha_0^3-2\alpha_0^4]J^{(1)2}\\
&+(\text{off-diagonal terms in }\mathcal{B}^{(1)\dagger},\ \mathcal{B}^{(1)},\ J^{(1)},\text{ and }\phi^{(1)})\,.
\end{aligned}\tag{13.41b}$$

Omitted in Eq. (13.41b) are terms that give corrections two orders higher than the RPA, while those included give the leading-order corrections to the RPA. The corresponding higher-order corrections in the expansion (13.11) of the angular momentum operator are given by

$$J^{(2)}=\left[\frac{1-\alpha_0^2}{2T_0\alpha_0^3}\right]^{1/2}(\mathcal{B}^{(1)\dagger}+\mathcal{B}^{(1)})J^{(1)}-2i\,[2T_0\alpha_0^3(1-\alpha_0^2)]^{1/2}(\mathcal{B}^{(1)\dagger}-\mathcal{B}^{(1)})\phi^{(1)}\,,\tag{13.42a}$$

$$\begin{aligned}
J^{(3)}=&-\frac{1}{32T_0^2\alpha_0^2}J^{(1)3}-\tfrac{1}{4}\alpha_0^2\{\phi^{(1)2},J^{(1)}\}\\
&-\frac{1}{16T_0\alpha_0}[(3-\alpha_0^2)(\mathcal{B}^{(1)\dagger 2}+\mathcal{B}^{(1)2})+2(3+\alpha_0^2)\mathcal{B}^{(1)\dagger}\mathcal{B}^{(1)}+3-8\alpha_0+\alpha_0^2]J^{(1)}+\tfrac{1}{2}i\alpha_0^2(\mathcal{B}^{(1)\dagger 2}-\mathcal{B}^{(1)2})\phi^{(1)}\,.
\end{aligned}\tag{13.42b}$$

At this point, Eqs. (13.22) can be readily solved for $S^{(3)}$ and $S^{(4)}$, from which $\tilde{S}^{(3)}$ and $\tilde{S}^{(4)}$ are immediately obtained with the aid of Eq. (13.25) as follows:

$$\tilde{S}^{(3)}=-\left[\frac{1-\alpha_0^2}{8T_0\alpha_0^3}\right]^{1/2}\{\phi,J\}(\mathcal{B}^\dagger+\mathcal{B})+i\,[2T_0\alpha_0^3(1-\alpha_0^2)]^{1/2}\phi^2(\mathcal{B}^\dagger-\mathcal{B})\,,\tag{13.43a}$$

$$\begin{aligned}
\tilde{S}^{(4)}=&\frac{1}{64T_0^2\alpha_0^2}\{\phi,J^3\}-\tfrac{1}{6}(1-\tfrac{3}{2}\alpha_0^2)\{\phi^3,J\}-\tfrac{1}{4}i\alpha_0^2\phi^2(\mathcal{B}^{\dagger 2}-\mathcal{B}^2)\\
&+\frac{1}{32T_0\alpha_0^3}\{\phi,J\}[(4-\alpha_0^2-\alpha_0^4)(\mathcal{B}^{\dagger 2}+\mathcal{B}^2)+2(4-\alpha_0^2+\alpha_0^4)\mathcal{B}^\dagger\mathcal{B}+4-\alpha_0^2-8\alpha_0^3+\alpha_0^4]\,.
\end{aligned}\tag{13.43b}$$

As mentioned previously, the solutions (13.43) are modulo arbitrary functions of J, ϕ, $\mathcal{B}^\dagger$, and $\mathcal{B}$. These functions have

been set equal to zero. They could have been chosen to diagonalize the Hamiltonian to the given order, but that can always be accomplished afterwards by an additional unitary transformation. Given the transformation (13.43), one may obtain the expansion of H in terms of J, $\mathcal{B}^\dagger$, and $\mathcal{B}$ [Eq. (13.26)] with the aid of Eqs. (13.27). The result, of course, must be independent of the angle variable ϕ because of the rotational invariance of H. The cancellation of ϕ-dependent contributions through quartic terms can be verified with the aid of Eqs. (13.43). The result then is the expansion (13.26) with

$$H^{(2)}=W_{\rm RPA}+\hbar\omega\mathcal{B}^\dagger\mathcal{B}+\hbar^2J^2/(2\mathcal{I}) , \tag{13.44}$$

where the frequency and rotational parameter are given by Eqs. (13.39) and (13.40) and

$$W_{\rm RPA}=-G^2T_0(1-\tfrac{1}{2}\alpha_0) . \tag{13.45}$$

The higher-order correction terms are

$$H'^{(3)}=\frac{1}{4}G^2\left[\frac{T_0(1-\alpha_0)^2}{2\alpha_0}\right]^{1/2}[(1-\alpha_0^2)(\mathcal{B}^{\dagger 3}+\mathcal{B}^3)+(3+\alpha_0^2)(\mathcal{B}^{\dagger 2}\mathcal{B}+\mathcal{B}^\dagger\mathcal{B}^2)]$$
$$+(1-\alpha_0)(3-\alpha_0)(\mathcal{B}^\dagger+\mathcal{B})]-\frac{1}{8}G^2\left[\frac{1-\alpha_0^2}{2T_0\alpha_0^3}\right]^{1/2}\left[\frac{2-\alpha_0^2}{\alpha_0^3}\right](\mathcal{B}^\dagger+\mathcal{B})J^2 , \tag{13.46a}$$

$$H'^4=\frac{G^2}{32\alpha_0^2}[2(3-5\alpha_0^2-3\alpha_0^4-3\alpha_0^6)\mathcal{B}^{\dagger 2}\mathcal{B}^2$$
$$+4(1-\alpha_0)^2(3+2\alpha_0-4\alpha_0^2-2\alpha_0^3-3\alpha_0^4)\mathcal{B}^\dagger\mathcal{B}-(1-\alpha_0)(1+9\alpha_0+6\alpha_0^2-10\alpha_0^3+5\alpha_0^4-3\alpha_0^5)]$$
$$+\frac{G^2}{32T_0\alpha_0^5}[2(1-\alpha_0^2)(6-\alpha_0^2+\alpha_0^4)\mathcal{B}^\dagger\mathcal{B}+(1-\alpha_0^2)(6-\alpha_0^2-4\alpha_0^3+\alpha_0^4)]J^2+\frac{G^2(1-\alpha_0^2)}{128T_0^2\alpha_0^4}J^4$$
$$+\text{omitted off-diagonal terms} . \tag{13.46b}$$

The higher-order terms (13.46a) and (13.46b) are not identical to the corresponding terms arising from the "exact expansion" in the action-angle representation, but, in fact, may be shown to be *equivalent by unitary transformation* (or by a suitable choice of the arbitrary functions). Moreover, comparison with the original expansion given by Eq. (13.41) shows that, although the pure vibrational terms can be obtained by a simple renaming of variables, the vibration-rotation coupling terms are substantially changed. With the RPA Hamiltonian (13.44) as the zero-order term, the next-order corrections in the T_0^{-1} expansion of the eigenvalues are obtained by treating $H'^{(3)}$ in second-order perturbation theory and, at the same time, $H'^{(4)}$ in first-order. The omitted off-diagonal terms of $H'^{(4)}$ do not contribute to this order. The resulting expansion for the eigenvalues $E_{n,M}$ (units of G^2) to the order of interest as a function of the eigenvalue n of the boson number operator $\mathcal{B}^\dagger\mathcal{B}$ and the angular momentum quantum number M is

$$E_{n,M}=-T_0^2(1-\tfrac{1}{2}\alpha_0^2)-T_0-\frac{3(1+\alpha_0^2)}{16\alpha_0^2}+\left[T_0\alpha_0+\frac{1}{\alpha_0}\right](n+\tfrac{1}{2})-\frac{(3-\alpha_0^2)}{4\alpha_0^2}(n+\tfrac{1}{2})^2$$
$$+\left[\frac{1}{8\alpha_0^2}-\frac{(1-\alpha_0^2)}{4T_0\alpha_0^4}\right]M^2+\frac{(1-\alpha_0^2)(6-\alpha_0^2)}{8T_0\alpha_0^5}(n+\tfrac{1}{2})M^2-\frac{(1-\alpha_0^2)^2}{32T_0^2\alpha_0^8}M^4 . \tag{13.47}$$

This expression is exact to the given order. Equation (13.47) was rearranged as a function of $n+\frac{1}{2}$ to facilitate comparison with semiclassical methods (which also simplifies the coefficients). In addition to the "B coefficient" of the M^4 term, which agrees with the self-consistent cranking model, there is also a quantal correction to the cranking moment of inertia in this order. The $(n+\frac{1}{2})M^2$ term can be interpreted either as an angular momentum dependence of the vibrational frequency or as an n dependence of the moment of inertia. All of these effects depend sensitively on the interference between the vibration-rotation term proportional to $(\mathcal{B}^\dagger+\mathcal{B})J^2$ and the anharmonic terms proportional to $\mathcal{B}^{\dagger 2}\mathcal{B}+\mathcal{B}^\dagger\mathcal{B}^2$ and $\mathcal{B}^\dagger+\mathcal{B}$ in $H^{(3)}$. In addition to the vibration-rotation effects, one picks up anharmonic vibrational corrections proportional to n^2 and a higher-order renormalization of the RPA frequency.

For the corresponding treatment of transition operators in this model, the reader is referred to the original paper of Marshalek (1982a). As in the case of the energies, complete agreement with the exact treatment was found.

Finally, it should be emphasized that the applicability of the Marshalek-Weneser method for two dimensions is not restricted to toy models, but can also be applied to the problem of *pairing rotation* in realistic cases, where

the role of J is played by $\tilde{N}-\langle\tilde{N}\rangle$, where $\tilde{N}$ is the particle number operator, and where 2ϕ is then called the gauge angle.

C. Three-dimensional rotation of axially symmetric systems: Bohr-Mottelson and Villars representations

The extension of the two-dimensional Marshalek-Weneser reconstruction formalism to the Abelian case of several commuting constants of motion is trivial. The generalization to the non-Abelian case is nontrivial, especially in the case of axially symmetric systems in which the intrinsic excitations carry good angular momentum $\hbar K$ along the symmetry axis. It is this situation that will concern us in this section. As shown elsewhere (Marshalek, 1982a), if $K=0$ for all vibrational modes, the extension of the formalism is quite straightforward. It is also fairly straightforward for the cases of nuclei at high spin (Marshalek, 1977) or isospin (Ginocchio and Weneser, 1968), although these applications have not been carried out beyond the RPA. The key to extending the reconstruction method to deformed axial systems is the understanding of the relation between what we call the Bohr-Mottelson and the Villars representations (Marshalek, 1987a).

The aim of the Marshalek-Weneser method is to transform the local microscopic boson expansion of the Hamiltonian and associated transition operators into the general global form of the Bohr-Mottelson (1975) representation for strongly deformed nuclei. In the Bohr-Mottelson representation, which was originally applied to phenomenological models, the Hamiltonian has the angular momentum expansion

$$H_{\rm BM}=\sum_{mn}\mathcal{H}_{mn}\tfrac{1}{2}\{I'^{m}_{-},I'^{n}_{+}\}\ , \tag{13.48}$$

where $I'_{\pm}\equiv I_1\pm iI_2, I_k$ $(k=1,2,3)$ are interpreted as the components of the total angular momentum along the principal axes of the nucleus, and the 3-axis is designated as the symmetry axis. The components of $\mathbf{I}$ are usually represented as differential operators in the space of Euler angles. The operators $\mathcal{H}_{mn}=\mathcal{H}^{\dagger}_{nm}$ are intrinsic operators, i.e., they commute with all components of $\mathbf{I}$ and arbitrary functions of the Euler angles. What makes the Bohr-Mottelson representation unique is the occurrence of an additional operator J_3, which commutes with all components of $\mathbf{I}$ and is interpreted as the *intrinsic angular momentum* along the symmetry axis. However, J_3 does not commute with the intrinsic coefficients, which have the property

$$[J_3,\mathcal{H}_{mn}]=(m-n)\mathcal{H}_{mn}\ . \tag{13.49}$$

Moreover, the Hamiltonian can have spurious eigenvectors, but the physical ones satisfy the requirement

$$(I_3-J_3)|\,\rangle=0\ ; \tag{13.50}$$

i.e., the component of $\mathbf{I}$ along the symmetry axis must coincide with the intrinsic angular momentum.

In the Villars representation, the Hamiltonian has the same general form as Eq. (13.48), but there is no intrinsic angular momentum. We write the Hamiltonian in the Villars representation as

$$H=\sum_{mn}\mathring{\mathcal{H}}_{mn}\tfrac{1}{2}\{J'^{m}_{-},J'^{n}_{+}\}\ , \tag{13.51}$$

where the $\mathring{\mathcal{H}}_{mn}=\mathring{\mathcal{H}}^{\dagger}_{nm}$ are intrinsic operators and $J'_{\pm}\equiv J_1\pm iJ_2$ are combinations of principal-axis components of angular momentum. Since the latter are scalars, Eq. (13.51) is manifestly rotationally invariant. The relation between the spherical-vector principal-axis and laboratory components is given as usual by

$$J'_k=\sum_{M=-1}^{1}D^{1\dagger}_{MK}(\varphi,\theta,\psi)J_M=\sum_{M=-1}^{1}J_M D^{1\dagger}_{MK}(\varphi,\theta,\psi)$$
$$(K=0,\pm1)\ , \tag{13.52}$$

where the $D^{1}_{MK}(\varphi,\theta,\psi)$ are the matrix elements of the spin-1 irreducible representation of the rotation group, parametrized by a set of Euler angles φ,θ,ψ. As shown by Villars (1966) and Mikhailov (1971), the intrinsic coefficients $\mathring{\mathcal{H}}_{mn}$ are expressable in a series involving multiple commutators of H with the $D^{1}_{MK}(\varphi,\theta,\psi)$ analogous to Eqs. (13.8) and (13.9) in the two-dimensional case, but this result, which has not been derived in a closed form to all orders as in the two-dimensional case, will not be explicitly utilized here. The reason for the different notation used for the angular momentum in Eqs. (13.48) and (13.51) will become apparent shortly.

The advantage of the Bohr-Mottelson representation (13.48) is that the intrinsic coefficients $\mathcal{H}_{mn}$ carry angular momentum in the sense of Eq. (13.49) and, therefore, may be expressed as functions of RPA-like vibrational quanta in the case of even-even nuclei. This option does not seem to be available in the case of the Villars representation, in which the intrinsic coefficients $\mathring{\mathcal{H}}_{mn}$ must commute with all components of the angular momentum, laboratory, and principal axes. Nevertheless, we shall show that the two representations are connected by a unitary transformation.

Within the Villars representation, one may proceed as follows. The Euler angle ψ describing a rotation about the symmetry axis is canonically conjugate to J_3, which may be expressed by

$$[J_3,\exp(i\psi)]=\exp(i\psi)\ . \tag{13.53}$$

The operator $\mathcal{H}_{mn}$ defined by

$$\mathcal{H}_{mn}\equiv\mathring{\mathcal{H}}_{mn}\exp[i(m-n)\psi] \tag{13.54}$$

then satisfies an equation of exactly the same form as (13.49), since J_3 commutes with $\mathring{\mathcal{H}}_{mn}$. Combining Eqs. (13.51) and (13.54), one may write Villars' Hamiltonian as

$$H=\sum_{mn}\mathcal{H}_{mn}\exp[i(n-m)\psi]\tfrac{1}{2}\{J'^{m}_{-},J'^{n}_{+}\}\ . \tag{13.55}$$

Although the coefficients $\mathcal{H}_{mn}$ in the Villars Hamiltonian

(13.55) carry angular momentum along the symmetry axis, as in the Bohr-Mottelson representation, they do not commute with any angular momentum components and therefore are not intrinsic operators with respect to **J**. Thus this simple substitution certainly does not produce the Bohr-Mottelson representation. Moreover, the form (13.55) also carries the unesthetic ψ-dependent exponential factors.

At this point, we pull a rabbit out of a hat, although the basis for the trick is indirectly hinted at by Bohr and Mottelson (1975). First, we introduce a redundant variable $\bar{\psi}$. That is, all operators such as H and the transition operators may be regarded as defined on an extended Hilbert space that is the direct product of the original (nuclear) Hilbert space with the space of periodic functions $e^{iK\bar{\psi}}$. To emphasize again the difference: the variable ψ is a function of the original (nuclear) degrees of freedom, whereas $\bar{\psi}$ is a dummy variable, destined, however, to assume the role of ψ. In the extended space, one may define the operator I_3 by

$$I_3 = i\frac{\partial}{\partial\bar{\psi}} , \tag{13.56}$$

which is canonically conjugate to $\bar{\psi}$, i.e.,

$$[I_3, \exp(i\bar{\psi})] = \exp(i\bar{\psi}) . \tag{13.57}$$

On course, I_3 and $\bar{\psi}$ both commute with all of the original degrees of freedom.

The reason for introducing the redundant variable is that it enables one to relate the Villars and Bohr-Mottelson representations by means of a unitary transformation defined on the extended Hilbert space. Before introducing this transformation, however, it is useful to display explicitly the angular momentum components in terms of the Euler-angle degrees of freedom. In the immediate context, the conventional *zyz* definition of the Euler angles would do perfectly well. On the other hand, as will be discussed later, this definition of the Euler angles is unsuitable for the small-angle expansion, since the expressions diverge when $\theta=0$. It will be seen that the ideal definition is the *xyz* definition (Pio, 1976), in which an arbitrary rotation R is represented by

$$R = \exp(-i\varphi J_x)\exp(-i\theta J_y)\exp(-i\psi J_z) . \tag{13.58}$$

With this definition, the angle ψ has essentially the same significance as it has in the conventional definition, describing a rotation about the symmetry axis. For the sake of economy, then, the *xyz* definition is adopted forthwith. In the space of these Euler angles, the laboratory components of **J** are represented by the differential operators

$$J_x = -i\frac{\partial}{\partial\varphi} , \tag{13.59}$$

$$J_z \pm iJ_y = -ie^{\mp i\varphi}\left[-\tan\theta\frac{\partial}{\partial\varphi} \pm i\frac{\partial}{\partial\theta} + \frac{i}{\cos\theta}J_3\right]$$

and the principal-axis components by

$$J'_\pm = J_1 \pm iJ_2 = -ie^{\mp i\psi}\left[\frac{1}{\cos\theta}\frac{\partial}{\partial\varphi} \pm i\frac{\partial}{\partial\theta} - i\tan\theta J_3\right] , \tag{13.60}$$

$$J_3 = -i\frac{\partial}{\partial\psi} .$$

In contrast to the standard definition of the Euler angles, there is no problem in expanding Eqs. (13.59) and (13.60) about $\theta=0$.

Consider now the following unitary transformation $\mathcal{U}$ defined on the extended Hilbert space:

$$\mathcal{U} = \exp(i\bar{\psi}J_3)\exp(-i\psi I_3) . \tag{13.61}$$

Since the two factors do not commute, the order is important. With the properties (13.53) and (13.57) taken into account, elementary calculations give, first of all,

$$\mathcal{U}J_3\mathcal{U}^\dagger = I_3 , \tag{13.62}$$

and for an arbitrary function $f(\psi)$

$$\mathcal{U}f(\psi)\mathcal{U}^\dagger = f(\psi+\bar{\psi}) . \tag{13.63}$$

This transformation has no effect on the other Euler-angle degrees of freedom:

$$\mathcal{U}g\left[\varphi, \frac{\partial}{\partial\varphi}, \theta, \frac{\partial}{\partial\theta}\right]\mathcal{U}^\dagger = g\left[\varphi, \frac{\partial}{\partial\varphi}, \theta, \frac{\partial}{\partial\theta}\right] , \tag{13.64}$$

for an arbitrary function g. As a consequence, the only effect of $\mathcal{U}$ on the laboratory components of **J** [Eq. (13.59)] is to replace J_3 by I_3 to give

$$\mathcal{U}J_k\mathcal{U}^\dagger = I_k \quad (k=x,y,z) , \tag{13.65}$$

where

$$I_x \equiv J_x = -i\frac{\partial}{\partial\varphi} , \tag{13.66}$$

$$I_z \pm iI_y \equiv -ie^{\mp i\varphi}\left[-\tan\theta\frac{\partial}{\partial\varphi} \pm i\frac{\partial}{\partial\theta} + \frac{i}{\cos\theta}I_3\right] .$$

The effect on the principal-axis components (13.60) is similar, but with an additional phase factor arising from the property (13.63) as follows:

$$\mathcal{U}J'_\pm\mathcal{U}^\dagger = e^{\mp i\psi}I'_\pm = I'_\pm e^{\mp i\psi} , \tag{13.67}$$

where

$$I'_\pm \equiv I_1 \pm iI_2 = -ie^{\mp i\bar{\psi}}\left[\frac{1}{\cos\theta}\frac{\partial}{\partial\varphi} \pm i\frac{\partial}{\partial\theta} - i\tan\theta I_3\right] . \tag{13.68}$$

In summary, what the transformation $\mathcal{U}$ accomplishes is to replace the components of **J**, which depend only on the nuclear degrees of freedom, with components of a vector **I** (apart from a possible phase factor) that have the

same commutation relations as the angular momentum but replace ψ and J_3 with the redundant variables $\bar{\psi}$ and I_3, respectively. The operator $\mathcal{H}_{mn}$, which carries angular momentum along the 3-axis and thereby fails to commute with all components of **J**, is easily seen to commute with all components of **I**, as compactly summarized by

$$[\mathcal{H}_{mn},\mathbf{I}]=0 \ . \tag{13.69}$$

Since $\mathcal{H}_{mn}$ also commutes with all Euler angles, it certainly qualifies as an intrinsic operator with respect to **I**. Moreover, since it satisfies Eq. (13.49), it also carries angular momentum along the symmetry axis, playing the role of the "intrinsic" angular momentum. In other words, $\mathcal{H}_{mn}$ has all the attributes necessary for the Bohr-Mottelson representation.

To clinch the case, it must be shown that the transformation (13.61) in fact maps operators and state vectors from the Villars to the Bohr-Mottelson representation. The Hamiltonian (13.51) or (13.55) is easily transformed with the use of Eqs. (13.63), (13.67) and the fact that $\mathring{\mathcal{H}}_{mn}$, being an intrinsic operator, is invariant under $\mathcal{U}$. The result is

$$\mathcal{U}H\mathcal{U}^{\dagger}=\sum_{mn}\mathcal{H}_{mn}\tfrac{1}{2}\{I'^{m}_{-},I'^{n}_{+}\}=H_{\mathrm{BM}} \ , \tag{13.70}$$

which meets all of the requirements for the Bohr-Mottelson representation (13.48) of the Hamiltonian.

Next, consider the state vectors. The introduction of a redundant degree of freedom is inevitably accompanied by spurious states that must be eliminated by a suitable subsidiary condition. Prior to the transformation $\mathcal{U}$, the subspace of physical states may be chosen to satisfy the condition

$$I_3|\rangle'=0 \ . \tag{13.71}$$

From the definition of $\mathcal{U}$ [Eq. (13.61)], it is easily found that

$$\mathcal{U}I_3\mathcal{U}^{\dagger}=I_3-J_3 \ . \tag{13.72}$$

Therefore transformation of Eq. (13.71) yields Eq. (13.50) where $|\rangle\equiv\mathcal{U}|\rangle'$ are the physical states after the transformation.

The treatment of H can be generalized to arbitrary transition operators. Thus, in the Villars representation, spherical tensor operators T^L_M can be written in the form

$$T^L_M=\sum_{K=-L}^{L}\hat{T}^L_K D^L_{MK}(\varphi,\theta,\psi) \ , \tag{13.73}$$

where $D^L_{MK}(\varphi,\theta,\psi)$ is a rotation matrix element corresponding to the spin-L irreducible representation, and where the $\hat{T}^L_K$, the principal-axis components of the spherical tensor, are scalars, and therefore, like H, may be expanded in powers of the angular momentum components:

$$\hat{T}^L_K=\sum_{mn}(\mathring{t}^L_K)_{mn}\tfrac{1}{2}\{J'^{m}_{-},J'^{n}_{+}\} \ , \tag{13.74}$$

where the coefficients $(\mathring{t}^L_K)_{mn}$ are intrinsic operators with respect to **J**. Upon applying the transformation (13.61) to the transition operator (13.73), one obtains with the aid of Eqs. (13.63)–(13.65) and the relation $D^I_{MK}(\varphi,\theta,\psi)=\exp(iK\psi)d^I_{MK}(\varphi,\theta)$, the result

$$\mathcal{U}T^L_M\mathcal{U}^{\dagger}=\sum_{K=-L}^{L}T'^L_K D^L_{MK}(\varphi,\theta,\bar{\psi}) \ , \tag{13.75}$$

where

$$T'^L_K=\sum_{mn}(t^L_K)_{mn}\tfrac{1}{2}\{I'^{m}_{-},I'^{n}_{+}\} \tag{13.76}$$

and

$$(t^L_K)_{mn}\equiv(\mathring{t}^L_K)_{mn}\exp[i(K+m-n)\psi] \ , \tag{13.77}$$

with the property

$$[J_3,(t^L_K)_{mn}]=(K+m-n)(t^L_K)_{mn} \ . \tag{13.78}$$

This shows that (t^L_K) carries angular momentum along the symmetry axis, but it is an intrinsic operator with respect to **I**. It may be noted that the rotation matrix in (13.75) involves the angle $\bar{\psi}$ in place of ψ. The representation defined by Eqs. (13.75)–(13.77) has all the properties required for a transition operator in the Bohr-Mottelson representation. Thus we conclude that

$$\mathcal{U}T^L_M\mathcal{U}^{\dagger}=(T^L_M)_{BM} \ . \tag{13.79}$$

We shall implement the relation between the Villars and Bohr-Mottelson representations in Sec. XIII.E below.

D. Three-dimensional rotations of axially symmetric systems: Taylor expansion of the Hamiltonian in the generic model

Just as in the two-dimensional case, the Hamiltonian can be subject to the boson Taylor expansion (13.1) about the mean-field solution. Since the stability of the axially symmetric mean-field extremum is neutral, it is always possible by means of a rotation to choose the z axis as the symmetry axis. The broken-symmetry generators are then the total angular momentum operators J_x and J_y, or, equivalently, $J_{\pm}=J_x\pm iJ_y$, which then have the Taylor expansions commencing with linear boson terms:

$$J_{\pm}=J^{(1)}_{\pm}+J^{(2)}_{\pm}+J^{(3)}_{\pm}+\cdots \ . \tag{13.80}$$

The unbroken symmetry generator J_z in the generalized Holstein-Primakoff expansion is a finite quadratic form $J_z=J^{(2)}_z$, as will be assumed here. The angular momentum components must obey, of course, the usual SU(2) commutation rules $[J_+,J_-]=2J_z$ and $[J_z,J_{\pm}]=\pm J_{\pm}$, and these must be satisfied order by order, as follows:

$$[J^{(1)}_{+},J^{(1)}_{-}]=2J^{(0)}_z=0 \ , \tag{13.81a}$$

$$[J^{(1)}_{+},J^{(2)}_{-}]+[J^{(2)}_{+},J^{(1)}_{-}]=2J^{(1)}_z=0 \ , \tag{13.81b}$$

$$\sum_{k=1}^{n+1}[J^{(n-k+2)}_{+},J^{(k)}_{-}]=2J^{(n)}_z \ (\text{any } n) \ . \tag{13.81c}$$

In Eq. (13.81a), $J_z^{(0)}$ is the mean-field average, which vanishes for all components of the angular momentum for an even-even nucleus. In addition, since $J_z=J_z^{(2)}$, the right-hand side of Eq. (13.81c) vanishes unless $n=2$ and for any n

$$[J_z,J_\pm^{(n)}]=\pm J_\pm^{(n)} . \tag{13.82}$$

Since H is rotationally invariant, for any component of angular momentum J_M one has $[H,J_M]=0$, which must also be fulfilled order by order. In the RPA order, this becomes

$$[H^{(2)},J_\pm^{(1)}]=0 , \tag{13.83}$$

and in higher orders

$$\sum_{k=2}^{n}[H^{(k)},J_\pm^{(n-k+1)}]=0 , \tag{13.84}$$

while for any order n

$$[H^{(n)},J_z]=0 . \tag{13.85}$$

Instead of invoking a specific model, the technique for three dimensions can be illustrated by starting with the most general Taylor expansion of H (as well as transition operators) that can be constructed from the RPA modes, commensurate with conservation laws. This has been dubbed the *generic model.* The reconstruction of the generic model then holds for any specific model, whether microscopic or phenomenological. The Hamiltonian of the generic model is constructed with the aid of the following assumptions: (i) the mean field is axially symmetric; (ii) H is Hermitian; (iii) H is even under time reversal; (iv) H is rotationally invariant, i.e., Eqs (13.83)–(13.85) apply. It is also implicitly assumed that the mean field has reflection symmetry, to preclude tunneling modes that would be inconsistent with a small-oscillation ansatz. For simplicity, Goldstone modes other than the rotational ones will be omitted.

We begin with the RPA. The diagonalized form of the RPA Hamiltonian $H^{(2)}$ for an axial system is given by (Thouless and Valatin, 1962; Kammuri, 1967)

$$H^{(2)}=W_{\rm RPA}+\sum_\mu \hbar\omega_\mu \mathcal{B}_\mu^{(1)\dagger}\mathcal{B}_\mu^{(1)}+\hbar^2 J_+^{(1)}J_-^{(1)}/(2\mathcal{I}) , \tag{13.86}$$

where the notation is the same as in the two-dimensional analog given by Eq. (13.18). The last term on the right-hand side of Eq. (13.86), which can also be written as $\hbar^2(J_x^{(1)2}+J_y^{(1)2})/(2\mathcal{I})$, corresponds to a pair of Goldstone modes, one for each broken symmetry, as implied by Eq. (13.83). Physically, of course, this term represents the collective rotation about an axis perpendicular to the symmetry axis with the moment of inertia $\mathcal{I}$. The normal-mode variables $J_+^{(1)}$ and $J_-^{(1)}$ commute with each other according to Eq. (13.81a), and with the $\mathcal{B}_\mu^{(1)},\mathcal{B}_\mu^{(1)\dagger}$ from the definition of normal modes. As in the two-dimensional case, these variables do not suffice to form a complete set. The set can be completed by adding a pair of mutually commuting angle variables $\phi_+^{(1)}$ and $\phi_-^{(1)}=\phi_+^{(1)\dagger}$ such that

$$[\phi_\pm^{(1)},J_\mp^{(1)}]=i,\quad [\phi_\pm^{(1)},J_\pm^{(1)}]=0 , \tag{13.87a}$$

$$[\phi_\pm^{(1)},\mathcal{B}_\mu^{(1)}]=0\ \ (\text{and H.c. Eq.}) . \tag{13.87b}$$

It will be seen later that the variables $\varphi^{(1)}$ and $\theta^{(1)}$ defined by

$$\phi_\pm^{(1)}=(\varphi^{(1)}\pm i\theta^{(1)})/2 \tag{13.88}$$

are the leading terms in the expansion of appropriately defined Euler angles φ and θ. From Eqs. (13.86) and (13.87), it follows that the RPA angle variables may be obtained by solving the linear inhomogeneous equation

$$[H^{(2)},\phi_\pm^{(1)}]=-i\hbar^2 J_\pm^{(1)}/(2\mathcal{I}) , \tag{13.89}$$

which is the Thouless-Valatin equation. From Eqs. (13.87a) and (13.89), the moment of inertia may be expressed in terms of a double commutator as

$$\begin{aligned}\hbar^2/(2\mathcal{I})&=[\phi_+^{(1)},[H^{(2)},\phi_-^{(1)}]]\\&=[\phi_-^{(1)},[H^{(2)},\phi_+^{(1)}]] .\end{aligned} \tag{13.90}$$

To complete the story of the RPA, consider the angular momentum component J_z, which, being purely quadratic in bosons, must take the form

$$J_z=\sum_\mu K_\mu \mathcal{B}_\mu^{(1)\dagger}\mathcal{B}_\mu^{(1)}+i\phi_+^{(1)}J_-^{(1)}-i\phi_-^{(1)}J_+^{(1)} . \tag{13.91}$$

The first term on the right follows from the fact that the RPA phonon $\mathcal{B}_\mu^{(1)\dagger}$ carries K_μ units of angular momentum along the z axis, while the next two terms are required in order to satisfy Eq. (13.81c) for $n=2$.

It should be noted that for $K_\mu\neq 0$, each vibrational mode is doubly degenerate, with creation operators distinguished by $\mathcal{B}_\mu^{(1)\dagger}$ and $\mathcal{B}_{-\mu}^{(1)\dagger}$. With the proper choice of phases, the two operators may be related by $\mathcal{B}_{-\mu}^{(1)\dagger}=T\mathcal{B}_\mu^{(1)\dagger}T^{-1}$ where T is the time-reversal operator. One has, of course, that $K_{-\mu}=-K_\mu$. In general, Σ_μ denotes summation over both signs.

The next task is the construction of the higher-order terms $H^{(3)},H^{(4)}$ in the Hamiltonian and $J_\pm^{(2)},J_\pm^{(3)}$ in the angular momentum operators, using as building blocks the RPA normal modes. The most general form of $H^{(3)}$ compatible with Hermiticity and invariance of H under both time reversal and arbitrary rotations is given by

$$
\begin{aligned}
H^{(3)}=&\tfrac{1}{2}\sum_{\mu\nu\lambda}h^{(3)}(00)_{\mu\nu\lambda}\mathcal{B}_{\mu}^{(1)\dagger}\mathcal{B}_{\nu}^{(1)\dagger}\mathcal{B}_{\lambda}^{(1)}+\text{H.c.}+\left[\tfrac{1}{6}\sum_{\mu\nu\lambda}h^{(3)}(00)'_{\mu\nu\lambda}\mathcal{B}_{\mu}^{(1)\dagger}\mathcal{B}_{\nu}^{(1)\dagger}\mathcal{B}_{\lambda}^{(1)\dagger}+\text{H.c.}\right]\\
&+\sum_{\mu}h^{(3)}(00)_{\mu}[\mathcal{B}_{\mu}^{(1)\dagger}+\mathcal{B}_{\mu}^{(1)}]+\left[J_{-}^{(1)}\sum_{\mu\nu}h^{(3)}(10)_{\mu\nu}\mathcal{B}_{\mu}^{(1)\dagger}\mathcal{B}_{\nu}^{(1)}+\text{H.c.}\right]\\
&+\tfrac{1}{2}J_{-}^{(1)}\sum_{\mu\nu}h^{(3)}(10)'_{\mu\nu}[\mathcal{B}_{\mu}^{(1)\dagger}\mathcal{B}_{\nu}^{(1)\dagger}-\mathcal{B}_{-\nu}^{(1)}\mathcal{B}_{-\mu}^{(1)}]+\text{H.c.}\\
&+J_{+}^{(1)}J_{-}^{(1)}\sum_{\mu}h^{(3)}(20)_{\mu}[\mathcal{B}_{\mu}^{(1)\dagger}+\mathcal{B}_{\mu}^{(1)}]+\left\{J_{-}^{(1)2}\sum_{\mu}h^{(3)}(20)'_{\mu}[\mathcal{B}_{\mu}^{(1)\dagger}+\mathcal{B}_{-\mu}^{(1)}]+\text{H.c.}\right\}\\
&+\tfrac{1}{2}i[\{\phi_{+}^{(1)},J_{-}^{(1)}\}+\{\phi_{-}^{(1)},J_{+}^{(1)}\}]\sum_{\mu}h^{(3)}(11)_{\mu}[\mathcal{B}_{\mu}^{(1)\dagger}-\mathcal{B}_{\mu}^{(1)}]+\left\{i\phi_{-}^{(1)}J_{-}^{(1)}\sum_{\mu}h^{(3)}(11)'_{\mu}[\mathcal{B}_{\mu}^{(1)\dagger}-\mathcal{B}_{-\mu}^{(1)}]+\text{H.c.}\right\}\\
&+\left\{i\phi_{-}^{(1)}\sum_{\mu\nu}h^{(3)}(01)_{\mu\nu}\mathcal{B}_{\mu}^{(1)\dagger}\mathcal{B}_{\nu}^{(1)}+\tfrac{1}{2}\sum_{\mu\nu}h^{(3)}(01)'_{\mu\nu}[\mathcal{B}_{\mu}^{(1)\dagger}\mathcal{B}_{\nu}^{(1)\dagger}+\mathcal{B}_{-\nu}^{(1)}\mathcal{B}_{-\mu}^{(1)}]\right\}+\text{H.c.}\\
&+\phi_{+}^{(1)}\phi_{-}^{(1)}\sum_{\mu}h^{(3)}(02)_{\mu}[\mathcal{B}_{\mu}^{(1)\dagger}+\mathcal{B}_{\mu}^{(1)}]+\left\{\tfrac{1}{2}\phi_{-}^{(1)2}\sum_{\mu}h^{(3)}(02)'_{\mu}[\mathcal{B}_{\mu}^{(1)\dagger}+\mathcal{B}_{-\mu}^{(1)}]+\text{H.c.}\right\}. \qquad (13.92)
\end{aligned}
$$

Equation (13.92) reflects the following notation for general coefficients: in $h^{(n)}(kl)$, n denotes the order of the polynomial, k the number of $J_{\pm}^{(1)}$ factors, and l the number of $\phi_{\pm}^{(1)}$ factors. The coefficients may be chosen to be real without loss of generality. They are arbitrary in the generic model except for certain restrictions arising from invariance requirements and exchange symmetries of indices. For example, the coefficient $h^{(3)}(00)_{\mu}$ vanishes unless $K_{\mu}=0$ as a consequence of J_z conservation, and $h^{(3)}(10)_{-\nu-\mu}=-h^{(3)}(10)_{\mu\nu}$ as a consequence of Hermiticity and time-reversal invariance. Since the full list of such restrictions is rather lengthy and not absolutely essential for the present discussion, the reader is referred to the literature (Marshalek, 1987b).

Continuing in the same vein, one may construct the most general form of the quartic polynomial $H^{(4)}$, which contains a large number of different kinds of terms (34, to be exact). Fortunately, for the purpose of evaluating the leading-order corrections to the RPA, only a small subset of these terms is actually needed. Thus it is sufficient to write

$$
\begin{aligned}
H^{(4)}=&\,h^{(4)}(40)[J_{+}^{(1)}J_{-}^{(1)}]^2+h^{(4)}(00)+\sum_{\mu\nu}h^{(4)}(00)_{\mu\nu}\mathcal{B}_{\mu}^{(1)\dagger}\mathcal{B}_{\nu}^{(1)}+\tfrac{1}{4}\sum_{\mu\nu\kappa\lambda}h^{(4)}(00)_{\mu\nu,\kappa\lambda}\mathcal{B}_{\mu}^{(1)\dagger}\mathcal{B}_{\nu}^{(1)\dagger}\mathcal{B}_{\kappa}^{(1)}\mathcal{B}_{\lambda}^{(1)}\\
&+J_{+}^{(1)}J_{-}^{(1)}[h^{(4)}(20)+\sum_{\mu\nu}h^{(4)}(20)_{\mu\nu}\mathcal{B}_{\mu}^{(1)\dagger}\mathcal{B}_{\nu}^{(1)}]+\text{dispensable terms}. \qquad (13.93)
\end{aligned}
$$

The coefficients are all real and subject to additional symmetry restrictions.

Next, we give the required higher-order terms in the generic expansion (13.80) of the broken-symmetry generators:

$$
\begin{aligned}
J_{+}^{(2)}=&\,J_{+}^{(1)}\sum_{\mu}j^{(2)}(10)_{\mu}[\mathcal{B}_{\mu}^{(1)\dagger}+\mathcal{B}_{\mu}^{(1)}]+J_{-}^{(1)}\sum_{\mu}j^{(2)}(10)'_{\mu}[\mathcal{B}_{\mu}^{(1)\dagger}+\mathcal{B}_{-\mu}^{(1)}]\\
&+i\phi_{+}^{(1)}\sum_{\mu}j^{(2)}(01)_{\mu}[\mathcal{B}_{\mu}^{(1)\dagger}-\mathcal{B}_{\mu}^{(1)}]+i\phi_{-}^{(1)}\sum_{\mu}j^{(2)}(01)'_{\mu}[\mathcal{B}_{\mu}^{(1)\dagger}-\mathcal{B}_{-\mu}^{(1)}]\\
&+\sum_{\mu\nu}j^{(2)}(00)_{\mu\nu}\mathcal{B}_{\mu}^{(1)\dagger}\mathcal{B}_{\nu}^{(1)}+\tfrac{1}{2}\sum_{\mu\nu}j^{(2)}(00)'_{\mu\nu}[\mathcal{B}_{\mu}^{(1)\dagger}\mathcal{B}_{\nu}^{(1)\dagger}-\mathcal{B}_{-\nu}^{(1)}\mathcal{B}_{-\mu}^{(1)}],\quad J_{-}^{(2)}=J_{+}^{(2)\dagger}, \qquad (13.94)
\end{aligned}
$$

and

$$
J_{+}^{(3)}=J_{+}^{(1)}[j^{(3)}(10)+\sum_{\mu\nu}j^{(3)}(10)_{\mu\nu}\mathcal{B}_{\mu}^{(1)\dagger}\mathcal{B}_{\nu}^{(1)}]+j^{(3)}(30)J_{+}^{(1)2}J_{-}^{(1)}+\text{dispensable terms},\quad J_{-}^{(3)}=J_{+}^{(3)\dagger}. \qquad (13.95)
$$

The coefficients in Eqs. (13.94) and (13.94) are assumed to be real and subject to further restrictions arising from time-reversal considerations and the commutator (13.82) with J_z, which are omitted for brevity. It is important to note, however, that the highest-order rotational invariance condition (13.84) implies relations between certain coefficients of $H^{(n)}$ and $J_{\pm}^{(n-1)}$; in particular, for $n=3$, one finds

$$
\begin{aligned}
h^{(3)}(11)_{\mu}&=\hbar\omega_{\mu}j^{(2)}(10)_{\mu}+\frac{\hbar^2}{2\mathcal{J}}j^{(2)}(01)_{\mu},\\
h^{(3)}(11)'_{\mu}&=\hbar\omega_{\mu}j^{(2)}(10)'_{\mu}+\frac{\hbar^2}{2\mathcal{J}}j^{(2)}(01)'_{\mu},\\
h^{(3)}(02)_{\mu}&=-\hbar\omega_{\mu}j^{2}(01)_{\mu},\\
h^{(3)}(02)'_{\mu}&=-\hbar\omega_{\mu}j^{2}(01)'_{\mu},\\
h^{(3)}(01)_{\mu\nu}&=\hbar(\omega_{\mu}-\omega_{\nu})j^{(2)}(00)_{\mu\nu},\\
h^{(3)}(01)'_{\mu\nu}&=\hbar(\omega_{\mu}+\omega_{\nu})j^{(2)}(00)'_{\mu\nu}.
\end{aligned} \qquad (13.96)
$$

It may be noted that only the coefficients of the $\phi_{\pm}^{(1)}$-

dependent terms of $H^{(3)}$ are fixed by the rotational invariance. It is precisely such angle-dependent terms that must be eliminated in the final reconstructed Hamiltonian.

E. Outline of the reconstruction method for axially symmetric systems

From Eq. (13.91), it follows that $[\mathcal{B}_\mu^{(1)},J_z]=K_\mu\mathcal{B}_\mu^{(1)}$. This result implies that neither the RPA phonons nor reconstructed phonons given by the expansion (13.18) can be chosen to commute with all the components of **J** (except when $K_\mu=0$). It therefore follows that if the intrinsic excitations are described by such phonons, the angular momentum **J** cannot be used *directly* to reconstruct a Bohr-Mottelson representation. For some time, this fact appeared to be an impasse to extending the Marshalek-Weneser method to three-dimensional rotation. But from the previous analysis in Sec. XIII.D, there is an obvious way out. One can first construct a Villars representation in the form of Eq. (13.55), with the $\mathcal{H}_{mn}$ as functions of the phonon operators. In this representation, the $\mathcal{H}_{mn}$ do not commute with components of the angular momentum **J** as implied by Eq. (13.49). In the second step, the unitary transformation (13.61) can be invoked to pass over to the Bohr-Mottelson representation (13.70). In effect, this step amounts to setting $\psi=0$ and replacing **J** by **I** in Eq. (13.55).

We are now in a position to outline a viable reconstruction scheme. The first step is to establish a correspondence between the known boson expansion of the laboratory components of **J** and their representation in Euler-angle space. As mentioned previously, the conventional *zyz* representation is unsuitable for a Taylor-series expansion about vanishing values of the angles, but the *xyz* representation may be used instead. With the choice of the *xyz* convention for the Euler angles, the laboratory components of **J** can be expressed in terms of the Euler angles and Hermitian canonically conjugate momenta p_φ, p_θ, and $p_\psi=J_3$ as follows:

$$J_x=p_\varphi \ , \tag{13.97a}$$

$$J_z\pm iJ_y=-\tan\theta\,\tfrac{1}{2}\{e^{\mp i\varphi},p_\varphi\}\pm ie^{\mp i\varphi}p_\theta+e^{\mp i\varphi}\sec\theta J_3 \ , \tag{13.97b}$$

in accord with Eq. (13.59). The first term on the right-hand side of Eq. (13.97b) has been symmetrized to ensure that J_y and J_z are manifestly Hermitian. The momenta p_φ and p_θ must satisfy

$$[\varphi,p_\varphi]=[\varphi,J_x]=i \ , \tag{13.98a}$$

$$[\theta,p_\theta]=i \ , \tag{13.98b}$$

whereas ψ satisfies Eq. (13.53) or, equivalently, $[\psi,J_3]=i$, but as is clear from Sec. XIII.C, ψ need not be determined explicitly. All other combinations of pairs from the set $(\varphi,\theta,\psi,p_\varphi,J_3)$ are required to commute.

We first describe briefly a viable scheme employing commutators. First, Eq. (13.97a) determines the expansion of p_φ as identical to that of J_x. The expansion of φ can then be obtained from the requirement that Eq. (13.98a) be fulfilled order by order commencing with the RPA order. Next, from the pair of equations (13.97b), p_θ is easily derived in the explicit form

$$p_\theta=\frac{i}{4}\{e^{-i\varphi},J_z-iJ_y\}-\tfrac{1}{4}\{e^{i\varphi},J_z+iJ_y\} \ . \tag{13.99}$$

Once the expansion of φ is obtained, that of p_θ is then fully determined, and the expansion of θ can subsequently be obtained from Eq. (13.98b). Finally, the expansion of the phonon operators can be obtained by requiring them to commute with the angle variables and their conjugate momenta. In this way, the expansions of a complete set of canonical variables can be determined, which are clearly related to the laboratory components through Eqs. (13.97a), (13.99) and the principal-axis components through Eq. (13.52). Inversion then allows the Hamiltonian and other operators to be expressed in terms of these canonical variables, thereby providing the Villars representation.

In practice, it proves convenient to introduce the complex combinations of Euler angles $\phi_\pm$ defined by

$$\phi_\pm\equiv(\varphi\pm i\theta)/2 \tag{13.100}$$

and the canonically momenta $p_\pm$ given by

$$p_\pm\equiv p_\varphi\pm ip_\theta \tag{13.101}$$

and satisfying

$$[\phi_\pm,p_\mp]=i, \quad [\phi_\pm,p_\pm]=0 \ , \tag{13.102}$$

which is equivalent to Eq. (13.98). From Eqs. (13.97a), (13.99), and (13.101), one readily finds for $p_\pm$ the expansion

$$p_\pm=p_\pm^{(1)}+p_\pm^{(2)}+p_\pm^{(3)}+\cdots \ , \tag{13.103}$$

where

$$\begin{aligned} &p_\pm^{(1)}=J_\pm^{(1)},p_\pm^{(2)}=J_\pm^{(2)} \ , \\ &p_\pm^{(3)}=J_\pm^{(3)}\pm\frac{i}{2}\{\phi_+^{(1)}+\phi_-^{(1)},J_z\} \\ &\qquad\mp\tfrac{1}{4}[\phi_+^{(1)}+\phi_-^{(1)}]^2[J_+^{(1)}-J_-^{(1)}] \ . \end{aligned} \tag{13.104}$$

Thus $p_\pm$ first differs from $J_\pm$ in the cubic terms.

Just as in the two-dimensional case, it is more convenient to use the formal unitary transformation method than the commutator method. The applicability of this method is based on the fact that the set of final variables $(\phi_\pm,p_\pm,\mathcal{B}_u,\mathcal{B}_u^\dagger)$ and the set of RPA variables $(\phi_\pm^{(1)},J_\pm^{(1)},\mathcal{B}_\mu^{(1)},\mathcal{B}_\mu^{(1)\dagger})$ involve the same mutual commutation relations, allowing the possibility of connecting the two sets by a canonical transformation. Specifically, the procedure is to seek a formal unitary transformation $\exp(iS)$ such that

$$\exp(iS)J_{\pm}^{(1)}\exp(-iS)=p_{\pm}^{(1)}+p_{\pm}^{(2)}+p_{\pm}^{(3)}+\cdots , \tag{13.105}$$

where the right-hand side, is given by Eqs. (13.104). The generator has an expansion commencing with cubic polynomial terms:

$$S=S^{(3)}+S^{(4)}+\cdots . \tag{13.106}$$

Equation (13.105) in the first two orders leads to the following linear inhomogeneous equations:

$$[iS^{(3)},J_{\pm}^{(1)}]=J_{\pm}^{(2)} , \tag{13.107}$$

$$[iS^{(4)},J_{\pm}^{(1)}]=-\tfrac{1}{2}[iS^{(3)},J_{\pm}^{(2)}]+J_{\pm}^{(3)}\pm\frac{i}{2}\{\phi_{+}^{(1)}+\phi_{-}^{(1)},J_z\} \mp\tfrac{1}{4}[\phi_{+}^{(1)}+\phi_{-}^{(1)}]^2[J_{+}^{(1)}-J_{-}^{(1)}] . \tag{13.108}$$

Just as in the two-dimensional case, the solutions for $S^{(n)}$ are arbitrary to the extent that any solution may be augmented by an arbitrary nth-order polynomial function of $J_{\pm}^{(1)}$, $\mathcal{B}_{\mu}^{(1)}$, and $\mathcal{B}_{\mu}^{(1)\dagger}$, all of which commute with $J_{\pm}^{(1)}$. This arbitrariness is entirely equivalent to that occurring in the commutator approach.

Once the expansion of S has been found, the expansion of $\phi_{\pm}$,

$$\phi_{\pm}=\phi_{\pm}^{(1)}+\phi_{\pm}^{(2)}+\cdots+\phi_{\pm}^{(n)}+\cdots , \tag{13.109}$$

and that of $\mathcal{B}_{\mu}$ and $\mathcal{B}_{\mu}^{\dagger}$, as in Eq. (13.18), can be generated directly by applying the unitary transformation to the corresponding RPA variables as follows:

$$\begin{aligned}\phi_{\pm}&=\exp(iS)\phi_{\pm}^{(1)}\exp(-iS)\\&=\phi_{\pm}^{(1)}+[iS^{(3)},\phi_{\pm}^{(1)}]+\cdots ,\\ \mathcal{B}_{\mu}&=\exp(iS)\mathcal{B}_{\mu}^{(1)}\exp(-iS)\\&=\mathcal{B}_{\mu}^{(1)}+[iS^{(3)},\mathcal{B}_{\mu}^{(1)}]+\cdots \text{ (and H.c. Eq.) .}\end{aligned} \tag{13.110}$$

The unitary transformation ensures that the final variables satisfy the same commutation relations as in the commutator method.

The transformations of an arbitrary function of the RPA variables $F(p_{\pm}^{(1)},\phi_{\pm}^{(1)},\mathcal{B}_{\mu}^{(1)\dagger},\mathcal{B}_{\mu}^{(1)})$ to the set $(p_{\pm},\phi_{\pm},\mathcal{B}_{\mu}^{\dagger},\mathcal{B}_{\mu})$ can be accomplished with the following formula analogous to Eq. (13.25) in the two-dimensional case:

$$F=\exp(-iS)\tilde{F}\exp(iS)=F(p_{\pm},\phi_{\pm},\mathcal{B}_{\mu}^{\dagger},\mathcal{B}_{\mu})-[i\tilde{S}^{(3)},F(p_{\pm},\phi_{\pm},\mathcal{B}_{\mu}^{\dagger},\mathcal{B}_{\mu})]+\cdots , \tag{13.111}$$

where

$$\tilde{F}\equiv\exp(iS)F(J_{\pm}^{(1)},\phi_{\pm}^{(1)},\mathcal{B}_{\mu}^{(1)},\mathcal{B}_{\mu}^{(1)\dagger})\exp(-iS)=F(p_{\pm},\phi_{\pm},\mathcal{B}_{\mu},\mathcal{B}_{\mu}^{\dagger}) , \tag{13.112}$$

i.e., each RPA variable is replaced by its transform: $\tilde{J}_{\pm}^{(1)}=p_{\pm}$, $\tilde{\phi}_{\pm}^{(1)}=\phi_{\pm}$, $\tilde{\mathcal{B}}_{\mu}^{(1)}=\mathcal{B}_{\mu}$, $\tilde{\mathcal{B}}_{\mu}^{(1)\dagger}=\mathcal{B}_{\mu}^{\dagger}$. In the case of H, one then obtains the expansion given by Eqs. (13.26) and (13.27), where, of course, the solutions of Eqs. (13.107) and (13.108) for $\tilde{S}^{(3)}$ and $\tilde{S}^{(4)}$ must now be used. In the subsequent step, $p_{\pm}$ should be eliminated in favor of the principal-axis components of **J**.

F. Reconstruction of the generic Hamiltonian

The solutions of Eqs. (13.107) and (13.108), transformed via Eq. (13.112) are readily found to be

$$\begin{aligned}i\tilde{S}^{(3)}=&-\tfrac{1}{2}i[\{\phi_{-},p_{+}\}+\{\phi_{+},p_{-}\}]\sum_{\mu}j^{(2)}(10)_{\mu}[\mathcal{B}_{\mu}^{\dagger}+\mathcal{B}_{\mu}]-i\phi_{-}p_{-}\sum_{\mu}j^{(2)}(10)'_{\mu}[\mathcal{B}_{\mu}^{\dagger}+\mathcal{B}_{-\mu}]-\text{H.c.}\\&+\phi_{+}\phi_{-}\sum_{\mu}j^{(2)}(01)]_{\mu}[\mathcal{B}_{\mu}^{\dagger}-\mathcal{B}_{\mu}]+\left[\tfrac{1}{2}\phi_{-}^2\sum_{\mu}j^{(2)}(01)'_{\mu}(\mathcal{B}_{\mu}^{\dagger}-\mathcal{B}_{-\mu})-\text{H.c.}\right]\\&-i\phi_{-}\sum_{\mu\nu}[j^{(2)}(00)_{\mu\nu}\mathcal{B}_{\mu}^{\dagger}\mathcal{B}_{\nu}+\tfrac{1}{2}j^{(2)}(00)'_{\mu\nu}(\mathcal{B}_{\mu}^{\dagger}\mathcal{B}_{\nu}^{\dagger}-\mathcal{B}_{-\nu}\mathcal{B}_{-\mu})]-\text{H.c.}\end{aligned} \tag{13.113}$$

and

$$\begin{aligned}i\tilde{S}^{(4)}=&\tfrac{1}{2}i\{\phi_{-},p_{+}\}[j^{(3)}(10)-\tfrac{1}{2}\sum_{\mu}j^{(3)}(10)_{\mu\mu}\\&-\tfrac{1}{2}\sum_{\mu\nu}[2j^{(3)}(10)_{\mu\nu}+2j^{(2)}(10)_{\mu}j^{(2)}(10)_{\nu}+j^{(2)}(10)'_{\mu}j^{(2)}(10)'_{\nu}+j^{(2)}(10)'_{-\mu}j^{(2)}(10)'_{-\nu}]\\&\times(\mathcal{B}_{\mu}^{\dagger}\mathcal{B}_{\nu}+\tfrac{1}{2}\delta_{\mu\nu})-\tfrac{1}{2}ij^{(3)}(30)\{\phi_{-},p_{+}^2p_{-}\}-\text{H.c. of all terms}\\&+\text{dispensable terms .}\end{aligned} \tag{13.114}$$

To Eqs. (13.113) and (13.114) arbitrary functions of p_{+}, p_{-}, $\mathcal{B}_{\mu}^{\dagger}$, and $\mathcal{B}_{\mu}$ may be added, but these are chosen to be zero here. In the case of $\tilde{S}^{(4)}$, we retain only contributions that are actually needed to evaluate the leading-order corrections

to the RPA. This abbreviation is consistent with that for $H^{(4)}$ [Eq. (13.93)] and $J_{\pm}^{(3)}$ [Eq. (13.95)].

We are now in a position to evaluate the expansions given by Eqs. (13.26) and (13.27). The result is

$$H_B = W_0 + H'^{(2)} + H'^{(3)} + H'^{(4)} + \cdots , \tag{13.115}$$

where

$$H'^{(2)} = W_{\mathrm{RPA}} + \sum_{\mu} \hbar\omega_{\mu} \mathcal{B}_{\mu}^{\dagger}\mathcal{B}_{\mu} + \frac{\hbar^2}{2\mathcal{J}} p_+ p_- , \tag{13.116a}$$

$$\begin{aligned} H'^{(3)} = {} & \tfrac{1}{2} \sum_{\mu\nu\lambda} h^{(3)}(00)_{\mu\nu\lambda} (\mathcal{B}_{\mu}^{\dagger}\mathcal{B}_{\nu}^{\dagger}\mathcal{B}_{\lambda} + \mathcal{B}_{\lambda}^{\dagger}\mathcal{B}_{\nu}\mathcal{B}_{\mu}) + \sum_{\mu} h^{(3)}(00)_{\mu} (\mathcal{B}_{\mu}^{\dagger} + \mathcal{B}_{\mu}) \\ & + \tfrac{1}{6} \sum_{\mu\nu\lambda} h^{(3)}(00)'_{\mu\nu\lambda} (\mathcal{B}_{\mu}^{\dagger}\mathcal{B}_{\nu}^{\dagger}\mathcal{B}_{\lambda}^{\dagger} + \mathcal{B}_{\lambda}\mathcal{B}_{\nu}\mathcal{B}_{\mu}) + \left[p_- \sum_{\mu\nu} \Gamma^{(3)}(10)_{\mu\nu} \mathcal{B}_{\mu}^{\dagger}\mathcal{B}_{\nu} + \mathrm{H.c.} \right] \\ & + \tfrac{1}{2} p_- \sum_{\mu\nu} \Gamma^{(3)}(10)'_{\mu\nu} (\mathcal{B}_{\mu}^{\dagger}\mathcal{B}_{\nu}^{\dagger} - \mathcal{B}_{-\nu}\mathcal{B}_{-\mu}) + \mathrm{H.c.} \\ & + p_+ p_- \sum_{\mu} \Gamma^{(3)}(20)_{\mu} (\mathcal{B}_{\mu}^{\dagger} + \mathcal{B}_{\mu}) + \left[p_-^2 \sum_{\mu} \Gamma^{(3)}(20)'_{\mu} (\mathcal{B}_{\mu}^{\dagger} + \mathcal{B}_{-\mu}) + \mathrm{H.c.} \right] , \end{aligned} \tag{13.116b}$$

with the coefficients given by

$$\begin{aligned} & \Gamma^{(3)}(10)_{\mu\nu} \equiv h^{(3)}(10)_{\mu\nu} - \frac{\hbar^2}{2\mathcal{J}} j^{(2)}(00)_{\mu\nu} , \quad \Gamma^{(3)}(10)'_{\mu\nu} \equiv h^{(3)}(10)'_{\mu\nu} - \frac{\hbar^2}{2\mathcal{J}} j^{(2)}(00)'_{\mu\nu} , \\ & \Gamma^{(3)}(20)_{\mu} \equiv h^{(3)}(20)_{\mu} - \frac{\hbar^2}{2\mathcal{J}} j^{(2)}(10)_{\mu} , \quad \Gamma^{(3)}(20)'_{\mu} \equiv h^{(3)}(20)'_{\mu} - \frac{\hbar^2}{2\mathcal{J}} j^{(2)}(10)'_{\mu} , \end{aligned} \tag{13.117}$$

and finally

$$\begin{aligned} H'^{(4)} = {} & W^{(4)} + \tfrac{1}{2} \sum_{\mu\nu} \Gamma^{(4)}(00)_{\mu\nu} (\mathcal{B}_{\mu}^{\dagger}\mathcal{B}_{\mu} + \tfrac{1}{2})(\mathcal{B}_{\nu}^{\dagger}\mathcal{B}_{\nu} + \tfrac{1}{2}) + \sum_{\mu} \Gamma^{(4)}(00)_{\mu} (\mathcal{B}_{\mu}^{\dagger}\mathcal{B}_{\mu} + \tfrac{1}{2}) + \Gamma^{(4)}(20) p_+ p_- + \Gamma^{(4)}(40) p_+^2 p_-^2 \\ & + p_+ p_- \sum_{\mu} \Gamma^{(4)}(20)_{\mu} (\mathcal{B}_{\mu}^{\dagger}\mathcal{B}_{\mu} + \tfrac{1}{2}) + \text{off-diagonal terms} + \frac{\hbar^2}{2\mathcal{J}} \left[\theta^2 p_{\varphi}^2 - 2\theta p_{\varphi} \sum_{\mu} K_{\mu} \mathcal{B}_{\mu}^{\dagger}\mathcal{B}_{\mu} - \tfrac{1}{2} \right] . \end{aligned} \tag{13.116c}$$

In Eq. (13.116c), only the diagonal terms, which are of order Ω^{-1} relative to the RPA, have been retained, whereas the off-diagonal terms, which lead to correction of relative order Ω^{-2}, have been suppressed. In addition to the constant $W^{(4)}$, a higher-order correction to the ground-state energy of no compelling interest in the present context, the other coefficients may be expressed in terms of original coefficients as follows:

$$\begin{aligned} \Gamma^{(4)}(00)_{\mu\nu} \equiv {} & \tfrac{1}{2} h^{(4)}(00)_{\mu\nu,\mu\nu} - 2j^{(2)}(00)_{\mu\nu} h^{(3)}(10)_{\mu\nu} \\ & - 2j^{(2)}(00)_{\nu\mu} h^{(3)}(10)_{\nu\mu} - j^{(2)}(00)'_{\mu\nu} h^{(3)}(10)'_{\mu\nu} - j^{(2)}(00)'_{-\mu-\nu} h^{(3)}(10)'_{-\mu-\nu} \\ & + \frac{\hbar^2}{4\mathcal{J}} \{ 4[j^{(2)}(00)_{\mu\nu}]^2 + [j^{(2)}(00)'_{\mu\nu}]^2 + [j^{(2)}(00)'_{-\mu-\nu}]^2 \} , \end{aligned} \tag{13.118a}$$

$$\begin{aligned} \Gamma^{(4)}(00)_{\mu} \equiv {} & h^{(4)}(00)_{\mu\mu} - \tfrac{1}{2} \sum_{\nu} h^{(4)}(00)_{\mu\nu,\mu\nu} \\ & + \frac{\hbar^2}{\mathcal{J}} \{ 2[j^{(2)}(10)_{\mu}]^2 - 2j^{(3)}(10)_{\mu\mu} + [j^{(2)}(10)'_{\mu}]^2 + [j^{(2)}(10)'_{-\mu}]^2 \} , \end{aligned} \tag{13.118b}$$

$$\Gamma^{(4)}(20) \equiv h^{(4)}(20) - \tfrac{1}{2} \sum_{\mu} h^{(4)}(20)_{\mu\mu} , \tag{13.118c}$$

$$\Gamma^{(4)}(40) \equiv h^{(4)}(40) - \frac{\hbar^2}{\mathcal{J}} j^{(3)}(30) , \tag{13.118d}$$

$$\begin{aligned} \Gamma^{(4)}(20)_{\mu} \equiv {} & h^{(4)}(20)_{\mu\mu} - 4j^{(2)}(10)_{\mu} h^{(3)}(20)_{\mu} - 4j^{(2)}(10)'_{\mu} h^{(3)}(20)'_{\mu} - 4j^{(2)}(10)'_{-\mu} h^{(3)}(20)'_{-\mu} \\ & + \frac{\hbar^2}{\mathcal{J}} \{ 2[j^{(2)}(10)_{\mu}]^2 + [j^{(2)}(10)'_{\mu}]^2 + [j^{(2)}(10)'_{-\mu}]^2 \} . \end{aligned} \tag{13.118e}$$

We remark that in the derivation of Eqs. (13.116)-(13.118), the rotational invariance conditions (13.84) or, equivalently (13.96) play an essential role in eliminating the dependence of the Hamiltonian on the Euler angles. The residual dependence in the last term of $H^{(4)}$ is due, of course, to the fact that the Hamiltonian has so far been expressed in terms of $p_{\pm}$ rather than the principal-axis components of **J**.

The next task then is to replace $p_\pm$ in H by principal-axis components of angular momentum. Beginning with the identity $J_1^2+J_2^2+J_3^2=J_x^2+J_y^2+J_z^2$, substituting on the right-hand side from Eqs. (13.97) and expanding in powers of the Euler angles, one may readily arrive at the result

$$J_1^2+J_2^2=\tfrac{1}{2}\{J'_+,J'_-\}\cong p_+p_-+\theta^2p_\varphi^2-2\theta p_\varphi J_3-\tfrac{1}{2}\ . \tag{13.119}$$

This result is valid through quartic terms. After multiplying Eq. (13.119) across by $\hbar^2/(2\mathcal{I})$, we see that the right-hand side, accounts for the rotation term in $H'^{(2)}$ [Eq. (13.116a)] as well as for the last Euler-angle-dependent term in $H'^{(4)}$ [Eq. (13.116c)]. That is, these two contributions to H may be replaced to the given order by the single contribution

$$\frac{\hbar^2}{2\mathcal{I}}[J_1^2+J_2^2]\ .$$

[To arrive at this result requires the identification

$$J_3=\sum_\mu K_\mu\mathcal{B}_\mu^\dagger\mathcal{B}_\mu\ , \tag{13.120}$$

which can be derived from Eq. (13.52)]. This replacement then removes the angle dependence in $H'^{(4)}$. In general, as mentioned earlier, $p_\pm$ can be related to the laboratory components of **J** using Eqs. (13.97a) (13.99), and (13.101), and subsequently to the principal-axis components through the inverse of Eqs. (13.52). It is then not difficult to show that

$$p_\pm\cong\exp(\pm i\psi)J'_\pm+\text{cubic terms}=J'_\pm\exp(\pm i\psi)+\text{cubic terms}\ , \tag{13.121}$$

where the $\exp(\pm i\psi)$ dependence arises from the rotation matrix in Eq. (13.52). Since H is expanded only through quartic terms, substitution of Eq. (13.121) in $H^{(3)}$ and $H^{(4)}$ does not require the use of the cubic terms, i.e., effectively one makes the replacement $p_\pm\rightarrow\exp(\pm i\psi)J'_\pm\cong J'_\pm\exp(\pm i\psi)$. With the elimination of $p_\pm$ in favor of the principal-axis components of **J**, H may finally be written as the Villars expansion

$$H=W_0+H''^{(2)}+H''^{(3)}+H''^{(4)}+\cdots\ , \tag{13.122}$$

where

$$H''^{(2)}=W_{\rm RPA}+\sum_\mu\hbar\omega_\mu\mathcal{B}_\mu^\dagger\mathcal{B}_\mu+\frac{\hbar^2}{2\mathcal{I}}(J_1^2+J_2^2)\ , \tag{13.123a}$$

$$\begin{aligned}H''^{(3)}=&\tfrac{1}{2}\sum_{\mu\nu\lambda}h^{(3)}(00)_{\mu\nu\lambda}(\mathcal{B}_\mu^\dagger\mathcal{B}_\nu^\dagger\mathcal{B}_\lambda+\mathcal{B}_\lambda^\dagger\mathcal{B}_\nu\mathcal{B}_\mu)+\sum_\mu h^{(3)}(00)_\mu(\mathcal{B}_\mu^\dagger+\mathcal{B}_\mu)+\tfrac{1}{6}\sum_{\mu\nu\lambda}h^{(3)}(00)'_{\mu\nu\lambda}(\mathcal{B}_\mu^\dagger\mathcal{B}_\nu^\dagger\mathcal{B}_\lambda^\dagger+\mathcal{B}_\lambda\mathcal{B}_\nu\mathcal{B}_\mu)\\&+J'_-e^{-i\psi}\sum_{\mu\nu}[\Gamma^{(3)}(10)_{\mu\nu}\mathcal{B}_\mu^\dagger\mathcal{B}_\nu+\tfrac{1}{2}\Gamma^{(3)}(10)'_{\mu\nu}(\mathcal{B}_\mu^\dagger\mathcal{B}_\nu^\dagger-\mathcal{B}_{-\nu}\mathcal{B}_{-\mu})]+\text{H.c.}\\&+(J_1^2+J_2^2)\sum_\mu\Gamma^{(3)}(20)'_\mu(\mathcal{B}_\mu^\dagger+\mathcal{B}_\mu)+\left[J'^2_-e^{-2i\psi}\sum_\mu\Gamma^{(3)}(20)'_\mu(\mathcal{B}_\mu^\dagger+\mathcal{B}_{-\mu})+\text{H.c.}\right]\ ,\end{aligned} \tag{13.123b}$$

$$\begin{aligned}H''^{(4)}=&W^{(4)}+\tfrac{1}{2}\sum_{\mu\nu}\Gamma^{(4)}(00)_{\mu\nu}(\mathcal{B}_\mu^\dagger\mathcal{B}_\mu+\tfrac{1}{2})(\mathcal{B}_\nu^\dagger\mathcal{B}_\nu+\tfrac{1}{2})+\sum_\mu\Gamma^{(4)}(00)_\mu(\mathcal{B}_\mu^\dagger\mathcal{B}_\mu+\tfrac{1}{2})\\&+\Gamma^{(4)}(20)(J_1^2+J_2^2)+\Gamma^{(4)}(40)(J_1^2+J_2^2)^2+(J_1^2+J_2^2)\sum_\mu\Gamma^{(4)}(20)_\mu(\mathcal{B}_\mu^\dagger\mathcal{B}_\mu+\tfrac{1}{2})+\text{off-diagonal terms}\ .\end{aligned} \tag{13.123c}$$

The final step in the reconstruction of the Hamiltonian is the application of the unitary transformation (13.61) in order to pass over to the Bohr-Mottelson representation, as discussed in Sec. XIII. In effect, this amounts to the replacements $\psi\rightarrow 0$ and $\mathbf{J}\rightarrow\mathbf{I}$ in the Villars Hamiltonian. Thus we immediately obtain

$$H_{\rm BM}\equiv\mathcal{U}H\mathcal{U}^\dagger=W_0+H_{\rm BM}^{(2)}+H_{\rm BM}^{(3)}+H_{\rm BM}^{(4)}+\cdots\ , \tag{13.124}$$

where

$$H_{\rm BM}^{(2)}=W_{\rm RPA}+\sum_\mu\hbar\omega_\mu\mathcal{B}_\mu^\dagger\mathcal{B}_\mu+\frac{\hbar^2}{2\mathcal{I}}(I_1^2+I_2^2)\ , \tag{13.125a}$$

$$\begin{aligned}H_{\rm BM}^{(3)}=&\tfrac{1}{2}\sum_{\mu\nu\lambda}h^{(3)}(00)_{\mu\nu\lambda}(\mathcal{B}_\mu^\dagger\mathcal{B}_\nu^\dagger\mathcal{B}_\lambda+\mathcal{B}_\lambda^\dagger\mathcal{B}_\nu\mathcal{B}_\mu)+\sum_\mu h^{(3)}(00)_\mu(\mathcal{B}_\mu^\dagger+\mathcal{B}_\mu)+\tfrac{1}{6}\sum_{\mu\nu\lambda}h^{(3)}(00)'_{\mu\nu\lambda}(\mathcal{B}_\mu^\dagger\mathcal{B}_\nu^\dagger\mathcal{B}_\lambda^\dagger+\mathcal{B}_\lambda\mathcal{B}_\nu\mathcal{B}_\mu)\\&+I'_-\sum_{\mu\nu}[\Gamma^{(3)}(10)_{\mu\nu}\mathcal{B}_\mu^\dagger\mathcal{B}_\nu+\tfrac{1}{2}\Gamma^{(3)}(10)'_{\mu\nu}(\mathcal{B}_\mu^\dagger\mathcal{B}_\nu^\dagger-\mathcal{B}_{-\nu}\mathcal{B}_{-\mu})]+\text{H.c.}\\&+(I_1^2+I_2^2)\sum_\mu\Gamma^{(3)}(20)_\mu(\mathcal{B}_\mu^\dagger+\mathcal{B}_\mu)+\left[I'^2_-\sum_\mu\Gamma^{(3)}(20)'_\mu(\mathcal{B}_\mu^\dagger\mathcal{B}_{-\mu})+\text{H.c.}\right]\ ,\end{aligned} \tag{13.125b}$$

$$H^{(4)}_{\rm BM}=W^{(4)}+\tfrac{1}{2}\sum_{\mu\nu}\Gamma^{(4)}(00)_{\mu\nu}(\mathcal{B}^\dagger_\mu\mathcal{B}_\mu+\tfrac{1}{2})(\mathcal{B}^\dagger_\nu\mathcal{B}_\nu+\tfrac{1}{2})+\sum_\mu\Gamma^{(4)}(00)_\mu(\mathcal{B}^\dagger_\mu\mathcal{B}_\mu+\tfrac{1}{2})$$
$$+\Gamma^{(4)}(20)(I_1^2+I_2^2)+\Gamma^{(4)}(40)(I_1^2+I_2^2)^2+(I_1^2+I_2^2)\sum_\mu\Gamma^{(4)}(20)_\mu(\mathcal{B}^\dagger_\mu\mathcal{B}_\mu+\tfrac{1}{2})+\text{off-diagonal terms}\ . \quad (13.125c)$$

The Bohr-Mottelson Hamiltonian (13.124) is more convenient for a perturbative treatment than the Villars Hamiltonian (13.122). In order to obtain the leading-order energy corrections to the RPA, one should treat $H^{(3)}_{\rm BM}$ in second order and add it to the diagonal matrix element of $H^{(4)}_{\rm BM}$. For details, see Marshalek (1987b).

G. Reconstruction of transition operators

A spherical tensor operator T^L_K may be expanded like H in a Taylor series in the RPA normal-mode variables $(J^{(1)}_\pm,\phi^{(1)}_\pm,\mathcal{B}^{(1)\dagger}_\mu,\mathcal{B}^{(1)}_\mu)$, and then transformed into an expansion in $(p_\pm,\phi_\pm,\mathcal{B}^\dagger_\mu,\mathcal{B}_\mu)$, using Eq. (13.111). The variables $p_\pm$, in turn, can be eliminated in favor of $J'_\pm$. The resulting form of T^L_K may be treated with the aid of the formula

$$D^L_{MK}(0,0,\psi)=e^{iK\psi}\delta_{MK}\ . \quad (13.126)$$

The principal-axis components of the spherical tensor in the Villars representation are obtained from Eq. (13.73) as follows:

$$\hat{T}^L_M=\lim_{\phi_\pm\to 0}T^L_M e^{-iM\psi}\ . \quad (13.127)$$

Upon applying the transformation to the Bohr-Mottelson representation, one obtains the result (13.75), where the principal-axis components $\hat{T}^L_K$ are obtained by making the replacements $\psi\to 0$ and $J_\pm\to I'_\pm$. For further details, see Marshalek (1987b).

XIV. NONPERTURBATIVE METHODS FOR DOING DYNAMICS AFTER MAPPING

A. Early work

In this section we consider nonperturbative shell-model applications in which the mapping from the fermion to the boson space is purely kinematic and all the dynamics are subsequently carried out in the boson space. This is to be contrasted with the applications considered in the succeeding two sections, where at least some preliminary dynamics are carried out in the fermion space. A further restriction to realistic applications makes the available subject matter rather sparse pickings, but there appears to be much room for further development. The basic formulas that condition what has and what can be done are, first of all, Eq. (9.10) for the mapping of the multipole operators and the three alternatives, Eqs. (9.29), (9.33), and (9.39), for the mapping of pair operators. In addition, there is the constraint that state vectors have all their components in the physical subspace. There are also the Dyson formulas (9.12) and (9.13). These play a role in some of the calculations reported in this section, but we shall nevertheless postpone most of the discussion of the application of the Dyson mapping until Sec. XVI. It is somewhat remarkable that the overwhelming number of applications of the BZM model have utilized the most complicated of the three alternatives, namely, Eq. (9.39). There are various historical and practical reasons for this, some of which are implicit in the discussions of the previous sections. It appears that although the alternatives (9.29) and (9.33) can be found in the early literature (Janssen *et al.*, 1971; Marshalek, 1989), they have been largely neglected in applications.

To our knowledge, the first attempt to utilize Eq. (9.33) occurs in a series of papers by Hirsekorn and Weigert (Hirsekorn and Weigert, 1976, 1977; Hirsekorn, 1980, 1981). They apparently rediscovered that Eq. (9.39) can be transformed into (9.33) provided one guaranteed to stay in the physical subspace. Under these circumstances, a general fermion shell-model Hamiltonian (with antisymmetrized $V_{\alpha\beta\gamma\delta}$),

$$H=\sum_{\alpha\beta}\epsilon_{\alpha\beta}a^\dagger_\alpha a_\beta+\tfrac{1}{4}\sum_{\alpha\beta\gamma\delta}V_{\alpha\beta\gamma\delta}a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma\ , \quad (14.1)$$

can be transformed into a Hermitian boson Hamiltonian that is essentially no more complicated in structure than Eq. (14.1). In order to guarantee (one of the main selling points) that their theory does not violate the Pauli principle, these authors expand their eigenstates in terms of the basis (9.44) (since the applications are all to closed-shell nuclei, the boson operators $b^\dagger_{ij}$ here refer to particle-hole excitations). Herein lies the rub. The calculation, except for certain approximations to be detailed, is indistinguishable in form or complexity from a shell-model diagonalization in the space of one and two particle-hole excitations. It is less general than the above because three-particle/one-hole and one-particle/three-hole matrix elements of $V_{\alpha\beta\gamma\delta}$ are omitted; it is more general because three-particle/three-hole amplitudes are included approximately by a factorization approximation into an antisymmetric product of one-particle/one-hole and two-particle/two-hole matrix elements.

The equations can be compared in their structure with an approximate version of the RPA (approximate since the latter contains some multiparticle, multihole excitations because of the ground-state correlation amplitude). The most important difference emphasized is that the new formalism contains corrected single-particle energies compared to HF values. It has been applied to the LMG model, to ^{4}He, and to ^{16}O (negative-parity states). Using relatively simple interactions, typical "good" agreement

with experiment is achieved. The formalism has also been applied successfully to ^{15}O and ^{17}O. No further efforts in this direction have been reported.

B. Recent work

We turn next to work (Kuchta, 1988a–1988e) based on a different philosophy. This work is rather fresh and contains a core methodology that certainly deserves further independent study. An essential part of the procedure is some imposition or test of the necessary Pauli principle restrictions. One of the problems encountered by a reader of this work is the lack of uniformity in the way this is done. For example, in 1988a, 1988b, and 1988e, the method reported is the inverse of the method used by Takada (fully described in Sec. XVI), whereas in 1988c and 1988d a method based on Park's operator (Park, 1987; also discussed in Sec. XVI) is utilized. However, Dobeš (1989) has argued that Kuchta's imposition of the Park constraint is inconsistent with the results reported in 1988c. In the opinion of the writers, the work of Kuchta, though it may well contain some elements of an important new approach, has not yet crystallized sufficiently to warrant a presentation of detailed results. We therefore limit ourselves to a brief description of the underlying philosophy.

To begin with, one rewrites Eq. (14.1) as

$$H=\sum_{\alpha\beta}\epsilon'_{\alpha\beta}a^{\dagger}_{\alpha}a_{\beta}+\tfrac{1}{4}\sum_{\alpha\beta\gamma\delta}V_{\alpha\beta\gamma\delta}(a^{\dagger}_{\alpha}a_{\gamma})(a^{\dagger}_{\beta}a_{\delta}) , \tag{14.2}$$

where

$$\epsilon'_{\alpha\beta}=\epsilon_{\alpha\beta}-\tfrac{1}{4}V_{\alpha\gamma\gamma\beta} . \tag{14.3}$$

It should be noted that this rearrangement is only one of four different possible multipole forms. Since the Hamiltonian has been expressed completely in terms of multipoles, why not, as Kuchta does, substitute the multipole mapping (9.10)? The mapped Hamiltonian is then in the simplest conceivable form. The objection to such a step, which has certainly been thought of many times before, is immediate for anyone who has been schooled in the Copenhagen tradition. We all know that the matrix elements $V_{\alpha\beta\gamma\delta}$ can be expanded either in the pairing form [which couples $(\alpha\beta)$ and $(\gamma\delta)$ to good angular momentum J] or in the multipole form in two ways [in one of these, one couples $(\alpha\gamma)$ and $(\beta\delta)$ to good angular momentum J]. We also know that the two expansions are complementary, in the sense that an interaction that has a narrow spread in J in one form has a broad distribution in J in the other. Finally, we have been taught (and many of us believe) that for the study of the collective aspects of nuclear structure the essential content of a realistic interaction can be represented by the sum of the lowest J contributions from *each* form of writing the interaction as in Eq. (11.18).

Therefore, the objection to Kuchta's procedure would appear to be obvious. Though his ansatz should give an excellent account of the multipole-multipole interaction, the pairing interactions will be severely underestimated! In this objection, we have made the tacit assumption that we drop the contributions of bosons with large values of J. But Kuchta does not do this. He makes an exact mapping and keeps all terms. Therefore the Hamiltonian is exact, in the shell-model sense. Approximation enters through the method of finding the spectra and wave functions. Here Kuchta adopts the program of Dukelsky *et al.* (Dukelsky, Dussel, Perazzo, Reich, and Sofia, 1984; Dukelsky, Perazzo, Reich, and Sofia, 1985), in which it is suggested that given a many-body boson Hamiltonian, one should initially investigate its properties by applying the analogs of the methods that have proved so popular and enduring for fermions, namely Hartree-Fock (→Hartree-Bose) followed by Tamm-Dancoff or random-phase approximation, followed by projection, etc.

As stated initially, all of this is worthy of further study, but the ultimate success of such an approach is dependent on finding a cogent method of dealing with the Pauli principle. In our opinion, Kuchta has not yet shown that he has solved this problem. Quite recently Li (1991) has suggested a new variational approach that combines Kuchta's suggestion for mapping the Hamiltonian with a new use of the Park operator. There is some relation to Li's previous work on the Pauli principle described in Sec. XVI.

C. Mapping of the Hamiltonian

One problem that cannot be sidestepped, as we have seen, is that of nonuniqueness in the mapping of the Hamiltonian. In the course of the discussions up to this point that have included dynamics, it was necessary to adopt some solution, explicit or implicit, to this problem. We first summarize these various approaches and then ask what can be done, if anything in a systematic approach.

Consider first the group-theoretical schematic models. In every case the Hamiltonian is presented uniquely as a polynomial in the generators. The mapping of the generators then determines the mapping of the Hamiltonian. Here there is no ambiguity because one uses exact mappings. A second, more widely applicable, class that includes the previous one was encountered in Secs. XI–XIII, where we assumed a form of the Hamiltonian that contained multipole and pairing parts clearly labeled as such. This was the Hamiltonian (11.18), mapped according to (11.19) so as to preserve the physics. Even for this case, more than one mapping procedure has appeared in the literature. If one maps perturbatively, with a small parameter, then clearly the mapping of the Hamiltonian is determined to a consistent order by the mapping of the generators. We shall encounter a fresh example of this point of view in Sec. XVII, which treats number-conserving mappings that are perturbatively valid near the limit of good seniority. On the other hand, the modified Marumori-Yamamura-Tokunaga mapping,

to be discussed in the next two sections, is not strictly perturbative. In such cases, an independent Marumori-Yamamara-Tokunaga mapping for the Hamiltonian, as discussed below, has been applied widely, since it was first used in the work of Otsuka *et al.* (Otsuka, Arima, and Iachello, 1978), especially in connection with studies of the interacting boson model. This work is described in Sec. XVIII.

In the general case represented by Eq. (14.1) we must admit, however, that the mapping of the Lie algebra does not in and of itself solve the mapping problem, and indeed two different solutions have already been described in this section. In Eq. (14.1) the "natural" ordering of the interaction term as products of pair annihilation and pair creation operators suggested the mapping utilized in Sec. XIV.A. On the other hand, Kuchta went out of his way to rewrite the interactions in the multipole form as described in Eqs. (14.2) and (14.3) so that he could take advantage of the relative simplicity of the mapping of the multipole operators.

Finally, we describe two methods for mapping the Hamiltonian that do not involve the direct substitution of the generators and that can be carried out in practice, at least through the first few orders. For the first, the modified Marumori-Yamamura-Tokunaga method already mentioned above, one deviates from the previous focus on the operators and emphasizes instead the mapping of states. Thus if $|A\rangle$, $|B\rangle$, ... denote the states in the shell-model space and $|A)$, $|B)$, ... the corresponding mapped space generated by a set of bosons $b_i, b_i^\dagger$, then we can determine H_B, an approximation to a boson Hamiltonian, from the requirements

$$\langle A|H|B\rangle = (A|H_B|B) \tag{14.4}$$

and, for example, the specification that H_B be written in some series form (normal form or otherwise) in the boson operator set $\{b_i, b_i^\dagger\}$. Of course, this is nothing other than the Marumori prescription. It is also the epitome of the shell-model point of view. If only a subspace of the space $|A\rangle$ is utilized, as will be the case in practice, then H_B is a polynomial perfectly accurate in the corresponding subspace of the boson space. The *philosophy* of the mapping method then calls for its application to a larger space, as a test of the validity of the approximate mapping method utilized. This method has been the most popular approach to the study of the microscopic derivation of the interacting boson model and therefore, as already mentioned, will play an extensive role in the work to be described in Sec. XVIII.

A second method of mapping seeks to preserve the equations of motion (RPA with exchange and higher RPA with exchange; Li and Klein, 1971; Marshalek and da Providência, 1973a). A systematic way of achieving this result is to adopt the equations of motion approach of Rowe 1970; Ring and Schuck, 1980). We illustrate the particle-hole case: Let $C_\alpha^\dagger|\mathrm{HF}\rangle$ be a complete set of particle-hole (p-h) states (including the state $|\mathrm{HF}\rangle$ itself). Thus $C_\alpha^\dagger$ corresponds to certain product of p-h operators. Let $(C_\alpha^\dagger)_B|0)$ be the mapped basis and C_α, $(C_\alpha)_B$ the corresponding Hermitian conjugate operators. The boson Hamiltonian H_B, as a polynomial, is defined up to an additive constant by the equations

$$\langle \mathrm{HF}|[C_\alpha,[H,C_\beta^\dagger]]|\mathrm{HF}\rangle = (0|[(C_\alpha)_B[H_B,(C_\beta^\dagger)_B]]|0) , \tag{14.5}$$

$$\langle \mathrm{HF}|[C_\alpha,[H,C_\beta]]|\mathrm{HF}\rangle = (0|[(C_\alpha)_B,[H_B,(C_\beta)_B]]|0) \tag{14.6}$$

and H.c. equations. [The additive constant is the HF approximation to the ground-state energy, but further contributions to this energy, the correlation energy, will arise from H_B. In connection with the latter there is a delicate problem of double counting, which has been discussed thoroughly in the literature (da Providência and Weneser, 1970)]. If the set $C_\alpha^\dagger$ is limited to a single (p-h) creation operator, then Eqs. (14.5) and (14.6) will yield a boson Hamiltonian quadratic in the boson operators. Diagonalization by a Bogoliubov transformation is equivalent to solving the RPA with exchange. This approach could be used to generalize the theory described in Secs. XI–XIII to a realistic interaction.

For both general methods described above an important object of the mapping exercise is to apply the result in a larger space than the subspace used in the derivation, i.e., a definite gain in dynamical power is anticipated. Of course this will require a property that we have emphasized repeatedly, namely, the appearance of a small parameter as we proceed to higher-order terms in the polynomial. Here, however, previous considerations should apply, the method being applicable with greatest confidence in the immediate vicinity of a stable mean-field configuration.

We mention briefly a further form of mapping, which has (possibly) less dynamical interest than the methods above, but may have some value as a way of characterizing or framing the results of a calculational procedure. In this approach, one has already done one's *best* calculation in the fermion space and one has accurate energies and matrix elements of various one-body operators within a finite subspace including the ground state. Without going into detail (for examples see Dreiss and Klein, 1969; Ginocchio and Talmi, 1980), assuming that one is describing collective motion, it is easy to invent a boson subspace and a polynomial representation of observables, in the sense of the phenomenological ideas described in Sec. V, that reproduces the theoretical calculations in the fermion space. In this sense, the calculation plays the role of the data and the "correspondence" the role of a phenomenological model. In such a representation the Hamiltonian is particularly simple in that it is a polynomial in diagonal operators; higher-order results are also easily predicted for all observables.

D. Iterated boson mappings

The application of the Hartee (Fock) approximation to a problem of interacting fermions implies that one is including only mean-field effects. The application of a Hartree (Bose) approximation following one of the boson mappings discussed in this paper implies that one is also including particle-hole or particle-particle correlations, depending on what kind of fermion pair has been mapped into the boson. It is intriguing to take this idea one step further and introduce a mapping from the bosons to boson pairs in the hope that a Hartree approximation at this stage could lead to a reasonable first treatment of four-particle correlations.

The most serious problem that arises in connection with this idea of iterated boson mappings is that the relative weight of unphysical states becomes greater at each step. As we shall see in Sec. XVI, the problem of separating physical from spurious states, though well understood from a theoretical point of view, is still not solved from a practical standpoint for realistic models, except for a procedure, to be described, that leans heavily on the shell-model formalism. For example, in the boson space, we can construct states that either satisfy the Pauli principle or transform under other representations of the permutation group. Most physical Hamiltonians will bring the more symmetric representations down in energy, and thus, in a variational calculation of the ground-state energy, for example, one may anticipate serious mixing of unphysical (symmetric states) with the physical antisymmetric state.

For this reason, in a first application of the idea of iterated mappings (Bijker, Pittel, and Dukelsky, 1989), application was made to a starting boson system that has the advantage of having a symmetric ground state. A further simplification was achieved by writing a typical one-plus-two particle boson Hamiltonian in the particle-hole form, so that the simple Hermitian mapping of particle-hole operators, identical for the Dyson mapping and for the generalized Holstein-Primakoff mapping, could be applied. The system chosen for study was an IBM-1 model with the simplified (nonphysical) Hamiltonian

$$H_{\mathrm{IBM}}=-(1-\chi)C_{2\mathrm{SU}(3)}+\chi C_{2\mathrm{O}(6)}\ , \tag{14.7}$$

where the strengths, determined by the parameter χ, taken in the range (0,1), multiply the appropriately identified second-order Casimir invariants. At one end of the range, $\chi=0$, we have a (deformed) boson mean field, and at the other, $\chi=1$, boson pair correlations dominate. In between there is a phase transition, which occurs at $\chi\cong 0.85$. The ground-state energy of the system described was calculated for 16 bosons in three Hartree-Bose approximations, namely, calculations for the original, for the once-mapped, and for the twice-mapped boson Hamiltonians, thus providing the possibility of describing mean-field, pair, and quartet correlations. The results are compared with the exact energy in Fig. 4. Besides the evident successive improvement upon iteration, it is equally evident that correlations of all orders play a role at the phase transition.

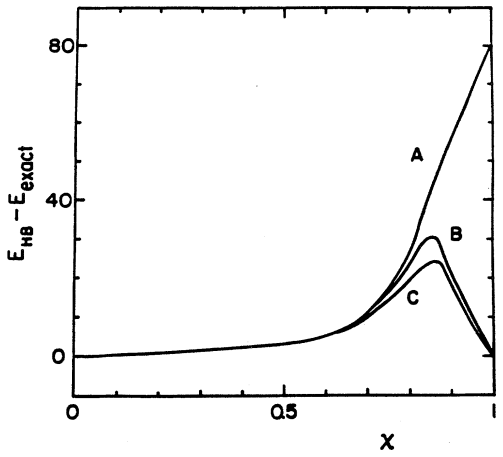

FIG. 4. Differences between the ground-state energies calculated in the Hartree-Bose approximation and by exact diagonalization for the boson system described in the text. Curve *A* refers to Hartree-Bose calculations carried out in the original interacting boson model boson space, curve *B* to calculations carried out in the pair-boson space, and curve *C* to calculations carried out in the quartet-boson space. From Bijker, Pittel, and Dukelsky, 1989.

A beginning has also been made on the fermion problem by Dukelsky and Pittel (1989). This paper contains a useful review of many of the issues discussed in Sec. XVI. It approaches the problem of iterated boson mappings, starting from a shell-model Hamiltonian, by considering a two-level pairing model, confining attention to the senority-zero subspace. For this model, the spurious state problem is avoided entirely on the fermion-to-boson mapping simply by choosing the size of the boson subspace appropriately, i.e., by making a one-to-one correspondence as was done for the corresponding model earlier in this review. Unphysical states are, however, unavoidable in the further iteration. The matter is dealt with in part by the use of the Park operator discussed in Sec. XVI. As stated above, the use of this constraint after the fermion-to-boson mapping has been called into question; this point is also discussed by Dukelsky and Pittel. However, these authors make a convincing argument that by adding to the Hamiltonian of the first iterated mapping a multiple of the Park operator, one can raise the energy of unphysical solutions compared to physical ones. There is, however, a delicate balance between that desirable end and an undesirable feature that this also weakens multiparticle correlations. It remains to be seen what can be done in more realistic cases.

XV. MAPPING OF TAMM-DANCOFF BOSONS

A. Introduction

Up to now we have studied boson mappings largely as a problem in kinematical transformation to a boson space. The dynamics is then to be studied exclusively in terms of bosons. It has been seen that this is a very natural procedure for the simplified models with which we began this review, but it is practically a fledgling field for realistic studies of nuclear properties. Much more practical attention has been paid to a procedure whereby one carries out at least minimal dynamical studies in the fermion space in order to choose the most important degrees of freedom determining the low-energy structure of nuclei. In the work to be summarized in this section, a uniform procedure can be discerned: after a Bogoliubov-Valatin transformation to quasiparticles, a Tamm-Dancoff calculation based on quasiparticle pairs allows one to select the most collective quadrupole excitation, although, in a more refined approach, some subset of the noncollective excitations is also retained. We refer to all of these excitations as phonons, though strictly we should reserve this name for the corresponding boson excitations. Orthonormal states constructed from the phonon operators define a subspace of the fermion shell-model space. One then maps this subspace onto a suitably defined boson space using either the commutator or the Marumori-Yamamura-Tokunaga method.

In fact, the commutators of any subset of the Tamm-Dancoff operators do not generally form a closed algebra. If we use the commutator method, we can satisfy the algebra only in an approximate sense, as an expansion in a suitable small parameter. As we shall show in Sec. XVII, an extended use of the commutator method can lead to consistency requirements on the number and type of included excitations. In the modified Marumori-Yamamura-Tokunaga method, which has been used widely for carrying through the mapping of a restricted subspace, no apparent inconsistency forcing a redefinition of the subspace can occur. On the other hand, one can be certain, on the basis of previous work, that some failure of the commutation relations must occur (Matsuyanagi, 1982). In either case, because the algebra is not rigorously satisfied, some violation of the Pauli principle is unavoidable. This observation also applies to the work reported in Sec. XVI.

In any event, once the mapping has been carried out, we are confronted with a polynomial boson Hamiltonian that represents an approximation to the original fermion Hamiltonian. Since orthonormal boson bases are simpler to construct than fermion bases, we have now hopefully a tractable problem of matrix diagonalization, though the physics sometimes dictates additional transformations of the Hamiltonian before this is done, as will be seen.

This section will be devoted mainly to the work of Tamura,[2] Kishimoto,[3] and their collaborators, who are the authors of one of the most extensive theoretical and practical efforts in this field. However, around 1980, this work came under attack on two fronts. On the one hand, questions were raised (Silvestre-Brac and Piepenbring, 1977b, 1979; Marshalek 1980a, 1980b) about the theoretical foundations of the work as so far reported (Kishimoto and Tamura, 1971, 1972, 1976). On the other hand, the accuracy of the results in describing experiments, in comparison with the interacting boson model, for instance, was also called into question (Arima, 1980; Arima, Yoshida, and Ginocchio, 1981). We shall comment more fully on this second point in Secs. XVII and XVIII in connection with our discussion of the foundations of the interacting boson model, but we wish to deal substantively with the first criticism in this section. Not to keep readers in suspense, let it be noted here that Tamura and Kishimoto responded effectively to all criticisms.

B. Summary of the model of Tamura and Kishimoto

Before providing a more detailed technical account of the work under review, we shall first make a reasonable assessment of the present status of the problem of applications, with the understanding that currently there are no theoretical objections to the basic formulation: (i) An extensive body of applications with overall impressive agreement with experiment has been reported, for Sm isotopes ($^{148-154}$Sm) (Tamura, Weeks, and Kishimoto, 1979), for $^{98-104}$Ru and $^{102-110}$Pd (Weeks and Tamura, 1980a), for $^{186-194}$Os and $^{188-198}$Pt (Weeks and Tamura, 1980b, 1980c), and for the Ge problem (Weeks, Tamura, Udagawa, and Hahne, 1981). Several of the ideas in these calculations were tested earlier (Kammuri and Kishimoto, 1976a, 1976b, 1978). Some results of this work will be reported at the end of this section. (ii) What actually entered these calculations of energy levels, $B(E2)$'s, static quadrupole moments, isomer shifts, and two-particle transfer amplitudes is never completely stated in any one of the references, but from the 1976 paper (Kishimoto and Tamura, 1976), the above quoted articles, and several conference reports (Tamura, Weeks, and Kishimoto, 1980; Tamura, 1982, 1983a) one can extract a largely complete and unambiguous statement of the in-

[2]The bulk of this section was written before the untimely death in October, 1988 of T. Tamura. Though his fame in the nuclear experimental community is based on his pioneering work on the coupled channel analysis of nuclear reactions, to the theoretical community he was equally known for the work described below, which was his primary interest during the last decade.

[3]It is devastating to note the more recent death of T. Kishimoto in April, 1990.

gredients of the calculations: The Hamiltonian is schematically of the form

$$H = H_{\rm sp} + H_{P0} + H_{P2} + H_{QQ} \ , \tag{15.1}$$

the sum of an independent-particle part $H_{\rm sp}$, monopole-pairing part H_{P0}, quadrupole pairing H_{P2}, and quadrupole-quadrupole interaction H_{QQ}. Considerable attention was paid to the (fixed) choice of single-particle energies for each set of nuclei, making use of various previous studies. It should be emphasized that these calculations exceed the conventional shell model in the number of j shells included, going both below and above the Fermi surface. For instance, the Sm calculations included eleven neutron levels and (generally) eight proton levels. The pairing parameters were fitted to odd-even mass differences, thus leaving in effect two parameters to adjust for each nucleus, an overall strength for each of H_{P2} and H_{QQ} (assuming equal strength for *pp*, *nn*, and *pn* components). These scale factors varied slowly with atomic number, within a range of roughly 10% and with a median very close to the value predicted by a famous self-consistency argument (Bes and Sorensen, 1969).

For the mapping, only a single collective quadrupole phonon was chosen initially. The first three orders of the boson expansion (see below) were retained, yielding a sixth-order boson Hamiltonian. This Hamiltonian contained nondiagonal quadratic pieces, so that a linear transformation diagonalizing this part was carried out first. This RPA-like transformation introduced strong ground-state correlations, which were very important in rescaling the spectrum to be in better accord with experiment than spectra found by previous workers. One further step was included to improve the physical content of the Hamiltonian. The mapped space was enlarged to include at most one noncollective phonon in any basis state in addition to a variable number of collective phonons. The collective-noncollective coupling was eliminated by second-order perturbation theory to yield an effective Hamiltonian (and other operators) defined in the collective subspace only. The size of this space was sufficient to guarantee numerical convergence of results. For the Ge calculations, where one has an unusually low-lying 0^+ excitation, a monopole-pairing phonon was also included.

Admittedly all the approximations involved in this approach have not been fully evaluated, but the results *published* are impressive. An *ad hoc* feature of the method is that the true Tamm-Dancoff collective excitation is replaced by the corresponding excitation in the strong-coupling limit, obtained by neglecting the spread of the single-particle spectrum. This choice probably also increases the collectivity. A possible source of error that one can pinpoint, that has been studied for a single-j model (Li, Pedrocchi, and Tamura, 1985) and for the SO(8) model (Li, Pedrocchi, and Tamura, 1986), is the BCS or number nonconserving approximation; here the error increases with increasing excitation. It is believed that this source of criticism can be removed without a major overhaul of the programs (Tamura, Li, and Pedrocchi, 1985).

Recently an improved version of the Kishimoto-Tamura program has been proposed (Sakamoto and Kishimoto, 1988). The main substantive changes are three: A larger array of noncollective phonons is included in the mapping for coupling, as described, to the collective degree of freedom; a procedure is developed for eliminating spurious states that enter because of number nonconservation; finally the Hamiltonian includes for self-consistency reasons, three- and four-body forces developed in other studies (Kishimoto *et al.*, 1983; Marshalek, 1984). Results of this extended version of the theory are awaited.

C. Method of mapping

Let us return briefly to the theoretical questions that arose in connection with the early formulations. Tamura and his co-workers responded to the criticisms with a series of papers (Tamura, Weeks, and Pedrocchi, 1981; Pedrocchi and Tamura, 1982; Kishimoto and Tamura, 1983; Pedrocchi and Tamura, 1983; Tamura, 1983b, 1983c). These papers contain roughly two types of material: much space is devoted to the independent discovery of ideas and results that were largely available in the literature, though often in a different guise. More important for our purposes, this work serves to clarify how, in fact, Tamura and Kishimoto applied boson expansions in their practical work, and shows that this application is in our present understanding, correct. The source of confusion was the following: In Kishimoto and Tamura (1972), a formally exact boson expansion was developed (equivalent to the full Marumori-Yamamura-Tokunaga mapping) in which there is no sign of a small parameter except for a single quantity $(1/x)$, where x may be thought of as the numerical coefficient multiplying the linear term in the boson expansion. It is now understood that, for vacuum states to map properly, this parameter *must* have the value unity. [For further discussion of this point, see Marshalek (1980b).] Therefore this series cannot be applied, it appears, to the physical problem. Instead, an approximate perturbative solution of the commutation relations with a different small parameter, the one established in Sec. IX, is utilized in Kishimoto and Tamura (1976) with a value of x chosen empirically to be near unity. That this procedure, which is almost consistent, is followed, is unambiguously stated on p. 345 of Kishimoto and Tamura, 1976. (All final calculations reported from 1980 on use the correct value $x=1$).

The upshot of the recent developments is finally that the current theoretical underpinning of the work being reported is the modified Marumori-Yamamura-Tokunaga mapping. Here modified means that one maps from a basis constructed by orthonormalization of products of quasifermion pairs, $A_a^\dagger$, which are chosen as Tamm-Dancoff pairs. Together with the pairs, there are suitably defined multipole operators $B_a^\dagger$ such that the

commutation relations, symbolically, take the form

$$[A_a, A_b^\dagger] = \delta_{ab} - \sum_p P_{a,b}^{(p)} B_p^\dagger \ , \tag{15.2}$$

$$[B_p^\dagger, A_a^\dagger] = \sum_b P_{a,b}^{(p)} A_b^\dagger \ , \tag{15.3}$$

where the three-index symbol $P_{a,b}^{(p)}$ depends on both kinematic and dynamic factors, but need not otherwise concern us in detail except that for fixed a, b, p it should, for large Ω, behave as Ω^{-1}. We now map onto a subspace of a boson space, assigning a boson $b_a^\dagger$ to every $A_a^\dagger$.

Consider the Hilbert space

$$|n;a\rangle = \prod_a \frac{1}{\sqrt{n_a!}} (b_a^\dagger)^{n_a} |0) \ , \quad b_a|0) = 0 \ . \tag{15.4}$$

We can always assign a subspace of the set (15.4) to be the physical boson space. In the language of Tamura, in this problem a subspace of this *ideal* boson space can be chosen as the physical boson space. This can be understood as follows: returning to the fermion space, let us build an orthonormal set. We start with $|0\rangle$ and the set $A_a^\dagger|0\rangle$. In general, different members of the set $A_a^\dagger A_b^\dagger|0\rangle$ are not mutually orthogonal; furthermore, because of Pauli principle restrictions, the states are not all linearly independent in the space of four particles. Next, we order the Tamm-Dancoff solutions according to some criterion (collectivity and/or energy of excitation) and assemble enough two-phonon states until we have a complete set. We map the state whose largest component is $A_a^\dagger A_b^\dagger|0\rangle$ to the state $b_a^\dagger b_b^\dagger|0)$, which we present symbolically as

$$A_a^\dagger A_b^\dagger|0\rangle \rightarrow b_a^\dagger b_b^\dagger|0) \ . \tag{15.5}$$

In fact we can consider the right-hand side to be a symbolic representation of the fermion state, as we have explained several times in connection with our study of SU(2) and SO(5) mappings. In general, the mapping as usually written,

$$(A_a^\dagger)_B = A_a^\dagger(b_b^\dagger, b_c) \ , \tag{15.6}$$

has an inverse, defined within the physical subspace only,

$$(b_a^\dagger)_F = b_a^\dagger(A_b^\dagger, A_c) \ , \tag{15.7}$$

that provides a construction of this orthonormal state. The procedure described may be extended in principle to multiboson states.

In practice, if we make a further approximation and limit ourselves to the mapping of one or two bosons, we normally assume that all mapped states are physical and in this we are unlikly to be far wrong. To find this mapping, it is perfectly acceptable and as useful as any other approach to employ the commutation relations in an approximate way not dissimilar to the method we shall describe in Sec. XVII. This has been applied, for instance, by Pedrocchi and Tamura (1984).

D. The norm matrix

We turn next to a brief account of the main technical problem associated with the application of the modified Marumori-Yamamura-Tokunaga mapping identified above as the method utilized in the work under discussion. This method was first developed and applied by several different groups (Lie and Holzwarth, 1975; Holzwarth, Janssen, and Jolos, 1976; Iwasaki, Sakata, and Takada, 1977). The basic element entering such calculations is a quantity called the *norm matrix,* defined below. This quantity also occurs as the fundamental entity in studies of $K=0^+$ states in deformed nuclei (Silvestre-Brac and Piepenbring, 1977a, 1978, 1979). The papers above for the most part develop methods for calculating the norm matrix for a single collective boson. Iwasaki *et al.* gave an exact recursion formula in the multiphonon case in a spherical representation, which is, however, cumbersome to apply. More recently, Silvestre-Brac and Piepenbring have extended and applied their analysis to the multiphonon case (Silvestre-Brac and Piepenbring, 1982; Jammari, Piepenbring, and Silvestre-Brac, 1983). (These authors work exclusively in shell-model space, however; i.e., in terms of fermions.) As we shall see below, the norm matrix is a construct in the fermion space that can play an essential role in shell-model calculations that have nothing to do with boson mappings. It also plays a basic role, however, in the modified Marumori-Yamamara-Tokunaga method.

Another important development—important because it involves a powerful method for improving the treatment of the pairing degree of freedom (Suzuki and Matsuyanagi, 1976) and because it bears on the microscopic foundations of the interacting boson model— also uses the modified Marumori method and involves a study of the properties of the norm matrix (Suzuki and Matsuyanagi, 1979; Suzuki, Fuyuki, and Matsuyanagi, 1979a, 1979b, 1981). This work has been reviewed previously (Matsuyanagi, 1982). We have also discussed some of the ideas in Secs. III, IV, and VI.

We turn then to some elementary remarks on how the norm matrix enters the problem and how one utilizes it. For practical details, we refer to the references already given. In the Marumori-Yamamura-Tokunaga method, the boson image O_B of a fermion operator O_F is given by the formula

$$O_B = U O_F U^\dagger \ , \tag{15.8}$$

where

$$U = \sum_{n=0}^{N} \sum_a |n,a)\langle n;a| \ . \tag{15.9}$$

Here U is a *modified* Marumori operator in the sense that the set $|n;a\rangle$, an orthonormal set of fermion states that should in principle span a complete irrep of the fermion algebra, is chosen in practice as a collective subspace in the fermion space. For the boson states, one takes *ideal* states,

$$|n;a)=\prod_a \frac{1}{\sqrt{n_a!}}(b_a^\dagger)^{n_a}|0) \ , \tag{15.10}$$

whereas, as described above, the states $|n;a\rangle$ must be constructed from elements that are not generally orthonormal, namely,

$$|n;a\rangle\rangle \equiv \prod_a \frac{1}{\sqrt{n_a!}}(A_a^\dagger)^{n_a}|0\rangle \ . \tag{15.11}$$

If the states (15.11) were at least linearly independent we could express the orthornormalized states in the form

$$|n;a\rangle = \sum_b (Z_n^{-1})_{a;b}|n;b\rangle\rangle \ ; \tag{15.12}$$

then the inverse relation

$$|n;a\rangle\rangle = \sum_b (Z_n)_{a;b}|n;b\rangle \tag{15.13}$$

would yield the definition

$$(Z_n^2)_{a,b}\delta_{mn} \equiv \langle\langle n;a|m;b\rangle\rangle \tag{15.14}$$

of the norm matrix Z^2. Equation (15.14) is well defined even if (15.11) is overcomplete, but in order for Eqs. (15.12) and (15.17) below to make sense, we must assume that we have preselected a linearly independent subspace of Eq. (15.11). How this is to be done is described after Eq. (15.17). [It may be noted that Z^2 is the Gram matrix of the set (15.11).]

Let us convince ourselves that any matrix element of the form

$$\langle\langle n;a|O_F|m;b\rangle\rangle \ , \tag{15.15}$$

where O_F is a polynomial in the generators $A_a^\dagger$, A_a, and $B_a^\dagger$ is a linear combination of norm matrix elements. It follows from the commutation relations (15.2) and (15.3) plus the relations $B_a|0\rangle = A_a|0\rangle = 0$ that this can always be arranged. It follows from the same properties that

$$(Z_n^2)_{ab} = \left\langle 0 \left| \prod_a \frac{(A_a)^{n_a}}{\sqrt{n_a!}} \prod_b \frac{(A_b^\dagger)^{n_b}}{\sqrt{n_b!}} \right| 0 \right\rangle \ , \tag{15.16}$$

where $n = \sum_{n_a} = \sum_{n_b}$ can itself be defined by recursion relations and the values of the simplest ($n = \sum_{n_a} = 1$) such quantities, is proved by commuting one factor A_a at a time to the right.

Finally, writing out Eq. (15.8) we have

$$\begin{aligned} O_B = UO_FU^\dagger &\equiv \sum_{n,m}\sum_{a,b}|n;a)\langle n;a|O_F|m;b\rangle(m;b| \\ &= \sum_{n,m}\sum_{a,a',b,b'}|n;a)(Z_n^{-1})_{a,a'}\langle\langle n;a'|O_F|m;b'\rangle\rangle(Z_m^{-1})_{b'b}(m;b| \ , \end{aligned} \tag{15.17}$$

which is then completely determined, insofar as numerical coefficients of fermionic origin are concerned, by the norm matrix. To have a definite mapping, we must put in an explicit boson expansion for the vacuum projector $|0)(0|$.

In the discussion just completed, we have assumed that the matrix Z_n has an inverse. This will be true provided we have preselected a linearly independent basis in the shell-model space. If this has not otherwise been done, a nonredundant basis may be selected by diagonalization of this matrix and discarding of the eigenstates with zero eigenvalues. At this juncture a nonsingular norm matrix can be defined, and we may proceed as described above. Further discussion of this procedure will be found in Sec. XVI.

E. Some results

We conclude this section by reporting some of the results published by Tamura and his associates. The references have already been given in the opening paragraph of Sec. XV.B, and therefore it suffices below simply to identify the nucleus under discussion. The results quoted were all obtained using a sixth-order boson Hamiltonian. The first question to be addressed is then that of the convergence of the numerical results. We may take for granted the trivial aspect of this question: that the size of the basis was chosen sufficiently large. The nontrivial aspect concerns the convergence of the boson expansion for the states included in the comparisons with experiment. This was the first question addressed in the Sm paper. It was tested by comparing spectra calculated with a fourth-order boson Hamiltonian (second-order boson expansion) with those calculated with a sixth-order boson Hamiltonian (third-order boson expansion). This comparison is shown for a series of Sm isotopes in Fig. 5. First of all, these figures already define the range of the states described by the program, namely, up to an excitation energy of about 2 MeV. For vibrational nuclei, this includes, as one sees from the first column, collective quadrupole states up to the three-phonon level. For deformed nuclei, the number of collective states included is considerably larger. The main conclusion from the figure is that for levels below 1 MeV, and for the vibrational nucleus in general, the fourth-order Hamiltonian already suffices, but for deformed nuclei, the positions of the high-lying levels are substantially modified by going to the next order. It is implied that changes coming from still higher orders will be smaller or even negligible, but strictly speaking this question is still open.

We turn next to what is perhaps the stickiest question in this whole business. What constitutes "good agreement" with experiment? Here, as a fiducial mark, let us remind the reader that the most elaborate and sophisti-

cated shell-model calculations (in the *s-d* shell) fit spectra with an average rms deviation of more than 100 KeV (Brown, 1987). What then should one expect from a theory of collective motion? On the one hand the Hamiltonian is much cruder. On the other hand, the configuration space is larger and we preselect more carefully the part of the spectrum to be studied. Tamura *et al.* consider energy levels agreeing within an average deviation of several hundred KeV and transition probabilities within a factor of two to be good agreement, but do much better for the most prominent collective states. Further, the trend with neutron or proton number will often be more impressive than absolute magnitudes. In what follows, we shall adopt these criteria. What is unambiguous is the relative merit (in purely numerical terms) of two competing theories. But, in fact, the only full competitor in the sense of being a microscopic theory with a similar starting Hamiltonian is the work of Kumar (Kumar and Baranger, 1967, 1968; Baranger and Kumar, 1968; Kumar, 1983, 1984). As explained in the Introduction, this theory falls outside the scope of the present work.

Turning to questions of detail, we show first some results for vibrational nuclei, for a series of Ru isotopes—in Fig. 6 some energy level comparisons, and in Table II,

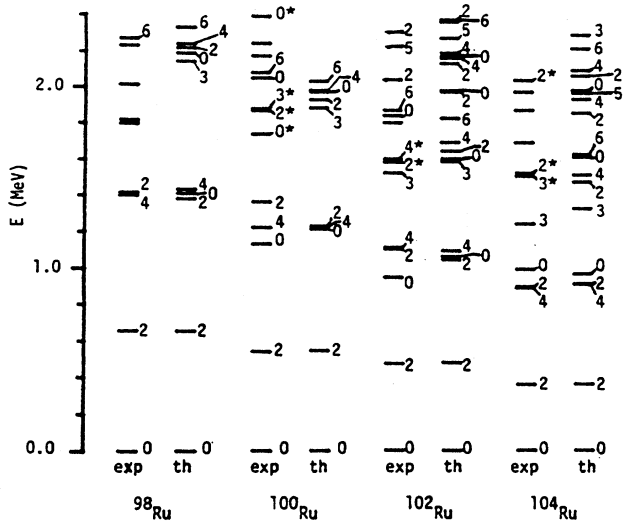

FIG. 6. Comparison of theoretical and experimental energy levels for $^{98-104}$Ru. Starred experimental levels are tentatively assigned. From Weeks and Tamura, 1980b.

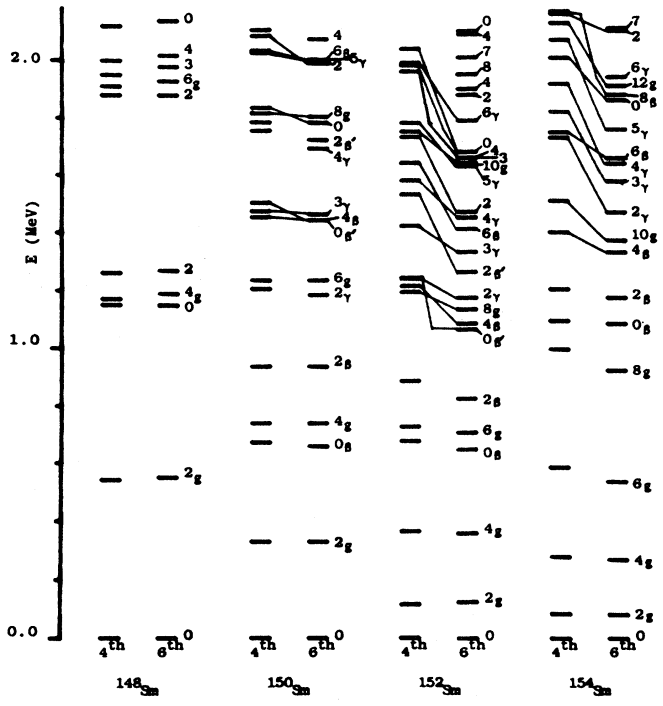

FIG. 5. Comparison of fourth- and sixth-order calculations for energy spectra of ^{148,150,152,154}Sm. Only the spins of the sixth-order spectra are labeled. The corresponding fourth-order levels, which differ significantly in energy or in ordering, are connected by straight lines to the corresponding sixth-order levels. From Tamura, Weeks, and Kishimoto, 1979.

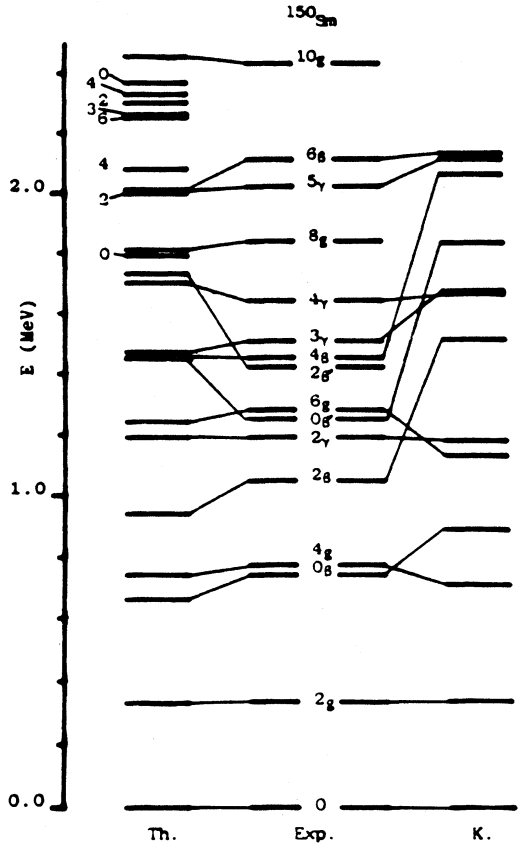

FIG. 7. Comparison of the sixth-order spectrum (Th) with experiment (Exp.), as well as with the result of Kumar (1973) (K) for ^{150}Sm. Only the experimental levels are labeled and are connected to corresponding levels in the theoretical spectra. Theoretical levels are also labeled when there is no counterpart in experiment. From Tamura, Weeks, and Kishimoto, 1979.

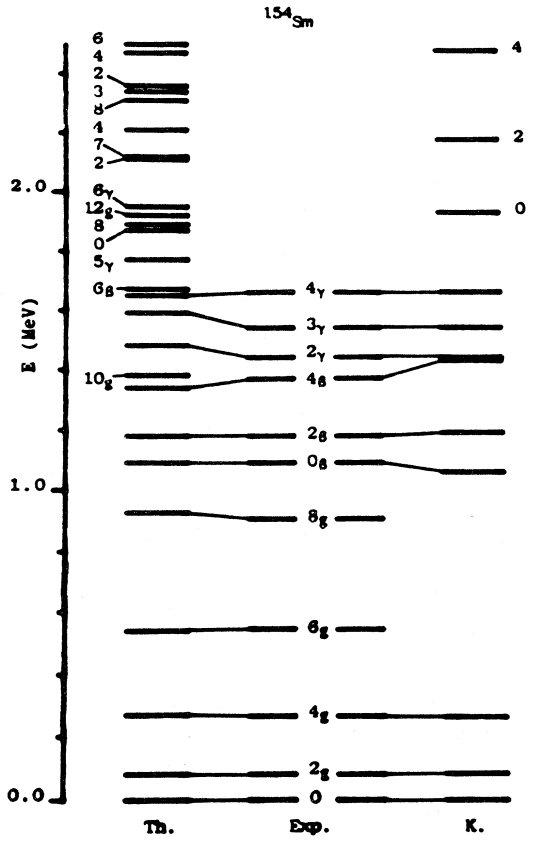

FIG. 8. Same as Fig. 7, except that this is for ^{154}Sm. From Tamura, Weeks, and Kishimoto, 1979.

$B(E2)$ values and quadrupole moments. Most of the latter agree within 20%, and the energies satisfy the criterion set above. Nevertheless it is worth pointing out that the relative ordering and splitting of the two-phonon levels is not given correctly. (These splittings are generally within 100 keV.) This at least suggests a very challenging problem, namely, to formulate a theory of collective motion so that at least the lowest-lying levels can be described with greater accuracy.

Turning to the Sm isotopes, we limit ourselves to two comparisons. In Fig. 7, one sees an energy level diagram for ^{150}Sm compared with boson expansion theory and the theory of Kumar. The same comparison is made for ^{154}Sm in Fig. 8. For the latter case, which concerns a well-deformed nucleus, both theories give excellent agreement, whereas in the former case, which involves a transitional nucleus, the boson expansion theory agrees rather better with experiment. The differences are not nearly as apparent when one examines the quadrupole transition probabilities and quadrupole moments, for which we refer the reader to the original paper. In any event, because it involves an adiabatic assumption, one expects the Kumar work to be most accurate for well-deformed nuclei. What remains somewhat of a mystery is why the boson expansion theory works so well in this limit.

The boson expansion theory has also proved successful in the interpretation of the properties of the γ-unstable isotopes of Os and Pt. In Table III, the energy levels of several isotopes of Os are presented. The energies of the ground-state band up to angular momentum eight are quite accurately given, those of excited bands less so. In Fig. 9, a selected set of branching ratios is compared with

TABLE II. $B(E2;I_i\rightarrow I_f)$ in units of 10^{-2} e^2b^2, quadrupole moments in units of eb, and magnetic moment g_R factor in units of μ_N, for $^{98-104}$Ru. An asterisk after the theoretical $B(E2)$ denotes that the matrix element is negative. From Weeks and Tamura, 1980b.

	^{98}Ru		^{100}Ru		^{102}Ru		^{104}Ru	
Transition	Exp.	Th.	Exp.	Th.	Exp.	Th.	Exp.	Th.
$2_1\rightarrow 0_1$	8.0±0.6	8.1	10.4±0.7	1.0	13.0±0.9	12.7	16.4±1.2	15.7
$4_1\rightarrow 2_1$	10.77±1.22	14.0	14.6±1.5	17.1	20.1±3.8	21.3*	21.7±0.4	25.1*
$6_1\rightarrow 4_1$		17.9*		21.5		25.7*		29.6*
$8_1\rightarrow 6_1$		20.2*		23.9		26.2*		31.5
$2_2\rightarrow 2_1$	14.7±2.5	11.0	9.10±1.55	12.0	11.7±1.5	14.5	12.3±1.9	9.9
$2_2\rightarrow 0_1$	0.15±0.02	0.21	0.37±0.05	0.33	0.33±0.04	0.40	0.55±0.06	0.69
$3_1\rightarrow 2_1$		0.46*		0.71*		0.82		1.20*
$2_3\rightarrow 0_1$		0.07*		0.12*		0.13*		0.09
$4_2\rightarrow 2_1$		0.13*		0.15		0.19		0.18*
$0_2\rightarrow 2_1$		9.9	9.55±1.45	11.6	9.95±1.45	12.7*	7.6±0.8	9.4*
$0_3\rightarrow 2_1$		0.51		0.55*		1.3*		2.3*
$3_1\rightarrow 4_1$		3.8		4.2		5.0		3.9*
$Q(2_1)$	−0.33±0.14	−0.33	−0.43±0.07	−0.42	−0.57±0.07	−0.48	−0.76±0.19	−0.69
	−0.20±0.09	−0.43±0.07	−0.4±0.1		−0.66±0.05			
			−0.40±0.12		−0.37±0.12		0.70±0.08	
					−0.68±0.06		−0.84±0.21	
							−0.63±0.20	
$Q(2_2)$		0.23		0.31		0.38		0.60
$Q(4_1)$		−0.50		−0.62		−0.69		−0.90
$g_R(2_1)$	0.30±0.17	0.39	0.42±0.03	0.36	0.34±0.06	0.34	0.29±0.04	0.30
e_{eff}		0.96		1.06		1.22		1.13

experiment. Despite the logarithmic scale, what is impressive here is the correct trend with neutron number.

The Ge isotopes are particularly interesting because of the occurrence of an anomalously low first excited 0+ state, reaching its minimum for ^{72}Ge. To correctly describe this and other properties of these nuclei requires an extension of the boson expansion theory, as so far formulated. Here one must take due note of the occurrence of a pairing vibration, as described at the beginning of Sec. IV, and include in the Hamiltonian the coupling between the quadrupole degrees of freedom, and (according to Tamura *et al.*) the neutron pairing vibration. When the collective Hamiltonian is diagonalized in a space including the additional excitation, good things happen, as exemplified by Fig. 10, which shows two important features. The first is the convincing correlation between the neutron energy gap and the importance of the pairing vibrational degree of freedom. The second is that several simpler forms of the theory simply do not work.

We have given only selected examples of a very detailed set of analyses, to which we must refer the reader

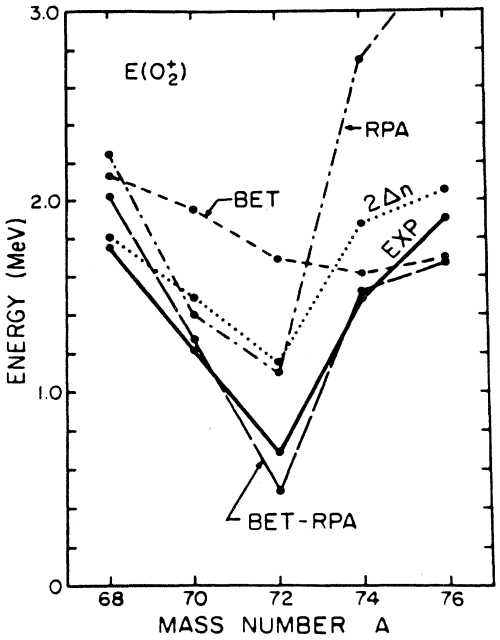

FIG. 10. Energies of the boson expansion theory: first excited 0^+ state (BET); the RPA neutron pairing vibration (RPA); twice the neutron gap ($2\Delta_N$); the coupled first excited 0^+ state (BET-RPA); and experiment for $^{68-76}$Ge. From Weeks, Tamura, Udegawa, and Hahne, 1981.

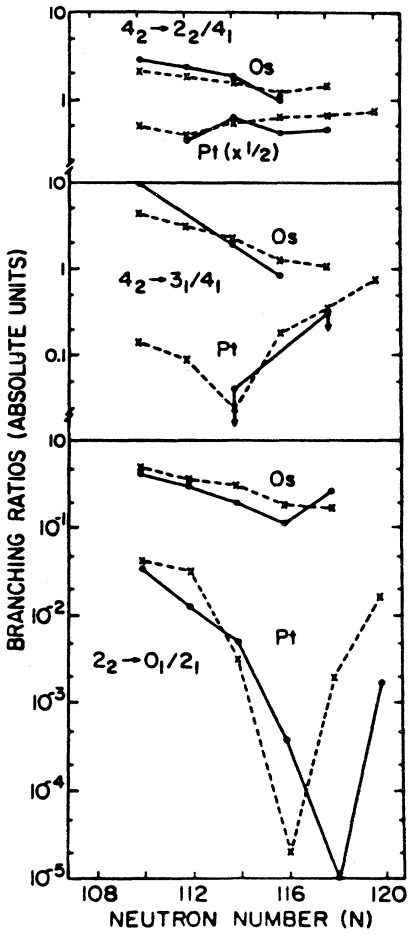

FIG. 9. Selected ratios of $B(E2)$'s. ×, theory; ●, experiment. Arrows indicate upper limits. From Weeks and Tamura, 1980c.

with a deeper interest in the subject. That the results are impressive is indubitable. Since boson expansion theory's natural domain is the description of vibrational nuclei, one must still wonder why it works so well for transitional and deformed nuclei. It is reasonable to surmise that the success of boson expansion theory for transitional and especially deformed nuclei can only be attributed to the *ad hoc* character of some of its elements. The incorporation of these elements, in particular, the strong-coupling phonon and the way ground-state correlations are included in the boson space, renders the connection with the shell model less than fully deductive at the present time. It was these puzzles that stimulated, at least in part, the work of Takada described in the next section.

XVI. APPLICATIONS OF THE GENERALIZED DYSON MAPPING. PROBLEM OF IDENTIFYING PHYSICAL SOLUTIONS

A. Introduction

Up to this point the generalized Dyson mapping has appeared as a convenient intermediate stop on the road to a unitary mapping. However, even the original authors of this mapping in nuclear physics (Janssen, Dönau, Frauendorf, and Jolos, 1971; Dönau and Janssen, 1973) gave serious thought to the practical application of the

mapping because of its superficial simplicity. Since then this beguiling method has attracted increasing attention. In this section, we first describe recent theoretical investigations concerned with implementing the Pauli principle in a practical manner. Such a discussion is long overdue, since it is germane to all mappings, Hermitian or non-Hermitian, though a number of treatments have emphasized the problem within the framework of Dyson mappings. We shall follow this discussion with a report on recent applications. But first we describe the theoretical problems that had to be addressed.

We have seen in Sec. XIV.A that, if one attempts to satisfy the requirements of the Pauli principle in a boson basis in too literal a sense, then one is simply reproducing all the technical difficulties of the original shell-model calculations, without any apparent gain. Let us consider a familiar example. In the shell-model space all pair creation operators represent independent excitation operators. However, the set of all products of such pairs creates a redundant basis in the four-particle space, because suitable antisymmetrization between pairs, as required will lead to linear dependencies. In the boson space, as we have noticed in the previous two sections, it is convenient not to worry about this problem. In general one introduces one boson per independent fermion pair. Therefore the full two-boson space will contain spurious elements. A number of authors have studied the four-particle shell-model problem by finding the eigenvalues of the Dyson-mapped Hamiltonian in the four-particle space (Ring and Schuck, 1977; Geyer and Lee, 1982). Two-particle/two-hole test cases have also been carried out (Ring and Schuck, 1974; Schuck, 1976; Shuck, Wittman, and Ring, 1976). In these investigations it was observed that the exact physical eigenvalues emerged unscathed (as verified by an exact shell-model diagonalization) but were accompanied by a set of spurious solutions that all occurred at the unperturbed energy. It has since been shown (Kim and Vincent, 1985) that this is a special property of the four-particle space and will not recur for larger numbers of particles. A warning was issued concerning the use of the four-particle test case for other theories involving redundant bases, such as that of Wu and Feng (1981). This difficulty can lead us back to the view that one should work with properly antisymmetrized states in the boson space (Gambhir, Nikam, *et al.*, 1985; Gambhir, Sheikh, *et al.*, 1985). Though such an approach is theoretically correct, we have made a reasonably strong case that it is not practical. For the remainder of the discussion we shall study the boson problem in a basis that has been chosen for technical con-

TABLE III. Energies in keV predicted by the boson expansion theory, with the corresponding experimental energies in parentheses for $^{186-194}$Os. From Weeks and Tamura, 1980c.

	^{186}Os	^{188}Os	^{190}Os	^{192}Os	^{194}Os
0_2	740(1061)	734(1086)	748(912)	677(956)	724
0_3	1167(1456)	1215(1478)	1117(1545)	964(1206)	1473
0_4	2038(1990)	1825(1704)	2053(1732)	1991	1521
0_5	2547	2497(1765)	2745	2603	2351
2_1	145(137)	156(155)	186(187)	206(206)	226(219)
2_2	671(767)	666(633)	607(558)	507(489)	601(656)
2_3	978	1011(1305)	1036(1114)	932	1082
2_4	1153	1243(1462)	1200(1435)	1048	1357
2_5	1328	1429(1807)	1269	1139	1816
2_6	1808	2079	2075	1980	1863
3_1	790(910)	853(790)	810(756)	732(690)	887
3_2	1287	1390(1620)	1290	1131	1659
4_1	449(434)	468(478)	551(548)	585(580)	606(601)
4_2	978(1070)	965(1012)	996(955)	908(909)	1040
4_3	1140(1195)	1271	1243	1143	1366
4_4	1321	1464(1279)	1383(1163)	1227(1070)	1521
4_5	1475(1321)	1541	1447	1296	1784
5_1	1180(1276)	1236(1181)	1246(1204)	1168(1140)	1357
5_2	1432	1685(1516)	1571	1418(1362)	1747
5_3	1688(1560)	1762	1653(1446)	1473	2134
6_1	886(860)	904(940)	1053(1050)	1091(1088)	1114
6_2	1249(1491)	1389(1425)	1391(1514)	1329(1360)	1558
6_3	1490	1669	1650	1533(1465)	1910
6_4	1943	1837	1977(1836)	1588	2023
7_1	1461	1627	1640	1579(1713)	1900
8_1	1345(1421)	1412(1514)	1623(1666)	1691(1708)	1726
10_1	1855(2069)	1987	2256	2345	2421

venience, what is often called the ideal boson basis.

We then ask how the Pauli principle can be satisfied within the chosen framework. Here we distinguish two cases. In the first, the boson basis is of sufficient size to include the entire physical subspace, but it contains as well spurious elements. For this problem, the literature contains three rigorous methods, which we shall describe below, for distinguishing the full complement of exact eigenstates of the Hamiltonian from the remaining unphysical solutions. However, this is not the case of prime interest. The more usual situation is the one in which the problem cannot be solved without some truncation of the Hilbert space. The cleanest examples of this occur in the recent works of Takada and his associates (Takada, 1985; Takada and Tazaki, 1986; Takada and Yamada, 1987; Tsukuma, Thorn, and Takada, 1987). Here one of the above methods (Takada, 1986) has been applied successfully, in that it has been established that the bases used in the above calculations, which are severely limited in size, are entirely nonspurious. (As we shall ultimately understand, this is by construction.) Thus there is at least one method applicable to both undercomplete and redundant bases.

A second method has been described by Geyer *et al.* (Geyer, Engelbrecht, and Hahne, 1986; see also Dobaczewski, 1981b; Hahne, 1981). This method has not yet been tested for "dirty cases," where the truncation corresponds at best to an approximately decoupled subspace. It is rigorous if the truncation corresponds to an exactly decoupled subspace, i.e., if a collective subalgebra of the original Lie algebra exists. The third method (Park, 1987) may be of limited value for the truncated case (Hahne, Geyer, and Engelbrecht, 1988). For a more optimistic approach, see Li (1991).

These methods are concerned with testing or sorting the results of a completed calculation. A different viewpoint has been developed by C. T. Li (1984b; see also Li, 1983, 1984a), who studies the problem of trying to salvage calculations made in a truncated space where an initial mixing of physical and spurious components has occurred. He describes an approximate method for projecting out the physical part of a state. The method is reminiscent of related methods for approximate projection of good particle number and angular momentum.

B. Theoretical methods for selecting physical states

We turn then to an account of the individual contributions. We start with the work of Takada (1986), but shall intermingle elements of the other methods. Let $H_{\rm HP}$ and H_D be the boson Hamiltonians in the Holstein-Primakoff Hermitian boson theory and the Dyson theory, respectively. If the Marumori mapping technique is employed, these are obtained in the form (see Sec. II).

$$H_{\rm HP}=P\tilde{H}_{\rm HP}P=\tilde{H}_{\rm HP}P,\quad H_D=P\tilde{H}_DP=\tilde{H}_DP\ , \tag{16.1}$$

where P is the projector onto the physical subspace, and the tilde operators are obtained by one of the algebraic methods described in earlier sections. Since P is technically unwieldly, it is discarded at this point, and in consequence one is immediately led back to the problem of spurious states.

Let $\{|i)\}$ be an orthonormal basis in the boson space. In the Hermitian boson theory, we need to consider only the eigenket $|\Psi_\lambda)$ belonging to an eigenvalue λ, whereas in the Dyson theory we have to consider both the eigenket $|\psi_\lambda)$ and the eigenbra $(\phi_\lambda|$. Expanding these eigenvectors as

$$|\Psi_\lambda)=\sum_i \alpha_i^{(\lambda)}|i)\ , \tag{16.2}$$

$$|\psi_\lambda)=\sum_i \beta_i^{(\lambda)}|i)\ , \tag{16.3}$$

$$(\phi_\lambda|=\sum_i \gamma_i^{(\lambda)*}(i|\ , \tag{16.4}$$

we thus encounter three eigenvalue problems,

$$\sum_j (h_{ij}^{\rm HP}-E_\lambda\delta_{ij})\alpha_j^{(\lambda)}=0\ , \tag{16.5}$$

$$\sum_j (h_{ij}^{D}-E_\lambda\delta_{ij})\beta_j^{(\lambda)}=0\ , \tag{16.6}$$

$$\sum_i \gamma_i^{(\lambda)*}(h_{ij}^{D}-E_\lambda\delta_{ij})=0\ , \tag{16.7}$$

where $h_{ij}^{\rm HP}=(i|\tilde{H}_{\rm HP}|j)$ and $h_{ij}^{D}=(i|\tilde{H}_D|j)$. In writing Eqs. (16.5), (16.6), and (16.7), we have begged one of the essential questions: apparently, we have assumed that all three problems yield the same set of eigenvalues. Actually it turns that that it does not matter whether or not they yield the same set of spurious eigenvalues, as long as one can identify these values. It is essential, however, that they yield the same physical spectrum.

At first sight, this does not appear to be a problem. We have only to remember the equation connecting the Hermitian and Dyson forms that we have utilized repeatedly for generators, namely,

$$\tilde{H}_{\rm HP}=S\tilde{H}_DS^{-1}\ , \tag{16.8}$$

where S is the unitarizing matrix that has been calculated for a number of cases, as in Secs. IX and X. Since the operators differ by a similarity transformation, they have the same spectrum. But now we must remember that the rules of the game have been changed. Equation (16.8) refers to an irrep of a Lie algebra, which means that we are dealing only with physical states. Now, with the hope of simplifying calculations, we have enlarged the space to include unphysical states in the boson space, and, as well, are utilizing the tilde Hamiltonian operators that may connect the physical and unphysical states in the enlarged space. With respect to the tilde operators, S will generally not be defined.

To help clarify the situation, it is useful at this point to turn to some observations of Geyer *et al.* (Geyer, Engelbrecht, and Hahne, 1986). Consider, for example, the ei-

genvalue problem (16.6) (the conclusions also apply to Eq. (16.7)]. Since the spectrum is independent of the choice of basis, let us choose one consisting of two orthonormal sets, $\{|i)\}$ belonging to the physical space (irrep) and $\{|\bar{i})\}$ belonging to the unphysical space. Let P and Q, $P+Q=1$, be the projection operators for the respective subspaces. Now $\tilde{H}_{\rm D}$ belongs by construction to the enveloping algebra of the Lie algebra associated with the irrep. Therefore $\tilde{H}_{\rm D}$ leaves the P space invariant. In general, as a result of its action on the Q space, there will be components in both subspaces. This means that the Hamiltonian matrix h_{ij} has the form

$$h=\begin{bmatrix} h_{PP} & h_{PQ} \\ 0 & h_{QQ} \end{bmatrix} . \tag{16.9}$$

[for the case of $\tilde{H}_{\rm HP}$, (16.9) simplifies further to a block-diagonal form. Appropriate modifications have then to be made in some of the statements below]. It follows that a subset of eigenvalues of dimensionality equal to that of the P space has associated eigenvectors confined to the latter space. The remaining "spurious" states may have components in both subspaces. For the physical subspace, Eq. (16.8) (or a suitable approximate form of it, if for some reason the P space is less than the full irrep) will guarantee that the physical eigenvalues will be the same irrespective of whether we use the Hermitian or the non-Hermitian calculation. It remains to specify how to sort the physical from the nonphysical solutions, since in practice we do not know how to choose a basis that leads to Eq. (16.9).

Toward this end, let us first consider the case in which all solutions are physical. Then we can use Eq. (16.8). If we apply that equation to Eqs. (16.5)–(16.7), assuming that S is a real, symmetric matrix, we can derive

$$\alpha_i^{(\lambda)}=S_{ij}\beta_j^{\lambda}=(S^{-1})_{ij}\gamma_j^{\lambda} , \tag{16.10}$$

or

$$\beta_i^{(\lambda)}=(S^2)_{ij}^{-1}\gamma_j^{(\lambda)} . \tag{16.11}$$

As shown by Dobaczewski (1981b), S, the square root of the norm matrix in the boson space, is the reciprocal of Z, the square root of the norm matrix in the fermion space. The latter matrix, associated in this way with a linearly independent basis in the fermion space, is nonsingular. As pointed out in Sec. XV, however, an extended Z matrix is also readily defined in a redundant fermion space; for this extended matrix the reciprocal connection with S is lost. As further noted, the zero eigenvalues of the extended Z matrix will identify the redundant states. Subsequently, a reduced norm matrix can be defined and the connection with S established. In fact, Takada and his associates use this method of excluding spurious states, so that they map only nonspurious states.

The approach just described provides a sound method for applying the Dyson mapping if one already has the necessary shell-model programs to perform the calculations. We may still ask if it is possible to solve the problem of identifying physical solutions entirely within the boson space. Perhaps the simplest way of separating the physical from the unphysical eigenvalues is by the method of Geyer *et al.* Let Θ be a physical operator distinct from the Hamiltonian, i.e., another operator in the enveloping algebra, chosen for convenience. Logical choices would include transition operators or a part of the Hamiltonian. Let $\{|\lambda)\}$ and $\{|\bar{\lambda})\}$ be the physical and unphysical eigenvectors, respectively. Then, since Θ leaves the P space invariant, it follows that $(\tilde{\lambda}|\Theta|\lambda)=0$ for fixed $\tilde{\lambda}$ and varying λ. We can thus identify and exclude the unphysical states $|\tilde{\lambda}\rangle$. As noted above, this method has not been used for "dirty" problems, but it certainly deserves a try.

There is still an element missing from both methods, as so far described. The method of Geyer *et al.* consists of the two steps: (i) solve Eq. (16.6) and (ii) select the physical solutions. In Takada's method, the order is reversed. What is missing is a prescription for calculating the matrix elements of physical operators other than the energy. To see the difficulty here, we note that the solution of the eigenvalue problems (16.6) and (16.7), considered jointly, provide a set of eigenvectors β_i^{λ} and an associated dual set $\gamma_i^{(\lambda)}$. These may be normalized according to

$$\gamma_i^{(\lambda)*}\beta_i^{(\lambda')}=\delta_{\lambda\lambda'} , \tag{16.12}$$

where only the orthogonality is *de rigeur*. But the relations (16.12) are invariant under compensating changes of scale, which we write as

$$|\bar{\psi}_\lambda)=k_\lambda|\psi_\lambda), \ (\bar{\phi}_\lambda|=k_\lambda^{-1}(\phi_\lambda| , \tag{16.13}$$

where the k_λ, k_λ^{-1} are chosen to be real. The correctly normalized states, which are assumed to be the barred states, are identified in principle by transforming from the unitary basis by means of the transformation S^{-1}, and consequently they satisfy Eq. (16.11).

To apply this observation, let O be the operator of interest and $O_{\rm HP}$ and O_D its representations connected by an equation similar to (16.8). By transforming the relation between the matrix of an operator and the matrix of its Hermitian conjugate, in a unitary basis, to the corresponding relation in a nonunitary basis, we find the equation

$$(\bar{\phi}_\lambda|O_D|\bar{\psi}_{\lambda'})=(\bar{\phi}_{\lambda'}|(O^\dagger)_D|\bar{\psi}_\lambda)^* . \tag{16.14}$$

Combining Eqs. (16.13) and (16.14) yields the relation

$$(k_\lambda/k_{\lambda'})^2=(\phi_\lambda|O_D|\psi_{\lambda'})/(\phi_{\lambda'}|(O^\dagger)_D|\psi_\lambda)^* \tag{16.15}$$

or

$$\begin{aligned}(\bar{\phi}_\lambda|O_D|\bar{\psi}_{\lambda'})&=[(\phi_\lambda|O_D|\psi_{\lambda'})(\phi_{\lambda'}|(O^\dagger)_D|\psi_\lambda)^*]^{1/2} \\ &\equiv(\Psi_\lambda|O_{\rm HP}|\Psi_{\lambda'}) .\end{aligned} \tag{16.16}$$

Thus we can calculate the observable matrix elements of any operator without having to know the "correct" normalizations for the non-Hermitian eigenvalue problem. Equation (16.16) was derived by several authors

(Gambhir and Basavaraju, 1979; Hahne, 1981; Li, 1983, 1984a).

But one can go further. If O is a Hermitian operator, then in Eq. (16.14) and sequel we may replace $(O^\dagger)_D$ by O_D. Moreover, the states need not be eigenstates of H but can be members of any pair of mutually dual bases, provided the basis states are eigenstates of the boson number operator (Takada, 1988). In this case we have the formula

$$(\bar{i}|H_D|\bar{j})=[(i|H_D|j)(j|H_D|i)^*]^{1/2} \equiv(i|H_{\rm HP}|j)\ , \tag{16.17}$$

so that the Hermitian Hamiltonian matrix can again be computed from the simpler matrix of the Dyson Hamiltonian taken between "simple" boson states. Since the diagonalization of this matrix will yield the unitary basis $|\Psi_\lambda)$ from which all other observables can be computed, this would appear to be a highly desirable approach, though not an indispensable one. This is, in fact, the preferred method of Takada *et al.*

We describe one additional element of the procedures actually utilized. Rewriting Eq. (16.13) in terms of components and inserting this into the square root of (16.15) gives

$$k_{\lambda\lambda'}\equiv\frac{k_\lambda}{k'_\lambda}=\left[\frac{\sum_{ij}\gamma'^{(\lambda)*}_i(i|O_D|j)\beta'^{(\lambda')}_j}{\sum_{ij}\gamma'^{(\lambda')*}_i(i|(O_D)^\dagger|j)\beta'^{(\lambda)}_j}\right]^{1/2}\ , \tag{16.18}$$

where the primed expansion coefficients refer to an arbitrary normalization of states. One now chooses a reference state $\lambda=0$ and writes $k_\lambda=k_0k_{\lambda0}$. From Eq. (16.11), one can then derive the equations

$$\sum_j(\tilde{Z}^{\,2})_{ij}\gamma'^{(\lambda)}_j=k^2_{\lambda0}\beta'^{(\lambda)}_i\ , \tag{16.19}$$

$$(\tilde{Z}^{\,2})_{ij}=(Z^2)_{ij}/k_0^2\ . \tag{16.20}$$

For fixed i and varying λ,j, these are a set of linear inhomogeneous equations for $(\tilde{Z}^{\,2})_{ij}$. Provided this matrix is nonsingular, the results of solving Eq. (16.19) can be used to rewrite the second form of (16.10) as

$$\alpha^{(\lambda)}_i=k^{-1}_{\lambda0}\sum_j\tilde{Z}_{ij}\gamma'^{(\lambda)}_j\ , \tag{16.21}$$

which should agree with the results obtained by diagonalizing Eq. (16.17). In the examples worked out by Takada and his associates and summarized below, this required concordance is achieved with high numerical accuracy. Actually, their method of choosing a basis, described in Sec. XVI.C below, guarantees this result, except for numerical errors. In practice, therefore, this procedure was a check on the numerical accuracy.

If $\tilde{Z}^{\,2}$ were singular, the last calculation described above would be useless, but precisely in that case, where there are redundant solutions, the method of Geyer *et al.* could be combined with the method associated with Eq. (16.17) to provide a tractable approach.

Before turning to the main applications, we first complete our account of the most recent theoretical contributions to the subject. We describe first the contribution of Park (1987). In the fermion space the operator (not to be confused with the previously occurring matrix with the same symbol)

$$\hat{S}\equiv\hat{N}^{\,2}_H-\hat{N}^{\,2}_{NH} \tag{16.22a}$$

is the zero operator, since

$$\hat{N}^{\,2}_H=\sum_{ij}a^\dagger_ia_ia^\dagger_ja_j \tag{16.22b}$$

and

$$\hat{N}^{\,2}_{NH}=\sum_i a^\dagger_ia_i-\sum_{ij}a^\dagger_ia^\dagger_ja_ia_j \tag{16.22c}$$

are different forms of the square of the number operator. The subscripts allude to the fact that one form remains self-adjoint under the Dyson mapping (9.10), (9.12), and (9.13), whereas the other does not. Under this mapping, $\hat{S}$ becomes

$$(\hat{S})_D=\hat{N}^{\,2}-\hat{N}-\hat{K}\ , \tag{16.23}$$

where

$$\hat{N}=\sum_{ik}b^\dagger_{ik}b_{ik} \tag{16.24}$$

is twice the fermion number,

$$\hat{K}=\sum_{ik}\mathcal{A}^\dagger_{ik}b_{ik}\ , \tag{16.25}$$

and $\mathcal{A}^\dagger_{ik}$ is defined in Eq. (9.14), in a slightly different notation.

The operator $(\hat{S})_D$ still vanishes in the physical boson space, but now can be applied to all boson states. If 2Ω is the number of single-particle levels, both $A^\dagger_{ik}$ and b_{ik} are tensors under U(2Ω), as emphasized in Sec. IX. Furthermore, $\hat{K}$ commutes with all the generators $N_{ij}=b^\dagger_{ij}b_{kj}$ and from its structure is necessarily a sum of $\hat{N}$ and the second-order Casimir invariant. In fact, rearranging $\hat{K}$ with the help of the commutation relations for the elementary bosons, we find that

$$(\hat{S})_D=(\hat{S})_B=\hat{N}^{\,2}-\hat{N}(2\Omega+1)+C_2\ , \tag{16.26}$$

where

$$C_2=\sum_{ik}N_{ik}N_{ki} \tag{16.27}$$

and can be calculated from the formula related to (9.16) (Perelomov and Popov, 1966)

$$\langle C_2\rangle_{[f]}=\sum_{i=1}^{2\Omega}f_i(f_i+2\Omega+1-2i)\ . \tag{16.28}$$

One can conclude from Eq. (16.28) that, for a given particle number, $\langle C_2\rangle$ has its minimum value for the an-

tisymmetric representation, and thus $(\hat{S})_B$ takes on expectation values larger than zero for spurious states, which may be mixtures of states belonging to any representation other than the antisymmetric one. Thus, in a calculation with a redundant basis, the operator $(\hat{S})_B$ would certainly serve to distinguish physical from unphysical states.

As pointed out by Hahne *et al.* (Hahne, Geyer, and Engelbrecht, 1988), when a truncation is made in the boson space, even if there is an invariant subspace under the Hamiltonian, the subspace will not be invariant under $\hat{S}$. The latter may be represented in the subspace by a multiple of the unit operator, but the multiple will not be zero and thus $\hat{S}$ in its approximate form will no longer clearly distinguish physical from unphysical states. Thus, to use Park's method, it appears that one cannot truncate the basis.

Earlier, C. T. Li (1984b) had suggested a use of the operator $\hat{S}$ that is not subject to the limitations stated above. Actually Li formulates the problem differently from any of the previous authors who considered the question: Given that the solutions fall into two sets, one consisting of states that are physical (or almost physical) and the other of states that are unphysical (or mostly unphysical), how does one identify the members of each set? Li takes for granted that we know the states that are largely physical. Assuming now that even the best of such states has some admixture of nonphysical components (this is not always true), he provides a technique for approximate projection of the physical part by a method analogous to approximate number projection (Li, 1983a). A description of Li's method follows. (As formulated, this method applies to number-conserving situations.)

Let $|\phi_n\rangle$ be an eigenstate of H_D for n bosons. We want to construct $P|\phi_n\rangle$, where P is the projector onto the physical subspace. It is not difficult to show (Li, 1984b) that P can be written formally as a power series in the operator $\hat{K}$, defined in Eq. (16.25). In the following we shall utilize the expectation value of this operator for the antisymmetric representation,

$$\langle \hat{K} \rangle = 2n(2n-1) . \tag{16.29}$$

We then consider a νth approximant,

$$P_\nu = \sum_{q=0}^{\nu} C_q(n) \hat{K}^q , \tag{16.30}$$

in the n-boson space and determine the coefficients $C_q(n)$ from the conditions

$$(\phi_n|\hat{K}^r P_\nu|\phi_n) = [2n(2n-1)]^r , \tag{16.31}$$

which, according to the properties of the operator $\hat{S}$, would be exact conditions if P_ν were replaced by P. For example, for the case $\nu=1$, we get two equations from Eq. (16.31) that have the solution

$$C_0 = \frac{\langle \hat{K}^2 \rangle - 2n(2n-1)\langle \hat{K} \rangle}{\langle \hat{K}^2 \rangle - \langle \hat{K} \rangle^2} , \tag{16.32}$$

$$C_1 = \frac{2n(2n-1) - \langle \hat{K} \rangle}{\langle \hat{K}^2 \rangle - \langle \hat{K} \rangle^2} , \tag{16.33}$$

where

$$\langle \hat{K}^q \rangle = \langle \phi_n | \hat{K}^q | \phi_n \rangle . \tag{16.34}$$

Finally, Li envisions that if $|\phi_n\rangle$ is actually a trial function with some free parameters, $P_\nu|\phi_n\rangle$ provides an improved trial function for a variational calculation of the associated energy.

The theoretical method described above was applied to the standard pairing model with single-particle energies and pairing force strength appropriate to the isotopes of Ni and of Sn, respectively. Ground-state energies and single-particle occupation probabilities compare favorably with the results of exact diagonalization. We refer the reader to Li's paper for details.

C. Applications

The most extensive set of applications of some of the above ideas has been carried out by Takada and his associates, who, in each application, have used the method described above [starting with Eq. (16.18)] to verify that their basis is nonspurious (never mentioning that this is guaranteed). This is the first such verification in realistic applications to collective motion. In three out of four initial applications the agreement with experiment is "promising," but (to be discussed) there is one acknowledged failure. This has led to a second round of work (Takada, Yamada, and Tsukuma, 1989; Yamada and Takada, 1989; Yamada, Takada, and Tsukuma, 1989), intended to cure some of the acknowledged deficiencies of the previous efforts. Below we give a brief account of these works.

The standard schematic collective Hamiltonian,

$$H = H_{sp} + H_{P0} + H_{P2} + H_{QQ} , \tag{16.35}$$

a sum of single-particle, monopole-pairing, quadrupole-pairing and quadrupole-quadrupole interactions, is utilized. The quasiparticle representation is then introduced. [The problem of spurious states associated with number nonconservation has been dealt with, especially in Takada and Tazaki (1986), and results in some modification of the formulas presented below. We refer the reader to the literature for these modifications.] If $A_M^{(2)\dagger}(ab)$ is the quadrupole pair formed from quasiparticle levels a and b, the most collective linear combination

$$X_{2M}^\dagger = \sum_{ab} \psi(ab) A_m^{(2)\dagger}(ab) \tag{16.36}$$

is determined from a Tamm-Dancoff calculation, and the collective subspace is limited, either to the space generated by these phonons or, as physical requirements suggest, to additional elements to be specified individually later.

The excitation (16.36) is, however, a uniform ingredient of all applications.

The collective space is thus the orthonormalized space of direct product states obtained from the operation of Eq. (16.36) upon the vacuum. In practice this construction is carried out as follows: Let us define the n-phonon (fermion) state vector with total spin (JM) and additional quantum number β as

$$|n\beta JM\rangle\rangle=\frac{1}{\sqrt{n}}\sum_{\beta_1 J_1}(d^{n-1}(\beta_1 J_1)d|\}d^n\beta J)\sum_{M_1M_2}\langle J_1M_1 2M_2|JM\rangle X^\dagger_{2M_2}|n-1\ \beta_1 J_1 M_1\rangle\rangle\ , \tag{16.37}$$

where $(d^{n-1}(\beta_1 J_1)d|\}d^n\beta J)$ is the d-boson ($J=2$) coefficient of fractional parentage. These multiphonon state vectors are not necessarily orthonormal, nor are they all necessarily linearly independent. (The double-bracket notation for these states matches the notation of the previous section.) To find an orthonormal set, one diagonalizes the norm matrix (called Z^2 in previous discussions),

$$\sum_{\beta'}M^{(n,J)}(\beta,\beta')u^{(n,J)}(\beta',\gamma)=M(nJ\gamma)u^{(n,J)}(\beta,\gamma)\ , \tag{16.38}$$

$$M^{(n,J)}(\beta,\beta')=\langle\langle n\beta J|n\beta' J\rangle\rangle\ . \tag{16.39}$$

We must have eigenvalues $M(nJ\gamma)\geq 0$. If there are any zero eigenvalues, these are excluded from the following construction. We can then define orthornormalized basis vectors as

$$|n\gamma JM\rangle=M^{-1/2}(nJ\gamma)\sum_\beta u^{(n,J)}(\beta,\gamma)|n\beta JM\rangle\rangle\ , \tag{16.40}$$

which satisfy the orthonormality relations

$$\langle n\gamma JM|n'\gamma' J'M'\rangle=\delta_{nn'}\delta_{\gamma,\gamma'}\delta_{JJ'}\delta_{MM'}\ . \tag{16.41}$$

Next, one defines the *orthonormalized* n-boson state vector $|n\beta JM))$ corresponding to the n-phonon state vector (16.37),

$$|n\beta JM))=\frac{1}{\sqrt{n}}\sum_{\beta_1 J_1}(d^{n-1}(\beta_1 J_1)d|\}d^n\beta J)\sum_{M_1M_2}\langle J_1M_1 2M_2|JM\rangle b^\dagger_{2M_2}|n-1\ \beta_1 J_i M_1))\ , \tag{16.42}$$

where $b^\dagger_{2M}$ is a d-boson creation operator. The n-boson state vector (16.42) is just the counterpart of the n-phonon state vector (16.37). Similarly, the counterpart of the orthonormalized n-phonon state vector (16.40) is defined by

$$|n\gamma JM)=\sum_\beta u^{(n,J)}(\beta,\gamma)|n\beta JM))\ , \tag{16.43}$$

which is also an orthonormalized n-boson state vector.

It is time to explain why these constructions have been carried out. It is to provide the necessary elements for the application of the modified Marumori mapping as described in Sec. XV. To obtain the Dyson form, Takada utilizes a Usui mapping, as discussed for example by Tamura (1983c). We shall not describe this construction because it is unnecessary for the derivation of the mapping, which can be obtained by algebraic means alone, as indicated below. What is important to notice concerning the construction described above is that, as long as the Hamiltonian is diagonalized in the nonsingular space of states (16.42), nonspuriosity (with respect to the Pauli principle) is guaranteed, and therefore the check based on the calculation of the Z^2 matrix outlined in connection with Eqs. (16.19) and (16.20) appears superfluous.

The previous discussion suggests a possible alternative method of calculation in which no use whatsoever is made of Eqs. (16.38) and (16.39): One utilizes the basis (16.42) in the boson space, computes H_D from the mapping (obtained by purely algebraic means below), calculates the Hermitian matrix from Eq. (16.17), and checks for physical solutions by the method of Geyer *et al.*

The approximate Dyson mapping needed in the present work can be derived by using the approximate algebra obtained by truncating to the collective space [keeping only $J=2$ and replacing $A^{(2)\dagger}_m(ab)\rightarrow\psi(ab)X^\dagger_{2M}$]. One then arrives at the approximate algebra

$$[X_{2M_1},X_{2M_2}]=\delta_{M_1M_2}+10(-1)^{M_2}\sum_{abc}\psi(ab)\psi(ca)\sum_J(-1)^J\begin{Bmatrix}J & 2 & 2\\ j_a & j_b & j_c\end{Bmatrix}\langle 2M_1 2-M_2|2M\rangle B^{(J)}_M(bc)\ , \tag{16.44}$$

$$[X_{2M_1},B^J_M(ab)]=2\sqrt{5(2J+1)}\sum_c\psi(ac)\psi(bc)(-1)^{j_a+j_c-J}\begin{Bmatrix}2 & 2 & J\\ j_a & j_b & j_c\end{Bmatrix}\langle 2M_1JM|2M_2\rangle X_{2M_2}\ , \tag{16.45}$$

$$[X_{2M_1},[X_{2M_2},X^{\dagger}_{2M_3}]]=-2\sum_{L=0,2,4} C_L\langle 2M_1 2M_2|LM\rangle\langle 2M_3 2M_4|LM\rangle X_{2M_4}\ , \tag{16.46}$$

where

$$C_L=50\sum_{abcd}\psi(ab)\psi(cd)\psi(ac)\psi(bd)\begin{Bmatrix} j_a & j_b & 2\\ j_c & j_d & 2\\ 2 & 2 & L\end{Bmatrix}. \tag{16.47}$$

It is to be emphasized that these relations do not define a Lie algebra. We can then apply the commutation relations order by order to derive the mapping

$$(X^{\dagger}_{2\mu})_D=b^{\dagger}_{2\mu}-\frac{1}{\sqrt{5}}\sum_L C_L\sqrt{2L+1}[[b^{\dagger}_2\otimes b^{\dagger}_2]_L\otimes b_2]_{2\mu}\ , \tag{16.48}$$

$$(X_{2\mu})_D=b_{2\mu}\ , \tag{16.49}$$

$$(B^{(2)\dagger}_{\mu}(ab))_D=10\sum_c(-1)^{j_a+j_c}\psi(ac)\psi(bc)\times\begin{Bmatrix}2 & 2 & 2\\ j_a & j_b & j_c\end{Bmatrix}[b^{\dagger}_2\otimes b_2]_{2\mu}\ , \tag{16.50}$$

$$(B^{\dagger}_{00}(aa))_D=\frac{2}{\sqrt{2j_a+1}}\sum_c\psi^2(ac)\sum_{\mu}b^{\dagger}_{2\mu}b_{2\mu}\ , \tag{16.51}$$

$$(A^{(2)\dagger}_{\mu}(ab))_D=\psi(ab)(X^{\dagger}_{2\mu})_D\ , \tag{16.52}$$

$$(A^{(2)}_{\mu}(ab))_D=\psi(ab)(X_{2\mu})_D\ . \tag{16.53}$$

Turning to results, in the first paper Takada (1985) applies the above formulas and method of calculation to the yrast bands of ^{74}Se, ^{114}Cd, and ^{126}Xe. No comparison with experiment is offered, but three methods of calculation are compared. First, a space of fixed maximum phonon number is chosen. The three calculations are direct diagonalization (in the corresponding fermion space), the Dyson method being tested, and a so-called SU(6) approximation, developed previously by Takada and his associates (Marumori, Takada, and Sakata, 1981; Takada, Kaneko, Sakata, and Tazaki, 1981; Takada, 1984), which is a form of the modified Marumori method (Lie and Holzwarth, 1975). We reproduce the results for ^{114}Cd, which are typical. Figure 11 shows the energies and Fig. 12 the ratio $B(E2;\ J+2\rightarrow J)/B(E2;2^+_1/0^+_1)$. On the basis of the results found, the authors register optimism concerning the basic validity of the method. The calculations utilize up to seven phonons, and it is verified that the results have converged for this number.

In Takada and Tazaki (1986) the formalism is extended to include pairing vibrations, so that the results may be applied to the famous low-lying 0^+ states in the Se and Ge isotopes which had previously been treated, among other ways, by the modified Marumori mapping (Iwasaki, Marumori, Sakata, and Takada, 1976; Sakata, Iwasaki, Marumori, and Takada, 1978; Tazaki, Takada, Kaneko, and Sakata, 1982; Weeks, Tamura, Udagawa, and Hahne, 1981); the last of these papers was described in the previous section. The collective space consists of the Tamm-Dancoff quadrupole phonon and pairing phonons for neutrons and protons. With the standard Hamiltonian, the experimental energies and quadrupole transitions can be reproduced with somewhat less precision than in the test examples previously discussed. In Figs. 13 and 14 the $E0$ transition elements between the first excited 0^+ and the ground state are shown for the two sets of isotopes. In each case there is one scale factor $\delta e_0=0.6e$. This paper, as well as the succeeding one, have the unpleasant feature that experimental and theoretical spectra are plotted in different figures, and for that reasons we have not reproduced the results.

Takada and Yamada (1987) extended the basic formalism once more to include the odd particle and applied it to the odd-mass isotopes of Rh, where the lowest single-particle state is $0g_{9/2}$ and the ground state is uniformly $\frac{7}{2}+$ (anomalous $J=j-1$ ground state) followed by $J=\frac{9}{2}+$ first excited states. The $J=\frac{7}{2}+$ arises mainly from the one-phonon, one-particle mixture, but only in the sense that the effects of two-phonon and three-phonon admixtures, which are non-negligible, cancel each other. The resemblance between the remainder of

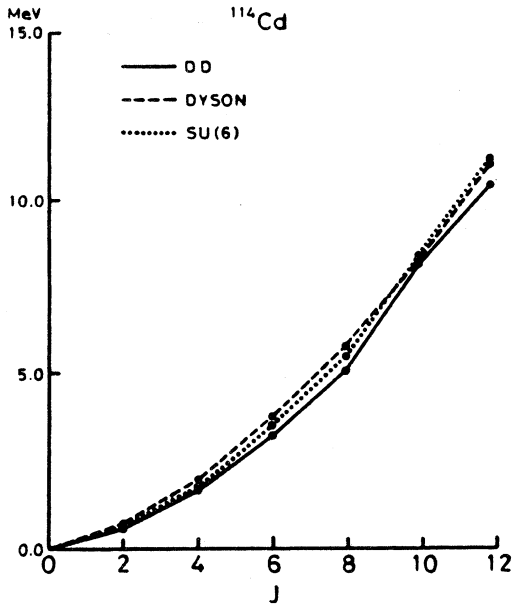

FIG. 11. Calculated energy eigenvalues of the even-spin yrast states in ^{114}Cd with the quadrupole force parameter $\chi_0=430$ MeV: solid line, direct diagonalization in the multiphonon subspace; dashed line, the Dyson boson theory; dotted line, the SU(6) model. From Takada, 1985.

the theoretical spectrum and the experimental spectrum is not impressive. The formalism for odd nuclei has recently been applied, as well, to the problem of the coupling of the quadrupole and pairing vibrational degrees of freedom (Yamada and Takada, 1988).

Takada regards all the previous works as promising. The one acknowledged failure (Tsukuma, Thorn, and Takada, 1987) is an attempt to describe the famous phase transition in the Sm isotopes, taking into account the coupling between the most collective 2+ Tamm-Dancoff boson and a set of noncollective bosons, in which the included basis states contain at most one of these noncollective excitations. Without the latter, theory and experiment are widely divergent. With the latter one does better, but the spectra remain too spread out. This phenomenon had, in fact, already been noted by Tamura (see Sec. XV), and he got around it in two ways: first, he chose his collective boson as the Tamm-Dancoff boson in the strong-coupling limit, where all single-particle levels are degenerate. This should and apparently does enhance the collectivity. Approximate inclusion of RPA correlations also has a major effect, perhaps the chief effect. Thus it appears that the straightforward method based on an "honest-to-goodness" Tamm-Dancoff boson is not sufficient for the problem at hand, no matter which specific boson mapping is adopted, and additional recipes are necessary, within the linear boson approach, to obtain agreement with experiment. The omission of g bosons may also be of some importance in this respect.

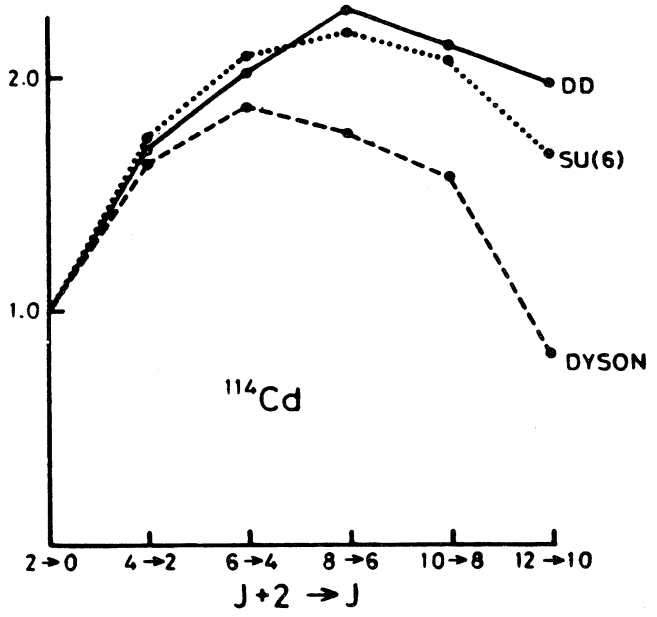

FIG. 12. Calculated values of the ratio $B(E2;J+2\rightarrow J)/B(E2;2_1^+\rightarrow 0_1^+)$ among the even-spin yrast states in ^{114}Cd: solid line, direct diagonalization in the multiphonon subspace; dashed line, the Dyson boson theory; dotted line, the SU(6) boson model. From Takada, 1985.

In the very latest work, the problem of the shape transition from spherical to quadrupole, observed in the Sm isotopes, is attacked with renewed vigor. This work is based on methods developed in several formal contributions (Shimizu and Takada, 1987; Takada and Shimizu, 1987; Takada, 1988). In the last of these papers, it is shown how an approximate Dyson mapping can be carried out when the most collective Tamm-Dancoff boson is replaced by the most collective RPA boson, further modified in order to be able to satisfy algebraic constraints imposed by the nature of the mapping. This procedure is dynamically equivalent to finding the lowest mode of the RPA subject to an additional constraint that in effect extends the range of validity of the theory beyond the usual point of breakdown, where the phase transition occurs.

There are two elements of this procedure that are worthy of special notice. The first concerns the modification of the RPA itself. Let $\mathbf{x}_\lambda$, $\mathbf{y}_\lambda$ be the usual

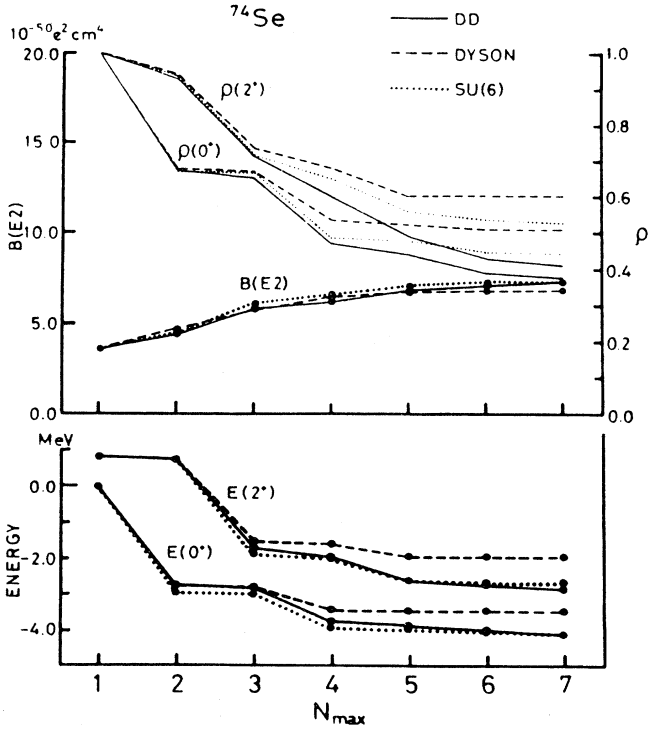

FIG. 13. Calculated energy eigenvalues of the 0_1^+ and 2_1^+ states and calculated values of $B(E2;2_1^+\rightarrow 0_1^+)$ in ^{74}Se: solid lines, direct diagonalization in the multiphonon subspace; dashed lines, the Dyson boson theory; dotted lines, the SU(6) boson model. The abcissa (N_{max}) denotes the maximum number of phonons (or bosons) in the truncated multiphonon (or boson) subspace. The quantity $\rho(0^+)$ denotes the square of the absolute value of the amplitude of the zero-phonon (or zero-boson) component contained in the 0_1^+ state, and the quantity $\rho(2^+)$ denotes that of the one-phonon (or one-boson) component contained in the 2_1^+ state. From Takada and Tazaki, 1986.

"forward-going" and "backward-going" amplitudes composing a solution of the RPA belonging to an eigenvalue ω_λ, and let $\lambda=0$ designate the most collective solution. In place of these amplitudes, we introduce coordinates and momenta according to the standard equations

$$\mathbf{x}_\lambda=\sqrt{1/2}(\sqrt{\omega_\lambda}\,\mathbf{q}_\lambda+i\sqrt{1/\omega_\lambda}\,\mathbf{p}_\lambda)\ , \tag{16.54a}$$

$$\mathbf{y}_\lambda=\sqrt{1/2}(-\sqrt{\omega_\lambda}\,\mathbf{q}_\lambda+i\sqrt{1/\omega_\lambda}\,\mathbf{p}_\lambda)\ . \tag{16.54b}$$

In the standard RPA these amplitudes are subject to the usual quasiboson normalization condition

$$\mathrm{Tr}[\mathbf{p}_\lambda^\dagger\mathbf{q}_{\lambda'}]=\tfrac{1}{2}\delta_{\lambda\lambda'}\ . \tag{16.55}$$

In the modified RPA, the trick is to replace the frequency of the lowest mode, which passes through zero (on the way to imaginary values) at the phase transition, by another parameter s, according to the equation

$$\omega_0\rightarrow s^2\ . \tag{16.56}$$

The parameter s is now to be chosen so as to extend the validity of a modified RPA formalism beyond the phase transition. In earlier work, Matsuo and Matsuyanagi (1985) had introduced the *ad hoc* condition

$$s^4=\frac{\mathrm{Tr}[\mathbf{qq}^\dagger]}{\mathrm{Tr}[\mathbf{pp}^\dagger]}\ . \tag{16.57}$$

A slightly different condition appears in a more natural fashion in the work under discussion. The aim here is to introduce an approximate Dyson mapping using bosons associated with the modified RPA mode. However, the conventional linear and trilinear structure of this mapping cannot be derived unless one can justify an approximate closed algebra of the structure recorded in Eqs. (16.44)–(16.46). In particular, it turns out that the analog of Eq. (16.46) can be achieved (only approximately in the general case) provided one chooses

$$s^4=\frac{\mathrm{Tr}[\mathbf{qp}^\dagger\mathbf{qq}^\dagger]}{\mathrm{Tr}[\mathbf{qp}^\dagger\mathbf{pp}^\dagger]}\ . \tag{16.58}$$

The result of this effort is an approximate Dyson mapping of the most collective mode including ground-state correlations. This improvement on the corresponding mapping of Tamm-Dancoff bosons turns out not to yield a sufficiently improved agreement with experiment. Some coupling to noncollective coordinates is necessary. A convenient way to include these effects within a framework that retains the ground-state correlations and that is something akin to the Dyson structure of the mapping is to utilize a formalism developed in Shimizu and Takada (1987) and Takada and Shimizu (1987). We refer the reader to these papers for details.

This formation has been applied to Sm (Takada, Yamada, and Tsukuma, 1989; Yamada and Takada, 1989; Yamada, Takada, and Tsukuma, 1989). Noncollective modes with $J=0,2,4$ were included. The overall agree-

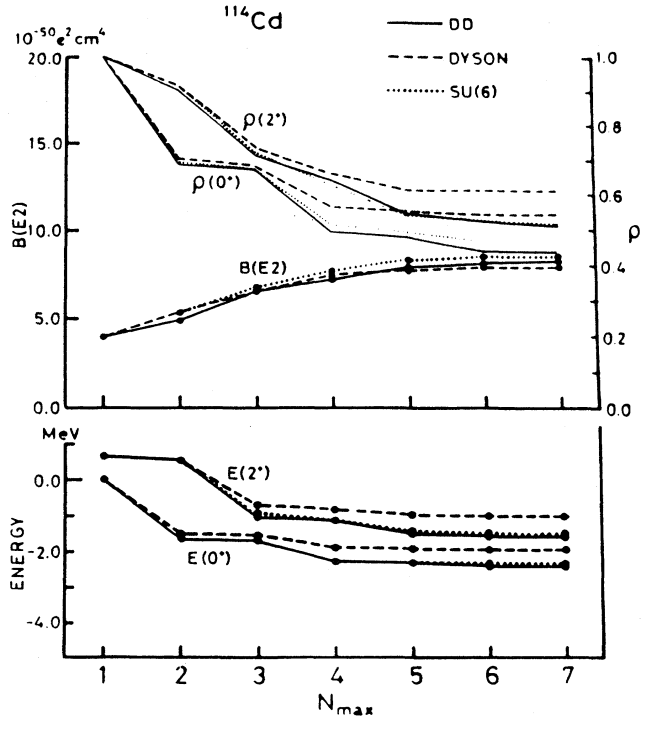

FIG. 14. Calculated energy eigenvalues of the 0_1^+ and 2_1^+ states and calculated values of $B(E2;2_1^+\rightarrow 0_1^+)$ in ^{114}Cd. See caption for Fig. 13. From Takada and Tazaki, 1986.

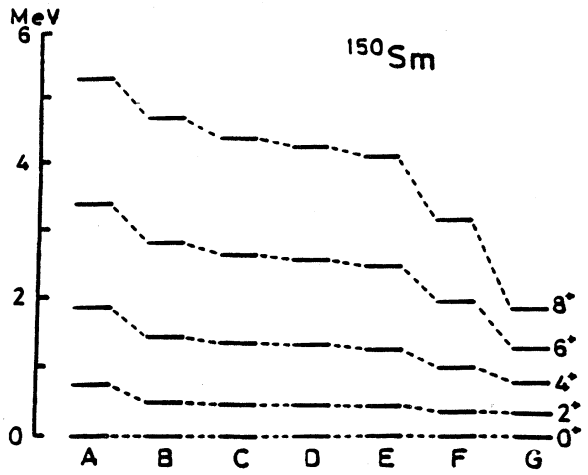

FIG. 15. Yrast levels of ^{150}Sm: *A*, calculated results with the Dyson boson mapping based on the quadrupole Tamm-Dancoff phonons; *B*, the first-order SCC method; *C*, results including the coupling effect arising from the $J=0$ (pairing-vibrational), noncollective phonon degrees of freedom; *D*, results including coupling effect from $J=2$ noncollective phonon degrees of freedom; *E*, results including coupling from $J=4$ degrees of freedom; *F*, result including all these coupling effects; *G*, the experimental data. From Yamada, Takada, and Tsukuma, 1989.

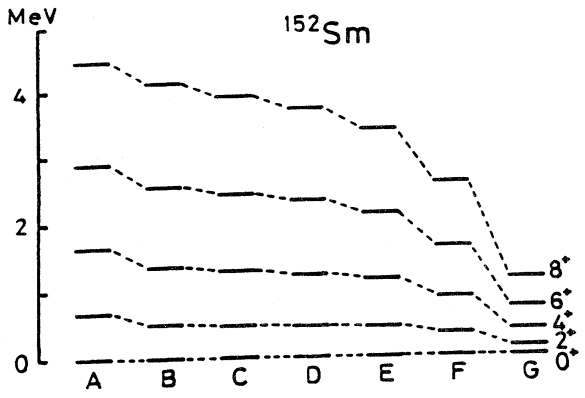

FIG. 16. Yrast levels of ^{152}Sm. For further explanation see caption for Fig. 15. From Yamada, Takada, and Tsukuma, 1989.

ment of energy levels and $B(E2)$ rates for the ground-state band was greatly improved over the previous results of this group. Fits are, in fact, impressive, except for ^{152}Sm, where the fluctuations associated with a phase transition may be at a maximum. The energy levels found by Yamada, Takada, and Tsukuma (1989) for ^{150}Sm and ^{152}Sm by a succession of approximations are compared with experiment in Figs. 15 and 16.

The application of the Dyson mapping to an RPA-like model has also been carried out for several exactly solvable models (Takada, Shimizu, and Thorn, 1988; Thorn, Shimizu, and Takada, 1988). The authors view these papers as contributions to the theory of large-amplitude collective motion, a subject discussed in Sec. XXI. The method applied is valid, however, only for the domain of anharmonic vibrations, and therefore complements the work described in Sec. XXI. We refer the reader to the literature cited for details. There have been some efforts, not without their interest, to apply the Dyson mapping to the derivation of the interacting boson model. These will be described in Sec. XVIII, which is devoted to this subject.

D. Other recent work

The study of nuclear structure using equations of motion methods related to Heisenberg matrix mechanics has had some currency in nuclear physics, mainly in the approach of Rowe (1970), Kerman and Klein (1962), and Klein (1983a, 1984b), but the bulk of this work has been only indirectly connected with bosons. Such efforts as exist to use the equations of motion in connection with unitary boson mappings have been described tersely in Sec. IV.C. Recently Pannert and Ring (1987) undertook to apply an equation of motion method after carrying out a Dyson mapping. Since this work is in its infancy, we confine ourselves to a qualitative description of its main elements.

(i) In the first step, the general Dyson mapping described in Sec. IX is carried out.

(ii) The second step is a nonlinear polynomial transformation to new bosons that are assumed to contain collective varieties among them. This generalizes the usual linear transformation that is utilized in most of the applications reported in this review. A restriction to cubic terms identifies the class of applications as anharmonic vibrations. Another restriction adopted by these authors, but not discussed by them, is the condition that the creation and annihilation operators of the collective modes be Hermitian conjugates. This implies that a similarity transformation to a unitary basis has been incorporated into the procedure.

(iii) A collective Hamiltonian is defined as the restriction of the full Hamiltonian to the collective subspace. In practice this means that, after the Hamiltonian is expressed in terms of the new bosons, terms containing the noncollective bosons are dropped. To obtain a well-defined problem, this has to be done before normal ordering, though at the same time this implies that some quantum fluctuations are being neglected.

(iv) Pannert and Ring describe this last step in a somewhat different fashion. They simply assume the existence of a collective Hamiltonian of the required form. Equations for the parameters of the collective Hamiltonian are then obtained by requiring that the full and collective Hamiltonians give the same equations of motion for the collective bosons. This is enforced by equating coefficients of the same operator polynomial, yielding nonlinear equations that determine the unknown parameters.

(v) Diagonalization of the collective Hamiltonian will yield the collective states. There is no discussion of other observables, but these can be studied by methods described earlier in this section.

The method described above was applied to the pairing Hamiltonian, considered as a model for the ground state of the even isotopes of Ni, Sn, and Pb. A second method of using the equations of motions, essentially equivalent to the mean-field method described below, was also applied. Rather impressive results were found for the ground-state energies up to midshell for the Ni and Sn cases by the use of a single collective boson, the equivalent of the Cooper pair. In the Pb case, however, the nonlinear equations have no solution unless a second mode is included.

We next report on a related project (Cambiaggio and Dukelsky, 1987) that in practice involves the application of the pairing Hamiltonian to the Sn isotopes. The first main difference from the work of Pannert and Ring is that after the Dyson mapping these authors recognize that, since they work in a nonunitary basis, there may be some advantage to maintaining the distinction between the bra and ket bases in their second transformation, which is, however, required to be linear (a second difference). The ground state is then taken to be a number-conserving coherent state, but, as stated, the col-

lective boson creation and annihilation operators, though canonically conjugate, are not Hermitian conjugate. Thus the bra and ket descriptions of these states remain distinct. To obtain these states, one applies the variational principle. One has here a non-Hermitian version of the boson mean-field method described in Sec. XIV, and, as stated above (without proof), one that is methodologically equivalent to one of the methods applied by Pannert and Ring. Some improvement in the results of the calculation are achieved by including a second boson obtained by a Tamm-Dancoff calculation based on the coherent ground state. The results of this paper play a role in the work of Dukelsky and Pittel (1989) described at the end of Sec. XIV.

Another recent work (Sheikh, 1988a) describes an algorithm for performing diagonalizations in the boson space after the usual Dyson mapping, utilizing an antisymmetric basis and thus avoiding the problem of spurious states. Though tentative claims for its utility in a general dynamic situation are put forth, its effectiveness is established only by solving once more the pairing problem for Ni and for Sn. We mention finally an investigation of diagonalization of a Dyson-mapped pairing plus quadrupole-quadrupole Hamiltonian (Vincent, Henry, and Kim, 1988). It was concluded that any truncation has an adverse influence both on the accuracy of the partial spectrum and on one's ability to separate mainly spurious states from those that are mainly physical. These authors emphasize, for example, that the physical states in the practical works described in this section and in the prior one cannot be spuriosity free, simply because they are based on approximate algebras that violate the Pauli principle. In a related investigation, Navrátil and Dobeš (1989) have applied the Dyson mapping to a system of four neutrons and four protons, each confined to a single-j shell and described by a standard schematic Hamiltonian. They show that the spurious eigenstates found by diagonalizing a Dyson-transformed Hamiltonian are so numerous at low energies as to render the procedure worthless. They observe, however, that application of a form of the seniority-dictated boson mapping described at length in the next section gives good results for the model near the good-seniority limit. This agrees with results to be described.

XVII. NUMBER-CONSERVING MAPPINGS

A. Introduction

In the previous sections it has been demonstrated that the BZM mapping can be applied to both vibrational and rotational physics provided that one introduces suitable quasiparticles from the beginning and maps the algebra of quasiparticle pairs and multipoles rather than the algebra constructed from the original shell-model fermion operators. As we have seen, this brings in from the beginning the problem of spurious states associated with symmetry breaking, and methods for dealing with these had to be developed. With the advent of the interacting boson model, attention was addressed to a new problem, that of finding useful mappings of number- and angular-momentum-conserving algebras. The present section is devoted to this problem.

Why is it that a perturbative boson expansion of the BZM type fails (or is likely to fail in any event) for such a problem? It was emphasized in Sec. IX that the BZM angular-momentum-coupled expansion has a small parameter, Ω^{-1}, the average level degeneracy, but in fact the small parameters are $(\langle \hat{n}_J \rangle / \Omega)$, where $\langle \hat{n}_J \rangle$ is an average occupation number of bosons carrying angular momentum J. For a spherical superconducting nucleus, we have $(\langle \hat{n}_0 \rangle / \Omega) \sim 1$. For this reason, we introduce quasiparticles, redefining the vacuum so that the inequality is satisfied if $\langle \hat{n}_J \rangle$ is the quasiparticle occupation number. For deformed nuclei, moreover, $\langle \hat{n}_2 \rangle$ becomes large, and we are forced to introduce deformed quasiparticles.

If we wish to eschew quasiparticles and the attendant formalism of broken symmetry, then the only alternative is to resum the large terms, so that for spherical superconducting nuclei the operator $(\hat{n}_0/\Omega)$, where $\hat{n}_0$ is the number operator for s $(J=0)$ bosons, does not appear in expanded form. It turns out that this is not difficult to do, at least in principle. The progress made on this problem to date will be the main topic treated in this section. On the other hand, when d-boson $(J=2)$ occupation effects become of comparable importance to s occupation effects, the *a priori* expectation is that the method described, which distinguishes the s boson from all others, will break down. A different approach to the deformed limit has been adopted, provisionally, that utilizes the symplectic shell-model algebra (L-S coupling) rather than the usual j-j coupled shell-model algebra. Our discussion of this case will omit the technical details because they are repetitive.

B. Method of Bonatsos, Klein, and Li applied to seniority-determined mapping

The physical basis for the method of this section will now be described. For the sake of simplicity, we provide the details only for the case of a single j level and at the end quote the work on the multilevel case. We shall rely in this subsection on the most extensive corpus of results in this area (Klein, Li, Cohen, and Vallières, 1983; Bonatsos, Klein, and Li, 1984; Bonatsos and Klein, 1987; Bonatsos, Peres Menezes, and Klein, 1987, 1988), obtained by an approach that was conceived as a variant of the commutator method.

Consider the angular momentum algebra as recorded in Eq. (9.49) specialized to a single value of j. In order to quote correct results, we note a change in the scale of the operators used by Bonatsos *et al.* (1984) compared to the notation of Sec. IX. These authors define

$$A^\dagger_{JM}=\sqrt{1/2}\sum_{mm'}(jjmm'|JM)a^\dagger_{jm}a^\dagger_{jm'}\ , \tag{17.1a}$$

$$B_{JM}=(2J+1)^{-1/2}\sum_{mm'}(jjmm'|JM)(-1)^{j+m'}a^\dagger_{jm}a_{jm'}\ . \tag{17.1b}$$

In the BZM mapping, essential use is made of the fact that there is a U($2j+1$) subalgebra, that the total fermion basis chosen is a direct sum of irreps of U($2j+1$), and that the mapping of the generators of this subgroup is "trivially" discovered. In the present case we utilize in like fashion the fact that there is an SU(2) subalgebra made up of the $J=0$ operators. We shall construct the fermion basis as the direct sum of irreps of this subalgebra. Of course, this is just the pairing algebra, and the basis is none other than a seniority basis (DeShalit and Talmi, 1963). Assuming that one maps onto a seniority basis in the boson space, which has as a subspace that of the Holstein-Primakoff mapping described in Sec. II, the mapping of the SU(2) subalgebra is fixed from the start and, as will be seen, sets restrictions equivalent to the Wigner-Eckart theorem on the mapping of the remaining generators. The flavor of the reasoning below is also conditioned by a desire to avoid the preliminary detailed construction of a fermion basis. Finally, we shall refer to Eq. (9.49) rather than record the commutation relations anew.

We impose one further restriction, which is physically rather than algebraically motivated, but which turns out to be algebraically sound. If the Hamiltonian were purely of the pairing type, then, as we know, we would need to consider only the SU(2) subalgebra, mapped by a single boson $(a^\dagger_0,a_0)$. We believe on good experimental grounds that near the seniority limit the main perturbing effect on seniority conservation comes from $J=2$ effects. We shall be looking for a mapping that sums $J=0$ effects but that is an expansion in Ω^{-1} otherwise. Can we satisfy the commutation relations or a subset of them to a given order in Ω^{-1} with only a limited number of other bosons? In particular, we first ask how far we can get with $J=0,2$ only, i.e., s and d bosons.

Thus we introduce six bosons, a scalar, $a^\dagger_0$, and a tensor of rank two, $a^\dagger_{2\mu}$. The destruction operator a_0 is also a scalar and, as is well known, $\tilde{a}_{2\mu}=(-1)^\mu a_{2-\mu}$ transforms like $a^\dagger_{2\mu}$. The associated number operators are $\hat{n}_0$ and $\hat{n}_2$. Following the procedure outlined above, we *impose* an exact solution for the SU(2) subalgebra, namely (N is the fermion number),

$$\hat{N}=\sqrt{2j+1}B^{(0)}_0=2\hat{n}_0+2\hat{n}_2\ , \tag{17.2}$$

$$A^\dagger_0=a^\dagger_0\sqrt{r}=(A_0)^\dagger\ , \tag{17.3}$$

$$r=1-(\hat{n}_0+2\hat{n}_2)/\Omega\ . \tag{17.4}$$

The generalization of these expressions to include bosons of other angular momentum is obvious. Next we consider the possible form for the $J=2$ pair operators, $A^\dagger_{2\mu}$. In general, this will involve an expansion, of which for the moment we consider only the leading terms. In order to construct the appropriate expression, we must specify tensor properties of the bosons, just as we did in Sec. IX. We first assign to $a^\dagger_{2\mu}$, in addition to its angular momentum properties, the property that it increase the seniority v by two units, i.e., it definitely adds two nucleons that are not coupled to $J=0$. On the other hand, $A^\dagger_{2\mu}$ has, as is well known (Lawson and MacFarlane, 1965), the selection rule $\Delta v=\pm 2,0$. To leading order we have the form

$$A^\dagger_{2\mu}=a^\dagger_{2\mu}f_1(n_0,n_2)+(a^\dagger_0)^2\tilde{a}_{2\mu}\tilde{f}_1(n_0,n_2)+a^\dagger_0[a^\dagger_2\otimes\tilde{a}_2]_{2\mu}f_2(n_0,n_2)\ . \tag{17.5}$$

In detail, this expression is determined by the following restrictions:

(i) Each term increases the fermion number by two.

(ii) Each term is a tensor of rank two under rotations.

(iii) Each term carries a definite seniority selection rule ($\Delta v=+2,-2,0$, respectively). Basically, the boson decomposition transforms the vectorial selection rules of seniority into a series of algebraic selection rules.

(iv) Terms with more than two d bosons are omitted. Here it is assumed that each d boson carries with it a factor $\Omega^{-1/2}$ (buried in the f functions) and thus we have the two leading terms in an expansion in $\Omega^{-1/2}$, or rather an expansion in powers of $(\hat{n}_2/\Omega)^{1/2}$.

We shall now briefly outline the further steps by which the functions f are determined. The method is analogous to that described in Sec. VII for the seniority SO(5) mappings. From the vanishing commutator (equivalent to applying the Wigner-Eckart theorem)

$$[A^\dagger_0,A^\dagger_{2\mu}]=0\ , \tag{17.6}$$

we obtain difference equations that determine the relations

$$f_1(n_0,n_2)=\sqrt{r}\sqrt{r-(1/\Omega)}\phi(n_2)\ , \tag{17.7}$$

$$\tilde{f}_1(n_0,n_2)=\tilde{\phi}(n_2)\ , \tag{17.8}$$

$$f_2(n_0,n_2)=\sqrt{r}\ \psi(n_2)\ , \tag{17.9}$$

which define the functions ϕ, $\tilde{\phi}$, and ψ. Next, using the commutator

$$[A_0,A^\dagger_{2\mu}]=10\begin{Bmatrix}0&2&2\\j&j&j\end{Bmatrix}B_{2\mu}\ , \tag{17.10}$$

involving a $6j$ symbol, we discover that the form of $B_{2\mu}$ is also determined by Eqs. (17.7)–(17.9). Furthermore the condition $B_{2\mu}=(-1)^\mu B^\dagger_{2-\mu}$ fixes

$$\tilde{\phi}(n_2)=-\Omega^{-1}\phi(n_2-1)\ . \tag{17.11}$$

We summarize the results found thus far by the more explicit expressions

$$A^{\dagger}_{2\mu}=a^{\dagger}_{2\mu}\sqrt{r}\,\sqrt{r-(1/\Omega)}\phi(n_2)-\frac{1}{\Omega}(a^{\dagger}_0)^2\tilde{a}_{2\mu}\phi(n_2-1)+a^{\dagger}_0[a^{\dagger}_2\otimes\tilde{a}_2]_{2\mu}\sqrt{r}\;\psi(n_2)\;, \tag{17.12}$$

$$10\begin{Bmatrix}0 & 2 & 2\\ j & j & j\end{Bmatrix}B_{2\mu}=-\frac{2}{\Omega}a^{\dagger}_{2\mu}a_0\sqrt{r}\;\phi(n_2)-\frac{2}{\Omega}a^{\dagger}_0\tilde{a}_{2\mu}\sqrt{r+(1/\Omega)}\phi(n_2-1)+[a^{\dagger}_2\otimes\tilde{a}_2]_{2\mu}[r-(n_0/\Omega)]\psi(n_2)\;. \tag{17.13}$$

To determine the remaining unknown functions ϕ and ψ, we have at our disposal the commutator

$$[A_{2\mu},A^{\dagger}_{2\nu}]=\delta_{\mu\nu}+10(-1)^{\mu}\sum_{J=0}^{4}\sqrt{2J+1}(22-\mu\nu|J\nu-\mu)\begin{Bmatrix}2 & 2 & J\\ j & j & j\end{Bmatrix}B_{J\nu-\mu}\;. \tag{17.14}$$

To use this expression we first drop the $J=3,4$ terms on the right. This is consistent with the approximation being described, since these terms play a role only in the next approximation. We already know B_0 [Eq. (17.2)] and the form of $B_{2\mu}$. The $B_{1\mu}$ operators close by themselves under commutation (angular momentum algebra) and are given by

$$B_{1\mu}=-\sqrt{1/3}\frac{\begin{Bmatrix}1 & 1 & 1\\ j & j & j\end{Bmatrix}}{\begin{Bmatrix}1 & 1 & 1\\ 2 & 2 & 2\end{Bmatrix}}[a^{\dagger}_2\otimes\tilde{a}_2]_{1\mu}\;. \tag{17.15}$$

For the rest, the information contained in Eq. (17.14) must be extracted with care. Equation (17.14), as approximated, is a linear combination of tensors of ranks zero, one, and two under rotations. The tensors of rank two are either linear in $a^{\dagger}_{2\nu-\mu}$ (or $\tilde{a}_{2\nu-\mu}$) or quadratic in these operators. From the linear dependence, we find by comparing both sides of Eq. (17.14),

$$\psi(n_2)=-\frac{2\sqrt{5}}{\Omega}\frac{\begin{Bmatrix}2 & 2 & 2\\ j & j & j\end{Bmatrix}}{\begin{Bmatrix}0 & 2 & 2\\ j & j & j\end{Bmatrix}}\frac{1}{r_2}\;, \tag{17.16}$$

$$r_2=1-\frac{2n_2}{\Omega}=r+\frac{n_0}{\Omega}\;. \tag{17.17}$$

This result is sound because, when we study the structure of higher-order corrections, we can show that further contributions to the commutator (17.14) linear in the d boson, will not occur. On the other hand, the coefficient of $[a^{\dagger}_2\otimes\tilde{a}_2]_{2\nu-\mu}$ will be modified, and therefore, to avoid inconsistencies, we must draw no conclusions from this term at the present level of approximation. Finally, from the spherically symmetric part of the commutator, we learn in a consistent order (omitting inconsistent higher-order terms),

$$\phi(n_2)=[r_2-(1/\Omega)]^{-1/2}\;. \tag{17.18}$$

We thus conclude that, to the given order, we have found a consistent solution involving only s and d bosons. The question that arises naturally is how we should utilize the result. We shall postpone this discussion until after we have given a summary of the extended results found by this method.

The next step is thus to calculate $A^{\dagger}_{2\mu}$ to the next order in $(n_2/\Omega)^{1/2}$, i.e., to include terms cubic in the d bosons. The methods have been established, but the details are considerably longer and more tedious to carry through. A major surprise (in view of the existing experience with the Marumori method, see below) was that a consistent solution could not be found unless g bosons $(a^{\dagger}_{4\mu},a_{4\mu})$ were included linearly in the expressions for $A^{\dagger}_{2\mu}$. More properly put, there occurs a major difference from the results of the Marumori-Yamamura-Tokunaga method, but one that should not have been a surprise after all: once seniority conservation is broken, all the angular momenta are coupled in products and commutators and must successively be brought in. We refer the reader to the literature (Bonatsos, Klein, and Li, 1984) for the details and results of this laborious calculation.

After the original single-j calculation, a calculation was carried out for two nondegenerate j shells with $|j_1-j_2|=2$ up to the same order of approximation as for the single-j case (Bonatsos and Klein, 1987). Both the requisite number of g bosons (3 for two levels) and an f boson ($J=3$) were required to obtain a consistent approximate mapping. Thus in this case as in the previous one, the mapping has been established to two orders in $(n_2/\Omega)^{1/2}$ or $(n_4/\Omega)^{1/2}$ beyond the leading order. However, for multilevel cases, interesting bosons with angular momentum other than zero and two already appear in leading order (Bonatsos, Peres Menezes, and Klein, 1987, 1988). For example, the physically interesting case $j_1=j_2-1=j_3-3$, the last shell having parity opposite to the other two, was analyzed in some detail because of the occurrence of negative-parity bosons, in particular, octupole bosons.

The results of this section for the single-j case have been applied recently (Peres Menezes, Yoshinaga, and Bonatsos, 1989). Spectra and quadrupole transition rates obtained from several approximate mappings were compared with the results of exact shell-model calculations. Since the mappings discussed in the next subsection were included in the comparisons, it is appropriate to delay further discussion until these have been described.

For the realistic case of a multi-j shell model, there is, in addition to the kinematic mapping, an associated dynamic problem, that of determining the *collective* $s, d, \ldots$ degrees of freedom in accordance with the philosophy of the interacting boson model. Here one possibility is the use of the trace variational principle, as de-

scribed in Sec. IV.B, or some nonlinear generalization thereof. Equivalently one may study the equations of motion in the boson space. So far, however, almost all semiplausible attempts other than those discussed in Sec. XIV have first done some dynamics in the fermion shell-model space followed by a mapping of only the collective degrees of freedom. These will be discussed in Sec. XVIII on the derivation of the interacting boson model.

C. Application of the Marumori, Yamamura, and Tokunaga (OAI) method

The paper that officially inaugurated the problem of using number-conserving mappings to derive the interacting boson model was that of Otsuka *et al.* (Otsuka, Arima, and Iachello, 1978). Though presented as a new mapping method, referred to in the literature as the OAI method, this paper uses essentially the approach of Marumori, Yamamura, and Tokunaga, with an additional prescription for the mapping of the Hamiltonian, that described in connection with Eq. (14.20). (A mapping of operators is defined so that the matrix elements of the boson image in the physical boson space equal the matrix elements of the corresponding fermion operators in the fermion space.)

A particularly noteworthy feature of the OAI paper, which confines itself to a single j shell, is that the structure of the lowest-lying, i.e., lowest-seniority states is elucidated, with particular attention paid to the method of constructing states of definite seniority involving operators carrying $J=2$. The difficulty here arises from the fact that $A^{\dagger}_{2\mu}$ can increase, decrease, or not change the seniority, as we have learned. Therefore a projection operator P is defined such that $PA^{\dagger}_{2\mu}$ only increases v by two. This means that this operator is proportional to $a^{\dagger}_{2\mu}$ when the latter is viewed as an operator in the fermion basis, a concept we have explained in the context of the SU(2) and SO(5) models. As described in the next paragraph, the OAI result in any order is simply related to the corresponding approximation obtained by the commutator method. For $j=\frac{29}{2}$ a comparison was made between multiparticle matrix elements of various operators calculated numerically in the fermion basis and their approximate values obtained by using the lowest-order boson mapping. This was done in order to justify the optimism of using the lowest-order images to construct a mapped Hamiltonian. The latter is guaranteed, according to the Marumori-Yamamura-Tokunaga method, to give exact matrix elements in a small subspace, but it is then used for diagonalizations in a considerably larger space, a procedure that is justified for small values of $(\hat{n}/\Omega)$.

We call attention to another distinction found in the literature, that between the OAI and OAIT expansions (Otsuka, Arima, Iachello, and Talmi, 1978). We can explain this distinction most simply by referring to Eqs. (17.12), (17.13), (17.17), and (17.18). The point is that the various scalar operators that we have been at pains to calculate, the so-called Pauli reduction factors, which are partial resummation effects, depend in general on $\hat{n}_0$ and $\hat{n}_2$. This identifies our result with the OAIT form. We can also rework the expansion by substituting everywhere $\hat{n}_0=\hat{n}-\hat{n}_2$ and subsequently expand the various square roots in powers of $(\hat{n}_2/\Omega)$. The resulting reduction factors depend only on $\hat{n}$. This is the OAI result. From a theoretical point of view, the OAI result is more consistent as an expansion, since all dependency on the d boson other than in the number operator has perforce appeared in power-series form. There is no *a priori* way of deciding which form will yield more accurate results. The distinction is of some conceptual importance because only the OAI form strictly leads to an interacting boson model Hamiltonian.

It should be remarked that the extension of the OAI or OAIT method to multi-j situations, which we would view as the parallel development to the commutator method described above, has not been carried out. Instead, further developments have involved preliminary dynamics in the fermion space, so that generally only a single boson of each included angular momentum is ultimately mapped to the boson space. This approach will be described in some detail in Sec. XVIII.

Actually, the first discussion of the Marumori-Yamamura-Tokunaga method for the seniority mapping was undertaken for a rather different reason (Li, Dreizler, and Klein, 1970, 1971). It was to discuss the problem of the apparent nonconvergence of the Marumori "expansion" even for the angular-momentum-coupled form. The purpose and results of this paper have already been discussed in Sec. XV.

We return to a discussion of the work of Peres Menezes *et al.* (1989). They study the standard pairing plus quadrupole-quadrupole interaction Hamiltonian,

$$H=-xA^{\dagger}_0A_0-5(1-x)B_2\cdot B_2\ , \tag{17.19}$$

where x varies from 1 (spherical limit) to 0 (deformed limit), and the operators are defined by Eq. (17.1) (the second term representing the standard scalar product in spherical components). In the work based on the commutator method, referred to below as the Bonatsos-Klein-Li approach, the point of view is taken that the image of the Hamiltonian is to be calculated directly from the image of the generators to a consistent order in the small parameters. In the OAI approach, as previously explained, a separate image of H is obtained directly by the Marumori-Yamamura-Tokunaga method. The difference in the two approaches can be understood by imagining that we evaluate a given matrix element of Eq. (17.19) by introducing a sum of intermediate states between the pair of generators. The OAI method, in effect, includes all possible intermediate states, so that it contains a selective sum of higher-order contributions. Whether this improves or worsens results is not *a priori* clear.

What emerges from a computation is that both kinds of mappings work well in the "vibrational" regime near

the seniority limit and neither works well in the deformed region. As an example, in Fig. 17, computed spectra are shown for the case $j=\frac{23}{2}$, six identical particles (three bosons), and $x=0.7$, near but not too near the seniority limit. Every binding energy has been pushed up by $15(1-x)/\Omega$, and the energy scale is arbitrary. The results in the various columns are: (a) exact shell model; (b) Bonatsos-Klein-Li (solid lines) *s-d* approximation and Suzuki-Fuyuki-Matsuyanagi (dotted line), defined as the same mapping of generators but the OAI mapping of the Hamiltonian (Suzuki, Fuyuki, and Matsuyanagi, 1979a); (c) *s-d-g* Bonatsos-Klein-Li; (d) OAI; (e) Zirnbauer and Brink (1982; see Sec. XVIII). Except for a small compression of the spectrum, the results of (c) are perhaps the most impressive. On the other hand, a corresponding plot for $x=0.3$, not reproduced, shows that all the seniority approximations fail in the deformed regime, but that (c) fails not disastrously.

A related calculation (Yoshinaga, 1989) involves a study of the Hamiltonian (17.19) with the OAI mapping, the object being a comparison of the *s-d* approximation with an *s-d-g* approximation. Qualitative conclusions are similar to those discussed above, the *s-d-g* approximation yielding a clearly superior result.

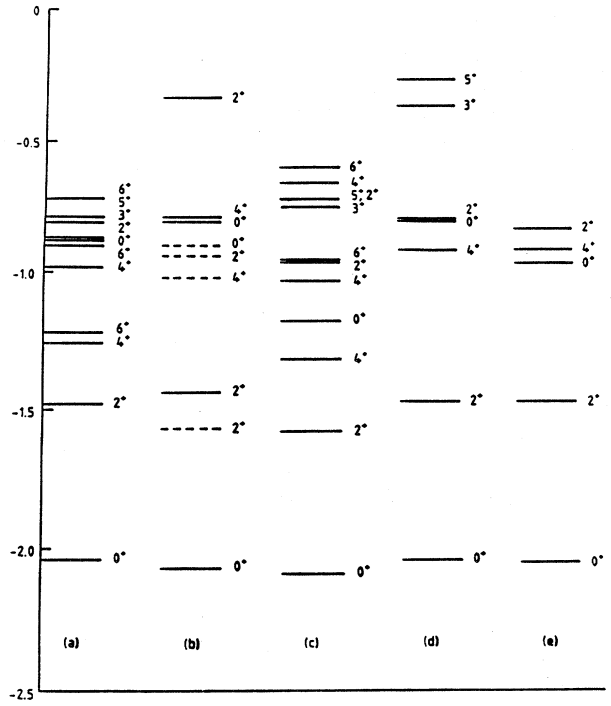

FIG. 17. Spectra of $j=\frac{23}{2}$, three bosons (six particles) and $x=0.7$ for (a) shell model; (b) *s-d* Bonatsos-Klein-Li (solid line) and Suzuki-Fuyuki-Matsuyanagi (dashed line); (c) *s-d-g* BKL; (d) Otsuka-Arima-Iachello and (e) Zirnbauer-Brink methods. Every binding energy is pushed up by $15(1-x)/\Omega$. The spins are calculated up to 6. The unit of energy is arbitrary. From Peres Menezes, Yoshinaga, and Bonatsos, 1989.

D. Li's method. Other methods

C. T. Li (1983) has developed a method for transforming the Dyson mapping into the OAI form of the seniority-based mapping. (Recall that the general method for unitarizing the Dyson mapping discussed previously transforms it to the BZM mapping.) We describe the procedure for the single-*j* case without giving detailed formulas. The first step is to map a standard pairing Hamiltonian, such as was studied in Sec. IV, utilizing the full Dyson mapping (9.12) and (9.13) in angular-momentum-coupled form. It is to be emphasized that under this mapping the representation of $A_0^\dagger$ contains pairs of all angular momenta and that therefore the mapped Hamiltonian is far from trivial, although it is limited to terms at most quartic in the bosons. Under this mapping, furthermore, the basis of kets is complicated, since it is determined by Eq. (9.13), whereas the corresponding basis of bras, determined by Eq. (9.12), is simple.

The relative simplicity of the basis of bras remains even after one introduces states with good angular momentum. This last point is one of the two secrets for the success of the calculation. It turns out that the eigenfunctions of seniority and angular momentum zero and of seniority and angular momentum two are readily computed in analytic form by diagonalizing the mapped Hamiltonian in the mapped bases.

The second ingredient of success is to apply Eq. (16.16) in order to obtain the matrix elements of the pair and multipole operators between states of low seniority in a *unitary* basis. The remainder of the calculation is then the same as the OAI one in that each matrix element found can be identified as a coefficient (reduction factor) in a boson expansion of the type we have described.

This result has not been extended fully to the multilevel case. As a first step in this direction, Li (1983) maps the approximate algebra defined by a single Cooper pair and a single Tamm-Dancoff quadrupole phonon. He leaves open whether the expansion coefficients are to be determined by a calculation in the fermion space or by one in the boson space.

Several other exact or approximate equivalents of the single-*j* results have been derived in the literature. For example, Li (1984a) starting with boson expansion theory, defined in Sec. XV, which (we remind the reader) operates in a quasiparticle space, introduces a sequence of successively more accurate approximations to exact number projection, analogous to the ideas described following Eq. (16.30), and thereby generates a series of boson expansions approximating better and better the number-conserving seniority mapping considered in this section. For the single-*j* model, the first step is sufficient to produce exact number conservation and a mapping that has been utilized to illustrate how the program of Tamura *et al.*, described in Sec. XV, can be improved (Li, Pedrocchi, and Tamura, 1985). The successful application of the same method to the SO(8) model (Li, Pedroc-

chi, and Tamura, 1986) will be described below. In detail, the method approximates the quantized Bogoliubov transformation introduced in Secs. III and IV.

Essentially the same results as found by Li can be obtained by the term-by-term bosonization method (Tamura, Li, and Pedrocchi, 1985).

Finally, Geyer (1986a) has applied a method due to Kim and Vincent (1987) to the unitarization of the Dyson mapping and to a simultaneous change of basis to a seniority basis. This method is based on a special property of the pairing Hamiltonian (that it is tridiagonal in an ideal boson basis). The case of more than one level has been treated successfully by Navrátil and Dobeš (1990).

E. The SO(8) Ginocchio model

In a seminal paper, Ginocchio (1980) discovered two methods of constructing subalgebras of a shell-model algebra that contained only fermion pairs carrying angular momentum zero and two. Of course, there are two elementary ways of doing this. In one of these, germane to the work of this section, one sees immediately that this requirement is satisfied for the single-j level with $j=\frac{3}{2}$, described by the algebra SO(8). Another solution, within the framework of symplectic algebras, can be found in the L-S coupled p shell, yielding the algebra Sp(6). In order to create interesting physics, however, one must somehow associate the above with subalgebras of shell-model algebras describing heavy nuclei. Ginocchio supplied the "trick" for doing this, in effect reinventing an idea already incorporated in the pseudo-SU(3) model (Arima, Harvey, and Shimizu, 1969; Arvieu, 1969; Hecht and Adler, 1969), though differently motivated. The current interest in these models may be attributed mainly to the roles that they play in the fermion dynamical symmetry model, a subject that will be discussed briefly in Sec. XVIII. In this section, we shall carry the account up to that point.

Starting with the usual j-j coupled shell model, one decomposes the angular momentum $\mathbf{j}$, into the sum of a pseudo-orbital and a pseudospin part, $\mathbf{j}=\mathbf{k}+\mathbf{i}$, and composes new fermion operators by means of the equation

$$b^{\dagger}_{km_k,im_i}=\sum_{jm}(kim_k m_i|jm)a^{\dagger}_{jm} \ . \tag{17.20}$$

The operators (17.20) are then combined in pairs so that the pseudo-orbital momentum is coupled to a total of K and the pseudospin is coupled to a total of I, the two angular momenta then coupled to a resultant of J. It turns out then that there are precisely two solutions to the problem specified by restricting the latter to $J=0$ and $J=2$, and it is perhaps not surprising that the resulting algebras are isomorphic to the simple algebras already specified. One solution is to choose $I=0$ and $k=1$ (for then K and J are confined to the values 0 and 2), and the other is to choose $K=0$ and $i=\frac{3}{2}$ (and then I and J have the values 0 and 2). These two possibilities generate precisely the algebras Sp(6) and SO(8). We shall now devote our attention exclusively to the latter case.

Written in terms of the original shell-model fermions, the generators of this algebra are the pair operators

$$S^{\dagger}=\frac{1}{2}\sum_{jm}(-1)^{j-m}a^{\dagger}_{jm}a_{j-m} \ , \tag{17.21}$$

$$D^{\dagger}_{\mu}=\sum_{j,j'}(-1)^{k+3/2+j}\{(2j+1)(2j'+1)\}^{1/2}\begin{Bmatrix} j & j' & 2 \\ \frac{3}{2} & \frac{3}{2} & k \end{Bmatrix}[a^{\dagger}_j a^{\dagger}_{j'}]^2_{\mu} \ , \tag{17.22}$$

and the multipole operators with $K=0$ and i coupled to $r=0, 1, 2, 3$,

$$P^r_{\mu}\equiv 2(2k+1)^{1/2}[b^{\dagger}_{k(3/2)}\tilde{b}_{k(3/2)}]^r_{0\,\mu}=2\sum_{j,j'}(-1)^{r+k+3/2+j}[(2j+1)(2j'+1)]^{1/2}\begin{Bmatrix} j & j' & r \\ \frac{3}{2} & \frac{3}{2} & k \end{Bmatrix}[a^{\dagger}_j\tilde{a}_{j'}]^r_{\mu} \ , \tag{17.23}$$

these formulas involving the standard Wigner $6j$ symbols. In terms of these definitions, the SO(8) algebra takes the form

$$[S,S^{\dagger}]=(\Omega-\hat{n})\equiv -2S_0 \ , \tag{17.24a}$$

$$[D_{\mu'},D^{\dagger}_{\mu}]=-2\delta_{\mu,\mu'}S_0+\sum_{t\ \text{odd}}(-1)^{\mu'}(22-\mu'\mu|t\mu-\mu')\begin{Bmatrix} j & j' & t \\ \frac{3}{2} & \frac{3}{2} & k \end{Bmatrix}P^t_{\mu-\mu'} \ , \tag{17.24b}$$

$$[D^{\dagger}_{\mu},S]=P^2_{\mu} \ , \tag{17.24c}$$

$$[P^r_{\mu},S^{\dagger}]=2\delta_{r,2}D^{\dagger}_{\mu}+2\delta_{r,0}\delta_{\mu,0}S^{\dagger} \ , \tag{17.24d}$$

$$[P^r_{\mu'},D^{\dagger}_{\mu}]=-4[5(2r+1)]^{1/2}(r2\mu'\mu|2\mu'+\mu)\begin{Bmatrix} 2 & 2 & r \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \end{Bmatrix}D^{\dagger}_{\mu+\mu'}+2(-1)^{\mu'}\delta_{r,2}\delta_{\mu-\mu}S^{\dagger} \ , \tag{17.24e}$$

$$[P^r_{\mu'},P^s_{\mu}]=2(-1)^{r+s}[(2r+1)(2s+1)]^{1/2}\sum_t(rs\mu'\mu|t\mu'+\mu)[1-(-1)^{r+s+t}]\begin{Bmatrix} r & s & t \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \end{Bmatrix}P^t_{\mu'+\mu} \ . \tag{17.24f}$$

The monopole-multipole operator is just twice the valence nucleon number,

$$\tfrac{1}{2}P^0=\hat{n}=\sum a^\dagger_{jm}a_{jm}=\Omega+2S_0\ . \tag{17.25}$$

The dipole operator P^1_μ is proportional to the total pseudospin operator $\hat{I}_\mu$,

$$P^1_\mu=\frac{2}{5^{1/2}}\hat{I}_\mu\ , \tag{17.26}$$

which, within the $(S^\dagger,D^\dagger)^N$ space, is the total angular momentum operator.

The most general rotationally invariant Hamiltonian that can be constructed from the generators of the algebra and that describes at most two-body interactions has the form

$$H=G_0S^\dagger S+G_2D^\dagger\cdot D+\sum_{r=1,2,3}b_rP^r\cdot P^r\ . \tag{17.27}$$

For a general choice of the coupling strengths, this Hamiltonian must be dealt with numerically. For special values, specified below, the eigenvalues of H can be given by purely group-theoretical considerations. These correspond to cases where the Hamiltonian can be written as a linear combination of the quadratic Casimir operators corresponding to one of the three possible chains of subalgebras available for the construction of a basis of states carrying good angular momentum. These are

$$\mathrm{SO}(8)\supset\mathrm{SO}(7)\supset\mathrm{SO}(5)\supset\mathrm{SO}(3)\ , \tag{17.28a}$$

$$\mathrm{SO}(8)\supset\mathrm{SO}(6)\supset\mathrm{SO}(5)\supset\mathrm{SO}(3)\ , \tag{17.28b}$$

$$\mathrm{SO}(8)\supset\mathrm{SO}(5)\otimes\mathrm{SU}(2)\supset\mathrm{SO}(5)\supset\mathrm{SO}(3)\ . \tag{17.28c}$$

If $b_2=G_2$ the group chain (17.28c) is realized as a dynamical symmetry; if $G_0=G_2$, the chain (17.28b) is realized as a dynamical symmetry; and if $b_2=G_0$, the chain (17.28a) is realized as a dynamical symmetry.

The bulk of the Ginocchio (1980) paper under discussion is, in fact, devoted to a calculation of the matrix elements of the generators for a single representation of SO(8) for each of the two chains (17.28b) and (17.28c). These were singled out in preference to (17.28a) because of their presumed relation to the γ-unstable and vibrational limits of IBM-1. The representation in question is that in which the states are linear combinations of states made up of monopole and quadrupole pairs (called favored pairs below),

$$|NN_d\rho JM\rangle=(S^\dagger)^{N-N_d}(D^\dagger)^{N_d}_{\rho JM}|0\rangle\ , \tag{17.29}$$

where N is the total number of pairs and N_d is the total number of quadrupole pairs, characterized by angular momentum quantum numbers and by an additional set of quantum numbers ρ necessary for a unique specification. It is expected that the most collective states are made up largely of states in this representation, characterized as the representation in which all the fermions are coupled in favored pairs. To distinguish this representation from those in which one or more fermions remain uncoupled, the quantum number u, a kind of generalized seniority, is utilized. It measures the number of fermions not coupled to favored pairs. To distinguish it from the usual seniority, the name *heritage* has been adopted.

We shall not repeat any of this analysis, except to remark that the results are obtained in the form of a product of two factors, the first depending on the quantum numbers of the Casimir operators of the chain and the other on a matrix element of a simple boson operator taken in a boson realization of the same basis. In other words, the matrix realization is only one step away from a boson mapping. That last step was taken in one of the initial applications of these ideas discussed below. We note, however, that the mathematical analysis of Ginocchio has been extended to the subchain of groups (17.28a) and to the representations with $u=1,2$ (Hecht, 1987b; Lü *et al.*, 1988). We shall discuss the mathematical basis for this extended work in Sec. XIX.

We begin the discussion of applications with a passing mention of a direct application of the SO(8) model (with a suitable choice of the degeneracy parameter Ω) to the properties of the low-lying states of the Sm isotopes (Arima, Ginocchio, and Yoshida, 1982). Except for the deformed nuclei, where it was necessary to add three-body forces, surprisingly good fits were obtained with a Hamiltonian of the simplest type in the algebra of SO(8)×SO(8) (to describe neutrons and protons).

In a work that has been more widely noticed and is, in any event, more germane to the purposes of this review (Arima, Yoshida, and Ginocchio, 1981), the boson mappings that are implied (with only a small additional effort required) by the matrix elements computed in the original Ginocchio paper were applied for the purpose of comparing the accuracy of interacting boson model Hamiltonians, obtained by one of these mappings from Eq. (17.27), with the corresponding mapping that would be used in the same problem in the boson expansion theory approach of Tamura, described in Sec. XV. In all, three mappings are used in this comparison, the unitary mappings through the chain (17.28b), also called the BZM mapping (obtained by us in one form in Sec. IX); a unitary mapping for the chain (17.28c), also called the OAI mapping; and the boson expansion theory mapping. The last is given from the first as a truncated series, whereas the first two are given in closed form, involving, however, square roots of operators that can only be evaluated by expansion, except at the special symmetry points, where one or the other of the mapped forms can be evaluated exactly. The comparison is weighted against boson expansion theory by making the comparison only at these symmetry points. Remembering that boson expansion theory does not conserve particle number, it nevertheless fares reasonably at low excitations, as might be expected. At excitations above a few MeV, it fares poorly compared to the number-conserving mappings.

This was not to be the final word on this subject, however. We recall that C. T. Li (1984a) had developed a

practical method for restoring number conservation to expansions based on quasiparticles. It involved a number of successive steps, each of which was designed to bring the system closer to exact number conservation. However, for the single-j shell model and for the SO(8) model, the first step was already sufficient to restore number conservation exactly. The result is an exact equivalent of the mapping involving the SO(6) chain, i.e., the BZM result. [This observation was not made in the paper in which the Li result was applied (Li, Pedrocchi, and Tamura, 1986)]. It remains to be seen, however, to what extent Li's ideas can be incorporated into the full boson expansion program.

The remainder of this discussion alludes briefly to work concerned with Dyson mappings of SO(8) and related algebras. The works involved attempt, in one manner or another, to exploit the simplicity of the Dyson mapping. The question Kim and Vincent (1988) ask is whether they can introduce a similarity transformation that unitarizes the Dyson Hamiltonian and still retains its one- and two-body character. They obtain a positive answer for SO(8), but cannot guarantee that the corresponding generators will be free of higher-order terms. It is well to emphasize that this work and the further papers mentioned below take a rather unconventional view toward boson mappings compared to the usual group-theoretical practitioners, whom we have taken as our model. The latter very clearly define the pair of bases involved in a boson mapping, namely, the basis in the fermion or other starting space and the boson basis of one's final focus. The work under discussion gives up this control for the sake of other aims. What may be missing, then, is the ability to specify the physical regime where the results apply.

In a paper related to SO(8), Geyer and Hahne (1981) first derive the Dyson form of the mapping corresponding to the SO(6)$\equiv$SU(4) subalgebra. This is the usual chain used for the Dyson mapping and is the only one that gives the standard simple result. [In fact, Arima, Yoshida, and Ginocchio (1981), in the paper described above, also give a Dyson mapping for the SO(5)$\times$SU(2) chain, where one can verify the truth of this assertion.] Geyer and Hahne then extend the mapping to include an odd particle. Finally, they ask the following amusing question: Suppose one carries out the Dyson mapping for a general shell-model algebra, but one with an arrangement of subshells allowing the decomposition into pseudo-orbital and pseudospin angular momenta leading to the SO(8) subalgebra. Suppose further that one carries out this recoupling and drops from the maps of the SO(8) generators all terms containing no s and d bosons. Does the remainder constitute a mapping of SO(8)? The result is affirmative. In a recent work (Kim and Vincent, 1988), one finds a general discussion of this phenomenon.

We call attention, finally, to the work of Kaup, who has discovered and studied a Schwinger mapping for SO(8) (Kaup, 1987, 1988; Kaup, Ring, and Nigam, 1988). Here we shall only describe the mapping itself, referring the reader to the literature cited for applications to the Hamiltonian (17.27). Kaup realized the SO(8) algebra by a generalization of the scheme he had already used for SO(4), described in Sec. VI. In addition to the physical angular momentum, one introduces a kinematically independent quasispin and two boson double tensors (where in the following two equations the first superscript and subscript represent the quasispin and its component and the second pair fixes the ordinary spin in the corresponding way).

$$b_{\pi 0}^{(10)}=(u,s,v)\ , \tag{17.30a}$$

$$b_{0\mu}^{(02)}=d_\mu\ , \tag{17.30b}$$

$$[b_i,b_j]=[b_i^\dagger,b^\dagger]=0,\quad [b_i,b_j^\dagger]=\delta_{ij}\ , \tag{17.31}$$

where in the last equation i,j clearly each represent a quartet of indices. Altogether there are eight bosons. This number is sufficient to permit a bilinear realization of the generators of SO(8) defined in Eqs. (17.21)–(17.23), according to the equations

$$S^\dagger=2^{1/2}(s^\dagger u-v^\dagger s)\ , \tag{17.32}$$

$$D_\mu^\dagger=2^{1/2}(d_\mu^\dagger u+v^\dagger\tilde{d}_\mu),\quad \tilde{d}_\mu=(-1)^\mu d_{-\mu}\ , \tag{17.33}$$

$$S_0=v^\dagger v-u^\dagger u\ , \tag{17.34}$$

$$P_\mu^{(2)}=2(d_\mu^\dagger s+s^\dagger\tilde{d}_\mu)\ , \tag{17.35}$$

$$P_\mu^{(1)}=2^{3/2}[d^\dagger\times\tilde{d}]_\mu^{(1)}\ , \tag{17.36}$$

$$P_\mu^{(3)}=2^{3/2}[d^\dagger\times\tilde{d}]_\mu^{(3)}\ . \tag{17.37}$$

The realization just given corresponds to the embedding of SO(8) into the larger group SU(8). To complete the mapping we must specify the irrep of SO(8)$\subset$SU(8) isomorphic to the "vacuum" irrep in the fermion space. This requires only that we specify the map of the vacuum state satisfying

$$S|\text{vac}\rangle=D_\mu|\text{vac}\rangle=P_\mu^{(k)}|\text{vac}\rangle=0\ . \tag{17.38}$$

These requirements are satisfied by the state

$$|\text{vac}\rangle=(\Lambda!)^{-1/2}(u^\dagger)^\Lambda|)\ , \tag{17.39}$$

where the round ket $|)$ denotes the boson vacuum state. The value of the integer Λ is $\Omega/2$, where Ω is the value of the average $\langle\text{vac}|SS^\dagger|\text{vac}\rangle$. We refer the reader to the literature cited for more of the mathematical details. It is clear that the Schwinger mapping can be quite useful in providing a convenient basis for diagonalizing Hamiltonians belonging to the set (17.27). Kaup has also carried out an extensive study of the associated model within the mean-field approximation, utilizing the formalism associated with the concept of coherent state. Again we refer the reader to the literature for details.

Recently, Scholtz and Geyer (1988) have given a general solution to the problem of embedding the physical shell-model algebra by a generalized Schwinger mapping into a larger unitary algebra. They despair that the re-

sults will be of practical value because of the large number of unphysical (supernumerary) bosons introduced in such a mapping.

F. Symplectic shell-model algebras

Interest in symplectic algebras has two distinct origins. One source is the development of the Sp(6,R) model of collective motion [for a review see (Rowe, 1985)]. This subject is considered outside the bounds of the present review and, in any event, has been described clearly in the review cited. Independently, interest has been regenerated in the compact unitary symplectic algebras Sp(2Λ), with Λ an integer, motivated by the problem of providing a microscopic basis for the interacting boson model. Discussion of an Sp(6) model was initiated by Ginocchio (1980) and has been followed by suggestions (Chen, Feng, and Wu, 1986; Wu *et al.*, 1986, 1987) that this symmetry may manifest itself in actual nuclear problems. The toy models associated with Sp(4) [isomorphic to SO(5)] have been reviewed in Sec. VII of this paper.

In further work stimulated by the continuing search for a microscopic basis of the interacting boson model for deformed nuclei, Bonatsos and Klein (1985, 1986; Bonatsos, Klein, and Zhang, 1986a; Wybourne, 1986) studied the shell-model algebras for identical nucleons coupled to intrinsic spin zero. These are the algebras Sp(2Λ), the value of Λ depending on the shell, which were analyzed through the chain Sp(2Λ)$\supset$U(Λ)$\supset$SU(3), with boson realizations constructed in an angular-momentum-coupled basis. This chain is the one utilized in the pseudo-SU(3) approximation applied to the deformed rare-earth nuclei (Draayer, Weeks, and Hecht, 1982) as a shell-model approximation and more recently as the basis for an algebraic treatment of deformed nuclei (Draayer and Weeks, 1983, 1984; Draayer and Rosensteel, 1985; Leschber and Draayer, 1986, 1987; Castaños, Draayer, and Leschber, 1988). Our interest is in a boson realization of this algebraic model, and in particular in a representation of the algebra containing the vacuum state of the shell model. An improved method for carrying out this program was described by Bonatsos, Klein, and Zhang (1986a). It should be remarked that this work was influenced by papers on the corresponding noncompact symplectic algebras (Deenen and Quesne, 1982, 1984, 1985; Castaños, Chacon, Moshinsky, and Quesne, 1985; Castaños, Kramer, and Moshinsky, 1985a, 1985b; Chacon and Moshinsky, 1987). Derivations of mappings for more general irreps (Hecht and Elliott, 1985; Hecht, 1985, 1988) have been carried out using the method of the vector coherent state described in Sec. XIX.

To apply these results, one needs the reduced matrix elements of the boson operators between irreps of U(Λ), the unitary subgroups, that occur in various cases. Considerable effort has gone into this problem. For example, very explicit results for the Sp(6)$\supset$U(3) reduction have been given recently by Hecht (1988), where references to earlier work can be found.

It may also be remarked that mappings of the square-root BZM form, analogous to Eq. (9.39) of Sec. IX, can also be derived for the symplectic algebras (Deenen and Quesne, 1982, 1984, 1985). In further work, a mapping of an isospin-invariant generalization of Sp(6) has been studied by Peres Menezes (Peres Menezes, Brink, and Bonatsos, 1989).

Quantitative applications of the results for the symplectic algebras remain to be carried out. Bonatsos (Bonatsos and Klein, 1986) has analyzed all the possible examples of pseudo-SU(3) shells that might occur for realistic shell-model calculations, constructing those linear combinations of the generators of the appropriate symplectic algebra which form irreducible tensors under SU(3). The most complicated case studied was the neutron (or proton) p-f-h shell corresponding to Sp(42). It turns out that there is an important point of fundamental difference between pseudo-SU(3) and the deformed SU(3) limit of the interacting boson model in how they assign the low-lying bands to irreps. In a qualitative analysis Bonatsos (Bonatsos and Klein, 1986) has argued that the experimental evidence may favor the pseudo-SU(3) analysis. At the very least, the question is open. In this connection the phenomenological algebraic analysis of Draayer *et al.* and its microscopic basis, quoted earlier, are of exceptional interest.

XVIII. MICROSCOPIC FOUNDATIONS OF THE INTERACTING BOSON MODEL

A. Introduction

Before turning our attention to the details of various methods that have been utilized in the attempt to justify the interacting boson model from a microscopic point of view, we include the briefest possible sketch of the elements of IBM-2 and IBFM. We have already described IBM-1 for present purposes in Sec. VIII.B. The IBM-2 is an extension of the IBM-1 that explicitly takes into account neutron and proton degrees of freedom, distinguished by the labels ν and π, respectively. For a review of the phenomenological aspects of these models, see Iachello and Arima (1987) and Bonatsos (1988). For a viewpoint on microscopic foundations that contrasts with the one to be presented in this section, see Iachello and Talmi (1987).

The IBM-2 Hamiltonian is given by

$$H_B=\sum_{\rho=\pi,\nu}\epsilon_\rho d_\rho^\dagger d_\rho+\kappa_B Q_\pi\cdot Q_\nu + \sum_\rho V_{\rho\rho}+\sum_J \zeta_J[(d_\pi^\dagger d_\nu^\dagger)^{(J)}(\tilde{d}_\pi\tilde{d}_\nu)^{(J)}]^{(0)} , \tag{18.1}$$

with

$$\tilde{d}_{\rho,\mu}=(-1)^\mu d_{\rho,-\mu} ,$$
$$Q_\rho=\kappa_\rho[d_\rho^\dagger s_\rho+s_\rho^\dagger\tilde{d}_\rho+\chi_\rho^H(d_\rho^\dagger\tilde{d}_\rho)^{(2)}] . \tag{18.2}$$

Here $d_\rho^\dagger$ is the creation operator for a d boson, carrying

angular momentum $L=2$, and $s^\dagger$ is the corresponding object for $L=0$. (In presentations of the phenomenological model, the factor κ_ρ is usually omitted, but it is convenient to retain it for microscopic investigations.) The interaction between bosons of the same kind, which describes at the same time the two-boson part of the Hamiltonian of IBM-1, has the general form

$$\begin{aligned} V_{\rho\rho} = & \tfrac{1}{2}\sum_J C_{\rho,J}[(d_\rho^\dagger d_\rho^\dagger)^{(J)}(\tilde{d}_\rho \tilde{d}_\rho)^{(J)}]^{(0)} \\ & + 2^{-1/2}\bar{v}_{\rho,2}[(d_\rho^\dagger d_\rho^\dagger)^{(2)}(\tilde{d}_\rho s_\rho)^{(2)} + \text{H.c.}]^{(0)} \\ & + \tfrac{1}{2}\bar{v}_{\rho,0}[(d_\rho^\dagger d_\rho^\dagger)^{(0)} s_\rho s_\rho + \text{H.c.}] \ . \end{aligned} \tag{18.3}$$

Finally the last term in Eq. (18.1) is an operator (the Majorana operator) (Scholten, 1983), introduced to guarantee that the low-lying states of IBM-2 resemble those of IBM-1. This operator will not receive any further attention in this review.

In Eq. (18.2) we have insisted on maintaining a distinction between the quadrupole operator appearing in the Hamiltonian and the boson $E2$ operator, which we write as

$$Q_\rho^B(E2) = e_{\rho,B}[d_\rho^\dagger s_\rho + s_\rho^\dagger \tilde{d}_\rho + \chi_\rho^Q (d_\rho^\dagger \tilde{d}_\rho)^{(2)}] \ . \tag{18.4}$$

For odd nuclei (Iachello and Scholten, 1979) one adds a fermion (actually a quasifermion) $(c_\alpha^\dagger, c_\alpha)$ to the interacting boson model. The additional fermion-boson interaction, which then defines the IBFM, is taken to be

$$\begin{aligned} V_{BF} = & \sum_a A_a (c_a^\dagger \tilde{c}_a)^{(0)}(d^\dagger \tilde{d})^{(0)} \\ & + \sum_{ab} \Gamma_{ab}[(s^\dagger \tilde{d} + d^\dagger s + \chi (d^\dagger \tilde{d})^{(2)})(c_a^\dagger \tilde{c})^{(2)}]^{(0)} \\ & + \sum_{abj} \Lambda_{ab}^j :[(c_a^\dagger \tilde{d})^{(j)}(\tilde{c}_b d^\dagger)^{(j)}]^{(0)}: \ . \end{aligned} \tag{18.5}$$

Here the dots indicate the normal product, the indices a and b denote single-particle states for the fermion, and j is the angular momentum for the coupled fermion and boson. In an effort to simplify the analysis of data, one often assumes simplified microscopically motivated relations for the parameters in Eq. (18.5), namely,

$$A_a = A_0(2j_a+1)^{1/2} \ , \tag{18.6}$$

$$\Gamma_{ab} = \Gamma_0(u_a u_b - v_a v_b)(a \| \frac{r^2}{b^2} Y_2 \| b) \ , \tag{18.7}$$

$$\Lambda_{ab}^j = 20\Lambda_0(2j+1)^{-1/2} Q_{ja} Q_{jb}(\alpha_j + \alpha_a)(\alpha_j + \alpha_b) \ , \tag{18.8}$$

$$Q_{ja} = (j \| \frac{r^2}{b^2} Y_2 \| a) \ . \tag{18.9}$$

Here b, occurring in the denominator in Eqs. (18.7 and 18.9), is the standard harmonic-oscillator parameter, not to be confused with the indices of the same name. Equation (18.7) is the conventional quasiparticle-phonon coupling in the BCS model, and relations (18.6) and (18.8) are approximations of expressions for matrix elements in a system of three valence nucleons (Talmi, 1981; Gelberg, 1983; Kaup, 1983). Finally, the parameters u_a and v_a are the well-known BCS parameters, with $\alpha_a = (v_a/u_a)$. The latter is the BCS approximation to the parameters that appears below in Eq. (18.11). We turn to an account of the efforts to derive these parameters from the starting point of the broken-pair approximation.

B. The broken-pair approximation

In the following we shall rely heavily (though not exclusively) on review papers of the group that has done the most extensive work in this area, Allaart, Bonsignori, Savoia, and Paar (1986a, 1986b); Allaart, Boeker, Bonsignori, Savoia, and Ghambhir (1988). (For early work, see Akhermans, Loriaux, Allaart, and Bonsigniori, 1983; Allaart and Bonsigniori, 1983.) For other applications of this method to the derivation of the IBM, see below and Duval and Barrett, 1981; Gambhir, Ring, and Schuck, 1982a 1982b, 1984; Sage, Goode, and Barrett, 1982; van Isacker *et al.*, 1986. For a brief survey, see Barrett *et al.*, 1989. Our interest in the broken-pair method is that it provides the soundest approximation to the shell-model description of vibrational nuclei that incorporates the essential physical ideas of superconductivity or generalized seniority (Talmi, 1971) without breaking the symmetries of the Hamiltonian. Since the collectivity is studied and extracted by calculations in the fermion space, this approximation is a natural tool for implementing the seniority-based OAI mapping (see Sec. XVII). As we shall see, a corresponding symmetry-conserving basis for mapping of deformed nuclei has not been established, and such work as has been done on this problem, although quite suggestive, is nevertheless much cruder in quality. We describe first the broken-pair or generalized-seniority approximation for semimagic nuclei.

The method is based, first, on the assumption that the ground state of the system with $2p$ nucleons is well approximated by a state of the form

$$|\psi_{2p}(v_g=0)\rangle = N_0 (S^\dagger)^p |0\rangle \ . \tag{18.10}$$

Here N_0 is a normalization constant, while $S^\dagger$ is the creation operator for a Cooper pair,

$$S^\dagger = \sum_a \alpha_a S_+(a) \ , \tag{18.11}$$

characterized by the coefficients α_a, which measure the distribution of pair strength over the single-particle states a, and $S_+(a)$ is the $j=0$ pair creation operator studied in Sec. IV, proportional to $A_i^\dagger$ defined in (4.2). The accuracy of the state (18.11) has been established by comparison with more accurate shell-model calculations (Allaart and Boeker, 1971) or even exact shell-model calculations (Gambhir, Rimini, and Weber, 1969, 1973), provided the coefficients α_a are determined by a variational calculation based on the state (18.10). Although there are spe-

cial circumstances under which these coefficients may be independent of particle number (Talmi, 1971), for all interactions used in practice a smooth variation with particle number is to be expected.

Starting with this generalized seniority, $v_g=0$ ($0bp$) state, one constructs the $v_g=2$ ($1bp$) states by replacing one $S^\dagger$ creation operator by a two-particle creation operator, leading to the configurations

$$A^\dagger_{JM}(ab)(S^\dagger)^{p-1}|0\rangle \ . \tag{18.12}$$

For $J=0$ we must restrict the set (18.12) to linear combinations orthogonal to Eq. (18.10). By convention, all of the configurations defined, even those with angular momentum zero, are said to carry generalized seniority, $v_g=2$. Similarly the $v_g=4$ states are constructed as linear combinations of

$$A^\dagger_{J_1M_1}(ab)A^\dagger_{J_2M_2}(cd)(S^\dagger)^{p-2}|0\rangle \ , \tag{18.13}$$

orthogonalized to the previously obtained set of $v_g\leq 2$ states. The construction of an orthonormal set of states, characterized by the v_g quantum number, requires the overlaps and nontrivial normalization factors of the various states, a matter beyond the purview of the present section, though the techniques are similar to those described briefly in Secs. XV and XVI.

The validity of the generalized seniority truncation is well established for semimagic nuclei and was also the basis (somewhat more obliquely) for the mean-field methods applied to vibrational nuclei in Sec. XII. The truncation scheme is based mainly on the empirical relation

$$E(v_g=2)\geq E(v_g=0)+v_g\bar{\Delta} \ , \tag{18.14}$$

where $\bar{\Delta}$ is half the energy gap between the ground state and the lowest (noncollective) state.

For odd nuclei, one may establish a similar classification that can be made the basis for a discussion of the interacting boson-fermion model. This will be considered in Sec. XVIII.G.

Rather complete calculations have been carried out for $v_g\leq 2$, but for higher seniority, further truncation has been necessary. A general local interaction was employed, including spin-dependent and tensor forces. We refer the reader to Allaart *et al.* (1988) for details.

We are now prepared to make contact with IBM-2 and to consider what features of the broken-pair approximation make it an excellent candidate for the problem at hand. But first let us remark on some limitations that cannot be overcome within the present framework. Both the shell-model approximation (for the most part) and the phenomenological boson model (so far) work within the confines of a single valence shell. This means that for all observables, and not only the Hamiltonian, we need effective operators. In favorable cases this involves mainly a rescaling of the simplest operator one can write down, introducing the so-called effective charge. We show in Fig. 18, taken from Allaart *et al.* (1986b), a case

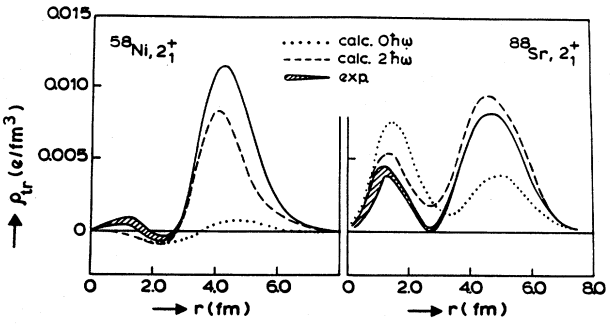

FIG. 18. The transition charge density for a neutron boson (2_1^+ in ^{58}Ni) and a proton boson (2_1^+ in ^{88}Sr). The calculations were performed with a broken-pair model in a $0\hbar\omega$ and in a large model space that included all $2\hbar\omega$ excitations. From Allaart, Bonsigniori, Savoia, and Paar, 1986b.

in which this cannot be done. What is shown is the experimentally deduced transition charge density for the transition from the ground state to the first excited 2+ state for two semimagic nuclei in comparison with a valence-shell calculation ($0\hbar\omega$) and with an augmented calculation including $2\hbar\omega$ excitations. The latter, though of small individual amplitude, add up coherently to give much of the required surface peaking. The implication is that we must anticipate similar problems for the nuclei of interest to us, which are not semimagic.

Let us consider, then, possible methods of adapting the broken-pair technology to this problem. We might start by doing the separate broken-pair calculations for neutrons and protons and then forming the direct product space for the combined system. Let

$$(D^\dagger_\rho)_\mu=\sum_{ab}\beta_\rho(ab)(A^{(2)\dagger}_\rho)_\mu \tag{18.15}$$

be that linear combination of the $v_g=2$ configurations which creates the lowest (collective) 2+ state for the respective semimagic nuclei. The next impulse would be to define an orthonormalized direct product basis made up of two excitations each [$S^\dagger$ and Eq. (18.15)] for neutrons and protons, constructed by means like the norm matrix method discussed in Secs. XV and XVI. Using such a basis, one could map a microscopic Hamiltonian onto the Hamiltonian (18.1) by the OAI method described in Sec. XVII. Here it is permissible to drop some terms of the phenomenological Hamiltonian, since there are, in any event, too many parameters to obtain a unique phenomenological fit to the data. In the ensuing discussion, we focus on the single-boson terms and the quadrupole-quadrupole interaction, though the fitting procedure also requires some combination of the Majorana parameters in order that the IBM-1 states, i.e., the states of maximum symmetry, occur much lower in energy than the states of mixed symmetry.

This naive method does not work. We emphasize two reasons for its failure. The first is that the quadrupole

neutron or proton excitations should depend on the presence of the other particle, i.e., on the neutron-proton interaction. This can be accounted for approximately (Pittel, Duval, and Barrett, 1982a, 1982b; Barrett, Duval, and Pittel, 1983; Druce *et al.*, 1987) by adjusting the single-particle energies of one species as a function of the number of the other. Since one needs only the ground state, the $2+$ excitation of each species, and the OAI mapping method to map the fermion quadrupole operator onto the form of Eq. (18.2), we consider this application immediately. The relevant formulas, one set each for neutron and proton, are

$$\kappa=\frac{1}{\sqrt{5}}\langle 0_1^+\|r^2Y_2\|2_1^+\rangle \tag{18.16}$$

and (p is the number of pairs)

$$\chi\kappa=p^{1/2}(5)^{-1/2}\langle 2_1^+\|r^2Y_2\|2_1^+\rangle\ . \tag{18.17}$$

In Fig. 19, from Barrett *et al.* (1983), we show some neutron results obtained by these authors for the shell $N=82-126$. The results are shown for several different choices of single-particle states that we shall not explain in detail. Except near the closed shells, where there is sensitivity to this choice, the trends are roughly consistent. More important, these parameters are in good agreement with parameters determined from phenomenological fits. This type of result must be accounted a success of the program under study.

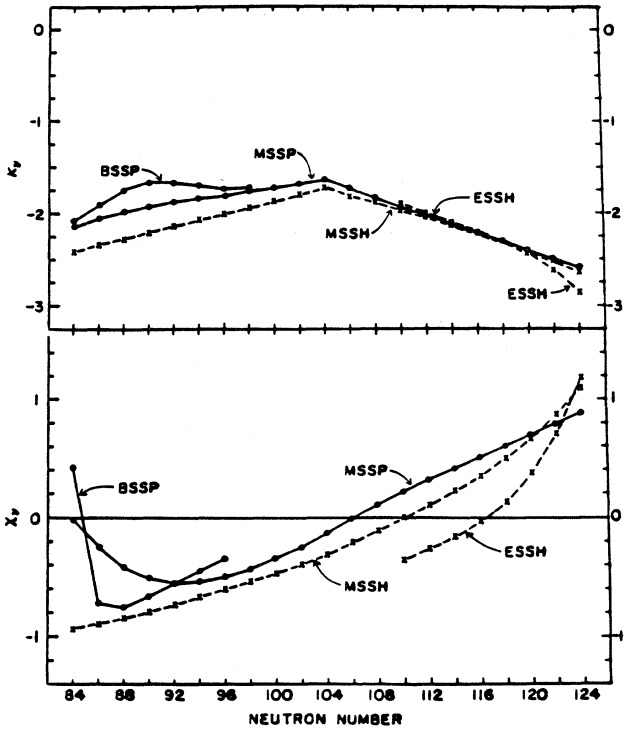

FIG. 19. The interacting boson model parameters κ_ν and χ_ν calculated for $N=82-126$ from the Otsuka-Arima-Iachello mapping procedure and the broken-pair approximation. The labels refer to different choices of single-particle energies described in the original paper. From Barrett, Duval, and Pittel, 1983.

The second reason for the failure of what might be called the naive method of mapping, i.e., any mapping procedure that utilizes only S and D degrees of freedom, will now be considered. If the spectra are compared with experiment for the parameters obtained by this fitting procedure, it is found generally that the correct order of low-lying levels is obtained, but the energy splittings are too large by a factor of two or more. This discrepancy has been attributed to the need to take into account the coupling to the multitude of states omitted from the mapping procedure, since the Pauli principle, as well as the dynamics, precludes any exact decoupling of these configurations from the included space. Various approximate renormalization schemes have been proposed and carried out (Pittel, Duval, and Barret, 1982a, 1982b; Scholten, 1983; van Egmond and Allaart, 1984; Allaart *et al.*, 1986a, 1986b). Particularly extensive calculations were performed by van Egmond. He obtained an improved version of the basic $2+$ state by diagonalization of the shell-model Hamiltonian within a model space of the old 2_1^+, $v_g=2$ state and a thousand $v_g\leq 4$ states formed from the direct product of the separate $v_g=2$ spaces for the neutrons and protons:

$$|D_\rho(S_\rho^\dagger)^{p_\rho-1}(S_{\rho'}^\dagger)^{p_{\rho'}}\rangle_{\rm ren} =a_0|D_\rho^\dagger(S_\rho^\dagger)^{p_\rho-1}(S_{\rho'}^\dagger)^{p_{\rho'}}\rangle+\sum_{i\in v_g\leq 4}a_i|i\rangle\ . \tag{18.18}$$

The main conclusions were the following:

(1) The coefficient a_0 does not become much smaller than 0.9, so that the dominant structure remains that built with S and D pairs.

(2) Typically the excitation energy of the state (18.18) is decreased by 30–50 %. About half of this effect is due to $J=4$ pairs (G pairs).

(3) The admixtures also reduce the calculated interaction parameter κ between bosons by about 40%.

(4) No indication was found for a force that pushes upwards states with building blocks that are antisymmetric for interchange of proton and neutron bosons, but instead a force was found that pushes the symmetric ones further downwards in energy. This yields an additional compression of rotational bands.

A typical result (Allaart *et al.*, 1986b) of this fitting procedure for ^{130}Ba is shown in Fig. 20. The crosses indicate mixed-symmetry states, not confirmed by experiment.

In conclusion, it is clear that, though there is a qualitative basis for optimism that the IBM-2 may be viewed as a phenomenology with a rational basis in the shell model, the uncertainties in the parameters of the latter, the fact that they are chosen by calculations made for near-closed-shell cases, as well as the possible sensitivity of detailed results to high-lying configurations, suggests that

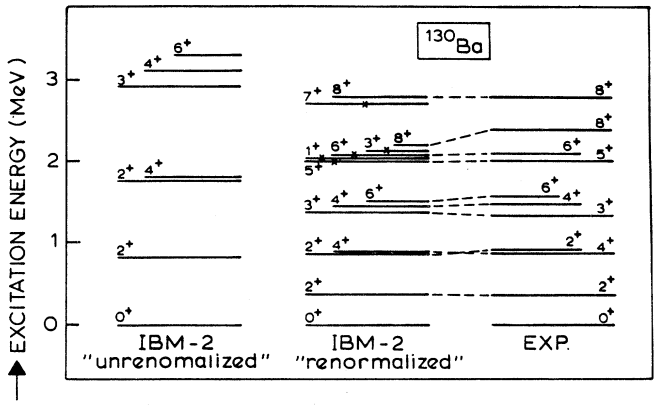

FIG. 20. Spectra of ^{130}Ba obtained in an IBM-2 calculation with boson parameters that were obtained by a mapping procedure based on the broken-pair approximation. The states marked with a cross distinguish those states obtained for the first time in an IBM-2 approximation as opposed to those that are common to IBM-1 and IBM-2. From Allaart, Bonsigniori, Savoia, and Paar, 1986b.

any expectation of detailed quantitative agreement is somewhat naive and, where such agreement is claimed, it is at least in part the result of phenomenological adjustment of parameters.

C. Unitary mapping for deformed states

The method described in the previous subsection is based on an approximation scheme valid only for spherical nuclei. As we have previously stated, this approximation scheme incorporates the same physics as the mean-field approximations erected on the basis of the spherical Hartree-Bogoliubov approach, but it has the distinct conceptual advantage of preserving the symmetries of the Hamiltonian. In choosing to discuss the problem of providing a microscopic foundation for the interacting boson model for deformed nuclei, we are faced with the situation that no analogous systematic approximation scheme has been applied to this problem. Note that we have been careful not to assert that no such approximation scheme exists, since we have already pointed out in Sec. XVII that the pseudo-SU(3) scheme fulfills the necessary criteria:

(i) If we choose the unitary-symplectic (pseudo-L-S coupled) form of the shell-model algebra, there is an SU(3) subalgebra.

(ii) In the strongly deformed limit, the irreps of this algebra define an energy ordering, i.e., a Hamiltonian belonging to the enveloping algebra of SU(3) will provide a zero-order description of the spectrum and other properties.

(iii) One can introduce symmetry-breaking terms into the Hamiltonian and study the mixing of SU(3) irreps required to fit the data.

What is missing so far is an investigation of how to extract from such calculations basic excitation operators that can be mapped onto the bosons of the IBM-1 or -2. It is far from clear how this is to be done or even if one can anticipate having success with a single set of bosons. What is clear is that only this type of approach will provide a method that can be viewed as fully complementary to the broken-pair approximation, in contrast to the ingenious but fragmentary approaches that we shall describe below. Here it is appropriate to mention once more the work of Draayer and associates (Leschber and Braayer, 1987). This program does respond to the problem of associating a successful phenomenology with the pseudo-SU(3) model, but it does so by algebraic methods that so far have not utilized concepts of boson mappings in an essential way. This work therefore falls outside the boundaries of this review. Nevertheless, for anyone interested in pursuing the investigation we are proposing, it would appear to be a good starting point.

It may be appropriate at this point to introduce a tangential suggestion that may have some significance for those familiar with the Kumar-Baranger (Kumar and Baranger, 1967, 1968; Baranger and Kumar, 1968) approach to the study of deformed nuclei and its further refinement by Kumar (1983, 1984). Without intending to be facetious, we note that insofar as the results of his program fit the data, one may assert that the calculations of Kumar (1983) are only a few, relatively straightforward steps away from a microscopic derivation of IBM-1. One would first have to fit the Bohr Hamiltonian for a given nucleus, as calculated by Kumar, to a polynomial or other convenient form in terms of the so-called Bohr-Mottelson bosons discussed in Sec. VIII. Interpreting the latter as Holstein-Primakoff bosons, we could then use the simple transformation (8.22) to Schwinger-type bosons utilized in the standard interacting boson model formulation. The resulting Hamiltonian would not generally be of the required polynomial form, but if this is deemed important, further adjustments could be made to the fitting procedure to yield such a form.

We turn finally to the mainstream of the actual work done involving unitary mappings. Work on the Dyson mapping will be considered in the following subsection. All the work about to be discussed is based on a single intrinsic state in the fermion space, a number-projected, axially-deformed Hartree-Bogoliubov state. This has the form

$$|\Lambda^N\rangle \equiv (\Lambda^\dagger)^{N/2}|0\rangle \ , \tag{18.19}$$

a condensate of Λ pairs, where each such pair can be decomposed in terms of pairs coupled to good angular momentum (the magnetic quantum number has the value zero and is suppressed),

$$\Lambda^\dagger = x_0 S^\dagger + x_2 D^\dagger + x_4 G^\dagger + \cdots \ . \tag{18.20}$$

In addition to the constants x_L that characterize the admixtures of different angular momenta, there are, for the multilevel shell model, additional structure constants

characterizing the admixture of elementary fermion pair operators, such as the coefficients $\beta(ab)$, defined in Eq. (18.15). After performing a self-consistent calculation based on the standard pairing-plus-quadrupole Hamiltonian, one examines the angular momentum makeup of the Λ pair and subsequently compares the values of observables calculated from the full wave function (18.19) with values obtained from approximate wave functions that retain only $L=0,2$ (S-D) truncation or $L=0,2,4$ (S-D-G) truncation. The uniform conclusion of all the investigations (Bes *et al.*, 1982; Maglione *et al.*, 1983, 1984; Pittel and Dukelsky, 1983; Maglione, Catara, *et al.*, 1984; Cohen, 1985; Pannert, Ring, and Gambhir, 1985) is that, even though the S-D truncation gives 80–90 % of the probability, accurate approximations to the intrinsic quadrupole moment, to the moment of inertia, and to pair-transfer matrix elements can only be obtained by including the G pair. Here, use of a number-projected trial function as in Eq. (18.19) is important, but these conclusions are not sensitive to whether angular momentum projection is done before or after the dynamics. It appears to be an unavoidable conclusion from these investigations that, even in the simplest conceivable problem, the mapping of the intrinsic state (18.19), which generates an approximate version of the ground-state rotational band, one must include an $L=4$ degree of freedom in the dynamics. We turn then to the description of the explicit proposals put forward for mapping this state or one of the approximations to it.

We consider first the method of Otsuka and Yoshinaga (1986). These authors proposed the mapping

$$|\Lambda^N\rangle \rightarrow |\lambda^N\rangle \ , \tag{18.21}$$

where

$$\lambda^\dagger = x_0^B s^\dagger + x_2^B d^\dagger + x_4^B g^\dagger + \cdots \tag{18.22}$$

is the intrinsic boson pair-creation operator and

$$|\lambda^N\rangle = (\lambda^\dagger)^{N/2}|0) \ . \tag{18.23}$$

Below, for the *normalized* intrinsic states, we employ the notation $|\Phi_F\rangle$ and $|\Phi_B\rangle$, respectively. The criterion for choosing the coefficients x_L^B is taken to be

$$\langle \Phi_F|R(\theta)|\Phi_F\rangle \cong \langle \Phi_B|R(\theta)|\Phi_B\rangle \ , \tag{18.24a}$$

where

$$R(\theta)=\exp(-i\theta J_y) \tag{18.24b}$$

is the rotation operator for a rotation about the (intrinsic) y axis, assumed, semiclassically, to be the axis of collective rotation, and θ is the usual polar angle. [Without taking a full detour to justify this assumption in detail, it may nevertheless be useful to remark that Eq. (18.24a) was motivated by a desire to have angular-momentum-projected results correspond in the two spaces (Ring and Schuck, 1980).] In an earlier communication (Otsuka, 1984), the mapping had been defined by choosing $x_L^B = x_L$. To a good approximation this is equivalent to satisfying the condition (18.24) at $\theta=0$. An improved solution is obtained by requiring also that the second derivative of (18.24) with respect to θ at $\theta=0$ be satisfied. Equivalently, this is the condition that the average angular momentum content be the same in the two intrinsic states. From calculated examples, for instance, for ^{158}Gd, with an assumed deformation parameter of 0.3, as shown in Fig. 21, the criterion (18.24) is well satisfied once it is fit near the origin.

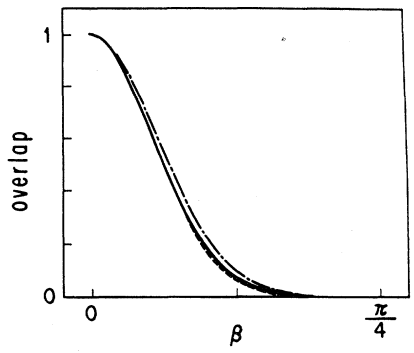

FIG. 21. The rotation matrix element as a function of the angle β for ^{158}Gd with $\delta=0.3$: solid line, the matrix element for the multi-Λ-pair system; dash-dotted line, bosons obtained with the same amplitude as the Λ pair ($x_L^B=x_L$); dashed line, boson obtained for optimized amplitudes. From Otsuka and Yoshinaga, 1986.

However, the mapping of operators is still not determined. For any number-conserving operator $\mathcal{Q}$, one would like to impose the condition that generalizes (18.24), namely,

$$\langle \Phi_F|\mathcal{Q}_F R(\theta)|\Phi_F\rangle \cong \langle \Phi_B|\mathcal{Q}_B R(\theta)|\Phi_B\rangle \ . \tag{18.25}$$

Again one makes the same compromise as was made for the overlap (unit operator), namely, one equates values and second derivatives with respect to angle, both evaluated at $\theta=0$. It turns out that these are not enough conditions, for example, to give a unique determination of the mapping of the quadrupole operator. This permits a creative adjustment that yields a mapped quadrupole operator agreeing well with its fermionic counterpart. The test of this agreement comes from a comparison of values of diagonal matrix elements in states of definite angular momentum, obtained by a standard projection procedure applied to each space.

By considering intrinsic states for neighboring values of N, we can also extend the procedure described above for number-conserving operators to pairing operators, with only a small loss of cogency. The results for the pairing energy are off by a factor of two, although the individual angular-momentum-projected pairing matrix elements are accurately reproduced. Improved procedures are promised for the future.

Before going on to describe an alternative procedure, we remark on an interesting and important side issue, the renormalization of g-boson effects. This subject was

treated previously for spherical nuclei by perturbative methods (Otsuka, 1981; Sage *et al.*, 1982; Scholten, 1983; van Egmond and Allaart, 1984). However, for deformed nuclei these renormalization effects are large enough that a nonperturbative treatment is necessary. Otsuka and Ginocchio (1985) have applied a unitary transformation to an IBM-2 Hamiltonian containing a description of the coupling between g bosons and the s-d space and have shown how to determine the parameters of this transformation so as to decouple the two spaces as much as possible.

We turn now to a description of an alternative partial solution (just as the previous work is at best a partial solution) to the problem of mapping the intrinsic Λ-pair state. This work is based on the following hypotheses (Dukelsky *et al.*, 1986; Dukelsky and Pittel, 1986a, 1986b):

(i) The values of the x_L^B mapping coefficients are to be chosen so that the matrix elements of the single-particle density operator are preserved.

(ii) This proposal is made definite by choosing the BZM mapping, wherein we recall from Eq. (9.10) that the elements of the density operators map as linear combinations of elements of the boson density operator. This makes it easy to satisfy (i) and guarantees that the values in the intrinsic state of one-body operators will be preserved under the mapping. This is verified for the quadrupole operator, as shown in Fig. 22, which illustrates a test calculation for a major shell of protons. An interesting result of this particular rule of mapping, as illustrated by the figure, is that, in contrast to the previous rule, where the importance of angular-momentum four effects in the fermion space is directly reflected in the boson space, this form of mapping yields S-D dominance.

The mapping of pair operators presents a special difficulty in view of the nontrivial square-root operators that occur and the fact that, since "spherical" bosons are used in the mapping, a straightforward expansion of the square root is not permissible. An alternative suggestion, the expansion of the square root about its deformed average, appears to provide only a partial solution to the problem at hand. Here, as well, further developments are awaited.

We close this discussion with remarks concerning an intrinsic limitation of both approaches summarized above. It is well known that the description of the ground-state band by a fixed intrinsic state is not an accurate basis for calculating the moment of inertia of such a band, which is well described by the self-consistent cranking formula (Thouless and Valatin, 1961). The latter corresponds to a intrinsic state that varies with angular frequency, this variation providing approximately for the restoration of rotational invariance, as demonstrated by several authors (Dreizler and Klein, 1965; Kamlah, 1968; Beck, Mang, and Ring, 1970). The point is that, even if one constructs a mapping procedure that duplicates the properties of the Hartree-Bogoliubov state, this is far from duplicating experiment, except for ground-state properties. We remain convinced that the difficult problem under discussion is still far from a satisfactory solution.

Precisely this question has been addressed in a recent paper (de Winter, Walet, and Brussaard, 1988b). These authors calculate number-projected, constrained Hartree-Fock-Bogoliubov states as input to separately done, i.e., one-variable, generator coordinate calculations for a β band, a γ band, and a $K^\pi = 1^+$ state (scissors mode). The model space was restricted to one major shell, $50 \le Z \le 82$, $82 \le N \le 126$, and specifically the nucleus ^{156}Gd was studied. We emphasize that the constrained solutions (cranking) provide a continuous manifold of intrinsic operators that, upon angular momentum projection, further provides a continuous manifold of $S, D, G, \ldots$, pairs. It is found that keeping the structure of each pair fixed and varying only their relative amplitudes in the intrinsic pair operator provides a poor approximation to the potential-energy surface and to the low-lying intrinsic states. Most of the inaccuracy can be

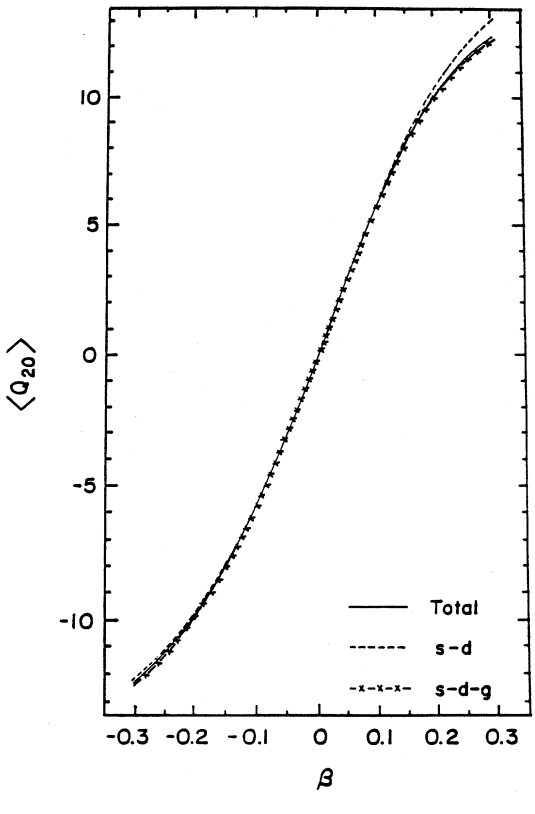

FIG. 22. Intrinsic quadrupole moment as a function of β. The solid curve (denoted Total) was obtained using either the Hartree-Fock-Bogoliubov intrinsic state or the full boson intrinsic state. The s-d and s-d-g curves were obtained by truncating to only s-d and s-d-g bosons, respectively. From Dukelsky and Pittel, 1986b.

repaired by including two fixed sets of pair excitations, which may roughly be described as ground-state pairs and β-band pairs. This raises a serious question. If the road to a microscopic interacting boson model description is through a band picture, then the clear implication is that in the deformed region single $s,d,\ldots$ degrees of freedom will not suffice for a phenomenological description. But there is the obverse side of the coin, namely, if the simple interacting boson model picture works, then the band picture is not necessarily the way to derive it.

D. L. M. Yang's method

The work to which we now turn is closest in spirit to a method suggested by Klein (1980b) for attacking the problem of microscopic foundations, namely, to first map a subspace of the fermion space, containing only the s and d degrees of freedom, and subsequently to do restricted dynamics in the boson space. We shall describe this work (Yang, 1983; Yang, Lu, and Zhou, 1984) in language that permits us to take advantage of notations that have been developed previously in this review without reproducing technical details. (The authors present their work in rather different terms.) One deals throughout with a seniority-based mapping. In addition to the conventional applications described below, a contribution to the problem of deformed intruder states in the Sn isotopes should be mentioned (Yang, Song, and Wang, 1986).

The first conceptual step is to consider a multilevel mapping of s and d degrees of freedom. As opposed to the method of Bonatsos (Bonatsos, Peres Menezes, and Klein, 1988), the technique of mapping utilized by Yang resembles most closely the OAIT method (Otsuka, Arima, Iachello, and Talmi, 1978) described in Sec. XVII. These two methods resemble each other in that the "coefficients" of the various terms in the expansion are operator functions of the occupation numbers of s and d bosons. As we have pointed out in Sec. XVII, they differ in how the Hamiltonian is mapped as well as in the property of the mapping of states that one is not forced in higher order to introduce additional degrees of freedom.

The next step is to construct the collective subspace in the boson space. This is done by linear transformations of the bosons that preserve number and angular momentum, i.e., they are the boson analogs of Eqs. (18.11) and (18.15). A technical difficulty appears because the mapping of the fermion pair and multipole operators contained square roots of the boson number operators of the individual levels. Under linear transformation of the boson operators, these become intractable. We faced this problem as early as Sec. IV, in the discussion of the quantized Bogoliubov transformation. (In fact, Yang's account makes use of this transformation.) The square roots involved are approximated as linear functions of the number of d bosons with coefficients depending on the total number of bosons, i.e., as a first-order expansion in the number of d bosons.

Using the procedure outlined above, the mapped operators, including the Hamiltonian, are determined to the order required to yield a formalism of the IBM type, except for the extra dependence on the number of d bosons, provided one has done appropriate dynamics to determine the correlated bosons. To solve the latter problem, it is natural to apply the Hartree-Bose method, first described in Sec. XIV. One is finally free to construct the eigenstates of the determined Hamiltonian in an IBM basis. The formalism can also be extended to odd nuclei.

Calculations have been carried out with a standard schematic Hamiltonian of the same form as used in Sec. XV. Because of the unavailability of suitable computers, the calculations are not sufficiently realistic, compared to some of the work discussed earlier in this section, for us to quote at this time. In effect, coupling strengths were adjusted unrealistically in order to reach agreement with experiment. We believe, however, that the method (still incomplete because the problem of renormalization effects of neglected parts of the space has not been considered) is a promising one that is deserving of further study.

A final comment of a technical nature is required. Yang would have us believe that he does the theory completely in the fermion space, in contrast to the description we have given of his work. But we have explained several times in this review that the orthonormal boson basis can be thought of, equivalently, as a basis in the fermion space. Yang's assertions must be understood in this sense.

E. Non-Hermitian mappings

We study next methods based on versions of the generalized Dyson mapping. The first work of this kind (Zirnbauer and Brink, 1982; Zirnbauer, 1984) was also the first actually to produce a spectrum from an IBM-like Hamiltonian, albeit a non-Hermitian one. If Zirnbauer had utilized a schematic Hamiltonian such as that discussed in Sec. XVI and preceding sections, we could have described his work as the Dyson mapping of the broken-pair approximation, in that the fermion S pair was determined as a number-conserving Cooper pair (albeit in an approximate manner), and the fermion D pair was determined as a one-phonon state in which the angular momentum can be carried either by a neutron pair or by a proton pair, in short, as a Tamm-Dancoff phonon. The details of the mapping can be understood by reference to the more recent work of Takada described in Sec. XVI. As we have explained there, when the Hamiltonian contains a sufficiently general two-body force, the mapping of the Lie algebra fails to determine a unique mapping of the Hamiltonian. In order for the mapping of the Hamiltonian to preserve both the pairing and the multipole properties of the interaction, a general procedure is to map so that the equation of motion, i.e., a set of commutators of generators with the Hamiltonian, are preserved

in a suitably chosen subpace. Alternatively, one may apply the OAI method of preserving the matrix elements of the Hamiltonian in this subspace. Zirnbauer applies a version of the latter method in which the space chosen is a six-parameter space of intrinsic states.

The calculation is based on the fermion intrinsic state

$$|\phi\rangle \equiv |\phi(\tau,\lambda)\rangle \equiv \exp\left[\tau S^\dagger + \sum_\mu \lambda_\mu D_\mu^\dagger\right]|0\rangle = \exp\left[\frac{1}{2}\sum_{12}\chi_{12}(\tau,\lambda)a_1^\dagger a_2^\dagger\right]|0\rangle , \quad (18.26)$$

where the numerical subscripts each stand for a complete set of single-particle labels, and the pair operators $S^\dagger$ and $D^\dagger$ are special cases of Eqs. (18.11) and (18.15). Since we want to consider different values of the parameters τ and λ in the bras and in the kets, we utilize subscripts L and R to render this distinction. To define the mapping procedure, we introduce the usual Dyson mapping to antisymmetric bosons,

$$a_1^\dagger a_2^\dagger \rightarrow b_{12}^\dagger - \sum_{1'2'} b_{11'}^\dagger b_{22'}^\dagger b_{1'2'} , \quad (18.27a)$$

$$a_2 a_1 \rightarrow b_{12} , \quad (18.27b)$$

$$a_1^\dagger a_2 \rightarrow \sum_3 b_{13}^\dagger b_{23} . \quad (18.27c)$$

The mapping is then defined in the first place by a correspondence of states

$$|\phi(\tau_R,\lambda_R)\rangle \rightarrow |\phi_B(\tau_R,\lambda_R)) = \exp\left[\frac{1}{2}\sum_{12}\chi_{12}(\tau_R,\lambda_R)\left(b_{12}^\dagger - \sum_{1'2'} b_{11'}^\dagger b_{22'}^\dagger b_{1'2'}\right)\right]|0) , \quad (18.28a)$$

$$\langle\phi(\tau_L,\lambda_L)| \rightarrow (\phi_B(\tau_L,\lambda_L)| = (0|\exp\left[\frac{1}{2}\sum_{12}\tilde{\chi}_{12}(\tau_L,\lambda_L)b_{12}\right] . \quad (18.28b)$$

To the prescription just given, we must add a mapping of operators. In particular, we seek a boson Hamiltonian H_B, to be determined by the requirement

$$\langle\phi(\tau_L,\lambda_L)|H|\phi(\tau_R,\lambda_R)\rangle = \langle\phi_B(\tau_L,\lambda_L)|H_B|\phi_B(\tau_R,\lambda_R)\rangle . \quad (18.29)$$

We are interested in finding an approximate version of Eq. (18.29) such that the boson Hamiltonian is a non-Hermitian version of an IBM-1 Hamiltonian,

$$H_B = \epsilon_s s^\dagger s + \epsilon_d d^\dagger\cdot\tilde{d} + u_0 s^\dagger s^\dagger ss + u_2 s^\dagger d^\dagger\cdot\tilde{d}s + (v_0^{(a)}s^\dagger s^\dagger d\cdot d + v_0^{(b)}d^\dagger\cdot d^\dagger ss) + (v_2^{(a)}s^\dagger d^\dagger\cdot(\tilde{d}\times\tilde{d})^{(2)} + v_2^{(b)}(d^\dagger\times d^\dagger)^{(2)}\cdot\tilde{d}s) + \sum_{L=0,2,4} c_L(d^\dagger\times d^\dagger)^{(L)}\cdot(\tilde{d}\times\tilde{d})^{(L)} . \quad (18.30)$$

It is, however, impossible to satisfy Eq. (18.29) by the form (18.30) unless the left-hand side of the former is suitably approximated. The procedure for doing this requires several steps that we give in outline only. One first demonstrates that the Hamiltonian overlap can be written in the form

$$\langle\phi(\tau_L,\lambda_L)|H|\phi(\tau_R,\lambda_R)\rangle = \langle\phi(\tau_L,\lambda_L)|\phi(\tau_R,\lambda_R)\rangle\{\langle 0|[B_L,H]\bar{B}_R^\dagger|0\rangle + \tfrac{1}{4}\langle 0|[B_L,[B_L,H]]\bar{B}_R^\dagger\bar{B}_R^\dagger|0\rangle\} = \langle\phi(\tau_L,\lambda_L)|\phi(\tau_R,\lambda_R)\rangle\times\{\langle 0|\bar{B}_L[H,B_R^\dagger]|0\rangle + \tfrac{1}{4}\langle 0|\bar{B}_L\bar{B}_L[[H,B_R^\dagger],B_R^\dagger]|0\rangle\} , \quad (18.31)$$

where the undefined quantities are

$$B_L = \frac{1}{2}\sum_{12}\tilde{\chi}_{12}(\tau_L,\lambda_L)a_1 a_2 , \quad (18.32a)$$

$$\bar{B}_R^\dagger = \frac{1}{2}\sum_{12}[\chi(\tau_R,\lambda_R)(1+\tilde{\chi}(\tau_L,\lambda_L)\chi(\tau_R,\lambda_R))^{-1}]_{12}a_1^\dagger a_2^\dagger . \quad (18.32b)$$

Now for the mapping (18.29) to be exact [with H_B given by Eq. (18.30)], in the sense that the energy eigenvalues are the same in the two spaces defined by the corresponding intrinsic states and a one-to-one correspondence between fermion and boson eigenstates can be established, it is necessary that the S-D subspace completely decouple from all other states. This happens if and only if $\langle 0|[B,H]$ and $\langle 0|[B,[B,H]]$ lie in the S-D subspace for any B that is a linear combination of S and D_μ (Ginocchio and Talmi, 1980). In situations where these stringent conditions are not satisfied exactly, one projects on the S-D subspace:

$$\langle\phi(\tau_L,\lambda_L|H|\phi(\tau_R,\lambda_R)\rangle \cong \langle\phi(\tau_L,\lambda_L)|H|\phi(\tau_R,\lambda_R)\rangle_{\text{proj}} \equiv \langle\phi(\tau_L,\lambda_L)|\phi(\tau_R,\lambda_R)\rangle\{\langle 0|[B_L,H]\hat{P}_{SD}\bar{B}_R^\dagger|0\rangle + \tfrac{1}{4}\langle 0|[B_L,[B_L,H]]\hat{P}_{SD}\bar{B}_R^\dagger\}|0\rangle . \quad (18.33)$$

The substitution of Eq. (18.33) for the left-hand side of (18.29) now provides a consistent condition that serves to determine the parameters of (18.30). For the technical details of this determination we refer the reader to the original papers.

The discussion above pertains to IBM-1. The extension to IBM-2 was made by adding a quadrupole-quadrupole interaction between neutrons and protons. The resulting Hamiltonian was applied to the low-lying levels of the samarium isotopes. The spectra found suffer from the same disease as was encountered by many early workers in the field, namely that the energy levels are much too far apart. This point has already been discussed several times, in this section in connection with the broken-pair approximation and previously in connection with our account of Tamura's program; it will not be dealt with further here. Both in this regard and because Zirnbauer was not aware of how to calculate transition rates within the Dyson formalism, this work, elegant though it is from the formal point of view, remains quite incomplete from the practical side.

Sugita (Sugita, Sugawara-Tanabe, and Arima, 1984) has studied the validity of the Zirnbauer mapping for a system of neutrons and protons, each moving in a single-j level and described by a standard schematic Hamiltonian, namely, pairing-plus-quadrupole interactions. Not surprisingly, it is found that the S-D truncation is appropriate for spherical or for weakly deformed systems, in that the Dyson procedure yields the same spectrum as a diagonalization of the Hamiltonian in the S-D subspace. For strong deformation, there are large deviations. Since neither of these calculations is exact, it is not clear whether either is accurate.

Next we discuss briefly recent work of Sambataro (1986, 1987, 1988a, 1988b; for related earlier work see Sambataro and Insolia, 1986; Sambataro, Schasser, and Brink, 1986). This work is confined to the single-j level and quadrupole-quadrupole interactions; it is not yet clear that it will work for more realistic systems. Inspired by the "simple correspondence" idea of Ginocchio and Talmi (1980), Sambataro first constructs a (non-Hermitian) boson Hamiltonian that reproduces exactly the appropriate set of low-lying levels of his model system for two and four particles (we discuss only even systems of one type of nucleon). As a general mapping procedure, he adopts the same criterion, in principle, as utilized by Zirnbauer, namely, the preservation of the Hamiltonian overlap between intrinsic states. The Hamiltonian that works for the few-particle system then fails this test, but he makes the interesting observation that by shifting the ground-state energy and rescaling his boson Hamiltonian, both changes depending on particle number and level degeneracy, he is able to satisfy his mapping criterion to good accuracy. So far, Sambataro's work is restricted to single-j descriptions and to quadrupole-quadrupole interactions.

An important critique of those mapping procedures for the quadrupole operator that omit all contributions but those of the collective operators has been made by de Winter (de Winter, Walet, and Brussaard, 1988a). Though the calculations are carried out within the framework of the Dyson mapping, the results are equally germane to Holstein-Primakoff work. De Winter *et al.* first carried out a number-projected HFB calculation for a model system with one kind of nucleon, in a model space representing a realistic version of the 50-82 major shell and with a schematic Hamiltonian consisting of pairing-plus-quadrupole-quadrupole forces. The strength of the latter was varied in order to control the deformation predicted by the model. The quantity chosen for study under the mapping was

$$\frac{\langle x|Q_0|x\rangle}{\langle x|x\rangle}\rightarrow\frac{(x|Q_B|x)}{(x|x)}\ , \tag{18.34}$$

where the operators on the left and right are the fermion quadrupole tensor and its mapped boson counterpart, respectively, and the states in question are

$$|x\rangle=|(S^\dagger+xD_0^\dagger)^N\rangle \tag{18.35}$$

in the fermion space, with the bra

$$(x|=((s+xd_0)^N| \tag{18.36}$$

in the boson space taking this simple form because of Eq. (18.27b). By the same token, because of Eq. (18.27a), the mapping of the kets is more complicated. The latter involves the mapping of collective linear combinations of both sides of Eq. (18.27a). Because of angular momentum coupling, bosons other than the collective ones will occur in the trilinear term. What these authors demonstrate rather convincingly is that, unless one includes some contributions other than the purely collective ones, the numerical requirement (18.34) cannot be satisfied. They show that in addition to the usual s and d bosons, they require an additional pair of bosons s' and d' in order to achieve their goal. The structure of these additional bosons is determined by the following algebraic argument: Let Z_{ab}^J be the expansion coefficients of the s and d bosons in terms of the elementary bosons of Eq. (18.27), and define a *fermion* pair by means of the same linear combination, $\Lambda_{JM}^\dagger=\frac{1}{2}Z_{ab}^J[a_a^\dagger a_b^\dagger]_M^J$. The calculation of a double commutator

$$[\Lambda_{L_3M_3},[\Lambda_{JM},\Lambda_{L_1M_1}^\dagger]]=\frac{1}{2}\sum_{L_2bd}Z_{bd}^{\prime L_2}[a_b a_d]^{(L_2)} \tag{18.37}$$

yields a set of primed coefficients that can be used to define primed bosons that almost commute with the unprimed ones and that prove most effective to solve the problem at hand. What we have here is a possible basis for a renormalization procedure.

F. Fermion dynamical symmetry model and the interacting boson model

The fermion dynamical symmetry model (Casten *et al.*, 1986; Chen, Feng, and Wu, 1986; Wu *et al.*, 1986; Wu *et al.*, 1987; Lü *et al.*, 1988) is a simplified shell mod-

el and, as such, does not strictly belong in this review. In fact, no effort will be made to do the model justice by entering very deeply into its confrontation with data. Rather we shall be concerned with broad technical issues, mainly the relation between this model and the interacting boson model, insofar as light is shed on the questions of interest by boson correspondences. The fermion dynamical symmetry model proposes a way of converting the Ginocchio models, SO(8) and Sp(6), into a useful phenomenological description of collective motion that operates in the original shell-model space. We have already remarked how the SO(8) model contains the vibrational and γ-soft limits of the interacting boson model. Furthermore, the Sp(6) model contains an SU(3) subalgebra that one might wish to associate with the deformed limit of the IBM. If it is assumed, however, that all the subshells participate in the representation, then because of Pauli principle restrictions, it can be shown that the most symmetric representation, that utilized in the deformed limit of the IBM, cannot occur. For this reason, Sp(6) was excluded initially as a candidate for the microscopic basis of the interacting boson model. What has altered the picture is the assumption that the unnatural parity level (which occurs in each major shell beyond the lowest few because of spin-orbit coupling) does not, in fact, participate in the scheme, but instead forms, in lowest approximation, a system with seniority zero. With this assumption, it then turns out that the remaining levels of every major shell can be analyzed and assigned as being either k-active [Sp(6)] or i-active [SO(8)]. For the heaviest shells, this assignment is unique, and otherwise both possibilities can occur. [The notion of k-active or i-active was essentially introduced in Sec. XVII, referring to either the pseudo-orbital (k) or the pseudospin (i) angular momentum's not being coupled in pairs. As indicated above, the former leads to Sp(6), the latter to SO(8) symmetry.]

Both of the algebras under discussion can be mapped onto an s-d boson space, and because of number conservation these mappings can always be expressed in terms of the generators of U(6). We may therefore ask in what sense the fermion dynamical symmetry model is more general than the IBM, and also in what sense, if any, it is less so. In this connection, we take what may not be a universally accepted view, but one that we have already espoused in Sec. VIII.B. Forgetting for the moment about g bosons and other necessary complications of real life, we take the point of view that IBM-1 is defined simply as physics in a symmetric representation of U(6), whose size N (the number of bosons) may or may not be determined by the nearest closed shell. While hoping for simplicity, of course, it then becomes the business of the experimentalist to tell us if the Hamiltonian can be restricted to two-boson terms or if transition operators are linear in the generators. In this sense, any physics contained in the fermion dynamical symmetry model can also be accommodated within our definition of the interacting boson model.

As is well known, U(6) has only three (linear) chains of subalgebras, ending with the physical angular momentum and corresponding to the vibration, γ-soft, and rotational limits, all of which have some experimental underpinning. On the other hand, the fermion dynamical symmetry model, in its SO(8) incarnation, has, in addition, an SO(7) chain. There may well be evidence for the occurrence of this symmetry in the Ru and Pd isotopes (Casten *et al.*, 1986). Another general type of evidence for the specific validity of the fermion dynamical symmetry model arises if one makes the assumption, for instance, that the quadrupole generator of the appropriate fermion dynamical symmetry algebra is to be used as the physical quadrupole operator in confronting experiment. Its boson map then contains unique Pauli reduction factors that can be tested against experiment. Though there may be some positive evidence for the model in this regard, it should be noted that any bona fide mapping from the shell model to the s-d boson space will contain Pauli reduction factors other than those that appear explicitly in the phenomenology of the interacting boson model, so that one must be careful to distinguish between evidence for the Pauli principle, and evidence for a specific fermion model. Nevertheless, the data can be read as suggesting that the fermion dynamical symmetry model may be, at the very least, a useful first approximation to the shell model. In this regard, there intrudes the most serious question concerning its basis: Where is the effect of the splitting of the single-particle levels?

To terminate this all-too-brief discussion, we return to one of the issues discussed above, where bosons intervene. In a recent contribution (Geyer, Hahne, and Scholtz, 1987), it was pointed out on the basis of a specific mapping that, since the SO(7) limit could be accommodated within the s-d boson framework, it was not clear what purpose, if any, is served by the fermion dynamical symmetry model. Though we were in danger of approaching such a viewpoint above, because our definition of IBM-1 renders the observation of Geyer *et al.* a truism, we nevertheless cannot follow them to their conclusion. To have a theory as relatively simple as the fermion dynamical symmetry model as a halfway house between the full complexities of the realistic shell model and the interacting boson model is manifestly of interest and importance. Furthermore, the fermion dynamical symmetry model can be viewed in the same light as the broken-pair approximation in that an entire major shell can be decomposed into subspaces of different heritage quantum number u (defined in Sec. XVII). In the sense that collectivity at low energies can be associated largely with the $u=0$ subspace, the scheme may have great advantages for approximate shell-model calculations in heavy nuclei (Novoselsky, Vallières, and Gilmore, 1988; Chen *et al.*, 1989; Wu and Vallières, 1989).

G. Interacting fermion-boson model

Though the IBM-1 is generally understood to be a phenomenological model, the interacting boson-fermion

model (IBFM) is often referred to as semimicroscopic. This can be understood if we elaborate on the very brief allusion to the derivation made in the introduction to this section. From the point of view that has been most consistently followed in this review, the most satisfactory method of founding a boson-fermion model would be to utilize a generalized quantized Bogoliubov transformation, as advocated in Sec. V, in which the s and d degrees of freedom are bosonized and the remaining degrees of freedom are treated as quasifermions. Precisely this approach, carried out approximately, can be found in the work of Scholten (Scholten and Dieperink, 1981). To the lowest order in the d bosons, the mapping of the original fermions $a^\dagger_{jm}$ onto s and d bosons and quasifermions $c^\dagger_{jm}$ is rendered by the formula

$$a^\dagger_{jm}=u_j c^\dagger_{jm}+\frac{v_j}{\sqrt{N}}(s^\dagger c_j)^{(j)}_m + \sum_{j'} u_j \bar{\beta}_{j'j}\left[\frac{10}{2j+1}\right]^{1/2}(d^\dagger \tilde{c}_{j'})^{(j)}_m - \sum_{j'}\frac{v_j}{\sqrt{N}}\bar{\beta}_{jj'}\left[\frac{10}{2j+1}\right]^{1/2}(s^\dagger \tilde{d} c^\dagger_{j'})^{(j)}_m . \quad (18.38)$$

In this formula, the coefficients $\bar{\beta}_{jj'}$ are normalized coefficients defining the collective D pair,

$$\bar{\beta}_{jj'}=\beta_{jj'}\Big/\left[\sum_{j''j'''}(\beta_{j''j'''})^2\right]^{1/2} . \quad (18.39)$$

Thus Eq. (18.38) is to be understood as an approximate mapping formula within the framework of the broken-pair approximation.

The initial application of Eq. (18.38) was to the quadrupole-quadrupole interaction as the major source of the terms of Eq. (18.5) depending on Γ_{ab}, the so-called direct interaction, and of the terms depending on Λ^j_{ab}, the exchange interaction. It was later remarked (Bijker and Scholten, 1985) that the quadrupole pairing interaction was an important source of the exchange interaction, or perhaps the most important source (Otsuka *et al.*, 1987; de Kock and Geyer, 1988). The form of the boson-fermion interaction actually used to confront experiment, as given in Eqs. (18.6)-(18.8), involves further simplifications, the main ones being that

$$\beta_{jj'}=(u_j v_{j'}+v_j u_{j'})Q_{jj'} \quad (18.40)$$

and

$$Q_{jj'}=\langle l\tfrac{1}{2}j\|Y^{(2)}\|l\tfrac{1}{2}j'\rangle . \quad (18.41)$$

The former follows from the assumption that the D pair carries the full $E2$ sum-rule strength (Scholten and Dieperink, 1981), whereas the latter assumes that the radial integrals in a given shell are equal, with a value that can be incorporated into an overall scale factor. The various constants that appear in Eqs. (18.6)-(18.8) are now taken as fitting parameters, and with this phenomenological viewpoint the model has been rather successful in fitting data. For a review, see Scholten (1985).

The most painstaking effort to check the IBFM from a microscopic broken-pair calculation for odd nuclei is found in a work of van Egmond and Allaart (1983), where the IBFM form (18.5) was applied to two isotopes of Xe, 125 and 131, without utilizing the special assumptions (18.6)-(18.8). A comparison of the microscopically computed parameters Γ_{ab} with those given by Eq. (18.7), shown in Fig. 23 is impressive. A corresponding comparison of the values of Λ^j_{ab} of Eq. (18.8) was given by these authors only for $a\neq b$, because the phenomenological fits do not include the term (18.6), and this prevents our disentangling the diagonal elements. The results, not reproduced, are quite reasonable.

The well-developed approach described above is, of course, applicable only to vibrational nuclei. A first effort to develop an IBFM formalism applicable to deformed nuclei has been reported (Wood and Morrison, 1988) that has detailed formulas applicable only to the s-d shell model. It is suggested, however, that the formalism may be extended to heavy deformed nuclei by means of the pseudo-SU(3) model.

An important critique of the IBFM methodology has appeared recently (Geyer and Morrison, 1989). In principle, the mapping of single-fermion operators should be

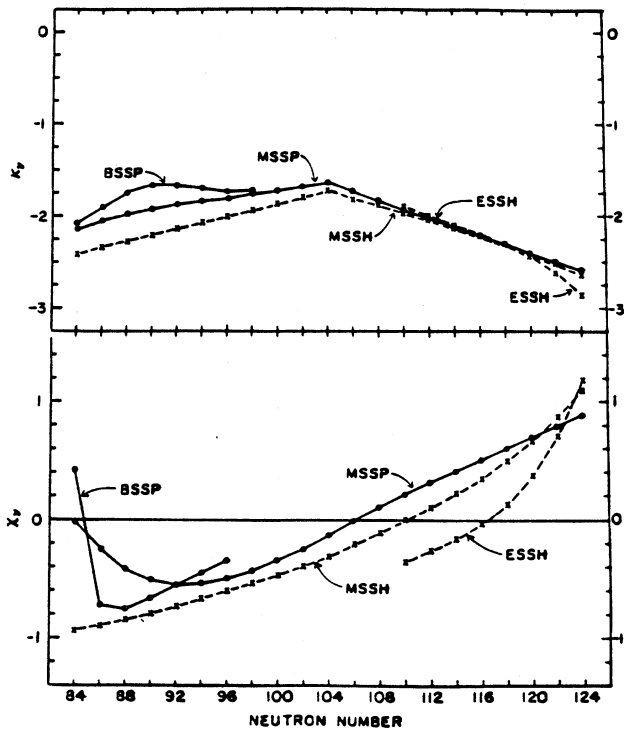

FIG. 23. Comparison of microscopically calculated values of Γ_{ab} (solid line) and those obtained from the formula used in the interacting boson-fermion model: $\Gamma_{ab}=\Gamma(a\|r^2/b^2Y_2\|b)(u_a u_b-v_a v_b)$; $\Gamma=0.13$ MeV (dashed line). From van Egmond and Allaart, 1983.

sufficiently accurate that one be able to recover the mapping of generators of the shell-model algebra by forming suitable products (and possibly taking linear combinations). This is true for the examples of the quantized Bogoliubov-Valatin transformation that appear in this review, but it is not true for Eq. (18.38). This remains an open problem, whose solution should be amenable to the methods described in Sec. XIX.

H. Other contributions

We conclude this section with a description of works that clearly have some relevance to the subject matter at hand, but do not fit neatly into any of the categories previously established. We start with a recent algebraic contribution (Kyrchev and Paar, 1986, 1987, 1988). A brief account of related work appeared earlier (Baktybaev and Strygin, 1981). The inspriation for this work goes back to a paper that contained an independent proposal of U(6) (IBM) ideas, with a derivation that was weakly microscopic and utilized the quasiparticle picture (Janssen, Jolos, and Dönau, 1974). By weakly microscopic, we mean that an algebraic argument was presented to justify the emergence of an SU(6) boson picture without in any way providing a reliable method of computing the parameters. The main purpose of the new work is to strengthen the algebraic arguments. Briefly, the treatment utilizes the quasiparticle RPA phonon creation operators $Q^{\dagger}_{\lambda\mu i}$, whose Hermitian conjugates annihilate the RPA vacuum. The three subscripts denote angular momentum, its z component, and a label to resolve solutions with the same angular momentum content. The collective operators are assumed to be those with $\lambda=2$, $i=1$. The quasiparticle shell-model algebra is written in terms of these operators. The subalgebra generated by the commutators involving the collective operators is then forced to close under itself by the common expedient of simply dropping the unwanted ("scattering") terms. This is trivial, but unfortunately what remains does not constitute a Lie algebra. The nontrivial step comes next in forcing the Q, the $Q^{\dagger}$, and their mutual commutators, 35 operators in all, to satisfy appropriate Jacobi identies. This leads to constraints on the RPA solutions that they do not necessarily satisfy. Kyrchev and Paar talk about redefining the dynamical problem so as to incorporate the new constraints, but how well this can be done remains a key question with their approach. This is, however, the same problem that arises in most recent work of Takada *et al.* discussed in Sec. XVI, who solved it approximately by the introduction of a single scaling parameter that modifies the definition of the most collective mode. In any event, the method of Kyrchev and Paar is still weakly microscopic, in the sense defined above, since it is difficult to see how even the modified RPA excitations will serve for the physics of all the regimes to which the interacting boson model is applicable, though one may be able to go beyond the point of breakdown of the usual RPA.

Still another novel attempt to derive the interacting boson model starting from the language of quasiparticles has been described in the recent literature (Ivanova, Jolos, and Pedrosa, 1989). The basic new theoretical contribution here is the derivation of approximations to the square-root operator (which occurs in the Holstein-Primakoff mapping) that avoid perturbative expansions, in part at least. It is not clear from this first account if this approach offers any advantage over the seniority-dictated mapping described in Sec. XVII.

It is important to mention, finally, a unique and precisely formulated program in the course of execution by Elliott, Evans, and their collaborators (Evans, Elliott, and Szpikowski, 1985; Elliott and Evans, 1987; Thompson, Elliott, and Evans, 1987; Elliott, Evans, and Van Isacker, 1988; Evans, Van Isacker, and Elliott, 1988). This work is devoted to the study of a mapping of a single j shell of neutrons and protons to the IBM-3 form. In IBM-3, because one is dealing with a situation in which charge independence is being enforced, it is necessary to adjoin a neutron-proton (ν-π) boson to the ν-ν and π-π bosons of IBM-2 to constitute altogether a $T=1$ isospin triplet. One then investigates—in the sense of the OAI—the mapping of an isospin-invariant seniority shell-model basis to a boson basis. Consider first the two-fermion system. Focusing on the s and d degrees of freedom only, one identifies the states that will have a boson counterpart and refers to these as the boson states; the exact shell-model energies of these states provide the single-boson energies of the boson Hamiltonian (specific calculations refer to the $f_{7/2}$ shell). Next, the OAI procedure and the chosen subspace fix the membership of the two-boson (four-fermion) states. An exact shell-model diagonalization in the four-fermion space will yield—if the seniority coupling scheme that underlies the procedure is germane to the shell-model Hamiltonian chosen—a number of eigenstates that are predominantly of bosonic type, and, from these, matrix elements of the boson-boson interaction can be deduced. Finally, the validity of the entire procedure is tested by comparing the exact spectrum and wave functions of the six-fermion system with the three-boson results derivable from the IBM-3 Hamiltonian. A corresponding program was also carried out for odd systems, thus defining IBFM-3. We remark, finally, that in studies of the nuclear s-d shell an additional ν-π, $T=0$ boson was introduced, thus defining IBM-4 (Halse, Elliott, and Evans, 1984).

XIX. BOSE-FERMI MAPPINGS AND QUANTIZED BOGOLIUBOV TRANSFORMATIONS: FURTHER EXAMPLES AND RELATION TO COHERENT-STATE METHODS

A. Survey of literature on generator coordinates and boson mappings

The remainder of this review is concerned with recent theoretical developments. Before turning to the main

business of this section, we feel duty-bound to mention some prior developments that are of more than tangential interest. As we have already pointed out as early as Sec. II and shall emphasize again below, Dyson mappings can be derived either algebraically, as we have done in this work, or by a method based on coherent states. There is at least one topic of interest to physics, however, for which the use of coherent states is almost indispensable, namely understanding the foundations of the method of generator coordinates and its connection with boson expansions. Though we shall not provide the technical details (because competent reviews exist, see below), a brief survey of the relevant literature is in order.

The method of generator coordinates was introduced into nuclear physics to treat collective motion (Griffin and Wheeler, 1957) and to restore symmetries broken by Hartree-Fock calculations (Peierls and Yoccoz, 1957; Peierls and Thouless, 1962). In this latter connection, the projection of states of good angular momentum from intrinsic deformed states and the projection of states of sharp particle number from BCS states remains a fundamental tool of the nuclear shell-model theorist (Ring and Schuck, 1980). A decisive turn may be said to have been taken with the paper of Jancovici and Schiff (1964), which introduced complex generator coordinates in a derivation of the RPA, and with the subsequent quasiboson interpretation of this work (Brink and Weiguny, 1968; Ui and Biedenharn, 1968; da Providência, 1970). This work is reviewed by Wong (1975). More recent developments in this field are reviewed by Reinhard and Goeke (1987).

Our main interest in this subject, however, is in the theoretical connection between generator coordinates and boson expansions. Here the ground-breaking papers include a series of works by Holzwarth (1968, 1969, 1970, 1971) which apply generator coordinate methods in a boson basis, microscopically derived, culminating in the derivation of a complete boson mapping (Holzwarth, 1972) that may be characterized loosely as a summed form of the Marumori mapping, non-normal ordered. Simple applications (da Providência, Urbano, and Ferreira, 1971; Caldeira, Pascoal, Ruivo, and Silva, 1972) and fundamental mapping theory (da Providência, 1974) were also carried out by da Providência and associates. Other seminal works (Janssen, Dönau, Frauendorf, and Jolos, 1971; Hage-Hassan and Lambert, 1972) contain complete derivations of the Dyson and Marumori mappings on the basis of the method of generator coordinates, understood as a mapping between two Hilbert spaces. The work of Janssen *et al.*, in particular, has been a rich source of inspiration to subsequent workers.

We have already sprinkled this review with some examples of the ideas involved in this approach (Secs. II and IV). We have indicated that the Dyson mapping can be obtained naturally in two ways, either algebraically, as we have mostly favored doing it, and as differential operators on generalized coherent states—which in nuclear physics means either Slater determinants or BCS states. A rigorous systematization of the latter approach has been given by Dobaczewski (1981a, 1981b, 1982), whose work, in our opinion, has not received the attention it deserves. An updated version of some of the ideas developed by Dobaczewski, influenced as well by the more recent work described below, can be found in Xu, Wang, and Yang (1987).

B. Survey of main ideas

One chief purpose of this section is to relate some of the methods of this review to a recent development of great practical significance in the theory of matrix representations of Lie algebras, including all representations of compact algebras and the analogous ladder representations of associated noncompact algebras. This development is associated with the rubric "vector coherent state theory" (Deenen and Quesne, 1984; Rowe, 1984; Rowe, Rosensteel, and Gilmore, 1985), and, as such, has already been reviewed in the monograph by Hecht (1987a). The accomplishments of this program to date are amply documented in Hecht's book as well as in the introductory sections of several recent papers (Rowe, Le Blanc, and Hecht, 1988; Hecht, 1989). The papers most directly linked to the contents of this section are those of Hecht, Le Blanc, and Rowe (1987); Le Blanc and Hecht (1987); and Le Blanc and Rowe (1987), though in many respects these papers develop detailed results well beyond anything we shall describe.

It is not our aim in this section to give a detailed account of this work, which includes generalizations of some of the results described in this review. Aside from limitations of space, our reason for avoiding most of the details of these papers is that they are not couched in the language of boson realizations, though the authors are fully aware of many direct and indirect relations. Instead, our purpose will be to describe and illustrate the elements of the method within a framework that is completely adapted to the needs of the nuclear shell model. We shall then find that the natural language for our purposes is that of boson-quasifermion mappings. In this respect, the work described in Sec. III.E, on the quantized Bogoliubov transformation, and in Sec. X represents the *equivalent* of fully worked out examples of the new technique. Though this work is not sufficiently complex to illustrate all the strengths of the method, it contains enough information to permit us to describe its elements in general form, illustrating the arguments with these examples. (Some readers may find this exposition a useful reprise.) Following these general arguments (and reinterpretations), we shall illustrate with two examples the case of U(n), which has already been fully worked out previously (Hecht, Le Blanc, and Rowe, 1987; Le Blanc and Hecht, 1987) and a special case of SO($2\Omega+1$) (Klein and Marshalek, 1989) that we shall present first.

The first point to recognize is that for the Lie algebras of interest to us, the generators can be divided into three subsets: raising (creation) operators, lowering (destruc-

tion) operators, and members of a core subalgebra. For SU(2), the most trivial case, each set has a sole member, J_+, J_-, J_0, respectively. The core subalgebra here is U(1). For the orthogonal algebra SO(4Ω), discussed in Sec. IX, the sets comprise the fermion pair creation operators, the pair annihilation operators, and the multipole operators, respectively. The latter span the core subalgebra U(2Ω). Note that in this case the raising operators all commute and are Hermitian-conjugate to the lowering operators. The elements of the core subalgebra are or can be chosen to be Hermitian. For the algebra SO(4Ω+1), studied in Sec. X, the core algebra remains the same, but the single-fermion creation operators are added to the set of raising operators and their Hermitian conjugates to the set of lowering operators. In constructing the boson-quasifermion mapping, we treat the single-fermion operators differently from the pair operators. This extra complication is quite natural from a physical point of view.

The next element to recall is the special (simple) character of most of the mappings discussed in this paper, namely, mappings in which we introduced one boson creation operator per raising operator. We also restrict further considerations to the mapping that utilizes the core subalgebra in the group chain that specifies the basis. Furthermore, we perform the mapping to a unitary basis in two steps, going through the intermediary of a generalized Dyson mapping. Concentrating on the latter, we have already emphasized that it can be derived in two ways. One approach makes use of the coherent state, leading to a realization in terms of differential operators acting on a space of holomorphic functions [cf. Eqs. (2.47)–(2.51)]. The other is the algebraic approach that we have emphasized in this review, which provides a standard, uniformly applicable procedure that has been described several times: first, lowering operators are chosen proportional to the appropriate boson destruction operator. Since the raising and lowering operators form irreducible tensors under the core subalgebra, this implies that the bosons are also such tensors. The generators of the core subalgebra are thus uniquely determined as bilinear forms in the bosons, and the raising operators are then necessarily a linear combination of linear and trilinear forms in the boson operators. The remarks above imply that this trilinear operator must have the form of a commutator of the boson creation operator with an invariant of the core subalgebra, called Λ, that is a linear combination of the quadratic Casimir operator and a polynomial of second degree in the linear Casimir operator. The operator Λ plays an essential technical role in the further development. [These observations, made independently by Rowe (1984) and by Klein, Cohen, and Li (1982), have been dubbed the "Toronto trick" by Hecht (1987a)]. The algebraic method just described is well defined and completely equivalent to the use of the coherent state.

The special class of boson realizations described above is referred to in the literature on vector coherent state methods as the class of collective realizations. They can be characterized by the property that the state of maximum weight (the vacuum state) is itself a one-dimensional irrep of the core subalgebra. The essential new observation is that all the irreps (of ladder type) can be constructed by letting the same set of boson operators act on the states of an arbitrary irrep of the core subalgebra. (These latter irreps are said to define the intrinsic space.) The proof of this assertion is that it can be shown that the direct product of the boson space with the intrinsic space provides the correct number of labels for a unique specification of the basis for a general irrep. This direct-product (commuting) character of the two spaces plays an essential role in the construction of mappings.

The Dyson mapping for the general irrep can now be constructed by allowing the generators of the algebra to act on vector-valued or generalized coherent states which are related to the scalar coherent state by the replacement of the single state of extreme weight by an irrep of the core subalgebra, the intrinsic representation. Though somewhat more complicated in detail, the structure of the generalized Dyson mapping is quite similar to the scalar case. The lowering operators are unchanged, but the generators of the core subalgebra are now a direct sum of the previous collective (bosonic) part and of an intrinsic part, in consequence of the direct-product character of the basis states. The raising operators must once more be commutators of the boson creation operators with group invariants of first and second order in the generators of the core subalgebra. As we have already seen in Sec. X, it is straightforward in the general case to derive the Dyson mapping by purely algebraic means; this point will be illustrated again in this section.

The unitarization of the Dyson mapping proceeds in the same manner as has been illustrated on numerous occasions in this paper, except that applications to considerably more complicated examples have been carried through. An important contribution of the new method is to provide a natural solution to the vexing problem of missing labels that occurs when a group chain whose choice is dictated by physical considerations does not provide enough labels associated with the Casimir invariants of the subgroups to fully specifiy the basis. We shall illustrate this point in subsection D for the famous case of SU(3).

The most important new point we have to contribute to this section is the observation that, for application to nuclear physics (and probably for other cases as well), there is a most natural realization of the intrinsic representations of the core subalgebra, namely, that in terms of quasifermions. This was certainly made completely explicit for the case of the quantized Bogoliubov transformation for the SU(2) algebra, which in the language of coherent states involves only the scalar case. It is amusing to observe, however, that the material presented in Secs. IX and X can be viewed in two different ways. From the standpoint of the algebra SO(4Ω+1), the mapping developed in Sec. X constitutes a single irrep and an

example of the scalar coherent state (or, in our approach, of its algebraic equivalent). On the other hand, from the point of view of the algebra of pairs only, the algebra SO(4Ω), these same results constitute an application of the vector coherent state, in that the realization of the pair operators comprises not only a representation based on the vacuum state as intrinsic state, but also representations based on one-fermion states, i.e., intrinsic states that form the defining representation of the core subalgebra U(2Ω). Within this framework, the mapping of the single-fermion operators is interpreted as that of special tensors under the smaller group and can furthermore be understood to define implicitly (i.e., in operator form) a class of Clebsch-Gordan series. This was also the role of the quantized Bogoliubov transformation and is an idea that we wish to emphasize as the basis for future developments. In any event, the observations recorded above will be illustrated fully in the two examples worked out in this section.

C. Boson-quasifermion realization of the particle-hole SO(2Ω+1) algebra

1. Introduction

The boson-quasifermion mappings of the particle-hole algebra were first derived independently by Marshalek (1980a, 1981) and by Geyer and Hahne (1980a, 1983) using traditional techniques. While both obtained the nonunitary generalized Dyson realization, Marshalek derived in addition the unitary generalized Holstein-Primakoff realization. Afterwards, Kuriyama and Yamamura (1981c, 1981d) showed that the same mappings can be derived by means of a Dirac-bracket quantization of the time-dependent Hartree-Fock self-consistent field equations. The present account (Klein and Marshalek, 1989) is intended to supersede previous work by providing derivations within the framework described in Sec. IX.B.

We start with a review of the properties of the fermion particle-hole SO(2Ω+1) algebra and its fermion carrier space and show that one may proceed through the group chain $U_h(\Omega_h)\times U_p(\Omega_p)\subset U(\Omega)\subset SO(2\Omega)\subset SO(2\Omega+1)$, the first subgroup corresponding to the core subalgebra, with Ω_h being the number of hole levels and Ω_p the number of particle levels.

Let $c_I, c^I\equiv c_I^\dagger$, $I=1,\ldots,\Omega$, denote a set of Ω fermion destruction and creation operators. We partition this into a subset of Ω_h destruction and creation operators $c_\mu, c^\mu\equiv c_\mu^\dagger$, $\mu=1,\ldots,\Omega_h$, associated with what we call *hole states,* and a set of Ω_p destruction and creation operators $c_i, c^i\equiv c_i^\dagger$, $i=\Omega_{h+1},\ldots,\Omega_p$, associated with what we call *particle states,* with the equality $\Omega=\Omega_h+\Omega_p$. Throughout this discussion, we adhere to the following conventions: greek indices denote hole states, lowercase roman indices denote particle states, and uppercase roman indices can be replaced consistently throughout an equation by either hole or particle values, i.e., there are really two equations, one for holes and one for particles. Unless stated otherwise, the summation convention for repeated indices is assumed to hold. Thus the fermion anticommutation relations are given by

$$\{c_I,c^J\}=\delta_I^J,\quad \{c^I,c^J\}=\{c_J,c_I\}=0\ , \tag{19.1a}$$

$$\{c_\mu,c^i\}=\{c_i,c^\mu\}=\{c^i,c^\mu\}=\{c_\mu,c_i\}=0\ . \tag{19.1b}$$

As is well known, the set of all bilinear fermion operators

$$\{c^Ic^J,c_Jc_I,c^ic^\mu,c_\mu c_i,\tfrac{1}{2}(c^Ic_J-c_Jc^I),c^ic_\mu,c^\mu c_i\} \tag{19.2}$$

spans the Lie algebra corresponding to the group SO(2Ω), while the set obtained by adjoining the linear fermion operators $i(c_I+c^I)/2$, $i(c_I-c^I)/2$ spans the algebra of SO(2Ω+1). We now introduce the notation

$$A_J^I\equiv c^Ic_J\ , \tag{19.3a}$$

$$R^{i\mu}\equiv c^ic^\mu,\quad R_{i\mu}\equiv c_\mu c_i\ , \tag{19.3b}$$

$$R^{IJ}\equiv c^Ic^J,\quad R_{IJ}\equiv c_Jc_I,\quad A_\mu^i\equiv c^ic_\mu,\quad A_i^\mu\equiv c^\mu c_i\ , \tag{19.3c}$$

$$A^I\equiv c^I,\quad A_I\equiv c_I\ . \tag{19.3d}$$

These operators have the following behavior under Hermitian conjugation:

$$A_I^J=A_J^{I\dagger}\ , \tag{19.4a}$$

$$R^{i\mu}=R_{i\mu}^\dagger,\quad A^I=A_I^\dagger,\quad R^{IJ}=R_{IJ}^\dagger,\quad A_i^\mu=A_\mu^{i\dagger}\ , \tag{19.4b}$$

as well as the antisymmetry properties

$$R^{\mu i}=-R^{i\mu}\ , \tag{19.5a}$$

$$R^{JI}=-R^{IJ},\quad R_{JI}=-R_{IJ}\ . \tag{19.5b}$$

The set of operators (19.3a) consists of the subsets A_ν^μ and A_j^i, the former generating the unitary subalgebra $U_h(\Omega_h)$ based on the hole levels and the latter the unitary subalgebra $U_p(\Omega_p)$ based upon the particle levels. The long list of commutation rules for the operators defined in Eq. (19.3) that follow from the anticommutation relations (19.1) and define the algebras and subalgebras of interest in this discussion will not be exhibited here. Instead, we shall simply identify the members of each set. Thus the operators (19.3a) span the group $U_h(\Omega_h)\times U_p(\Omega_p)$. It is this subalgebra that will be chosen as the core subalgebra. The remaining generators in Eq. (19.3) are three sets of ladder operators, consisting of the particle-hole creation and destruction operators (19.3b), the two-particle transfer operators (19.3c), and the one-particle transfer operators (19.3d). When we add the first set to the core algebra, we arrive at the algebra U(Ω), the further addition of the second set brings us to the algebra SO(2Ω), and the addition of the remaining set yields the algebra SO(2Ω+1).

An orthonormal basis for the 2^Ω-dimensional fermion vector space $\mathcal{H}_F$ is provided by the set consisting of the

normalized vacuum $|0\rangle$ (the closed-shell system), satisfying

$$c_i|0\rangle = c_\mu|0\rangle = 0 \ , \tag{19.6}$$

for all i and μ, together with the vectors

$$|i_1\mu_1\cdots i_{N_B}\mu_{N_B}\rangle = \prod_{n=1}^{N_B} c^{i_n}c^{\mu_n}|0\rangle \ , \tag{19.7a}$$

$$|\nu_1\cdots\nu_{n_h}i_1\mu_1\cdots i_{N_B}\mu_{N_B}\rangle = \prod_{n=1}^{n_h} c^{\nu_n}|i_1\mu_1\cdots i_{N_B}\mu_{N_B}\rangle \ , \tag{19.7b}$$

$$|j_1\cdots j_{n_p}i_1\mu_1\cdots i_{N_B}\mu_{N_B}\rangle = \prod_{n=1}^{n_p} c^{j_n}|i_1\mu_1\cdots i_{N_B}\mu_{N_B}\rangle \ , \tag{19.7c}$$

which span the subspaces with an equal number N_B of particles and holes and subspaces with an excess n_h of holes and n_p of particles. In Eqs. (19.7b) and (19.7c) it is understood that in the case $N_B=0$, the ket on the right-hand side becomes the vacuum $|0\rangle$. The fermion space $\mathcal{H}_F$ carries the solitary spinor irrep of SO($2\Omega+1$), while the subspaces with even and odd particle numbers separately carry the two $2^{\Omega-1}$-dimensional spinor irreps of the subgroup SO(2Ω). The fermion space also carries $\Omega+1$ antisymmetric irreps of the subgroup U(Ω), each of which may be labeled by N_p-N_h, the difference in the number of particles and holes. These irreps may be further decomposed into a total of $(\Omega_h+1)(\Omega_p+1)$ antisymmetric irreps of the subgroup $U_h(\Omega_h)\times U_p(\Omega_p)$, each labeled by the number of particles and the number of holes.

Of central importance in our later analysis are the Casimir invariants of $U_h(\Omega_h)$ and $U_p(\Omega_p)$. In terms of the generators, the corresponding Casimir operators of order k are given by (summation convention)

$$\mathcal{C}_h^{(k)} = A^{\mu_1}_{\mu_2}A^{\mu_2}_{\mu_3}\cdots A^{\mu_k}_{\mu_1}, \quad \mathcal{C}_p^{(k)} = A^{i_1}_{i_2}A^{i_2}_{i_3}\cdots A^{i_k}_{i_1} \ . \tag{19.8}$$

Upon inserting the fermion realization (19.3a) and performing a trivial rearrangement, one obtains for Eq. (19.8) the diagonal forms

$$\begin{aligned}\mathcal{C}_h^{(k)} &= \hat{N}_h(\Omega_h-\hat{N}_h+1)^{k-1} \ , \\ \mathcal{C}_p^{(k)} &= \hat{N}_p(\Omega_p-\hat{N}_p+1)^{k-1} \ ,\end{aligned} \tag{19.9}$$

where $\hat{N}_h$ and $\hat{N}_p$ are the hole and particle number operators

$$\hat{N}_h = c^\mu c_\mu, \quad \hat{N}_p = c^i c_i \ . \tag{19.10}$$

Each of the number operators (19.10) commutes with all the generators of $U_h(\Omega_h)\times U_p(\Omega_p)$, thereby providing labels for the antisymmetric irreps. The difference $\hat{N}_p-\hat{N}_h$ commutes with all the generators of U(Ω), thereby providing a label for its irreps.

Our aim is to map the algebra discussed above, together with the finite-dimensional fermion carrier space $\mathcal{H}_F$, into a subspace of a Hilbert space, denoted here by $\mathcal{I}$. The space $\mathcal{I}$ is generated by the boson destruction and creation operators $B_{i\mu}, B^{i\mu}\equiv B^\dagger_{i\mu}$, respectively, together with a set of what we call quasifermion destruction and annihilation operators, $a_\mu, a^\mu\equiv a^\dagger_\mu$ associated with hole levels and $a_i, a^i\equiv a^\dagger_i$ associated with particle levels. The bosons $B_{i\mu}$ and $B^{i\mu}$, which replace the degrees of freedom of particle-hole pairs $c_\mu c_i$ and $c^i c^\mu$, respectively, obey the Heisenberg-Weyl algebra

$$[B_{j\nu}, B^{i\mu}] = \delta^i_j\delta^\mu_\nu, \quad [B^{i\mu}, B^{j\nu}] = [B_{j\nu}, B_{i\mu}] = 0 \ . \tag{19.11}$$

The quasifermions, which are to represent *valence* particles or holes, are assumed to commute with all the boson operators,

$$[a^I, B_{i\mu}] = [B^{i\mu}, a_I] = [a^I, B^{i\mu}] = [B_{i\mu}, a_I] = 0 \ , \tag{19.12}$$

and also obey an algebra to be discussed presently.

To describe the physical subspace of the Hilbert space $\mathcal{I}$, one may introduce the orthonormal basis spanned by the vectors.

$$\prod_I (a^I)^{n_I}\prod_{i\mu}(n_{i\mu}!)^{-1/2}(B^{i\mu})^{n_{i\mu}}|0) \ , \tag{19.13}$$

where $|0)$ is the normalized vacuum state (tensor product of the quasifermion and boson vacua), satisfying

$$a_\mu|0) = a_i|0) = B_{i\mu}|0) = 0 \ . \tag{19.14}$$

The physical subspace is obtained, first, by forming antisymmetric combinations of the vectors (19.13). Second, since a particle-hole combination $c^i c^\mu$ is represented by a boson $B^{i\mu}$, the quasifermions (thus the reason for the name) are required to satisfy the constraint $a^i a^\mu = 0$. It turns out that, although this constraint modifies the anticommutation relations of the quasifermions, the products that look like generators of the core subalgebra, as defined in Eq. (19.3a), also act like generators in that they satisfy the core subalgebra. This is, of course, necessary if they are to be used to realize the intrinsic space.

These assertions can be proved in two ways. A method that works in all known examples is to *assume* that the core subalgebra is satisfied when the appropriate bilinear forms in the quasifermions are chosen as generators. One further assumes that the commutators of single quasifermions with these generators are the same as if they were ordinary fermions. (In fact, this assumption implies the preceding one.) One then succeeds in constructing a Dyson mapping of the algebra, provided that, in addition to these assumptions, one has anomalous anticommutators, which in the present instance have the form

$$\{a_\nu, a^\mu\} = \delta^\mu_\nu Q_p, \quad \{a_j, a^i\} = \delta^i_j Q_h \ , \tag{19.15a}$$

$$\{a^I, a^J\} = \{a_J, a_I\} = 0 \ . \tag{19.15b}$$

Here, the operators Q_h and Q_p are defined by the equations

$$Q_h \equiv 1 - a^\mu (1+\hat{n}_h)^{-1} a_\mu \ , \qquad Q_p \equiv 1 - a^i (1+\hat{n}_p)^{-1} a_i \ , \tag{19.16}$$

(summation convention!) where $\hat{n}_h$ and $\hat{n}_p$ are quasihole and quasiparticle number operators. The Hermitian operators (19.16) satisfy

$$Q_h^2 = Q_h, \quad Q_p^2 = Q_p, \quad Q_h Q_p = Q_p Q_h \ . \tag{19.17}$$

Q_h is the projector to the subspace of $\mathcal{I}$ having zero quasiholes, while Q_p is the projector to the subspace having zero quasiparticles. For the present example, the anticommutation relations (19.15) were first established using the projector character of Eq. (19.16). The argument is given in all of the papers cited above, but will not be reproduced here.

2. The generalized Dyson mapping

In this subsection we shall use the symbols defined in Eq. (19.3) to denote the Dyson images of the quantities that these symbols previously denoted. The first task is to find the mapping of the $U_h(\Omega_h) \times U_p(\Omega_p)$ core subalgebra. This can be accomplished with the aid of the additional requirement that under the core subalgebra the boson operators $B^{i\mu}$, $B_{i\mu}$ transform like particle-hole operators and the quasifermion operators a^I, a_I transform like one-particle transfer operators. These requirements, coupled with the definitions of the vacuum state, yield the *unique* solution

$$A^\mu_\nu = B^{i\mu} B_{i\nu} + a^\mu a_\nu, \quad A^i_j = B^{i\mu} B_{j\mu} + a^i a_j \ . \tag{19.18}$$

For construction of the particle-hole operators, we need the maps of the quadratic Casimir invariants of the core subalgebra. By substituting the generators (19.18) into Eqs. (19.9), we find

$$\mathcal{C}_h^{(2)} = \Omega_h (\hat{N}_B + \hat{n}_h) + \overline{\mathcal{C}}_h^{(2)} \ , \qquad \mathcal{C}_p^{(2)} = \Omega_p (\hat{N}_B + \hat{n}_p) + \overline{\mathcal{C}}_p^{(2)} \ , \tag{19.19}$$

where $\hat{N}_B$ is the boson number operator

$$\hat{N}_B = B^{i\mu} B_{i\mu} \ , \tag{19.20}$$

and $\overline{\mathcal{C}}_h^{(2)}$, $\overline{\mathcal{C}}_p^{(2)}$ are the two-body parts of the respective operators, given by

$$\overline{\mathcal{C}}_h^{(2)} = \overline{\mathcal{C}}_B^{(2)} + a^\mu a^\nu a_\mu a_\nu + 2 B^{i\mu} B_{i\nu} a^\nu a_\mu \ , \qquad \overline{\mathcal{C}}_p^{(2)} = \overline{\mathcal{C}}_B^{(2)} + a^i a^j a_i a_j + 2 B^{i\mu} B_{j\mu} a^j a_i \ , \tag{19.21}$$

with $\overline{\mathcal{C}}_B^{(2)}$, the common boson part, given by

$$\overline{\mathcal{C}}_B^{(2)} = B^{i\mu} B^{j\nu} B_{j\mu} B_{i\nu} \ . \tag{19.22}$$

Given the realization (19.18) of the core subalgebra, the images of the particle-hole pairs $R^{i\mu}$ and $R_{i\mu}$ are determined by the requirements that they be irreducible tensors under $U_h(\Omega_h) \times U_p(\Omega_p)$ and transform like $B^{i\mu}$ and $B_{i\mu}$, respectively. Such tensors can be systematically generated by calculating commutators of $B^{i\mu}$ and $B_{i\mu}$ with the Casimir operators or combinations thereof. Thus, for example, from Eqs. (19.21) and (19.22) one obtains the following additional tensors with the required transformation properties:

$$\tfrac{1}{2}[B^{i\mu}, \overline{\mathcal{C}}_B^{(2)}] = -B^{i\nu} B^{j\mu} B_{j\nu} \ ; \tag{19.23a}$$

$$\tfrac{1}{2}[B^{i\mu}, \overline{\mathcal{C}}_h^{(2)} - \overline{\mathcal{C}}_B^{(2)}] = -B^{i\nu} a^\mu a_\nu \ , \qquad \tfrac{1}{2}[B^{i\mu}, \overline{\mathcal{C}}_p^{(2)} - \overline{\mathcal{C}}_B^{(2)}] = -B^{j\mu} a^i a_j \ . \tag{19.23b}$$

These additional tensors suffice to provide the simple mappings (see below)

$$R_{i\mu} = B_{i\mu} \ , \tag{19.24a}$$

$$R^{i\mu} = B^{i\mu} + \tfrac{1}{2}[B^{i\mu}, \overline{\mathcal{C}}_h^{(2)} + \overline{\mathcal{C}}_p^{(2)} - \overline{\mathcal{C}}_B^{(2)}] = B^{i\mu} - B^{i\nu} B^{j\mu} B_{j\nu} - B^{i\nu} a^\mu a_\nu - B^{j\mu} a^i a_j \ . \tag{19.24b}$$

The mapping of $U(\Omega)$ contained in Eqs. (19.18) and (19.24) is isomorphic to the result that we could have found by the method of vector coherent states. The content of the latter appears in our formalism in two places: in the introduction of the intrinsic space as a space of quasifermions and in the specification of Eq. (19.24a). The scale chosen for the latter has the advantage that it normalizes the image vector of a one-particle/one-hole state.

It is convenient to discuss next the mapping of the single-fermion creation and annihilation operators [Eq. (19.3c)]. These must transform like c^I and c_I, respectively, under the core subalgebra. The simplest such operators are a^μ and $B^{i\mu} a_i$, which transform like c^μ, and a^i and $B^{i\mu} a_\mu$, which transform like c^i, as well as the corresponding H.c. operators. More complex operators of this type can be constructed by taking commutators with Casimir invariants, as, for example,

$$\tfrac{1}{2}[a^\mu, \overline{\mathcal{C}}_h^{(2)}] = -a^\nu A^\mu_\nu, \quad \tfrac{1}{2}[a^i, \overline{\mathcal{C}}_p^{(2)}] = -a^j A^i_j \ . \tag{19.25}$$

The addition of these last tensors provides a sufficient set for construction of the required mapping. We refer the reader to Klein and Marshalek (1989) for the details. The final expressions for the Dyson images of the single-fermion operators are given by

$$\begin{aligned} A^\mu &= (a^\mu - a^\nu A^\mu_\nu)(1+\hat{n}_h)^{-1} - B^{i\mu} a_i \\ &= (a^\mu - a^\nu A^\mu_\nu)(1+\hat{n})^{-1} - B^{i\mu} a_i \ , \\ A^i &= (a^i - a^j A^i_j)(1+\hat{n}_p)^{-1} + B^{i\mu} a_\mu \ , \\ &= (a^i - a^j A^i_j)(1+\hat{n})^{-1} + B^{i\mu} a_\mu \ ; \end{aligned} \tag{19.26a}$$

$$A_\mu = a_\mu - B_{i\mu} a^i (1+\hat{n}_p)^{-1} = a_\mu - B_{i\mu} a^i (1+\hat{n})^{-1} \ , \qquad A_i = a_i + B_{i\mu} a^\mu (1+\hat{n}_h)^{-1} = a_i + B_{i\mu} a^\mu (1+\hat{n})^{-1} \ . \tag{19.26b}$$

The creation operators can be rewritten as

$$A^{\mu}=(a^{\mu}+\tfrac{1}{2}[a^{\mu},\overline{\mathcal{C}}_h^{(2)}])(1+\hat{n})^{-1}-B^{i\mu}a_i\ ,$$
$$A^{i}=(a^{i}+\tfrac{1}{2}[a^{i},\overline{\mathcal{C}}_p^{(2)}])(1+\hat{n})^{-1}+B^{i\mu}a_{\mu}\ . \tag{19.27}$$

It is worth remarking that these results were not completely determined by the commutation relations of the fermion operators with the generators of $U(\Omega)$. There remained, naturally, a single overall scaling function depending on $\hat{n}$. This was fixed by the requirement that particles and quasiparticles be equivalent when no quasiholes are present and that holes and quasiholes be equivalent when no quasiparticles are present. The same result is found from the requirement that the fermion anticommutation relations be satisfied, which can also be used to derive or verify Eqs. (19.15). There is a subtlety here, however, that must be remarked on. Whereas all the commutation relations used up to now could be satisfied in the full direct-product space of bosons and quasifermions, it is perhaps not surprising to find that the anticommutators must be restricted to the appropriately antisymmetrized physical subspace, namely, the correct image from the fermion space. This is true not only for the anticommutation relations themselves but also for all other algebraic relations, such as those discussed below, that distinguish between fermions and bosons.

Equipped with the Dyson images $R^{i\mu}$, $R_{i\mu}$, A^I, and A_I, one can construct the Dyson mappings of all other fermion operators, in particular for the remaining generators of $SO(2\Omega+1)$, the elementary pair-transfer operators. For these one may take

$$R^{IJ}=\tfrac{1}{2}[A^I,A^J],\quad R_{IJ}=\tfrac{1}{2}[A_J,A_I]\ , \tag{19.28a}$$
$$A^i_{\mu}=\tfrac{1}{2}[A^i,A_{\mu}],\quad A^{\mu}_I=\tfrac{1}{2}[A^{\mu},A_i]\ , \tag{19.28b}$$

these expressions possessing the virtue of automatically fulfilling commutation relations and antisymmetry requirements.

In summary, we have obtained a Dyson realization of the algebra $U(\Omega)$ as well as of the core subalgebra $U_h(\Omega_h)\times U_p(\Omega_p)$. This result was obtained by methods equivalent to the utilization of the full machinery of the vector coherent state. With the inclusion of a convenient normalization requirement, the realizations of one-particle transfer operators with the correct tensor properties under $U(\Omega)$ are also uniquely determined. This also holds for higher tensors, such as the two-particle transfer operators. But so far, nothing has been said about the mutual commutators of the two-particle transfer operators, the commutators of the one- with the two-particle transfer operators, and, except parenthetically, the fermion anticommutation relations. These additional commutation rules must be satisfied if one is to have a realization of the full $SO(2\Omega+1)$ algebra. Now it is a straightforward exercise to check that, in general, they are not identically satisfied, as we have already remarked above for the anticommutation relations. However, it can be shown that they are satisfied in the finite-dimensional physical subspace, which is all that is needed. For the proof of this and the further assertions needed to round out the discussion of the present example, we refer the reader once more to the original work (Klein and Marshalek, 1989).

It should be apparent to the reader that there is a considerable resemblance between the methods presented so far in this discussion and those described in Secs. IX and X. The emphasis here has been on their equivalence to the method of the vector coherent state. The essential relationships have been established in carrying the discussion through the derivation of the Dyson mapping and in emphasizing its essential tensorial structure. We shall therefore also omit the discussion of the unitarization of the mapping, which is the final step of the established procedure, especially since we shall describe that procedure once more for the further example considered in the next subsection.

D. Boson-quasifermion mapping for generalized Lipkin model

1. Dyson mapping for U(n)

We consider once more n single-particle levels, each of degeneracy Ω, that we label $0,1,\ldots,n-1$. The vacuum state is taken to be the one in which the level 0 is fully occupied, and, as has been done previously, we utilize a particle-hole formalism to describe excitations of this state, in immediate generalization of the description of the simple Lipkin model found in Sec. II. We take the algebra in the form

$$J_{+k}=\sum_{m=1}^{\Omega}\alpha^{\dagger}_{mk}\beta^{\dagger}_m=(J_{-k})^{\dagger}\ , \tag{19.29}$$
$$J_{0k}=-\frac{1}{2}\Omega+\frac{1}{2}\sum_m(\alpha^{\dagger}_{mk}\alpha_{mk}+\beta^{\dagger}_m\beta_m)\ , \tag{19.30}$$
$$N_{kl}=\sum_m\alpha^{\dagger}_{mk}\alpha_{ml}\ , \tag{19.31}$$
$$N_{\beta}=\sum_m\beta^{\dagger}_m\beta_m\ . \tag{19.32}$$

Though it is convenient to utilize all of the operators (19.29)-(19.32) in the discussion, it is obvious that the set (19.30) is a linear combination of the sets (19.31) and (19.32). For each value of $k=1,\ldots,n-1$, the triple $J_{\pm k}$, J_{0k} generates an SU(2) subalgebra, whereas N_{kl} are the generators of a $U(n-1)$ algebra associated with all levels exclusive of the lowest one. For the application we have in mind, the core subalgebra is the direct sum $U(n-1)\oplus U(1)$, associated with the generators (19.31) and (19.32) respectively. The raising and lowering operators are those given in Eq. (19.29).

We shall realize the mapping we are after in terms of bosons B_k, $B^{\dagger}_k$, $k=1,\ldots,n-1$, as well as a set of quasifermions, $a^{\dagger}_{mk}$ and $b^{\dagger}_m$, that replace the fermions, one to one, although their algebra will be different because of the imposed constraints

$$\sum_m a^{\dagger}_{mk}b^{\dagger}_m=0\ \ (\text{and H.c. Eq.})\ . \tag{19.33}$$

Nevertheless the operators

$$\mathcal{N}_{kl}=\sum_m a^\dagger_{mk}a_{mk} \tag{19.34}$$

also generate U($n-1$), despite the altered algebra. This assertion is ultimately established as a self-consistent one. What follows is a summary of the work of Klein and Walet (1989).

Utilizing ideas that have been explained repeatedly, one can now establish easily that the following formulas provide a boson-quasifermion mapping:

$$N_{kl}=B^\dagger_k B_l+\mathcal{N}_{kl}\ , \tag{19.35}$$

$$N_\beta=\hat{n}_b+\hat{N}_B\ , \tag{19.36}$$

$$J_{-k}=B_k\ , \tag{19.37}$$

$$J_{0k}=-\tfrac{1}{2}\Omega+\tfrac{1}{2}\hat{N}_B+\tfrac{1}{2}\hat{N}_k+\tfrac{1}{2}(\hat{n}_k+\hat{n}_b)\ , \tag{19.38}$$

$$J_{+k}=B^\dagger_k\{\Omega-\hat{N}_B-\hat{n}_k-\hat{n}_b\}-\sum_{l\neq k}B^\dagger_l\mathcal{N}_{kl}\ , \tag{19.39}$$

where

$$\hat{N}_k=B^\dagger_k B_k\ , \tag{19.40a}$$

$$\hat{N}_B=\sum_k\hat{N}_k\ , \tag{19.40b}$$

$$\hat{n}_k=\mathcal{N}_{kk}\ , \tag{19.40c}$$

and

$$\hat{n}_b=\sum_m b^\dagger_m b_m\ . \tag{19.40d}$$

Little need be said concerning the derivation of these formulas, which can all be written down by inspection, with the sole exception of Eq. (19.39); the latter can be derived from the appropriate SU(2) subalgebra.

Next one verifies that Eq. (19.39) can be rewritten in the form

$$J_{+k}=[B^\dagger_k,\Lambda]\ , \tag{19.41}$$

where

$$\Lambda=\tfrac{1}{2}C_2(\mathrm{U}(n-1))-\hat{N}_B[\Omega-n_b+\tfrac{1}{2}(n-1)] \tag{19.42}$$

and

$$C_2(\mathrm{U}(n-1))=\sum_{ik}N_{ik}N_{ki} \tag{19.43}$$

is the quadratic Casimir operator.

2. Basis and special algebraic relations for $n=3$

In the further exposition we specialize to the case $n=3$, since this is the only case for which we have so far derived the mapping of single-fermion operators in the form of a quantized Bogoliubov transformation. In this subsection we develop some auxiliary tools that are vital for the achievement of this goal.

For U(3), the mapping is to a final basis that will be constructed from the intermediate basis

$$|n_a,\mathcal{S},\mathcal{M}_\mathcal{S},n_b,\mathcal{L},\mathcal{M}_\mathcal{L}\rangle \equiv\prod_{i,j,k}a^\dagger_{m_i1}a^\dagger_{m_j2}b^\dagger_{m_k}(B^\dagger_2)^{N_{B2}}(B^\dagger_1)^{N_{B1}}|\Omega\rangle\ . \tag{19.44}$$

Here, the notation $|\Omega\rangle$ refers to the reference state with the lowest level fully occupied. The six quantities chosen to label the basis in question, specified on the left-hand side of Eq. (19.44), have the following significance. The first three are U(2) labels associated with the intrinsic representation. Here n_a is the eigenvalue of the operator [cf. Eq. (19.40c)]

$$\hat{n}_a=\sum_k\hat{n}_k\ , \tag{19.45}$$

the linear Casimir invariant, whereas the other two are angular momentum and magnetic quantum numbers, respectively. Next comes the U(1) label, associated with the bottom level, and finally we have the angular momentum and magnetic quantum numbers associated with the collective U(2). For the problem of unitarization of the Dyson mapping, we shall find it convenient to introduce a basis in which the square of the total angular momentum,

$$\vec{\mathcal{J}}=\vec{\mathcal{S}}+\vec{\mathcal{L}}\ , \tag{19.46}$$

is diagonal. The new basis is specified as $|n_b,n_a,\mathcal{J},\mathcal{M},\mathcal{S},\mathcal{L}\rangle$, where $\mathcal{M}$ is a total magnetic quantum number. Note that the label $\mathcal{L}$, in contrast to the others, is not associated with a subalgebra of the original U(3).

We next turn to the derivation of some essential identities associated with the mapping of the quadratic Casimir invariant of the core subalgebra (summation convention used below),

$$C_2(\mathrm{U}(2))=N_{kl}N_{lk}=\tfrac{1}{2}\hat{N}^{\,2}_\alpha+2\vec{\mathcal{J}}^{\,2}\ , \tag{19.47}$$

where

$$\hat{N}_\alpha\equiv N_{kk} \tag{19.48}$$

and

$$\vec{\mathcal{J}}^{\,2}=\tfrac{1}{2}(N_{21}N_{12}+N_{12}N_{21})+\tfrac{1}{4}(N_{11}-N_{22})^2\ . \tag{19.49}$$

With the help of Eq. (19.46) and the equations

$$\hat{N}_\alpha=\hat{n}_a+\hat{N}_B \tag{19.50}$$

and

$$\vec{\mathcal{L}}^{\,2}=\tfrac{1}{2}\hat{N}_B(\tfrac{1}{2}\hat{N}_B+1)\ , \tag{19.51}$$

we can transform Eq. (19.47) into

$$C_2(\mathrm{U}(2))=\tfrac{1}{2}\hat{n}^{\,2}_a+\hat{N}_B(\hat{N}_B+1)+\hat{n}_a\hat{N}_B+2\vec{\mathcal{S}}^{\,2}+4\vec{\mathcal{S}}\cdot\vec{\mathcal{L}}\ . \tag{19.52}$$

Another form obtained by direct substitution of Eq. (19.35) into the defining expression is

$$C_2(\mathrm{U}(2))=\mathcal{N}_{qp}\mathcal{N}_{pq}+2\mathcal{N}_{pq}B^\dagger_qB_p+\hat{N}_B(\hat{N}_B+1)\ . \tag{19.53}$$

Further profitable comparison of Eqs. (19.52) and (19.53) can be made by noting that, in analogy with Eq. (19.47), one can write

$$\mathcal{N}_{pq}\mathcal{N}_{qp}=\tfrac{1}{2}\hat{n}_a^{\,2}+2\vec{\mathcal{S}}^2=\hat{n}_a(\hat{n}_a+1)+2\Sigma\ , \tag{19.54}$$

where

$$\Sigma\equiv\vec{\mathcal{S}}^2-\tfrac{1}{2}\hat{n}_a(\tfrac{1}{2}\hat{n}_a+1)=\hat{K}(\hat{K}+\hat{n}_a+1) \tag{19.55}$$

and

$$\hat{K}\equiv\hat{\mathcal{S}}-\tfrac{1}{2}\hat{n}_a\ , \tag{19.56a}$$

$$\hat{\mathcal{S}}(\hat{\mathcal{S}}+1)\equiv\vec{\mathcal{S}}^2\ . \tag{19.56b}$$

[The quantity in Eq. (19.56a) has been introduced because it vanishes for one-rowed representations of U(2).] Incorporating Eq. (19.54) into a comparison of Eqs. (19.52) and (19.53), we find a relation that will prove useful in the unitarization procedure, namely,

$$2\mathcal{N}_{qp}B_p^\dagger B_q=\hat{N}_B\hat{n}_a+4\vec{\mathcal{S}}\cdot\vec{\mathcal{L}}\ , \tag{19.57}$$

where the last scalar product can be evaluated in the standard way, in the chosen basis, by means of the relation

$$4\vec{\mathcal{S}}\cdot\vec{\mathcal{L}}=2\vec{\mathcal{J}}^2-\vec{\mathcal{S}}^2-\vec{\mathcal{L}}^2\ . \tag{19.58}$$

The remainder of the current discussion will be devoted to the study of relations needed in the derivation of the mappings for the single-fermion operators. In the following we shall utilize the notation introduced for quasifermions, but, according to our assumptions, the same relations hold *a fortiori* for fermions, since the most basic manipulations depend on the validity of the commutation relation

$$[a_{mk},\mathcal{N}_{pq}]=a_{mq}\delta_{kp} \tag{19.59}$$

and the H.c. relation, from which it also follows that the $\mathcal{N}_{pq}$ are generators of U(2).

Thus, by commuting a quasifermion annihilation (creation) operator with Eq. (19.54), we derive

$$\mathcal{N}_{qk}a_{mq}=\hat{n}_a a_{mk}+[a_{mk},\Sigma] \tag{19.60a}$$

$$+\text{H.c. relation}\ . \tag{19.60b}$$

By examing each side directly for the different possible independent choices of available indices, we can verify either of the relations

$$\mathcal{N}_{pk}\mathcal{N}_{qp}=(\hat{n}_a-1)\mathcal{N}_{qk}+\delta_{qk}(\hat{n}+\Sigma)\ , \tag{19.61a}$$

$$\mathcal{N}_{qp}\mathcal{N}_{pk}=(\hat{n}_a+1)\mathcal{N}_{qk}+\delta_{qk}\Sigma\ , \tag{19.61b}$$

the other following from a simple commutation. By contraction of the remaining free indices, we are naturally led back to Eq. (19.54). [We cannot overemphasize that the preceding as well as the following relations are all restricted to U(2).]

Finally, by repeated application of Eqs. (19.59) and (19.61), we can derive the useful identities

$$a_{mr}^\dagger\mathcal{N}_{qk}a_{mq}=a_{mp}^\dagger\mathcal{N}_{rp}a_{mk}=(\hat{n}_a-1)\mathcal{N}_{rk}+\delta_{rk}\Sigma\ , \tag{19.62a}$$

$$a_{mp}^\dagger\mathcal{N}_{rp}\mathcal{N}_{qk}a_{mq}=a_{mq}^\dagger\mathcal{N}_{pq}\mathcal{N}_{rp}a_{mk}=[(\hat{n}_a-1)^2+\Sigma]\mathcal{N}_{rk}+(\hat{n}_a-3)\Sigma\delta_{rk}\ , \tag{19.62b}$$

$$a_{ms}^\dagger\mathcal{N}_{ps}\mathcal{N}_{rp}\mathcal{N}_{qk}a_{mq}=[(\hat{n}_a-1)^3+2(\hat{n}_a-2)\Sigma]\mathcal{N}_{rk}+[(\hat{n}_a^{\,2}-4\hat{n}_a+7)\Sigma+\Sigma^2]\delta_{rk}\ . \tag{19.62c}$$

3. Mapping of fermion operators

The first step of this procedure is to satisfy the commutation relations of the fermion operators *vis à vis* the generators of U(3). The relevant commutation relations are

$$[\alpha_{mk},N_{lp}]=\delta_{kl}\alpha_{mp}\ , \tag{19.63a}$$

$$[\beta_m,N_{lp}]=0\ , \tag{19.63b}$$

$$[\alpha_{mk},J_{-l}]=0\ , \tag{19.64a}$$

$$[\beta_m,J_{-l}]=0\ , \tag{19.64b}$$

$$[\alpha_{mk},J_{+l}]=\delta_{lk}\beta_m^\dagger\ , \tag{19.65a}$$

$$[\beta_m,J_{+l}]=-\alpha_{ml}^\dagger\ . \tag{19.65b}$$

We shall first consider the mapping of the operator α_{mk}. We want to write the latter as a linear combination of operators in the mapped space that themselves satisfy Eq. (19.63a) (and decrease fermion number by unity). From our previous studies of the quantized Bogoliubov transformation, we see that the candidates a_{mk} and $b_m^\dagger B_k$ are obvious charter members of the required set. Further members are obtained by commutation of these with quadratic invariant operators associated with the collective, intrinsic, and total U(2) algebras. This procedure adds the operators $a_{mp}B_p^\dagger B_k$, $\mathcal{N}_{pk}a_{mp}$, and $b_m^\dagger B_p\mathcal{N}_{pk}$. Of these, the first can be ruled out because it violates Eq. (19.64a). Further commutation with the invariant operators will not produce any additional independent operators in consequence of the identifies (19.61). We can thus write

$$\alpha_{mk}=ea_{mk}+fb_m^\dagger B_k+gb_m^\dagger\mathcal{N}_{pk}B_p+h\mathcal{N}_{pk}a_{mp}\ , \tag{19.66a}$$

where e, f, g, h can depend on the invariants other than $\hat{N}_B$, which is ruled out because Eq. (19.64a) would be violated. We choose the set $\hat{n}_b$, $\hat{n}_a$, Σ. The latter arguments will be suppressed below except when it is necessary to indicate a shift of value from a standard one.

By a similar argument, we can arrive at the representation

$$\beta_m=Eb_m-Fa_{mp}^\dagger B_p-Ga_{mp}^\dagger\mathcal{N}_{qp}B_q\ , \tag{19.66b}$$

where E, F, G are once more invariant operator func-

tions. Equations (19.63) and (19.64) have thus been satisfied. The only remaining *condition* to be satisfied is Eq. (1965a) for $k \neq l$. The study of this condition leads to the relation

$$f=(\Omega-\hat{n}+2)g \ , \tag{19.67}$$

$$\hat{n}=\hat{n}_a+\hat{n}_b \ . \tag{19.68}$$

With the aid of Eqs. (19.59) and (19.61), we find that Eqs. (19.65) now consistently determine the remaining mappings,

$$\beta_m^\dagger=\{f[\Omega-\hat{n}_b+1]-g(\hat{n}_a)+\Sigma\}b_m^\dagger - fb_m^\dagger\hat{N}_B-gb_m^\dagger B_p^\dagger B_q\mathcal{N}_{qp}-eB_p^\dagger a_{mp}-hB_q^\dagger\mathcal{N}_{pq}a_{mp} \ , \tag{19.69a}$$

$$\alpha_{mk}^\dagger=EB_k^\dagger b_m+\{F[\Omega-\hat{n}_b]-G[2(\hat{n}_a-1)+\Sigma]\}a_{mk}^\dagger +[G(\Omega-\hat{n}+3)-F]a_{mp}^\dagger\mathcal{N}_{kp} -Fa_{mp}^\dagger B_k^\dagger B_p-Ga_{mp}^\dagger\mathcal{N}_{qp}B_k^\dagger B_q \ . \tag{19.69b}$$

The unknown invariant functions that appear in Eqs. (19.66) and (19.69) must now be chosen to be consistent with the standard fermion anticommutation relations. We shall give in outline the most succinct line of argument we have been able to find to produce the needed results.

We start with the assertion that without loss of generality the functions e and E can be chosen to be unity. It follows from the nonvanishing anticommutators that a different choice for e rescales F and G, whereas a different choice of E rescales f and g. In particular, it can be shown that the nonvanishing anticommutators among the *quasifermion* operators are invariant under this rescaling. Next, we notice that we can determine f and g immediately by the requirement [cf. Eq. (19.36)]

$$N_\beta(\hat{N}_B=0)=\hat{n}_b \ . \tag{19.70}$$

This leads to the condition that the coefficient of $b_m^\dagger$ in Eq. (19.69a) must be unity. Coupled with the relation (19.67), we then find

$$g=(D_1)^{-1}(D_2)^{-1} \ , \tag{19.71a}$$

$$D_1=\Omega-\hat{n}-\hat{K}+1 \ , \tag{19.71b}$$

$$D_2=\Omega-\hat{n}_b+\hat{K}+2 \ . \tag{19.71c}$$

Notice that D_1 and D_2 go over into one another under the transformation $\hat{K}\rightarrow-\hat{K}-\hat{n}_a-1$, which corresponds to the replacement $\hat{S}\rightarrow-\hat{S}-1$, leaving the square of the intrinsic angular momentum invariant.

We can take immediate advantage of the information contained in Eq. (19.71) to determine the anticommutator $\{a_{mk},b_{m'}^\dagger\}$ by studying the vanishing anticommutator $\{a_{mk},\beta_{m'}^\dagger\}$. This anticommutator contains the same information as $\{\alpha_{mk},\alpha_{m'k'}\}$, but is algebraically simpler to manipulate than the latter, which has been used as an *a posteriori* check of the final results of the calculations. In this investigation, as in all succeeding ones of a similar nature, we argue that if a vanishing operator is itself a linear combination of a number of independent nontrivial operators, then the coefficients of each of these must vanish. We thereby eventually establish the result

$$\{a_{mk},b_{m'}^\dagger\}=fb_m^\dagger a_{m'k}+g\mathcal{N}_{pk}b_m^\dagger a_{m'p} \ , \tag{19.72}$$

where f and g are the functions specified by Eqs. (19.67) and (19.71).

A strategem for finding two of the remaining three unknown functions, namely, F and G, can be worked out next. First we take the Hermitian conjugate of Eq. (19.72) and move the operators f and g through to the left. The technique for doing this is implied by the identities (19.61) but will not be discussed here. The result is

$$\{b_m,a_{m'k}^\dagger\}=\tilde{f}a_{mk}^\dagger b_{m'}+\tilde{g}a_{mp}^\dagger b_{m'}\mathcal{N}_{kp} \ , \tag{19.73}$$

where

$$\tilde{g}=g, \quad \tilde{f}=(\Omega-\hat{n}+3)g \ . \tag{19.74}$$

One can now guess that

$$\tilde{g}=G, \quad \tilde{f}=F \ . \tag{19.75}$$

That this is indeed the correct identification is verified by a study of the anticommutator $\{\beta_m,\beta_{m'}\}$, which leads to an alternative to Eq. (19.73), involving F and G, that is compatible with the latter, provided Eq. (19.75) holds.

The relation between F and G expressed in Eqs. (19.74) and (19.75) leads to the vanishing of the coefficient of $a_{mp}^\dagger\mathcal{N}_{kp}$ in the expression (19.69b) for $\alpha_{mk}^\dagger$ and thus to a significant simplification of subsequent manipulations. In fact, the evaluation [cf. Eq. (19.50)] of the requirement

$$\hat{N}_\alpha(\hat{N}_B=0)=\hat{n}_a \tag{19.76}$$

yields the value zero for the remaining unknown operator coefficient h. Thus all the operator coefficients have been chosen consistently or determined.

The remaining nontrivial quasifermion anticommutators, each having its source in the obviously related fermion anticommutator, now follow rather straightforwardly and have the forms

$$\{b_m,b_{m'}^\dagger\}=\delta_{mm'}-Fa_{mp}^\dagger a_{m'p}-Ga_{mp}^\dagger\mathcal{N}_{qp}a_{m'q} \ , \tag{19.77}$$

$$\{a_{mk},a_{m'k'}^\dagger\}=\delta_{mm'}\delta_{kk'}-f\delta_{kk'}b_m^\dagger b_{m'}-g\mathcal{N}_{k'k}b_m^\dagger b_{m'} \ . \tag{19.78}$$

We summarize the results found in this subsection: the fermion mappings have been simplified to the forms

$$\alpha_{mk}=a_{mk}+fb_m^\dagger B_k+gb_m^\dagger\mathcal{N}_{pk}B_p \ , \tag{19.79a}$$

$$\beta_m=b_m-Fa_{mp}^\dagger B_p-Ga_{mp}^\dagger B_q\mathcal{N}_{qp} \ , \tag{19.79b}$$

$$\beta_m^\dagger=b_m^\dagger-B_p^\dagger a_{mp}-fb_m^\dagger\hat{N}_B-gb_m^\dagger B_p^\dagger B_q\mathcal{N}_{qp} \ , \tag{19.79c}$$

$$\alpha_{mk}^\dagger=a_{mk}^\dagger+B_k^\dagger b_m-Fa_{mp}^\dagger B_k^\dagger B_p-Ga_{mp}^\dagger\mathcal{N}_{qp}B_k^\dagger B_q \ . \tag{19.79d}$$

This mapping, together with the anticommutators (19.72), (19.73), (19.77), and (19.78), is completely determined by the operator functions f, g, F, and G given in Eq. (19.67), (19.71), (19.74), and (1975). We note that the structure of all the quasifermion anticommutation relations is compatible with the basic assumption (19.59) as well as with the concomitant relations

$$[a_{mk},\hat{N}_\beta]=0\ , \tag{19.80a}$$

$$[b_m,\mathcal{N}_{qp}]=0\ , \tag{19.80b}$$

all proofs making use of the constraint (19.33).

It can be verified that the mapping of the fermion operators leads back to the mapping of the generators exhibited in Eqs. (19.35)–(19.39).

4. Unitarization of the Dyson mapping

As has been done many times in this review, we introduce a unitarizing operator S and its square V and carry through a procedure that utilizes the three equations

$$S(\Theta)_D S^{-1}=(\Theta)_U\ , \tag{19.81}$$

$$(\Theta^\dagger)_D=V^{-1}(\Theta)_D^\dagger V\ , \tag{19.82}$$

$$(\Theta^\dagger)_U=S^{-1}(\Theta)_D^\dagger S\ , \tag{19.83}$$

where, in the following, Θ will refer either to one of the generators or to one of the fermion operators. A subscript D or U will be necessary to distinguish Dyson from unitarized mappings. We recall that Eq. (19.81) is the definition of S and that Eqs. (19.82) and (19.83) are consequences of that definition. Because the generators of the core subalgebra already satisfy the required Hermiticity conditions, it follows that S commutes with all these generators, which have the form of a direct sum of a collective and of an intrinsic part. It follows that S can be chosen to be an operator-valued function of the invariants (for the remainder of the discussion the hats on number and angular momentum operators will be suppressed, the context serving to distinguish them),

$$S=S(n_b,N_B,n_a,\mathcal{S},\mathcal{I})\ . \tag{19.84}$$

We apply Eq. (19.82) first to the raising operators (19.41); the equation in question has the form

$$[B_k^\dagger,\Lambda]=V^{-1}B_k^\dagger V\ . \tag{19.85}$$

Taking nonvanishing matrix elements in the coupled basis defined after Eq. (19.46), cancelling the reduced matrix element, and taking the (positive) square root of both sides, yields the two recursion relations

$$S^{-1}(N_B,\mathcal{I})S(N_B-1,\mathcal{I}+\tfrac{1}{2})=\sqrt{D_+}\ , \tag{19.86a}$$

$$S^{-1}(N_B,\mathcal{I})S(N_B-1,\mathcal{I}-\tfrac{1}{2})=\sqrt{D_-}\ , \tag{19.86b}$$

where

$$\begin{aligned}D_+&=\Lambda(N_B-1,\mathcal{I}+\tfrac{1}{2})-\Lambda(N_B,\mathcal{I})\\&=\Omega+\mathcal{I}+2-\tfrac{1}{2}(n_a+N_B+2n_b)\ ,\end{aligned} \tag{19.87a}$$

$$\begin{aligned}D_-&=\Lambda(N_B-1,\mathcal{I}-\tfrac{1}{2})-\Lambda(N_B,\mathcal{I})\\&=\Omega-\mathcal{I}+1-\tfrac{1}{2}(n_a+N_B+2n_b)\ .\end{aligned} \tag{19.87b}$$

In these equations, only those labels that change between initial and final states are indicated explicitly. We shall follow this practice for the remainder of the section.

Since there are only two reduced matrix elements, we can therefore write, with a special choice of independent operators,

$$(J_{+k})_U=\Phi_1(n_b,N_B,n_a,\mathcal{S},\mathcal{I})B_k^\dagger+\Phi_2[B_k,\Lambda]\ . \tag{19.88}$$

Entering this form into Eq. (19.83), we find

$$(2\mathcal{I}+1)\Phi_2=(\sqrt{D_+}-\sqrt{D_-})\ , \tag{19.89a}$$

$$(2\mathcal{I}+1)\Phi_1=\sqrt{D_+D_-}(\sqrt{D_+}-\sqrt{D_-})\ . \tag{19.89b}$$

We repeat the above processes for the slightly more involved case of the single fermion operators. We first define the invariant operators [cf. Eqs. (19.51)–(19.58)]

$$\begin{aligned}\Lambda_1(N_B,n_a,\mathcal{S},\mathcal{I})&\equiv\mathcal{N}_{qp}B_p^\dagger B_q\\&=\tfrac{1}{2}N_Bn_a+\vec{\mathcal{I}}^2-\vec{\mathcal{S}}^2-\vec{\mathcal{L}}^2\ ,\end{aligned} \tag{19.90}$$

$$\Lambda_2(n_a,\mathcal{S})\equiv n_a-\tfrac{1}{4}n_a^2-\vec{\mathcal{S}}^2\ . \tag{19.91}$$

We observe that with the help of these operators, Eqs. (19.79) may be written,

$$\alpha_{mk}=\alpha_{mk}+fb_m^\dagger B_k+g[b_m^\dagger B_k,\Lambda_1]\ , \tag{19.92a}$$

$$\alpha_{mk}^\dagger=\alpha_{mk}^\dagger+B_k^\dagger b_m+F[a_{mk}^\dagger,\Lambda_1]+G[[a_{mk}^\dagger,\Lambda_1],\Lambda_2]\ , \tag{19.92b}$$

$$\beta_m=b_m-Fa_{mp}B_p-G[a_{mp}^\dagger B_p,\Lambda_2]\ , \tag{19.93a}$$

$$\beta_m^\dagger=b_m^\dagger-fB_p^\dagger a_{mp}-fb_m^\dagger N_B-gb_m^\dagger\Lambda_1\ . \tag{19.93b}$$

If we apply Eqs. (19.82)–(19.92), we find six nontrivial conditions, leading in a straightforward way to the relations

$$S^{-1}(n_b,N_B,\mathcal{I})S(n_b+1,N_B-1,\mathcal{I}+\tfrac{1}{2})=\left[\frac{(\Omega-n-K)(\Omega-n_b+K+1)}{\Omega-n-\mathcal{I}+\frac{1}{2}(n_a-N_B)}\right]^{1/2}\equiv(L_+)^{-1}\ , \tag{19.94a}$$

$$S^{-1}(n_b,N_B,\mathcal{I})S(n_b+1,N_B-1,\mathcal{I}-\tfrac{1}{2})=\left[\frac{(\Omega-n-K)(\Omega-n_b+K+1)}{\Omega+n+1+\mathcal{I}+\frac{1}{2}(n_a-N_B)}\right]^{1/2}\equiv(L_-)^{-1}\ , \tag{19.94b}$$

$$S^{-1}(n_a,\mathcal{S},\mathcal{I})S(n_a-1,\mathcal{S}+\tfrac{1}{2},\mathcal{I}+\tfrac{1}{2}) = \left[\frac{\Omega+\mathcal{I}-n_b-\frac{1}{2}(n_a+N_B)+2}{\Omega-n_b+K+2}\right]^{1/2} \equiv M_{++} , \tag{19.95a}$$

$$S^{-1}(n_a,\mathcal{S},\mathcal{I})S(n_a-1,\mathcal{S}+\tfrac{1}{2},\mathcal{I}-\tfrac{1}{2}) = \left[\frac{\Omega-\mathcal{I}-n_b-\frac{1}{2}(n_a+N_B)+1}{\Omega-n_b+K+2}\right]^{1/2} \equiv M_{+-} , \tag{19.95b}$$

$$S^{-1}(n_a,\mathcal{S},\mathcal{I})S(n_a-1,\mathcal{S}-\tfrac{1}{2},\mathcal{I}+\tfrac{1}{2}) = \left[\frac{\Omega+\mathcal{I}-n_b-\frac{1}{2}(n_a+N_B)+2}{\Omega-n-K+1}\right]^{1/2} \equiv M_{-+} , \tag{19.95c}$$

$$S^{-1}(n_a,\mathcal{S},\mathcal{I})S(n_a-1,\mathcal{S}-\tfrac{1}{2},\mathcal{I}-\tfrac{1}{2}) = \left[\frac{\Omega-\mathcal{I}-n_b-\frac{1}{2}(n_a+N_B)+1}{\Omega-n-K+1}\right]^{1/2} \equiv M_{--} . \tag{19.95d}$$

Since there are six reduced matrix elements, we look for a form of $(\alpha^\dagger_{mk})_U$ that will simplify the algebra arising in the subsequent study of Eq. (19.84). The form

$$(\alpha^\dagger_{mk})_U = X_1 a^\dagger_{mk} + 2X_2[a^\dagger_{mk},\mathcal{S}] + 2X_3[a^\dagger_{mk},\mathcal{I}] + 4X_4[[a_{mk},\mathcal{S}],\mathcal{I}] + Y_1 B^\dagger_k b_m + Y_2[B^\dagger_k b_m,\Lambda_1] , \tag{19.96}$$

in terms of the six operator-valued functions X_i, $i=1,\ldots,4$ and Y_i, $i=1,2$ easily yields the solution

$$\begin{aligned} M_{++} &= X_1+X_2+X_3+X_4 , \\ M_{+-} &= X_1+X_2-X_3-X_4 , \\ M_{-+} &= X_1-X_2+X_3-X_4 , \\ M_{--} &= X_1-X_2-X_3+X_4 , \\ (2\mathcal{I}+1)Y_2 &= (L_+-L_-) ; \end{aligned} \tag{19.97a}$$

$$(2\mathcal{I}+1)Y_2 = (\mathcal{I}+\tfrac{1}{2}n_a-\tfrac{1}{2}N_B)L_+ + (\mathcal{I}+1-\tfrac{1}{2}n_a+\tfrac{1}{2}N_B)L_- . \tag{19.97b}$$

It only remains for us to carry out the same calculations for the fermion hole operators. If we apply Eqs. (19.82)–(19.93) we encounter two reduced matrix elements and two relations (remembering that the value of $\mathcal{I}$ cannot change),

$$S^{-1}(n_b)S(n_b-1) = (1-fN_B-g\Lambda_1)^{1/2} , \tag{19.98}$$

$$S^{-1}(n_a,N_B,\mathcal{S})S(n_a+1,N_B-1,\mathcal{S}+\tfrac{1}{2}) = \left[\frac{(\Omega-n+2)(\Omega-n-K)}{(\Omega-n-K+1)(\Omega-n_b+K+2)}\right]^{1/2} . \tag{19.99}$$

With the ansatz

$$(\beta^\dagger_m)_U = \Psi_1 b^\dagger_m + \Psi_2 B^\dagger_p a_{mp} , \tag{19.100}$$

to be used in conjunction with Eq. (19.83), we easily find

$$\Psi_1 = (1-fN_B-g\Lambda_1)^{1/2} , \tag{19.101a}$$

$$\Psi_2 = -\left[\frac{(\Omega-n+2)}{(\Omega-n-K)(\Omega-n-K+1)(\Omega-n_b+K+2)}\right]^{1/2} . \tag{19.101b}$$

The interpretation of the core subalgebra in terms of quasifermions, with the possibility of applications to core-particle coupling models, is a subject that is in its infancy, with much work remaining to be done both in mathematical development and in physical application.

XX. FORMALLY EXACT QUANTUM VARIATIONAL PRINCIPLES FOR COLLECTIVE MOTION BASED ON BOSON TRIAL FUNCTIONS

A. Introduction

The purpose of this section is to describe a formally complete quantum theory of nuclear collective motion that utilizes the concept of boson mapping in an essential way (Klein, Marumori, and Une, 1982, 1984; Klein and Tanabe, 1984). Though motivated by ideas developed by Marumori and his associates (Marumori, 1977; Marumori, Hayashi, Tomoda, Kuriyama, and Maskawa, 1980; Marumori, Maskawa, Sakata, and Kuriyama, 1980), the implementation is to be distinguished from this previous work.

Generalized coherent states of the form

$$|\Psi(\underline{\beta},\underline{\beta}^*,t)\rangle = \exp[\mathbf{b}^\dagger\cdot\underline{\beta}(t)-\underline{\beta}^\dagger(t)\cdot\mathbf{b}]|\Phi_0\rangle \tag{20.1}$$

have been used widely as trial states for the variational principle of the time-dependent Schrödinger equation. Here $\underline{\beta}^\dagger(t)$ is a row matrix,

$$\underline{\beta}^\dagger(t) = \{\beta^*_\alpha(t)\} = [\beta^*_1(t),\beta^*_2(t),\ldots] , \tag{20.2}$$

$\mathbf{b}^\dagger$ is a corresponding operator-valued row matrix, and $\underline{\beta}(t)$ and $\mathbf{b}$ are the associated column matrices. If, for example, $b^\dagger_\alpha$ is a particle-hole creation operator and the index α runs over a complete set associated with a fixed Slater determinant $|\Phi_0\rangle$, then Eq. (20.1) is itself, according to Thouless' theorem, an arbitrary determinant, and the equation that follows from the variational principle is time-dependent Hartree-Fock. By further specialization we can derive the random-phase approximation or adiabatic time-dependent Hartree-Fock approximation which provides a foundation for the study of large-amplitude collective motion. Here we shall be concerned with another interpretation of Eq. (20.1), which arises if we take $b_\alpha, b^\dagger_\alpha$ as boson operators that may, in practice, be complicated functions of fermion pair operators, with $|\Phi_0\rangle$ the vacuum for these bosons,

$$b_\alpha|\Phi_0\rangle=0\ , \tag{20.3}$$

or at least of a subset of them and of any other boson degrees of freedom of the system not explicitly contained in the set **b**. If the vector space constructed from products of the $b_\alpha^\dagger$ operating on $|\Phi_0\rangle$ is invariant under the action of the Hamiltonian H, it follows that these states span a subspace, equally well spanned by a subset of eigenstates of H. The main purpose of the developments that follow is to show how this subspace can be studied with the help of a variational principle. In fact, it is well known (Marshalek and Weneser, 1970; Klein, 1972) (and will emerge again in this section) that a collective Hamiltonian, valid in the classical limit, can be extracted completely from the associated theory.

In this section, we go beyond these results to a formally complete quantum theory by applying the time-dependent variational principle to a state that bears a superficial resemblance to Eq. (20.1), but which is a formally *exact* solution of the time-dependent Schrödinger equation, namely (E_0 is the ground-state energy),

$$|\Psi(\underline{\beta},\underline{\beta}^*,t)\rangle=\exp[i(H-E_0)t]\times\exp[\mathbf{b}^\dagger\cdot\underline{\beta}-\underline{\beta}^\dagger\cdot\mathbf{b}]|\Phi\rangle\ . \tag{20.4}$$

In effect, Eq. (20.4) replaces the time-dependent parameters in Eq. (20.1) by time-dependent Heisenberg operators $\mathbf{b}(t)$ and $\mathbf{b}^\dagger(t)$.

Though Eq. (20.4) is more difficult to work with than Eq. (20.1), it turns out, nevertheless, to be quite tractable. For illustrative purposes, we shall carry out the formal manipulations in this section with a single collective coordinate. The basic approach, which exploits the assumption that an operator pair $b,b^\dagger$ exists such that

$$\exp[i(H-E_0)t]|\Psi(\beta,\beta^*)\rangle\equiv|\Psi(\beta,\beta^*,t)\rangle \tag{20.5}$$

lies in the same subspace as $|\Psi(\beta,\beta^*)\rangle$ itself, has been called the "invariance principle of the Schrödinger equation" (Rowe and Basserman, 1976; Marumori, 1977). In Eq. (20.5), we are allowing a more general form than (20.4) (see Sec. XX.G). The results that can be derived from Eq. (20.5) contain as a special case those that follow from Eq. (20.4). Furthermore, states of the form (20.5) are a special case of what we shall call intrinsic states. We turn to a discussion of this concept.

Consider a subset of eigenstates $|\Psi_n\rangle$, $n=0,1,\ldots N$, of the many-body system described by the Hamiltonian H, where the possibility $N\to\infty$ is also admitted. Under the action of H, the subspace generated by $|\Psi_n\rangle$ is invariant. Because of the special structures we have in mind, we shall call the space in question a decoupled collective subspace. Restricting this space to a single degree of freedom, we assume that the states $|\Psi_n\rangle$ can be expanded in terms of a set of oscillatorlike states.

$$|p\rangle=\frac{(b^\dagger)^p}{\sqrt{p!}}|0\rangle,\quad b|0\rangle=0\ , \tag{20.6}$$

where $b,b^\dagger$ satisfy the commutation relations

$$[b,b^\dagger]=\Gamma\ . \tag{20.7}$$

Here, if we are truly dealing with bosons, Γ is the unit operator. To include the situation of fundamental interest to us, the nuclear shell model, Γ will, in general, be a projection operator onto a finite set of the states (20.6), and $b,b^\dagger$ may be defined as (generally rather complicated) functions of fermion pair or particle-hole operators, as we have seen in Secs. II and VII.

We wish then to determine the properties of the space generated by $|\Psi_n\rangle$, supposing that $n=0$ corresponds to the exact ground state. Because of assumptions (20.6) and (20.7), regarding $|\Psi_n\rangle$, there exists a Hamiltonian $H_C(b^\dagger;b)$, where the notation $(b^\dagger;b)$ implies normal ordering, which has the same excitation spectrum as H in the collective subspace. By adjusting an additive constant, we can choose

$$H_C(b^\dagger;b)|\Psi_0\rangle=0\ . \tag{20.8}$$

We may thus write

$$H=H_C(b^\dagger;b)+H_{in}\ , \tag{20.9}$$

where H_{in}, the intrinsic Hamiltonian, must satisfy

$$H_{in}|\Psi_n\rangle=E_0|\Psi_n\rangle\ , \tag{20.10}$$

with E_0 the ground-state energy, i.e., H_{in} is completely degenerate in the collective subspace, or effectively a multiple of the unit operator in that space,

$$[b,H_{in}]=[b^\dagger,H_{in}]=0\ . \tag{20.11}$$

Our goal is to develop a means of decomposing H into the form (20.9). The techniques to accomplish this must also provide the means for transforming any other operator of interest. Thus, if H is some shell-model Hamiltonian, a natural inference from Eq. (20.9) is that we shall be seeking a boson mapping that will express the fermion pairs in terms of the $b,b^\dagger$ and other less collective canonical pairs. We can expect this mapping or series of mappings to be determined by both the kinematical criterion (20.7) and the dynamical ones (20.8)–(20.11).

In any event, we may view the problem as the determination of the collective variables $b,b^\dagger$. As a dynamical criterion, we shall initially apply the variational principle of the time-dependent Schrödinger equation to study the wave packet that evolves in time from a special initial state in the collective subspace, of the form

$$|\Psi(\beta,\beta^*)\rangle=U^{-1}(\beta,\beta^*)|\Psi_0\rangle\ , \tag{20.12}$$

where

$$U^{-1}=e^G,\quad G=\beta b^\dagger-\beta^* b\ , \tag{20.13}$$

which resembles a coherent state, except that $|\Psi_0\rangle$, as the exact ground state, need not be the vacuum for the operator b. We shall show that, from Eqs. (20.12) and (20.13) and the general time-dependent variational principle, one can derive a number of useful time-independent variational principles, several quite familiar. In the ma-

nipulations, the role of the operator U as a displacement operator,

$$Ub^{\dagger}U^{-1}=b^{\dagger}+\beta^{*},\quad UbU^{-1}=b+\beta\ , \tag{20.14}$$

leads to a great simplification. Other reasons for choosing this form will emerge. It should be noted, however, that Eq. (20.14) is strictly valid only when the operator Γ that appears on the right-hand side of Eq. (20.7) can be replaced by unity. This possible source of error in the following account remains to be investigated.

B. Time-independent variational principles from the time-dependent variational principle

We study the state [differing by a phase from Eq. (20.4)]

$$|\Psi(\beta,\beta^{*},t)\rangle=\exp(iHt)U^{-1}(\beta,\beta^{*})|\Psi_0\rangle\ , \tag{20.15}$$

which is a solution of the time-dependent Schrödinger equation (with the reversed sign of $\sqrt{-1}$) and which, according to our assumptions, belongs to the invariant subspace, provided the operators $b,b^{\dagger}$ and the state $|\Psi_0\rangle$ have been properly chosen. [The reason for this choice of sign is to have the conventional sign in Eq. (20.19) below.] By utilizing Eqs. (20.7)-(20.11), we can write Eq. (20.15) as

$$|\Psi(\beta,\beta^{*},t)\rangle=\exp(iE_0t)U^{-1}(\beta,\beta^{*},t)|\Psi_0\rangle\ , \tag{20.16}$$

where

$$U^{-1}(\beta,\beta^{*},t)=e^{G(t)}\ , \tag{20.17}$$

$$G(t)=\beta b^{\dagger}(t)-\beta^{*}b(t)\ , \tag{20.18}$$

and

$$b^{\dagger}(t)=\exp(iH_Ct)b\exp(-iH_Ct)\ . \tag{20.19}$$

We characterize the state (20.16) by means of the time-dependent variational principle

$$\delta\int_{t_1}^{t_2}dt\,\langle\Psi(\beta,\beta^{*},t)|[\bar{H}+i\partial_t]|\Psi(\beta,\beta^{*},t)\rangle=\delta\int_{t_1}^{t_2}dt\,\langle\Psi_0|U(\beta,\beta^{*},t)[H+i\partial_t]U^{-1}(\beta,\beta^{*},t)|\Psi_0\rangle=0\ , \tag{20.20}$$

where

$$\bar{H}=H+E_0 \tag{20.21}$$

is used to eliminate an irrelevant phase factor. The name *invariance principle of the Schrödinger equation* has been applied particularly to the form (20.20) of the variational principle.

We next manipulate the variational form so as to eliminate the time dependence. The main tool for this transformation is a well-known formula that permits us to calculate the time derivative of $U^{-1}(\beta,\beta^{*},t)\equiv U^{-1}(t)$,

$$\frac{d}{dt}U^{-1}(t)=\frac{d}{dt}e^{G(t)}=\int_0^1 d\nu\,e^{G(t)\nu}[dG(t)/dt]e^{G(t)(1-\nu)}=\int_0^1 d\nu\,e^{G(t)\nu}[dG(t)/dt]e^{-G(t)\nu}U^{-1}\ . \tag{20.22}$$

With the aid of the multiple commutator expansion of $\exp(A)b\exp(-A)$, the equation of motion,

$$dG(t)/dt=-i[G(t),H_C(t)]\ , \tag{20.23}$$

and the formula

$$[b,H_C]=\partial H_C/\partial b^{\dagger}\ , \tag{20.24}$$

together with its Hermitian conjugate, one can readily derive the result

$$\int_0^1 d\nu\,e^{G\nu}(dG/dt)e^{-G\nu}=i\sum_{n=1}^{\infty}(-1)^{n+1}(1/n!)(\beta\partial/\partial b+\beta^{*}\partial/\partial b^{\dagger})^{n}H_C \tag{20.25}$$

$$=i\{H_C(b^{\dagger};b)-H_C(b^{\dagger}-\beta^{*},b-\beta)\}\ . \tag{20.26}$$

In these equations we have suppressed the explicit time dependence, i.e., set $t=0$, since it is a trivial consequence that once the forms (20.25) or (20.26) have been reached, the time dependence embodied in the time-development operators cancels out between operators and state vectors. Thus the time integration yields (t_2-t_1), which is divided out.

One thus derives two forms of the variational principle. From Eq. (20.25), one obtains (as form I)

$$0=\delta\langle\Psi(\beta,\beta^{*})|\left\{H-\sum_{n=1}^{\infty}(-1)^{n+1}(1/n!)(\beta\partial/\partial b+\beta^{*}\partial b^{\dagger})^{n}H_C\right\}|\Psi(\beta,\beta^{*})\rangle\ . \tag{20.27}$$

For the special case $H_C=\omega b^{\dagger}b$, this coincides with a result given previously in the literature (Rowe and Basserman, 1976). To derive a second form, one utilizes Eq. (20.14) followed by Eq. (20.8), in order to observe that

$$\langle\Psi(\beta,\beta^{*})|H_C(b^{\dagger}-\beta^{*};b-\beta)|\Psi(\beta,\beta^{*})\rangle$$

$$=\langle\Psi_0|H_C(b^{\dagger};b)|\Psi_0\rangle=0\ . \tag{20.28}$$

In consequence of Eq. (20.28), the second term of Eq.

(20.26) disappears, and one is left with the variational principle (form II)

$$0=\delta\langle\Psi(\beta,\beta^*)|[H-H_C(b^\dagger;b)]|\Psi(\beta,\beta^*)\rangle$$
$$=\delta\langle\Psi(\beta,\beta^*)|H_{in}|\Psi(\beta,\beta^*)\rangle\ . \tag{20.29}$$

To reach form II, we have utilized the specific form of the trial state. This seems a little surprising, since II is not only a familiar variational principle for a special intrinsic state, but also a variational characterization of any intrinsic state. In Sec. XX. E we return to this question and show that a derivation can be given for any intrinsic state. This is also true for variational principle III derived below.

We shall find variational principle form II extremely useful in application, and we therefore turn to a study of the admissible variations for it. Initially we take the view that

$$\delta|\Psi(\beta,\beta^*)\rangle=\delta U^{-1}(\beta,\beta^*)|\Psi_0\rangle\ , \tag{20.30}$$

i.e., we are seeking to optimize the choice of the collective operators $b,b^\dagger$ for a given ground state. Writing

$$\delta U^{-1}=e^{G+\delta G}-e^G\ , \tag{20.31}$$

we must then distinguish two cases:

(i) δG commutes with G, i.e., with b and $b^\dagger$. Such variations correspond to δG in the space of operators kinematically independent of b and $b^\dagger$. Remarkably, such variations can be obtained equivalently from a variation of the ground-state vector $|\Psi_0\rangle$ with fixed U^{-1}, namely,

$$\delta|\Psi_0\rangle=\delta G|\Psi_0\rangle\ . \tag{20.32}$$

(δG is, of course, skew-Hermitian.) This can be seen easily, since with $[\delta G,G]=0$, Eq. (20.31) may be written

$$\delta U^{-1}=\delta G U^{-1}=U^{-1}\delta G\ , \tag{20.33}$$

and, in consequence, for Eq. (20.30),

$$\delta|\Psi(\beta,\beta^*)\rangle=U^{-1}\delta G|\Psi_0\rangle=U^{-1}\delta|\Psi_0\rangle\ . \tag{20.34}$$

Second, since we have, together with Eq. (20.33),

$$\delta U=-U\delta G=-\delta G U\ , \tag{20.35}$$

we may write from variational principle II

$$0=\langle\Psi_0|[\delta G,(UHU^{-1}-UH_CU^{-1})]|\Psi_0\rangle$$
$$=\langle\Psi_0|[\delta G,UHU^{-1}]|\Psi_0\rangle\ , \tag{20.36}$$

since δG commutes with UH_CU^{-1}.

(ii) $[\delta G,G]\neq 0$. We may still write

$$\delta U^{-1}=\delta G_L U^{-1}=U^{-1}\delta G_R\ , \tag{20.37}$$

$$\delta U=-U\delta G_L=-\delta G_R U\ , \tag{20.38}$$

where $\delta G\neq\delta G_L\neq\delta G_R$, in general. Nevertheless, as we vary δG over a complete set of noncommuting variations, we expect δG_L and δG_R to form a complete set. Using the version involving δG_R, this implies first that Eq. (20.34) holds for all variations, and thus our care in distinguishing what is to be varied turns out in practice to be unnecessary. We have learned, however, that we may vary U^{-1} or $|\Psi_0\rangle$ and that varying both is redundant. Finally, the first form of Eq. (20.36) holds for all G.

We may summarize the results of this discussion by stating a third form of the variational principle, which can be given in two forms, namely,

$$0=\langle\Psi_0|[\delta G,(UHU^{-1}-UH_CU^{-1})]|\Psi_0\rangle$$
$$=\langle\Psi(\beta,\beta^*)|[\delta G,(H-H_C)]|\Psi(\beta,\beta^*)\rangle\ . \tag{20.39}$$

The second form of III may also be considered to be a trivial consequence of form II.

C. Elementary applications to vibrations

Before continuing with the theoretical development, it may be useful to illustrate the ideas presented so far with elementary examples based on the two-level *LMG* (Lipkin) model that was the core of Sec. III and the various n-level generalizations of it (Li, Klein, and Dreizler, 1970; Meshkov, 1971) studied in Secs. VIII and XIX. For the latter, returning to the notation of Sec. VIII, we consider n single-particle levels, each with the same degeneracy N. The operator $\alpha_{ir}^\dagger$ creates a "nucleon" in level r ($r=1,\ldots,n$), sublevel i ($i=1,\ldots,N$); α_{ir} is the corresponding destruction operator. The number-conserving bilinear sums

$$A_r^s=(A_s^r)^\dagger=\sum_{i=1}^N\alpha_{ir}^\dagger\alpha_{is} \tag{20.40}$$

are generators for U(n), whereas the operators

$$J_0^{(k)}=\tfrac{1}{2}[A_{k+1}^{k+1}-A_1^1]\ , \tag{20.41}$$

$$J_+^{(k)}=(J_-^k)^\dagger=A_{k+1}^1,\quad k=1,\ldots,n-1\ , \tag{20.42}$$

together with the remaining A_{r+1}^{s+1}, $r\neq s$, generate SU(n).

We study a very special Hamiltonian within the algebra of SU(n),

$$H=\epsilon\sum_{k=1}^{n-1}\{\eta_kJ_0^{(k)}+(f/2N)[(J_+^{(k)})^2+(J_-^{(k)})^2]\}\ , \tag{20.43}$$

$$\sum\eta_k=1,\quad \eta_{k+1}\geq\eta_k\ , \tag{20.44}$$

which is a sum of a single-particle term and a "monopole-monopole" interaction. This class of models has the virtue that it can be studied profitably using either the Lie algebra or, as is our current interest, a mapping to bosons.

We confine our attention to the standard problem in which the number of nucleons is N, the degeneracy of each level. The ground state then belongs to the symmetric representation $(N,0,\ldots,0)\equiv(N)$ of SU(n). For this representation, we can map to a space of $n-1$ bosons by means of a generalized Holstein-Primakoff transformation, such as was utilized in Sec. VIII,

$$(J_-^k)^\dagger = J_+^{(k)} = a_k^\dagger \sqrt{N-\hat{n}} \ , \tag{20.45}$$

$$J_0(k) = -\tfrac{1}{2}(N-\hat{n}) + \tfrac{1}{2} a_k^\dagger a_k \ , \tag{20.46}$$

$$A_{k+1}^{l+1} = a_k^\dagger a_l \ (k,l=1,\ldots,n-1) \ , \tag{20.47}$$

$$A_1^1 = N-\hat{n} \ , \tag{20.48}$$

$$n = \sum_{k=1}^{n-1} a_k^\dagger a_k \ , \tag{20.49}$$

where

$$[a_k, a_l^\dagger] = \delta_{kl} \ . \tag{20.50}$$

With the help of Eqs. (20.45)–(20.49), Eq. (20.43) becomes

$$H = H_0 + H_1' \ , \tag{20.51}$$

$$H_0 = -\frac{1}{2}\epsilon(N-\hat{n}) + \frac{1}{2}\sum_k \eta_k a_k^\dagger a_k \ , \tag{20.52}$$

$$H_1' = \frac{1}{2} f\epsilon \sum_k \{a_k^\dagger a_k^\dagger [1-(\hat{n}+1)/N]^{1/2} \times [1-(\hat{n}/N)^{1/2}] + \text{H.c.}\} \ . \tag{20.53}$$

To the first two terms in powers of N^{-1}, the interaction H_1' may be replaced by the simple polynomial H_1,

$$H_1 = \frac{1}{2} f\epsilon \left[1-\frac{1}{2N}\right] \sum_k (a_k^\dagger a_k^\dagger + a_k a_k) - \frac{f\epsilon}{2N}[a_k^\dagger a_k^\dagger \hat{n} + \hat{n} a_k a_k] \ . \tag{20.54}$$

We shall work with the sum of Eqs. (20.52) and (20.54). The discussion that follows is applicable only to the vibrational regime.

Let us first consider the case $n=2$. Then the effective Hamiltonian takes the form utilized in Sec. III,

$$H = -\frac{1}{2}\epsilon N + \epsilon a^\dagger a + \frac{1}{2} f\epsilon \left[1-\frac{1}{2N}\right](a^\dagger a^\dagger + aa) - (f\epsilon/2N)[a^\dagger a^\dagger a^\dagger a + a^\dagger aaa] \ . \tag{20.55}$$

Of course, from the dynamical point of view, this model is trivial. There is no subspace to decouple. Up to an additive constant required to ensure the condition (20.8), Eq. (20.55) is already the collective Hamiltonian H_C. It seems that the only reasonable procedure at this point is to diagonalize it numerically. Nevertheless, something can be learned from this model by proceeding along the lines suggested by the theory developed in the preceding subsections. For example, how is the boson a related to the boson b of those sections? The point is that this question does not have a unique answer. There are instead several possibly interesting answers:

(i) $a=b$ as already stated.

(ii) As we shall see below there is some simplification in the theoretical structure if we define b to annihilate the exact ground state,

$$b|\Psi_0\rangle = 0 \ . \tag{20.56}$$

Combined with Eq. (20.8) this further requires that

$$[b, H_C]|\Psi_0\rangle = 0 \ . \tag{20.57}$$

Equations (20.56) and (20.57) imply that as a function of $b, b^\dagger$, H_C has no "dangerous diagrams," i.e., no terms of the form $b^p + (b^\dagger)^p$ for any (integer) value of p. Thus H_C will have the form

$$H_C = h_{11} b^\dagger b + h_{31}[(b^\dagger)^3 b + \text{H.c.}] + \cdots \ . \tag{20.58}$$

To find the relationship between a and b, it suffices to write

$$a^\dagger = \sum_{r=0,1,\ldots} [x_{2r+1}(b^\dagger)^{2r+1} + y_{2r+1}(b)^{2r+1}] \ . \tag{20.59}$$

We may choose the x_{2r+1} and y_{2r+1} to be real. In detail, consider the approximation which includes $r=0,1$ only. We then have four coefficients to determine. From the commutation relation

$$[a, a^\dagger] = 1 \ , \tag{20.60}$$

we can conclude, consistently, by substituting Eq. (20.59),

$$1 = x_1^2 - y_1^2 + 6(x_3^2 - y_3^2) \ , \tag{20.61}$$

$$0 = x_1 x_3 - y_1 y_3 \ . \tag{20.62}$$

For the remaining conditions, substitute Eq. (20.59) and its Hermitian conjugate into Eq. (20.55) abd reorder into normal form. The result can be written

$$H = \sum_{r,s} g_{rs} (b^\dagger)^r (b)^s \equiv E_0 + H_C \ . \tag{20.63}$$

By comparing Eq. (20.63) with Eq. (20.58), (which has little, at first sight, to do with the variational theory presented above, but see below), we conclude that, for those terms in H and H_C that are sufficiently well approximated,

$$g_{rs} = h_{rs} \quad (r \neq 0, \ s \neq 0) \ . \tag{20.64}$$

We also have

$$g_{00} = E_0 \equiv H_{in} \ . \tag{20.65}$$

The remaining conditions that determine x_{2r+1} and y_{2r+1} come from the values of r and s in Eq. (20.64) for which h_{rs} in Eq. (20.58) vanishes, namely,

$$g_{20} = g_{40} = 0 \ . \tag{20.66}$$

These are the conditions for the vanishing of the "dangerous diagrams." On the other hand, if $h_{rs} \neq 0$, we have a definition of h_{rs}, rather than a condition.

This procedure, as stated above, is clearly of interest only in the vibrational or weak-coupling regime. In this regime, as can easily be checked for the particular case under study,

$$(x_{2r+1}/x_{2r-1}) \sim (y_{2r+1}/y_{2r-1}) = O(N^{-1}) \tag{20.67}$$

and

$$h_{rs}=O(N^{1-(1/2)(r+s)}) , \tag{20.68}$$

i.e., the expansion in powers of $b,b^\dagger$ converges because the operators are $O(1)$ and the successive coefficients decrease in the stated way.

Now how do the variational principles of the previous subsections enter, if at all? In place of the reasoning following Eq. (20.63), let us apply the variational principle II, Eq. (20.29). We calculate

$$\begin{aligned} E_0&=\langle\Psi(\beta,\beta^\dagger)|[H-H_C]|\Psi(\beta,\beta^*)\rangle \\ &\equiv\sum(g_{rs}-h_{rs})(\beta^*)^r(\beta)^s \end{aligned} \tag{20.69}$$

and, treating β and β^* as variational parameters, we have

$$\partial E_0/\partial\beta^*=\partial E_0/\partial\beta=0 . \tag{20.70}$$

The resulting double power series can only vanish if Eqs. (20.64) and (20.66) are satisfied. The special condition (20.65) is not included. However, (20.70) now reduces (20.69) to (20.65).

Since all conditions necessary to determine the transformation (20.59) have been found, there should be no further variational requirements. Nevertheless, we cannot help wondering whether the expansion coefficients x_r and y_r can be treated as variational parameters so that we would have, for example,

$$0=\partial E_0/\partial x_r=\sum(\partial g_{ts}/\partial x_r)(\beta^*)^t(\beta)^s, \quad \text{and} \quad x_r\leftrightarrow y_r . \tag{20.71}$$

For the simplest possible example, $x_3=y_3=0$, it is elementary to verify that the conditions (20.71), which can also be understood as the equation $\delta E_0=\delta g_{00}=0$, together with (20.61), imply $g_{20}=0$. This result should generalize, as will be evident from the further considerations.

The same results as follow from Eq. (20.70) can be derived from variational principle III, Eq. (20.39). Choosing $G=(b-\beta)$ or its Hermitian conjugate, we find that the calculation equivalent to (20.70) is

$$\langle\Psi(\beta,\beta^*)|[b,(H-H_c)]|\Psi(\beta,\beta^*)\rangle=0 , \tag{20.72}$$

leading to the same consequences.

This is about all we can learn from the two-level model, unless we want to define b as the boson in the presentation in which H is diagonal,

$$H=E_0+h_1b^\dagger b+h_2(b^\dagger)^2b^2+\cdots . \tag{20.73}$$

We still have Eq. (20.56), but Eq. (20.59) has to be replaced by a more general expansion. The same principles apply, however, to the determination of the expansion coefficients.

To augment our knowledge, we turn next to the three-level model and two bosons a_k. In order to reach the understanding sought in the simplest possible terms, we restrict our study of the decoupling problem to quadratic terms and thus introduce two bosons b_1 and b_2, each satisfying Eq. (20.56), related to a_1 and a_2 by the equations ($k=1,2$; $\lambda=1,2$)

$$a_k^\dagger=x_{k\lambda}b_\lambda^\dagger+y_{k\lambda}b_\lambda , \tag{20.74a}$$

$$a_k=x_{k\lambda}b_\lambda+y_{k\lambda}b_\lambda^\dagger , \tag{20.74b}$$

$$b_\lambda=x_{k\lambda}a_k-y_{k\lambda}a_k^\dagger , \tag{20.75a}$$

$$b_\lambda^\dagger=x_{k\lambda}a_k^\dagger-y_{k\lambda}a_k . \tag{23.75b}$$

The eight coefficients $x_{k\lambda}$ and $y_{k\lambda}$ satisfy four kinematical constraints following from the canonical commutation relations (summation convention),

$$\delta_{kk'}=x_{k\lambda}x_{k'\lambda} , \tag{20.76}$$

$$0=x_{1\lambda}y_{2\lambda}-y_{1\lambda}x_{2\lambda} . \tag{20.77}$$

Substituting Eq. (20.74) into H we find that the sum of Eqs. (20.52) and (20.54) yields the form

$$\begin{aligned} H&=\sum g_{r_1r_2,s_1s_2}(b_1^\dagger)^{r_1}(b_2^\dagger)^{r_2}(b_1)^{s_1}(b_2)^{s_2} \\ &\equiv E_0+H_C . \end{aligned} \tag{20.78}$$

Proceeding as in the sequel to Eq. (20.63), we find that the conditions that determine the remaining transformation coefficients are the vanishing of the terms proportional to $(b_1^\dagger)^2$, $(b_2^\dagger)^2$, $b_1^\dagger b_2^\dagger$, and $b_1^\dagger b_2$ (plus H.c. in every case). These are the four conditions:

$$g_{20,00}=g_{02,00}=g_{11,00}=g_{10,01}=0 . \tag{20.79}$$

If $g_{10,10}<g_{01,01}$, as we suppose, after Eqs. (20.76), (20.77), and (20.79) are solved, we set further terms depending on $b_2,b_2^\dagger$ in H, such as $b_2^\dagger b_2$, to zero, in order to be able to satisfy Eq. (20.78), since H_C depends only on $b_1,b_1^\dagger$. Though this last step appears to be arbitrary, it is equivalent (as we shall now see) to what the variational principles require for maximal decoupling of the two degrees of freedom.

Consider variational principle II, Eq. (20.29). As in Eq. (20.69), we now find

$$E_0=\sum(g_{r0,s0}-h_{rs})(\beta^*)^r(\beta)^s . \tag{20.80}$$

Here Eq. (20.70) yields, in a consistent order, only one dynamical condition, namely,

$$g_{20,00}=g_{00,20}=0 , \tag{20.81}$$

and we are missing three conditions. This tells us that we were correct to pay attention to the arguments associated with Eq. (20.71). The appropriate form appears to be

$$0=\delta E_0-\sum_{\alpha=1}\Lambda_\alpha\delta\mathcal{N}_\alpha=\delta g_{00,00}-\sum\Lambda_\alpha\delta\mathcal{N}_\alpha . \tag{20.82}$$

Here $\mathcal{N}_\alpha=0$ are the four kinematical constraints (20.76) and (20.77), and Λ_α are associated Lagrange multipliers. Since there are eight variables, when the Λ_α are eliminated, we obtain four conditions that must be equivalent to Eq. (20.79). In this case, Eq. (20.80) is redundant. It thus appears that for variational principle II, the procedures based on Eq. (20.71) or Eq. (20.82) are the natural ones for the case in which one first evaluates the expectation

value of $(H-H_C)$ and then considers c-number variations.

Though the ground-state variational principle (20.82) suffices, as we know from experience, to fix the harmonic approximation, it is almost certain that when we go beyond this approximation the optimum determination of the transformations must involve the "β dependence," i.e., must bring in excited eigenstates. How to make this determination within the present framework is a question worthy of further study, though it can be avoided in practice by utilizing the method described below.

If we turn to variational principle III, Eq. (20.39), where we utilize q-number variations, the direct derivation of Eq. (20.79) is the natural outcome. To be specific, we have, in a consistent approximation,

$$H=g_{00,00}+g_{10,10}b_1^\dagger b_1+g_{01,01}b_2^\dagger b_2 +g_{20,00}[(b_1^\dagger)^2+(b_1)^2]+g_{02,00}[(b_2^\dagger)^2+(b_2)^2] +g_{11,00}[b_1^\dagger b_2^\dagger+b_2 b_1]+g_{10,01}[b_1^\dagger b_2+b_2^\dagger b_1] , \quad (20.83)$$

$$H_C=h_{11}b_1^\dagger b_1 . \quad (20.84)$$

We apply variational principle III with the variations $\delta G=(b_1-\beta)$, b_2, and $(b_2)^2$ taken in turn and find easily that these yield the four conditions (20.79). For instance,

$$0=\langle\Psi(\beta,\beta^*)|[b_2,(H-H_c)]|\Psi(\beta,\beta^*)\rangle =g_{11,00}\beta^*+g_{10,01}\beta , \quad (20.85)$$

which yields two of the conditions.

The considerations of this section apply only to the vibrational regime. The extension to large-amplitude collective motion can be carried out (Klein, Marumori, and Une, (1984), but will not be recorded here because the alternative approach described in the next section has been developed much more thoroughly.

D. Reconstruction of the collective Hamiltonian from the intrinsic state

The aim here is to formalize the procedures of the previous section, as well as to consider one generalization, necessary in the long run. We describe the reasoning within the context of the three-level model, in order to be concrete. We have previously written H in the form

$$H=H(b_1^\dagger,b_2^\dagger;b_1,b_2) \quad (20.86)$$

and imposed the condition

$$b_i|\Psi_0\rangle=0 . \quad (20.87)$$

In the regime of large-amplitude collective motion, as well as under certain approximate circumstances, where we do not insist on using the exact ground state as reference state, it is inconvenient to impose condition (20.87) on the collective mode. We retain it, however, for the noncollective modes. Under these more general circumstances, with $b_1=b$, we have, assuming

$$H_C=\sum h_{rs}(b^\dagger)^r(b)^s \quad (20.88)$$

and using Eq. (20.14),

$$\langle\Psi(\beta,\beta^*)|H_C|\Psi(\beta,\beta^*)\rangle=\langle\Psi_0|H_C(b^\dagger+\beta^*;b+\beta)|\Psi_0\rangle =\sum\tilde{h}_{rs}(\beta^*)^r\beta_s , \quad (20.89)$$

where by straightforward expansion we find

$$\tilde{h}_{rs}=\sum_{n_1,n_2}h_{r+n_1,s+n_2}\frac{(r+n_1)!(s+n_2)!}{r!n_1!s!n_2!} \times\langle\Psi_0|(b^\dagger)^{n_1}b^{n_2}|\Psi_0\rangle . \quad (20.90)$$

In the vibrational domain, the series (20.90) should be rapidly convergent in consequence of condition (20.68). Therefore, given $\tilde{h}_{rs}$, Eq. (20.90) can be solved by iteration for the h_{rs}, starting from the approximation $h_{rs}\cong\tilde{h}_{rs}$.

Of course, the determination of the h_{rs} must trace back to the properties of the given Hamiltonian. In fact, we have

$$H_C(\beta^\dagger;\beta)\equiv\langle\Psi(\beta,\beta^*)|H|\Psi(\beta,\beta^*)\rangle-\langle\Psi_0|H|\Psi_0\rangle . \quad (20.91)$$

As we have seen in the preceding subsection, once the "b bosons" are introduced, H will take the general form

$$H=\sum g_{rs}(b^\dagger)^r b^s+H'\equiv E_0+H_C+H' , \quad (20.92)$$

where H' contains all dependence on the noncollective bosons and therefore includes coupling between the collective and noncollective spaces. As long as we retain Eq. (20.87) for the noncollective bosons, we must have the equations

$$0=\langle\Psi(\beta,\beta^*)|H'|\Psi(\beta,\beta^*)\rangle \equiv\langle\Psi(\beta,\beta^*)|[\delta G,H']|\Psi(\beta,\beta^*)\rangle , \quad (20.93)$$

where the second condition holds for δG completely within the collective subspace. Both conditions (20.93) follow from (20.87) and the dual condition $\langle\Psi_0|b_i^*=0$ for the noncollective degrees of freedom.

Another important conclusion to be drawn from the preceding subsection, as well as from the remarks just made, is that all that can be learned from variations within the collective subspace using variational principle III can be learned from linear variations,

$$\delta G=\epsilon b^\dagger-\epsilon^* b . \quad (20.94)$$

This could be gleaned from our examples, but follows in general from variational principle II, since the latter is equivalent to Eq. (20.70). We can thus conclude that the most general set of variations consists of the direct sum of Eq. (20.94) and variations that commute with the collective operators. From variational principle III with $\delta G=$ Eq. (20.94), we learn that

$$\tilde{g}_{rs}=\tilde{h}_{rs} . \quad (20.95)$$

As pointed out previously, these conditions fall into

two sets. Where $h_{rs}\neq 0$, they are determinations of h_{rs} in terms of the known (or to be computed) quantities g_{rs}. If $h_{rs}=0$, $g_{rs}=0$ is a condition for the determination of the b_1 bosons. Further conditions are obtained by choosing δG outside the collective subspace or, as in Eq. (20.82), by applying energy minimization conditions.

E. Cranking variational principle

We are finally in a position to derive yet another form of variational principle. We shall deal directly with the case

$$b|\Psi_0\rangle\neq 0 \ . \tag{20.96}$$

Details simplify when the right-hand side of Eq. (20.96) vanishes, but the same general forms will hold. We write

$$\langle\Psi(\beta,\beta^*)|H_C(b^\dagger;b)|\Psi(\beta,\beta^*)\rangle=\tilde{H}_C(\beta^*,\beta) \equiv\sum\tilde{h}_{rs}(\beta^*)^r\beta^s \ . \tag{20.97}$$

We have from variational principle II, Eq. (20.29),

$$0=\delta\langle\Psi(\beta,\beta^*)|H|\Psi(\beta,\beta^*)\rangle-\delta\tilde{H}_C(\beta^*,\beta) \ . \tag{20.98}$$

Since the $\tilde{h}_{rs}$ are not variational quantities, the second term on the right can be rewritten

$$\begin{aligned}\delta\tilde{H}_C(\beta^*,\beta)&=(\partial\tilde{H}_C/\partial\beta^*)\delta\beta^*+(\partial\tilde{H}_C/\partial\beta)\delta\beta\\&\equiv\lambda\delta\beta^*+\lambda^*\delta\beta\\&=\delta\langle\Psi(\beta,\beta^*)|[\lambda b^\dagger+\lambda^* b]|\Psi(\beta,\beta^*)\rangle \ .\end{aligned} \tag{20.99}$$

In the last form of Eq. (20.99) we recognize that λ and λ^* are to remain fixed and thus play the role of Lagrange multipliers. We have thus transformed the variational principle into form IV:

$$\begin{aligned}0&=\delta\langle\Psi(\beta,\beta^*)|[H-\lambda b^\dagger-\lambda^* b]|\Psi(\beta,\beta^*)\rangle\\&=\delta\langle\Phi(\lambda,\lambda^*)|[H-\lambda b^\dagger-\lambda^* b]|\Phi(\lambda,\lambda^*)\rangle\\&\equiv\delta[\langle H\rangle_{\lambda,\lambda^*}-\lambda\langle b^\dagger\rangle_{\lambda,\lambda^*}-\lambda^*\langle b\rangle_{\lambda,\lambda^*}] \ ,\end{aligned} \tag{20.100}$$

where we have written

$$\Phi(\lambda,\lambda^*)\equiv\Psi(\beta,\beta^*) \ . \tag{20.101}$$

Applications of form IV (as of form I) will not be considered in this section; see Klein, 1972, however.

F. Classical equations of motion

In contrast to our completely quantum starting point in the trial state [Eq. (20.5)], one encounters most frequently in the literature a trial state in which $G(t)$ [Eqs. (20.18) and (20.19)] has been replaced by

$$G_{cl}(t)=\beta(t)b^\dagger-\beta^*(t)b \ . \tag{20.102}$$

Under these conditions, if we imagine that H has been put into the form of Eq. (20.92), we find by evaluating Eq. (20.20),

$$0=(t_2-t_1)\delta E_0 +\delta\int_{t_1}^{t_2}dt[\tilde{H}_C(\beta^*;\beta)-i\dot{\beta}^*(t)\beta(t)+i\dot{\beta}(t)\beta^*(t)] \ , \tag{20.103}$$

where $\tilde{H}_C$ is discussed below but is temporarily understood to be given by the last form of Eq. (20.89). Thus varying with respect to $\beta(t)$ and $\beta^*(t)$ yields Hamilton's classical equations of motion (note again the unconventional sign of i)

$$i\dot{\beta}^*=\frac{\partial\tilde{H}_C}{\partial\beta} \ , \tag{20.104a}$$

$$-i\dot{\beta}=\frac{\partial\tilde{H}_C}{\partial\beta^*} \ . \tag{20.104b}$$

To make further progress, let us suppose that H_C has the form appropriate to vibrations,

$$H_C=H_C(b^\dagger;b)=h_1b^\dagger b+h_2(b^\dagger)^2b^2+\cdots \ . \tag{20.105}$$

If Eq. (20.87) is satisfied for all degrees of freedom including b, then

$$\tilde{H}_C(\beta^*,\beta)=H_C(\beta^*,\beta) \ , \tag{20.106}$$

and from Eq. (20.104) we find, for example, that

$$\begin{aligned}-i\dot{\beta}(t)&=[h_1+2h_2n+\cdots]\beta(t)\\&\equiv\omega(n)\beta(t)\end{aligned} \tag{20.107}$$

and that

$$n(t)=n(0)=\beta^*(t)\beta(t) \tag{20.108}$$

is a classical constant of the motion, corresponding to the quantum operator $\hat{n}=b^\dagger b$. Equation (20.107) has the solution

$$\beta(t)=e^{i\omega(n)t}\beta(0) \ . \tag{20.109}$$

Classically n is a function of the energy E, so that we may write

$$\begin{aligned}\omega(n)&=h_1+2h_2n+\cdots\\&\equiv\bar{\omega}(E)\\&=\omega_0+\omega_1E+\cdots \ .\end{aligned} \tag{20.110}$$

Furthermore, from Eq. (20.105),

$$E(n)=h_1n+h_2n(n-1)+\cdots \ . \tag{20.111}$$

From Eqs. (20.110) and (20.111) together we conclude

$$\omega_0=h_1, \quad \omega_1h_1=2h_2, \cdots \ . \tag{20.112}$$

Since in practice a classical calculation would give us $\bar{\omega}(E)$ (see below), the calculation outlined above shows that if indeed we could obtain H_C in the form (20.105), with the associated conditions on $|\Psi_0\rangle$, a classical calcu-

lation would allow us to determine the parameters of the collective Hamiltonian. In reality, we cannot know the Hamiltonian in the form (20.105) without having diagonalized H, in which event we hardly need the aftermath of classical mechanics. It is nevertheless an amusing observation that, in a suitable representation, we can obtain exact quantum results from a classical Hamiltonian. Another way of saying the same thing is that in this case the quantum theory reduces to a classical theory. This can be seen by studying the matrix element

$$\beta(t)=\langle\beta,\beta^*|b(t)|\beta,\beta^*\rangle = \langle\beta,\beta^*|e^{+i\omega(\hat{n})t}b|\beta,\beta^*\rangle \ , \tag{20.113}$$

since the quantum analog of Eq. (20.109) holds. Furthermore, Eq. (20.113) now yields Eq. (20.109).

In practice we usually find ourselves studying the classical equations of motion in a representation in which H_C is not diagonal in form and in which the condition $b|\Psi_0\rangle=0$ is not truly satisfied. Nevertheless, the considerations of Sec. XX.D lead us to the conclusion that solution of the classical equations of motion by a Fourier series will yield $\bar{\omega}(E)$ from which a set of quantities $h_1,h_2,\ldots,$ can be constructed from Eqs. (20.110) and (20.111). These can be associated with the corresponding quantities discussed in Sec. XX.D and thus identified with the true quantities up to corrections of order Ω^{-1}, in general.

Furthermore, if we are not in the representation in which H is diagonal, then the exact equation of motion for $\beta(t)$ no longer coincides with the classical equation. Solutions of the latter yield essential information on the collective Hamiltonian. Higher-order corrections to the classical equation undoubtedly contain important dynamical information. However, the exact connection with the fully quantum methods developed earlier in this section remains to be investigated, perhaps through variational principle I.

G. Relation to the trace variational principle

In this subsection, we refine the previous analysis by proving first that variational principles II and III are true for any intrinsic state, i.e., any state in the decoupled subspace. We shall then show how the trace variational principle, described in Sec. IV, is contained in the extended framework.

In the following we utilize Eqs. (20.9)–(20.11), with Eq. (20.10) generalized to any linear combination of $|\Psi_n\rangle$, namely, for an arbitrary state, $|\Psi_{in}(b,b^\dagger)\rangle$. From these equations and the associated assumptions it follows that

$$|\Psi_{in}(b,b^\dagger,t)\rangle\equiv\exp(iH_ct)|\Psi_{in}(b,b^\dagger)\rangle \tag{20.114}$$

is an exact solution of the time-dependent Schrödinger equation (with reversed sign of $\sqrt{-1}$). When the state (20.114) is utilized as a trial state in the variational principle (20.20), the time dependence is observed to cancel out because of the relation

$$[H_C,(H-H_C)]=[H_C(b,b^\dagger),H_{in}]=0 \ . \tag{20.115}$$

As a consequence, we obtain variational principle II,

$$\delta\langle\Psi_{in}(b,b^\dagger)|[H-H_C]|\Psi_{in}(b,b^\dagger)\rangle \ , \tag{20.116}$$

from which follows immediately variational principle III,

$$\langle\Psi_{in}|[\delta G,(H-H_C)]|\Psi_{in}\rangle=0 \ , \tag{20.117}$$

where $\delta|\Psi_{in}\rangle=i\delta G|\Psi_{in}\rangle$. Though we now have a more general basis for these variational principles, it should be emphasized that the applications in Secs. XX.C and XX.D utilized the properties of the coherent state. In the following discussion we aim to show that useful consequences can be obtained with different choices of the trial state.

We assume as before that the decoupled subspace is spanned by a set of exact eigenstates of H, $|\Psi_n\rangle$, $n=1,\ldots,N$, and choose the intrinsic state to be a normalized unweighted sum of all these states,

$$|\Psi_{in}(b,b^\dagger)\rangle=N^{-1/2}\sum_n|\Psi_n\rangle \ . \tag{20.118}$$

Upon substitution into variational principle II, we find that

$$\delta\,\mathrm{Tr}(H-H_C)=0 \ , \tag{20.119}$$

where Tr is the trace over the collective subspace and where we have discarded a factor N^{-1}. Below we study two formal applications of Eq. (20.119). The first establishes the basis for the application described in Sec. IV.

Toward this end, consider a Hamiltonian with q boson degrees of freedom $(a_i,a_i^\dagger)$, $i=1,\ldots,q$, $H=H(a_i,a_i^\dagger)$ and a transformation (that might but need not be linear in practice)

$$a_i=f_i(b_j,b_j^\dagger) \ , \tag{20.120}$$

$$b_i=f_i^{-1}(a_j,a_j^\dagger) \ , \tag{20.121}$$

where $b_1\equiv b$ is the collective boson. Thus

$$H(a_i,a_i^\dagger)=\bar{H}(b_i,b_i^\dagger;c_\nu,c_\nu^*) \ , \tag{20.122}$$

where c_ν are a set of unknown transformation coefficients that determine the explicit form of the transformation (20.120). Since the trace operation is invariant under a unitary transformation within the space of states considered, we choose to evaluate the trace using the basis states (20.6). The result

$$\mathrm{Tr}H=\mathrm{Tr}\bar{H}\equiv F(c_\nu,c_\nu^*) \tag{20.123}$$

is a stationary c-number functional F, that is to be varied subject to constraints on the transformation coefficients following from the requirement that Eq. (20.120) preserve the boson commutation relations. This is precisely the form of variational principle proposed and implemented in Sec. IV. (In this application, H_C is not varied and

therefore can be dropped.)

As a second application, we shall see that Eq. (20.119) has the same consequences as the variational principle

$$\delta \operatorname{Tr}\left[H - H_C \sum_i [a_i, a_i^\dagger]\right], \tag{20.124}$$

where the elements to be varied are the *operators* a_i, $a_i^\dagger$, and H_C is treated as a Lagrange multiplier matrix. The variational principle (20.124) has been studied previously (Klein, Li, and Vassanji, 1980) as a basis for a formulation of Heisenberg matrix mechanics. The equations of motion that follow from Eq. (20.124) using the cyclic invariance of the trace operation, are

$$\frac{\partial H}{\partial a_i^\dagger} = [a_i, H_C] \text{ and H.c. Eq.} \tag{20.125}$$

On the other hand, the corresponding derivation for Eq. (20.119) replaces the right-hand side of Eq. (20.125) by the corresponding partial derivative. In consequence of the standard formula (20.24), these two forms are equivalent. Second, since the collective Hamiltonian H_C is the generator of time displacements within the decoupled subspace, the right-hand side can be replaced by the time derivative $\dot{a}_i$, which yields a standard form of Heisenberg's equations of motion.

We remark finally that still other intrinsic states have been utilized in dynamical calculations involving nuclear models (Dasso, Krejs, Klein, and Chattopadhyay, 1973). The examples given nevertheless suffice to indicate that we have at hand a dynamical tool of some flexibility.

XXI. BOSONS AND THE THEORY OF LARGE-AMPLITUDE COLLECTIVE MOTION

A. Introduction

The formulation of a quantum theory of large-amplitude collective motion presupposes the presence of a general quantum theory of collective motion as a suitable starting point. One such theory has been described in the previous section. A number of other methods, including the generator coordinate method (Reinhard and Goeke, 1987), the equation of motion method (Klein, 1983b, 1984a, 1984b), and the Born-Oppenheimer method (Villars, 1984), have proved to be useful starting points for the development of a description of large-amplitude collective motion. Toward this end, the most important part of the theory is its classical limit, although the quantum formulations play an essential role in providing a basis for computing quantum corrections. The justification for the emphasis on the classical limit is that, in order to develop a large-amplitude mode, many degrees of freedom $\mathcal{N}$ must participate in the motion. In an expansion in $\mathcal{N}^{-1} = \hbar/(\hbar\mathcal{N})$, the correspondence principle assures us that the leading term is the classical limit. This section will be largely devoted to a study of that limit and the part that bosons can play both in the basic formulation and in selected applications.

To begin this exploration, it is not in fact necessary for us to describe any of the quantum theories referred to above. All the results we need have been derived in Sec. XI. There it was shown that after the BZM mapping from the shell-model space to the boson space, a simple c-number substitution—equivalent to taking the average in a coherent state and neglecting quantum fluctuations—reduces the equations of motion to the form of the time-dependent Hartree-Bogoliubov equations or, if one neglects pairing as we shall do, to the time-dependent Hartree equations. In Sec. XIV, we indicated how the exchange terms could be included, so that one could arrive at the time-dependent Hartree-Fock equations. A second result of major interest to us is that the time-dependent Hartree-Fock equations are equivalent to a set of classical Hamilton's equations, with the Hartree-Fock functional playing the role of Hamiltonian. Again a proof for the Hartree case was given explicitly, and the elegant treatment of the Hartree-Fock case by Yamamura and Kuriyama (1987) was outlined. In the course of this section we shall provide yet other proofs, but the main point allowing us to proceed has been vouchsafed, encouraging us to study large-amplitude collective motion in the classical domain. Thus, in the next subsection, we shall describe a recent, useful formulation of this theory (Do Dang and Klein, 1985; Bulgac, Klein, and Do Dang, 1987a, 1987b; Do Dang, Bulgac, and Klein, 1987). Following that, we shall describe some applications to simple boson models (Umar and Klein, 1986; Bulgac, Klein, and Do Dang, 1988). Other applications are conveniently studied within the usual time-dependent Hartree-Fock framework. A classical boson mapping of the BZM type then makes a useful tool for carrying out this transformation. We shall actually derive the formulas necessary to apply the previously developed theory within the framework provided by the time-dependent Hartree-Fock approximation in the adiabatic limit. This is as far as we shall carry the matter, since algorithms necessary for realistic applications are in a state of development and, though some elementary applications have been carried out (Bulgac, Klein, Walet, and Do Dang, 1989; Walet, Klein, Do Dang, and Bulgac, 1990), this work is in an early stage of development.

It would take us too far afield to give a detailed account of previous and concurrent work that has influenced the ideas to be described below. Particular mention should be made, however, of the work of Marumori and his associates (Marumori, 1977; Marumori, Hayashi, Tomoda, Kuriyama, and Maskawa, 1980; Marumori, Maskawa, Sakata, and Kuriyama, 1980); Rowe and Basserman (1976), and Kuriyama and Yamamura (Kuriyama and Yamamura, 1984a 1984b; Yamamura, Kuriyama, and Iida, 1984; Yamamura and Kuriyama, 1984, 1986a, 1986b, 1987a, 1987c).

B. Classical theory of decoupled motion

The study of collective motion in the classical domain is based on the idea of decoupled motion, i.e., motion

confined, under the Hamiltonian flow, to a submanifold of the total phase space. As has been amply documented in the references on the quantum theory of collective motion, this idea is the classical analog of the notion of an invariant subspace of the Hilbert space that we studied in one form in the previous section. Furthermore, a complete and usable theory of decoupled motion exists only for the case in which the classical Hamiltonian is quadratic in the momenta, i.e., is equivalent to a Lagrangian system. We shall therefore discuss only this case, following the work of Bulgac, Do Dang, and Klein cited above. From the references cited in these works, the interested reader can trace all the germane literature, both theoretical developments and the much smaller number of applications.

We study a classical system with N canonical pairs ξ^α and π_α (the single-particle coordinates) described by the Hamiltonian

$$H=\tfrac{1}{2}\pi_\alpha B^{\alpha\beta}(\xi)\pi_\beta+V(\xi)\ , \tag{21.1}$$

and consider the transformation to "collective" coordinates by means of a point canonical transformation,

$$\xi^\alpha=g^\alpha(q^1,\ldots,q^N)\ , \tag{21.2}$$

$$\pi_\alpha=f^\mu_{,\alpha}p_\mu \tag{21.3}$$

with inverse

$$q^\mu=f^\mu(\xi)\ , \tag{21.4}$$

$$p_\mu=g^\alpha_{,\mu}\pi_\alpha\ , \tag{21.5}$$

where the comma indicates partial derivative.

We assume that in the new coordinate set we can identify a decoupled surface, defined as follows: We divide the set q into two subsets, q^i, $i=1,\ldots,K$ and q^a, $a=K+1,\ldots,N$, and suppose this division to be such that if at time $t=0$ both $q^a=0$ (by convention) and $\dot q^{\,a}=0$, then $q^a(t)=0$. Such motions evolve on a K-dimensional submanifold

$$\xi^\alpha=g^\alpha(q^1\cdots q^K)\equiv g^\alpha(q)\ , \tag{21.6}$$

designated as the surface Σ. In geometrical terms, if the system point is initially on Σ and the initial velocity is on $T\Sigma$, the tangent plane to Σ at the given point, then, provided the subsequent motion of the system is confined to this surface, Σ is said to be decoupled. It is as if the system had imposed on itself a set of holonomic constraints.

Before deriving the conditions that characterize such a motion, let us note that under the point transformation (21.2) and (21.3) the Hamiltonian becomes

$$H(\xi,\pi)=\overline{H}(q,p)=\tfrac{1}{2}p_\mu\overline{B}^{\,\mu\nu}(q)p_\nu+\overline{V}(q)\ , \tag{21.7}$$

where

$$\overline{B}^{\,\mu\nu}=f^\mu_{,\alpha}B^{\alpha\beta}f^\nu_{,\beta}\ . \tag{21.8}$$

Moreover, the chain-rule relations

$$f^\mu_{,\alpha}g^\alpha_{,\nu}=\delta^\mu_\nu \tag{21.9}$$

permit us to reexpress Eq. (21.8) in the form

$$\overline{B}^{\,\mu\nu}g^\alpha_{,\nu}=B^{\alpha\beta}f^\mu_{,\beta}\ . \tag{21.10}$$

The conditions that characterize Σ are derived most readily from the equations of motion for the set q^a,p_a. These are

$$\dot q^{\,a}=\partial\overline{H}/\partial p_a=\overline{B}^{\,ai}p_i+\overline{B}^{\,ab}p_b\ , \tag{21.11}$$

$$-\dot p_a=\partial\overline{H}/\partial q^a=\overline{V}_{,a}+\tfrac{1}{2}\overline{B}^{\,ij}_{,a}p_ip_j+\overline{B}^{\,bi}_{,a}p_ip_b+\tfrac{1}{2}\overline{B}^{\,bc}_{,a}p_bp_c\ . \tag{21.12}$$

The requirements $\dot q^{\,a}=\dot p_a=0$ can be compatible with the requirements $q^a=p_a=0$ only if the equations

$$\overline{B}^{\,ai}p_i=0\ , \tag{21.13}$$

$$\overline{V}_{,a}+\tfrac{1}{2}p_ip_j\overline{B}^{\,ij}_{ia}=0 \tag{21.14}$$

are satisfied, as one sees from Eqs. (21.11) and (21.12). Equations (20.13) and (20.14) are equivalent to three sets of conditions, provided none of the p_i are constants of the motion, for in that case (21.14) yields two independent conditions, and altogether we have

$$\overline{B}^{\,ai}=0\ , \tag{21.15}$$

$$\overline{V}_{,a}=0\ , \tag{21.16}$$

$$\overline{B}^{\,ij}_{,a}=0\ . \tag{21.17}$$

The physical significance of these equations is apparent. The first tells us that we may choose the mass tensor to be block diagonal, i.e., it restricts the choice of noncollective coordinates at each point. The remaining two equations then demand the absence of both "real" and "geometric" (centrifugal) forces orthogonal to the decoupled surface. In the event that there are additional constants of the motion, the necessary modifications (Bulgac, Klein, and Do Dang, 1987b) can be made, but will not be discussed here.

It follows readily from the decoupling conditions that the Hamiltonian that governs the motion on Σ, the "collective" Hamiltonian, is the value of $\overline{H}$, [Eq. (21.7)] on the surface.

Though Eqs. (21.15)–(21.17) are the most physically transparent form of the decoupling conditions, it has proven of value to transform them in order to obtain a constructive procedure applicable to a wide range of applications. The first, trivial stage of transformation is to replace (21.15)-(21.17) by the equivalent set,

$$B^{\alpha\beta}f^i_{,\beta}=\overline{B}^{\,ij}g^\alpha_{,j}\ , \tag{21.18}$$

$$V_{,\alpha}=\overline{V}_{,i}f^i_{,\alpha}\ , \tag{21.19}$$

$$\overline{B}^{\,ij}_{,\alpha}=\overline{B}^{\,ij}_{,k}f^k_{,\alpha}\ . \tag{21.20}$$

Of these relations, Eqs. (21.19) and (21.20) are chain-rule relations that have been simplified by the imposition of Eqs. (21.16) and (21.17), respectively, whereas Eqs. (21.18) is a simplified version of Eq. (21.10) obtained by

remembering Eq. (21.15). Geometrically, Eq. (21.18) states that the quantities $g^{\alpha}_{,i}$ and $f^{i}_{,\alpha}$ are equivalent sets of basis vectors on Σ, and Eq. (21.19) affirms that the gradient of V lies in $T\Sigma$.

Next we study the transformation of the above conditions into the form to be utilized. The basic idea that underlies the following considerations is that we should be able to construct the surface $\xi^{\alpha}=g^{\alpha}(q)$ provided we can specify the tangent plane at each point. We shall do this by constructing a complete set of basis vectors for the tangent plane from the "ingredients" of the given Hamiltonian.

To carry out this program, we define a sequence of single-index point functions according to the definitions

$$X^{(0)}\equiv V(\xi)=\overline{V}(q) , \tag{21.21}$$

$$X^{(1)}\equiv V_{,\alpha}B^{\alpha\beta}V_{,\beta}=\overline{V}_{,\mu}\overline{B}^{\mu\nu}\overline{V}_{,\nu} , \tag{21.22}$$

$$\vdots$$

$$X^{(\sigma+1)}\equiv X^{(\sigma)}_{,\alpha}B^{\alpha\beta}X^{(\sigma)}_{,\beta}=\overline{X}^{(\sigma)}_{,\mu}\overline{B}^{\mu\nu}\overline{X}^{(\sigma)}_{,\nu} . \tag{21.23}$$

Then, for $\sigma\neq\tau$, we define a sequence of double-index point functions,

$$X^{(\sigma\tau)}\equiv X^{(\sigma)}_{,\alpha}B^{\alpha\beta}X^{(\tau)}_{,\beta}=\overline{X}^{(\sigma)}_{,\mu}\overline{B}^{\mu\nu}\overline{X}^{(\tau)}_{,\nu} . \tag{21.24}$$

Thus the single-index sequence is constructed with the help of the mass tensor, here in its role as metric tensor, by forming the gradient of the previous point function and then calculating the length of the new vector. The double-index scalars are mixed scalar products of gradients. By finding the gradients of these, we can form still additional sequences of point functions, all of which are subsumed under the considerations that follow.

The basic assertion is that, for a decoupled surface, the gradient of every scalar, either defined above or alluded to, is a vector field that lies in the tangent plane to Σ. The proof depends on induction. We first note that, according to the fundamental decoupling condition (21.16), the gradient of $X^{(0)}$ lies in the tangent plane, i.e., $\overline{X}^{(0)}_{,a}=0$. Now let us assume that $\overline{X}^{(\sigma)}_{,a}=0$ and show that, in consequence of this statement and all the remaining decoupling conditions, $\overline{X}^{(\sigma+1)}_{,a}=0$. We simply compute

$$\begin{aligned}\overline{X}^{(\sigma+1)}_{,a}&=2\overline{X}^{(\sigma)}_{,\mu a}\overline{X}^{(\sigma)}_{,\nu}\overline{B}^{\mu\nu}+\overline{X}^{(\sigma)}_{,\mu}\overline{X}^{(\sigma)}_{,\nu}\overline{B}^{\mu\nu}_{,a}\\&=2\overline{X}^{(\sigma)}_{,ba}\overline{X}^{(\sigma)}_{,i}\overline{B}^{bi}+2\overline{X}^{(\sigma)}_{,ia}\overline{X}^{(\sigma)}_{,j}\overline{B}^{ij}+\overline{X}^{(\sigma)}_{,i}\overline{X}^{(\sigma)}_{,j}\overline{B}^{ij}_{,a}\\&=0 .\end{aligned} \tag{21.25}$$

In passing to the second line of Eq. (21.25), we have used only the statement $\overline{X}^{(\sigma)}_{,a}=0$; in order to obtain zero overall, we have then used Eqs. (21.15) and (21.17) in the first and third terms, respectively, whereas the second term vanishes because $\overline{X}^{(\sigma)}_{,ai}=0$. The vanishing of the gradients of the multiply-indexed scalars follows from the same mode of proof.

The results obtained can be summarized in two equivalent forms. Let us suppose for the moment that all the point functions of interest have been arranged into a linear array designated $X^{(\sigma)}$, in the notation used previously only for the single-index scalars. In the same way that Eq. (21.16) implied Eq. (21.19), we have more generally

$$X^{(\sigma)}_{,\alpha}=\overline{X}^{(\sigma)}_{,i}f^{i}_{,\alpha} . \tag{21.26}$$

By using $B^{\alpha\beta}$ in the entire space and $\overline{B}^{ij}$ on Σ to raise indices, and by remembering Eqs. (21.18), we can convert Eq. (21.26) to the form

$$X^{(\sigma),\alpha}=\overline{X}^{(\sigma),i}g^{\alpha}_{,i} . \tag{21.27}$$

Equations (20.26) and (20.27) both state, one in covariant, the other in contravariant form, that the vector fields in question lie in the tangent plane to Σ.

The remainder of this section is devoted to the application of the theorem to examples that represent cases where the decoupling is not exact. In those cases, the best one can do is to define a collective submanifold as the solution of a suitably chosen subset of the ensemble of conditions derived for exact decoupling. The most complicated case that has been studied (Bulgac, Klein, and Do Dang, 1987) is the case $K=2$, i.e., a two-dimensional manifold. Here one utilizes the simplest point functions $V=X^{(0)}$, $U=X^{(1)}$, and $T=X^{(01)}$ in the notation of Eq. (20.24). For example, Eq. (20.27) becomes in this case

$$V^{,\alpha}=\overline{V}^{,i}g^{\alpha}_{,i} , \tag{21.28}$$

$$U^{,\alpha}=\overline{U}^{,i}g^{\alpha}_{,i} , \tag{21.29}$$

$$T^{,\alpha}=\overline{T}^{,i}g^{\alpha}_{,i} . \tag{21.30}$$

In the following, we shall treat several one-dimensional cases, where we need only the first two of the above equations.

C. Application to an SU(3) model

As a first application (Umar and Klein, 1986) we consider the three-level generalization of the Lipkin model described and utilized in Sec. XX. For the purposes of this account, it suffices to pass immediately to our working Hamiltonian, obtained from Eqs. (20.51)–(20.54), in the polynomial approximation, by a standard linear transformation from boson creation and annihilation operators to coordinates and momenta. In dimensionless form and with parameters defined below, we have

$$h=(H/\epsilon J)=t+v , \tag{21.31}$$

where

$$t=\frac{1}{4}\sum_k(\eta_k+1)p_k^2+\frac{1}{2}\bar{f}\underline{p}^2-\frac{1}{8}(\underline{p}^2)^2 , \tag{21.32}$$

$$v=-\left[1+\frac{n}{4J}\right]+\frac{1}{4}\sum_k(\eta_k+1)x_k^2 -\frac{1}{2}\bar{f}\underline{x}^2+\frac{1}{8}f(\underline{x}^2)^2 . \tag{21.33}$$

These equations contain a number of parameters from the original model: There are n levels leading to $n-1$ canonical pairs (the summations over k running from 1 to $n-1$); ϵ is an overall scale factor for single-particle energies; $\eta_k \leq \eta_{k+1}$ is an ordering parameter for these levels with $\sum \eta_k = 1$; $2J$ is the degeneracy of each level; and

$$\underline{x}^2 = \sum_k (x_k)^2, \quad \underline{p}^2 = \sum_k (p_k)^2 , \tag{21.34}$$

$$\bar{f} = f(1+n/4J) . \tag{21.35}$$

The commutation relations for x, p are

$$[x_k, p_l] = (i/J)\delta_{kl} . \tag{21.36}$$

Here J^{-1} plays the role of $\hbar$. It is thus seen that the model describes coupled quartic oscillators. Further discussion will be confined to the case $n=3$.

Since the starting Hamiltonian (21.31)–(21.33) is quartic in the momenta, the first step in applying the theory of large-amplitude collective motion is to identify a range of coupling strengths f for which the adiabatic approximation is valid. This is done by looking for phase transitions within the model associated with critical points of the potential where $v_i \equiv (\partial v/\partial x_i)$ vanishes. From Eq. (21.33), we find the conditions (let $\eta_2 = 1-\eta_1$)

$$[\tfrac{1}{2}(1+\eta_1) - \bar{f} + \tfrac{1}{2} f \underline{x}^2] x_1 = 0 , \tag{21.37}$$

$$[\tfrac{1}{2}(2-\eta_1) - \bar{f} + \tfrac{1}{2} f \underline{x}^2] x_2 = 0 . \tag{21.38}$$

[Note that $\eta_1 \leq \frac{1}{2}$, $(2-\eta_1) \geq (1+\eta_1)$.] Defining two critical values of $\bar{f}$, namely,

$$f_1 = \tfrac{1}{2}(1+\eta_1) , \tag{21.39}$$

$$f_2 = \tfrac{1}{2}(2-\eta_1) , \tag{21.40}$$

we obtain the following solutions:

(I) $\bar{f} < f_1$. The only solution is

(a) $x_1 = x_2 = 0$ (absolute minimum of v) . (21.41)

(II) $f_2 > \bar{f} > f_1$. There are two solutions,

(a) $x_1 = x_2 = 0$ (saddle point) ,

(b) $x_2 = 0$, $(x_1)^2 = 2[\bar{f} - \frac{1}{2}(1+\eta_1)]/f$

(symmetric minima) . (21.42)

(III) $\bar{f} > f_2$. There are three solutions,

(a) $x_1 = x_2 = 0$ (local maximum) ,

(b) $x_2 = 0$, $x_{1c}^2 = 2[\bar{f} - \frac{1}{2}(1+\eta_1)]/f$

(symmetric minima) ,

(c) $x_1 = 0$, $x_{2c}^2 = 2[\bar{f} - \frac{1}{2}(2-\eta_1)]/f$

(symmetric saddle points) . (21.43)

The above presupposes that $\eta_1 \neq 0.5$. This limiting case is discussed in Umar and Klein (1986), but will not be considered here.

These results are sufficient to indicate the domain of validity of the adiabatic approximation. In region I, where quantum fluctuations about $x_1 = x_2 = 0$ are governed by Eq. (21.36), we have the usual relationships for small vibrations, namely $\langle x_k \rangle \sim \langle p_k \rangle$, and each, in consequence of Eq. (21.36), is $0(1/\sqrt{J})$. (Here $\langle\ \rangle$ indicates a dominant nonvanishing matrix element.) Under these conditions, the adiabatic approximation is not valid for any of the coordinates. In region II, on the other hand, the same remarks used in region I apply to x_2 and p_2; however, now $\langle x_1 \rangle \sim 1$, consequently $\langle p_1 \rangle \sim J^{-1}$, and therefore this canonical pair is in the large-amplitude, small-momentum regime. In region III there is also, as we shall see, an appropriate collective coordinate, which may be taken to be an angle around a closed, almost elliptical, path passing through the points described by (b) and (c) in Eq. (21.43). Along such a path the collective coordinate is on the average 0(1) and the corresponding momentum is small. Though the theory described in Sec. XXI.B can thus be applied to both regions II and III, we shall give only the more interesting results for region III.

One is seeking to decouple one collective coordinate from a system with two degrees of freedom. For this purpose one need apply only Eqs. (21.28) and (21.29), and the hypersurface Σ is simply a collective path,

$$x^\alpha = g^\alpha(q), \quad \alpha = 1,2 , \tag{21.44}$$

in an obvious change of notation. Since Eqs. (21.28) and (21.29) require the gradients of V and of U to be parallel to the tangent to the collective path, they must necessarily be parallel to each other. This is expressed by the vanishing of a two-by-two determinant,

$$\begin{vmatrix} V^1 & V^2 \\ U^1 & U^2 \end{vmatrix} = 0 . \tag{21.45}$$

The solution (solutions) of this equation is (are) most naturally obtained as a relationship between the two original coordinates, and we therefore choose

$$x^1 = q , \tag{21.46}$$

$$x^2 = g(x^1) , \tag{21.47}$$

the latter representing the collective path. In Fig. 24, the contours of the potential-energy function $v(x^1, x^2)$, in region III, are shown for a convenient choice of parameters, $J=14$, $\eta_1 = 0.1$, $f = 2.5$. The relevant solution of Eq. (21.45) is actually a closed valley connecting the minima to the saddle points. One quadrant of this path is shown for various values of f in Fig. 25. For values not too near the transition point between the regions, it is well approximated by an ellipse, and for very large values of f it approaches a circle.

According to the theory presented in Sec. XXI.B, the collective potential energy follows directly from a knowledge of the collective path, but the computation of the collective mass requires also the tangent vector to the path. Here some care is required in understanding exact-

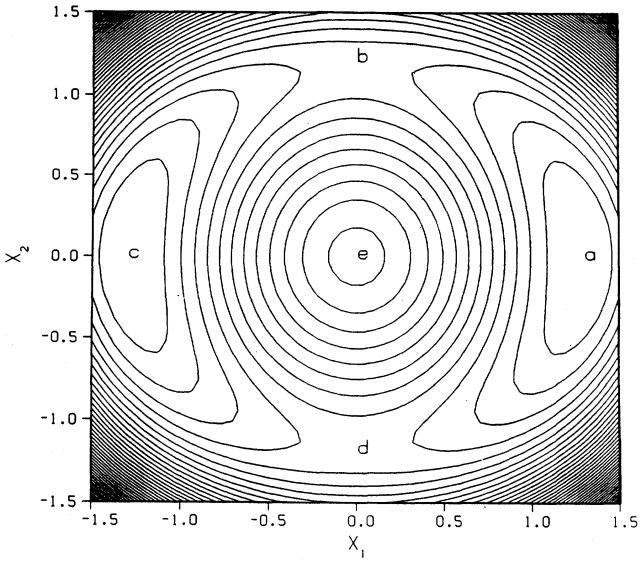

FIG. 24. Contours of the potential $v(x_1,x_2)$ in region III, with $\mathcal{N}=28$, $\eta_1=0.1$, and $f=2.5$. The points (a) and (c) are the symmetric minima, points (b) and (d) are saddle points, and point (e) is a local minimum. From Umar and Klein, 1986.

ly what has emerged from the calculation. The collective path found is the locus of all points where the gradients of the two point functions are parallel. Only when the decoupling is exact is this common direction also the direction of the tangent to the path. With two different sets of directions, we can compute two different collective masses and compare them. This will be done for the next example, but here we shall simply report results obtained by the use of the tangent vector.

The remaining work for this example is an elementary exercise in one-dimensional quantum mechanics. To quantize the system, the polar angle was introduced as coordinate. There is an ambiguity in the quantization of the kinetic energy, since the collective mass is position dependent, but differences are of relative order J^{-2} in the *potential energy*. A comparison of a subset of the exact eigenvalues of Eq. (21.31), obtained by a matrix diagonalization, with the excitation energies predicted by the one-dimensional Schrödinger equation derived by the decoupling procedure, is shown in Fig. 26 for two different quantizations of the kinetic energy. It is possible to go further and include the missing degree of freedom approximately, so as to obtain absolute energies. For these details, we refer the reader to the original work.

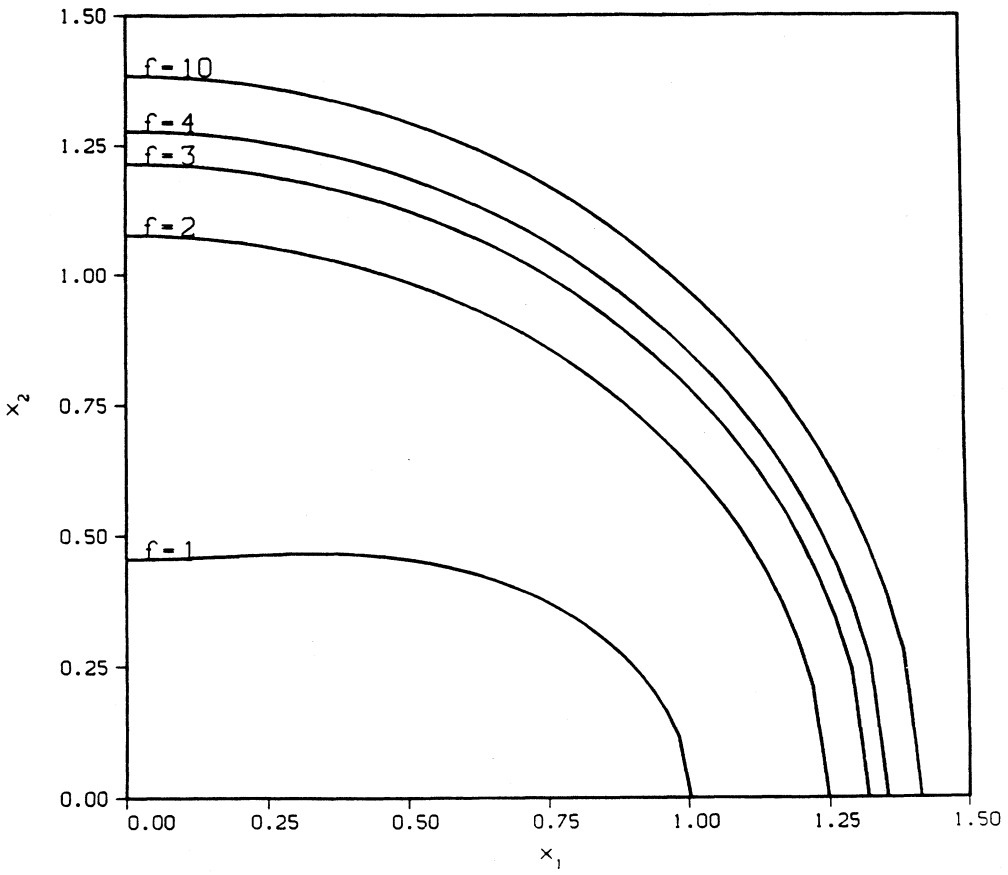

FIG. 25. The collective path $x_2(x_1)$ in the quadrant ($x_2>0$, $x_1>0$), for various values of the coupling strength f. The values of $\mathcal{N}=28$ and $\eta_1=0.1$ are held fixed. From Umar and Klein, 1986.

D. Application to a model with tunneling

The system to be studied is described by the Hamiltonian

$$H=\sum_{i=1}^{N}\frac{1}{2}(p_i^2+q_i^2)+\kappa\left[\sum_{i=1}^{N}q_i\right]\left[\sum_{i=1}^{N}\sigma_{zi}\right]-\lambda\left[\sum_{i=1}^{N}\sigma_{xi}\right]^2 , \tag{21.48}$$

where, for reasons explained by Arve *et al.* (1987), the parameters are chosen to have the values $\kappa=0.006\,403$, $\lambda=0.0005$, and $N=40$. It is easy to see that only the center of mass motion is coupled to the total spin, and the internal motion is unaffected by the interaction. We take advantage of this observation to introduce the new variables

$$q=\frac{1}{\sqrt{N}}\sum_{i=1}^{N}q_i , \tag{21.49a}$$

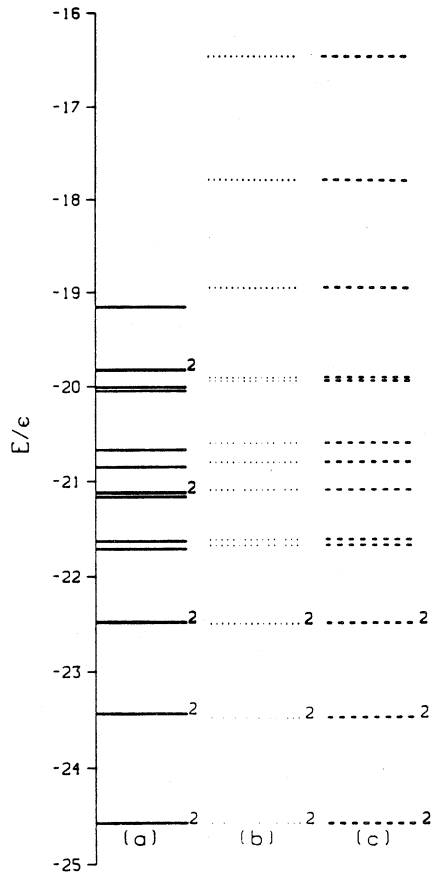

FIG. 26. Comparison of the eigenvalues of the exact diagonalization, shown in column (a), with the eigenvalues of the collective Hamiltonian shown in column (b). Parameter values are $\mathcal{N}=28$, $\eta=0.1$, and $f=2.5$ (region III). Column (c) corresponds to the eigenvalues arising from the case of a modified kinetic-energy quantization. From Umar and Klein, 1986.

$$p=\frac{1}{\sqrt{N}}\sum_{i=1}^{N}p_i , \tag{21.49b}$$

$$S_k=\tfrac{1}{2}\sum_{i=1}^{N}\sigma_{ik},\quad k=x,y,z . \tag{21.50}$$

Then the interesting part of the Hamiltonian (21.48) reads

$$H'=\tfrac{1}{2}p^2+\tfrac{1}{2}q^2+2\kappa\sqrt{N}\,qS_z-4\lambda S_x^2 . \tag{21.51}$$

Henceforth we replace H' by H. We shall also set $N=2J$, where J is the value of the angular momentum for the band of states that includes the ground state.

In order to study the Hamiltonian (21.51) by the methods of this section, we have to take two steps: we must first map the spin system to a system of bosons, and second we must go to the classical limit. For the first step, it is convenient to utilize the mapping of the SU(2) algebra onto a pair of quantum action-angle variables Π and ξ, according to formulas identical, except for notation, with those derived in Sec. III.C,

$$\begin{aligned}S_+=(S_-)^\dagger=S_z+iS_x&\\ =\exp(\tfrac{1}{2}i\xi)\sqrt{(J+\tfrac{1}{2})^2-\Pi^2}\exp(\tfrac{1}{2}i\xi)&\\ =\exp(i\xi)\sqrt{J(J+1)-\Pi(\Pi+1)}\ ,&\end{aligned} \tag{21.52}$$

$$S_y=\Pi , \tag{21.53}$$

where

$$[\Pi,\exp(i\xi)]=\exp(i\xi) . \tag{21.54}$$

From these formulas, we obtain approximate formulas for S_z and S_z^2, required for the evaluation of Eq. (21.51), by expanding in powers of $1/J$, assuming Π to be no larger than of order unity. These formulas are

$$S_z=\sqrt{J(J+1)}\cos(\xi)-\frac{1}{4\sqrt{J(J+1)}}\{\cos(\xi),\Pi^2\} , \tag{21.55}$$

$$S_x=\sqrt{J(J+1)}\sin(\xi)-\frac{1}{4\sqrt{J(J+1)}}\{\sin(\xi),\Pi^2\} , \tag{21.56a}$$

$$S_x^2=J(J+1)\sin^2(\xi)-\tfrac{1}{4}\{\sin(\xi),\{\sin(\xi),\Pi^2\}\} , \tag{21.56b}$$

where the braces stand for the anticommutator of the corresponding operators. The Hamiltonian (21.51) thus becomes

$$H=V(q,\xi)+T , \tag{21.57}$$

$$\begin{aligned}V(q,\xi)=\tfrac{1}{2}q^2+2\kappa q\sqrt{2J^2(J+1)}\cos(\xi)&\\ -4\lambda J(J+1)\sin^2(\xi)\ ,&\end{aligned} \tag{21.58}$$

$$T=\tfrac{1}{2}p^2-\kappa q\frac{1}{\sqrt{2(J+1)}}\{\cos(\xi),\Pi^2\}$$
$$+\lambda\{\sin(\xi),\{\sin(\xi),\Pi^2\}\}\ . \tag{21.59}$$

The above expression for H contains correctly the classical limit plus the corrections of order $1/J$. The choice of variables made in Eqs. (21.52)–(21.54) has the great advantage that Π remains small compared to J in the domain of interest to us, as already assumed in Eqs. (21.55) and (21.56).

Passing to the classical limit, we may rewrite the kinetic energy as

$$T_{\text{class}}=\tfrac{1}{2}p^2+\tfrac{1}{2}B(q,\xi)\Pi^2\ , \tag{21.60}$$

where

$$B(q,\xi)=8\lambda\sin^2(\xi)-\kappa q\frac{4}{\sqrt{2(J+1)}}\cos(\xi)\ . \tag{21.61}$$

The profile of the potential-energy function, $V(q,\xi)$, is displayed in Fig. 27, where one can easily see a *valley*, the precise course of which is defined below in Eq. (21.66). We take ξ in the range $0\le\xi\le 2\pi$. We then compute that V is minimum at $\xi=n\pi$ and $q=-2\kappa\sqrt{2J^2(J+1)}\cos(\xi)$, with

$$V_{\text{min}}=-4\kappa^2J^2(J+1)\ , \tag{21.62}$$

and V has saddle points at $q=0$, $\xi=(2n+1)\pi/2$, with

$$V_{\text{saddle}}=-4\lambda J(J+1)\ . \tag{21.63}$$

We thus identify a potential barrier of height

$$V_0=4J(J+1)(J\kappa^2-\lambda)\ . \tag{21.64}$$

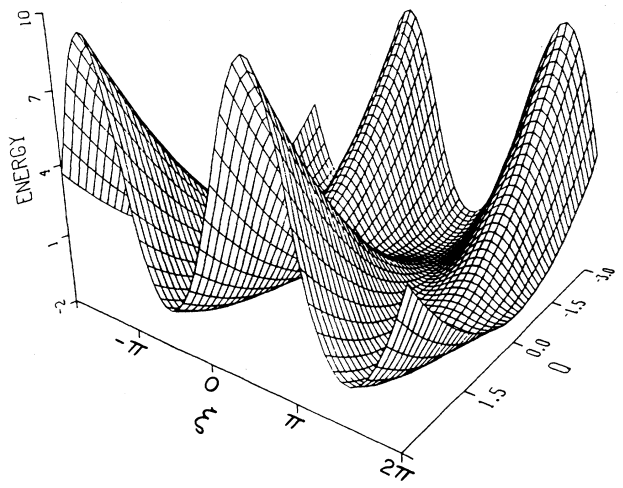

FIG. 27. The profile of the potential-energy surface $V(q,\xi)$ given by Eq. (23.58) associated with a model for tunneling. From Bulgac, Do Dang, and Klein, 1988.

We now turn to the machinery used to calculate a collective path. We construct the point function

$$U(q,\xi)=\tfrac{1}{2}(V_q^2+BV_\xi^2) \tag{21.65}$$

and solve the equations

$$V_q-\omega U_q=0\ , \tag{21.66a}$$
$$V_\xi-\omega U_\xi=0\ , \tag{21.66b}$$

where by subscripts we denote the corresponding partial derivatives and where ω is a Lagrange multiplier. These are the equations of a valley on the potential-energy surface as seen from the standpoint of a metric defined by the classical kinetic energy (21.60).

We parametrize the collective path by choosing a collective coordinate x defined by the equations

$$\xi=x\ , \tag{21.67a}$$
$$q=g(x)\ . \tag{21.67b}$$

The resulting collective potential energy that emerges from the calculation is computed from the formula

$$\bar{V}(x)=V(q(x),\xi(x))\ . \tag{21.68}$$

Turning to a consideration of the collective mass, as we have explained in connection with the previous example, the theory provides two distinct formulas, which agree only when there is exact decoupling. The mass that is determined by the tangent to the path is given by the formula

$$\bar{B}^{-1}=\left[\frac{d\xi}{dx}\right]^2\frac{1}{B}+\left[\frac{dq}{dx}\right]^2=\frac{1+Bg_x^2}{B}\ , \tag{21.69}$$

where

$$\begin{bmatrix}1 & 0\\ 0 & \dfrac{1}{B}\end{bmatrix}\ , \tag{21.70}$$

is the inverse of the metric tensor associated with Eq. (21.59). [Equation (21.69) is the covariant form of Eq. (21.8)]. Since the decoupling is not exact, we can calculate an alternative formula for the mass by replacing the components of the tangent vector in Eq. (21.8) by the components of a corresponding vector parallel to grad$V$. This gives the formula

$$\tilde{B}=\frac{V_q^2+BV_\xi^2}{(V_qg_x+V_\xi)^2}\ . \tag{21.71}$$

As a measure of the goodness of decoupling, we choose the fractional difference of the two masses, as expressed by the equation

$$D=\frac{\tilde{B}-\bar{B}}{\bar{B}}\ . \tag{21.72}$$

The numerical result is that the quantity D is never larger than 0.000 12 on any point of the collective path.

TABLE IV. The mean excitation energies and the decimal logarithms of the splittings of the energy levels below the barrier for the tunneling model introduced by Arve *et al.*, 1987. The first two columns list the exact and approximate mean excitation energies of the doublets. The remaining columns record the decimal logarithms of the energy splittings of the first five levels. From Bulgac, Klein, and Do Dang, 1988.

Energy Arve *et al.* (1987)	Energy (Large-amplitude collective motion)	Exact Arve *et al.* (1987)	Imaginary time-dependent Hartree-Fock Arve *et al.* (1987)	Large-amplitude collective motion
G.S.	G.S.	13.00	13.00	12.89
0.154	0.154	8.96	8.57	8.89
0.289	0.288	5.92	5.39	5.87
0.403	0.401	3.51	2.85	3.51
0.489	0.487	1.92	1.00	1.92

This means that, in the present case, the collective branch of the spectrum is almost completely decoupled from the noncollective one.

The last point that we shall discuss is the quantization of the collective Hamiltonian in order to find the collective states. This procedure is standard. We choose the form

$$H=\tfrac{1}{2}\{\Pi,\{\Pi,\bar{B}\}\}+\bar{V}(\xi)\ , \tag{21.73}$$

where $[\xi,\Pi]=i$ are the collective coordinate and momenta, $0\leq\xi\leq 2\pi$, and periodic boundary conditions are imposed. The results obtained by solving Eq. (21.73) are compared with the exact results and with results obtained by other approaches (Arve *et al.*, 1987) in Table IV. Of course one obtains a spectrum of doublets characteristic of the tunneling process. In the first two columns we list the exact and approximate mean excitation energies of the doublets. The resulting decimal logarithms of the energy splittings for the first five levels are given in the remaining columns of the table.

For all the levels below the barrier, the agreement with the exact result from Arve *et al.* (1987) is very good. The fact that both the mean excitation energies and the splittings are well reproduced means that the formalism presented is accurate for the description of both the classically allowed and the forbidden regions. The values of the mean excitation energies are defined by the collective mass and potential energy in the classically allowed region, while the splittings depend on the properties of the collective Hamiltonian in the classically forbidden region.

The theory applied above to the decoupling of one degree of freedom from a two-dimensional system has also been applied to the decoupling of two degrees of freedom from a three-dimensional system (Bulgac, Klein, and Do Dang, 1987b). Furthermore, it appears to be completely feasible to write down a coupled system with many degrees of freedom, with application to tunneling problems in molecular systems, and to derive an effective Hamiltonian for the tunneling (Walet, Klein, and Do Dang, 1989). The problems of nuclear physics are, however, more difficult and at the present moment seem to require that one transcribe the previous theory back into time-dependent Hartree-Fock language. This part of the theory has a clear connection with boson mappings and is the last subject that we shall describe in this section, since algorithms for applying these ideas to realistic examples are still in the course of development.

E. Transcription to time-dependent Hartree-Fock theory

We take the time-dependent Hartree-Fock equations in the form

$$i\dot{\rho}_{ab}=[\mathcal{H},\rho]_{ab}\ , \tag{21.74}$$

where

$$\mathcal{H}_{ab}=h_{ab}+V_{acbd}\rho_{dc} \tag{21.75}$$

and

$$h_{ab}=h^*_{ba},\ V_{abcd}=-V_{bacd}=-V_{abdc}=V^*_{cdab} \tag{21.76}$$

are the traditional elements of a nuclear Hamiltonian. The labels $a,b,\ldots$ refer to a complete orthonormal set of single-particle functions ϕ_a; this set will be further subdivided into a set h occupied in the reference Slater determinant and an unoccupied set p. We have

$$\rho=\sum_h \phi_h\phi^*_h,\ (\rho^2)_{ab}=\rho_{ab}\ . \tag{21.77}$$

The most convenient choice of basis for exhibiting the canonical structure of Eq. (21.74) is the one in which ρ is instantaneously diagonal. In this basis Eq. (21.74) is equivalent to the pair of equations

$$\begin{aligned} i\dot{\rho}_{ph}&=\mathcal{H}_{ph}=(\delta H/\delta\rho_{hp})\ ,\\ i\dot{\rho}_{hp}&=-\mathcal{H}_{hp}=-(\delta H/\delta\rho_{ph})\ , \end{aligned} \tag{21.78}$$

where

$$H=W_{\mathrm{HF}}[\rho]=h_{ab}\rho_{ba}+\tfrac{1}{2}V_{abcd}\rho_{ca}\rho_{db}\ , \tag{21.79}$$

the Hartree-Fock functional, serves as "Hamiltonian." Remarkably Eqs. (21.78) are already in Hamiltonian form, where we identify ρ_{ph} and $\rho_{hp}=\rho^*_{ph}$ as complex

canonical variables. We may introduce real canonical coordinates ξ and π,

$$\rho_{ph}=\frac{1}{\sqrt{2}}(\xi^{ph}+i\pi_{ph}) ,$$
$$\rho_{hp}=\frac{1}{\sqrt{2}}(\xi^{ph}-i\pi_{ph})=\frac{1}{\sqrt{2}}(\xi^{hp}+i\pi_{hp}) . \tag{21.80}$$

The canonical coordinates (21.80) that provide such a concise proof of the canonicity of the time-dependent Hartree-Fock equations are, however, not suitable for the study of the adiabatic limit. Note, in particular, that Eqs. (20.80) vanish at the particular instant of their utilization, though of course their derivatives do not. We describe briefly the alternative method that we utilize, based on a classical version of the Holstein-Primakoff boson mapping (Blaizot and Marshalek, 1978).

The formulas

$$\rho_{ph}=[\beta(1-\beta^{\dagger}\beta)^{1/2}]_{ph} , \tag{21.81}$$
$$\rho_{hp}=[(1-\beta^{\dagger}\beta)^{1/2}\beta^{\dagger}]_{hp} \tag{21.82}$$
$$\rho_{pp'}=[\beta\beta^{\dagger}]_{pp'} , \tag{21.83}$$
$$\rho_{hh'}=[1-\beta^{\dagger}\beta]_{hh'} \tag{21.84}$$

define a mapping from the elements of the density matrix in an arbitrary basis onto a set of complex numbers β_{ph}, which, together with their complex conjugates, are a set of complex canonical variables, convenient for the discussion of vibrational degrees of freedom. For the study of large-amplitude motion, we introduce real canonical variables ξ and π, distinct from the variables in Eq. (20.80), according to the standard formulas

$$\beta=\frac{1}{\sqrt{2}}(\xi+i\pi), \quad \beta^{\dagger}=\frac{1}{\sqrt{2}}(\tilde{\xi}-i\tilde{\pi}) , \tag{21.85}$$

where tilde means transposed. We have

$$\xi^{ph}=\xi^{hp}, \quad \pi_{ph}=-\pi_{hp} . \tag{21.86}$$

In this transcription the classical Hamiltonian is given by the formula

$$H(\xi,\pi)=W_{\mathrm{HF}}(\rho(\xi,\pi)) . \tag{21.87}$$

In the adiabatic limit by far the most convenient representation appears to be the one in which $\rho(\xi,\pi=0)\equiv\rho^{(0)}(\xi)$ *is diagonal.* This choice greatly simplifies the ensuing formulas.

In order to expand $H(\xi,\pi)$ in powers of π to second-order terms, we first expand $\rho(\xi,\pi)$,

$$\rho(\xi,\pi)=\rho^{(0)}(\xi)+\rho^{(1a)}(\xi)\pi_a+(1/2)\rho^{(2ab)}(\xi)\pi_a\pi_b+\cdots , \tag{21.88}$$

where the different orders are subject to the well-known constraints following from the idempotency of ρ,

$$\rho^{(0)}=\rho^{(0)2} , \tag{21.89}$$
$$\rho^{(1a)}=\rho^{(0)}\rho^{(1a)}+\rho^{(1a)}\rho^{(0)} , \tag{21.90}$$
$$\rho^{(2ab)}=\rho^{(0)}\rho^{(2ab)}+\rho^{(2ab)}\rho^{(0)}+\rho^{(1a)}\rho^{(1b)}+\rho^{(1b)}\rho^{(1a)} . \tag{21.91}$$

Next, if we carry out a formal expansion of Eq. (21.81), after having substituted Eq. (21.85), we find to second order

$$\sqrt{2}\rho=\xi(1-\tfrac{1}{2}\xi^2)^{1/2}+i\pi(1-\tfrac{1}{2}\xi^2)^{1/2} -\tfrac{1}{4}\xi\pi^2[1-\tfrac{1}{2}\xi^2)]^{-1/2} . \tag{21.92}$$

The consequences of Eq. (21.92) are, first, that in the representation in which $\rho^{(0)}$ is diagonal we have

$$(\rho^{(0)})_{ph}=[\xi(1-\tfrac{1}{2}\xi^2)^{1/2}]_{ph}=0 , \tag{21.93}$$

and therefore $\xi^{ph}=\xi^{hp}=0$. It follows that Eq. (21.92) reduces to

$$\sqrt{2}\rho_{ph}=i\pi_{ph}+O(\pi^3) . \tag{21.94}$$

Upon comparison with Eq. (21.88), we conclude that

$$\rho_{ph}^{(1p'h')}=-\rho_{hp}^{(1p'h')}=\frac{1}{\sqrt{2}}i\delta_{pp'}\delta_{hh'} , \tag{21.95}$$

$$\rho_{ph}^{(2p'h'p''h'')}=0 . \tag{21.96}$$

The adiabatic expansion of the Hamiltonian leading to the form (21.1) can be obtained by collecting the above results, substituting into Eq. (21.87), and expanding to the required order. A more elegant derivation is to expand directly in powers of π_{ph} and then notice that, for variation about the chosen representation, we have from Eq. (21.92)

$$\sqrt{2}\,\delta\rho=\delta\xi+i\delta\pi , \tag{21.97}$$

which is also a consequence of Eq. (21.80). Following this latter procedure and remembering Eq. (21.97), we can write

$$H(\xi,\pi)=W_{\mathrm{HF}}[\rho(\xi,\pi)]$$
$$=W[\rho(\xi,0)]+\tfrac{1}{2}\pi_{ph}\pi_{p'h'}(\delta^2W/\delta\pi_{ph}\delta\pi_{p'h'})$$
$$\equiv V(\xi)+\tfrac{1}{2}\pi_{\alpha}B^{\alpha\beta}(\xi)\pi_{\beta} , \tag{21.98}$$

where the linear term vanishes,

$$\frac{\delta W}{\delta\pi_{ph}}=\frac{i}{\sqrt{2}}\left[\frac{\delta W}{\delta\rho_{ph}}-\frac{\delta W}{\delta\rho_{hp}}\right]$$
$$=\frac{i}{\sqrt{2}}(\mathcal{H}_{hp}-\mathcal{H}_{ph})=0 , \tag{21.99}$$

since the matrix elements of $\mathcal{H}$ can be chosen real if the system under study is time-reversal invariant. Furthermore

$$\frac{\delta^2 W}{\delta\pi_{ph}\delta\pi_{p'h'}} = B^{php'h'} = -\frac{1}{2}\left[\frac{\delta^2 W}{\delta\rho_{ph}\delta\rho_{p'h'}} - \frac{\delta^2 W}{\delta\rho_{ph}\delta\rho_{h'p'}} - \frac{\delta^2 W}{\delta\rho_{hp}\delta\rho_{p'h'}} + \frac{\delta^2 W}{\delta\rho_{hp}\delta\rho_{h'p'}}\right] . \tag{21.100}$$

The evaluation of Eq. (21.100) yields the formula

$$B^{php'h'} = \tfrac{1}{2}\delta_{hh'}(\mathcal{H}_{pp'} + \mathcal{H}_{p'p}) - \tfrac{1}{2}\delta_{pp'}(\mathcal{H}_{hh'} + \mathcal{H}_{h'h}) + \tfrac{1}{2}(V_{ph'hp'} + V_{hp'ph'} - V_{pp'hh'} - V_{hh'pp'}) . \tag{21.101}$$

The preceding formula simplifies if we consider either separable interactions in the Hartree approximation or Skyrme interactions in conjunction with spin- and isospin-saturated systems, for in those cases the last set of terms depending explicitly on the two-body matrix elements cancels. The remaining formulas of the transcription will, for the sake of simplicity, apply only to these cases. It is straightforward to elaborate formulas corresponding to the general case.

We shall restrict further attention to the case of a single collective coordinate. Thus the point function U takes the form

$$U = \mathcal{H}_{ph} B^{php'h'} \mathcal{H}_{p'h'} . \tag{21.102}$$

More explicitly, with the help of the simplified form of Eq. (21.101), we can write the resulting expression as a trace,

$$U = \mathrm{Tr}\{\rho\mathcal{H}(1-\rho)\mathcal{H}(1-\rho)\mathcal{H} - (1-\rho)\mathcal{H}\rho\mathcal{H}\rho\mathcal{H}\} , \tag{21.103}$$

where in all such formulas we henceforth mean the density matrix in the limit of vanishing momenta, i.e., the first term of Eq. (21.88).

Our final task is to transcribe Eqs. (21.28) and (21.29), which determine the collective path. In practice it is convenient in the nuclear case to utilize the covariant form of the equations, since Eq. (21.28) will then be thoroughly familiar as the equation of the standard cranking method. It is also natural to make a change in the notation, this change being partly defined by writing the point transformation equations (21.2) and (21.4) in the form

$$\xi^{ph} = \xi^{ph}(Q^i) , \tag{21.104}$$

$$Q^i = Q^i(\xi) . \tag{21.105}$$

For a collective path, the superscript i becomes superfluous. We also set

$$\delta V/\delta\rho_{hp} = \mathcal{H}_{ph} , \tag{21.106}$$

$$\delta U/\delta\rho_{hp} \equiv \mathcal{H}^{(1)}_{ph} , \tag{21.107}$$

$$\delta Q/\delta\rho_{hp} \equiv f_{ph} . \tag{21.108}$$

With this nomenclature Eqs. (21.28) and (21.29) take the concise forms

$$\mathcal{H}_{ph} = \lambda f_{ph} , \tag{21.109}$$

$$\mathcal{H}^{(1)}_{ph} = \mu f_{ph} . \tag{21.110}$$

where $\lambda = dV/dQ$ and $\mu = dU/dQ$. Each of these equations is of the cranking form, differing in the structure of the cranking Hamiltonians and in the definition of the cranking parameters, but both driven by the same cranking operator f. The cranking operator that accomplishes this heavy burden is no longer freely at our disposal, but must be a self-consistent solution of the two sets of conditions.

The specification of the contents of Eq. (21.110) is not yet complete, since we must evaluate the additional cranking Hamiltonian $\mathcal{H}^{(1)}$. We shall do so for the choice

$$V_{abcd} = \sum_\sigma \kappa_\sigma (q_\sigma)_{ac}(q_\sigma)_{bd} , \tag{21.111}$$

where q_σ is a single-particle operator and κ_σ an associated interaction strength. One then calculates

$$\begin{aligned}\mathcal{H}^{(1)} &= \mathcal{H}(1-\rho)\mathcal{H}(1-\rho)\mathcal{H} - 2\mathcal{H}\rho\mathcal{H}(1-\rho)\mathcal{H} - 2\mathcal{H}(1-\rho)\mathcal{H}\rho\mathcal{H} + \mathcal{H}\rho\mathcal{H}\rho\mathcal{H} \\ &\quad + \sum_\sigma \kappa_\sigma q_\sigma \mathrm{Tr}[\rho q_\sigma(1-\rho)\mathcal{H}(1-\rho)\mathcal{H} + \rho\mathcal{H}(1-\rho)q_\sigma(1-\rho)\mathcal{H} + \rho\mathcal{H}(1-\rho)\mathcal{H}(1-\rho)q_\sigma \\ &\qquad - (1-\rho)q_\sigma\rho\mathcal{H}\rho\mathcal{H} - (1-\rho)\mathcal{H}\rho q_\sigma\rho\mathcal{H} - (1-\rho)\mathcal{H}\rho\mathcal{H}\rho q_\sigma] \\ &= \mathcal{H}^{(1)}_{NL} + \mathcal{H}^{(1)}_{L} .\end{aligned} \tag{21.112}$$

Here the terms proportional to one of the q_σ have been designated by the subscript L, for local, and the remaining terms recognized as nonlocal. This formula is not as forbidding as it appears, since its evaluation involves straightforward matrix multiplication in a shell-model basis.

The formalism described above has been applied successfully (Bulgac, Klein, Do Dang, and Walet, 1989; Walet, Klein, Do Dang, and Bulgac, 1990) to the model described in Sec. XXI.D (with obvious modifications for bo-

sons) and to an exactly soluble model for nuclear monopole vibrations (T. Suzuki, 1973). More realistic applications are under study.

XXII. THERMAL BOSON MAPPINGS

A. Introduction

In this section we shall give an introductory account of a rather new area of application of boson mappings to nuclear physics. This development was stimulated by the accumulation of experiments on highly excited nuclear systems where, for instance, the concept of temperature-dependent excitations appears to be germane (Abrikosov, Gor'kov, and Dzyaloshinskii, 1963). Standard temperature-dependent equation of motion methods (Sugawara-Tanabe and Tanabe, 1986; Tanabe and Sugawara-Tanabe, 1986; Tanabe, 1988) and linear-response theory (Ring *et al.*, 1984) have been applied to these problems. By contrast, the techniques to be described in this section remain to be applied to realistic problems in nuclear physics, though there is nothing to prevent their application other than the problems mentioned in the earlier pages of this review, once the foundations of the method have been understood. Our main goal in this section is to provide and illustrate this understanding. The account that follows is based largely on the work of Walet and Klein (1990), which was stimulated by the prior work of Hatsuda (1989). The latter contains a complete and satisfactory formal account of a method for the utilization of temperature-dependent bosons. An application to the Lipkin model, however, raised some issues that were treated more fully by Walet and Klein. This explains the order of the exposition. Following an account of the results of Hatsuda and Walet and Klein, we shall discuss prior work (da Providência and Fiolhais, 1985; Brajczewska, Fiolhais, and da Providência, 1986), based on a different mapping procedure. We finally mention a most recent paper, which develops a time-dependent Hartree-Fock theory of mixed states with the aid of the thermofield formalism (Yamamura, da Providência, Kuriyama, and Fiolhais, 1989).

In the following, thermal boson mappings will appear within the framework of the method of thermofield dynamics (Takahashi and Umezawa, 1975; Umezawa, Matsumoto, and Tachiki, 1982; Umezawa, 1984). In this method, it is shown that the use of thermal averages, i.e., mixed states, in the standard version of statistical mechanics, can be replaced by computation of an expectation value with respect to the vacuum state of a doubled Hilbert space, whose precise definition is temperature dependent. After the briefest resumé of the fundamentals of thermofield dynamics, we shall study the application to the Lipkin model, though most results translate to more general cases. With the help of this model, we shall show how to apply thermal Hartree-Fock, thermal RPA, and thermal boson expansion techniques to nuclear models. The most important point that we shall attempt to elucidate, given a Hamiltonian with a dynamical symmetry, is the way this symmetry is necessarily broken by the thermal vacuum state, requiring boson mappings of a larger algebra. A further effect that will be studied is the dependence of the result on the symmetry properties assumed for the heat bath.

B. Thermofield dynamics

The idea behind thermofield dynamics is to define a thermal vacuum $|0(\beta)\rangle$ such that the thermal expectation value of any operator,

$$\langle \hat{O} \rangle_\beta \equiv \mathrm{tr}(\exp(-\beta H)\hat{O})/\mathrm{tr}(\exp(-\beta H)) , \tag{22.1}$$

equals the expectation value with respect to the thermal state,

$$\langle \hat{O} \rangle_\beta = \langle 0(\beta)|\hat{O}|0(\beta)\rangle , \tag{22.2}$$

where, as usual, $\beta=(kT)^{-1}$. Using an eigenvalue-eigenvector decomposition for the Hamiltonian H, we can thus write in the standard way

$$\langle 0(\beta)|\hat{O}|0(\beta)\rangle = \left[\sum_n \exp(-\beta E_n)\langle n|\hat{O}|n\rangle\right] \Big/ \sum_n \exp(-\beta E_n) . \tag{22.3}$$

The problem in the construction of $|0(\beta)\rangle$ is that, when thermal averages are taken, only diagonal matrix elements of the operator O enter. From this one can infer that the state $|0(\beta)\rangle$ cannot be defined in the original Hilbert space, i.e., as a linear combination of the eigenvectors $|n\rangle$ only, but if we multiply each state $|n\rangle$ with a state $|\tilde{n}\rangle$, chosen from a second, independent, Hilbert space and endowed with the same orthogonality properties as $|n\rangle$, we can write

$$|0(\beta)\rangle = \frac{\sum_n \exp(-(\beta/2)E_n)|n\rangle|\tilde{n}\rangle}{\left[\sum_n \exp(-\beta E_n)\right]^{1/2}} . \tag{22.4}$$

If we require that the Kubo-Martin-Schwinger condition hold (Kubo, 1957; Martin and Schwinger, 1959), namely,

$$\langle 0(\beta)|A(t)B(t')|0(\beta)\rangle = \langle 0(\beta)|B(t')A(t+i\beta)|0(\beta)\rangle , \tag{22.5}$$

we find some extra conditions on the second space. These are satisfied when we require that we have a Hamiltonian and operator structure in the "tilde-conjugated" space similar to that in the original space. Introducing a tilde conjugation operator that maps from the original Fock space $\mathcal{F}$ to the new one $\tilde{\mathcal{F}}$, we can prove the following rules:

$$\begin{aligned}
&\widetilde{AB}=\tilde{A}\,\tilde{B}\\
&c_1\widetilde{A}+c_2B=c_1^*\,\tilde{A}+c_2^*\tilde{B}\\
&\widetilde{A}^\dagger=\tilde{A}^\dagger\\
&\tilde{\tilde{A}}=\pm A\\
&\widetilde{|0(\beta)\rangle}=|0(\beta)\rangle\ ,
\end{aligned}\tag{22.6}$$

where in the penultimate equation the plus sign (minus sign) refers to bosonic (fermionic) operators. From these relations we can derive that the Schrödinger equation in the original space translates to a Schrödinger equation in the doubled space, but now with Hamiltonian

$$\mathcal{H}=H-\tilde{H}\ .\tag{22.7}$$

This Hamiltonian is generally unbounded from both below and above, and the thermal state is one of the states at eigenvalue zero (note that all states that are symmetric under tilde conjugation occur at this eigenvalue).

There is an important difference between thermofield dynamics and the usual ($T=0$) quantum mechanics. If the Hamiltonian has a dynamical symmetry $\mathcal{G}\subset\mathcal{S}$, where $\mathcal{S}$ is the shell-model algebra of all bilinear operators, the thermal Hamiltonian has the direct product group $\mathcal{G}\otimes\tilde{\mathcal{G}}$ as dynamical symmetry group. In general, the thermal vacuum breaks this dynamical symmetry, since it usually has components in several irreps of $\mathcal{G}\otimes\tilde{\mathcal{G}}$. We shall define the minimal group $\mathcal{R}$ such that

$$\mathcal{G}\otimes\tilde{\mathcal{G}}\subseteq\mathcal{R}\subseteq\mathcal{S}\otimes\tilde{\mathcal{S}}\tag{22.8}$$

and $|0(\beta)\rangle\in\mathcal{R}$ to be the relevant symmetry group. This contains those collective excitations that are strongly connected to the thermal vacuum. Clearly this in general entails collective operators not present in the Hamiltonian. The corresponding modes will be shown to be massless in the thermodynamic limit, in analogy to the Goldstone modes found in dynamical symmetry breaking.

In general, we are not able to calculate the eigenstates exactly and have to resort to some form of mean-field theory. Suppose that we have a Hartree-Fock-Bogoliubov–type mean field, with quasiparticles $b_i^\dagger$ (creation) and b_i (annihilation) that obey Fermi statistics. The Hamiltonian can always be reduced to

$$H=H_{\rm MF}+V_{\rm res}\ ,\tag{22.9}$$

where $H_{\rm MF}$ is the (thermal) mean-field Hamiltonian

$$H_{\rm MF}=\sum_i\epsilon_i(\beta)b_i^\dagger b_i\ .\tag{22.10}$$

If we wish to remain within the framework of mean-field theory, we should use a mean-field form for the thermal vacuum. In a grand canonical average in a mean-field approximation, the sum in Eq. (22.4) can be carried out and the result written as

$$\begin{aligned}
|0(\beta)\rangle=&Z(\beta)^{-1/2}\exp[-\beta(H_{\rm MF}-\mu\hat{N})/2]\\
&\times\exp\left[\sum_i b_i^\dagger\tilde{b}_i^\dagger\right]|0\rangle|\tilde{0}\rangle\ .
\end{aligned}\tag{22.11}$$

The operator b_i no longer annihilates the thermal vacuum, since we calculate that

$$b_i|0(\beta)\rangle=e^{-\beta\epsilon_i/2}\tilde{b}_i^\dagger|0(\beta)\rangle\ ,\tag{22.12}$$

but

$$a_i=\sqrt{1-f(\epsilon_i)}\,b_i-\sqrt{f(\epsilon_i)}\,\tilde{b}_i^\dagger\tag{22.13}$$

does! (Note: $f(\omega)=1/[\exp(\beta\omega)+1]$, the usual Fermi statistics factor.)

Equation (22.13) is referred to as the transformation to thermal quasiparticles. It illustrates the extreme ease of applying thermofield dynamics to the grand canonical ensemble, where we can use a thermal Bogoliubov-Valatin transformation to take into account the effect of temperature. Before illustrating the use of the grand canonical ensemble as the basis for a thermal boson mapping in the Lipkin model, we shall first consider a case in which restrictions imposed on the heat bath render the canonical ensemble the one of choice.

Let us consider a mean-field theory for such a case. Let $H_{\rm MF}$ again denote the mean-field Hamiltonian,

$$H_{\rm MF}=\sum_i\epsilon_i b_i^\dagger b_i\ .\tag{22.14}$$

In this case we find

$$|0(\beta)\rangle=Z(\beta)^{-1/2}\exp(-H_{\rm MF}\beta/2)\sum_n|n\rangle|\tilde{n}\rangle\ ,\tag{22.15}$$

and the sum over n is now over a set of orthogonal states that spans the subspace with fixed particle number N, $\hat{N}|n\rangle=N|n\rangle$. In this case, it is not possible to carry out the Bogoliubov-Valatin transformation. Nevertheless, as we illustrate below, if we can identify a dynamical symmetry group $\mathcal{G}$ associated with the Hamiltonian, we can use this to construct the states needed in the expansion of the thermal vacuum. In these models a coherent state on the direct product group $\mathcal{G}\otimes\tilde{\mathcal{G}}$ is very convenient in the evaluation of the sum over states in Eq. (22.15), which may lead to simple expressions for the thermal Hamiltonian. This will be illustrated in the next section for the Lipkin model.

C. Lipkin-Meshkov-Glick model in the restricted canonical ensemble

We turn once more to the model described in Sec. III.A, with Hamiltonian given by Eq. (3.5). We find it convenient here to use lowercase letters for the quasi-spin operators and to employ a particle rather than a particle-hole formalism. This entails the definitions

$$\begin{aligned}
&j_0=\frac{1}{2}\sum_{p=1}^{\Omega}(c_{+p}^\dagger c_{+p}-c_{-p}^\dagger c_{-p})=\sum_p j_{0p}\ ,\\
&j_+=\sum_{p=1}^{\Omega}c_{+p}^\dagger c_{-p}=\sum_p j_{+p},\quad j_-=(j_+)^\dagger\ ,
\end{aligned}\tag{22.16}$$

where $c_{+p}^\dagger$ ($c_{-p}^\dagger$) creates a fermion in the upper (lower) level, and the vacuum is the completely filled lower shell,

$$|0\rangle=\prod_{p=1}^{\Omega} c^{\dagger}_{-p}|-\rangle\ , \tag{22.17}$$

where we use $|-\rangle$ to denote the state with both levels empty.

In terms of the individual spin-$\frac{1}{2}$ operators $j_{\alpha p}$, the Hamiltonian reads

$$H=\epsilon\sum_{p} j_{0p}-\frac{1}{2}V\sum_{p\neq q}(j_{+p}j_{+q}+j_{-p}j_{-q})\ , \tag{22.18}$$

where we use the fact that we cannot have more than one particle-hole excitation in each substate $-p$, $+p$. If we perform the standard mean-field approximation at $T=0$, we find

$$H_{\rm MF}=\sum_{p}[\epsilon j_{0p}-V(\Omega-1)(\Delta j_{+p}+\Delta^{*}j_{-p})]\ , \tag{22.19}$$

with $\Delta=\langle j_{+}\rangle$. There is a well-known self-consistency condition for a deformed solution ($\Delta\neq 0$),

$$\frac{\sqrt{1+4[V(\Omega-1)\Delta]^2}}{V(\Omega-1)}=1\ . \tag{22.20}$$

We now assume that no correlations other than the ones discussed above will appear at finite temperature.

Since one usually considers only those states in which half the available sites are filled ($N=\Omega$), it is natural to ask whether the canonical ensemble can be profitably employed in this problem. It turns out that this task can be performed with least effort if we allow only for excitation and deexcitation between levels of the same index p, which is a very strong restriction on the behavior of the bath. This corresponds to taking into account the excitations indicated in Fig. 28(a), but leaving out effects of operators outside the group SU(2), arising from diagrams such as Fig. 28(b).

Turning to the construction of the mean field, we shall follow the derivation for the $T=0$ case given in Ring and Schuck (1980) as closely as possible. The Hartree-Fock approximation for the pth particle-hole substate reads

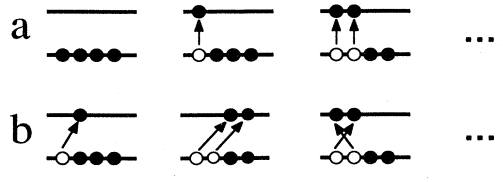

FIG. 28. A schematic indication of the various excitations possible in the thermal Lipkin model. The two lines indicate the upper and lower level, the black blobs indicate particles and the white blobs indicate holes in the lower level. In the set of figures labeled (a) only excitations from upper to lower levels with the same index p are allowed, corresponding to the definition of a restricted heat bath. In the second set (b), excitations that change the value of p are also permitted, corresponding to a general heat bath. From Walet and Klein, 1990.

$$H_{{\rm MF}p}/\epsilon=j_{0p}-\chi(\Delta j_{+p}+\Delta^{*}j_{-p}) \tag{22.21}$$

with

$$\chi=\frac{V}{\epsilon}(\Omega-1),\quad \Delta=\langle 0(\beta)|j_{+q}|0(\beta)\rangle\ . \tag{22.22}$$

Here we use the subscript p to indicate that the operator acts only in the two-dimensional space $+p$, $-p$. The total mean-field Hamiltonian is a sum over all the substates p. Since all the mean-field Hamiltonians are the same, irrespective of the index, the value of Δ is independent of q.

Having defined the mean field, we can now use Eq. (22.15) to define the thermal vacuum. To simplify the formulas, we drop the normalization factor contained in the definition of this state; when calculating thermal expectation values, we have thus to divide by the norm. This has the additional advantage that the partition function $Z(\beta)$ can be evaluated as $Z(\beta)=\langle 0(\beta)|0(\beta)\rangle$. The state $|0(\beta)\rangle$ can be written as

$$\begin{aligned}|0(\beta)\rangle&=\exp\left[-\sum_{p}H_{{\rm MF}p}\beta/2\right]\sum_{n}|n\rangle|\tilde{n}\rangle\\&=\exp\left[-\sum_{p}H_{{\rm MF}p}\beta/2\right]\prod_{p=1}^{\Omega}(|00\rangle_p+|11\rangle_p)\\&=\prod_{p=1}^{\Omega}\exp(-H_{{\rm MF}p}\beta/2)(|00\rangle_p+|11\rangle_p)\\&\equiv\prod_{p=1}^{\Omega}\exp(-H_{{\rm MF}p}\beta/2)|D\rangle\rangle_p\ .\end{aligned} \tag{22.23}$$

The notation $|ij\rangle_p$ denotes a state with i particle-hole excitations in substate p in the ordinary space, and j particle-hole excitations in substate p in the tilde-conjugated space. Using this last equation, we find

$$\Delta=\frac{{}_p\langle\langle D|e^{-H_{{\rm MF}p}\beta/2}j_{+p}e^{-H_{{\rm MF}p}\beta/2}|D\rangle\rangle_p}{{}_p\langle\langle D|e^{-H_{{\rm MF}p}\beta}|D\rangle\rangle_p}\ . \tag{22.24}$$

It is not difficult to evaluate this quantity; one finds

$$\Delta=\frac{\chi\Delta^{*}}{R}\tanh(\beta' R)\ , \tag{22.25}$$

where we have introduced the dimensionless "reduced thermal energy"

$$\beta'=\beta\epsilon \tag{22.26}$$

and the factor

$$R=\sqrt{1+4\chi^2|\Delta|^2}\ . \tag{22.27}$$

From Eq. (22.25) we can derive two conditions. First, we see that Δ should be real, and second, we see that self-consistency is required for a solution with $\Delta\neq 0$,

$$1=\frac{\chi}{R}\tanh(\beta' R/2)\ . \tag{22.28}$$

Using the fact that $R\geq 1$, we can derive that a deformed solution ($\Delta\neq 0$) will only occur when

$$\chi \tanh(\beta'/2) > 1 \ , \tag{22.29}$$

from which we infer that we only find a phase transition for $\chi > 1$, at $\beta' = 2\tanh^{-1}(1/\chi)$.

Further detailed calculations will be reported only for the normal phase. What is usually referred to as the normal phase of the Lipkin model is the high-temperature phase where $\Delta = 0$. In that case the first part of the Hamiltonian (ϵj_0) plays the role of a mean field. Using this we can rewrite the definition of the thermal vacuum, Eq. (22.4), as

$$|0(\beta)\rangle = \exp(-\beta' j_0/2)\exp\left[\sum_p j_{+p}\tilde{j}_{+p}\right]|0\rangle|\tilde{0}\rangle \ . \tag{22.30}$$

Since j_{+p}, etc., are operators within the $j=\frac{1}{2}$ representation of SU(2), what we have shown is that in the normal phase the mean-field approximation can be constructed from states that are direct products of N independent $j=\frac{1}{2}$ representations. This is formally equivalent to a noninteracting system of spins in a homogeneous external magnetic field, a problem that has been discussed within the thermofield method by M. Suzuki (1985). The thermal state (22.30) can be rewritten as

$$|0(\beta)\rangle = \prod_{p=1}^{N} (u + v j_{+p}\tilde{j}_{+p})|0\rangle|\tilde{0}\rangle \ . \tag{22.31}$$

Following Suzuki we now use the fact that the thermal vacuum is unitarily equivalent to the original vacuum, to show that we can define a new set of thermal spin operators of which $|0(\beta)\rangle$ is an eigenstate, namely

$$\begin{aligned} J_{\pm p} &= u j_{\pm p} + 2v j_{0p}\tilde{j}_{\mp p} \ , \\ \tilde{J}_{\pm p} &= u\tilde{j}_{\pm p} + 2v\tilde{j}_{0p} j_{\mp p} \ , \\ J_{0p} &= u^2 j_{0p} - v^2\tilde{j}_{0p} - uv(j_{+p}\tilde{j}_{+p} + j_{-p}\tilde{j}_{-p}) \ , \\ \tilde{J}_{0p} &= u^2\tilde{j}_{0p} - v^2 j_{0p} - uv(j_{+p}\tilde{j}_{+p} + j_{-p}\tilde{j}_{-p}) \ , \end{aligned} \tag{22.32}$$

where $J_-|0(\beta)\rangle = \tilde{J}_-|0(\beta)\rangle = 0$, and

$$\begin{aligned} u &= 1/\sqrt{\exp(-\beta\epsilon/2)+1} \ , \\ v &= \sqrt{1-u^2} \ . \end{aligned} \tag{22.33}$$

These transformations between the operators have some very peculiar but useful properties: Even though they are nonlinear in the generators of SU(2)×SU(2), the final form again spans an SU(2)×SU(2) algebra. Such a transformation can be found only for the $j=\frac{1}{2}$ spinor irrep of SU(2).

We also need the inverse relations, which will be useful to calculate the thermal Hamiltonian,

$$\begin{aligned} j_{\pm p} &= uJ_{\pm p} - 2v J_{0p}\tilde{J}_{\mp p} \ , \\ \tilde{j}_{\pm p} &= u\tilde{J}_{\pm p} - 2v\tilde{J}_{0p} J_{\mp p} \ , \\ j_{0p} &= u^2 J_{0p} - v^2\tilde{J}_{op} + uv(J_{+p}\tilde{J}_{+p} + J_{-p}\tilde{J}_{-p}) \ , \\ \tilde{j}_{0p} &= u^2\tilde{J}_{0p} - v^2 J_{0p} + uv(J_{+p}\tilde{J}_{+p} + J_{-p}\tilde{J}_{-p}) \ . \end{aligned} \tag{22.34}$$

The nonlinear transformation (22.34) is a linear transformation for the operators

$$J_{\alpha p}, \ \tilde{J}_{\alpha p}, \ Y_{\alpha\beta p} = J_{\alpha p}\tilde{J}_{\beta p} \ . \tag{22.35}$$

These 15 operators close under commutation and form the algebra SU(4). When we sum over the spin index p, we thus obtain operators that are the complete equivalent of the operators in the Wigner supermultiplet classification of nuclear states (Wigner, 1937) and thus generate the group SU(4). As a clarification of this result, consider the fact that each substate $(-p,+p)$ containing one particle is a $j=\frac{1}{2}$ state. In constructing the thermal state, we take a direct product of such substates in the original space and in the tilde space. If we identify the original space with the spin in the Wigner model and the tilde space with isospin, it becomes clear that the present procedure parallels Wigner's original construction.

From the transformation (22.34), using the notation (22.35), we find that the Hamiltonian becomes

$$\begin{aligned} \mathcal{H} = \epsilon(J_0 - \tilde{J}_0) - \frac{V}{2}(u^2[J_+J_+ + J_-J_- - \tilde{J}_+\tilde{J}_+ - \tilde{J}_-\tilde{J}_-] + 4v^2[Y_{0-}Y_{0-} + Y_{0+}Y_{0+} - Y_{-0}Y_{-0} - Y_{+0}Y_{+0}] \\ - 2uv[\{J_+, Y_{0-}\} + \{J_-, Y_{0+}\} - \{\tilde{J}_+, Y_{-0}\} - \{\tilde{J}_-, Y_{+0}\}]) \ . \end{aligned} \tag{22.36}$$

The commutation rules of the basic operators are most readily given in the Cartesian form

$$\begin{aligned} &[J_i, J_j] = i\epsilon_{ijk}J_k \ , \\ &[\tilde{J}_i, \tilde{J}_j] = i\epsilon_{ijk}\tilde{J}_k \ , \\ &[\tilde{J}_i, J_j] = 0 \ , \\ &[J_i, Y_{jk}] = i\epsilon_{ijl}Y_{lk} \ , \\ &[\tilde{J}_i, Y_{jk}] = i\epsilon_{ikl}Y_{jl} \ , \\ &[Y_{ij}, Y_{kl}] = i\epsilon_{ikm}J_m\delta_{jl} + i\epsilon_{jlm}\tilde{J}_m\delta_{ik} \ . \end{aligned} \tag{22.37}$$

The operators that give a nonzero result on the vacuum are the step-up operators J_+, $\tilde{J}_+$, Y_{0+}, Y_{+0}, and Y_{++}, and the diagonal operators J_0, $\tilde{J}_0$, and Y_{00} (with eigenvalues $-\Omega/2$, $-\Omega/2$, and $\Omega/4$).

We now try to build an SU(4) irrep on the extremal state $|0(\beta)\rangle$. Although the state $|0(\beta)\rangle$ has a complex structure in terms of the lowercase generators of SU(4), it has a simple structure in terms of the uppercase generators, with SU(2)×SU(2) labels $J=\tilde{J}=\Omega/2$. Since we are considering states with Ω particles, the sum of all the rows of the SU(4) Young tableau is at most Ω. Using Littlewood's analysis (Littlewood, 1950) of the group

reduction $SU(4)\supset SU(2)\times SU(2)$, we find that the only SU(4) irrep containing a state with the quantum numbers of $|0(\beta)\rangle$ is the one-row irrep $\{\Omega\}$. In order to understand the problem, it is also useful to know the spin and "tildespin" quantum numbers of the states contained in this irrep. Again using Littlewood's rules, we can find the $SU(2)\times SU(2)$ labels

$$\begin{array}{c} SU(4)\supset SU(2)\times SU(2) \\ \{\Omega\} \quad \downarrow \quad \sum_{j=0}^{\Omega/2}\{j\}\times\{j\} \end{array} . \tag{22.38}$$

The boson mapping for this irrep is not too difficult to construct using commutator techniques often used previously in this review. We introduce two bosons (B and $\tilde{B}$) for the Holstein-Primakoff mapping of the two SU(2) algebras, and one extra boson ($C=\tilde{C}$) that decreases both J and $\tilde{J}$ by one. Using the commutation relations, we can now derive the form of all operators. The details that follow should be sufficiently familiar in outline so that some readers may wish to skip beyond Eq. (22.50).

We emphasize that we consider here only one specific boson realization of the Wigner supermultiplet algebra, corresponding to the case in which we have the SU(4) irrep $\{\Omega\}$ with the states labeled by the $SU(2)\times SU(2)$ quantum numbers

$$|J,m_J;\tilde{J},m_{\tilde{J}}\rangle=|\Omega/2-j,m;\Omega/2-j,\tilde{m}\rangle . \tag{22.39}$$

The selection rules for the bosons introduced in the previous paragraph are thus

$$\begin{aligned} B^\dagger|\Omega/2-j,m;\Omega/2-j,\tilde{m}\rangle &=|\Omega/2-j,m+1;\Omega/2-j,\tilde{m}\rangle , \\ \tilde{B}^\dagger|\Omega/2-j,m;\Omega/2-j,\tilde{m}\rangle &=|\Omega/2-j,m;\Omega/2-j,\tilde{m}+1\rangle , \\ C^\dagger|\Omega/2-j,m;\Omega/2-j,\tilde{m}\rangle &=|\Omega/2-j-1,m+1;\Omega/2-j-1,\tilde{m}+1\rangle . \end{aligned} \tag{22.40}$$

We first use the Holstein-Primakoff mapping for SU(2), in an obvious notation,

$$\begin{aligned} J_+ &=B^\dagger(\Omega-2N_C-N_B)^{1/2} , \\ J_0 &=-\Omega/2+N_C+N_B , \\ J_- &=(J_+)^\dagger , \end{aligned} \tag{22.41}$$

and a similar expression for $\tilde{J}$ (obtained by replacing B by $\tilde{B}$). It is easy to give the most general form of the step-up operator Y_{++} compatible with all selection rules, namely,

$$\begin{aligned} Y_{++}=&B^\dagger\tilde{B}^{\dagger}a_1(N_B,N_{\tilde{B}},N_C)+C^\dagger a_2(N_B,N_{\tilde{B}},N_C) \\ &+B^{\dagger 2}\tilde{B}^{\dagger 2}Ca_3(N_B,N_{\tilde{B}},N_C) . \end{aligned} \tag{22.42}$$

We can determine the dependence on N_B and $N_{\tilde{B}}$ from the Wigner-Eckart theorem. Thus it follows that

$$\begin{aligned} a_1(N_B,N_{\tilde{B}},N_C)&=r(N_1)r(N_2)b_1(N_C) , \\ a_2(N_B,N_{\tilde{B}},N_C)&=r(N_1)r(N_1+1)r(N_2)r(N_2+1)b_2(N_C) , \\ a_3(N_B,N_{\tilde{B}},N_C)&=b_3(N_C) , \end{aligned} \tag{22.43}$$

where we have introduced the shorthand notation

$$r(N)=(\Omega-N)^{1/2},\quad N_1=2N_C+N_B,\quad N_2=2N_C+N_{\tilde{B}} . \tag{22.44}$$

All operators $Y_{\alpha\beta}$ can now be determined by laddering down with J_- and $\tilde{J}_-$. We can calculate difference relations for the three functions b by evaluating the commutator of Y_{++} with Y_{--},

$$[Y_{++},Y_{--}]=J_0+\tilde{J}_0 . \tag{22.45}$$

We need only a boundary condition at $N_C=0$. This is provided by $b_1(0)=1/\Omega$ as follows from the condition

$$\begin{aligned} \langle 0|J_-\tilde{J}_-Y_{++}|0\rangle&=\Omega^2 b_1(0) \\ &=\langle 0|4Y_{00}|0\rangle=\Omega . \end{aligned} \tag{22.46}$$

We ultimately find

$$\begin{aligned} b_1(N_C)&=\frac{\Omega+2}{(\Omega-2N_C)(\Omega-2N_C+2)} , \\ b_2(N_C)&=b_3(N_C+1) , \\ b_3(N_C)&=\frac{1}{\Omega-2N_C+2}\left[\frac{\Omega-N_C+2}{(\Omega-2N_C+3)(\Omega-2N_C+1)}\right]^{1/2} \end{aligned} \tag{22.47}$$

We can now derive that

$$\begin{aligned} Y_{0+}=&-\tfrac{1}{2}[J_-,Y_{++}] \\ =&\tilde{B}^\dagger(-\Omega/2+N_B+N_C)r(N_2)b_1(N_C) \\ &+\tilde{B}C^\dagger r(N_1)r(N_2)r(N_2+1)b_2(N_C) \\ &-B^\dagger\tilde{B}^{\dagger 2}Cr(N_1-1)b_3(N_C) , \end{aligned} \tag{22.48}$$

from which we can obtain Y_{0-}, Y_{+0}, Y_{-0} by either complex or tilde conjugation;

$$\begin{aligned} Y_{-+}=&[J_-,Y_{0+}] \\ =&B\tilde{B}^\dagger r(N_1-1)r(N_2)b_1(N_C) \\ &-B^2C^\dagger r(N_2)r(N_2+1)b_2(N_C) \\ &-\tilde{B}^{\dagger 2}Cr(N_1-2)r(N_1-1)b_3(N_C) , \end{aligned} \tag{22.49}$$

from which Y_{+-} follows by tilde conjugation, and

$$Y_{00}=-\tfrac{1}{2}[\tilde{J}_-,Y_{0+}]$$
$$=(-\Omega/2+N_B+N_C)(-\Omega/2+N_{\tilde{B}}+N_C)b_1(N_C)$$
$$+B\tilde{B}C^\dagger r(N_1)r(N_2)b_2(N_C)$$
$$+B^\dagger\tilde{B}^\dagger Cr(N_1-1)r(N_2-1)b_3(N_C)\ . \quad (22.50)$$

Using this boson realization we can construct a boson expansion, assuming that the expectation value of all boson number operators in the states of interest is much smaller than Ω. Including all terms up to order $1/\Omega$, we find for Eq. (22.36) ($\chi_0=\Omega V/\epsilon$)

$$\mathcal{H}/\epsilon=N_B-N_{\tilde{B}}-\tfrac{1}{2}\chi_0$$
$$\left\{[B^2+B^{\dagger 2}-\tilde{B}^2-\tilde{B}^{\dagger 2}]\left[u^2\left[1-\frac{4N_C+1}{\Omega}\right]-v^2\left[1+\frac{4N_C+1}{2\Omega}\right]\right]+\frac{v^2}{\Omega}[(B^2+B^{\dagger 2})N_{\tilde{B}}-(\tilde{B}^2+\tilde{B}^{\dagger 2})N_B]\right.$$
$$-\frac{4uv}{\Omega}[-\tilde{B}^{\dagger 2}\tilde{B}B+\tilde{B}^\dagger B^\dagger B^2+B^{\dagger 2}B\tilde{B}-B^\dagger\tilde{B}^\dagger\tilde{B}^2]-\frac{v^2}{\sqrt{\Omega}}[B\tilde{B}^\dagger-B^\dagger\tilde{B}][C-C^\dagger]$$
$$\left.-\frac{4uv}{\sqrt{\Omega}}[(B^2-\tilde{B}^2)C^\dagger-(B^{\dagger 2}-\tilde{B}^{\dagger 2})C]\right\}. \quad (22.51)$$

From this equation we can read off that the correct expansion parameter is $1/\sqrt{\Omega}$. The boson expansion includes effects of the finite size of the system. In the thermodynamic limit we obtain the Hamiltonian

$$\mathcal{H}/\epsilon=N_B-N_{\tilde{B}}-\frac{\chi_0(u^2-v^2)}{2}[B^2+B^{\dagger 2}-\tilde{B}^2-\tilde{B}^{\dagger 2}]\ . \quad (22.52)$$

We wish to diagonalize this last Hamiltonian while retaining the antisymmetry under tilde conjugation. Since the Hamiltonian can be separated into a piece containing only B and $B^\dagger$ and its tilde conjugate, we can apply techniques that are commonly used for deriving the RPA from a boson expansion and write

$$\begin{bmatrix}B\\B^\dagger\end{bmatrix}=\begin{bmatrix}\alpha&\beta\\\beta&\alpha\end{bmatrix}\begin{bmatrix}A\\A^\dagger\end{bmatrix} \quad (22.53)$$

and an identical transformation for $\tilde{B}$ and $\tilde{B}^\dagger$ to remove the dangerous terms B^2, etc. This leads to the usual RPA-type Hamiltonian

$$\mathcal{H}/\epsilon=\tfrac{1}{2}(\omega-1)+\omega N_A-\tfrac{1}{2}(\omega-1)-\omega N_{\tilde{A}}+(\text{higher order}) \quad (22.54)$$

where

$$\omega=\sqrt{1-\chi_0^2(u^2-v^2)^2}\ . \quad (22.55)$$

The constant in $\mathcal{H}$, which is canceled by its tilde conjugate, gives a $1/\Omega$ correction to the ground-state energy when calculated in the RPA.

Corresponding results have been derived for the deformed case, but these will not be presented here.

We conclude this discussion with some numerical examples. We take two cases with $\chi_0=\frac{1}{2}$ and 2. Only the

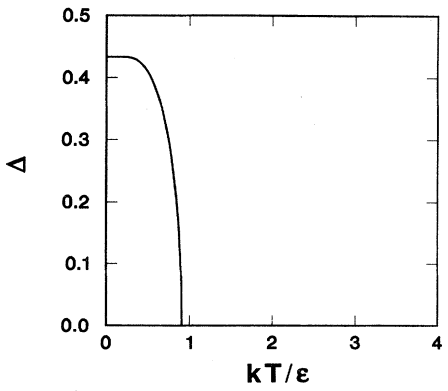

FIG. 29. The deformation parameter Δ vs temperature for $\chi_0=2$. From Walet and Klein, 1990.

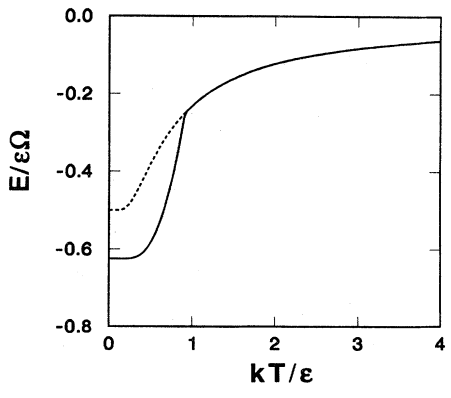

FIG. 30. The ground-state energy vs temperature for $\chi_0=2$ (solid line) and $\chi_0=\frac{1}{2}$ (dashed line). From Walet and Klein, 1990.

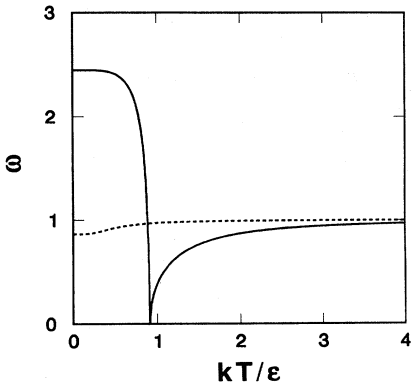

FIG. 31. The RPA approximation for the excitation energy of the first excited state for $\chi_0=2$ (solid line) and $\chi_0=\frac{1}{2}$ (dashed line). From Walet and Klein, 1990.

larger of the two has a deformed solution; in Fig. 29 we give the temperature dependence of the deformation Δ for this case. If we look at the ground-state energy, Fig. 30, where the solid line indicates the larger value of χ_0 and the dotted the smaller value, we see that the derivative of the ground-state energy is singular across the phase transition. The model thus exhibits a second-order phase transition, very similar to a paramagnetic-to-ferromagnetic transition. The most interesting quantity is the thermal excitation energy, Fig. 31. This quantity is very smooth for $\chi_0=\frac{1}{2}$ (dotted line) and becomes zero at the phase transition for $\chi_0=2$. For very large temperatures both RPA frequencies approach 1, indicating that for very high temperatures only the single-particle energy determines the excitation energies.

D. Grand canonical ensemble

If we lift the symmetry restrictions imposed on the heat bath, it is still possible to carry through a calculation based on the canonical ensemble (Walet and Klein, 1989), but the manipulations become more cumbersome. They will not be described here. Instead we turn immediately to the grand canonical ensemble.

For the grand canonical ensemble the thermal state becomes

$$|0(\beta)\rangle \propto \exp[-\beta(H_{\rm MF}-\mu\hat{N})/2] \times \exp\left[\sum_i (c^\dagger_{+i}\tilde{c}^\dagger_{+i}+c_{-i}\tilde{c}_{-i})\right]|0\rangle|\tilde{0}\rangle \ . \quad (22.56)$$

Here we again define the vacuum of the Lipkin model as the completely filled lower shell. The two terms in the rightmost exponential in Eq. (22.56) thus correspond to creating a particle in the upper level and annihilating a particle in the lower level.

Since the Hartree approximation to the Hamiltonian commutes with the number operator $\hat{N}$, we can use a method very similar to that discussed for the canonical ensemble to calculate the partition function $Z(\beta)$. We evaluate all the individual contributions separately as in Eq. (22.4) and find ($\mu'=\mu/\epsilon$)

$$ {}_p\langle 0(\beta)|e^{-\beta(H_{{\rm MF}p}-\mu'\hat{N}_p)}|0(\beta)\rangle_p = e^{-2\mu'\beta'}+1+e^{-\mu'\beta'}\,{}_p\langle\langle D|e^{-H_{{\rm MF}p}\beta}|D\rangle\rangle_p \ . \quad (22.57)$$

Here the first two terms come from states that are completely filled or completely empty. The remainder of the calculation parallels that necessary in the case of the restricted canonical ensemble. We now find that

$$Z(\beta)=(\cosh\beta'\mu'+\cosh\beta' R)^\Omega e^{-\mu'\beta'\Omega} \ . \quad (22.58)$$

The self-consistency condition for Δ now reads

$$\Delta=\frac{\chi\Delta^*}{R}\frac{\sinh\beta' R}{\cosh\beta'\mu'+\cosh\beta' R} \ . \quad (22.59)$$

The only operators entering the Hamiltonian are the four generators of U(2),

$$N_{\alpha\beta}\equiv\sum_i c^\dagger_{\alpha i}c_{\beta i} \ , \quad (22.60)$$

($\alpha,\beta=\pm$), and their tilde conjugate $\tilde{N}_{\alpha\beta}$. Looking at the operator in the rightmost exponent of Eq. (22.56), we can easily see that the algebra closes if we add eight more operators,

$$M^\dagger_{\alpha\beta}\equiv\sum_i c^\dagger_{\alpha i}\tilde{c}^\dagger_{\beta i} \ , \quad (22.61)$$

and their Hermitian conjugates M. We thus have a total of 16 operators that span the algebra U(4), which could have been expected from the results obtained in the previous section.

The linear Casimir operator of this algebra is the *difference* between the number of particles and its tilde conjugate,

$$C^1=\hat{N}-\hat{\tilde{N}}=N_{++}+N_{--}-\tilde{N}_{++}-\tilde{N}_{--} \ . \quad (22.62)$$

Since the thermal vacuum is symmetric under tilde conjugation, we find

$$C^1|0(\beta)\rangle=0 \ , \quad (22.63)$$

so that any U(4) irrep that has any overlap with the vacuum must have the sum of its row-lengths zero! This does not imply that this is the irrep $\{0\}$, however, but that it is a mixed-symmetry irrep with rows of both positive and negative length, with sum zero (Weyl, 1946). The reader who is not familiar with such irreps may find it useful to consider a single representative of such an irrep as a tensor that has two types of indices, some that transform covariantly and others that transform contravariantly. The usual representation of a Lie algebra then amounts to a tensor with covariant indices only; the negative-length row indicates that there are also indices that transform contravariantly.

Let us now introduce the thermal quasiparticles

$$
\begin{aligned}
&a^\dagger_{-i}=uc^\dagger_{-i}+v\tilde{c}_{-i},\quad a_{-i}=uc_{-i}+v\tilde{c}^\dagger_{-i}\ ,\\
&\tilde{a}^\dagger_{-i}=u\tilde{c}^\dagger_{-i}+vc_{-i},\quad \tilde{a}_{-i}=u\tilde{c}_{-i}+vc^\dagger_{-i}\ ,\\
&a^\dagger_{+i}=u'c^\dagger_{+i}+v'\tilde{c}_{+i},\quad a_{+i}=u'c_{-i}+v'\tilde{c}^\dagger_{+i}\ ,\\
&\tilde{a}^\dagger_{+i}=u'\tilde{c}^\dagger_{+i}+v'c_{+i},\quad \tilde{a}_{+i}=u'\tilde{c}_{+i}+v'c^\dagger_{+i}\ .
\end{aligned}
\tag{22.64}
$$

Here we have different weight factors for the upper and lower levels,

$$
\begin{aligned}
&u=1/\sqrt{1+e^{-\beta(\epsilon/2+\mu)}},\quad v=\sqrt{1-u^2}\ ,\\
&u'=1/\sqrt{1+e^{-\beta(\epsilon/2-\mu)}},\quad v'=\sqrt{1-u'^2}\ .
\end{aligned}
\tag{22.65}
$$

Since the thermal Bogoliubov transformation (22.15) is unitary, the relevant group is still U(4). The thermal vacuum has the very simple form

$$
|0(\beta)\rangle=\prod_p a^\dagger_{-p}\tilde{a}^\dagger_{-p}|-\rangle\ , \tag{22.66}
$$

where $|-\rangle$ is the state with no particles (empty upper and lower shells). The group operators when expressed in terms of the new fermion operators will now be indicated by script letters such as $\mathcal{N}_{++}=\sum a^\dagger_{+p}a_{+p}$. Let us now analyze what kind of U(4) weights are associated with the individual operators, since the weights are additive and will thus indicate the relevant irrep(s) contained in $|0(\beta)\rangle$. We follow an analysis similar to that given on pages 270–271 of Wybourne's book (Wybourne, 1974), from which we also derive our notation.

The maximal set H_i of commuting operators can be chosen as $H_1=\mathcal{N}_{--}$, $H_2=\mathcal{N}_{++}$, $H_3=-\tilde{\mathcal{N}}_{++}$ and $H_4=-\tilde{\mathcal{N}}_{--}$ (the minus signs insure that the linear Casimir operator is the sum of these four operators). From the nonzero commutation relation of $a^\dagger_{-p}$ and $\tilde{a}^\dagger_{-p}$ with the operators H_i,

$$
[H_1,a^\dagger_{-p}]=a^\dagger_{-p},\quad [H_4,\tilde{a}^\dagger_{-p}]=-\tilde{a}^\dagger_{-p}\ , \tag{22.67}
$$

we find that $a^\dagger_{-p}$ belongs to the weight-space $(1,0^3)$ and that $\tilde{a}^\dagger_{-p}$ belongs to the weight-space $(0^3,-1)$. Since the weights are additive, we find from Eq. (22.66) that the thermal state thus has unique weight $(\Omega,0^2,-\Omega)$, and since the absolute sum of the column length must not be larger than the total particle number 2Ω, the U(4) irrep built on this state is $\{\Omega,0^2,-\Omega\}$.

We next outline a derivation of the boson realization associated with this irrep. The main thread of the argument resumes after Eq. (22.74). Since the techniques are very similar to those used in Sec. XXII.C, we shall give very few details of the derivation.

The generators of the group are $\mathcal{N}_{\alpha\beta}$, $\tilde{\mathcal{N}}_{\alpha\beta}$, $\mathcal{M}^\dagger_{\alpha\beta}$, and $\mathcal{M}_{\alpha\beta}$. The nonzero commutation relations are

$$
\begin{aligned}
&[\mathcal{N}_{\alpha\beta},\mathcal{N}_{\gamma\delta}]=\delta_{\beta\delta}\mathcal{N}_{\alpha\gamma}-\delta_{\alpha\delta}\mathcal{N}_{\gamma\beta}\ ,\\
&[\tilde{\mathcal{N}}_{\alpha\beta},\tilde{\mathcal{N}}_{\gamma\delta}]=\delta_{\beta\delta}\tilde{\mathcal{N}}_{\alpha\gamma}-\delta_{\alpha\delta}\tilde{\mathcal{N}}_{\gamma\beta}\ ,\\
&[\mathcal{M}^\dagger_{\alpha\beta},\mathcal{M}_{\gamma\delta}]=\delta_{\alpha\gamma}\mathcal{N}_{\beta\gamma}+\delta_{\beta\delta}\mathcal{N}_{\alpha\gamma}-\delta_{\alpha\gamma}\delta_{\beta\delta}\Omega\ ,\\
&[\mathcal{M}^\dagger_{\alpha\beta},\mathcal{N}_{\gamma\delta}]=\delta_{\alpha\delta}\mathcal{M}^\dagger_{\gamma\beta}\ ,\\
&[\mathcal{M}^\dagger_{\alpha\beta},\tilde{\mathcal{N}}_{\gamma\delta}]=\delta_{\beta\delta}\mathcal{M}^\dagger_{\alpha\gamma}
\end{aligned}
\tag{22.68}
$$

and two relations involving $\mathcal{M}$ that can be obtained from the last given above by Hermitian conjugation. The operators $\mathcal{N}$ form the U(2) algebra corresponding to J, and $\tilde{\mathcal{N}}$ to $\tilde{J}$.

It follows that we need a minimum of four bosons to map the algebra. These are the bosons B and $\tilde{B}$ similar to those introduced for the mapping of SU(4) and two more bosons, C and D. Here the effect of $C^\dagger$ is simultaneous creation of a particle in both the normal- and tilde-conjugate upper level, and $D^\dagger$ annihilates a similar pair in the lower level. The Holstein-Primakoff mapping for the operators $\mathcal{N}$ is again readily given:

$$
\begin{aligned}
&\mathcal{N}_{+-}=J_+=B^\dagger\sqrt{\hat{J}}\\
&\mathcal{N}_{-+}=J_-=\sqrt{\hat{J}}\ B\\
&\mathcal{N}_{++}-N_{--}=2J_0=-\hat{J}+2N_B\\
&\hat{J}=\Omega-(N_C+N_D)/2\ .
\end{aligned}
\tag{22.69}
$$

We can derive similar relations for $\tilde{\mathcal{N}}$ by replacing B by $\tilde{B}$ in all the above equations.

From the selection rules we can easily derive that [again $r(N)=\sqrt{\Omega-N}$]

$$
\mathcal{M}^\dagger_{++}=r(N_B+N_C+N_D)r(N_{\tilde{B}}+N_C+N_D)a_1(N_C,N_D)C^\dagger+a_2(N_C,N_D)DB^\dagger\tilde{B}^\dagger\ . \tag{22.70}
$$

where a_1 and a_2 are functions of N_C and N_D only. We can also derive that

$$
\begin{aligned}
\mathcal{M}^\dagger_{-+}&=[J_-,\mathcal{M}^\dagger_{++}]\\
&=r(N_B+N_C+N_D)a_2(N_C,N_D)DB^\dagger-r(N_{\tilde{B}}+N_C+N_D-1)a_1(N_C,N_D)C^\dagger B\ ,\\
\mathcal{M}^\dagger_{--}&=[J_-,[\tilde{J}_-,\mathcal{M}^\dagger_{++}]]\\
&=r(N_B+N_C+N_D)r(N_{\tilde{B}}+N_C+N_D)a_2(N_C,N_D)D+a_1(N_C,N_D)C^\dagger B\tilde{B}\ .
\end{aligned}
\tag{22.71}
$$

Equating commutation relations between the various $\mathcal{M}$'s, using an heuristic form for

$$
\mathcal{N}_{++}+\tilde{\mathcal{N}}_{++}=N_B+N_{\tilde{B}}+2N_C,\quad \mathcal{N}_{--}+\tilde{\mathcal{N}}_{--}=-N_B-N_{\tilde{B}}-2N_D+2\Omega\ , \tag{22.72}
$$

we find recursion relations for the quantities a_1 and a_2. From these relations together with $a_2(0,0)=\sqrt{1/\Omega}$, we derive

$$a_1(N_C,N_D)=\left[\frac{\Omega-N_C+2}{(\Omega-N_C-N_D+2)(\Omega-N_C-N_D+1)}\right]^{1/2},\quad a_2(N_C,N_D)=\left[\frac{\Omega-N_D+1}{(\Omega-N_C-N_D+1)(\Omega-N_C-N_D)}\right]^{1/2}. \tag{22.73}$$

This can be used to give the realization of all the operators $\mathcal{M}^\dagger$, $\mathcal{M}$. Since we have already given the form of those operators above, one should just substitute the explicit values for a_1 and a_2, Eq. (22.73). It remains only to give the explicit form of the diagonal operators $\mathcal{N}_{\alpha\alpha}$, which are

$$\mathcal{N}_{++}=N_C+N_B,\quad \mathcal{N}_{--}=\Omega-N_D-N_B,\quad \tilde{\mathcal{N}}_{++}=N_C+N_{\tilde{B}},\quad \tilde{\mathcal{N}}_{--}=\Omega-N_D-N_{\tilde{B}}\ . \tag{22.74}$$

Expanding the realization of the operators in $1/\Omega$, we find for the normal phase that

$$\begin{aligned}\mathcal{H}/\epsilon=&N_B-N_{\tilde{B}}-\chi_0(uu'-vv')(B^{\dagger 2}[1-(N_B+N_C+N_D+\tfrac{1}{2})/\Omega]^{1/2}+[1-(N_B+N_C+N_D+\tfrac{1}{2})/\Omega]^{1/2}B^2\\&-\{\tilde{B}^{\dagger 2}[1-(N_B+N_C+N_D+\tfrac{1}{2})/\Omega]^{1/2}+[1-(N_B+N_C+N_D+\tfrac{1}{2})/\Omega]^{1/2}\tilde{B}^{2}\})\\&-\chi_0/\sqrt{\Omega}\{(u'vD-uv'C)[B^\dagger(uu'B^\dagger-vv'\tilde{B})-\tilde{B}^\dagger(uu'\tilde{B}^\dagger-vv'B)]\\&-(uv'D-u'vC)[B^\dagger(uu'B^\dagger-vv'\tilde{B})-\tilde{B}^\dagger(uu'\tilde{B}^\dagger-vv'B)]\\&+(uv'D^\dagger-u'vC^\dagger)[\tilde{B}(uu'B^\dagger-vv'\tilde{B})-B(uu'\tilde{B}^\dagger-vv'B)]\\&-(u'vD^\dagger-uv'C^\dagger)[\tilde{B}(uu'B^\dagger-vv'\tilde{B})-B(uu'\tilde{B}^\dagger-vv'B)]\}\ .\end{aligned} \tag{22.75}$$

In the limit $\mu=0$, which corresponds to $\langle N\rangle_\beta=\Omega$, we find that the terms of order $\Omega^{-1/2}$ disappear. This corresponds to the case studied by Hatsuda (1989), who used the symmetry to classify the thermal state as belonging to an SO(5) irrep.

This is once more a manifestation of the fact that it is the thermal vacuum and not the thermal Hamiltonian that determines the minimal symmetry group of the thermal dynamics. The extra symmetry found in the limit $\mu\to 0$ is a clear manifestation of this fact; we find that one mode exactly decouples in this special case. This implies that we should make a smaller choice of dynamical symmetry group, in this case SO(5) instead of U(4). The extra symmetry is also reflected by the Bogoliubov-Valatin transformation, where $\mu=u'$ for $\mu=0$.

From inspection we find that, for $\mu=0$, the thermodynamic limit of Eq. (22.75) gives the same result as Eq. (22.51) thus showing that the RPA frequencies are identical. This is similar to the usual situation in which the canonical and grand canonical ensembles give identical results in the thermodynamical limit, and can be traced to the fact that symmetry-breaking fluctuations are suppressed by at least a factor $(1/\sqrt{\Omega})$.

E. Comments on other work

The work by Hatsuda takes an approach similar to that described in this section as far as application to the Lipkin model is concerned. (In fact it was this work that stimulated the research we have described.) The major difference between it and the present approach is that it deals only with the grand canonical ensemble. However, in application to the Lipkin model, where we have given a general expression, valid for any value of the thermodynamic potential μ, the derivation in Hatusuda's paper is valid only for the special case $\mu=0$. For this special case the algebra generated by the thermal vacuum appears to be SO(5), since the number operator is no longer contained in the thermal weight factor. Our general result is that the relevant algebra is U(4).

In the first part of this paper, Hatsuda carried through a Hartree-Fock-Bogoliubov and RPA analysis for the general shell-model case within the framework of thermofield dynamics. This development has not been bosonized and therefore falls outside the scope of this review.

Hatsuda raises another interesting question, namely, the commutability of the two processes, introduction of thermofield dynamics and bosonization. We have utilized the order thermofield dynamics followed by bosonization. It is reasonable to suppose, as does Hatsuda, that properly done, the order does not matter. However, Hatusuda does not verify this supposition when he tests it on the Lipkin model. The reason for this discrepancy appears to be that in mapping the Lipkin model he maps *only the representation containing the ground state of the system,* namely, the representation with which we begin this review. This would be the correct choice if one imposed on the heat bath the condition that it allow transitions only within this band. This is much more restrictive even than what we have termed the restricted canonical ensemble and appears to be the origin of the results found in this case, differing from those described above. One can map all irreps by introducing a boson-quasifermion mapping, as has been fully described for the case of SU(2). It remains an open problem to carry out the transition to thermofield dynamics utilizing this representation.

We turn finally to the earlier work of da Providência and Fiolhais (1985). The method employed by these authors is very different from that used in the present work,

even though the claim is made that they have a procedure that leads to the thermal Hartree-Fock energy functional. Let us first take a closer look at the form of the density operator D used by them: If we employ Eqs. (6.3) and (6.4) of da Providência and Fiolhais, we find that D is equivalent to the temperature one-body density matrix as employed in the present work,

$$D = C \exp[\alpha(\cos\theta j_0 + \sin\theta j_-)] \ . \tag{22.76}$$

This shows that da Providência and Fiolhais make the same assumptions about the heat bath as are made explicitly in the present work in the case of the restricted canonical ensemble. However, expression (6.5) in da Providência and Fiolhais, $E = \mathrm{tr}(DH)$, is not the thermal Hartree-Fock energy functional. Straightforward calculations show that, if we use the states of the restricted canonical ensemble,

$$\begin{aligned} E_{\mathrm{HF}}/(\epsilon\Omega) &= \langle j_{0p} \rangle_\beta - \frac{V(\Omega-1)}{2\epsilon}(\langle j_+ \rangle_\beta^2 + \langle j_- \rangle_\beta^2) \\ &= -\tfrac{1}{2}\cos\theta P - \sin^2\theta \frac{V}{4\epsilon}(\Omega-1)P^2 \ . \end{aligned} \tag{22.77}$$

What appears to be evaluated in da Providência and Fiolhais is the expectation value of the square, $\langle j_x^2 \rangle$ instead of the square of the expectation value. This result contains corrections of order $1/\Omega$ relative to the Hartree-Fock result. Note, however, that the value of $\langle j_y^2 \rangle$ that is not evaluated by da Providência and Fiolhais is of the same order of magnitude as these corrections. We have not been able to reproduce their Eq. (6.5), which leads to an implausible normal-to-deformed phase transition at large temperature. Even if such a transition is present, it should be rejected as due to finite size $[\mathcal{O}(1/\Omega)]$ effects. These objections to the detailed treatment of the Lipkin model do not apply to their subsequent work on the Heisenberg ferromagnet (Brajczewska, Fiolhais, and da Providência, 1986).

The boson expansion introduced in da Providência and Fiolhais bears no resemblance to our work. As an example of their procedure, consider once more the Lipkin model. There they introduce a modified Holstein-Primakoff mapping of SU(2) that requires the algebra to be satisfied exactly, as usual. However, the boson vacuum, instead of being the map of the fermion vacuum, is required to reproduce the known thermal averages in a mean-field approximation. Thus it is a kind of thermal vacuum, but there is no precise sense of broadening of the original symmetry. These ideas certainly deserve (and require) further investigation. Whatever the meaning of the mapping, it does not correspond to an irrep of a Lie algebra, the idea to which this review has been restricted. From the results obtained, we have already surmised that this procedure may be equivalent to the use of a restricted canonical ensemble, but this remark remains a conjecture rather than a conclusion. In a recent note (da Providência and Fiolhais, 1990), the computations for the Lipkin model have been corrected and the detailed prescription upon which the results are based clarified. Questions concerning the foundation of the method remain, however.

It is clear that the study of thermal boson mappings is in its infancy. It would be of considerable interest to work out the details for models containing richer physics than the Lipkin model.

XXIII. SUMMARY AND OUTLOOK

In this review, we have presented three distinct types of subject matter:

(i) an essentially complete survey of all known boson and boson-quasifermion mappings of compact Lie algebras that have appeared in the literature of nuclear physics;

(ii) an essentially complete survey of viable and/or recent applications of the above algebras to models ranging from standard toy models of varying complexity to the fundamental problem of collective motion within the framework of a realistic shell model;

(iii) a potpourri of other uses of boson mappings both for the elucidation of concepts and for unusual applications. Let us review each of these categories in somewhat greater detail.

A. Methods and varieties of boson mappings

The problem of realization of the generators of a compact Lie algebra as operator-valued functions of a suitable set of boson operators becomes a problem in nuclear physics when the generators are realized first as bilinear fermion operators that form the building blocks of observables within the framework of the shell model. For the study of the latter, a fundamental task is the construction of bases for irreps of the associated algebras. The fundamental technical problem addressed in this review is the mapping of such finite-dimensional vector spaces onto subspaces of boson (boson-quasifermion) spaces, called physical subspaces. There are a number of distinguishable aspects of this problem that need to be considered before the variety of results found in the literature can be understood:

(a) A mapping is always from a definite fermion basis with defined diagonal operators to a definite boson or boson-quasifermion basis with the same (mapped) or possibly different diagonal operators. Thus, two boson mappings of the same algebra may differ because either the fermion basis or the mapped basis involved is different. Though this observation may appear to be trivial, it is remarkable how often it has been overlooked, especially in the early literature, leading to incomplete or wrong results.

(b) Mappings may differ even if the same starting and ending bases are utilized. This is because the only requirement on a mapping is that it reproduce the correct matrices of an irrep when the generators act within the physical subspace. Mappings between the same bases

may therefore be distinguished by the action of the generators on the basis vectors of the unphysical subspace.

(c) Several different derivations of the same mapping are known in a number of cases. The continued search for simplified techniques is relevant because there are physically important mappings that are not known in closed form.

(d) The most general physically motivated mapping is a boson-quasifermion mapping, which also provides an important new technique in the theory of group representations.

Instead of following the historical development of the subject, which was described in Sec. I, we chose to try to illuminate the four elements identified above by starting with the simplest non-Abelian algebra, SU(2). Actually element (a) was not encountered in full generality until Sec. VI, where we studied the algebra SO(4), or even more clearly, until Sec. VII, where the effects of choosing different bases, each with special interest for different applications, was emphasized for the algebra SO(5) [=Sp(4)]. For SU(2), we always mapped the standard irrep in which $\vec{J}^2$ and J_z are diagonal. Even here, and for the simplest subclass of mappings, those without quasifermions, i.e., the irreps containing the physical vacuum of the given nuclear model, we described four mappings—the Holstein-Primakoff, the Marumori-Yamamura-Tokunaga, the Dyson, and the Schwinger mappings. Of these, the first three have a structural relationship that has made them all of value for microscopic studies of collective motion. On the other hand, the Schwinger mapping, which is the most useful of the lot for reproducing all the mathematical apparatus of the algebra, has not played any role in the microscopic theories of collective motion, though it has proved of value in phenomenological studies.

We described how a Holstein-Primakoff mapping can be found by direct transcription if one is given the matrices of an irrep. The Marumori-Yamamura-Tokunaga mapping is then the normal-ordered product of the Holstein-Primakoff mapping with the projection operators onto the physical subspace. Whereas the Holstein-Primakoff generators, when acting between unphysical states, will generally yield values that contradict required Hermiticity properties of these operators, the Marumori-Yamamura-Tokunaga mapping, by definition requires that these matrix elements vanish. The Dyson and Schwinger mappings were also described fully.

It is often not possible to implement the simple definition of a Holstein-Primakoff mapping, namely, that based on direct transcription. This is because for most examples of physical interest we do not have explicit forms for the matrices of the required irreps. For such cases, we described a general algebraic technique for obtaining approximate, or, in special instances, exact boson mappings. In this method special emphasis is placed on the tensor character of the bosons with respect to the various subalgebras used to characterize the basis. Using this knowledge, we can then combine the bosons algebraically and compile lists of those that transform in the same way as the various generators. The generators themselves can then be represented as linear combinations of these basic tensors, each multiplied by an unknown scalar operator. For the simple algebras treated early in the review, there are only a small number of such terms, and by applying the commutation relations one can obtain difference equations for the scalar operators that prove readily solvable, as shown particularly by the examples in Secs. II, VI, and VII. In more complicated cases, where exact results are not presently achievable, if the basis were chosen in some sensible way related to the physics one was trying to describe, one could arrange the terms in powers of a small parameter and solve the commutation relations to some prescribed power in that parameter. The best pure example of this technique is contained in Sec. XVII.

In the general case, that is the best one can do at the moment for deriving Holstein-Primakoff mappings. In the special case when the physics dictates the use of a basis that includes the core subalgebra, as defined in Sec. XIX, in the chain of subalgebras specifying the basis, a more powerful special technique is available. This consists first of carrying out the Dyson mapping. This is the simplest part of the procedure and can be done either by the purely algebraic technique used throughout this review or through the intermediary of coherent states, as we illustrated in a few instances. The second part of the procedure is a systematic technique for transforming to a unitary basis. This method solves, at least in principle, the problem of deriving a Holstein-Primakoff mapping, not only for the pure boson mappings currently under discussion, but also for the boson-quasifermion mappings that we shall discuss below. For Lie algebras of sufficiently large order, detailed calculations remain to be carried out. Illustrations of this technique are to be found, for example, in Secs. II, VII, IX, X, and XXI.

In Sec. XVII, we have discussed briefly two other techniques found in the literature for deriving Holstein-Primakoff mappings from Dyson mappings, where the aim was to provide an alternative to the derivation featured there of the seniority-determined mapping, i.e., the object was to carry out a unitarizing similarity transformation that also changed the quantum numbers of the basis. Finally, for the examples of SU(2) and SO(4), we showed how to introduce quantum action-angle variables into the Holstein-Primakoff mappings, transformations that are convenient for the treatment of strong-coupling and semiclassical situations.

We also showed that there is an important difference between the kind of approximation achieved by the commutator method employed to derive approximate Holstein-Primakoff mappings and the method used to derive approximate Marumori-Yamamura-Tokunaga mappings. In the former case all matrix elements are guaranteed to be correct up to a certain order in a small parameter, whereas for the latter a subset of elements is guaranteed to be exact. A further modification of the

Marumori-Yamamura-Tokunaga method was introduced (the OAI method) in connection with the derivation of the interacting boson model: in mapping operators such as the Hamiltonian that are polynomials in the generators, one does not map the product of operators as products of the individual maps, but instead exactly calculates the matrix element of the product in the same subspace as was previously used for the generators. If one thinks of the completeness relation, this corresponds to a selective summation of higher-order terms, removing this method further from the perturbative realm. Other problems relating to the mapping of polynomials in the generators were discussed in Sec. XIV.C.

Turning to the discussion of boson-quasifermion mappings, although we discussed in Sec. X a case where we bosonized all bifermion operators, leaving just one unpaired fermion, we argued that in general the physics dictated the bosonization of at most a subalgebra of the full shell-model algebra. For purposes of concise expression we refer to these degrees of freedom as paired. The original shell-model subspace can be decomposed into subspaces containing varying numbers of paired and unpaired fermions, the previous mappings discussed referring to the subspace with no unpaired fermions. When we seek to generalize the mappings to include unpaired fermions, we must be cognizant of the fact that the original fermion operators are not kinematically independent of the bosons. It was both useful and interesting to introduce objects called quasifermions, which are kinematically independent of the bosons, sharing all the properties of ordinary fermions except that their anticommutation relations must be modified to reflect the constraint preventing them from combining into the pairs represented by the bosons. It turns out that when one has finished carrying through the mapping to this "hybrid" space, including the mapping of single-fermion operators, one has invented a new technique for obtaining all irreps of the subalgebra. Furthermore, in the mappings of the single-fermion operators, one has the fundamental tool for computing the matrix elements of all shell-model tensors outside the subalgebra. This is a fundamental tool of shell-model technology.

We return below to the physical uses of the hybrid mapping. Concerning the technical means of achieving these extended mappings, we have given one example of the application of a brute-force algebraic means, the example of SO(4) in Sec. VI. Otherwise, all the examples have utilized an extension of the unitarized Dyson technique that is our alternative to the method of the vector coherent state.

B. Applications to simple models

Bosons first appeared in nuclear physics within the framework of the Bohr-Mottelson description of the quadrupole degree of freedom of the nuclear surface. The unified model later recognized the relation between this phenomenological collective description and the underlying single-particle degrees of freedom and of their interaction, as described by core-particle coupling models. The study of boson and boson-quasifermion mappings in nuclear physics was stimulated by the need to understand the success of these models, as well as the collective description of giant resonances, from a more fundamental point of view, here taken to be the many-particle shell model. However, progress along these lines was delayed until the development of suitable many-body theory for the description of nuclear vibrations and the introduction of the concepts of the theory of superconductivity. Considerably later, the study of boson mapping was reinvigorated by the success of the various versions of the interacting boson model. These core problems will be discussed below. We first consider the variety of applications of the simpler models that were developed to explain how boson mappings work or to treat specialized problems. We list and describe these applications:

(i) *Vibrations.* This is the application *par excellence* of boson mapping. Applications to the Lipkin model, to pairing vibrations, and to the SO(4) and SO(5) models were described. The criterion for the applicability of the concept, that the number of phonons excited be small compared to the total degeneracy of the collective subspace, was most clearly in evidence for these models. From the well-known theory of the harmonic oscillator, this is also the domain where the kinetic and potential energies are of comparable size.

(ii) *Phase transitions at zero temperature.* The concept of phase transition for a finite system at zero temperature arises from the observation of a rapid change in the nature of the ground-state wave function as a result of a small change in some parameter of the system. The most outstanding realistic example is the spherical-to-deformed phase transition and its inverse, which occur in a number of places in the periodic table. This can be simulated in model systems by changing the ratio of two coupling strengths, thus determining the dominance of one or another term in the Hamiltonian. Though not a necessity, it is nevertheless a convenience to study this phenomenon within the boson framework. A number of examples were described, beginning with a phase transition in the Lipkin model, going on to the superconducting phase transition, and culminating in the SO(4) and SO(5) models of the spherical-to-deformed phase transition.

(iii) *Strong-coupling limit.* Several examples were given of the study of the strong-coupling limit, defined as the domain where the potential energy of collective motion dominates over the kinetic energy. To study this domain, it is useful to transform from boson operators either to canonical variables or to action-angle variables. In this review, we emphasized the latter approach for the simpler systems, describing the transformation in detail for the SU(2) and SO(4) cases. In particular, for the latter case, we gave an account of an application to a system with collective modes corresponding to rotations about a

fixed axis and vibrations in a plane perpendicular to that axis. Aside from the intrinsic interest of the system, the results were used to establish the validity of a perturbative method later applied to realistic systems, since for the SO(4) model the perturbative method and the method based on exact action-angle variables were shown to yield the same results to a specified order. Though we shall return to this point below, it is worth remarking here that the work on large-amplitude collective motion, described in Sec. XXI, also belong to the strong-coupling category.

(iv) *Semiclassical limit.* The same transformations to canonical or action-angle variables may be useful in the study of the semiclassical limit of simple models. As explained, this domain is more difficult to study than either the vibrational or the strong-coupling regime, and results of high quality have been reported only for an SU(2) model.

(v) *Phenomenological applications.* In general, the application of boson concepts to phenomenology falls outside the scope of this review, for the reason that they concern mainly the introduction of bosons, without any direct concern with their origin in mappings of Lie algebras. We have, however, made two exceptions to this exclusionist policy because, though concerned with phenomenology, each exception makes use of a boson mapping. One problem concerned the connection between the interacting boson model and the Bohr-Mottelson descriptions of the quadrupole degrees of freedom. We showed that the most direct way of exhibiting this relationship is as a mapping between the Holstein-Primakoff and the Schwinger forms of boson realization of the completely symmetric representation of SU(6). A second application was made to the study of the triaxial rotor at high angular momentum. Here the largest body of useful results was obtained by the application of Holstein-Primakoff mappings with expansions carried out about the well-known configurations of dynamical equilibrium corresponding to rotation about a principal axis. However, an elegant application of the Schwinger mapping to this problem was also described.

C. Mainstream applications

We can divide these applications into bins according to several criteria. The first is the form of the shell model employed. In older work, mainly aimed at deriving (or replacing) the unified model, it seemed advisable to incorporate the pairing correlations from the very beginning and thus to carry out the boson mapping procedures starting from the quasiparticle form of the shell model. On the other hand, much recent work, influenced by the advent of the interacting boson model, has taken the original number-conserving form of the shell model as a starting point. A second basis for classification was whether the application was perturbative or nonperturbative. A third distinction was whether the application was to experiment or to the clarification of theoretical concepts.

(i) *Quasiparticle shell model. Perturbative methods.* This heading refers to the material presented in Secs. XII and XIII for which the theoretical basis was provided in Secs. IX and X. The expansion in a relevant small parameter takes place only after an appropriate mean-field configuration has been established in the shell-model space, describing either a spherical or a deformed superconductor. The mapping is of the BZM type. Initially the expansion parameter measuring the validity of the mapping is the ratio of the average number of quasiparticles to the average degeneracy of a shell. After the introduction of correlated bosons by means of an RPA calculation that permits the identification of the collective degrees of freedom, one subsequently recognizes two expansion parameters; for a theory of anharmonic vibrations to work, the ratio of the number of phonons to an average degeneracy parameter must be small for low-energy states and, for the coupling of collective and noncollective degrees of freedom to be subject to a perturbative treatment, the ratio of the collective phonon energy to twice the minimum quasiparticle energy must be small. Given the validity of these conditions, the method described is the most systematic microscopic method available within the boson framework for the study of vibrations in the spherical regime or of rotations and vibrations in the deformed regime, including the proper restoration of the broken symmetries. Unfortunately, the method has not been widely disseminated. Applications to the vibrational case, described in this review, were carried out long ago. The theory for the deformed case, described in the text, has only recently been completed. As presently constituted, the method is not applicable to transitional nuclei.

(ii) *Quasiparticle shell model. Nonperturbative methods.* This category encompasses the main parts of Secs. XV and XVI, referring to the work of Tamura, Kishimoto, and their associates on the one hand and of Takada and his associates on the other. Together with the classical work of Kumar and Baranger and of Kumar, these constitute the major applied programs to realize the concepts of the unified model from a microscopic starting point. As opposed to the perturbative approach, where one maps uncorrelated bifermion degrees of freedom, here one first does Tamm-Dancoff or RPA dynamics in the fermion space and then carries out an approximate BZM-type mapping, in one case, or a closely related approximate Dyson mapping, in the other. Furthermore, one maps only a few degrees of freedom, namely, the most collective quadrupole phonon and selected other degrees of freedom. The coupling of these additional degrees of freedom is eliminated by a perturbational procedure, leaving an effective Hamiltonian describing only the collective degrees of freedom. The nonperturbative aspect enters at this juncture, in that the collective Hamiltonian is diagonalized within its appropriate subspace rather than treated as a quadratic form plus anharmonic perturbations. From the results found, it is not ruled out that, even though one begins with spherical quasiparti-

cles, this method has the capability of describing transitional and deformed nuclei. In the course of this work, one must face the problem that the introduction of correlated bosons may well mix parts of the unphysical boson space with the physical space. A full discussion of this problem and a description of all suggestions for dealing with it led to the conclusion that the most satisfactory method presently available is built into the mapping procedure utilized in this nonperturbative approach.

(iii) *Derivation of the interacting boson model.* In Sec. XVIII we surveyed all the work designed to provide a microscopic foundation for the interacting boson model and the interacting boson-fermion model. In the end, we also rendered some judgments. Since the new and the previous phenomenologies, properly expanded, are mathematically equivalent (this is not a judgment about which approach is more fruitful to apply to a specific problem), one can take the extreme point of view that a derivation of one is a derivation of the other. We have not taken this viewpoint. Rather we have almost insisted that, insofar as a new task has been defined, it requires derivations to be carred out within the framework of a shell model, with number conservation and rotational invariance preserved. Applying this criterion, we have concluded that some understanding has been achieved of the origin of the interacting boson model for spherical (vibrational) nuclei by the approximate mapping of shell-model bases associated with the broken-pair (generalized seniority) coupling scheme. The most elaborate work of this type reported is qualitatively convincing, but quantitatively there remain differences with the most successful form of the phenomenology, even after one has pushed the technology as far as seems feasible at the current time. At the same time, we found all efforts to derive the interacting boson model for deformed nuclei to be pale imitations of the microscopic theory of the unified model. We have suggested that the only ideologically pure way of approaching this subject is through the pseudo-SU(3) model, through which meaningful contact has already been made with the geometrical and other aspects of the unified model. Finally, we also pointed out in this section and briefly in Sec. XIX that the current forms of the interacting boson-fermion model can be interpreted as approximate version of boson-quasifermion mappings. and furthermore that present technology leaves room for improvement.

(iv) *Dynamics in boson space.* All the successful applied programs reported above share the feature that at least some of the dynamics is carried out in the fermion space before any approximate mapping is carried out. For some simplified models, in Secs. IV.B and VII.E and for realistic models in Sec. XIV, we have reported methods that propose mappings of uncorrelated bifermion operators, defined as products of spherical shell-model operators, leaving all dynamics to be carried out in the boson space. It is too early to render a judgement concerning the promise of this work. For realistic situations, it appears to be difficult to avoid the problem of intrusion of unphysical states into the low-energy part of the spectrum, and until this problem is solved this avenue for possible progress remains closed.

(v) *Classical limit of boson mapping.* In Sec. XI we described applications of boson mappings of a character quite distinct from anything else presented above, applications that appear quite formal, but that, in fact, form the theoretical basis for important practical efforts still underway. The main theoretical result reviewed in this section was the proof that the classical limit of the mapped Heisenberg equations of motion are of the Hamiltonian form, the classical limit of the boson creation and annihilation comprising pairs of (complex) canonical coordinates. The most important application of this result is the method of first deriving a classical collective Hamiltonian and then requantizing. For one thing, it can be argued that, for well-developed collective motion, the errors necessarily attendant upon ambiguities in requantization are quite tolerable and, in some cases, even controllable. For another, it leads to unique methods of studying the large-amplitude (strong-coupling) limit, in particular, as described in Sec. XXI. One of the main goals of the latter work is to provide a self-consistent generalization of the Kumar-Baranger theory.

D. Recent work and open problems

Our summary thus far has included some recent work, but not all that has been reported in the review. In the listing that follows, we wish to emphasize that there is no dearth of interesting problems remaining, both of a technical and of an applied nature. In addition to the items listed below, we note the importance of efforts to deepen our understanding of the older programs that have already proven their worth, in particular the work described in Secs. XII, XV, XVI and some of the work described in XVII.

(i) *Boson-fermion mappings and the quantized Bogoliubov-transformation.* The most important new technical development that has emerged in the course of this review is the understanding of the connection between the method of the vector coherent state in the theory of group representations and the concept and utilization of boson-quasifermion mappings in nuclear physics. The appellation "quantized Bogoliubov transformation" refers to the special example of the mapping of single-fermion operators to the hybrid space furnished by the algebra SU(2). The importance of such formulas both for the general tensor analysis of group algebras and for the problem of core-particle coupling has been stressed in this review. Though a few additional examples have been developed and reported in the body of this paper, the problem remains largely untouched.

(ii) *Application of the pseudo-*SU(3) *scheme to derivation of the interacting boson model.* This point has been made before, but we believe that the absence of a believable foundation of the interacting boson model in the de-

formed regime represents the most important lacuna in the structure of this formalism. We have also expressed the opinion several times in this review that the appropriate tool for such a study is at hand, the symplectic shell model in pseudo-SU(3) mode.

(iii) *Iterated boson mappings and dynamics in boson space.* We emphasized the current impasse concerning proposals to carry out all dynamics in the boson space, opinions differing about whether one can avoid the intertwining of physical and unphysical parts of the boson space. In Sec. XIV.D we reported a proposal to utilize iterated boson mappings in order to ameliorate this difficulty. This suggestion has another possible virtue that may make it even more important to pursue: it may represent a method of studying more-than-two-fermion correlations.

(iv) *Variational principles based on boson trial functions.* The material described in Sec. XX represents a foundation for a quantum theory of collective motion with a well-defined classical limit, on the one hand, and a systematic method of including quantum corrections, on the other. It has hardly been developed thus far and is worthy of further study.

(v) *Thermal boson mappings.* The method proposed in Sec. XXII for the application of boson mappings to systems at finite temperature also cries out for further development.

More generally, our review has been concerned mainly with the problem of collective motion near the ground state of a many-body system. Though this problem can hardly be said to have been "solved," the focus of much effort in nuclear-structure physics has shifted to the study of systems with high spin and/or finite temperature, and one may well ask if there is any future for the methods of boson mappings in these domains. We have already indicated that one can deal with finite temperature, but, as always, these problems are never simpler to solve than the corresponding problems at zero temperature. Even for zero temperature, but high spin, one encounters the coupling of collective and individual degrees of freedom; the natural machinery with which to deal with these problems is at hand in the form of boson-quasifermion mappings.

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REFERENCES

Abrikosov, A. A., L. P. Gor'kov, and I. E. Dzyaloshinskii, 1963, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Englewood Cliffs, N.J.).

Agassi, D., 1968, "Validity of the BCS and RPA approximations in the pairing-plus-monopole shell model," Nucl. Phys. A **116**, 49.

Akkermans, J. N. L., E. Loriaux, K. Allaart, and G. Bonsignori, 1983, "Correlated-pair structure in nuclei and possible relation to IBM-2," Z. Phys. A **313**, 213.

Allaart, K., and E. Boeker, 1971, "Comparison between the BCS and FBCS formalisms as approximations to the low seniority shell model," Nucl. Phys. A **168**, 630.

Allaart, K., and E. Boeker, 1972, "FBCS for odd nuclei and the inverse gap equations: Application to $N=50$ isotones," Nucl. Phys. A **198**, 33.

Allaart, K., E. Boeker, G. Bonsigniori, M. Savoia, and Y. K. Gambhir, 1988, "The broken pair model for nuclei and its recent applications," Phys. Rep. C **169**, 209.

Allaart, K., and G. Bonsigniori, 1983, "Microscopic study of the interaction between like bosons in IBM-2," Phys. Lett. B **124**, 1.

Allaart, K., G. Bonsigniori, M. Savoia, and V. Paar, 1986a, "Construction of microscopic boson states and their relevance for IBM-1," Nucl. Phys. A **458**, 412.

Allaart, K., G. Bonsigniori, M. Savoia, and V. Paar, 1986b, "Nuclear pair states and the interacting boson model," in *Microscopic Approaches to Nuclear Structure Calculations,* edited by A. Covello (Italian Physical Society Conference Proceedings, Bologna), Vol. 4, p. 237.

Arima, A., 1980, "Relation of the interacting boson model to the shell model, Nucl. Phys. A **347**, 339.

Arima, A., J. Ginocchio, and N. Yoshida, 1982, "The structure of samarium isotopes in a schematic and quadrupole pairing model," Nucl. Phys. A **384**, 112.

Arima, A., M. Harvey, and K. Shimizu, 1969, "Pseudo LS coupling and pseudo SU(3) coupling scheme," Phys. Lett. B **30**, 517.

Arima, A., and F. Iachello, 1976a, "Interacting boson model of collective states. I. The vibrational limit," Ann. Phys. (N.Y.) **99**, 253.

Arima, A., and F. Iachello, 1976b, "Interacting boson model of collective states. II. The rotational limit," Phys. Rev. C **14**, 761.

Arima, A., and F. Iachello, 1978, "Extension of the interacting boson model to odd A nuclei," Ann. Phys. **111**, 201.

Arima, A., and F. Iachello, 1979, "Interacting boson model of collective nuclear states. IV. The 0(6) model," Ann. Phys. (N.Y.) **123**, 468.

Arima, A., N. Yoshida, and J. N. Ginocchio, 1981, "A Test of Boson Mappings of Fermion Systems," Phys. Lett. B **101**, 209.

Arve, P., G. F. Bertsch, J. Negele, and G. Puddhu, 1987, "Model for tunneling in many-particle systems," Phys. Rev. C **36**, 2018.

Arvieu, R., 1969, "Properties of single closed-shell nuclei in a simple model," in *Nuclear Structure and Nuclear Reactions: Proceedings of the International School of Physics "Enrico Fermi,"* Course XL, edited by M. Jean and R. A. Ricci (Academic, New York/London), p. 629.

Arvieu, R., and M. Veneroni, 1960, "Quasi-particles and collective states of spherical nuclei," C. R. Acad. Sci. **250**, 2155.

Badea, M., and A. A. Raduta, 1979, "Boson representation of

the asymmetric rotator," Rev. Roum. Phys. **24**, p. 743.
Baktybaev, K. B., and D. P. Strygin, 1981, "Microscopic substantiation of the interacting-boson model," Izv. Akad. Nauk. SSSR, Ser. Fiz. **45**, 1981.
Banville, M., and P. A. Simard, 1970a, "Etude de la transcription de la quasi particle dans l'espace idéal," Can. J. Phys. **48**, 819.
Banville, M., and P. A. Simard, 1970b, "The boson method and the antisymmetric problem," Nucl. Phys. A **158**, 51.
Banville, M., and P. A. Simard, 1973, "Antisymmetric boson in nuclear structure," Can. J. Phys. **51**, 40.
Baranger, M., 1960, "Extension of the shell model for heavy spherical nuclei," Phys. Rev. **120**, 957.
Baranger, M., and K. Kumar, 1968, "Nuclear deformations in the pairing-plus-quadrupole model. (IV). Theory of collective motion," Nucl. Phys. A **122**, 241.
Bardeen, J., L. N. Cooper, J. R. Schieffer, 1957, "Theory of Superconductivity," Phys. Rev. **108**, 1175.
Barrett, B. R., C. H. Druce, S. Pittel, and P. H. Duval, 1989, " A microscopic interpretation of the interacting boson model," in *Shell Model and Nuclear Structure,* edited by A. Covello (World Scientific, Singapore), p. 421.
Barrett, B. R., P. D. Duval, and S. Pittel, 1983, "A microscopic determination of the interacting boson model," in *Collective Bands in Nuclei,* Progress in Particle and Nuclear Physics, Vol. 9, edited by D. Wilkinson (Pergamon, Oxford), p. 539.
Beck, H. J. Mang, and P. Ring, 1970, "Symmetry conserving Hartree-Fock-Bogoliubov theory and its application to collective nuclear rotation," Z. Phys. **231**, 26.
Belyaev, S. T., and V. G. Zelevinsky, 1962, "Anharmonic effects of quadrupole oscillations of spherical nuclei," Nucl. Phys. **39**, 582.
Bes, D. R., R. A. Broglia, E. Maglioni, and A. Vitturi, 1982, "Nilsson and interacting-boson-model pictures of deformed nuclei," Phys. Rev. Lett. **48**, 1001.
Bes, D. R., and R. A. Sorensen, 1969, "The pairing-plus quadrupole model," in *Advances in Nuclear Physics,* Vol. 2, edited by M. Baranger and E. Vogt (Plenum, New York), p. 129.
Bijker, R., S. Pittel, and J. Dukelsky, 1989, "Boson expansions for systems of interacting bosons," Phys. Lett. B **219**, 5.
Bijker, R., and O. Scholton, 1985, "Relation between boson-fermion approximation model and dynamical boson-fermion symmetries," Phys. Rev. C **32**, 591.
Birbrair, B. L., 1976, "Goldstone theorem in the self-consistent cranking model." Nucl. Phys. A **257**, 445.
Birkhoff, G. D., 1966, *Dynamical Systems* (American Mathematical Society, New York).
Blaizot, J. P., and E. R. Marshalek, 1978a, "Boson expansions and quantization of time-dependent self-consistent fields. (I). Particle-hole excitations," Nucl. Phys. A **309**, 422.
Blaizot, J. P., and E. R. Marshalek, 1978b, "Boson expansions and quantization of time-dependent self-consistent fields. (II). Relation between finite and infinite boson expansions," Nucl. Phys, A **309**, 453.
Blaizot, J. P., and G. Ripka, 1986, *Quantum Theory of Finite Systems* (MIT Press, Cambridge).
Blatt, J., and T. Matsubara, 1958, "Quasi-chemical equilibrium theory, Part II," Prog. Theor Phys. **20**, 553.
Bogoliubov, N. N., 1958, "A new method in the theory of superconductivity, I," Zh. Eksp. Teor. Phys. **34**, 58 [Sov. Phys. JETP **7**, 41 (1958)].
Bohr, A. 1952, "The coupling of nuclear surface oscillations to the motion of individual nucleons," K. Dan. Vidensk. Selsk. Mat. Fys. Medd. **26**, No. 14.
Bohr, A., and B. R. Mottelson, 1953, "Collective and individual aspects of nuclear structure," K. Dans. Vidensk. Selsk. Mat. Fys. Medd. **27**, No. 16.
Bohr, A., and B. R. Mottelson, 1975, *Nuclear Structure,* Vol. II (Benjamin, New York).
Bohr, A., and B. R. Mottelson, 1982, "On the spectrum of ^{168}Er," Phys. Scr. **25**, 28.
Bonatsos, D., 1988, *Interacting Boson Models of Nuclear Structure* (Oxford University Press, Oxford).
Bonatsos, D., and A. Klein, 1985, "Exact boson mappings for nuclear neutron (proton) p shell with the symmetry $SO(7)\supset SU(3)$," Phys. Rev. C **31**, 992.
Bonatsos, D., and A. Klein, 1986, "Exact boson mappings for nuclear neutron (proton) shell-model algebras having SU(3) subalgebras," Ann. Phys. (N.Y.) **169**, 61.
Bonatsos, D., and A. Klein, 1987, "Extended commutator method for boson mapping in the seniority scheme," Nucl. Phys. A **469**, 253.
Bonatsos, D., A. Klein, and C. T. Li, 1984, "Commutator method for boson mapping in the seniority scheme," Nucl. Phys. A **425**, 521.
Bonatsos, D., A. Klein and Q.-Y. Zhang, 1986a, "Simplified boson mappings of symplectic shell model algebras," Phys. Rev. C **34**, 686.
Bonatsos, D., A. Klein, and Q.-Y. Zhang, 1986b, "New forms for the Belyaev-Zelevinsky-Marshalek boson realization of shell model algebras $SO(2\sum_i(2j_i+1))$," Phys. Lett. B **175**, 249.
Bonatsos, D., D. Peres Menezes and A. Klein, 1987, "Octupole bosons of negative parity in multi-level seniority boson mappings," J. Phys. G. **14**, L45.
Bonatsos, D., D. Peres Menezes, and A. Klein, 1988, "Extended commutator method for boson mapping in the seniority scheme, many non-degenerate j shells," Nucl. Phys. A **474**, 381.
Bortignon, P. F., R. A. Broglia, D. R. Bes, and R. Liotta, 1977, "Nuclear Field Theory," Phys. Rep. C **30**, 305.
Brajczewska, M., C. Fiolhais, and J. da Providência, 1986, "Thermal boson expansion for the Heisenberg ferromagnet," Physica A **137**, 196.
Brink, D. M., A. F. R. De Toledo-Piza, and A. K. Kerman, 1965, "Interval rules and intensity ratio in vibrating spherical nuclei," Phys. Lett. **19**, 413.
Brink, D. M., and A. Weiguny, 1968, "The generator-coordinate method and the random-phase approximation," Phys. Lett. B **26**, 497.
Broglia, R. A., M. Diebel, S. Frauendorf, and M. Gallardo, 1986, "Role of the ground-state correlations induced by pairing vibrations in strongly rotating nuclei," Phys. Lett. B **166**, 252.
Broglia, R. A., R. Liotta, and V. Paar, 1972, "Quadrupole moments of the 2_2^+ state of proton-closed-shell nuclei," Phys. Lett. B **38**, 480.
Broglia, R. A., and B. Sorensen, 1968, "Comparison between a boson expansion method and the exact solution in a two level model," Nucl. Phys. A **110**, 241.
Brown, B. A., 1987, "Achievements and goals of recent shell-model calculations," in *Proceedings of the International Nuclear Physics Conference, Harrogate,* Institute of Physics Conference Series No. 86, edited by J. L. Durell, J. M. Irvine, and G. C. Morrison (Institute of Physics, Bristol), p. 119.
Bulgac, A., A. Klein, and D. Do Dang, 1987a, "Uniqueness of the collective submanifold in the adiabatic theory of large-amplitude collective motion," Phys. Lett. B **191**, 217.

Bulgac, A., A. Klein, and G. Do Dang, 1987b, "Application of a theory of large-amplitude collective motion to a generalized landscape model," Phys. Rev. C **36**, 2672.

Bulgac, A., A. Klein, and G. Do Dang, 1988, "Decoupling a collective degree of freedom from a model for tunneling in many-particle systems," Phys. Rev. C **37**, 2156.

Bulgac, A., A. Klein, N. R. Walet, and G. Do Dang, 1989, "Adiabatic time-dependent Hartree-Fock theory in the generalized valley approximation," Phys. Rev. C **40**, 945.

Caldeira, M. H., H. Pascoal, M. D. C. Ruivo, and E. M. Silva, 1972, "Collective vibrations of intrinsic excited states," Nucl. Phys. A **197**, 265.

Cambiaggo, M. C., and J. Dukelsky, 1987, "Mean-field approximation and the collective transformation for Dyson boson Hamiltonians," Phys. Lett. B **197**, 479.

Carruthers, P., and N. N. Nieto, 1968, "Phase and angle variables in quantum mechanics," Rev. Mod. Phys. **40**, 41.

Carruthers, P., and F. Zachariasen, 1983, "Quantum collision theory with phase-space distributions," Rev. Mod. Phys. **55**, 245.

Castaños, O., E. Chacon, M. Moshinsky, and C. Quesne, 1985, "Boson realization of Sp(4). I. The matrix formulation," J. Math. Phys. **26**, 2107.

Castaños, O., J. P. Draayer, and Y. Leschber, 1988, "Shape variables and the shell model," Z. Phys. A **329**, 33.

Castaños, O., P. Kramer, and M. Moshinsky, 1985a, "Generating kernel for the boson realization of symplectic algebras," J. Phys. A **18**, L493.

Castaños, O., P. Kramer, and M. Moshinsky, 1985b, "Boson realization of Sp(4,*R*). II. The generating kernel formulation," J. Math. Phys. **4**, 604.

Casten, R. F., C.-L. Wu, D. H. Feng, J. N. Ginocchio, and X.-L. Han, 1986, "Empirical evidence for an SO(7) fermion dynamical symmetry in nuclei," Phys. Rev. Lett. **56**, 2578.

Catara, F., A. Insolia, E. Maglioni, and A. Vitturi, 1984, "Description of the even samarium isotopes in the collective pair approximation," Phys. Rev. C **29**, 1916.

Chacon, E. P., and M. Moshinsky, 1987, "Matrix representation of the generators of sympectic algebras," J. Phys. A **20**, 4595.

Chattopadhyay, P. K., A. Klein, and F. Krejs, 1972, "An exactly soluble shell model with $R(5)$ symmetry exhibiting two-dimensional vibrations and rotations," Phys. Lett. B **42**, 315.

Chattopadhyay, P. K., A. Klein, and F. Krejs, 1974, "Boson-like expansions without bosons: Application to several exactly soluble shell models with collective features," Nucl. Phys. A **229**, 509.

Chen, J.-Q., D. H. Feng, and C.-L. Wu, 1986, "Correspondence between the interacting boson model and the fermion dynamical symmetry model of nuclei," Phys. Rev. C **34**, 2269.

Chen, J.-Q., A. Novoselsky, M. Vallières, and R. Gilmore, 1989, "New approach to multishell calculations in multiple angular momentum coupling schemes," Phys. Rev. C **39**, 1088.

Cohen, T. D., 1985, "Self-consistency and boson mappings for deformed nuclei," Phys. Lett. B **158**, 1.

Cohen, T. D., and A. Klein, 1982, "Derivation and test of accuracy of an IBM-like Hamiltonian for a model with SO(5)×SO(5) symmetry," Nucl. Phys. A **390**, 1.

Da Providência, J., 1970, "Remark on the relation between the generator-coordinate method and the random-phase approximation," Nucl. Phys. A **157**, 358.

Da Providência, J., 1974, "On the connection between the generator coordinate method and boson expansions for odd-particle systems," Nucl. Phys. A **224**, 262.

Da Providência, J., and C. Fiolhais, 1985, "Mean-field theories with mixed states and associated boson expansions," Nucl. Phys. A **435**, 190.

Da Providência, J., and C. Fiolhais, 1990, "A note on thermal bosons," Nucl. Phys. A **516**, 53.

Da Providência, J., J. Urbano, and L. Ferreiro, 1971, "Corrections to the Gaussian overlap approximation: A new boson expansion," Nucl. Phys. A **170**, 129.

Da Providência, J., and J. Weneser, 1970, "Nuclear ground-state correlations and boson expansions," Phys. Rev. C **1**, 825.

Das, T. K., R. M. Dreizler, and A. Klein, 1970, "Phenomenological analysis of quasirotational spectra and possible evidence for higher-phonon states," Phys. Rev. C **2**, 632.

Dasso, C., and A. Klein, 1973, "An intrinsic state generating the ground-state band of a deformed nucleus; illustration by an exactly soluble model," Nucl. Phys. A **210**, 443.

Dasso, C., and A. Klein, 1974, "The Coriolis anti-pairing transition in a simplified shell model with the symmetry of R(5)×R(5)," Nucl. Phys. A **222**, 445.

Dasso, C., F. Krejs, A. Klein, and P. K. Chattopadhyay, 1973, "Validity of an algebraic-variational approach to the theory of collective motion for an exactly soluble shell model with R(5) symmetry," Nucl. Phys. A **210**, 429.

Deenen, J., M. Moshinsky and T. H. Seligman, 1980, "Canonical transformations to action and angle variables and their representation in quantum mechanics. III. The general problem," Ann. Phys. (N.Y.) **127**, 458.

Deenen, J., and C. Quesne, 1982, "Dynamical group of microscopic collective states. II. Boson representations in d dimensions," J. Math. Phys. **23**, 2004.

Deenen, J., and C. Quesne, 1984, "Partially coherent states of the real symplectic group," J. Math. Phys. **25**, 2354.

Deenen, J., and C. Quesne, 1985, "Boson representations of the real symplectic group and their applications to the nuclear collective model," J. Math. Phys. **26**, 2705.

de Kock, E. A., and H. B. Geyer, 1988, "Interaction strengths in the interacting boson-fermion model," Phys. Rev. C **38**, 2887.

De Shalit, A., and T. Talmi, 1963, *Nuclear Shell Theory* (Academic, New York).

de Winter, L. C., N. R. Walet, and and P. J. Brussard, 1988a, "Boson image of the quadrupole operator in deformed nuclei," Nucl. Phys. A **483**, 295.

de Winter, L., C., N. R. Walet, and P. J. Brussard, 1988b, "Collective pair structure of $K^{\pi}=0^{+},1^{+},2^{+}$ bands in deformed nuclei," Nucl. Phys. A **486**, 235.

Dieperink, A. E. L., O. Scholten, and F. Iachello, 1980, "Clasical limit of the interacting boson model," Phys. Rev. Lett. **44**, 1747.

Dirac, P. A. M., 1950, "Generalized Hamiltonian dynamics," Can. J. Math. **2**, 129.

Dobaczewski, J., 1981a, "A unification of boson expansion theories. (I) Functional representations of fermion states," Nucl. Phys. A **369**, 213.

Dobaczewski, J., 1981b, "A unification of boson expansion theories. (II) Boson expansions as provided by the functional representation method," Nucl. Phys. A **369**, 237.

Dobaczewski, J., 1982, "A unification of boson expansion theories. (III) Applications," Nucl. Phys. A **380**, 1.

Dobeš, J., 1989, "Can the boson condensate ground state of ^{16}O be a physical one?" Phys. Lett. B **222**, 315.

Do Dang, G., A. Bulgac, and A. Klein, 1987, "Determination of the collective Hamiltonian in a self-consistent theory of large-amplitude collective motion," Phys. Rev. C **36**, 2661.

Do Dang, G., and A. Klein, 1985, "Is there a consistent theory

of large-amplitude collective motion?" Phys. Rev. Lett. **55**, 2265.
Dönau, F., and D. Janssen, 1973, "Boson description of collective states. (II) Description of odd vibrational nuclei," Nucl. Phys. A **209**, 109.
Donnelly, T. W., and J. D. Walecka, 1975, "Electron scattering and nuclear structure," Ann. Rev. Nucl. Sci. **25**, 329.
Draayer, J. P., and G. Rosensteel, 1985, "U(3)→R(3) integrity-basis spectroscopy," Nucl. Phys. A **439**, 61.
Draayer, J. P., and K. J. Weeks, 1983, "Shell-model description of the low-energy structure of strongly deformed nuclei," Phys. Rev. Lett. **51**, 1422.
Draayer, J. P., and K. J. Weeks, 1984, "Towards a shell-model description of the low-energy structure of deformed nuclei," Ann. Phys. (N.Y.) **156**, 41.
Draayer, J. P., K. J. Weeks, and K. T. Hecht, 1982, "Strength of the $Q_\pi \cdot Q_\nu$ interaction and the strong coupled pseudo-SU(3) limit," Nucl. Phys. A **381**, 1.
Dreiss, G. J., and A. Klein, 1969, "Algebra of currents as a complete dynamical scheme in the nuclear many-body problem," Nucl. Phys. A **139**, 81.
Dreizler, R. M., and A. Klein, 1965, "Interpretations of higher rotational states of deformed nuclei and foundations of the self-consistent cranking model," Phys. Rev. Lett. **15**, 893.
Dreizler, R. M., and A. Klein, 1973, "Concept of transition-operator boson and its application to an exactly soluble model," Phys. Rev. C **7**, 512.
Druce, C. H., S. Pittel, B. R. Barrett, and P. D. Duval, 1987, "The interacting boson model microscopic calculations for the mercury isotopes," Ann. Phys. (N.Y.) **176**, 441.
Dukelsky, J., C. G. Dussel, R. P. J. Perazzo, S. L. Reich, and H. M. Sofia, 1984, "A self-consistent description of systems with many interacting bosons," Nucl. Phys. A **425**, 93.
Dukelsky, J., R. P. J. Perazzo, S. L. Reich, and H. M. Sofia, 1985, "Self-consistent picture of the interplay of quadrupole and octupole degrees of freedom in many-boson systems," Phys. Lett. B **158**, 361.
Dukelsky, J., and S. Pittel, 1986a, "Microscopic approach to the interacting boson model in the deformed region," in *Microscopic Approaches to Nuclear Structure Calculations,* edited by A. Covello (Italian Physical Society Publications, Bologna), Vol. 4. p. 285.
Dukelsky, J., and S. Pittel, 1986b, "Boson mapping of one-body operators in deformed nuclei," Nucl. Phys. A **456**, 75.
Dukelsky, J., and S. Pittel, 1989, "The iterative boson expansion procedure for fermion systems," Phys. Rev. C **39**, 2001.
Dukelsky, J., S. Pittel, H. M. Sofia, and C. LIma, 1986, "Boson mapping of one-body operators in deformed nuclei," Nucl. Phys. A **456**, 75.
Duval, P. D., and B. R. Barret, 1981, "Shell-model determination of the interacting boson model for two nondegenerate j shells," Phys. Rev. C **24**, 1272.
Dyson, F. J., 1956, "General theory of spin-wave interactions," Phys. Rev. **102**, 1217.
Eichler, J., and M. Yamamura, 1972, "Quadrupoling and pairing in the shell model," Nucl. Phys. A **182**, 33.
Eisenberg, J. M., and W. Greiner, 1970, *Nuclear Models* (North-Holland, Amsterdam), Vol. I.
Elliott, J. P., 1958a, "Collective motion in the nuclear shell model. I. Classification schemes for states of mixed configurations," Proc. R. Soc. A **245**, 128.
Elliott, J. P., 1958b, "Collective motion in the nuclear shell model. II. The introduction of intrinsic wave functions," Proc. R. Soc. A **245**, 562.
Elliott, J. P., and J. A. Evans, 1987, "Isospin in a boson approximation for particles and holes," Phys. Lett. B **195**, 1.
Elliott, J. P., J. A. Evans, P. Van Isacker, 1988, "An isospin-invariant form of the interacting boson model for odd nuclei (IBFM-3)," Nucl. Phys. A **481**, 245.
Evans, J. A., J. P. Elliott, and S. Szpikowski, 1985, "An IBM analysis of a single j shell of neutrons and protons," Nucl. Phys. A **435**, 317.
Evans, J. A., and N. Kraus, 1971, "An exact solution of the pairing-plus monopole Hamiltonian using a boson representation of the group Sp(4)," Phys. Lett. B **37**, 455.
Evans, J. A., P. Van Isacker, and J. P. Elliott, 1988, "A test for the interacting boson-fermion model in a single j shell of neutrons and protons," Nucl. Phys. A **489**, 269.
Ferreira, P. L., J. A. Castilho Alcarás, and V. C. Aguilera Navarro, 1964, "States of nuclear quadrupole vibrations and the two-phonon triplets," Phys. Rev. **136**, 1243B.
Gambhir, Y. K., and G. Basavaraju, 1979, "Effective operators and the truncation of shell model configuration space," Pramana **13**, 47.
Gambhir, Y. K., R. S. Nikam, C. R. Sarma, and J. A. Sheikh, 1985, "Boson basis realization for shell-model fermion problem," J. Math. Phys. **26**, 2067.
Gambhir, Y., K., A. Rimini, and T. Weber, 1969, "Number-conserving approximation to the shell model," Phys. Rev. **188**, 1573.
Gambhir, Y. K., A. Rimini, and T. Weber, 1973, "Analysis of (p,t) and (t,p) reaction data for Ni isotopes with structure wave functions of the broken pair approximation," Phys. Rev. C **7**, 1454.
Gambhir, Y. K., P. Ring, and P. Schuck, 1982a, "Microscopic determination of the interacting boson model parameters," Phys. Rev. C **25**, 2858.
Gambhir, Y. K., P. Ring, and P. Schuck, 1982b, "Microscopic investigation of the structure of the bosons and the Hamiltonian of the interacting boson model," Nucl. Phys. A **384**, 37.
Gambhir, Y. K., P. Ring, and P. Schuck, 1984, "Microscopic formulation of interacting fermion-boson systems," Nucl. Phys. A **423**, 35.
Gambhir, Y. K., J. A. Sheikh, P. Ring, and P. Schuck, 1985, "Physical basis states in the boson expansion theories," Phys. Rev. C **31**, 1519.
Garbaczewski, P., 1977, "The method of boson expansions in quantum theory," Phys. Rep. C **36**, 65.
Gelberg, A., 1983, "On the exchange term of the interacting boson-fermion Hamiltonian," Z. Phys. A **310**, 117.
Geyer, H. B. 1986a, "Boson mapping of the shell-model algebra obtained from a seniority-dictated similarity transformation," Phys. Rev. C **34**, 2373.
Geyer, H. B., 1986b, "Boson realization for Sp(4) obtained from seniority-dictated similarity transformations," Phys. Lett. B **182**, 111.
Geyer, H. B., C. A. Engelbrecht, and F. J. W. Hahne, 1986, "Adequacy of the boson basis and the identification of spurious states," Phys. Rev. C **33**, 1041.
Geyer, H. B., and F. J. W. Hahne, 1980a, "Ideal fermions in boson expansions," Phys. Lett. B **90**, 6.
Geyer, H. B., and F. J. W. Hahne, 1980b, "Pairing and surface bosons and the group Sp(4)," Phys. Lett. B **97**, 173.
Geyer, H. B., and F. J. W. Hahne, 1981, "Unveiling s- and d-bosons in even- and odd-A nuclei," Nucl. Phys. A **363**, 45.
Geyer, H. B., and F. J. W. Hahne, 1983, "Ideal fermions in boson mappings," S. Afr. J. Phys. **6**, 67.
Geyer, H. B., F. J. W., Hahne, and F. G. Scholtz, 1987,

"Dynamical symmetries of the interacting-boson model from a non-U(6) viewpoint," Phys. Rev. Lett. **58**, 459.
Geyer, H. B., and S. Y. Lee, 1982, "Microscopic structure of an interacting boson model in terms of the Dyson boson mapping," Phys. Rev. C **26**, 642.
Geyer, H. B., and I. Morrison, 1989, "Seniority mapping of single-fermion operators," Phys. Rev. C **40**, 2383.
Gilmore, R., 1970, "Spectrum of Casimir invariants for the simple classical Lie groups," J. Math. Phys. **11**, 1855.
Gilmore, R., 1972, "Geometry of symmetrized states," Ann. Phys. (N.Y.) **74**, 391.
Gilmore, R., 1974, *Lie Groups, Lie Algebras and Some of Their Applications* (Wiley, New York).
Ginocchio, J. N., 1980, "A schematic model for monopole and quadrupole pairing in nuclei," Ann. Phys. (N.Y.) **125**, 234.
Ginocchio, J. N., and M. W. Kirson, 1980a, "Relationship between the Bohr collective Hamiltonian and the interacting boson model," Phys. Rev. Lett. **44**, 1744.
Ginocchio, J. N., and M. W. Kirson, 1980b, "An intrinsic state for the interacting boson model and its relationship to the Bohr-Mottelson model," Nucl. Phys. A **350**, 31.
Ginocchio, J. N., and I. Talmi, 1980, "On the correspondence between fermion and boson states and operators," Nucl. Phys. A **337**, 431.
Ginocchio, J., and J. Weneser, 1968, "Isospin invariance and the pairing force problem," Phys. Rev. **170**, 859.
Goshen, G., and H. J. Lipkin, 1968, "On the application of the group Sp(4) or R(5) to nuclear structure," in *Spectroscopic and Group Theoretical Methods in Physics,* edited by F. Bloch, S. G. Cohen, A. De-Shalit, S. Sanibursky, and I. Talmi (Wiley Interscience, New York/North-Holland, Amsterdam), p. 245.
Griffin, J. J., and J. A. Wheeler, 1957, "Collective motion in nuclei by the method of generator coordinates," Phys. Rev. **108**, 312.
Gulshani, P., 1979a, "Oscillator-like coherent states of an asymmetric top," Phys. Lett. A **71**, 13.
Gulshani, P., 1979b, "Generalized Schwinger boson realizations and the oscillator-like coherent states of the rotation groups and the asymmetric top," Can. J. Phys. **57**, 998.
Gustavson, G. F., 1966, "On constructing formal integrals of a Hamiltonian system near an equilibrium point," Astron. J. **71**, 670.
Hage-Hassan, M., and M. Lambert, 1972, "La méthode de la fonction génératice et les développements en quasibosons," Nucl. Phys. A **188**, 545.
Hahne, F. J. W., 1981, "Finite and infinite boson expansions," Phys. Rev. C **23**, 2305.
Hahne, F. J. W., H. B. Geyer, and C. A. Engelbrecht, 1988, "Comment on algebraic analysis of physical and spurious states in Dyson boson mapping," Phys. Rev. C **37**, 885.
Hahne, F. J. W., and A. Klein, 1989, "Generalization of the quantized Bogoliubov-Valatin transformation," Phys. Lett. B **229**, 1.
Hahne, F. J. W., and F. G. Scholtz, 1988a, "Vibrations and rotations in a schematic proton-neutron interacting boson model," J. Phys. G **14**, 1059.
Hahne, F. J. W., and F. G. Scholtz, 1988b, "Vibrations and rotations for the proton-neutron interacting boson model," Nucl. Phys, A **483**, 23.
Halse, P., J. P. Elliott, and J. A. Evans, 1984, "An application of the interacting boson model to the first half of the *sd* shell. (I) Even-even nuclei," Nucl. Phys. A **417** 301.
Hasegawa, M., and N. Kanesaki, 1980a, "Modification of quasi-particle theory in spherical nuclei I," Prog. Theor. Phys. **63**, 469.
Hasegawa, M., and M. Kanesaki, 1980b, "Modifications of quasi-particle theory in spherical nuclei, II," Prog. Theor. Phys. **63**, 1984.
Hasegawa, M., and N. Kanesaki, 1980c, "Modification of quasi-particle theory in spherical nuclei, III," Prog. Theor. Phys. **64**, 897.
Hatsuda, T., 1989, "Mean-field theory and boson expansion at finite temperature on the basis of the thermo field dynamics," Nucl. Phys. A **492**, 187.
Hecht, K. T., 1965a, "Some simple R_5 Wigner coefficients and their application," Nucl. Phys. **63**, 177.
Hecht, K. T., 1965b, "Five-dimensional quasi-spin. The n,T dependence of shell-model matrix elements in the seniority scheme," Phys. Rev. **139**, 794B.
Hecht, K. T., 1967, "Five-dimensional quasispin. Exact solution of a pairing Hamiltonian in the J-T scheme," Nucl. Phys. A **102**, 11.
Hecht, K. T., 1985, "Coherent-state theory for the *LST* quasi-spin group," Nucl. Phys. A **444**, 189.
Hecht, K. T., 1987a, *The Vector Coherent State Method and Its Application to Problems of Higher Symmetries,* Lecture Notes in Physics Vol. 290 (Springer, New York).
Hecht, K. T., 1987b, "Vector coherent state theory for the *S,D* fermion pair algebra," Nucl. Phys. A **475**, 276.
Hecht, K. T., 1988, "Vector coherent state theory for the $SP(6)\supset U(3)$ branch of the *s-d* fermion pair algebra," Nucl. Phys. A **484**, 61.
Hecht, K. T., 1989, "Wigner coefficients for the proton-neutron quasispin group: an application of vector coherent state techniques," Nucl Phys. A **493**, 29.
Hecht, K. T., and A. Adler, 1969, "Generalized seniority for favored $J\neq 0$ pairs in mixed configurations," Nucl. Phys. A **137**, 129.
Hecht, K. T., and J. P. Elliott, 1985, "Coherent-state theory for the proton-neutron quasispin group," Nucl. Phys. A **438**, 29.
Hecht, K. T., R. Le Blanc, and D. J. Rowe, 1987, "New perspectives on the U(n) Wigner-Racah calculus: I. Vector coherent state and construction of Gel'fand bases," J. Phys. A **20**, 2241.
Hirsekorn, R.-P., 1980, "Approximate solution of the n-fermion problem, application to the ^{16}O nucleus," Z. Phys. A **297**, 141.
Hirsekorn, R.-P., 1981, "Consistent second-order approximation solution of the n-fermion problem," Nucl. Phys. A **370**, 269.
Hirsekorn, R.-P., and L. J. Weigert, 1976, "A Pauli-exclusion-principle satisfying approximation for the n-fermion problem," Phys. Lett. **65**, 12.
Hirsekorn, R.-P., and L. J. Weigert, 1977, "A new boson expansion for the solution of the n-fermion problem with an application to ^4He," Nucl. Phys. A **288**, 439.
Högaasen-Feldman, J., 1961, "A study of some approximations of the pairing force," Nucl. Phys. **28**, 258.
Holstein, T., and H. Primakoff, 1940, "Field dependence of the intrinsic domain magnetization of a ferromagnet," Phys. Rev. **58**, 1098.
Holzwarth, G., 1968, "A microscopic theory of nuclear collective motion," Nucl. Phys. A **113**, 448.
Holzwarth, G., 1969, "Microscopy aspects of the nuclear collective model, II," Nucl. Phys. A **133**, 161.
Holzwarth, G., 1970, "The transition spherical→deformed for $K=0$ states in even nuclei," Nucl. Phys. A **156**, 511.
Holzwarth, G., 1971, "On the treatment of anharmonicities in the collective spectra of intermediate nuclei," Nucl. Phys. A

174, 97.
Holzwarth, G., 1972, "The connection between the generator coordinate method and bose expansions," Nucl. Phys. A **185**, 268.
Holzwarth, G., D. Janssen, and R. V. Jolos, 1976, "On the validity of the boson method for transitional nuclei," Nucl. Phys. A **261**, 2.
Holzwarth, G., and T. Yukawa, 1974, "Choice of the constraining operator in the constrained Hartree-Fock method," Nucl. Phys. A **219**, 125.
Iachello, F., and A. Arima, 1987, *The Interacting Boson Model* (Cambridge University Press, Cambridge).
Iachello, F., and O. Scholton, 1979, "Interacting boson-fermion model of collective states in odd-A nuclei," Phys. Rev. Lett. **43**, 679.
Iachello, F., and I. Talmi, 1987, "Shell-model foundations of the interacting boson model," Rev. Mod. Phys. **59**, 339.
Ichihashi, H., and M. Yamamura, 1978, "On the classical interpretation of Schwinger boson representation for the quantized rotator," Prog. Theor. Phys. **60**, 753.
Ichimura, M., 1969, "On the quasispin formalism," in *Progress in Nuclear Physics,* Vol. 10, edited by D. M. Brink and J. H. Mulvey (Pergamon, Oxford), p. 307.
Ivanova, S. P., R. V. Jolos, and R. Pedrosa, 1989, "Approximate treatment of the Holstein-Primakoff-type boson mapping," J. Phys. G **15**, 55.
Iwasaki, M., 1981, "Note on the canonical quantization of the time-dependent Hartree-Fock equation," Prog. Theor. Phys. **65**, 2042.
Isawaki, S. O., T. Marumori, F. Sakata, and K. Takada, 1976, "Structure of the anomalous $0^{+'}$ excited states in spherical even-even nuclei with N or $Z \approx 40$," Prog. Theor. Phys. **56**, 1140.
Iwasaki, S., F. Sakata, and K. Takada, 1977, "A new method for microscopic description of the so-called 'many phonon' states in spherical even-even nuclei. I," Prog. Theor. Phys. **57**, 289.
Jammari, K., R. Piepenbring, and B. Sylvestre-Brac, 1983, "Anharmonicities in $K^{\pi}=0^{+}$ and $K^{\pi}=0^{-}$ coupled vibrational spectra of deformed nuclei discussed in a simple model," Phys. Rev. C **28**, 2136.
Jancovici, B., and D. H. Schiff, 1964, "On the collective vibrations of a many-fermion system," Nucl. Phys. **58**, 678.
Janssen, D., 1977, "Coherent states of the quantum mechanical top," Yad. Fiz. **25**, 897 [Sov. J. Nucl. Phys. **25**, 479 (1977)].
Janssen, D., F. Dönau, S. Frauendorf, and R. V. Jolos, 1971, "Boson description of collective states. (I) Derivation of the boson transformation for even-fermion systems," Nucl. Phys. A **172**, 145.
Janssen, D., R. V. Jolos, and F. Dönau, 1974, "An algebraic treatment of the nuclear quadrupole degree of freedom," Nucl. Phys. A **224**, 93.
Johnson, R. E. A. Klein, and R. M. Dreizler, 1968, "Relation between the time-dependent Hartree-Fock method and boson expansion methods in a two level-model," Ann. Phys. (N.Y.) **49**, 496.
Jolos, R. V., 1969, "Collective modes associated with nuclear pair correlation," Phys. Lett. B **30**, 390.
Jolos, R. V., 1971, "Pairing correlations and collective 0^{+} states of nuclei I," Theor. Math. Phys. **6**, 291.
Jolos, R. V., F. Dönau, V. G. Kartavenko, and D. Janssen, 1973, "Pairing correlations and collective 0^{+} states of nuclei II." Theor. Math. Phys. **40**, 70.
Kamlah, A., 1968, "An approximation for rotation-projected expectation values of the energy for deformed nuclei and a derivation of the cranking variational equation," Z. Phys. **216**, 52.
Kammuri, T., 1967, "Nuclear collective motion in the random-phase approximation," Prog. Theor. Phys. **37**, 1131.
Kammuri, T., and T. Kishimoto, 1976a, "Boson description of nuclear collective motion. I," Z. Phys. A **276**, 51.
Kammuri, T., and T. Kishimoto, 1976b, "Boson description of nuclear collective motion. II. Application to octupole vibration," Z. Phys. A **278**, 41.
Kammuri, T., and T. Kishimoto, 1978, "Boson description of nuclear collective motion. III. Coupled system of the quadrupole and octupole vibrations," Z. Phys. A **288**, 204.
Kaup, U., 1983, "On the microscopic origin of the Pauli term in the interacting boson fermion model (IBFM)," in *Collective Bands in Nuclei,* Progress in Particle and Nuclear Physics, Vol. 9, edited by D. Wilkinson (Pergamon, Oxford), p. 561.
Kaup, U., 1987, "Schwinger mapping and the interacting boson model," Phys. Lett. **185**, 249.
Kaup, U., 1988, "Boson coherent states and intrinsic Hamiltonian in a microscopic SO(8) model," Phys. Rev. Lett. **60**, 909.
Kaup, U., and P. Ring, 1987, "Supersymmetry in the interacting boson fermion model as a formal property of a Schwinger-type boson-fermion mapping," Nucl. Phys. A **462**, 455.
Kaup, U., P. Ring, and R. S. Nigam, 1988, "A microscopic boson model for collective transition nuclei: Schwinger representation of the SO(8) shell-model algebra," Nucl. Phys. A **480**, 222.
Keller, J., 1958, "Corrected Bohr-Sommerfeld quantum conditions for nonseparable systems," Ann. Phys. (N.Y.) **4**, 180.
Kerman, A., and A. Klein, 1962, "The description of rotating nuclei," Phys. Lett. **1**, 1985.
Kim, G.-K., and C. M. Vincent, 1985, "Inadequacy of four-particle tests of Dyson boson mapping," Phys. Rev. C **32**, 1776.
Kim, G.-K., and C. M. Vincent, 1987, "Deriving an IBM model from the fermion SO(8) model," Phys. Rev. C **35**, 1517.
Kim, G.-K., and C. M. Vincent, 1988, "Skeleton boson realizations of collective subalgebras," Phys. Rev. C **37**, 2176.
Kishimoto, T., and T. Tamura, 1971, "Coefficients of fractional parentage of many-boson systems," Nucl. Phys. A **163**, 100.
Kishimoto, T., and T. Tamura, 1972, "Description of nuclear collective motions in terms of the boson expansion technique (I). Formulation," Nucl. Phys. A **192**, 246.
Kishimoto, T., and T. Tamura, 1976, "Description of nuclear collective motions in terms of the boson expansion technique (II). Additional formulation and numerical calculations," Nucl. Phys. A **270**, 317.
Kishimoto, T., and T. Tamura, 1983, "Bosonization of fermion operators as linked-cluster expansion," Phys. Rev. C **27**, 341.
Kishimoto, T., T. Tamura, and T. Kammuri, 1973, "A theory for coupling between fundamental excitation modes in nuclei," Prog. Theor. Phys. Suppl. **74** and **75**, 170.
Kisslinger, L. S., and R. A. Sorensen, 1963, "Spherical nuclei with simple residual forces," Rev. Mod. Phys. **35**, 853.
Kleber, M., 1969, "On the convergence of Marumori's boson expansion for pairing vibrations," Phys. Lett. B **30**, 588.
Klein, A., 1963, "Invariant operators of the unitary unimodular group in n dimensions," J. Math. Phys. **4**, 1283.
Klein, A., 1972, "Theories of Collective Motion" in *Dynamical Structure of Nuclear States,* edited by D. J. Rowe (University of Toronto, Toronto), p. 38.
Klein, A., 1980a, "Examples of the relationship between the

shell model and the Bohr collective Hamiltonian: The multilevel extension of the Lipkin-Meshkov-Glick model," Phys. Lett. B **95**, 327.
Klein. A., 1980b, "Perspectives in the theory of nuclear collective motion," Nucl. Phys A **347**, 3.
Klein, A., 1982, "Boson mappings in nuclear physics, a brief and prejudiced survey," in *Group Theoretical Methods in Physics,* Lecture Notes in Physics Vol. 180, edited by M. Serdaroglu and E. Inonu (Springer, Berlin), p. 422.
Klein, A., 1983a, "Algebraic methods for a direct calculus of observables in the theory of nuclear band structure," in *Progress in Particle and Nuclear Physics,* Vol. 10, edited by D. Wilkinson (Pergamon, Oxford), p. 39.
Klein, A., 1983b, "Equation of motion approach to large-amplitude collective motion: Coupled bosons," Nucl. Phys. A **410**, 74.
Klein, A., 1984a, "Equations of motion approach to a quantum theory of large-amplitude collective motion," Nucl. Phys. A **431**, 90.
Klein, A., 1984b, "Revised generalized density matrix method for the study of nuclear collective motion," Phys. Rev. C **30**, 1680.
Klein, A., T. D. Cohen, and C. T. Li, 1982, "Boson mappings for schematic nuclear models with the symmetry of SO(5)," Ann. Phys. (N.Y.) **141**, 382.
Klein, A., R. M. Dreizler, and R. E. Johnson, 1968, "Concept of ideal collective coordinate as the foundation for a phenomenological theory of nuclear collective motion," Phys. Rev. **171**, 1216.
Klein, A., and C. T. Li, 1981, "WKB quantization of pseudospin Hamiltonians," Phys. Rev. Lett. **46**, 895.
Klein, A., C. T. Li, T. D. Cohen, and M. Vallières, 1983, "Phenomenological concepts of nuclear collective motion and their possible microscopic foundation," in *Collective Bands in Nuclei,* Progress in Particle and Nuclear Physics, Vol. 9, edited by D. Wilkinson (Pergamon, Oxford), p. 183.
Klein, A., C. T. Li, and M. Vallières, 1981, "Applications of generalized Holstein-Primakoff transformations to problems of nuclear collective motion," *Recent Progress in Many-Body Theories,* in Lecture Notes in Physics, No. 142 (Springer, Berlin/New York), p. 76.
Klein, A., C. T. Li, and M. Vallières, 1982a, "Relationship between the interacting boson model and the Bohr collective Hamiltonian," Phys. Scr. **25**, 452.
Klein, A., C. T. Li, and M. Vallières, 1982b, "Relationship between the Bohr-Mottelson model and the interacting boson model," Phys. Rev. C **25**, 2733.
Klein, A., C. T. Li, and M. F. Vassanji, 1980, "Variational principles for particles and fields in Heisenberg matrix mechanics," J. Math. Phys. **21**, 2521.
Klein, A., and E. R. Marshalek, 1988, "A new derivation of the Marshalek-Okubo realization of the shell-model algebra $SO(2\nu+1)$ for even and odd systems with ν single-particle levels," Z. Phys. A **329**, 441.
Klein, A., and E. R. Marshalek, 1989, "On the boson-quasifermion realization of the particle-hole $SO(2\Omega+1)$ algebra," J. Math. Phys. **30**, 219.
Klein, A., T. Marumori, and T. Une, 1982, "Formally exact quantum variational principles for collective motion based on the invariance principle of the Schrödinger equation," Phys. Lett. B **109**, 237.
Klein, A., T. Marumori, and T. Une, 1984, "Formally exact quantum variational principles for collective motion based on the invariance principle of the Schrödinger equation," Phys. Rev. C **29**, 240.
Klein, A., H. Rafelski, and J. Rafelski, 1981, "IBM Hamiltonian and classical limit for an exactly soluble model with the symmetry of O(5)," Nucl. Phys. A **355**, 189.
Klein, A., and K. Tanabe, 1984, "Variational principles for collective motion: Relation between invariance principle of the Schrödinger equation and the trace variational principle," Phys. Lett. B **135**, 255.
Klein, A., and M. Vallières, 1980, "Relationship between the Bohr collective Hamiltonian and the interacting-boson model," Phys. Rev. Lett. **46**, 586.
Klein, A., and M. Vallières, 1981, "New method for studying the microscopic foundations of the interacting boson model," Phys Lett. B **98**, 5.
Klein, A., and M. Vallières, 1982, "Use of a boson mapping to elucidate the relationship between the IBM and the Bohr collective Hamiltonian and between a simplified shell model and the Bohr collective Hamiltonian," in *Contemporary Research Topics in Nuclear Physics,* edited by D. H. Feng, M. Vallières, M. W. Guidry, and L. L. Riedinger (Plenum, New York), p. 487.
Klein, A., and N. R. Walet, 1990, "Generalization of the quantized Bogoliubov-Valatin transformation and relationship to the method of the vector coherent state: The case of U(3)," Nucl. Phys. A **515**, 207.
Klein, A., and Q.-Y. Zhang, 1986, "Closed, analytic, boson realizations for Sp(4)," J. Math. Phys. **27**, 1987.
Kleinert, H., 1977, "Field theory of collective excitations," Phys. Lett. B **69**, 9.
Krumlinde, J., and E. R. Marshalek, 1973, "A solvable model for describing quasirotational bands in transitional nuclei," Lett. Nuovo Cimento **7**, 679.
Krumlinde, J., and Z. Szymanski, 1971, "A simple model for nuclear rotation at high angular momenta," Phys. Lett. B **36**, 157.
Krumlinde, J., and Z. Szymanski, 1973, "Discussion of the structure of nuclear rotation at high angular momentum" Ann. Phys. (N.Y.) **79**, 201.
Kubo, R., 1957, "Statistical mechanical theory of irreversible processes. I. General theory and simple applications to magnetic and conduction problems," J. Phys. Soc. Jpn. **12**, 570.
Kuchta, R., 1988a, "Electron scattering from ^{20}Ne and ^{24}Mg in a microscopic boson model," Nucl. Phys. A **483**, 92.
Kuchta, R., 1988b, "Microscopic boson description of proton-proton systems: Application to elastic and inelastic electron scattering from ^{18}O and ^{20}Ne," Phys. Rev. C **38**, 1460.
Kuchta, R., 1988c, "Nuclear structure of ^{16}O in a mean-field boson approach," Phys. Lett. B **212**, 264.
Kuchta, R., 1988d, "Microscopic mean-field boson apporoach to the shape transition in Sm isotopes," Nucl. Phys. A **489**, 1.
Kuchta, R., 1988e, "Microscopic foundations of the interacting boson model in the *sd*-shell nuclei," Z. Phys. A **331**, 243.
Kumar, K., 1983, "Dynamic deformation model," in *Collective Bands in Nuclei,* Progress in Particle and Nuclear Physics Vol. 9, edited D. Wilkinson (Pergamon, Oxford), p. 233.
Kumar, K., 1984, *Nuclear Models and the Search for Unity in Nuclear Physics* (Universitetsforlaget, Oslo).
Kumar, K., and M. Baranger, 1967, "Complete numerical solution of Bohr's collective Hamiltonian," Nucl. Phys. A **92**, 608.
Kumar, K., and M. Baranger, 1968, "Nuclear deformation in the pairing-plus-quadrupole model. (V). Energy levels and electromagnetic moment of the W, Os, and Pt nuclei," Nucl. Phys. A **122**, 273.
Kuriyama, A., and M. Yamamura, 1981a, "A note on

specification of collective path," Prog. Theor. Phys. **65**, 759.
Kuriyama, A., and M. Yamamura, 1981b, "Utility of Dirac quantization of classical system involving both collective and independent-particle degrees of freedom," Prog. Theor. Phys. **65**, 1094.
Kuriyama, A., and M. Yamamura, 1981c, "Boson expansion for many-fermion system as a canonical theory with constraints. I," Prog. Theor. Phys. **66**, 2130.
Kuriyama, A., and M. Yamamura, 1981d, "Boson expansion for many-fermion system as a canonical theory with constraints. II", Prog. Theor. Phys. **66**, 2147.
Kuriyama, A., and M. Yamamura, 1981e, "A microscopic theory of collective and independent-particle motions," Prog. Theor. Phys. **65**, 550.
Kuriyama, A., and M. Yamamura, 1984a, "Canonical formulation of time-dependent Hartree-Fock method," Prog. Theor. Phys. **71**, 122.
Kuriyama, A., and M. Yamamura, 1984b, "Canonical coordinate system suitable for adiabatic treatment of collective motion," Prog. Theor. Phys. **71**, 973.
Kyrchev, G., and V. Paar, 1986, "Deductive inference of nuclear boson models (IBM, TQM, FQM) based on SU(6) dynamical symmetry," Ann. Phys. (N.Y.) **170**, 257.
Kyrchev, G., and V. Paar, 1987, "IBFM form of SU(6/M) supergenerators for microscopic foundations," Phys. Lett. B **195**, 107.
Kyrchev, G., and V. Paar, 1988, "Microscopic approach to enforced SU(6) symmetry in random-phase approximation," Phys. Rev. C **37**, 838.
Lawson, R. D., and M. H. MacFarlane, 1965, "The quasi-spin formalism and the dependence of nuclear matrix elements on particle number," Nucl. Phys. **66**, 80.
Leacock, R. A., and M. Padget, 1987, "Quantum action-angle variable analysis of boson systems," Am. J. Phys. **55**, 261.
Le Blanc, R., and K. T. Hecht, 1987, "New perspective on the U(n) Wigner-Racah calculus: II. Elementary reduced Wigner coefficients for U(n)," J. Phys. A **20**, 4613.
Le Blanc, R., and D. J. Rowe, 1987, "Heisenberg-Weyl algebras of symmetric and antisymmetric bosons," J. Phys. A **20**, L681.
Leschber, Y., and J. P. Draayer, 1986, "Microscopic limit of the microscopic SU(3)⊃SO(3) integrity basis interaction," Phys. Rev. C **33**, 749.
Leschber, Y., and J. P. Draayer, 1987, "Algebraic realization of rotational dynamics," Phys. Lett. B **190**, 1.
Li, C. T., 1983, "New derivation of the seniority scheme boson mapping of the shell-model algebra," Phys. Lett. B **120**, 251.
Li, C. T., 1984a, "Approximate projection of physical states in the Dyson boson description of nuclear collective motion," Nucl. Phys. A **417**, 37.
Li, C. T., 1984b, "Approximate projection of physical states in the Dyson boson description of nuclear collective motion," Phys. Rev. C **29**, 2309.
Li, C. T., 1991, "Optimal auxiliary Hamiltonians for truncated boson calculations by means of a maximal-decoupling variational principle," preprint, Taiwan National University.
Li, C. T., V. G. Pedrocchi, and T. Tamura, 1985a, "Application of the Kishimoto-Tamura boson expansion to a single-j shell model," Phys. Rev. C **32**, 1745.
Li, C. T., V. G. Pedrocchi, and T. Tamura, 1986, "Application of a number-conserving boson expansion theory to Ginocchio's SO(8) model," Phys. Rev. C **33**, 1762.
Li, S. Y., R. M. Dreizler, and A. Klein, 1970, "Effect of the Pauli principle on the representation of nucleon pair operators by means of bosons," Phys. Lett. B **32**, 169.
Li, S. Y., R. M. Dreizler, and A. Klein, 1971, "Boson expansions for fermion pair operators: The single j level," Phys. Rev. C **4**, 1571.
Li, S. Y., and A. Klein, 1971, "Application of the Marumori boson expansion to the problem of particle-hole excitation in closed-shell nuclei," Phys. Rev. C **3**, 1871.
Li, S. Y., A. Klein, and R. M. Dreizler, 1970, "Boson expansions for an exactly soluble model of interacting fermions with SU(3) symmetry," J. Math. Phys. **11**, 975.
Lie, S. G., and G. Holzwarth, 1975, "Application of the boson expansion method to even Se and Ru isotopes," Phys. Rev. C **12**, 1035.
Lipkin, H. J., N. Meshkov, and A. Glick, 1965, "Validity of many-body approximation methods for a soluble model," Nucl. Phys. **62**, 188.
Littlewood, D. E., 1950, *The Group Theory of Characters* (Clarendon, Oxford).
Louck, J. D., and L. C. Biedenharn, 1970, "Canonical unit adjoint tensors in U(n)," J. Math. Phys. **11**, 2368.
Lü, Z.-M., X.-W. Pan, J.-Q. Chen, X.-G. Chen, and D. H. Feng, 1988, "Correspondence between the fermion dynamical symmetry model and the interacting boson model in wave functions," Phys. Rev. C **37**, 2789.
Maglione, E., F. Catara, A. Insolia, and A. Vitturi, 1983, "Microscopic structure of monopole and quadrupole bosons," Nucl. Phys. A **397**, 102.
Maglione, E., and A. Vitturi, 1984, "On the boson mapping of fermion collective pairs," Nucl. Phys. A **430**, 158.
Maglione, E., A. Vitturi, F. Catara, and A. Insolia, 1983, "Microscopic description of β-band in the collective pair approximation," Nucl. Phys. A **411**, 181.
Maleev, S. V., 1958, "Scattering of slow neutrons in ferromagnets," Zh. Eksp. Teor. Fiz. **33**, 1010 [Sov. Phys. JETP **6**, 776 (1958)].
Marshalek, E. R., 1971a, "On the relation between Belyaev-Zelevinsky and Marumori boson expansions," Nucl. Phys. A **161**, 401.
Marshalek, E. R., 1971b, "Microscopic description of quasirotational bands," Phys. Rev. C **3**, 1710.
Marshalek, E. R., 1973, "A new boson expansion for odd-particle systems," Phys. Lett. B **44**, 5.
Marshalek, E. R., 1974a, "Perturbative boson expansions to all orders for even and odd nuclei, I. Formal properties," Nucl. Phys. A **224**, 221.
Marshalek, E. R., 1974b, "Perturbative boson expansions to all orders for even and odd nuclei, II. Illustrative examples," Nucl. Phys. A **232**, 24.
Marshalek, E. R., 1975a, "Boson expansions and the cranking model," in *Problems of Vibrational Nuclei,* edited by G. Alaga, V. Paar, and L. Sips (North-Holland, Amsterdam), p. 108.
Marshalek, E. R., 1975b, "Boson representation for the quantized oscillator," Phys. Rev. C **11**, 1426.
Marshalek, E. R., 1976b, "Large-amplitude collective motion in the Lipkin model," Phys. Rev. C **14**, 342.
Marshalek, E. R., 1977, "The RPA at high spin and conservation laws," Nucl. Phys. A **275**, 416.
Marshalek E. R., 1980a, "How microscopic boson models work," Nucl. Phys. A **347**, 253.
Marshalek, E. R., 1980b, "Chimerical boson expansions," Phys. Lett. B **95**, 337.
Marshalek, E. R., 1981, "Microscopic particle-vibrator model. A new fermion boson expansion," Nucl. Phys. A **357**, 398.
Marshalek, E. R., 1982a, "Boson expansions and broken symmetry," Ann. Phys. (N.Y.) **143**, 191.

Marshalek, E. R., 1982b, "Approximate solution of the particle-rotor model for decoupled bands," Phys. Rev. C **26**, 1678.
Marshalek, E. R., 1983, "Boson-fermion expansions," Prog. Theor. Phys. Suppl. **74** and **75**, 89.
Marshalek, E. R., 1984, "Constant-volume constraint and many-body forces in the Nilsson model," Phys. Rev. C **29**, 640.
Marshalek, E. R., 1987a, "Reconstituting rotational band structure from deformed boson expansions: Generic treatment," Phys. Rev. C **35**, 1900.
Marshalek, E. R., 1987b, "Reconstituting rotational band structure from deformed boson expansions: II. Model test," Phys. Rev. C **36**, 2538.
Marshalek, E. R., 1988, "Finite Hermitian alternatives to the Dyson Hamiltonian," Phys. Rev. C **38**, 2961.
Marshalek, E. R., and J. da Providência, 1973a, "Belyaev-Zelevinsky-type expansion about the Hartree-Fock minimum," II Nuovo Cimento B **14**, 32.
Marshalek, E. R., and J. da Providência, 1973b, "Sum rules, random-phase approximation, and constrained self-consistent fields," Phys. Rev. C **7**, 2281.
Marshalek, E. R., and G. Holzwarth, 1972, "Boson expansions and Hartree-Bogoliubov theory," Nucl. Phys. A **191**, 438.
Marshalek, E. R., and M. Sabato, 1971, "Static quadrupole moments of vibrational states of spherical nuclei in the self-consistent cranking approximation," Phys. Rev. C **4**, 1006.
Marshalek, E. R., and M. Sabato, 1972, "Problems with perturbation treatments of anharmonic nuclear oscillations," Phys. Rev. C **5**, 1130.
Marshalek, E. R., and J. Weneser, 1969, "Nuclear rotation and the random-phase approximation," Ann. Phys. (N.Y.) **53**, 568.
Marshalek, E. R., and J. Weneser, 1970, "Nuclear rotation and boson expansions, I," Phys. Rev. C **2**, 1682.
Martin, P. C., and J. Schwinger, 1959, "Theory of many-particle systems," Phys. Rev. **115**, 1342.
Marumori, T., 1960, "On collective motion in even-even spherical nuclei," Prog. Theor. Phys. **24**, 331.
Marumori, T., 1977, "A microscopic theory of large-amplitude collective motion," Prog. Theor. Phys. **57**, 112.
Marumori, T., A. Hayashi, T. Tomoda, A. Kuriyama, and T. Maskawa, 1980, "Concept of a collective subspace associated with the invariance principle of the Schrödinger equation," Prog. Theor. Phys. **63**, 1576.
Marumori, T., T. Maskawa, F. Sakata, and A. Kuriyama, 1980, "Self-consistent collective-coordinate method for the large-amplitude nuclear collective motion," Prog. Theor. Phys. **64**, 1294.
Marumori, T., K. Takada, and F. Sakata, 1981, "Present status of the microscopic study of low-lying collective states in spherical and transitional nuclei," Prog. Theor. Phys. Suppl. **71**, Chap. 1.
Marumori, T., M. Yamamura, and A. Tokunaga, 1964, "On the 'anharmonic effects' on the collective oscillations in spherical nuclei. I," Prog. Theor. Phys. **31**, 1009.
Marumori, T., M. Yamamura, A. Tokunaga, and K. Takada, 1964, "On the 'anharmonic effects' on the collective oscillations in spherical nuclei, II," Prog. Theor. Phys. **32**, 726.
Matsuo, M., and K. Matsuyanagi, 1985, "Microscopic description of anharmonic gamma-vibrations by means of the self-consistent coordinate method. I," Prog. Theor. Phys. **74**, 1227.
Matsuo, M., and K. Matsuyanagi, 1986, "Microscopic description of anharmonic gamma-vibrations by means of the self-consistent coordinate method. II," Prog. Theor. Phys. **76**, 93.
Matsuo, M., and K. Matsuyanagi, 1987, "Microscopic description of anharmonic gamma-vibrations by means of the self-consistent coordinate method. III," Prog. Theor. Phys. **78**, 591.
Matsuyanagi, K., 1982, "A mode-mode coupling theory of nuclear anharmonicities," in *Nuclear Physics,* edited by C. H. Dasso, R. A. Broglia, and A. Winther (North-Holland, Amsterdam), p. 29.
Meshkov, N., 1971, "Mixing of collective states in an exactly soluble three-level model," Phys. Rev. C **3**, 2214.
Meyer-ter-Vehn, J., 1979, "Exact removal of spurious states in RPA calculations," Z. Phys. A **289**, 319.
Migdal, A. B., 1967, *Theory of Finite Fermi Systems and Applications to Atomic Nuclei* (Wiley, Interscience, New York).
Mihailov, V. M., 1971, "Collective interpretation of conservation laws and the description of the rotation of deformed nuclei," Izv. Akad. Nauk. SSSR Ser. Fiz. **35**, 794.
Moshinsky, M., and T. H. Seligman, 1978, "Canonical transformations to action and angle variables and their representations in quantum mechanics," Ann. Phys. (N.Y.) **114**, 243.
Moshinsky, M., and T. H. Seligman, 1979, "Canonical transformations to action and angle variables and their representations in quantum mechanics. II. The Coulomb problem," Ann. Phys. (N.Y.) **120**, 402.
Moszkowski, S. A., 1958, "Connection between nuclear shell model and unified model," Phys. Rev. **110**, 403.
Navrátil, P., and J. Dobeš, 1989, "Model calculations using boson mappings," Phys. Rev. C **40**, 2371.
Navrátil, P., and J. Dobeš, 1990, "Microscopic seniority-boson quadrupole operator from the similarity transformed Dyson boson operator," Nucl. Phys. A **507**, 340.
Newton, R., 1980, "Quantum action-angle variables for harmonic oscillators," Ann. Phys. (N.Y.) **124**, 327.
Nilsson, S. G., 1955, "Binding states of individual nucleons in strongly deformed nuclei," K. Dan. Vidensk. Selsk. Mat.-Fys. Medd. **29**, No. 16.
Nodvik, J. S., 1969, "Generalized Schrödinger representations," Ann. Phys. (N.Y.) **51**, 251.
Novoselsky, A., M. Vallières, and R. Gilmore, 1988, "Matrix elements of shell-model Hamiltonians in multiple angular momentum coupling schemes," Phys. Rev. C **38**, 1440.
Okubo, S., 1974a, "Remark on the transition-operator boson," Phys. Rev. C **9**, 1188.
Okubo, S., 1974b, "Nonassociative algebra in nuclear boson-expansion theory," Phys. Rev. C **10**, 2045.
Okubo, S., 1974c, "General boson expansion theory and Lie algebras B_n and D_n," Phys. Rev. C **10**, 2048.
Okubo, S., 1975, "Symmetric representations of the U(n) group and exactly solvable nuclear model," J. Math. Phys. **16**, 528.
Oliveira, C. R. de, and C. P. Malta, 1984, "Action-angle variables for the harmonic oscillator: Ambiguity spin × duplication spin," Ann. Phys. (N.Y.) **155**, 447.
Otsuka, T., 1981, "Microscopic basis of the proton-neutron interacting boson model," Phys. Rev. Lett. **46**, 710.
Otsuka, T., 1984, "Independent-pair property of condensed coherent fermion pairs and derivation of the IBM quadrupole operator," Phys. Lett. B **138**, 1.
Otsuka, T., A. Arima, and F. Iachello, 1978, "Nuclear shell model and interacting bosons," Nucl. Phys. A **309**, 1.
Otsuka, T., A. Arima, F. Iachello, and I. Talmi, 1978, "Shell-model description of interacting bosons," Phys. Lett. B **76**, 139.
Otsuka, T., A. Arima, and N. Yoshida, 1982, "Dominance of monopole and quadrupole pairs in the Nilsson model," Phys.

Rev. Lett. **48**, 387.
Otsuka, T., and J. N. Ginocchio, 1985, "Renormalization of g-boson effects in the interacting-boson model," Phys. Rev. Lett. **55**, 276.
Otsuka, T., N. Yoshida, P. van Isacker, A. Arima, and O. Scholten, 1987, "Shell-model evidence for the exchange term in the interacting boson-fermion Hamiltonian," Phys. Rev. C **35**, 328.
Otsuka, T., and N. Yoshinaga, 1986, "Fermion-boson mapping for deformed nuclei," Phys. Lett. B **168**, 1.
Pang, S. C., A. Klein, and R. M. Dreizler, 1968, "Study of boson expansion methods in an exactly soluble two-level shell model," Ann. Phys. (N.Y.) **49**, 477.
Pannert, W., and P. Ring, 1987, "Collective motion of fermions and boson expansions (I)," Nucl. Phys. A **465**, 379.
Pannert, W. P., P. Ring, and Y. K. Gambhir, 1985, "An analysis of angular momentum-projected Hartree-Fock-Bogoliubov wave functions in terms of interacting bosons," Nucl. Phys. A **443**, 189.
Park, P., 1987, "Algebraic analysis of physical and spurious states in Dyson boson mapping," Phys. Rev. C **35**, 807.
Pedrocchi, V. G., and T. Tamura, 1982, "Hermitian and non-Hermitian boson expansions based on the commutator method," Prog. Theor. Phys. **68**, 820.
Pedrocchi, V. G., and T. Tamura, 1983, "Boson expansion based on the extended commutator method in the Tamm-Dancoff representtaion," Phys. Rev. C **28**, 410.
Peierls, R. E., and J. Yoccoz, 1957, "The collective model of nuclear rotation," Proc. Phys. Soc. London, Ser. A **70**, 381.
Peiersl, R. E., and D. J. Thouless, 1962, "Variational approach to collective motion," Nucl. Phys. **38**, 154.
Perelomov, A. M., 1972, "Coherent states for arbitrary Lie groups," Commun. Math. Phys. **26**, 222.
Perelomov, A. M., 1977, "Generalized coherent states and some of their applications," Usp. Fiz. Nauk. **123**, 23 [Sov. Phys. Usp. **20**, 703].
Perelomov, A. M., 1986, *Generalized Coherent States and Their Applications* (Springer, New York).
Perelomov, A. M., and V. S. Popov, 1966, "Casimir opreators for U(n) and SU(n)," Yad. Fiz. **3**, 924 [Sov. J. Nucl. Phys. **3**, 676 (1966)].
Peres Menezes, D., D. M. Brink, and D. Bonatsos, 1989, "Exact boson mappings for the proton-neutron nuclear p-shell with the symmetry Sp(12)$\supset$U(6)," Nucl. Phys. A **494**, 186.
Peres Menezes, D., and D. Bonatsos, 1989, "Exact boson mappings for the nuclear single $j=\frac{3}{2}$ shell of protons and neutrons with the symmetry SO(16)$\supset$U(8)," Nucl. Phys. A **499**, 29.
Peres Menezes, D., N. Yoshinaga, and D. Bonatsos, 1989, "Application of the commutator method for boson mapping," Phys. Lett. B **221**, 103.
Piepenbring, R. B., 1983, "Application of the multiphonon method to the description of octupole vibrations in even deformed nuclei with $222 \leq A \leq 228$," Phys. Rev. C **27**, 2968.
Piepenbring, R., B. Silvestre-Brac, and Z. Szymánski, 1980, "Anharmonicities in vibrational spectra of deformed nuclei discussed in a simple model. I," Nucl. Phys. A **349**, 77.
Piepenbring, R., B. Silvestre-Brac, and Z. Szymański, 1982, "Anharmonicities in vibrational spectra of deformed nuclei discussed in a simple model. II. Approximate treatment for the projection on a good particle number," Nucl. Phys. A **378**, 77.
Pio, R. L., 1966, "Euler angle transformations," IEEE Trans. Autom. Control **AC-11**, 707.
Pittel, S., and J. Dukelsky, 1983, "The structure of S and D pairs of the interacting boson model," Phys. Lett. B **128**, 9.
Pittel, S., P. D. Duval, and B. R. Barrett, 1982a, "Particle-hole ambiguity in the parameters of the microscopic interacting boson model," Phys. Rev. C **25**, 2834.
Pittel, S., P. D. Duval, and B. R. Barrett, 1982b, "The microscopic interacting boson model for nondegenerate orbits," Ann. Phys. (N.Y.) **168**, 144.
Quesne, C., 1986, "Coherent states of the real symplectic group in a complex analytic representation," J. Math. Phys. **27**, 428.
Rădută, A. A., 1987, "The description of the quadrupole collective motion of a proton-neutron interacting system within a generalized coherent state model," in *Symmetries and Semiclassical Features of Nuclear Dynamics,* edited by A. A. Rădută (Springer, New York), p. 255.
Rădută, A. A., V. Ceausescu, A. Gheorghe, and M. S. Popa, 1984, "Semiclassical treatment of the interaction between individual and quadrupole collective degrees of freedom," Nucl. Phys. A **427**, 1.
Ratna Raju, R. D., J. P. Draayer, and K. T. Hecht, 1973, "Search for a coupling scheme in heavy deformed nuclei: The pseudo SU(3) model," Nucl. Phys. A **202**, 433.
Reinhardt, H., 1975, "On the foundation of nuclear field theory," Nucl. Phys. A **251**, 317.
Reinhardt, H., 1977, "Nuclear Field Theory," Nucl. Phys. A **298**, 77.
Reinhard, P.-G., and K. Goeke, 1987, "The generator coordinate method and quantized collective motion in nuclear systems," Rep. Prog. Phys. **50**, 1.
Ring, P., L. M. Robledo, J. L. Egido, and M. Fales, 1984, "Microscopic theory of the isovector dipole resonance at high angular momenta," Nucl. Phys. A **429**, 261.
Ring, P., and P. Schuck, 1974, "A numerical comparison of three different approaches to the weak or intermediate coupling model for 2-particle–1-hole states," Z. Phys. **269**, 323.
Ring, P., and P. Schuck, 1977, "Mode coupling and Dyson's finite boson expansion in nuclei," Phys. Rev. C **16**, 801.
Ring, P., and P. Schuck, 1980, *The Nuclear Many-Body Problem* (Springer, New York).
Rowe, D. J., 1970, *Nuclear Collective Motion* (Methuen and Co., London), Chap. 13.
Rowe, D. J., 1984, "Coherent state theory of the noncompact symplectic group," J. Math. Phys. **25**, 2662.
Rowe, D. J., 1985, "Microscopic theory of the nuclear collective model," Rep. Prog. Phys. **48**, No. 10.
Rowe, D. J., and R. Basserman, 1976, "Coherent state theory of large-amplitude collective motion," Can. J. Phys. **54**, 1941.
Rowe, D. J., and J. Carvalho, 1986, "Boson expansions of Lie algebras: The fermion pair algebra," Phys. Lett. B **175**, 243.
Rowe, D. J., and F. Iachello, 1983, "Group theoretical models of giant resonance splittings in deformed nuclei," Phys. Lett. B **130**, 231.
Rowe, D. J., R. Le Blanc, and K. T. Hecht, 1988, "Vector coherent state theory and its application to the orthogonal groups," J. Math. Phys. **29**, 287.
Rowe, D. J., G. Rosensteel, and R. Carr, 1984, "Analytical expressions for the matrix elements of the non-compact symplectic algebra," J. Phys. A **17**, L399.
Rowe, D. J., G. Rosensteel, and R. Gilmore, 1985, "Vector coherent state representation theory," J. Math. Phys. **26**, 2787.
Ruggeri, G. J., 1988, "Dirac bracket revisited," J. Math. Phys. **29**, 362.
Sage, K. A., P. R. Goode, and B. R. Barrett, 1982, "Microscopic calculation of the effects of the g boson on the interacting boson model Hamiltonian," Phys. Rev. C **26**, 668.

Sakamoto, H., and T. Kishimoto, 1988, "Microscopic analysis of nuclear collective motion in terms of the boson expansion (I)," Nucl. Phys. A **486**, 1.
Sakata, F., S. Iwasaki, T. Marumori, and K. Takada, 1978, "Structure of excited states in Se-Isotopes," Z. Phys. A **286**, 195.
Sambataro, M., 1986, "Boson images of fermion operators," Phys. Rev. Lett. **57**, 1554.
Sambataro, M., 1987, "Boson representations of fermion systems," Phys. Rev. C **35**, 1530.
Sambataro, M., 1988a, "Microscopic derivation of boson-fermion Hamiltonians," Phys. Rev. C **37**, 2186.
Sambataro, M., 1988b, "Boson representations of fermion systems: Proton-neutron systems," Phys. Rev. C **37**, 2199.
Sambataro, M., and A. Insolia, 1986, "On the collective pair approximation for low-energy nuclear spectra," Phys. Lett. B **166**, 259.
Sambataro, M., H. Schasser, and D. M. Brink, 1986, "Microscopic calculation of IBM parameters by using intrinsic states," Phys. Lett. B **167**, 145.
Sawada, K., 1957, "Correlation energy of an electron gas at high density," Phys. Rev. **106**, 372.
Scholten, O., 1983, "Microscopic calculations for the interacting boson model," Phys. Rev. C **28**, 1783.
Scholten, O., 1985, "Single particle degrees of freedom in the interacting boson model," in *Progress in Particle and Nuclear Physics,* Vol. 14, edited by A. Faessler (Pergamon, Oxford), p. 189.
Scholten, O., and A. E. Dieperink, 1981, in *Interacting Bose-Fermion Systems in Nuclei,* edited by F. Iachello (Plenum, New York), p. 343.
Scholtz, F. G., and H. B. Geyer, 1988, "A generalized Schwinger boson mapping with a physical subspace," Phys. Lett. B **201**, 403.
Scholtz, F. G., and F. J. W. Hahne, 1987, "Geometrical representation of interacting boson models I. U(3) model," Phys. Rev. C **36**, 2076.
Schuck, P., 1976, "Mode coupling theory for the description of two-particle–two-hole states of ^{208}Pb," Lett. Nuovo Cimento **17**, 107.
Schuck, P., R. Wittman, and P. Ring, 1976, "The nuclei ^{212}Pb, ^{212}Bi, ^{212}Po, ^{212}At, ^{212}Rn in a microscopic weak-coupling approach," Z. Phys. **279**, 31.
Schütte, D., and K. Bleuler, 1968, "Pairing and deformation in a solvable model and the validity of different approximation methods," Nucl. Phys. A **119**, 221.
Schwinger, J., 1965, "On angular momentum," in *Quantum Theory of Angular Momentum,* edited by L. Biedenharn and H. Van Dam (Academic, New York), p. 229.
Sheikh, J. A., 1988, "Fermion calculations in the boson space using the Dyson boson mapping," Phys. Rev. C **37**, 1295.
Shimizu, Y. R., and K. Takada, 1987, "Nonunitary realization of the collective coordinate method," Prog. Theor. Phys. **77**, 1192.
Silvestre-Brac, B., and R. Piepenbring, 1977a, "Multiphonon $K^\pi=0^+$ in even-even deformed nuclei. I. Calculation of the norms," Phys. Rev. C **16**, 1638.
Silvestre-Brac, B., and R. Piepenbring, 1977b, "Real and complex boson expansions in even-even deformed nuclei," Nucl. Phys. A **288**, 408.
Silvestre-Brac, B., and R. Piepenbring, 1978, "Multiphonon $K^\pi=0^+$ in even-even deformed nuclei. II. Calculation of matrix elements of a general Hamiltonian," Phys. Rev. C **17**, 364.
Silvestre-Brac, B., and R. Piepenbring, 1979, "Multiphonon $K^\pi=0^+$ states in even-even deformed nuclei. III. Comparison with boson expansions in a realistic case," Phys. Rev. C **20**, 1161.
Silvestre-Brac, B., and R. Piepenbring, 1982, "Multiphonon theory: Generalized Wick's theorem and recursion formulas," Phys. Rev. C **26**, 2640.
Simard, P. A., 1967, "Boson approximation and antisymmetry," Can. J. Phys. **45**, 3241.
Simard, P. A., 1969a, "Transcription de la quasi-particle dans l'espace idéal et la méthode des bosons," Can. J. Phys. **47**, 103.
Simard, P. A., 1969b, "Construction d'une base antisymétrique dans l'espace idéal," Can. J. Phys. **47**, 2645.
Simard, P. A., and M. Banville, 1971, "Boson method not violating the Pauli principle applied to the three-quasi-particle system," Can. J. Phys. **49**, 3122.
Simard, P. A., and M. Banville, 1972, "Nuclear structure in ideal space," Phys. Lett. B **38**, 461.
Sørensen, B., 1966a, "Phenomenological description of anharmonic vibrations," Phys. Lett. **23**, 274.
Sørensen, B., 1966b, "Boson description of fermion systems," Phys. Lett. **23**, 274.
Sørensen, B., 1967, "On the description of fermion systems in boson representations," Nucl. Phys. A **97**, 1.
Sørensen, B., 1968a, "Self-consistent treatment of collective vibrations in terms of boson expansions," Prog. Theor. Phys. **39**, 1468.
Sørensen, B., 1968b, "On the description of fermion systems in boson representation. II," Nucl. Phys. A **119**, 65.
Sørensen, B., 1969, "Neutron pairing states in doubly even nuclei," Nucl. Phys. A **134**, 1.
Sørensen, B., 1970a, "On the description of fermion states in boson representation. III," Nucl. Phys. A **142**, 392.
Sørensen, B., 1970b, "On the description of fermion states in boson representation. IV," Nucl. Phys. A **142**, 411.
Sørensen, B., 1971, "Can phenomenological particle-particle and particle-hole forces be distinguished?" Phys. Lett. B **35**, 8.
Sørensen, B., 1973, "On the description of fermion systems in boson representation. V. A. boson variational approach to collective motion," Nucl. Phys. A **217**, 505.
Speth, J., E. Werner, and W. Wild, 1977, "Theory of finite Fermi systems and applications to the lead region," Phys. Rep. **33**, 127.
Sugawara-Tanabe, K., and K. Tanabe, 1973, "The Yrast states in the asymmetric rotor model," Nucl. Phys. A **208**, 317.
Sugawara-Tanabe, K., and K. Tanabe, 1986, "Thermal and rotational effect of giant dipole resonances in rotating nuclei at high temperature," Prog. Theor. Phys. **76**, 1272.
Sugita, M., K. Sugawara-Tanabe, and A. Arima, 1984, "Dyson boson mapping in *S-D* subspace," Phys. Lett. B **148**, 8.
Suzuki, M., 1985, "Thermo field dynamics of quantum spin systems," J. Stat. Phys. **42**, 1047.
Suzuki, T., "New giant resonances," 1973, Nucl. Phys. A **217**, 182.
Suzuki, T., M. Fuyuki, and K. Matsuyanagi, 1979a, "Dynamical interplay of pairing and quadrupole modes in transitional nuclei. I," Prog. Theor. Phys. **61**, 1682.
Suzuki, T., M. Fuyuki, and K. Matsuyanagi, 1979b, "Dynamical interplay of pairing and quadrupole modes in transitional nuclei. II," Prog. Theor. Phys. **62**, 690.
Suzuki, T. M. Fuyuki, and K. Matsuyanagi, 1981, "Dynamical interplay of pairing and quadrupole modes in transitional nuclei. III," Prog. Theor. Phys. **65**, 1667.
Suzuki, T., and K. Matsuyanagi, 1976, "Interplay of pairing and intrinsic modes of excitation in nuclei. I," Prog. Theor. Phys.

56, 1156.
Suzuki, T., and K. Matsuyanagi, 1979, "Property of many-phonon norm matrix," Prog. Theor. Phys. **61**, 1082.
Takada, K., 1984, "Quasi-particle shell-model calculation within multi-phonon subspace and SU(6) boson model," Nucl. Phys. A **431**, 16.
Takada, K., 1985, "Dyson-type boson mapping and SU(6) boson model," Nucl. Phys. A **439**, 489.
Takada, K., 1986, "Hermitian treatment of Dyson boson theory," Phys. Rev. C **34**, 750.
Takada, K., 1988, "Hermitian treatment of Dyson boson theory, revisited," Phys. Rev. C **38**, 2450.
Takada, K., K. Kaneko, F. Sakata, and S. Tazaki, 1981 "A new microscopic method for describing the elementary modes of excitation in the intrinsic subspace," Prog. Theor. Phys. Suppl. **71**, Chap. 3.
Takada, K., and Y. R. Shimizu, 1987, "Biunitary representation of the self-consistent collective coordinate method for large-amplitude collective motion," Nucl. Phys. A **468**, 11.
Takada, K., Y. R. Shimizu, and H. Thorn, 1988, "Application of the Dyson-type non-unitary representation of the self-consistent method to simple models. (I). SU(3) model," Nucl. Phys. A **485**, 189.
Takada, K., and S. Tazaki, 1986, "Application of Dyson boson mapping to the analysis of mode-mode coupling in Ge and Se isotopes," Nucl. Phys. A **448**, 56.
Takada, K., and K. Yamada, 1987, "Application of the Dyson mapping to odd-mass nuclei," Nucl. Phys. A **462**, 561.
Takada, K., K. Yamada, and H. Tsukuma, 1989, "Microscopic study of the shape phase transition in the Sm isotopes. (I)," Nucl. Phys. A **496**, 224.
Talmi, I., 1971, "Generalized seniority and structure of semi-magic nuclei," Nucl. Phys. A **172**, 1.
Talmi, I., 1981, in *Interacting Bose-Fermion Systems,* edited by F. Iachello (Plenum, New York), p. 229.
Takahashi, Y., and H. Umezawa, 1975, "Thermo field dynamics," in *Collective Phenomena* (Gordon and Breach, New York), p. 55.
Tamura, T., 1982, "Boson expansion theories: Formalism and applications," in *Dynamics of Nuclear Collective Motion,* edited by K. Ogawa and K. Tanabe (Institute for Nuclear Studies, Tokyo University), p. 400.
Tamura, T., 1983a, "Aspects of boson expansion theories," Prog. Theor. Phys. Suppl. **74**, and **75**, 282.
Tamura, T., 1983b, "Term-by-term bosonization method," Phys. Rev. C **28**, 2154.
Tamura, T., 1983c, "Reformulation of Dyson's boson expansion theory," Phys. Rev. C **28**, 2480.
Tamura, T., C.-T. Li, and V. G. Pedrocchi, 1985, "Boson expansion theory in the seniority scheme," Phys. Rev. C **32**, 2129.
Tamura, T., K. J. Weeks, and T. Kishimoto, 1979, "Sixth-order boson expansion calculations applied to samarium isotopes," Phys. Rev. C **20**, 307.
Tamura, T., K. J. Weeks, and T. Kishimoto, 1980, "Analysis of nuclear collective motion in terms of the boson expansion theory," Nucl. Phys. A **347**, 359.
Tamura, T., K. J. Weeks, and V. G. Pedrocchi, 1981, "Exact and approximate aspects of the boson expansion theories," Phys. Rev. C **23**, 1297.
Tanabe, K., 1973, "The explicit determination of the linear boson transformation coefficients," J. Math. Phys. **14**, 618.
Tanabe, K., 1988, "Extended thermal random-phase approximation equation for nuclear collective excitation at finite temperature," Phys. Rev. C **37**, 2802.
Tanabe, K., and K. Sugawara-Tanabe, 1971, "Triaxiality in nuclear rotational state," Phys. Lett. B **34**, 575.
Tanabe, K., and K. Sugawara-Tanabe, 1976, "Microscopic theory of the rotational states of asymmetrically deformed nuclei in the Hartree-Fock-Bogoliubov approximation," Phys. Rev. C **14**, 1963.
Tanabe, K., and K. Sugawara-Tanabe, 1986, "The thermal linear response theory for the dissipative high-spin quantum states," Prog. Theor. Phys. **76**, 356.
Tazaki, S., K. Takada, K. Kaneko, and F. Sakata, 1982, "Dynamical interplay between pairing and quadrupole correlations," Prog. Theor. Phys. Suppl. **71**, Chap. 4.
Thompson, M. J., J. P. Elliott, and J. A. Evans, 1987, "A microscopically determined IBM-3 Hamiltonian," Phys. Lett. B **195**, 511.
Thorn, H., Y. R. Shimizu, and K. Takada, 1988, "Application of the Dyson-type non-unitary representation of the self-consistent collective-coordinate method to simple models. (II). O(4) model," Nucl. Phys. A **486**, 221.
Thouless, D. J., 1960, "Stability conditions and nuclear rotations in the Hartree-Fock theory," Nucl. Phys. **21**, 225.
Thouless, D. J., 1961, "Vibrational states of nuclei in the random-phase approximation," Nucl. Phys. **22**, 78.
Thouless, D. J., and J. G. Valatin, 1962, "Time-dependent Hartree-Fock equations and rotational states of nuclei," Nucl. Phys. **31**, 211.
Tsukuma, H., H. Thorn, and K. Takada, 1987, "Application of the Dyson boson mapping to the analysis of the phase transition at $N=88-90$," Nucl. Phys. A **466**, 70.
Überall, H., 1971, *Electron Scattering from Complex Nuclei* (Academic, New York).
Ui, H., 1968, "SU(1,1) quasi-spin formalism of the many-boson system in a spherical field," Ann. Phys. (N.Y.) **49**, 69.
Ui, H., 1970, "Quantum mechanical rigid rotation with an arbitrary deformation. I. Dynamical group approach to quadratically deformed body," Prog. Theor. Phys. **44**, 153.
Ui, H., and L. C. Biedenharn, 1968, "A remark on the generator-coordinate method and the quasi-boson approximation," Phys. Lett. B **27**, 608.
Umar, A. S., and A. Klein, 1986, "Application of a self-consistent theory of large-amplitude collective motion to the generalized Meshkov-Glick-Lipkin model," Nucl. Phys. A **458**, 246.
Umezawa, H., 1984, "Methods of quantum field theory in condensed matter physics," Prog. Theor. Phys. Suppl. **80**, 26.
Umezawa, H., H. Matsumoto, and M. Tachiki, 1982, *Thermo Field Dynamics and Condensed States* (North-Holland, Amsterdam).
Umezawa, M., 1963, "Invariant quantities in semi-simple groups (I)," Nucl. Phys. **48**, 111.
Umezawa, M., 1964a, "Invariant quantities in semi-simple groups (II)," Nucl. Phys. **53**, 54.
Umezawa, M., 1964b, "Invariant quantities in semi-simple groups (III)," Nucl. Phys. **57**, 65.
Usui, T., 1960, "Excitations in a high-density electron gas. I," Prog. Theor. Phys. **23**, 787.
Valatin, J. G., 1961, "Generalized Hartree-Fock method," Phys. Rev. **122**, 1012.
Vallières, M., A. Klein, and R. M. Dreizler, 1973, "Algebraic approach to the theory of collective motion: spherical to deformed transition in the single-j shell," Phys. Rev. C **7**, 2188.
Van Egmond, A., and K. Allart, 1983, "Analysis of IBFA-model assumptions with the broken-pair model," Nucl. Phys. A **394**, 173.

Van Egmond, A., and K. Allart, 1984, "Do we understand IBM parameters?" Nucl. Phys. A **425**, 275.
Van Isacker, P., S. Pittel, A. Frank, and P. D. Duval, 1986, "Microscopic study of configuration mixing in the interacting boson model," Nucl. Phys. A **451**, 202.
Vassanji, M., and A. Klein, 1978, "Algebraic approach to vibrational collective motion in a model with both collective and non-collective degrees of freedom," Phys. Rev. C **17**, 755.
Villars, F. M. H., 1965, "Elementary quantum theory of nuclear collective rotation," Nucl. Phys. **74**, 353.
Villars, F., 1977, "Adiabatic time-dependent Hartree-Fock theory in nuclear physics," Nucl. Phys. A **285**, 269.
Villars, F., 1984, "Nuclear collective motion and the Born-Oppenheimer approximation," Nucl. Phys. A **420**, 61.
Vincent, C. M., L. J. Henry, and G.-K. Kim, 1988, "Six-fermion tests of Dyson boson mappings," Phys. Rev. C **39**, 1583.
Walet, N. R., and A. Klein, 1990, "Thermal boson expansions and dynamical symmetry," Nucl. Phys. A **510**, 261.
Walet, N. R., A. Klein, and G. Do Dang, 1989, "Reaction paths and generalized valley approximation," J. Chem. Phys. **91**, 2848.
Walet, N. R., A. Klein, G. Do Dang, and A. Bulgac, 1990, "Generalized valley approximation applied to a schematic model of the monopole excitation," Phys. Rev. C **41**, 318.
Weeks, K. J., and T. Tamura, 1980a, "Description of collective states in ^{190}Os and ^{196}Pt in terms of the boson expansion theory," Phys. Rev. Lett. **44**, 533.
Weeks, K. J., and T. Tamura, 1980b, "Boson expansion description of collective states in Ru and Pd isotopes," Phys. Rev. C **22**, 888.
Weeks, K. J., and T. Tamura, 1980c, "Boson expansion description of collective states in osmium and platinum isotopes," Phys. Rev. C **22**, 1323.
Weeks, K. J., T. Tamura, T. Udagawa, and F. J. W. Hahne, 1981, "Coupling of collective quadrupole and monopole pairing vibrations in the Ge nuclei," Phys. Rev. C **24**, 703.
Weyl, H., 1946, *The Classical Groups* (Princeton University, Princeton).
Wigner, E., 1937, "Symmetry of the nuclear Hamiltonian and the spectroscopy of nuclei," Phys. Rev. **51**, 106.
Williams, R. D., and S. E. Koonin, 1982, "Semi-classical quantization of the shell model," Nucl. Phys. A **391**, 72.
Wong, C. W., 1975, "Generator coordinate methods in nuclear physics," Phys. Rep. **15C**, 283.
Wood, L. D., and I. Morrison, 1988, "A Bose-Fermi map in the SU(3) limit of the IBA," Int. J. Mod. Phys. A **3**, 1475.
Wu, C. L., and D. H. Feng, 1981, "The boson-fermion hybrid representation formulation, I," Ann. Phys. (N.Y.) **135**, 166.
Wu, C. L., and D. H. Feng, 1982, "The composite particle representation theory: I. The composite particle representation transformation," Commun. Theor. Phys. **1**, 705.
Wu, C. L., and D. H. Feng, 1983, "The composite particle representation theory: II. The composite particle approximation," Commun. Theor. Phys. **2**, 811.
Wu, C. L., D. H. Feng, X.-G. Chen, J. Q. Chen, and M. W. Guidry, 1986, "Fermion dynamical symmetries and the nuclear shell model," Phys. Lett. B **168**, 313.
Wu, C. L., D. H. Feng, X.-G. Chen, J. Q. Chen, and M. W. Guidry, 1987, "Fermion dynamical symmetry model of nuclei: basis, Hamiltonian, and symmetries," Phys. Rev. C **36**, 1157.
Wu, H., and M. Vallières, 1989, "Numerical realization of the fermion dynamical symmetry model," Phys. Rev. C **39**, 1066.
Wybourne, B. G., 1974, *Classical Groups for Physicists* (Wiley, New York).
Wybourne, B. G., 1986, "Comment on 'Exact boson mappings for the nuclear neutron (proton) p shell with the symmetry $SO(7)\supset SU(3)$,'" Phys. Rev. C **33**, 1103.
Xu, G., S. Wang, and Y. Yang, 1987, "Generator coordinate approach to the dynamic group representation," Phys. Rev. C **36**, 2095.
Yamada, K., and K. Takada, 1988, "Coupling effect between quadrupole phonon and pairing vibration in odd-mass nuclei," Nucl. Phys. A **480**, 143.
Yamada, K., and K. Takada, 1989, "Microscopic study of the shape phase transition in the Sm isotopes. (III)," Nucl. Phys. A **503**, 53.
Yamada, K., Y. Takada, and H. Tsukuma, 1989, "Microscopic study of the shape phase transition in the Sm isotopes. (II)," Nucl. Phys. A **496**, 239.
Yamamura, M., 1965, "On the phonon-quasiparticle interactions in spherical odd nuclei," Prog. Theor. Phys. **33**, 199.
Yamamura, M., 1980, "A possible quantization of time-dependent Hartree-Bogoliubov theory based on the $SO(2N+1)$ algebra," Prog. Theor. Phys. **63**, 486.
Yamamura, M., 1983, "An extension of TDHF and boson-fermion expansion," Prog. Theor. Phys. Suppl. **74** and **75**, 271.
Yamamura, M., J. da Providência, A. Kuriyama, and C. Fiolhais, 1989, "Equation of collective submanifold for mixed states," Prog. Theor. Phys. **81**, 1198.
Yamamura, M., and A. Kuriyama, 1984, "Generalization of equation of collective submanifold," Prog. Theor. Phys. **71**, 752.
Yamamura, M., and A. Kuriyama, 1986a, "Equations of collective submanifold for large-amplitude collective motion and its coupling with intrinsic degrees of freedom. I," Prog. Theor. Phys. **75**, 272.
Yamamura, M., and A. Kuriyama, 1986b, "Equations of collective submanifold for large-amplitude collective motion and its coupling with intrinsic degrees of freedom. II," Prog. Theor. Phys. **75**, 583.
Yamamura, M., and A. Kuriyama, 1987a, "Equations of collective submanifold for large-amplitude collective motion and its coupling with intrinsic degrees of freedom. III," Prog. Theor. Phys. **77**, 94.
Yamamura, M., and A. Kuriyama, 1987b, "Classical image of many-fermion system and its canonical quantization," in *Symmetries and Semiclassical Features of Nuclear Dynamics,* Lecture Notes in Physics, No. 279, edited by A. A. Raduta (Springer, Berlin/New York), p. 44.
Yamamura, M., and A. Kuriyama, 1987c, "Time-dependent Hartree-Fock method and its extension," Prog. Theor. Phys. Suppl. **93**, 1.
Yamamura, M., A. Kuriyama, and S. Iida, 1984, "Unique specification of collective submanifold," Prog. Theor. Phys. **71**, 109.
Yamamura, M., and S. Nishiyama, 1977, "An *a priori* quantized time-dependent Hartree-Bogoliubov theory," Prog. Theor. Phys. **56**, 124.
Yamamura, M., T. Suzuki, and H. Ichihashi, 1978, "Schwinger boson representation for the quantized rotator," Prog. Theor. Phys. **60**, 197.
Yamamura, M., A. Tokunaga, and T. Marumori, 1967, "On applicability of the random-phase approximation to the collective excitation in spherical even nuclei. II," Prog. Theor. Phys. **37**, 336.
Yang, L. M., 1983, "Microscopic investigation of the interacting boson model," in *Collective Bands in Nuclei,* Progress in Particle and Nuclear Physics, Vol. 9, edited by D. Wilkinson

(Pergamon, Oxford), p. 147.
Yang, L. M., D. H. Lu, and Z. N. Zhou, 1984, "Microscopic investigation of IBM and IBFM," Nucl. Phys. A **421**, 229c.
Yang, L. M., T. Song, and X. H. Wang, 1986, "Microscopic treatment of shape coexistence in even-even nuclei and related extension of IBM," Phys. Lett. B **175**, 6.
Yoshinaga, N., 1989, "Sdg interacting boson Hamiltonian in the seniority scheme," Nucl. Phys. A **493**, 323.
Zirnbauer, M. R, 1984, "Microscopic approach to the interacting boson model. (II). Extension to neutron-proton systems and applications," Nucl. Phys. A **419**, 241.
Zirnbauer, M. R., and D. M. Brink, 1982, "Microscopic approach to the interacting boson model," Nucl. Phys. A **384**, 1.
Zylicz, J., 1986, "A review of experimental evidence for octupole deformation," in *Nuclear Structure, Reactions and Symmetries,* edited by R. Meyer and V. Paar (World Scientific, Singapore), Vol. 1, p. 79.