# Coherent states: Theory and some applications

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In this review, a general algorithm for constructing coherent states of dynamical groups for a given quantum physical system is presented. The result is that, for a given dynamical group, the coherent states are isomorphic to a coset space of group geometrical space. Thus the topological and algebraic structure of the coherent states as well as the associated dynamical system can be extensively discussed. In addition, a quantum-mechanical phase-space representation is constructed via the coherent-state theory. Several useful methods for employing the coherent states to study the physical phenomena of quantum-dynamic systems, such as the path integral, variational principle, classical limit, and thermodynamic limit of quantum mechanics, are described.

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## I. INTRODUCTION

In the past two and a half decades, developments in the field of coherent states and their applications have been breathtaking. Yet, the idea of creating a coherent state for a quantum system was conceived well before that. In fact, *back in 1926*, Schrödinger first proposed the concept of what is now called "coherent states" (Schrödinger, 1926) in connection with the classical states of the quantum harmonic oscillator. Thus the coherent states were invented immediately after the birth of quantum mechanics. However, between 1926 and 1963, activities in this field remained dormant. It was not until some thirty-five years after Schrödinger's pioneering paper that the first modern and specific application was made by Glauber and Sudarshan (Glauber, 1963a, 1963b, 1963c; Sudarshan, 1963) and launched this fruitful and important field of study. In his two seminal papers, in which the term *coherent states* was first coined, Glauber constructed the eigenstates of the annihilation operator of the harmonic oscillator in order to study the electromagnetic correlation functions, a subject of great importance in quantum optics.

Roughly at the same time as Glauber and Sudarshan, Klauder (1963a, 1963b) developed a set of continuous states in which the basic ideas of coherent states for arbitrary Lie groups were contained. Some ten years after the work of Glauber, Sudarshan, and Klauder, the complete construction of coherent states of Lie groups, with various properties similar to the harmonic-oscillator coherent states, was achieved (Perelomov, 1972; Gilmore, 1972, 1974). The basic theme of this development was to connect the coherent states intimately with the dynamical group for each physical problem. For example, when one includes only creation, annihilation, identity and number-conserving operators as generators, the harmonic-oscillator system is known to possess the Heisenberg-Weyl $H_4$ dynamical group.[1] Then, as we shall show in this review, it is very natural to construct the Glauber coherent states via a one-to-one correspondence with the geometrical coset space $H(4)/U(1)\otimes U(1)$. Since all physical problems formulated in the quantum-mechanical language have a dynamical group (although at times the group may be too large to be useful), an important outcome of this recognition is that the coherent states need not be restricted to the harmonic oscillator but can be generalized to all types of physical problems. Furthermore, the fact that coset spaces are known to have natural geometrical properties means that the coherent states must also be endowed with natural geometry.

A large body of work on the coherent states has by now appeared. This vast literature was recently exhaustively collected, catalogued, and classified by Klauder and Skagerstam (1985). It is of course not our intention to repeat here the excellent work of these two editors. Nor will we discuss at length the mathematical usefulness of the coherent states as a tool for the study of the unitary representations of Lie groups, since an excellent expository book now exists on the subject by Perelomov (1986). The primary motivation of this review is to give a comprehensive, yet didactic, algorithm for the construction of coherent states for a given physical system. It is our intention to show that such states can provide a fundamental framework in which global dynamical properties of quantum systems can be interpreted. Throughout the review we shall "sprinkle" discussions of the geometrical properties of the coherent states and their physical interpretations, a subject which we believe has not yet received sufficient attention in the literature but which is clearly taking on increasing importance in the attempt to understand and address the fundamental aspects of quantum mechanics.

[1]The technical details of the harmonic-oscillator group $H_4$ and its algebra $h_4$ are discussed in Sec. II B 1 of this review.

In Sec. II, a full account of the field (i.e., harmonic-oscillator) coherent states is given. However, our discussion will have a slightly different emphasis since it centers on how the field coherent states can be constructed from the dynamical group of the system: the $H_4$ Heisenberg-Weyl group. As emphasized by Glauber, there are three equivalent ways to construct the coherent states for such a dynamical group. The first is to define them as eigenstates of the annihilation operator. The second is to define them by the application of a "displacement" operator on the vacuum state. The third is to consider the coherent states as quantum states with a minimum uncertainty relationship. We shall show that such coherent states are in one-to-one correspondence with the coset space $H_4/\mathrm{U}(1)\otimes\mathrm{U}(1)$, the flat complex plane. We shall also discuss many of the well-known properties of coherent states via this path. In Sec. III, a general theory of coherent states is presented. This section presents the rationale for generalizing the field coherent states to other dynamical systems via the "displacement" concept. This generalization was proposed simultaneously but separately by Perelomov and Gilmore, as was also indicated by Klauder (1963). We shall discuss the technical details and differences between the methods proposed by these two authors. Throughout this review, we have adopted the method of Gilmore and provide extensive discussions on the geometrical properties of coherent states. The coherent-state representations of quantum mechanics and quantum-statistical mechanics are presented in Sec. IV. In particular, we shall show how variational and path-integral methods can be naturally carried out within the coherent-state theory. The important concepts of classical limit and thermodynamics will also be discussed. In Sec. V, we treat the properties of the so-called "squeezed states" in the same vein. This is applied to quantum optics, and to the calculation of the $S$ matrix in molecular dynamics. The technique presented in this section is also applicable to all bosonic systems. Section VI discusses the fermion coherent states for atomic and nuclear systems. In fact, basic ideas of fermion coherent states in terms of the $c$-number representation were presented as early as 1960 by Klauder (1960). The dynamical groups of these states are $\mathrm{U}(n)$ and $\mathrm{SO}(2n)$ compact Lie groups. Applications of the coherent-state method to the time-independent and time-dependent mean-field theories of atomic and nuclear systems are also discussed in this section. In Sec. VII the concept of vector coherent states is reviewed. Primary applications of such states are to compute various matrix elements. Readers who are interested in these applications should consult the recently published book of Hecht (1987). Finally, conclusions are given in Sec. VIII.

## II. FIELD COHERENT STATES

### A. Motivation

Since the now well-known "Glauber" coherent states for the electromagnetic field have been the source of inspiration for the development of coherent states in the past two and a half decades, it is worthwhile emphasizing here their historical as well as pedagogical importance.

Glauber (1963b, 1963c) showed that such states are enormously useful for describing the physics of quantum optics. Physically, they turn out to be eigenstates of the coherence (correlation) function of the electromagnetic field. Consequently, he named these states *field coherent states.*

#### 1. Glauber's definitions of field coherent states

According to Glauber (1963c), the field coherent states can be constructed starting from any one of three mathematical definitions.

*Definition 1:* The coherent states $|\alpha\rangle$ are eigenstates of the harmonic-oscillator annihilation operator $a$,

$$a|\alpha\rangle=\alpha|\alpha\rangle\ , \tag{2.1}$$

where $\alpha$ is a complex number.

*Definition 2:* The coherent states $|\alpha\rangle$ can be obtained by applying a displacement operator $D(\alpha)$ on the vacuum state of the harmonic oscillator,

$$|\alpha\rangle=D(\alpha)|0\rangle\ , \tag{2.2a}$$

where the displacement operator $D(\alpha)$ is defined as

$$D(\alpha)=\exp(\alpha a^{\dagger}-\alpha^{*}a)\ . \tag{2.2b}$$

*Definition 3:* The coherent states $|\alpha\rangle$ are quantum states with a minimum-uncertainty relationship,

$$(\Delta p)^2(\Delta q)^2=(\tfrac{1}{2})^2\ . \tag{2.3a}$$

where the coordinate and momentum operators $(\hat{q},\hat{p})$ are defined as

$$\hat{q}=\frac{1}{\sqrt{2}}(a+a^{\dagger})\ , \tag{2.3b}$$

$$\hat{p}=\frac{1}{i\sqrt{2}}(a-a^{\dagger}) \tag{2.3c}$$

and

$$(\Delta f)^2\equiv\langle\alpha|(\hat{f}-\langle\hat{f}\rangle)^2|\alpha\rangle\ , \tag{2.3d}$$

$$\langle\hat{f}\rangle\equiv\langle\alpha|\hat{f}|\alpha\rangle\ . \tag{2.3e}$$

It is worth pointing out that Definition 3 is by no means unique because Eq. (2.3a) does not provide a unique solution for $(\Delta p,\Delta q)$. Such non-uniqueness is graphically depicted in Fig. 1, where (a) represents the uncertainty circle for the usual field coherent state and (b) represents the uncertainty ellipse for the so-called squeezed states, which will be discussed in Sec. V. For the field coherent states, $\Delta p=\Delta q=\frac{1}{2}$.

Glauber's original approach was entirely motivated by the physical consideration of factorizing to all orders the

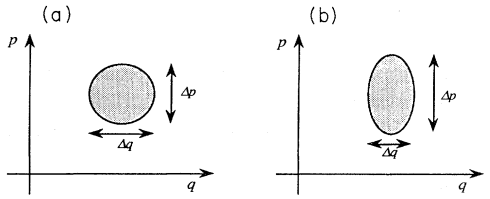



FIG. 1. The uncertainty picture in the coherent states.

electromagnetic field correlation functions. To this end, he constructed the field coherent states by using the harmonic-oscillator algebra. Glauber concluded that the same field coherent states are obtained from the three mathematical definitions. Of course, not all physical systems are describable by the harmonic oscillators. Therefore, ever since the appearance of Glauber's papers there has been a real need to generalize these field coherent states to other systems which may possess different dynamical properties. We shall specifically discuss this point later on. Suffice it to mention at this point that the generalization of each of the mathematical definitions, unlike the case of the field coherent states, will not result in equivalent coherent states.

### 2. Hamiltonian structure of the field system

In order to discuss the dynamical properties of a quantum system, the starting point is the Hamiltonian of the system. This is because the Hamiltonian and its Hilbert space completely determine the dynamics of the quantum-mechanical system. In quantum optics, the Hamiltonian of the system with interaction between an atomic system and the electromagnetic field can be taken as follows:

$$H=\sum_k \hbar\omega_k a_k^\dagger a_k+\sum_\alpha \varepsilon\sigma_0^{(\alpha)} + \sum_{k,\alpha}\gamma_{k\alpha}\left[\frac{\sigma_+^{(\alpha)}}{\sqrt{N}}a_k+\frac{\sigma_-^{(\alpha)}}{\sqrt{N}}a_k^\dagger\right], \tag{2.4}$$

where $\hbar\omega_k$ is the energy of the field mode $k$, and $\gamma_{k\alpha}$ are the coupling coefficients between the atomic system and the electromagnetic field. One of the crucial assumptions made in the construction of the Hamiltonian given by Eq. (2.4) is that each of the $N$ atoms, labeled by the index $\alpha$, is a two-level system and therefore its dynamical variables are just the usual "spin" operators $\{\sigma_0^{(\alpha)},\sigma_+^{(\alpha)},\sigma_-^{(\alpha)}\}$. Normally, one considers the coupling strength $\gamma_{k\alpha}$ as a constant, i.e., $\gamma_{k\alpha}=\gamma$. If we regard the atomic system as a classical source (i.e., treat the spin operators $\sigma^{(\alpha)}$ as $c$ numbers), Eq. (2.4) can be reduced to

$$\begin{aligned}H^F&=\sum_k \hbar\omega_k a_k^\dagger a_k+\sum_\alpha\langle\varepsilon\sigma_0^{(\alpha)}\rangle \\ &\quad+\gamma\sum_{k,\alpha}\left[\frac{\langle\sigma_+^{(\alpha)}\rangle}{\sqrt{N}}a_k+\frac{\langle\sigma_-^{(\alpha)}\rangle}{\sqrt{N}}a_k^\dagger\right] \\ &=\sum_k \hbar\omega_k a_k^\dagger a_k+\sum_k[\lambda_k(t)a_k^\dagger+\lambda_k^*(t)a_k]+\text{constant} \\ &=\sum_k H_k^F+\text{constant}\ ,\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}H_k^F&=\hbar\omega_k a_k^\dagger a_k+\lambda_k(t)a_k^\dagger+\lambda_k^*(t)a_k \\ &=H_0+H_{\text{inter}}\ .\end{aligned} \tag{2.6}$$

In Eq. (2.6), the operator $H_0$ mimics the free electromagnetic field (or free harmonic oscillator) and $H_{\text{inter}}$ describes the interaction between the electromagnetic field and the external time-dependent source. Thus the optical system is modeled as a harmonic-oscillator system in an external field. Starting from the semiclassical Hamiltonian (2.6), we can construct the field coherent states in a more compact and elegant manner: *the group-theoretical method*. This method, as we shall see in the next section, usefully generalizes the concept of coherent states to arbitrary Lie groups. At the same time, the physical interpretations of the coherent states become transparent.

## B. Group-theoretic construction of the field coherent states

In this section, we shall discuss the field coherent states in the group-theoretical language. The most transparent way to discuss this is within the context of the Hamiltonian (2.6). We illustrate here an algorithm to produce the coherent states from a knowledge of the algebraic structure of the Hamiltonian, the Hilbert space on which the Hamiltonian acts, and an extremal state of the operator algebra.

### 1. Inputs from the Hamiltonian

Based on the Hamiltonian of the system given by Eq. (2.6), one finds immediately the following three obvious properties:

(a) Algebraic structure (commutation relations). The Hamiltonian (2.6) is a linear combination of harmonic-oscillator operators $a^\dagger, a$ and $\hat{n}=a^\dagger a$. In this case, we have considered a single-mode field because the total Hamiltonian, Eq. (2.5), is the algebraic summation of single-mode Hamiltonians. These operators $\{\hat{n},a^\dagger,a\}$ and the unit operator $I$ obey the following familiar commutation relations:

$$\begin{aligned}&[\hat{n},a^\dagger]=+a^\dagger,\quad [\hat{n},I]=0\ ,\\ &[\hat{n},a]=-a,\quad [a^\dagger,I]=0\ ,\\ &[a,a^\dagger]=+I,\quad [a,I]=0\ .\end{aligned} \tag{2.7}$$

The set of operators $\{\hat{n},a^\dagger,a,I\}$ spans a Lie algebra, denoted as $h_4$. The corresponding Lie group is the well-known Heisenberg-Weyl group $H_4$ (Weyl, 1928).

(b) Hilbert space. The Hilbert space (Fock space) for $H_4$ is spanned by the number eigenstates $\{|0\rangle,|1\rangle,|2\rangle,\ldots,|n\rangle,\ldots\}$:

$$\hat{n}|n\rangle=n|n\rangle \tag{2.8a}$$

and

$$|n\rangle=\frac{(a^\dagger)^n}{(n!)^{1/2}}|0\rangle\ . \tag{2.8b}$$

(c) Extremal state. Since $H_0$ is proportional to the particle number operator $\hat{n}$, the energy eigenstates of $H_0$ are $|n\rangle$:

$$H_0|n\rangle=\hbar\omega n|n\rangle\ . \tag{2.9}$$

Therefore the ground state of $H_0$ is the field vacuum state $|0\rangle$. This state is called an extremal state in this algorithm.

From these three results, we shall extract the coherent states with the help of a subgroup and its coset space.

### 2. Three steps to the coherent states

We obtain the coherent states in three steps as follows:

(a) Stability subgroup. This is the subgroup which leaves the extremal state invariant. For $H_4$, this is $U(1)\otimes U(1)$ with an algebra spanned by $\{\hat{n},I\}$. The stability subgroup consists of all operations $h$ of the form

$$h=e^{i(\delta\hat{n}+\varphi I)}\ . \tag{2.10a}$$

Thus

$$h|0\rangle=|0\rangle e^{i\varphi}\ . \tag{2.10b}$$

(b) Coset space. The coset space with respect to the stability subgroup will provide the operators to construct the coherent states. In the $H_4$ example with the stability subgroup $U(1)\otimes U(1)$, the coset $H_4/U(1)\otimes U(1)$ is the set of elements $\Omega$ providing a unique decomposition for any element $g\in H_4$,

$$g=Dh\ . \tag{2.11}$$

A typical coset representative in the coset space $H_4/U(1)\otimes U(1)$ is

$$D(\alpha)=\exp(\alpha a^\dagger-\alpha^* a)\ , \tag{2.12}$$

and the complex parameter $\alpha$ is arbitrary. The argument in the exponential operator (2.12) is anti-Hermitian and corresponds to a finite transformation in the complex plane ($c\rightarrow c+\alpha$) (see Fig. 2).

(c) Coherent states. These are defined by the actions of the coset elements on the extremal state. Using Eqs. (2.10)–(2.12), it is obvious that any group transformation element $g$ of $H_4$ acting on the unperturbed ground state

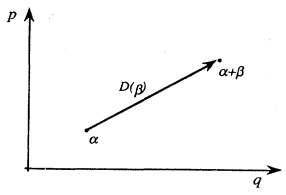



FIG. 2. The geometry of the displacement operator.

$|0\rangle$ can be factorized as follows:

$$g|0\rangle=D(\alpha)h|0\rangle$$
$$=D(\alpha)|0\rangle e^{i\varphi}\equiv|\alpha\rangle e^{i\varphi},\quad g\in H_4,\ h\in U(1)\otimes U(1)\ ,$$
$$D(\alpha)\in H_4/U(1)\otimes U(1) \tag{2.13}$$

Here $|\alpha\rangle$ is the coherent state. It is identical to one of the definitions of the coherent states given by Glauber. However, since the construction has been carried out in a manifestly group-theoretical formulation, generalization to systems governed by Hamiltonians with other dynamical groups now becomes straightforward.

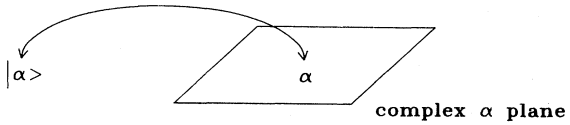
## C. Properties of the field coherent states

The field coherent states have a number of useful properties, which we summarize below. Each of these properties has a group-theoretical interpretation that can be generalized. We shall point out the group interpretation of each of these properties.

### 1. Geometric structure

There are three aspects of the geometric structure to be considered:

(a) Complex structure. From the above group-theoretical definition of the field coherent states one sees that the displacement operator [coset representative of $H_4/U(1)\otimes U(1)$ in the language of group theory] is a finite transformation operator in the complex $\alpha$ plane, that is, *one-to-one correspondence between the coherent states $|\alpha\rangle$ and points in the complex $\alpha$ plane:*




This mapping is continuous, i.e., for any given $\varepsilon$, there exists a $\delta$ such that

$$||\alpha\rangle-|\alpha'\rangle|<\delta \tag{2.14a}$$

$$\text{if } |\alpha-\alpha'|<\varepsilon\ . \tag{2.14b}$$

Distances are determined with respect to in the intrinsic metrics. That is, the metric on the left side of Eq. (2.14b) is determined from the Hilbert-space inner product,

while the one on the right is determined from the metric in the complex plane. Furthermore, the action of any group transformation $g=\exp[i(\gamma a^\dagger+\beta a+\eta n+\delta I)]$ ($\in H_4$) on a point $\alpha$ in the plane $H_4/\mathrm{U}(1)\otimes\mathrm{U}(1)$ is given by

$$g\alpha\rightarrow\alpha'=e^{i\eta}\alpha+(\gamma/\eta)(e^{i\eta}-1)\ . \tag{2.15}$$

This equation is derived in the Appendix.

(b) Metric and measure. In order to understand the geometry of the field coherent state, we should calculate the explicit forms of the metric and measure of $H_4/\mathrm{U}(1)\otimes\mathrm{U}(1)$. This can be carried out as follows: any group transformation of $H_4/\mathrm{U}(1)\otimes\mathrm{U}(1)$ on the $\alpha$ plane is given by

$$D(\beta)|\alpha\rangle=|\alpha+\beta\rangle e^{i\psi}\ , \tag{2.16a}$$

$$\psi=\mathrm{Im}(\beta\alpha^*)\ . \tag{2.16b}$$

This shows that $H_4/\mathrm{U}(1)\otimes\mathrm{U}(1)$ is a flat space, and the metric of the coset space $H_4/\mathrm{U}(1)\otimes\mathrm{U}(1)$ is diagonal. The measure of this space is $d\mu=d\alpha\,d\alpha^*$.

(c) Symplectic structure. The above results show that $H_4/\mathrm{U}(1)\otimes\mathrm{U}(1)$ is a complex space with explicit metric and therefore mathematically it must have a symplectic structure (Arnold, 1978). This structure is $H_4$ invariant and can clearly be presented by making the standard transformation from complex to phase-space coordinates:

$$\alpha=\frac{1}{\sqrt{2}}(q+ip),\quad \alpha^*=\frac{1}{\sqrt{2}}(q-ip)\ . \tag{2.17}$$

Then the standard two-form of $H_4/\mathrm{U}(1)\otimes\mathrm{U}(1)$ is $\omega^2=dp\wedge dq$, whose Poisson bracket is

$$\{f,g\}=\frac{\partial f}{\partial q}\frac{\partial g}{\partial p}-\frac{\partial g}{\partial q}\frac{\partial f}{\partial p}\ , \tag{2.18}$$

where the functions $f,g$ are entire functions defined on the complex $\alpha$ plane. The physical interpretation of the parameter space $H_4/\mathrm{U}(1)\otimes\mathrm{U}(1)$ is that it provides a phase-space structure for the field system.

### 2. The Baker-Campbell-Hausdorff formula

The Baker-Campbell-Hausdorff (BCH) formula (or disentangling theorem) allows for rearrangements in the ordering of the exponential operator products [for a derivation using the so-called faithful matrix representation method (Gilmore, 1974b)],

$$\begin{aligned}\exp(\alpha a^\dagger-\alpha' a)&=\exp(\tfrac{1}{2}\alpha\alpha')\exp(-\alpha' a)\exp(\alpha a^\dagger)\\&=\exp(-\tfrac{1}{2}\alpha\alpha')\exp(\alpha a^\dagger)\exp(-\alpha' a)\ .\end{aligned} \tag{2.19}$$

This is extremely useful in dealing with practical problems, as we shall see later.

### 3. Hilbert-space properties

There are two properties to be noted:

(a) Non-orthogonality. By direct calculation, we have

$$\langle\alpha|\alpha'\rangle=\exp[\alpha^*\alpha'-\tfrac{1}{2}(\alpha^*\alpha+\alpha'^*\alpha')]\ , \tag{2.20a}$$

$$|\langle\alpha|\alpha'\rangle|^2=\exp(-|\alpha-\alpha'|^2)\ . \tag{2.20b}$$

This shows that the coherent states $|\alpha\rangle$ are not orthogonal, but normalized.

(b) Over-completeness. For the field coherent states, Glauber showed that the resolution of the identity in terms of coherent states is not unique. A common and useful resolution (see Secs. II.C.4 and II.C.5 below) is

$$\int|\alpha\rangle\frac{d^2\alpha}{\pi}\langle\alpha|=I\ . \tag{2.21}$$

Since the coherent states are labeled by a continuous index in a Hilbert space that has a countable basis, they are over-complete.

### 4. Hilbert-space expansions

Expansions in a Hilbert space may be performed in a number of ways.

(a) Coherent state in terms of diagonal states. The field coherent states can be expanded in terms of the eigenstates of the particle number operator $|n\rangle$, known as Fock states:

$$\begin{aligned}|\alpha\rangle&=D(\alpha)|0\rangle\\&=\exp(-\tfrac{1}{2}\alpha^*\alpha)\sum_0^\infty(\alpha a^\dagger)^n(n!)^{-1}|0\rangle\\&=\exp(-\tfrac{1}{2}\alpha^*\alpha)\sum_0^\infty(\alpha)^n(n!)^{-1/2}|n\rangle\ .\end{aligned} \tag{2.22}$$

(b) Arbitrary state expanded in terms of coherent states. Any arbitrary state $|\Psi\rangle$ can be expressed in terms of the states $|n\rangle$ in the form

$$|\Psi\rangle=\sum_n c_n|n\rangle=\sum_n c_n\frac{(a^\dagger)^n}{(n!)^{1/2}}|0\rangle\ , \tag{2.23}$$

where $\sum_n|c_n|^2=1$. Using the over-completeness of the field coherent states, we can also expand $|\Psi\rangle$ in the coherent state:

$$|\Psi\rangle=\int|\alpha\rangle f(\alpha^*)e^{-|\alpha|^2/2}\frac{d^2\alpha}{\pi}\ , \tag{2.24}$$

where the analytical function $f(\alpha^*)$, the coherent-state representation of $|\Psi\rangle$, is

$$f(\alpha^*)=\langle\alpha|\Psi\rangle e^{|\alpha|^2/2}=\sum_n c_n\frac{(\alpha^*)^n}{(n!)^{1/2}} \tag{2.25}$$

and $e^{-|\alpha|^2}d^2\alpha/\pi$ is the measure in the entire function space $L^2(\phi)$ (Bargmann, 1961):

$$\langle\Psi_1|\Psi_2\rangle=\int f_1(\alpha)f_2(\alpha^*)e^{-|\alpha|^2}\frac{d^2\alpha}{\pi}<\infty\ . \tag{2.26}$$

As an example, suppose $|\Psi\rangle=|n\rangle$; the corresponding coherent-state representation is

$$f_n(\alpha^*)=\frac{(\alpha^*)^n}{(n!)^{1/2}} . \tag{2.27}$$

The entire functions $\{f_n(\alpha^*),\ n=0,1,\ldots,\infty\}$ span $L^2(\phi)$.

(c) General operators in terms of coherent states. A general quantum-mechanical operator $B$ in the domain of coherent states (Klauder and Skagerstam, 1985) may be expressed in terms of its matrix elements connecting states with fixed number of quanta:

$$B=\sum_{n,m}|n\rangle B_{nm}\langle m| . \tag{2.28}$$

The coherent-state representation of this operator can easily be found as follows:

$$B=\int|\alpha\rangle\langle\alpha|B|\alpha'\rangle\langle\alpha'|\frac{d^2\alpha\,d^2\alpha'}{\pi^2}$$

$$=\int|\alpha\rangle B(\alpha^*,\alpha')\langle\alpha'|\exp(-\tfrac{1}{2}|\alpha|^2-\tfrac{1}{2}|\alpha'|^2)\frac{d^2\alpha\,d^2\alpha'}{\pi^2} , \tag{2.29}$$

where

$$B(\alpha^*,\alpha')=\langle\alpha|B|\alpha'\rangle\exp(\tfrac{1}{2}|\alpha|^2+\tfrac{1}{2}|\alpha'|^2)$$

$$=\sum_{n,m}B_{nm}(n!m!)^{-1/2}(\alpha^*)^n(\alpha')^m . \tag{2.30}$$

It is obvious that the function $B(\alpha^*,\alpha)$ completely determines the operator $B$ and is an analytic function of the complex variables $\alpha^*$ and $\alpha$. Hence the density operator $\rho$ may be represented by means of a function of two complex variables in the coherent-state representation, $R(\alpha^*,\alpha')$:

$$R(\alpha^*,\alpha')=\langle\alpha|\rho|\alpha'\rangle\exp(\tfrac{1}{2}|\alpha|^2+\tfrac{1}{2}|\alpha'|^2)$$

$$=\sum_{n,m}\langle n|\rho|m\rangle(n!m!)^{-1/2}(\alpha^*)^n(\alpha')^m . \tag{2.31}$$

Conversely,

$$\rho=\int|\alpha\rangle R(\alpha^*,\alpha')\langle\alpha'|\exp(-\tfrac{1}{2}|\alpha|^2-\tfrac{1}{2}|\alpha'|^2)\frac{d^2\alpha\,d^2\alpha'}{\pi^2} . \tag{2.32}$$

The normalization condition of the density operator, $\mathrm{Tr}\rho=1$, requires that $R(\alpha^*,\alpha')$ satisfy the following equation:

$$\int R(\alpha^*,\alpha')\exp(\alpha'^*\alpha-|\alpha|^2-|\alpha'|^2)\frac{d^2\alpha\,d^2\alpha'}{\pi^2}=1 . \tag{2.33a}$$

From Eq. (2.31), we see that $R(\alpha^*,\alpha')$ is an entire function of $\alpha^*$, and therefore Eq. (2.33a) can be reduced to a simpler form:

$$\int R(\alpha^*,\alpha)e^{-|\alpha|^2}\frac{d^2\alpha}{\pi}=1 . \tag{2.33b}$$

5. Phase-space distributions

The phase-space distribution functions have been widely used to discuss the quantum-classical correspondence (Hillery *et al.*, 1984) and for the calculation of the quantum average values of mechanical quantities in the classical phase space. In the coherent-state representation, there are three convenient distribution functions: the $P$ distribution, the $Q$ distribution, and the Wigner distribution.

(a) $P$ representation. Under suitable conditions, the coherent-state representation of the operator, Eq. (2.29), can be reduced to the diagonal matrix form

$$B=\int|\alpha\rangle B_P(\alpha,\alpha^*)\langle\alpha|\frac{d^2\alpha}{\pi} . \tag{2.34}$$

By the same token, it may be possible to reduce the density operator $\rho$ to a diagonal matrix within the coherent-state representation,

$$\rho=\int|\alpha\rangle P(\alpha)\langle\alpha|\frac{d^2\alpha}{\pi} , \tag{2.35}$$

where $P(\alpha)$ is called the $P$ representation of the density matrix (or the distribution function representing the density matrix). Let $\chi_n(\eta)$ be the normally ordered characteristic function (Hillery *et al.*, 1984)

$$\chi_n(\eta)=\mathrm{Tr}(\rho e^{\eta a^\dagger}e^{-\eta^* a}) . \tag{2.36}$$

Then $P(\alpha)$ is just the Fourier transform of $\chi_n(\eta)$,

$$P(\alpha)=\int e^{\eta^*\alpha-\alpha^*\eta}\chi_n(\eta)\frac{d^2\eta}{\pi} . \tag{2.37}$$

The statistical average of an operator $B$ can be obtained easily:

$$\mathrm{Tr}(B\rho)=\mathrm{Tr}\int|\alpha\rangle P(\alpha)\langle\alpha|B\frac{d^2\alpha}{\pi}$$

$$=\int P(\alpha)\langle\alpha|B|\alpha\rangle\frac{d^2\alpha}{\pi} . \tag{2.38}$$

It should be pointed out that the $P$ distribution always exists for suitable bounded operators.

(b) $Q$ representation. There is a corresponding $Q$ representation $B_Q(\alpha^*,\alpha)$ for bounded operators, which can be determined uniquely by its diagonal matrix element in the coherent states:

$$B\rightarrow B_Q(\alpha^*,\alpha)=\langle\alpha|B|\alpha\rangle$$

$$=e^{-|\alpha|^2}\sum_{n,m}B_{nm}(n!m!)^{-1/2}(\alpha^*)^n(\alpha)^m , \tag{2.39}$$

where $B_{nm}$ has been given by Eq. (2.28). The $Q$ representation of the density operator is defined as

$$Q(\alpha)=\langle\alpha|\rho|\alpha\rangle , \tag{2.40}$$

and the statistical average of any operator is

$$\mathrm{Tr}(B\rho)=\mathrm{Tr}\int|\alpha\rangle B_P(\alpha,\alpha^*)\langle\alpha|\rho\frac{d^2\alpha}{\pi}$$
$$=\int B_P(\alpha,\alpha^*)\langle\alpha|\rho|\alpha\rangle\frac{d^2\alpha}{\pi}$$
$$=\int B_P(\alpha,\alpha^*)Q(\alpha)\frac{d^2\alpha}{\pi}\ , \tag{2.41}$$

where the function $B_P(\alpha,\alpha^*)$ is the $P$ representation of the operator $B$. Likewise, $Q(\alpha)$ is easily expressed by a characteristic function, the antinormal characteristic function $\chi_a(\eta)$,

$$\chi_a(\eta)=\mathrm{Tr}(\rho e^{-\eta^* a}e^{\eta a^\dagger})\ , \tag{2.42}$$

and $Q(\alpha)$ is the Fourier transform of $\chi_a(\eta)$,

$$Q(\alpha)=\int e^{\eta^*\alpha-\alpha^*\eta}\chi_a(\eta)\frac{d^2\eta}{\pi}\ . \tag{2.43}$$

(c) Wigner representation ($W$ representation) of the coherent states. Wigner was the first to propose the construction of a phase-space distribution function from quantum-mechanical wave functions (i.e., a Wigner distribution). The Wigner distribution function of the density operator in ordinary phase space is defined as (Wigner, 1932)

$$W(p,q)=\int_{-\infty}^{\infty}dy\langle q-y|\rho|q+y\rangle e^{2ipy}\ . \tag{2.44}$$

In the coherent-state representation the Wigner distribution function is (Hillery *et al.*, 1984; Fan, 1984)

$$W(\alpha)=\int\langle\alpha-\alpha'|\rho|\alpha+\alpha'\rangle\exp(\alpha\alpha'^*-\alpha^*\alpha')\frac{d^2\alpha'}{\pi}\ , \tag{2.45}$$

where the relation between $(\alpha,\alpha^*)$ and $(p,q)$ is given by Eq. (2.17). As in the case of the $P$ and $Q$ representations, the Wigner distribution $W(\alpha)$ can be found from the symmetrized characteristic function $\chi_s(\alpha)$,

$$\chi_s(\eta)=\mathrm{Tr}(\rho e^{-\eta^* a+\eta a^\dagger}) \tag{2.46}$$

and

$$W(\alpha)=\int e^{\eta^*\alpha-\alpha^*\eta}\chi_s(\eta)\frac{d^2\eta}{\pi}\ . \tag{2.47}$$

The statistical average value of $B$ is

$$\mathrm{Tr}(B\rho)=\int B_W(\alpha,\alpha^*)W(\alpha)\frac{d^2\alpha}{\pi}\ , \tag{2.48}$$

where $B_W(\alpha,\alpha^*)$ is the Wigner distribution of $B$ in the coherent-state representation.

According to the BCH formula (2.19), we have

$$e^{|\eta|^2/2}\chi_a(\eta)=\chi_s(\eta)=e^{-|\eta|^2/2}\chi_n(\eta)\ . \tag{2.49}$$

Therefore it is not difficult to prove the relationships among the $Q$, $P$, and $W$ distributions:

$$W(\alpha)=2\int P(\alpha')e^{-2|\alpha-\alpha'|^2}\frac{d^2\alpha'}{\pi}\ , \tag{2.50a}$$

$$Q(\alpha)=2\int W(\alpha')e^{-2|\alpha-\alpha'|^2}\frac{d^2\alpha'}{\pi}\ , \tag{2.50b}$$

$$Q(\alpha)=\int P(\alpha')e^{-|\alpha-\alpha'|^2}\frac{d^2\alpha'}{\pi}\ , \tag{2.50c}$$

and

$$(A,B)=\mathrm{Tr}(A^\dagger B)=\int A_P^*(\alpha,\alpha^*)B_Q(\alpha^*,\alpha)\frac{d^2\alpha}{\pi}$$
$$=\int A_Q^*(\alpha,\alpha^*)B_P(\alpha^*,\alpha)\frac{d^2\alpha}{\pi}$$
$$=\int A_W^*(\alpha^*,\alpha)B_W(\alpha,\alpha^*)\frac{d^2\alpha}{\pi}\ . \tag{2.50d}$$

This inner product can also be expressed in terms of the $Q$ and $P$ distributions using a nonlocal inner product,

$$(A,B)=\int A_Q^*(\alpha,\alpha^*)e^{|\alpha-\alpha'|^2}B_Q(\alpha',\alpha'^*)\frac{d^2\alpha\,d^2\alpha'}{\pi^2}$$
$$=\int A_P^*(\alpha,\alpha^*)e^{-|\alpha-\alpha'|^2}B_P(\alpha',\alpha'^*)\frac{d^2\alpha\,d^2\alpha'}{\pi^2}\ . \tag{2.50e}$$

### 6. Generating functions

In quantum optics, it is often necessary to calculate correlation functions with matrix elements of the following forms:

normal form: $\langle\alpha|(a^\dagger)^m(a)^n|\alpha\rangle$ , (2.51a)

antinormal form: $\langle\alpha|(a)^n(a^\dagger)^m|\alpha\rangle$ , (2.51b)

symmetric form: $\langle\alpha|S(a,a^\dagger)|\alpha\rangle$ . (2.51c)

Using definition (2.1), one can easily derive the matrix element (2.51a), whereas the general forms of (2.51b) and (2.51c) are more complicated. One way to calculate (2.51b) and (2.51c) is to rewrite the functions $(a)^n(a^\dagger)^m$ and $S(a,a^\dagger)$ in the normal form by using the normal product technique and then use (2.1) again; another way is to find the generating function, which will be given next.

Using the BCH formulas, we find the generating function for an antinormally ordered product,

$$\langle\alpha|\exp(\gamma a)\exp(\delta a^\dagger)|\alpha\rangle=\exp(-|\alpha|^2)\langle 0|\exp(\alpha^* a)\exp(\gamma a)\exp(\delta a^\dagger)\exp(\alpha a^\dagger)|0\rangle$$
$$=\exp(-|\alpha|^2)\langle 0|\exp[(\alpha+\delta)a^\dagger]\exp[(\alpha^*+\gamma)(\alpha+\delta)]\exp[(\alpha^*+\gamma)a]|0\rangle$$
$$=\exp(-|\alpha|^2)\exp[(\alpha^*+\gamma)(\alpha+\delta)]\ . \tag{2.52}$$

Similarly, the generating function for the symmetric operator products is constructed by replacing $\exp(\gamma a)\exp(\delta a^\dagger)$ with $\exp(\gamma a+\delta a^\dagger)$ in the above expression.

The normal, antinormal, and symmetric forms of (2.51) will give the corresponding $P$, $Q$, and $W$ distribution functions (Hillery *et al.*, 1984).

### 7. $D$ algebra

A very beautiful application of field coherent states in quantum optics is the reformulation of the entire laser theory in terms of $c$-number differential equations (Haken, 1970). This procedure is carried out by means of the so-called $D$ algebra (Gilmore *et al.*, 1975).

The $D$ algebra is a mapping of quantum observables into a differential form, so that they act as differential operators on the continuous parametrizing coset space of the coherent states. The $D$ algebra of an operator $B$ is defined as (Gilmore *et al.*, 1975)

$$B|\alpha\rangle\equiv D^k(B)|\alpha\rangle\ , \tag{2.53}$$

$$\langle\alpha|B\equiv D^b(B)\langle\alpha|\ . \tag{2.54}$$

It is obvious that $D^k(B)$ and $D^b(B)$ satisfy the adjoint relation

$$D^k(B)=[D^b(B)]^*\ . \tag{2.55}$$

On the other hand, the coherent-state projector $|\alpha\rangle\langle\alpha|$ provides a basis in which most physically reasonable operators may be expanded. The $D$ algebras on projectors are

$$B|\alpha\rangle\langle\alpha|\equiv D^l(B)|\alpha\rangle\langle\alpha|\ , \tag{2.56a}$$

$$|\alpha\rangle\langle\alpha|B\equiv D^r(B)|\alpha\rangle\langle\alpha|\ , \tag{2.56b}$$

and

$$D^r(B)=[D^l(B)]^*\ . \tag{2.57}$$

A theorem for $D$ algebras can be found as follows:

$$AB|\alpha\rangle=D^k(A)D^k(B)|\alpha\rangle\ , \tag{2.58a}$$

$$AB|\alpha\rangle\langle\alpha|=D^l(A)D^l(B)|\alpha\rangle\langle\alpha|\ . \tag{2.58b}$$

Thus the differential form of an arbitrary operator can be obtained from the fundamental differential form of the operators $a$ and $a^\dagger$:

$$D^k(a)=\alpha,\quad D^k(a^\dagger)=\frac{1}{2}\alpha^*+\frac{\partial}{\partial\alpha}\ , \tag{2.59a}$$

$$D^l(a)=\alpha,\quad D^l(a^\dagger)=\alpha^*+\frac{\partial}{\partial\alpha}\ . \tag{2.59b}$$

One can prove that the $D$-algebra representation preserves the commutation relation of the respective operators:

$$[D^k(a),D^k(a^\dagger)]=1,\quad [D^l(a),D^l(a^\dagger)]=1,\quad \text{etc.}\ , \tag{2.60a}$$

$$[D^k(a),D^b(a^\dagger)]=0\ , \tag{2.60b}$$

$$[D^k(A),D^k(B)]=D^k([A,B])\ , \tag{2.60c}$$

$$[D^b(A),D^b(B)]=-D^b([A,B])\ . \tag{2.60d}$$

### D. Physical interpretation of the field coherent states

The field coherent states are widely used in quantum optics. In fact, the fundamental theory of quantum optics was constructed with such states. We shall not repeat discussions of the theory here, since they can be found in standard textbooks on quantum optics (e.g., Klauder and Sudarshan, 1968). Moreover, there are many applications in field theory, for example, in eliminating the infrared divergence of quantum electrodynamics in all orders of perturbation theory (Chung, 1965) and in nontopological soliton calculations (Wilets, 1989).

In this section, we discuss only the original physical interpretation of the field coherent states as proposed by Schrödinger (1926). In the absence of external fields [i.e., $\lambda(t)=0$ for Eq. (2.6)], the time evolution of a field coherent state is

$$\begin{aligned}|\alpha(t)\rangle&=e^{-iH_0t/\hbar}|\alpha\rangle\\&=e^{-|\alpha|^2/2}\sum_n\frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}}|n\rangle\\&=|\alpha e^{-i\omega t}\rangle\ .\end{aligned} \tag{2.61}$$

By calculating the expectation values of the position and momentum operators of the harmonic oscillator, one obtains

$$\langle\hat{q}\rangle=\frac{1}{\sqrt{2}}\langle\alpha|a+a^\dagger|\alpha\rangle=\frac{1}{\sqrt{2}}(\alpha+\alpha^*)\ , \tag{2.62}$$

$$\langle\hat{p}\rangle=\frac{-i}{\sqrt{2}}\langle\alpha|a-a^\dagger|\alpha\rangle=\frac{-i}{\sqrt{2}}(\alpha-\alpha^*)\ . \tag{2.63}$$

Therefore the coherent states $|\alpha(t)\rangle$ ($\alpha(t)=(1/\sqrt{2})[q(t)+ip(t)]$) are minimum-uncertainty quantum states $\{q(t),p(t)\}$, which follow the classical motion of a harmonic oscillator:

$$\begin{bmatrix}q(t)\\p(t)\end{bmatrix}=\sqrt{2}\begin{bmatrix}\mathrm{Re}\\\mathrm{Im}\end{bmatrix}\alpha(t)=\begin{bmatrix}\mathrm{Re}\\\mathrm{Im}\end{bmatrix}[q(0)+ip(0)]e^{i\omega t}\ . \tag{2.64}$$

*This is the original motivation for Schrödinger's proposing such states.*

In the presence of an interaction [$\lambda(t)\neq0$], the equation of motion is

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}|\varphi(t)\rangle&=H(t)|\varphi(t)\rangle\\&=[\hbar\omega a^\dagger a+\lambda(t)a^\dagger+\lambda^*(t)a]|\varphi(t)\rangle\ .\end{aligned} \tag{2.65}$$

Suppose at $t=0$ the system is in the vacuum state $|0\rangle$. Then

$$|\varphi(t)\rangle = T\exp\left[-\frac{i}{\hbar}\int dt\,(\hbar\omega a^\dagger a+\lambda(t)a^\dagger+\lambda^*(t)a)\right]|0\rangle$$
$$=\exp[\alpha(t)a^\dagger-\alpha^*(t)a]|0\rangle e^{i\eta(t)}$$
$$=|\alpha(t)\rangle e^{i\eta(t)}\,, \tag{2.66}$$

where $T$ is the time-ordering operator and

$$\alpha(t)=-ie^{-i\omega t}\int_0^t\lambda^*(\tau)e^{i\omega\tau}d\tau\,, \tag{2.67a}$$

$$\eta(t)=-\tfrac{1}{2}\omega t-\int_0^t \mathrm{Re}[\lambda(\tau)\alpha(\tau)]d\tau\,. \tag{2.67b}$$

This shows that the system will remain at all times in a coherent state. Furthermore, the following selection rule exists: if the initial state is a coherent state (including the extremal state) and the Hamiltonian is linear in the operators of $H_4$, then the state will evolve into a coherent state, i.e., "*once a coherent state, always a coherent state.*" Therefore, under the condition of Eq. (2.6), if the system is initially in a coherent state, this quantum state will evolve according to a classical trajectory. Moreover, like a true classical particle, this state will never spread, since $\Delta x=\Delta p=$constant. From this point of view, the field coherent states provide a natural framework in which to discuss the relation between quantum and classical mechanics. The classical limit of quantum correlation functions for systems with many degrees of freedom has been extensively discussed by Hepp (1974).

It is worth noting that the study of the manifestation of classical chaos in quantum systems has attracted much attention lately. A major concern of this study is to seek the correspondence between classical and quantum mechanics. Obviously, the coherent states seem to be a natural bridge for studying the quantum-classical correspondence. For example, using the $Q$ representation [also known as the Husimi representation (Husimi, 1940)], one finds that there are precise patterns of classical trajectories in the wave functions. The $Q$ distribution follows the classical periodic orbits (Chang and Shi, 1986; Davis, 1988; Raden and Prange, 1988). Furthermore, for the classical unstable periodic orbits, there are "scars" (Heller, 1984) in the $Q$ distribution (Waterland *et al.*, 1988). The subject is still in its infancy, but we envision that these coherent states will play a vital role in such studies (Zhang, Feng, and Yuan, 1990; Zhang, Feng, Yuan, and Wang, 1989).

## III. GENERAL THEORY OF COHERENT STATES

### A. Generalizing the concept of the coherent state

In Sec. II, two versions of the field coherent states were discussed: one version is Glauber's original description and the other is the more "modern" group-theoretic description. It must be emphasized that, for the field coherent states, the latter description contains all the information given by Glauber's three definitions of field coherent states. We also noticed that by exploring the geometrical (coset) structure of the displacement operator $D(z)$ of Definition 2 [see Eq. (2.2a)], we were able to explore further the topological structure of these states. Of course, for the field coherent states, the topological structure is rather trivial. Thus there is no obvious advantage in the group-theoretic description. However, as we shall now see, the advantage will become manifest once we generalize the concept of coherent states to systems described by dynamical groups different from $H_4$.

Although the harmonic oscillator is important and useful, one often encounters other dynamical systems that cannot be so described. For example, it is usually not a good approximation for an interacting many-fermion system. Therefore it is quite natural to inquire whether there are states that can preserve most of the useful properties of the field coherent states and be utilized to describe and simplify other quantum systems. How does one generalize the concept of field coherent states in order to describe quantum systems with other dynamical groups?

In order to generalize the field coherent states, one should begin by generalizing, if possible, the three definitions, as we have pointed out in the last section. In doing so, one finds that (a) the different generalizations will lead to different results; (b) some of the definitions in fact cannot be generalized. We illustrate the possible generalization procedure here.

#### 1. Generalization of Definition 1 (eigenstates of the lowering operator)

At first glance it would seem attractive to use the eigenstates of the lowering operator as a basis for defining coherent states for arbitrary dynamical systems because eigenvalue equations play a prominent role in quantum theory. However, in this case, the eigenvalue equations are non-Hermitian and the eigenvalues are complex. This approach was adopted by Barut and Girardello (1971) in their discussions about the "new coherent states" associated with the spectrum generating algebra su(1,1). As has been emphasized (Gilmore, 1974a), the adoption of this definition to generalize the concept of the coherent state has two major drawbacks, the first mathematical and the second physical: (a) Coherent states cannot be defined in Hilbert spaces of finite dimensionality in this way. In particular, this would preclude the construction of coherent states for compact Lie groups (Perelomov, 1972). Furthermore, states defined in this way have few useful properties and in particular they are not computationally useful. (b) The states so defined do not correspond to physically realizable states, except under the special circumstance that the commutator of the annihilation operator (or lowering step operator) and its Hermitian adjoint is a multiple of the identity operator. Therefore, under these conditions one restricts oneself to the electromagnetic field. The

generalization of Glauber's Definition 1 to general dynamic systems appears not to be widely applicable. One limiting case in which they appear to be useful is in the construction of wave packets that follow a classical elliptical trajectory in very highly excited states of the hydrogen atom, in a description using the dynamical group SO(4,2) (McAnally and Bracken, 1989).

### 2. Generalization of Definition 3 (states of minimum uncertainty)

According to Glauber's third definition, coherent states are also minimum-uncertainty states. This is essentially the original motivation of Schrödinger in his construction of wave packets which follow the motion of a classical particle while retaining their shapes. Minimum-uncertainty states are those which saturate the inequality $\Delta X^2 \Delta Y^2 \geq |\frac{1}{2}\langle [X,Y]\rangle|^2$, where $X$ and $Y$ are operators in a Lie algebra. The generalization along this direction was carried out and named "intelligent" states by Aragone and collaborators (Aragone *et al.*, 1974, 1976). Nieto and co-workers (Nieto and Simmons, 1978; 1979; Nieto, Simmons, and Gutschick, 1981; Nieto, 1983) have extensively discussed such states for various dynamical systems. However, this generalization has several limitations. First of all, these coherent states can only be constructed for the classically integrable systems in which there exists a set of canonical coordinates and momenta such that the respective Hamiltonians can be reduced to quadrature. This condition requires a flatness condition on the Lie algebra, which reduces the commutation relations to those of the standard photon creation and annihilation operators. Secondly, the wave packets with minimum uncertainty are not unique (see Fig. 1). Different ones may have different properties. Such states may also be incomplete, or even if they are complete it is not certain that a resolution of unity of the standard form exists (Klauder and Skagerstam, 1985). Thus minimum-uncertainty states appear to have few, if any, useful properties. For example, minimum-uncertainty states do not evolve into minimum-uncertainty states when they are driven by a Hamiltonian linear in the generators in the Lie group.

### 3. Generalization of Definition 2 (displacement operator acting on a reference state)

The discussion above shows that in attempting to generalize the concept of the coherent state, it may be more useful to adopt the definition in terms of a displaced reference state as a point of departure (Gilmore, 1972, 1974a; Perelomov, 1972, 1977). We shall now present the group-theoretic construction algorithm, paralleling the treatment of $H_4$ in Sec. II.B.

## B. General definition of coherent states: An algorithm

Consider an arbitrary quantum-dynamic system. In general, the Hamiltonian and transition operators $\{A\}$ of the system can be expressed in terms of a complete set of operators $\{T_i\} \equiv \mathbf{g}$,

$$H = H(T_i), \quad A = A(T_i) , \tag{3.1}$$

where the completeness of an operator set means that $\{T_i\} \equiv \mathbf{g}$ spans a closed algebra, i.e., the commutator of any two operators $T_i, T_j \in \mathbf{g}$ still belongs to $\mathbf{g}$: $[T_i, T_j] \in \mathbf{g}$.

In most applications of quantum mechanics, the Hamiltonian of the system can be simplified into the following generic form of linear and quadratic functions of operators $\{T_i\}$ under a mean-field approximation:

$$H = \sum_i d_i T_i \tag{3.2}$$

or

$$H = \sum_i c_i T_i + \sum_{i,j} c_{ij} T_i T_j . \tag{3.3}$$

For example, in the case of Eq. (2.4), the Hamiltonian has the form of Eq. (3.3), in which the quadratic operator terms are constructed from the set $\{a^\dagger a, a^\dagger, a, I\}$ and $\{\sigma_0, \sigma_+, \sigma_-\}$. On the other hand, when the atomic system is treated classically, the Hamiltonian of Eq. (2.4) can be reduced to Eq. (2.5), in which there are only linear operator terms constructed from $\{a^\dagger a, a^\dagger, a, I\}$. Another example is the many-body fermion (boson) system in which the Hamiltonian under a mean-field approximation can generally be expressed as (Fetter and Walecka, 1971)

$$H = \sum_{ij} k_{ij} a_i^\dagger a_j + \sum_{ijlm} V_{ijlm} a_i^\dagger a_j^\dagger a_m a_l \quad (i,j,l,m = 1, \ldots n) , \tag{3.4}$$

where $a_i^\dagger$, $a_i$ are the creation and annihilation fermion (boson) operators. The Hamiltonian given by Eq. (3.4) is of the generic type of Eq. (3.3) with the operator set $\{a_i^\dagger a_j, a_i^\dagger a_j^\dagger, a_i a_j\}$ spanning a closed algebra so(2n)[sp(2n)] for fermions (bosons) (Gilmore, 1974b; Wybourne, 1974)].

### 1. Three inputs

We now proceed to define the general coherent states. Just as in Sec. II, three inputs are required in the construction.

(a) Dynamical group $G$ with its algebra $\mathbf{g}$. The algebra $\mathbf{g}$ is determined by the operator algebraic properties of a quantum system. According to Eq. (3.1), the algebra $\mathbf{g}$ is spanned by operators $\{T_i\}$ closed under commutation:

$$[T_i, T_j] = \sum_k C_{ij}^k T_k . \tag{3.5}$$

In Eq. (3.5), $C_{ij}^k$ are the structure constants of $\mathbf{g}$. If $\mathbf{g}$ is a semisimple Lie algebra, then it is more convenient to transform $\{T_i\}$ into the standard Cartan basis $\{H_i, E_\alpha, E_{-\alpha}\}$:

$$[H_i, H_j]=0 , \tag{3.6a}$$

$$[H_i, E_\alpha]=\alpha_i E_\alpha , \tag{3.6b}$$

$$[E_\alpha, E_{-\alpha}]=\alpha^i H_i , \tag{3.6c}$$

$$[E_\alpha, E_\beta]=N_{\alpha;\beta} E_{\alpha+\beta} , \tag{3.6d}$$

and Eq. (3.2) can be rewritten as

$$H=\sum_i \varepsilon_i H_i+\sum_\alpha (\lambda_\alpha E_\alpha+\lambda_\alpha^* E_{-\alpha}) . \tag{3.7}$$

(b) Hilbert space $V^\Lambda$. For a given Hamiltonian of Eq. (3.1), the physical state (Hilbert) space $V^\Lambda$ carries a unitary irreducible representation $\Gamma^\Lambda$ of the group $G$.

(c) Reference state. This is a state $|\Phi_0\rangle \equiv |\text{ref}\rangle$ within the Hilbert space $V^\Lambda$, which can be normalized to unity: $\langle \Phi_0|\Phi_0\rangle = 1$.

It must be emphasized that the choice of the reference state is in principle arbitrary. However, for a given dynamic system, construction of a useful set of coherent states depends strongly on the choice of the reference state $|\Phi_0\rangle$. The state $|\Phi_0\rangle$ will determine not only the structure of the coherent states but also the structure of the phase space of the dynamic system, as we shall discuss in Sec. III.B.3.b.

### 2. Three outputs

The group-theoretic algorithm generates the coherent states in three steps, utilizing the stability subgroup and its coset.

(a) Maximum stability subgroup. A subgroup of $G$ that consists of all the group elements $h$ that will leave the reference state invariant up to a phase factor is the maximum-stability subgroup $H$. Formally,

$$h|\Phi_0\rangle = |\Phi_0\rangle e^{i\phi(h)}, \quad h\in H . \tag{3.8}$$

The phase factor is unimportant here because we shall generally take the expectation value of any operators in the coherent state.

(b) Quotient or coset space $G/H$. For every element $g\in G$, there is a unique decomposition of $g$ into a product of two group elements, one in $H$ and the other in the quotient $G/H$:

$$g=\Omega h, \quad g\in G, \quad h\in H, \quad \Omega\in G/H . \tag{3.9}$$

In other words, we can obtain a unique coset space for a given $|\Phi_0\rangle$.

(c) Coherent states $|\Lambda,\Omega\rangle$. Based on Sec. II.B.2, one sees that the action of an arbitrary group element $g\in G$ on $|\Phi_0\rangle$ is given by

$$g|\Phi_0\rangle = \Omega h|\Phi_0\rangle = \Omega|\Phi_0\rangle e^{i\phi(h)} . \tag{3.10}$$

The combination $\Omega|\Phi_0\rangle$, rewritten as

$$|\Lambda,\Omega\rangle \equiv \Omega|\Phi_0\rangle , \tag{3.11}$$

is *the general group definition of the coherent states.* Equation (3.11) guarantees that this definition of the coherent states is in one-to-one correspondence with the coset space $G/H$ ($\Omega\in G/H$). Therefore the coherent states preserve all the algebraic and topological properties of the coset space $G/H$.

### 3. Remarks

#### *a. Difference between Gilmore's and Perelomov's constructions*

The above constructions of the coherent states for any arbitrary dynamic systems are carried out almost entirely from purely mathematical considerations by Perelomov and Gilmore independently (Gilmore, 1972, 1974a; Perelomov, 1972). The idea behind this definition was first proposed by Klauder (1963) a decade earlier. The coherent states constructed in this algorithm depend on the choice of $G$, $V^\Lambda$, and $|\Phi_0\rangle$. Gilmore has presented a detailed discussion of the various possible constructions of the coherent states using the above algorithm and their different properties (Gilmore, 1974a).

First, the group $G$ may be an arbitrary dynamic symmetry group in general (Gilmore, 1974a). We may also impose sufficient additional structure on $G$ to make it a finite-dimensional Lie group (Perelomov, 1972), or we may impose further additional structure and demand that $G$ be compact.

Second, the unitary irreducible representation $\Gamma^\Lambda$ may be arbitrary (Perelomov, 1972). But we may also demand that $\Gamma^\Lambda$ be a square integrable by requiring $G$ to be finite dimensional and choose only its square-integrable representations (Gilmore, 1974a). By requiring that $G$ be compact we find only finite-dimensional unitary irreducible representations.

TABLE I. Possibilities available for $\{G, V^\Lambda, |\Phi_0\rangle\}$ in constructing coherent states.

| Level of structure | Refers to | Severity of constraints 1 | 2 | 3 |
|---|---|---|---|---|
| $A$ | $G$ | dynamical symmetry group | Lie group | compact |
| $B$ | $V^\Lambda$ | arbitrary unitary irreducible rep. | square-integrable space | finite-dimensional |
| $C$ | $\|\Phi_0\rangle$ | arbitrary, normalized to unity | eigenstate of unperturbed $H_0$ | highest (lowest) state of $V^\Lambda$ |

TABLE II. Differences between Gilmore's and Perelomov's definitions.

| | Gilmore | Perelomov |
|---|---|---|
| $G$ | $A1$ | $A2$ |
| $V^\Lambda$ | $B2$ | $B1$ |
| $\|\Phi_0\rangle$ | $C3$ | $C1$ |

Finally, the reference state $|\Phi_0\rangle$ may be an arbitrary state in $V^\Lambda$ (Perelomov, 1972), but by imposing additional constraints, we could demand that it be the eigenstate of some unperturbed Hamiltonian. Moreover, we can impose yet more structure and demand that it be an extremal state annihilated by a maximal subset of the algebra $\mathbf{g}$ of $G$ (Gilmore, 1974a). If $G$ is semisimple, then $|\Phi_0\rangle$ will be the highest (lowest)-weight state of $\Gamma^\Lambda$ (Gilmore, 1974a).

All possibilities listed above are summarized in Table I.

This discussion shows that there are several choices for constructing coherent states. The definitions of the general coherent states, established by Gilmore and Perelomov, are summarized in Table II.

### *b. Physical choice of the reference state and physical interpretation of the coherent states*

In the algorithm for constructing coherent states, the choice of the maximum stability subgroup $G$ is often well defined for a given physical system because it, or the corresponding algebra $\mathbf{g}$, is completely determined by the algebraic properties of the Hamiltonian and transitional operators. In practice, $G$ is often a Lie group. The second input of the algorithm is also well determined since physics requires the carrier space of the unitary irreducible representation $V^\Lambda$ of $G$ to be the physical state space of interest. The third input of the algorithm, however, is more arbitrary but is of vital importance for constructing physically sensible coherent states. It will ultimately decide, on the one hand, the structure of the coherent states, and, on the other hand, the topological structure of the dynamical system, as will become clear later. It is usually most useful to choose the reference state $|\Phi_0\rangle$ to be an unperturbed physical ground state. After all, in the absence of interaction, when the Hamiltonian of Eq. (3.1) consists only of linear terms [see Eq. (3.7)], the physical ground state is mathematically an "extremal state." An "extremal state," at least for a discrete spectrum, is defined to be the highest-weight[2] state $|\Lambda,\Lambda\rangle$ of the irrep of the Lie group $G$,

$$E_\alpha|\Lambda,\Lambda\rangle=0, \quad \alpha\in\text{positive root set of } G\ . \tag{3.12}$$

[2]We could just as well choose the lowest-weight state. For clarity, we assume the highest-weight state in the following formulas.

With this choice, the coset space as well as the coherent states (3.11) are constructed by the exponential map of those shift-down (shift-up) operators which, when acting on the extremal state, produce a nonzero result ($E_{-\alpha}|\Lambda,\Lambda\rangle\neq 0$), and their conjugate operators.

Summarizing the above discussions, in practical applications, the three important inputs in the construction of the coherent states are the Lie group $G$, its square-integrable irrep $\Gamma^\Lambda$, and the extremal state $|\text{ext}\rangle$ of its carrier space.

Up to now, we have not been specific in our discussions of the coherent state $|\Lambda,\Omega\rangle$. Since the dynamical groups in practice are often Lie groups, we shall confine our discussions to such groups.

In any $l$-rank $m$-dimensional semisimple Lie algebra $\mathbf{g}$, there are two types of basic operators in the Cartan basis, $H_i$ and $E_\alpha$ [see Eqs. (3.6)]. The operators $H_i$ may be taken as diagonal in any unitary irreducible representation $\Gamma^\Lambda$, while $E_\alpha$ are the "shift operators." If $\Gamma^\Lambda(\mathbf{g})$ is Hermitian, then $H_i^\dagger=H_i$, and $E_\alpha^\dagger=E_{-\alpha}$. Every group element $g\in G$ can be written as the exponential of an anti-Hermitian complex linear combination of $H_i$ and $E_\alpha$. If $\Lambda$ is the highest weight in the representation space $V^\Lambda$, the highest-weight state $|\Lambda,\Lambda\rangle$ which is often the ground state of the unperturbed Hamiltonian, is

(i) annihilated by all the shift-up operators $E_\alpha$ with $\alpha>0$,

$$E_\alpha|\Lambda,\Lambda\rangle=0, \quad \alpha>0\ ; \tag{3.13}$$

(ii) mapped into itself by all diagonal operators $H_i$,

$$H_i|\Lambda,\Lambda\rangle=\Lambda_i|\Lambda,\Lambda\rangle\ ; \tag{3.14}$$

(iii) annihilated by some shift-down operators $E_\alpha$ with $\alpha<0$, not by other $E_\beta$ with $\beta<0$:

$$E_\alpha|\Lambda,\Lambda\rangle=0, \quad \text{some } \alpha<0\ , \tag{3.15a}$$

$$E_\beta|\Lambda,\Lambda\rangle=|\Lambda,\Lambda+\beta\rangle\times\text{factor}, \quad \text{some } \beta<0\ . \tag{3.15b}$$

*The coherent states $|\Lambda,\Omega\rangle$ are explicitly written as Eq. (3.11), with a generalized "displacement operator" in one-to-one correspondence to the coset representatives $\Omega$ of $G/H$,*

$$\Omega=\exp\left[\sum_\beta{}' \eta_\beta E_\beta-\text{H.c.}\right], \tag{3.16}$$

*acting on* $|\text{ext}\rangle=|\Lambda,\Lambda\rangle$. In Eq. (3.16), the parameters $\eta_\beta$ are complex numbers, and H.c. means Hermitian conjugation, $\sum'_\beta$ is restricted to those shift operators which obey Eq. (3.15b) and excludes those which obey (3.15a).

Based on the above general definition and analysis of the coherent states, we can now discuss their properties in detail.

## C. Properties of the coherent states

### 1. Geometrical structure of $|\Lambda,\Omega\rangle$

The coherent states $|\Lambda,\Omega\rangle$ are in one-to-one correspondence with the coset representatives $\Omega\in G/H$,

$$|\Lambda,\Omega\rangle \leftrightarrow G/H \ , \tag{3.17}$$

i.e., $|\Lambda,\Omega\rangle$ and $G/H$ are *topologically equivalent.* This crucial development is discussed throughout this review. This topological interpretation of the coherent states provides a vivid description of the properties of a physical system. An important property is that the geometry of the coherent states depends sensitively on $G$ and $V^{\Lambda}$. As an example, consider the SU(3) group. For this group, the generators in the Cartan standard basis are $\{H_1,H_2,E_\alpha,E_\beta,E_{\alpha+\beta},E_{-\alpha},E_{-\beta},E_{-\alpha-\beta}\}$, where $\alpha,\beta$ are the two fundamental positive roots and the root space diagram is shown in Fig. 3.

The two fundamental representations of SU(3) are shown in Fig. 4, where $|f_i\rangle$ $(i=1,2)$ are the highest-weight states of these two representations. Each of these requires two variable quantum numbers to label the bases of its carrier space. Correspondingly, because $E_{-\beta}|f_1\rangle=0$ the maximum-stability subgroup which leaves $|f_1\rangle$ invariant is U(2) with $\{H_1,H_2,E_\beta,E_{-\beta}\}$ as generators. Thus the geometry of coherent states for the two fundamental representations of SU(3) is SU(3)/U(2). On the other hand, for any arbitrary irrep of SU(3), the highest-weight $\Lambda$ can always be expressed as $\Lambda=\sum_{i=1}^{2}\mu_i f_i$. If $\mu_1$ (or $\mu_2$)$=0$, the coherent states are isomorphic to SU(3)/U(2). However, when $\mu_1\neq0$ and $\mu_2\neq0$, the maximum-stability subgroup is $U(1)\otimes U(1)$ with generators $\{H_1,H_2\}$. Therefore it is obvious that the corresponding geometry of the coherent states is not SU(3)/U(2) but $SU(3)/U(1)\otimes U(1)$. A physical interpretation of these cases will be given in Sec. VI.C.5.

The above discussions can be extended to a general $l$-rank $m$-dimensional semisimple Lie group. The reason is that the highest-weight $\Lambda$ can always be expressed in terms of the highest weights $f_i$ of the $l$ fundamental irreducible representations of $\mathbf{g}$,

$$\Lambda=\sum_{i=1}^{l}\mu_i f_i \ , \tag{3.18}$$

where $\mu_i$ are non-negative integers. Therefore it is not difficult to find the negative roots $\beta$ for which $E_\beta|f_i\rangle\neq0$ for each fundamental representation. From this one can easily determine the dimensionality of the geometric space $G/H$ which parametrizes $|f_i,\Omega\rangle$. If $\Lambda$ is the sum

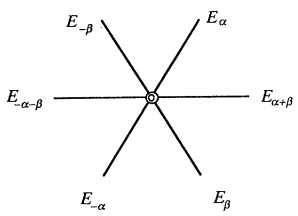



FIG. 3. Root space diagram of the su(3) algebra.

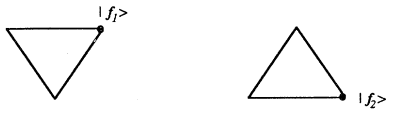



FIG. 4. The weight diagram of the two fundamental representations of su(3).

of two or more distinct fundamental weights, then the sum over shift-down operators in (3.16) includes all operators that do not annihilate any weight $f_i$ that occurs in (3.18) with $\mu_i\neq0$. Since the parameter space of the coherent states $(G/H)$ depends sensitively on the space $V^{\Lambda}$, it can be defined as a geometric space of the quantum-dynamic system. Furthermore, we can obtain an explicit structure of this geometric space for any Lie group. For non-semisimple Lie groups, the above procedure still holds; an application related to squeezed states will be given in Table V.

### *a. Complex structure*

For $G$, the coherent states of Eq. (3.11) can be explicitly expressed [see Eq. (3.16)] as

$$|\Lambda,\Omega\rangle=\exp\sum_{\beta}{}'(\eta_\beta E_\beta-\eta_\beta^* E_{-\beta})|\text{ext}\rangle \ , \tag{3.19}$$

which is in one-to-one correspondence with the coset space $G/H$. Let $\mathbf{g}=\mathbf{k}\oplus\mathbf{p}$, where $\mathbf{k}$ is the Lie algebra of $H$ and $\mathbf{p}=\eta_\beta E_\beta-\eta_\beta^* E_{-\beta}$ is its orthogonal complement. We shall explicitly consider only those cases in which the decomposition of semisimple Lie algebra $\mathbf{g}=\mathbf{k}\oplus\mathbf{p}$ satisfies the Cartan decomposition:

$$[\mathbf{k},\mathbf{k}]\subset\mathbf{k}, \quad [\mathbf{k},\mathbf{p}]\subset\mathbf{p}, \quad [\mathbf{p},\mathbf{p}]\subset\mathbf{k} \ . \tag{3.20}$$

When $G$ is compact, its defining matrix representation has the following form:

$$\mathrm{R}(\mathrm{k}) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad \mathrm{R}(\mathrm{p}) = \begin{bmatrix} 0 & \eta \\ -\eta^* & 0 \end{bmatrix}. \tag{3.21}$$

The matrices $\mathbf{R}(\mathbf{k})$ and $\mathbf{R}(\mathbf{p})$ are skew symmetric and real: $a^t=-a$, $b^t=-b$. For noncompact $G$, we have

$$\mathrm{R}(\mathrm{k}) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad \mathrm{R}(\mathrm{p}) = \begin{bmatrix} 0 & \eta \\ +\eta^* & 0 \end{bmatrix} \tag{3.22}$$

where $\mathbf{R}(\mathbf{k})$ is a skew-symmetric matrix and $\mathbf{R}(\mathbf{p})$ is a symmetric matrix. Thus $G/H$ is a symmetric space. It can be expressed in a matrix form as follows (Hua, 1963; Helgason, 1978). For compact $G$,

$$G/H\rightarrow\begin{bmatrix} \sqrt{I-zz^\dagger} & z \\ -z^\dagger & \sqrt{I-z^\dagger z} \end{bmatrix}, \quad \text{compact} \tag{3.23a}$$

with

$$z=\eta\frac{\sin\sqrt{\eta^\dagger\eta}}{\sqrt{\eta^\dagger\eta}} . \tag{3.23b}$$

For a noncompact Lie group $G$ the representation is

$$G/H\rightarrow\begin{bmatrix}\sqrt{I+zz^\dagger} & z\\ +z^\dagger & \sqrt{I+z^\dagger z}\end{bmatrix}, \quad \text{noncompact} \tag{3.24a}$$

with

$$z=\eta\frac{\sinh\sqrt{\eta^\dagger\eta}}{\sqrt{\eta^\dagger\eta}} . \tag{3.24b}$$

The results given by Eqs. (3.23b) (a sine function) and (3.24b) (a sine hyperbolic function) are clear in manifesting the compactness and noncompactness of the respective geometry. In Eqs. (3.23) and (3.24), $z$ is an $m\times n$ complex matrix with $m$ and $n$ the dimensions of $k$ and $p$. If we explicitly introduce a complex projective representation $\tau$ of $G/H$,

$$\tau=z(1\pm z^\dagger z)^{-1/2} , \tag{3.25}$$

where the $+$ $(-)$ sign corresponds to the noncompact (compact) case, then any group transformation $g$ acting on $G/H$ must be a holomorphic transformation,

$$\tau'=g\tau=(A\tau+B)(C\tau+D)^{-1} \tag{3.26}$$

where

$$g=\begin{bmatrix}A & B\\ C & D\end{bmatrix}\in G . \tag{3.27}$$

### b. Riemannian structure of the coset space

The standard metric of $G/H$ can be found (Hua, 1963) from Eq. (3.26) by restricting the group transformation on $G/H$. Here, we shall express it in the form

$$ds^2=\sum_{\alpha,\beta}g_{\alpha\beta}d\tau_\alpha d\tau_\beta^* , \tag{3.28}$$

where $\alpha$ and $\beta$ are summed over all the matrix elements of $\tau$ given by Eq. (3.25). In terms of coherent states (Zhang, Feng, and Yuan, 1989), $g_{\alpha\beta}$ is given by

$$g_{\alpha\beta}=\frac{\partial^2 F(\tau,\tau^*)}{\partial\tau_\alpha\partial\tau_\beta^*} . \tag{3.29}$$

The function $F(\tau,\tau^*)$ is obtained from the unnormalized form of the coherent states:

$$\begin{aligned}|\Lambda,\Omega\rangle&=\exp\left[\sum_\beta{}'\,\eta_\beta E_\beta-\text{H.c.}\right]|\text{ext}\rangle\\ &=N^{-1/2}(\tau,\tau^*)\exp\left[\sum_\beta{}'\,\tau_\beta E_\beta\right]|\text{ext}\rangle\\ &=N^{-1/2}(\tau,\tau^*)\|\Lambda,\tau\rangle\end{aligned} \tag{3.30a}$$

and

$$F(\tau,\tau^*)=\ln N(\tau,\tau^*)=\ln(\langle\Lambda,\tau\|\Lambda,\tau\rangle) . \tag{3.30b}$$

When $G/H$ is a symmetric space, the normalization constant $N(\tau,\tau^*)$ can easily be calculated. For example, the field coherent states (2.2a) and (2.2b) can be rewritten in terms of the unnormalized form by using the BCH formulas (2.19):

$$|\alpha\rangle=e^{-|\alpha|^2/2}\exp(\alpha a^\dagger)|0\rangle=e^{-|\alpha|^2/2}\|\alpha\rangle . \tag{3.31}$$

Then the function $F(\alpha,\alpha^*)$ is

$$F(\alpha,\alpha^*)=-\alpha\alpha^* \tag{3.32}$$

and the metric of the coset space $H_4/\mathrm{U}(1)\otimes\mathrm{U}(1)$ is obtained directly from Eq. (3.29),

$$g_{ij}=\delta_{ij} , \tag{3.33}$$

which is identical to the transformation of Eq. (2.16). Other examples will be presented in subsequent sections.

Similarly, the measure of the coset space is (Hua, 1963)

$$d\mu=\text{const}\times\det(g_{\alpha\beta})\prod_\alpha\frac{dz_\alpha dz_\alpha^*}{\pi} . \tag{3.34}$$

### c. Symplectic structure

From the definition of Eq. (3.19), the coherent states have a natural sympletic structure when the reference state is chosen as an extremal state (Onofri, 1975; Simon, 1980). First of all, the dimension of the coset space $G/H$ is even. Furthermore, the nondegenerate closed 2-form (Helgason, 1978) of $G/H$ can be written explicitly as

$$\omega=i\sum_{\alpha,\beta}g_{\alpha\beta}d\tau_\alpha\wedge d\tau_\beta^* . \tag{3.35}$$

Therefore the Poisson bracket on $G/H$ is

$$\{f,g\}=-i\sum_{\alpha,\beta}{}'\,g^{\alpha\beta}\left[\frac{\partial f}{\partial\tau_\alpha}\frac{\partial g}{\partial\tau_\beta^*}-\frac{\partial g}{\partial\tau_\alpha}\frac{\partial f}{\partial\tau_\beta^*}\right] \tag{3.36}$$

where the prime in the sum is defined immediately after Eq. (3.16). By transforming $\tau$ to the $z$ parameters of Eqs. (3.23b) and (3.24b), one obtains the so-called diagonal form of the Poisson bracket:

$$\{f,g\}=-i\sum_\beta{}'\left[\frac{\partial f}{\partial z_\beta}\frac{\partial g}{\partial z_\beta^*}-\frac{\partial g}{\partial z_\beta}\frac{\partial f}{\partial z_\beta^*}\right] . \tag{3.37}$$

Performing the transformation

$$z_\beta=\frac{1}{\sqrt{2}}(q_\beta+ip_\beta),\quad z_\beta^*=\frac{1}{\sqrt{2}}(q_\beta-ip_\beta) , \tag{3.38}$$

we obtain as in classical mechanics (Goldstein, 1980) the standard form of the symplectic structure:

$$\{f,g\}=\sum_\beta{}'\left[\frac{\partial f}{\partial q_\beta}\frac{\partial g}{\partial p_\beta}-\frac{\partial g}{\partial q_\beta}\frac{\partial f}{\partial p_\beta}\right] . \tag{3.39}$$

### 2. The Baker-Campbell-Hausdorff formula

There are some additional group-theoretic formulas that are useful for coherent-state calculations. They are the famous BCH formulas, which are defined as the analytic isomorphism connecting simple exponentials of the algebra elements and products of exponentials (Helgason, 1978). We already had an example with Eq. (2.19) for the $H_4$ coherent states. We shall shortly discuss SU(2), which has the BCH formula Eq. (3.96). In general, the exponentiated element given formally by Eq. (3.16) can be reexpressed as the following product:

$$\begin{aligned}\exp\sum_{\beta}{}'(\eta_\beta E_\beta-\eta_\beta^* E_{-\beta})&=\left[\exp\sum_{\alpha}\tau_\alpha E_\alpha\right]\left[\exp\sum_i\gamma_i H_i\right]\left[\exp\sum_\beta-\tau_\beta^* E_{-\beta}\right]\\&=\left[\exp\sum_\beta-\tau_\beta^* E_{-\beta}\right]\left[\exp\sum_i-\gamma_i H_i\right]\\&\quad\times\left[\exp\sum_\alpha\tau_\alpha E_\alpha\right].\end{aligned}\tag{3.40}$$

The "breaking up" of the exponentials of operators is crucial to the technical development of coherent states. The relation between $\eta$ and $\tau$ can be derived from the matrix representation of $G$. For example, if $G/H$ is an appropriate symmetric space, the useful representative of (3.40) is

$$\begin{bmatrix}\sqrt{I\pm zz^\dagger} & z\\ \pm z^\dagger & \sqrt{I\pm z^\dagger z}\end{bmatrix}=\begin{bmatrix}1&\tau\\0&1\end{bmatrix}\begin{bmatrix}\exp\gamma_1&0\\0&\exp\gamma_2\end{bmatrix}\begin{bmatrix}1&0\\\pm\tau^\dagger&1\end{bmatrix}.\tag{3.41}$$

We then have

$$\begin{aligned}\tau&=z(I\pm z^\dagger z)^{-1/2},\\ \exp\gamma_1&=(I\pm zz^\dagger)^{-1/2},\\ \exp\gamma_2&=(I\pm z^\dagger z)^{1/2}.\end{aligned}\tag{3.42}$$

### 3. Hilbert-space properties

Just as in the field coherent states, two important algebraic properties are maintained by the generalized coherent states:

(a) Non-orthogonality. The coherent states $|\Lambda,\Omega\rangle$ defined by Eq. (3.10) are generally non-orthogonal (except for a set of measure zero),

$$\begin{aligned}\langle\Lambda,\Omega|\Lambda,\Omega'\rangle&=\langle\Phi_0|\Omega^{-1}\Omega'|\Phi_0\rangle\\&=\langle\Phi_0|\Omega''|\Phi_0\rangle e^{i\psi}\neq 0,\quad(\text{all }\Omega\neq\Omega'\in G/H)\end{aligned}\tag{3.43}$$

but normalized to unity,

$$\begin{aligned}\langle\Lambda,\Omega|\Lambda,\Omega\rangle&=\langle\Phi_0|\Omega^{-1}\Omega|\Phi_0\rangle\\&=\langle\Phi_0|\Phi_0\rangle=1.\end{aligned}\tag{3.44}$$

(b) Over-completeness. From definition (3.10), the coherent states $\{|\Lambda,\Omega\rangle\}$ span an invariant subspace of $V^\Lambda$ under $\Omega\in G/H$. On the other hand, $V^\Lambda$ is an invariant space under $G$; therefore $|\Lambda,\Omega\rangle$ must span the entire space $V^\Lambda$, which means that $|\Lambda,\Omega\rangle$ is complete. Furthermore, consider the operator

$$O=\int|\Lambda,\Omega\rangle d\mu(G/H)\langle\Lambda,\Omega|,\tag{3.45}$$

where $d\mu(G/H)=d\mu(\Omega)$ is the group-invariant measure of $G$. One can prove that the operator $O$ is invariant under the transformation of $G$,

$$\begin{aligned}gO&=\int g|\Lambda,\Omega\rangle d\mu(\Omega)\langle\Lambda,\Omega|g^{-1}g\\&=\int|\Lambda,\Omega'\rangle e^{i\phi'(h)}d\mu(\Omega)e^{-i\phi'(h)}\langle\Lambda,\Omega|g=Og.\end{aligned}\tag{3.46}$$

We have made use of the group-measure-invariance property $d\mu(\Omega)=d\mu(\Omega')$ in deriving the results given by Eq. (3.46). Then, according to Schurr's lemma, $O$ must be proportional to the identity operator, since the space is invariant under $G$. Hence, with an appropriately normalized measure $d\mu$, we can get

$$\int|\Lambda,\Omega\rangle d\mu(\Omega)\langle\Lambda,\Omega|=I.\tag{3.47}$$

Equations (3.43) and (3.47) indicate that $|\Lambda,\Omega\rangle$ are over-complete. Such over-completeness of $|\Lambda,\Omega\rangle$ is particularly transparent for any compact Lie groups in which $\Gamma^\Lambda$ is a unitary irreducible representation and $V^\Lambda$ finite dimensional. Since $|\Lambda,\Omega\rangle$ are also continuous (with continuous variable $\Omega$) they must be over-complete. As we have seen in the case for the field coherent states, many of the most useful properties of the coherent states are in fact derived from their over-completeness property. However, it is worth noting that care must be exercised about the uniqueness when using over-completeness properties.

### 4. Hilbert-space expansions

We consider two types of expansions.

(a) Expansion of the coherent states in terms of diagonal states. Suppose a complete set of orthonormal states of $V^\Lambda$ are denoted by $|\Lambda,\lambda\rangle$, where

$$\sum_\lambda|\Lambda,\lambda\rangle\langle\Lambda,\lambda|=I\tag{3.48}$$

and $\lambda$ is the usual Gel'fand-Tsetlein pattern which indexes all the basic states in $V^\Lambda$, including weight and multiplicity. In this case the coherent states can be expressed in terms of this basis of $V^\Lambda$:

$$|\Lambda,\Omega\rangle=\sum_\lambda f_\lambda^\Lambda(\Omega)|\Lambda,\lambda\rangle N^{-1/2}(\Omega)\tag{3.49}$$

where $N(\Omega)$ is given by Eq. (3.30a) and

$$f^{\Lambda}_{\lambda}(\Omega)=\langle \Lambda,\lambda|\Lambda,\Omega\rangle N^{1/2}=\langle \Lambda,\lambda\|\Lambda,\tau\rangle \tag{3.50}$$

is an analytical (entire) function on $G/H$.

(b) Expansion of arbitrary state $|\Psi\rangle$. Any arbitrary state $|\Psi\rangle\in V^{\Lambda}$ can be expanded in the coherent-state representation by using the over-complete relationship

$$|\Psi\rangle=\int |\Lambda,\Omega\rangle f_{\Lambda}(\Omega)N^{-1/2}(\Omega)d\mu(\Omega)\ , \tag{3.51}$$

where $f_{\Lambda}(\Omega)$ is the expansion coefficient (continuous in $\Omega$) defined on the coset space $G/H$. Since $|\Lambda,\Omega\rangle$ is over-complete, the expansion of Eq. (3.51) is obviously not unique. This means that $f_{\Lambda}(\Omega)$ and $f_{\Lambda}(\Omega)+f'_{\Lambda}(\Omega)$ will both satisfy Eq. (3.51) if the following condition is obeyed:

$$\int \langle \Lambda,\Omega|\Lambda,\Omega'\rangle f'_{\Lambda}(\Omega')N^{1/2}(\Omega)N^{-1/2}(\Omega')d\mu(\Omega')=0 \tag{3.52a}$$

or

$$\int \langle \Lambda,\tau\|\Lambda,\tau'\rangle f'_{\Lambda}(\tau')N^{-1}(\tau')d\mu(\tau')=0 \tag{3.52b}$$

where $d\mu(\tau)\equiv d\mu(\Omega)$. Hence the unique expansion of $|\Psi\rangle$ places a requirement on $f_{\Lambda}(\Omega)$, namely, that it satisfy

$$\int \langle \Lambda,\Omega|\Lambda,\Omega'\rangle f_{\Lambda}(\Omega')N^{1/2}(\Omega)\times N^{-1/2}(\Omega')d\mu(\Omega')=f_{\Lambda}(\Omega) \tag{3.53a}$$

or

$$\int \langle \Lambda,\tau\|\Lambda,\tau'\rangle f_{\Lambda}(\tau')N^{-1}(\tau')d\mu(\tau')=f_{\Lambda}(\tau)\ . \tag{3.53b}$$

It follows from Eq. (3.51) that the Hilbert space $V^{\Lambda}$ is spanned by the eigenstates $f_{\Lambda}(\Omega)$ which satisfy an (integral) eigenvalue equation with the coherent-state overlap $\langle \Lambda,\Omega|\Lambda,\Omega'\rangle$ serving as the kernel (Klauder, 1963). Using Eqs. (3.51) and (3.53), we get

$$f_{\Lambda}(\Omega)\equiv f_{\Lambda}(\tau)=\langle \Lambda,\tau\|\Psi\rangle\ . \tag{3.54}$$

Moreover, the eigensolution of the integral equation (3.53) is given by Eq. (3.50), which constructs the basis vectors of the function space $L^2(G/H)$. The scalar product on $L^2(G/H)$ is

$$(f_1,f_2)=\int f_1^*(\Omega)f_2(\Omega)N^{-1}(\Omega)d\mu(\Omega)\ , \tag{3.55}$$

i.e., $N^{-1}(\Omega)d\mu(\Omega)$ is the measure on the function space $L^2(G/H)$.

## 5. Phase-space distributions

There are two useful properties of the phase-space distribution: one is the association of a function on the phase space with an operator on the Hilbert space and the other is the computation of Hilbert-space averages by taking integrals over the phase space. The coherent states provide a natural phase-space structure for a given quantum system and three useful phase-space distributions based on the coherent-state representation, which we now discuss.

A general operator $B$ (see Sec. II.C.4) that acts on the invariant space $V^{\Lambda}$ can be expanded in terms of coherent states:

$$B=\int |\Lambda,\Omega\rangle\langle \Lambda,\Omega|B|\Lambda,\Omega'\rangle\langle \Lambda,\Omega'|d\mu(\Omega)d\mu(\Omega')\ . \tag{3.56}$$

Equation (3.56) may be reduced to the three special representations that are widely used in discussions of the analogous classical phase-space distributions and calculations of the statistical average of the original operators. These are the $P$, $Q$, and $W$ representations.

(a) $P$ representation. The operator $B$ can be expressed in a diagonal form,

$$B=\int |\Lambda,\Omega\rangle B_P(\Lambda,\Omega)\langle \Lambda,\Omega|d\mu(\Omega)\ . \tag{3.57}$$

This is known as the $P$ representation of the operator $B$. However, this expansion is of course generally unphysical.

The $P$ representation of the density operator is defined as

$$\rho=\int |\Lambda,\Omega\rangle P(\Lambda,\Omega)\langle \Lambda,\Omega|d\mu(\Omega)\ . \tag{3.58}$$

Then the statistical average of an operator is given by

$$\mathrm{Tr}(\rho B)=\int P(\Lambda,\Omega)\langle \Lambda,\Omega|B|\Lambda,\Omega\rangle d\mu(\Omega)\ . \tag{3.59}$$

(b) $Q$ representation. The operator $B$ that maps an invariant space $V^{\Lambda}$ into itself has a $Q$ representation in terms of the coherent states $|\Lambda,\Omega\rangle$:

$$B\rightarrow B_Q(\Lambda,\Omega)=\langle \Lambda,\Omega|B|\Lambda,\Omega\rangle\ . \tag{3.60}$$

The $Q$ representation of $B$ in $V^{\Lambda}$ is obviously unique. The $Q$ representation of the density operator is

$$\rho\rightarrow Q(\Lambda,\Omega)=\langle \Lambda,\Omega|\rho|\Lambda,\Omega\rangle\ . \tag{3.61}$$

Correspondingly, the statistical average of an operator is given by

$$\mathrm{Tr}(\rho B)=\int Q(\Lambda,\Omega)B_P(\Lambda,\Omega)d\mu(\Omega)\ . \tag{3.62}$$

(c) $W$ representation. The Wigner distribution on the phase-space structure of coherent states is required to satisfy the following two conditions:

$$\text{(i)}\quad B\leftrightarrow B_W(\Lambda,\Omega)\ , \tag{3.63a}$$

$$\text{(ii)}\quad (A,B)=\mathrm{Tr}_{\Lambda}(A^{\dagger}B)=\int d\mu(\Omega)A_W^*(\Lambda,\Omega)B_W(\Lambda,\Omega)\ . \tag{3.63b}$$

Here $A$ and $B$ are operators acting on the Hilbert space $V^{\Lambda}$. The statistical average of $B$ is

$$\mathrm{Tr}(\rho B)=\int d\mu(\Omega)W(\Lambda,\Omega)B_W(\Lambda,\Omega)\ , \tag{3.64}$$

where $W(\Lambda,\Omega)$ is the Wigner function of the density operator $\rho$.

The functions $B_P(\Lambda,\Omega)$, $B_Q(\Lambda,\Omega)$, and $B_W(\Lambda,\Omega)$ are defined over the geometric space $G/H$, and as such can be expanded in terms of the irreducible harmonic functions on $G/H$. This expansion is especially simple when $B$ is some component of an irreducible tensor operator, and then its $P$, $Q$, and $W$ representations are proportional to the corresponding irreducible harmonic functions. The relationship among these three distribution functions of an operator can be found as follows:

$$B_W(\Lambda,\Omega)=\int d\mu(\Omega')K_\Lambda(\Omega,\Omega')B_P(\Lambda,\Omega') , \tag{3.65}$$

$$B_Q(\Lambda,\Omega)=\int d\mu(\Omega')K_\Lambda(\Omega,\Omega')B_W(\Lambda,\Omega') , \tag{3.66}$$

where $K(\Omega,\Omega')$ satisfies

$$\begin{aligned} K*K &\equiv \int d\mu(\Omega_1)K_\Lambda(\Omega,\Omega_1)K_\Lambda(\Omega_1,\Omega') \\ &=|\langle\Lambda,\Omega|\Lambda,\Omega'\rangle|^2 . \end{aligned} \tag{3.67}$$

### 6. Generating functions

By using the BCH formula, one can easily derive the generating functions, which are powerful and widely used to calculate expectation values of various complicated operators in the coherent states. A general form of the generating function is

$$f(\Lambda,\Omega;\beta,\alpha)=\langle\Lambda,\Omega|\exp(\beta\cdot H+\alpha\cdot E)|\Lambda,\Omega\rangle , \tag{3.68}$$

where $\{\beta,\alpha\}$ are parameters. Then the expectation values of a complex operator can be obtained by differentiating with respect to $\{\beta,\alpha\}$.

### 7. $D$ algebra

Extensive applications of the over-completeness of the coherent states, Eq. (3.47), lead to the final results that appeared in the integral forms over the coset representative of $G/H$. However, it is commonly preferable to use the differential evaluations in practical calculations. Thus the coherent-states integral form can be easily transformed into a differential form by utilizing the coherent-state $D$ algebra in the same manner as the field coherent-states $D$ algebra in Sec. II.

The $D$ algebra for generalized coherent states is similar to that for the field coherent states: the mapping of quantum observables $B$ onto a differential form which acts on the parameter space of the coherent states. The usual definitions have the following four forms:

$$B|\Lambda,\Omega\rangle=D^k(B)|\Lambda,\Omega\rangle , \tag{3.69a}$$

$$\langle\Lambda,\Omega|B=\langle\Lambda,\Omega|D^b(B) , \tag{3.69b}$$

$$B|\Lambda,\Omega\rangle\langle\Lambda,\Omega|=D^l(B)|\Lambda,\Omega\rangle\langle\Lambda,\Omega| , \tag{3.69c}$$

$$|\Lambda,\Omega\rangle\langle\Lambda,\Omega|B=|\Lambda,\Omega\rangle\langle\Lambda,\Omega|D^r(B) . \tag{3.69d}$$

It is obvious that $D^k(B)$, $D^b(B)$ and $D^l(B)$, $D^r(B)$ satisfy the adjoint relationships

$$D^b(B)=[D^k(B)]^* , \tag{3.70a}$$

$$D^r(B)=[D^l(B)]^* . \tag{3.70b}$$

Some detailed calculations can be found in Gilmore *et al.* (1975) and Zhang (1987).

## D. An example: Atomic coherent states

### 1. Construction of the SU(2) coherent states

Starting from Eq. (2.1), if the electromagnetic field can be approximated as a classical field, i.e., if $\{a,a^\dagger\}$ are considered as $c$ numbers, then Eq. (2.1) is reduced to a many-atomic system:

$$\begin{aligned} H^A &= \langle H\rangle_F \\ &= \left\langle \sum_k \hbar\omega_k a_k^\dagger a_k \right\rangle + \Delta E \sum_\alpha \sigma_0^{(\alpha)} \\ &\quad + \frac{\lambda}{\sqrt{N}} \sum_{\alpha,k} (\langle a_k\rangle\sigma_+^{(\alpha)} + \langle a_k^\dagger\rangle\sigma_-^{(\alpha)}) \\ &= \sum_\alpha [\Delta E\sigma_0^{(\alpha)} + \gamma(t)\sigma_+^{(\alpha)} + \gamma^*(t)\sigma_-^{(\alpha)}] . \end{aligned} \tag{3.71}$$

The coherent states for this system can now be constructed with the general algorithm described above. To begin with, we define three inputs for the construction of the coherent states from the Hamiltonian of Eq. (3.71):

(a) Group structure (commutation relationship). Since the Hamiltonian (3.71) is constructed by single-atomic operators $\{\sigma_+^{(\alpha)},\sigma_-^{(\alpha)},\sigma_0^{(\alpha)}\}$, these operators are kinematically independent and obey the usual SU(2) commutation relations:

$$[\sigma_0^{(\alpha)},\sigma_\pm^{(\alpha')}]=\sigma_\pm^{(\alpha)}\delta_{\alpha\alpha'}, \quad \text{etc} . \tag{3.72}$$

Therefore one can always define the two-level many-atom operators $J_+$, $J_-$, and $J_0$ as

$$J_0=\sum_\alpha \sigma_0^{(\alpha)}, \quad J_\pm=\sum_\alpha \sigma_\pm^{(\alpha)} , \tag{3.73}$$

which are closed under the su(3) commutation relations (the usual angular momentum algebra):

$$[J_0,J_\pm]=\pm J_\pm, \quad [J_+,J_-]=2J_0 . \tag{3.74}$$

The corresponding covering Lie group is SU(2). In terms of $J_+$, $J_-$, and $J_0$, the Hamiltonian (3.71) can be rewritten as

$$H^A=\Delta EJ_0+\gamma(t)J_+ +\gamma^*(t)J_-=H_0+H_1 . \tag{3.75}$$

It then follows that the dynamical group of the two-level many-atomic system is SU(2).

(b) Hilbert space. The Hilbert space of SU(2) is the space $\{|jm\rangle, m=-j,-j+1,\ldots,j-1,j;\ j=\text{integer or half-integer}\}$. Here $|jm\rangle$ are the simultaneous eigenstates (also known as Dicke states in the literature) of the

SU(2) Casimir operators $J^2$ and $J_0$:

$$J^2|jm\rangle = j(j+1)|jm\rangle \ , \tag{3.76a}$$

$$J_0|jm\rangle = m|jm\rangle \ . \tag{3.76b}$$

Moreover, the states $|jm\rangle$ can be obtained by applying the shift-up operator $J_+$ to the lowest-weight state $|j\ -j\rangle$ $(j+m)$ times:

$$|jm\rangle = \begin{bmatrix} 2j \\ j+m \end{bmatrix}^{1/2} \frac{(J_+)^{j+m}}{(j+m)!}|j\ -j\rangle \ . \tag{3.77}$$

(c) Extremal state. Since the energy eigenstates of $H_0$ are $|jm\rangle$,

$$H_0|jm\rangle = \Delta E m|jm\rangle \ , \tag{3.78}$$

therefore the extremal state (i.e., the ground state of $H_0$) is the lowest-weight state of SU(2): $|j\ m=-j\rangle$ with $E_{\min} = -\Delta E j$.

Applying the algorithm, we obtain the following.

(i) Maximal stability subgroup. By the extremal state $|j\ -j\rangle$, we can find the suitable subgroup of the two-level atomic system as U(1):$\{J_0\}$, which leaves $|j\ -j\rangle$ invariant, i.e.,

$$h|j\ -j\rangle = |j\ -j\rangle e^{i\varphi}, \quad h\in \mathrm{U}(1) \tag{3.79}$$

and the general form of the subgroup element $h$ can be expressed as

$$h = \exp(i\alpha J_0), \quad \varphi = -\alpha j \ . \tag{3.80}$$

(ii) Coset space. We can obtain a unique coset decomposition from group theory with respect to the stability subgroup U(1),

$$g = \Omega h \ , \tag{3.81}$$

where $g\in$SU(2), and $\Omega$ is the coset representative of SU(2)/U(1):

$$\mathrm{SU}(2)/\mathrm{U}(1): \quad \Omega(\zeta) = \exp(\zeta J_+ - \zeta^* J_-) \ . \tag{3.82}$$

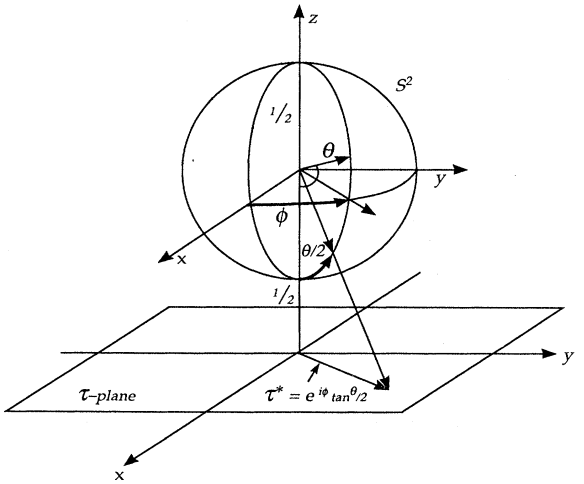



FIG. 5. Geometry of SU(2) coherent states. The SU(2) coherent states map the two-dimensional sphere $S^2$ onto the complex plane. This is compact when one includes the point at infinity.

It is easy to prove that the geometry of SU(2)/U(1) is a two-dimensional sphere $S^2$ (usually called a Bloch sphere; see Fig. 5):

$$J_+ \to \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad J_- \to \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad J_0 \to \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} , \tag{3.83}$$

then

$$\Omega(\zeta) \to \mathbf{\Omega} = \exp\begin{bmatrix} 0 & \zeta \\ -\zeta^* & 0 \end{bmatrix} = \begin{bmatrix} \cos|\zeta| & \frac{\zeta}{|\zeta|}\sin|\zeta| \\ -\frac{\zeta^*}{|\zeta|}\sin|\zeta| & \cos|\zeta| \end{bmatrix} . \tag{3.84}$$

Thus the parameter $\zeta$ can be rewritten as

$$\zeta = \frac{\theta}{2} e^{-i\phi} \quad (0\le\theta\le\pi,\ 0\le\phi\le 2\pi) \ . \tag{3.85}$$

(iii) Coherent states. Following the general procedure, we see that any group transformation $g$ of SU(2) acting on the extremal state $|j\ -j\rangle$ can be expressed as follows:

$$\begin{aligned} g|j\ -j\rangle &= \Omega(\zeta)h|j\ -j\rangle \\ &= \Omega(\zeta)|j\ -j\rangle e^{i\varphi} \equiv |j,\zeta\rangle e^{i\varphi} \ , \\ g\in\mathrm{SU}(2), &\quad h\in\mathrm{U}(1), \Omega(\zeta)\in\mathrm{SU}(2)/\mathrm{U}(1) \end{aligned} \tag{3.86a}$$

so that

$$|j,\zeta\rangle \equiv \Omega(\zeta)|j\ -j\rangle = \exp(\zeta J_+ - \zeta^* J_-)|j\ -j\rangle \tag{3.86b}$$

is the group definition of the atomic coherent states (Arecchi *et al.*, 1972). It is also often referred to as the spin coherent state (Radcliffe, 1971).

## 2. Properties of the atomic coherent states

### a. Geometrical structure

By the definition of $|j,\zeta\rangle$, parameters $\zeta$ and $\zeta^*$ are the geometrical conjugate variables of $J_+, J_-$. On the other hand, there is a one-to-one correspondence between $|j,\zeta\rangle$ and SU(2)/U(1),

$$|j,\zeta\rangle \leftrightarrow S^2 \ . \tag{3.87}$$

Therefore the geometrical structure of the atomic coherent states is SU(2)/U(1) or $S^2$. The metric of this space can be found from the unnormalized form of Eq. (3.86b),

$$|j,\zeta\rangle = \frac{1}{(1+|\tau|^2)^j}\exp(\tau J_+)|j\ -j\rangle \ , \tag{3.88}$$

which can be calculated using the BCH formulas of SU(2) [see Eq. (3.96) below] and

$$\tau = \tan\frac{\theta}{2} e^{-i\phi} \ . \tag{3.89}$$

Then the function $F(\tau,\tau^*)$ is

$$F(\tau,\tau^*) = \ln(1+|\tau|^2)^{2j} \tag{3.90}$$

and

$$d\mu=\frac{2j+1}{4\pi}\frac{d\tau d\tau^*}{(1+|\tau|^2)^2} . \tag{3.91}$$

It is not difficult to find the canonical form of the parameters on SU(2)/U(1), i.e., Eq. (3.84):

$$z=\tau(1+\tau\tau^*)^{-1/2}=\zeta\frac{\sin|\zeta|}{|\zeta|}=\sin\frac{\theta}{2}e^{-i\phi}$$

and (3.92)

$$z^*=\sin\frac{\theta}{2}e^{i\phi} .$$

By taking the transformation

$$z=\frac{1}{\sqrt{4j}}(q+ip) \quad \text{and} \quad z^*=\frac{1}{\sqrt{4j}}(q-ip) , \tag{3.93}$$

we can show easily that the symplectic structure of the geometrical space is

$$\{f,g\}=\frac{\partial f}{\partial q}\frac{\partial g}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} . \tag{3.94}$$

It also can be shown (Zhang, Martens, *et al.*, 1988) that the algebraic structure of SU(2) generators, Eq. (3.74), is preserved for the expectation values of the generators in the coherent states under Poisson brackets on its geometrical space,

$$\{\langle J_i\rangle,\langle J_j\rangle\}=\varepsilon_{ijk}\langle J_k\rangle . \tag{3.95}$$

Unlike the harmonic-oscillator case, the geometrical space of the atomic coherent states is compact.

### b. BCH formulas

A large number of BCH formulas can be derived for the Lie group SU(2) (Gilmore, 1974b). Some particularly useful BCH formulas are

$$\begin{aligned}&\exp(\zeta J_+-\zeta^*J_-)\\&\quad=\exp(\tau J_+)\exp[\ln(1+\tau^*\tau)J_0]\exp(-\tau^*J_-)\\&\quad=\exp(-\tau^*J_-)\exp[-\ln(1+\tau^*\tau)J_0]\exp(\tau J_+) ,\end{aligned} \tag{3.96}$$

where $\tau$ is given by Eq. (3.89). As a concrete example, we give a detailed derivation of Eq. (3.84) by the method of matrix representations of the group. Using the matrix representation of $J_+$, $J_-$, and $J_0$, we have

$$\begin{aligned}\exp(\tau J_+)\exp(\beta J_0)\exp(-\tau^*J_-)&\rightarrow\begin{bmatrix}1&\tau\\0&1\end{bmatrix}\begin{bmatrix}\exp(\beta/2)&0\\0&\exp(-\beta/2)\end{bmatrix}\begin{bmatrix}1&0\\-\tau^*&1\end{bmatrix}\\&=\begin{bmatrix}\exp(\beta/2)-|\tau|^2\exp(-\beta/2)&\tau\exp(-\beta/2)\\-\tau^*\exp(-\beta/2)&\exp(-\beta/2)\end{bmatrix} .\end{aligned} \tag{3.97}$$

By requiring the right side of Eq. (3.84) to be equal to Eq. (3.97), one obtains

$$\tau=\frac{\zeta\sin|\zeta|}{|\zeta|\cos|\zeta|}=\tan\frac{\theta}{2}e^{-i\phi} \tag{3.98}$$

and

$$\beta=-2\ln\cos|\zeta|=\ln(1+\tau^*\tau) . \tag{3.99}$$

Thus we prove the BCH formula of Eq. (3.96) by using the general procedure of Eqs. (3.41)–(3.43). We have often emphasized that these BCH formulas are exceedingly useful in coherent-state calculations. For example, the unnormalized coherent states of SU(2), Eq. (3.30), can be directly obtained by using Eq. (3.96).

### c. Hilbert-space properties

*Non-orthogonality.* The atomic coherent states are in general not orthogonal, except for antipodal points:

$$\begin{aligned}\langle j,\zeta'|j,\zeta\rangle=&[\cos\tfrac{1}{2}(\theta-\theta')\cos\tfrac{1}{2}(\phi-\phi')\\&-i\cos\tfrac{1}{2}(\theta+\theta')\sin\tfrac{1}{2}(\phi-\phi')]^{2j}e^{ij(\phi'-\phi)} ,\end{aligned} \tag{3.100a}$$

$$|\langle j,\zeta'|j,\zeta\rangle|^2=\left[\frac{1+\hat{n}(\Omega')\cdot\hat{n}(\Omega)}{2}\right]^{2j}=\cos^{4j}\frac{\Theta}{2} . \tag{3.100b}$$

Here $\hat{n}(\Omega)$ is the unit vector from the center to the point $(\theta\phi)$ on the surface of the Bloch sphere, and $\Theta$ is the angle between the $(\theta\phi)$ and $(\theta'\phi')$,

$$\cos\Theta=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi') . \tag{3.100c}$$

*Over-completeness.* Within any SU(2)-invariant subspace the identity operator may be resolved with respect to either the diagonal or the coherent states. It should be pointed out that the resolution of the identity operator in terms of coherent states is certainly not unique because of the over-completeness properties. One can easily show that

$$\frac{2j+1}{4\pi}\int d\Omega|j,\xi\rangle\langle j,\xi|=\frac{2j+1}{4\pi}\int d\Omega\sum_{m,m'}\begin{bmatrix}2j\\ j\pm m\end{bmatrix}^{1/2}\begin{bmatrix}2j\\ j\pm m'\end{bmatrix}^{1/2}(\cos\tfrac{1}{2}\theta)^{2j-m'-m}(\sin\tfrac{1}{2}\theta)^{2j+m'+m}e^{i(m'-m)\phi}|jm'\rangle\langle jm|$$
$$=\frac{2j+1}{2}\int_0^{\pi}\sin\theta\,d\theta\sum_m\begin{bmatrix}2j\\ j\pm m\end{bmatrix}(\cos\tfrac{1}{2}\theta)^{2j-2m}(\sin\tfrac{1}{2}\theta)^{2j+2m}|jm\rangle\langle jm|$$
$$=\sum_m|jm\rangle\langle jm|=1\ , \tag{3.101}$$

where $d\Omega=\sin\theta\,d\theta\,d\phi$ is the solid-angle volume element at $(\theta\phi)$ on $S^2$, while $d\mu=(1/4\pi)d\Omega$ is the invariant measure of SU(2)/U(1), and $2j+1$ is the dimension of the Hilbert space in which the coherent states live.

### d. Hilbert-space expansions

(i) Expansion of coherent states. The coherent state can be expanded in terms of the Dicke states (3.77), since they form a complete set of orthonormal states. This expansion is further facilitated by the BCH formula (3.96):

$$|j,\xi\rangle=\exp(\xi J_+-\xi^*J_-)|j\ -j\rangle$$
$$=(1+\tau^*\tau)^{-j}\exp(\tau J_+)|j\ -j\rangle$$
$$=(1+\tau^*\tau)^{-j}\sum_{n=0}^{\infty}\frac{(\tau J_+)^n}{n!}|j\ -j\rangle$$
$$=\sum_{m=-j}^{+j}|jm\rangle\begin{bmatrix}2j\\ j+m\end{bmatrix}^{1/2}(\cos\tfrac{1}{2}\theta)^{j-m}$$
$$\times[\exp(-i\phi)\sin\tfrac{1}{2}\theta]^{j+m}\ . \tag{3.102}$$

(ii) Expansion of an arbitrary Dicke state in terms of the atomic coherent states. Since the atomic coherent states form a complete basis, Eq. (3.101), any arbitrary Dicke space can be expanded in terms of $|jm\rangle$ as follows:

$$|c\rangle=\sum_m c_m|jm\rangle$$
$$=\frac{2j+1}{4\pi}\int d\Omega\sum_m c_m|j,\xi\rangle\langle j,\xi|jm\rangle$$
$$=\frac{2j+1}{4\pi}\int d\Omega\frac{f(\tau^*)}{[1+|\tau|^2]^j}|j\xi\rangle \tag{3.103a}$$

where

$$f(\tau^*)=\sum_m c_m\begin{bmatrix}2j\\ j+m\end{bmatrix}^{1/2}(\tau^*)^{j+m}=(1+|\tau|^2)^j\langle j,\xi|c\rangle \tag{3.103b}$$

is the coherent-state representation of $|c\rangle$ and is expressed as a polynomial of degree $2j$. It is worth pointing out that, although $|j,\xi\rangle$ is over-complete, the function $f(\tau^*)$ of $|c\rangle$ defined by (3.103b) is nevertheless unique. Furthermore, the scalar product of the two states in the coherent-state representation is

$$\langle c'|c\rangle=\frac{2j+1}{4\pi}\int\frac{d\Omega}{(1+|\tau|^2)^{2j}}[f'(\tau^*)]^*f(\tau^*)\ , \tag{3.104}$$

where $(2j+1)d\Omega/4\pi(1+|\tau|^2]^{2j}$ is the measure on the Hilbert space $L_2(S^2)$ of square-integrable functions $f(\tau^*)$ defined by Eq. (3.103b).

### e. Phase-space distributions

By use of Eq. (3.101), one can expand any operator $B$ acting on Dicke space in the atomic coherent-state representation as

$$B=\sum_{mm'}|jm\rangle B_{mm'}\langle jm'|$$
$$=\left[\frac{2j+1}{4\pi}\right]^2\int d\Omega\,d\Omega'|j,\xi\rangle\langle j,\xi|B|j,\xi'\rangle\langle j,\xi'|\ , \tag{3.105}$$

where $B_{mm'}=\langle jm|B|jm'\rangle$. In particular, there are three useful representations that can be used to express an operator in terms of a phase-space distribution function.

(i) $P$ representation. The integral kernel in Eq. (3.105) can always be written in the diagonal form

$$B=\frac{2j+1}{4\pi}\int d\Omega|j,\xi\rangle B(\xi)\langle j,\xi|\ , \tag{3.106}$$

where $B(\xi)$ is a $c$-number function defined on $S^2$. Generally speaking, the function $B(\xi)$ is not unique. On the other hand, since any operator in SU(2) space can always be expanded in terms of symmetrized spherical tensor operators $Y_m^L(J)$, a suitable $B(\xi)$ can easily be expressed in terms of tensorial function (Gilmore, 1976),

$$B=\sum_{L=0}^{2j}\sum_{M=-L}^{L}c_{LM}Y_M^L(J)$$
$$\rightarrow B(\xi)=\sum_{L=0}^{2j}\sum_{M=-L}^{L}c_{LM}\frac{(2j+L+1)!}{(2j+1)!2^L}Y_M^L(\theta\phi)\ , \tag{3.107}$$

where $Y_m^L(\theta\phi)$ are spherical harmonics. Likewise, the density operator $\rho$ can be expressed in the $P$ representation

$$\rho=\frac{2j+1}{4\pi}\int d\Omega|\xi\rangle P(\xi)\langle\xi|\ , \tag{3.108a}$$

with the normalization

$$\int d\Omega P(\xi)=1\ , \tag{3.108b}$$

where $P(\zeta)$ is the distribution function in the $P$ representation.

(ii) $Q$ representation. Since the $Q$ representation is defined as the expectation value of the operator in the coherent states and therefore may easily be taken as symmetrized tensor operators, then (Gilmore, 1976)

$$B=\sum_{L=0}^{2j}\sum_{M=-L}^{L}c_{LM}Y_M^L(J)$$
$$\rightarrow\langle j,\zeta|B|j,\zeta\rangle=\sum_{L=0}^{2j}\sum_{M=-L}^{L}c_{LM}\frac{(2j)!}{(2j-L)!2^L}Y_M^L(\theta\phi)\ . \tag{3.109}$$

(iii) $W$ representation. Following the general discussion, the $W$ distribution function can be found as follows (Gilmore, 1986):

$$B=\sum_{L=0}^{2j}\sum_{M=-L}^{L}c_{LM}Y_M^L(J)$$
$$\rightarrow B_W=\sum_{L=0}^{2j}\sum_{M=-L}^{L}c_{LM}\left[\frac{(2j)!(2j+L+1)!}{(2j-L)!2^{2L}(2j+1)!}\right]^{1/2}\times Y_M^L(\theta\phi)\ . \tag{3.110}$$

### f. Generating functions

The useful generating functions in the atomic coherent states are

$$\langle\zeta|\exp(\alpha_-J_-)\exp(\alpha_0J_0)\exp(\alpha_+J_+)|\zeta\rangle=(1+|\tau|^2)^{-2j}[\exp(-\tfrac{1}{2}\alpha_0)+\exp(\tfrac{1}{2}\alpha_0)(\tau^*+\alpha_-)(\tau+\alpha_+)]^{2j} \tag{3.111a}$$

and

$$\langle\zeta|\exp(\alpha_+J_+)\exp(\alpha_0J_0)\exp(\alpha_-J_-)|\zeta\rangle=(1+|\tau|^2)^{-2j}[\exp(\tfrac{1}{2}\alpha_0)\tau^*\tau+\exp(-\tfrac{1}{2}\alpha_0)(\alpha_+\tau^*+1)(\alpha_-\tau+1)]^{2j}\ . \tag{3.111b}$$

The matrix element of any complicated operator in the atomic coherent states can easily be obtained by simple derivative computations of these generating functions.

### g. D algebra

The $D$ algebra of the atomic coherent states has been studied in detail with extensive applications in quantum optics (Narducci *et al.*, 1974, 1975). In this section, we shall present only some of the results. For example,

$$J_+|j,\zeta\rangle\langle j,\zeta|=D_{J_+}(\zeta)|j,\zeta\rangle\langle j,\zeta|=e^{i\varphi}\left[j\sin\theta+\cos^2\left(\frac{\theta}{2}\right)\frac{\partial}{\partial\theta}+\frac{i}{2}\cot\left(\frac{\theta}{2}\right)\frac{\partial}{\partial\varphi}\right]|j,\zeta\rangle\langle j,\zeta|\ , \tag{3.112a}$$

$$J_-|j,\zeta\rangle\langle j,\zeta|=D_{J_-}(\zeta)|j,\zeta\rangle\langle j,\zeta|=e^{-i\varphi}\left[j\sin\theta-\sin^2\left(\frac{\theta}{2}\right)\frac{\partial}{\partial\theta}-\frac{i}{2}\tan\left(\frac{\theta}{2}\right)\frac{\partial}{\partial\varphi}\right]|j,\zeta\rangle\langle j,\zeta|\ , \tag{3.112b}$$

$$J_0|j,\zeta\rangle\langle j,\zeta|=D_{J_0}(\zeta)|j,\zeta\rangle\langle j,\zeta|=\left[-j\cos\theta+\frac{1}{2}\sin\theta\frac{\partial}{\partial\theta}+\frac{i\partial}{2\partial\varphi}\right]|j,\zeta\rangle\langle j,\zeta|\ , \tag{3.112c}$$

and

$$|j,\zeta\rangle\langle j,\zeta|J_+=D_{J_-}^*(\zeta)|j,\zeta\rangle\langle j,\zeta|\ ,$$
$$|j,\zeta\rangle\langle j,\zeta|J_-=D_{J_+}^*(\zeta)|j,\zeta\rangle\langle j,\zeta|\ , \tag{3.113}$$
$$|j,\zeta\rangle\langle j,\zeta|J_0=D_{J_0}^*(\zeta)|j,\zeta\rangle\langle j,\zeta|\ .$$

This $D$-algebra calculus gives a remarkable simplification in deriving the Fokker-Planck equation encountered in quantum optics (Narducci *et al.*, 1974).

## IV. QUANTUM-MECHANICS AND QUANTUM-STATISTICAL-MECHANICS REPRESENTATIONS OF THE COHERENT STATES

In this section, the quantum and quantum-statistical-mechanics representations of the coherent states will be discussed. We shall present some general coherent-state methods for the study of quantum dynamics. The applications to various dynamical systems will be discussed in Secs. V and VI.

### A. Coherent-state representation of quantum theory

#### 1. Stationary Schrödinger equation

For an isolated (i.e., conservative) quantum system, the quantum-dynamical equation is the stationary Schrödinger equation

$$H|\Psi\rangle=E|\Psi\rangle \ . \tag{4.1}$$

Using the coherent-state representation of the wave function $|\Psi\rangle$ of Eq. (3.51) and the over-completeness property of the coherent states of Eq. (3.47), it is not difficult to show that the coherent-state representation of Eq. (4.1) is

$$\int \langle \Lambda,\Omega|H|\Lambda,\Omega'\rangle f_\Lambda(\Omega')N^{1/2}(\Omega)N^{-1/2}(\Omega')d\mu(\Omega') = Ef_\Lambda(\Omega) \tag{4.2a}$$

or

$$\int \langle \Lambda,\tau\|H\|\Lambda,\tau'\rangle f_\Lambda(\tau')d\mu'(\tau')=Ef_\Lambda(\tau) \ , \tag{4.2b}$$

where $\|\Lambda,\tau\rangle$ represents the unnormalized form of $|\Lambda,\Omega\rangle$. The unique solution of Eq. (4.2) must satisfy Eq. (3.53).

On the other hand, if we regard $|\Psi\rangle$ as a trial function, then Eq. (4.2) can also be obtained by the variational principle:

$$\delta\langle\Psi|H|\Psi\rangle=\delta\left[\int \langle \Lambda,\Omega|H|\Lambda,\Omega'\rangle f^*_\Lambda(\Omega)f_\Lambda(\Omega') \times N^{-1/2}(\Omega)N^{-1/2}(\Omega')d\mu(\Omega')d\mu(\Omega)\right] =0 \ , \tag{4.3}$$

where the variation is carried out with respect to the function $f^*_\Lambda(\Omega)$. Thus Eq. (4.2) is a generalization of the generating coordinate method of Hill and Wheeler (1953). Since the trial function $|\Psi\rangle$ is an arbitrary state in the Hilbert space, the variational procedure shows that it is an exact quantum-mechanical approach.

Furthermore, Eq. (4.2) can be expressed in terms of the differential equation

$$D^b(H)f_\Lambda(\Omega)=Ef_\Lambda(\Omega) \ , \tag{4.4}$$

where $D^b(H)$ is the bra $D$ algebra of the Hamiltonian,

$$D^b(H)\langle\Lambda,\Omega|=\langle\Lambda,\Omega|H \ . \tag{4.5}$$

Hence Eqs. (4.2) and (4.4) are equivalent descriptions of the stationary Schrödinger equation in terms of integral and differential equations defined on the geometrical coset space. For illustration, let us again consider the SU(2) symmetry, with a Hamiltonian of the form of Eq. (3.3) (Lipkin, Meshkov, and Glick, 1965),

$$H=\varepsilon J_0+\frac{V}{2}(J_+J_++J_-J_-)+\frac{W}{2}(J_+J_-+J_-J_+) \ . \tag{4.6}$$

If we denote $|E_i\rangle$ as the eigenstate of $H$ with eigenvalue $E_i$, then the coherent-state representation of $|E_i\rangle$ is given by Eq. (3.86b) and satisfies Eq. (4.2) or (4.4). The $D$ algebra of $H$ is then

$$D^b(H)=\varepsilon D^b(J_0)+\frac{V}{2}[D^b(J_+)D^b(J_+)+D^b(J_-)D^b(J_-)] +\frac{W}{2}[D^b(J_+)D^b(J_-)+D^b(J_-)D^b(J_+)] \ , \tag{4.7}$$

where

$$D^b(J_0)=\tau\frac{d}{d\tau}-j \ , \tag{4.8a}$$

$$D^b(J_+)=2j\tau-\tau^2\frac{d}{d\tau} \ , \tag{4.8b}$$

$$D^b(J_-)=\frac{d}{d\tau} \ . \tag{4.8c}$$

According to Eqs. (4.7) and (4.8), the Schrödinger equation of Eq. (4.4) now becomes a differential equation defined on $S^2$. An interesting result is that the above angular momentum $D$ algebra (4.8) is precisely the boson expansion of the angular momentum. In fact, the $D$ algebra of (4.5) is the most general form of the so-called Dyson boson expansion (Garbaczewski, 1978; Dobaczewski, 1982; Zhang, 1987; Klein and Marshalek, 1990).

### 2. The coherent-states mean-field variational approach

Coherent states form a continuous complete subset of the Hilbert space. Thus, if we take them as trial wave functions, the following variation can be carried out:

$$\delta\langle\Lambda,\Omega|H|\Lambda,\Omega\rangle=\delta H_Q(Q,P)=0 \ , \tag{4.9}$$

where

$$Q\sim\mathrm{Re}(z) \quad \text{and} \quad P\sim\mathrm{Im}(z) \tag{4.10}$$

and $z$ is given by Eqs. (3.23) and (3.24). Equation (4.9) constitutes the static mean-field approach of the problem. When the coherent states span the entire Hilbert space of the problem, Eq. (4.9) will give rise to the exact quantum solution.

In general, using Eq. (4.9), one obtains not only the best approximation for the solution of the ground state by a coherent state, but also an elegant way to study the ground-state phase transitions. The procedure is as follows: (i) Calculate the potential of Eq. (4.10), i.e., the $Q$ representation of $H$ in the reduced real parameter space; (ii) minimize the potential function over the entire parameter space. This simple procedure is widely utilized to study the geometrical structure and critical properties of dynamical systems (Gilmore and Feng, 1978a, 1978b, 1978c; Feng and Gilmore, 1980, 1982; Feng, Gilmore, and Narducci, 1980; Feng *et al.*, 1981; Dieperink, 1983; Zhang *et al.*, 1987; Zhang, Wu, *et al.*, 1988; Zhang, 1988d). An example will be given in Sec. IV.C.

### 3. Time-dependent Schrödinger equation

The time-dependent Schrödinger equation is

$$i\frac{d}{dt}|\Psi\rangle=H|\Psi\rangle \ . \tag{4.11}$$

The solution of Eq. (4.11) can formally be expressed as

$$|\Psi(t)\rangle = T\left[\exp\left[\int_{t_0}^{t} -iH(\tau)d\tau\right]|\Psi(t_0)\rangle\right] \tag{4.12}$$

where $T$ is the time-ordering operator.

If the Hamiltonian is a linear function of the generators of the dynamical (Lie) group $G$ and its initial state is either the extremal state $|\Phi_0\rangle$ (which is also a coherent state) of $H_0$ or an arbitrary coherent state $|\Psi(t_0)\rangle = \Omega(t_0)|\Phi_0\rangle$, then Eq. (4.12) can *exactly* be solved:

$$|\Psi(t)\rangle = \Omega(t)|\Psi(t_0)\rangle e^{i\varphi(t)} , \tag{4.13}$$

where

$$\varphi(t) = i\int_{t_0}^{t} \langle \Psi(t_0)|\Omega(\tau)\left[i\frac{\partial}{\partial\tau} - H\right]\Omega(\tau)|\Psi(t_0)\rangle d\tau \tag{4.14}$$

and $\Omega(t)$ is determined by the Hamilton equations

$$\begin{aligned} ig_{ij}\frac{d\tau_j}{dt} &= \frac{\partial H_Q(\tau,\tau^*)}{\partial\tau_i^*} , \\ -ig_{ij}\frac{d\tau_j^*}{dt} &= \frac{\partial H_Q(\tau,\tau^*)}{\partial\tau_i} . \end{aligned} \tag{4.15}$$

In Eq. (4.15), the metric $g_{ij}$ and the coordinates $\tau_i$ are given by Eqs. (3.29) and (3.30) and $H_Q(\tau,\tau^*)$ is the expectation value of $H$ in the coherent state of Eq. (3.13). Equations (4.14) and (4.15) are obtained by directly substituting Eq. (4.13) into Eq. (4.11). If $G/H$ satisfies the Cartan decomposition of Eq. (3.20), then Eq. (4.15) can be rewritten in canonical form:

$$\begin{aligned} \frac{dq_j}{dt} &= \frac{\partial H_Q(q,p)}{\partial p_i} , \\ \frac{dp_i}{dt} &= -\frac{\partial H_Q(q,p)}{\partial q_i} , \end{aligned} \tag{4.16}$$

where the canonical coordinates $(q,p)$ are given by Eq. (3.38).

The above results show that in this special case the quantum and classical dynamics can be described in the same equation. The time evolution of the quantum wave function follows the classical trajectory of Eq. (4.16). It is worth pointing out that the phase in Eq. (4.14) contains two parts, the second part being the normal dynamical phase while the first is a geometrical phase related to the so-called Berry phase (Berry, 1984; Kuratsuji, 1988).

However, when the Hamiltonian is a nonlinear function of the generators of the dynamical group, it is in general difficult to obtain the exact solution of Eq. (4.12). In this case, some approximate methods must be developed in order to proceed. It turns out that one such approximation is the path-integral method, which also happens to be very convenient to use within the coherent-state formalism. In the next section we shall discuss this case.

## 4. Path-integral formalism of quantum theory in terms of coherent states

### *a. Coherent-state formalism of the path integral*

The standard path integral (Feynman, 1948; Feynman and Hibbs, 1965) is derived by expressing the time propagator $\exp(-iHt)$ as a product of the form $[\exp(-iHt/N)]^N$ and then inserting the resolution of identity in between terms in the product. A key input in the path integral is that the resolution of identity is usually expressed in terms of the coordinate $|x\rangle$ or the momentum $|p\rangle$ state. Hence the path-integral formalism when the system has an explicit phase-space structure.

The fact that the coherent states provide an elegant and nontrivial continuous basis and always possess a resolution of identity means that the path-integral formalism can always be applied to group-theoretic Hamiltonians. This important point was first recognized by Klauder (1960, 1979) and is now widely used in many situations (see the many papers collected in Klauder and Skagerstam, 1985). In particular, it has been used in the study of the interacting many-body problem (Blaizot and Orland, 1981; Koonin, 1982; Kuratsuji and Suzuki, 1983). Here we shall briefly outline the use of coherent states in this formalism.

The basic theory is as follows: One begins by evaluating the time propagation operator $\exp[iH(t_2-t_1)]$ between an initial ($|\Lambda,\Omega_i\rangle$) and a final ($\langle\Lambda,\Omega_f|$) coherent state,

$$K(\Omega_f,t_2;\Omega_i,t_1) = \langle\Lambda,\Omega_f|e^{-iH(t_2-t_1)}|\Lambda,\Omega_i\rangle . \tag{4.17}$$

One then sets $\varepsilon=(t_2-t_1)/(N+1)$, followed by inserting $N$ times the resolution of identity of Eq. (3.47) between $t_2$ and $t_1$. Finally, one lets $N\to\infty$. With this procedure, one can show that

$$K(\Omega_f,t_2;\Omega_i,t_1) = \int \mathcal{D}[d\mu(\Omega(t))]e^{iS[\Omega(t)]} , \tag{4.18}$$

where $\mathcal{D}[d\mu(\Omega(t))]$ is the functional measure of the path integral and $S[\Omega(t)]$ the action functional, which can be explicitly expressed as

$$\mathcal{D}[d\mu(\Omega(t))] = \prod_{t_1\le t\le t_2} d\mu(\Omega(t)) , \tag{4.19}$$

$$S[\Omega(t)] = \int_{t_1}^{t_2} dt\left[\left\langle\Lambda,\Omega(t)\left|i\frac{d}{dt}\right|\Lambda,\Omega(t)\right\rangle - H_Q(\Omega(t))\right] . \tag{4.20}$$

The initial and final boundary conditions for Eq. (4.18) are

$$\Omega(t_1)=\Omega_i, \quad \Omega(t_2)=\Omega_f . \tag{4.21}$$

In principle, Eq. (4.18) can describe the system's time evolution, the energy spectra of the bound states, and the $S$ matrix (Gutzwiller, 1971; Schulman, 1981). The action functional of Eq. (4.20) can be explicitly expressed in

terms of the Lagrangian functional (Kan *et al.*, 1979a, 1979b),

$$S[\Omega(t)]=\int_{t_1}^{t_2}\mathcal{L}\left[\tau(t),\frac{\partial\tau}{\partial t}\right]dt\ , \tag{4.22}$$

where the Lagrangian functional in the coherent-state phase space is defined as

$$\mathcal{L}\left[\tau(t),\frac{\partial\tau(t)}{\partial t}\right]=\frac{i}{2}\mathrm{Tr}g_{ij}\left[\tau_i^*\frac{\partial\tau_j}{\partial t}-\tau_i\frac{\partial\tau_j^*}{\partial t}\right]-H_Q(\tau,\tau^*)\ , \tag{4.23}$$

and

$$H_Q(\tau(t),\tau^*(t))=H_Q(\Omega(t)) \tag{4.24}$$

is the Hamiltonian functional of the system.

### *b. Stationary phase approximation*

As is well known, the integration of Eq. (4.18) is mathematically ill defined. This difficulty can be circumvented by making certain approximations in which the integration can be avoided. One such approximation is the stationary phase approximation.

The stationary phase approximation expands the action $S$ in Eq. (4.18) around stationary paths. Such paths are determined by the variation of the action functional,

$$\delta S[\Omega(t)]=\delta\int\left[\left\langle\Lambda,\Omega(t)\left|i\frac{d}{dt}\right|\Lambda,\Omega(t)\right\rangle-H_Q(\Omega(t))\right]dt=0\ . \tag{4.25}$$

It is interesting to note that Eq. (4.25) is a time-dependent mean-field equation for the many-body system. In other words, the coherent-state path-integral formalism includes naturally the time-dependent mean-field theory as a classical limit. Furthermore, from the canonical complex structure of the Lagrangian functional of Eq. (4.23), we see that Eq. (4.25) leads immediately to the classical-like equations of motion

$$ig_{ij}\frac{d\tau_j}{dt}=\frac{\partial H_Q(\tau,\tau^*)}{\partial\tau_i^*}\ ,\qquad -ig_{ij}\frac{d\tau_j^*}{dt}=\frac{\partial H_Q(\tau,\tau^*)}{\partial\tau_i}\ , \tag{4.26}$$

which formally are the same as Eq. (4.15). If we used the standard canonical form of Eq. (3.39), Eq. (4.26) would be reduced to the same form as Eq. (4.16).

When the system is a many-body system, then Eq. (4.26) provides the most general time-dependent Hartree-Fock or time-dependent Hartree-Fock-Bogoliubov dynamical equations (see Sec. VI.C.). Furthermore, if the semiclassical motion of Eq. (4.26) is quasi-periodic, then requantization will generally give a good approximate solution to the original problem. In the special case where the Hamiltonian is a linear function of generators of the dynamical group, the solutions of Eq. (4.26) are in fact exact quantum solutions, as shown in the last subsection.

The above stationary phase approximation has been applied by many authors to some solvable models, such as the Lipkin-Meshkov-Glick (1965) model (for example, Krieger, 1977; Hoodboy and Negele, 1978; Kan *et al.*, 1979b; Levit *et al.*, 1980a; Feng and Gilmore, 1980).

## B. Coherent-states representation of thermodynamics

### 1. Coherent-state representation of the partition function

Another important application of the coherent-state theory is thermodynamics. The thermodynamic properties associated with the Hamiltonian of a system can be determined from the partition function $Z(\beta)$ ($\beta=1/kT$) or from the free energy $F(\beta)$:

$$e^{-\beta F(\beta)}=Z(\beta)=\mathrm{Tr}e^{-\beta H}\ . \tag{4.27}$$

In Eq. (4.27) the trace is carried out over the entire Hilbert space. Since $H$ [Eqs. (3.1) or (3.2)] is constructed from the generators of the Lie algebra **g**, the trace operation is simplified considerably. First, the entire Hilbert space is decomposed into **g**-invariant subspaces $V^\Lambda$. Each subspace is represented by $\Lambda$ with degeneracy $Y_\Lambda$. Since the operators $T_i$ have nonzero matrix elements only within an irreducible subspace, and the matrix elements are identical in all equivalent (same $\Lambda$) representations, the trace in Eq. (4.27) becomes

$$\mathrm{Tr}e^{-\beta H}=\sum_\Lambda Y_\Lambda\mathrm{Tr}_\Lambda e^{-\beta H}\ . \tag{4.28}$$

Here $\mathrm{Tr}_\Lambda$ means that the trace is restricted to any invariant subspace with $\Lambda$. In the coherent-state representation, according to Eq. (3.47), this trace is expressed in terms of a coherent-state integral (Lieb, 1973; Rasetti, 1975):

$$\mathrm{Tr}_\Lambda e^{-\beta H}=\int d\mu(\Omega)\langle\Lambda,\Omega|\exp(-\beta H)|\Lambda,\Omega\rangle\ . \tag{4.29}$$

In principle, most of the thermodynamic properties of the system are contained in Eq. (4.29). However, except for some special cases, this trace cannot be computed in closed form. When that happens, only approximate results can be obtained. One of the usual approximate methods is again via path integrals. One begins with the formal expression

$$\mathrm{Tr}_\Lambda e^{-\beta H}=\int\mathcal{D}[d\mu(\Omega)]\times\exp\left[-\int_0^\beta dt\left[\left\langle\Lambda\Omega(t)\left|\frac{d}{dt}\right|\Lambda\Omega(t)\right\rangle-H_Q(\Omega(t))\right]\right]\ . \tag{4.30}$$

Another very useful approximation is that the upper and lower bounds on this trace can be computed and simply expressed in terms of the coherent-state representation. This will be discussed next.

### 2. The Lieb-Berezin inequalities

In 1973, using atomic coherent states, Lieb (1973; Lieb and Loss, 1990) and Berezin (1975a, 1975b) derived an ingenious thermodynamic inequality of the partition function for quantum spin systems. These inequalities (now known as Lieb-Berezin inequalities) give both the upper and lower bounds to the quantum free energy (or ground-state energy) in terms of two classical free energies (or ground-state energies). Furthermore, it has been shown that such inequalities are also available to coherent states for all compact Lie groups (Simon, 1980).

For the lower bound of the quantum partition function, according to the Peierls-Bogoliubov inequality, $\langle\Phi|e^X|\Phi\rangle \geq \exp\langle\Phi|X|\Phi\rangle$ for any normalized $\langle\Phi|\in V^\Lambda$ and self-adjoint $X$, we have

$$\begin{aligned}\mathrm{Tr}_\Lambda e^{-\beta H} &= \int d\mu(\Omega)\langle\Lambda,\Omega|e^{-\beta H}|\Lambda,\Omega\rangle \\ &\geq \int d\mu(\Omega)\exp(-\beta\langle\Lambda,\Omega|H|\Lambda,\Omega\rangle) \\ &= \int d\mu(\Omega)e^{-\beta H_Q(\Omega)} ,\end{aligned} \tag{4.31}$$

where $H_Q(\Omega)$ is the $Q$ representation of $H$.

For the upper bound of the quantum partition function, let $|c_i\rangle$, $i=1,\ldots n,\ldots,$ be a complete set of eigenstates of $H$. We then have

$$\begin{aligned}\langle c_i|\exp(-\beta H)|c_i\rangle &= \exp(-\beta\langle c_i|H|c_i\rangle) \\ &= \exp\left[-\beta\int H_P|\langle\Lambda,\Omega|c_i\rangle)|^2 d\mu(\Omega)\right] \\ &\leq \int \exp(-\beta H_P)|\langle\Lambda,\Omega|c_i\rangle|^2 d\mu(\Omega) ,\end{aligned} \tag{4.32}$$

where $H_P(\Omega)$ is the $P$ representation of $H$. By taking the trace of Eq. (4.32), one obtains

$$\mathrm{Tr}_\Lambda e^{-\beta H} \leq \int d\mu(\Omega)e^{-\beta H_P(\Omega)} . \tag{4.33}$$

The above derivation was given by Lieb in his unpublished work, as quoted in a paper of Simon (1980). Combining Eqs. (4.31) and (4.33), we have the Lieb-Berezin (thermodynamic) inequalities

$$\int d\mu(\Omega)e^{-\beta H_Q(\Omega)} \leq \mathrm{Tr}_\Lambda e^{-\beta H} \leq \int d\mu(\Omega)e^{-\beta H_P(\Omega)} . \tag{4.34}$$

Using these inequalities, one can construct a number of approximate descriptions of a quantum-dynamic system in the coherent-state representation. This is obviously extremely useful because the upper and lower bounds to the quantum free energy have been widely used to study qualitative statistical behavior as well as thermodynamic phase transitions of the systems. By using Eq. (4.34), one finds that the upper and lower bounds of the partition function and the free-energy functions are

$$\begin{aligned}\sum_\Lambda Y_\Lambda \int d\mu(\Omega)e^{-\beta H_Q(\Omega)} &\leq Z(\beta) \\ &= e^{-F(\beta)} \leq \sum_\Lambda Y_\Lambda \int d\mu(\Omega)e^{-\beta H_P(\Omega)} .\end{aligned} \tag{4.35}$$

### 3. Thermodynamic functions in terms of coherent states

At finite temperature a quantum-mechanical system is in a statistical mixture of states and may conveniently be represented by a density operator, which is defined to be $\rho=[1/Z(\beta)]e^{-\beta H}$. Then using the results of the previous section we have

$$\rho = \sum_\Lambda \int \sum_{\alpha=1}^{Y_\Lambda} |\Lambda,\Omega\rangle P(\Lambda,\Omega)\langle\Lambda,\Omega|d\mu(\Omega) , \tag{4.36}$$

where $\alpha=1,2,\ldots,Y_\Lambda$ indexes the number of distinct invariant subspaces $\Lambda$ that occur in the decomposition of the entire Hilbert space.

Applying the Lieb-Berezin inequalities, we can show that the free energy is

$$F(\beta) \leq \min_{\Omega,\Lambda} \Phi(\Lambda,\Omega,\beta) , \tag{4.37}$$

where

$$\Phi(\Lambda,\Omega,\beta) = H_Q(\Lambda,\Omega) - \beta^{-1}\ln Y_\Lambda . \tag{4.38}$$

### 4. Zero-temperature limit

In the zero-temperature limit, the free energy may be identified as the ground-state energy. Therefore the inequality (3.46) can be used to construct the upper and lower bounds of the ground-state energy in the limit $\beta\to\infty$. Explicitly,

$$E_g = \lim_{\beta\to\infty} F(\beta) . \tag{4.39}$$

Using the formula (Gilmore and Feng, 1978b)

$$\lim_{\beta\to\infty}\left[-\frac{1}{\beta}\ln\int e^{-\beta f(\Omega)d\mu(\Omega)}\right] = \min_{\Omega,\Lambda} f(\Omega) \tag{4.40}$$

we have

$$\min_{\Omega,\Lambda} H_Q(\Omega) \geq E_g \geq \min_{\Omega,\Lambda} H_P(\Omega) . \tag{4.41}$$

Equation (4.41) shows that the upper and lower bounds of the ground-state energy are determined by minimizing the $Q$ and $P$ representations of the Hamiltonian. The upper bound of the ground-state energy is easy to understand from the variational principle in which the coherent states are taken as trial wave functions. A de-

tailed study of Eq. (4.41) for the Lipkin-Meshkov-Glick model was made by Gilmore and Feng (1978a). The lower bound for a Coulomb Hamiltonian was studied by Lieb (1981) and Thirring (1981).

### C. Classical limits

#### 1. $N\rightarrow\infty$ limit

As discussed in Sec. III, for a general system with dynamical symmetry group $G$, if the Hamiltonian is a linear superposition of the generators of $G$ and the initial state is a coherent state, the state will forever remain a coherent state. In this case, the time evolution of a quantum system resembles that of a classical particle. Unfortunately, without additional constraints there exist very few realistic systems that behave in this manner.

On the other hand, one normally assumes that the classical limit of the quantum theory is achieved by letting $\hbar\rightarrow 0$. This limit of $\hbar$ demands that the uncertainties of the coordinate and momentum vanish in a physical state. In addition, expectation values of the operators satisfy $\langle AB\rangle = A_{\rm cl}B_{\rm cl}+O(\hbar)$.

For a quantum system with dynamical symmetry group $G$, $\hbar\rightarrow 0$ is in fact equivalent to the $N\rightarrow\infty$ limit (where $N$ is the "particle" number). This equivalence can be shown to be within the framework of the coherent states (Simon, 1980; Yaffe, 1982). For example, the classical limit of a quantum spin system [with dynamical symmetry group SU(2)] has become a powerful tool for the rigorous study of the ground-state and thermodynamic critical properties. In this example, since $N=2j$, the classical limit of spin is known to be $j\rightarrow\infty$ (Lieb, 1973). In this section, we shall review the classical limit descriptions of coherent states.

For spin systems, the Hamiltonian can be written as a linear superposition of spherical tensor operators. Then the $Q$ and $P$ representations are obtained by replacing spherical tensor operators with spherical harmonics. For example, when the Hamiltonian is linear in the spin operators, Eq. (4.34) becomes

$$Z^{\rm cl}(\beta,j)\leq Z(\beta)\leq Z^{\rm cl}(\beta,j+1)\ . \tag{4.42}$$

When $N=2j\rightarrow\infty$, the quantum partition function per particle as well as the $Q$ and $P$ representations of the Hamiltonian $H/N$ are exactly reduced to the classical form by suitable weight functions [see Eqs. (3.107) and (3.109)]. This conclusion can be extended to any compact dynamical group and to a noncompact group with square-integrable Hilbert space (Gilmore, 1979; Simon, 1980; Yaffe, 1982). For a Hamiltonian of the form

$$H(T_i)\equiv NH(T_i/N)\ , \tag{4.43}$$

where $T_i$ are the generators of a Lie group defined in Eq. (3.1), we have

$$\lim_{N\rightarrow\infty}H_P(\Lambda,\Omega)/N=\lim_{N\rightarrow\infty}H_Q(\Lambda,\Omega)/N$$
$$=H(\langle T_i/N\rangle_\Lambda)=H_{\rm cl}(\Lambda,\Omega)\ . \tag{4.44}$$

Equation (4.44) tells us that the limit on the left side is obtained by replacing the generators that appear in $H$ with their classical limit. The classical limit of the generators is given by (Gilmore, 1979)

$$\left\langle\frac{T_j}{N}\right\rangle_\Lambda=\sum_{i=1}^{l}\left[\lim_{N\rightarrow\infty}\frac{\mu_i}{N}\right]\langle f_i,\Omega|T_j|f_i,\Omega\rangle\ . \tag{4.45}$$

Here $|f_i,\Omega\rangle$ are the coherent states for the $i$th fundamental representation of $g$, and $\mu_i$ are determined from Eq. (3.18). The classical limit of the free energy is given by Eq. (4.37). Consequently the classical limit of any arbitrary operator is determined by its $Q$ representation.

#### 2. Ground-state-energy phase transitions

Ground-state-energy phase transitions can occur if the interaction parameters $c_{ij}$ in Eq. (3.2) are sufficiently strong in the thermodynamic limit $N\rightarrow\infty$. Such phase transitions can be studied easily as follows: the bounds on the ground-state energy are determined from Eq. (4.41) when the interaction parameters $c_{ij}$ are zero. The $\Omega_0$ that minimizes the bounds is usually a point of high symmetry in $G/H$ under minimal assumptions about the symmetry of the Hamiltonian $H$. As the parameters $c_{ij}$ increase slowly, the minima of the bounds remain unchanged at $\Omega_0$. At some critical value of $c_{ij}$, the minimum value changes and the point $\Omega$ in $G/H$ which minimizes the bound(s) is no longer $\Omega_0$. A ground-state-energy phase transition is said to occur when the interaction parameters cross their critical values. The phase transition is second order if the point $\Omega$ that minimizes the bounds departs from $\Omega_0$ continuously as a function of increasing interaction parameters. Otherwise, it is first order.

In the thermodynamic limit, the determination of the ground-state critical properties of Eq. (3.2) may also be reduced to a simple algorithm. It is not necessary to calculate $H_P(\Lambda,\Omega)$, since only the equality of Eq. (4.41) is satisfied in this limit. The algorithm is as follows:

(i) Write down the Hamiltonian of the physical system $H$ as

$$H/N=H(T_i/N)\ . \tag{4.46}$$

(ii) Construct the classical limit by

$$H_c(\Omega)=H_c(\langle\Lambda,\Omega|T_i/N|\Lambda,\Omega\rangle)\ . \tag{4.47}$$

(iii) Study how the minima of $H_c$ change as a function of changing interaction parameters.

Step (i) of this algorithm depends on the dynamic structure of the physical system. Step (ii) depends on Lie group theory. Step (iii) depends on catastrophe theory (see Gilmore, 1981). The importance of this algorithm is that the problem of determining the collective coopera-

tive phenomena of the ground state for a complicated system under a mean-field assumption is reduced to a very simple procedure.

The ground-state phase transition for finite $N$ can also be studied in this framework. In this case, the critical values of the interaction parameters are not precisely defined, as the $P$ and $Q$ representations of $H$ differ slightly and the critical values of the $c_{ij}$ for these two representations also differ slightly. Moreover, $N$ as an internal parameter of the system is also a control parameter for the structural phase transition of the system. A practical difficulty is that the $P$ representation of $H$ can be calculated precisely only for a pseudospin Hamiltonian; we do not yet know how to do this for an arbitrary group. However, the minimum of the $Q$ representation is a much better approximation to the ground-state energy than is the minimum of the $P$ representation of $H$ from the point of view of the mean-field theory. Therefore, for finite $N$ it is sufficient to study the ground-state critical properties by studying how the minimum value of $H_Q(\Lambda,\Omega)$, and the $\Omega$ which minimizes it, depends on the interaction parameters or the internal parameters. This procedure is identical with that discussed for the potential function of systems derived from the coherent-state path integral.

We use this extended algorithm here to study the ground-state critical properties for a system of $N$ identical nucleons, each of which has an internal dynamical group SU($r$). We assume that the single-particle energies are $\varepsilon_1<\varepsilon_2<\cdots<\varepsilon_r$. We have chosen the following Hamiltonian to study the phase-transition problem:

$$H/N=\sum_{i=1}^{r}\varepsilon_i\frac{H_i}{N}+\begin{cases}\frac{1}{2}\chi\left[\left[\frac{E_{1j}}{N}\right]^2+\left[\frac{E_{j1}}{N}\right]^2\right], & 1<j \\ \frac{1}{2}\chi\left[\left[\frac{E_{ij}}{N}\right]^2+\left[\frac{E_{ji}}{N}\right]^2\right], & 1<i<j\le r\end{cases} \tag{4.48a}$$

(4.48b)

The parameter $\chi$ ($>0$) measures the strength of the quadrupole interaction between the appropriate levels. Since the physics depends on whether the quadrupole interaction is between the ground state and an excited state (4.48a) or between two excited states (4.48b), we treat these two interactions separately. The notations $H_i$, $E_{ij}$ for the su($r$) operators are standard. The Hamiltonian (4.48) may be regarded as the multilevel extension of the Lipkin-Meshkov-Glick two-level pseudospin Hamiltonian.

The ground state of the SU($r$) Hamiltonian above lies in the fully symmetric representation. In this representation the classical limits of the su($r$) operators are known as

$$\sum_{i=1}^{r}\varepsilon_i\frac{H_i}{N}\to\varepsilon_1\cos^2\frac{\theta}{2}+\sum_{i=2}^{r}\varepsilon_i\left[\frac{\theta_i}{\theta}\sin\frac{\theta}{2}\right]^2, \tag{4.49a}$$

$$\frac{1}{2}\chi\left[\left[\frac{E_{1j}}{N}\right]^2+\left[\frac{E_{j1}}{N}\right]^2\right]\to\frac{1}{2}\chi\left[\frac{\theta_j}{\theta}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]^2(e^{2i\phi_j}+e^{-2i\phi_j}), \tag{4.49b}$$

$$\frac{1}{2}\chi\left[\left[\frac{E_{ij}}{N}\right]^2+\left[\frac{E_{ji}}{N}\right]^2\right]\to\frac{1}{2}\chi\left[\frac{\theta_i\theta_j}{\theta}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]^2(e^{2i(\phi_j-\phi_i)}+e^{-2i(\phi_j-\phi_i)}), \tag{4.49c}$$

where $(\theta/2)e^{i\phi_j}$ is the order parameter for the level pair $(1j)$ and $(\theta/2)^2=\sum_{i=2}^{r}(\theta_i/2)^2\le\pi^2$. The ground-state energy per nucleon is determined by minimizing the classical limits of Eq. (4.49).

For concreteness, we consider three-level systems with quadrupole interactions between the ground and first excited state. Then from Eqs. (4.48) and (4.49) the function to be determined is

$$F-\varepsilon_1=(\varepsilon_2-\varepsilon_1)\left[\frac{\theta_2}{\theta}\sin\frac{\theta}{2}\right]^2+(\varepsilon_3-\varepsilon_1)\left[\frac{\theta_3}{\theta}\sin\frac{\theta}{2}\right]^2+\frac{1}{2}\chi\left[\frac{\theta_2}{\theta}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]^2(e^{2i\phi_2}+e^{-2i\phi_2}). \tag{4.49d}$$

By inspection, the minimum occurs for $\theta_3=0$, $\phi_3$ arbitrary, $e^{2i\phi_2}=-1$, and $\theta_2$ determined by minimizing

$$F-\varepsilon_1\to(\varepsilon_2-\varepsilon_1)\sin^2(\theta/2)-\chi(\tfrac{1}{2}\sin\theta)^2. \tag{4.50}$$

A standard bifurcation analysis shows that, when $\chi<\Delta_2=\varepsilon_2-\varepsilon_1$, the minimum occurs for $\theta=0$ and $F-\varepsilon_1=0$. For $\chi>\Delta_2$, the minimum occurs for $\cos\theta=\Delta_2/\chi$, and $F-\varepsilon_1=-(\chi-\Delta_2)^2/4\chi^2$. A second-order ground-state-energy phase transition occurs at $\chi=\varepsilon_2-\varepsilon_1$. For a general discussion see Gilmore and Feng (1979; Gilmore, 1981).

Recently, the coherent-state method in the large-$N$ limit has been extensively used to study the nonperturbative behaviors of quantum chromodynamics (Brown and Yaffe, 1986; Dickens *et al.*, 1988).

### 3. Thermodynamic phase transitions

Finite-temperature thermodynamic phase transitions can occur in physical systems modeled by Hamiltonians of the form (3.3) if the interaction parameters are sufficiently large. Such thermodynamic phase transitions may be studied by essentially the same methods described in the previous section, both for finite $N$ and in the ther-

modynamic limit.

In the case of finite $N$, the upper and lower bounds on the free energy $F(\beta)$ may be obtained from the bounds (4.35) on the partition function $Z(\beta)$. If $\Omega_0$ minimizes these bounds in the absence of interactions, then $\Omega_0$ will also minimize these bounds in the presence of an interaction at sufficiently high temperature. Assume that at $T=0$, $\Omega\neq\Omega_0$ minimizes these bounds; then as $T$ decreases below some critical temperature $T_c$ the point $\Omega\in G/H$ which minimizes the bound(s) is no longer $\Omega_0$. A thermodynamic phase transition is said to occur at $T=T_c$. The phase transition is second order if the $\Omega$ that minimizes the bound(s) is continuously connected to $\Omega_0$ as a function of temperature. Otherwise it is first order.

In the thermodynamic limit, the thermodynamic critical properties associated with the Hamiltonian $H$ (3.2) reduce to a simple algorithm:

(i) Write down the Hamiltonian per particle as a function of the generators of the dynamical group $T_i\in\mathbf{g}$,

$$H/N=H(T_i/N)\ . \tag{4.51}$$

(ii) Replace the generators by their classical limits, the $Q$ representation of $T_i$ in each of the fundamental representation of $\mathbf{g}$. This converts $H/N$ into a classical Hamiltonian $H_{\rm cl}$. The free energy is $\Phi(\beta)=H_{\rm cl}-(1/\beta)\ln Y_\Lambda$.

(iii) Study how the minima of $\Phi(\beta)$ change as a function of changing temperature.

The three steps of this algorithm depend on physical intuition, Lie group theory, and catastrophe theory, respectively. The importance of this algorithm is that it reduces to a very simple procedure the problem of determining the cooperative finite-temperature properties of the dynamic system.

## V. SQUEEZED STATES

### A. Introduction

The uncertainty principle places a damper on the enthusiasm with which quantum engineers approach the problem of coding and transmitting information by optical means. Specifically, the quantum noise inherent in a beam of light places a limit on the information capacity of an optical beam.

It is therefore useful to see if there is any way to "beat" the uncertainty principle. In fact, the uncertainty principle is a statement about areas in phase space, and noise levels in different quadratures are statements about intersections of uncertainty ellipses with these axes. Any procedure that can deform, or squeeze, the uncertainty circle to an ellipse can in principle be used for noise reduction in one of the quadratures. Such squeezing does not violate the uncertainty principle; rather, it places the larger uncertainty in a quadrature not involved in the information transmission process.

A procedure for squeezing the error ellipse was first discussed systematically by Yuen (1976). This procedure involved applying a classical source to drive two-photon emission and absorption processes in much the same way that single-photon processes can be used to generate a coherent state of the electromagnetic field. The states produced by this process were originally called "two-photon coherent states" since they were so closely analogous to the usual (one-photon) field coherent states. The mathematical properties had been discussed earlier by Stoler (1970, 1971), who called them "minimum-uncertainty packets" and by Lu (1971, 1972), who called them "new coherent states." A method of optical communication using two-photon coherent states was proposed by Yuen and Shapiro (1978, 1980) and by Shapiro, Yuen, and Mata (1979). The term "squeezed states" was introduced by Hollenhorst (1979). Squeezed states of the electromagnetic field have been detected (Slusher *et al.*, 1985; Kimble and Hall, 1986; Shelby *et al.*, 1986; Wu *et al.*, 1986). At present, the principal potential applications of squeezed states are in the field of optical communications and "quantum nondemolition experiments" designed for the detection of gravity waves (Walls, 1983). Squeezed states of molecular systems have also been discussed (Gazdy and Micha, 1985; Gilmore and Yuan, 1987, 1989).

In this section we describe these two-photon coherent states and summarize their applications in optical communications and in molecular dynamics.

### B. Single-mode two-photon systems

#### 1. Two-photon algebra

Squeezed states are obtained by driving two-photon processes with a classical source. The basic field mode Hamiltonian describing two-photon processes in a single mode is

$$H=h\omega(a^\dagger a+\tfrac{1}{2})+\lambda(t)a^{\dagger 2}+\lambda^*(t)a^2\ . \tag{5.1}$$

The operators in the Hamiltonian span the Lie algebra su(1,1) and generate a restricted class of squeezed states. More generally, it is desirable to squeeze a field coherent state. The Hamiltonian for such processes is

$$H=h\omega(a^\dagger a+\tfrac{1}{2})+f_2(t)a^{\dagger 2}+f_2^*(t)a^2+f_1(t)a^\dagger+f_1^*(t)a\ . \tag{5.2}$$

The sequence in which the processes of coherent-state formation and squeezing occur is governed by the time dependence of the functions $f_2(t),f_1(t)$. They can be constructed by following the general procedure given in Sec. III.

The two-photon (non-semisimple) algebra $h_6$ of Eq. (5.2) (single-mode) is spanned by the six operators

$$h_6:\ \{n=a^\dagger a+\tfrac{1}{2},a^{\dagger 2},a^2,a^\dagger,a,I\}\ , \tag{5.3}$$

with commutation relations

$$[n,a^{\dagger 2}]=2a^{\dagger 2},\quad [n,a^2]=-2a^2,\ \text{etc}\ . \tag{5.4}$$

These commutation relations are summarized by a root space diagram (Fig. 6). It should be recalled that root space diagrams exist for non-semisimple algebras as well as semisimple algebras. The two-photon algebra is not semisimple (but see below).

The root space diagram is interpreted as follows. Each of the six operators is plotted in the two-dimensional space $(r,d)$, where $r$ is defined by

$$[n+\tfrac{1}{2},Op]=rOp\ , \tag{5.5}$$

$r=2,1,0,-1,-2$ for $a^{\dagger 2}$, $a^{\dagger}$, $I$, $a$, $a^2$, and $d$ is the degree of the operator: $d=(2,2,2)$, $(1,1)$, $(0)$ for $(a^{\dagger 2},n+\frac{1}{2},a^2)$, $(a^{\dagger},a)$, $(I)$. The two subalgebras $h_4=(n+\frac{1}{2},a^{\dagger},a,I)$ and $\mathrm{su}(1,1)=(a^{\dagger 2},n+\frac{1}{2},a^2)$ are indicated in Fig. 6. It can be seen that the root space diagram for the two-photon algebra is a subset of the root space diagram $C_2$ for the symplectic algebra $\mathrm{sp}(4)=\mathrm{sp}(2+2)$ (cf. Fig. 6 with Fig. 8.3 in Gilmore, 1974b). That is, the two-photon algebra is a subalgebra of sp(4).

## 2. Representation of the two-photon algebra

In physical processes, the two-photon algebra acts on the harmonic-oscillator Fock space. Matrix elements of the operators (5.3) are easily computed with respect to the standard number states $|n\rangle$ which form a basis for this space.

However, many computational simplifications can be effected by reordering these operators before computations are actually carried out. The required disentangling relations are most easily determined in a faithful matrix representation of $h_6$, which is smaller than the Fock-space representation. In the present case the smallest faithful representation consists of $4\times 4$ matrices. This matrix representation is given explicitly by

$$\begin{aligned} &n+\tfrac{1}{2}\rightarrow E_{22}-E_{33},\quad I\rightarrow -2E_{41}\ ,\\ &a^{\dagger 2}\rightarrow 2E_{23},\quad a^{\dagger}\rightarrow E_{21}-E_{43}\ ,\\ &a^2\rightarrow -2E_{32},\quad a\rightarrow -E_{31}-E_{42}\ , \end{aligned} \tag{5.6}$$

and

$$\eta(n+\tfrac{1}{2})+\delta I+Ra^{\dagger 2}+La^2+ra^{\dagger}+la \xrightarrow{M(h_6)} \begin{bmatrix} 0 & 0 & 0 & 0\\ r & \eta & 2R & 0\\ -l & -2L & -\eta & 0\\ -2\delta & -l & -r & 0 \end{bmatrix}\ . \tag{5.7}$$

This representation can be obtained from the $4\times 4$ matrix representation of $\mathrm{sp}(4)\supset h_6$. It should be observed that this finite-dimensional representation is not Hermitian, and its exponential is not unitary. The validity of group multiplication within representations is independent of the metric properties of these representations (e.g., Hermiticity, unitarity) acting within specific spaces. That is, this representation is useful for carrying out exponential operator products, such as occur when integrating equations of motion, but it is not directly useful for computing Hilbert-space matrix elements.

## 3. Realization of the two-photon group

To illustrate the usefulness of this small matrix representation, we set up the machinery to carry out a number of disentangling theorems that will subsequently be useful. The general unitary evolution operator $U(t)$ can be written in a number of ways. One is as the exponential of a linear superposition of the elements in the Lie algebra $h_6$:

$$\begin{aligned} U(t)=\exp[&\eta''(t)(n+\tfrac{1}{2})+R''(t)a^{\dagger 2}+r''(t)a^{\dagger}\\ &+L''(t)a^2+l''(t)a+\delta''(t)I]\ . \end{aligned} \tag{5.8}$$

In the $4\times 4$ matrix representation the exponential of Eq. (5.7) can be computed explicitly. The resulting matrix is given in Table III.

A different realization is more convenient for computations of squeezed states in optical communications. This involves products of three separate exponential operators. These exponentials involve the two-photon operators $a^{\dagger 2},a^2$, the single-photon operators $a^{\dagger},a$, and the diagonal operators $n+\frac{1}{2},I$. For example, a useful parametrization of the unitary evolution operator $U(t)$ is

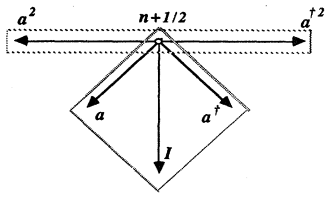



FIG. 6. Root space diagram of the two-photon algebra, in which there are three useful subalgebras. These are the su(1,1) algebra (with generators $n+\frac{1}{2}$, $a^2$, and $a^{\dagger 2}$); the single-photon algebra $h_4$ (with generators $n+\frac{1}{2}$, $a$, $a^{\dagger}$, and $I$); and the Heisenberg-Weyl algebra $h_3$ (with generators $a$, $a^{\dagger}$, and $I$). These three algebras are simple, solvable, and nilpotent, respectively. All are noncompact.

$$U(t)=\exp(ra^{\dagger}+la)\exp(Ra^{\dagger 2}+La^{2})\exp[\eta(n+\tfrac{1}{2})+\delta I]\ . \tag{5.9}$$

Other realizations of the unitary operator are obtained by changing the order of the exponential operators in Eq. (5.9). The 4×4 matrix representation of this operator product is obtained by straightforward matrix multiplication and is given in Table III.

Yet a third realization of the unitary evolution operator is particularly convenient for molecular dynamics calculations. This realization involves products of three exponential operators. In this normal-order realization, however, all annihilation operators are placed on the right, all creation operators on the left, and the diagonal operators in the middle:

$$U(t)=\exp(R'a^{\dagger 2}+r'a^{\dagger})\times\exp[\eta'(n+\tfrac{1}{2})+\delta' I]\exp(L'a^{2}+l'a)\ . \tag{5.10}$$

The 4×4 matrix representation of this realization is also obtained by simple matrix multiplication. It is also presented in Table III.

Analytic Baker-Campbell-Hausdorff relations or disentangling theorems for the parameters $(R,L,r,l,\eta,\delta)$, $(R,L,r,l,\eta,\delta)'$, and $(R,L,r,l,\eta,\delta)''$ are simply obtained by equating the matrix representations of these three operator realizations.

### C. Applications of single-mode squeezed states

#### 1. Integrating the equations of motion

The Hamiltonian equations of motion can be converted to equations of motion for the group parameters $(R,L,r,l,\eta,\delta)^{(',\,'')}$ in any of the realizations of $H_6$, in particular, the three described in the previous section. Such a set of equations has been derived for squeezed states of a single field mode by Yuen (1976) and for one- and two-mode squeezed states of molecular systems by Gazdy and Micha (1985). These equations of motion are complex and not easy to generalize to systems with a multiplicity of modes. Further, although the equations for the coefficients of the one- and two-photon creation and annihilation operators as well as the identity operator $(R,L,r,l,\delta)$ can be integrated in terms of the remaining

TABLE III. The faithful matrix representations of three realizations of $H_6$.

$U(t)$

$$e^{\eta''(n+1/2)+R''a^{\dagger 2}+r'a^{\dagger}+L''a^{2}+l''a+\delta''I}=\begin{bmatrix} 1 & 0 & 0 \\ \dfrac{e^{M}-I}{M}\begin{bmatrix} r'' \\ -l'' \end{bmatrix} & e^{M} & 0 \\ -2\delta''+(-l'',-r'')\dfrac{e^{M}-I-M}{M^{2}}\begin{bmatrix} r'' \\ -l'' \end{bmatrix} & (-l'',-r'')\dfrac{e^{M}-I}{M} & 1 \end{bmatrix}$$

$$e^{ra^{\dagger}+la}e^{Ra^{\dagger 2}+La^{2}}e^{\eta(n+1/2)+\delta I}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ r & \cosh\theta e^{\eta} & 2R\dfrac{\sinh\theta}{\theta}e^{-\eta} & 0 \\ -l & -2L\dfrac{\sinh\theta}{\theta}e^{\eta} & \cosh\theta e^{-\eta} & 0 \\ -2\delta & -le^{\eta}\cosh\theta-2Lr\dfrac{\sinh\theta}{\theta} & -re^{-\eta}\cosh\theta-2Rl\dfrac{\sinh\theta}{\theta} & 1 \end{bmatrix}$$

$$e^{R'a^{\dagger 2}+r'a^{\dagger}}e^{\eta'(n+1/2)+\delta' I}e^{L'a^{2}+l'a}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2R'l'e^{-\eta'} & e^{\eta'}-4L'R'e^{-\eta'} & 2R'e^{-\eta'} & 0 \\ -l'e^{-\eta'} & -2L'e^{-\eta'} & e^{-\eta'} & 0 \\ -2\delta'+r'l'e^{-\eta'} & 2r'L'e^{-\eta'} & -r'e^{-\eta'} & 1 \end{bmatrix}$$

$$M=\begin{bmatrix} \eta'' & 2R' \\ -2L'' & -\eta'' \end{bmatrix},$$

$$e^{M}=\cosh\theta I_{2}+M\frac{\sinh\theta}{\theta},\quad \theta^{2}=\eta''^{2}-4L''R''\ .$$

parameter $\eta$, the equation for $\eta$ cannot be integrated in closed form, as it is reducible to a Riccati equation. Thus one is ultimately forced to a numerical integration of the equations of motion.

We prefer numerical integration of the equations of motion from the outset followed by extraction of the parameter values $(R,L,r,l,\eta,\delta)$ using the results of the disentangling theorems presented in the last section. This approach is made possible by the mathematical theorems which underly the disentangling theorems.

The equations of motion for the unitary evolution operator

$$i\frac{\partial}{\partial t}U(t,t_0)=H(t)U(t,t_0) \tag{5.11}$$

with initial conditions

$$U(t_0,t_0)=I \tag{5.12}$$

are in fact equations of motion for an element, $U(t,t_0)$, in the group $H_6$, since the Hamiltonian $H(t)$ of Eq. (5.2) is an element of the Lie algebra $h_6$. As a result, these equations can be integrated in the group itself (Yuen, 1976; Gazdy and Micha, 1985), in the unitary representation acting on the Hilbert space, which is the normal quantum-mechanical procedure, or in any other faithful representation. In the latter case the greatest economy of effort comes about by choosing the smallest faithful finite-dimensional representation, given in Eq. (5.7). The equations of motion are then

$$i\frac{\partial}{\partial t}M[U(t,t_0)]=M[H(t)]M[U(t,t_0)]\ , \tag{5.13}$$

with the initial condition

$$M[U(t_0,t_0)]=I_4\ . \tag{5.14}$$

The matrix integration can be carried out numerically. The resulting $4\times4$ matrix, after any time interval, has the form

$$M[U(t,t_0)]=\begin{bmatrix} 1 & 0 & 0 & 0 \\ M_{21} & M_{22} & M_{23} & 0 \\ M_{31} & M_{32} & M_{33} & 0 \\ M_{41} & M_{42} & M_{43} & 1 \end{bmatrix}\ . \tag{5.15}$$

This matrix contains nine (complex) pieces of information. Since there are only six independent Lie group parameters, the $4\times4$ matrix contains three redundant pieces of information, which can be, and have been, used as a check on the accuracy of the numerical integration.

Once the integration has been completed, the group parameters can be extracted by comparison of Eq. (5.15) with the $4\times4$ matrix representation of any of the realizations of the group given in Table III. In the limit $t_0\to-\infty$, $t\to+\infty$, the group parameters for the $S$ matrix are obtained.

### 2. Operator expectation values

The computation of matrix elements and expectation values of operators is greatly facilitated by the disentangling theorems. These can be used to construct generating functions for operator expectation values. The expectation value of an operator $X$ (e.g., $a^\dagger$) in the state $|\Psi(t)\rangle=U(t,t_0)|0\rangle$ is

$$\langle X\rangle=\langle 0|U^\dagger(t,t_0)XU(t,t_0)|0\rangle \tag{5.16a}$$

$$=\frac{\partial}{\partial\gamma}\langle 0|U^\dagger(t,t_0)e^{\gamma X}U(t,t_0)|0\rangle|_{\gamma=0}\ . \tag{5.16b}$$

The matrix element in Eq. (5.16a) depends on the choice of representation (i.e., Fock space), while the product of exponential operators in Eq. (5.16b) is representation independent. This product can therefore be carried out explicitly in any faithful representation, for example, in the representation Eq. (5.9),

$$\begin{array}{ccc} U^\dagger(t,t_0)e^{\gamma X}U(t,t_0) & \to & M^{-1}[U(t,t_0)]M[e^{\gamma X}]M[U(t,t_0)] \\ \downarrow & & \downarrow \\ e^{Ra^{\dagger 2}+ra^\dagger}e^{\eta(n+1/2)+\delta I}e^{La^2+la} & \leftarrow & M[e^{Ra^{\dagger 2}+ra^\dagger}]M[e^{\eta(n+1/2)+\delta I}]M[e^{La^2+la}]\ . \end{array}$$

The computation follows the solid arrows, and it results in an expression valid in any representation, in particular, on Fock space. The matrix element of the disentangling operator product is

$$\langle 0|e^{Ra^{\dagger 2}+ra^\dagger}e^{\eta(n+1/2)+\delta I}e^{La^2+la}|0\rangle=e^{\eta(\gamma)/2+\delta(\gamma)}\ . \tag{5.17}$$

This is the generating function for the moments of the operator $X$:

$$\langle X^n\rangle=\left[\frac{\partial}{\partial\gamma}\right]^n e^{\eta(\gamma)/2+\delta(\gamma)}\Big|_{\gamma=0}\ . \tag{5.18}$$

Since $\eta(0)=\delta(0)=0$, the lowest moments are

$$\begin{aligned} \langle X\rangle&=\tfrac{1}{2}\eta'(0)+\delta'(0)\ , \\ \langle X^2\rangle&=(\tfrac{1}{2}\eta+\delta)''|_{\gamma=0}+(\langle X\rangle)^2\ , \\ \langle\Delta X^2\rangle&=\langle X^2\rangle-(\langle X\rangle)^2=(\tfrac{1}{2}\eta+\delta)''|_{\gamma=0}\ . \end{aligned} \tag{5.19}$$

In Table IV we summarize the generating functions for moments of the creation and annihilation operators $a^\dagger,a$ ($\langle e^{\gamma a^\dagger+\gamma^* a}\rangle$) in three different parametrizations of the coherent states for $H_6/\mathrm{U}(1)\otimes\mathrm{U}(1)$. These are: $|\alpha\beta\rangle=D(\alpha)S(\beta)|0\rangle$, $|\beta\alpha\rangle=S(\beta)D(\alpha)|0\rangle$, and

TABLE IV. Generating functions for moments of the creation and annihilation operators $F(\gamma^*,\gamma)=\langle\exp(\gamma a^\dagger+\gamma^* a)\rangle$ in three different parametrizations of the coherent states for $H_6/U(1)\otimes U(1)$.

| $\|\Psi\rangle$ | $\log F(\gamma,\gamma^*)$ |
|---|---|
| $D(\alpha)S(\beta)\|0\rangle$ | $\gamma^*\alpha+\frac{1}{2}\gamma^{*2}\beta\sinh\|\beta\|/\|\beta\|+\text{H.c.}+\frac{1}{2}\gamma^*\gamma(\sinh^2\|\beta\|+\cosh^2\|\beta\|)$ |
| $S(\beta)D(\alpha)\|0\rangle$ | $\gamma^*(\alpha\cosh\|\beta\|+\alpha^*\beta\sinh\|\beta\|/\|\beta\|)+\frac{1}{2}\gamma^{*2}\beta\sinh\|\beta\|/\|\beta\|+\text{H.c.}$ |
| $e^{Ra^{\dagger 2}+ra^\dagger-R^*a^2-r^*a}\|0\rangle$ | $\gamma^*\dfrac{r+2r^*R}{1-4R^*R}+\dfrac{1}{2}\gamma^{*2}\dfrac{2R}{1-4R^*R}+\text{H.c.}$ |

$$|Rr\rangle=\exp(Ra^{\dagger 2}+ra^\dagger-R^*a^2-r^*a)|0\rangle\ ,$$

where $D(\alpha)=\exp(\alpha a^\dagger-\alpha^* a)$ and $S(\beta)=\exp\frac{1}{2}(\beta a^{\dagger 2}-\beta^* a^2)$.

Computation of $S$-matrix elements between Fock states is even simpler. The $4\times4$ matrix (5.15) is related by disentangling theorems to the normally ordered product

$$\{(5.15)\}\rightarrow e^{Ra^{\dagger 2}+ra^\dagger}e^{\eta(n+1/2)+\delta I}e^{La^2+la}\ . \tag{5.20}$$

The matrix element of particular interest for molecular dynamics calculations is the excitation amplitude between the initial ground state $|0\rangle$ and a final state with $n$ quanta, $|n\rangle$. This matrix element is

$$\langle n|\exp(Ra^{\dagger 2}+ra^\dagger)\exp[\eta(n+\tfrac{1}{2})+\delta I]\exp(La^2+la)|0\rangle=\langle n|\exp(Ra^{\dagger 2}+ra^\dagger)|0\rangle\exp(\eta/2+\delta)\ . \tag{5.21}$$

The remaining matrix element is a finite sum:

$$\langle n|\exp(Ra^{\dagger 2}+ra^\dagger)|0\rangle=\sum_{n_2=0}^{[n/2]}\frac{r^{n-2n_2}}{(n-2n_2)!}\frac{R_2^n}{n_2!}\sqrt{n!}\ . \tag{5.22}$$

### 3. Squeezed states in quantum optics

In this section we show how squeezed states can be used to reduce uncertainty in measurements of the electromagnetic field. This is done by expressing the electromagnetic field in terms of the single-mode creation and annihilation operators. The expectation values of these operators then determine the time-dependent mean value of the electromagnetic field. Measurement uncertainties are determined by computing the variance of these operators. These expectation values are computed for

(i) a pure (single-photon) coherent state $|\alpha\rangle=D(\alpha)|0\rangle$,

(ii) a pure squeezed state $|\beta\rangle=S(\beta)|0\rangle$,

(iii) a general squeezed state $|\alpha\beta\rangle=D(\alpha)S(\beta)|0\rangle$.

The $A$ vector potential for a single mode of the electromagnetic field can be constructed in terms of the single-mode creation and annihilation operators as

$$A(x,t)=\frac{1}{i}\left[\frac{4\pi\hbar c^2}{2V\omega_k}\right]^{1/2}(ae^{i(kx-\omega t)}-a^\dagger e^{-i(kx-\omega t)})\ . \tag{5.23}$$

Since position dependence is not important in the discussion to follow, it will be suppressed below. The electromagnetic field operator

$$E=-\frac{1}{c}\frac{\partial A}{\partial t}$$

is

$$E(t)=\left[\frac{4\pi\hbar\omega}{2V}\right]^{1/2}(ae^{-i\omega t}+a^\dagger e^{i\omega t})\ . \tag{5.24}$$

It is useful to express the creation and annihilation operators in terms of two Hermitian operators $X$ and $Y\,(=P)$ as

$$\begin{aligned}a&=\frac{X+iY}{\sqrt{2}}\\ a^\dagger&=\frac{X-iY}{\sqrt{2}}\end{aligned}\qquad\begin{aligned}X&=(a+a^\dagger)/\sqrt{2}\ ,\\ Y&=(a-a^\dagger)/\sqrt{2i}\ .\end{aligned} \tag{5.25}$$

The Hermitian operators then have the properties

$$[X,Y]=i \tag{5.26}$$

$$\Delta X\Delta Y\geq\tfrac{1}{4}\ , \tag{5.27}$$

where Eq. (5.27) follows from (5.26) by the Schwartz inequality. The Hermitian operators $X, Y$ describe the two quadrature phases of the electromagnetic field. They can in principle be separately measured using a phase-sensitive detection technique, such as homodyne detection (Slusher *et al.*, 1985). For an electromagnetic field in a pure (one-photon) coherent state, the zero-point fluctuations in the two quadratures are equal. However, in a squeezed state, the zero-point fluctuation in one of the quadratures is amplified, while the fluctuation in the other quadrature is deamplified. As a result, there is a reduction in the quantum noise level associated with

measurements in the second quadrature.

The electric field operator is

$$E=2\lambda(X\cos\omega t+Y\sin\omega t)\ . \tag{5.28}$$

A pure coherent state $|\alpha\rangle$ can be formed by applying a classical current to the vacuum. This is represented by a Hamiltonian of the form (5.2) with $f_2(t)=0$, $f_1(t)\neq 0$. Once a coherent state $|\alpha\rangle$ has been created and $f_1(t)=0$, the coherent state evolves in time with simple statistical properties. The electric field and variance are

$$\langle E(t)\rangle=2\lambda(\langle X\rangle\cos\omega t+\langle Y\rangle\sin\omega t)\ , \tag{5.29a}$$

$$\langle\Delta E^2(t)\rangle=(2\lambda)^2(\cos\omega t,\sin\omega t)\times\begin{bmatrix}\Delta X^2 & \Delta\{XY\}\\ \Delta\{YX\} & \Delta Y^2\end{bmatrix}\begin{bmatrix}\cos\omega t\\ \sin\omega t\end{bmatrix}\ , \tag{5.29b}$$

where

$$\Delta\{XY\}=\langle\tfrac{1}{2}(XY+YX)\rangle-\langle X\rangle\langle Y\rangle\ . \tag{5.30}$$

In a pure coherent state $|\alpha\rangle=D(\alpha)|0\rangle$, the electric field and variance are

$$\langle E(t)\rangle/2\lambda=(\mathrm{Re}\alpha,\mathrm{Im}\alpha)\begin{bmatrix}\cos\omega t\\ \sin\omega t\end{bmatrix}\ , \tag{5.31}$$

$$\langle\Delta E^2(t)\rangle=(2\lambda)^2(\cos\omega t,\sin\omega t)\times\begin{bmatrix}\Delta X^2 & \Delta\{XY\}\\ \Delta\{YX\} & \Delta Y^2\end{bmatrix}\begin{bmatrix}\cos\omega t\\ \sin\omega t\end{bmatrix}\ , \tag{5.32}$$

where the covariance matrix is

$$\begin{bmatrix}\Delta X^2 & \Delta\{XY\}\\ \Delta\{YX\} & \Delta Y^2\end{bmatrix}=\frac{1}{4}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\ . \tag{5.33}$$

These results are presented graphically in Fig. 7. The electric field oscillates sinusoidally. It may be visualized as the projection onto the $X$ axis of a point moving on a circle of radius $|\alpha|$, moving clockwise with angular velocity $\omega$. The initial condition is the point with coordinate $\alpha=(\langle X\rangle,\langle Y\rangle)=(\mathrm{Re}\alpha,\mathrm{Im}\alpha)$. This is shown in Fig. 7(a). The covariance matrix is a multiple of the identity, indicating that the error ellipse is a circle. This is shown in Fig. 7(b). The results of Figs. 7(a) and 7(b) are combined in Fig. 7(c), which shows the time dependence of the electric field together with its uncertainty (shaded region). The uncertainty in the measurement is constant, independent of the value of the electric field and independent of the quadrature that is measured.

A pure squeezed state $|\beta\rangle$ can be found by applying a Hamiltonian of the form (5.2) with $f_1(t)=0$, $f_2(t)\neq 0$ to the vacuum. Once the squeezed state $|\beta\rangle$ has been found and $f_2(t)=0$, it evolves in time with simple statistical properties. The electric field is

$$\langle\beta|E|\beta\rangle=0\ . \tag{5.34}$$

The covariance matrix is

$$\begin{bmatrix}\Delta X^2 & \Delta\{XY\}\\ \Delta\{YX\} & \Delta Y^2\end{bmatrix}=\frac{\cosh^2|\beta|+\sinh^2|\beta|}{4}I_2+\frac{\sinh|\beta|\cosh|\beta|}{2|\beta|}\begin{bmatrix}\mathrm{Re}\beta & \mathrm{Im}\beta\\ \mathrm{Im}\beta & -\mathrm{Re}\beta\end{bmatrix}\ . \tag{5.35}$$

The uncertainty ellipse described by this covariance matrix is shown in Fig. 8. The semimajor and semiminor axes have lengths $\frac{1}{2}e^{\pm|\beta|}$. The semimajor axis makes an angle $\psi$ with the $X$ axis, where $\psi$ is defined by $\beta=|\beta|e^{i\psi}$. The variance in electric-field measurements is given by the quadratic form Eq. (5.35). This has a particularly convenient interpretation as the projection onto the $X$ axis of the error ellipse, which is rotating about its center in the clockwise direction with an angular velocity $\omega$. This interpretation is illustrated in Fig. 8(b).

Homodyne detection in phase with the quadrature of the minor axis leads to reduction of the quantum zero-point fluctuation below the level of a standard coherent state, for example, the ground state of the electromagnetic field.

The average value of the electric-field operator and its variance can be computed in the same way when the field is in a generalized squeezed state,

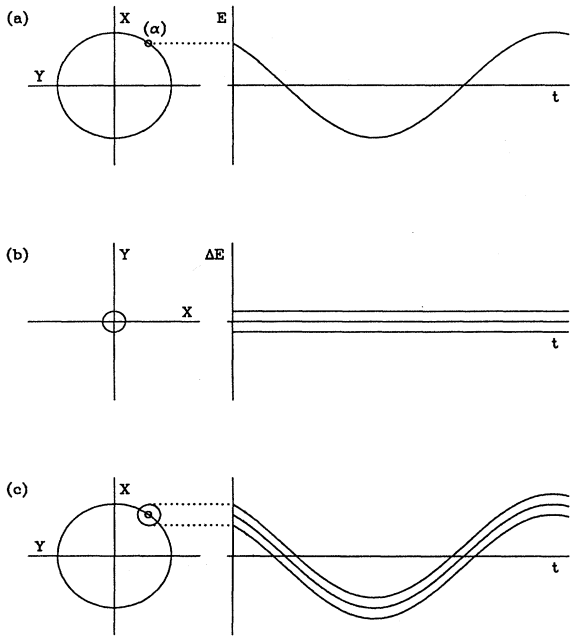



FIG. 7. Time-dependent field-coherent states: (a) Evolution of states with the initial condition $\alpha=(\langle X\rangle,\langle Y\rangle)=(\mathrm{Re}\alpha,\mathrm{Im}\alpha)$; (b) the time dependence of the uncertainty (described by covariance matrix) in the coherent states; (c) the combination of (a) and (b).

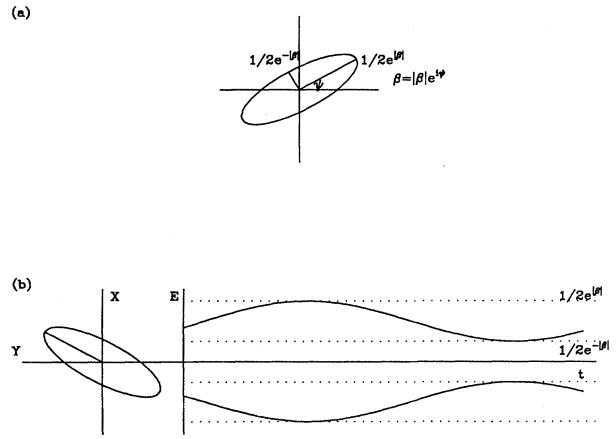



FIG. 8. Uncertainty ellipse of squeezed states: (a) Uncertainty ellipse described by the covariance matrix of Eq. (5.35); (b) the time dependence of the uncertainty ellipse.

$$|\alpha\beta\rangle = D(\alpha)S(\beta)|0\rangle \ . \tag{5.36}$$

In this case the field expectation value and its variance are given by previously described results. We find

$$\langle \alpha\beta|E|\alpha\beta\rangle = \langle \alpha 0|E|\alpha 0\rangle = 2\lambda(\mathrm{Re}\alpha\ \mathrm{Im}\alpha)\begin{bmatrix}\cos\omega t\\ \sin\omega t\end{bmatrix}, \tag{5.37}$$

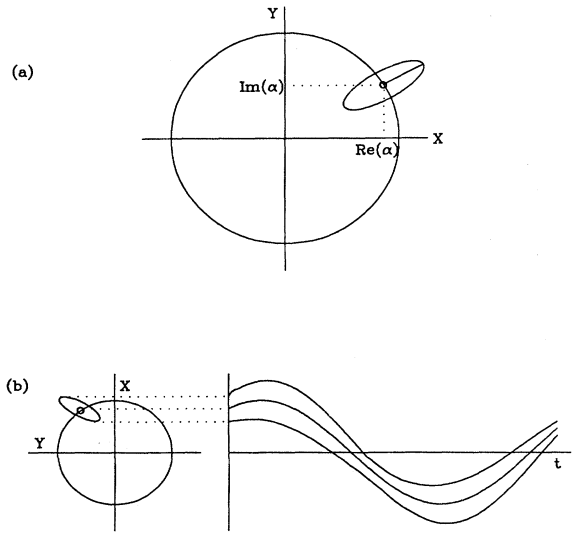



FIG. 9. Field expectation values in squeezed states: (a) The electric-field expectation value and variance in field measurements (uncertainty) in generalized squeezed states; (b) the time dependence of field measurements and corresponding variance.

$$\begin{aligned}\langle \alpha\beta|\Delta E^2|\alpha\beta\rangle &= \langle 0\beta|\Delta E^2|0\beta\rangle \\ &= (2\lambda)^2(\cos\omega t\ \sin\omega t) \\ &\quad\times\begin{bmatrix}\Delta X^2 & \Delta\{XY\}\\ \Delta\{YX\} & \Delta Y^2\end{bmatrix}\begin{bmatrix}\cos\omega t\\ \sin\omega t\end{bmatrix}.\end{aligned} \tag{5.38}$$

That is, the electric-field expectation value in the generalized squeezed state $|\alpha\beta\rangle$ is the same as for the pure coherent state $|\alpha\rangle$, and can therefore be represented graphically as in Fig. 9(a). The variance in field measurements is precisely that determined for a pure squeezed state and is therefore as illustrated in Fig. 9(b). The first two moments of the electric-field operator are illustrated graphically in Fig. 9(b). In the figure the most likely values of the electric-field measurements are those which occur within the error ellipse. This is centered at the position $(\langle X\rangle,\langle Y\rangle)=(\mathrm{Re}\alpha,\mathrm{Im}\alpha)$ associated with the mean value of the electric field for a pure coherent state $|\alpha,0\rangle$. The error ellipse itself is that associated with the pure squeezed state $|0\beta\rangle$. The time evolution of $\langle E\rangle$ and its uncertainty is obtained by projecting the error ellipse and its center onto the $X$ axis as the ellipse rotates rigidly around the circle $|\alpha|=$constant. The ellipse rotates uniformally around its center with angular frequency $\omega$ as its center rotates uniformly about the origin with the same angular velocity (Walls, 1983).

### 4. Squeezed states in molecular dynamics

Squeezed states occur naturally in the description of molecular processes. They occur when the Hamiltonian describing the collision is expanded in a power series of the local position operators and the expansion is truncated at quadratic terms. The Hamiltonian that remains is then a linear superposition of operators which close under commutation. If the collision is collinear and only one internal vibrational mode is present, the algebra describing the collision is the two-photon algebra $h_6$ of Eq. (5.3). As a result, the disentangling theorems of Table III are immediately applicable to the description of this molecular collision problem. These methods have been used to describe excitation of vibrational states, but not rotational or electronic states, in molecular collisions (Gilmore and Yuan, 1987, 1989).

To make this discussion explicit, we assume that a diatomic molecule, such as $O_2$, NO, or $N_2$ is bombarded with an inert atom such as He. The Hamiltonian describing this collision is

$$H = H_t + H_p + H_{\mathrm{int}} \ , \tag{5.39}$$

where $H_t$ ($H_p$) is the Hamiltonian describing the target (projectile) and $H_{\mathrm{int}}$ is the interaction Hamiltonian

$$H_{\mathrm{int}}(R,q) = V(R,q) - V(\infty,q) \ . \tag{5.40}$$

Here $R$ is a coordinate describing the target-projectile separation and $q$ are internal coordinates describing displacement from equilibrium of the atoms within the mol-

ecule. In the limit of large $R$, $H_{\rm int}(R,q)$ is zero, but as the projectile approaches the target the interaction increases.

To describe the collision, we make a semiclassical ansatz. That is, we assume the coordinate $R$ is classical and follows a classical trajectory $R(t)$. The interaction Hamiltonian is then expanded up to second order in the displacement operators,

$$H_{\rm int}=U_0(t)+\sum_i U_i(t)(q_i-q_i^0)+\tfrac{1}{2}U_{ij}(q-q^0)_i(q-q^0)_j + \text{higher-order terms} . \tag{5.41}$$

The higher-order terms are neglected. The resulting Hamiltonian then has the form of Eq. (5.2). This can be reexpressed in terms of creation and annihilation operators using the relation $q_i=\sqrt{\hbar/m\omega}(a_i+a_i^\dagger)/\sqrt{2}$. In the case of a collision of an atom with a diatomic molecule, only one vibrational mode is present, and the total Hamiltonian reduces to

$$H=\left[\hbar\omega+\frac{\hbar}{m\omega}B(t)\right](a^\dagger a+\tfrac{1}{2})+\sqrt{\hbar/2m\omega}\,A(t)(a+a^\dagger) +(\hbar/2m\omega)B(t)(a^2+a^{\dagger 2}) . \tag{5.42}$$

We are particularly interested in computing the probability that a target molecule, initially in its ground vibrational state, is excited into its $n$th vibrational state during the collision.

This probability is evaluated by computing the $S$ matrix for the collision. The $S$ matrix is computed as previously described. That is, both the Hamiltonian and the time-dependent unitary evolution operator are mapped onto the faithful but nonunitary $4\times4$ matrix representation (5.6). The equations of motion are then integrated over the course of the collision. This integration is carried out numerically, resulting in a $4\times4$ matrix with the structure of Eq. (5.15). The group-theoretic parameters for the $S$ matrix are then extracted from the $4\times4$ matrix $M(S)$ in the realization (5.10), which is particularly convenient for computing transition probabilities.

Since the description is semiclassical, we balance the initial and final kinetic energies to enforce energy conservation. That is, to compute the excitation probabilities of the $n$th state from the ground state, the time dependence of the coordinate $R$ is chosen so that

$$\frac{M}{2}[\dot R^{\,2}(\infty)-\dot R^{\,2}(-\infty)]+\Delta V=0 . \tag{5.43}$$

The coordinate $R(t)$ is computed classically, with the loss of energy spread smoothly over the collision. Once $R(t)$ is chosen numerical integration of Eq. (5.13) is straightforward.

After the $S$-matrix parameters have been extracted from the numerical integration, the $S$ matrix can be constructed explicitly,

$$S=\exp(Ra^{\dagger 2}+ra^\dagger)\exp[\eta(n+\tfrac{1}{2})+\delta I]\exp(La^2+la) . \tag{5.44}$$

The matrix element of $S$ between the initial ground state $|0\rangle$ and the final excited state $\langle n|$ is

$$\langle n|S|0\rangle=\sum_{n_2=0}^{[n/2]}\frac{r^{\,n-2n_2}}{(n-2n_2)!}\frac{R_2^{\,n}}{n_2!}\sqrt{n!}\exp(\eta/2+\delta) . \tag{5.45}$$

Matrix elements between an initial excited state $|n_i\rangle$ and a final excited state $\langle n_f|$ are also easily computed:

$$\langle n_f|S|n_i\rangle=\sum_{k=0}^{\min(n_f,n_i)}C(n_f,k;rR)C(n_i,k;lL)e^{\eta(k+1/2)+\delta} \tag{5.46}$$

where

$$C(n_i,k;lL)=\sum_{m=0}^{[(n_i-k)/2]}\frac{L^m}{m!}\frac{l^{\,n_i-k-2m}}{(n_i-k-2m)!}\sqrt{n_i!/k!} . \tag{5.47}$$

We mention explicitly that a different classical trajectory is integrated for each different excitation, in keeping with the energy conservation requirement.

In Fig. 10 we show the probability of excitation from the ground state to the $n$th excited state in the collision

$$\mathrm{He}+\mathrm{H_2}\rightarrow\mathrm{He}+\mathrm{H_2^*} \tag{5.48}$$

for $n=0$ to $n=6$. These results are compared with calculations by Gazdy and Micha (1985) made using different computational procedures.

### D. Multimode two-photon squeezed states

#### 1. Multimode two-photon algebra

The procedure described in the previous sections can be extended from the single-mode case to the multimode case. On the mathematical side, this involves extending the analysis from a subgroup of sp(2+2) to a subgroup of sp($2n+2$), where $n$ is the number of modes present. On the physical side, the interpretation for quantum optics differs slightly from that of molecular dynamics. In quantum optics, this extension allows the description of coherent two-photon processes in which the photon can occur in any of $n$ modes. In molecular dynamics, it allows description of collinear molecular scattering processes which involve vibrational excitation into $n$ normal modes distributed in an arbitrary fashion between the incident and target molecules.

The "two-photon" algebra is spanned by the operators

$$a_i^\dagger a_j^\dagger,\quad a_i^\dagger a_j+\tfrac{1}{2}\delta_{ij},\quad a_i a_j,\quad a_i^\dagger, a_j, I\ . \tag{5.49}$$

The single-mode operators $a_i^\dagger, a_j$ obey standard boson commutation relations $[a_i, a_j^\dagger]=\delta_{ij}$, from which the commutation relations of all other operators in the two-photon algebra are easily derived.

The two-photon algebra (5.49) can be regarded as a graded algebra, with the grading $d$ defined by the discussion directly following Eq. (5.5):

$$\begin{array}{llll} d=2 & a_i^\dagger a_j^\dagger & a_i^\dagger a_j+\frac{1}{2}\delta_{ij} & a_i a_j \\ d=1 & a_i^\dagger & a_j & \\ d=0 & I & . & \end{array} \tag{5.50}$$

This algebra has dimension $(2n+1)\times(n+1)$. The generators can be identified with a subset of generators of the Lie group $\mathrm{Sp}(2n+2)$ as follows:

$$\begin{array}{ll} a_i^\dagger a_j+\frac{1}{2}\delta_{ij} & E_{e_i-e_j} \quad (H_i \text{ if } i=j) \\ a_i^\dagger a_j^\dagger & E_{e_i+e_j} \\ a_i a_j & E_{-e_i-e_j} \\ a_i^\dagger & E_{e_i-e_{n+1}} \\ a_i & E_{-e_i-e_{n+1}} \\ I & E_{-2e_{n+1}} \quad . \end{array} \tag{5.51}$$

The generators that must be deleted from $\mathrm{Sp}(2n+2)$ to obtain the two-photon algebra are $E_{\pm e_i+e_{n+1}}$ and $H_{n+1}$. This already suggests that the smallest faithful matrix representation of the two-photon algebra (5.49) is a $(2n+2)\times(2n+2)$ matrix with the bordered structure shown in Eq. (5.7). That is, this representation of the algebra will have an entire row and column (top and right, by convention) equal to zero.

The smallest faithful matrix representation of the $n$-mode two-photon algebra for the case $n=2$ is explicitly

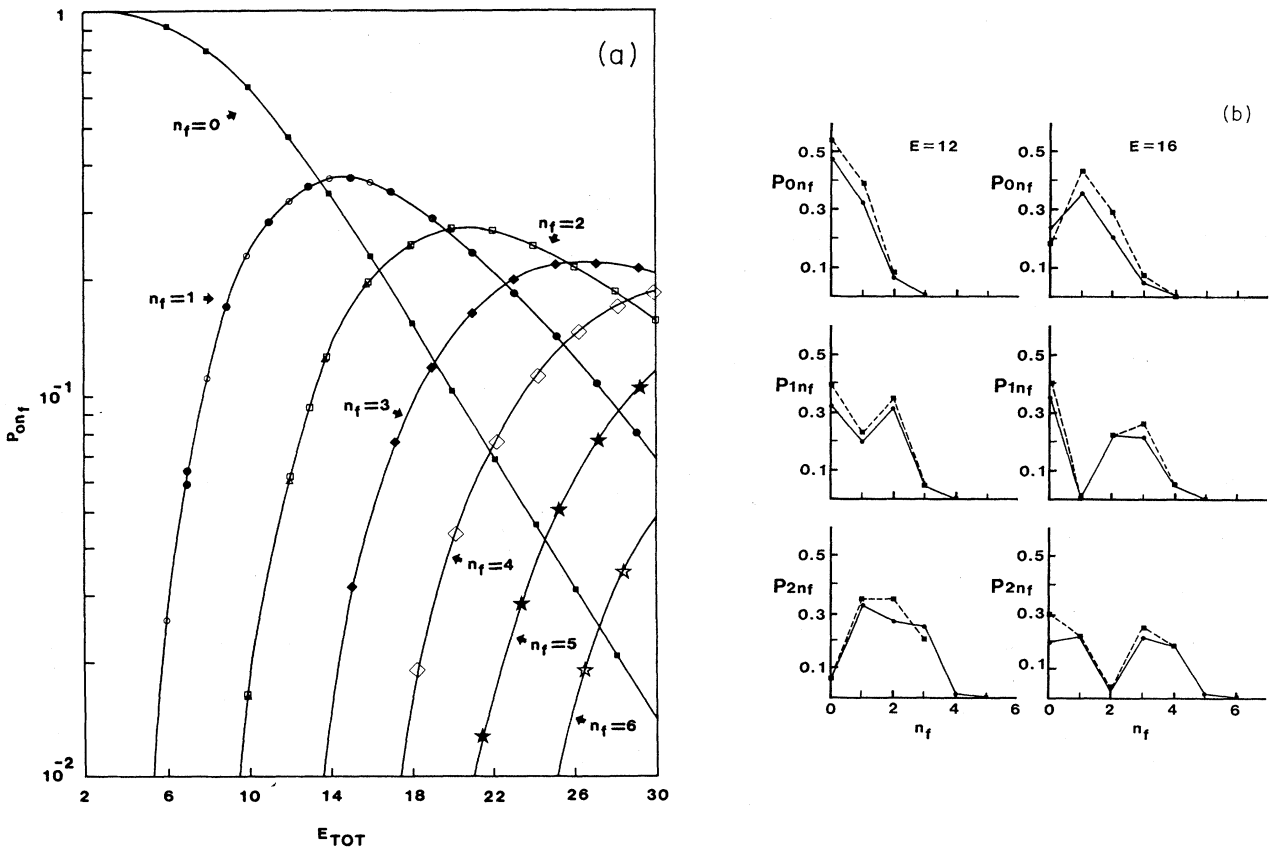



FIG. 10. Transition probabilities for excitations of $H_2$ by He scattering: (a) Transition probabilities $P_{0\to n_f}$ ($n_f=0$ to 6), plotted as a function of the total energy $E_{\mathrm{tot}}$ of the He+$H_2$ system. All curves are drawn smoothly through the data points (Gilmore and Yuan, 1987). ●, and △ denote results obtained by Gazdy and Micha (1985); (b) transition probabilities $P_{n_i\to n_f}$ plotted as a function of $n_f$ for $E=12$ and 16. The three plots on the left are for $E=12$ and those on the right for $E=16$ (Gilmore and Yuan, 1987).

$$D_{ij}(a_i^\dagger a_j+\tfrac{1}{2}\delta_{ij})+R_{ij}(a_i^\dagger a_j^\dagger)+L_{ij}(a_i a_j)+r_i a_i^\dagger+l_i a_i+\delta I \xrightarrow{n=2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ r_1 & D_{11} & D_{12} & R_{12} & 2R_{11} & 0 \\ r_2 & D_{21} & D_{22} & 2R_{22} & R_{21} & 0 \\ -l_2 & -L_{21} & -2L_{22} & -D_{22} & -D_{12} & 0 \\ -l_1 & -2L_{11} & -L_{12} & -D_{21} & -D_{11} & 0 \\ -2\delta & -l_1 & -l_2 & -r_2 & -r_1 & 0 \end{bmatrix} . \tag{5.52}$$

It should be obvious that $L_{ij}=L_{ji}$ and $R_{ij}=R_{ji}$, but $D_{ij}\neq D_{ji}$ in general. In the general $n$-mode case the $(2n+2)\times(2n+2)$ matrix representation of this algebra is

$$\xrightarrow{n} \begin{bmatrix} 0 & 0 & 0 & 0 \\ r & D & 2R & 0 \\ -l & -2L & -\tilde{D} & 0 \\ -2\delta & -\tilde{l} & -\tilde{r} & 0 \end{bmatrix} , \tag{5.53}$$

where the square matrices $D$, $R$, $L$ are analogs of those in (5.52), "tilde" indicates reflection in the minor diagonal, and $\tilde{R}=R$, $\tilde{L}=L$.

Disentangling theorems of the type discussed in Sec. V.B.3 (cf. Eqs. (5.8) and (5.10)] can be constructed without difficulty with the aid of the representation (5.53). For example, the analog of Eq. (5.10) is

$$\exp(R_{ij}a_i^\dagger a_j^\dagger+r_i a_i^\dagger)\exp[D_{ij}(a_i^\dagger a_j+\tfrac{1}{2}\delta_{ij})+\eta I]\exp(L_{ij}a_i a_i+l_i a_i) \xrightarrow{n} \begin{bmatrix} 1 & 0 & 0 & 0 \\ r-2\tilde{R}e^{-\tilde{D}}l & e^{D}-4\tilde{R}e^{-\tilde{D}}L & 2\tilde{R}e^{-\tilde{D}} & 0 \\ -e^{-\tilde{D}}l & -e^{-\tilde{D}}2L & e^{-\tilde{D}} & 0 \\ -2\delta+\tilde{r}e^{-\tilde{D}}l & -\tilde{l}+2\tilde{r}e^{-\tilde{D}}L & -\tilde{r}e^{-\tilde{D}} & 1 \end{bmatrix} . \tag{5.54}$$

The analogs of Eqs. (5.8) and (5.9) are similarly constructed. In fact, they can be constructed by inspection of the corresponding matrix results given in Table I. That is, appropriate indices are given to the scalars ($l,r$ in Table I$\rightarrow l_i,r_i$, while $\eta,R,L$ in Table I$\rightarrow D_{ij},R_{ij},L_{ij}$), and contractions are carried out on the appropriate indices.

The point of constructing these realizations of the $n$-mode two-photon algebra is again to simplify computation. That is, the equation of the motion for this algebra can be integrated following the procedure described in Sec V.C.1. The only change is that the $4\times4$ matrix representation used in Eqs. (5.13)–(5.15) is replaced by the $(2n+2)\times(2n+2)$ matrix representation (5.53). The output of a numerical integration code is a $(2n+2)\times(2n+2)$ matrix with the block structure given by the matrix on the right-hand side of (5.15), where $M_{ij}$ $(i,j=2,3)$ are $n\times n$ matrices and $M_{rs}$ ($r$ or $s=1$ or 4) are $n\times1$ or $1\times n$ matrices. Again, the parameters $(D_{ij},R_{ij},L_{ij},r_i,l_i,\delta)$ for the $S$ matrix are extracted from the numerical integration [right-hand side of Eq. (5.15)] by a matrix comparison.

### 2. Multimode squeezed states

The Hamiltonian describing the preparation of the electromagnetic field in a quantum state through one- and two-photon processes driven by classical sources is

$$H(t)=\sum_i \hbar\omega_i(a_i^\dagger a_i+\tfrac{1}{2})+\sum_{ij}[f_{ij}(t)a_i^\dagger a_j^\dagger+\text{H.c.}] + \sum_{ij} g_{ij}(t)(a_i^\dagger a_j+\tfrac{1}{2}\delta_{ij})+\sum_i [h_i(t)a_i^\dagger+\text{H.c.}] . \tag{5.55}$$

The dynamic group for this Hamiltonian is the two-photon group. As a consequence of the general theorem on coherent states, under this Hamiltonian a two-photon coherent state, in particular, the ground state of the electromagnetic field, will evolve into a two-photon coherent state.

The most general $n$-mode two-photon coherent state is of the form $\exp(R_{ij}a_i^\dagger a_j^\dagger+r_i a_i^\dagger-\text{H.c.})|0\rangle$. As before, there are a number of ways to parametrize these coherent states. The simplest involve successive applications of the $n$-mode squeezing operator $S(\beta)$ and the generalized displacement operator $D(\alpha)$,

$$D(\alpha)=\exp(\alpha_i a_i^\dagger-\alpha_i^* a_i) , \tag{5.56a}$$

$$S(\beta)=\exp\tfrac{1}{2}(\beta_{ij}a_i^\dagger a_j^\dagger-\beta_{ij}^* a_i a_j) . \tag{5.56b}$$

In fact, the most useful parametrization is

$$|\alpha\beta\rangle\equiv D(\alpha)S(\beta)|0\rangle . \tag{5.57}$$

The statistical properties of the electric-field operator

$E(t)$ can be computed when the electromagnetic field is in the squeezed state $|\alpha\beta\rangle$. The squeezed state is prepared from a Hamiltonian of the form (5.2) with $f_1(t)=0$, $f_2(t)\neq 0$. We analyze the properties of the field after the state has been prepared and the driving terms $f_1(t), f_2(t)$ have returned to zero. The electric field in the many-mode case is

$$E(t)=2\lambda\sum_i \sqrt{\omega_i}(a_i e^{-i\omega_i t}+a_i^\dagger e^{i\omega_i t}) , \tag{5.58}$$

where we have assumed a small frequency dispersion. The expectation value of the electric-field operator is

$$\langle\alpha\beta|E|\alpha\beta\rangle=\langle\alpha 0|E|\alpha 0\rangle=2\lambda\sum_i \sqrt{\omega_i}(\alpha_i e^{-i\omega_i t}+\alpha_i^* e^{i\omega_i t}) . \tag{5.59}$$

Expressing this in terms of in-phase and out-of-phase quadrature $\alpha_i=(x_i+iy_i)$ as before, we find

$$\langle\alpha\beta|E|\alpha\beta\rangle=2\lambda\sum_i \sqrt{\omega_i}(x_i\cos\omega_i t+y_i\sin\omega_i t) . \tag{5.60}$$

That is, the centroid of the distribution, or expectation value of the electric-field operator, evolves in time as it would for $n$ uncoupled modes.

The variance of this operator is given by

$$\langle\alpha\beta|(E-\bar{E})^2|\alpha\beta\rangle=\langle 0\beta|E^2|0\beta\rangle . \tag{5.61}$$

This is computed by computing the effect of the squeezing operator on the one-photon creation and annihilation operators,

$$S^{-1}(\beta)\begin{bmatrix}a^\dagger\\ a\end{bmatrix}S(\beta)=e^{-Ad(\beta)}\begin{bmatrix}a^\dagger\\ a\end{bmatrix}=\begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}a^\dagger\\ a\end{bmatrix} , \tag{5.62}$$

where

$$A=\tilde{D}=\cosh\sqrt{\beta\beta^*},\quad B=\beta\frac{\sinh\sqrt{\beta\beta^*}}{\sqrt{\beta\beta^*}}=C^* . \tag{5.63}$$

Then the covariance matrix is ($\omega_i=\omega$)

$$\begin{bmatrix}\Delta X^2 & \Delta\{XY\}\\ \Delta\{YX\} & \Delta Y^2\end{bmatrix}$$
$$=\tfrac{1}{4}(\cosh^2\sqrt{\beta\beta^*}+\sinh\sqrt{\beta\beta^*})I_2$$
$$+\frac{1}{4}\begin{bmatrix}M\beta+\beta^*M & \frac{1}{i}(M\beta-\beta^*M)\\ \frac{1}{i}(M\beta-\beta^*M) & -(M\beta+\beta^*M)\end{bmatrix} , \tag{5.64}$$

where

$$M=\frac{\sinh\sqrt{\beta\beta^*}\cosh\sqrt{\beta\beta^*}}{\sqrt{\beta\beta^*}} . \tag{5.65}$$

Thus the variance of a multimode electromagnetic field is ($\lambda^2=4\pi\hbar\omega/2V$)

$$\langle\alpha\beta|\Delta E^2|\alpha\beta\rangle=\langle 0\beta|\Delta E^2|0\beta\rangle$$
$$=(2\lambda)^2(\cos\omega t\,\sin\omega t)\times\begin{bmatrix}\Delta X^2 & \Delta\{XY\}\\ \Delta\{YX\} & \Delta Y^2\end{bmatrix}\begin{bmatrix}\cos\omega t\\ \sin\omega t\end{bmatrix} . \tag{5.66}$$

### 3. Vibrational excitation in molecular collisions

When the molecules involved in the collinear scattering described in Sec. V.C.4 are more complex than a diatomic and a single atom, then typically more than one internal vibrational degree of freedom can be excited. In this case, the Hamiltonian (5.42) has the more complicated structure

$$H(t)=\sum_i A_i(t)(a_i^\dagger a_i+\tfrac{1}{2})+\sum_{i<j}D_{ij}(t)(a_i^\dagger+a_i)(a_j^\dagger+a_j)$$
$$+\sum_i B_i(t)(a_i^\dagger+a_i)+\sum_i C_{ii}(t)(a_i^{\dagger 2}+a_i^2) . \tag{5.67}$$

Once again, this operator is a linear superposition of generators in the $k$-mode two-photon algebra; here $k$ is the number of excitable internal vibrational modes. The time dependence of the parameters in the Hamiltonian is constructed using the semiclassical ansatz, and the operator structure describing the internal degrees of freedom is obtained by truncating the momentum-independent potential, typically a sum of Morse potentials, beyond quadratic displacements from equilibrium.

We are particularly interested in determining the excitation probabilities during the collision. The initial state is assumed to be the ground vibrational state $|\underline{0}\rangle$. The final state will be a coherent state, since $H(t)$ is linear in the generators of the two-photon algebra,

$$|\text{final}\rangle=\sum_{\underline{n}}|\underline{n}\rangle A(\underline{n}) , \tag{5.68}$$

where $|\underline{n}\rangle=|n_1,n_2,\ldots,n_k\rangle$ and in particular the ground state is $|\underline{0}\rangle=|0_1,0_2,\ldots,0_k\rangle$.

The probability amplitudes are $S$-matrix elements,

$$A(\underline{n})=\langle\underline{n}|S|\underline{0}\rangle . \tag{5.69}$$

The $S$ matrix is constructed numerically as described previously in Secs. V.C.1 and V.D.1. Specifically, the equations of motion (5.13) are integrated numerically in the faithful $(2k+2)\times(2k+2)$ matrix representation. The output of the numerical integration is then compared with the $(2k+2)\times(2k+2)$ matrix representation of some convenient realization of the $S$ matrix. The most convenient realization is the "normally ordered" representation (5.10), whose faithful representation is given explicitly in Table III. The redundancy in the information content in these matrices is used to verify the accuracy of the numerical integration.

Once the $S$-matrix parameters have been computed, the transition probability amplitudes are easily found:

$$A(n)=\langle \underset{\sim}{n}|\exp(R_{ij}a_i^\dagger a_j^\dagger+r_i a_i^\dagger)\exp[D_{ij}(a_i^\dagger a_j+\tfrac{1}{2}\delta_{ij})]\exp(L_{ij}a_i a_j+l_i a_i)|\underset{\sim}{0}\rangle$$
$$=\langle \underset{\sim}{n}|\exp(R_{ij}a_i^\dagger a_j^\dagger+r_i a_i^\dagger)|\underset{\sim}{0}\rangle\langle \underset{\sim}{0}|\exp[D_{ij}(a_i^\dagger a_j+\tfrac{1}{2}\delta_{ij})]|\underset{\sim}{0}\rangle\langle \underset{\sim}{0}|\exp(L_{ij}a_i a_i+l_i a_i)|\underset{\sim}{0}\rangle \ . \tag{5.70}$$

The last matrix element is 1. The middle matrix element is $e^{(1/2)\mathrm{Tr}D+\delta}$. The first matrix element is

$$\left\langle \underset{\sim}{n}\left|\sum\prod_{m_{ij}}\frac{(R_{ij}a_i^\dagger a_j^\dagger)^{m_{ij}}}{(m_{ij})!}\prod_{m_i}\frac{(r_i a_i^\dagger)^{m_i}}{m_i!}\right|\underset{\sim}{0}\right\rangle . \tag{5.71}$$

The operator $a_i^\dagger$ occurs to the power

$$2m_{ii}+m_i+\sum_{j\neq i}m_{ij} \ . \tag{5.72}$$

The matrix element is nonzero only when this sum is $n_i$ for each $i$. As a result,

$$A(\underset{\sim}{n})=\sum_{m_{ij}}\prod_{ij}\frac{(R_{ij})^{m_{ij}}}{m_{ij}!}\frac{(r_i)^{n_i-2m_{ii}-\sum m_{ij}}}{(n_i-2m_{ii}-\sum m_{ij})!}\sqrt{n_i!}$$
$$\times e^{(1/2)\mathrm{Tr}D+\delta} \ . \tag{5.73}$$

Although this sum may appear complicated, in practice it involves relatively few terms, particularly for the low-lying excitations that are generally of interest. It should also be recalled that use of the disentangling theorem allows us to calculate exactly all intermediate processes involving excitation and deexcitation which are represented by the sum implicit in the computation of

$$\langle \underset{\sim}{n}|\exp(R'_{ij}a_i^\dagger a_j^\dagger+r'_i a_i^\dagger-\mathrm{H.c.})|\underset{\sim}{0}\rangle$$

by a power-series expansion of the exponential and resummation of all nonzero matrix elements. The disentangling theorem, as represented by the computation of the $S$ matrix in normally ordered form, reduces this infinite summation to a sum (5.73) over a finite and typically small number of terms.

These methods have been used to compute excitation probabilities in the collision $N_2+O_2$, which involves two vibrational modes, and the collisions $N_2+O_2$ and $NO+CO_2$, which involve five modes, only three of which are excitable in the collinear collisions (Gilmore and Yuan, 1989).

In summary, squeezed states are coherent states of the two-photon algebra. They have already been useful in describing electromagnetic processes and scattering of molecular species. However, we expect these states to find even wider applications in the future for several reasons. First, the two-photon algebra is a subalgebra of the symplectic algebra. This algebra, in turn, plays a basic role in the formulation of classical mechanics. Any attempt to demonstrate a close relation between classical and quantum mechanics is likely to do so through use of unitary representations of the symplectic group and specifically through the coherent states associated with the group. Squeezed states will evolve in another useful, important, and related direction as well. The dynamic group for the $n$-dimensional harmonic oscillator, as $u(n+1)$, is readily enlarged to $sp(2n+2)$ by inclusion of double-photon creation and annihilation operators. This enlarged algebra can be used to relate harmonic oscillators with frequency $\omega$ to harmonic oscillators with a different frequency. Algebraic techniques employing $sp(4,R)$ have already been used to compute overlaps and matrix elements of important operators (dipole moment, kinetic energy, potential energy) between arbitrary excited states of oscillators with two different frequencies separated by a distance $d$ in one dimension (Katriel, 1970). The two-photon algebra generated by the creation operators $a_i^\dagger$, the angular momentum operators $L_i=a_j^\dagger a_k-a_k^\dagger a_j$ ($i,j,k=1,2,3$ cycle), and the scalar creation operator $a^\dagger\cdot a^\dagger$, plus their adjoints and commutators has been used to compute overlaps and operator matrix elements between three-dimensional oscillator states with different frequencies at different centers (Katriel *et al.*, 1990). Such multicenter Gaussian overlap integrals have multiple uses in molecular and nuclear physics (Rowe, 1985).

## VI. FERMION COHERENT STATES

There are two fundamental types of particles in nature: bosons and fermions. These two types of particles have different commutation relationships for their basic operators and different statistical properties for the corresponding states. Bosons satisfy the standard canonical commutation relations and Bose-Einstein statistics, while fermions obey the anticommutation relations and Fermi-Dirac statistics. These differences are explicitly represented by different group structures and the associated Hilbert spaces. Generally speaking, realistic quantum systems usually couple these two types of particles. However, since the bosonic and fermionic algebras commute, their coherent states can be separately constructed. In Sec. V, we constructed the boson coherent states. In this section, we shall discuss fermions only.

The archetypical fermion systems are of course the atomic and nuclear systems. However, except for the hydrogen atom and the deuteron, such systems usually consist of a large number of interacting particles (of the order of $10^2$), which would render exact many-body solutions an impossibility. Thus much of our understanding of the quantum dynamics of fermions has been obtained via some (time-honored) approximation schemes. One such scheme is the variational principle, whose cardinal input is of course the choice of the trial wave function. The two criteria of a "good" trial wave function are that it maximize the number of quantum correlations and that it be simple to use. The coherent states constructed from

the system's group structure are, as we have argued, the best candidates to satisfy such requirements.

In this section we shall construct various fermion coherent states for different fermion algebraic structures. The corresponding geometrical space will include explicitly the Pauli principle and Fermi-Dirac statistics. General many-body methods are also discussed via the coherent-state theory.

### A. Single-fermion system

#### 1. Algebraic structure of the single fermion

First let us consider a single-fermion system. Suppose the system's Hamiltonian can be expressed as a function of the creation and annihilation operators $a^\dagger$ and $a$ of a single fermion which satisfies the standard anticommutation relations

$$[a,a^\dagger]_+=1, \quad [a,a]_+=[a^\dagger,a^\dagger]_+=0 . \tag{6.1}$$

Then such a system possesses an SU(2) dynamic group (Klauder, 1960). Generators of this SU(2) are $\{a,a^\dagger,a^\dagger a-\frac{1}{2}\}$ with the following algebraic structure:

$$\begin{aligned}&[a^\dagger,a]=2(a^\dagger a-\tfrac{1}{2}) ,\\ &[a^\dagger a-\tfrac{1}{2},a]=-a ,\\ &[a^\dagger a-\tfrac{1}{2},a^\dagger]=a^\dagger .\end{aligned} \tag{6.2}$$

Obviously, the operators $\{a^\dagger,a,a^\dagger a-\frac{1}{2}\}$ are in one-to-one correspondence with the angular momentum operators $\{J_+,J_-,J_0\}$ with spin $\frac{1}{2}$. Thus the coherent states of the single-mode fermion can be constructed by following the general algorithm of Sec. III.

#### 2. Single-fermion coherent states

Since the Hilbert space of the single fermion contains only two states, it can be realized by the space of the simplest irrep $\Gamma(j=\frac{1}{2})$ of SU(2), where $j$ is the angular momentum quantum number, i.e., spin. The basis vectors of $\Gamma(j=\frac{1}{2})$ are $\{|0\rangle=|\frac{1}{2},-\frac{1}{2}\rangle$ and $|1\rangle=|\frac{1}{2},\frac{1}{2}\rangle\}$. The fermion coherent states are then constructed as follows:

$$\begin{aligned}|\tfrac{1}{2},\zeta\rangle&=\exp(\zeta a^\dagger-\zeta^* a)|\tfrac{1}{2},-\tfrac{1}{2}\rangle\\ &=\sin(\theta/2)e^{-i\varphi}|\tfrac{1}{2},\tfrac{1}{2}\rangle+\cos(\theta/2)|\tfrac{1}{2},-\tfrac{1}{2}\rangle ,\end{aligned} \tag{6.3}$$

where $\zeta=(\theta/2)e^{-i\phi}$. Equation (6.3) shows that the group definition of the fermion coherent states possesses a natural topological space, SU(2)/U(1), and thus all the results of Sec. III.D are also applicable to the fermion coherent states with $j=\frac{1}{2}$. For instance, the completeness of the fermion coherent states is

$$\frac{1}{2\pi}\int d\Omega|\tfrac{1}{2},\zeta\rangle\langle\tfrac{1}{2},\zeta|=1 , \tag{6.4}$$

where $d\Omega=\sin\theta\, d\theta\, d\varphi$, and one of the useful BCH formulas can be expressed as

$$\begin{aligned}\exp(\zeta a^\dagger-\zeta^* a)&=\exp(\tau a^\dagger)\exp[\ln(1+\tau^*\tau)(a^\dagger a-\tfrac{1}{2})]\exp(-\tau^* a)\\ &=\exp(-\tau^* a)\exp[-\ln(1+\tau^*\tau)(a^\dagger a-\tfrac{1}{2})]\exp(\tau a^\dagger) .\end{aligned} \tag{6.5}$$

The relationship of $\tau$ and $(\theta\varphi)$ is given by Eq. (3.89).

The coherent states of a spin-$\frac{1}{2}$ fermion field in terms of $c$ numbers and the associated path-integral formalism were discussed by Klauder (1960).

### B. Finite many-fermion system

Now let us consider a system of $r$ single-fermion states. The creation $(a_i^\dagger)$ and annihilation $(a_j)$ operators of these states (e.g., the electrons in the atomic system or the nucleon in a nucleus) satisfy the following anticommutation relations:

$$[a_i,a_j^\dagger]_+=\delta_{ij}, \quad [a_i,a_j]_+=[a_i^\dagger,a_j^\dagger]_+=0 , \tag{6.6}$$

where $1\le i,\ j\le r$. Several algebras can be constructed from these operators. This means that different coherent states related to the corresponding algebra can be established and have different properties.

#### 1. Fermion u($r$) algebra

It is well known that the $r^2$ operators $\{a_i^\dagger a_j|1\le i,j\le r\}$ span Lie algebra u($r$). Their commutation relations are

$$[a_i^\dagger a_j,a_k^\dagger a_l]=\delta_{jk}a_i^\dagger a_l-\delta_{il}a_k^\dagger a_j . \tag{6.7}$$

If we let $H_i=a_i^\dagger a_i$, then the set $\{H_i|i=1,\ldots,r\}$ spans the maximum Abelian (Cartan) subgroup of u($r$). The commutation relations in the Cartan standard bases are

$$[H_i,a_j^\dagger a_k]=(\delta_{ij}-\delta_{ik})a_j^\dagger a_k . \tag{6.8}$$

It is obvious that the roots $e_i-e_j$ $(1\le i,j\le r)$ of this $r$-level fermion u($r$) algebra span the root space $A_{r-1}$.

For the fermion u($r$) algebra, the Hilbert space of interest carries a fully antisymmetric representation

$$\begin{aligned}\Lambda&=\{\lambda_1,\lambda_2,\ldots,\lambda_r\}=\{1,\ldots,1,0,\ldots,0\}\\ &\equiv\{1^k,0^{r-k}\}\end{aligned}$$

of u($r$). The basic states in this Hilbert space are the set

$\{|n_1,n_2,\ldots,n_r\rangle\}$, with $n_i=0,1$ and $\sum n_i=k$. There are $C_k^r=r!/k!(r-k)!$ states in this space.

Although not directly related to the fermion irrep of u($r$), the faithful matrix representation of the u($r$) element is also useful, as we shall see in the subsequent discussions and calculations. In the faithful matrix representation, every generator of u($r$) corresponds to an $r\times r$ matrix. Explicitly, we have

$$a_i^\dagger a_j \leftrightarrow E_{ij} , \tag{6.9}$$

where $E_{ij}$ is an $r\times r$ matrix with $+1$ in the $i$th row and $j$th column and zeros otherwise. For the fermion U($r$) with fully antisymmetric representation $\{1^k,0^{r-k}\}$, the extremal state may be taken as $|\text{ext}\rangle=|1,1,\ldots 1,0,\ldots 0\rangle$ [which in fact is the unperturbed ground state of the many-body Hamiltonian, Eq. (3.3), in the Hilbert space of $\{1^k,0^{r-k}\}$]. Thus

$$a_i^\dagger a_j|\text{ext}\rangle=0 \quad (1\le i\ne j\le k \text{ or } k+1\le i,j\le r) . \tag{6.10}$$

These operators $\{a_i^\dagger a_j|1\le i\le k,\ k+1\le j\le r\}$ span a subalgebra u($k$)$\oplus$u($r-k$) of u($r$). The corresponding Lie group U($k$)$\otimes$U($r-k$) is just the stability subgroup of U($r$) which leaves the extremal states $|\text{ext}\rangle$ invariant. Hence the coherent states of the fermion U($r$) group can be defined as

$$|\Lambda,\Omega\rangle=\exp\sum_{\substack{k+1\le i\le r\\ 1\le j\le k}}(\eta_{ij}a_i^\dagger a_j-\text{H.c.})|\text{ext}\rangle . \tag{6.11}$$

These coherent states have a natural topological coset space U($r$)/U($k$)$\otimes$U($r-k$). By comparing these states with the boson (photon) coherent states of the U($r$) group, we can see clearly the difference between the topological spaces of the boson and fermion systems. [Note, for the boson system, the topological space is U($r$)/U(1)$\otimes$U($r-1$), since the extremal state is the boson ground state in which all particles are in the lowest state. The physical reason for this difference is that they satisfy different statistical properties.]

Since the fermion U($r$) coherent states are defined according to the general algorithm given in Sec. III, all the properties listed there are preserved for Eq. (6.11). Here we shall discuss the details only of the symplectic structure, the BCH formula, and the completeness relation. These properties are useful in practical applications.

*The symplectic structure:* The topological space U($r$)/U($k$)$\otimes$U($r-k$) of the fermion U($r$) coherent states is a symmetric space [a $(r-k)\times k$ dimensional complex manifold (Helgason, 1978)]. Its symplectic structure is found as follows: In the faithful matrix representation of the U($r$), the coset representative of U($r$)/U($k$)$\otimes$U($r-k$) is

$$\exp\sum_{\substack{k+1\le i\le r\\ 1\le j\le k}}(\eta_{ij}a_i^\dagger a_j-\text{H.c.}) \rightarrow \begin{bmatrix}\sqrt{I_{r-k}-zz^\dagger} & z\\ -z^\dagger & \sqrt{I_k-z^\dagger z}\end{bmatrix} \tag{6.12}$$

with

$$z=\eta\frac{\sin\sqrt{\eta^\dagger\eta}}{\sqrt{\eta^\dagger\eta}} , \tag{6.13}$$

where $z$ and $\eta$ are the $(r-k)\times k$ complex matrix. The explicit form of $\eta$ can be obtained from the faithful matrix representation of the operator

$$\sum_{\substack{k+1\le i\le r\\ 1\le j\le k}}(\eta_{ij}a_i^\dagger a_j)$$

by using Eq. (6.9).

If we introduce the projected coset representative as

$$\tau\equiv\frac{z}{\sqrt{I_k-z^\dagger z}} , \tag{6.14}$$

then a group transformation

$$g=\begin{bmatrix}A & B\\ C & D\end{bmatrix}\in\text{U}(r)$$

acting on the coset space U($r$)/U($k$)$\times$U($r-k$) can be explicitly expressed as (Hua, 1963)

$$\tau'=(A\tau+B)(C\tau+D)^{-1} . \tag{6.15}$$

By confining the transformation $g$ onto $G/H$: $g\in G/H$, we can find the metrics $g_{ij}$ of such a space from Eq. (6.15). The result is

$$ds^2=\sum_{\alpha,\beta}g_{\alpha\beta}d\tau_\alpha d\tau_\beta^*=\sum_{\alpha,\beta}\frac{\partial^2 F}{\partial\tau_\alpha\partial\tau_\beta^*}d\tau_\alpha d\tau_\beta^* , \tag{6.16}$$

where $\tau_\alpha\equiv\tau_{ij}$, and

$$F=\ln\det(I_k+\tau^\dagger\tau) . \tag{6.17}$$

The measure of this space is (Hua, 1963)

$$d\mu_\Lambda(\tau,\tau^*)=\frac{\dim V^\Lambda}{\text{Vol}[\text{U}(r)/\text{U}(r-k)]}\times[\det(I_k+\tau^\dagger\tau)]^{-r}\prod_\alpha d\tau_\alpha d\tau_\alpha^* . \tag{6.18}$$

Equations (6.15) and (6.17) show further that U($r$)/U($k$)$\times$U($r-k$) is a Kaehlerian manifold (Helgason, 1978) which possesses explicit symplectic structure with a nondegenerate closed two-form:

$$\omega=\frac{i}{2}\sum_{\alpha,\beta}g_{\alpha\beta}d\tau_\alpha\wedge d\tau_\beta^* . \tag{6.19}$$

The function $F$ defined by Eq. (6.17) is called the Kaehler potential of this Kaehlerian manifold.

It is worth pointing out here that although the above discussion of the geometrical structure seems only to depend on the properties of U($r$)/U($k$)$\otimes$U($r-k$) and to be independent of the fermion representation space, it will be seen in the next section that the Kaehler potential [Eq.

(6.18)] which determines this geometrical structure is easily obtained from the coherent states of Eq. (6.11) and thereby intimately relates to the fermion representation.

*The BCH formula:* The BCH formula of U($r$) can easily be obtained by using the matrix representation of its group element. For example, the following BCH formula of U($r$) is often used in the application of the fermion U($r$) coherent states:

$$\exp\sum_{\substack{k+1\le i\le r\\1\le j\le k}}(\eta_{ij}a_i^\dagger a_j-\text{H.c.})=\exp\sum_{\substack{k+1\le i\le r\\1\le j\le k}}(\tau_{ij}a_i^\dagger a_j)\exp\sum_{\substack{1\le i,j\le k\\k+1\le i,j\le r}}(\lambda_{ij}a_i^\dagger a_j)\exp\sum_{\substack{1\le i\le k\\k+1\le j\le r}}(-\tau_{ij}^*a_i^\dagger a_j)\ . \tag{6.20}$$

The relation between $(\tau_{ij},\lambda_{ij})$ and $(\eta_{ij})$ can be found explicitly in the corresponding matrix representation of the above identity relation,

$$\exp\sum_{\substack{k+1\le i\le r\\1\le j\le k}}(\tau_{ij}a_i^\dagger a_j)\exp\sum_{\substack{1\le i,j\le k\\k+1\le i,j\le r}}(\lambda_{ij}a_i^\dagger a_j)\exp\sum_{\substack{1\le i\le k\\k+1\le j\le r}}(-\tau_{ij}^*a_i^\dagger a_j)\to\begin{bmatrix}I_{r-k}&\tau\\0&I_k\end{bmatrix}\begin{bmatrix}\exp\lambda_1&0\\0&\exp\lambda_2\end{bmatrix}\begin{bmatrix}I_{r-k}&0\\-\tau^\dagger&I_k\end{bmatrix}$$

$$=\begin{bmatrix}\exp\lambda_1-\tau\exp\lambda_2\tau^\dagger&\tau\exp\lambda_2\\-\exp\lambda_2\tau^\dagger&\exp\lambda_2\end{bmatrix}, \tag{6.21}$$

where the matrices $\tau$, $\lambda_1$, and $\lambda_2$ are the nonzero matrix block in the faithful matrix representation of the operators

$$\sum_{\substack{k+1\le i\le r\\1\le j\le k}}(\tau_{ij}a_i^\dagger a_j),\qquad\sum_{k+1\le i,j\le r}(\lambda_{ij}a_i^\dagger a_j)\ ,$$

and

$$\sum_{1\le i,j\le k}(\lambda_{ij}a_i^\dagger a_j)\ ,$$

respectively. Comparing the matrices in Eqs. (6.9) and (6.21), one obtains the following relations:

$$\tau=z(I_k-z^\dagger z)^{-1/2}\ , \tag{6.22a}$$

$$\exp\lambda_1=(I_{r-k}-zz^\dagger)^{-1/2},\quad\exp\lambda_2=(I_k-z^\dagger z)^{1/2}\ . \tag{6.22b}$$

Using this BCH formula, we can rewrite the coherent states (6.11) as

$$|\Omega\rangle=\frac{1}{N(\tau,\tau^*)^{1/2}}\exp\sum_{\substack{k+1\le i\le r\\1\le j\le k}}(\tau_{ij}a_i^\dagger a_j)|\text{ext}\rangle$$

$$\equiv\frac{1}{N(\tau,\tau^*)^{1/2}}\|\tau\rangle\ , \tag{6.23}$$

where $N(\tau,\tau^*)$ is the normalization

$$N(\tau,\tau^*)=\det(I_k+\tau^\dagger\tau) \tag{6.24}$$

and $\|\tau\rangle$ is the unnormalized form of Eq. (6.17). It is interesting to note that (as we have pointed out in the general theory of the coherent states in Sec. III) the Kaehler potential of the Kaehlerian manifold $\mathrm{U}(r)/\mathrm{U}(k)\times\mathrm{U}(r-k)$ can be obtained from the unnormalized coherent states $\|\tau\rangle$:

$$F(\tau,\tau^*)=\ln N(\tau,\tau^*)=\ln(\langle\tau\|\tau\rangle)\ . \tag{6.25}$$

*The completeness relation:* According to the general theorem of completeness, coherent states [see Sec. III and Eq. (6.11)] satisfy the completeness relation

$$\int|\Omega\rangle d\mu(\tau,\tau^*)\langle\Omega|=I\ , \tag{6.26}$$

where the measure $d\mu(\tau,\tau^*)$ is given by Eq. (6.18). Therefore any $|\Psi\rangle$ in the Hilbert space of the irrep $[1^k,0^{r-k}]$ can be expanded in terms of the coherent states given by Eqs. (6.11) or (6.23):

$$|\Psi\rangle=\int\|\tau\rangle f(\tau)N^{-1}(\tau,\tau^*)d\mu(\tau,\tau^*) \tag{6.27}$$

where $f(\tau)\in L^2[\mathrm{U}(r)/\mathrm{U}(k)\otimes\mathrm{U}(r-k),\mu']$ is an analytical function of $\tau$ and $L^2[\mathrm{U}(r)/\mathrm{U}(k)\otimes\mathrm{U}(r-k),\mu']$ is the square-integrable functional Hilbert space with measure $d\mu'=N^{-1}(\tau)d\mu(\tau)$,

$$\int f_1^*(\tau)f_2(\tau)d\mu'(\tau,\tau^*)<\infty\ . \tag{6.28}$$

### 2. Fermion so($2r$) algebra

The $r(2r-1)$ operators $\{a_i^\dagger a_j-\frac{1}{2}\delta_{ij}\ (1\le i,j\le r),\ a_ia_j,\ a_i^\dagger a_j^\dagger\ (1\le i\ne j\le r)\}$ span the Lie algebra so($2r$). The commutation relations are

$$[a_i^\dagger a_j-\tfrac{1}{2}\delta_{ij},a_k^\dagger a_l-\tfrac{1}{2}\delta_{kl}]=\delta_{jk}(a_i^\dagger a_l-\tfrac{1}{2}\delta_{il})-\delta_{il}(a_k^\dagger a_j-\tfrac{1}{2}\delta_{jk})\ , \tag{6.29a}$$

$$[a_i^\dagger a_j-\tfrac{1}{2}\delta_{ij},a_k^\dagger a_l^\dagger]=\delta_{jk}a_i^\dagger a_l^\dagger-\delta_{jl}a_i^\dagger a_k^\dagger\ , \tag{6.29b}$$

$$[a_ia_j,a_k^\dagger a_l^\dagger]=\delta_{ik}(a_l^\dagger a_j-\tfrac{1}{2}\delta_{lj})+\delta_{lj}(a_k^\dagger a_i-\tfrac{1}{2}\delta_{ki})-\delta_{li}(a_k^\dagger a_j-\tfrac{1}{2}\delta_{kj})-\delta_{ki}(a_l^\dagger a_i-\tfrac{1}{2}\delta_{li})\ . \tag{6.29c}$$

In the Cartan standard basis, we have

$$[H_i,a_ja_k]=-(\delta_{ij}+\delta_{ik})a_ja_k\ , \tag{6.30a}$$

$$[H_i,a_j^\dagger a_k^\dagger]=(\delta_{ij}+\delta_{ik})a_j^\dagger a_k^\dagger\ , \tag{6.30b}$$

$$[H_i,a_j^\dagger a_k]=(\delta_{ij}-\delta_{ik})(a_j^\dagger a_k-\tfrac{1}{2}\delta_{jk})\ , \tag{6.30c}$$

where $H_i=a_i^\dagger a_i-\frac{1}{2}$. It is obvious that the roots $\pm e_i\pm e_j$ span the root space $D_r$.

The Hilbert space of interest for fermion so($2r$) algebra comprises the carrier spaces of the two spinor representations $[\frac{1}{2},\frac{1}{2},\ldots,\pm\frac{1}{2}]$, where the $+$ $(-)$ sign corresponds to the even (odd) $n=\sum_{i=1}^r n_i$. The basis states in this Hilbert space are $|n_1,n_2,\ldots,n_r\rangle$ with $n_l=0,1$. There are $2^{r-1}$ states in both cases of $[\frac{1}{2},\frac{1}{2},\ldots,\pm\frac{1}{2}]$.

Similar to the fermion u($r$) case, the faithful matrix representation so($2r$) is more useful in practical calculations. The faithful matrix representations of generators of SO($2r$) are $2r\times 2r$ matrices. Explicitly, they are

$$a_i^\dagger a_j-\tfrac{1}{2}\delta_{ij}\leftrightarrow E_{ij}-E_{r+jr+i}\ , \tag{6.31a}$$

$$a_i^\dagger a_j^\dagger\leftrightarrow E_{ir+j}-E_{jr+i}\ , \tag{6.31b}$$

$$a_i a_j\leftrightarrow E_{r+ij}-E_{r+ji}\ , \tag{6.31c}$$

where $1\le i,j\le r$, and $E_{ij}$ is a $2r\times 2r$ matrix with $+1$ in the $i$th column and $j$th row and zeros otherwise. Thus the set of all diagonal matrices $H_{ii}=E_{ii}-E_{r+ir+i}$ is the Cartan subalgebra of so($2r$).

For the two spinor representations $[\frac{1}{2},\frac{1}{2},\ldots,\pm\frac{1}{2}]$ of the fermion Spin($2r$) group, where Spin($2r$) is the covering group of algebra so($2r$), the extremal states may be taken as

$$|\text{ext}\rangle=|0\rangle=|0,0,\ldots,0\rangle \quad \text{for } [\tfrac{1}{2},\tfrac{1}{2},\ldots,\tfrac{1}{2}] \tag{6.32a}$$

and

$$|\text{ext}\rangle=|1\rangle=|1,0,\ldots,0\rangle \quad \text{for } [\tfrac{1}{2},\tfrac{1}{2},\ldots,-\tfrac{1}{2}]\ , \tag{6.32b}$$

respectively. In fact, these are the unperturbed ground states of the many-body Hamiltonian of Eq. (3.3), for the even and odd systems in the Hilbert space of $[\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}]$ and $[\frac{1}{2},\frac{1}{2},\ldots,-\frac{1}{2}]$. Thus one finds that

$$a_i^\dagger a_j|0\rangle=0 \quad (1\le i,j\le r) \tag{6.33a}$$

or

$$\begin{aligned} &a_1^\dagger a_1|1\rangle=|1\rangle,\quad a_i^\dagger a_j|1\rangle=0 \quad (2\le i,j\le r)\ ,\\ &a_1^\dagger a_i^\dagger|1\rangle=0,\quad a_i a_1|1\rangle=0 \quad (2\le i\le r)\ . \end{aligned} \tag{6.33b}$$

Clearly, both sets of operators, $\{a_i^\dagger a_j-\frac{1}{2}\delta_{ij}|1\le i,j\le r\}$ and $\{a_1^\dagger a_1-\frac{1}{2},\ a_i^\dagger a_j-\frac{1}{2}\delta_{ij}\ (2\le i,j\le r),\ a_1^\dagger a_1,\ a_1 a_i\ (2\le i\le r)\}$, span a subalgebra u($r$) of so($2r$). The corresponding Lie groups U($r$) are the stability subgroup of Spin($2r$), which leaves the extremal states $|0\rangle$ of irreps $[\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}]$ ($|1\rangle$ for $[\frac{1}{2},\frac{1}{2},\ldots,-\frac{1}{2}]$) invariant. Again, following the general algorithm, the coherent states of the fermion so($2r$) algebra can easily be constructed. For simplicity, we shall discuss here only the coherent states of the irrep $[\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}]$. The same procedure is also applicable for the irrep $[\frac{1}{2},\frac{1}{2},\ldots,-\frac{1}{2}]$.

Since the stability subgroup of Spin($2r$) in $[\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}]$ is U($r$) (generated by operators $\{a_i^\dagger a_j|1\le i,j\le r\}$), the coherent states are isomorphic to the coset space Spin($2r$)/U($r$),

$$|\Omega\rangle=\exp\sum_{1\le i\ne j\le r}(\eta_{ij}a_i^\dagger a_j^\dagger-\text{H.c.})|\text{ext}\rangle\ , \tag{6.34}$$

where $\Omega=\exp\sum_{1\le i\ne j\le r}(\eta_{ij}a_i^\dagger a_j^\dagger-\text{H.c.})$ is the general form of the coset representative of Spin($2r$)/U($r$). Thus the coherent states of Eq. (6.34) have the natural topological structure of Spin($2r$)/U($r$) from which most properties can be deduced.

Just as in the fermion U($r$) coherent states, the generic properties of coherent states are also available for the fermion Spin($2r$) coherent states. Here we shall discuss in detail only three main properties: the symplectic structure, the BCH formula, and the completeness relation.

*The symplectic structure:* The topological space Spin($2r$)/U($r$) of the fermion Spin($2r$) coherent states with irrep $[\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}]$ is also a symmetric space—an $r\times r$ dimensional complex manifold. Since the coset space Spin($2r$)/U($r$) is isomorphic to SO($2r$)/U($r$) (Helgason, 1978), the symplectic structure of Spin($2r$)/U($r$) can be found from SO($2r$)/U($r$) simply by following the general procedure discussed in Sec. III. In the faithful matrix representation of the SO($2r$), the coset representative of SO($2r$)/U($r$) is

$$\exp\sum_{1\le i\ne j\le r}(\eta_{ij}a_i^\dagger a_j^\dagger-\text{H.c.})\rightarrow\begin{bmatrix}\sqrt{I_r-zz^\dagger} & z\\ -z^\dagger & \sqrt{I_r-z^\dagger z}\end{bmatrix} \tag{6.35}$$

with

$$z=\eta\frac{\sin\sqrt{\eta^\dagger\eta}}{\sqrt{\eta^\dagger\eta}}\ , \tag{6.36}$$

where $z$ and $\eta$ are the $r\times r$ antisymmetric complex matrix. The explicit form of $\eta$ is the matrix representation of the operator $\sum_{1\le i\ne j\le r}(\eta_{ij}a_i^\dagger a_j^\dagger)$, which can be found by using Eq. (6.31).

Again, by introducing the projected coset representation

$$\tau\equiv\frac{z}{\sqrt{I_r-z^\dagger z}}\ , \tag{6.37}$$

we can explicitly express a group transformation

$$g=\begin{bmatrix}A & B\\ C & D\end{bmatrix}\in\text{SO}(2r)$$

on the coset space SO($2r$)/U($r$) as

$$\tau'=(A\tau+B)(C\tau+D)^{-1}\ . \tag{6.38}$$

Thus the Riemannian metrics $g_{ij}$ of this space may be found from the non-normalized form $\|\tau\rangle$ of coherent states, Eq. (6.34):

$$\begin{aligned}|\Omega\rangle&=\exp\sum_{1\le i\ne j\le r}(\eta_{ij}a_i^\dagger a_j^\dagger-\text{H.c.})|0\rangle\\ &=N^{-1/2}(\tau,\tau^*)\exp\sum_{1\le i\ne j\le r}(\eta_{ij}a_i^\dagger a_j^\dagger)|0\rangle\\ &=N^{-1/2}(\tau,\tau^*)\|\tau\rangle\end{aligned} \tag{6.39}$$

where

$$N(\tau,\tau^*)=\det(I_r+\tau^\dagger\tau)^{1/2} . \tag{6.40}$$

The result is

$$ds^2=\sum_{\alpha,\beta} g_{\alpha\beta}d\tau_\alpha d\tau_\beta^*=\sum_{\alpha,\beta}\frac{\partial^2 F}{\partial\tau_\alpha\partial\tau_\beta^*}d\tau_\alpha d\tau_\beta^* \tag{6.41}$$

with

$$F=\ln N(\tau,\tau^*)=\tfrac{1}{2}\ln\det(I_r+\tau^\dagger\tau) . \tag{6.42}$$

The measure of this space is

$$d\mu_\Lambda(\tau,\tau^*)=\frac{\dim V^\Lambda}{\mathrm{Vol}[\mathrm{Spin}(2r)/\mathrm{U}(r)]}\times[\det(I_r+\tau^\dagger\tau)]^{-2r}\prod_\alpha d\tau_\alpha d\tau_\alpha^* . \tag{6.43}$$

Equations (6.38) and (6.41) also show that Spin$(2r)$/U$(r)$ is a Kaehlerian manifold (Helgason, 1978) possessing explicit symplectic structure with a closed nondegenerate 2-form

$$\omega=\frac{i}{2}\sum_{\alpha,\beta} g_{\alpha\beta}d\tau_\alpha\wedge d\tau_\beta^* , \tag{6.44}$$

and the function $F$, Eq. (6.42), is the Kaehler potential of this Kaehlerian manifold.

*The BCH formula:* The BCH formula of Spin$(2r)$ is obtained via the faithful matrix representation of SO$(2r)$. For instance, the following BCH formula of SO$(2r)$ is very useful:

$$\exp\sum_{l\le i\ne j\le r}(\eta_{ij}a_i^\dagger a_j^\dagger-\mathrm{H.c.})=\exp\sum_{1\le i\ne j\le r}(\tau_{ij}a_i^\dagger a_j)\exp\sum_{1\le i,j\le r}(\lambda_{ij}a_i^\dagger a_j-\tfrac{1}{2}\delta_{ij'})\exp\sum_{1\le i\ne j\le r}(-\tau_{ij}^* a_i a_j) . \tag{6.45}$$

The right side of Eq. (6.45) can be rewritten in the faithful matrix representation of SO$(2r)$ as

$$\begin{bmatrix} I_r & \tau \\ 0 & I_r \end{bmatrix}\begin{bmatrix} \exp\lambda & 0 \\ 0 & \exp-\lambda^T \end{bmatrix}\begin{bmatrix} I_r & 0 \\ -\tau^\dagger & I_r \end{bmatrix} = \begin{bmatrix} \exp\lambda-\tau\exp\lambda^T\tau^\dagger & \tau\exp\lambda^T \\ -\exp\lambda^T\tau^\dagger & \exp\lambda^T \end{bmatrix} \tag{6.46}$$

where the matrices $\tau$ and $\lambda$ are the nonzero matrix blocks in the so$(2r)$ faithful matrix representation of the operators

$$\sum_{1\le i,j\le r}(\tau_{ij}a_i^\dagger a_j^\dagger) \quad\text{and}\quad \sum_{1\le i,j\le r}(\lambda_{ij}a_i^\dagger a_j-\tfrac{1}{2}\delta_{ij'}) ,$$

respectively. Thus, comparing Eqs. (6.35) and (6.46), one obtains the following relations:

$$\tau=z(I_r-z^\dagger z)^{-1/2} , \qquad \exp\lambda=(I_r-zz^\dagger)^{-1/2} . \tag{6.47a}$$

However, Eq. (6.47a) gives only the SO$(2r)$ BCH formula. In order to obtain the Spin$(2r)$ BCH formula from those of SO$(2r)$, one notes that the subgroup U$(r)$ of Spin$(2r)$ is double-valued in the SO$(2r)$ representation. Therefore, for the Spin$(2r)$ case, $\lambda$ must be replaced by $\lambda/2$, and the correct relationship between $z_{ij}$ and $\lambda_{ij}$ with $\eta_{ij}$ for the Spin$(2r)$ case is

$$\tau=z(I_r-z^\dagger z)^{-1/2} , \qquad \exp(\lambda/2)=(I_r-zz^\dagger)^{-1/2} . \tag{6.47b}$$

The unnormalized coherent states of Eq. (3.39) can easily be obtained by the above relationship.

*The completeness relation:* From the general theorem of completeness, the coherent states of Eq. (6.34) satisfy the completeness relation

$$\int|\Omega\rangle d\mu(\tau,\tau^*)\langle\Omega|=I , \tag{6.48}$$

where the measure $d\mu(\tau,\tau^*)$ is given by Eq. (6.43). Thus any state $|\Psi\rangle$ in the Hilbert space of the irrep $[\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}]$ can be expanded in terms of the coherent states (6.34) or (6.39),

$$|\Psi\rangle=\int\|\tau\rangle f(\tau)d\mu'(\tau,\tau^*) , \tag{6.49}$$

where $f(\tau)\in L^2[\mathrm{Spin}(2r)/\mathrm{U}(r),\mu']$ is an analytical function of $\tau$ and $L^2[\mathrm{Spin}(2r)/\mathrm{U}(r),\mu']$ the Hilbert space of square-integrable functions with measure $d\mu'=N^{-1}(\tau)d\mu(\tau)$:

$$\int f_1^*(\tau)f_2(\tau)d\mu'(\tau,\tau^*)<\infty . \tag{6.50}$$

### 3. Fermion so$(2r+1)$ algebra

When we add "single" operators $\{a_i,a_j^\dagger|1\le i,j\le r\}$ to the generators of so$(2r)$: $\{a_i^\dagger a_j-\frac{1}{2}\delta_{ij},\ a_ia_j,\ a_i^\dagger a_j^\dagger\}$, they span the maximum dynamical Lie algebra of the $r$-mode fermion system, i.e., so$(2r+1)$. The commutation relations are given by Eqs. (6.29) plus

$$[a_i^\dagger,a_j]=2(a_i^\dagger a_j-\tfrac{1}{2}\delta_{ij}) , \tag{6.51a}$$

$$[a_i^\dagger a_j-\tfrac{1}{2}\delta_{ij},a_k]=-\delta_{ik}a_j , \tag{6.51b}$$

$$[a_i^\dagger a_j-\tfrac{1}{2}\delta_{ij},a_k^\dagger]=\delta_{jk}a_i . \tag{6.51c}$$

Moreover, the operators $(H_i=a_i^\dagger a_j-\frac{1}{2};\ i=1,\ldots,r)$ span the maximum Abelian (Cartan) subalgebra of so$(2r+1)$, and the commutation relations in the Cartan standard bases are (6.30) plus the following:

$$[H_i,a_k]=-\delta_{ik}a_k , \tag{6.52a}$$

$$[H_i,a_k^\dagger]=\delta_{ik}a_i^\dagger\ , \tag{6.52b}$$

$$[H_i,a_j^\dagger a_k]=(\delta_{ij}-\delta_{ik})(a_j^\dagger a_k-\tfrac{1}{2}\delta_{jk})\ . \tag{6.52c}$$

The root space is spanned by $\pm e_i$, and $\pm e_i\pm e_j$, and corresponds to the Lie algebra $B_r$.

We shall now give the matrix representation of the fermion so$(2r+1)$ algebra. The Hilbert space of interest for the fermion so$(2r+1)$ algebra comprises the carrier spaces of the single spinor representation $[\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}]$ with basic states $|n_1,n_2,\ldots,n_r\rangle$, where $n_i=0,1$. There are $2^r$ states in total.

The matrix representation of the generators of SO$(2r+1)$ in the faithful matrix representation of so$(2r+1)$ is a $(2r+1)\times(2r+1)$ matrix. The explicit form is

$$a_i^\dagger\leftrightarrow E_{i0}-E_{0r+i}\ , \tag{6.53a}$$

$$a_i\leftrightarrow E_{0i}-E_{r+i0}\ , \tag{6.53b}$$

$$a_i^\dagger a_j-\tfrac{1}{2}\delta_{ij}\leftrightarrow E_{ij}-E_{r+jr+i}\ , \tag{6.53c}$$

$$a_i^\dagger a_j^\dagger\leftrightarrow E_{ir+j}-E_{jr+i}\ , \tag{6.53d}$$

$$a_i a_j\leftrightarrow E_{r+ij}-E_{r+ji}\ , \tag{6.53e}$$

where $1\le i,j\le r$, $E_{ij}$ is a $(2r+1)\times(2r+1)$ matrix with $+1$ in the $i$th row and $j$th column and zeros otherwise. Again, the set of all diagonal matrices $H_i=E_{ii}-E_{r+ir+i}$ is the Cartan subalgebra of so$(2r+1)$. Let $H=\sum_{i=1}^r\lambda_i H_i$; we then have

$$[H,E_{i0}-E_{0r+i}]=\lambda_i(E_{i0}-E_{0r+i})\ , \tag{6.54a}$$

$$[H,E_{0i}-E_{r+i0}]=-\lambda_i(E_{0i}-E_{r+i0})\ , \tag{6.54b}$$

$$[H,E_{ij}-E_{r+jr+i}]=(\lambda_i-\lambda_j)(E_{ij}-E_{r+jr+i})\ , \tag{6.54c}$$

$$[H,E_{ir+j}-E_{jr+i}]=(\lambda_i+\lambda_j)(E_{ir+j}-E_{jr+i})\ , \tag{6.54d}$$

$$[H,E_{r+ij}-E_{r+ji}]=-(\lambda_i+\lambda_j)(E_{r+ij}-E_{r+ji})\ . \tag{6.54e}$$

Thus the roots of so$(2r+1)$ are

$$\pm\lambda_i,\quad \pm\lambda_i\pm\lambda_j(i<j,\ 1<i,j\le r)\ . \tag{6.55}$$

For the spinor representation $[\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}]$ of the so$(2r+1)$ algebra, the extremal state may be taken as

$$|\text{ext}\rangle=|0\rangle=|0,0,\ldots,0\rangle\ , \tag{6.56}$$

which is the unperturbed ground state of the many-body Hamiltonian, Eq. (3.3), in a Hilbert space of $[\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}]$. Thus the maximum suitable subgroup of so$(2r+1)$ that will leave the extremal states $|0\rangle$ invariant is U$(r)$, whose generators are $\{a_i^\dagger a_j-\frac{1}{2}\delta_{ij}|1\le i,j\le r\}$. Following the same procedure as in the u$(r)$ and so$(2r)$ cases, we can easily construct the coherent states of the fermion so$(2r+1)$ algebra, which is isomorphic to the coset space Spin$(2r+1)$/U$(r)$.

### C. Applications of the fermion coherent states

We shall now briefly discuss the usage of coherent states in the study of many-body fermion systems. For most finite fermion systems (e.g., nuclei, atoms, and molecules), under suitable mean-field approximations the shell structure will always emerge. Suppose the respective shell structure is denoted by $r$ single-particle (s.p.) levels, each with s.p. annihilation and creation operators $a_j$ and $a_j^\dagger(j=1,\ldots,r)$ respectively. Then, when we restrict the Hamiltonian to having only one- and two-body interaction terms, it can take on the form of Eq. (3.3), i.e.,

$$H=\sum_{j=1}^r\varepsilon_j a_j^\dagger a_j+\tfrac{1}{4}\sum_{ijkl}V_{ijkl}a_i^\dagger a_j^\dagger a_l a_k=H_0+H_{\text{inter}}\ . \tag{6.57}$$

It is well known that the maximum dynamical group for such a system is SO$(2r+1)$, as was discussed in Sec. VI.B.2. If we do not include in this discussion the transfer of a single particle, then the dynamical symmetry group can be reduced to SO$(2r)$. Furthermore, if the pairing correlation is neglected, then the dynamical group of the system is further reduced to U$(r)$. However, the pairing correlation is usually important in any collective cooperative phenomena of the many-body system, and therefore we shall consider here the SO$(2r)$ dynamical group of Eq. (6.57). The same procedure can be applied directly to the U$(r)$ dynamic group. The algebraic structure and coherent states of SO$(2r)$ have been discussed in detail in the last section. In this section, we shall present their applications in many-body mean-field dynamics.

#### 1. Finite many-fermion Hartree-Fock-Bogoliubov theory

Let us first consider the even fermion spinor irrep of SO$(2r)$, which is $[\frac{1}{2},\ldots,\frac{1}{2}]$. The corresponding coherent states are defined as

$$|\Psi\rangle=T|0\rangle=\exp\sum_{j<k}(\eta_{jk}^*a_j^\dagger a_k^\dagger-\eta_{jk}a_k a_j)|0\rangle\ , \tag{6.58}$$

where $|0\rangle$ is the vacuum state of Eq. (6.57) and $T$ is a coset representative of SO$(2r)$/U$(r)$,

$$T=\exp\sum_{j<k}(\eta_{jk}^*a_j^\dagger a_k^\dagger-\eta_{jk}a_k a_j)\ . \tag{6.59}$$

Since U$(r)$ is the maximum stability subgroup of SO$(2r)$ for $|0\rangle$, Eq. (6.58) is the most general transformation of the vacuum state under the SO$(2r)$ group space. The corresponding transformation for the single-particle operators is

$$\begin{bmatrix}\alpha_j\\ \alpha_j^\dagger\end{bmatrix}=T\begin{bmatrix}a_j\\ a_j^\dagger\end{bmatrix}T^{-1}\ . \tag{6.60}$$

It is not difficult to show that Eq. (6.60) is the most gen-

eral form of the well-known Hartree-Fock-Bogoliubov (HFB) quasiparticle transformation,

$$\begin{bmatrix} \alpha \\ \alpha^\dagger \end{bmatrix} = \mathbb{T}^\dagger \begin{bmatrix} a \\ a^\dagger \end{bmatrix} = \begin{bmatrix} U^\dagger & V^\dagger \\ V^T & U^T \end{bmatrix} \begin{bmatrix} a \\ a^\dagger \end{bmatrix} , \tag{6.61}$$

where

$$\mathbb{T} = \begin{bmatrix} U & -V^\dagger \\ V & U^T \end{bmatrix}$$

is the faithful matrix representative of $T$, and

$$U^\dagger = \cos\sqrt{\eta^\dagger\eta} = U, \quad V^* = \eta \frac{\sin\sqrt{\eta^\dagger\eta}}{\sqrt{\eta^\dagger\eta}} = -V^\dagger . \tag{6.62}$$

Hence it is obvious that the vacuum state of the Bogoliubov quasiparticles is the coherent state of Eq. (6.58). Finally, the quasiparticles can be obtained by solving the variational equation

$$\delta\langle\Psi|H'|\Psi\rangle = \delta\langle\Psi|H-\lambda N|\Psi\rangle = 0 , \tag{6.63}$$

where $N$ and $\lambda$ are the particle number and the chemical potential, respectively. Under the HFB approximation, Eq. (6.63) will immediately lead to the well-known HFB equations

$$\begin{bmatrix} v & \Delta \\ -\Delta^* & -v^* \end{bmatrix} \begin{bmatrix} U_i \\ V_i \end{bmatrix} = E_i \begin{bmatrix} U_i \\ V_i \end{bmatrix} , \tag{6.64}$$

where the Hartree-Fock potential $v$ and the pair potential $\Delta$ are defined by

$$v_{ij} = \varepsilon_{ij} - \lambda\delta_{ij} + \Gamma_{ij} , \tag{6.65a}$$

$$\Delta_{ij} = \tfrac{1}{2} \sum_{i'j'} V_{iji'j'} \kappa_{i'j'} . \tag{6.65b}$$

The matrix $\Gamma$ is known in the literature as the "deformed" potential and is defined as

$$\Gamma_{ij} = \sum_{i'j'} V_{ii'jj'} \rho_{i'j'} . \tag{6.66}$$

The density matrix $\rho_{ij}$ and the pair tensor $\kappa_{ij}$ in Eqs. (6.65b) and (6.66) are the matrix elements $\langle\Psi|a_i^\dagger a_j|\Psi\rangle$ and $\langle\Psi|a_i a_j|\Psi\rangle$, respectively.

If pairing correlation is neglected, then the dynamic group of the system is $\mathrm{U}(r)$. In this case, the quasiparticle transformation of Eqs. (6.58) and (6.60) is restricted to the coset $\mathrm{U}(r)/\mathrm{U}(r-k)\otimes\mathrm{U}(k)$ [i.e., Eq. (6.12)]. Therefore the above procedure is equivalent to the mean-field Hartree-Fock theory.

### 2. Symmetry-constrained HFB theory

In nuclear physics, it is well known that the basic concept of the quasiparticle is to represent the ground state as a quasiparticle vacuum (Bogoliubov, 1953). The starting point of the HFB theory is a linear unitary transformation of the single basic states. On the other hand, the coherent-states description of the HFB theory given above is based entirely on the dynamical group of the Hamiltonian, which means that it begins with the dynamical properties, i.e., the Hamiltonian, of the system and attempts to find the most general unitary transformation under the *dynamical symmetry constraint*. We refer to this procedure as the symmetry-constrained Hartree-Fock-Bogoliubov approach (Zhang, Feng, Wu, Wu, and Ginocchio, 1989). Since the symmetry-constrained HFB approach is based entirely on the dynamic symmetry group structure of the system, it is always valid for any system that has a dynamic group constraint. The general algorithm of the symmetry-constrained HFB theory can be expressed as follows:

When the Hamiltonian of the system is constrained by the dynamical group, its corresponding Hilbert space is sharply reduced to an irrep of the group, and therefore the general Bogoliubov quasiparticle transformation must be restricted to a unitary transformation within the group. To be more concrete, suppose $G$ is the dynamical symmetry group of some system, and $g$ is a unitary general transformation operator of $G$; then, within the constraint of the dynamical symmetry, the Bogoliubov transformation of the bare vacuum state $|0\rangle$ to a quasiparticle vacuum state $|\Psi\rangle$ must be given simply as

$$|\Psi\rangle = g|0\rangle = Th|0\rangle = T|0\rangle e^{i\varphi(h)} , \tag{6.67}$$

where $h \in H$ is the maximum stability subgroup of $G$ that keeps the bare vacuum state $|0\rangle$ invariant up a phase factor $\varphi$ and $T \in G/H$. According to the coherent-state theory (i.e., Sec. IV), the quaisparticle vacuum state (6.67) is precisely the coherent state of $G/H$. Thus the Bogoliubov quasiparticle transformation

$$\begin{bmatrix} \alpha_j \\ \alpha_j^\dagger \end{bmatrix} = T \begin{bmatrix} a_j \\ a_j^\dagger \end{bmatrix} T^{-1} \tag{6.68}$$

is uniquely restricted on the coset $G/H$. Using elementary group theory, we can easily determine the corresponding matrix form of Eq. (6.68) (the Bogoliubov quasiparticle transformation matrix) from the matrix representation of $G$. Then the HFB solutions can be found by use of the variational equation

$$\delta H'_Q(T) = 0 , \tag{6.69}$$

where

$$H'_Q(T) = \langle\Psi|H-\lambda N|\Psi\rangle = \langle 0|T^{-1}(H-\lambda N)T|0\rangle$$

is the $Q$ representative of $H-\lambda N$ in the coherent states of the dynamical group $G$. Computation of $H'_Q$ is straightforward using group theory. The final result is that $H'_Q(T)$ is an analytical function defined over $G/H$. Thus, under the dynamical symmetry constraint, the coherent-state variational method will completely determine the HFB theory. Further, Eq. (6.69) corresponds to finding the stable solutions of the function $H'_Q(T)$. These can be carried out by means of the standard stability theory (Huseyin, 1986) or catastrophe theory (Gilmore, 1981).

The symmetry-constrained HFB approach originates from the dynamics of the system, i.e., the Hamiltonian. Therefore, if $G$ is smaller than SO($2r$) under the dynamical constraint of the Hamiltonian (or the dynamical truncation of the Hilbert space—that is, if the number of single-particle bases of the Hilbert space is invariant under this truncation), then even though the final results of the HFB and symmetry-constrained HFB theories are equivalent, the symmetry-constrained approach possesses some advantages over the original HFB theory. These advantages are as follows: (1) The HFB theory may be extended to any system whose Hamiltonian is constrained by some dynamical group $G$; (2) for a given Hamiltonian, it is very easy and natural to find the HFB state of the most general transformation, i.e., the coherent states of $G$; (3) since the symmetry-constrained transformation is restricted to lie within the dynamical group $G$, the HFB transformation matrix can be found directly to be the matrix representation of the coset space $G/H$, which is obviously much simpler than the original one; (4) the symmetry-constrained approach provides a natural dynamic, geometric (topological) structure of the system, which is the coset space $G/H$; (5) calculations are greatly simplified via the matrix representation of group theory.

### 3. Applications to nuclear collective states

The first study of geometry and phase transitions in nuclear physics, although it was made via a toy model (Lipkin, Meshkov, and Glick, 1965), was in 1978 (Gilmore and Feng, 1978). In this study, the SU(2) coherent states were utilized. An extension of this type of study was soon after applied to nuclear structure physics to study the geometrical and phase-transitional problems. In that effort, the focus was on an investigation of the structural behavior of the phenomenological interacting-boson model, which was proposed by Arima and Iachello in the mid 1970s (Arima and Iachello, 1975, 1976). The coherent state used in this study was the U(6)/U(5) ten-dimensional coset space (Dieperink *et al.*, 1980; Ginocchio and Kirson, 1980a, 1980b; Feng *et al.*, 1981). In the past five years, a realistic algebraic model for collective states in nuclei has been proposed (C. L. Wu *et al.*, 1986, 1987). It is called the fermion dynamical symmetry model. The symmetry-constrained HFB theory described in Sec. VI.C.2 has been extensively applied to this model by Zhang *et al.* (Zhang *et al.*, 1987; Zhang, Feng, and Ginocchio, 1988; Zhang, Wu, *et al.*, 1988). Since the techniques discussed above are explicitly employed in these studies, we shall only briefly discuss them here.

#### *a. The algebraic structure*

The fermion dynamical symmetry model is discussed extensively in the literature (C. L. Wu *et al.*, 1986, 1987), to which the interested reader is referred. The basis of the model is a symmetry-dictated truncation scheme of the spherical shell model. The central idea is that for low-lying states of nuclei, once the valence levels are specified, the fermion dynamical symmetry model uniquely links normal-parity levels in a major shell with symmetry SO(8) or Sp(6) (Ginocchio, 1980). For the intruder or abnormal-parity level, the model prescribes a quasispin symmetry [$\mathcal{SU}(2)$]. For any nucleus, there are two valence major shells, one for neutrons and one for protons. Therefore, in reality, the symmetry for realistic nuclei must be of the type $\mathrm{SO}^{\pi}(8)\otimes\mathrm{SO}^{\nu}(8)\otimes\mathcal{SU}^{\pi}(2)\otimes\mathcal{SU}^{\nu}(2)$, $\mathrm{SO}^{\pi}(8)\otimes\mathrm{Sp}^{\nu}(6)\otimes\mathcal{SU}^{\pi}(2)\otimes\mathcal{SU}^{\nu}(2)$, or $\mathrm{Sp}^{\pi}(6)\otimes\mathrm{Sp}^{\nu}(6)\otimes\mathcal{SU}^{\pi}(2)\otimes\mathcal{SU}^{\nu}(2)$, where $\pi$ and $\nu$ denote protons and neutrons, respectively.

Thus in essence the fermion dynamic symmetry model is based on $\mathrm{SO}(8)\otimes\mathcal{SU}(2)$ and $\mathrm{Sp}(6)\otimes\mathcal{SU}(2)$ dynamic group structures. These symmetries are constructed from the truncation of the shell-model space via the so-called $k-i$ decomposition of the single nucleon level ($a^{\dagger}_{jm}$) for each major shell. The generators of SO(8) and Sp(6) are the monopole and the quadrupole pairing operators together with the multipole [$l=0,1,2,(3)$][SO(8)] operators within the normal levels:

(a) SO(8) case,

$$
\begin{aligned}
&S^{\dagger}=\sqrt{\Omega_{k3/2}/2}\,[b^{\dagger}_{k3/2}b^{\dagger}_{k3/2}]^{00}_{00},\quad S=(S^{\dagger})^{\dagger},\\
&D^{\dagger}_{\mu}=\sqrt{\Omega_{k3/2}/2}\,[b^{\dagger}_{k3/2}b^{\dagger}_{k3/2}]^{02}_{0\mu},\quad D_{\mu}=(D^{\dagger}_{\mu})^{\dagger},\\
&P_{l\mu}=\sqrt{\Omega_{k3/2}/2}\,[b^{\dagger}_{k3/2}\tilde{b}_{k3/2}]^{02}_{0\mu},\quad l=0,1,2,3\ ;
\end{aligned}
\tag{6.70a}
$$

(b) Sp(6) case,

$$
\begin{aligned}
&S^{\dagger}=\sum_i\sqrt{\Omega_{1i}/2}\,[b^{\dagger}_{1i}b^{\dagger}_{1i}]^{00}_{00},\quad S=(S^{\dagger})^{\dagger},\\
&D^{\dagger}_{\mu}=\sum_i\sqrt{\Omega_{1i}/2}\,[b^{\dagger}_{1i}b^{\dagger}_{1i}]^{20}_{\mu 0},\quad D_{\mu}=(D^{\dagger}_{\mu})^{\dagger},\\
&P_{l\mu}=\sum_i\sqrt{\Omega_{1i}/2}\,[b^{\dagger}_{1i}\tilde{b}_{1i}]^{20}_{\mu 0},\quad l=0,1,2\ ,
\end{aligned}
\tag{6.70b}
$$

while the generators of $\mathcal{SU}(2)$ are the pairing quasispin operators in single abnormal level $j$:

$$
\mathcal{S}^{\dagger}=\left[\frac{2j+1}{2}\right]^{1/2}[a^{\dagger}_{jm}a^{\dagger}_{j-m}]^{0}_{0},\quad \mathcal{S}=(\mathcal{S}^{\dagger})^{\dagger},
\tag{6.70c}
$$

where

$$
b^{\dagger}_{km_k im_i}=\sum_j\langle jm|km_k im_i\rangle a^{\dagger}_{jm}
\tag{6.71}
$$

and

$$
\Omega_1=\begin{cases}2(2k+1)\ \text{ for SO(8)}\ ,\\ \sum_i\frac{3}{2}(2i+1)\ \text{ for Sp(6)}\ .\end{cases}
\tag{6.72}
$$

According to the model, the Hamiltonian is now a function of these generators, constrained by the fundamental symmetries of the nucleus: rotational invariance and particle number conservation. For even-even nuclei, the effective interactions between nucleons are monopole

and quadrupole pairing and multipole ($l=0,1,2,3$) interactions. For illustrative purposes, we shall confine our discussion to particles in the normal-parity levels.

A possible Hamiltonian of the system is as follows:

$$H_{\rm coll}=H_G^{\pi}+H_G^{\nu}+H^{\nu\pi}\ , \tag{6.73}$$

$$H_G^{\sigma}=\begin{cases} G_0^{\sigma}C_{\rm SU(2)}^{\sigma}+b_2^{\sigma}C_{\rm SO(6)}^{\sigma} & \text{for } G=\text{SO(8)} \\ G_0^{\sigma}C_{\rm SU(2)}^{\sigma}+b_2^{\sigma}C_{\rm SU(3)}^{\sigma} & \text{for } G=\text{Sp(6)} \end{cases} \quad (\sigma=\pi,\nu) \tag{6.74a}$$

and

$$H^{\nu\pi}=b_2^{\nu\pi}P_2^{\nu}\cdot P_2^{\pi} \tag{6.74b}$$

where $G_0^{\sigma}$ and $b^{\sigma}$ are the strengths of pairing-pairing and quadrupole-quadrupole interactions between neutron-neutron or proton-proton, and $C_G$ represents the second-order Casimir operator of subgroup $G$ in the fermion dynamic symmetry model:

$$C_{\rm SU(2)}\equiv S^{\dagger}S\ , \tag{6.75a}$$

$$C_{\rm SO(6)}=\sum_{l=1}^{3}P^l\cdot P^l\ , \tag{6.75b}$$

$$C_{\rm SU(3)}=\sum_{l=1}^{2}P^l\cdot P^l\ . \tag{6.75c}$$

The term $H^{\nu\pi}$ represents the quadrupole-quadrupole neutron-proton interaction.

### b. Energy surface

The geometry of even-even nuclei can be realized naturally via the coherent-state theory [i.e., $\text{SO}^{\pi}(8)\otimes\text{SO}^{\nu}(8)$, $\text{SO}^{\pi}(8)\otimes\text{Sp}^{\nu}(6)$, or $\text{Sp}^{\pi}(6)\otimes\text{Sp}^{\nu}(6)$ coherent states as

$$|\eta\rangle=T^{\pi\nu}(\eta)|0\rangle\ , \tag{6.76a}$$

where

$$T^{\pi\nu}(\eta)=\exp\sum_{\sigma}\left[\eta_{00}^{\sigma}S^{\sigma\dagger}+\sum_{\mu}\eta_{2\mu}^{\sigma}D_{\mu}^{\sigma\dagger}-\text{H.c.}\right] =T(\eta^{\nu})T(\eta^{\pi}) \tag{6.76b}$$

is the coset representative of $G/H$, $G=\text{SO}^{\pi}(8)\otimes\text{SO}^{\nu}(8)$, $\text{SO}^{\pi}(8)\otimes\text{Sp}^{\nu}(6)$, or $\text{Sp}^{\pi}(6)\otimes\text{Sp}^{\nu}(6)$, and $H=\text{U}^{\pi}(4)\otimes\text{U}^{\nu}(4)$, $\text{U}^{\pi}(4)\otimes\text{U}^{\nu}(3)$, or $\text{U}^{\pi}(3)\otimes\text{U}^{\nu}(3)$ while $|0\rangle$ the doubly-closed shell state. Since the coherent states are the direct products of neutron and proton coherent states, we can discuss separately the SO(8) and Sp(6) geometries whose coherent states are isomorphic to SO(8)/U(4) and Sp(6)/U(3), respectively:

$$|\eta\rangle=T^{\sigma}(\eta)|0\rangle=\exp\left[\eta_{00}^{\sigma}S^{\sigma\dagger}+\sum_{\mu}\eta_{2\mu}^{\sigma}D_{\mu}^{\sigma\dagger}-\text{H.c.}\right]|0\rangle \quad (\sigma=\pi,\nu)\ . \tag{6.77}$$

Following the procedure given in Sec. VI.B.2, we find that the faithful matrix representation of $T(\eta)$ has the general form of Eqs. (6.35) and (6.36):

$$T(\eta)\rightarrow\begin{bmatrix}\sqrt{I-zz^{\dagger}} & z\\ -z^{\dagger} & \sqrt{I-z^{\dagger}z}\end{bmatrix} \tag{6.78}$$

with

$$z=\eta\frac{\sin\sqrt{\eta^{\dagger}\eta}}{\sqrt{\eta^{\dagger}\eta}} \equiv\begin{bmatrix} 0 & \sqrt{2}z_{22} & \sqrt{2}z_{21} & z_{20}+z_{00}\\ -\sqrt{2}z_{22} & 0 & z_{20}-z_{00} & \sqrt{2}z_{2-1}\\ -\sqrt{2}z_{21} & z_{00}-z_{20} & 0 & \sqrt{2}z_{2-2}\\ -z_{00}-z_{20} & -\sqrt{2}z_{2-1} & -\sqrt{2}z_{2-2} & 0\end{bmatrix} \tag{6.79a}$$

for the SO(8) case, and

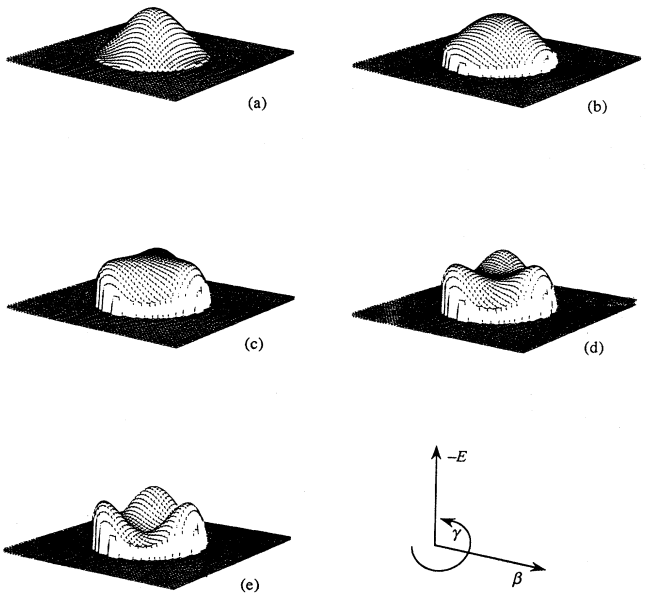



FIG. 11. A three-dimensional plot of the Sp(6) energy surface of the fermion dynamic symmetry model as functions of deformation parameters $\beta$ and $\gamma$ for several values of the control parameter $G_0/b_2$: (a) $G_0/b_2=\infty$ ($b_2=0.0$ MeV); (b) $G_0/b_2=2.0$ ($b_2=0.04$ MeV); (c) $G_0/b_2=1.0$ ($b_2=0.06$ MeV); (d) $G_0/b_2=0.5$ ($b_2=0.08$ MeV); (e) $G_0/b_2=0.0$ ($b_2=0.06$ MeV). The origin of the coordinate system is in the center of each plot. For all cases, $\Omega_1=21$ and $n_1=12$. The energy surfaces have the same behavior for fixed ratio $G_0/b_2<1$ while changing the particle number $n_1$ from 0 to $2\Omega_1/3$ (Zhang, Wu, *et al.*, 1988).

$$z \equiv \begin{bmatrix} \sqrt{3}z_{22} & \sqrt{3/2}z_{21} & z_{00}+\sqrt{1/2}z_{20} \\ \sqrt{3/2}z_{21} & \sqrt{2}z_{20}-z_{00} & \sqrt{3/2}z_{2-1} \\ z_{00}+\sqrt{1/2}z_{20} & \sqrt{3/2}z_{2-1} & \sqrt{3}z_{2-2} \end{bmatrix} \tag{6.79b}$$

for the Sp(6) case. Using these representations, we can compute the $Q$ representation, i.e., the expectation value of the Hamiltonian of Eq. (6.73) evaluated in the coherent states of Eq. (6.76). The result will give us the energy surface. Minimizing the energy surface is equivalent to solving the Hartree-Fock-Bogoliubov (HFB) mean-field dynamics. Generally speaking, this energy surface [SO(8) or Sp(6)] exists in a twelve-dimensional parameter space. However, since we are only interested in the ground-state intrinsic properties of nuclei, for which time-reversal and triaxial symmetries are good symmetries, this will reduce the twelve parameters in $T(\eta)$ to three. Furthermore, by requiring that the expectation of the number of nucleons within the normal levels have the desired value

$$\langle \eta^\sigma | n_1^\sigma | \eta^\sigma \rangle = n_1^\sigma \quad (\sigma=\pi,\nu) , \tag{6.80}$$

we ensure that the energy surface is merely a function of two parameters. These two parameters can be related to the usual geometric parameters $\{\beta,\gamma\}$ (Bohr and Mottelson, 1975) via the intrinsic quadrupole moment,

$$\mathcal{P}_{2\mu}^\sigma = \langle \eta^\sigma | P_{2\mu}^\sigma | \eta^\sigma \rangle \begin{cases} -2\Omega_1^\sigma \chi_{00}^\sigma \beta^\sigma \cos\gamma^\sigma , & \mu=0 \\ 0, & \mu=\pm1 \\ -\sqrt{2}\Omega_1^\sigma \chi_{00}^\sigma \beta^\sigma \sin\gamma^\sigma , & \mu=\pm2 \end{cases} \text{for SO(8)}$$
$$\begin{cases} -2\Omega_1^\sigma (\chi_{00}^\sigma \beta^\sigma \cos\gamma^\sigma - \frac{1}{2\sqrt{2}}\beta^{\sigma 2}\cos 2\gamma^\sigma) , & \mu=0 \\ 0 , & \mu=\pm1 \\ -\sqrt{2}\Omega_1^\sigma (\chi_{00}^\sigma + \frac{1}{\sqrt{2}}\beta^\sigma \cos\gamma^\sigma)\beta^\sigma \sin\gamma^\sigma , & \mu=\pm2 \end{cases} \text{for Sp(6)} \tag{6.81}$$

where

$$\chi_{00}^\sigma = \pm\sqrt{n_1^\sigma/2\Omega_1^\sigma - \beta^{\sigma 2}} . \tag{6.82}$$

The sign $-$ $(+)$ corresponds to the case of $n_1^\sigma < (>) \Omega_1^\sigma$, where $\Omega_1^\sigma$ is given by Eq. (6.72). The energy surface is now finally a function of the control parameter $\{n\}$ and the deformed parameters $\{\beta,\gamma\}$:

$$\mathcal{H}_G^\sigma(n_1^\sigma,\beta^\sigma,\gamma^\sigma) = \sum_s g_s \mathcal{H}_{Gs}^\sigma(n_1^\sigma,\beta^\sigma,\gamma^\sigma) , \tag{6.83a}$$

$$\mathcal{H}^{\nu\pi}(n_1^\nu,\beta^n,\gamma^\nu;c^\pi,n_1^\pi,\beta^\pi,\gamma^\pi)$$
$$= 4\Omega_1^\pi \Omega_1^\nu b_2^{\pi\nu} \beta^\pi \beta^\nu \begin{cases} \chi_{00}^\pi \chi_{00}^\nu \cos(\gamma^\nu - \gamma^\pi) & \text{for } SO^\pi(8)\otimes SO^\nu(8) , \\ \chi_{00}^\pi [\chi_{00}^\nu \cos(\gamma^\nu-\gamma^\pi) - \frac{1}{2\sqrt{2}}\beta^\nu \cos(2\gamma^\nu+\gamma^\pi)] & \text{for } SO^\pi(8)\otimes Sp^\nu(6) , \\ \chi_{00}^\pi \chi_{00}^\nu \cos(\gamma^\nu-\gamma^\pi) + \frac{1}{8}\beta^\pi\beta^\nu \cos(2\gamma^\nu-2\gamma^\pi) - \frac{1}{2\sqrt{2}}[\chi_{00}^\pi \beta^\nu \cos(2\gamma^\nu+\gamma^\pi) + \chi_{00}^\nu \beta^\pi \cos(\gamma^\nu+2\gamma^\pi)] & \\ & \text{for } Sp^\pi(6)\otimes Sp^\nu(6) . \end{cases} \tag{6.83b}$$

TABLE V. The stiffness coefficients of the energy surface for various subgroups of SO(8) and Sp(6).

| $G$ | $g_s$ | $A_\sigma$ | $B_\sigma$ | $C_\sigma$ | $D_\sigma$ | $\Delta_\sigma$ |
|---|---|---|---|---|---|---|
| SO(8) | SU(2) | $2\Omega_1^\sigma(\Omega_1^\sigma-2)$ | $-n_1^\sigma(\Omega_1^\sigma-2)$ | 0 | $\frac{n_1^\sigma}{4}\left[\Omega_1^\sigma - \frac{n_1^\sigma}{2} + \frac{n_1^\sigma}{\Omega_1^\sigma}\right]$ | $f_1^\sigma$ |
| | SO(6) | $-4\Omega_1^\sigma(\Omega_1^\sigma+3)$ | $2n_1^\sigma(\Omega_1^\sigma+3)$ | 0 | $5n_1^\sigma(2\Omega_1^\sigma-n_1^\sigma)/4\Omega_1^\sigma$ | 0 |
| Sp(6) | SU(2) | $\frac{7}{2}\Omega_1^\sigma\left[\frac{\Omega_1^\sigma}{3}-1\right]$ | $-2n_1^\sigma\left[\frac{\Omega_1^\sigma}{3}-1\right]$ | $2\sqrt{2}\Omega_1^\sigma\left[\frac{\Omega_1^\sigma}{3}-1\right]$ | $\frac{n_1^\sigma}{6}\left[\Omega_1^\sigma - \frac{n_1^\sigma}{2} + \frac{3n_1^\sigma}{2\Omega_1^\sigma}\right]$ | $f_2^\sigma$ |
| | SU(3) | $-\frac{7}{4}\Omega_1^\sigma(2\Omega_1^\sigma-1)$ | $n_1^\sigma(2\Omega_1^\sigma-1)$ | $-\sqrt{2}\Omega_1^\sigma(2\Omega_1^\sigma-1)$ | $5n_1^\sigma(2\Omega_1^\sigma-n_1^\sigma)/4\Omega_1^\sigma$ | 0 |

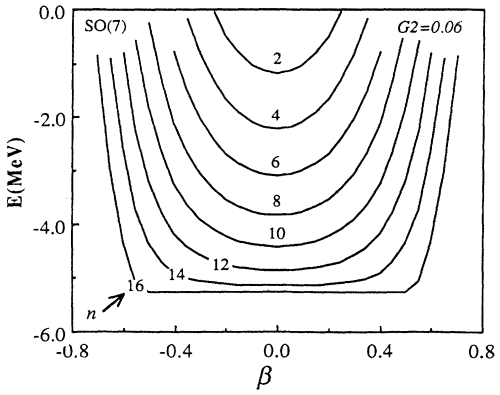



FIG. 12. The $\beta$-softness behavior in the SO(7) dynamical symmetry of the fermion dynamic symmetry model. The energy curves are $\gamma$ soft (Zhang *et al.*, 1987).

The coefficients $g_s$ in Eq. (6.83a) are the interaction strengths of Eqs. (6.74), the subscript $s$ denotes the possible subgroup, and

$$\mathcal{H}^{\sigma}_{Gs}(n^{\sigma}_1,\beta^{\sigma},\gamma^{\sigma})=A_{\sigma}\beta^{\sigma 4}+B_{\sigma}\beta^{\sigma 2} +C_{\sigma}\beta^{\sigma 3}\chi^{\sigma}_{00}\cos 3\gamma+D_{\sigma}+\Delta_{\sigma} \tag{6.84}$$

where the coefficients $A_\sigma$, $B_\sigma$, $C_\sigma$, $D_\sigma$, and $\Delta_\sigma$ are listed in Table V for each subgroup of SO(8) and Sp(6).

In Table V, the functions $f_1$ and $f_2$ are given as

$$f^{\sigma}_1=\tfrac{1}{2}(\Omega^{\sigma}_1)^2(n^{\sigma}_1/2\Omega^{\sigma}_1-2\beta^{\sigma}) \times\sqrt{(1-n^{\sigma}_1/2\Omega^{\sigma}_1)^2-4\beta^{\sigma 2}(n^{\sigma}_1/2\Omega^{\sigma}_1-\beta^{\sigma 2})}\ , \tag{6.85a}$$

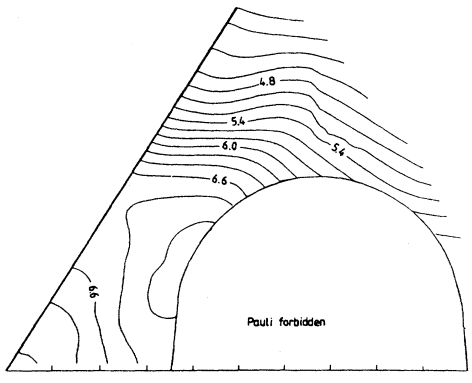



FIG. 13. Contour plot of the Sp(6) energy surface on the $\beta$-$\gamma$ plane ($\beta\geq 0$ and $0\geq\gamma\geq 60°$), with $G_0=0.04$ MeV and $b_2=0.06$ MeV; $\Omega_1=21$ and $n_1=20$. The contours range from $-7.00$ to $-3.00$ MeV. The contour interval of energy is 0.20 MeV. The empty region in the $\beta$-$\gamma$ plane is forbidden by the Pauli principle (Zhang, Feng, Wu, and Guidry, 1989).

$$f^{\sigma}_2=(\Omega^{\sigma}_1/3)^2\sum_{i<j}2s^{\sigma}_i s^{\sigma}_j\sqrt{(1-s^{\sigma 2}_i)(1-s^{\sigma 2}_j)}\ , \tag{6.85b}$$

and

$$s^{\sigma}_i=\chi^{\sigma}_{00}+\sqrt{2}\beta^{\sigma}\cos[\gamma^{\sigma}+(i-2)\pi/3],\quad i=1,2,3\ . \tag{6.85c}$$

An example of a three-dimensional plot of the energy surface in the fermion dynamic symmetry model is shown in Fig. 11. By minimizing the energy surface, we can get the exact HFB solution for the ground state.

In these investigations, several interesting properties of nuclear collective motions are revealed. One of them is the discovery of the existence of a $\beta$ soft mode (Zhang, Feng, and Ginocchio, 1987) (see Fig. 12); another is the effect of the Pauli principle on the geometric shape of nuclei (see Fig. 13) (Zhang, Feng, Wu, and Guidry, 1989). Finally, the existence of a narrow window of oblate shape in rare-earth nuclei is naturally explained (Wu *et al.*, 1989).

### 4. Time-dependent mean-field theory

The time-dependent mean-field theory has been widely used to study the so-called large-amplitude collective motions in many-body systems, especially in nuclear physics (Kerman and Koonin, 1976). However, the framework of the mean-field theory is essentially classical. Hence it does not extract quantum-mechanical information, such as the energy spectra of bound states (Kuratsuji and Suzuki, 1983). Furthermore, the time-dependent mean-field theory is a perturbation theory. There is no obvious systematic expansion beyond the mean-field level. However, it has been noted by Blaizot and Orland (1981) that many of the difficulties of this theory can be overcome by the path-integral formalism (Negele and Orland, 1987).

As we have already discussed in Sec. IV, for a quantum system with dynamic group $G$, the coherent-state theory provides a natural framework in which to use the path-integral formalism. Furthermore, we see that under the stationary phase approximation, the time-dependent mean-field theory provides a natural semiclassical limit of the path-integral formalism. Therefore, using fermion coherent states [of the U($r$) or SO($2r$) group], one can immediately derive the time-dependent Hartree-Fock equation (Blaizot and Orland, 1981; Kuratsuji and Suzuki, 1983) or the time-dependent Hartree-Fock-Bogoliubov equations. We shall briefly discuss the construction of these equations here.

When we substitute the coherent states of U($r$) [or SO($2r$)] from Eq. (6.11) [or Eq. (6.34)] into the action functional of Eq. (4.20), then the variation of the action will give rise to the time-dependent Hartree-Fock (or Hartree-Fock-Bogoliubov) equations. Since the coherent states of Eq. (6.11) [or Eq. (6.34)] have a natural symplectic structure given by Eq. (6.19) [or (6.44)], the time-dependent Hartree-Fock (or HFB) are identical to the

Hamiltonians

$$\frac{dq_{ij}}{dt}=\frac{\partial H_c(p,q)}{\partial p_{ij}}\ ,$$
$$\frac{dp_{ij}}{dt}=-\frac{\partial H_c(p,q)}{\partial q_{ij}}\ , \tag{6.86}$$

where the local canonical coordinates $(q,p)$ are

$$\frac{1}{\sqrt{2}}(q_{ij}+ip_{ij})=z_{ij}\ . \tag{6.87}$$

The matrix $z$ is given by Eq. (6.13) [or (6.36)] for U($r$) [or SO(2$r$)]. The Hamiltonian $H_c(q,p)$ is the expectation value of the many-fermion Hamiltonian operator $H$ of Eq. (6.57), evaluated in the coherent states of Eq. (6.11) [or Eq. (6.34)]. In principle, solving Eq. (6.86) one may obtain the bound-state spectra as well as quantum fluctuations (Blaizot and Orland, 1981; Kuratsuji and Suzuki, 1983). Of course, Eq. (6.86) is generally nonlinear in nature. The nonlinearity arises because of the mean-field approximation. The study of the behavior of such nonlinearities for many-fermion systems has recently attracted much attention. A favorite model for the study of this problem is the three-level extension of the two-level Lipkin model, which will be discussed in the next section (Gilmore and Feng, 1979; Williams and Koonin, 1982; Meredith *et al.*, 1988; Zhang, Feng, and Yuan, 1989).

### 5. Application to the three-level Lipkin model

A three-level Lipkin model may be the "next best thing" to the full-blown many-body problem. Its Hamiltonian is given (see Li *et al.*, 1971; Meredith *et al.*, 1988; Zhang, Feng, and Yuan, 1989) by

$$H=\sum_{i=0}^{2}\varepsilon_i E_{ii}+\tfrac{1}{2}\sum_{i\neq j}V(E_{ij})^2\ . \tag{6.88}$$

For each of the three levels, there is an imposed $\mathcal{N}$-fold degeneracy. Such an imposition is strictly for mathematical convenience. The operator $E_{ij}$ is defined as

$$E_{ij}=\sum_{k=1}^{\mathcal{N}}a_{ik}^{\dagger}a_{jk},\quad i,j=0,1,2\ . \tag{6.89}$$

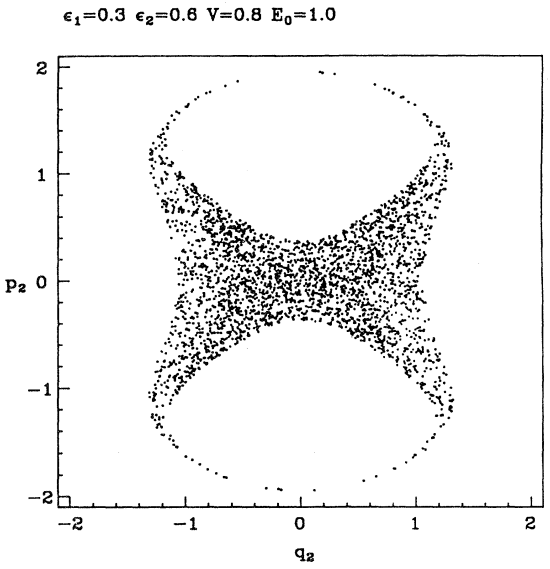



FIG. 14. The Poincaré surfaces of section for a three-level nonintegrable system in the classical dynamics ($N\to\infty$) with the parameters $\varepsilon_1=0.3$, $\varepsilon_2=0.6$, $E_0=1.0$, and $V=0.8$. The calculation is performed under the scaling of the phase space of two units.

The operator $a_{ik}^{\dagger}(a_{jk})$ is the single-fermion creation (annihilation) operator in the model.

For simplicity, we shall first consider the situation in which the total particle number $N$ is $\mathcal{N}$. For the more general cases, according to the given procedure, we find that the mean-field dynamics phase space is SU(3)/U(2). Such a space is determined by the SU(3) and its irrep in the extremal state,

$$|0\rangle=\prod_{k=1}^{N}a_{0k}^{\dagger}|\underline{0}\rangle\ , \tag{6.90}$$

where $|\underline{0}\rangle$ denotes the vacuum of the unperturbed Hamiltonian. By choosing the canonical variables

$$\frac{1}{\sqrt{2N}}(q_i+ip_i)=\theta_i\frac{\sin\theta}{\theta}e^{-i\phi_i}\ , \tag{6.91}$$

where $\theta$ is given in Eq. (4.49), we have

$$\mathcal{H}(q,p)=\frac{e_1}{2}(p_1^2+q_1^2)+\frac{e_2}{2}(p_2^2+q_2^2)+\frac{V}{4}\left[1-\frac{1}{N}\right][(p_1^2+p_2^2)^2-(q_1^2+q_2^2)^2+(q_1^2-p_1^2)(q_2^2-p_2^2)$$
$$+4q_1q_2p_1p_2+2N(q_1^2+q_2^2-p_1^2-p_2^2)]\ . \tag{6.92}$$

By taking the limiting case of $N$ $(=\chi)\to\infty$, we can reduce Eq. (6.92) to the "classical limit," as was discussed in Sec. IV.C.

This mean-field dynamical equation is in general nonlinear and nonintegrable, and its numerical solutions exhibit chaotic behavior, as is shown in Fig. 14. From the study of this schematic many-body system, we see that mean-field dynamics could provide a route towards the exploration of chaotic motions in realistic systems.

Furthermore, when $2\mathcal{N}>N>\mathcal{N}$ the extremal state, which is a key to determining the quantum phase space, is not Eq. (6.90) but

$$|0\rangle = \prod_{k=1}^{N-\mathcal{N}} \prod_{l=1}^{\mathcal{N}} a_{1k}^{\dagger} a_{0l}^{\dagger} |\underline{0}\rangle \ . \tag{6.93}$$

Correspondingly, the elementary excitation operators are $(E_{ij}, i>j)$, and the quantum phase space for such a case is given by the coset space SU(3)/U(1)⊗U(1), which is a six-dimensional manifold. This conclusion can also be obtained directly from the irrep space of U(3), which we have discussed in Sec. III.C.1. For the case $2\mathcal{N}>N>\mathcal{N}$, the irrep space is the nondegenerate irrep space $(\mathcal{N}, N-\mathcal{N}, 0)$. However, in the case of $N\leq\mathcal{N}$, the irrep space corresponds to the degenerate irrep of U(3): $(N,0,0)$, whose geometric spaces are different. Physically, this difference is a manifestation of the Pauli principle in the geometry. When $3\mathcal{N}\geq N\geq 2\mathcal{N}$ the structure of the quantum phase space is the same as for $N\leq\mathcal{N}$.

### D. A short comment on Grassmann variables

In field theory, fermion coherent states were also constructed by using the language of Grassmann variables (Berezin, 1966). These fermion coherent states greatly facilitated computations in the fermion path-integral formalism in quantum field theory (see, Negele and Orland, 1987). On the other hand, if one is interested in the geometrical properties of fermion coherent states, then the Grassmannian approach will offer less transparancy. In fact, the discussions presented in Sec. VI.C shows that fermion systems do possess ordinary but nontrivial geometrical spaces and can indeed be parametrized by $c$-numbers. All the properties of fermions can be realized or resolved from such continuous parameters. For example, the path integral of a spin system is based on the SU(2) coherent states, which has the kinematical action

$$S = j\hbar \int (1-\cos\theta) d\phi \ . \tag{6.93}$$

The integration is the exact form of the Dirac monopole potential and its gauge invariance can naturally be carried out for the various integer and half-integer spin values. However, the reader is cautioned that when using $c$-number fermion coherent states to describe fermion dynamics, care must be exercised to choose the local coordinate system properly. Otherwise, some physical properties of fermions may be lost. For the spin system, for example, the commonly used local canonical variables for two spheres $S^2$,

$$q=\phi, p=\cos\theta \ , \tag{6.94}$$

is in fact an inappropriate choice, since such a local coordinate system cannot describe the half-integer spin case (Nielson and Rohrlich, 1988). In any case, it appears that introduction of Grassmann variables may not be an absolute necessity for the meaningful portrayal of the geometry of a fermion system.

## VII. VECTOR COHERENT STATES

As described in Sec. III, the ingredients for constructing coherent states include a group $G$, a Hilbert space $V^{\Lambda}$ on which $G$ acts through a unitary irreducible representation $\Gamma^{\lambda}(G)$, and an extremal state $|\Lambda,\text{ext}\rangle$ in $V^{\Lambda}$. These structures are sufficient to define uniquely the subgroup $H$, required for the explicit construction of normalized states on the coset space $G/H$.

The question can be raised: what happens if one attempts to construct coherent states using a subgroup $K$ (a subgroup of $G$) that is different from the natural subgroup $H$ determined by $G$, $V^{\Lambda}$, and $|\Lambda,\text{ext}\rangle$? Two particular cases will be discussed, depending on the relation between $K$ and $H$. The first case, $H\supset K$, is essentially trivial. The second, $K\supset H$, is nontrivial. This case gives rise to mathematical structures called vector coherent states, proposed some time ago (Castanos, Chacon, and Moshinsky, 1984; Deenen and Quesne, 1984; Rowe, 1984, 1985; Rowe, Rosensteel, and Gilmore, 1985; Quesne, 1986). Both cases will be illustrated in the context of $G=$SU(3) and its degenerate and nondegenerate representations. Many detailed discussions of the matrix elements of various algebras using vector-coherent-state techniques are summarized in a book recently published by Hecht (1987). We shall not discuss the material contained in Hecht's book. Rather, we shall emphasize the relationship of the vector coherent states viewed from the context of the algorithm proposed in this review. A third possibility, $K\not\supseteq H$ and $H\not\supseteq K$, will not be discussed.

### A. The case $G\supset H\supset K$

In this case, both $G$ and $H$ have coset decompositions, and we can write

$$g=\Omega h, \quad \Omega\in G/H, \quad h\in H \ , \tag{7.1a}$$

$$h=\omega k, \quad \omega\in H/K, \quad k\in K \ . \tag{7.1b}$$

The prescription for constructing coherent states then becomes

$$g|^{\Lambda}_{\text{ext}}\rangle = \Omega\omega k|^{\Lambda}_{\text{ext}}\rangle = \Omega\omega|^{\Lambda}_{\text{ext}}\rangle e^{i\phi(k)} \ , \tag{7.2a}$$

$$\Omega\omega|^{\Lambda}_{\text{ext}}\rangle = \Omega|^{\Lambda}_{\text{ext}}\rangle e^{i\phi(\omega)} \ . \tag{7.2b}$$

In short, the coherent states defined on $G/K$ are simply those defined on $G/H$ up to a phase factor. The phase factor is a function of $\omega\in H/K$. That is, the coset space $G/K=\Omega\omega$ consists of a subspace $(H/K)$ with "ineffective" variables and a subspace of variables $(G/H)$ whose action is transitive. Furthermore, the group multiplication property $h=\omega k$ and representation property $h|^{\Lambda}_{\text{ext}}\rangle = |^{\Lambda}_{\text{ext}}\rangle e^{i\phi(h)}$ $(h\in H)$ ensure the composition property $e^{i\phi(\omega)}e^{i\phi(k)}=e^{i\phi(h=\omega k)}$.

In addition to the sequential coset representation of group operations $g=\Omega\omega k$ there is a sequential decomposition of measure in $G$ into a product of measures of the

quotients $G/H$, $H/K$, and $K$,

$$d\mu(g)=d\mu(\Omega)d\mu(h)=d\mu(\Omega)d\mu(\omega)d\mu(k) . \quad (7.3)$$

The properties of coherent states that depend on measure, and in particular depend on the group properties of $G$ when the subgroup $H$ is integrated out (e.g., resolution of the identity), hold for the coherent states associated with $G/K$. For example, the resolution of the identity in any space, $V^{\Lambda}$, is a group-theoretical property,

$$I=\frac{\dim(\Lambda)}{\mathrm{Vol}(G)}\int g|^{\Lambda}_{\mathrm{ext}}\rangle\langle^{\Lambda}_{\mathrm{ext}}|g^{-1}d\mu(g) . \quad (7.4)$$

Under the decomposition $g=\Omega h$ and after integration over $d\mu(h)$, this reduces to the completeness relation for coherent states [Eq. (3.47)]. Under the decomposition $g=\Omega\omega k$ and after integration over $d\mu(k)$ this reduces to a similar expression, but for a subgroup $K$,

$$I=\frac{\dim(\Lambda)}{\mathrm{Vol}(G/K)}\int \Omega\omega|^{\Lambda}_{\mathrm{ext}}\rangle\langle^{\Lambda}_{\mathrm{ext}}|(\Omega\omega)^{-1}d\mu(\Omega)d\mu(\omega) . \quad (7.5)$$

The standard expression (3.47) is obtained from Eq. (7.4) by integrating out the $K$ dependence using (7.3).

Example: As we have mentioned in Sec. III, the group SU(3) has two classes of representations: the degenerate representations $(\lambda_1,0)$ and $(\lambda_1,\lambda_1)$, and generic or nondegenerate representation $(\lambda_1,\lambda_2)$, $\lambda_1>\lambda_2>0$. The degenerate representations have a maximal-stability subgroup $H=\mathrm{U}(2)$, while the nondegenerate representations have $H=\mathrm{U}(1)\otimes\mathrm{U}(1)$. For the degenerate representations we may choose $K=\mathrm{U}(1)\otimes\mathrm{U}(1)\subset H=\mathrm{U}(2)$. The coherent states for (2,0) of SU(3) exist in one-to-one correspondence with the four-dimensional space SU(3)/U(2). However, the coherent states for (2,0) for the decomposition $K=\mathrm{U}(1)\otimes\mathrm{U}(1)$ are defined on the six-dimensional quotient space $\mathrm{SU}(3)/\mathrm{U}(1)\otimes\mathrm{U}(1)$. In this space two of the dimensions are essentially trivial, or dummy, directions, in that elements $[\omega\in\mathrm{U}(2)/\mathrm{U}(1)\otimes\mathrm{U}(1)]$ produce only a phase change when applied to $|^{2\ 0}_{\mathrm{ext}}\rangle$. We should like to make two remarks concerning this example.

First, the Lie groups of rank greater than 1 have many classes of (non)degenerate representations. For the unitary groups $A_n\sim\mathrm{SU}(n+1)$ the unitary irreducible representations $(\lambda_1,\lambda_2,\ldots,\lambda_n)$ obey $(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n\geq 0)$. All representations with strict inequality are nondegenerate. One or more equalities among the $\lambda_i$ produces a degenerate class of representations. Each degenerate class of representations has a uniquely defined maximal-stability subgroup. The number of subgroups $H$ for $\mathrm{SU}(n+1)$ is equal to the number of partitions of $n$ (number of representations of the permutation group $P_n$ and number of its classes). More than one inequivalent class of degenerate representations may have the same subgroup $H$. For example, the degenerate classes $(\lambda_1=\lambda_2>\lambda_3)$ and $(\lambda_1>\lambda_2=\lambda_3)$ of SU(4) have distinct subgroups $\mathrm{U}(2)\otimes\mathrm{U}(1)$.

Second, for some purposes it may be useful to construct coherent states for all classes of representations of $G$ on the same quotient space. This must be done by choosing the "smallest" maximal subgroup $H$, that is, the intersection of the stability subgroup for all classes of representations. This is exactly the Cartan subgroup $e^{i(H\cdot\varphi)}$, which has dimension $l$ for a rank $l$ simple group: $H=\mathrm{U}(1)\otimes\mathrm{U}(1)\cdots\otimes\mathrm{U}(1)$. If this is chosen as $K$, then $H\neq K$ for any class of representation that is partially degenerate. The quotient space $G/K$ then has two complementary directions. These are the subspace $H/K$ of "ineffective" directions producing a phase change on $|^{\Lambda}_{\mathrm{ext}}\rangle$, and the space $G/H$ which acts transitively on $|^{\Lambda}_{\mathrm{ext}}\rangle$ and whose points parametrize the standard coherent states $|^{\Lambda}_{\mathrm{ext}}\rangle$.

### B. The case $G\supset K\supset H$

Coset decompositions of the type described in Eq. (7.1) also hold in the case

$$g=\Theta k,\quad \Theta\in G/K,\quad k\in K , \quad (7.6a)$$

$$k=\theta h,\quad \theta\in K/H,\quad h\in H . \quad (7.6b)$$

The prescription for constructing coherent states then becomes

$$g|^{\Lambda}_{\mathrm{ext}}\rangle=\Theta\theta h|^{\Lambda}_{\mathrm{ext}}\rangle=\Theta\theta|^{\Lambda}_{\mathrm{ext}}\rangle e^{i\phi(h)} . \quad (7.7)$$

At this stage there is a new development which takes us out of the framework of the usual coherent states. Since $H$ is the maximal-stability subgroup of $|^{\Lambda}_{\mathrm{ext}}\rangle$ and $K\supset H$, the group operation $\theta$ maps the extremal state $|^{\Lambda}_{\mathrm{ext}}\rangle$ onto a state that differs from $|^{\Lambda}_{\mathrm{ext}}\rangle$ by more than a phase factor: $|\langle^{\Lambda}_{\mathrm{ext}}|\theta|^{\Lambda}_{\mathrm{ext}}\rangle|<1$.

More generally, the set of states $\theta|^{\Lambda}_{\mathrm{ext}}\rangle$, $\theta\in K/H$ forms a closed subspace in $V^{\Lambda}$ if $K$ is a closed subgroup of $G$. The most useful cases occur when (a) the subspace $k|^{\Lambda}_{\mathrm{ext}}\rangle$ is a proper subspace of $V^{\Lambda}$ and (b) the state $|^{\Lambda}_{\mathrm{ext}}\rangle$ is an extremal state of some irreducible representation of $K$.

In this typical case, we have

$$k|(^{\Lambda\ \lambda}_{\mathrm{ext}})\rangle=|(^{\Lambda\ \lambda}_{m})\rangle D^{\lambda}_{m\ \mathrm{ext}}(k),\quad k\in K , \quad (7.8)$$

where $m$ is a discrete-state label that indexes states in the subspace $V^{\lambda}\subset V^{\Lambda}$, which is invariant under $K$. With this decomposition, the canonical coherent-state construction assumes the form

$$\begin{aligned} g|^{\Lambda}_{\mathrm{ext}}\rangle=\Theta\theta h|^{\Lambda}_{\mathrm{ext}}\rangle&=\Theta\theta|(^{\Lambda\ \lambda}_{\mathrm{ext}})\rangle e^{i\phi(h)}\\ &=\Theta|(^{\Lambda\ \lambda}_{m})\rangle D^{\lambda}_{m\ \mathrm{ext}}(\theta)e^{i\phi(h)}\\ &=|(^{\Lambda\ \ \lambda}_{\Theta\ \ m})\rangle D^{\lambda}_{m\ \mathrm{ext}}(\theta)e^{i\phi(h)} . \end{aligned} \quad (7.9)$$

As for the usual coherent states, since $H$ is the maximal-stability subgroup of $G$, $e^{i\phi(h)}$ is a phase factor, or a one-dimensional representation of the subgroup $H$.

The vector coherent states are the states

$$|(^{\Lambda\ \ \lambda}_{\Theta\ \ m})\rangle . \quad (7.10)$$

They are *vectors* because their components are indexed by a discrete-state label $m$. This identifies a basis vector on the Hilbert space $V^\lambda$, which carries the unitary representation $D^\lambda(K)$. They are *coherent* because they are indexed by a continuous index $\Theta$, which labels a point in the coset space $G/K$.

The larger $K$ is, the larger the dimension of the weight label $m$ and the smaller the dimension of the coset space $\Theta=G/K$. The tradeoff in dimension is given by the relation

$$\dim(G/K)+2\dim(m)=\dim(G/H)\ . \tag{7.11}$$

The two familiar limits in the tradeoff are the following:

(i) $K=H$, which recovers the usual coherent states that have no discrete index. Thus

$$\dim(m)=0 \quad \text{and} \quad \dim(G/K)=\dim(G/H)\ . \tag{7.12}$$

(ii) $K=G$, which recovers the standard unitary irreducible representation $D^\lambda(K)=D^\Lambda(G)$. In this case there is no continuous index, thus $\dim(m)=\frac{1}{2}\dim(G/H)$, which reduces to the well-known condition that the number of discrete-state labels required to identify states in a carrier space for representations of $G$ is equal to half the dimensionality of the coset $G/H$, where $H$ is the usual subgroup. This is the Peter-Weyl completeness theorem.

It can be seen from Eq. (7.11) that the vector coherent states conveniently interpolate between the generalized coherent states of $G$ and the representations of $G$ in a discrete basis.

The sequential coset and invariant measure decompositions (7.1) and (7.3) for $G\supset H\supset K$ also hold for the case $G\supset K\supset H$. This allows us to convert all the standard integral expressions for generalized coherent states to corresponding expressions for vector coherent states. For example, the resolution of the identity is given by

$$\begin{aligned} I&=\frac{\dim(\Lambda)}{\mathrm{Vol}(G/H)}\int g|^\Lambda_{\rm ext}\rangle\langle^\Lambda_{\rm ext}|g^{-1}d\mu(g)\\ &=\frac{\dim(\Lambda)}{\mathrm{Vol}(G/H)}\int \Theta k|^\Lambda_{\rm ext}\rangle\langle^\Lambda_{\rm ext}|(\Theta k)^{-1}d\mu(\Theta k)\\ &=\frac{\dim(\Lambda)}{\mathrm{Vol}(G/H)}\int \Theta|^\Lambda_m\,{}^\lambda_{\rm ext}\rangle D^\lambda_{{\rm ext}\,m}(k)D^{\lambda *}_{{\rm ext}\,m'}(k)\\ &\quad\times\langle^\Lambda_{m'}\,{}^\lambda_{\rm ext}|\Theta^{-1}d\mu(\Theta)d\mu(k)\ . \end{aligned} \tag{7.13}$$

The integral over the subgroup $K$ is evaluated using the orthogonality relations, leading to

$$I=\frac{\dim(\Lambda)/\dim(\lambda)}{\mathrm{Vol}(G/K)}\int\sum_m |^\Lambda_\Theta\,{}^\lambda_m\rangle\langle^\Lambda_\Theta\,{}^\lambda_m|d\mu(\Theta)\ . \tag{7.14}$$

This is, of course, what one would expect for the resolution of the identity in terms of vector coherent states.

The vector coherent states are also nonorthogonal, except in the degenerate case $G=K$, when continuous indices are not present in their definitions, and the vector coherent states (7.10) reduce to the orthogonal basis in $V^\Lambda$. Computation of the overlaps of the vector coherent state is facilitated by the relation

$$|^\Lambda_\Theta\,{}^\lambda_m\rangle=\frac{\dim(\lambda)}{\mathrm{Vol}(K)}\int\Theta k|^\Lambda_{\rm ext}\,{}^\lambda_{\rm ext}\rangle D^\lambda_{{\rm ext}\,m}(k^{-1})d\mu(k)\ . \tag{7.15}$$

This result, together with the general overlap expression

$$\langle^\Lambda_{\rm ext}\,{}^\lambda_{\rm ext}|g^{-1}g'|^\Lambda_{\rm ext}\,{}^\lambda_{\rm ext}\rangle=D^\lambda_{{\rm ext}\,{\rm ext}}(g^{-1}g')\ , \tag{7.16}$$

provides a simple expression for the derived overlap

$$\begin{aligned} \langle^\Lambda_\Theta\,{}^\lambda_m|^\Lambda_\Theta\,{}^\lambda_{m'}\rangle&=\left[\frac{\dim(\lambda)}{\mathrm{Vol}(K)}\right]^2\\ &\quad\times\int\int D^\lambda_{m\,{\rm ext}}(k)D^\lambda_{{\rm ext}\,{\rm ext}}(k^{-1}\Theta^{-1}\Theta'k')\\ &\quad\times D^\lambda_{{\rm ext}\,m'}(k'^{-1})d\mu(k^{-1})d\mu(k')\ . \end{aligned} \tag{7.17}$$

For $K=H$, $\dim(\lambda)=1$ and $D^{(\lambda)}_{m\,{\rm ext}}(\Theta'k')=D^\lambda_{m\,{\rm ext}}(\Theta')e^{i\phi(k')}$, so this reduces to the standard expression [Eq. (3.43)] for the overlap of generalized coherent states. For $K=G$, $\Theta=$identity, and the result reduces to $D^\lambda_{m\,m'}(g^{-1}g')$.

Example: We consider again the case $G=\mathrm{SU}(3)$, but this time consider the nondegenerate representations $(\lambda_1,\lambda_2)$ with $\lambda_1>\lambda_2>0$. Then $H=\mathrm{U}(1)\otimes\mathrm{U}(1)$, and we take $K=\mathrm{U}(2)$. This situation is shown in Fig. 15. The extremal state of the representation (3,1) has been identified in Fig. 15(a). This is also an extremal state of the representation $D^{1/2}[\mathrm{U}(2)]$. We therefore have vector

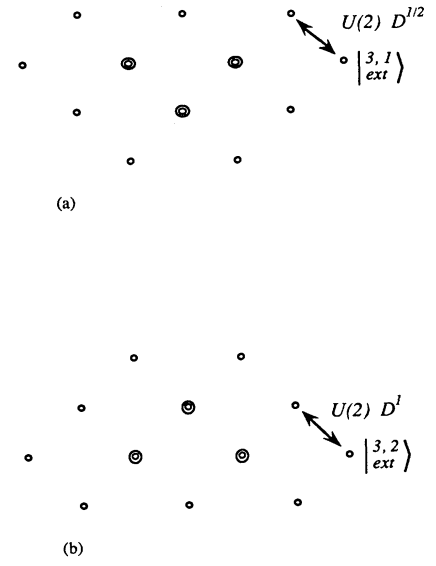



FIG. 15. Root space diagram of SU(3) representations: (a) The representation of (3,1) of SU(3) with the extremal state identified, showing the action of the subgroup $K=\mathrm{U}(2)$ on the highest state, leading to the $D^{1/2}$ representation of SU(2); (b) the representation (3,2) of SU(3), showing the action of the same u(2) subgroup as in (a). The representation of U(2) is now $D^1$ of dimension 3.

coherent states of SU(3) with respect to the subgroup U(2) based on the spin-$\frac{1}{2}$ representation. The vector coherent states are

$$\Theta\theta|^{(31)\ 1/2}_{(31)\ 1/2}\rangle=\sum_{m=1/2}^{-1/2}|^{(31)\ 1/2}_{\Theta\ \ m}\rangle D^{1/2}_{m\ 1/2}[\theta]\ , \tag{7.18}$$

where

$$\theta\in \mathrm{U}(2)/\mathrm{U}(1)\otimes\mathrm{U}(1)\ . \tag{7.19}$$

For the representation (3,2), which is conjugate to (3,1), the highest-weight state is also the highest-weight state $m=1$ of the $j=1$ representation of U(2). The vector coherent states are

$$\Theta\theta|^{(32)\ 1}_{(32)\ 1}\rangle=|^{(32)\ 1}_{\Theta\ \ m}\rangle D^{1}_{m\ 1}[\theta]\ . \tag{7.20}$$

In both cases the continuous index $\Theta$ is defined by a point in the coset space SU(3)/U(2), and the discrete index labels basis vectors in the irreducible representations of the subgroup $K=\mathrm{U}(2)$. In the vector coherent state (7.20), two of the six continuous indices in the usual coset space SU(3)/U(1)⊗U(1) have been combined into a single discrete index ($m$). This index, together with the four continuous indices, uniquely describes vector coherent states in the smaller coset space SU(3)/U(2).

Concerning this example, we note that vector coherent states have been particularly useful when the subgroup $K$ is maximal in the sense of the reductive decomposition of Eq. (3.20). Two cases that have been studied in detail are $\mathrm{Sp}(2n,R)\supset\mathrm{U}(n)$ and $\mathrm{SO}(2n)\supset\mathrm{U}(n)$. In the first case $G$ is noncompact, whereas in the second $G$ is compact. Vector coherent states have been useful for computing matrix elements of shift operators in $\mathrm{Sp}(2n,R)$ and $\mathrm{SO}(2n)$, which connect distinct representations of the subgroup $\mathrm{U}(n)$. These matrix elements are analytically available when the representation $\Gamma^{\Lambda}$ [of $\mathrm{Sp}(2n,R)$ or $\mathrm{SO}(2n)$] is simply reducible under restriction to $\mathrm{U}(n)$. This occurs, for example, in degenerate tower representations of $\mathrm{Sp}(2n,R)$ and in some of the spinor representations of $\mathrm{SO}(2n)$. These are precisely the representations required in describing the relation between Lie theory and parastatistics commutation relations.

## VIII. CONCLUSIONS AND DISCUSSION

We have presented a general algorithm for constructing the coherent states for an arbitrary quantum system. This algorithm depends on the system's algebraic structure (dynamic group) and its Hilbert space (one of the algebra's unitary irreps). The algebra comes from the system's dynamic properties, i.e., the Hamiltonian and all possible transition operators. Thus the coherent states constructed via this path provide a fundamental framework for a quantum system.

However, for a given dynamic group and a particular associated irrep, one can in principle construct many inequivalent sets of coherent states (Klauder, 1963; Perelomov, 1972), depending on the choice of reference state. When one selects the extremal state (i.e., the lowest- or highest-weight state of the irrep) for *compact* Lie groups as the reference state (Gilmore, 1974a), the resultant set of coherent states has the most useful geometry. This choice of the extremal state is also applicable to any *noncompact* Lie group with a *discrete* irrep. In this case the extremal state is the ground state. Since any realistic quantum system with a noncompact dynamic group usually includes a subset of bound states in its Hilbert space, the Hilbert space must carry a discrete irrep. From this viewpoint, the algorithm presented in this review is generic for providing a geometry for a given quantum system.

The geometry of the coherent states is a *coset space*. The explicit structure of this geometric space is based upon the topological structure of the dynamic group. One of its very useful and important features is the existence of a *symplectic structure*. The symplectic structure is a consequence of the choice of extremal state in the Hilbert space as a reference state $|\mathrm{ref}\rangle$, which is built into Gilmore's definition of a coherent state. The existence theorem of symplectic structure can be traced back to the pioneering works of Cartan (1935), Bergmann (1947), and Hua (1958). An explicit proof of such symplectic structures for coherent states was made by Onofri (1975). Similar discussions, although not cast explicitly in the language of coherent states, can also be found in Berezin (1975b). When the coset space is symmetric (Helgason, 1978), a general explicit canonical form of the symplectic structure can be constructed in terms of a suitable local coordinate system (Zhang, Feng, and Yuan, 1990). However, when the coset space is not symmetric, this general explicit canonical form is lacking. Indeed, the symplectic structure of the coherent states provides a phase space (Arnold, 1978) of an arbitrary quantum system. The physical significance of this phase space could be the correspondence principle, expounded by Dirac as far back as 1925, linking quantum and classical mechanics (Dirac, 1925). As a result, constructing phase space from the quantum system could be regarded as an inverse procedure of geometric quantization (Kostant, 1970; Kirillov, 1976). Furthermore, for certain coset spaces (i.e., for symmetric ones), the geometry of the coherent states also has a complex structure. In such cases, the coset space is a Kaehler manifold (Helgason, 1978) from which Berezin's quantization (Berezin, 1974; 1975a; Perelomov, 1986) can be realized.

By using the geometric structure and the overcompleteness of the coherent states, this review provides a representation of the dynamic group on the coset space (see Sec. III). There are two parts in the construction of this representation: (i) The irrep space (i.e., the system's physical state space) is a Hilbert space of entire functions with the coset space as its domain. Pioneering works on this subject were by Cartan (1935) and Bergmann (1947). Application of coherent state theory was discussed by Bargmann (1961), Klauder (1963), and Perelomov (1986). (ii) The operators and density function were expressed in

terms of three phase-space distributions, i.e., the $P$, $Q$, and Wigner distributions (Wigner, 1932; Glauber, 1963c; Gilmore, 1976). The first two distributions are also known as contravariant and covariant symbols (Berezin, 1972), or upper and lower symbols (Lieb, 1973; Simon, 1980), respectively. All these representations are in integral form. Conversion of these integral representations into differential representations gives rise to the $D$ algebra (Glauber, 1963c; Narducci *et al.*, 1974; Gilmore *et al.*, 1975), a useful tool for practical calculations.

The above-discussed geometric and algebraic properties of the coherent states have played a vital role in providing an interpretation of many phenomena. In fact, as we have seen throughout this review, quantum-mechanical and quantum-statistical mechanics can be recast in terms of the coherent states, with two important consequences. First of all, it provides naturally a general form of the two commonly utilized quantum-mechanical methods: the variational principle and the path-integral formalism. The variational principle gives the various static mean-field dynamics, such as the Hartree-Fock and the Hartree-Fock-Bogoliubov theories (Zhang, Feng, Wu, Wu, and Ginocchio, 1989), boson expansion theory (Dobaczewski, 1982; Rowe, 1984; Moshinsky, 1985; Klein and Marshalek, 1990), and the Hill-Wheeler generating coordinate method (Hill and Wheeler, 1953; see also Klauder, 1963). This is useful to discuss the critical and collective phenomena. The path integral, on the other hand, naturally derives the time-dependent mean-field dynamics, which is in fact a classical limit of the path integral via the stationary phase approximation (Klauder, 1979; Blaizot and Orland, 1981; Kuratsuji and Suzuki, 1983; Negele and Orland, 1987). It is useful to discuss the detailed dynamical processes. Second, the coherent-state formalism of quantum statistics has been most useful in the study of thermodynamic phenomena. In particular, the thermodynamic inequalities derived by Lieb (1973) for spin systems and their extension to arbitrary compact Lie groups by Gilmore (1979) and Simon (1980) provide an elegant description of thermodynamic phase transitions, structural phase transitions and classical limits (Gilmore and Feng, 1978a; 1978b; Yaffe, 1982).

For illustrations, we constructed in detail the coherent states of the two main classes of quantum systems, bosons and fermions. Examples considered in this review for the boson case are the single and multiple two-photon algebras. The coherent states constructed are known in the literature as "squeezed states" (Yuen, 1976; Hollenhorst, 1979; Walls, 1983). Applications of these states to quantum optics and molecular dynamics have been demonstrated (Gilmore and Yuan, 1987, 1989). However, the results presented here can also be extended to other bosonic systems. For the fermion case, the coherent states come from the fermion u($r$) and so($2r$) algebras (Gilmore and Feng, 1983; Suzuki, 1983). In fact, for a single fermion, the coherent states are indeed the SU(2) coherent states with spin $\frac{1}{2}$ (Klauder, 1960). The detailed structure of the phase space for these fermion systems was discussed in this review and application was made to nuclear collective motions (Zhang *et al.*, 1987; Zhang, Feng, and Ginocchio, 1988; Zhang, Wu, *et al.*, 1988; Zhang, Feng, Wu, and Guidry, 1989).

Finally, the concept of vector coherent states is mentioned in Sec. VII. These states, as we have emphasized, have been useful in the computation of various matrix elements (Hecht, 1988). Unfortunately, the relationship between the geometry of these states and the dynamic system is unclear at this point.

It is our regret that other important applications of the coherent states to condensed matter and field-theoretical problems are not included in this review. We note that many of the early applications of coherent states were in fact in condensed-matter studies. Their primary role was to explain the dynamic mechanisms of superconductivity and superfluidity (see Anderson, 1958; Cummings and Johnston, 1966; Langer, 1968, 1969; Srinivasan, 1976). There is also a large body of work using the coherent states to study the various collective phenomena in condensed matter (Carruthers and Dy, 1966; Nieto, 1968, 1969; Feldman and Kahn, 1970; Kano, 1974; Ghosh *et al.*, 1977; Haldane, 1983; Klauder, 1984; Fradkin and Stone, 1987). Coherent states have also been widely used in field theory and particle physics. A collection of these works can be found in Klauder and Skagerstam (1985).

More recently, the study of nonlinearity and chaotic motions in quantum systems has attracted a great deal of interest. It is not surprising that the coherent-state theory is also a useful tool for such studies. In particular, it has already played a role in the search for manifestations of classical chaos in quantum mechanics (Eckhardt, 1988). These investigations (i) reveal the patterns of classical trajectories in the quantum wave functions (Chang and Shi, 1986; Davis, 1988; Raden and Prange, 1988) and (ii) manifest the "scar" phenomenon (Heller, 1984) in wave packets (Waterland *et al.*, 1988). Furthermore, since the geometry of the coherent states provides a fundamental framework for connecting classical and quantum mechanics (Zhang, Feng, Yuan, and Wang, 1989; Zhang, Feng, and Yuan, 1990), it is a powerful tool for the quantitative analysis of semiclassical chaotic motion (Williams and Koonin, 1982; Meredith *et al.*, 1988. The geometry of the coherent states preserves most of the quantum properties, such as the Pauli principle, the internal quantum degrees of freedom, and the statistical properties of microscopic particles (Zhang, Feng, Wu, and Guidry, 1989). It could conceivably be the tool to reveal many of the fundamental effects of pure quantum properties in semiclassical chaotic motion.

It needs to be mentioned that there have been recent attempts to construct coherent states for supergroups (Bars and Gunaydin, 1983; Aragone and Zypman, 1986; Balantekin *et al.*, 1988). These rather mathematical constructions still await physical applications and will not be discussed here.

Throughout this review, we have focused on the construction of the coherent states and on connecting their

geometric structures to quantum systems. However, a physical realization of the coherent states in a *realistic* dynamic system is certainly important and meaningful. Realizations to date are the squeezed states in quantum optics (Slusher *et al.*, 1985; Kimble and Hall, 1986; Shelby *et al.*, 1986; L. A. Wu *et al.*, 1986). Many discussions have centered on the realization of coherent states in anharmonic oscillating systems and other general potential systems (Nieto and Simmons, 1979; Nieto, 1983). Unfortunately there are no measurements at hand for such studies. Perhaps the most exciting experimental development to date is the observation of Keplerian-like wave-packet motion in hydrogen atom systems (Parker and Stroud, 1986, 1987). Whether such a wave packet is in fact a coherent state is still unanswered and therefore is an exciting challenge for the future.

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## APPENDIX

In this appendix we outline the derivation of the group transformations of $H_4$ in the coherent-state representation. We begin with the following matrix representation of $h_4$:

$$a\rightarrow\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix},\quad a^\dagger\rightarrow\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix},$$
$$\hat{n}\rightarrow\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\quad I\rightarrow\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}. \tag{A1}$$

The coset representative has the following form:

$$D(\alpha)\rightarrow\begin{bmatrix}1&-\alpha^*&-\frac{1}{2}|\alpha|^2\\0&1&\alpha\\0&0&1\end{bmatrix}, \tag{A2}$$

where $\alpha$ can take on any value of the one-dimensional complex $C^1$ and hence $H_4/\mathrm{U}(1)\otimes\mathrm{U}(1)$ is isomorphic with $C^1$. A group unitary transformation $g=\exp i(\gamma a^\dagger+\beta a+\eta\hat{n}+\delta I)$, where the unitarity of $g$ requires that $\beta=\gamma^*$ and $\eta$ and $\delta$ be real, on a point $x$ in $H_4/\mathrm{U}(1)\otimes\mathrm{U}(1)$ can be obtained as follows:

$$\begin{aligned}gD(\alpha)=D(\alpha')h\quad[h\in\mathrm{U}(1)\otimes\mathrm{U}(1)]&\rightarrow\exp\begin{bmatrix}0&i\gamma^*&i\delta\\0&i\eta&i\gamma\\0&0&0\end{bmatrix}\exp\begin{bmatrix}0&-\alpha^*&0\\0&0&\alpha\\0&0&0\end{bmatrix}\\&=\begin{bmatrix}1&-a^*+\frac{\gamma^*}{\eta}(e^{i\eta}-1)&\theta\\0&e^{i\eta}&ae^{i\eta}+\frac{\gamma}{\eta}(e^{i\eta}-1)\\0&0&1\end{bmatrix}\\&=\exp\begin{bmatrix}0&-\alpha'^*&0\\0&0&\alpha'\\0&0&0\end{bmatrix}\exp\begin{bmatrix}0&0&i\delta'\\0&i\eta'&0\\0&0&0\end{bmatrix}-\begin{bmatrix}1&-\alpha'^*e^{i\eta'}&i\delta'-\frac{1}{2}|\alpha'|^2\\0&e^{i\eta'}&\alpha'\\0&0&1\end{bmatrix},\end{aligned} \tag{A3}$$

where

$$\theta=i\delta+\frac{|\gamma|^2}{\eta^2}(e^{i\eta}-i\eta-1)+\frac{\gamma^*\alpha}{\eta}(e^{i\eta}-1)-\tfrac{1}{2}|\alpha|^2\ . \tag{A4}$$

Hence

$$\alpha'=\alpha e^{i\eta}+\frac{\gamma}{\eta}(e^{i\eta}-1)\ , \tag{A5a}$$

which is Eq. (2.15). When we restrict $g=D(\beta)\in H_4/\mathrm{U}(1)\otimes\mathrm{U}(1)$, then Eq. (5a) becomes

$$\alpha'\equiv\alpha+\beta \tag{A5b}$$

which gives Eq. (2.16a).

The above technique is very useful in applications of coherent states and the procedure can be directly extended to general cases of a dynamical group $G$, as we have shown in the text.

## REFERENCES

Anderson, P. W., 1958, Phys. Rev. **110**, 827.
Aragone, C., E. Chalbaud, S. Salamo, and J. L. Tani, 1974, J. Phys. A 7, L149.

Aragone, C., E. Chalbaud, and S. Salamo, 1976, J. Math. Phys. **17**, 1963.
Aragone, C., and F. Zypman, 1986, J. Phys. A **19**, 2267.
Arecchi, F. T., E. Courtens, R. Gilmore, and H. Thomas, 1972, Phys. Rev. A **6**, 2211.
Arima, A., and F. Iachello, 1975, Phys. Rev. Lett. **35**, 1069.
Arima, A., and F. Iachello, 1976, Ann. Phys. (NY) **99**, 253.
Arnold, V. I., 1978, *Mathematical Methods of Classical Mechanics* (Springer, New York).
Balantekin, A. B., H. A. Schmitt, and B. R. Barrett, 1988 J. Math. Phys. **29**, 1634.
Bargmann, V., 1961, Commun. Pure Appl. Math. **14**, 187.
Bars, I., and M. Gunaydin, 1983, Commun. Math. Phys. **91**, 31.
Barut, A. O., and L. Girardello, 1971, Commun. Math. Phys. **21**, 41.
Berezin, F. A., 1966, *The Method of Second Quantization* (Academic, New York).
Berezin, F. A., 1972, Sov. Math. Izv. **36**, 1134.
Berezin, F. A., 1974, Sov. Math. Izv. **38**, 1116.
Berezin, F. A., 1975a, Sov. Math. Izv. **39**, 363.
Berezin, F. A., 1975b, Commun. Math. Phys. **40**, 153.
Bergmann, S., 1947, *Sur les fonctions orthogonales de plusieurs variables complexes avec les applications à la théorie des fonctions analytiques,* Memor Sci. Math. No. 106 (Gauthier-Villars, Paris).
Berry, M. V., 1984, Proc. R. Soc. London, Ser. A **392**, 45.
Blaizot, J. P., and H. Orland, 1981, Phys. Rev. C **24**, 1740.
Bogoliubov, N. B., 1958a, Zh. Eksp. Teor. Fiz. **34**, 58 [Sov. Phys. JETP **7**, 41 (1958)].
Bogoliubov, N. B., 1958b, Nuovo Cimento **7**, 795.
Bohr, A., and B. R. Mottelson, 1975, *Nuclear Structure,* Vol. II (Benjamin, New York).
Brown, F. B., and G. Yaffe, 1986, Nucl. Phys. B **271**, 267.
Cahill, K. E., 1965, Phys. Rev. B **138**, 1566.
Carruthers, P., and K. S. Dy, 1966, Phys. Rev. **147**, 214.
Carruthers, P., and M. M. Nieto, 1965, Am. J. Phys. S. **33**, 537.
Cartan, E., 1935, *Sur les domaines bornés homogènes de l'espace de n variables complexes,* Hamb. Abh. **11**, 116.
Castanos, O., E. Chacon, and M. Moshinsky, 1984, J. Math. Phys. **25**, 1211.
Chand, P., 1978, Phys. Lett. A **67**, 99.
Chang, S. J., and K. J. Shi, 1986, Phys. Rev. A **34**, 7.
Chung, V., 1965, Phys. Rev. **140**, 1110B.
Cummings, F. W., and R. J. Johnston, 1966, Phys. Rev. **151**, 105.
Davis, M. J., 1988, J. Phys. Chem. **92**, 3124.
Deenen, J., and C. Quesne, 1984a, J. Math. Phys. **25**, 1638.
Deenen, J., and C. Quesne, 1984b, J. Math. Phys. **25**, 2354.
Dicke, R. H., 1954, Phys. Rev. **93**, 99.
Dickens, T. A., U. T. Lindqwister, W. R. Somsky, and L. G. Yaffe, 1988, Nucl. Phys. B **309**, 1.
Dieperink, A. E. L., 1983, Prog. Part. Nucl. Phys. **9**, 121.
Dieperink, A. E. L., O. Scholten, and F. Iachello, 1980, Phys. Rev. Lett. **44**, 1747.
Dirac, P. A. M., 1925, Proc. R. Soc. London, Ser. A **109**, 642.
Dobaczewski, J., 1982, Nucl. Phys. A **369**, 213 (Part I); 237 (Part II).
Eckhardt, B., 1988, Phys. Rep. **163**, 205.
Fan, H. Y., and T. N. Ruan, 1984, Commun. Theor. Phys. (China) **3**, 345.
Fan, H. Y., H. R. Zaidi, and R. J. Klauder, 1987, Phys. Rev. D **35**, 1831.
Feldman, A., and A. H. Kahn, 1970, Phys. Rev. B **1**, 4584.
Feng, D. H., and R. Gilmore, 1980, Phys. Lett. B **90**, 327.
Feng, D. H., and R. Gilmore, 1982, Phys. Rev. C **26**, 1244.
Feng, D. H., R. Gilmore, and S. R. Deans, 1981, Phys. Rev. C **23**, 1254.
Feng, D. H., R. Gilmore, and L. M. Narducci, 1979, Phys. Rev. C **19**, 1119.
Feng, D. H., R. Gilmore, and L. M. Narducci, 1980, Ann. Israel Phys. Soc. **3**, 214.
Fetter, A. L., and J. D. Walecka, 1971, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York).
Feynman, R. P., 1948, Rev. Mod. Phys. **20**, 367.
Feynman, R. P., and A. R. Hibbs, 1965, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York).
Fowler, G. N., N. Stelte, and R. M. Weiner, 1979, Nucl. Phys. A **319**, 349.
Fradkin, E., and M. Stone, 1987, Phys. Rev. B **36**, 5291.
Garbaczewski, P., 1978, Phys. Rep. **36**, 65.
Gazdy, B., and D. A. Micha, 1985, J. Chem. Phys. **82**, 4926.
Gazdy, B., and D. A. Micha, 1985, J. Chem. Phys. **82**, 4937.
Ghosh, G., B. Dutta-Roy, and D. Bhaumik, 1977, Plasma Phys. **19**, 1051.
Gilmore, R., 1972, Ann. Phys. (NY) **74**, 391.
Gilmore, R., 1974a, Rev. Mev. de Fisica **23**, 142.
Gilmore, R., 1974b, *Lie Groups, Lie Algebras and Some of Their Applications* (Wiley, New York).
Gilmore, R., 1974c, J. Math. Phys. **15**, 2090.
Gilmore, R., 1976, J. Phys. A **9**, L65.
Gilmore, R., 1979, J. Math Phys. **20**, 891.
Gilmore, R., 1980, in *Symmetries in Science,* edited by B. Gruber and R. S. Millman (Plenum, New York), p. 105.
Gilmore, R., 1981, *Catastrophe Theory for Scientists and Engineers* (Wiley, New York).
Gilmore, R., 1986, in *The Physics of Phase Space,* Lecture Notes in Physics, Vol. 278, edited by Y. S. Kim and W. W. Zachary (Springer, Berlin), p. 211.
Gilmore, R., C. M. Bowden, and L. M. Narducci, 1975, Phys. Rev. A **12**, 1019.
Gilmore, R., and D. H. Feng, 1978a, Phys. Lett. B **76**, 26.
Gilmore, R., and D. H. Feng, 1978b, Nucl. Phys. A **301**, 189.
Gilmore, R., and D. H. Feng, 1978c, Phys. Rev. C **19**, 1119.
Gilmore, R., and D. H. Feng, 1979, Phys. Lett. B **85**, 155.
Gilmore, R., and D. H. Feng, 1983, Prog. Part. Nucl. Phys. **9**, 479.
Gilmore, R., and J. M. Yuan, 1987, J. Chem. Phys. **86**, 130.
Gilmore, R., and J. M. Yuan, 1989, J. Chem. Phys. **91**, 917.
Ginocchio, J. N., 1980, Ann. Phys. (NY) **126**, 234.
Ginocchio, J. N., and M. W. Kirson, 1980a, Phys. Rev. Lett. **44**, 1744.
Ginocchio, J. N., and M. W. Kirson, 1980b, Nucl. Phys. A **350**, 31.
Glauber, R. J., 1963a, Phys. Rev. Lett. **10**, 277.
Glauber, R. J., 1963b, Phys. Rev. **130**, 2529.
Glauber, R. J., 1963c, Phys. Rev. **131**, 2766.
Glauber, R. J., 1966, Phys. Lett. **21**, 650.
Goldstein, H., 1980, *Classical Mechanics* (Addison-Wesley, Reading, MA/Palo Alto/London).
Gutzwiller, M. C., 1971, J. Math. Phys. **12**, 343.
Haken, M., 1970, *Handbuch der Physik,* Vol. XXV/2 (Springer, Berlin).
Haldane, F. D. M., 1983, Phys. Lett. A **93**, 464.
Hecht, K. T., 1987, *The Vector Coherent State Method and Its Application to Problems of Higher Symmetries,* Lecture Notes in Physics, Vol. 290 (Springer, Berlin).
Helgason, S., 1978, *Differential Geometry, Lie Groups and Symmetric Space* (Academic, New York).

Heller, E. J., 1984, Phys. Rev. Lett. **53**, 1515.
Hepp, K., 1974, Commun. Math. Phys. **35**, 265.
Hepp, K., and E. H. Lieb, 1973, Phys. Rev. A **8**, 2517.
Hill, D. L., and J. A. Wheeler, 1953, Phys. Rev. **89**, 1102.
Hillery, M., R. F. O'Connell, M. O. Scully, and E. P. Wigner, 1984, Phys. Rep. **106**, 121.
Hollenhorst, H. N., 1979, Phys. Rev. D **19**, 1669.
Hoodboy, R., and J. Negele, 1978, Phys. Rev. C **18**, 2380.
Hougoh, M., 1977, J. Math. Phys. **18**, 2081.
Hua, L. K., 1963, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains,* Translations of Mathematical Monographs, Vol. 6 (Am. Math. Society, Providence).
Huseyin, K., 1986, Ed., *Multiple Parameter Stability Theory and its Applications* (Clarendon, New York/Oxford).
Husimi, K., 1940, Proc. Phys. Math. Soc. Jpn. **22**, 264.
Kan, K.-K., 1981, Phys. Rev. A **24**, 2831.
Kan, K.-K., J. J. Griffin, P. C. Lichtner, and M. Dworzecka, 1979a, Nucl. Phys. A **332**, 109.
Kan, K.-K., P. C. Lichtner, M. Dworzecka, and J. J. Griffin, 1979b, Phys. Rev. C **21**, 1098.
Kano, Y., 1974, J. Phys. Soc. Jpn. **36**, 649 (Part I); **37**, 310 (Part II).
Kano, Y., 1976, Phys. Lett. A **56**, 7.
Katriel, J., 1970, J. Phys. B **3**, 1315.
Katriel, J., A. Novoselsky, M. Vallières, and R. Gilmore, 1990 (unpublished).
Kerman, A. K., and S. E. Koonin, 1976, Ann. Phys. (NY) **100**, 332.
Kimble, H. J., and J. L. Hall, 1986, in *Quantum Optics IV,* edited by J. D. Harvey and D. F. Walls (Springer, Berlin), p. 58.
Kirillov, A. A., 1976, *Elements of the Theory of Representations* (Springer, Berlin).
Klauder, J. R., 1960, Ann. Phys. **11**, 123.
Klauder, J. R., 1963, J. Math. Phys. **4**, 1055 (Part I); 1058 (Part II).
Klauder, J. R., 1979, Phys. Rev. D **19**, 2349.
Klauder, J. R., 1978, in *Path Integrals and their Applications in Quantum Statistical and Solid State Physics,* Proceedings of the NATO Advanced Study Institute, Antwerp, 1977, edited by G. I. Papadopoulos and J. T. Devreese (Plenum, New York), p. 5.
Klauder, J. R., 1984, Phys. Rev. A **29**, 2036.
Klauder, J. R., and B. S. Skagerstam, 1985, *Coherent States, Applications in Physics and Mathematical Physics* (World Scientific, Singapore).
Klauder, J. R., and E. C. G. Sudarshan, 1968, *Fundamentals of Quantum Optics* (Benjamin, New York), p. 178.
Klein, A., and E. Marshalek, 1990, "Boson realizations of Lie algebras with applications to nuclear physics," University of Pennsylvania preprint UPR-report-0065NT.
Koonin, S. E., 1982, in *Nuclear Theory, 1981: Proceedings of the Nuclear Theory Summer Workshop,* Santa Barbara, 1981, edited by G. F. Bertsch (World Scientific, Singapore), p. 183.
Kostant, B., 1970, in *Lectures in Modern Analysis and Applications III,* Lecture Notes in Mathematics, Vol. 170, edited by C. T. Taam (Springer, Berlin), p. 87.
Krieger, S. J., 1977, Nucl. Phys. A **276**, 12.
Kuratsuji, H., 1988, Phys. Rev. Lett. **61**, 1687.
Kuratsuji, H., and T. Suzuki, 1980, Phys. Lett. B **92**, 19.
Kuratsuji, H., and T. Suzuki, 1983, Prog. Theor. Phys. Suppl. **74&75**, 209.
Langer, J. S., 1968, Phys. Rev. **167**, 183.
Levit, S., J. W. Negele, and Z. Paltiel, 1980a, Phys. Rev. C **21**, 1603.
Levit, S., J. W. Negele, and Z. Paltiel, 1980b, Phys. Rev. C **22**, 1979. Li, S. Y., R. M. Dreizler, and A. Klein, 1971, Phys. Rev. C **4**, 1571.
Lieb, E. H., 1973, Commun. Math. Phys. **31**, 327.
Lieb, E. H., 1973, Commun. Math. Phys. **62**, 35.
Lieb, E. H., 1973, Rev. Mod. Phys. **53**, 603.
Lieb, E. H., and M. Loss, 1990 (unpublished).
Lipkin, H. J., N. Meshkov, and A. J. Glick, 1965, Nucl. Phys. **62**, 188.
Lu, E. Y. C., 1971, Lett. Nuovo Cimento **2**, 1241.
Lu, E. Y. C., 1971, Lett. Nuovo Cimento **3**, 585.
McAnally, D. S., and A. J. Bracken, 1989, Phys. Rev. A **37**, 2304.
Meredith, D. C., S. E. Koonin, and M. R. Zirnbauer, 1988, Phys. Rev. A **37**, 3499.
Moshinsky, M., 1985, J. Phys. A **18**, L1.
Moyal, J. E., 1949, Proc. Cambridge Philos. Soc. **45**, 99.
Narducci, L. M., C. M. Bowden, V. Bluemel, G. P. Garrazana, and R. A. Tuft, 1975, Phys. Rev. A **11**, 973.
Narducci, L. M., C. M. Bowden, and C. A. Coulter, 1974, Phys. Rev. A **9**, 829.
Narducci, L. M., D. H. Feng, R. Gilmore, and G. S. Agarwal, 1978, Phys. Rev. A **18**, 1571.
Negele, J. W., and H. Orland, 1987, *Quantum Many-Particle Systems,* Frontiers in Physics, Vol. 68 (Addison-Wesley, Reading, MA).
Ni, G. J., and Y. P. Wang, 1983, Phys. Rev. D **27**, 969.
Nielson, H. B., and D. Rohrlich, 1988, Nucl. Phys. B **299**, 471.
Nieto, M. M., 1968, Phys. Rev. **167**, 416.
Nieto, M. M., 1969, Nuovo Cimento B **62**, 95.
Nieto, M. M., 1980, Phys. Rev. D **22**, 391.
Nieto, M. M., 1983, in *Group Theoretical Methods in Physics,* Vol. 2, edited by M. A. Markov (Nauka, Moscow), p. 174.
Nieto, M. M., and L. M. Simmons, Jr., 1978, Phys. Rev. Lett. **41**, 207.
Nieto, M. M., and L. M. Simmons, Jr., 1979, Phys. Rev. D **20**, 1321 (Part I); 1332 (Part II); 1342 (Part III).
Nieto, M. M., L. M. Simmons, Jr., and V. P. Gutschick, 1981, Phys. Rev. D **23**, 927.
Ohnuki, Y., and T. Kashiwa, 1978, Prog. Theor. Phys. **60**, 548.
Onofri, E., 1975, J. Math. Phys. **16**, 1087.
Onofri, E., 1980, in *Functional Integration,* edited by J.-P. Antoine and E. Triapegui (Plenum, New York), p. 121.
Parker, J., and C. R. Stroud, Jr., 1986, Phys. Rev. Lett. **56**, 716.
Parker, J., and C. R. Stroud, Jr., 1987, Phys. Rev. **35**, 2806.
Perelomov, A. M., 1972, Commun. Math. Phys. **26**, 222.
Perelomov, A. M., 1975, Commun. Math. Phys. **44**, 197.
Perelomov, A. M., 1977, Usp. Fiz. Nauk **123**, 23 [Sov. Phys. Usp. **20**, 703 (1977)].
Perelomov, A. M., 1986, *Generalized Coherent States and Their Applications* (Springer, Berlin).
Quesne, C., 1986, J. Math. Phys. **25**, 1638.
Radcliffe, J. M., 1971, J. Phys. A **4**, 313.
Raden, G., and R. E. Prange, 1988, Phys. Rev. Lett. **61**, 1691.
Rajagopal, A. K., and J. T. Marshall, 1982, Phys. Rev. A **26**, 2977.
Rasetti, M., 1975, Int. J. Theor. Phys. **14**, 1.
Ring, P., and P. Schuk, 1980, *The Nuclear Many-Body Problem* (Springer, Berlin).
Rowe, D. J., 1984, J. Math. Phys. **25**, 2262.
Rowe, D. J., 1985, Rep. Prog. Phys. **48**, 1419.
Rowe, D. J., G. Rosensteel, and R. Gilmore, 1985, J. Math. Phys. **26**, 2787.

Rowe, D. J., A. Ryman, and G. Rosensteel, 1980, Phys. Rev. A **22**, 2362.
Schrödinger, E., 1926, Naturwissenschaften **14**, 664.
Schulman, L. S., 1981, *Techniques and Applications of Path Integration* (Wiley, New York).
Shapiro, J. H., H. P. Yuen, and J. A. Machado Mata, 1979, IEEE Trans. Inf. Theor. **IT-25**, 179.
Shelby, R. M., M. D. Levenson, S. H. Perlmutter, R. G. Devoe, and D. F. Walls, 1986, Phys. Rev. Lett. **57**, 691.
Simon, B., 1980, Commun. Math. Phys. **71**, 247.
Slusher, R. E., L. W. Hollberg, B. Yurke, J. C. Mertz, and J. F. Valley, 1985, Phys. Rev. Lett. **55**, 2409.
Srinivasan, V., 1976, Prog. Theor. Phys. **55**, 939.
Stoler, D., 1970, Phys. Rev. D **1**, 3217.
Stoler, D., 1971, Phys. Rev. D **4**, 1925.
Sudarshan, E. C. G., 1963, Phys. Rev. Lett. **10**, 277.
Suzuki, T., 1983, Nucl. Phys. A **398**, 557.
Takahashi, Y., and F. Shibata, 1975, J. Phys. Soc. Jpn. **38**, 656.
Thirring, W., 1981, Commun. Math. Phys. **79**, 1.
Thouless, D. J., 1960, Nucl. Phys. **21**, 225.
Walls, D. F., 1983, Nature (London) **306**, 141.
Waterland, R. L., J. M. Yuan, C. C. Martens, R. E. Gillilan, and W. P. Reinhardt, 1988, Phys. Rev. Lett. **61**, 2733.
Weissman, Y., 1983, J. Phys. A **16**, 2693.
Weyl, H., 1928, *Gruppentheorie und Quantenmechanik* (Hirzel, Leipzig).
Wigner, E. P., 1932, Phys. Rev. **40**, 749.
Wilets, L., 1989, *Non-topological Solitons* (World Scientific, Singapore).
Williams, R. D., and S. E. Koonin, 1982, Nucl. Phys. A **391**, 72.
Wu, C. L., D. H. Feng, X. G. Chen, J. Q. Chen, and M. W. Guidry, 1986, Phys. Lett. B **186**, 313.
Wu, C. L., D. H. Feng, X. G. Chen, J. Q. Chen, and M. W. Guidry, 1987, Phys. Rev. C **37**, 1157.
Wu, C. L., W. M. Zhang, D. H. Feng, M. W. Guidry, and L. L. Riedinger, 1989 (unpublished).
Wu, L. A., H. J. Kimble, J. L. Hall, and H. Wu, 1986, Phys. Rev. Lett. **57**, 2520.
Wybourne, B. G., 1974, *Classical Groups for Physicists* (Wiley, New York).
Yaffe, L. G., 1982, Rev. Mod. Phys. **54**, 407.
Yuen, H. P., 1976, Phys. Rev. A **13**, 2226.
Yuen, H. P., and J. H. Shapiro, 1978, IEEE Trans. Inf. Theor. **IT-24**, 657.
Yuen, H. P., and J. H. Shapiro, 1980, IEEE Trans. Inf. Theor. **IT-26**, 78.
Zak, J., 1980, Phys. Rev. B **21**, 3345.
Zhang, W. M., 1987, Nucl. Phys. A **467**, 422.
Zhang, W. M., D. H. Feng, and J. N. Ginocchio, 1987, Phys. Rev. Lett. **59**, 2032.
Zhang, W. M., D. H. Feng, and J. N. Ginocchio, 1988, Phys. Rev. C **37**, 1281.
Zhang, W. M., D. H. Feng, C. L. Wu, H. Wu, and J. N. Ginocchio, 1989, Nucl. Phys. A **505**, 7.
Zhang, W. M., D. H. Feng, C. L. Wu, and M. W. Guidry, 1989, J. Phys. G **20**, L115.
Zhang, W. M., D. H. Feng, and J. M. Yuan, 1989, Phys. Rev. A (in press).
Zhang, W. M., D. H. Feng, J. M. Yuan, and S. J. Wang, 1989, Phys. Rev. A **40**, 438.
Zhang, W. M., C. C. Martens, D. H. Feng, and J. M. Yuan, 1988, Phys. Rev. Lett. **61**, 2167.
Zhang, W. M., C. L. Wu, D. H. Feng, J. N. Ginocchio, and M. W. Guidry, 1988, Phys. Rev. C **38**, 1475.