Tunneling times: a critical review

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The old question of "How long does it take to tunnel through a barrier?" has acquired new urgency with the advent of techniques for the fabrication of semiconductor structures in the nanometer range. For the restricted problem of tunneling in a scattering configuration, a coherent picture is now emerging. The dwell time τ_D has the status of an exact statement of the time spent in a region of space, averaged over *all* incoming particles. The phase times τ_T^φ and τ_R^φ are defined separately for transmitted and reflected particles. They are asymptotic statements on *completed* scattering events and include self-interference delays as well as the time spent in the barrier. Consequently, neither the dwell time nor the phase times can answer the question of how much time a transmitted (alternatively, reflected) particle spent in the barrier region. Our discussion of this question relies on a few simple criteria: (1) The average duration of a physical process must be real. (2) Since transmission and reflection are mutually exclusive events, the times τ_T and τ_R spent in the barrier region are, *if they exist*, conditional averages. Consequently, they must obey the identity $\tau_D = T\tau_T + R\tau_R$, where T and R are the transmission and reflection probabilities, respectively. The existence of this identity distinguishes tunneling in a scattering configuration from tunneling out of a metastable state. (3) Any proposed τ_T and τ_R must meet every requirement that can be constructed from τ_D. On the basis of (2), the naively extrapolated phase times, as well as the Büttiker-Landauer time, must be rejected. The local Larmor times, as introduced by Baz', satisfy (2), but not every criterion of type (3). The local Larmor clock is therefore unreliable. The *asymptotic* Larmor clock shows the phase times, as it should. Finally, the inverse characteristic frequency of an oscillating barrier cannot always be defined. It is shown *not* to represent the duration of the tunneling process. This leaves the dwell time and the phase times as the only well-established times in this context. It also leaves open the question of the length of time a transmitted particle spends in the barrier region. It is not clear that a generally valid answer to this question exists.

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I. INTRODUCTION

The question loosely formulated as "How much time does tunneling take?" is not new. Early answers were given in the 1940s and 1950s (Eisenbud, 1948; Bohm, 1951; Wigner, 1955), with alternatives proposed in the 1960s (Smith; 1960, Baz', 1966a, 1966b; Rybachenko, 1966). The prospect of high-speed devices based on tunneling structures in semiconductors (see, for example, Capasso *et al.*, 1986) has, in recent years, brought new urgency to the problem. An understanding of the time-dependent aspects of tunneling is clearly required for the construction of a kinetic theory for such systems. The simple question of tunneling times seems a natural one from which to start. But simplicity can be deceptive: Tunneling times have continued to be controversial throughout the 1980s.

Ideally, a review on tunneling times should lay the conceptual foundation for the subject, clarify real and apparent contradictions, and point out current misconceptions. It should discuss tunneling during escape from localized states as well as in a scattering context. Furthermore, it should include effects of dimensionality, of elastic and inelastic processes, of induced space charge, and other many-body effects. Potentially different characteristic times for various aspects of tunneling processes should be defined and discussed and their respective ranges of validity carefully circumscribed. Finally, the ultimate review should relate all this to existing numerical and real experiments and point out remaining challenges.[1]

The scope of this review is much more modest. The recent theoretical controversy on tunneling times has evolved around seemingly simple notions on idealized one-dimensional models in a scattering configuration. Within this limited universe, conflicting answers have claimed general validity. In our opinion, a coherent picture of this restricted set of problems is emerging, and a correspondingly restricted review can, it is hoped, serve the useful purpose of removing some of the obstacles to further progress.

In particular, we shall argue that two different tunneling times are well established and have well-defined and complementary meanings. One of them, the dwell time, provides an exact statement (in the context of scattering of particles with fixed energy) of the time spent in any finite region of space, averaged over all incoming particles. In this sense the dwell time can serve as a reference point in any discussion on tunneling times. The weakness of the dwell time is that it is a total average, with no distinction between different scattering channels.[2]

The other well-established concept is that of the asymptotic phase times describing completed scattering events involving wave packets with a narrow energy distribution. Distinct phase times do exist for transmission and reflection. On the other hand, each of the asymptotic phase times includes, beyond free-particle motion outside the barrier, two contributions that cannot be disentangled: the time spent in the barrier region, and the self-interference delay during the approach to the barrier. Since the status of the phase times is rather subtle, and since, on this issue, much confusion exists in the literature, we discuss them extensively here.

A sizable literature exists on the mathematical aspects of phase and dwell ("sojourn") times in the context of scattering in three dimensions. [For a review, see Martin (1981). For recent work, see, for example, Amrein and Cibils (1987).] With one exception (Jaworski and Wardlaw, 1988a), this has apparently been overlooked in the recent controversy over tunneling times.

With their complementary strengths and weaknesses, none of the above times can answer a more precise version of the question from which we started: "When a particle in a scattering configuration, and with given energy, tunnels through (or, is reflected from) a barrier, how much time did it, on the average, spend in the barrier region?" Several answers with a claim to general validity have been proposed: the extrapolated phase times (Collins *et al.*, 1987; Hauge *et al.*, 1987; Teranishi *et al.*, 1987), the time resulting from the Stevens procedure (Stevens, 1980, 1983, 1984), various Larmor times (Baz', 1966a, 1966b; Rybachenko, 1966; Büttiker, 1983), a newly proposed complex "time" (Sokolovski and Baskin, 1987), and the Büttiker-Landauer time (Büttiker and Landauer, 1982, 1985, 1986; Büttiker, 1983).

We stress that we consider scattering configurations only, not escape from localized states. Let the wave number of the incoming beam be k. We denote by $\tau_T(b,a;k)$ the average time spent in a barrier region (b,a) (see Fig. 1) by particles that are ultimately transmitted. Similarly, the time spent in the barrier region by reflected particles is denoted by $\tau_R(b,a;k)$. The claim of each of the above conflicting proposals is that of providing a well-defined general procedure by which $\tau_T(b,a;k)$ and $\tau_R(b,a;k)$ can be calculated, for arbitrary k and barrier potential $V(x)$. These times represent *conditional* averages over mutually exclusive events. One can clearly determine, without interfering with the tunneling process itself, whether a particle is ultimately transmitted or reflected. The dwell time $\tau_D(b,a;k)$, on the other hand, is the *total* average over all incoming particles of the time spent in the barrier region. The conditional averages must therefore, *if they exist,* obey the probabilistic rule $\tau_D = T\tau_T + R\tau_R$, where $T(k)$ and $R(k)=1-T(k)$ are the transmission and reflection probabilities, respectively. This requirement plays a crucial role as a consistency check in our discussion. We do not know of a corresponding requirement on the tunneling time in the context of escape from a localized state. Although the two types of tunneling processes are expected to be physically similar, they are therefore not equivalent; results derived in one context cannot automatically be used in the other.

In addition to this probabilistic requirement, we insist that the average duration of a process be a real quantity. The fact that complex "times" have proved useful in, for example, discussions of adiabaticity (Sokolovski and

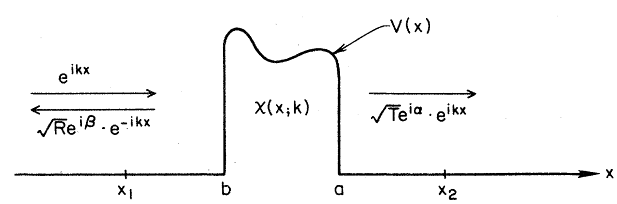

FIG. 1. Stationary scattering configuration. An arbitrary barrier $V(x)$ is confined to the x interval (b,a). For later use, a larger interval (x_1,x_2), containing the barrier region, is indicated.

[1]We restrict ourselves to a few unsystematic pointers here: There is a huge literature on the escape from local states. Recent reviews are those of Hänggi (1986) and Leggett *et al.* (1987). Similarly, a large number of papers consider effects of dimensionality, elastic and inelastic processes. A qualitative overview can be found in Capasso *et al.* (1986). Dynamic space-charge effects were first discussed in a tunneling context by Jonson (1980). For an example of recent work, see Persson and Baratoff (1988). Finally, several experiments have been proposed and performed recently, in which time aspects of tunneling can be studied fairly directly. Examples are Guéret *et al.* (1988), Lucas *et al.* (1988), and Martinis *et al.* (1988).

[2]We use the term "channel" in the simple meaning of distinct final states (transmission and reflection in one dimension).

Hänggi, 1988) does not imply that they are candidates for quantities defining the duration of a physical process.

With these requirements and with further consistency tests, we in turn consider the above-mentioned proposals for tunneling times. We conclude that they all suffer from one logical flaw or another, flaws sufficiently serious that they must be rejected as candidates for a generally valid answer to the question posed. We stress that this does not necessarily imply that these times are meaningless. Some of the proposals could well represent partial answers, valid under given circumstances. This possibility will not be explored here.

In our attempt to convince the reader of the soundness of the above conclusions, we shall, whenever possible, supplement general arguments with simple examples. We restrict ourselves to one dimension throughout. Calculational details that can be found in the literature will not be reproduced. We make no attempt at mathematical rigor. On the other hand, we have made an extra effort to be clear, also in circumstances where a complete understanding is lacking. Apparent discrepancies will be removed, and remaining difficulties pointed out. Only passing reference will be made to numerical work (Barker, 1985; Ravaioli *et al.*, 1985; Jauho and Nieto, 1986; Collins *et al.*, 1987; Jauho and Jonson, 1989a, 1989b; Leavens and Aers, 1989a), in spite of its obvious intrinsic value. Our main concern is conceptual clarification, and even though numerical work has played an important role as a guide for thought, it has not been sufficiently forceful, in this context, to close arguments.

In Sec. II the main contenders for tunneling times are presented. The exact results that must be respected by any interpretation are stated in Sec. III. Section IV is devoted to a discussion of the status and physical meaning of the phase times, whereas Sec. V presents an alternate view of oscillating barriers. In Sec. VI the reliability of Larmor clocks is discussed. We collect our conclusions in Sec. VII.

II. THE MAIN CONTENDERS

In this section we briefly summarize the arguments leading to some of the expressions for tunneling times proposed in the literature. Most of these concern elastic tunneling, but we shall also discuss the interesting case, introduced by Büttiker and Landauer, of tunneling through an oscillating barrier.

A. Elastic tunneling

We consider (see Fig. 1) the one-dimensional case of an arbitrary time-independent barrier $V(x)$ localized on the interval (b,a). We shall assume that the stationary scattering problem at any energy, $E=\hbar^2k^2/2m$, has been solved exactly, with a space-independent effective mass m for simplicity. The wave function has the form

$$\psi(x;k)=\begin{cases} e^{ikx}+\sqrt{R}\,e^{i\beta}e^{-ikx}, & x<b, \\ \chi(x;k), & b<x<a, \\ \sqrt{T}\,e^{i\alpha}e^{ikx}, & a<x. \end{cases} \tag{2.1}$$

Here $R(k)$ and $T(k)=1-R(k)$ are the reflection and transmission probabilities, respectively, and $\beta(k)$ and $\alpha(k)$ are the corresponding phase shifts. All these quantities will be considered as known functions of k, as will the wave function $\chi(x;k)$ in the barrier region. For the textbook example of a rectangular barrier, everything can be calculated analytically. This special case will be used extensively for illustrative purposes. Consequently we have, for easy reference, collected some results for the rectangular barrier in Appendix A. For later convenience, we let $b<0$ and $a>0$ (see Fig. 1) and introduce the arbitrary interval (x_1,x_2), which, except in special cases, will be assumed to contain the interval (b,a).

B. The phase times

Let a wave packet with wave numbers distributed sharply[3] around a given k impinge on the barrier. The rough argument giving the phase times can be formulated as follows. The transmitted packet is described by a wave packet dominated by a small set of Fourier components of the form

$$\sqrt{T(k)}\exp[i\alpha(k)+ikx-iE(k)t/\hbar] \ . \tag{2.2}$$

Follow the peak $x_p(t)$ of the packet. In the stationary phase approximation it is given by

$$\frac{d\alpha}{dk}+x_p(t)-\frac{1}{\hbar}\frac{dE}{dk}t=0 \ . \tag{2.3}$$

The tunneling process clearly causes a spatial delay, $\delta x=\alpha'(k)=d\alpha/dk$, and a temporal delay,[4]

$$\delta\tau_T=\hbar\frac{d\alpha}{dE}=\frac{1}{v(k)}\frac{d\alpha}{dk} \ , \tag{2.4}$$

where $v(k)=\hbar^{-1}dE/dk=\hbar k/m$ is the group velocity. For reasons that will become clear later, we prefer to use the term "phase time" for the *total* time $\tau_T^{\varphi}(x_1,x_2;k)$ spent between $x_1<b$, to the left of the barrier, and $x_2>a$, to the right of it. From Eq. (2.4) it reads

[3]The concept of a transmission time for a wave packet is precise only to the extent that the packet is sharp in k space. As an illuminating contrast, consider a packet impinging on a double barrier with two resonances of different lifetimes. For a packet broad enough in its energy distribution to cover both resonances, a characterization of the tunneling delay in terms of a single (average) transmission time is, if formally possible, not very meaningful from a physical point of view.

[4]The concept of a time delay is more subtle than one would think [see Hauge *et al.* (1987), Jaworski and Wardlaw (1988a), and Sec. IV.E].

$$\tau_T^{\varphi}(x_1,x_2;k)=\frac{1}{v(k)}[x_2-x_1+\alpha'(k)] \; . \tag{2.5}$$

Equation (2.5) is our expression for the classic phase time for transmitted particles (Eisenbud, 1948; Bohm, 1951; Wigner, 1955). The reasoning for reflected particles is analogous, except that delays must be defined with respect to a reference, which we choose to be ideal reflection at $x=0$. The total phase time for reflection is independent of this choice and reads

$$\tau_R^{\varphi}(x_1,x_2;k)=\frac{1}{v(k)}[-2x_1+\beta'(k)] \; . \tag{2.6}$$

Against this derivation one could object (Büttiker and Landauer, 1982, 1985, 1986) that the peak is not a reliable characteristic of packets distorted during the tunneling process. Furthermore, it is not clear how to obtain corrections for wave packets of finite but small width, σ, in k space. In several recent derivations these objections have been met (Barker, 1986; Hauge *et al.*, 1987; Teranishi *et al.*, 1987; Falck and Hauge, 1988; Jaworski and Wardlaw, 1988a). For example, in Hauge *et al.* and Teranishi *et al.*, the positions of wave packets are identified with their centers of gravity, a much sturdier characteristic than the peak. Explicit corrections to (2.5) and (2.6) have been derived by Hauge *et al.* These corrections are formally of $O(\sigma^2)$, but as we shall see in Sec. IV.E, the dominant among them is really of $O(\sigma)$.

There is another objection against the derivation sketched above. It does not reveal the *asymptotic* status of the phase times: the fact that $b-x_1$ and x_2-a must both be large compared to the spatial extent, roughly $\sim\sigma^{-1}$, of the wave packets. This aspect of the phase times will be discussed thoroughly in Sec. IV. Since the asymptotic character is not apparent in Eqs. (2.5) and (2.6), it is tempting to linearly extrapolate back to the barrier region and introduce

$$\Delta\tau_T^{\varphi}(b,a;k)=\frac{1}{v(k)}[a-b+\alpha'(k)] \; , \tag{2.7}$$

$$\Delta\tau_R^{\varphi}(b,a;k)=\frac{1}{v(k)}[-2b+\beta'(k)] \; . \tag{2.8}$$

In itself, there is nothing objectionable about this extrapolation, and we shall use it repeatedly. However, as will be shown in Sec. IV, it is not correct to attribute to the extrapolated phase times the physical meaning of the time spent in the barrier region, as was done, for example, by Hauge *et al.* (1987) and by Falck and Hauge (1988).

A word of warning here: The term "phase times" has been used in the literature for several related, but different, quantities. In this review, we shall stick to the terminology that we believe most clearly reflects the underlying realities. The term *phase times* will be reserved for the asymptotic quantities $\tau_T^{\varphi}(x_1,x_2;k)$ and $\tau_R^{\varphi}(x_1,x_2;k)$ of Eqs. (2.5) and (2.6), which are associated with completed collisions on the (large) interval (x_1,x_2) of wave packets, narrow in k space. (As will be argued in Sec. IV, the addition of overbars, $\bar{\tau}_T^{\varphi}$ and $\bar{\tau}_R^{\varphi}$, which denote an average with respect to a narrow wave packet, is asymptotically irrelevant.) The term *extrapolated* phase times will always be used for the quantities of (2.7) and (2.8). These are not the same as the phase time *delays* [such as Eq. (2.4)], since they include the barrier contributions $(a-b)/v$ and $-2b/v$.

For later reference, we quote the results for a rectangular barrier of height $V_0=\hbar^2k_0^2/2m$ and width d, for the opaque case, i.e., for $T(k)\sim\exp(-2\kappa d)\ll 1$ [where $\kappa^2=k_0^2-k^2=2m(V_0-E)/\hbar^2$]. The phase times (2.7) and (2.8), extrapolated to the barrier region $(-d/2,d/2)$, give (see Appendix A)

$$\Delta\tau_T^{\varphi}(d;k)=\Delta\tau_R^{\varphi}(d;k)\simeq\frac{2m}{\hbar k\kappa} \; . \tag{2.9}$$

In other words, the linearly extrapolated phase times grow inversely with k for small energies. One can view (2.9) as the finite limit of these phase times as the thickness d goes to infinity at fixed k and κ.

C. The Stevens procedure

Stevens (1980, 1983) has suggested that the motion of wave packets with a sharp front in x space can be followed through the barrier region if one focuses on the front of the main packet. In an approximate analytic treatment of an otherwise exact formalism, he found that when a packet with a sharp front enters the barrier, the position of the front remains recognizable and moves through the barrier with a velocity $\hbar\kappa/m$. As a result, one obtains the transmission time (2.18).

Quite apart from the problematic association of particle position with the front of a wave packet, several authors have questioned these findings. Collins *et al.* (1987) and Teranishi *et al.* (1987) found that the terms kept by Stevens are dominated by terms he neglected. In addition, Stevens's claims do not stand up under the numerical scrutiny of Jauho and Jonson (1989a). Although the front of the initial packet used was sharp, no distinct front traveling through the barrier could be detected.

We no longer consider the approximate version of Stevens's procedure, intriguing as it was, a serious candidate for the calculation of the duration of tunneling.

D. The dwell time

Smith (1960) introduced the dwell time τ_D as a measure of the time, averaged over all scattering channels, spent by a particle in a region of space. In our one-dimensional case, the dwell time was first introduced by Büttiker (1983) and is defined as

$$\tau_D(x_1,x_2;k)=\frac{1}{v(k)}\int_{x_1}^{x_2}dx\,|\psi(x;k)|^2 \; , \tag{2.10}$$

with the understanding that the incoming flux is $v(k)$, as in Eq. (2.1). For a rectangular opaque barrier, the dwell time in the barrier itself becomes (see Appendix A)

$$\tau_D(d;k) \simeq \frac{2mk}{\hbar\kappa k_0^2} . \quad (2.11)$$

In sharp contrast to the extrapolated phase times (2.9), the dwell time vanishes with k for small energies.

E. The local Larmor times

Baz' (1966a) suggested an appealing thought experiment (see Fig. 2) to measure separately the time spent in different scattering channels: take advantage of the constant Larmor precession of a spin in a homogeneous magnetic field by covering the region of interest with an infinitesimal field $\mathbf{B}=B_0\hat{z}$ (so that only first-order effects need be retained). With the incident spin-$\frac{1}{2}$ particles polarized in the x direction, the time spent in the field region should be proportional to the averaged spin component $\langle s_y \rangle$ of the particles emerging in the given scattering channel. Rybachenko (1966) used the idea of Baz' for the one-dimensional case of concern to us. With the magnetic field covering the (x_1,x_2) interval, the Larmor time for transmission becomes[5]

$$\tau_{yT}^L(x_1,x_2;k) = \lim_{\omega_L \to 0} \langle s_y \rangle_T / (-\tfrac{1}{2}\hbar\omega_L) , \quad (2.12)$$

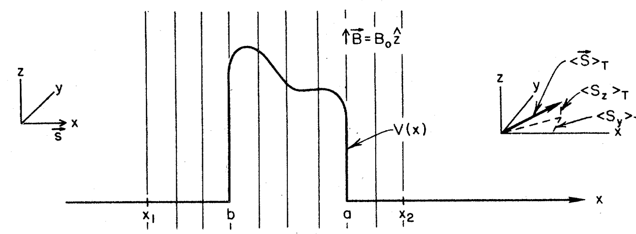

FIG. 2. Larmor clock. A weak homogeneous magnetic field, $\mathbf{B}=B_0\hat{z}$, covers the x interval (x_1,x_2). The spin $\mathbf{s}$ of particles entering the field region are polarized in the $\hat{x}$ direction. After transmission beyond $x=x_2$, the spin has developed small (average) y and z components, $\langle s_y \rangle_T$ and $\langle s_z \rangle_T$.

where $\omega_L = gqB_0/2m$. (Here g is the gyromagnetic ratio and q the charge of the particle.) The reflection time τ_{yR}^L is similarly defined with $\langle s_y \rangle_R$ replacing $\langle s_y \rangle_T$.

Rybachenko (1966) derived formulas for τ_{yT}^L and τ_{yR}^L for the general case envisioned in Fig. 2, with the magnetic field spanning the interval (x_1,x_2) and with $x_1<b$, $x_2>a$. In our notation, they read (Falck and Hauge, 1988),

$$\tau_{yT}^L(x_1,x_2;k) = \frac{1}{v}(x_2-x_1+\alpha') + \frac{\sqrt{R}}{2kv}[\sin(\beta-2kx_1)-\sin(2\alpha-\beta+2kx_2)] , \quad (2.13)$$

$$\tau_{yR}^L(x_1,x_2;k) = \frac{1}{v}(-2x_1+\beta') + \frac{\sqrt{R}}{2kv}[\sin(\beta-2kx_1)-\sin(2\alpha-\beta+2kx_2)] + \frac{1}{2kv\sqrt{R}}[\sin(\beta-2kx_1)+\sin(2\alpha-\beta+2kx_2)] . \quad (2.14)$$

In both cases, the Larmor times equal the phase times plus terms oscillating with kx_1 or kx_2. The amplitudes of the oscillating terms become large for small energies. For a rectangular barrier, $\tau_D(d)=\tau_{yT}^L(d)=\tau_{yR}^L(d)$ (see Appendix A), and consequently, the qualitative difference between the extrapolated phase times and τ_D carries over to the Larmor times τ_{yT}^L and τ_{yR}^L.

It is not only the spin components in the plane orthogonal to the field that are affected (see Fig. 2). The spin, originally polarized in the x direction, has a probability of $\frac{1}{2}$ of being spin-up or spin-down with respect to the field. As pointed out by Büttiker (1983), the spin-up component will be preferentially transmitted [except under those special circumstances when $dT(k)/dk<0$]. As a result, $\langle s_z \rangle_T = O(B_0) = O(\omega_L)$. In fact, with opaque barriers, $\langle s_z \rangle_T \gg \langle s_y \rangle_T$. This led Büttiker to introduce the Larmor "time"

$$\tau_{zT}^L(x_1,x_2;k) = \lim_{\omega_L \to 0} \langle s_z \rangle_T / (\tfrac{1}{2}\hbar\omega_L) , \quad (2.15)$$

with a corresponding equation for τ_{zR}^L. The basis for an interpretation of these objects as times intrinsically characteristic of the tunneling process is not clear. Nevertheless, let us note that for an opaque rectangular barrier, one has

$$\tau_{zT}^L(d;k) \simeq \frac{md}{\hbar\kappa} , \quad (2.16)$$

different from both the extrapolated phase time (2.9) and the dwell time $\tau_D=\tau_{yT}^L$ of (2.11). Büttiker (1983) finally introduced a third set of Larmor "times," which we shall call the Büttiker-Landauer time[6] τ^{BL}. For transmission, it is defined as

$$(\tau_T^{BL})^2 = (\tau_{yT}^L)^2 + (\tau_{zT}^L)^2 , \quad (2.17)$$

and similarly for τ_R^{BL}. Since, with opaque barriers, $\langle s_z \rangle_T \gg \langle s_y \rangle_T$,

$$\tau_T^{BL} \simeq \frac{md}{\hbar\kappa} . \quad (2.18)$$

[5]Since only terms linear in ω_L contribute to the Larmor times, orbital effects of $O(\omega_L^2)$, which would render a one-dimensional treatment impossible, can be ignored.

[6]Büttiker (1983) used the term τ_x^L, rather than τ^{BL}. This notation tends to confuse the issues involved [see Collins *et al.* (1987), Büttiker and Landauer (1988), Falck and Hauge (1988), and Lowe and Collins (1988)].

F. A complex time

In a recent elegant paper, Sokolovski and Baskin (1987) proposed a formal generalization of the classical time concept to the quantum domain and explored its consequences. For one-dimensional motion, their expression for the classical time spent in a region Ω reads

$$\tau_{\rm cl}^{\Omega}[x(t)]=\int_{t_i}^{t_f}dt\int_{\Omega}dx\ \delta(x-x(t))\ , \tag{2.19}$$

where $x(t)$ is the classical path from the initial point $x_i(t_i)$ to the final one $x_f(t_f)$. A natural quantum generalization of (2.19) is the path-integral average

$$\tau^{\Omega}(x_i,t_i;x_f,t_f)=\langle\tau_{\rm cl}^{\Omega}[x(\ )]\rangle_{\rm paths}\ , \tag{2.20}$$

in which $x(\ )$ is an arbitrary path between the given end points. Sokolovski and Baskin showed that, in general, τ^{Ω} is complex. For the scattering process described in Fig. 1 and Eq. (2.1), it is given by

$$\tau_T^{\Omega}=i\hbar\int_{\Omega}dx\frac{\delta\ln A}{\delta V(x)},\quad \tau_R^{\Omega}=i\hbar\int_{\Omega}dx\frac{\delta\ln B}{\delta V(x)}\ . \tag{2.21}$$

Here $A=\sqrt{T}e^{i\alpha}$ and $B=\sqrt{R}e^{i\beta}$ are the scattering amplitudes for transmission and reflection, respectively; Ω can be identified with the interval (b,a) or, alternatively, the interval (x_1,x_2); and $\delta/\delta V(x)$ is a functional derivative. The authors found a close connection between τ^{Ω} and the local Larmor times,

$$\mathrm{Re}\tau_T^{\Omega}=\tau_{yT}^{L},\quad \mathrm{Im}\tau_T^{\Omega}=-\tau_{zT}^{L}\ . \tag{2.22}$$

A corresponding connection was found for τ_R^{Ω}. A complex Larmor time equivalent to τ_T^{Ω} (apart from an arbitrarily chosen sign), was introduced by Leavens and Aers (1987a; see also Leavens and Aers, 1988).

As stressed by Sokolovski and Baskin (1987), there is nothing unique about their quantum generalization (2.20). The requirement that τ^{Ω} reduce to $\tau_{\rm cl}^{\Omega}$ in the classical limit is met by any number of alternatives to (2.20). One trivial example: replace the right-hand side of (2.20) by its real part. This would identify τ^{Ω} with τ_y^L. A second example: replace the right-hand side of (2.20) by its absolute value. This would amount to an identification of τ^{Ω} with $\tau^{\rm BL}$.

Elegant as it is, the Sokolovski and Baskin procedure remains purely formal. It can only be given physical content through connections to clearly defined model situations. The relations (2.22) to the local Larmor times provide one example of this. We shall give another example below.

A similar, but not identical, complex "time" had been introduced previously by Pollak and Miller (1984) and Pollak (1985) and associated with the kinetics of chemical reactions. For a discussion of the relation between these complex times, see Leavens and Aers (1987a).

There is no doubt that complex quantities with the dimension of time can be useful theoretical objects. For an example, see Sokolovski and Hänggi (1988), where adiabatic criteria are discussed. Nevertheless, common sense dictates that to the question of the duration of a tunneling process, the answer, if it exists at all, must be a *real* time.

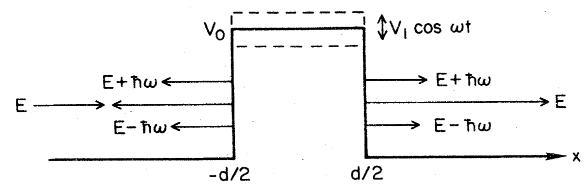

FIG. 3. Rectangular oscillating barrier. In the barrier region $(-d/2,d/2)$, the scattered particles can absorb or emit modulation quanta $\hbar\omega$, resulting in sidebands with energies $E+n\hbar\omega$. The first sidebands are indicated.

G. The Büttiker-Landauer time

So far, only time-independent barriers have been considered. However, the papers that stimulated much of the recent interest in tunneling times were those of Büttiker and Landauer (1982, 1985, 1986), in which they studied tunneling through a rectangular barrier with a small oscillating component added to the height (see Fig. 3),

$$V(t)=V_0+V_1\cos\omega t\ . \tag{2.23}$$

The incident particles, with energy E, can absorb or emit modulation quanta $\hbar\omega$ during the tunneling process, leading to the appearance of sidebands with energies $E+n\hbar\omega$. To first order in V_1, only the neighboring sidebands with energies $E\pm\hbar\omega$ appear. Büttiker and Landauer showed that for an opaque barrier, with frequencies not too high ($\hbar\omega$ small with respect to E and V_0-E), the relative sideband intensities are

$$I_{\pm1}^{T}(\omega)\equiv|A_{\pm1}(\omega)/A_0|^2$$
$$=\left[\frac{V_1}{2\hbar\omega}\right]^2\left[\exp\left[\pm\omega\frac{md}{\hbar\kappa}\right]-1\right]^2, \tag{2.24}$$

where $A_{\pm1}$ are the sideband transmission amplitudes, and A_0 is the transmission amplitude of the unperturbed problem.[7]

Clearly, $\hbar\kappa/md$ is the characteristic frequency separating the low- and high-frequency domains in (2.24), and Büttiker and Landauer identified $md/\hbar\kappa$ with the traversal time for tunneling. For opaque barriers, $\tau_T^{\rm BL}$ of (2.18) has this form. More generally, on the basis of the $\omega\to0$ limit of $I_{\pm l}^{T}(\omega)$ for an arbitrary rectangular barrier,

[7]The *technical* critique, voiced by Collins *et al.* (1987), of the work by Büttiker (1983) and Büttiker and Landauer is unfounded. See the comment by Büttiker and Landauer (1988) and the reply by Lowe and Collins (1988). See also Leavens and Aers (1988) and Støvneng and Hauge (1989). The *interpretational* controversy is the subject of this review.

Büttiker and Landauer identified their tunneling time with τ_T^{BL}, as defined by (2.17).

Elsewhere (Støvneng and Hauge, 1989), we have generalized this model to arbitrary $V_0(x)$. We find a simple, general connection between the $\omega \to 0$ limit of the relative sideband amplitudes $A_{\pm 1}$ and $B_{\pm 1}$, and Sokolovski and Baskin's complex "time" τ^Ω, namely,

$$A_{\pm 1}(0)/A_0 = -i\frac{V_1}{2\hbar}\tau_T^\Omega, \quad B_{\pm 1}(0)/B_0 = -i\frac{V_1}{2\hbar}\tau_R^\Omega \; . \tag{2.25}$$

These connections provide an alternate interpretation of τ^Ω in terms of a concrete physical model. Clearly, the relative sideband intensities are, quite generally, given by $|\tau^\Omega|$, i.e., by τ^{BL}. Whether this fact warrants an interpretation of τ^{BL} as the duration of the tunneling process is a different matter. We shall return to this point in Sec. V.

III. SOME EXACT RESULTS

In Sec. II several sharply contrasting results for an opaque barrier were quoted: the extrapolated phase time $\Delta\tau_T^\varphi$ of Eq. (2.9); the dwell time τ_D of (2.11), which in this case equals the local Larmor time τ_{yT}^L; and the Büttiker-Landauer time τ_T^{BL}, essentially given by τ_{zT}^L of Eq. (2.17). Clearly, these times describe, at best, different aspects of the tunneling process. Before attempting an interpretation that brings the various proposals into perspective, we shall, in this section, list some exact results that any such interpretation must respect.

A. The dwell time

In Hauge *et al.* (1987), erroneous statements were made on the status of the dwell time. The source of the confusion will be made clear in Sec. IV.B. As pointed out by Leavens and Aers (1989a), the methods of Smith (1960) and Hauge *et al.* (1987) can be used to prove that the dwell time (2.10), for arbitrary x_1 and x_2, is precisely what it claims to be: the average time spent by particles on the interval (x_1,x_2), when no attempt is made to distinguish between scattering channels. Aside from standard technicalities, the proof by Leavens and Aers contains one conceptual step that can be considered open to challenge. For this reason we present their arguments in Appendix B. We stress, however, that we are in complete agreement with Leavens and Aers on this point and shall henceforth treat the dwell time (2.10) as an exact statement of the time spent on the interval (x_1,x_2), averaged over all incoming particles.

The interval (x_1,x_2) may be large or small, it may include or exclude the barrier, or it may be an infinitesimal one deep inside the barrier, as it is for the cases studied by Kotler and Nitzan (1988) and Leavens and Aers (1988, 1989b). This versatility makes the dwell time a useful tool for testing some of the general claims made in this field. If the dwell time, in addition, could be distributed over scattering channels, there would not be much room for discussion.

B. Mutually exclusive events

We consider the status of the dwell time as well established. However, it is not *a priori* clear that the total average represented by τ_D can be distributed over the different scattering channels in a unique manner. This might be possible quite generally, or only under restrictive conditions such as those given in Sec. IV.B. In any case, a necessary (but not sufficient) requirement for meaningful general expressions for τ_R and τ_T is

$$\tau_D = R\tau_R + T\tau_T \; . \tag{3.1}$$

This is a direct consequence of the fact that an incident particle, in the one-dimensional tunneling configuration of Fig. 1, ends up as either reflected or transmitted. These mutually exclusive events, in the sense of Feynman and Hibbs (1965), exhaust all possibilities. There is no obvious candidate for a fundamental relation like (3.1) in the context of tunneling out of a localized state. The problems of defining tunneling times in the context of scattering configurations and in the context of escape from localized states are in this sense qualitatively different.

We have deliberately left vague the specification of the various times in Eq. (3.1), but will return to specific cases below. However, behind the statement (3.1) lies the premise that all three times are related to the same well-defined physical situation. Section IV.B will show that this proviso is not as trivial as it seems.

C. Three identities

Sokolovski and Baskin (1987) proved the following identity connecting the dwell time τ_D and the complex times τ^Ω:

$$\tau_D(x_1,x_2;k) = R(k)\tau_R^\Omega(x_1,x_2;k) + T(k)\tau_T^\Omega(x_1,x_2;k) \; . \tag{3.2}$$

In our use of Eq. (3.2), we shall assume that the interval $(x_1,x_2)=\Omega$ either covers the barrier region (b,a), i.e., $x_1<b$, $x_2>a$, or does not overlap with it. Taking the real and imaginary parts of (3.2), using (2.17), one finds

$$\tau_D(x_1,x_2;k) = R(k)\tau_{yR}^L(x_1,x_2;k) + T(k)\tau_{yT}^L(x_1,x_2;k) \; , \tag{3.3}$$

$$0 = R(k)\tau_{zR}^L(x_1,x_2;k) + T(k)\tau_{zT}^L(x_1,x_2;k) \; . \tag{3.4}$$

From the definition (2.15) of τ_z^L one recognizes (3.4) as an equation of conservation of angular momentum. This was already pointed out by Büttiker (1983). The identity (3.3) was derived independently by Falck and Hauge (1988).

A third identity, closely related to one found by Smith (1960), is (Hauge *et al.*, 1987)

$$\tau_D(x_1,x_2;k)=R(k)\tau_R^{\varphi}(x_1,x_2;k)+T(k)\tau_T^{\varphi}(x_1,x_2;k)+\frac{\sqrt{R}}{kv}\sin(\beta-2kx_1)\ . \tag{3.5}$$

Identities (3.3) and (3.5) are straightforward consequences of the Schrödinger equation. Their interpretation is less trivial, however. We come back to this in Secs. IV and VI.

D. Symmetry relations

For arbitrary symmetric potentials $V(-x)=V(x)$, it can be shown that (Falck and Hauge, 1988)

$$\tau_R^{\varphi}(-x,x;k)=\tau_T^{\varphi}(-x,x;k)\ . \tag{3.6}$$

Similarly, one has

$$\tau_{yR}^{L}(-x,x;k)=\tau_{yT}^{L}(-x,x;k)=\tau_D(-x,x;k)\ , \tag{3.7}$$

where the last equality follows from (3.3). For the special case of a rectangular barrier, Eq. (3.7) was found first by Büttiker (1983). As a contrast, Leavens and Aers (1987a) showed, by an explicit example, that Eq. (3.7) breaks down for *asymmetric* potentials.

IV. THE STATUS OF THE PHASE TIMES

A. Asymptotic versus local times

One source of confusion in this field has been the lack of distinction between times, such as the dwell time $\tau_D(x_1,x_2;k)$, that can be defined locally (for arbitrary x_1 and x_2), and times that are essentially asymptotic in character. The phase times $\tau_R^{\varphi}(x_1,x_2;k)$ and $\tau_T^{\varphi}(x_1,x_2;k)$ are in the latter category. They are derived (Barker, 1986; Hauge *et al.*, 1987; Teranishi *et al.*, 1987; Falck and Hauge, 1988; Jaworski and Wardlaw, 1988a) as asymptotic characteristics for *completed* scattering events involving wave packets narrow (width σ) in k space. Such packets are necessarily wide (width $\gtrsim\sigma^{-1}$) in x space, and the asymptotic phase times, consequently, can only be derived for $b-x_1\gg\sigma^{-1}$ and $x_2-a\gg\sigma^{-1}$. Typical results from numerical experiments (with a fairly large σ) are sketched in Fig. 4. Since in Eqs. (2.5) and (2.6), the phase times are linear in x_1 and x_2, it is tempting to linearly extrapolate them back to the scattering region (b,a), as was done in Eqs. (2.7) and (2.8). This formal device does not alter the status of the phase times: they remain asymptotic results.

B. Consistency with Eq. (3.1)

On the basis of the apparent contradiction between (3.1) and (3.5), Hauge *et al.* (1987) erroneously concluded that the standard interpretation of the dwell time must be wrong. Leavens and Aers (1989a) subsequently proved that the dwell time does indeed give the time, averaged over all scattering channels, spent in any correspondingly given region of space. They also pointed out the error in the argument of Hauge *et al.*: Those authors had attached *physical* significance to the extrapolation from (2.5) and (2.6) to (2.7) and (2.8) by assuming the motion to be that of free particles everywhere outside the barrier region. This is correct on the far side of the barrier. The dwell time calculated for an interval beyond $x=a$ confirms (see Sec. VI) that the particle moves with constant velocity $v(k)$. During the *approach* to the barrier, however, the incoming part of the wave function interferes with the reflected part, and free particle motion cannot be assumed all the way up to the barrier.

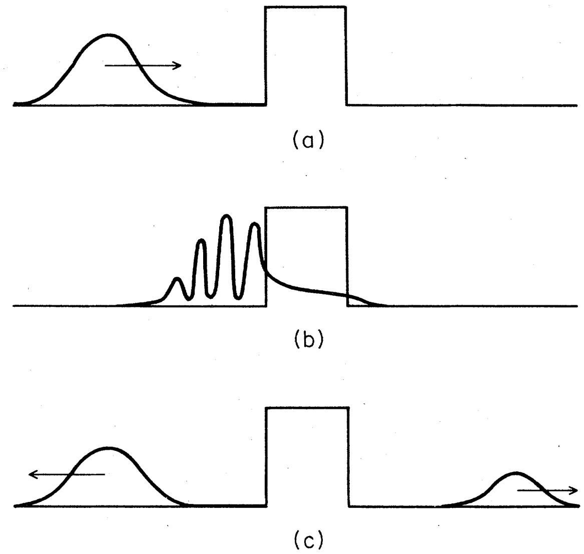

FIG. 4. Sketch of typical results for $|\psi(x,t)|^2$ from numerical solutions of the scattering problem: (a) the initial wave packet moving towards the barrier, (b) strong interference effects as the packet moves into the barrier region, (c) the final state, transmitted and reflected wave packets leave the barrier region.

Because of the way the phase times are constructed, with completed scattering events within (x_1,x_2), one can use free motion for both incoming and reflected particles when $x<x_1$. Furthermore, averaging the identity (3.5) over an initial packet of the small k width, σ, one finds, when averaged quantities are distinguished by a bar,

$$\bar{\tau}_D(x_1,x_2;k)=\bar{R}(k)\bar{\tau}_R^{\varphi}(x_1,x_2;k)+\bar{T}(k)\bar{\tau}_T^{\varphi}(x_1,x_2;k)+\left[\frac{\sqrt{R}}{kv}\right]\frac{1}{\sigma}\int_{\sigma}dk\,\sin(\beta-2kx_1)+O(\sigma)\ . \tag{4.1}$$

Clearly, the direct average of (3.5) gives, e.g., $\overline{R\tau_R^{\varphi}}$ rather than $\bar{R}\bar{\tau}_R^{\varphi}$, which appears in Eq. (4.1). However, the difference between these expressions is easily shown to be of $O(\sigma)$. Since $|x_1| \gg \sigma^{-1}$, the last integral in (4.1) is negligible. In fact, the definition of the phase times as pertaining to completed collisions is tantamount to a *requirement* that the integral in (4.1) be negligible. Thus for the phase times one has[8]

$$\bar{\tau}_D(x_1,x_2;k)=\bar{R}(k)\bar{\tau}_R^{\varphi}(x_1,x_2;k)+\bar{T}(k)\bar{\tau}_T^{\varphi}(x_1,x_2;k)\ , \tag{4.2}$$

in agreement with Eq. (3.1). It is this type of asymptotic relation that is, under precisely specified conditions, proven in the mathematical literature on scattering in three dimensions (see Martin, 1981). As will become clear in the following subsection, the overbar is essential on $\bar{\tau}_D$. For the remaining quantities in (4.2), one would make asymptotically negligible errors, of $O(\sigma)$, by removing the overbars.

We have thus shown that when the asymptotic nature of the phase times is taken into account and the corresponding integration over narrow packet has been carried out, the apparent conflict between Eqs. (3.1) and (3.5) no longer exists. On the other hand, the conflict between (3.1) and (3.5) is real, when used with $x_1=b$ and $x_2=a$ and with the extrapolated phase times (2.7) and (2.8). In conclusion, Eq. (3.5) can be used to attack *neither* the status of the dwell time *nor* that of the asymptotic phase times. It *can* be used (Leavens and Aers, 1989a) to show that the extrapolated phase times do not represent the time spent in the barrier region.

C. Physical interpretation

We shall now carry the argument of Leavens and Aers (1989a) one step further and cast the identity (3.5) in a form that suggests an appealing picture for the physical content of phase times.

The dwell time on the interval $(-L,x_1)$, to the left of the barrier, clearly diverges as $L\to\infty$. Subtraction of the time due to free motion of incoming and reflected particles leaves a finite (positive or negative) *excess* dwell time[9] $\Delta\tau_D(x<x_1;k)$. At fixed k it can be calculated explicitly:

$$\begin{aligned}\Delta\tau_D(x<x_1;k)&\\ &=\lim_{L\to\infty}\frac{1}{v}\int_{-L}^{x_1}dx\left[|e^{ikx}+\sqrt{R}\,e^{i\beta}e^{-ikx}|^2-(1+R)\right]\\ &=-\frac{\sqrt{R}}{kv}\sin(\beta-2kx_1)\ ,\end{aligned} \tag{4.3}$$

where we, in the limit $L\to\infty$, have neglected[10] the term $\sim\sin(\beta+2kL)$. Equation (4.3) shows that the oscillating term in (3.5) is a dwell time associated with interference effects in front of the barrier. The phase times do not oscillate with k. Subtraction of pieces corresponding to free-particle motion on the intervals (x_1,b) and (a,x_2) amounts to extrapolation of the phase times back to the barrier region. One easily finds, using (4.3), that Eq. (3.5) can be written

$$\begin{aligned}\Delta\tau_D(x<b;k)&+\tau_D(b,a;k)\\ &=R(k)\Delta\tau_R^{\varphi}(b,a;k)+T(k)\Delta\tau_T^{\varphi}(b,a;k)\ ,\end{aligned} \tag{4.4}$$

where the linearly extrapolated phase times are those of Eqs. (2.7) and (2.8).

Equation (4.4) suggests a simple physical interpretation of these phase times. The second term on the left-hand side gives the time spent in the barrier itself. The first term represents the delay (positive or negative) due to interference between the incoming and reflected waves. Consistent with this interpretation is the fact that no interference term comes from the far side of the barrier. However, one should realize that no distinction between reflected and transmitted particles was made in the above argument, as it pertains to the left-hand side of (4.4). In general, no such distinction *can* be made in the context of the dwell time. This important reservation mars an otherwise appealing interpretation.

D. Two clear-cut examples

There are two simple but illuminating cases for which the above reservation becomes irrelevant. The first is that of an infinitely thick barrier. In that case, when $E<V_0$, all particles are reflected and no ambiguity with respect to scattering channels exists. Let, for simplicity, the barrier be rectangular, spanning the interval $(0,d)$ with $d\to\infty$, and with height V_0. For $E<V_0$, one clearly has that $\exp(-2\kappa d)\to 0$; the results for an opaque barrier apply. Using the relations (2.9), (2.11), (4.3), with $x_1=0$, and Eq. (A3), one easily shows that

$$\begin{aligned}&\Delta\tau_R^{\varphi}=\frac{2}{v\kappa},\quad \tau_D(x>0)=\frac{E}{V_0}\Delta\tau_R^{\varphi}\ ,\\ &\Delta\tau_D(x<0)=\frac{V_0-E}{V_0}\Delta\tau_R^{\varphi}\ .\end{aligned} \tag{4.5}$$

These simple results shed light on some of the controversial issues regarding tunneling times. The apparent contradiction between the extrapolated phase time, increas-

[8]It is interesting to note that Smith (1960), for the very similar case considered by him, dispensed with the oscillating term by arbitrarily averaging over x_1. In our opinion, only a wave-packet argument can consistently lead to Eq. (4.2).

[9]Note the close correspondence between the definition of the excess dwell time and that of the linearly extrapolated phase times.

[10]This can be justified formally when one considers the stationary case as the $\sigma\to 0$ limit of an approach based on wave packets and takes the $L\to\infty$ limit first. In the mathematical literature, the corresponding limit is "weak" and appeal is made to the Riemann-Lebesgue lemma.

ing with $v^{-1} \sim k^{-1}$ for small energies, and the dwell time in the barrier, decreasing like $E/v \sim k$, is resolved. The extrapolated phase time, extracted from a description of the completed scattering event, consists of two contributions: the time spent in the barrier region, $\tau_D(x>0)$, and the delay in front of the barrier, $\Delta\tau_D(x<0)$, due to self-interference. For small energies, the latter is clearly the dominant one.

It is worth noting that $\tau_D(x>0)$, according to the definition (2.10), essentially measures the particle density in the barrier. Since all particles are reflected here, a given one may or may not enter the barrier and contribute to $\tau_D(x>0)$. How to proceed from (4.5) to a statement about typical velocities of those particles that do enter the barrier is therefore not clear.

The second example is that of a δ-function barrier. Nontrivial results for time-dependent versions of this model have been established recently by Teranishi *et al.* (1987), Elberfeld and Kleber (1988), and by Scheitler and Kleber (1988). In this context, we shall need only simple results for the stationary case. In the equations for the general rectangular barrier in Appendix A, let $d \to 0$ and $V_0 \to \infty$ in such a way that $V_0 d = (\hbar^2 k_0^2/2m)d \equiv \hbar c_0$ is kept constant. This δ-function limit yields, with $v(k)=\hbar k/m$,

$$T(k)=\frac{1}{1+(c_0/v)^2}\ , \tag{4.6}$$

$$\Delta\tau_T^{\varphi}=\Delta\tau_R^{\varphi}=\frac{\hbar c_0/mv^3}{1+(c_0/v)^2}\ . \tag{4.7}$$

First, the transmission probability (4.6) is less than unity and depends on the strength of the δ function. But more important is the fact that the extrapolated phase times (4.7) can be identified unambiguously as self-interference delays here, since no time is spent in the barrier region in this special case. Such delays were observed by Elberfeld and Kleber in the case of a sharply defined k.[11] For small energies, Eq. (4.7) gives $\Delta\tau^{\varphi} \simeq (\hbar/mc_0)v^{-1} = (2/k_0^2 d)v^{-1}$, which agrees with (4.5), when $k_0^2 d$ takes the role of κ.

One interesting consequence of (4.7), in conflict with a naive classical picture, should be noted. The delay due to self-interference is the same for reflected particles, which pass through the interference region twice, and for transmitted particles passing only once. One might be tempted to conjecture that this is true for symmetric barriers in general. Such a conjecture, combined with Eqs. (3.6) and (4.4), would imply that both reflected and transmitted particles spend the dwell time in the barrier region. However, the precise equality of these two delays most likely reflects the special nature of the δ function. Consequently, we can see no reason to trust the above conjecture.

[11]That the interference effect could not be seen with wave packets of moderate extension in x space is not surprising, in the light of the discussion in Sec. IV.B.

E. Scattering of packets with finite width

In this subsection we shall discuss the effects of a small, but finite, k width on the scattering process. Our aim is to further clarify the status, not apparent from the derivation in Sec. II.B, of the asymptotic phase times and their extrapolated counterparts. The essential content of our remarks is contained in the thorough discussion by Jaworski and Wardlaw (1988a). The simplified style will be closer to that of Hauge *et al.* (1987).

Two technically very different derivations (Falck and Hauge, 1988; Jaworski and Wardlaw, 1988a) of the asymptotic phase times for wave packets narrow in k space lead to essentially the same[12] results,

$$\tau_T^{\varphi}(x_1,x_2;k)=\overline{T(k)v(k)^{-1}[x_2-x_1+\alpha'(k)]}/\overline{T}(k)\ , \tag{4.8}$$

$$\tau_R^{\varphi}(x_1,x_2;k)=\overline{R(k)v(k)^{-1}[-2x_1+\beta'(k)]}/\overline{R}(k)\ . \tag{4.9}$$

In (4.8) and (4.9), the overbars denote an average with respect to the initial wave packet, i.e., with respect to the distribution $P_I(k)=|\varphi(k)|^2/2\pi$, where $\varphi(k)=\int dx\, e^{-ikx}\psi(x,t=0)$ is the Fourier transform of the initial packet. Both derivations are similar in one crucial respect: They invoke *completed* scattering events. Consequently, $b-x_1$ and x_2-a must both be sufficiently large to accommodate wave packets of spatial extent $\sim\sigma^{-1}$. Formally, the results (4.8) and (4.9) apply to any initial packet, wide or narrow in k. However, to make sure that one does not average over physically quite

[12]The expressions found by Jaworski and Wardlaw (1988a) focus on transmission and reflection *delays,* with free-particle motion of the initial wave packet used as a reference. The initial packet is prepared far from the barrier, and the arrival times for transmitted and reflected particles are measured by idealized detectors at x_2 and x_1, respectively. Their formulas have a quite different appearance, but they can be shown to correspond closely to Eqs. (4.8) and (4.9). The minor differences that do exist can be traced to the fact that the Gedanken experiments basic to the two calculations, while similar, are not identical. The results of Hauge *et al.* (1987) and Teranishi *et al.* (1987) are (before extrapolation) similar to (4.8) and (4.9), but only asymptotically equivalent, as $\sigma\to 0$. See also Jaworski and Wardlaw (1988b). Barker's (1986) derivation dispenses with the interference terms in a manner equivalent to an asymptotic argument, plus subsequent linear extrapolation. It seems fair to say that all these derivations of the phase times yield asymptotically equivalent results as $\sigma\to 0$.

different processes,[13] and since this review focuses on scattering at (essentially) given energy, we require that σ be small in the sense that

$$\sigma/k_I \ll 1, \quad \sigma T'(k_I)/T(k_I) \ll 1 , \tag{4.10}$$

with $k_I = \bar{k}$.

One can rewrite (4.8) in terms of the probability distribution associated with the *transmitted* packet, $P_T(k) = P_I(k)T(k)/\bar{T}(k)$,

$$\bar{\tau}_T^{\varphi}(x_1,x_2;k) = \int dk\, P_T(k) \frac{m}{\hbar k}[x_2 - x_1 + \alpha'(k)] . \tag{4.11}$$

With $P_I(k)$, and thus $P_T(k)$, sharply peaked in the sense of (4.10), one can expand (4.11) to get

$$\bar{\tau}_T^{\varphi}(x_1,x_2;k) = \tau_T^{\varphi}(x_1,x_2;k_T)\left[1 + \frac{\sigma^2}{k_T^2}\right] + \frac{m}{\hbar k_T}\left[-\frac{\alpha''(k_T)}{k_T} + \tfrac{1}{2}\alpha'''(k_T)\right]\sigma^2 + O(\sigma^3) . \tag{4.12}$$

With the σ^2's defined as the corresponding mean-square deviations, we can interchangeably use the widths of the initial, $\sigma_I \equiv \sigma$, and transmitted, σ_T, packets, since their difference is of $O(\sigma^3)$. The two averages are related by

$$k_T = k_I\left[1 + \frac{T'(k_I)\sigma^2}{k_I T(k_I)}\right] + O(\sigma^3) . \tag{4.13}$$

Similar equations can be written for the reflected packet. For example, the average k is given by

$$k_R = k_I\left[1 - \frac{T'(k_I)\sigma^2}{k_I R(k_I)}\right] + O(\sigma^3) , \tag{4.14}$$

where we used $R' = -T'$. [As a check, to $O(\sigma^2)$, (4.13) and (4.14) give $k_I = Tk_T + Rk_R$.]

All the corrections to τ_T^{φ} in Eq. (4.12) are formally of $O(\sigma^2)$. This is clearly also the actual order of the last two corrections in (4.12), since their coefficients are of $O(1)$. However, the correction $\tau_T^{\varphi} k_T^{-2}\sigma^2$ is larger, due to the fact that the derivations of (4.8) invoke completed scattering events and therefore require $x_2 - x_1$, and thus τ_T^{φ}, to be large, at least of $O(\sigma^{-1})$. The leading correction to τ_T^{φ} is therefore of $O(|x_2 - x_1|\sigma^2) = O(\sigma)$. It vanishes linearly[14] as the k width of the wave packet tends to zero.

In the strict limit $\sigma \to 0$, as the corrections to τ_T^{φ} and τ_R^{φ} vanish, the asymptotic phase times tend to infinity. However, their linearly extrapolated counterparts tend to the finite limits,[15] $\Delta\tau_T^{\varphi}$ and $\Delta\tau_R^{\varphi}$. This should be clear from Eq. (4.4). The physical content of the above statements is this: When the wave packets stretch out in x space as $\sigma \to 0$, the self-interference delays contained in $\Delta\tau_T^{\varphi}$ and $\Delta\tau_R^{\varphi}$ do *not* grow beyond bounds, but tend to finite limits. It is worth repeating, however, that, except in special cases like the one discussed in Sec. IV.D, one cannot, in the asymptotic results $\Delta\tau_T^{\varphi}$ and $\Delta\tau_R^{\varphi}$, separate contributions from self-interference delays and time spent in the barrier region.

To give a feeling for some of the issues involved when the wave packets have finite width, Fig. 5 gives a schematic representation in the (t,x) plane of the transmission event. The barrier appears as the hatched region between $x = b$ and $x = a$. The initial packet starts at x_0 and moves, on the average,[16] along the solid line characterized by k_I. Its extrapolation into and beyond the barrier region is dotted. The transmitted particle emerges along the dot-dashed line of slope k_T, which is extrapolated backwards as a dotted one. Figure 5 immediately shows that the delay time, defined with free-particle motion as a reference, is a cumbersome concept here. It will depend on x_2, due to the difference, $k_T - k_I = O(\sigma^2)$, in slopes. The total transmission time, given by Eq. (4.8), is free of this difficulty.

Inspection of Eq. (4.8) for the transmission time shows, with reference to Fig. 5, that the transmitted particle, prior to the collision, should be represented by the dashed line, of slope k_T, and not by the solid line. (The time $\bar{\tau}_T^{\varphi}$ therefore roughly equals $t_2 - t_1$.) As a result of this, $\bar{\tau}_T^{\varphi}$ is invariant with respect to a displacement[17] of the barrier. This important subtlety was pointed out by Jaworski and Wardlaw (1988a). It can be interpreted[18] as follows: The fact that the particle is transmitted, not reflected, amounts to a *measurement*. This measurement adds to our information about the particle. Consequently one should, after the fact, think of the transmitted parti-

[13]An example of how different physical mechanisms can conspire to give startling and potentially confusing results is found in Fig. 3 of Leavens and Aers (1989a). By respecting both inequalities of (4.10), one avoids such pitfalls.

[14]To make sure that $x_2 - x_1 \gg \sigma^{-1}$ as $\sigma \to 0$, one could, of course, let $x_2 - x_1 = c\sigma^{-(1+s)}$, where c is a constant and $0 < s < 1$. This would make the leading correction in (4.12) of $O(|x_2 - x_1|\sigma^2) = O(\sigma^{1-s})$. It would still vanish with σ, albeit slower than linearly.

[15]Except, perhaps, in very special limiting cases [see Garcia-Calderon and Rubio (1989)].

[16]Since it is v^{-1}, not v, that appears under the integral sign in Eqs. (4.8) and (4.9), it would, strictly speaking, be more appropriate to base the discussion on k^{-1} rather than k. For small σ, the corresponding corrections would be the same as those in (4.13) and (4.14), except for a sign change. The message of Fig. 5 would remain the same.

[17]The displacement is arbitrary as long as the condition of a completed scattering event within (x_1, x_2) is not violated.

[18]In Jaworski and Wardlaw's view, the invariance inherent in Eq. (4.8) favors the statistical interpretation of quantum mechanics.

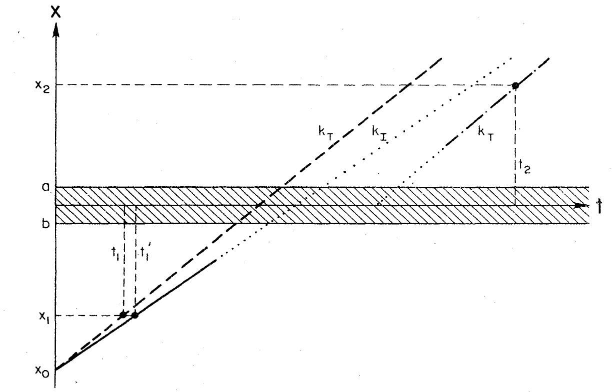

FIG. 5. Center-of-mass motion in the (t,x) plane. The hatched area between $x=b$ and $x=a$ represents the barrier region. The solid line emanating from $(0,x_0)$ represents the center-of-mass motion of the initial packet. Its extrapolation is dotted. The dot-dashed line on the far side of the barrier represents the center-of-mass motion of the transmitted packet, with a dotted backward extrapolation. The dashed line from $(0,x_0)$ gives the center-of-mass motion of the initial packet, filtered through the transmission probability $T(k)$. It is parallel to the dot-dashed line. The difference k_T-k_I is exaggerated for clarity.

cle as being represented by the initial packet *filtered*[19] through the transmission probability $T(k)$, even prior to the collision.

The center-of-mass clock, discussed by Hauge *et al.* (1987) and by Teranishi *et al.* (1987), takes the initial packet, i.e., the solid line in Fig. 5, as representing the transmitted particle prior to the collision. As a result $\bar{\tau}_T^\varphi$ is set roughly equal to t_2-t_1', rather than to t_2-t_1. As explained above, this is inconsistent with a correct interpretation of quantum mechanics (even though the numerical error vanishes in the $\sigma\rightarrow 0$ limit). The center-of-mass clock, however, *is* relevant for explaining the results of numerical "experiments," in which the act of observation has been trivialized: It amounts to nothing more than the reading of computer printouts.

All the points made above apply equally to the case of reflection. However, there is one additional point to be made: The reflection time $\bar{\tau}_R^\varphi$ of Eq. (4.9) depends on the position of the barrier. Clearly, if the barrier is moved down the line, it takes longer to get there and back.

Finally, Fig. 5 shows that the asymptotic phase times are sharp, with absolute corrections of $O(\sigma)$. The slopes of the various lines differ by $O(\sigma^2)$. Their lengths are of $O(\sigma^{-1})$. Absolute uncertainties are therefore of $O(\sigma)$.

F. Are the asymptotic times, in principle, sharp?

The considerations above showed that, theoretically, the asymptotic phase times are sharp within a margin of $O(\sigma)$. But is a correspondingly sharp *measurement* of the asymptotic times possible, even in principle? Localized ideal counters (Jaworski and Wardlaw, 1988a) measuring arrival times will (at least in a naive interpretation) produce a distribution of the asymptotic tunneling times reflecting the *spatial* extent of the wave packets. Consequently, they could yield a distribution of tunneling times of width $\sim\sigma^{-1}$. If no other experiments were conceivable, the status of asymptotic times would be doubtful. However, as shown in Falck and Hauge (1988; see Sec. VI below), the *asymptotic* Larmor clock yields Eq. (4.8) as well. A precise measurement of $\langle s_y \rangle$ for a transmitted spin-$\frac{1}{2}$ particle, initially polarized in the x direction, would therefore give the phase time. At first sight, since single measurements of s_y must yield $\hbar/2$ or $-\hbar/2$, this also seems to result in a wide distribution of times. One could avoid this by following the suggestion of Baz' (1966b) and letting the spin be classical. However, even for a quantum-mechanical spin-$\frac{1}{2}$ particle, the spin components are given by expressions of a form indicated by (2.12) and (6.1). That is, the uncertainty in the components, and therefore in the direction of the spin, is given by the uncertainty in k. Operationally, nothing prevents us from determining that direction (see Fig. 2) with an idealized Stern-Gerlach apparatus. Thus the width of the distribution[20] of phase times around their mean is proportional to σ, not to σ^{-1} [see also Sokolovski and Baskin (1987) on this point].

V. AN ALTERNATE VIEW OF THE OSCILLATING BARRIER

In a separate publication (Støvneng and Hauge, 1989), we have discussed in detail the oscillating-barrier problem introduced by Büttiker and Landauer (1982, 1985, 1986). Here we focus on those aspects that are important for the corresponding time interpretation.

The result (2.24) shows convincingly that, for an opaque barrier, the relative sideband intensities $I_{\pm 1}^T(\omega)$ are governed by a single characteristic frequency [at least over the range of frequencies in which (2.24) is valid]. In particular, the low-frequency limit gives

$$I_{\pm 1}^T(0)=\left[\frac{V_1\tau_T^{\mathrm{BL}}}{2\hbar}\right]^2 , \tag{5.1}$$

[19]The fact that both the solid and the dashed lines in Fig. 5 emanate from x_0 is therefore, in general, a minor oversimplification.

[20]The above arguments for a sharp distribution apply to the *asymptotic* phase times τ_T^φ and τ_R^φ. It is possible that a deeper understanding of the issue of *local* tunneling times can emerge from a study of their distribution, to the extent that this concept can be given a meaningful definition. Tantalizing results in this direction can be found in the works of Leavens (1988), Persson (1988), and Schulman and Ziolkowski (1989). However, since we find fault with all suggested procedures for calculating the *average* of local times, it seems premature to comment on their distribution.

which can be taken as a general definition of $\tau^{\rm BL}$ (Büttiker and Landauer, 1985), equivalent to that of (2.17). On the other hand, the relative sideband asymmetry reads

$$F(\omega)\equiv\frac{I^T_{+1}(\omega)-I^T_{-1}(\omega)}{I^T_{+1}(\omega)+I^T_{-1}(\omega)}=\tanh(\omega\tau_T^{\rm BL})\ . \qquad (5.2)$$

This result for opaque barriers leads naturally to the following definition of the characteristic frequency ω_c separating the high- from the low-frequency domain:[21]

$$|F(\omega_c)|\equiv\tanh 1=0.76\ldots, \qquad (5.3)$$

which yields $\omega_c=(\tau_T^{\rm BL})^{-1}$. This shows that, with opaque barriers, the sideband asymmetry is governed by the same quantity, $\tau_T^{\rm BL}$, that determines the $\omega\to 0$ limit. It is tempting, then, to identify $\tau_T^{\rm BL}$ as the duration of the tunneling process itself.

On closer examination, however, the rationale for this identification becomes less clear. Büttiker and Landauer (1985) derived exact expressions for the sideband amplitudes $A_{\pm1}(\omega)$ and $B_{\pm1}(\omega)$ to first order in V_1, and for general rectangular barriers. The full expressions are complicated and a detailed discussion of them difficult. For opaque barriers, $I^T_{\pm1}(\omega)$ simplifies to (2.24). In the δ-function limit, $d\to 0$, $V_0\to\infty$, $V_1\to\infty$, with $V_0 d=\hbar c_0$ and $V_1 d=\hbar c_1$ constants, the expressions again simplify. One finds

$$I^T_{\pm1}(\omega)=\left[\frac{c_1}{2(v^2+c_0^2)^{1/2}}\right]^2\left[1\pm\frac{\hbar\omega}{E+\frac{1}{2}mc_0^2}\right]^{-1}, \qquad (5.4)$$

with

$$\frac{c_1}{2(v^2+c_0^2)^{1/2}}=\frac{V_1}{2\hbar}\frac{d}{(v^2+c_0^2)^{1/2}}=\frac{V_1}{2\hbar}\tau_T^{\rm BL}\ . \qquad (5.5)$$

Comparison between (5.4) and (5.5) shows that, whereas the characteristic frequency ω_c separating the high- and low-frequency domains remains finite, the Büttiker-Landauer time $\tau_T^{\rm BL}$ vanishes as $d\to 0$. The two concepts are therefore qualitatively different, in spite of the fact that they coincide in the case of opaque barriers.[22] Equation (5.4) also shows that ω_c^{-1} cannot, in general, be interpreted as the duration of the tunneling process. Regardless of whether it is given by $\tau_T^{\rm BL}$, the duration of the tunneling process must vanish in this case. Note that the sign of the sideband asymmetry in (5.4) is the opposite of that in (2.24).

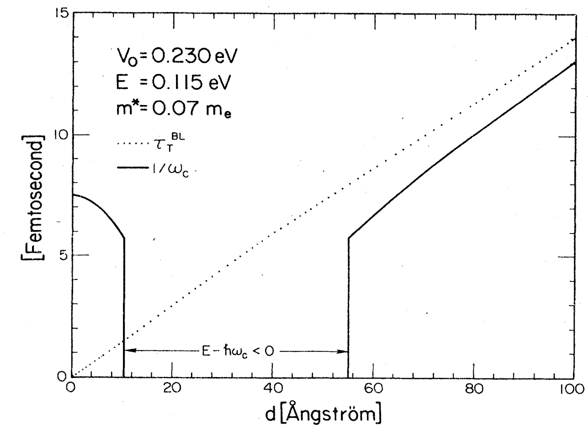

FIG. 6. Comparison between the inverse characteristic frequency ω_c^{-1} and the Büttiker-Landauer time for transmission, $\tau_T^{\rm BL}$, in a GaAs/AlGaAs/GaAs structure. We define ω_c by $|(I^T_{+1}-I^T_{-1})/(I^T_{+1}+I^T_{-1})|_{\omega=\omega_c}=\tanh 1=0.76\ldots$. For our choice of energy, $E=0.115$ eV, ω_c is not defined in the d interval (11 Å, 55 Å). For thick barriers, $\tau_T^{\rm BL}-\omega_c^{-1}=O(d^{-1})$. As $d\to\infty$, $\tau_T^{\rm BL}\simeq md/\hbar\kappa$. When $d\to 0$, on the other hand, $\tau_T^{\rm BL}\simeq md/\hbar k$. With the choice $E=V_0/2$, one has $k=\kappa$, and the two limiting slopes of $\tau_T^{\rm BL}(d)$ coincide. Even so, $\tau_T^{\rm BL}(d)$ is not a straight line.

One could object to our use of the δ-function barrier in the arguments above on the grounds that it is a nonphysical special case. We have therefore investigated barriers of increasing thickness d and with typical GaAs, AlGaAs parameter values, i.e., $m=0.07m_e$ (where m_e is the bare electron mass), and $V_0=0.23$ eV. For the somewhat arbitrary choice $E=V_0/2=0.115$ eV, $\tau_T^{\rm BL}$ was determined numerically from Eq. (A7) as a function of d. The exact expression for $A_{\pm1}(\omega)$ found by Büttiker and Landauer (1985) was used, together with the definition (5.3), to determine ω_c^{-1}, also as a function of d. The results are shown in Fig. 6. In a d interval (11 Å, 55 Å) no solution for ω_c exists. This can be understood simply. The δ-function result (5.4) indicates that thin barriers have a sideband asymmetry sign opposite of that for opaque ones. This implies that the ω dependence of the asymmetry is very weak for an intermediate range of d values. To reach the relative asymmetry $\tanh 1=0.76\ldots$, which defines ω_c, one needs to go to large ω. However, the lower sideband exists only as long as $\hbar\omega<E$ (Jauho and Jonson, 1989b). Definition (5.3) therefore does not yield an ω_c for barriers of intermediate thickness.[23] One could try to repair this by changing definition (5.3) somewhat, but the qualitative conclusion remains unchanged: Since the asymmetry is very weak for intermediate d, no ω_c can

[21]Clearly, nothing dramatic happens at precisely $\omega=\omega_c$. A number of alternate definitions yielding characteristic frequencies of the same order of magnitude are equally plausible.

[22]Our results at this point are consistent with the general ones by Sokolovski and Hänggi (1988).

[23]Since free-particle motion outside the barrier is assumed, realism dictates that the energy should not exceed the optical phonon threshold, $E\approx 0.04$ eV. At this threshold ω_c, as defined by Eq. (5.3), does not exist in the d interval (7 Å, 165 Å). This interval includes essentially all barriers of physical interest.

be meaningfully defined in that range. The identification of ω_c^{-1} with the duration of the tunneling process cannot be generally valid.

How, then, should the results for the opaque barrier be viewed? Büttiker and Landauer (1985) stress the energy sensitivity as a dominating feature of tunneling through such barriers. They provide the following appealing picture of the sideband asymmetry of Eq. (2.24). To leading order, one has for the transmission probability $T(E\pm\hbar\omega)\simeq T(E)\exp(\pm 2\omega md/\hbar\kappa)$. Assume now that the upper sideband is dominated by particles that, upon entering the barrier, immediately absorb a modulation quantum $\hbar\omega$ and, with this added energy, proceed to tunnel through the entire barrier. This picture is consistent with (2.24). The corresponding picture for the lower sideband would be one in which the modulation quantum is emitted as the particles leaves the barrier. Again, this is consistent with (2.24): The lower sideband is governed by $[\exp(-\omega md/\hbar\kappa)-1]^2\simeq 1$, i.e., by $T(E)$ rather than by $T(E-\hbar\omega)$.

The precise duration of the tunneling process never enters the above argument. The precise energy sensitivity of $T(E)$ does. One might be tempted to link the two by a relation of the type $\tau\Delta E\sim\hbar$. However, the example of the δ-function barrier and the results in Fig. 6 show that no straightforward general relation exists between the duration of the process and the characteristic frequency separating the high- and low-frequency domains. We propose instead that (2.24) should simply be accepted as an expression of the nonlinear response of the tunneling probability to the available energy quantum.

The above arguments alone do not exclude the possibility that the Büttiker-Landauer times τ_T^{BL} and τ_R^{BL} are generally valid expressions for the duration of tunneling processes. However, this is ruled out by their failure to obey relation (3.1).

Both the inverse characteristic frequency and the Büttiker-Landauer times fail to satisfy necessary conditions on general expressions for the duration of tunneling processes. That does not imply that they are uninteresting quantities. For example, since ω_c characterizes the coupling between the tunneling process and other degrees of freedom, it is, in contexts in which it can be meaningfully defined [see, for example, Bruinsma and Bak (1986)], expected to play a more important role than the intrinsic tunneling times.

VI. THE LARMOR TIMES

A. The asymptotic Larmor clock

The basis for the Larmor clock, as sketched in Sec. II.D, is the statement that inside a homogeneous magnetic field a spin will precess at a constant rate. Outside it will not precess at all. This is an elementary truth in classical physics. However, in quantum mechanics the situation is complicated by the fact that "inside/outside" statements, in the context of scattering states with fixed k, violate the uncertainty principle. The premise for the local Larmor clock, as introduced by Baz' (1966a), is therefore an unproven extension of classical ideas into the quantum realm.

On the other hand, the Larmor clock can also be based on wave packets (Falck and Hauge, 1988). That clock was shown to be reliable in an asymptotic sense, with a magnetic field covering an interval (x_1,x_2) sufficiently wide to accommodate completed scattering events. For the transmission time from x_1 to x_2, through a barrier with transmission probability $T(k)$, this asymptotic Larmor clock gives the result (4.8),

$$\tau_T^{\mathrm{AL}}=\overline{T(k)v(k)^{-1}[x_2-x_1+\alpha'(k)]}/\overline{T}(k)=\overline{\tau}_T^{\varphi}\ , \qquad (6.1)$$

with the analogous equation for the asymptotic reflection time τ_R^{AL} given by (4.9).

B. The local Larmor clock

Let us take a closer look at the local[24] Larmor times. There is no disagreement that a finite z component of transmitted or reflected spins is a result of the energy sensitivity of the tunneling probability, and that $\langle s_z\rangle_T \gg \langle s_y\rangle_T$ with opaque barriers. Nor is there disagreement that, for the case discussed in Sec V, the characteristic frequency is determined by $\tau_T^{\mathrm{BL}}\simeq\tau_{zT}^{L}$ when the barrier is opaque. In spite of this, no forceful argument has, in our opinion, been advanced for the general identification of τ_{zT}^{L} (or τ_{zR}^{L}) with the *intrinsic duration* of a tunneling process. On the contrary, for the case of the asymptotic Larmor clock (Falck and Hauge, 1988), it is easy to see that τ_{zT}^{L} is unrelated to the time aspects of the completed process. Furthermore, the τ_z^L's do not obey a relation of the form (3.1), but rather the conservation law (3.4). These facts do not exclude the possibility that in a carefully delimited set of circumstances τ_{zT}^{L} has the meaning of the duration of a tunneling process. So far, however, no argument exists that, with explicit limitations, shows this.

These remarks carry over to the Büttiker-Landauer times. In general, τ_T^{BL} and τ_R^{BL} are hybrid quantities, reflecting a combination of time and energy aspects of the tunneling process.[25] They violate (3.1) and therefore cannot be general expressions for tunneling times.

This leaves us with the reading of the local Larmor clock τ_y^L, as introduced by Baz' (1966a). In the context of stationary scattering states, the reliability of this clock remains conjectural. A conjecture can be correct, of course, and τ_y^L *does* meet one stringent test: The identity (3.3) has precisely the form (3.1) required by transmission

[24]We use the term *local* as opposed to *asymptotic* in this review. Both Kotler and Nitzan (1988) and Leavens and Aers (1988, 1989b) use the term in a stronger sense (see Sec. VI.D).

[25]It is precisely that combination, in fact, which determines the sideband intensities of Sec. V in the $\omega\to 0$ limit.

and reflection being mutually exclusive events. What to make of the symmetry relation (3.7) is less clear.

The identity (3.3) shows only that a necessary, not a sufficient, condition has been met. Closer inspection of the Rybachenko (1966) formulas for τ_{yT}^L and τ_{yR}^L, (2.13) and (2.14), does not strengthen one's confidence in these local Larmor times. How can the reflection time depend on the position, at $x_2>a$, of the far end of the magnetic-field region, when a reflected particle never moves beyond the barrier? What do the sine terms in (2.13) and (2.14) represent?

Partial answers to these questions can be found in the rederivation of the Rybachenko formulas by Falck and Hauge (1988). There the sine terms are depicted as results of new events made possible by the $O(\omega_L)$ reflection probability inherent in any step in the magnetic field. As examples, the term $\sim R^{1/2}\sin(2\alpha-\beta+2kx_2)$ in τ_{yT}^L of Eq. (2.13) represents the event sketched in Fig. 7(a), whereas the term $\sim R^{1/2}\sin(\beta-2kx_1)$ in τ_{yR}^L of Eq. (2.14) represents the event of Fig. 7(b). The manner in which these terms contribute to τ_y^L is rather subtle, however. They do *not* give the duration of the events. Qualitatively, this is seen as follows: The field splitting, of $O(\omega_L)$, of the transmission amplitudes and free propagators of the direct process carries the information of the Larmor precession. In the events of Fig. 7, the reflections at x_2 and x_1 already give factors of $O(\omega_L)$; the Larmor splitting, of total $O(\omega_L^2)$, is therefore neglected for consistency. Then what do these extra terms represent? Let us consider some simple examples.

C. Three simple examples

There are very few cases in which the predictions of the local Larmor clock can be tested against exact results. The reason is obvious. The dwell time is the only exact statement available about local times. For cases in which $R=0$, or $T=0$, the identity (3.3) shows that $\tau_y^L=\tau_D$. Our two first examples are of this kind.

(1) Free particle ($R=0$). In this trivial case,

$$\tau_{yT}^L(x_1,x_2;k)=\tau_D(x_1,x_2;k)=(x_2-x_1)/v(k)\ .$$

Edge reflections at x_2 and x_1, like those in Fig. 7, only contribute to $\langle s_y\rangle_T$ to $O(\omega_L^2)$ and are thus irrelevant.

(2) The infinitely wide barrier of Sec. IV.D ($T=0$). Let the magnetic field cover an interval (x_1,x_3) entirely to the left of the barrier. The corresponding local Larmor time can then be found as a difference between two expressions of the form (2.14), with $R=1$,

$$\begin{aligned}\tau_{yR}^L(x_1,x_3;k)&=\frac{2}{v}(x_3-x_1)+\frac{1}{kv}[\sin(\beta-2kx_1)\\&\qquad-\sin(\beta-2kx_3)]\\&=\tau_D(x_1,x_3;k)\ ,\end{aligned}\qquad(6.2)$$

where the last equality can be checked by direct integration, similar to Eq. (4.3). This result shows that the sine terms are of physical relevance here: They take care of the interference effects not contained in the first term.

None of these examples address the crucial question, since the identity (3.3) guarantees the accuracy of the local Larmor clock when $R=0$ or $T=0$. The question is whether the local Larmor times represent a reliable *distribution* of the average τ_D over the two scattering channels when $0<T<1$. We therefore turn to our third example.

(3) An arbitrary barrier with the interval (x_3,x_2) entirely to the right of it.[26] Subtraction of two versions of (2.13) gives

$$\begin{aligned}\tau_{yT}^L(x_3,x_2;k)&=\frac{1}{v}(x_2-x_3)\\&\quad+\frac{\sqrt{R}}{2kv}[\sin(2\alpha-\beta+2kx_3)\\&\qquad-\sin(2\alpha-\beta+2kx_2)]\ .\end{aligned}\qquad(6.3)$$

The dwell time, on the other hand, gives

$$\tau_D(x_3,x_2;k)=\frac{T}{v}(x_2-x_3)\ .\qquad(6.4)$$

Equation (6.4) gives the time spent on (x_3,x_2), averaged over all incoming particles. For this special case one can, in addition, unambiguously define a dwell time τ_D^c on (x_3,x_2), *conditional* on the particles having tunneled through:

$$\tau_D^c=\frac{\tau_D}{T}=\frac{1}{v}(x_2-x_3)\ .\qquad(6.5)$$

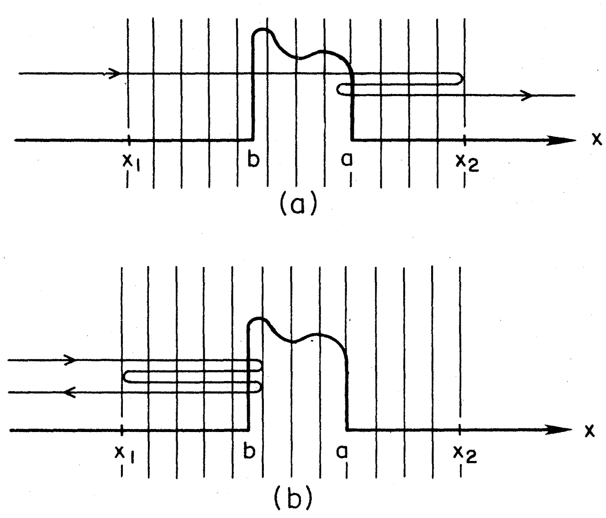

FIG. 7. Two interference terms contributing to the local Larmor times: (a) an extra reflection from the right end, at $x=x_2$, of the magnetic-field region, (b) an extra reflection at the left end, at $x=x_1$, of the field region.

[26]This example has been found independently by Leavens and Aers (1989b).

Clearly, the local Larmor time, if reliable, should coincide with this conditional dwell time. Equation (6.3) shows that it does not. Here, the sine terms associated with Fig. 7(a) introduce spurious effects, of the same order as the conditional dwell time, due to the $O(\omega_L)$ reflectivity at the field steps at x_2 and x_3. Identity (3.3) is still obeyed, saved by equally spurious contributions (oscillating between positive and negative values) to $\tau_{yR}^L(x_3,x_2;k)$.

Rather than take the above example as evidence against the reliability of the local Larmor clock, Leavens and Aers (1989b) speculate that it could be a manifestation of quantum nonlocality. In our opinion, there is no need for such speculations, since the mechanism behind the oscillating terms has been identified as the $O(\omega_L)$ reflectivity of the field steps at x_3 and x_2; this mechanism has been shown (Falck and Hauge, 1988) to quantitatively account for the spurious terms.

D. Some further calculations

Let us, for the moment, pretend that no conceptual difficulties exist in connection with the local Larmor times τ_y^L and τ_z^L. Returning to the Sokolovski-Baskin formulas (2.21), we note that the spatial region Ω is arbitrary. Furthermore, the form of (2.21) implies that the contributions to the complex "times" τ_T^Ω and τ_R^Ω from different spatial regions are additive. From this it follows that contributions to $\tau^{\rm BL}=|\tau^\Omega|=|\tau_y^L-i\tau_z^L|$ are, in general, *not* additive. Exceptions are circumstances under which the imaginary part of τ^Ω completely dominates the real part (or vice versa). Kotler and Nitzan (1988) considered such a case and studied the connection between a $\tau^{\rm BL}$, associated with a region deep inside the barrier, and the corresponding contribution to the transmission probability.

Now let $d\Omega=dx$ be a differentially small region somewhere within the barrier. Using the relations (2.22), one can construct the corresponding differential Larmor times $d\tau_{yT}^L$ and $d\tau_{zT}^L$. From these one can formally construct inverse speeds, $v_{yT}^{-1}=d\tau_{yT}^L/dx$ and $v_{zT}^{-1}=d\tau_{zT}^L/dx$, that vary from point to point inside the barrier. Leavens and Aers (1988, 1989b) discuss these concepts analytically and numerically. They compare results for a symmetric double barrier at resonance, when τ_T^Ω is real, with results for a corresponding single barrier, for which τ_T^Ω is complex. In spite of the fact that the Larmor "times" τ_z^L do not obey (3.1), whereas τ_y^L do, Leavens and Aers find that it is v_{zT}, not v_{yT}, which shows the more "reasonable" behavior. (For detailed argumentation, the reader is referred to the papers in question.) In a single barrier, v_{zT} is approximately constant through the barrier, whereas v_{yT} is strongly x dependent. In fact, it becomes exponentially large in the middle of the barrier, exceeding the speed of light already for barriers with a thickness of about 6 Å in their numerical example. They also find that even when the Schrödinger equation is replaced by the Dirac equation, v_{yT}, averaged over the barrier, can exceed the speed of light for sufficiently thick barriers. This is contrasted with the symmetric double barrier at resonance (when τ_T^Ω is real): For that case Leavens and Aers prove, on the basis of the Dirac equation, that $v_{yT}<c$.

Intriguing as these results may be, their relevance is less than clear. As the discussion above (Secs. VI.B and VI.C) shows, the concepts on which the results are based are of dubious validity in this context.

E. Status

We have found one clear-cut example showing that, in the context of scattering states with fixed k, the local Larmor time τ_y^L is not always reliable. On the other hand, the identity (3.3) guarantees the validity of τ_y^L for cases in which $R=0$ or $T=0$. Whether there are circumstances beyond these special cases in which the local Larmor clock can nevertheless be trusted is not known. It is therefore not clear how to interpret results that take the validity of the local Larmor clock, and its Büttiker-Landauer generalization, as their basic premise (Leavens and Aers, 1987b, 1988, 1989b; Kotler and Nitzan, 1988).

VII. CONCLUSIONS

In this review we have discussed the issue of tunneling times in the context of scattering in one dimension. Various general procedures, proposed over the years for calculating the transmission and reflection times in this context, have been critically examined. Our conclusions are as follows.

(1) The dwell time τ_D of Eq. (2.10) offers an exact local statement on the time spent in any region of space, averaged over *all* incoming particles. This status, discussed in Appendix B, makes the dwell time a useful tool for checking general claims in the field. On the other hand, the dwell time cannot distinguish between reflected and transmitted particles.

(2) Since reflection and transmission are mutually exclusive events, transmission and reflection times are, *if they exist,* conditional averages. Consequently, they must obey a probabilistic rule of the form (3.1), $\tau_D=T\tau_T+R\tau_R$, where $T=1-R$ is the probability of transmission. This relation plays a key role in our discussion. It does not apply in the context of escape from a localized metastable state. Although physically similar, the tunneling-time issue in that context is not identical to the one considered in this review. We do not comment on it here.

(3) All asymptotic treatments of tunneling in which completed scattering events are considered give results that converge on the classic phase times (2.5) and (2.6) as the width of the wave packets, σ, in k space tends to zero.

The phase times obey the identity (3.5), which seems to contradict the basic probabilistic rule (3.1). However, this contradiction is only apparent. When the status of the phase times, as relating to completed scattering events, is taken into account and the corresponding integration is performed over wave packets narrow in k space, (3.5) turns into (3.1) within corrections of $O(\sigma)$. The rule (3.1) is therefore obeyed in the appropriate asymptotic sense.

However, if the *extrapolated* phase times (2.7) and (2.8) are interpreted as the time spent in the barrier region, the conflict between the corresponding version of (3.5) and (3.1) becomes real. The reason for this conflict is that the extrapolated phase times are not determined only by the time spent in the barrier. They also include the delays due to self-interference during the approach to the barrier. These two aspects of phase times cannot, in general, be separated. An exception is offered by the illuminating examples discussed in Sec. IV.D. The first of these removes the apparent contradiction between the extrapolated phase times and the dwell time for opaque barriers.

(4) The dwell time and the asymptotic phase times provide reliable and complementary information on time aspects of the tunneling process. However, neither can answer the more specific question: When a particle in a scattering context, and with a given energy, has tunneled through (alternatively, has been reflected from) a barrier, how much time did it, on the average, spend in the barrier region?

The various candidates for general answers to this question have also been critically examined. All have been found to suffer from one logical flaw or another, flaws sufficiently serious that they must be rejected.

(5) The extrapolated phase times, (2.7) and (2.8), and the Büttiker-Landauer times, defined by (2.17) or, equivalently, by (5.1), both fail to satisfy the necessary requirement (3.1).

(6) The approximate version of the procedure for calculating tunneling times, suggested by Stevens, does not stand up under analytic and numerical scrutiny.

(7) We agree with the interpretation proposed by Büttiker and Landauer of their results for the opaque, oscillating barrier, except with regard to the duration of the tunneling process. We have shown, by the example of the δ-function barrier, that no direct relation exists between that duration and the characteristic frequency of an oscillating barrier. For barriers of intermediate thickness, a characteristic frequency cannot even be meaningfully defined.

(8) Although the complex "time" introduced by Sokolovski and Baskin has proved useful, for example, in clarifying relations between various suggested tunneling times, it cannot be taken seriously as a quantum generalization of the "classical" concept of time. The duration of a process, quantum or not, must be a real quantity.

(9) The local Larmor times τ_y^L of (2.12) *do* obey the probabilistic rule (3.1). Consequently, their claim in this context is that of distributing the dwell time correctly between transmitted and reflected particles. We have constructed a counterexample showing that this claim cannot always be true. The region of validity of the local Larmor clock is, as a consequence, unknown.

The reliability of the asymptotic Larmor clock, on the other hand, has been demonstrated. However, it does not produce essential new information: It shows, as it should, the phase times.

(10) The logical status of the main contenders for tunneling-time expressions is summarized in Table I. Although a number of apparent paradoxes have been

TABLE I. Logical status of the different tunneling times. For definitions, references to the text are given. Direct relations between different times are pointed out. Requirements met are listed in the three right-hand columns. Irrelevance on the basis of preceding entries is denoted by a dash. All candidates below the dashed line fail one requirement or other. The two survivors—the dwell time and the asymptotic phase times—have complementary strengths and weaknesses.

	Characteristics				Requirements		
	Definition equation	Relation	Distinct transmission and reflection	Local definition possible	Real	Obey (3.1)[a]	Consistent with τ_D always
Dwell time, τ_D	(2.10)	–	No	Yes	Yes	–	–
Asymptotic phase times, τ^φ	(2.5), (2.6)	–	Yes	No	Yes	Yes[b]	Yes,[b] Sec. IV.B
Extrapolated phase times, $\Delta\tau^\varphi$	(2.7), (2.8)	–	Yes	Yes[c]	Yes	No	–
Local Larmor, τ_y^L	(2.12)	–	Yes	Yes	Yes	Yes	No, Sec. VI.C
Local Larmor, τ_z^L	(2.15)	–	Yes	Yes	Yes	No	–
Büttiker-Landauer, τ^{BL}	(2.17), (5.1)	$\tau^{BL}=\lvert\tau^\Omega\rvert$	Yes	Yes –	Yes	No	–
Sokolovski-Baskin, τ^Ω	(2.20)	$\tau^\Omega=\tau_y^L-i\tau_z^L$	Yes	Yes	No	Yes	No, Sec. VI.C

[a](3.1) reads $\tau_D=T\tau_T+R\tau_R$.

[b]After appropriate average over wave packet has been taken.

[c]By formal extrapolation of τ^φ.

resolved and a clearer picture of the relative merits of the various proposed tunneling times has emerged, the specific question quoted under point (4) remains open. It is not clear that a general answer to this question exists.

(11) This does not necessarily imply that the various proposals for the duration of a tunneling process are uninteresting or experimentally irrelevant. For example, the inverse characteristic frequency of an oscillating barrier is, when it exists, clearly important; it gives the characteristic time scale of the coupling between the tunneling process and other degrees of freedom. The fact that this time scale has not been convincingly identified with the duration of the tunneling process does not detract from its importance.

In one special case the extrapolated phase time, the dwell time, the local Larmor time, and the Büttiker-Landauer time all agree: For a symmetric double-barrier structure at resonance, they all reduce to a result that essentially equals the lifetime of the metastable state.

(12) One final comment: At this stage one could choose to continue the search for a general answer to the question posed under point (4). Alternatively, one could turn to tunneling experiments now in progress with the aim of thoroughly understanding the temporal aspects of the individual experiments. At the present time, the latter strategy seems to us the more promising one.

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APPENDIX A

We present for reference (Büttiker, 1983; Büttiker and Landauer, 1985) some results for the rectangular barrier of height V_0 and width d. A particle with energy $E=\hbar^2k^2/2m < V_0$ has the tunneling probability $T(k)$:

$$T(k)=4k^2\kappa^2/D=1-R(k) , \tag{A1}$$

where

$$D=4k^2\kappa^2+k_0^2\sinh^2(\kappa d) , \tag{A2}$$

and $V_0-E=\hbar^2(k_0^2-k^2)/2m=\hbar^2\kappa^2/2m$. With the barrier located at $(0,d)$, the scattering phases are

$$\begin{aligned}\alpha(k)+kd&=\beta(k)+\frac{\pi}{2}\\ &=-\tan^{-1}\left[\frac{\kappa^2-k^2}{2k\kappa}\tanh(\kappa d)\right] .\end{aligned} \tag{A3}$$

[This is the only result sensitive to the location of the barrier. A shift to $(-d/2,d/2)$ would amount to $\beta\to\beta-kd$ in (A3)]. The linearly extrapolated phase times (2.7) and (2.8) read, with $d=a-b$,

$$\begin{aligned}\Delta\tau_T^{\varphi}(d;k)&=\Delta\tau_R^{\varphi}(d;k)\\ &=\frac{m}{\hbar k\kappa D}[2\kappa dk^2(\kappa^2-k^2)+k_0^4\sinh(2\kappa d)] ,\end{aligned} \tag{A4}$$

with D given by (A2). The dwell time in the barrier region is

$$\tau_D(d;k)=\frac{m}{\hbar\kappa D}[2\kappa d(\kappa^2-k^2)+k_0^2\sinh(2\kappa d)] . \tag{A5}$$

The Larmor times $\tau_{yT}^L=\tau_{yR}^L$ coincide with the dwell time (A5) in this symmetric case. The quantity τ_{zT}^L reads

$$\tau_{zT}^L=\frac{mk_0^2}{\hbar\kappa^2D}[(\kappa^2-k^2)\sinh^2(\kappa d)+(\kappa dk_0^2/2)\sinh(2\kappa d)] , \tag{A6}$$

and, finally, the expression for the Büttiker-Landauer time for transmission is

$$\begin{aligned}\tau_T^{\mathrm{BL}}=\frac{m}{\hbar\kappa^2D^{1/2}}[&(\kappa^2-k^2)\kappa^2d^2+k_0^4(1+\kappa^2d^2)\sinh^2\kappa d\\ &+k_0^2\kappa d(\kappa^2-k^2)\sinh(2\kappa d)]^{1/2} .\end{aligned} \tag{A7}$$

APPENDIX B

In the following we sketch a derivation of Büttiker's Eq. (2.10) for the dwell time. This derivation is due to Leavens and Aers (1989a). While we agree with Leavens and Aers at this point, their derivation has been (privately) criticized as essentially amounting to no more than a plausible definition. Consequently, we highlight the step where disagreement is possible.

Consider a particle described by a wave packet $\psi(x,t)$ scattered off some barrier $V(x)$ on the x interval (b,a). Initially, the packet is somewhere to the left of the barrier. Eventually, the reflected and transmitted parts of the packet will move off towards $-\infty$ and $+\infty$, respectively. The probability of finding the particle on an arbitrary fixed interval (x_1,x_2) at time t is

$$P(t;x_1,x_2)=\int_{x_1}^{x_2}dx\,|\psi(x,t)|^2 , \tag{B1}$$

when the packet is assumed to be normalized to unity. The statement (B1) is nothing but the standard probabilistic interpretation of quantum mechanics. We now take one crucial step beyond this and assert that the average time spent on (x_1,x_2) by the particle described by the packet $\psi(x,t)$ is

$$\overline{\tau}_D(x_1,x_2)=\int_0^\infty dt\,P(t;x_1,x_2)\ . \tag{B2}$$

If some probability measure over possible trajectories is given, one can, in classical mechanics, derive an equation for the probability distribution over times spent on the interval (x_1,x_2). From the first moment of this distribution, a relation corresponding to (B2) can be derived. However, in quantum mechanics, a path-integral formulation based on probability *amplitudes* does not, in general, produce a probability distribution for the time spent on (x_1,x_2). On the other hand, a statement on the average does not require all the information contained in a full probability distribution. The fact that an unambiguous distribution cannot be constructed does not imply that a statement about the average is false.

Our position, and that implicit in the derivation of Leavens and Aers, is that when $P(t;x_1,x_2)$ is reliable, so is (B2). Equation (B2) is a purely probabilistic statement, independent of the details of the theory underlying $P(t;x_1,x_2)$. We warn the reader, however, that this view is not universally accepted.

If (B2) is accepted, what remains in the derivation of Eq. (2.10) for the dwell time are technicalities, which we now sketch.[27] Insert (B2) into (B1) and decompose the initial wave packet into scattering states, so that

$$\psi(x,t)=\int\frac{dk}{2\pi}\varphi(k)\psi(x;k)\exp(-i\hbar k^2t/2m)\ , \tag{B3}$$

where $\psi(x;k)$ is given by (2.1). When $\psi(x,0)$ is assumed to be (essentially) zero for $x>x_1$, the lower limit of integration in (B2) can be replaced by $-\infty$. Thus, replacing the integrals over k and k' by integrals over $Q=(k+k')/2$ and $q=k-k'$, one has

$$\begin{aligned}\overline{\tau}_D(x_1,x_2)&\\ &=\int_{-\infty}^{\infty}dt\int_{x_1}^{x_2}dx\int\frac{dQ\,dq}{2\pi}\varphi^*(Q-q/2)\varphi(Q+q/2)\\ &\quad\times\psi^*(x;Q-q/2)\psi(x;Q+q/2)\\ &\quad\times e^{-i\hbar Qqt/m}\ .\end{aligned} \tag{B4}$$

Integration over t gives $2\pi(m/\hbar Q)\delta(q)$. Finally, integration over q and relabeling $Q\rightarrow k$ lead to

$$\overline{\tau}_D(x_1,x_2)=\int\frac{dk}{2\pi}|\varphi(k)|^2\frac{1}{v(k)}\int_{x_1}^{x_2}dx\,|\psi(x;k)|^2\ . \tag{B5}$$

Since $\overline{\tau}_D(x_1,x_2)$ is an average over an arbitrary initial wave packet with probability distribution $|\varphi(k)|^2/2\pi$ over wave numbers, the average time spent on the interval (x_1,x_2) for particles in a scattering state $\psi(x;k)$ is

$$\tau_D(x_1,x_2;k)=\frac{1}{v(k)}\int_{x_1}^{x_2}dx\,|\psi(x;k)|^2\ , \tag{B6}$$

which is Büttiker's Eq. (2.10) for the dwell time.

[27]The rigorously inclined reader will enjoy modifying the argument to suit personal taste.

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