The geometry of string perturbation theory

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This paper is devoted to recent progress made towards the understanding of closed bosonic and fermionic string perturbation theory, formulated in a Lorentz-covariant way on Euclidean space-time. Special emphasis is put on the fundamental role of Riemann surfaces and supersurfaces. The differential and complex geometry of their moduli space is developed as needed. New results for the superstring presented here include the supergeometric construction of amplitudes, their chiral and superholomorphic splitting and a global formulation of supermoduli space and amplitudes.

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I. INTRODUCTION

Local quantum field theory offers a remarkably successful description of the electromagnetic, weak, and strong interactions of the particles thus far observed. The standard electroweak theory of Glashow, Weinberg, and Salam, together with quantum chromodynamics, accounts extremely well for the vast amounts of high-energy particle accelerator data that have accumulated over the past forty years. Unification of quarks and leptons and of these three fundamental forces has been proposed by Georgi, Quinn, and Weinberg (1974), and several models have been constructed, amongst them the unified theories of Georgi and Glashow (1974) and Pati and Salam (1974). Though experimental evidence for the predicted decay of baryons in these theories is still lacking, there is a widespread belief that some type of unification should take place. As far as the physics of elementary particles is concerned, local quantum field theory thus provides a consistent and predictable framework.

Nature has provided us with one more force, however, that of gravitational attraction. The theory of general relativity accounts for this force, at a non-quantum-mechanical level, as a manifestation of the curved geometry of space-time. General relativity has been well tested on the cosmic scale, but has not yet been incorporated in a consistent scheme based on local quantum field theory. An overview of some of the attempts at the quantization of gravity may be found in Hawking and Israel (1979). Better yet, a natural goal would be to unify the four fundamental forces of nature into a single consistent and predictive quantum theory. Though supergravity theories pioneered by Freedman, van Niewenhuizen, and Ferrara (1976) and by Deser and Zumino (1976a) seemed at one time good candidates for such a unification within the framework of conventional quantum field theory, there are still problems with their consistent quantization.

In a key development, Neveu and Scherk (1972) found an effective Yang-Mills theory present in *dual models,* and Yoneya (1973) and Scherk and Schwarz (1974) argued that dual models with their "string" interpretation automatically contained a massless spin-2 particle, coupling precisely as the graviton couples in general relativity. The picture of elementary particles, and in particular the graviton, as pointlike objects with no internal structure could then be traded in for a theory in which elementary particles are thought of as one-dimensional curves with infinitesimal thickness, or so-called *strings.* Strings interact by joining and splitting. A unification of all forces along these lines was proposed by Scherk and Schwarz (1975). The length scale of such strings is set by the only scale characteristic of quantum gravity: the Planck length, which is on the order of 10^{-33} cm. It was also discovered that standard fermions and gauge bosons are automatically present in fermionic versions of the dual string models such as those found by Ramond (1971) and Neveu and Schwarz (1971). Furthermore, certain truncations of this model were shown to exhibit a supersymmetric spectrum by Gliozzi, Scherk, and Olive (1975, 1976), and the full supersymmetry was subsequently proven in one of the first papers on modern string theory by Green and Schwarz (1981). Soon thereafter the famous type-I theory of open and closed superstrings and the type-II A and B theories of closed superstrings only were identified by Green and Schwarz (1982). The type-I string possesses gauge symmetry from the outset, but the type-II string does not. For the type-II string, Witten (1983, 1985c) argued that serious problems arise if one wants to keep chiral fermion multiplets after compactification to four dimensions. In 1983, Alvarez-Gaumé and Witten showed that rather generic anomalies in gauge and gravitational symmetries cancel for the type-II superstring. The discovery of the absence of anomalies in the type-I superstring with gauge group O(32) by Green and Schwarz (1984) sparked a great deal of excitement about the phenomenological possibilities of that theory. The anomaly cancellation mechanism also allowed a gauge group $E_8 \times E_8$, and a theory with this symmetry seemed to be even more promising phenomenologically. A new type of string theory that encompasses this possibility—called the heterotic string—was soon discovered by Gross, Harvey, Martinec, and Rohm (1985a, 1985b). As these string models only seem to make sense in higher dimensions, it is usually assumed that the ground state rolls up in a tiny compact space in all but four dimensions, an idea going back to Kaluza (1921) and Klein (1926) and revived more recently in Cremmer and Scherk (1977). Promising compactifications and their phenomenological implications were discussed early on by Candelas, Horowitz, Strominger, and Witten (1985). Not only may superstrings contain the right particles, they also present strong evidence for being consistent, unitary, and predictable quantum theories of all particles and forces in nature. In a sense these string theories appear even healthier than quantum field theory itself, as calculations of scattering amplitudes do not seem to require renormalization, they are just finite. At least those are the indications gotten from analyses to tree level and sometimes to one-loop order in string perturbation theory.

It is obviously an important question whether the indications of one-loop finiteness and unitarity persist to all orders in perturbation theory. Most of the present review will be explaining the general framework for perturbatively calculating scattering amplitudes in string theories as we understand it today. One might compare this program with the derivation of the Feynman rules in a quantum field theory, to any order in perturbation theory. It has become clear that string theory offers a challenge with sometimes intricate but generally beautiful mathematical concepts, and we shall acquaint the reader gradually with the geometry that enters the perturbative methods, along with the physical ideas involved. Perhaps string perturbation theory does not provide us with sufficient flexibility and insight into questions of compactification and symmetry breaking, and a more general scheme is needed. Many attempts in this direction have been undertaken, and though their discussion would take us too far away from the mainstream of this review, we shall periodically indicate connections with such investigations.

Because of the almost unique nature of consistent string theories and the occurrence of surprising anomaly cancellation mechanisms we may expect a very simple but fundamental principle to underlie their existence. Such a principle remains to be fully uncovered. There is, however, a recurrent theme that sharply distinguishes strings and pointlike particle theories. With pointlike particles, there is a geometrical distinction between free propagation of particles and their interaction. The dynamics of the freely moving particle and of the interaction of several particles are separate components of the theory: in particular, the smooth world lines of free propagation experience a "singular" joining at the interaction point. The nature of the interaction is an additional input in the theory. An interaction occurs at a geometric point, and if it were observed from a different Lorentz frame, geometrically speaking the point of interaction would be unaltered [Fig. 1(a)]. In a theory of say, closed strings, formulated in a Lorentz-covariant way, two strings may touch at one instant and merge into one string, but the interaction point is not "geometric," as observation from different Lorentz frames will lead to different geometric locations of the interaction point [Fig. 1(b)]. The local dynamics of the string does not depend on whether there are interactions or not. In a Lorentz-covariant formulation, the action of the interacting string is the same as that of the free string. The topology of the worldsheet swept out by the strings alone is able to inform us that the strings interact. Thus the interaction appears global and "smeared out." This was known already in the days of the old dual models; there one noticed that the form factor of a string indicated nothing hard to scatter off, and this is clearly important for its nice short-distance properties.

From the point of view presented above, string theories describe surfaces moving in a target space-time, with no local interaction on the worldsheet. String interactions result from nontrivial topology of the surface; in particular, connectedness is related to the degree of interaction, boundary curves to initial and final strings, and the number of handles to the number of loops in an analogous dual or Feynman diagram representation. The formulation in which this topological and geometrical character of string amplitudes is manifest is that of Polyakov (1981a, 1981b), originally proposed mainly as a model of random surfaces. It provides a natural framework for maintaining reparametrization and conformal invariance, which are crucial symmetries of string theories, and further elucidates the role of the critical dimension. Actually, the conformal invariance properties in two dimensions are very restrictive, as was realized by Kadanoff (1969) and Polyakov (1969), who used it to de-

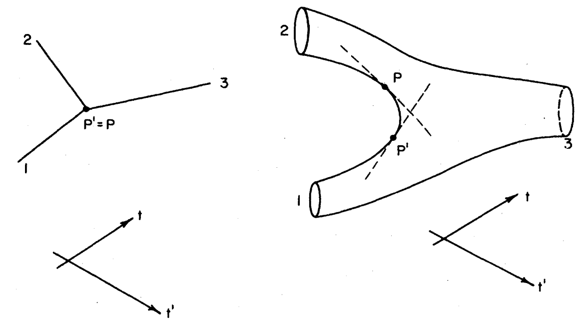


FIG. 1. Interactions of particles and of strings: (a) The point of interaction to two pointlike particles is geometrical and independent of the Lorentz frame of observation; (b) the point of interaction of two strings is not geometrical and depends on the Lorentz frame of observation.

velop the conformal bootstrap program. More recently, conformal field theory in two dimensions has been the scene of an intense independent development, sparked by the work of Belavin, Polyakov, and Zamolodchikov (1984) and its unitary restriction discovered by Friedan, Qiu, and Shenker (1984) and constructed explicitly by Goddard, Kent, and Olive (1986).

Though the table of contents should facilitate the reader's access to this review, we should like to sketch the broad outlines of our approach. We shall be considering only closed-string theories, first the closed oriented bosonic model of Virasoro (1969) and Shapiro (1970), generalizing the original open-string model of Veneziano (1968), then the type-II and heterotic superstrings. In many ways, the modifications required for open strings are of a purely technical nature, and we shall provide a few key references to the work on open strings when appropriate.

In keeping with manifest Lorentz invariance, we use the Polyakov formulation. Scattering amplitudes are evaluated perturbatively in the loop expansion. To order of h loops, the answer reduces to an integral over moduli space (or supermoduli space for superstrings) of an integrand consisting of determinants of certain operators and correlation functions. Great effort is devoted to evaluating these quantities and the moduli measure explicitly, first with the help of real geometry of moduli space, then with the help of complex geometry, leading us to make contact with the more algebraic conformal field theory formulation. Scattering amplitudes in the light-cone gauge have been investigated by Mandelstam (1973a, 1973b, 1974a, 1974b, 1974c, 1986a, 1986b).

A word about references may be in order. Within the field of string perturbation theory about flat Minkowski or Euclidean space-time, we have attempted to include many published references and preprints that are of direct relevance to the approach adopted here. Unfortunately, however, it has become exceedingly difficult to keep track of all the literature, and we present our apologies to those authors who feel their work has not been appropriately referenced. At places where we discuss connections with separate fields of investigation such as string field theory, propagation in nontrivial backgrounds and compactifications, open strings, universal moduli space, Grassmannians, etc., we shall quote only some of the earliest papers.

Finally, a number of reviews have been published over the past years. Some of the earlier work appears in Alessandrini, Amati, Le Bellac, and Olive (1971), Schwarz (1973), Frampton (1974), Mandelstam (1974a), Rebbi (1974), Veneziano (1974), and, perhaps the most accessible, Scherk (1975).

More recent reviews are those of Schwarz (1982), Green (1983), a reprint collection by Schwarz (1986), a book of Green, Schwarz, and Witten (1987), and a number of conference proceedings, including conferences held at Argonne (edited by Bardeen and White, 1985), Santa Barbara (edited by Green and Gross, 1986), and San Diego (edited by Yau, 1987). Polyakov's viewpoint and strings in other contexts than grand unification are in his book: Polyakov (1987b).

II. THE CLOSED ORIENTED BOSONIC STRING

The evolution of a closed string sweeps out a worldsheet, which is a two-dimensional surface embedded in a target space-time. The worldsheet is bounded by the position curves of the initial and final strings, and its handles indicate the creation and annihilation of virtual pairs. Thus the worldsheet is similar to a Feynman diagram in which propagator lines are replaced by cylinders and a loop now corresponds to a handle [Fig. 2(a)]. In this review we shall consider only S matrix elements, for which the initial and final strings are on shell and set at infinity. Under conformal transformations, which are the crucial symmetries of the theory, such a worldsheet can be transformed to a compact surface with a number n of points removed corresponding to the external string states. Such points are called punctures [Fig. 2(b)]. In

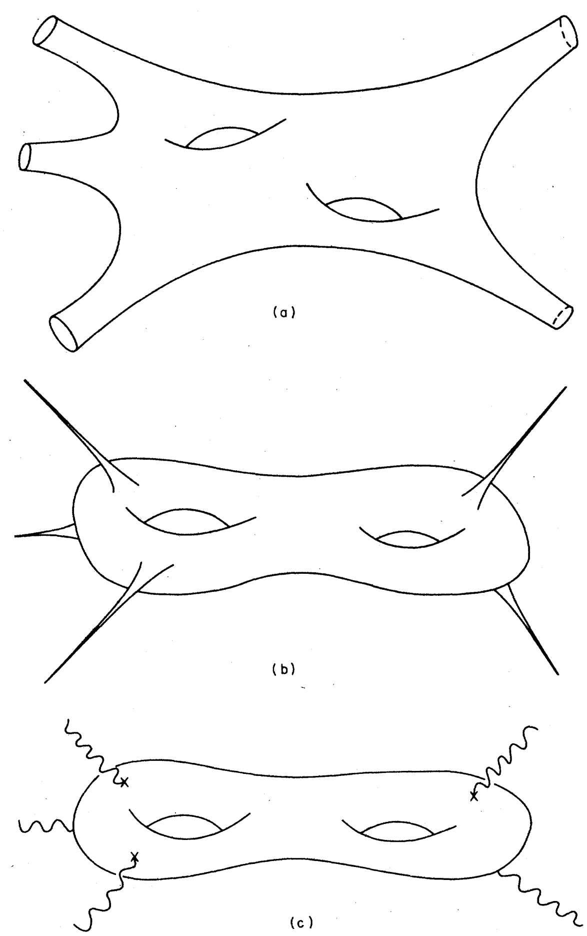


FIG. 2. The five-point function to two-loop ($h=2$) order, with incoming and outgoing strings represented as (a) full boundary curves; (b) punctures; (c) vertex operators.

the path-integral quantization procedure, the scattering amplitude is obtained by summing over all surfaces with n punctures and integrating at the punctures against the wave functions of the string states. Alternatively, we can rely on a string analog of the Lehmann-Symanzik-Zimmermann (LSZ) reduction formalism of quantum field theory, which gives on-shell scattering amplitudes in terms of vacuum expectation values of a time-ordered product of fields. The worldsheet is viewed then as a compact surface without punctures, but with insertions of local operators with the quantum numbers of the external string states. These operators are called vertex operators. In this formulation, the amplitude is obtained by summing over all compact surfaces and over all possible locations of the vertex operators [Fig. 2(c)]. The equivalence between the two formulations, together with the relation between the wave functions and vertex operators, will be discussed in detail later in Sec. II.L, and for the time being we shall adopt the vertex operator approach.

For closed oriented strings, the worldsheet is a compact orientable surface. At the h-loop level, there is topologically speaking only one such surface, which is a sphere with h handles. The number h is often referred to as the genus of the surface. Equivalently, we can classify the topology of the surface M by its Euler characteristic $\chi(M)$, defined as

$$\chi(M)=f-e+v \ ,$$

where f, e, and v are, respectively, the number of faces, edges, and vertices of any triangulation of M. The relation between $\chi(M)$ and h is readily seen to be

$$\chi(M)=2-2h \ . \tag{2.1}$$

In the presence of a metric g_{mn} on the surface M, the Gauss-Bonnet theorem asserts that $\chi(M)$ can be evaluated from the Gaussian scalar curvature R,

$$\chi(M)=\frac{1}{2\pi}\int_M d^2\xi\sqrt{g}R \ . \tag{2.2}$$

This formula can also be viewed as a topological constraint on the curvature of a surface of given genus.

We have already mentioned that conformal invariance plays a key role in string theories, and this issue will be discussed in detail as it emerges again and again in the review. It may be helpful to note at this point that two surfaces that are topologically equivalent may still not be equivalent as surfaces with complex structures. It is the space of complex structures on a given topological surface—the moduli space—which lies at the center of string perturbation theory.

As we progress, more facts about geometry of surfaces will be introduced as we need them.

A. Classical strings

A natural reparametrization-invariant action is the geometrical area, as proposed by Nambu (1970) and Goto (1971):

$$I_{\rm NG}(x^\mu)=T\int_M d^2\xi\sqrt{h} \ . \tag{2.3}$$

Here $\xi^m=(\xi^1,\xi^2)$ are coordinates on M, and $x^\mu(\xi)$, $\mu=1,\ldots,d$ describe the propagation of a string in a space-time of dimension d. The metric $G_{\mu\nu}(x)$ in space-time should ultimately arise dynamically as excitations of the $x^\mu(\xi)$, but in string perturbation theory it is taken just as a background metric which satisfies the string equations of motion. The embedding x^μ then induces a metric h_{mn} on the worldsheet given by

$$h_{mn}=\partial_m x^\mu \partial_n x^\nu G_{\mu\nu}(x) \ , \tag{2.4}$$

and $h=\det(h_{mn})$ (see Fig. 3). Finally T has dimensions of inverse length squared (or equivalently mass squared) and is called the string tension. It is simply related to the Regge slope parameter α' of dual-model theory by $T=1/2\alpha'$.

The field equations for x^μ implied by the Nambu-Goto action have two constraints expressing the vanishing of the worldsheet stress tensor. These constraints may be obtained as field equations directly if an intrinsic metric g_{mn} independent of h_{mn} is introduced. This leads to the formulation of Polyakov (1981a). Its action is that of a σ model with space-time as the target Riemannian manifold and the key property of reparametrization invariance on the worldsheet:

$$I_0(x^\mu,g_{mn})=\frac{T}{8\pi}\int_M d^2\xi\sqrt{g}g^{mn}\partial_m x^\mu \partial_n x^\nu G_{\mu\nu}(x) \ . \tag{2.5}$$

Classically the Nambu-Goto and the Polyakov actions lead to identical dynamics. Quantum mechanically it is not known whether the corresponding theories are equivalent, mainly because the string theory obtained from $I_{\rm NG}$ is hard to quantize unambiguously. The main advantages of the Polyakov action are that there is a clear distinction between the intrinsic geometry g_{mn} of the worldsheet and its embedding in space-time, and that the action is quadratic in x^μ's if $G_{\mu\nu}$ is the flat Euclidean metric. This is the case we shall study in detail. Henceforth, we shall set the string tension to unity: $T=1$.

The classical symmetries of Eq. (2.5) are as follows.

(i) The group Diff(M) of differentiable reparametriza-

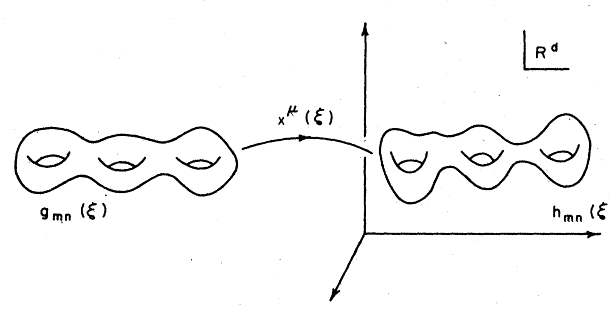


FIG. 3. The worldsheet: with intrinsic metric g_{mn} (left); embedded in the target space-time with the induced metric h_{mn} (right).

tions, or diffeomorphisms of M. Their action on the coordinates of the surface is given by $\xi^m \to \xi'^m(\xi)$, and the action on the metric is also familiar from general relativity:

$$g_{mn}(\xi) \to g'_{mn}(\xi') = \frac{\partial \xi^p}{\partial \xi'^m} \frac{\partial \xi^q}{\partial \xi'^n} g_{pq}(\xi) . \tag{2.6}$$

Diffeomorphisms connected to the identity form the smaller group $\mathrm{Diff}_0(M)$ and are generated by continuous vector fields $\delta v^m = \xi'^m - \xi^m$. The corresponding infinitesimal changes in the fields are

$$\delta g_{mn} = \nabla_m(\delta v_n) + \nabla_n(\delta v_m), \quad \delta x^\mu = \delta v^m \partial_m x^\mu . \tag{2.7}$$

(ii) The group Weyl(M) of all rescalings of the metric by (M-dependent) positive real functions. These transformations do not move the points of M and act infinitesimally as

$$\delta g_{mn} = 2\delta\sigma g_{mn}, \quad \delta x^\mu = 0 . \tag{2.8}$$

(iii) For flat target space-time, $G_{\mu\nu}$ is the Minkowski metric $\eta_{\mu\nu}$. The group of Poincaré transformations in the target space-time is

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + x_0^\mu, \quad \eta_{\mu\nu}\Lambda^\mu{}_\kappa \Lambda^\nu{}_\lambda = \eta_{\kappa\lambda} . \tag{2.9}$$

Most of the time, we shall assume analytic continuation to imaginary time, so that the metric of the target space-time is Euclidean; it would then be more appropriate to call this the group of isometries of flat d space.

It is useful to remark that some diffeomorphisms preserve the angles and are thus conformal reparametrizations. On the other hand, as the Weyl transformations merely rescale the metric, all Weyl transformations are conformal.

The action (2.5) had actually been considered before Polyakov in the context of two-dimensional supergravity by Brink *et al.* (1976) and by Deser and Zumino (1976b).

Notice that both the actions I_{NG} and I_0 involve only the intrinsic geometry of the string, with no reference to the extrinsic curvature experienced by the string. This appears to be the appropriate setting for a string model of elementary particles. However, if string theory is to be viewed as an effective theory of flux tubes in QCD or of the Ising model, the extrinsic curvature corrections should be taken into account. Such a model has recently been proposed by Polyakov (1986), but we shall not discuss it here.

We shall also constantly make use of standard differential and Riemannian geometry. Some key formulas are collected in Appendix A. Fuller accounts can be found in Spivak for differential geometry, Weinberg (1972) for general relativity, and Bott and Tu (1983) for topological aspects.

B. Quantization

Quantization may be performed by summing in the functional integral over all closed compact surfaces, as originally proposed by Hsue, Sakita, and Virasoro (1970) and by Gervais and Sakita (1971b). In the Polyakov formulation, this corresponds to treating both the x^μ and the worldsheet metrics g_{mn} as two-dimensional quantum fields. For flat Euclidean space-time, the x^μ are free fields and their path integrals Gaussian, so the crucial part will be the path integral over metrics g_{mn}.

The functional integral approach requires the addition to the Polyakov action of all possible renormalization counterterms consistent with the symmetries of the theory. In general, Weyl invariance is broken upon quantization in view of the conformal anomaly,[1] so we must include Weyl-noninvariant counterterms as well, and the most general local action compatible with reparametrization invariance is

$$I(x^\mu, g_{mn}) = I_0(x^\mu, g_{mn}) + \lambda\chi(M) + \mu_0^2 \int_M d^2\xi \sqrt{g} . \tag{2.10}$$

The Weyl invariance lost in the action because of μ_0^2 can actually be restored in the critical dimension $d=26$. We shall derive this crucial fact in detail later, and for the moment restrict our discussion to why we should have Weyl invariance at all. The standard philosophy is roughly as follows. Weyl and reparametrization invariance make up for three degrees of freedom, exactly the number in the metric g_{mn}. The only true degrees of freedom are then those of the d x^μ fields, with of course the two constraints implied by the equations of motion of g_{mn}. Thus with Weyl invariance the quantum string has $(d-2)$ degrees of freedom, precisely the number of the classical string. Note that the requirement of having the same number of quantum and classical degrees of freedom is assumed from the start in the light-cone formulation of Goddard *et al.* (1973) and Mandelstam (1973a, 1973b, 1974). Actually Polyakov (1981a) originally proposed his model precisely with the objective of obtaining a consistent quantum theory without Weyl invariance. The scale factor develops, then, an effective dynamics that is described by the Liouville theory. Ultimately, the constraint of Weyl invariance must be analyzed in the light of the unitarity of the theory, as we shall discuss in Sec. II.G.

Now the physical quantities of interest are the partition function

$$Z = \sum_{h=0}^{\infty} \int \frac{Dg_{mn} Dx^\mu}{\mathcal{N}} e^{-I(x,g)} \tag{2.11}$$

(which can be identified with the space-time integral of the target space-time cosmological constant) and on-shell scattering amplitudes, obtained by inserting vertex operators V_i

$$\langle V_{i_1}(k_1^\mu) \cdots V_{i_p}(k_p^\mu) \rangle = \sum_{h=0}^{\infty} \int \frac{Dg_{mn} Dx^\mu}{\mathcal{N}} e^{-I(x,g)} \times V_{i_1}(k_1^\mu) \cdots V_{i_p}(k_p^\mu) . \tag{2.12a}$$

[1]With dimensional regularization, for example, Weyl symmetry would be destroyed away from two dimensions.

Here $\mathcal{N}$ denotes a normalization factor to be specified later on. In Sec. VIII a detailed discussion of vertex operators for on-shell physical particles will be presented. For the time being, it may be sufficient to say that they are typically of the form

$$V(k^\mu, x^\mu(\xi)) = P(\varepsilon, Dx^\mu(\xi)) e^{ik\cdot x(\xi)} , \tag{2.12b}$$

where $P(\varepsilon, Dx^\mu)$ is a polynomial expression in the derivatives of x and ε is a polarization tensor. This form is dictated by the symmetries of the action, and a key requirement is Weyl invariance after inclusion of anomalies. The lowest mass levels will turn out to be

$$\begin{aligned} &k^2=2, \quad V(k^\mu)=e^{ik\cdot x} , \\ &k^2=0, \quad V(k^\mu)=\varepsilon_{\mu\nu} g^{mn}\partial_m x^\mu \partial_n x^\nu e^{ik\cdot x} , \end{aligned} \tag{2.12c}$$

where the first corresponds to the tachyon, and the second corresponds to $\varepsilon_{\mu\nu}$ symmetric traceless, the graviton; $\varepsilon_{\mu\nu}$ antisymmetric, the antisymmetric tensor field; $\varepsilon_{\mu\nu}$ pure trace part, the dilaton.

The integration measures Dg_{mn} and Dx^μ are determined by requirements of symmetry and locality. The construction of Dg_{mn} will be discussed in the next section. For Dx^μ, the measure is completely determined once one has a metric function on the space of small variations δx^μ, so that one can measure lengths and angles and hence volumes. This metric on the space of embeddings x^μ is unique due to Poincaré and reparametrization invariance,

$$\|\delta x^\mu\|^2 = \int_M d^2\xi \sqrt{g}\, \delta x^\mu \delta x^\mu . \tag{2.13}$$

It induces an inner product, which we denote by $\langle \delta x_1 | \delta x_2 \rangle$. Note that it is not Weyl invariant, a property providing another explanation for the Weyl anomaly. Since, however, the measure involves a product over an infinite number of variables, there may be some ambiguity in defining it from Eq. (2.13). This is resolved by the principle of ultralocality as stated by Polchinski (1986), which asserts that since the measure is a pointwise, reparametrization-invariant product over the worldsheet, any ambiguity must also be a reparametrization-invariant pointwise product. In particular, no derivatives should occur, and the only ambiguity can reside in a factor of the form

$$\exp\left[-\mu_1^2 \int_M d^2\xi \sqrt{g}\right] \tag{2.14}$$

for some constant μ_1^2. In particular, no constant other than 1 in front of the exponential is allowed, since this could not be written as a pointwise reparametrization-invariant product over the surface. Upon substitution into functional integrals, Eq. (2.14) results in just a shift in the counterterm μ_0^2 in Eq. (2.10). Ultimately, in the critical dimension $d=26$, the net counterterm will be fixed by requiring Weyl invariance, so the measure associated with Eq. (2.13) will in effect be unique. Henceforth we shall assume that such a counterterm has been fixed, and we shall not exhibit the area term any longer. This argument applies equally well to the case of any space-time metric $G_{\mu\nu}(x)$ as long as it is independent of the derivatives of x^μ.

The main issue, then, is to evaluate the functional integrals in Eqs. (2.11) and (2.12a). The integral in x^μ is easily performed once we have specified the measure arising from Eq. (2.13). First, we write the action I_0 using the scalar Laplacian Δ_g on the Riemann surface:

$$I_0(x^\mu, g_{mn}) = \frac{1}{8\pi} \langle x \mid \Delta_g x \rangle , \tag{2.15}$$

where

$$\Delta_g = -\frac{1}{\sqrt{g}} \partial_m \sqrt{g} g^{mn} \partial_n . \tag{2.16}$$

Next, the x variable is divided into the constant zero mode x_0^μ of the Laplacian and all other modes x'^μ orthogonal to it: $x^\mu = x_0^\mu + x'^\mu$ with $\langle x_0 | x' \rangle = 0$. Upon splitting the functional integral accordingly, we have

$$\begin{aligned} \int Dx^\mu e^{-I_0(x,g)} &= \int Dx_0^\mu \int Dx'^\mu e^{-\langle x' | \Delta_g x' \rangle / 8\pi} \\ &= (\det{}'\Delta_g)^{-d/2} \\ &\quad \times \int dX_0^\mu \int Dx'^\mu e^{-\|x'\|^2/8\pi} . \end{aligned} \tag{2.17}$$

With the principle of ultralocality, one deduces that the Gaussian integral

$$\int Dx^\mu e^{-\|x\|^2/8\pi} \tag{2.18}$$

is a local product over the worldsheet, with no derivatives of the metric entering. Consequently it must be the exponential of the worldsheet area, which may be absorbed into the μ_0^2 coefficient in Eq. (2.10). Let us now split up this integral as we did before:

$$\begin{aligned} 1 &= \int Dx^\mu e^{-\|x\|^2/8\pi} \\ &= \int Dx_0^\mu \int Dx'^\mu e^{-\|x_0\|^2/8\pi - \|x'\|^2/8\pi} \\ &= \left[\frac{8\pi^2}{\int_M d^2\xi \sqrt{g}} \right]^{d/2} \int Dx'^\mu e^{-\|x'\|^2/8\pi} . \end{aligned} \tag{2.19}$$

If, in addition, we note that $\int Dx_0^\mu = \Omega$, the volume of space-time, then we finally obtain

$$\int Dx^\mu e^{-I_0(x,g)} = \Omega \left[\frac{8\pi^2}{\int_M d^2\xi \sqrt{g}} \det{}'\Delta_g \right]^{-d/2} . \tag{2.20}$$

If, instead of considering the contribution to the partition function as we did above, we also have a sequence of vertex operators, then the space-time volume element should be replaced by the total momentum conservation δ function, resulting from the x_0^μ integral over the $\exp(ik\cdot x)$ factors in the vertex operators (2.12b).

We turn to the more difficult task of integrating over metrics in Eqs. (2.11) and (2.12a) in the next section.

C. Worldsheet metrics and deformations of conformal classes

We now fix the topology, i.e., the number of handles of the worldsheet M. Let $\mathcal{M}=\{g_{mn} \text{ on } M\}$ be the space of positive worldsheet metrics. An infinitesimal deformation δg_{mn} of a metric is a symmetric two-tensor, and the natural norm for δg_{mn} is

$$\|\delta g_{mn}\|^2=\int_M d^2\xi\sqrt{g}\,(cg^{mn}g^{pq}+g^{mp}g^{nq})\delta g_{mn}\delta g_{pq} \ . \tag{2.21}$$

The arbitrary constant c will not appear in any physical answer, so we shall set it to zero. Associated with the norm is an inner product on the space of metric deformations, which will be denoted by $\langle \delta g_1 \mid \delta g_2 \rangle$. The measure Dg_{mn} will be the one associated with Eq. (2.21) (for $c=0$), with the usual harmless ambiguity which ultralocality fixes to be of the form (2.14). To determine the form of the final answer for Eqs. (2.11) and (2.12), we assume momentarily that all possible Weyl and reparametrization anomalies will cancel and ask whether all modes of g_{mn} can be gauged away with the help of reparametrizations and Weyl transformations. The number of degrees of freedom is the same (3 for g_{mn}, 1 for Weyl, and 2 for reparametrizations), and it is a classic theorem of Gauss that in any simply connected patch on the surface the metric can indeed be made Euclidean by such transformations. Whenever the topology is nontrivial, however, the reparametrizations of different patches need not match and there may be topological obstructions. To see this, we note that the joint action of reparametrizations and Weyl transformations on the metric is given by Eqs. (2.7) and (2.8),

$$\delta g_{mn}=(2\delta\sigma+\nabla^p\delta v_p)g_{mn}+(P_1\delta v)_{mn} \ , \tag{2.22}$$

where the operator P_1 sends vectors into symmetric traceless two-tensors,

$$(P_1\delta V)_{mn}=\nabla_m\delta v_n+\nabla_n\delta v_m-g_{mn}\nabla^p\delta v_p \ , \tag{2.23}$$

and describes the traceless piece of the deformation coming from reparametrization by the vector field δv^m. Clearly the total trace piece can always be eliminated without topological obstruction by a Weyl transformation alone. Thus the only metric deformations δg_{mn} that are not gotten by reparametrization and Weyl transformation are in $(\mathrm{Range}P_1)^\perp$. This means that any deformation δg_{mn} is given by the decomposition orthogonal under Eq. (2.21):

$$\{\delta g_{mn}\}=\{\delta\sigma g_{mn}\}\oplus\mathrm{Range}P_1\oplus\mathrm{Ker}P_1^\dagger \ , \tag{2.24}$$

where the action of $P_1^\dagger$ on symmetric traceless two-tensors is given by

$$(P_1^\dagger\delta g)_m=-2\nabla^n\delta g_{mn} \tag{2.25}$$

and we have used the result that, under the inner product $\langle \ | \ \rangle$, we have the identification

$$(\mathrm{Range}P_1)^\perp=\mathrm{Ker}P_1^\dagger \ . \tag{2.26}$$

The first two spaces on the right-hand side of Eq. (2.24) consist of modes that can be gauged away by combined Weyl and reparametrization symmetries. The dimension of the remaining space is finite, and we shall now determine it.

The way to determine the number of zero modes of these operators is to appeal to an index theorem, which gives the difference between the number of zero modes of the operator and its adjoint in terms of a topological invariant. The problem is then reduced to a similar one for the adjoint, which often may be solved by independent methods such as vanishing theorems. In the present case, zero modes of P_1 are just reparametrizations inducing only trace changes in the metric, in other words, conformal Killing vectors; the topological invariant is the Euler characteristic, and the index theorem reduces to the following version of the Riemann-Roch theorem:

$$\dim\mathrm{Ker}P_1-\dim\mathrm{Ker}P_1^\dagger=3\chi(M) \ . \tag{2.27}$$

This relation will actually follow from the short-time heat-kernel expansion in Sec. II.F, as we shall see later. For the sphere, the conformal Killing transformations form the Möbius group

$$z\mapsto(az+b)/(cz+d), \quad \begin{bmatrix} a & b \\ c & d\end{bmatrix}\in\mathrm{SL}(2,C) \ ,$$

so that $\dim\mathrm{Ker}P_1=6$; for the torus, it is the group of translations that has dimension 2. For genus ≥ 2, there are no conformal Killing vectors on a surface without boundary. It is easy to provide a proof for the case of metrics of constant negative curvature R. As we shall indicate in the next section, any metric on a surface of genus ≥ 2 can be brought back to this case by a Weyl transformation, and the dimensions of these kernels are not changed. A conformal Killing vector δv^m satisfies $(P_1\delta v)_{mn}=0$, and upon differentiation one gets

$$\nabla^q\nabla_q\delta v^p=-R\,\delta v^p \ . \tag{2.28}$$

Integrating versus δv_p over the surface, one finds

$$\|\nabla^q\delta v^p\|^2-R\,\|\delta v^p\|^2=0 \ , \tag{2.29}$$

so that $\delta v^p=0$ for $R<0$. Using the index theorem (2.27) and the above counting of conformal Killing vectors, we conclude that

$$\dim\mathrm{Ker}P_1^\dagger=\begin{cases}0, & h=0 \ ,\\ 2, & h=1 \ ,\\ 6h-6, & h\geq 2 \ .\end{cases} \tag{2.30}$$

Thus we expect the partition function and scattering amplitudes to reduce to finite-dimensional integrals over spaces of the corresponding dimensions. Elements of $\mathrm{Ker}P_1^\dagger$ are called real *quadratic differentials* or *moduli deformations,* and parametrize infinitesimal deformations of conformal classes of metrics.

The space of conformal classes of metrics is a vast subject in the mathematics literature, going back as far as Riemann. For modern texts, we shall refer systematically to the books by Schiffer and Spencer (1954), Ahlfors (1966), Siegel (1970, 1971, 1973), Griffiths and Harris (1978), Abikoff (1980), Farkas and Kra (1980), Beardon (1983), and the survey articles of Bers (1972, 1981). In the physics literature, moduli parameters appear implicitly in dual-model diagrams. A lucid geometric account in this early phase is that of Alessandrini (1971) and Alessandrini and Amati (1971). For light-cone diagrams, moduli parameters were essentially introduced by Mandelstam (1973a, 1973b). The above approach to quadratic differentials appears in Alvarez (1983); the detailed mathematical treatment is given by Fischer and Tromba (1984a, 1984b, 1984c).

D. Teichmüller and moduli spaces

In the absence of anomalies, the string path integrals in Eqs. (2.11) and (2.12) should reduce to integrals over the space of inequivalent metrics under the combined Weyl and reparametrization symmetries. The discussion in the preceding section has shown that this space is locally a finite-dimensional manifold of dimension 0, 2, and $6h-6$ when h is 0, 1, and ≥ 2, respectively. We still need a global description, taking into account the fact that Diff(M) acts on the space of metrics by isometries but Weyl(M) does not.

A natural way to do this is to make use of the key fact that for any metric g_{mn} on M there exists a unique scaling factor $e^{2\sigma}$ such that $\hat{g}_{mn}=e^{-2\sigma}g_{mm}$ has constant curvature $R_{\hat{g}}=1$ when $h=0$, $R_{\hat{g}}=0$ and Area$(\hat{g})=1$ when $h=1$, and $R_{\hat{g}}=-1$ when $h\geq 2$. [Note that the sign of the curvature must be consistent with the Gauss-Bonnet theorem of Eq. (2.2).] This is equivalent to the fact that the Liouville equation,

$$\Delta_g \sigma = R_g - R_{\hat{g}} e^{-2\sigma} , \tag{2.31}$$

admits a unique solution. Thus

$$\mathcal{M}_{\text{const}} = \{\hat{g};\ R_{\hat{g}} = \text{const as above}\} \tag{2.32}$$

is a well-defined global slice for the Weyl group without any complication of the type discussed by Gribov (1978). It naturally carries the metric (2.21), since $\mathcal{M}_{\text{const}}$ is a subspace of $\mathcal{M}$. We may now define Teichmüller $\mathcal{T}_h$ and the moduli space $\mathcal{M}_h$ of Riemann surfaces of genus h by

$$\begin{aligned} \mathcal{T}_h &= \mathcal{M}_{\text{const}}/\text{Diff}_0(M) , \\ \mathcal{M}_h &= \mathcal{M}_{\text{const}}/\text{Diff}(M) = \mathcal{T}_h/\text{MCG}_h , \end{aligned} \tag{2.33}$$

with

$$\text{MCG}_h = \text{Diff}(M)/\text{Diff}_0(M) .$$

Teichmüller space $\mathcal{T}_h$ will turn out to be a complex manifold topologically equivalent to $(\mathbf{R}_+\times\mathbf{R})^{3h-3}$. The mapping class group MCG_h is a discrete group, which acts holomorphically with fixed points, however. Thus moduli space will have the structure of an orbifold. A more detailed discussion of some of these issues will be taken up in Sec. IV. As defined above, both Teichmüller and moduli space come equipped with a natural metric given by Eq. (2.21). The reason is that Diff(M) acts isometrically on $\mathcal{M}_{\text{const}}$ so that the natural metric on $\mathcal{M}_{\text{const}}$ can be pulled back to either space under the action of this isometry. This metric on Teichmüller and moduli space is called the Weil-Petersson metric.

To conclude this section we now verify that the tangent space to Teichmüller and moduli space is the space of quadratic differentials $\text{Ker}P_1^\dagger$, as may be expected. In fact under any deformation δg_{mn} of metrics (see Appendix A) the curvature changes by

$$\delta R = -\tfrac{1}{2}g^{mn}\delta g_{mn}R - \tfrac{1}{2}\nabla^p\nabla_p(g^{mn}\delta g_{mn}) + \tfrac{1}{2}\nabla^m\nabla^n(\delta g_{mn}) . \tag{2.34}$$

This shows that a deformation δg_{mn} in $\text{Ker}P_1^\dagger$ does not change the curvature and hence is tangent to $\mathcal{M}_{\text{const}}$. Combining this with Eq. (2.24) yields (see Fig. 4)

$$T_{\hat{g}}(\mathcal{M}_{\text{const}}) = \{\nabla_m(\delta v_n) + \nabla_n(\delta v_m)\} \oplus \text{Ker}P_1^\dagger$$

and, in particular,

$$T_{\hat{g}}(\mathcal{M}_h) = \text{Ker}P_1^\dagger . \tag{2.35}$$

Thus the Weil-Petersson metric can be described as follows: to determine the norm of a tangent vector to Teichmüller or moduli space, represent it by a quadratic differential δg_{mn}. Then its norm is given by Eq. (2.21), taken with respect to a metric $\hat{g}$ of constant curvature.

Other slices for $\mathcal{M}_h$ will also prove useful in treatments of string path integrals. For example, one can choose instead metrics that are flat everywhere with Dirac singularities for the curvature at a finite number of points, so that the Gauss-Bonnet relation is satisfied. From the point of view of complex analysis, one can parametrize

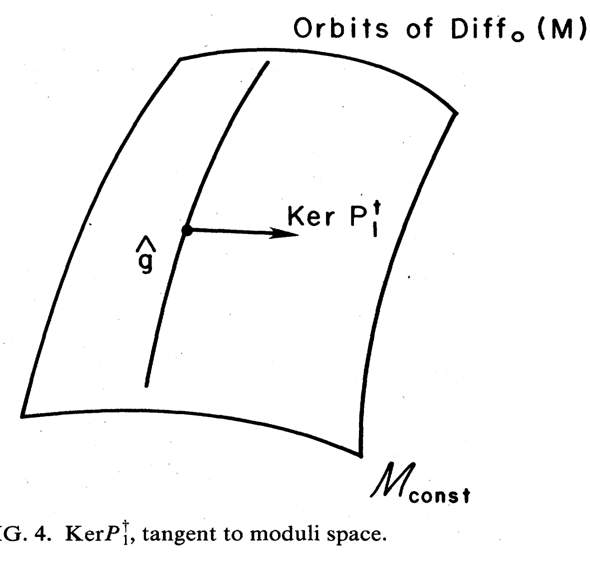


FIG. 4. $\text{Ker}P_1^\dagger$, tangent to moduli space.

moduli by period matrices (see Sec. VI.D). Different explicit parametrizations of moduli space will be discussed in Sec. IV.

It may be helpful to discuss at this point the distinction between Teichmüller and moduli spaces as far as string amplitudes are concerned. String amplitudes will reduce to integrals over Teichmüller space in the absence of $\mathrm{Diff}_0(M)$ anomalies, which are just perturbative gravitational anomalies. Elements in the mapping class group can be viewed as "large" reparametrizations. Only in the absence of global gravitational anomalies will string amplitudes reduce to integrals over moduli space.

For the bosonic string we can adopt throughout manifestly reparametrization-invariant methods of regularization such as ζ-function and short-time cutoff heat-kernel regularization. Thus neither perturbative nor global gravitational anomalies will occur, and physical quantities can all be expressed as integrals over moduli space. For fermonic strings, absence or cancellation of such anomalies is a highly nontrivial matter, and the famous constraints of modular invariance are just the requirement of absence of global gravitational anomalies when large reparametrizations act on various choices of homology bases. We shall discuss these issues in greater detail when they arise later.

E. Complex structures, tensors, covariant derivatives, and differentials

With Weyl and reparametrization invariance, the key geometric object on the worldsheet is not really the metric g_{mn}, but rather the complex structure $J_m{}^n$ it defines,

$$J_m{}^n=\sqrt{g}\,\varepsilon_{mp}g^{pn}\ , \tag{2.36}$$

with $\varepsilon_{12}=-\varepsilon_{21}=1$. We can readily check that

$$J_m{}^nJ_n{}^p=-\delta^p_m,\quad \nabla_pJ_m{}^n=0\ . \tag{2.37}$$

We see that the above definition of moduli space [Eq. (2.33)] is equivalent to the definition of moduli space as the set of equivalence classes under $\mathrm{Diff}(M)$ of the space of complex structures:

$$\mathcal{M}_h=\{J_m{}^n;J_m{}^pJ_p{}^n=-\delta_m{}^n\}/\mathrm{Diff}(M)\ .$$

We may also define holomorphic and antiholomorphic functions on M by the Cauchy-Riemann equations

$$J_m{}^n\partial_nf=i\partial_mf,\quad J_m{}^n\partial_n\bar f=-i\partial_m\bar f\ . \tag{2.38}$$

In any local coordinate patch, one can render the metric conformally flat by a reparametrization, so that $ds^2=2g_{z\bar z}dz\,d\bar z$, at least locally. This choice of coordinates exhibits the residual invariance under analytic reparametrizations $z\to z'(z)$, where z' is an analytic function of z. Thus M can be covered by coordinate charts with holomorphic transition functions and is consequently a Riemann surface. Since $J_m{}^n$ is Weyl invariant, and since it transforms as a tensor under general coordinate transformations, it characterizes the metric up to Weyl transformations and reparametrizations. This produces a one-to-one correspondence between points in moduli space and complex structures on M.

In the presence of a complex structure, tensors on M can be decomposed into tensors of weight (m,n), with m lower z and n lower $\bar z$ indices. The behavior of a tensor under an analytic coordinate transformation $z\to z'(z)$ is given by

$$T_{\underbrace{zz\cdots z}_{m}\,\underbrace{\bar z\bar z\cdots\bar z}_{n}}\to\left[\frac{\partial z}{\partial z'}\right]^m\left[\frac{\partial\bar z}{\partial\bar z'}\right]^nT_{\underbrace{zz\cdots z}_{m}\,\underbrace{\bar z\bar z\cdots\bar z}_{n}}\ . \tag{2.39}$$

Thus the invariant quantity characterizing such a tensor is

$$T_{\underbrace{zz\cdots z}_{m}\,\underbrace{\bar z\bar z\cdots\bar z}_{n}}(dz)^m(d\bar z)^n\ ; \tag{2.40}$$

this will also be called a tensor of weight (m,n). In particular, the space of tensors of weight $(m,0)$ will be denoted $\mathbf{T}^m$, and the space $\mathbf{T}^1$ is often referred to as (sections of) the canonical bundle K of the Riemann surface M. On a tensor $T(dz)^n$ in $\mathbf{T}^n$ the covariant derivative decomposes into

$$\nabla_pT(dz)^nd\xi^p=\nabla^n_zT(dz)^{n+1}+\nabla^n_{\bar z}T(dz)^nd\bar z\ . \tag{2.41}$$

The first term above defines an operator

$$\nabla^n_z:\ \mathbf{T}^n\mapsto\mathbf{T}^{n+1}\ ,$$

$$\nabla^n_z(T(dz)^n)=(g_{z\bar z})^n\frac{\partial}{\partial z}((g^{z\bar z})^nT)(dz)^{n+1}\ . \tag{2.42}$$

The second term, on the other hand, depends only on the conformal class of the metric and defines the key operator of the theory, namely the Cauchy-Riemann operator $\bar\partial_n=\nabla^n_{\bar z}$,

$$\bar\partial_n:\ \mathbf{T}^n\mapsto\{(n,1)\ \text{tensors}\}\ ,$$

$$\bar\partial_n(T(dz)^n)=\frac{\partial T}{\partial\bar z}(dz)^nd\bar z\ . \tag{2.43}$$

The $\bar\partial_n$ operators are intrinsically associated with moduli parameters and, in fact, holomorphically depend on them. Consequences for string amplitudes of this crucial property will be discussed at length in Sec. VII.

When we use the metric to change tensor weights from (m,n) to $(m-n,0)$, the operator $\bar\partial_n$ goes into an operator[2]

$$\nabla^z_n:\ \mathbf{T}^n\mapsto\mathbf{T}^{n-1},\quad \nabla^z_n(T(dz)^n)=g^{z\bar z}\frac{\partial}{\partial\bar z}T(dz)^{n-1}\ . \tag{2.44}$$

On each tensor space, there exists a unique inner prod-

[2]The operators ∇^n_z and ∇^z_n differ from those introduced by Alvarez's (1983) only because our $\mathbf{T}^n$ is Alvarez's $\mathcal{T}^{-n}$.

uct of tensor fields $T_1, T_2 \in \mathbf{T}^n$,

$$\langle T_1 \mid T_2 \rangle = \int d^2z \sqrt{g}\, (g^{z\bar{z}})^n T_1^* T_2 \ , \tag{2.45}$$

and one can obtain the adjoint operator in the usual way:

$$(\nabla_n^z)^\dagger = -\nabla_z^{n-1} \ . \tag{2.46}$$

We shall also make use of the Laplace operators

$$\Delta_n^{(+)} = -2\nabla_{n+1}^z \nabla_z^n, \quad \Delta_n^{(-)} = -2\nabla_z^{n-1}\nabla_n^z \ . \tag{2.47}$$

The operator $\Delta_n^{(-)}$ is exactly $2\bar{\partial}_n^\dagger \bar{\partial}_n$, while $\Delta_n^{(+)}$ will correspond to $2\bar{\partial}_{n+1}\bar{\partial}_{n+1}^\dagger$ after identification of $(n,0)$ forms with $(n+1,1)$ forms.

To make contact with the space of "real" two-component tensors, we set $\mathbf{T}^n \oplus \mathbf{T}^{-n} = \mathbf{S}^n$, and the covariant derivatives on this space act by

$$\begin{aligned} &P_n: \ \mathbf{S}^n \to \mathbf{S}^{n+1}, \quad P_n = \nabla_z^n \oplus \nabla_{-n}^z \ , \\ &P_n^\dagger: \ \mathbf{S}^{n+1} \to \mathbf{S}^n, \quad P_n^\dagger = -(\nabla_{n+1}^z \oplus \nabla_z^{-n-1}) \ . \end{aligned} \tag{2.48}$$

It is easy to see that P_1 is the operator of Eq. (2.23). Similarly, the Laplacian on scalars introduced in Eq. (2.16) is given by

$$\Delta_g = \Delta_0^{(+)} = \Delta_0^{(-)} \ . \tag{2.49}$$

It should be borne in mind that even though these operators were first defined on tensor fields with n integer, one may in fact generalize this construction so as to allow for spinors and spinor tensors for which n is a half-integer. A proper definition of some sign ambiguities requires the notion of spin structure, which will be introduced in Secs. III.A and VI.F.

Zero modes of these operators are of great interest in string theory, since they are potential sources of anomalies. First of all, we have the following generalization of Eq. (2.27):[3]

$$\dim \operatorname{Ker}\nabla_z^n - \dim \operatorname{Ker}\nabla_{n+1}^z = \tfrac{1}{2}(2n+1)\chi(M) \ . \tag{2.50}$$

This formula will be proven at the end of Sec. II.F. Since ∇_{-n}^z is the complex conjugate of ∇_z^n, we may restrict our attention to the case $n \geq -\frac{1}{2}$. By an argument similar to that given in Sec. II.C to show that $\operatorname{Ker}P_1 = 0$ for $h \geq 2$, one shows that for $h \geq 2$

$$\begin{aligned} &\dim \operatorname{Ker}\nabla_z^n = 0, \quad \text{for } n \geq 1 \ , \\ &\dim \operatorname{Ker}\nabla_z^0 = 1 \ , \\ &\dim \operatorname{Ker}\nabla_n^z = \dim \operatorname{Ker}\bar{\partial}_n \\ &\qquad = (2n-1)(h-1) \quad \text{for } n \geq \tfrac{3}{2} \ , \\ &\dim \operatorname{Ker}\nabla_1^z = \dim \operatorname{Ker}\bar{\partial}_1 = h \ . \end{aligned} \tag{2.51}$$

For the torus, the dimension of every kernel is exactly one (except for spinors where there are no zero modes for even-spin structures):

$$\dim \operatorname{Ker}\nabla_z^n = 1, \quad \dim \operatorname{Ker}\nabla_n^z = 1 \ . \tag{2.52}$$

For the sphere, there are no holomorphic forms, so that

$$\begin{aligned} &\dim \operatorname{Ker}\nabla_z^n = 2n+1 \quad \text{for } n \geq -\tfrac{1}{2} \ , \\ &\dim \operatorname{Ker}\nabla_n^z = 0 \quad \text{for } n \geq \tfrac{1}{2} \ . \end{aligned} \tag{2.53}$$

As one can see, these dimensions involve only topological information. Note that, in the case of differentials of weight $\frac{1}{2}$, the index theorem yields no information. To obtain the dimensions for the torus we have just used Liouville's theorem, whereas for the sphere we used Lichnerowicz's theorem on the absence of harmonic spinors on the sphere. For genus $h \geq 2$, topological information is insufficient to determine the number of holomorphic $\frac{1}{2}$ differentials. In Sec. VI we shall see that indeed the number of holomorphic $\frac{1}{2}$ differentials depends on the spin structure for $h \geq 1$ and also on the moduli for $h \geq 3$.

It is convenient to single out now the differentials of special significance in string theory. We shall encounter holomorphic 1-forms or Abelian differentials of the first kind ω_I belonging to $\operatorname{Ker}\nabla_1^z$, whose integrals are the standard Abelian integrals of the first kind. There are h of these, and they generate the first cohomology group of the Riemann surface. One also has meromorphic 1-forms with one double pole (Abelian differentials of the second kind) or with two simple poles of opposite residues (Abelian differentials of the third kind). There is a differential in each case, and it is unique up to addition of holomorphic differentials. These facts about meromorphic forms require the full version of the Riemann-Roch theorem (cf. Sec. VI.C), and their explicit constructions in terms of the prime form will be given in Sec. VI.F. Next we have the holomorphic quadratic differentials belonging to $\operatorname{Ker}\nabla_2^z = \operatorname{Ker}\bar{\partial}_2$, of which there are $3h-3$ (complex ones). Together with their complex conjugates, they span the space of Teichmüller deformations, introduced in Sec. II.C and shown to be identical to the tangent space to moduli space $\mathcal{M}_h$. Finally, for fermionic strings, of special importance will be the holomorphic $\frac{1}{2}$-differentials, which are just zero modes of the Dirac operator, the meromorphic $\frac{1}{2}$-differentials, which will be used to construct fermion propagators, and the $2h-2$ complex $\frac{3}{2}$-differentials, which will make up the odd variables of supermoduli space (cf. Secs. III.E, III.F, and III.G).

To conclude this subsection, we introduce the concepts of Beltrami differentials and quasiconformal vector fields. Beltrami differentials span the space dual to holomorphic quadratic differentials. Thus they are differential forms of weight $(-1,1)$, of the form $\mu = \mu_{\bar{z}}{}^z d\bar{z}(dz)^{-1}$ and $\mu_z{}^z = 0$, and they can be integrated versus quadratic differentials $\phi = \phi_{zz}(dz)^2$:

$$\langle \mu \mid \phi \rangle = \int_M d^2z\, \mu_{\bar{z}}{}^z \phi_{zz} \ . \tag{2.54}$$

[3]Dimensions of kernels of operators with complex indices are understood to be complex dimensions, whereas those of kernels of real operators are understood to be real.

Note that the pairing depends only on the conformal class and not on a particular choice of metric.

Beltrami differentials provide a natural parametrization of the metrics on the Riemann surface. If $d\hat{s}^2=\rho(z)\,|\,dz\,|^2$ is a metric on M, any other metric can be written as

$$ds^2=g_{mn}d\xi^m d\xi^n=\tilde{\rho}(z)\,|\,dz+\mu\, d\bar{z}\,|^2 , \qquad (2.55)$$

with $\tilde{\rho}$ real and positive and μ a suitable Beltrami differential. The role of Beltrami differentials is best explained in terms of the associated Beltrami equation

$$\partial_{\bar{z}}w=\mu_{\bar{z}}{}^z\partial_z w . \qquad (2.56)$$

At least for sufficiently small μ, this equation may be solved perturbatively in μ, and a solution is known always to exist locally. It can be written as

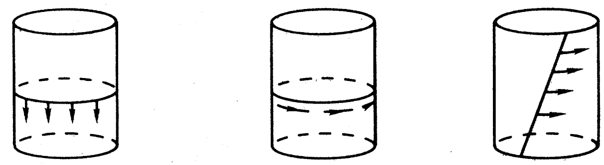
$$w(z,\bar{z})=z+v^z+O(v^z)^2 , \qquad (2.57)$$

where the vector field v^z is defined locally by

$$\nabla_{\bar{z}}v^z=\mu_{\bar{z}}{}^z . \qquad (2.58)$$

Since $ds^2=\tilde{\rho}(z)\,|\,dw/dz\,|^{-2}\,|\,dw\,|^2$, Eq. (2.57) means that the metric ds^2 comes from the metric $d\hat{s}^2$ by a Weyl transformation and a local reparametrization, and we have just restated the familiar fact that locally all conformal structures are the same. The Beltrami equation (2.56) takes real meaning only when we consider it in a global context. Indeed, whether it admits a global solution would tell us whether ds^2 and $d\hat{s}^2$ define the same conformal structure. There are several ways of expressing this more concretely: we could view Eq. (2.56) as defining a family of reparametrizations v^z on local coordinate patches, which, however, may not match. Whether they do can be measured by a vector-valued Cech cohomology class (see, for example, Sec. VI.A). This is the point of view of Kodaira-Spencer deformation theory; or, using uniformization (Sec. IV.A), we can represent M as cosets $\tilde{M}/\Gamma$ and $\tilde{M}/\hat{\Gamma}$ and solve Eq. (2.56) for a solution v^z which may not transform equivariantly; or, finally, consider solutions of the Beltrami equation which may have discontinuities. The last approach is the one we shall often adopt, with the vector fields v^z admitting jump discontinuities along closed curves on the surface M when μ deforms the complex structure. Vector fields of this type can be chosen to induce shifts, stretches, and twists. The corresponding transformations $z\to w(z,\bar{z})$ are called *quasiconformal transformations,* and we can in this way parametrize all deformations of complex structures by Beltrami differentials.

As an example of such vector fields, we may consider a piece of a surface that is a cylinder with Euclidean metric. Such configurations occur in Fenchel-Nielsen coordinates, in the light-cone diagrams of Mandelstam (1974a, 1974b, 1974c), or in the closed-string sector of the open-string field theory of Witten (1986a, 1986b), and they were explicitly given in D'Hoker and Giddings (1987). In Fig. 5 we illustrate the contours of discontinuity for the quasiconformal vector fields generating stretches (a), twists (b), or shifts (c) of the cylinder. Their analytic expressions are



(a) (b) (c)

FIG. 5. Quasiconformal vector fields v: (a) generating stretches; (b) generating twists; (c) generating shifts of the cylinder.

$$\begin{aligned} v^z_{(a)}&=\frac{z+\bar{z}}{2a}H[a-\mathrm{Re}(z)] ,\\ v^z_{(b)}&=i\frac{z+\bar{z}}{2a}H[a-\mathrm{Re}(z)] ,\\ v^z_{(c)}&=\frac{z-\bar{z}}{2\alpha}+i\frac{\theta(z+\bar{z})}{2\Delta\tau} , \end{aligned} \qquad (2.59)$$

where α is the radius of the cylinder, $\Delta\tau$ is its height, a is a parameter specifying the location of the discontinuity, and H is the Heaviside function.

It may be helpful at this point to clarify the tensor structure of moduli space. If we represent a conformal structure m by choosing a representative metric $g_{z\bar{z}}$, there is very little difference between Beltrami differentials and quadratic differentials, since we can raise and lower indices using $g_{z\bar{z}}$ to pass back and forth between the two notions. However, if we do not make such a choice, it is the Beltrami differentials that should be viewed as tangent vectors to moduli. In fact, Eq. (2.56) shows how to deform holomorphic structures without any choice of metrics. Since we still have to take into account reparametrization invariance, we see that

$$T(\mathcal{M}_h)=\frac{\{\text{Beltrami differentials}\}}{\{\text{Range}\partial_{\bar{z}} \text{ on vectors}\}} .$$

The pairing (2.54) exhibits, then, the quadratic differentials as cotangent vectors to moduli space. We now have a different way of explaining why this distinction disappears when a metric $g_{z\bar{z}}$ on the worldsheet is chosen. Such a metric provides a pairing on tensors, and hence on the tangent space to moduli at m. With this pairing, covariant and contravariant tensors on moduli space can be identified. This is why quadratic differentials appeared earlier [cf. Eq. (2.35)] as tangents to moduli.

F. Determinants and Weyl anomalies

We now study the behavior of the determinants of the Laplacians $\Delta_n^{(\pm)}$ under a Weyl scaling. These operators in general can have zero modes, which require special care. Let

$$N_n^+ = \dim \mathrm{Ker}(\nabla_{n+1}^z)^\dagger = \dim \mathrm{Ker}\Delta_n^{(+)} ,$$
$$N_n^- = \dim \mathrm{Ker}\nabla_n^z = \dim \mathrm{Ker}\Delta_n^{(-)} , \tag{2.60}$$

and let ϕ_j be a basis for $\mathrm{Ker}\nabla_n^z$ and ψ_α a basis for $\mathrm{Ker}(\nabla_{n+1}^z)^\dagger$. From Eqs. (2.42) and (2.44), it is evident that $N_n^\pm$ do not change under Weyl transformations. In fact, when one changes the metric from $\hat{g}_{mn}$ to $g_{mn} = e^{2\sigma}\hat{g}_{mn}$, the ϕ_j's do not change, whereas $\psi_\alpha = e^{2n\sigma}\hat{\psi}_\alpha$.

The regularized determinants can be defined by the heat-kernel, short-time cutoff procedure:

$$\ln \det{}'\Delta_n^{(\pm)} = -\int_\varepsilon^\infty \frac{dt}{t}(\mathrm{Tr}e^{-t\Delta_n^{(\pm)}} - N_n^\pm) . \tag{2.61}$$

Deletion of the zero modes from the determinant is indicated by a prime and requires the subtraction of the constants $N_n^\pm$, which makes the integral converge at $t=\infty$. Upon performing an infinitesimal Weyl transformation $\delta\sigma$, we have

$$\delta \ln \det{}'\Delta_n^{(\pm)} = \int_\varepsilon^\infty dt\, \mathrm{Tr}(\delta\Delta_n^{(\pm)} e^{-t\Delta_n^{(\pm)}}) .$$

The changes in the covariant derivatives and Laplacians follow from

$$\begin{aligned} \delta\nabla_n^z &= -2\delta\sigma\nabla_n^z , \\ \delta\nabla_z^n &= 2n\,\delta\sigma\nabla_z^n - 2n\,\nabla_z^n\delta\sigma , \\ \delta\Delta_n^{(-)} &= 2(n-1)\delta\sigma\Delta_n^{(-)} + 4n\,\nabla_z^{n-1}\delta\sigma\nabla_n^z , \\ \delta\Delta_n^{(+)} &= -2\delta\sigma\Delta_n^{(+)} - 4n\,\nabla_{n+1}^z\delta\sigma\nabla_z^n - 2n\,\Delta_n^{(+)}\delta\sigma , \end{aligned} \tag{2.62}$$

so that, for example,

$$\mathrm{Tr}(\delta\Delta_n^{(-)}e^{-t\Delta_n^{(-)}}) = \mathrm{Tr}[2(n-1)\delta\sigma\Delta_n^{(-)}e^{-t\Delta_n^{(-)}} - 2n\,\delta\sigma\Delta_{n-1}^{(+)}e^{-t\Delta_{n-1}^{(+)}}] . \tag{2.63}$$

Here we have used the rearrangement formula $e^{-AB}A = Ae^{-BA}$. Thus we get

$$\delta \ln \det{}'\Delta_n^{(-)} = \int_\varepsilon^\infty dt\, \mathrm{Tr}[2(n-1)\delta\sigma\Delta_n^{(-)}e^{-t\Delta_n^{(-)}} - 2n\,\delta\sigma\Delta_{n-1}^{(+)}e^{-t\Delta_{n-1}^{(+)}}] .$$

The t integral is easily carried out, and one obtains

$$\delta \ln \det{}'\Delta_n^{(-)} = -2(n-1)\mathrm{Tr}\delta\sigma e^{-t\Delta_n^{(-)}}\Big|_\varepsilon^\infty + 2n\,\mathrm{Tr}\delta\sigma e^{-t\Delta_{n-1}^{(+)}}\Big|_\varepsilon^\infty . \tag{2.64}$$

As $t\to\infty$, the heat kernels reduce to the projection operators onto $\mathrm{Ker}\nabla_n^z$ and $\mathrm{Ker}(\nabla_n^z)^\dagger$, respectively. Thus we have

$$\lim_{t\to\infty} \mathrm{Tr}\delta\sigma e^{-t\Delta_n^{(-)}} = \sum_{j=1}^{N_n^-} \langle \phi_j \mid \delta\sigma\phi_j \rangle . \tag{2.65}$$

On the other hand, the change in the finite-dimensional determinants of products of zero modes is given by

$$\begin{aligned} \delta \ln \det\langle \phi_j \mid \phi_k \rangle &= \delta\, \mathrm{tr} \ln\langle \phi_j \mid \phi_k \rangle \\ &= \mathrm{tr}\delta\langle \phi_j \mid \phi_k \rangle \\ &= \sum_{j=1}^{N_n^-} (2-2n)\langle \phi_j \mid \delta\sigma\phi_j \rangle , \end{aligned} \tag{2.66}$$

where we have used the fact that the basis ϕ_j was chosen orthonormal. With the analogous result for the zero modes ψ_α, we find

$$\delta \ln \frac{\det{}'\Delta_n^{(-)}}{\det\langle \phi_j \mid \phi_k \rangle \det\langle \psi_\alpha \mid \psi_\beta \rangle} = 2(n-1)\mathrm{Tr}\delta\sigma e^{-\varepsilon\Delta_n^{(-)}} - 2n\,\mathrm{Tr}\delta\sigma e^{-\varepsilon\Delta_{n-1}^{(+)}} . \tag{2.67}$$

From the short-time expansions of the heat kernels, derived in Appendix B.

$$\begin{aligned} \mathrm{Tr}\delta\sigma e^{-\varepsilon\Delta_n^{(+)}} &= \frac{1}{4\pi\varepsilon}\int_M d^2\xi\sqrt{g}\,\delta\sigma + \frac{1+3n}{12\pi}\int_M d^2\xi\sqrt{g}\,R\,\delta\sigma + O(\varepsilon) , \\ \mathrm{Tr}\delta\sigma e^{-\varepsilon\Delta_n^{(-)}} &= \frac{1}{4\pi\varepsilon}\int_M d^2\xi\sqrt{g}\,\delta\sigma + \frac{1-3n}{12\pi}\int_M d^2\xi\sqrt{g}\,R\,\delta\sigma + O(\varepsilon) , \end{aligned} \tag{2.68}$$

one finally obtains

$$\delta \ln \frac{\det{}'\Delta_n^{(-)}}{\det\langle \phi_j \mid \phi_k \rangle \det\langle \psi_\alpha \mid \psi_\beta \rangle} = -\frac{1}{2\pi\varepsilon}\int_M d^2\xi\sqrt{g}\,\delta\sigma - \frac{6n^2-6n+1}{6\pi}\int_M d^2\xi\sqrt{g}\,R\,\delta\sigma . \tag{2.69}$$

Putting all together and integrating the infinitesimal Weyl transformation using Eq. (2.31), one finds

$$\frac{\det{}'\Delta_n^{(\pm)}}{\det\langle \phi_j \mid \phi_k \rangle_g \det\langle \psi_\alpha \mid \psi_\beta \rangle_g} = \frac{\det{}'\hat{\Delta}_n^{(\pm)}}{\det\langle \hat{\phi}_j \mid \hat{\phi}_k \rangle_{\hat{g}} \det\langle \hat{\psi}_\alpha \mid \hat{\psi}_\beta \rangle_{\hat{g}}} \times e^{-2(6n^2\pm 6n+1)S_L(\sigma)} . \tag{2.70}$$

Here the Liouville action is

$$S_L(\sigma) = \frac{1}{12\pi}\int_M d^2\xi\sqrt{\hat{g}}\,[\tfrac{1}{2}\hat{g}^{mn}\partial_m\sigma\partial_n\sigma + \mu^2(e^{2\sigma}-1) + R_{\hat{g}}\sigma] . \tag{2.71}$$

In particular, the formulas needed for bosonic string theory are given by

$$\begin{aligned} \left[\frac{\det P_1^\dagger P_1}{\det\langle \phi_j \mid \phi_k \rangle_g}\right]^{1/2} &= \left[\frac{\det \hat{P}_1^\dagger \hat{P}_1}{\det\langle \hat{\phi}_j \mid \hat{\phi}_k \rangle_{\hat{g}}}\right]^{1/2} e^{-26S_L(\sigma)} , \\ \left[\frac{8\pi^2\det{}'\Delta_g}{\int_M d^2\xi\sqrt{g}}\right]^{1/2} &= \left[\frac{8\pi^2\det{}'\Delta_{\hat{g}}}{\int_M d^2\xi\sqrt{\hat{g}}}\right]^{1/2} e^{-S_L(\sigma)} . \end{aligned} \tag{2.72}$$

In the expression for the scalar Laplacian, we have deleted the determinant of holomorphic Abelian differentials ω_I, because it is Weyl invariant all by itself, as can be seen from its explicit expression:

$$\det\langle \omega_I \mid \omega_J \rangle_g = \det\int_M d^2\xi\sqrt{g}\,g^{z\bar z}\{(\omega_I^*)_{\bar z}(\omega_J)_z + \text{c.c.}\}$$
$$= \det\langle \omega_I \mid \omega_J \rangle_{\hat g} \ . \qquad (2.73)$$

We are now also in a position to prove the Riemann-Roch theorem [Eq. (2.50)]. First we have

$$\dim \operatorname{Ker}\nabla_z^n - \dim \operatorname{Ker}\nabla^z_{n+1} = \dim \operatorname{Ker}\Delta_n^{(+)} - \dim \operatorname{Ker}\Delta_{n+1}^{(-)}$$
$$= \operatorname{Tr} e^{-\varepsilon\Delta_n^{(+)}} - \operatorname{Tr} e^{-\varepsilon\Delta_{n+1}^{(-)}} \ . \qquad (2.74)$$

Letting $\varepsilon \to 0$, we recover Eq. (2.50) in view of (2.68).

A standard reference on conformal anomalies is Coleman and Jackiw (1971), and a discussion of global and local conformal symmetry is given by Polchinski (1988a). The original calculation of the above Weyl anomaly is due to Polyakov (1981a) and has been clarified by Di Vecchia *et al.* (1982a, 1982b), Friedan (1982), Fujikawa (1982), Alvarez (1983), and Ambjørn *et al.* (1986); the articles of Di Vecchia *et al.*, Alvarez, and Ambjørn *et al.* also treat the case with boundaries. The first careful account of the crucial zero-mode factors is that of Alvarez (1983). Different aspects of Weyl invariance in string theory were treated by Fujikawa (1987) and Tanii (1987).

G. Amplitudes as integrals over moduli space for $h \geq 2$

We finally come to a detailed discussion of cancellation of Weyl anomalies. We shall deal with the case $h \geq 2$ first and present the cases of the torus and the sphere in the next section. The only modification will involve the presence of conformal Killing vectors.

1. The quantum measure and conformal invariance

To carry out the Dg integral we parametrize the space of metrics by $g = \exp(\delta v)e^{2\sigma}\hat g$, with $\hat g$ in a slice $\hat S$ transversal to the orbits of Weyl(M) and of Diff$_0$(M). Such a slice may be taken, for example, within $\mathcal{M}_{\text{const}}$, which guarantees right away that it is transverse to Weyl(M), by the uniqueness arguments of Sec. II.D. Here $\exp(\delta v)$ denotes integrated elements of Diff$_0$(M). [Recall that, for a vector field δv, the action $\exp(\delta v)$ on a metric is to replace its value at a given point on M by its value at the point on the integral curve of δv, a unit of time away.]

The change of variables $g \to (\sigma, v, \hat g)$ requires a Jacobian which can be evaluated from the decomposition (2.24).

It will be helpful to keep the following picture in mind. A Riemannian manifold, parametrized by a set of coordinates $x_1, \ldots, x_n$, is endowed with the standard volume element $\sqrt{g}\,d^nx$, which may be viewed as the volume (with respect to this metric) of the n coordinate vectors. In our case, we have the coordinates σ and v, which are functions on M, together with real coordinates m_j, $j = 1, \ldots, 6h-6$ for $\hat S$. Each element of $\hat S$ is a metric $\hat g(m)$, and tangents to $\hat S$ are symmetric two-tensors f_j defined by

$$\delta g(m) = \sum_{j=1}^{3h-3} \delta m_j \hat f_j \ .$$

Thus the coordinate vectors along $\hat S$ are $\delta\sigma\hat g$, $\hat P_1\delta v$, and $\hat f_j$. Since we are interested in computing the Jacobian at an arbitrary point in $\mathcal{M}$, we shall apply the diffeomorphism $\exp(\delta v)$ and the Weyl rescaling $e^{2\sigma}$ under which $\hat f_j$ scales as $f_j = e^{2\sigma}\hat f_j$. The measure is

$$Dg_{mn} = \operatorname{Vol}_g(g\delta\sigma, P_1(\delta v), f_j)D\sigma\, Dv\, dm \ , \qquad (2.75)$$

and P_1, whose definition [Eq. (2.23)] requires a metric, is always chosen with respect to g. Using the orthogonal decomposition (2.24) of δg_{mn}, we see that the first two entries are orthogonal, and that the last one may be restricted to the orthogonal projection of f_j onto Ker$P_1^\dagger$ (see Fig. 6). When we use the orthogonality, the volume then decomposes into a product, and we obtain

$$Dg_{mn} = \operatorname{Vol}_g(g\delta\sigma)\operatorname{Vol}_g(P_1\delta v)$$
$$\times \operatorname{Vol}_g(f_j \perp \text{proj. Ker}P_1^\dagger)D\sigma\, Dv\, dm \ .$$

Ultralocality implies that the first factor is proportional to a factor of the type (2.14) and may be ignored. Moreover,

$$\operatorname{Vol}_g(P_1\delta v) = (\det P_1^\dagger P_1)^{1/2} \ .$$

Finally, recalling that Ker$P_1^\dagger$ was spanned by basis vectors ϕ_j,

$$\operatorname{Vol}_g(f_j \perp \text{proj. Ker}P_1^\dagger) = \frac{\det\langle f_j \mid \phi_k\rangle_g}{\det\langle \phi_j \mid \phi_k\rangle_g^{1/2}} \ .$$

Putting all together, we have

$$Dg_{mn} = \frac{\det\langle f_j \mid \phi_k\rangle_g}{\det\langle \phi_j \mid \phi_k\rangle_g^{1/2}}(\det P_1^\dagger P_1)^{1/2}D\sigma\, Dv\, dm \ . \qquad (2.76)$$

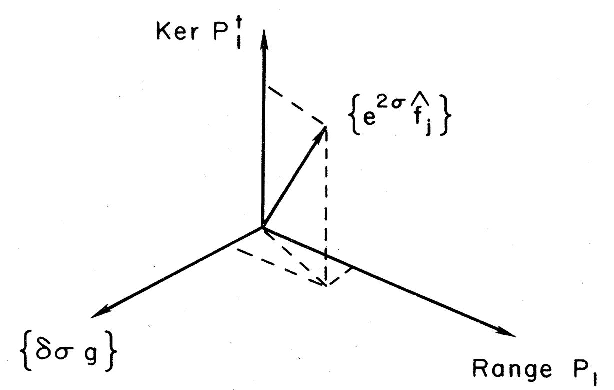


FIG. 6. Orthogonal decomposition of $f_j = e^{2\sigma}\tilde f_j$.

The Weyl dependence of this measure is readily exhibited if we notice that

$$\langle f_j \mid \phi_k \rangle_g = \langle \hat{f}_j \mid \phi_k \rangle_{\hat{g}} = \langle \mu_j \mid \phi_k \rangle ,$$

where

$$\mu_{j\bar{z}}^{z} = \hat{g}^{z\bar{z}} \hat{f}_{j\bar{z}\bar{z}}$$

are the Beltrami differentials corresponding to moduli deformations along the slice $\hat{S}$. Using the Weyl rescaling formulas of (2.72), we get

$$Dg_{mn} = \left[\frac{\det \hat{P}_1^\dagger \hat{P}_1}{\det \langle \phi_j \mid \phi_k \rangle_{\hat{g}}} \right]^{1/2} \det \langle \mu_j \mid \phi_k \rangle \times e^{-26 S_L(\sigma)} D\sigma \, Dv^m \prod_{j=1}^{6h-6} dm_j \ . \tag{2.77}$$

This is the desired result for $h \geq 2$, and we shall see that only a slight extension of it is necessary for $h=0,1$; the extension leaves the Weyl independence unaltered. We now discuss the choice of the normalization factor $\mathcal{N}$ in Eqs. (2.11) and (2.12). There are three possibilities.

(a) The metric $G_{\mu\nu}$ in $I(x,g)$ is flat Euclidean, in which case the x^μ integration may be carried out, and we find for genus h

$$\langle V_{i_1}(k_1) \cdots V_{i_n}(k_n) \rangle_h = \int_{\mathcal{T}_h} \prod_{j=1}^{6h-6} dm_j \frac{\det \langle \mu_j \mid \phi_k \rangle}{\det \langle \phi_j \mid \phi_k \rangle_{\hat{g}}^{1/2}} \int \frac{D\sigma \, Dv}{\mathcal{N}} e^{-\lambda\chi} (\det \hat{P}_1^\dagger \hat{P}_1)^{1/2} \left[\frac{8\pi^2}{\int d^2\xi \sqrt{g}} \det{}' \hat{\Delta} \right]^{-d/2} \times e^{-(26-d) S_L(\sigma)} \langle\langle V_{i_1}(k_1) \cdots V_{i_n}(k_n) \rangle\rangle \ . \tag{2.78}$$

For $d=26$, we have Weyl invariance, and it is natural to set that $\mathcal{N} = \text{Vol}(\text{Diff}(M)) \times \text{Vol}(\text{Weyl}(M))$. Note that the inserted vertex operators are constructed in such a way that possible Weyl anomalies are required to cancel (see Sec. VIII). Upon integrating out the reparametrization vector fields, we produce a factor of $\text{Vol}(\text{Diff}_0(M))$, which partially cancels the analogous factor in $\mathcal{N}$ and, in view of Eq. (2.33), reduces the integral from $\mathcal{T}_h$ to $\mathcal{M}_h$:

$$\langle V_{i_1}(k_1) \cdots V_{i_n}(k_n) \rangle_h = e^{-\lambda\chi} \int_{\mathcal{M}_h} \prod_{j=1}^{6h-6} dm_j \frac{\det \langle \mu_j \mid \phi_k \rangle}{\det \langle \phi_j \mid \phi_k \rangle_{\hat{g}}^{1/2}} (\det \hat{P}_1^\dagger \hat{P}_1)^{1/2} \times \left[\frac{8\pi^2}{\int_M d^2\xi \sqrt{\hat{g}}} \det{}' \Delta_{\hat{g}} \right]^{-13} \langle\langle \hat{V}_{i_1}(k_1) \cdots \hat{V}_{i_n}(k_n) \rangle\rangle \ . \tag{2.79}$$

Here $\langle\langle \ \rangle\rangle$ denotes the expectation value where only the x integral has been performed.

If we choose $\hat{S}$ to be a $6h-6$ dimensional slice within $\mathcal{M}_{\text{const}}$ and transversal to the orbits of $\text{Diff}_0(M)$, the Weil-Peterson measure $d(\text{WP})$ is related to the measure $\prod dm_j$ on the slice $\hat{S}$ by

$$d(\text{WP}) = \frac{\det \langle \mu_j \mid \phi_k \rangle}{\det \langle \phi_j \mid \phi_k \rangle_{\hat{g}}^{1/2}} \prod_{j=1}^{6h-6} dm_j \ , \tag{2.80}$$

and we conclude

$$\langle V_{i_1}(k_1) \cdots V_{i_n}(k_n) \rangle_h = e^{-\lambda\chi} \int_{\mathcal{M}_h} d(\text{WP}) (\det \hat{P}_1^\dagger \hat{P}_1)^{1/2} \left[\frac{8\pi^2}{\int_M d^2\xi \sqrt{\hat{g}}} \det{}' \Delta_{\hat{g}} \right]^{-13} \langle\langle \hat{V}_{i_1}(k_1) \cdots \hat{V}_{i_n}(k_n) \rangle\rangle \ . \tag{2.81}$$

Observe that, unlike in gauge theories, the normalization factor $\mathcal{N}$ depends on the metric and has been placed inside the functional integrals. The justification for the above procedure may be given by appealing to the principle of ultralocality. Define a norm on vector fields δv^m analogous to Eqs. (2.13) and (2.21),

$$\|\delta v^m\|^2 = \int_M d^2\xi \sqrt{g} \, g_{mn} \delta v^m \delta v^n \ , \tag{2.82}$$

and consider the integral

$$\int Dv^m e^{-\lambda \|\delta v^m\|^2} \ . \tag{2.83}$$

The principle of ultralocality implies that this integral must be given by an expression of the form

$$\exp \left[-\mu_2^2(\lambda) \int_M d^2\xi \sqrt{g} \right] \tag{2.84}$$

for some function $\mu_2^2(\lambda)$. As $\lambda \to 0$, Eq. (2.83) tends to $\text{Vol}(\text{Diff}_0(M))$, while (2.84) simply leads to a renormalization of the area term in Eq. (2.10) analogous to the one discussed in Eq. (2.14). Thus $\text{Vol}(\text{Diff}_0(M))$ is in effect irrelevant. The same argument applies to $\text{Vol}(\text{Weyl}(M))$. As for the "volume" of the mapping class group (=cardinality of MCG_h), it does not depend on g_{mn} but only on the topology. The only nontrivial volume element in $\mathcal{N}$ is that of the mapping class group, and it can thus be pulled out of under the integration, just as in the case of gauge theories, with the difference that the group is now discrete. The net effect, as mentioned above, is to reduce the integral over Teichmüller space to one over a fundamental domain for the mapping class group, which is the same as moduli space.

Critical dimensions of string theory in a flat Euclidean or Minkowski background have been obtained in a variety of ways over the years. In the light-cone gauge, it arises by insisting on Lorentz invariance, as pointed out by Brink and Nielsen (1973), Goddard *et al.* (1973), and Mandelstam (1974a, 1974b, 1974c). In the covariant approach, it appears by insisting on decoupling from the scattering amplitudes of negative norm states, as analyzed, for example, by Brower and Thorn (1971). Notice that, in our case, Weyl invariance will eliminate vertex operators producing unphysical states as well. In Sec. II.I we shall discuss how the critical dimension arises in a treatment with ghosts.

The above formulas for the Polyakov measure were obtained by D'Hoker and Phong (1986a) and independently by Moore and Nelson (1986). They provide a starting point for calculations of covariant multiloop amplitudes in the bosonic string.

In the days of dual models, multiloop amplitudes were considered by Kaku and Yu (1970), Lovelace (1970), Alessandrini (1971), Alessandrini and Amati (1971), and Kaku and Scherk (1971). These constructions were based on the assumption that the on-shell scattering vertices could actually be used as off-shell internal vertices, and unphysical states were generally not projected out. These shortcomings have been overcome more recently by Montonen (1974) and Neveu and West (1987a, 1987b) and through the introduction of ghost fields by Di Vecchia, Frau, *et al.* (1987a, 1987b) and Petersen and Sidenius (1987). It seems, however, that a precise definition of the integration region for moduli space is not available in these approaches. In the light-cone gauge, Mandelstam (1986a, 1986b) has also obtained explicit formulas for multiloop amplitudes. Finally, open-string amplitudes in the Polyakov string have been dealt with, for example, by Boulware and Newman (1986) and Burgess and Morris (1987a, 1987b).

From the above point of view, it is natural to divide by the volume of the mapping class group. This choice cannot really be justified within the context of the Polyakov ansatz; a further physical principle is required. This principle is the unitarity of the scattering amplitudes. To investigate unitarity, one may compare the Polyakov results with those of the manifestly unitary interacting string picture of Mandelstam (1973a, 1974a, 1986a, 1986b). Such a study was carried out by D'Hoker and Giddings (1987), and it was found that dividing out exactly once by the volume of the mapping class group leads to unitarity of all scattering amplitudes. For the bosonic string, unitarity is of course formal because of the presence of the tachyon. We shall come back to this question in Sec. V.G.

(b) $I(x^\mu,g)$ does not lead to a Weyl-invariant theory, for example, when d is different from 26. The $D\sigma$ integral is then not redundant, and the value $\mathcal{N}=\mathrm{Vol}(\mathrm{Diff}(M))$ is presumably the correct choice if a unitary theory exists at all. Since all determinants are real, there are no global gravitational anomalies, and we may factor out the mapping class group by restricting integrals from Teichmüller to moduli space:

$$\langle V_{i_1}(k_1)\cdots V_{i_n}(k_n)\rangle=\sum_{h=0}^{\infty}\int_{\mathcal{M}_h}d(\mathrm{WP})\int D\sigma\, Dx^\mu(\det\hat{P}_1^\dagger\hat{P}_1)^{1/2}V_{i_1}(k_1)\cdots V_{i_n}(k_n)e^{-26S_L(\sigma)}e^{-I(x,\hat{g})}\ . \tag{2.85}$$

No satisfactory quantization of the Liouville model has been achieved to date for closed strings, despite attempts by D'Hoker, Freedman, and Jackiw (1983) and Braaten *et al.* (1984). In the case of open strings alone, it seems possible to obtain a tachyon-free string theory in special dimensions 1, 7, 13, 19, and 25 with discrete mass spectrum, as discussed by Gervais and Neveu (1982, 1983, 1984, 1986), Marnelius (1983), and Bilal and Gervais (1987a, 1987b). An interesting proposal has been made very recently by Polyakov (1987a) based on SL(2,**R**) current algebra.

(c) For general background metric $G_{\mu\nu}(x)$ (and possibly antisymmetric tensor and dilaton fields), insisting upon Weyl invariance for the string tree-level theories leads to equations for the background fields and for the dimension of space-time. These equations were studied by Friedan (1980), Lovelace (1984, 1986), Callan *et al.* (1985), Fradkin and Tseytlin (1985a, 1985b), Callan, Klebanov, and Perry (1986), and Fridling and Jevicki (1986). The critical dimension $d=26$ emerges then as the condition of vanishing of the first coefficient of the dilation β function. It is possible that the dimension of space-time is dynamical, as suggested by Polyakov (1986). The solutions to these equations provide possible spaces for consistent string propagation. Higher string loop effects will in general again spoil the Weyl invariance, even if these background equations are satisfied. Fischler and Susskind (1986a, 1986b) have, however, argued that such effects should be understood as loop corrections to the string background equations of motion. Explicit examples of how this might happen have been presented for the open-string case by Callan *et al.* (1987, 1988).

2. Scalar Green's function and amplitudes

The Green's function is defined by $G(z,w)=\langle x(z)x(w)\rangle$, and in locally conformal coordinates z, with metric $ds^2=\rho\, dz\, d\bar{z}$, satisfies

$$\int d^2z\sqrt{g}\,G(z,w)=0\ ,$$

$$\partial_z\partial_{\bar{z}}G(z,w)=-2\pi\delta(z-w)+\frac{2\pi g_{z\bar{z}}}{\int d^2z\sqrt{g}}\ , \tag{2.86}$$

$$\partial_z\partial_{\bar{w}}G(z,w)=2\pi\delta(z-w)-\pi\sum_{I,J}\omega_I(z)(\mathrm{Im}\Omega)^{-1}_{IJ}\bar{\omega}_J(w)\ .$$

The "period matrix" Ω_{IJ} is the matrix of periods of the

Abelian integrals associated with ω_I. It will be defined in detail in Sec. VI.D. It is invariant under $\mathrm{Diff}_0(M)$ and $\mathrm{Weyl}(M)$, and characterizes the conformal structure of the surface. In Eq. (2.86), the additional terms besides the δ functions result from projections on the spaces of zero modes of ∇_0^z and ∇_1^z, respectively. They break Weyl invariance and, in fact, under scalings $g=e^{2\sigma}\hat{g}$ the two-point functions will transform as

$$G(z,w)=\hat{G}(z,w)-\frac{1}{\int d^2z\sqrt{g}}\int d^2v\sqrt{g}\,[\hat{G}(z,v)+\hat{G}(v,w)]+\frac{1}{\left[\int d^2z\sqrt{g}\,\right]^2}\int\int d^2v\,d^2y\sqrt{g(v)}\sqrt{g(y)}\hat{G}(v,y)\ . \qquad (2.87)$$

For coincident points, we may again regularize $G(z,w)$ by a heat kernel with small-time cutoff procedure. The regularized Green's function $G_R(z,z)$ at coincident points will satisfy a similar scaling law:

$$G_R(z,z)=\hat{G}_R(z,z)+2\sigma(z)-\frac{2}{\int d^2z\sqrt{g}}\int d^2v\sqrt{g}\,\hat{G}(z,v)+\frac{1}{\left[\int d^2z\sqrt{g}\,\right]^2}\int\int d^2v\,d^2y\sqrt{g(v)}\sqrt{g(y)}\hat{G}(v,y)\ , \qquad (2.88)$$

with the key additional term $\sigma(z)$ on the right-hand side.

The field x itself does not have a definite conformal dimension, but derivatives of x as well as $\exp(ikx)$ do, and our task is to determine the (Weyl-invariant) forms of their correlation functions, taking into account proper renormalization for composites such as $\exp(ikx)$. For example,

$$\langle \partial_z x\,\partial_w x\rangle=\partial_z\partial_w G(z,w)$$

is Weyl invariant in view of Eq. (2.87), and $\partial_{\bar z}x$ has conformal dimension (1,0). More subtle are correlation functions of the operator

$$V_k(z)=\rho^{k^2/2}e^{ikx(z)}\ . \qquad (2.89)$$

Since the exponential should be viewed as normal ordered, we can replace the Green's function at points (z,z) and (w,w) by their regularizations, and we arrive at

$$\begin{aligned}\langle V_k(z)V_{k'}(w)\rangle&=\delta(k+k')F(z,w)^{-k^2}\ ,\\ F(z,w)&=[\rho(z)\rho(w)]^{-1/2}\\ &\quad\times\exp[-G(z,w)+\tfrac{1}{2}G_R(z,z)+\tfrac{1}{2}G_R(w,w)]\ .\end{aligned} \qquad (2.90)$$

The crucial feature of $F(z,w)$ is that it is Weyl invariant, as can be deduced from Eqs. (2.87) and (2.88) and behaves like $|z-w|^2$ for z near w. Thus the conformal dimension of the vertex operator $V_k(z)$ is well defined and equal to $(k^2/2,k^2/2)$.

Although the scalar Green's function $G(z,w)$ depends in a more complicated way on the metric, the function $F(z,w)$ can actually be written explicitly in terms of the prime form. In fact, Eq. (2.90) shows that $F(z,w)$ is a single-valued real symmetric expression, transforming in each variable as a $(-\frac{1}{2},-\frac{1}{2})$ differential. Furthermore, it satisfies the equation

$$\partial_z\partial_{\bar z}\ln F(z,w)=2\pi\delta(z-w)-\pi\sum_{I,J}\omega_I(z)(\mathrm{Im}\Omega)^{-1}_{IJ}\bar{\omega}_J(z)\ .$$

All these properties characterize $F(z,w)$ as

$$F(z,w)=\exp\left[-2\pi\,\mathrm{Im}\int_w^z\omega(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\int_w^z\omega\right]\times|E(z,w)|^2\ . \qquad (2.91)$$

Here, Ω is again the period matrix, and $E(z,w)$ is a holomorphic $(-\frac{1}{2},0)$ form in z and w with a single zero at $z=w$, called the prime form. We shall define it in detail in Sec. VI and give a representation for it in terms of ϑ functions. At present, we need only the above-mentioned properties.

The above Green's function allows us to evaluate the scattering amplitudes very explicitly. Consider first the insertion of an exponential factor, universal to all vertex operators, leaving its position on the surface free. Thus we deal with multiple insertions of $V_k(z)$ of Eq. (2.89),

$$\langle\langle V_{k_1}(z_1)\cdots V_{k_n}(z_n)\rangle\rangle=(2\pi)^{26}\delta(k)\exp\left[-\tfrac{1}{2}\sum_{i\neq j=1}^{n}k_i\cdot k_j G(z_i,z_j)+\sum_{i=1}^{n}\tfrac{1}{2}k_i^2[\ln\rho(z_i)-G_R(z_i,z_i)]\right]\ . \qquad (2.92)$$

Using Eq. (2.90) one may recast this solely in terms of F,

$$\langle\langle V_{k_1}(z_1)\cdots V_{k_n}(z_n)\rangle\rangle=(2\pi)^{26}\delta(k)\prod_{i<j}F(z_i,z_j)^{k_i\cdot k_j}\ . \qquad (2.93)$$

In the special case of the tachyon, we have

$$V_k=\int d^2z\ V_k(z),\quad V_k(z)=\sqrt{g}\,(z)e^{ik\cdot x(z)}\ ,$$

with $k_\mu k^\mu=2$, and the above formula may be applied directly.

For massless particles, one should rather start from the generating function for amplitudes with one derivative on x,

$$V_k^*(z,\zeta)=\exp[ik\cdot x(z)+\zeta_\mu\partial_z x^\mu(z)+\bar{\zeta}_\mu\partial_{\bar z}x^\mu(z)]\ , \qquad (2.94)$$

so that amplitudes may be gotten from the above by retaining only terms linear in ζ and $\bar\zeta$. Since $k^2=0$, no ρ-dependent prefactor occurs. Correlation functions of V_k^*

may be worked out as easily as those of V_k. Let us just notice here that such correlation functions are already implicit in Eq. (2.93). Indeed, it suffices to replace ξ_μ by the difference between two momenta, taking their insertion points infinitesimally far apart, so Eq. (2.93) may be viewed as a generating function for all amplitudes. Its Weyl invariance guarantees the Weyl invariance of the original vertex operators, as will be explained in Sec. VIII.

To conclude this section, we discuss the role of internal loop momenta. In accord with the radial quantization procedure, the momentum operator measuring the momentum flowing through a contour C is given by

$$P_C^\mu = \oint_C \frac{dx}{2\pi} \partial_z x^\mu(z) . \tag{2.95}$$

The internal (or loop) momenta may be defined as the momenta flowing through the A_I-homology cycles (see Fig. 10 below)

$$P_I^\mu = \oint_{A_I} \frac{dz}{2\pi} \partial_z x^\mu(z) , \tag{2.96}$$

and amplitudes at fixed internal momenta (and fixed genus h) may be introduced by inserting δ functions in the functional integral:

$$\langle\langle V_1 \cdots V_n \rangle\rangle(p_I^\mu) = \left[\frac{8\pi^2 \det' \Delta_g}{\int_M d^2\xi \sqrt{g}} \right]^{13} \int Dx \prod_{I,\mu} \delta\left[p_I^\mu - \oint_{A_I} \frac{dz}{2\pi} \partial_z x^\mu \right] V_1 \cdots V_n e^{-I_0(x)} . \tag{2.97}$$

These amplitudes produce Weyl(M)- and Diff$_0$(M)-invariant amplitudes after the Faddeev-Popov ghost determinant has been taken into account, and as long as $V_1 \cdots V_n$ are physical vertex operators. However, we do not have invariance under the full mapping class group because a choice of homology basis has been made. The full amplitude is of course obtained after integrating over p,

$$\langle\langle V_1 \cdots V_n \rangle\rangle = \int dp_I^\mu \langle\langle V_1 \cdots V_n \rangle\rangle(p_I^\mu) . \tag{2.98}$$

For the special case of exponential insertions, Eq. (2.97) is easily evaluated, and we get

$$\langle\langle V_1 \cdots V_n \rangle\rangle(p_I^\mu) = (\det \operatorname{Im}\Omega)^{13} \left| \exp\left[i\pi p_I^\mu \Omega_{IJ} p_J^\mu + 2\pi i p_I^\mu \sum_i k_i^\mu \int_P^{z_i} \omega_I \right] \right|^2 \prod_{i<j} |E(z_i,z_j)|^{2k_i \cdot k_j} . \tag{2.99}$$

Later on we shall more fully explore the meaning of the above formulas in function of the holomorphic structure of moduli space.

The important observation that regularization produces a factor of $\sigma(z)$ leading to well-defined conformal dimensions is due to Polyakov (1981a). It is the starting point for determining the mass spectrum of the string by requiring Weyl invariance, an issue that will be discussed at length in Sec. VIII. The above careful discussion of scaling laws for two-point functions taking into account zero modes and global issues is due to Verlinde and Verlinde (1987a). They also point out that Eq. (2.90) can be inverted, producing a formula of type (2.87) for the Green's function, with $\hat{G}(z,w)$ on the right-hand side replaced by $\ln F(z,w)$. The basic ideas and some examples of radial quantization are in Fubini, Hanson, and Jackiw (1973). Internal momenta are of course familiar from the dual-model theories, but in the above form they were rediscovered by Verlinde and Verlinde (1987b).

H. Amplitudes for tree and one-loop level

As was explained in Sec. II.C, the tree and one-loop cases do not follow the pattern exhibited for $h \geq 2$. The main complication is that there now exist conformal Killing vectors ψ_α belonging to KerP_1. Thus the operators P_1 and $P_1^\dagger P_1$ should be acting only on the reparametrizations in $(\mathrm{Ker}P_1)^\perp$. It is convenient to treat the cases $h=0$ and 1 separately.

1. Tree-level amplitudes

For $h=0$, we have six real conformal Killing vectors and no moduli parameters. The measure (2.76) must be modified to

$$Dg_{mm} = (\det' P_1^\dagger P_1)^{1/2} D' v^m D\sigma , \tag{2.100}$$

where the prime on $D'v^m$ denotes the fact that it is restricted to $(\mathrm{Ker}P_1)^\perp$. Under a Weyl transformation we obtain from Eq. (2.70) that

$$\left[\frac{\det' P_1^\dagger P_1}{\det\langle \psi_\alpha | \psi_\beta \rangle_g} \right]^{1/2} = \left[\frac{\det' \hat{P}_1^\dagger \hat{P}_1}{\det\langle \hat{\psi}_\alpha | \hat{\psi}_\beta \rangle_{\hat{g}}} \right]^{1/2} e^{-26 S_L(\sigma)} \tag{2.101}$$

and hence

$$Dg_{mn} = (\det' \hat{P}_1^\dagger \hat{P}_1)^{1/2} \left[\frac{\det\langle \psi_\alpha | \psi_\beta \rangle_g}{\det\langle \hat{\psi}_\alpha | \hat{\psi}_\beta \rangle_{\hat{g}}} \right]^{1/2} \times e^{-26 S_L(\sigma)} D\sigma \, D' v^m . \tag{2.102}$$

Note that the above ratio of finite-dimensional determinants is precisely the ratio of volumes[4] of $\mathrm{Ker}P_1$ and $\mathrm{Ker}\hat{P}_1$, so that we have for the analog of Eq. (2.77)

$$Dg_{mn}=\frac{1}{\mathrm{Vol}(\mathrm{Ker}\hat{P}_1)}(\det{}'\hat{P}_1^\dagger\hat{P}_1)^{1/2}e^{-26S_L(\sigma)}D\sigma\, Dv^m\ , \tag{2.103}$$

where all reparametrizations v^m are now included in Dv^m. Assuming that we work in the critical dimension $d=26$ and adopt the normalization factor $\mathcal{N}=\mathrm{Vol}(\mathrm{Diff}(M))\times\mathrm{Vol}(\mathrm{Weyl}(M))$ as in Eq. (2.78), we have a general formula for tree-level scattering amplitudes:

$$\langle V_{i_1}(k_1^\mu)\cdots V_{i_n}(k_n^\mu)\rangle=ce^{-2\lambda}\langle\langle V_{i_1}(k_1^\mu)\cdots V_{i_n}(k_n^\mu)\rangle\rangle \times\frac{1}{\mathrm{Vol}(\mathrm{Ker}\hat{P}_1)}\ . \tag{2.104}$$

Here the symbol $\langle\langle\ \rangle\rangle$ denotes again the fact that the functional integral over x alone has been performed. The determinants of $\Delta_{\hat{g}}$ and $\hat{P}_1^\dagger\hat{P}_1$ are constants, since there are no moduli parameters left, and their contribution has been denoted by c. This constant has been computed by Weisberger (1987a).

Of course, $\mathrm{Vol}(\mathrm{Ker}\hat{P}_1)$ is infinite for the sphere, so the above expression can be nonzero only if the vacuum expectation value of the vertex operators involves a similar infinite factor. This can indeed happen because the vertex operators are integrals over the sphere of local functions:

$$V_j(k_j^\mu)=\int_M d^2z_j\sqrt{g}\,U_j[\varepsilon_j,Dx]e^{ik_j^\mu x_\mu}\ , \tag{2.105}$$

where U_j is a polynomial in derivatives of x and depends linearly on the polarization tensor ε_j (see Sec. VIII for details). Since V_j is reparametrization invariant, it is in particular invariant under the group $\mathrm{PSL}(2,C)=\mathrm{SL}(2,C)/\{\pm 1\}$ of conformal Killing transformations, acting on the coordinates z_j of the compactified plane (i.e., the sphere) by Möbius transformations

$$z_j\mapsto\frac{az_j+b}{cz_j+d}\quad\text{with } ad-bc=1\ . \tag{2.106}$$

This group acts freely on all the z_j's, and we may fix three arbitrary distinct points with the help of a unique Möbius transformation in $\mathrm{PSL}(2,C)$ and factor out the $\mathrm{PSL}(2,C)$-invariant volume. The latter is constructed by recalling that, under the Möbius transformation of Eq. (2.106), we have

$$dz_j\mapsto\frac{dz_j}{(cz_j+d)^2}\ ,\qquad z_i-z_j\mapsto\frac{z_i-z_j}{(cz_i+d)(cz_j+d)}\ , \tag{2.107}$$

so that the volume element on $\mathrm{Ker}\hat{P}_1$ is a constant times

$$d\mu=\frac{d^2z_id^2z_jd^2z_k}{|z_i-z_j|^2|z_j-z_k|^2|z_k-z_i|^2}\ , \tag{2.108}$$

where i,j,k denote any three distinct points among $1,2,\ldots,n$. The fact that three such points must be fixed, and the appearance of the difference factors in Eq. (2.108), are familiar from dual-model calculations.

To see how the volume of the conformal Killing group is factored out, it is instructive to compute the scattering amplitude for tachyonic particles only. In this case we have $U_j[\varepsilon_j,Dx]=\hat{\varepsilon}$ in Eq. (2.105) with $\hat{\varepsilon}$ constant and $k_\mu k^\mu=2$. We need the Green's function $G(z,z')$ for scalars on the sphere. Due to the zero mode of Δ, we can invert Δ only on the space of functions orthogonal to constants

$$\Delta_z G(z,z')=4\pi\delta^2(z,z')-\frac{4\pi}{\int_M d^2z\sqrt{g}}\ . \tag{2.109}$$

With the standard metric on the sphere

$$g_{mn}=\frac{2\delta_{mn}}{(1+|z|^2)^2} \tag{2.110}$$

it is easily checked that

$$G(z,z')=-\ln\frac{|z-z'|^2}{(1+|z|^2)(1+|z'|^2)}\ . \tag{2.111}$$

Evaluating the contractions over x fields in Eq. (2.104), we get

$$\langle\langle V(k_1)\cdots V(k_n)\rangle\rangle=\hat{\varepsilon}^n(2\pi)^{26}\delta(k)\prod_{j=1}^n\int\frac{2\,d^2z_j}{(1+|z_j|^2)^2}\exp\left[-\tfrac{1}{2}\sum_{i,j=1}^n k_i\cdot k_jG(z_i,z_j)\right]\ , \tag{2.112}$$

where $k=k_1+\cdots+k_n$ is the total momentum. From $k_i^2=2$ and $k=0$, it follows that the $(1+|z|^2)$ factors cancel out and we have

$$\langle\langle V(k_1)\cdots V(k_n)\rangle\rangle=\hat{\varepsilon}^n(2\pi)^{26}\delta(k)\prod_{j=1}^n\int d^2z_j\prod_{i<j}|z_i-z_j|^{2k_i\cdot k_j}\exp\left[\sum_{i=1}^n G(z_i,z_i)\right]\ . \tag{2.113}$$

The singularity that arises from considering the Green's function at coincident points should be regularized in a $\mathrm{PSL}(2,C)$-invariant way. This requires setting $G(z_i,z_i)$ to a constant independent of z_i. This constant arises once for

[4]More accurately it should be understood as the volume of the corresponding Lie group.

each vertex operator and thus effectively modifies $\hat{\varepsilon}$ to $\varepsilon=\hat{\varepsilon}e^{G(z_i,z_i)}$. The remaining expression in Eq. (2.113) is PSL$(2,C)$ invariant and thus divergent due to the infinite volume of Ker$\hat{P}_1$. We now fix three points z_{n-2},z_{n-1},z_n and isolate the invariant measure associated with them, as given in Eq. (2.108):

$$\langle\langle V(k_1)\cdots V(k_n)\rangle\rangle=\varepsilon^n(2\pi)^{26}\delta(k)\int d\mu\left[|z_{n-2}-z_{n-1}|^2|z_{n-1}-z_n|^2|z_n-z_{n-2}|^2 \times\prod_{j=1}^{n-3}\int d^2z_j\prod_{1\le i<j\le n}|z_i-z_j|^{2k_i\cdot k_j}\right]. \tag{2.114}$$

It is not hard to see that the object in large parentheses is PSL$(2,C)$ invariant all by itself upon transformation of all z_j with $j=1,2,\ldots,n$, so that the first integral yields Vol(Ker$\hat{P}_1$). It is customary to fix $z_{n-2}=0$, $z_{n-1}=1$, $z_n=\infty$, and one then obtains with the help of Eq. (2.104)

$$\langle V(k_1)\cdots V(k_n)\rangle=ce^{-2\lambda}\varepsilon^n(2\pi)^{26}\delta(k)\prod_{j=1}^{n-3}\int d^2z_j\prod_{1\le i<j\le n-1}|z_i-z_j|^{2k_i\cdot k_j}. \tag{2.115}$$

Introducing the generalized Mandelstam variables[5]

$$s_{ij}=-(k_i+k_j)^2$$

we see that Eq. (2.115) is absolutely convergent only if each s_{ij} is below the tachyon threshold,

$$\mathrm{Re}(s_{ij})<-2\ ,$$

and if

$$1<\sum_{j=1,j\neq i}^{n-1}\left[1+\frac{s_{ij}}{4}\right].$$

The function elsewhere is defined by analytic continuation, which will introduce imaginary components to the amplitude, so that it can obey the correct factorization properties. Note that, for real s_{ij}, the amplitude (2.115) cannot be correct for all ranges of s_{ij}, otherwise the full amplitude would be real, which is inconsistent with unitarity.

These closed-string amplitudes are analogous to those obtained by Koba and Nielsen (1969) for the open string. For the three-point function we have

$$\langle V(k_1)V(k_2)V(k_3)\rangle=ce^{-2\lambda}\varepsilon^3(2\pi)^{26}\delta(k)\ , \tag{2.116}$$

whereas for the four-point function we find the manifestly dual amplitude of Shapiro (1970) and Virasoro (1969),

$$\begin{aligned}\langle V(k_1)\cdots V(k_4)&=ce^{-2\lambda}\varepsilon^4(2\pi)^{26}\delta(k)\int d^2z_1\,|z_1|^{2k_1\cdot k_2}|1-z_1|^{2k_1\cdot k_3}\\&=ce^{-2\lambda}\varepsilon^4\pi(2\pi)^{26}\delta(k)\frac{\Gamma(-1-s/2)\Gamma(-1-t/2)\Gamma(-1-u/2)}{\Gamma(2+s/2)\Gamma(2+t/2)\Gamma(2+u/2)}\ ,\end{aligned} \tag{2.117}$$

where the Mandelstam variables for the four-particle amplitude are as usual denoted by $s=-(k_1+k_2)^2$, $t=-(k_2+k_3)^2$, $u=-(k_1+k_3)^2$. Identification of the amplitude with a combination of Γ functions has also freed us from the necessity for separate analytic continuation. The Γ function exhibits all the required factorization properties. Factorization in the s channel at the tachyon pole $s\sim-2$ imposes an additional relation on the constants λ and ε (recall that c was in principle calculated),

$$(ce^{-2\lambda}\varepsilon^3)^2=8\pi^2ce^{-2\lambda}\varepsilon^4$$

so that (2.118)

$$\varepsilon^2=\frac{8\pi^2}{c}e^{2\lambda}\ ,$$

and the normalization ε of the vertex operator is completely determined by the unique coupling constant λ, as pointed out by Weinberg (1985).

Our analysis has tacitly assumed that at least three vertex operators were inserted. When no vertex operator is inserted one should replace $(2\pi)^{26}\delta(k)$ by the volume Ω of space-time; we have $\langle\langle 1\rangle\rangle=1$, and by virtue of Eq. (2.104) the full amplitude vanishes. Physically, this indicates that the space-time cosmological constant vanishes at tree level. When one vertex is inserted, only one point on the sphere can be fixed, and after fixing that point, one should no longer factor out Vol(Ker$\hat{P}_1$), but rather the volume of the isotropy subgroup leaving that point invariant. If one chooses the fixed point at infinity, then this subgroup is generated by translations, rotations, and dilations in the plane and still has infinite volume under the PSL$(2,C)$-invariant measure on this group, which can be parametrized by two points z_1,z_2 in the plane. The invariant volume element is then $d^2z_1d^2z_2/|z_1$

[5]Our convention for s has a $-$ sign, because k_i is really Euclidean. Upon analytic continuation to Minkowski space-time, s is the usual Mandelstam variable.

$-z_2|^4$, whose integral indeed diverges. Thus the one-point function vanishes. Physically, this indicates that flat space-time is a tree-level solution to the string equations of motion. Finally, if two vertex operators are inserted, one should fix two points, and it remains to divide out by the volume of the isotropy subgroup leaving two points invariant. When one fixes these points at zero and infinity, the group is that of rotations and dilations and again has infinite volume, so that the two-point function also vanishes to tree level. Physically this means that tree-level mass and wave-function corrections are absent.

Finally, as is well known from dual-model theory, the amplitude (2.115) completely factorizes, a procedure that may be used to compute amplitudes for particles other than tachyons.

Tree-level amplitudes in the Polyakov formulation were studied by Nepomechie (1982) and Aoymama, Dhar, and Namazie (1986), and a prescription for linking their calculation to that of open strings (especially convenient for amplitudes of particles with spin) was suggested by Kawai, Lewellen, and Tye (1986).

2. One-loop-level amplitudes

For $h=1$, we have two real moduli τ_1,τ_2 and two real conformal Killing vectors, and the measure (2.77) is thus modified to

$$Dg_{mn}=(\det{}'P_1^\dagger P_1)^{1/2}\frac{\det\langle\mu_j\,|\,\phi_k\rangle}{\det\langle\phi_j\,|\,\phi_k\rangle^{1/2}}D\sigma\, D'v^m d^2\tau\ , \tag{2.119}$$

where the prime on $D'v^m$ denotes the omission of the two conformal Killing vectors. Under Weyl rescaling, any metric g can be mapped into a flat metric $\hat g$ with unit area, and in view of Eq. (2.70) the measure (2.119) becomes

$$Dg_{mn}=\left[\frac{\det{}'\hat P_1^\dagger\hat P_1}{\det\langle\phi_j\,|\,\phi_k\rangle_{\hat g}}\right]^{1/2}\det\langle\mu_j\,|\,\phi_k\rangle \times\left[\frac{\det\langle\psi_\alpha\,|\,\psi_\beta\rangle_g}{\det\langle\hat\psi_\alpha\,|\,\hat\psi_\beta\rangle_{\hat g}}\right]^{1/2}e^{-26S_L(\sigma)}D\sigma\, D'v^m d^2\tau\ . \tag{2.120}$$

Now Eq. (2.80) may be used to rewrite the τ integral as the Weil-Petersson measure, and the conformal Killing determinants of Eq. (2.120) may be handled as for the case of the sphere. Thus we obtain

$$Dg_{mn}=\frac{1}{\mathrm{Vol}(\mathrm{Ker}\hat P_1)}(\det{}'\hat P_1^\dagger\hat P_1)^{1/2}e^{-26S_L(\sigma)} \times D\sigma\, Dv^m d(\mathrm{WP})\ . \tag{2.121}$$

The standard representation of the torus is by a parallelogram in the complex plane, with sides 1 and $\tau=\tau_1+i\tau_2$ and $\mathrm{Im}(\tau)>0$, periodic boundary conditions, and the Euclidean metric. This slice is actually not of unit area (instead $\int d^2\xi\sqrt{g}=2\tau_2$), but because of Weyl invariance this choice is equivalent. The space of all tori obtained this way spans Teichmüller space and is parametrized by τ in the complex upper half-plane $\mathbf{H}=\{\tau=\tau_1+i\tau_2;\tau_2>0\}$. In Sec. IV.A, we shall describe an explicit construction of the Weil-Petersson measure for a slice of unit area yielding

$$d(\mathrm{WP})=2\frac{d^2\tau}{\tau_2^2}\ . \tag{2.122}$$

The torus obtained in this fashion is equivalent under Diff(M) to any torus with modular parameter $\tau'=(a\tau+b)/(c\tau+d)$, $ad-bc=1$, and a,b,c,d integers. These transformations form a group PSL$(2,Z)$. However the group of "large" diffeomorphisms, i.e., the mapping class group (or modular group for the torus), in addition includes the transformation that flips the sign of both sides of the parallelogram, corresponding to the element $-I$ of SL$(2,Z)$. Thus the full modular group should be taken to be SL$(2,Z)=\mathrm{MCG}_1$. Moduli space is obtained from Teichmüller space by identification under the mapping class group. For simplicity, we still identify it with the fundamental domain $\mathcal{M}_1$ of PSL$(2,R)$,

$$\mathcal{M}_1=\{\tau=\tau_1+i\tau_2 \text{ with } \tau_2>0,-\tfrac{1}{2}\le\tau_1\le\tfrac{1}{2},|\tau|\ge1\}\ , \tag{2.123}$$

represented in Fig. 7 on the condition of including a fac-

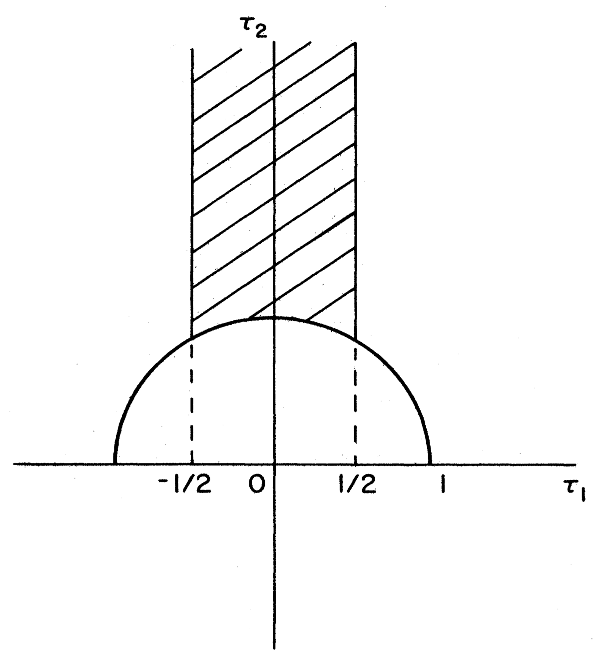


FIG. 7. A fundamental domain for PSL(2,Z), representing the torus.

tor of $\frac{1}{2}$ whenever we replace an integral over moduli space by an integral over $\mathcal{M}_1$. The Weil-Petersson measure $d(\mathrm{WP})$ is clearly invariant under MCG_1, so it may be projected down onto moduli space $\mathcal{M}_1$. More on this subject will be said in Sec. IV.B.

We are now in a position to write down the expression for one-loop scattering amplitudes:

$$\langle V_{i_1}(k_1)\cdots V_{i_n}(k_n)\rangle=\int_{\mathcal{M}_1}\frac{d^2\tau}{\tau_2^2}(\det{}'\hat{P}_1^\dagger\hat{P}_1)^{1/2}\left[\frac{4\pi^2}{\tau_2}\det{}'\Delta_{\hat{g}}\right]^{-13}\langle\langle V_{i_1}(k_1)\cdots V_{i_n}(k_n)\rangle\rangle\frac{1}{\mathrm{Vol}(\mathrm{Ker}\hat{P}_1)}\ . \qquad (2.124)$$

The determinants are evaluated in Sec. V.A, and one finds

$$(\det{}'\hat{P}_1^\dagger\hat{P}_1)^{1/2}=\det{}'2\Delta_{\hat{g}}=\tfrac{1}{2}\det{}'\Delta_{\hat{g}} \qquad (2.125)$$

and

$$\det{}'\Delta_{\hat{g}}=\tau_2^2\,|\eta(\tau)|^4\ , \qquad (2.126)$$

where the Dedekind η function is defined in Appendix E [Eq. (E9)]. The volume of $\mathrm{Ker}(\hat{P}_1)$ is easily obtained, since conformal Killing vectors on the torus are again constants, and one finds $\mathrm{Vol}(\mathrm{Ker}\hat{P}_1)=2\tau_2$, so that

$$\frac{d^2\tau}{\tau_2^2}(\det{}'\hat{P}_1^\dagger\hat{P}_1)^{1/2}\left[\frac{4\pi^2}{\tau_2}\det{}'\Delta_{\hat{g}}\right]^{-13}\frac{1}{\mathrm{Vol}(\mathrm{Ker}\hat{P}_1)}=\frac{d^2\tau}{8\pi^2\tau_2^2}\frac{1}{(4\pi^2\tau_2)^{12}}\,|\eta(\tau)|^{-48}\ . \qquad (2.127)$$

With the help of the transformation law of $\eta(\tau)$ given in Eq. (2.45), it is easy to check that Eq. (2.127) is invariant under $\mathrm{SL}(2,Z)$ as expected, since the calculation was manifestly reparametrization invariant throughout. The one-loop cosmological constant follows immediately,

$$\Lambda_{h=1}=\int_{\mathcal{M}_h}\frac{d^2\tau}{8\pi^2(\tau_2)^2}\frac{1}{(4\pi^2\tau_2)^{12}}\,|\eta(\tau)|^{-48}\ ,$$

and it is divergent due to the presence of the tachyon.

To obtain the scattering amplitudes, one needs the Green's function $G(z,z';\tau)$ satisfying Eq. (2.109), but now for the torus. It may be obtained by the method of images or, equivalently, from the translation properties of the standard ϑ_1 function for the torus, defined in Appendix C,

$$\begin{aligned}\vartheta_1(z+1,\tau)&=-\vartheta_1(z,\tau)\ ,\\ \vartheta_1(z+\tau,\tau)&=e^{-i\pi\tau-2\pi iz}\vartheta_1(z,\tau)\ .\end{aligned} \qquad (2.128)$$

The only zero of $\vartheta_1(z,\tau)$ as a function of z is at $z=0$, and hence

$$-\ln\left|\frac{\vartheta_1(z-z',\tau)}{\vartheta_1'(0,\tau)}\right|^2 \qquad (2.129)$$

satisfies the Laplace equation everywhere, except at $z=z'$, where it has the correct short-distance singularity. It can be made single valued on the torus by the addition of a function quadratic in $z-z'$:

$$G(z,z';\tau)=-\ln\left|\frac{\vartheta_1(z-z',\tau)}{\vartheta_1'(0,\tau)}\right|^2-\frac{\pi}{2\tau_2}(z-\bar{z}-z'+\bar{z}')^2\ . \qquad (2.130)$$

Thus it is the unique candidate for the Green's function on the torus, and indeed satisfies Eq. (2.109). Using a translation-invariant cutoff when $z'\to z$, one finds that $G(z,z;\tau)$ is independent of z and τ. It may be used to rescale the coupling constant $\hat{\varepsilon}$ to ε, exactly as in the case of the tree-level amplitudes.

As an example, we present the explicit expression for the one-loop amplitudes for the scattering of tachyonic states only. The measure is given by Eq. (2.124), and the vertex operator part yields

$$\begin{aligned}\langle\langle V(k_1)\cdots V(k_n)\rangle\rangle&=\varepsilon^n(2\pi)^{26}\delta(k)\prod_{j=1}^n\int d^2z_j\exp\left[-\sum_{i<j}k_i\cdot k_jG(z_i,z_j;\tau)\right]\\ &=(2\pi)^{26}\delta(k)\varepsilon^n\prod_{j=1}^n\int d^2z_j\prod_{i<j}F(z_i,z_j)^{k_i\cdot k_j}\ ,\end{aligned} \qquad (2.131)$$

where we use the function F defined in Sec. II.G:

$$F(z_i,z_j)=\exp\left[\frac{\pi(z_i-\bar{z}_i-z_j+\bar{z}_j)^2}{2\tau_2}\right]\left|\frac{\vartheta_1(z_i-z_j,\tau)}{\vartheta_1'(0,\tau)}\right|^2\ . \qquad (2.132)$$

Putting all together, we obtain

$$\langle V(k_1)\cdots V(k_n)\rangle=\delta(k)\varepsilon^n\int_{\mathcal{M}_1}\frac{d^2\tau}{2\tau_2^2}\frac{1}{(\tau_2)^{12}}\,|\eta(\tau)|^{-48}\prod_{j=1}^n\int d^2z_j\prod_{i<j}F(z_i,z_j)^{k_i\cdot k_j}\ . \qquad (2.133)$$

These amplitudes are divergent for all values of momenta, due to the presence of the factor $|\eta(\tau)|^{-48}$, which contributes a factor exponential in τ_2 as $\tau_2 \to \infty$. Ultimately, this is connected with the presence of the tachyon. Other manifestations of the instability of Minkowski space-time are the fact that the one-loop dilaton tadpole [computed, say, from Eq. (2.133) by factorization] is nonvanishing, indicating that the Minkowski space-time is not a solution to the string equations of motion to this order. Fischler and Susskind (1986a, 1986b) suggested that de Sitter space-time, on the other hand, does solve the equations of motion to this order.

In the dual model of open strings, one-loop amplitudes were considered by Gross, Neveu, Scherk, and Schwarz (1970). The closed bosonic string amplitudes to one-loop order were computed by Shapiro (1972), who also correctly identified the fundamental domain for moduli space (up to the above-mentioned factor of 2). In the new era, they were reevaluated first by Polchinski (1986) and subsequently by D'Hoker and Phong (1986a, 1986b) and by Panda (1987). Investigations for open strings are found in Cohen *et al.* (1987), Varughese and Weisberger (1987), and Weisberger (1987a, 1987b), where it is argued that the cylinder graph with boundary conditions may be used as an off-shell propagator for closed strings.

I. Formulation with ghosts

In the presence of an infinite-dimensional symmetry, factoring out the symmetry group and enforcing the correct measure in loop amplitudes can also be accomplished by introducing Faddeev-Popov ghosts. Our previous discussion shows that for the first-quantized bosonic string this is not strictly necessary, and any scattering amplitude can be computed without appealing to ghosts. However, a ghost formulation will put at our disposal the powerful tools of conformal field theory and exhibit the key Becchi-Rouet-Stora-Tyutin (BRST) invariance. These are also crucial ingredients in the construction of a full-fledged second-quantized string field theory, as was realized by Siegel (1985), Banks and Peskin (1986), Siegel and Zwiebach (1986, 1987), Witten (1986a, 1986b), and Neveu, Nicolai, and West (1986). Furthermore, in fermionic string theories, ghosts will be indispensable, as they will couple to fermion emission vertices.

Following the standard Faddeev-Popov procedure, we replace the gauge parameter δv^z for reparametrization invariance by an anticommuting ghost field c^z. Introducing its conjugate antighost field b_{zz}, we can now write down the reparametrization ghost action:

$$I_{\rm gh}(b,c)=\frac{1}{2\pi}\int d^2z\sqrt{g}\, b_{zz}\nabla^z c^z+\text{c.c.}\ , \tag{2.134}$$

which is Weyl and reparametrization invariant. The Weyl symmetry again will be anomalous, since the natural metrics on the ghost field space,

$$\begin{aligned}\|c\|^2&=\int d^2z\sqrt{g}\, g_{z\bar z}c^z c^{\bar z}\ ,\\ \|b\|^2&=\int d^2z\sqrt{g}\,(g^{z\bar z})^2 b_{zz}b_{\bar z\bar z}\ ,\end{aligned} \tag{2.135}$$

are not Weyl invariant. We also have an important global symmetry generated by

$$\begin{aligned}&c^z\to e^{-i\theta_z}c^z,\quad b_{zz}\to e^{+i\theta_z}b_{zz}\ ,\\ &c^{\bar z}\to e^{+i\theta_{\bar z}}c^{\bar z},\quad b_{\bar z\bar z}\to e^{-i\theta_{\bar z}}b_{\bar z\bar z}\ .\end{aligned} \tag{2.136}$$

Even though $c^{\bar z}$ is formally the complex conjugate of c^z, their analogs in Minkowski conventions would be independent. The metrics on the ghost space, however, are only invariant under a U(1) subgroup of Eq. (2.136), generated by $\theta_z=\theta_{\bar z}$. Thus the ghost number current

$$j_z=-b_{zz}c^z \tag{2.137}$$

can be expected to be anomalous in the full quantum theory. Indeed, in Appendix B, a heat-kernel regularization shows that

$$\nabla^z j_z=-\tfrac{3}{2}R\ . \tag{2.138}$$

The integrated version of this anomaly agrees with the index theorem of Eq. (2.50), which asserts that

$$8(c\ \text{zero modes})-\#(b\ \text{zero modes})=\tfrac{3}{2}\chi(M)\ . \tag{2.139}$$

Now recall that the gauge-fixing operators P_1 and $P_1^\dagger$ decompose as $P_1=\nabla_z^1\oplus\nabla_{-1}^z$, $P_1^\dagger=-(\nabla_2^z\oplus\nabla_z^{-2})$, so that the Faddeev-Popov determinant $(\det P_1^\dagger P_1)^{1/2}$ naively can be represented as

$$\int D(b\bar b c\bar c)e^{-I_{\rm gh}(b,c)}\ . \tag{2.140}$$

However, in the presence of zero modes this functional integral would vanish. For convenience, let us restrict our discussion to the case of genus $h\geq 2$, the case of the torus requiring straightforward modifications. In this case ∇_{-1}^z has no zero mode, while its adjoint ∇_2^z admits $3h-3$ zero modes, namely the holomorphic quadratic differentials (cf. Sec. II.E). To absorb these zero modes, we need $3h-3$ insertions. Thus the key nonvanishing functional integral of interest is

$$\int D(b\bar b c\bar c)\prod_{i=1}^{3h-3}b(z_i)\bar b(z_i)e^{-I_{\rm gh}(b,c)}\ , \tag{2.141a}$$

which can be evaluated to be

$$(\det{}'P_1^\dagger P_1)^{1/2}\frac{|\det\phi_k(z_j)|^2}{\det\langle\phi_j|\phi_k\rangle}\ , \tag{2.141b}$$

where ϕ_j are any basis of $3h-3$ holomorphic quadratic differentials. If we substitute this in the expression for the Polyakov string measure (2.79), we obtain

$$\begin{aligned}Z_B=\int dm_1\cdots dm_{6h-6}&\left|\frac{\det\langle\mu_j|\phi_k\rangle}{\det\phi_k(z_j)}\right|^2\\ &\times\int D(b\bar b c\bar c)Dx'\\ &\times\prod_{i=1}^{3h-3}b(z_i)\bar b(z_i)e^{-[I_{\rm gh}(b,c)+I_m(x)]}\ .\end{aligned} \tag{2.142}$$

In this expression it should be understood that for each value of the $6h-6$ moduli parameters $m_1, \ldots, m_{6h-6}$ characterizes a background metric with respect to which all the functional integrals and finite-dimensional determinants are evaluated. The "matter" action is just the Polyakov action [Eq. (2.5)] with the worldsheet metric as background:

$$I_m(x) = \frac{1}{4\pi} \int d^2z\, \partial_z x^\mu \partial_{\bar z} x^\mu \, . \tag{2.143}$$

Finally Dx' denotes omission of the constant zero mode, and $z_1, \ldots, z_{3h-3}$ are arbitrary points on the surface M.

Other useful formulations of the string amplitudes are also readily derivable from Eq. (2.141). In particular,

$$\left| \int Db\, Dc \prod_{i=1}^{3h-3} \langle \mu_i \mid b \rangle e^{-I_{\mathrm{gh}}(b,c)} \right|^2 = \frac{(\det' P_1^\dagger P_1)^{1/2}}{\det\langle \phi_j \mid \phi_k \rangle^{1/2}} \det\langle \mu_j \mid \phi_k \rangle \, , \tag{2.144}$$

where we have used real quadratic differentials on the right-hand side, to conform with Eq. (2.79). Thus the bosonic string amplitude can also be written as

$$Z_B = \int dm_1 \cdots d\bar m_{3h-3} W \bar W \tag{2.145}$$

with

$$W = \int Db\, Dc\, Dx' \prod_{i=1}^{3h-3} \langle \mu_i \mid b \rangle e^{-I_{\mathrm{gh}}(b,c) - I_m(x)} \, . \tag{2.146}$$

We have used the standard notation for the pairing between the b field and the Beltrami differentials

$$\langle \mu \mid b \rangle = \int d^2z\, \mu^z_{\bar z} b_{zz} \, . \tag{2.147}$$

Here the integral in x is assumed to have been split as well into holomorphic and antiholomorphic parts, and we have kept in W the holomorphic one. How this can be done exactly requires careful treatment and will be taken up in Sec. VII.

The above offers a remarkably simple procedure for guessing the right measure: simply insert the right number of b's to absorb the zero modes, and pair off with the tangents to the slice for moduli. We shall see that this procedure generalizes to the superstring as well.

The above formulas simplify to some extent if we choose to represent moduli space by slices with Beltrami differentials $\mu^z_{i,\bar z}$ admitting jump discontinuities. As observed earlier in Sec. II.E, a smooth Beltrami differential cannot be represented as $\nabla_{\bar z} v^z$ for a smooth vector field if it deforms the conformal structure. However, deformations can be achieved with discontinuous vector fields and Beltrami differentials. Thus let our slice satisfy

$$\mu^z_{i,\bar z} = \nabla_{\bar z} v_i^z \, , \tag{2.148}$$

where v_i^z are quasiconformal vector fields, i.e., v_i^z are smooth vector fields with a unit jump δv^z across a closed contour C_i. Contours generating a basis of quasiconformal deformations can be chosen in a variety of ways, for which we refer, for example, to the discussion of Mandelstam diagrams (Sec. IV.G) and Fenchel-Nielsen coordinates (Sec. IV.E) of deformations. If we substitute them into fermionic functional integrals of the form (2.144), we note that the $3h-3$ insertions $b(z_j)$ can effectively be viewed as holomorphic, because all $3h-3$ factors are required to produce zero modes to compensate for the ghost number anomaly. Thus the insertion becomes

$$\int d^2z \sqrt{g}\, g^{z\bar z} \mu^z_{i\bar z} b_{zz} = \int d^2z \sqrt{g}\, \nabla^z v_i^z b_{zz} = \oint_{C_i} dz\, b_{zz} \delta v_i^z = \oint_{C_i} dz\, b_{zz} \tag{2.149}$$

and Eq. (2.146) reduces to

$$W = \int Db\, Dc\, Dx' \left[\prod_{i=1}^{3h-3} \oint_{C_i} dz\, b_{zz} \right] e^{-[I_{\mathrm{gh}}(b,c) + I_m(x)]} \, . \tag{2.150}$$

If the worldsheet is viewed as a surface with punctures, one may use the insertion of the operator identity

$$\oint_{C_z} dw\, b_{ww}(w) c^z(z) = 1 \, , \tag{2.151}$$

where C_z is a small contour surrounding the point with coordinate z. If we choose the points z to coincide with the punctures (i.e., vertex operator insertions), then the ghost formulation of Eq. (2.79) reads

$$\langle V_{i_1}(k_1) \cdots V_{i_n}(k_n) \rangle = \sum_{h=0}^{\infty} e^{-\lambda\chi} \int_{\mathcal{M}_h} [dm] \int Dx^\mu \int Dc\, D\bar c\, Db\, D\bar b\, e^{-I_m(x) - I_{\mathrm{gh}}(b,c)} \times [\bar V_{i_1}(k_1) \cdots \bar V_{i_n}(k_n)] \prod_{j=1}^{3h-3+n} \oint_{C_j} dz\, b_{zz} \oint_{C_j} d\bar z\, b_{\bar z\bar z} \, . \tag{2.152}$$

Here the modified vertex operators are given by

$$\bar V(k) = \int d^2z\, c^z c^{\bar z} U(\varepsilon, x) e^{ikx} \, , \tag{2.153}$$

where the vertex operator without ghosts reads

$$V(k) = \int d^2\xi \sqrt{g}\, U(\varepsilon, g, x) e^{ikx} \, . \tag{2.154}$$

The crucial role of ghosts in formulating string theories emerged first out of Polyakov's original work (1981a). The b,c system was explored further in Friedan (1984) and string partition functions and amplitudes expressed in terms of ghost insertions in Friedan, Martinec and Shenker (1986). Equation (2.152) in terms of insertions of contour integrals was proposed by Martinec (1986) and Giddings and Martinec (1986), who derived it from a slightly different formalism of extended path integrals instead of ghost insertions that we adopted here. An alternative ghost action including the square of the ghost current was considered in Freedman and Warner (1986a, 1986b) and Freedman *et al.* (1987). Relations with the harmonic gauge are discussed in Freedman, Latorre, and Pilch (1988).

J. Conformal field theory

In the previous section we have presented string amplitudes in terms of correlation functions of the matter fields x^μ and the reparametrization ghosts b_{zz} and c^z. These are basic examples of conformal field theories, i.e., theories invariant under conformal transformations. In two dimensions conformal invariance is an especially powerful constraint, and we give now a brief discussion of the properties of these conformal fields.

The discussion will actually be clearer from a more general point of view, so we consider a theory of chiral fermions $b(dz)^n, c(dz)^{1-n}$ with action

$$I(b,c)=\frac{1}{2\pi}\int d^2z\sqrt{g}\,b\nabla^z_{1-n}c \ . \tag{2.155}$$

Classically the theory is invariant under Weyl transformations, so the stress tensor

$$T_{mn}=-\frac{4\pi}{\sqrt{g}}\delta I/\delta g^{mn}$$

is traceless and given by[6]

$$T_{zz}=-nb\partial_z c+(1-n)(\partial_z b)c \ . \tag{2.156}$$

In particular, the equations of motion imply that T_{zz} is holomorphic. Quantum mechanically, the Weyl symmetry is anomalous and will prevent the stress tensor from being simultaneously covariant and holomorphic. This can readily be seen from Ward identities for reparametrization invariance. Indeed, if we insist on reparametrization invariance, the correlation function

$$Z_F(z_1,\ldots,w_N)=\int D(bc)e^{-I(b,c)}\prod_{i=1}^M b(z_i)\prod_{i=1}^N c(w_i) \tag{2.157}$$

will transform by

$$\delta Z_F=\sum_{i=1}^M (n\nabla_{z_i}v^{z_i}+v^{z_i}\nabla_{z_i})Z_F + \sum_{i=1}^N [(1-n)\nabla_{w_i}v^{w_i}+v^{w_i}\nabla_{w_i}]Z_F \tag{2.158}$$

under a reparametrization $z\to z+v^z$. On the other hand, we can also write

$$\begin{aligned}\delta Z_F&=\frac{1}{4\pi}\int d^2z\sqrt{g}\,\delta g^{zz}\Big\langle T_{zz}\prod_{i=1}^M b(z_i)\prod_{i=1}^N c(w_i)\Big\rangle + \int d^2z\sqrt{g}\,\delta g^{\bar z\bar z}\left[\frac{1}{\sqrt{g}}\frac{\delta Z_F}{\delta g^{\bar z\bar z}}\right]\\ &=\frac{1}{2\pi}\int d^2z\sqrt{g}\,v^z\nabla^z\Big\langle T_{zz}\prod_{i=1}^M b(z_i)\prod_{i=1}^N c(w_i)\Big\rangle + \int d^2z\sqrt{g}\,v^z\nabla^{\bar z}\left[\frac{1}{\sqrt{g}}\frac{\delta Z_F}{\delta g^{\bar z\bar z}}\right] .\end{aligned} \tag{2.159}$$

The variation of Z with respect to the trace of g is determined by the conformal anomaly. As in the case of the reparametrization ghosts, there will be a fermion number violation, measured by the index of ∇^z_{1-n}. If, say, $n\geq 2$, ∇^z_{1-n} will have no zero modes, while $(\nabla^z_{1-n})^\dagger$ will have $\Upsilon=(h-1)(2n-1)$ zero modes. The only nonvanishing correlation function (2.157) must satisfy $M=N+\Upsilon$ and can then be expressed as

$$Z_F=\frac{\det'\nabla^z_n}{\det\langle\phi_a\,|\,\phi_b\rangle^{1/2}}\sum_{i_1,\ldots,i_I}(-1)^{\sum i_j}\det\phi_k(z_{i_j}) \times \det_{l\neq i_1,\ldots,i_I} G(z_l,w_j) \ , \tag{2.160}$$

where $\phi_1,\ldots,\phi_I$ are zero modes and $G(z,w)$ is a propagator for ∇^z_n. The zero modes and propagators are unchanged under Weyl scalings, so

$$\frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{z\bar z}}\ln Z_F=-\frac{1}{2}\frac{\delta}{\delta\sigma}\ln\left[\frac{\det'\nabla^z_n}{\det\langle\phi_a\,|\,\phi_b\rangle^{1/2}}\right] =\frac{c_n}{24\pi}Rg_{z\bar z} \ , \tag{2.161a}$$

where the central charge $-2c_n$ is given by

$$c_n=6n^2-6n+1 \tag{2.161b}$$

in view of Eq. (2.69). Equating the two expressions (2.158) and (2.159) for the variation of Z_F under reparametrizations gives

$$\frac{1}{2\pi}\nabla^z\Big\langle T_{zz}\prod_{i=1}^M b(z_i)\prod_{i=1}^N c(w_i)\Big\rangle+\frac{c_n}{24\pi}\nabla_z RZ_F=\sum_{i=1}^M [n\nabla_{z_i}\delta(z,z_i)+\delta(z,z_i)\nabla_{z_i}]Z_F + \sum_{i=1}^N [(1-n)\nabla_{w_i}\delta(z,w_i)+\delta(z,w_i)\nabla_{w_i}]Z_F \ .$$

[6]In this expression, the Christoffel symbols have canceled out.

This is an equation for $\langle T_{zz}\prod_1^M b(z_i)\prod_1^N c(w_i)\rangle$ which shows in particular that it is not meromorphic in z, with the obstruction arising precisely from the conformal anomaly. The equation can be solved using any propagator orthogonal to the zero modes $\phi_1,\ldots,\phi_{3h-3}$ of ∇_2^z. Such a propagator will satisfy

$$\begin{aligned}&\nabla^z G_{zz}^{\xi}=2\pi\delta(\xi,z)\ ,\\ &\nabla^{\xi}G_{zz}^{\xi}=-2\pi\delta(\xi,z)+2\pi\sum_{a=1}^{3h-3}g^{\xi\bar\xi}\mu_{a,\bar\xi}^{\xi}\phi_{a,zz}\ ,\end{aligned}\tag{2.162}$$

where the $\mu_{a,\bar z}^{z}$ are the dual basis of Beltrami differentials

$$\int d^2z\sqrt{g}\,g^{z\bar z}\mu_{a,\bar z}^{z}\phi_{b,zz}=\delta_{ab}\ .$$

The first Ward identity for reparametrization invariance takes the form

$$\left\langle T_{zz}\prod_1^M b(z_i)\prod_1^N c(w_i)\right\rangle-\sum_{a=1}^{3h-3}\phi_{a,zz}\int d^2\xi\sqrt{g}\,g^{\xi\bar\xi}\mu_{a,\bar\xi}^{\xi}\left\langle T_{\xi\xi}\prod_1^M b(z_i)\prod_1^N c(w_i)\right\rangle$$
$$=-\frac{c_n}{24\pi}Z_F\int d^2\xi\sqrt{g}\,G_{zz}^{\xi}\partial_{\xi}R+\sum_{i=1}^{M}(n\nabla_{z_i}G_{zz}^{z_i}+G_{zz}^{z_i}\nabla_{z_i})Z_F+\sum_{i=1}^{N}[(1-n)\nabla_{w_i}G_{zz}^{w_i}+G_{zz}^{w_i}\nabla_{w_i}]Z_F\ .\tag{2.163}$$

A second Ward identity is obtained by differentiating the first with respect to $g^{-1/2}\delta/\delta g^{ww}$. The result is

$$\left\langle T_{zz}T_{ww}\prod_1^M b(z_i)\prod_1^N c(w_i)\right\rangle-\sum_{a=1}^{3h-3}\phi_{a,zz}\int d^2\xi\sqrt{g}\,g^{\xi\bar\xi}\mu_{a,\bar\xi}^{\xi}\left\langle T_{\xi\xi}T_{ww}\prod_1^N b(z_i)\prod_1^N c(w_i)\right\rangle$$
$$=-\frac{c_n}{6}Z_F\nabla_w^3G_{zz}^{w}+\left[\frac{c_n}{24\pi}\int d^2\xi\sqrt{g}\,G_{zz}^{\xi}\partial_{\xi}R+G_{zz}^{w}\nabla_w+2(\nabla_wG_{zz}^{w})-\sum_1^M(n\nabla_{z_i}G_{zz}^{z_i}+G_{zz}^{z_i}\nabla_{z_i})\right.$$
$$\left.-\sum_1^N[(1-n)\nabla_{w_i}G_{zz}^{w_i}+G_{zz}^{w_i}\nabla_{w_i}]\right]\left\langle T_{ww}\prod_1^M b(z_i)\prod_1^N c(w_i)\right\rangle\ .\tag{2.164}$$

We can now read off operator product expansions by looking at short-distance $z-z_1$, $z-w_1$, $z-w$ singularities. Since the Green's function $G_{zz}^{z_1}$ is equal to $1/(z-z_1)$ up to smooth terms, the first and second Ward identities lead to

$$\begin{aligned}&T_{zz}b(\xi)\sim\left[\frac{n}{(z-\xi)^2}+\frac{1}{z-\xi}\partial_{\xi}\right]b(\xi)\ ,\\ &T_{zz}c(\xi)\sim\left[\frac{1-n}{(z-\xi)^2}+\frac{1}{z-\xi}\partial_{\xi}\right]c(\xi)\ ,\\ &T_{zz}T_{ww}\sim\left[\frac{2}{(z-w)^2}+\frac{1}{z-w}\partial_w\right]T_{ww}\\ &\qquad-\tfrac{1}{6}c_n(\partial_w-\Gamma_{ww}^{w})\partial_w(\partial_w+\Gamma_{ww}^{w})\frac{1}{z-w}\ .\end{aligned}\tag{2.165}$$

Introducing the local counterterm $\partial_w\Gamma_{ww}^{w}-\frac{1}{2}(\Gamma_{ww}^{w})^2$ and the chiral stress tensor

$$T_{zz}^{\text{chi}}=T_{ww}-\tfrac{1}{6}c_n[\partial_w\Gamma_{ww}^{w}-\tfrac{1}{2}(\Gamma_{ww}^{w})^2]\ ,$$

we can rewrite Eq. (2.165) as

$$\begin{aligned}&T_{zz}^{\text{chi}}b(\xi)\sim\left[\frac{n}{(z-\xi)^2}+\frac{1}{z-\xi}\partial_{\xi}\right]b(\xi)\ ,\\ &T_{zz}^{\text{chi}}c(\xi)\sim\left[\frac{1-n}{(z-\xi)^2}+\frac{1}{z-\xi}\partial_{\xi}\right]c(\xi)\ ,\\ &T_{zz}^{\text{chi}}T_{ww}^{\text{chi}}\sim\frac{-c_n}{(z-w)^4}+\left[\frac{2}{(z-w)^2}+\frac{1}{z-w}\partial_w\right]T_{ww}^{\text{chi}}\ ,\end{aligned}\tag{2.166}$$

where this time T_{zz}^{chi} is holomorphic. Note that as a result of the third operator product expansion (OPE) above, T_{zz}^{chi} does not transform as a rank-two tensor. Rather, under a holomorphic reparametrization, T_{zz}^{chi} will transform with a Schwarzian derivative:

$$\begin{aligned}&T_{zz}^{\text{chi}}=T_{ww}^{\text{chi}}\left[\frac{dw}{dz}\right]^2-\frac{c_n}{6}S(w,z)\ ,\\ &S(w,z)=\left[\frac{d^3w}{dz^3}\right]\Big/(dw/dz)-\frac{3}{2}\left[\frac{d^2w}{dz^2}\Big/\frac{dw}{dz}\right]^2\ .\end{aligned}\tag{2.167}$$

This is yet another way of expressing the fact that the conformal anomaly prevents simultaneous holomorphicity and covariance.

In the above discussion we chose to maintain manifest covariance. If we had chosen instead to maintain manifest holomorphicity, we could have defined the chiral stress tensor by the following normal ordering procedure:

$$T_{zz}^{\text{chi}}=\lim_{w\to z}\left[-nb(w)\partial c(z)+(1-n)\partial b(w)c(z)+\frac{1}{(w-z)^2}\right]\ .\tag{2.168}$$

A routine calculation will again lead to the transformation law (2.167). Henceforth by stress tensor we shall actually designate the chiral one, which we denote simply by T_{zz}. Similarly all composite operators requiring regularization will be normal ordered as in Eq. (2.168), by splitting points and subtracting the singular part of the OPE.

The stress tensor can be viewed as generator of local conformal transformations

$$\delta_\varepsilon b(\xi)=\oint_{C_\xi}\frac{dz}{2\pi i}\varepsilon(z)T_{zz}b(\xi)\ . \qquad (2.169)$$

As such it will give rise to a Virasoro algebra with central charge exactly the coefficient of the conformal anomaly. In fact, if we introduce the Virasoro generators

$$L_m=\oint_{C_0}\frac{dz}{2\pi i}z^{m+1}T_{zz}\ , \qquad (2.170)$$

the matrix elements will be given by

$$\begin{aligned}\langle [L_m,T_{zz}]\rangle &=\oint_{C_{0,z}}\frac{dw}{2\pi i}w^{m+1}\langle T_{ww}T_{zz}\rangle\\ &\quad-\oint_{C_0}\frac{dw}{2\pi i}w^{m+1}\langle T_{ww}T_{zz}\rangle\\ &=\oint_{C_z}\frac{dw}{2\pi i}w^{m+1}\langle T_{ww}T_{zz}\rangle\ . \qquad (2.171)\end{aligned}$$

Here $C_{0,z}$ is a curve enclosing both 0 and z (see Fig. 8). Substituting in Eq. (2.171) the OPE of (2.166) yields

$$[L_m,L_p]=(m-p)L_{m+p}+\frac{-c_n}{6}(m^3-m)\delta_{m+p,0}\ . \qquad (2.172)$$

Equations (2.166), (2.169), and (2.172) are the local equations characterizing a conformal field theory. In the case at hand the b,c fields are primary fields of conformal weights n and $(1-n)$, respectively. They live on a Riemann surface M (more precisely are sections of the line bundles K^n and K^{1-n} where K is the canonical bundle of M), and the global version of the operator product expansions is given by the Ward identities (2.163) and (2.164). The negative sign in front of c_n is due to the quantization of b and c as fermions; it would be absent if b and c were quantized as bosons. This will be the case of the superghosts of Secs. III and VIII.

The theory of b,c fields is actually completely characterized by its current algebra. As pointed out before [Eq. (2.134)] for the reparametrization ghosts, the theory admits a symmetry $b\to e^{i\theta}b, c\to e^{-i\theta}c$. The (chiral) fermion number current $j_z=-bc$ is anomalous and satisfies

$$\nabla_{\bar z}j_z=-\tfrac{1}{2}(2n-1)\sqrt{g}R\ . \qquad (2.173)$$

This can be seen by heat-kernel regularization exactly as in Eq. (2.138), which corresponds to $n=2$. Integrating this relation gives back the violation of fermionic number $\Upsilon=(2n-1)(h-1)$ determined earlier through index theorems [cf. Eq. (2.139)]. From the short-distance expansion[7]

$$b(z)c(w)\sim\frac{1}{z-w} \qquad (2.174)$$

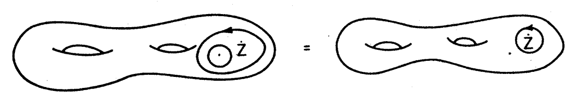


FIG. 8. Deformation of contour integrals.

it is easy to derive the OPE's:

$$\begin{aligned}j_z b(w)&\sim\frac{-1}{z-w}b(w)\ ,\\ j_z c(w)&\sim\frac{1}{z-w}c(w)\ , \qquad (2.175)\\ j_z j_w&\sim\frac{1}{(z-w)^2}\ .\end{aligned}$$

Finally the stress tensor T_{zz} can be recovered from the number current,

$$T_{zz}=\tfrac{1}{2}j_z^2+\tfrac{1}{2}Q\partial_z j_z\ , \qquad (2.176)$$

and will satisfy the OPE

$$T_{zz}j_w\sim\frac{-Q}{(z-w)^3}+\frac{1}{(z-w)^2}j_z\ , \qquad (2.177)$$

where $Q=(2n-1)$.

We turn now to bosons. The free scalar fields x^μ offer the simplest example of a bosonic conformal field theory, with the action that of the bosonic Polyakov string, the stress tensor equal to

$$T_{zz}=-\tfrac{1}{2}(\partial_z x^\mu)^2\ , \qquad (2.178)$$

and its central charge given by $c^x=d$. Strictly speaking, we have $\langle x^\mu(z)x^\nu(w)\rangle\sim-\delta^{\mu\nu}\ln|z-w|^2$, so that x^μ does not have a conformal dimension. This is no hindrance, however, since fields built out of $\exp(ik^\mu x_\mu)$ and $\partial_z x^\mu,\partial_{\bar z}x^\mu$ do have well-defined dimensions, as we saw in Sec. II.G. Chiral scalar fields with gravitational anomalies will be defined in Sec. VII.B.

More sophisticated theories arise when the bosonic fields φ are multiple valued and possibly coupled to a background charge Q. Indeed, one of the fundamental features of two-dimensional quantum field theory is the Bose-Fermi correspondence, and the bosonization of the chiral fermions b and c of ranks n and $1-n$ discussed above will lead precisely to such theories. Recall that for free bosons

$$\begin{aligned}\langle :e^{i\varphi(z)}::e^{-i\varphi(w)}:\rangle &=e^{\langle\varphi(z)\varphi(w)\rangle}\\ &=|z-w|^{-2}\\ &=\langle b(z)c(w)\bar b(z)\bar c(w)\rangle\ , \qquad (2.179)\end{aligned}$$

so we expect $\exp[i\varphi(z)]$ to correspond to a fermion bilinear $b\bar b$. With this we can exhibit a bosonic action that will reproduce the current algebra of the b,c system:

$$I(\varphi)=\frac{1}{4\pi}\int d^2z(\partial_z\varphi\partial_{\bar z}\varphi-iQ\sqrt{g}R\varphi)\ . \qquad (2.180)$$

Here $Q=2n-1$. The necessity of including the curva-

[7]In such relations it should always be assumed that Υ insertions have been made to absorb zero modes and ensure a meromorphic propagator.

ture term can be seen from several points of view. For example, the symmetry $b \to e^{i\theta} b$, $c \to e^{-i\theta} c$ should correspond to the shift $\varphi \to \varphi + \theta$, in which case the curvature term will produce the correct anomaly $\Upsilon = (2n-1)(h-1)$. The local version of this statement is that the current $j_z = -i\partial_z \varphi$ will satisfy the same equation as the fermion number current,

$$\partial_{\bar z}(-i\partial_z\varphi) = \frac{-Q}{2}\sqrt{g}R \ . \tag{2.181}$$

Moreover, the additional term $-iQ\partial_z^2\varphi/2$ that arises then in the stress tensor

$$T_{zz} = -\tfrac{1}{2}(\partial_z\varphi)^2 - \frac{i}{2}Q\partial_z^2\varphi \tag{2.182}$$

is precisely the modification needed to ensure that T_{zz} obey the Virasoro algebra with the correct central charge $1-3Q^2 = c_n$. Similarly we can check that the operator product expansions

$$T_{zz} j_w \sim \frac{-Q}{(z-w)^3} + \frac{1}{(z-w)^2} j_z \ , \qquad j_z j_w \sim \frac{1}{(z-w)^2} \ , \tag{2.183}$$

also reproduce Eqs. (2.175) and (2.177).

We can now confirm that vertex operators correspond to fermion bilinears by considering the OPE

$$T_{zz} e^{iq\varphi(w)} \sim \frac{q(q+Q)/2}{(z-w)^2} e^{iq\varphi(w)} + \frac{1}{z-w}\partial_w e^{iq\varphi(w)} \ , \qquad j_z e^{iq\varphi(w)} \sim \frac{-q}{z-w} e^{iq\varphi(w)} \tag{2.184}$$

The background charge has shifted the conformal weight of $\exp(iq\varphi)$ from $q^2/2$ to $q(q+Q)/2$. Thus q should be taken to be 1 and -1 for $b\bar b$ and $c\bar c$, in agreement with Eq. (2.179).

We have up to this point discussed only the formal and local aspects of the bosonic theories, unlike the fermionic theories for which we started from global formulas before examining singularities to obtain the local ones. Global issues do play an important role here, however. In fact, the operators $\exp(\pm i\varphi)$ suggest that φ is an angle. The correct statement on a worldsheet with nontrivial topology is that $d\varphi$ is a closed but in general not exact form, so that φ is a multiple-valued function. This requires the action $I(\varphi)$ to be suitably modified so as to be well defined and the path integral $D\varphi$ over all configurations to be correctly interpreted, as well. As usual with solitons, $d\varphi$ can be characterized up to an exact form by its winding numbers, so $D\varphi$ decomposes into a sum over soliton sectors indexed by winding numbers. The required machinery to do this as well as compute correlations and establish bosonization will be developed in Sec. VII. We postpone until then a careful study of global issues.

The program of using conformal invariance to classify critical points of statistical systems was pioneered by Polyakov (1969). The importance of conformal (primary) fields in string theory was recognized early on by Gervais and Sakita (1971a) and Andrić and Gervais (1972). The foundations of modern conformal field theory were laid out by Belavin, Polyakov, and Zamolodchikov (1984). Unitary conformal field theories with $c<1$ were classified by Friedan, Qiu, and Shenker (1984) and constructed explicitly by Goddard, Kent, and Olive (1986). Unitary $c=1$ models were studied by Dijkgraaf, Verlinde, and Verlinde (1988). Operator product expansions for the ghost system of the bosonic string appear in Friedan (1984). Our treatment here is an adaption to the higher-loop case of his arguments. Global versions of Ward identities are also derived by Eguchi and Ooguri (1987) and Sonoda (1987a). The stress tensor as a projective connection is studied in Alvarez and Windey (1987) and Dugan and Sonoda (1987). Bosonization of higher-spin free fermions b,c is due to Marnelius (1983) and Friedan, Martinec, and Shenker (1986). The corresponding bosonic system coupled to a background charge had appeared earlier in the work of Dotsenko and Fateev (1984). The importance of modular invariance in conformal field theory was stressed by Cardy (1986), Gepner and Witten (1986), Itzykson and Zuber (1986), Capelli, Itzykson, and Zuber (1987) and Gepner (1987a, 1987b).

K. Becchi-Rouet-Stora-Tyutin (BRST) invariance

We have seen in Sec. II.I that the Polyakov model for the bosonic string can be represented as a sum over moduli parameters of the full theory including ghosts with action

$$I_{\text{tot}} = \frac{1}{2\pi}\int d^2z\,(\tfrac{1}{2}\partial_z x^\mu \partial_{\bar z} x^\mu + b_{zz}\nabla_{\bar z} c^z + b_{\bar z\bar z}\nabla_z c^{\bar z}) \ . \tag{2.185}$$

In this formulation, the cancellation of the conformal anomaly in the critical dimension $d=26$ corresponds to the cancellation of the central charge in the total stress tensor,

$$T_{zz}^{\text{tot}} = T_{zz}^{x} + T_{zz}^{\text{gh}} \ . \tag{2.186}$$

From Eqs. (2.156), (2.168), and (2.178), the stress tensors for the ghost and matter parts are given by the following ordering prescription:

$$T_{zz}^{x} = \lim_{w\to z} -\frac{1}{2}\left[\partial_z x^\mu \partial_w x^\mu + \frac{d}{(z-w)^2}\right] \ ,$$

$$T_{zz}^{\text{gh}} = \lim_{w\to z}\left[-2b(w)\partial_z c(z) - \partial_w b(w) c(z) + \frac{1}{(w-z)^2}\right] \ .$$

The resulting central charges are $c^{\text{gh}} = -c_2 = -13$ and $c^x = dc_0/2 = d/2$, so that T_{zz}^{tot} is now a globally defined holomorphic rank-2 tensor. This is the property allowing decoupling of physical states by Virasoro gauge conditions.

As usual, the total action incorporating Faddeev-Popov ghosts exhibits a new symmetry, known as BRST symmetry:

$$\delta x^\mu = \lambda c^z \partial_z x^\mu, \quad \delta c^z = -\lambda c^z \nabla_z c^z ,$$
$$\delta b_{zz} = -\lambda[-\tfrac{1}{2}\partial_z x^\mu \partial_z x^\mu + c^z \nabla_z b_{zz} + 2(\nabla_z c^z) b_{zz}] . \tag{2.187}$$

The parameter λ is infinitesimal and Grassmann valued. The fact that these transformations indeed generate a symmetry is read off from the transformations of the matter and ghost parts separately,

$$\delta \partial_z x^\mu \partial^z x^\mu = \lambda(\nabla^z c^z)\partial_z x^\mu \partial_z x^\mu + \nabla_z(\lambda c^z \partial_z x^\mu \partial^z x^\mu) ,$$
$$\delta b_{zz} \nabla^z c^z = -\lambda \partial_z x^\mu \partial_z x^\mu \nabla^z c^z - \nabla_z(\lambda c^z b_{zz} \nabla^z c^z) .$$

The BRST current is

$$j_z^{\mathrm{BRST}} = c^z T_{zz}^x + \tfrac{1}{2} c^z T_{zz}^{\mathrm{gh}} + \tfrac{3}{2}(\nabla_z)^2 c^z . \tag{2.188}$$

The ordering prescription is again taken to be

$$j_z^{\mathrm{BRST}} = c^z T_{zz}^x + \tfrac{1}{2}\lim_{w\to z}\left[c^w T_{zz}^{\mathrm{gh}} + \frac{1}{(z-w)^2} c^w - \frac{2}{z-w}\partial_w c^w \right] + \tfrac{3}{2}(\nabla_z)^2 c^z .$$

In view of the transformation laws (2.167) for the stress tensors, the BRST current will transform as a genuine holomorphic rank-1 tensor. The corresponding BRST charge is given by the contour integral

$$Q_{\mathrm{BRST}} = \oint \frac{dz}{2\pi i} j_z^{\mathrm{BRST}} , \tag{2.189}$$

where the contour surrounds insertions. From operator product expansions, we can deduce that

$$[Q_{\mathrm{BRST}}, x^\mu] = c^z \partial_z x^\mu ,$$
$$\{Q_{\mathrm{BRST}}, c^z\} = c^z \nabla_z c^z , \tag{2.190}$$
$$\{Q_{\mathrm{BRST}}, b_{zz}\} = T_{zz}^{\mathrm{tot}} .$$

As already noted in Eq. (2.174) it is essential in deriving Eqs. (2.190) to have meromorphic propagators. This is automatic in string theory, since the string measure incorporates exactly the right number of insertions to absorb the effects of the ghost zero modes.

At the classical level Q_{BRST} is just a Grassmann quantity that squares to 0. At the quantum level the operator statement $Q_{\mathrm{BRST}}^2 = 0$ holds only in the critical dimension $d = 26$. In this case the Virasoro gauge conditions for physical states translate into

$$Q_{\mathrm{BRST}} \,|\, \mathrm{phys}\rangle = 0 , \tag{2.191}$$

and states of the form

$$Q_{\mathrm{BRST}} \,|\, \mathrm{anything}\rangle \tag{2.192}$$

are spurious and decouple from physical processes. In other words, physical states should rather be viewed as BRST cohomology classes, i.e., elements of the coset space $\mathrm{Ker}Q_{\mathrm{BRST}}/\mathrm{Image}Q_{\mathrm{BRST}}$.

In the Polyakov path-integral formulation of strings, the decoupling of spurious states (2.192) translates into the fact that amplitudes with an insertion of the BRST current around N arbitrary vertex insertions can be written as total derivatives on moduli space. More precisely, let $V_i(w_i)$ be vertices of conformal dimensions (λ_i, λ_i), let $C_{w_1,\ldots,w_N}$ be a contour surrounding $w_1,\ldots,w_N$, and parametrize moduli space by coordinates m_j,

$$\delta g_{\bar z \bar z} = \sum_{j=1}^{3h-3} \delta m_j g_{z\bar z} \mu_{j\bar z}^z , \tag{2.193}$$

where $\mu_{j\bar z}^z$ are $3h-3$ Beltrami differentials. Then

$$\prod_{l=1}^N d^2 w_l \sqrt{g}\,(g^{w_l \bar w_l})^{\lambda_l} \left\langle \prod_{j=1}^{3h-3} \langle \mu_j \,|\, b_{zz}\rangle \oint_{C_{w_1,\ldots,w_N}} j_w^{\mathrm{BRST}} dw \prod_{l=1}^N V(w_l) \right\rangle$$
$$= \sum_{k=1}^{3h-3} \frac{\partial}{\partial m_k}\left[(-)^k \prod_{l=1}^N d^2 w_l \sqrt{g}\,(g^{w_l \bar w_l})^{\lambda_l} \left\langle \prod_{j\neq k} \langle \mu_j \,|\, b_{zz}\rangle \prod_{l=1}^N V(w_l) \right\rangle \right] . \tag{2.194}$$

This relation between BRST invariance and total derivatives on moduli space can be easily seen from the OPE's (2.190) and a deformation-of-contours argument when the Beltrami differentials $\mu_{j\bar z}^z$ are generated by quasiconformal vector fields as in Sec. II.I. In this setup the insertions w_l remain separated from the supports of $\mu_{j\bar z}^z$. We can deform the contour $C_{w_1,\ldots,w_N}$ away from the insertions and pull it off the worldsheet, leaving in the process only the residues at the $3h-3$ insertions of b_{zz}. By the third equation in (2.190) the residue is exactly an insertion of the stress tensor T_{zz}. Such a term $\langle \mu_k \,|\, T_{zz}\rangle$ accounts for the piece of the right-hand side of Eq. (2.194) that arises when $\partial/\partial m_k$ lands on the action. In general the vertices V_l will depend on the moduli parameters. Their variations with respect to the trace of the metric cancel the variations of the volume forms in Eq. (2.194). Finally their variations with respect to δg^{zz} proper are Dirac functions supported only at w_l. They will vanish when paired with Beltrami differentials μ_k arising from quasiconformal deformations, since these are supported along disjoint contours.

This argument requires modifications if the supports of the Beltrami differentials cover the whole surface. The reason is that insertions of b_{zz} resulting from $\langle \mu \,|\, b_{zz}\rangle$, insertions of $V(w_l)$ as well as points on the contour $C_{w_1,\ldots,w_N}$, may come arbitrarily close together, invalidating operator product expansions such as (2.165) and (2.190), where only two points come close. Ignoring this effect would cause us to miss the variations with respect to moduli of the vertex operators.

A thorough justification of deformations-of-contours arguments along these lines seems quite involved at this point. Instead, we shall present an alternative argument, which does not rely on analyticity, and generalizes easily to superstrings. In this formulation, the basic object is the generating functional including ghosts and sources

$$Z[x^*,b^*,c^*]=\int D(xbc)e^{-I_{\rm tot}(x,b,c)+I_s}\ , \qquad (2.195)$$

where the source Lagrangian is given by

$$I_s=\int_M d^2z\sqrt{g}\,(x^{*\mu}x^{\mu}+b^{*zz}b_{zz}+c_z^*c^z)\ . \qquad (2.196)$$

The BRST charge Q, the partition function, and the correlation functions can all be expressed in terms of combinations of the operators,

$$\begin{aligned}\hat{x}&=\frac{1}{\sqrt{g}}\frac{\delta}{\delta x^*},\quad \hat{b}_{zz}=\frac{1}{\sqrt{g}}\frac{\delta}{\delta b^{*zz}}\ ,\\ \hat{c}^{\,z}&=\frac{1}{\sqrt{g}}\frac{\delta}{\delta c_z^*},\quad \hat{g}_{zz}=\frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{zz}}\ ,\end{aligned} \qquad (2.197)$$

acting on the generating functional. For example, the partition function is given by

$$Z_h=\int_{\mathcal{M}_h}\prod_{j=1}^{3h-3}d^2m_j\langle\mu_j\mid\hat{b}\rangle Z[x^*,b^*,c^*]\Bigg|_{*=0} \qquad (2.198)$$

and the expectation value of a general operator $V(\hat{x},\hat{b},\hat{c},g)$ is

$$\langle V\rangle=\int\prod_{j=1}^{3h-3}d^2m_j\langle\mu_j\mid\hat{b}\rangle V(\hat{x},\hat{b},\hat{c},g)Z[x^*,b^*,c^*]\ . \qquad (2.199)$$

The BRST transform of an operator V is then defined by

$$\delta V=[\lambda Q_{\rm BRST},V(\hat{x},\hat{b},\hat{c},g)]\ , \qquad (2.200)$$

with the BRST operator given by

$$Q_{\rm BRST}=\int d^2z\sqrt{g}\,(x^*\hat{c}^{\,z}\partial_z\hat{x}-2b^{*zz}\hat{g}_{zz}-c^*\hat{c}^{\,z}\nabla_z\hat{c})\ .$$

The OPE of Eq. (2.194) is replaced by

$$[\lambda Q_{\rm BRST},\hat{b}_{zz}]=\lambda\hat{g}_{zz}\ , \qquad (2.201)$$

and pulling through of the BRST contour can be justified by the following commutation rules for the BRST charge:

$$\begin{aligned}\{\langle\mu_k\mid\hat{b}\rangle,Q_{\rm BRST}\}&=-2\frac{\partial}{\partial m_k}\\&\quad+2\int d^2z_k\int d^2z\sqrt{g}\,b^{*zz}\frac{\delta\mu_{k\bar{z}}^{z}}{\delta g^{zz}}\hat{b}_{z_kz_k}\ ,\\ \Big[\prod_{j\neq k}\langle\mu_j\mid\hat{b}\rangle,\{\langle\mu_k\mid\hat{b}\rangle,Q_{\rm BRST}\}\Big]&=0\ .\end{aligned} \qquad (2.202)$$

The decoupling of spurious states in this formalism can now be established. First observe that Ward identities for BRST invariance can be stated as

$$Q_{\rm BRST}Z[x^*,b^*,c^*]=0 \qquad (2.203)$$

for all values of the sources, and in the critical dimension. To show this, one must use the Ward identities for reparametrization invariance, as well as for Weyl invariance. The Weyl Ward identities are anomalous in general, but in the critical dimension a cancellation of the matter and ghost contributions reduces them to the naive Ward identities, which are the ones needed to prove Eq. (2.203). Furthermore, in an expectation value

$$\langle\delta_{\rm BRST}V\rangle=\int\prod_j d^2m_j\langle\mu_j\mid\hat{b}\rangle[\lambda Q_{\rm BRST},V]\times Z[x^*,b^*,c^*]\Big|_{*=0} \qquad (2.204)$$

we may replace the commutator $[\lambda Q_{\rm BRST},V]$ by $\lambda Q_{\rm BRST}V$ and permute $Q_{\rm BRST}$ through all b insertions to obtain a total derivative on moduli. This establishes Eq. (2.194)

Strictly speaking, BRST invariance is at this point purely formal, since in principle it could be broken by contributions from the boundary of moduli space. A geometric discussion of the boundary of moduli space is provided in Sec. IV.H. For the bosonic string the amplitudes diverge, and a proper discussion of BRST invariance will require some renormalization (e.g., Fischler and Susskind, 1986a, 1986b; Seiberg, 1987; Sen, 1987). For superstrings where amplitudes are expected to be finite, whether the boundary of moduli space does contribute is a major issue, here as well as in questions of supersymmetry breaking.

The original BRST invariance of gauge-fixed Yang-Mills theories was introduced by Becchi, Rouet, and Stora (1976) and Tyutin (1975). That the BRST operator of string theory is nilpotent exactly in the critical dimension is due to Kato and Ogawa (1983), who also gave the interpretation of physical states as BRST cohomology classes. BRST invariance of multiloop amplitudes and deformation of contour arguments were stressed by Friedan, Martinec, and Shenker (1986). Arguments along these lines based on special meromorphic propagators are given in Sonoda (1987b). The setup in the functional language with external sources which we presented here to establish BRST invariance for the bosonic string is due to Mansfield (1987). The corresponding Ward identities are also given in Cohen, Gomez, and Mansfield (1986).

The requirement that spurious states decouple is what led originally to the discovery of the critical dimension, and the fact that this decoupling can be carried out consistently was one of the great successes of dual-model theories. It was established by Brower and Thorn (1971), Del Giudice, Di Vecchia, and Fubini (1972), and Goddard and Thorn (1972). The BRST formulation can of course be used to recapture many properties of the dual models in the operator formalism. A BRST proof of the no-ghost theorem is given by Freeman and Olive (1986), Frenkel, Garland, and Zuckerman, (1986), Spiegelglas (1987), and Thorn (1987). A proof based on the Kac (1983) determinant was given by Thorn (1984).

L. Formulation on surfaces with punctures

The main formula (2.79) for the scattering amplitudes of n particles was derived in the vertex operator formalism. In the introduction to this section, we saw that one can also formulate string perturbation theory on surfaces with punctures and wave functions. We now compare the two formulations.

At the h-loop level, the worldsheet for the scattering of n particles is a surface[8] M^* with h handles and n punctures $\xi_1,\ldots,\xi_n$. Quantization can be carried out as for Eq. (2.12),

$$\langle V_1(k_1)\cdots V_n(k_n)\rangle^* = \int \frac{Dg_{mn}}{\mathcal{N}^*}\int Dx^\mu W_1(\xi_1)\cdots \times W_n(\xi_n)e^{-I(x)} . \tag{2.205}$$

This time the normalization $\mathcal{N}^*$ should be taken to be

$$\mathcal{N}^* = \mathrm{Vol}(\mathrm{Diff}(M^*))\times \mathrm{Vol}(\mathrm{Weyl}(M^*)) \tag{2.206}$$

and $W_i(\xi_i)$ are wave functions evaluated at the punctures. In the critical dimension $d=26$, the amplitude (2.205) reduces to an integral over the moduli space $\mathcal{M}_{h,n}$ of Riemann surfaces of genus h and with n punctures:

$$\langle V_1(k_1)\cdots V_n(k_n)\rangle^* = \int_{\mathcal{M}_{h,n}}[dm^*]\frac{\det\langle \mu_\alpha\mid\phi_\beta\rangle}{\det\langle\phi_\alpha\mid\phi_\beta\rangle^{1/2}}(\det{}^*P_1^\dagger P_1)^{1/2}\left[\frac{8\pi^2}{\int_M d^2\xi\sqrt{g}}\det{}'\Delta_g\right]^{-13}\langle\langle W_1(\xi_1)\cdots W_n(\xi_n)\rangle\rangle . \tag{2.207}$$

In appearance Eq. (2.207) is very similar to Eq. (2.79) of the vertex operator formalism. However, the definitions of the parameters m, Beltrami differentials μ_α, and quadratic differentials ϕ_β and $\det{}^*P_1^\dagger P_1$ in Eq. (2.207) have to be adapted to the fact that the worldsheet M^* is now viewed as having punctures. First a moduli parameter m^* for $\mathcal{M}_{h,n}$ will consist of a moduli parameter for $\mathcal{M}_h=\mathcal{M}_{h,0}$ and of n points on the surface. Thus m^* should correspond to $m_1,\ldots,m_{6h-6}$ and $\xi_1,\ldots,\xi_n$ of n points on the surface and the complex dimension of $\mathcal{M}_{h,n}$ is

$$\dim\mathcal{M}_{h,n}=3h-3+n .$$

(A more precise geometric description of $\mathcal{M}_{h,n}$ as a fiber bundle over moduli space can be given by Teichmüller universal curve constructions, which are treated in Sec. IV, but we shall not need it here.) The number of Beltrami differentials μ_α is correspondingly increased to $3h-3+n$. We can choose the slice representing $\mathcal{M}_{h,n}$ so that the first $\{\mu_j\}_{j=1,\ldots,3h-3}$ Beltrami differentials arise from a slice representing $\mathcal{M}_h$, while the remaining $\{\mu_p\}_{p=3h-3+1,\ldots,3h-3+n}$ are generated by vector fields v_p which move the punctures by a unit displacement. Similarly the ϕ_β's are now holomorphic quadratic differentials on the surface M^*. They can be divided into earlier differentials $\{\phi_j\}_{j=1,\ldots,3h-3}$, which are holomorphic on the whole surface M, and n meromorphic differentials $\{\phi_p\}_{p=3h-3+1,\ldots,3h-3+n}$ with each ϕ_p having a simple pole[9] at ξ_p. (Such ϕ_p's exist in view of the Riemann-Roch theorem, which we shall discuss later in Sec. VII.C.) Finally, $\det{}^*P_1^\dagger P_1$ is the determinant of the operator $P_1^\dagger P_1$ restricted to the subspace of vector fields that vanish at the punctures. The reason is that the "small" diffeomorphisms of the punctured surface m^* are the small diffeomorphisms of the full surface M that leave the punctures fixed, and those are generated only by vector fields in the above restricted subspace.

We can take the vector fields v_p to be smooth and supported in a neighborhood of small size δ around each puncture. Since $\mu_p=\nabla_{\bar z}v_p^z$, it follows readily that $\langle\mu_p\mid\phi_j\rangle=0$. If the meromorphic differentials ϕ_p are chosen so that $\langle\phi_j\mid\phi_p\rangle=0$, we shall have

$$\frac{\det\langle\mu_\alpha\mid\phi_\beta\rangle}{\det\langle\phi_\alpha\mid\phi_\beta\rangle^{1/2}}=\frac{\det\langle\mu_j\mid\phi_k\rangle}{\det\langle\phi_j\mid\phi_k\rangle^{1/2}}\times\frac{\det\langle\mu_p\mid\phi_q\rangle}{\det\langle\phi_p\mid\phi_q\rangle^{1/2}} . \tag{2.208}$$

On the other hand, the Faddeev-Popov determinants are related by

$$\det{}^*P_1^\dagger P_1=\det P_1^\dagger P_1\frac{\det\langle\phi_p\mid\phi_q\rangle\det\langle v_p\mid v_q\rangle}{[\det\langle\phi_p\mid\mu_q\rangle]^2} . \tag{2.209}$$

In view of the support of v_p, it is easy to see that

$$\det\langle v_p\mid v_q\rangle^{1/2}=\delta^{2n}\prod_{i=1}^n g(\xi_p)+O(\delta^{2n+1}) . \tag{2.210}$$

Combining Eq. (2.208) with (2.209) and (2.210) and absorbing the factor $\delta^2 g(\xi_p)$ into a redefinition of the wave function, we arrive at

[8]Objects considered on the punctured surface will be denoted with an asterisk.

[9]Meromorphic differentials with a simple pole are precisely dual to the reparametrization vector fields with a simple zero at the puncture, so one need not consider differentials with poles of higher order.

$$\langle V_1(k_1)\cdots V_n(k_n)\rangle^* = \int_{\mathcal{M}_{h,0}} [dm] \frac{\det\langle \mu_j \mid \phi_k\rangle}{\det\langle \phi_j \mid \phi_k\rangle^{1/2}} (\det P_1^\dagger P_1)^{1/2} \left[\frac{8\pi^2}{\int d^2\xi \sqrt{g}} \det{}'\Delta_g \right]^{-13} \times \int d^2\xi_1 \sqrt{g(\xi_1)} \cdots \int d^2\xi_n \sqrt{g(\xi_n)} \langle\langle W(\xi_1)\cdots W(\xi_n)\rangle\rangle \ . \tag{2.211}$$

It remains to express the wave functions W in terms of the vertex operators for on-shell particle emission V. One starts by considering a surface where the puncture is replaced with a boundary of finite size. The amplitude computed by the insertion of a vertex operator is the same as the one computed on the surface with a boundary component and a wave function gotten by doing the path-integral operator over a disc D (of radius δ), including the vertex operator, and fitting into the boundary, as indicated in Fig. 9. Thus the wave function equals the integral over the disc with the vertex operator inserted. This is easily computed, and we obtain

$$W[x(\sigma)] = \int^{x(\sigma)} Dx^\mu P(\varepsilon, Dx) e^{ik_\mu x^\mu(\xi)} e^{-I[x]} \ , \tag{2.212}$$

where the vertex operator was of the form

$$V(\varepsilon,k,x) = P(\varepsilon,\partial x)(\xi) e^{ik_\mu x^\mu(\xi)} \ . \tag{2.213}$$

Splitting $x^\mu(z,\bar z)$ into a harmonic piece $\tilde x^\mu(z,\bar z)$ with boundary values $x^\mu(\sigma)$ and a fluctuation $y^\mu(z,\bar z)$, we find

$$W[x(\sigma)] = \int Dy^\mu P[\varepsilon, \partial\tilde x^\mu + \partial y^\mu] e^{ik_\mu \tilde x^\mu + ik_\mu y^\mu} \exp\left[-\frac{1}{8\pi} \int d^2\xi\, \partial_m y^\mu \partial^m y_\mu \right] \exp\left[-\frac{1}{8\pi} \oint dn^m \tilde x^\mu \partial_m \tilde x_\mu \right] . \tag{2.214}$$

Now since vertex operators are constructed so that they are normal ordered, we should not contract two legs on the same vertex. Hence if we let $\delta\to 0$, the Gaussian factor in Eq. (2.214) tends to 1, and we recover the desired relation between vertex operators and wave functions at punctures:

$$W[x(\sigma)] = P[\varepsilon, \partial\tilde x^\mu](0) e^{ik_\mu \tilde x^\mu(0)} \times \exp\left[-\frac{1}{8\pi} \oint dn^m \tilde x^\mu \partial_m \tilde x^\mu \right] . \tag{2.215}$$

The above arguments are due to D'Hoker and Giddings (1987).

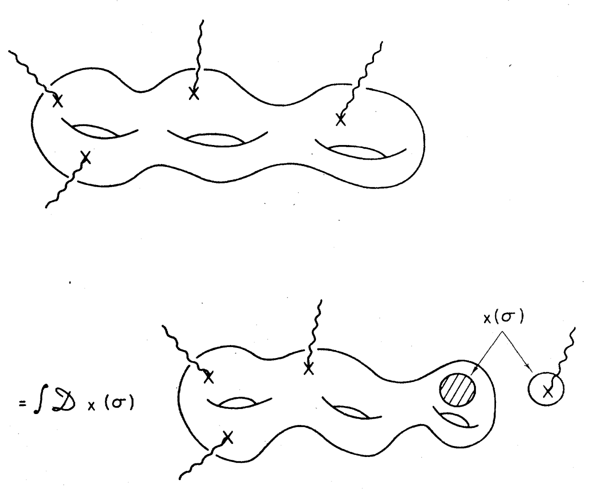


FIG. 9. Relation between amplitudes computed by inserting vertex operators and by giving wave functionals on boundary components.

III. CLOSED ORIENTED FERMIONIC STRINGS

Soon after the discovery of bosonic strings, it was realized that worldsheet spinors (fermions) ψ^μ carrying a space-time vector index μ could also be incorporated in the theory. As the number of negative norm states is now doubled compared to the bosonic string, an additional local symmetry is required to decouple these states. Local supersymmetry, discovered in this context by Gervais and Sakita (1971b , 1971c), is the appropriate invariance to do this, and so from a geometric point of view the starting point for the fermionic string is two-dimensional supergravity as developed by Zumino (1974), Brink, Di Vecchia, and Howe (1976), and by Deser and Zumino (1976b). For a general reference to supersymmetry and supergravity, we refer the reader to Ferrara and Fayet (1977), van Nieuwenhuizen (1981), Wess and Bagger (1983), Gates *et al.* (1984), Ferrara (1987), and West (1987).

Whereas the original model of Ramond (1971) contains space-time fermions, the model of Neveu and Schwarz (1971) also incorporates space-time bosons. In both models, one has worldsheet local supersymmetry, and space-time Lorentz invariance is manifest. These theories are consistent only in ten space-time dimensions. Once the Lorentz-covariant form is known, one may construct the associated light-cone gauge formulation. The light-cone Ramond-Neveu-Schwarz (RNS) formulation was used in a major development by Gliozzi, Scherk, and Olive (1976), who suggested that the even G parity sector of the Neveu-Schwarz theory together with a chiral truncation of the Ramond theory—the so-called GSO projection—yields a space-time supersymmetric spectrum. It took several years before Green and Schwarz (1981) proved the presence of a genuine supersymmetry by constructing the supercharge using the fermion emission vertex of the dual model. This fermion vertex had been introduced by

Thorn (1971) and by Mandelstam (1973b) in the light-cone formulation. Once the presence of supersymmetry was established, Green and Schwarz (1982) discovered a light-cone reformulation different from the Ramond-Neveu-Schwarz theory. Here, only physical bosonic space-time vectors and fermionic space-time spinors are present, no GSO projection is needed, and space-time supersymmetry is manifest. It is known as the Green-Schwarz formulation. Superstrings are classified into three groups: type I, type II, and heterotic.

Type-I superstring theories contain both open and closed unoriented strings. The open-string sector can support non-Abelian gauge fields when one attaches non-Abelian charges to the ends of the string. Mathematically, such charges are incorporated through the Chan-Paton rule, but factorization and duality limit the gauge groups to be orthogonal or symplectic. Ultimately it was discovered by Green and Schwarz (1984) that only O(32) can yield an anomaly-free theory, and thus the type-I superstring is unique. In a Minkowski space-time low-energy limit, it reduces to an $N=1$ supergravity plus Yang-Mills theory.

Type-II superstrings contain only closed oriented strings, and the only freedom left is the relative parity of the two gravitinos, producing the nonchiral type-IIA and the chiral type-IIB theories. In a Minkowski space-time low-energy limit, these theories reduce to $N=2$ supergravity without Yang-Mills multiplet.

Heterotic strings contain closed oriented strings only and are obtained as a hybrid (hence the nomenclature) between the type-II superstring and the closed oriented bosonic string. This hybrid is possible because on closed oriented worldsheets left- and right-movers are independent degrees of freedom, except for their collective momentum, to the point that one-half of one string theory can be replaced by that of another string theory. The 16 extra dimensions of the bosonic string component are compactified and yield a $\mathrm{Spin}(32)/Z_2$ or $E_8\times E_8$ gauge groups only. In a Minkowski space-time low-energy limit, it reduces to $N=1$ supergravity plus Yang-Mills theory.

In this section we shall derive the basic formulas for loop amplitudes for any of the above closed fermionic strings. We shall always be interested in theories with manifest space-time Lorentz invariance, and hence work with the covariant RNS and Polyakov formulations. As a drawback, space-time supersymmetry will not be manifest. From the worldsheet point of view, type-II theories are formulated as $N=1$ two-dimensional supergravity with "matter" multiplets x^μ and ψ^μ, and a supergravity multiplet consisting of a zweibein e_m^a and a two-dimensional spin-$\frac{3}{2}$ gravitino field χ_m. The worldsheet matter action for these fields reads

$$I_m=\frac{1}{4\pi}\int_M d^2\xi\sqrt{g}\,[\tfrac{1}{2}g^{mn}\partial_m x^\mu\partial_n x_\mu +\psi^\mu\gamma^m\partial_m\psi_\mu-\psi^\mu\gamma^a\gamma^m\chi_a\partial_m x_\mu -\tfrac{1}{4}\psi^\mu\gamma^a\gamma^b\chi_a(\chi_b\psi_\mu)]+\lambda\chi(M)\ . \tag{3.1}$$

Heterotic strings, on the other hand, correspond to $N=\frac{1}{2}$ supergravity with the same position and supergravity multiplets, except that ψ^μ and χ_m are of definite chirality:

$$\gamma^z\psi^\mu=0,\quad \gamma^{\bar z}\chi_m=0\ . \tag{3.2}$$

In addition there are internal degrees of freedom, which we represent by a fermionic variable ψ^a also of definite chirality:

$$\gamma^{\bar z}\psi^a=0,\quad a=1,2,\ldots,P\ .$$

For heterotic strings, the corresponding worldsheet action is

$$I_m=\frac{1}{4\pi}\int_M d^2\xi\sqrt{g}\,(\tfrac{1}{2}g^{mn}\partial_m x^\mu\partial_n x_\mu+\psi^\mu\gamma^{\bar z}e_{\bar z}{}^m\partial_m\psi_\mu -\psi_+^\mu\chi_{\bar z}^+e_z{}^m\partial_m x_\mu+\psi^a\gamma^z e_z{}^m\partial_m\psi^a) +\lambda\chi(M)\ . \tag{3.3}$$

Note that the internal degrees of freedom described here by ψ^a may alternatively be introduced as so-called left-moving bosonic fields $\dot{x}^a$. This is the approach originally taken by Gross *et al.* (1985a).

Spinors on a worldsheet of nontrivial topology must be appropriately defined. Indeed their phase shifts under parallel transport around closed loops are half of those of vectors and hence are ambiguous. We shall see that for closed oriented strings there are exactly 2^{2h} consistent choices of phase shifts for a worldsheet of genus h. Each choice is called a spin structure.

A crucial issue for fermionic strings is the assignment of spin structures. In the functional quantization formalism, the GSO projection for the type-II string is enforced by separating spinors of left chirality from spinors of right chirality, assigning each group independent spin structures ν and $\bar\nu$, and summing over ν and $\bar\nu$. This is the natural prescription to avoid global anomalies, since no spin structure is preferred, and the mapping class group will interchange them. That the spin structures within each group must be the same is a requirement of space-time Lorentz invariance. For the heterotic string, $\mathrm{Spin}(32)/Z_2$ symmetry forces the spin structures of all 32 ψ^a's to be identical, whereas $O(16)\times O(16)$ possibly extended to $E_8\times E_8$ requires the spin structures to be the same within each group of 16 ψ^a's, although the spin structures for the two groups need not be equal (see Witten, 1985b; D'Hoker and Phong, 1986d; and Seiberg and Witten, 1986).

In practice this principle of splitting left- from right-movers requires more specific prescriptions. In fact, actions are formulated with a Minkowski signature on g_{mn}, and we analytically continue to Euclidean signature. In the Minkowski metric ψ^μ,ψ^a,χ_m are Majorana-Weyl spinors. In the Euclidean metric, however, there are no Majorana-Weyl spinors, and the two chiral components of a Majorana spinor are complex conjugates of one another and must carry the same spin structure. To get around this difficulty, we start from a real spinor (sum of a Weyl spinor and its complex conjugate) and have to separate only upon quantization the contributions of the complex-conjugate factors. Each factor may then be thought of as the contribution of one Majorana-Weyl fermion. A major difficulty in this task is caused by the contributions of the bosonic fields x^μ and the terms $\chi\bar{\chi}\psi_+\psi_-$, which must be separated as well. We shall see in Sec. III.K below that this separation can only be enforced by introducing internal loop momenta p_I^μ, and contributions of left and right spinors after assignment of independent spin structures must be matched at the same value of p_I^μ. The precise prescriptions are given in Eqs. (3.196)–(3.201). In Secs. VII.F and VII.G we shall discuss their relations with the holomorphic structure of string amplitudes on supermoduli space.

When dealing with supersymmetric theories in general and with two-dimensional supergravity in particular, one may either use the component field formalism, in terms of the fields defined above, or group different component fields that transform into one another under supersymmetry transformations into the same multiplet or superfield. The superfield formalism is more appropriate for cancellation of local anomalies and enforcing the correct quantum measure. The natural setting for superstrings is $N=1$ supergeometry, and the analog of Riemann surfaces and moduli space will be super Riemann surfaces and supermoduli space. The structure of supermoduli space and its relation to moduli space are of great importance, and we shall explore them in this section as well as in Sec. VII. The superfield approach will be taken as a starting point in Secs. III.B, III.D–III.J, and III.L, and the component field formalism will be related to it in Secs. III.C, III.K, and III.M–III.P.

There are also string theories with larger worldsheet supersymmetry classified by Ademollo *et al.* (1976a). There is the $N=2$ superstring constructed by Ademollo *et al.* (1976b) for which a locally supersymmetric formulation was given by Brink and Schwarz (1977), which is critical in two (complex) space-time dimensions; it was recently explored by Cohn (1987) and D'Adda and Lizzi (1987). There is an $N=4$ theory constructed by Ademollo *et al.* (1976c) whose covariant formulation is due to Pernici and van Niewenhuizen (1986) and that is critical in -2 (quaternionic) dimensions. In the covariant formulation these string theories involve also a nondynamical gauge field on the worldsheet. String theories with gauge fields on the worldsheet have also been considered by Tomboulis (1987) and Porrati and Tomboulis (1988).

A compendium of standard conventions and reference formulas, including the Dirac matrices, is given in Appendix A.

A. Spinors on a Riemann surface

Before constructing the amplitudes for fermionic strings, it is useful to recall some standard terminology of the theory of Riemann surfaces needed for a proper definition of fermions on the surface. The *first homology group* of a compact surface M without boundaries and with h handles is given by

$$H^1(M)=\mathbf{Z}^{2h} . \tag{3.4}$$

A canonical basis for this group is provided by closed curves A_I and B_I, $I=1,2,\ldots,h$, with canonical intersection matrix

$$\#(A_I,A_J)=0,\quad \#(A_I,B_J)=\delta_{IJ},\quad \#(B_I,B_J)=0 . \tag{3.5}$$

Recall that the intersection form is antisymmetric. An example of such an assignment of A and B curves is given in Fig. 10. The choice of canonical basis is clearly not unique. If (A_I,B_I) is a canonical basis, then so is (A_I',B_I') with

$$B_I'=B_{IJ}A_J+A_{IJ}B_J,\quad A_I'=D_{IJ}A_J+C_{IJ}B_J , \tag{3.6}$$

where the $(2h\times 2h)$-dimensional matrix

$$M=\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

belongs to the symplectic group with integer coefficients $Sp(2h,\mathbf{Z})$. This group is the so-called *Siegel modular* group or simply modular group. One may think of it as being generated by 2π twists about A and B cycles. Such 2π twists about a closed curve are usually called *Dehn twists.*

Actually, the modular group is a subgroup of the mapping class group encountered earlier. To generate the mapping class group by Dehn twists, one needs twists about A and B cycles, but also about curves "linking consecutive handles" D_n as indicated in Fig. 10.

The quotient of the mapping class group by the modular group is the so-called *Torelli group*, which no longer acts on the homology basis.

Using the canonical homology decomposition in A and B cycles, we may cut the surface apart and represent it as a simply connected region of the plane—the fundamental region—on which sides are pairwise identified. As indicated in Fig. 11, it is convenient to perform this cutting process loop by loop, so that the boundary consists of unions of segments $A_IB_IA_I^{-1}B_I^{-1}$. Conversely, having such a fundamental region, one may reassemble the surface loop by loop, as shown in four stages in Fig. 12.

We now come to spin structures. In the Introduction

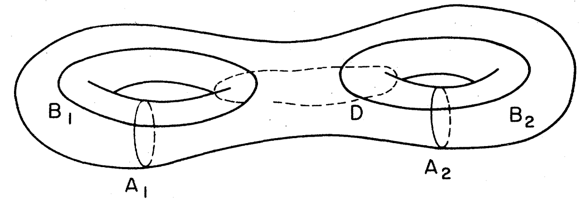


FIG. 10. A genus-2 surface with its canonical homology basis, generated by closed curves A_I and B_I. The Dehn twist D has also been indicated.

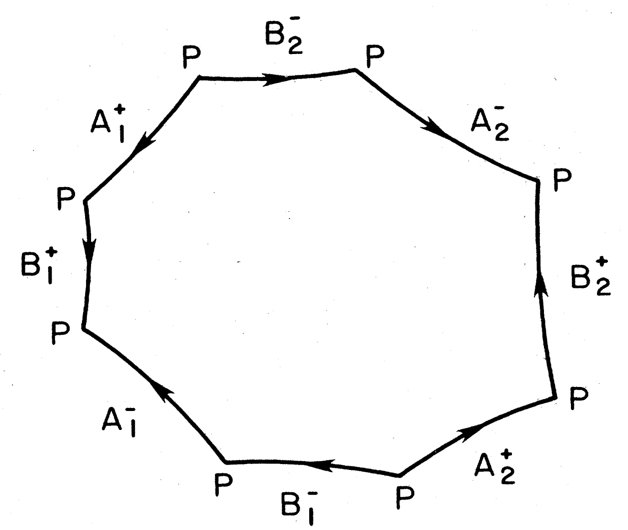


FIG. 11. A genus-2 surface cut along canonical homology cycle. All cycles pass through a common point P.

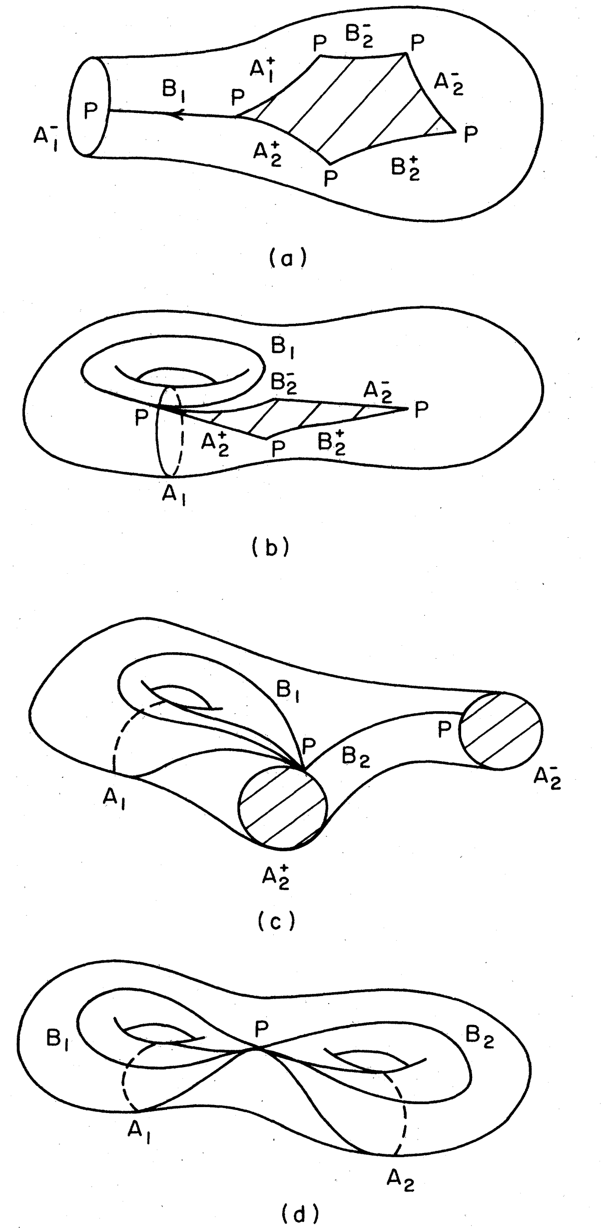


FIG. 12. Reconstruction of the genus-2 surface from the cut representation: (a) gluing B_1^+ and B_1^-; (b) gluing A_1^+ and A_1^-; (c) gluing B_2^+ and B_2^-; (d) gluing A_2^+ and A_2^-.

we already mentioned that the phase shift of a spinor after parallel transport along a closed curve should be half that of a vector and is ambiguous. Thus spinors exist only on manifolds for which a consistent choice of phases along all closed curves can be made. In general there is a topological obstruction to doing this, which is the second Stiefel-Whitney class. For oriented surfaces, however, this class vanishes and spin structures can be visualized as follows. If we fix a reference spin structure ν, the phase shifts around each of the homology cycles A_I and B_I of any other spin structure will differ from those of ν by 0 or π. Thus there are altogether 2^{2h} different spin structures [see Atiyah (1971) and Dabrowski and Percacci (1986), and the discussion in Sec. VI.F].

Each spin structure defines a distinct class of spinors, which does not interact with the others, and a corresponding Dirac operator. This implies immediately a natural classification of spin structures into even and odd ones, corresponding to the parity of the number of zero modes of the Dirac operator. It will be seen in Secs. V.C and VI.F that spin structures can be more conveniently expressed in terms of multipliers or theta characteristics, and that generically the number of Dirac zero modes is always 0 to 1. A diffeomorphism of the worldsheet M may transform a spin structure ν into a different one ν'. Since the parity of Dirac zero modes is invariant, Diff(M) will preserve the parity of the spin structure. It is an important fact that within each parity they can actually all be permuted under the mapping class group. (If we represent spin structures by theta characteristics, this will follow at once from the transformation law of theta functions; see Sec. VI.E and Appendix E.) This property will fix the relative phases of Dirac determinants within each group, and the relative phases between the two parities themselves will ultimately be determined from factorization requirements.

B. $N=1$ supergravity, supercomplex structures, and super Riemann surfaces

Locally, $N=1$ superspace is parametrized by two real $\xi^m=(\xi^1,\xi^2)$ or one complex coordinate $\xi=(1/\sqrt{2})(\xi^1+i\xi^2)$ and two real odd coordinates $\theta^\mu=(\theta^1,\theta^2)$, or one complex odd coordinate $\theta=(1/\sqrt{2})(\theta^1+i\theta^2)$ and its complex conjugate $\bar\theta$. These coordinates are collected into one supercoordinate $z^M=(\xi,\bar\xi;\theta,\bar\theta)$, where the index M is a coordinate or Einstein index. Correspondingly, we have the partial derivatives $\partial_M=(\partial/\partial\xi,\partial/\partial\bar\xi;\partial/\partial\theta,\partial/\partial\bar\theta)$. We shall also use a local U(1) frame with indices $A=(z,\bar z;+,-)$, where z and $\bar z$ refer to the *vector* representation of the U(1) frame group and $+$ and $-$ refer to the *spinor* representation. The corresponding lower-case latin and greek letters correspond to the even and odd parts of these coordinates, respectively.

The $N=1$ supergravity multiplet consists of the superzweibein $E_M{}^A$ and the U(1) superconnection Ω_M, from which a U(1)-covariant superderivative $\mathcal{D}_M$ may be constructed. When this derivative acts on U(1) tensors V of weight n, it is given by[10]

$$\mathcal{D}_M^n V=\partial_M V+in\Omega_M V \ . \tag{3.7}$$

In particular, on one-forms and vector fields we have

$$\mathcal{D}_M V_A=\partial_M V_A+\Omega_M E_A{}^B V_B \ ,$$
$$\mathcal{D}_M V^A=\partial_M V^A-(-)^{mb}V^B E_B{}^A\Omega_M \ ,$$

where

$$E_a{}^b=\varepsilon_a{}^b,\quad E_a{}^\beta=E_\alpha{}^b=0,\quad E_\alpha{}^\beta=\tfrac{1}{2}(\gamma_5)_\alpha{}^\beta \ .$$

In differential form notation we have

$$\mathcal{D}^n=dz^M\mathcal{D}_M^n=d+in\Omega \text{ with } \Omega=dz^M\Omega_M \ ,$$

and d stands for the ordinary differential $d=dz^M\partial_M$. We shall mostly be using the covariant derivatives with U(1) indices $\mathcal{D}_A^n=E_A{}^M\mathcal{D}_M^n$, because they are manifestly super-reparametrization invariant. It will also prove useful to employ the real operators

$$\mathcal{P}_n=\mathcal{D}_+^n\oplus\mathcal{D}_-^{-n} \tag{3.8}$$

acting on the direct sum of superfields of U(1) weights n and $-n$, analogous to the operators P_n of Sec. II. We also introduce the Laplacians

$$\Box_n^{(+)}=\mathcal{D}_-^{n+1/2}\mathcal{D}_+^n,\quad \Box_n^{(-)}=\mathcal{D}_+^{n-1/2}\mathcal{D}_-^n \ , \tag{3.9}$$

so that (as we shall see later)

$$\mathcal{P}_n^\dagger\mathcal{P}_n=\Box_n^{(+)}\oplus\Box_{-n}^{(-)} \ .$$

The Laplacian on scalar superfields will be denoted by $\Box_0=-\Box_0^{(+)}=\Box_0^{(-)}$.

Torsion $T_{BC}{}^A$ and curvature R_{AB} tensors are defined by

$$[\mathcal{D}_A,\mathcal{D}_B]=T_{AB}{}^C\mathcal{D}_C+inR_{AB} \ , \tag{3.10}$$

where [,] is understood to be a commutator except when both A and B are spinor indices in which case it is an anticommutator. The supergeometry may be specified by imposing the standard torsion constraints

$$T_{ab}{}^c=T_{\alpha\beta}{}^\gamma=0,\quad T_{\alpha\beta}{}^c=2(\gamma^c)_{\alpha\beta} \ . \tag{3.11a}$$

[10] $U(1)$ weights are normalized so that $n=1$ (-1) for a lower z $(\bar z)$ index; in flat space, $U(1)$ indices then agree with the conventions used in Sec. II.

Equivalently—and more usefully—we may replace Eq. (3.11a) by the constraint that the curvature $R_{\alpha\beta}$ be proportional to $(\gamma_5)_{\alpha\beta}$, instead of $T_{ab}{}^c=0$. Thus Eq. (3.11a) is equivalent to

$$R_{\alpha\beta}=-i(\gamma_5)_{\alpha\beta}R_{+-},\quad T_{\alpha\beta}{}^\gamma=0,\quad T_{\alpha\beta}{}^c=2(\gamma^c)_{\alpha\beta} \ . \tag{3.11b}$$

Another way of looking at the latter constraints is that they entirely specify the commutation relations between $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$, so that with the help of Eq. (3.10) all components of torsion and curvature can be computed in terms of the single scalar superfield R_{+-}. Thus, Eq. (3.11) implies the torsion formulas

$$\begin{aligned} &T_{\beta c}{}^a=0 \ ,\\ &T_{b\gamma}{}^\alpha=-\frac{i}{2}(\gamma_b)_\gamma^\alpha R_{+-} \ ,\\ &T_{bc}{}^\alpha=-\frac{i}{2}\varepsilon_{bc}(\gamma_5)^{\alpha\beta}\mathcal{D}_\beta R_{+-} \ ,\end{aligned} \tag{3.12}$$

and the curvature formulas

$$\begin{aligned} &R_{b\gamma}=i(\gamma_5\gamma_b)_\gamma{}^\delta\mathcal{D}_\delta R_{+-} \ ,\\ &R_{ab}=-\frac{i}{2}\varepsilon_{ab}\mathcal{D}^\alpha\mathcal{D}_\alpha R_{+-}-\tfrac{1}{2}\varepsilon_{ab}(R_{+-})^2 \ .\end{aligned} \tag{3.13}$$

All other components vanish. As a consequence of the torsion constrains, one may express the components of the superconnection in terms of the superzweibein:

$$\Omega_+=2iE_+{}^M(\partial_M E_+{}^N)E_N{}^+,\quad \Omega_z=E_+{}^M\partial_M\Omega_+ \ . \tag{3.14}$$

1. Symmetries

The supergeometry is invariant under transformations that preserve the torsion constraints. We list them below, together with their infinitesimal versions expressed in terms of the infinitesimal changes in the superzweibein $H_A{}^B$,

$$H_A{}^B=E_A{}^M\delta E_M{}^B \ . \tag{3.15}$$

Thus the symmetries of the supergeometry are as follows.

(i) Local U(1) transformations, forming a group sU(1). These are generated by a real superfield L acting by

$$\begin{aligned} &E_M^\pm=e^{\pm(i/2)L}\hat E_M^\pm,\quad \mathcal{D}_+^n=e^{-i(n+1/2)L}\hat{\mathcal{D}}_+^n e^{inL} \ ,\\ &E_M{}^z=e^{iL}\hat E_M{}^z,\quad \mathcal{D}_-^n=e^{-i(n-1/2)L}\hat{\mathcal{D}}_-^n e^{inL} \ ,\\ &E_M{}^{\bar z}=e^{-iL}\hat E_M{}^{\bar z},\quad \Omega_M=\hat\Omega_M+\partial_M L \ ,\end{aligned} \tag{3.16}$$

and infinitesimal transformations given by

$$\delta E_M{}^A=-E_M{}^B E_B{}^A\delta L,\quad H_A{}^B=-\delta L E_A{}^B \ . \tag{3.17}$$

(ii) Super-reparametrizations, forming a group sDiff(M). The infinitesimal ones are generated by super vector fields δV^M and are given by

$$H_A{}^B = \mathcal{D}_A \delta V^B - \delta V^C T_{CA}{}^B + \delta V^C \Omega_C E_A{}^B . \qquad (3.18)$$

(iii) Super Weyl transformations, forming a group sWeyl(M). These are generated by a real scalar superfield Σ,

$$E_M^a = e^{\Sigma} \hat{E}{}_M^{\ a} ,$$
$$E_M^\alpha = e^{\Sigma/2} [\hat{E}{}_M^{\ \alpha} + \hat{E}{}_M^{\ a} (\gamma_a)^{\alpha\beta} \hat{\mathcal{D}}_\beta \Sigma] , \qquad (3.19)$$

and induce the following transformation laws on the superconnection, supercurvature, and superderivatives:

$$\Omega_M = \hat{\Omega}_M + \hat{E}_M^a \varepsilon_a{}^b \hat{\mathcal{D}}_b \Sigma + \hat{E}{}_M^{\ \alpha} (\gamma_5)_\alpha{}^\beta \hat{\mathcal{D}}_\beta \Sigma ,$$
$$R_{+-} = e^{-\Sigma} (\hat{R}_{+-} - 2i \hat{\mathcal{D}}_+ \hat{\mathcal{D}}_- \Sigma) ,$$
$$\mathcal{D}_+^n = e^{(n-1/2)\Sigma} \hat{\mathcal{D}}{}_+^{\ n} e^{-n\Sigma} , \qquad (3.20)$$
$$\mathcal{D}_-^n = e^{-(n+1/2)\Sigma} \hat{\mathcal{D}}{}_-^{\ n} e^{+n\Sigma} .$$

The infinitesimal form of Eq. (3.19) reads

$$H_a{}^b = \delta\Sigma \delta_a{}^b , \quad H_\alpha{}^\beta = \tfrac{1}{2} \delta\Sigma \delta_\alpha{}^\beta ,$$
$$H_\alpha{}^a = 0 , \quad H_a{}^\alpha = (\gamma_a)^{\alpha\beta} \mathcal{D}_\beta \delta\Sigma . \qquad (3.21)$$

It will be useful to keep in mind that not all H's are independent due to the torsion constraints (3.11). The simplest set of independent deformations is $H_+{}^+$, $H_-{}^-$, and $H_\alpha{}^b$. The other components can then be calculated using the torsion constraints. To first order in H, we have the general formula

$$\delta T_{AB}{}^C = -H_A{}^D T_{DB}{}^C + T_{AB}{}^D H_D{}^C + (-)^{ab} H_B{}^D T_{DA}{}^C$$
$$- \mathcal{D}_A H_B{}^C + (-)^{ab} \mathcal{D}_B H_A{}^C + \psi_A E_B{}^C$$
$$- (-)^{ab} \psi_B E_A{}^C ,$$

where $\psi_A = E_A{}^M \delta\Omega_M$. These imply

$$H_z^z = \mathcal{D}_+ H_+{}^z + 2H_+{}^+ ,$$
$$H_z{}^{\bar z} = \mathcal{D}_+ H_+{}^{\bar z} ,$$
$$H_+{}^- = -\tfrac{1}{2} \mathcal{D}_+ H_-{}^{\bar z} - \tfrac{1}{2} \mathcal{D}_- H_+{}^{\bar z} ,$$
$$H_z{}^- = \mathcal{D}_+ H_+{}^- + \frac{i}{2} R_{+-} H_+{}^{\bar z} , \qquad (3.22)$$
$$H_z^+ = \mathcal{D}_+ H_+{}^+ + \mathcal{D}_+ H_-{}^- + \mathcal{D}_- H_+{}^- + \frac{i}{2} R_{+-} H_-{}^{\bar z} ,$$
$$\delta\Omega_+ = -i \mathcal{D}_{\bar z} H_+{}^{\bar z} - i R_{+-} H_-{}^{\bar z} + \tfrac{1}{2} (\mathcal{D}_- H_+{}^{\bar z}) \Omega_- - H_+{}^{\bar z} \Omega_{\bar z} .$$

2. Supercomplex structures

By analogy with two-dimensional geometry we introduce a supercomplex structure

$$J_M{}^N = E_M{}^a \varepsilon_a{}^b E_b{}^N + E_M{}^\alpha (\gamma_5)_\alpha{}^\beta E_\beta{}^N , \qquad (3.23)$$

which is a super-reparametrization tensor, and a local U(1) scalar. The main properties of $J_M{}^N$ are

$$J_M{}^N J_N{}^P = -\delta_M{}^P \qquad (3.24)$$

and the fact that it depends only on the *superconformal class* of $E_A{}^M$, i.e., it is invariant under the super Weyl transformations of (iii).

The almost complex structure $J_M{}^N$ of Eq. (3.23) may be used to define complex or, in this case, superholomorphic coordinates on the surface, provided this almost complex structure is integrable. This is actually a consequence of the superconformal flatness of two-dimensional supergeometry. A direct check of integrability illustrating the role of the torsion constraints is obtained by introducing the following one-forms:

$$\zeta^M = dz^M - i\, dz^N J_N{}^M ,$$
$$\bar\zeta^M = dz^M + i\, dz^N J_N{}^M . \qquad (3.25)$$

ζ^M by itself has only two independent components, in view of Eq. (3.25). The almost complex structure $J_M{}^N$ is integrable provided

$$d\zeta^M \equiv 0 \ (\mathrm{mod}\ \zeta^N) . \qquad (3.26)$$

Using the explicit expression for $J_M{}^N$ in Eq. (3.23), as well as the definition of the torsion $T_{BC}{}^A$ of the $N=1$ supergeometry, we get

$$d\zeta^M = -\tfrac{1}{2} \bar\zeta^P E_P{}^{\bar z} \zeta^Q E_Q{}^- (T_{-\bar z}{}^+ E_+{}^M + T_{-\bar z}{}^z E_z{}^M)$$
$$- \tfrac{1}{4} \bar\zeta^P E_P{}^- \bar\zeta^Q E_Q{}^- (T_{--}{}^+ E_+{}^M + T_{--}{}^z E_z{}^M)$$
$$(\mathrm{mod}\ \zeta^N) , \qquad (3.27)$$

which indeed yields Eq. (3.26) with the help of the torsion constraints (3.11) and their consequences (3.12). Conversely, a supergeometry will support a complex structure only when the above torsion constraints are satisfied.

Thus we may define superholomorphic and superantiholomorphic functions by

$$J_M{}^N \mathcal{D}_N f = i \mathcal{D}_M f , \quad J_M{}^N \mathcal{D}_N \bar f = -i \mathcal{D}_M \bar f , \qquad (3.28)$$

or, equivalently,

$$\mathcal{D}_- f = 0 , \quad \mathcal{D}_+ \bar f = 0 .$$

The supersurface together with a supercomplex structure $J_M{}^N$ will be called a super Riemann surface, although strictly speaking the geometry of the supersurface is not Riemannian, i.e., there is no metric for superspace. One can verify that a super Riemann surface admits an atlas of coordinate patches whose transition functions are superholomorphic. This approach provides an alternative definition of a super Riemann surface.

3. Flat and conformally flat superspace

Flat $N=1$ superspace is given by the superzweibein

$$E_m{}^a = \delta_m{}^a , \quad E_m{}^\alpha = 0 ,$$
$$E_\mu{}^a = (\gamma^a)_\mu{}^\beta \theta_\beta , \quad E_\mu{}^\alpha = \delta_\mu{}^\alpha , \qquad (3.29)$$

and the superderivatives take the simple form

$$\mathcal{D}_+ = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}, \quad \mathcal{D}_- = \frac{\partial}{\partial\bar\theta} + \bar\theta\frac{\partial}{\partial\bar z} . \tag{3.30}$$

Equivalently, flat superspace is characterized by $R_{+-}=0$. Locally every supergeometry is superconformal [i.e., equivalent under a super Weyl and local U(1) transformation] to flat supergeometry. One can easily see this directly from the equations characterizing superreparametrizations and by using the analogous result for ordinary geometry, or by evaluating the supercomplex structure tensor $J_M{}^N$ of Eq. (3.23). Locally, then, Eq. (3.28) is solved by

$$f = f_0(z) + \theta f_1(z) ,$$

where f_0 and f_1 are holomorphic in the ordinary sense.

Globally, however, there may be topological obstructions, and it will be necessary to introduce supermoduli space, i.e., the space of inequivalent superconformal structures. We shall take up this issue in Secs. III.E and III.G.

A complete local analysis of $N=1$ two-dimensional supergravity is due to Howe (1979). In particular, the fact that any two-dimensional supergeometry is locally superconformally flat is due to him. The superfield formalism and variations $H_A{}^B$ were used by Martinec (1983) to compute the super Weyl anomaly. The supercomplex structure $J_M{}^N$ was introduced by D'Hoker and Phong (1987a), who also showed that its integrability (vanishing of the Nijenhuis tensor) is a consequence of the Wess-Zumino torsion constraints. Interpretations of Wess-Zumino constraints as reductions of G structures were subsequently given by Giddings and Nelson (1988a).

The alternative approach to super Riemann surfaces through superholomorphic function theory and charts as in Eq. (3.28) was developed by Friedan (1986), Baranov, Frolov, and Schwarz (1987), and Crane and Rabin (1987). That the two classes of super Riemann surfaces coincide was proved by Giddings and Nelson (1988a).

C. Component formalism for $N=1$ supergravity

To obtain a better understanding of supergeometry and supergravity, it should be useful to discuss the associated component formulation. The passage from the superfield to the component language requires the elimination of the auxiliary fields required by the superfields. This is usually accomplished by fixing the Wess-Zumino gauge for the superzweibein. This gauge is defined by the condition that in the expansion in powers of θ we have[11]

$$\begin{aligned} &E_\mu{}^\alpha \sim \delta_\mu{}^\alpha + \theta^\nu e^{*\alpha}_{\nu\mu}, \quad E_\mu{}^a \sim \theta^\nu e^{**a}_{\nu\mu} , \\ &e^{*\alpha}_{\nu\mu} = e^{*\alpha}_{\mu\nu}, \quad e^{**a}_{\nu\mu} = e^{**a}_{\mu\nu} , \end{aligned} \tag{3.31}$$

[11]This is the correct choice provided—as we have—the gamma matrices are taken to be symmetric.

up to higher-order terms. A superzweibein can always be brought to this gauge by a super-reparametrization, which is obtained through algebraic equations alone. The main ingredients of the supergeometry can then be derived from the Bianchi identities and the torsion constraints. The results for their full θ expansions are

$$\begin{aligned} &E_m{}^a = e_m{}^a + \theta\gamma^a\chi_m - \frac{i}{2}\theta\bar\theta e_m{}^a A , \\ &E_m{}^\alpha = -\tfrac{1}{2}\chi_m{}^\alpha - \frac{i}{2}\theta^\beta(\gamma_m)_\beta{}^\alpha A - \tfrac{1}{2}\theta^\beta(\gamma_5)_\beta^\alpha\omega_m \\ &\qquad + i\theta\bar\theta[\tfrac{1}{2}(\gamma_m)^{\alpha\beta}\Lambda_\beta - \tfrac{3}{8}\chi_m^\alpha A] , \\ &E_\mu{}^a = (\gamma^a)_\mu{}^\beta\theta_\beta , \\ &E_\mu{}^\alpha = \delta_\mu{}^\alpha(1 + i\theta\bar\theta A/4) , \\ &\mathrm{sdet} E_M{}^A = e\left[1 + \tfrac{1}{2}\theta\gamma^n\chi_n - \frac{i}{2}\theta\bar\theta A + \tfrac{1}{8}\theta\bar\theta\varepsilon^{mn}\chi_m\gamma_5\chi_n\right] , \end{aligned} \tag{3.32}$$

where

$$\begin{aligned} &\omega_m = -e_m{}^a\varepsilon^{pq}\partial_p e_q{}^a - \tfrac{1}{2}\chi_m\gamma_5\gamma^p\chi_p , \\ &\Lambda = -i\gamma_5\varepsilon^{mn}D_m\chi_n - \tfrac{1}{2}\gamma^m\chi_m A , \\ &D_m\chi_n = \partial_m\chi_n + \tfrac{1}{2}\omega_m\gamma_5\chi_n . \end{aligned} \tag{3.33}$$

Notice that since $E_\mu{}^\alpha$ is basically the Kronecker symbol between a μ and an α index, the distinction between U(1) and Einstein spinor indices is lost in Wess-Zumino gauge, and θ may be written either with α or μ indices and transforms as a spinor under U(1). It is also useful to record the spinor components of the inverse superzweibeins,

$$\begin{aligned} &E_\alpha{}^\mu = \delta_\alpha{}^\mu + \tfrac{1}{2}\theta^\gamma\gamma^m_{\alpha\gamma}\chi_m{}^\mu + i\theta\bar\theta\hat e_\alpha{}^\mu , \\ &E_\alpha{}^m = \theta^\beta\gamma^m_{\alpha\beta} + \tfrac{1}{2}\theta\bar\theta(\gamma^n\gamma^m)_\alpha{}^\gamma\chi_{n\gamma} , \end{aligned} \tag{3.34}$$

where

$$\hat e^\mu_\alpha = \tfrac{1}{4}\delta_\alpha{}^\mu A + \frac{i}{2}(\gamma^m\gamma_5)_\alpha{}^\mu\omega_m - \frac{i}{4}(\gamma^n\gamma^m)_\alpha{}^\gamma\chi_{n\gamma}\chi_m{}^\mu . \tag{3.35}$$

For the superconnection and supercurvature we have

$$\begin{aligned} &\Omega_m = \omega_m + \frac{i}{2}\theta\gamma_5\chi_m A - i\theta\gamma_5\gamma_m\Lambda + i\theta\bar\theta\hat\omega_m , \\ &\Omega_\mu = \frac{i}{2}(\gamma_5)_\mu{}^\beta\theta_\beta A , \\ &R_{+-} = A + \theta^\alpha\Lambda_\alpha + i\theta\bar\theta C , \\ &C = R + \frac{i}{2}\chi_a\gamma^a\Lambda + \frac{i}{8}\varepsilon^{ab}\chi_a\gamma_5\chi_b A + \tfrac{1}{2}A^2 , \end{aligned} \tag{3.36}$$

where

$$\hat\omega_n = -\tfrac{1}{2}A\,\omega_n - \tfrac{1}{2}\chi_m\gamma_5\gamma_n\gamma^m\Lambda - \tfrac{1}{2}e_n^a\varepsilon_a{}^b e_b{}^m\partial_m A ,$$

with R the curvature of the connection ω_m appearing in Eq. (3.33):

$$R=\varepsilon^{mn}\partial_m\omega_n \ . \tag{3.37}$$

Thus the supergravity multiplet $E_M{}^A$ reduces to a zweibein $e_m{}^a$, a gravitino field $\chi_m{}^\alpha$, and an auxiliary field A which will not appear in the component Lagrangian. Wess-Zumino gauge is left invariant under a subgroup of all super-reparametrizations and local U(1) transformations, given by

$$\delta V^m=\delta v^m-\theta\gamma^m\zeta-\tfrac{1}{2}\theta\bar\theta\chi_n\gamma^m\gamma^n\zeta \ ,$$

$$\delta V^\mu=\zeta^\mu+\tfrac{1}{2}\theta^\alpha(\gamma_5)_\alpha{}^\mu l-\tfrac{1}{2}\theta\gamma^n\zeta\chi_n{}^\mu+i\theta\bar\theta\hat\zeta^\mu \ ,$$

$$L=l-\frac{i}{2}A\theta\gamma_5\zeta+\omega_n\theta\gamma^n\zeta+i\theta\bar\theta\hat l \ ,$$

where we have used the abbreviations

$$\hat\zeta=\frac{i}{2}\gamma_5\gamma^n\zeta\omega_n+\frac{i}{4}\chi_n(\chi_m\gamma^n\gamma^m\zeta)-\tfrac{1}{2}\zeta A \ ,$$

$$\hat l=\frac{1}{2}\zeta\gamma_5\Lambda-\frac{i}{2}\omega_n\chi_m\gamma^n\gamma^m\zeta+\tfrac{1}{4}\zeta\gamma_5\gamma^n\chi_n A \ .$$

It is now straightforward to translate the symmetries of the superzweibein into component language as well. The super-reparametrizations relevant to the component language are those that preserve the Wess-Zumino gauge up to local U(1) and super Weyl transformations. They decompose into reparametrization invariance and an $N=1$ supersymmetry. Super Weyl transformations will take us out of this gauge, so the component transformations written below are obtained only after compensation by a super-reparametrization and a local U(1) transformation taking us back to Wess-Zumino gauge.

(i) Local U(1) symmetry forming the group sU(1):

$$\delta e_m{}^a=l\varepsilon^a{}_b e_m{}^b \ ,$$

$$\delta\chi_m=-\tfrac{1}{2}l\gamma_5\chi_m \ ,$$

$$\delta A=0 \ ,$$

$$\delta\omega_m=\partial_m l \ .$$

(ii) Reparametrizations, forming Diff(M):

$$\delta e_m{}^a=\delta v^n\partial_n e_m{}^a+e_n{}^a\partial_m\delta v^n \ ,$$

$$\delta\chi_m=\delta v^n\partial_n\chi_m+\chi_n\partial_m\delta v^n \ ,$$

$$\delta A=\delta v^n\partial_n A \ ,$$

$$\delta\omega_m=\delta v^n\partial_n\omega_m+\omega_n\partial_m\delta v^n \ .$$

(iii) Local $N=1$ supersymmetry:

$$\delta e_m{}^a=\zeta\gamma^a\chi_m \ ,$$

$$\delta\chi_m=-2D_m\zeta-iA\gamma_m\zeta \ ,$$

$$\delta A=\zeta\Lambda \ ,$$

$$\delta\omega_m=i\zeta\gamma_m\gamma_5\Lambda+\frac{i}{2}\zeta\gamma_5\chi_m A \ ,$$

$$\delta\Lambda=-\tfrac{1}{2}\gamma^m\zeta(\partial_m A+\tfrac{1}{2}\chi_m\Lambda)-i\zeta C \ ,$$

$$\delta C=\frac{i}{2}(\chi_p\gamma^m\gamma^p\zeta)(\partial_m A+\tfrac{1}{2}\chi_m\Lambda)-i\zeta\gamma^m D_m\Lambda$$
$$-\tfrac{1}{2}\zeta\gamma^p\chi_p C-\tfrac{1}{2}A\zeta\Lambda \ .$$

(iv) Weyl transformations, forming the group Weyl(M):

$$\delta e_m{}^a=\delta\sigma e_m{}^a \ ,$$

$$\delta\chi_m=\tfrac{1}{2}\delta\sigma\chi_m \ .$$

(v) Super Weyl scalings:

$$\delta e_m{}^a=0 \ ,$$

$$\delta\chi_m=\gamma_m\delta\lambda \ .$$

Finally we note that the "super Euler number" reduces to the standard Euler number

$$\chi(M)=\frac{i}{2\pi}\int_M d^2\mathbf{z}\,ER_{+-}=\frac{1}{2\pi}\int_M d^2\xi\sqrt{g}R \ , \tag{3.38}$$

where the volume element on the superworldsheet is given by

$$d^2\mathbf{z}\,E=d^2\xi\,d\theta\,d\bar\theta\,\mathrm{sdet}E_M{}^A \ . \tag{3.39}$$

The topology of the super Riemann surface is just that of its "body" component when it is viewed as a De Witt (1983) supermanifold, and hence the topological classification is again by the number of handles, when the surface has no boundaries.

The passage to Wess-Zumino gauge and the construction of the super Weyl symmetry has been carried out by Howe (1979). The formulas of Howe have been reproduced here in Euclidean signature for convenience.

D. Path integrals for the RNS superstring

The superspace action for the Ramond-Neveu-Schwarz string model is obtained by coupling scalar "position" superfields X^μ, $\mu=1,\ldots,d=10$ to two-dimensional $N=1$ supergravity. The matter action is then given by

$$I_m=\frac{1}{8\pi}\int d^2\mathbf{z}\,E\mathcal{D}^\alpha X^\mu\mathcal{D}_\alpha X_\mu+\lambda\chi(M)$$
$$=\frac{1}{4\pi}\int d^2\mathbf{z}\,E\mathcal{D}_-X^\mu\mathcal{D}_+X_\mu+\lambda\chi(M) \ . \tag{3.40}$$

We may decompose X^μ into components: $X^\mu=x^\mu+\theta^\alpha\psi^\mu_\alpha+i\theta\bar\theta F^\mu$, where x^μ and ψ^μ may be identified with the fields occurring in Eq. (3.1) and F^μ is an auxiliary field. The action (3.40) actually coincides with Eq. (3.1) in Wess-Zumino gauge except for a term F^2. The symmetries (i), (ii), (iii) of Sec. III.B of supergravity will become symmetries of I_m when X^μ is assigned the corresponding transformation laws: X^μ is a local U(1), super Weyl, and super-reparametrization scalar. In addition, I_m is evidently invariant under space-time Poincaré transformations if the target space-time is flat Minkowskian. Imposing the above symmetries, we find that the action (3.40) is unique.

To quantize the theory we integrate e^{-I_m} over all supergeometries $(E_M{}^A, \Omega_M)$ satisfying the torsion constraints and over all superfields X^μ, and we sum over all possible topologies of the super Riemann surface. Recalling that this reduces to the sum over the number of handles just as in the bosonic case, we may conjecture the contribution to the partition function at h string loops,

$$Z_h = \int DE_M{}^A D\Omega_M DX^\mu \delta(T) \exp(-I_m[X^\mu, E_M{}^A]) , \tag{3.41}$$

with the topology of the worldsheet fixed at h handles. The delta function enforcing the torsion constraints [Eq. (3.11)] is denoted by $\delta(T)$. It involves only algebraic equations, which are linear in Ω_M, so that the Ω_M integral may be ignored once the torsion constraints have been enforced.

Similarly, scattering amplitudes are obtained by integrating the product of e^{-I_m} by a number of vertex operators, exactly as in the bosonic case. We shall not reproduce the corresponding formulas here.

The integral assumes the existence of a local U(1), super-reparametrization-invariant measure, not depending on derivatives. The unique choice for DX^μ comes from the metric

$$\|\delta X^\mu\|^2 = \int_M d^2\mathbf{z}\, E \delta X^\mu \delta X_\mu . \tag{3.42}$$

We can carry out the integration over X^μ in Eq. (3.41) since it is Gaussian. As will be shown in Sec. III.E, however, the operator $\mathcal{D}_+\mathcal{D}_-^{(0)} = \Box_0$ has zero modes. First, there is the constant superfield corresponding to constant x^μ. For odd-spin structure, there will also be a Dirac zero mode for ψ^μ, but how many zero modes remain for $\Box_0$ may depend on the superconformal class. If there are odd zero modes of $\Box_0$ (analogous to Dirac zero modes), then the partition function of Eq. (3.41) must vanish—though of course correlation functions may be nonvanishing. Thus the proper formula is obtained by omitting only the constant zero mode, so that we obtain

$$Z_h = \Omega \int DE_M{}^A D\Omega_M \delta(T) \left[\frac{8\pi^2}{\int d^2\mathbf{z}\, E} \operatorname{sdet}'\Box_0 \right]^{-d/2} . \tag{3.43}$$

Here Ω is the volume of space-time, and the prime denotes omission of the translation zero mode. Note that the superfield X^μ depends on the spin structure, and hence so does the superdeterminant.

The integration over supergeometries is considerably more complicated. Since there are torsion constraints, we have the choice of using the first- or second-order formalisms (see de Witt and Freedman, 1983). In the first-order formalism, all 16 components of $E_M{}^A$ and all 4 of Ω_M are integrated over, subject to the torsion constraints, which may be represented by the use of Lagrange multipliers. Alternatively, in the second-order formalism, dependent degrees of freedom are completely eliminated by use of the torsion constraints, and the integration measure is restricted to the independent components only. Though elimination of dependent degrees of freedom can conveniently be achieved only if simultaneously a gauge condition is imposed (like Wess-Zumino gauge), the dependent infinitesimal variations of the supergeometry are easily determined, as was done in Eq. (3.22). To write down the metric on the space of supergeometries, infinitesimal variations are all that is needed; thus to construct the natural measure on the independent components of $\delta E_M{}^A$ and $\delta\Omega_M$, we recall that the only independent components of $H_A{}^B = E_A{}^M \delta E_M{}^B$ are $H_\alpha{}^\alpha$, $\gamma_5 H = (\gamma_5)_\beta{}^\alpha H_\alpha{}^\beta$, and $H_\alpha{}^b$, the other components being given by Eq. (3.22). The expression for $\delta\Omega_M$ can be determined from Eq. (3.14). Note that these relations involve superderivatives of the independent components. Thus, in order to obtain a metric consistent with locality on the worldsheet, it is necessary to construct it in terms of independent fields only. This metric on $H_A{}^B$ should be of the form

$$\|\delta E_M{}^A\|^2 = \int d^2\mathbf{z}\, E[\varepsilon^{\alpha\beta} H_\alpha{}^a H_\beta{}^a + c_1 H_\alpha{}^\alpha H_\beta{}^\beta + c_2(\gamma_5 H)(\gamma_5 H)] , \tag{3.44}$$

where c_1 and c_2 are undetermined numerical constants, analogous to c in Eq. (2.21). The measure on $DE_M{}^A$ will always be understood as coming from this metric. Associated with Eq. (3.44) is a quadratic form, constructed in the standard way, and denoted by $\langle H_1 \mid H_2 \rangle$.

Though super-reparametrization and local U(1) invariant, $\|\delta E_M{}^A\|$ fails to be super Weyl invariant, which will give rise to the super Weyl anomaly, as we shall see later on. Super Weyl invariance is recovered for the full amplitude, as the anomaly from the matter determinants and Faddeev-Popov ghosts cancel in the critical dimension $d=10$ and in the case of the heterotic string for gauge groups of rank 16. The same will hold true for possible (perturbative) gravitational and holomorphic anomalies arising in connection with the chiral Dirac determinants, as will be shown in Sec. VII. Of course, as higher string loop effects are considered and surfaces of nontrivial topology are used, there may be global reparametrization (or modular) anomalies. In the case of heterotic strings, for example, they give rise to further restriction to the gauge group $\mathrm{Spin}(32)/Z_2$ and $E_8 \times E_8$.

After all these symmetry groups have been factored out, we should be left with a (finite-dimensional) integral over the space of supergeometries that are inequivalent under any of these transformations, and we are now going to identify this space, first locally in Sec. III.E and then globally in Sec. III.G.

E. Deformations of supercomplex structures

The effect on $H_A{}^B$ of combined super-reparametrization δV^M and U(1) and super Weyl transformations δL and $\delta\Sigma$ is completely described by the action on the independent components of $H_A{}^B$ which were identified in Sec. III.B:

$$H_\alpha{}^\alpha=\delta\Sigma+\mathcal{D}_\alpha\delta V^\alpha ,$$
$$(\gamma_5)_\alpha{}^\beta H_\beta{}^\alpha=\delta L+(\gamma_5)_\alpha{}^\beta\mathcal{D}_\beta\delta V^\alpha-\delta V^C\Omega_C , \quad (3.45)$$
$$H_\alpha{}^b=\mathcal{D}_\alpha\delta V^b+2(\gamma^b)_{\alpha\gamma}\delta V^\gamma .$$

This shows that $H_\alpha{}^\alpha$ and $\gamma_5 H$ can be completely eliminated without any topological obstruction through a super Weyl and local U(1) transformation. Since anything proportional to a γ matrix can also be eliminated from $H_\alpha{}^b$ in a purely algebraic fashion, it is natural to introduce

$$(\mathcal{P}_1\delta V)_\alpha{}^b=-(\gamma_c\gamma^b)_\alpha{}^\beta\mathcal{D}_\beta\delta V^c \quad (3.46)$$

in analogy with Eq. (2.23). Upon isolating the various components we obtain

$$(\mathcal{P}_1\delta V)_-{}^z=\mathcal{D}_-\delta V^z, \quad (\mathcal{P}_1\delta V)_-{}^{\bar z}=0 , \quad (3.47)$$

and their complex-conjugate expressions. We observe that the only nonremovable H's are those $H_\alpha{}^b$'s not in the range of $\mathcal{P}_1$. At this stage in the bosonic case, we concluded that the metric deformations δg_{mn} not in the range of P_1 must belong to the orthogonal complement of the range of P_1. This step assumes that the metric $\|\delta g\|^2$ is nondegenerate and (positive) definite.

For the superstring case, we see that the metric defined in Eq. (3.44) is nondegenerate but fails to be definite (i.e., there exist $H\neq 0$ with $\|H\|=0$). When the metric is nondefinite, there may in general be elements belonging to both Range$\mathcal{P}_1$ and $(\text{Range}\mathcal{P}_1)^\perp$, and the sum of these two spaces need not span the full space of deformations $H_\alpha{}^b$.

To analyze the structure of the complement of Range$\mathcal{P}_1$, let us investigate the intersection of Range$\mathcal{P}_1$ and $(\text{Range}\mathcal{P}_1)^\perp$. Introducing the natural metric

$$\|\delta V\|^2=\int_M d^2z\, E\,\delta V^\alpha\delta V_\alpha \quad (3.48)$$

on the space of tensor fields of weight $n\oplus -n$, we readily derive the identity

$$(\text{Range}\mathcal{P}_1)^\perp=\text{Ker}\mathcal{P}_1^\dagger , \quad (3.49)$$

where

$$(\mathcal{P}_1^\dagger H)^a=(\gamma_b\gamma^a)^{\beta\alpha}\mathcal{D}_\beta H_\alpha{}^b . \quad (3.50)$$

Now assume that $H\in(\text{Range}\mathcal{P}_1)\cap(\text{Range}\mathcal{P}_1)^\perp$, then we have with the help of Eq. (3.49) that $H=\mathcal{P}_1\delta V$ and $\mathcal{P}_1^\dagger H=0$. Combining both, we obtain $\mathcal{P}_1^\dagger\mathcal{P}_1\delta V=0$, and so there must be an element δV *not* in Ker$\mathcal{P}_1$ which belongs, however, to Ker$\mathcal{P}_1^\dagger\mathcal{P}_1$. Conversely, if the kernels are equal, then such elements $\delta V\neq 0$ can belong to Ker$\mathcal{P}_1^\dagger\mathcal{P}_1$, and the intersection between Range$\mathcal{P}_1$ and (Range $\mathcal{P}_1)^\perp$ must be trivial:

$$\text{Ker}\mathcal{P}_1=\text{Ker}\mathcal{P}_1^\dagger\mathcal{P}_1 \Longleftrightarrow (\text{Range}\mathcal{P}_1)\cap(\text{Range}\mathcal{P}_1)^\perp=\{0\} . \quad (3.51)$$

Equivalently, this means that the inner product $\langle\ |\ \rangle$ remains nondegenerate upon restriction to Range$\mathcal{P}_1$. Consider the element $H_1=\mathcal{P}_1\delta V_1$ and $H_2=\mathcal{P}_1\delta V_2$ of Range$\mathcal{P}_1$, and compute their inner product:

$$\langle H_1\,|\,H_2\rangle=\langle\mathcal{P}_1\delta V_1\,|\,\mathcal{P}_1\delta V_2\rangle$$
$$=\langle\delta V_1\,|\,\mathcal{P}_1^\dagger\mathcal{P}_1\delta V_2\rangle . \quad (3.52)$$

If this inner product vanishes for all δV_1, then $\delta V_2\in\text{Ker}\mathcal{P}_1^\dagger\mathcal{P}_1$, by nondegeneracy of $\langle\ |\ \rangle$ on the space of all δV's. If the inner product $\langle\ |\ \rangle$ is to remain nondegenerate upon restriction to Range$\mathcal{P}_1$, then we must also have $H_2=0$, so that (3.51) holds. Thus the issue here is the relation between Ker$\mathcal{P}_1$ and Ker$\mathcal{P}_1^\dagger\mathcal{P}_1$. As will become clear during our subsequent discussion, the case of the torus is truly exceptional, and we shall treat it separately later on.

For $h\geq 2$ and $h=0$, it will be shown in Sec. III.F that Ker$\mathcal{P}_1$=Ker$\mathcal{P}_1^\dagger\mathcal{P}_1$, so that the intersection between Range$\mathcal{P}_1$ and $(\text{Range}\mathcal{P}_1)^\perp$ is the null vector only and the sum of Range$\mathcal{P}_1$ and $(\text{Range}\mathcal{P}_1)^\perp$ spans the full space of γ-traceless $H_\alpha{}^b$'s. Putting everything together we obtain the orthogonal decomposition

$$\{H_A{}^B\}=\{\delta\Sigma\}\oplus\{\delta L\}\oplus\text{Range}\mathcal{P}_1\oplus\text{Ker}\mathcal{P}_1^\dagger . \quad (3.53)$$

The elements of Ker$\mathcal{P}_1$ will be termed *superconformal Killing vectors* and those of Ker$\mathcal{P}_1^\dagger$ *super moduli deformations* or *holomorphic superquadratic differentials.*

To gain further insight into the nature of the super moduli deformations of Ker$\mathcal{P}_1^\dagger$, we rewrite $\mathcal{P}_1^\dagger$ componentwise

$$(\mathcal{P}_1^\dagger\Phi)^z=\mathcal{D}_+\Phi^z_-, \quad (\mathcal{P}_1^\dagger\Phi)^{\bar z}=\mathcal{D}_-\Phi^{\bar z}_+ , \quad (3.54)$$

and make contact with Wess-Zumino gauge by setting

$$\Phi_+{}^{\bar z}=\phi_0+\theta\phi_++\bar\theta\phi_-+i\theta\bar\theta\phi_1 . \quad (3.55)$$

The result is

$$(\mathcal{P}_1^\dagger\Phi)^{\bar z}=\phi_-+\theta\left[-i\phi_1+\frac{3i}{4}A\phi_0\right]+\bar\theta(D_{\bar z}\phi_0+\tfrac{1}{2}\chi_{\bar z}{}^+\phi_+)$$
$$+\theta\bar\theta\Big[-D_{\bar z}\phi_+-\tfrac{1}{2}\chi_{\bar z}{}^+D_z\phi_0+\frac{3i}{2}\Lambda_-\phi_0$$
$$+iA\phi_-+\tfrac{1}{4}\chi_z{}^-\chi_{\bar z}{}^+\phi_-\Big] . \quad (3.56)$$

The changes $\delta E_M{}^A$ solving these equations will in general take us out of Wess-Zumino gauge, and a compensating super-reparametrization and U(1) transformation is needed, which, however, will not change the number of supermoduli. Under the hypothesis that the space of inequivalent supergeometries (to be termed supermoduli space later) is a supermanifold, we can determine its dimension at any point, and in particular at $(e_m{}^a,\chi_m)$ satisfying $\chi_z{}^-=0$ and $A=0$, so that

$$P_1^\dagger\phi_+=0 \quad\text{and}\quad P_{1/2}^\dagger\phi_0=0 \quad (3.57)$$

where P_1, $P_{1/2}$ are the operators (2.48) familiar from the component formalism. The index theorem and a simple counting of the number of conformal Killing vectors and spinors in each case yield the dimension of the vector spaces $\mathrm{Ker}P_1^\dagger$ and $\mathrm{Ker}P_{1/2}^\dagger$ [cf. Eqs. (2.50) and (2.51)],

$$\dim \mathrm{Ker}\mathcal{P}_1^\dagger = \begin{cases}(0,0), & h=0,\\ (6h-6,4h-4), & h\geq 2.\end{cases} \tag{3.58}$$

Here the two integers denote, respectively, the dimensions for the even and the odd coordinates. More generally, operators $\mathcal{P}_n$ acting on superfields of arbitrary weight $n>0$ can be introduced and expressed in terms of the U(1)-covariant derivatives $\mathcal{D}_-^n$. In Wess-Zumino gauge they will admit expansions similar to Eq. (3.56) [see Eq. (3.66) below], and the previous arguments will show that the number of zero modes is given by

$$\dim(\mathrm{Ker}\mathcal{P}_n) = \begin{cases}(4n+2,4n), & h=0,\\ (0,0), & h\geq 2,\end{cases}$$

$$\dim(\mathrm{Ker}\mathcal{P}_n^\dagger) = \begin{cases}(0,0), & h=0,\\ ((4n+2)(h-1),4n(h-1)), & h\geq 2.\end{cases}$$

For the case of the torus with $h=1$, it will be clear that the arguments given in Sec. III.F in support of the direct sum decomposition of Eq. (3.53) break down. In short, the reason is that the natural choice for constant curvature on the torus is zero curvature, so that the auxiliary field A vanishes and (3.51) does not hold. Actually, the natural metric $\|H\|$ becomes degenerate on the torus. Thus we would like to analyze the supermoduli problem in a way that does not depend on this metric. Ultimately we are interested in describing and parametrizing those geometries which cannot be interrelated by super-reparametrizations, local U(1), or super Weyl transformations, and we shall now attack this issue directly.

We start by considering the full supergeometry with the torsion constraints. First, by a super Weyl transformation, we fix the curvature R_{+-} to zero; the fact that this can always be done will be shown in Sec. III.F. For the torus, $R_{+-}=0$ cannot be chosen in a unique way since this slice is left invariant under harmonic super Weyl scalings satisfying

$$\mathcal{D}_+\mathcal{D}_-^{(0)}\Sigma_0=0\ . \tag{3.59}$$

The condition $R_{+-}=0$ is super-reparametrization and local U(1) invariant, and this is exactly what is needed to fix Wess-Zumino gauge, which we now do. In components, the zero-curvature condition becomes

$$\begin{aligned} &A=0\ ,\\ &\Lambda=-i\gamma_5\varepsilon^{mn}D_m\chi_n=0\ ,\\ &C=R=0\ ,\end{aligned} \tag{3.60}$$

where the components of the curvature were introduced in Eq. (3.36). Note that in view of Eq. (3.37) the last condition implies

$$\partial_m\omega_n-\partial_n\omega_m=0\ , \tag{3.61}$$

where ω_m is the only nonvanishing component of the connection Ω_M. The remaining symmetries of this slice are now local supersymmetry, local U(1) invariance, and ordinary reparametrizations, whose actions were listed in Sec. III.C.

However, on this slice, the form of the infinitesimal versions of these transformations may be considerably simplified. One finds that the effect of a reparametrization v^n, a supersymmetry ζ, and a local U(1) transformation l is given by

$$\begin{aligned} &\delta e_m{}^a=D_m(v^n e_n{}^a)+\tilde l\varepsilon^a{}_b e_m{}^b+\tilde\zeta\gamma^a\chi_m\ ,\\ &\delta\chi_m=-2D_m\tilde\zeta-\tfrac12\tilde l\gamma_5\chi_m\ ,\\ &\delta\omega_m=\partial_m\tilde l\ ,\end{aligned} \tag{3.62}$$

where we have introduced special combinations of local U(1) and supersymmetry transformations, defined by

$$\begin{aligned} &\tilde l=l+v^n\omega_n\ ,\\ &\tilde\zeta=\zeta-\tfrac12 v^n\chi_n\ .\end{aligned}$$

The action of the combined three symmetries is particularly simple; in fact it is global and triangular in the following sense. The (modified) local U(1) transformation $\tilde l$ acts globally on all three fields in a well-known way. The supersymmetry $\tilde\zeta$ no longer acts on ω_m, in contrast with ζ itself. This implies that the supersymmetry also integrates to a global action on χ_m, since the connection $D_m=\partial_m+\frac12 i\omega_m$ is invariant under $\tilde\zeta$ transformations. Finally, ordinary reparametrizations act only on $e_m{}^a$, and again their global action may be exploited to choose a *global* gauge for the "supertorus." Since ω_m satisfies Eq. (3.61), local U(1) transformations $\tilde l$ will eliminate all degrees of freedom of ω_m, except for the constant ones. Note that constant $\tilde l$'s have not been used to do so. Thus ω_m is constant, and this is unchanged by supersymmetry transformations $\tilde\zeta$.

We model the torus by a square with sides of unit length and opposite sides identified. If we assume that not all components of ω_m are multiples of 2π, so that D_m acting on spinors has no zero modes, then all components of χ_m may be eliminated via supersymmetry transformations $\tilde\zeta$. Similarly, all components of $\delta e_m{}^a$ are eliminated and $e_m{}^a$ may be chosen constant. Then, however, we must have $\omega_m=0$ by its very definition in Eq. (3.33), which is in contradiction with the original assumption, and hence all components of ω_m must be multiples of 2π. By redefining all fields by multiplications by a simple function, we may set $\omega_m=0$ without modifying the original boundary conditions. At $\omega_m=0$, the remaining components[12] of $e_m{}^a$ and χ_m are easily found.

For even-spin structure, D_m has no zero modes on spin fields, and we may set $\chi_m=0$ by supersymmetry and $e_m{}^a$ constant by reparametrization. There remain two translations (or conformal Killing vectors), a constant

[12]We count the number of real components here.

U(1), and constant Weyl transformation, the latter two eliminating two of the four degrees of freedom of $e_m{}^a$. In total we are left with two ordinary moduli, no odd moduli, and two translations as residual symmetries.

For odd-spin structure, D_m has zero modes on spinors, and we may set χ_m and $e_m{}^a$ only to a constant, but not necessarily to zero. There remain two translations and two constant supersymmetries (superconformal Killing spinors), a constant U(1) and Weyl transformation, and two constant super Weyl transformations as residual symmetries. The constant super Weyl transformations are eliminated by making the constant χ_m γ-traceless, and the U(1) and Weyl are used to restrict $e_m{}^a$ to two components. In total we are left with two moduli, two odd moduli, two translations, and two supersymmetries as residual symmetries.

To conclude, we obtain the decomposition

$$\{H_A{}^B\}=\{\delta\Sigma\}\oplus\{\delta L\}\oplus\{\mathrm{Range}\mathcal{P}_1\}$$
$$\oplus\{2 \text{ moduli } \delta e_m{}^a\}\oplus\{\text{odd moduli}\}\ , \qquad (3.63)$$

where {odd moduli} is zero for even-spin structure and parametrized by γ-traceless, constant χ_m for odd-spin structure.

Early investigations of supermoduli parameters and their role in superstring perturbation theory are those of D'Hoker and Phong (1986b), Friedan, Martinec, and Shenker (1986), Moore, Nelson, and Polchinski (1986), and Chaudhuri, Kawai, and Tye (1987).

F. Null spaces of superderivatives and Laplacians

In this section we examine the structure of null spaces of superderivatives $\mathcal{D}_\pm^n$ and their associated Laplacians $\Box_n^{(\pm)}$, as well as the relation between these null spaces. Questions relating to these issues have already come up in Secs. III.D and III.E with regard to the scalar Laplacian $\Box_0$ and the Faddeev-Popov operator $\mathcal{P}_1^\dagger\mathcal{P}_1$ and will be essential to the analysis of the super Weyl and superholomorphic anomalies later on.

To gain insight into the behavior of $\mathrm{Ker}\mathcal{D}_-^n$ and $\mathrm{Ker}\Box_n^{(-)}$, we note that the relation between the two kernels does not depend on super-reparametrizations or local U(1) transformations. Thus we may simplify the analysis by working in Wess-Zumino gauge and by choosing a slice for which χ_m is γ-traceless:

$$\chi_z{}^+=\chi_{\bar z}{}^-=0\ . \qquad (3.64)$$

We shall see that generically the relation between these kernels also does not depend on super Weyl rescalings. We introduce the field V of U(1) weight n, and its complex conjugate $\bar V$ of U(1) weight $-n$:

$$\begin{aligned} V&=V_0+\theta V_+ +\bar\theta V_- +i\theta\bar\theta V_1\ ,\\ \bar V&=\bar V_0+\theta\bar V_+ +\bar\theta\,\bar V_- +i\theta\bar\theta\,\bar V_1\ , \end{aligned} \qquad (3.65)$$

so that the U(1) weights of V_0, V_+, V_-, and V_1 are n, $n+\frac12$, $n-\frac12$, and n, respectively. (Or course these discrepancies arise because we choose Wess-Zumino gauge.) We easily find that

$$\begin{aligned}(\mathcal{D}_-^n V)=V_- +\theta\left[-iV_1+\frac{i}{2}nAV_0\right] &+\bar\theta(D_{\bar z}V_0+\tfrac12\chi_{\bar z}{}^+V_+)\\ &+\theta\bar\theta\left[\frac{i}{4}(2n+1)AV_- -\tfrac14\chi_{\bar z}{}^+\chi_z{}^-V_-\right.\\ &\left.-\tfrac12\chi_{\bar z}{}^+D_zV_0-D_{\bar z}V_+ +in\Lambda_-V_0\right]\ .\end{aligned} \qquad (3.66)$$

To compute $\Box_n^{(-)}V$, it is useful to evaluate

$$\begin{aligned}\int d\bar\theta\, d\theta\, E\mathcal{D}_+^{-n}\bar V\mathcal{D}_-^n V=e\Big[&D_z\bar V_0D_{\bar z}V_0-\bar V_+D_{\bar z}V_+-\bar V_-D_zV_- +\tfrac12\chi_{\bar z}{}^+\bar V_+D_zV_0+\tfrac12\chi_{\bar z}{}^+D_z\bar V_0V_+ +\tfrac12\chi_z{}^-\bar V_-D_{\bar z}V_0\\ &+\tfrac12\chi_z{}^-D_{\bar z}\bar V_0V_- +\left[\bar V_1-\frac{n}{2}A\bar V_0\right]\left[V_1-\frac{n}{2}AV_0\right]+inA\bar V_+V_- +\tfrac14\chi_z{}^-\chi_{\bar z}{}^+\bar V_+V_-\\ &-\tfrac14\chi_z{}^-\chi_{\bar z}{}^+\bar V_-V_+ +in\bar V_+\Lambda_-V_0+in\bar V_0\Lambda_+V_-\Big]\ .\end{aligned} \qquad (3.67)$$

The vanishing of $\Box_n^{(-)}V$ can then be gotten by variation with respect to $\bar V$ and one obtains

$$\begin{aligned} &D_zD_{\bar z}V_0+\frac{n}{2}A\left[V_1-\frac{n}{2}AV_0\right]-in\Lambda_+V_- +\tfrac12D_z(\chi_{\bar z}{}^+V_+)+\tfrac12D_{\bar z}(\chi_z{}^-V_-)=0\ ,\\ &D_zV_- +\tfrac12\chi_z{}^-D_{\bar z}V_0-\tfrac14\chi_z{}^-\chi_{\bar z}{}^+V_+=0\ ,\\ &D_{\bar z}V_+ +inAV_- +in\Lambda_-V_0+\tfrac12\chi_{\bar z}{}^+D_zV_0+\tfrac14\chi_z{}^-\chi_{\bar z}{}^+V_-=0\ ,\\ &V_1-\frac{n}{2}AV_0=0\ . \end{aligned} \qquad (3.68)$$

These equations are still rather formidable, and we shall take the following approach. We consider the case of zero gravitino field $\chi=0$ first, so that the equations reduce to

$$\begin{aligned} &D_z D_{\bar z} V_0 = 0 \quad \text{or} \quad P_n^\dagger P_n V_0 = 0 \;, \\ &D_z V_- = 0 \quad \text{or} \quad P_{n-1/2}^\dagger V_- = 0 \;, \\ &D_{\bar z} V_+ + inAV_- = 0 \quad \text{or} \quad P_{n-1/2} V_+ + inAV_- = 0 \;, \\ &V_1 - \frac{n}{2} A V_0 = 0 \;. \end{aligned} \tag{3.69}$$

These equations should now be compared with those obtained from $\mathcal{D}_-^n V=0$ in Eq. (3.66) at $\chi=0$, for which we find

$$\begin{aligned} &D_{\bar z} V_0 = 0 \quad \text{or} \quad P_n V_0 = 0 \;, \\ &V_- = 0 \;, \\ &D_{\bar z} V_+ = 0 \quad \text{or} \quad P_{n-1/2} V_+ = 0 \;, \\ &V_1 - \frac{n}{2} A V_0 = 0 \;. \end{aligned} \tag{3.70}$$

The first and the last equations of (3.69) and (3.70) are clearly equivalent.

Although the second and third equations in (3.69) and (3.70) seem different at first sight, we shall now show that, generically, they will also be equivalent. Indeed, let us derive an expression for the number of solutions $V_- \neq 0$ to (3.69). From $D_z^{n-1/2} V_- = 0$, we have

$$V_- = \sum_{\alpha=1}^{N} p_\alpha \phi_\alpha \;,$$

where ϕ_α span a basis for $\operatorname{Ker} D_z^{n-1/2}$, and N is its dimension. In order for the third equation of (3.69) to be consistent, AV_- must be in $\operatorname{Range} D_{\bar z}^{n+1/2}$, or equivalently it must be orthogonal to $\operatorname{Ker} D_z^{n-1/2}$. Thus the coefficients $p_\alpha \in \mathbf{C}$ must satisfy

$$\sum_{\alpha=1}^{N} \langle \phi_\beta | A\phi_\alpha \rangle p_\alpha = 0 \;,$$

$$\langle \phi_\beta | A\phi_\alpha \rangle = \int d^2 z \sqrt{g}\, \bar\phi_\beta A \phi_\alpha \;,$$

and the number of nonzero solutions V_- to Eq. (3.69) must be $\#(V_- \neq 0) = \dim \operatorname{Ker} \langle \phi_\beta | A\phi_\alpha \rangle$. Generically, the matrix $\langle \phi_\beta | A\phi_\alpha \rangle$ will be nondegenerate and thus $\#(V_- \neq 0)=0$. For example, this will be the case when A is any positive function.

Thus, for $h \neq 1$ and $n \neq 0$, we have established the validity of $\operatorname{Ker}\mathcal{D}_-^n = \operatorname{Ker}\square_n^{(-)}$ at least at the special point $\chi=0$. What happens when $\chi \neq 0$? In this case, we shall assume that χ results from a finite-dimensional space (parametrized by Grassmann-valued odd moduli) and we shall assume that χ is linear in these odd moduli ζ^a. Clearly, then, the different unknowns will be functions of ζ^a, but of course since there are a finite number of Grassmannian ζ's, these functions are just polynomials of bounded degree. It is not hard to see that one could expand

$$V_i = V_i^{(0)} + V_i^{(2)} + \cdots \;, \quad i=0,1,2,\ldots \;, \tag{3.71}$$

where the superscript denotes the degree of homogeneity in ζ. Now from the previous arguments when $\chi=0$, we know that for $h \geq 2$ and $n<0$ or $h=0$ and $n>0$, there are no solutions to order (0): $V_i^{(0)}=0$. But if this is so, the equation for $V_i^{(2)}$ is the same as for $V_i^{(0)}$, since all the perturbation terms are of order χ at least, and so on. One finds that V_i must identically vanish as soon as $V_i^{(0)}$ has to vanish.

We shall now discuss the above relations between null spaces for different genera. At this point, it is appropriate to deduce some generalizations that will prove fundamental later on. For $h \geq 2$ and A generic one has

$$\begin{aligned} &\operatorname{Ker}\square_n^{(+)} = \operatorname{Ker}\mathcal{D}_-^{n+1/2}\mathcal{D}_+^n = \operatorname{Ker}\mathcal{D}_+^n = 0, \quad n \geq \tfrac{1}{2} \;, \\ &\operatorname{Ker}\square_n^{(-)} = \operatorname{Ker}\mathcal{D}_+^{n-1/2}\mathcal{D}_-^n = \operatorname{Ker}\mathcal{D}_-^n = 0, \quad n \leq -\tfrac{1}{2} \;. \end{aligned} \tag{3.72}$$

As for the kernel of the square of the Laplacian $\mathcal{D}_-^{n+1/2}\mathcal{D}_+^n$ for $n \leq -1$,

$$\mathcal{D}_-^{n+1/2}\mathcal{D}_+^n \mathcal{D}_-^{n+1/2}\mathcal{D}_+^n V = 0 \;, \tag{3.73}$$

we can deduce using Eq. (3.72) that $\mathcal{D}_+^n \mathcal{D}_-^{n+1/2}\mathcal{D}_+^n V=0$, and with the help of Eq. (3.72) again, we find $\mathcal{D}_+^n V=0$, which implies that

$$\operatorname{Ker}(\mathcal{D}_-^{n+1/2}\mathcal{D}_+^n)^2 \subset \operatorname{Ker}\mathcal{D}_+^n \;. \tag{3.74}$$

Since one manifestly also has the inclusion in the opposite sense, these kernels are in fact equal to one another, even though they need not be empty. Of course, one has an analogous statement for the other Laplacian. Putting these conclusions together, we have

$$\begin{aligned} &\operatorname{Ker}(\square_n^{(+)})^2 = \operatorname{Ker}(\mathcal{D}_-^{n+1/2}\mathcal{D}_+^n)^2 = \operatorname{Ker}\mathcal{D}_+^n, \quad n \leq -1 \;, \\ &\operatorname{Ker}(\square_n^{(-)})^2 = \operatorname{Ker}(\mathcal{D}_+^{n-1/2}\mathcal{D}_-^n)^2 = \operatorname{Ker}\mathcal{D}_-^n, \quad n \geq 1 \;. \end{aligned} \tag{3.75}$$

In the case of the sphere $h=0$, the situation is precisely reversed. It is the $\mathcal{D}_-^n$ that have no zero modes for positive n, and it is readily established that

$$\begin{aligned} &\operatorname{Ker}\mathcal{D}_+^n = 0, \quad n \leq -\tfrac{1}{2} \;, \\ &\operatorname{Ker}\mathcal{D}_-^n = 0, \quad n \geq \tfrac{1}{2} \;, \end{aligned} \tag{3.76}$$

and similarly for their squares. By analogy with the higher-genus case, this implies the following identities between kernels of Laplacians:

$$\begin{aligned} &\operatorname{Ker}(\mathcal{D}_-^{n+1/2}\mathcal{D}_+^n)^2 = \operatorname{Ker}\mathcal{D}_+^n, \quad n \geq \tfrac{1}{2} \;, \\ &\operatorname{Ker}(\mathcal{D}_+^{n-1/2}\mathcal{D}_-^n)^2 = \operatorname{Ker}\mathcal{D}_-^n, \quad n \leq -\tfrac{1}{2} \;. \end{aligned} \tag{3.77}$$

For the torus $h=1$, the nongeneric choice $A=0$ is natural from several points of view, as was already noted at the end of Sec. III.E. For $A=0$ and flat metric, a direct inspection shows that

$$\operatorname{Ker}(\square_n^{(+)})^2 = \{ V = V_0 + \theta V_+ + \bar\theta V_- + i\theta\bar\theta V_1 \;; \; V_0, V_\pm, V_1 \text{ constants} \} \;,$$

$$\operatorname{Ker}(\square_n^{(+)}) = \operatorname{Ker}(\square_n^{(+)})^2 \cap \{ V_1 = 0 \} \;,$$

$$\mathrm{Ker}\mathcal{D}_+^n=\mathrm{Ker}(\Box_n^{(+)})\cap\{V_+=0\}\ .$$

On real fields, V_- equals V_+ and will vanish in the last case. For even-spin structure, all constant spinors vanish as well.

For $n=0$, the above arguments do not apply. The A term is absent in the third equation of (3.69), and at $\chi=0$, V_- is a Dirac zero mode. For odd-spin structure, there exists at least one such zero mode, and so $\mathrm{Ker}\Box_0^{(-)}\neq\mathrm{Ker}\mathcal{D}_-^0$. Whereas $\mathrm{Ker}\mathcal{D}_+^0$ reduces to constant superfields (when acting on real superfields), $\mathrm{Ker}\Box_0^{(+)}$ depends on moduli through the dependence of the number of Dirac zero modes on moduli, but may also depend on the odd moduli. However, the following argument will show that again $\mathrm{Ker}(\Box_0)^2=\mathrm{Ker}\Box_0$ generically. Consider the equation that must be satisfied by an element of $\mathrm{Ker}(\Box_0)^2$ not in $\mathrm{Ker}\Box_0$:

$$\Box_0V=c+\theta\eta+\bar\theta\bar\eta \tag{3.78}$$

with c constant and η a holomorphic spinor. For $\chi=0$, one readily finds that $\eta=0$, and integration over the surface must yield zero because $\Box_0$ is a derivative, so that

$$0=\int d^2\mathbf{z}\,E\Box_0V=c\int d^2\mathbf{z}\,E\ . \tag{3.79}$$

Now, generically, the area $\int d^2\mathbf{z}\,E$ will not vanish, though of course it need not be of definite sign. For constant-curvature geometries, indeed the area cannot vanish because of the Gauss-Bonnet formula for the Euler number of Eq. (3.38), when $h\neq1$, and similarly the area will not vanish on any regular geometry. If that is so, then the constant must vanish and $V\in\mathrm{Ker}\Box_0$. We have thus established that

$$\mathrm{Ker}(\Box_0)^2=\mathrm{Ker}\Box_0\ . \tag{3.80}$$

It will also be useful to simplify $\mathrm{Ker}(\Box_{1/2}^{(-)})^2$. Consider one of its elements V,

$$\mathcal{D}_+^0\mathcal{D}_-^{1/2}\mathcal{D}_+^0\mathcal{D}_-^{1/2}V=0\ . \tag{3.81}$$

Multiplying to the left by $\mathcal{D}_-^{1/2}$ and using Eq. (3.80), we get

$$\mathcal{D}_-^{1/2}\mathcal{D}_+^0\mathcal{D}_-^{1/2}V=0\ . \tag{3.82}$$

The spurious solutions satisfy

$$\mathcal{D}_-^{1/2}V=\mathrm{const}\ , \tag{3.83}$$

and upon integrating over the supersurface, as in Eq. (3.79), we find again that $\mathcal{D}_-^{1/2}V=0$. Hence we conclude

$$\mathrm{Ker}(\Box_{1/2}^{(-)})^2=\mathrm{Ker}\mathcal{D}_-^{1/2}\ . \tag{3.84}$$

The nongeneric slices are always easily treated as limits of generic slices.

G. Supermoduli space and its complex structure

In Sec. III.E, we identified the infinitesimal changes in the supergeometry of a super Riemann surface with Teichmüller deformations, spanning $\mathrm{Ker}\mathcal{P}_1^\dagger$. The space of supergeometries of genus h, satisfying the torsion constraints (3.11) inequivalent under the symmetry groups $\mathrm{sDiff}_0(M)$, $\mathrm{sWeyl}(M)$, and sU(1) is super Teichmüller space

$$s\mathcal{T}_h=\frac{\{E_M{}^A,\Omega_M\ \text{satisfying (3.11)}\}}{\{\mathrm{sDiff}_0(M)\times\mathrm{sWeyl}(M)\times\mathrm{sU}(1)\}}\ . \tag{3.85}$$

The quotient of the full super-reparametrization group $\mathrm{sDiff}(M)$ by $\mathrm{sDiff}_0(M)$ is the ordinary mapping class group MCG_h (acting on surface with spin structures) so that we may define *supermoduli* space as

$$\begin{aligned}s\mathcal{M}_h&=s\mathcal{T}_h/\mathrm{MCG}_h\ ,\\ \mathrm{MCG}_h&=\mathrm{sDiff}(M)/\mathrm{sDiff}_0(M)\\ &=\mathrm{Diff}(M)/\mathrm{Diff}_0(M)\ .\end{aligned} \tag{3.86}$$

The complex nature of $s\mathcal{M}_h$ can be seen by viewing it as the space of *superconformal classes.* Indeed, recall that the complex structure on a super Riemann surface $J_M{}^N$, introduced in Eq. (3.23), is unchanged under sWeyl(M) and sU(1) and that it is a tensor under sDiff(M). Thus we have

$$s\mathcal{M}_h=\{J_M{}^N\}/\mathrm{sDiff}(M)\ , \tag{3.87}$$

where $J_M{}^NJ_N{}^P=-\delta_M{}^P$, and it is understood that $J_M{}^N$ arises from a supergeometry satisfying the torsion constraints (3.11). There are now natural holomorphic coordinates on $s\mathcal{M}_h$, as can be seen by exhibiting a natural complex structure on it. The tangent space at $J_M{}^N$ can be identified with

$$T(s\mathcal{M}_h)=\{J_M{}^N;J_M{}^N\delta J_N{}^P+\delta J_M{}^NJ_N{}^P=0\}\ , \tag{3.88}$$

on which there is a natural map

$$\mathcal{J}:\ T(s\mathcal{M}_h)\to T(s\mathcal{M}_h),\quad \mathcal{J}(\delta J_M{}^N)=J_M{}^P\delta J_P{}^N \tag{3.89}$$

whose square is minus the identity

$$\mathcal{J}^2(\delta J_M{}^N)=\mathcal{J}(J_M{}^P\delta J_P{}^N)=-\delta J_M{}^N\ . \tag{3.90}$$

Thus $\mathcal{J}$ is an almost complex structure on $s\mathcal{M}_h$. It is actually integrable, as can be seen by considering the following one-forms:

$$\begin{aligned}\Gamma_M{}^N&=dJ_M{}^N-i\mathcal{J}(dJ_M{}^N)\ ,\\ \bar\Gamma_M{}^N&=dJ_M{}^N+i\mathcal{J}(dJ_M{}^N)\ .\end{aligned} \tag{3.91}$$

The exterior derivatives are easily obtained,

$$d\Gamma_M{}^N=\frac{i}{4}(\Gamma_M{}^P\wedge\bar\Gamma_P{}^N+\bar\Gamma_M{}^P\wedge\Gamma_P{}^N)\ , \tag{3.92}$$

and the almost complex structure is integrable provided $d\Gamma$ vanishes where $\Gamma=0$, which is obviously the case. Notice that this integrability condition uses only the fact that $J_M{}^N$ itself is a complex structure on the super Riemann surface; it does not further depend on the torsion constraints. One concludes that $s\mathcal{M}_h$ is a supercomplex ($V-$) manifold.

A perhaps more concrete description of supermoduli space may be given in terms of constant-(super)curvature geometries. The key step in analogy with the bosonic case is the choice of a slice for sWeyl(M) that generalizes constant curvature. The correct choice is

$$R_{+-}=\text{const} . \tag{3.93}$$

This slice is clearly invariant under super-reparametrizations and local U(1) transformations. it also has the advantage of implying that all components of the torsion and curvature are constant, as one can readily deduce from Eqs. (3.11)–(3.13). A simple interpretation of Eq. (3.93) can be obtained in Wess-Zumino gauge. Recall that in this gauge R_{+-} expanded in powers of θ is given by Eq. (3.36), so that A is constant, $\Lambda_\alpha=0$, and $C=0$. Finally we can also argue that (3.93) is indeed a slice, in the sense that any supergeometry $E_M{}^A$ can be brought back to a supergeometry $\hat{E}_M{}^A$ satisfying Eq. (3.93) by a unique super Weyl transformation. In fact Eq. (3.20) shows that the parameter Σ of the transformation must satisfy a super Liouville equation,

$$2i\mathcal{D}_+\mathcal{D}_-\Sigma+R_{+-}-e^{\Sigma}\hat{R}_{+-}=0 . \tag{3.94}$$

This equation is locally soluble, and there is no topological obstruction besides the Euler characteristic.

By restricting ourselves to the gauge slice $R_{+-}=\text{const}$, we have eliminated the action of the super Weyl group. To factor out the remaining symmetries we simply pass to cosets. More precisely, consider $dz^M\Omega_M$ as living in the space of one-forms modulo exact forms, and set $s\mathcal{M}_{\text{const}}$ to be the space of constant R_{+-} supergeometries modulo all local U(1) transformations. We can now define supermoduli space as the coset space

$$s\mathcal{M}_h=s\mathcal{M}_{\text{const}}/\text{sDiff}(M) . \tag{3.95}$$

From the orthogonal decomposition of $\{H_A{}^B\}$ given in Eq. (3.53), it is evident that $s\mathcal{M}_h$ is a supermanifold whose tangent space at each supergeometry is

$$T(s\mathcal{M}_h)=\text{Ker}\mathcal{P}_1^\dagger , \tag{3.96}$$

so that its dimension is also given by Eq. (3.58) for $h\neq 1$, whereas for $h=1$, the tangent space is $\{2 \text{ even moduli } e_m{}^a\}\oplus\{\text{odd moduli}\}$.

The holomorphic structure $\mathcal{J}$ on supermoduli space and its integrability are due to D'Hoker and Phong (1987a).

H. Determinants, super Weyl and local U(1) anomalies

In order to reduce the string path integrals over supergeometries to integrals over supermoduli space, one needs the behavior of the superdeterminants of the covariant derivatives with respect to super Weyl and local U(1) transformations. We start by considering the Laplacians $\Box_n^{(+)}$ and $\Box_n^{(-)}$ of Eq. (3.9).

The local part of the super Weyl anomaly has been evaluated by Martinec (1983). The zero modes for superdeterminants are, however, a nontrivial issue, since the nonpositivity of the norms could cause the kernels of $(\mathcal{D}_+\mathcal{D}_-)^2$, $\mathcal{D}_+\mathcal{D}_-$, and $\mathcal{D}_-$ to be distinct (cf. Sec. III.F). Each of these spaces has its own transformation law under super Weyl scalings, so it is important to determine which one will combine with $\text{sdet}(\mathcal{D}_+\mathcal{D}_-)^2$ to produce a *local* anomaly. Another consequence of the nonpositivity of the norms is that the Laplacians $\Box_n$ need not be diagonalizable. In addition, even though they are the product of an operator times its adjoint, they need not be positive. In fact, writing $\Box_n$ in components, it is clear that besides the standard Laplacians acting on ordinary functions, there is also a piece behaving like a first-order differential operator, so that the spectrum in general extends from $-\infty$ to $+\infty$. The square of $\Box_n$ is still not a positive operator in general, but is at least bounded from below. Strictly speaking, the last property has been shown only on surfaces of constant curvature by Aoki (1988), but is is clear that a continuous super Weyl transformation may alter the lower bound, but will not send it to $-\infty$. The heat kernel $\exp[-t(\Box_n^{(\pm)})^2]$ may thus tend to infinity as $t\to\infty$ in an exponential fashion, and ζ-function regularization cannot be applied to define the corresponding superdeterminants. We now provide a detailed analysis of these issues. We define the superdeterminant through an exponential regulator, depending on a complex parameter s,

$$\begin{aligned}\ln\delta_n^{(\pm)}(s)&=\ln\text{sdet}[(\Box_n^{(\pm)})^2+s]\\&=-\int_\varepsilon^\infty\frac{dt}{t}e^{-ts}\text{sTr}e^{-t(\Box_n^{(\pm)})^2} ,\end{aligned} \tag{3.97}$$

which converges absolutely for Re(s) sufficiently large and $\varepsilon>0$. Throughout the complex s plane, $\delta_n^{(\pm)}(s)$ is defined by analytic continuation. For constant-curvature supergeometries $\delta_n^{(\pm)}$ is meromorphic throughout **C**, and this is enough to argue that $\delta_n^{(\pm)}$ will be meromorphic for any supergeometry, as will become clear through the super Weyl anomaly calculation. Thus, around $s=0$, $\delta_n^{(\pm)}$ will in general have the following behavior:

$$\delta_n^{(\pm)}(s)=s^{N_n^{\pm}}\text{sdet}'(\Box_n^{(\pm)})^2+O(s^{N_n^{\pm}+1}) , \tag{3.98}$$

where $N_n^{\pm}$ are positive or negative integers, formally corresponding to the difference between the number of even zero modes and odd zero modes. This relation defines the superdeterminant of $(\Box_n^{(\pm)})^2$, whereas the superdeterminant of $\Box_n^{(\pm)}$ itself is the square root

$$\text{sdet}'(\Box_n^{(\pm)})^2\equiv(\text{sdet}'\Box_n^{(\pm)})^2 . \tag{3.99}$$

To examine the behavior under super Weyl transformations of the determinants in Eq. (3.99), we determine the super Weyl change of $\delta_n^{(\pm)}$ and analytically continue to $s=0$. We shall restrict attention to $\Box_n^{(+)}$ and quote the results for $\square_n^{(-)}$ at the end:

$$\delta \ln \delta_n^{(+)}(s) = 2\int_\varepsilon^\infty dt\, e^{-ts} \times \mathrm{sTr}(\delta\square_n^{(+)}\square_n^{(+)} e^{-t(\square_n^{(+)})^2}) \ . \tag{3.100}$$

The changes of the superderivatives are given by

$$\begin{aligned} \delta\mathcal{D}_+^n &= (n-\tfrac{1}{2})\delta\Sigma\mathcal{D}_+^n - n\mathcal{D}_+^n\delta\Sigma \ , \\ \delta\mathcal{D}_-^n &= -(n+\tfrac{1}{2})\delta\Sigma\mathcal{D}_-^n + n\mathcal{D}_-^n\delta\Sigma \ , \\ \delta\square_n^{(\pm)} &= (-1\mp n)\delta\Sigma\square_n^{(\pm)} \pm 2n\mathcal{D}_\mp^{n\pm 1/2}\delta\Sigma\mathcal{D}_\pm^n \mp n\square_n^{(\pm)}\delta\Sigma \ , \end{aligned} \tag{3.101}$$

so that[13]

$$\begin{aligned} \mathrm{sTr}(\delta\square_n^{(+)}\square_n^{(+)} e^{-t(\square_n^{(+)})^2}) &= -(2n+1)\mathrm{sTr}\delta\Sigma(\square_n^{(+)})^2 e^{-t(\square_n^{(+)})^2} - 2n\ \mathrm{sTr}\delta\Sigma(\square_{n+1/2}^{(-)})^2 e^{-t(\square_{n+1/2}^{(-)})^2} \\ &= (2n+1)\frac{\partial}{\partial t}\mathrm{sTr}\delta\Sigma e^{-t(\square_n^{(+)})^2} + 2n\frac{\partial}{\partial t}\mathrm{sTr}\delta\Sigma e^{-t(\square_{n+1/2}^{(-)})^2} \end{aligned} \tag{3.102}$$

Inserting this result into Eq. (3.100), one finds

$$\delta \ln \delta_n^{(+)} = 2\int_\varepsilon^\infty dt\, e^{-ts}\left[(2n+1)\frac{\partial}{\partial t}\mathrm{sTr}\delta\Sigma e^{-t(\square_n^{(+)})^2} + 2n\frac{\partial}{\partial t}\mathrm{sTr}\delta\Sigma e^{-t(\square_{n+1/2}^{(-)})^2}\right] \tag{3.103}$$

Integrating by parts yields

$$\begin{aligned} \delta \ln \delta_n^{(+)} &= 2e^{-ts}[(2n+1)\mathrm{sTr}\delta\Sigma e^{-t(\square_n^{(+)})^2} + 2n\ \mathrm{sTr}\delta\Sigma e^{-t(\square_{n+1/2}^{(-)})^2}]\,\Big|_\varepsilon^\infty \\ &\quad + 2s\int_\varepsilon^\infty dt\, e^{-ts}[(2n+1)\mathrm{sTr}\delta\Sigma e^{-t(\square_n^{(+)})^2} + 2n\ s\mathrm{Tr}\delta\Sigma e^{-t(\square_{n+1/2}^{(-)})^2}] \ . \end{aligned} \tag{3.104}$$

Since the expression is defined for $\mathrm{Re}(s)$ sufficiently large, the contribution from infinity in the first term cancels out, and the remaining traces of heat kernels are well defined at $s=0$. In the second term, the only nonzero contribution can arise in the limit where $s\to 0$ if the integral produces a simple pole at $s=0$. To see whether this happens, we remark that the general form of the contribution to the supertraces is $t^p e^{-\lambda t}$, where p and λ are arbitrary and independent of s. Substituted into the integral in (3.104), we find that such a contribution produces

$$s\int_\varepsilon^\infty dt\, e^{-ts} t^p e^{-\lambda t} = \frac{s}{(s+\lambda)^{p+1}}\Gamma(p+1) + O(\varepsilon) \ . \tag{3.105}$$

One notices that, whatever the value of p, a nonzero result is produced as $s\to 0$ only if $\lambda=0$. In the trace of the superheat kernel, this results from the zero modes, so that $p=0$ as well. Collecting these results, we get

$$\begin{aligned} \delta \ln \mathrm{sdet}'\square_n^{(+)} &= \tfrac{1}{2}\lim_{s\to 0} \ln\delta_n^{(+)}(s) \\ &= -(2n+1)\mathrm{sTr}\delta\Sigma e^{-\varepsilon(\square_n^{(+)})^2} - 2n\ \mathrm{sTr}\delta\Sigma e^{-\varepsilon(\square_{n+1/2}^{(-)})^2} + (2n+1)\mathrm{sTr}\delta\Sigma\,|_{\mathrm{Ker}(\square_n^{(+)})^2} + 2n\ \mathrm{sTr}\delta\Sigma\,|_{\mathrm{Ker}(\square_{n+1/2}^{(-)})^2} \ . \end{aligned} \tag{3.106}$$

The terms involving ε on the right-hand side of Eq. (3.106) are local functions of $\delta\Sigma$ in the limit where $\varepsilon\to 0$, and their expressions can be gotten from a short-time expansion of the super heat kernel, which is derived in Appendix C:

$$\begin{aligned} \mathrm{sTr}\delta\Sigma e^{-\varepsilon(\square_n^{(+)})^2} &= -i\frac{1+2n}{4\pi}(-)^{2n}\int d^2\mathbf{z}\, ER_{+-}\delta\Sigma + O(\varepsilon) \ , \\ \mathrm{sTr}\delta\Sigma e^{-\varepsilon(\square_n^{(-)})^2} &= +i\frac{1-2n}{4\pi}(-)^{2n}\int d^2\mathbf{z}\, ER_{+-}\delta\Sigma + O(\varepsilon) \ . \end{aligned} \tag{3.107}$$

Notice that, due to worldsheet supersymmetry, there is no term behaving like $1/\varepsilon$, as we had in the case of the bosonic string.

The traces of $\delta\Sigma$ restricted to the kernels of zero modes are familiar from the bosonic case, but much more care is needed for the case of the superstring, due to the fact that the kernel of the square of an operator may be different from the kernel of the operator itself, as we have seen in Sec. III.F.

For $h\geq 2$ and $n\geq\frac{1}{2}$, we have $\mathrm{Ker}(\square_n^{(+)})^2=0$ according to Eq. (3.72), and that $\mathrm{Ker}(\square_{n+1/2}^{(-)})^2=\mathrm{Ker}\mathcal{D}_-^{n+1/2}$ according to Eq. (3.75). The remaining trace can be linked to the change in the finite-dimensional determinant of elements[14] $\Phi_J\in\mathrm{Ker}\mathcal{D}_-^{n+1/2}$, using the fact that they scale as

[13]Note that the analogous calculation could have been performed using local U(1) transformations of the superderivatives. At this point one would have found that the contributions cancel and that the determinants are invariant.

[14]Henceforth J,K stand for mixed indices $J=(j,a), K=(k,b)$, etc., where $j=1,\ldots,3h-3$ and $a=1,\ldots,2h-2$.

$\Phi_J = e^{-(n+1/2)\Sigma}\hat{\Phi}_J$:

$$\delta \ln \text{sdet}\langle \Phi_J \mid \Phi_K \rangle = \text{sTr}\,\delta\langle \Phi_J \mid \Phi_K \rangle = -2n \sum_J \langle \Phi_J \mid \delta\Sigma \mid \Phi_J \rangle \ . \quad (3.108)$$

Putting these together, we find (for $h \geq 2$ and $n \geq \frac{1}{2}$)

$$\delta \ln \frac{\text{sdet}\square_n^{(+)}}{\text{sdet}\langle \Phi_J \mid \Phi_K \rangle} = i\frac{4n+1}{4\pi}(-)^{2n}\int d^2\mathbf{z}\, E R_{+-}\delta\Sigma \ . \quad (3.109)$$

It is straightforward to see that the same arguments apply for $h=0$, provided that $n \leq -\frac{1}{2}$.

For $h \geq 2$ and $n \leq -1$, exactly the opposite situation is produced, and we have according to Eq. (3.72) that $\text{Ker}(\square_{n+1/2}^{(-)})^2 = 0$ and $\text{Ker}(\square_n^{(+)})^2 = \text{Ker}\mathcal{D}_+^n$. The remaining trace can now be linked to the change in the finite-dimensional determinant of elements $\Psi_\alpha \in \text{Ker}\mathcal{D}_+^n$, which scale as $\Psi_\alpha = e^{n\Sigma}\hat{\Psi}_\alpha$. Thus

$$\delta \ln \text{sdet}\langle \Psi_\alpha \mid \Psi_\beta \rangle = \text{sTr}\,\delta\langle \Psi_\alpha \mid \Psi_\beta \rangle = -(2n+1)\sum_\alpha \langle \Psi_\alpha \mid \delta\Sigma \mid \Psi_\alpha \rangle \ , \quad (3.110)$$

and putting all together, we find

$$\delta \ln \frac{\text{sdet}\square_n^{(+)}}{\text{sdet}\langle \Psi_\alpha \mid \Psi_\beta \rangle} = i\frac{4n+1}{4\pi}(-)^{2n}\int d^2\mathbf{z}\, E R_{+-}\delta\Sigma \ . \quad (3.111)$$

Similarly for the sphere, this formula will hold for $n \geq \frac{1}{2}$.

The cases $n=0$ and $n=-\frac{1}{2}$ are symmetrical, so we shall limit our discussion to $n=0$. The novelty here is that one of the finite-dimensional traces is absent from Eq. (3.106), the other one being taken over $\text{Ker}(\square_0)^2$. Though $\text{Ker}(\square_0)^2$ might be larger than $\text{Ker}\square_0$, it was argued in Eq. (3.80) that this is not the generic case. Since the zero modes of $\square_0$ are super Weyl invariant, we readily deduce that Eq. (3.111) holds, but now with $\Psi_\alpha \in \text{Ker}\square_0$, which may be larger than $\text{Ker}\mathcal{D}_+^0$. Similarly, since $\Phi_J \in \text{Ker}\mathcal{D}_-^{1/2}$ scales as $\Phi_J = e^{-\Sigma/2}\hat{\Phi}_J$, it is clear that $\text{sdet}\langle \Phi_J \mid \Phi_K \rangle$ is super Weyl invariant, in analogy with the finite-dimensional determinant over inner products of holomorphic Abelian differentials in the bosonic string. We might be tempted to call the $\frac{1}{2}$ differentials $\Phi_J \in \text{Ker}\mathcal{D}_-^{1/2}$ *holomorphic super Abelian differentials.*

We may now collect all the above results for $h \geq 2$ or $h=0$ in one formula, and also integrate the infinitesimal $\delta\Sigma$'s to finite super Weyl transformations:

$$\ln \frac{\text{sdet}'\square_n^{(+)}}{\text{sdet}\langle \Phi_J \mid \Phi_K \rangle \text{sdet}\langle \Psi_\alpha \mid \Psi_\beta \rangle} = \ln \frac{\text{sdet}\hat{\square}_n^{(+)}}{\text{sdet}\langle \hat{\Phi}_J \mid \hat{\Phi}_K \rangle \text{sdet}\langle \hat{\Psi}_\alpha \mid \hat{\Psi}_\beta \rangle} - (1+4n)S_{sL}(\Sigma) \ , \quad (3.112)$$

where the local super Weyl anomaly is given by

$$S_{sL}(\Sigma) = \frac{1}{4\pi}\int d^2\mathbf{z}\, \hat{E}(\hat{\mathcal{D}}_+\Sigma\hat{\mathcal{D}}_-\Sigma - i\hat{R}_{+-}\Sigma) \quad (3.113)$$

and $\Phi_J \in \text{Ker}\mathcal{D}_-^{n+1/2}$, except when $n=-\frac{1}{2}$, where $\Phi_J \in \text{Ker}\square_0^{(-)}$ and $\Psi_\alpha \in \text{Ker}\mathcal{D}_+^n$, except for $n=0$, where $\Psi_\alpha \in \text{Ker}\square_0^{(+)}$. Similarly, we can derive the super Weyl anomaly for $\square_n^{(-)}$ and find

$$\ln \frac{\text{sdet}'\square_n^{(-)}}{\text{sdet}\langle \Phi_J \mid \Phi_K \rangle \text{sdet}\langle \Psi_\alpha \mid \Psi_\beta \rangle} = \ln \frac{\text{sdet}\hat{\square}_n^{(-)}}{\text{sdet}\langle \hat{\Phi}_J \mid \hat{\Phi}_K \rangle \text{sdet}\langle \hat{\Psi}_\alpha \mid \hat{\Psi}_\beta \rangle} - (1-4n)S_{sL}(\Sigma) \ . \quad (3.114)$$

Here $\Phi_J \in \text{Ker}\mathcal{D}_-^n$, except for $n=0$, where it belongs to $\text{Ker}\square_0^{(-)}$ and $\Psi_\alpha \in \text{Ker}\mathcal{D}_+^{n-1/2}$, except for $n=\frac{1}{2}$, where it belongs to $\text{Ker}\square_0^{(+)}$.

For the torus and a generic slice, it is clear that Eqs. (3.112)–(3.114) hold as well. If, on the other hand, the nongeneric slice $A=0$ is chosen, one should rather consider $\text{sdet}'(\square_n^{(\pm)})^2$ and divide by the determinants of inner products of $\text{Ker}(\square_n^{(\pm)})^2$.

I. Amplitudes as integrals over supermoduli

With the above analysis of the space of supergeometries, it is now easy to carry out the $DE_M{}^A$ integral. We shall limit ourselves to the case $h \geq 2$ and treat the sphere and the torus in Secs. III.L and III.M. In parallel with the bosonic case, we introduce a slice S of dimension $(6h-6, 4h-4)$, transversal to the action of $\text{sDiff}_0(M)$ within the space of supergeometries. We parametrize the space of supergeometries by

$$E_M{}^A = e^V e^\Sigma e^{iL}\hat{E}_M{}^A \ , \quad (3.115)$$

with $\hat{E}_M{}^A$ in S and the exponentials representing the actions of the various symmetry groups. If m_J are coordinates for the slice S, $\hat{F}_J$ are the corresponding coordinate vectors in $T(s\mathcal{M}_{\text{const}})$, and Φ_J is a basis for $\text{Ker}\mathcal{P}_1^\dagger$, then the measure is obtained from the calculation of the Jacobian factor associated with the change of variables from $E_M{}^A$ to Σ, L, V, and m_J. With the orthogonal decomposition of Eq. (3.53), this Jacobian can easily be worked out, and we find

$$DE_M{}^A = (\text{sdet}\mathcal{P}_1^\dagger + \mathcal{P}_1)^{1/2}\frac{\text{sdet}\langle e^{iL}e^{-\Sigma/2}\hat{F}_J \mid \Phi_K \rangle_E^2}{\text{sdet}\langle \Phi_J \mid \Phi_K \rangle_E} \times D\Sigma DL DV^M \prod dm_J \ . \quad (3.116)$$

The subscript in the inner product indicates which superzweibein is used in the pairing of tensors.

The super Weyl dependence of the various ingredients of Eq. (3.116) may be calculated in analogy with the bosonic string case. First one uses the fact that

$$\mathrm{Ker}\mathcal{P}_1^\dagger=\mathrm{Ker}\mathcal{D}_+^1\oplus\mathrm{Ker}\mathcal{D}_-^{-1} \tag{3.117}$$

and that if $\Phi_J\in\mathrm{Ker}\mathcal{D}_\pm^{\pm1}$ then $\hat{\Phi}_J=e^{(3/2)\Sigma}\Phi_J\in\mathrm{Ker}\hat{\mathcal{D}}_\pm^{\pm1}$. Similar properties are easily derived for the local U(1) transformations using Eqs. (3.15)–(3.17). As a result, one finds that

$$\mathrm{sdet}\langle e^{iL}e^{-\Sigma/2}\hat{F}_J\mid\Phi_K\rangle_E=\mathrm{sdet}\langle\mu_J\mid\hat{\Phi}_K\rangle\ , \tag{3.118}$$

where the inner product on the right-hand side is now evaluated with respect to the supergeometry $\hat{E}_M{}^A$. The μ_J are dual super Beltrami differentials, in the sense of Sec. III.J below. Their bosonic analog appeared in Secs. II.E and II.G. Next, we recall from Sec. III.H the super Weyl scalings relevant to superstring theory:

$$\begin{aligned}\frac{\mathrm{sdet}\mathcal{P}_1^\dagger\mathcal{P}_1}{\mathrm{sdet}\langle\Phi_J\mid\Phi_K\rangle}&=\frac{\mathrm{sdet}\hat{\mathcal{P}}_1^\dagger+\hat{\mathcal{P}}_1}{\mathrm{sdet}\langle\hat{\Phi}_J\mid\hat{\Phi}_K\rangle}e^{-10S_{sL}(\Sigma)}\ ,\\ \frac{\mathrm{sdet}'\Box_0}{\mathrm{sdet}\langle\Psi_\alpha\mid\Psi_\beta\rangle}&=\frac{\mathrm{sdet}'\hat{\Box}_0}{\mathrm{sdet}\langle\hat{\Psi}_\alpha\mid\hat{\Psi}_\beta\rangle}e^{-S_{sL}(\Sigma)}\ ,\end{aligned} \tag{3.119}$$

where $\Psi_\alpha\in\mathrm{Ker}\Box_0$.

The first and second equations show that the nonlocal Σ dependence cancels out of Eq. (3.116). The local dependence on the super Weyl scaling Σ is canceled out by putting contributions of the Faddeev-Popov and matter determinants together, provided the dimension of space-time is $d=10$. Since we are dealing with the type-II string here, a potential local U(1) anomaly is canceled between left- and right-movers on the worldsheet. For the heterotic string, the absence of the local U(1) anomaly will put further constraints on the theory, which will be explained in Sec. III.N and amount to requiring the gauge group to have rank 16. Vertex operators will be determined so that the above symmetries of the measure are preserved, after all anomalous contributions have been taken into account.

Since the combined measure will be invariant under super-reparametrizations, local U(1), and super Weyl transformations, it really runs over the coset space of all $N=1$ supergeometries by these symmetries. The remaining coset space coincides precisely with that of all supercomplex structures, and was termed supermoduli space in Sec. III.G. Thus the domain of integration will be supermoduli space. The measure becomes

$$DE_M{}^A=(\mathrm{sdet}\hat{\mathcal{P}}_1^\dagger\hat{\mathcal{P}}_1)^{1/2}e^{-5S_{sL}(\Sigma)}D\Sigma DLDV^M \times\frac{\mathrm{sdet}\langle\mu_J\mid\hat{\Phi}_K\rangle}{\mathrm{sdet}\langle\hat{\Phi}_J\mid\hat{\Phi}_K\rangle}\prod_J dm_J\ . \tag{3.120}$$

Now the last equation in (3.119) shows that in the critical dimension $d=10$ the local super Weyl anomaly $S_{sL}(\Sigma)$ disappears as well, to yield the formula

$$Z_h=\Omega\int_{s\mathcal{M}_h}\prod_J dm_J\frac{\mathrm{sdet}\langle\mu_J\mid\hat{\Phi}_K\rangle}{\mathrm{sdet}\langle\hat{\Phi}_J\mid\hat{\Phi}_K\rangle}\left[\frac{8\pi^2\mathrm{sdet}'\hat{\Box}_0}{\int d^2\mathbf{z}\,E}\right]^{-5}\times(\mathrm{sdet}\hat{\mathcal{P}}_1^\dagger\hat{\mathcal{P}}_1)^{1/2}\ . \tag{3.121}$$

As in the bosonic string, if we choose a slice within $s\mathcal{M}_{\mathrm{const}}$, this measure is manifestly a coset measure on $s\mathcal{M}_h$, which can be termed the super Weil-Petersson measure,

$$d(\mathrm{sWP})=\frac{\mathrm{sdet}\langle\mu_J\mid\hat{\Phi}_K\rangle}{\mathrm{sdet}\langle\hat{\Phi}_J\mid\hat{\Phi}_K\rangle}\prod_J dm_J\ . \tag{3.122}$$

We shall often refer to the right-hand side of Eq. (3.122) as the super Weil-Petersson measure, even when the slice does not lie within $s\mathcal{M}_{\mathrm{const}}$. Such slices, e.g., those that depend holomorphically on supermoduli parameters, will be important later.

We conclude this section by noting that on-shell scattering amplitudes may be reduced in the same way to integrals over supermoduli by insertion of the proper vertex operators, as discussed in Sec. VIII. For the case of bosonic vertex operator insertions, one finds in general

$$\langle V_1(k_1)\cdots V_n(k_n)\rangle_h=\int_{s\mathcal{M}_h}\prod_J dm_J\frac{\mathrm{sdet}\langle\mu_J\mid\hat{\Phi}_K\rangle}{\mathrm{sdet}\langle\hat{\Phi}_J\mid\hat{\Phi}_K\rangle}\left[\frac{8\pi^2\mathrm{sdet}'\hat{\Box}_0}{\mathrm{sdet}\langle\hat{\Psi}_\alpha\mid\hat{\Psi}_\beta\rangle}\right]^{-5}(\mathrm{sdet}\hat{\mathcal{P}}_1^\dagger\hat{\mathcal{P}}_1)^{1/2}\langle\langle V_1(k_1)\cdots V_n(k_n)\rangle\rangle_{\hat{E}}\ , \tag{3.123}$$

where $\langle\langle\ \rangle\rangle$ stands for the fact that only the X^μ integral has been carried out (including the integration over all X^μ zero modes).

J. Formulation with superghosts

In this section the Faddeev-Popov determinant, together with the finite-dimensional determinants involving super Beltrami and superquadratic differentials, is recast in terms of a functional integral over superghost fields, and a local action is obtained on the worldsheet. The goal ultimately is to derive a formulation in Wess-Zumino gauge closely related to that of conformal field theory. This will be fully achieved in the next section, III.K.

Before deriving the superghost expression, we need a better insight into the nature of superquadratic and super Beltrami differentials.

1. Superquadratic and super Beltrami differentials

Holomorphic superquadratic differentials Φ_J are U(1) tensors of weight $\frac{3}{2}$ and are solutions to

$$\mathcal{D}_-^{3/2}\Phi_J=0 \tag{3.124}$$

Recall that in Wess-Zumino gauge $\Phi_J=\phi_J^0+\theta\phi_J^1+(3i/4)A\phi_J^0\theta\bar\theta$ satisfies the equation

$$D_{\bar z}\phi_J^0+\tfrac{1}{2}\chi_{\bar z}^+\phi_J^1=0\ ,$$
$$D_{\bar z}\phi_J^1+\tfrac{1}{2}\chi_{\bar z}^+D_z\phi_J^0-\tfrac{3}{2}\Lambda_-\phi_J^0=0\ . \quad (3.125)$$

When $\chi=0$, ϕ_J^0 and ϕ_J^1 are holomorphic $\frac{3}{2}$ and quadratic differentials, respectively, so in that particular case we may set $\phi_j^0=\phi_a^1=0$. The remaining components ϕ_a^0 and ϕ_j^1 are then the standard holomorphic differentials, and their number is in accord with index calculations. They are also naturally even Grassmann-valued elements. Away from $\chi=0$, the same number of solutions to Eq. (3.125) exists, and here ϕ_a^0 and ϕ_j^1 are even, whereas ϕ_j^0 and ϕ_1^1 are odd Grassmann elements. Putting all together, we have $5h-5$ holomorphic superquadratic differentials Φ_J, $3h-3$ of which are odd (Φ_j) and $2h-2$ of which are even (Φ_a).

Super Beltrami differentials μ_K with $K=(k,b)$ are dual to holomorphic super quadratic differentials and may be normalized as

$$\langle \mu_K \mid \Phi_J\rangle=\delta_{KJ}\ , \quad (3.126)$$

so that there are again $5h-5$ μ_K's, $3h-3$ of which are odd (μ_k) and $2h-2$ of which are even (μ_a). More generally, super Beltrami differentials may also be viewed as inequivalent small deformations of the supergeometry of a super Riemann surface, belonging to the tangent space to supermoduli $T(s\mathcal{M}_h)$. (See the analogous discussion for the bosonic case in Secs. II.D and II.E). It will be convenient to introduce coordinates m_K for supermoduli space; m_k should be thought of as ordinary even moduli and m_b as odd moduli. The small deformations inequivalent under U(1), super Weyl, and super-reparametrizations could be parametrized by the component $H_-{}^z$ (and its complex conjugate) according to Eq. (3.45). Thus the super Beltrami differentials μ_K may naturally be defined as

$$\mu_K=(H_-{}^z)_K=E_-{}^M\frac{\partial E_M{}^z}{\partial m_K}\ . \quad (3.127)$$

It follows that super Beltrami differentials satisfy the integrability condition given by

$$\frac{\partial\mu_K}{\partial m_L}+(-)^{KL}\frac{\partial\mu_L}{\partial m_K}=0\ . \quad (3.128)$$

It is instructive to look at this structure in Wess-Zumino gauge, where we have

$$H_-{}^z=\bar\theta(e_{\bar z}{}^m\delta e_m{}^z-\theta\delta\chi_{\bar z}{}^+)\ . \quad (3.129)$$

There is also a contribution from a Weyl transformation of the form $e_z{}^m\delta e_m{}^z$ which has been omitted from Eq. (3.129) since it does not induce a motion in supermoduli space. Thus, in Wess-Zumino gauge, the super Beltrami differential may be decomposed as

$$\mu_K=\bar\theta(\mu_K^1+\theta\mu_K^0)\ ,$$

where

$$\mu_K^1=e_{\bar z}{}^m\frac{\partial e_m{}^z}{\partial m_K},\quad \mu_K^0=-\frac{\partial\chi_{\bar z}^+}{\partial m_K}\ . \quad (3.130)$$

Clearly, μ_k^1 and μ_b^0 are even and correspond (for $\chi=0$) to the ordinary Beltrami differentials.[15]

From the duality of μ_K and Φ_J, it follows that their components are also naturally dual,

$$\langle\mu_K\mid\Phi_J\rangle=\langle\mu_K^0\mid\Phi_J^0\rangle+\langle\mu_K^1\mid\Phi_J^1\rangle\ , \quad (3.131)$$

and for $\chi=0$ the ordinary Beltrami differentials μ_k^1 and μ_b^0 are dual to the holomorphic quadratic and $\frac{3}{2}$ differentials, respectively.

Finally, we introduce *super-quasiconformal vector fields* associated with *superquasiconformal transformations.* The superderivative of a super-quasiconformal vector field is to be identified with the super Beltrami differential, which lies in $T(s\mathcal{M}_h)$, and may be viewed as a deformation of the supercomplex structure,

$$(\mu_K)_-{}^z=\mathcal{D}_-V_K^z\ . \quad (3.132)$$

It is again useful to restrict our attention to the case of Wess-Zumino gauge, and with Eq. (3.132) we find that V_K^z must be of the form

$$V_K^z=V_K^1+\theta V_K^0-\frac{i}{2}\theta\bar\theta A V_K^1$$

with

$$\mu_K^0=D_{\bar z}V_K^0+\tfrac{1}{2}\chi_{\bar z}{}^+D_zV_K^1+i\Lambda_-V_K^1\ ,$$
$$\mu_K^1=D_{\bar z}V_K^1+\tfrac{1}{2}\chi_{\bar z}{}^+V_K^0\ . \quad (3.133)$$

For $\chi=0$, V_k^1 reduces to the ordinary quasiconformal vector field.

Super-quasiconformal transformations W can be defined as satisfying the *super Beltrami equation*

$$\mathcal{D}_-W=\mu\mathcal{D}_zW \quad (3.134)$$

for a general super Beltrami differential $\mu=\sum\zeta_K\mu_K$. When $\chi=0$, it contains the ordinary Beltrami equation for the body component of W.

2. Superghost expression for superdeterminants

To represent the Faddeev-Popov superdeterminants, we introduce a ghost superfield C of U(1) weight -1 and an antighost superfield B of U(1) weight $\frac{3}{2}$, as well as their complex conjugates $\bar C$ and $\bar B$. We shall also assign ghost charge 1 to C and $\bar B$ and -1 to $\bar C$ and B. The relevant superghost action is

$$I_{\rm sgh}(C,B)=\frac{1}{2\pi}\int d^2\mathbf{z}\,E(B\mathcal{D}_-C+\bar B\mathcal{D}_+\bar C)\ . \quad (3.135)$$

[15] μ_b^0 has also been termed a super Beltrami differential in the literature. We shall, however, reserve this name for μ_K.

Clearly, $I_{\rm sgh}$ is super-reparametrization, local U(1), and super Weyl invariant, provided B and C scale as $C=e^{\Sigma}\hat{C}$ and $B=e^{-(3/2)\Sigma}\hat{B}$. We introduce functional measures DC and DB through the metrics

$$\|\delta C\|^2=\int d^2{\bf z}\,E\delta\bar{C}\delta C\ ,\qquad \|\delta B\|^2=\int d^2{\bf z}\,E\delta\bar{B}\delta B\ , \tag{3.136}$$

each of which is invariant under ghost number rotations, super-reparametrizations, and U(1) transformations, but not under super Weyl rescalings. If we discard integrations over zero modes (denoted by primed fields), we have in a straightforward manner[16]

$$\int D(B'\bar{B}'C\bar{C})e^{-I_{\rm sgh}(C,B')}=(\text{sdet}\mathcal{P}_1^\dagger\mathcal{P}_1)^{1/2}\ . \tag{3.137}$$

This integral involving the first-order action $I_{\rm sgh}$ on the odd (C) and even (B) superfields may be understood by considering a toy example. Take the case of an odd $(C=c+\theta\gamma)$ and an even $(B=\beta+\theta b)$ supervariable. The ordinary integral is easily evaluated, and we find

$$\int dB\,\exp\left[i\int d\theta\,BC\right]=\int db\,d\beta\,e^{i(bc-\beta\gamma)}=2\pi i\delta(C)\ , \tag{3.138}$$

where $\delta(C)=\delta(c)\delta(\gamma)$. Thus, carrying out the B' and $\bar{B}'$ integrals in Eq. (3.137), one finds

$$\int D(B'\bar{B}')e^{-I_{\rm sgh}(C,B')}=\delta(\mathcal{D}_-C)\delta(\mathcal{D}_+\bar{C})\ , \tag{3.139}$$

so that the C and $\bar{C}$ integrals produce precisely the Jacobian factor as given by Eq. (3.137).

To obtain a representation including the finite-dimensional determinants as well, we should integrate over the zero modes of B. This can be done by adding to the ghost action the coupling of B to super Beltrami differentials, since these are dual to the zero modes. To do so we introduce variables ζ_K (ζ_k's are odd, ζ_b's even) and evaluate the integral

$$\prod_K\int d^2\zeta_K\int D(B\bar{B}C\bar{C})\exp\left[-I_{\rm sgh}(C,B)+\sum_K\zeta_K\langle\mu_K\,|\,B\rangle+\bar{\zeta}_K\langle\bar{\mu}_K\,|\,\bar{B}\rangle\right] \tag{3.140}$$

in two different ways. First, by separating $B=B_0+B'$ into the zero-mode contribution B_0 and the non-zero-mode contribution B', we see that the term involving μ_K precisely couples only to B_0, whereas $I_{\rm sgh}$ depends only on B'. Thus the B_0 and B' integrals separate. The B' integral produces the infinite-dimensional superdeterminant as in Eq. (3.137). In the B_0 integral, we may decompose B_0 onto Φ_J (suitably normalized): $B_0=\sum_J\beta_J\Phi_J$, and since B_0 is even, β_a is odd and β_j even. The B_0 and ζ integrals then reduce to

$$\prod_{J,K}\int d\zeta_K d\beta_J\exp\left[\sum_{J,K}\zeta_K\langle\mu_K\,|\,\Phi_J\rangle\beta_J\right]=\frac{\text{sdet}\langle\mu_K\,|\,\Phi_J\rangle}{\text{sdet}\langle\Phi_K\,|\,\Phi_J\rangle^{1/2}} \tag{3.141}$$

after restoring the normalization. Our second way of evaluating Eq. (3.140) is to carry out the ζ integral first. Putting all together, we have

$$\left[\frac{\text{sdet}\mathcal{P}_1^\dagger\mathcal{P}_1}{\text{sdet}\langle\Phi_J\,|\,\Phi_K\rangle}\right]^{1/2}\text{sdet}\langle\mu_K\,|\,\Phi_J\rangle=\int D(B\bar{B}C\bar{C})e^{-I_{\rm sgh}(C,B)}\prod_K|\delta(\langle\mu_K\,|\,B\rangle)|^2\ . \tag{3.142a}$$

Since for $K=k$, $\langle\mu_K\,|\,B\rangle$ is odd, the δ function reduces to a linear function, so that equivalently

$$\prod_K\delta(\langle\mu_K\,|\,B\rangle)=\prod_k\langle\mu_k\,|\,B\rangle\prod_b\delta(\langle\mu_b\,|\,B\rangle)\ . \tag{3.142b}$$

Thus we arrive at a general formula for the scattering amplitudes in terms of the superghosts,[17]

$$\langle V_1(k_1)\cdots V_n(k_n)\rangle_h=\int_{s\mathcal{M}_h}d^2m_K\int D(XBC)\langle\langle V_1(k_1)\cdots V_n(k_n)\rangle\rangle\prod_b|\delta(\langle\mu_b\,|\,B\rangle)|^2\prod_k|\langle\mu_k\,|\,B\rangle|^2e^{-I}\ . \tag{3.143}$$

Here, I is the full action $I=I_m+I_{\rm sgh}$. As compared with the ghost formulation of the bosonic string, an unexpected novelty arises here. Whereas $\langle\mu_k\,|\,B\rangle$ amounts to an insertion of the operator B, the factors $\delta(\langle\mu_b\,|\,B\rangle)$ give rise to a new type of nonlocal insertion. We shall come back to this issue when dealing with the component formulation.

Further reformulation is possible when representing super Beltrami differentials in terms of super-quasiconformal vector fields, through Eq. (3.132). Remarking that the B field is effectively holomorphic, we have

$$\langle\mu_K\,|\,B\rangle\sim\int d^2{\bf z}\,E\mathcal{D}_-(BV_K)\ .$$

Super-quasiconformal vector fields may be viewed as

[16]For the sake of definiteness, we shall only consider the case $h\geq2$, where C has no zero modes. Otherwise, the C integration must be similarly restricted.

[17]Henceforth, we use the notation $d^2m_K=\prod_K d\bar{m}_K dm_K$.

"super-reparametrizations" with a discontinuity δV_K across a contour C_K. In that case, the inner products further reduce to (for $B=\beta+\theta b+\cdots$)

$$\langle \mu_K \mid B \rangle = \oint_{C_K} dz(\beta\delta V_K^0 + b\delta V_K^1) + \oint_{C_K} d\bar{z}\, \chi_{\bar{z}}{}^{+}\beta\delta V_K^1 \ . \qquad (3.144)$$

We shall not make use of this formulation at present, and just point out that it should find use when dealing with the equivalence between the Polyakov first-quantized superstring, as we have discussed here, and Witten's string field-theoretic formulation of the superstring.

3. BRST symmetry

We begin by discussing the stress tensor. Supergeometry is specified by only six independent fields, and thus there are only six independent components of the stress tensor, defined through an infinitesimal change in the total action,

$$I = \frac{1}{2\pi}\int d^2\mathbf{z}\, E(\tfrac{1}{2}\mathcal{D}_- X^\mu \mathcal{D}_+ X_\mu + B\mathcal{D}_- C + \bar{B}\mathcal{D}_+\bar{C}) \ , \qquad (3.145)$$

$$\delta I \equiv \frac{1}{2\pi}\int d^2\mathbf{z}\, E(H_+{}^+ T_+{}^+ + H_+{}^z T_z{}^+ + H_-{}^z T_z{}^- + \text{c.c.}) \ . \qquad (3.146)$$

The full action I is U(1) and super Weyl invariant, so we must have $T_+{}^+=0$, and since it is invariant under super-reparametrizations $\delta V^\pm$, we also have $T_z{}^+=0$ at the classical level.[18] These symmetries will also be implemented at the quantum level (in the critical dimension), so we shall completely ignore the components $T_+{}^+$ and $T_z{}^+$ and set them to zero. We shall also denote $T_z{}^- = T$ and call this the stress tensor. Invariance under super-reparametrizations δV^z implies that T is conserved,

$$\mathcal{D}_-^{3/2} T = 0 \ . \qquad (3.147)$$

It is sometimes convenient to consider the matter (T_m) and superghost (T_{sgh}) contributions separately; they are given by

$$\begin{aligned} T &= T_m + T_{\text{sgh}} \ , \\ T_m &= -\tfrac{1}{2}\mathcal{D}_+ X^\mu \mathcal{D}_+^2 X_\mu \ , \\ T_{\text{sgh}} &= -C\mathcal{D}_+^2 B + \tfrac{1}{2}\mathcal{D}_+ C\mathcal{D}_+ B - \tfrac{3}{2}(\mathcal{D}_+^2 C)B \ , \end{aligned} \qquad (3.148)$$

and are classically conserved.

Once the local gauge symmetries have been fixed and Faddeev-Popov ghosts introduced, the presence of the original symmetries is revealed by the existence of BRST symmetry. The total action I is indeed invariant under

$$\begin{aligned} \delta X^\mu &= \lambda C\mathcal{D}_+^2 X^\mu - \tfrac{1}{2}\lambda\mathcal{D}_+ C\mathcal{D}_+ X^\mu + \text{c.c.} \ , \\ \delta C &= \lambda C\mathcal{D}_+^2 C - \tfrac{1}{4}\lambda\mathcal{D}_+ C\mathcal{D}_+ C \ , \\ \delta B &= -\lambda T \ , \end{aligned} \qquad (3.149)$$

where λ is an odd constant parameter. Associated with this symmetry is a current of weight $\frac{1}{2}$,

$$j_{\text{BRST}} = C(T_m + \tfrac{1}{2}T_{\text{sgh}}) - \tfrac{3}{4}\mathcal{D}_+(C(\mathcal{D}_+ C)B) \ , \qquad (3.150)$$

which is conserved: $\mathcal{D}_- j_{\text{BRST}}=0$. It was pointed out by Friedan, Martinec, and Shenker (1986) that the superghost system by itself also possesses an additional U(1) symmetry, making it into an $N=2$ superconformal algebra. The associated U(1) current

$$j_{\text{U}(1)} = 2(\mathcal{D}_+ B)C + 3B\mathcal{D}_+ C \qquad (3.151)$$

is conserved: $\mathcal{D}_{\bar{z}} j_{\text{U}(1)} = 0$.

The presence of BRST symmetry implies the existence of certain Ward identities for the correlation functions, assuming that these are taken with respect to a (physical) BRST-invariant vacuum. In the case of the bosonic string we presented two somewhat distinct methods for handling these Ward identities. In the first one, the BRST charge was written as a line integral and analyticity properties of the correlation functions were used to "pull off the contour" and rewrite the full contribution as a total derivative over moduli space. The second method did not rely on such analyticity properties and has a wider range of applicability, though in the case where the correlation functions possess analyticity properties, these are not readily translated into this langauge.

For the superstring, as we shall see explicitly in Sec. VII, the superghost correlation functions are meromorphic, but they possess in addition to the expected poles some spurious poles, which in general have to be taken into account in the analyticity arguments before correct conclusions can be drawn. Maybe the use of supercontour integrals and superanalyticity on the superworldsheet can get around this problem. For the time being, we shall formulate the BRST Ward identities using the more general functional treatment, where no analyticity properties are assumed. One derives such identities on the generating functional, and by differentiating with respect to the sources, one can obtain them for any correlation function.

The starting point is the generating functional

$$Z(X^*,B^*,C^*) = \int D(XBC)\exp[-I(X,B,C) + I_s(X,B,C;X^*,B^*,C^*)] \ , \qquad (3.152)$$

where I is the total action of Eq. (3.145) and I_s couples the external sources X^*, B^*, and C^* to the fields X, B, and C in a super-reparametrization and local-U(1)-invariant way:

$$I_s = \int d^2\mathbf{z}\, E(X^*X + B^*B + C^*C + \bar{B}^*\bar{B} + \bar{C}^*\bar{C}) \ . \qquad (3.153)$$

Correlation functions are obtained by taking successive functional derivatives of Z. We introduce the notation[19]

[18]Compare with the bosonic string where Weyl invariance implies that $T_{z\bar{z}}=0$.

[19]Henceforth, we suppress the μ index on the X field.

$$\hat{X}=\frac{1}{E}\frac{\delta}{\delta X^*}, \quad \hat{B}=\frac{1}{E}\frac{\delta}{\delta B^*}, \quad \hat{C}=\frac{1}{E}\frac{\delta}{\delta C^*} . \qquad (3.154)$$

We shall also need the functional derivative with respect to the supergeometry changes $H_-{}^z$,

$$\hat{H}=\frac{1}{E}\frac{\delta}{\delta H_-{}^z} . \qquad (3.155)$$

For example, the partition function, according to Eq. (3.143), becomes[20]

$$Z_h=\int_{s\mathcal{M}_h} d^2m_K \prod_K |\delta(\langle \mu_K | \hat{B}\rangle)|^2 Z(X^*,B^*,C^*)\Big|_{*=0} , \qquad (3.156)$$

so that the operators with hats effectively play the role of the quantum operators associated with the fields.

The BRST Ward identities are derived on the assumption that the measure $D(XBC)$ is invariant under BRST transformations (3.149), which will be true at the quantum level only in the critical dimension. We then define the BRST operator

$$\lambda\hat{Q}_{\mathrm{BRST}}Z(X^*,B^*,C^*) =\int_{s\mathcal{M}_h} d^2m_K \int D(XBC)(\delta_{\mathrm{BRST}}I_s)e^{-I+I_s} . \qquad (3.157)$$

A little algebra gives

$$\hat{Q}_{\mathrm{BRST}}=\int d^2\mathbf{z}\, E[X^*(\hat{C}\mathcal{D}_+^2\hat{X}-\tfrac{1}{2}\mathcal{D}_+\hat{C}\mathcal{D}_+\hat{X})-B^*\hat{H} -C^*(\hat{C}\mathcal{D}_+^2\hat{C}-\tfrac{1}{4}\mathcal{D}_+\hat{C}\mathcal{D}_+\hat{C})+\mathrm{c.c.}] , \qquad (3.158)$$

which yields almost the same BRST transformation laws for the operators $\hat{X}$, $\hat{B}$, and $\hat{C}$ as given in Eq. (3.149):

$$\begin{aligned}\delta\hat{X}&=[\hat{X},\lambda\hat{Q}_{\mathrm{BRST}}]\\ &=\lambda\hat{C}\mathcal{D}_+^2\hat{X}-\tfrac{1}{2}\lambda\mathcal{D}_+\hat{C}\mathcal{D}_+\hat{X}+\mathrm{c.c.} ,\\ \delta\hat{C}&=[\hat{C},\lambda\hat{Q}_{\mathrm{BRST}}]=\lambda\hat{C}\mathcal{D}_+^2\hat{C}-\tfrac{1}{4}\lambda\mathcal{D}_+\hat{C}\mathcal{D}_+\hat{C} ,\\ \delta\hat{B}&=[\hat{B},\lambda\hat{Q}_{\mathrm{BRST}}]=-\lambda\hat{H} .\end{aligned} \qquad (3.159)$$

As an interesting application, we may evaluate the BRST behavior of an insertion occurring in the expressions for the amplitudes[21]

$$[\langle \mu_K | \hat{B}\rangle,\lambda\hat{Q}_{\mathrm{BRST}}]=\lambda \left|(-)^K\frac{\partial}{\partial m_K}+\int d^2\mathbf{z}\, E_z \int d^2\mathbf{w}\, E_w B^*(z)[\hat{H}(z)\mu_K(w)]\hat{B}(w)\right| , \qquad (3.160)$$

where we have used the fact that

$$\langle \mu_K | \hat{H}\rangle=\frac{\partial}{\partial m_K} . \qquad (3.161)$$

The (anti) commutator of this object with another insertion vanishes in view of the integrability conditions (3.128),

$$[\langle \mu_L | \hat{B}\rangle,[\langle \mu_K | \hat{B}\rangle,\lambda\hat{Q}_{\mathrm{BRST}}]]=\lambda(-)^{L+1}\left\langle\frac{\partial\mu_K}{\partial m_L}+(-)^{KL}\frac{\partial\mu_L}{\partial m_K}\middle|\hat{B}\right\rangle=0 . \qquad (3.162)$$

We also have

$$[\delta(\langle \mu_K | \hat{B}\rangle),\lambda\hat{Q}_{\mathrm{BRST}}]=[\langle \mu_K | \hat{B}\rangle,\lambda\hat{Q}_{\mathrm{BRST}}]\delta'(\langle \mu_K | \hat{B}\rangle) , \qquad (3.163)$$

where the ordering of δ' and [ ] on the right-hand side is immaterial in view of Eq. (3.162). With the help of Eq. (3.162) once more, we can now permute the BRST operator through all insertions,

$$\left[\prod_K \delta(\langle \mu_K | \hat{B}\rangle),\lambda\hat{Q}_{\mathrm{BRST}}\right]=\sum_{K'=1}^{5h-5}[\langle \mu_{K'} | \hat{B}\rangle,\lambda\hat{Q}_{\mathrm{BRST}}]\prod_{K=1}^{K'-1}\delta(\langle \mu_K | \hat{B}\rangle)\delta'(\langle \mu_{K'} | \hat{B}\rangle)\prod_{K=K'+1}^{5h-5}\delta(\langle \mu_K | \hat{B}\rangle) . \qquad (3.164)$$

To deal with scattering amplitudes, we have to insert vertex operators for physical states. Furthermore, we must show that a total BRST change in any vertex operator—which simply amounts to a gauge transformation in field theory langauge—produces a vanishing contribution.

The physical vertex operators for the emission or absorption of bosonic particles in the functional formulation can be taken to be super-reparametrization-, local-U(1)-, and super-Weyl-invariant vertex operators of the Polyakov string (to be discussed fully in Sec. VIII) without any B or C insertions, and they are thus of the form $V_i(\hat{X},H)$. It is not hard to see that they are automatically BRST invariant (in the critical dimension) in the following sense:

$$[\lambda\hat{Q}_{\mathrm{BRST}},V_i(\hat{X},H)]Z(X^*,B^*,C^*)=0 . \qquad (3.165)$$

To show this we need the Ward identities of the generating functional under super-reparametrizations, local U(1), and

[20]The subscript $*=0$ sets all sources to zero.
[21]$(-)^K=1$ when $K=k$ and -1 when $K=b$.

super Weyl symmetry. All of these are nonanomalous in the critical dimension, as long as the sources remain orthogonal to the zero modes of their corresponding fields. For example, it is straightforward to derive the super-reparametrization Ward identity

$$\int d^2\mathbf{z}\, E(\mathcal{D}_- V^z[\hat{H},V_i] - \tfrac{1}{2}\mathcal{D}_+ V^z V_i'\hat{X} - V^z\mathcal{D}_+^2 V_i'\hat{X} + \tfrac{1}{2}V^+\mathcal{D}_+ V_i'\hat{X})Z(X^*,B^*,C^*)=0 \tag{3.166}$$

valid for arbitrary fields V^z and V^+. For later convenience, we have added the effect of a supplementary super Weyl and local U(1) transformation of the second term. Evaluating Eq. (3.165) explicitly, we get

$$\int d^2\mathbf{z}\, E[-B^*[\hat{H},V_i] - (\hat{C}\mathcal{D}_+^2\hat{X} - \tfrac{1}{2}\mathcal{D}_+\hat{C}\mathcal{D}_+\hat{X})V_i'(\hat{X},H)]Z(X^*,B^*,C^*)=0 \ . \tag{3.167}$$

Finally, we use the Schwinger-Dyson equation,

$$(B^* - \mathcal{D}_-\hat{C})Z(X^*,B^*,C^*)=0 \ ,$$

in order to replace B^* in the first term of Eq. (3.167) by $\mathcal{D}_-\hat{C}$. Furthermore, since V^z and V^+ were arbitrary in Eq. (3.166), we may choose $V^z=\lambda\hat{C}$ and $V^+=\lambda\mathcal{D}_+\hat{C}$ and add Eq. (3.166) to (3.167). The exact cancellation shows that Eq. (3.165) holds, so that any super-reparametrization-, local U(1)-, and super-Weyl-invariant vertex V_i is also BRST invariant.

To show decoupling of BRST charges, let us consider the amplitude with $n-1$ physical (BRST-invariant) vertex operators $V_1,\ldots,V_{n-1}$ and one insertion of the BRST transform of an arbitrary operator V_n,

$$\langle V_1\cdots V_{n-1}[\lambda Q_{\mathrm{BRST}},V_n]\rangle_h = \int_{s\mathcal{M}_h} d^2m_K \prod_K |\delta(\langle \mu_K|\hat{B}\rangle)|^2 \hat{V}_1\cdots\hat{V}_{n-1}[\lambda\hat{Q}_{\mathrm{BRST}},\hat{V}_n]Z(X^*,B^*,C^*)\Big|_{*=0} \ . \tag{3.168}$$

The BRST invariance of $\hat{V}_i$, $i=1,\ldots,n-1$ and of the generating functional allows us to move $\lambda\hat{Q}_{\mathrm{BRST}}$ just to the right of all δ-function insertions. With the help of Eq. (3.164), we can bring the resulting commutator $[\langle\mu_{K'}|\hat{B}\rangle, \lambda\hat{Q}_{\mathrm{BRST}}]$ completely to the left. But now the sources should be set to zero, and only the derivative with respect to m_K remains from Eq. (3.160), so that

$$\langle V_1\cdots V_{n-1}[Q_{\mathrm{BRST}},V_n]\rangle_h = \int_{s\mathcal{M}_h} d^2m_K \sum_{K'=1}^{5h-5}\frac{\partial}{\partial m_{K'}}W_{K'} \tag{3.169}$$

with

$$W_{K'} = \prod_{K=1}^{K'-1}\delta(\langle\mu_K|\hat{B}\rangle)\delta'(\langle\mu_{K'}|\hat{B}\rangle)\prod_{K=K'+1}^{5h-5}\delta(\langle\mu_K|\hat{B}\rangle)\prod_{K=1}^{5h-5}\overline{\delta(\langle\mu_K|\hat{B}\rangle)}\hat{V}_1\cdots\hat{V}_n Z(X^*,B^*,C^*)\Big|_{*=0}$$

Thus the insertion of BRST changes in arbitrary operators produces total derivatives on supermoduli space. The total contributions then arise only from evaluating $W_{K'}$ at the boundary of moduli space. If the string theory satisfies all its equations of motion, i.e., the background space-time is a solution to the "string field equations of motion," then such contributions may be expected to vanish. However, when this is not the case cancellation may be required with effects on surfaces of different topology.

The use of superfield superghosts was proposed by Friedan, Martinec, and Shenker (1986) and further developed by Martinec (1987).

K. Chiral splitting in the component formalism

Though the expressions for the amplitudes obtained in the previous section are complete, one may wish to render them yet more explicit by working in the component formalism. Actually this is where the calculation for these amplitudes was performed in the first place. In this section we shall treat the case of the type-II superstring, postponing the discussion of the heterotic string to Sec. III.N.

Upon choosing Wess-Zumino gauge, we find that the superspace action (3.40) reduces to Eq. (3.1). We shall recall it here for convenience and display its dependence on complex (chiral) fields explicitly. We shall also drop the term proportional to the Euler characteristic, as well as the one involving the auxiliary field F,

$$I_m = I_x + I_\psi + I_m^1 + I_m^2 \ ,$$

where

$$\begin{aligned}
I_x &= \frac{1}{4\pi}\int_M d^2\xi\sqrt{g}\,D_z x^\mu D_{\bar z}x^\mu \ ,\\
I_\psi &= \frac{1}{4\pi}\int_M d^2\xi\sqrt{g}\,(-\psi_+^\mu D_{\bar z}\psi_+^\mu - \psi_-^\mu D_z\psi_-^\mu) \ ,\\
I_m^1 &= \frac{1}{4\pi}\int_M d^2\xi\sqrt{g}\,(\chi_{\bar z}{}^+\psi_+^\mu D_z x^\mu + \chi_z{}^-\psi_-^\mu D_{\bar z}x^\mu) \ ,\\
I_m^2 &= \frac{1}{8\pi}\int_M d^2\xi\sqrt{g}\,\chi_z{}^-\chi_{\bar z}{}^+\psi_+^\mu\psi_-^\mu \ .
\end{aligned} \tag{3.170}$$

This matter action could now be considered as a supergravity theory in its own right. For string theory, quantization would require integrating over the x^μ, ψ^μ, g_{mn}, and $\mathcal{X}_m$ fields in a reparametrization-invariant, local supersymmetric, and Weyl-invariant way. The difficulty is that it is impossible to define a workable measure for the component fields that is local and supersymmetric; indeed, the framework in which local supersymmetry is manifest is precisely superspace. Thus, instead of taking the action (3.170) as our starting point and quantizing it directly, we shall rather begin with the superspace formulation of the previous sections and project it down onto Wess-Zumino gauge. Since it is most convenient to perform such gauge choices in a local quantum field theory, we see that the superghost formalism is most practical in this respect. Notice that the choice of the Wess-Zumino gauge can always be implemented in a purely algebraic way. After this has been done, the symmetries are those described in Sec. III.C.

We now restrict the superghost action $I_{\rm sgh}(C,B)$ to Wess-Zumino gauge as well. To this end, we decompose the ghost superfields into components,

$$\begin{aligned} B&=\beta+\theta b+\bar\theta B_2+i\theta\bar\theta B_3\ ,\\ C&=c+\theta\gamma+\bar\theta C_2+i\theta\bar\theta C_3\ . \end{aligned} \tag{3.171}$$

We also restrict $\mathcal{X}_m$ to be γ-traceless, as may be done in the critical dimension where the super Weyl anomaly cancels. One then finds

$$I_{\rm sgh}=\frac{1}{2\pi}\int_M d^2\xi\sqrt{g}\left[-iB_3C_2+B_2\left[iC_3+\frac{i}{2}Ac\right]+b(D_{\bar z}c+\tfrac{1}{2}\mathcal{X}_{\bar z}{}^{+}\gamma)+\beta\left[\frac{3i}{4}AC_2+\tfrac{1}{2}\mathcal{X}_{\bar z}{}^{+}D_zc+D_{\bar z}\gamma+i\Lambda_{-}c\right]\right]+\text{c.c.} \tag{3.172}$$

It remains to evaluate the contribution from the δ functions on $\langle\mu_K\mid B\rangle$ in Eq. (3.142) to have the full ghost expression. Since in Wess-Zumino gauge μ_k is given by Eq. (3.129), we see that B_2 and B_3 (and their complex conjugates) never contribute to these inner products, and we have

$$\langle\mu_K\mid B\rangle=\langle\mu_K^1\mid b\rangle+\langle\mu_K^0\mid\beta\rangle\ . \tag{3.173}$$

Thus in the full B-C integrals in Eq. (3.142), the fields B_2, B_3, C_2, C_3 and their complex conjugates are auxiliary and never carry any derivatives. They may be integrated out explicitly, and ultralocality here says that the only effect will be a super area term, whose coefficient is determined by super Weyl symmetry and is thus immaterial.

We end up with the following expression for the super Faddeev-Popov determinants in Wess-Zumino gauge:

$$\left[\frac{\text{sdet}\mathcal{P}_1^\dagger\mathcal{P}_1}{\text{sdet}\langle\Phi_J\mid\Phi_K\rangle}\right]^{1/2}\text{sdet}\langle\mu_K\mid\Phi_J\rangle=\int D(bc\beta\gamma)e^{-I_{\rm sgh}}\prod_k|\langle\mu_k\mid B\rangle|^2\prod_b|\delta(\langle\mu_b\mid B\rangle)|^2\ , \tag{3.174}$$

where it is now understood that the field B in the products is restricted to $B=\beta+\theta b$, the superghost action takes on the simplified form

$$I_{\rm sgh}=I^0_{\rm sgh}+I^1_{\rm sgh}\ ,$$

where

$$\begin{aligned} I^0_{\rm sgh}&=\frac{1}{2\pi}\int_M d^2\xi\sqrt{g}\,(bD_{\bar z}c+\beta D_{\bar z}\gamma+\text{c.c.})\ ,\\ I^1_{\rm sgn}&=-\frac{1}{2\pi}\int_M d^2\xi\sqrt{g}\,(\mathcal{X}_{\bar z}{}^{+}S_{\rm gh}+\mathcal{X}_z{}^{-}\bar S_{\rm gh})\ , \end{aligned} \tag{3.175}$$

and the ghost supercurrent that is the θ-independent piece of the super stress tensor $T_{\rm sgh}$ of Eq. (3.148) is given by

$$S_{\rm gh}=\tfrac{1}{2}b\gamma-\tfrac{3}{2}\beta D_zc-(D_z\beta)c\ . \tag{3.176}$$

Actually, we shall sometimes make use of the full current

$$S=-\tfrac{1}{2}\psi_+^\mu D_zx^\mu+S_{\rm gh}\ , \tag{3.177}$$

which is only the $\mathcal{X}$-independent part of the full supercurrent (the θ-independent component of the stress tensor T). We shall see later on that it is, however, all we need.

We are now in a position to express the general superstring amplitude to h-loop order ($h\geq 2$) as an integral over supermoduli, formulated in components

$$\langle V_1(k_1)\cdots V_n(k_n)\rangle_h=\int_{s\mathcal{M}_h}d^2m_K\int D(x\psi bc\beta\gamma)\prod_k|\langle\mu_k\mid B\rangle|^2\prod_b|\delta(\langle\mu_b\mid B\rangle)|^2V_1(k_1)\cdots V_n(k_n)e^{-I}\ , \tag{3.178}$$

where $I=I_m+I_{\rm sgh}$ is the total action in components and $B=\beta+\theta b$.

1. Chiral splitting of the matter integrals

In Sec. II.B, we have seen that physical vertex operators (for bosonic particles) do not depend on ghosts or superghosts. Hence the amplitude (3.178) exactly as in Eq. (3.174) is "chirally split" in terms of the chiral ghost fields $bc\beta\gamma$ and $\bar b\bar c\bar\beta\bar\gamma$ in the sense that there is no coupling between the opposite chiralities of these fields. This will be a crucial property in defining both the heterotic and type-II strings and will manifest itself under the form of superholomorphic factorization, as we shall see in Sec.

VII. However, it is clear that this chiral factorization does not manifestly hold for the matter part. Of course, this could not have been expected in the first place, since the x field is real and not chiral. Furthermore, the term I_m^2, bilinear in χ, couples ψ_+ to ψ_- and seems to spoil chirality. One of the main tasks of this section will be to formulate a modified version of chiral splitting which holds for the full amplitude.

To display chiral splitting of the matter part of the functional integrals, one must integrate out the x field in any amplitude. For simplicity we shall not consider full vertex operators, but just insert the universal factors $e^{ik\cdot X}$ required by translation invariance, located at different points on the surface. This may be thought of as a tachyon operator whose position is not yet integrated over. It is only technically harder to deal with the insertion of full vertex operators. In Wess-Zumino gauge where auxiliary fields have been integrated out, we have

$$e^{ik\cdot X(z)}=e^{ik\cdot x(z)}e^{ik\cdot\theta\psi_+(z)}e^{ik\cdot\bar\theta\psi_-(z)} . \tag{3.179}$$

It is clear that the dependence on ψ_+ and ψ_- is already chirally split, so we shall deal with it later on. Notice that the second and third exponentials on the right-hand side are complex conjugates of one another only when k^μ is purely imaginary. Of course, physically k^μ is rather a real vector, but we shall also see later on that from several points of view k^μ should be analytically continued to imaginary values.

Thus we are ultimately interested in the integral

$$\mathcal{A}_x=\int Dx\prod_{i=1}^n e^{ik_i^\mu x^\mu(z_i)}e^{-I_x-I_m^1-I_m^2} , \tag{3.180}$$

leaving the Dirac fermion $\psi_\pm$ integrals for later. We have, however, included the I_m^2 term here, because it will naturally cancel some of the x integrals. The Green's function for the x field is

$$G(z,w)=\langle x(z)x(w)\rangle , \tag{3.181}$$

which is, however, not Weyl invariant as explained in Sec. II.G, and it is appropriate to define the Weyl-invariant combination $F(z,w)$,

$$\begin{aligned}-\ln F(z,w)=&\ G(z,w)+\tfrac{1}{2}\ln\rho(z)+\tfrac{1}{2}\ln\rho(w)\\&-\tfrac{1}{2}G_R(z,z)-\tfrac{1}{2}G_R(w,w) .\end{aligned} \tag{3.182}$$

Furthermore, recall that $F(z,w)$ has a very simple decomposition,

$$\begin{aligned}\ln F(z,w)=&\ \ln|E(z,w)|^2\\&-2\pi\,\mathrm{Im}\int_z^w\omega_I(\mathrm{Im}\Omega)^{-1}_{IJ}\mathrm{Im}\int_z^w\omega_J ,\end{aligned} \tag{3.183}$$

where $E(z,w)$ is the prime form and Ω_{IJ} the period matrix.

The Gaussian integral is now easily performed, and one gets

$$\mathcal{A}_x=(2\pi)^{10}\delta(k)\left[\frac{8\pi^2\det'\Delta}{\int_M d^2\xi\sqrt{g}}\right]^{-5}e^{\mathcal{K}^0+\mathcal{K}+\mathcal{K}'} , \tag{3.184}$$

where

$$\begin{aligned}\mathcal{K}^0&=-\tfrac{1}{2}\sum_{ij}k_i^\mu k_j^\mu\langle x(z_i)x(z_j)\rangle ,\\ \mathcal{K}&=-i\sum_i k_i^\mu\langle x^\mu(z_i)I_m^1\rangle ,\\ \mathcal{K}'&=\tfrac{1}{2}\langle I_m^1I_m^1\rangle-I_m^2 .\end{aligned} \tag{3.185}$$

The next step is to single out the ingredients that are not manifestly split. They will be expressed in terms of correlation functions of the field σ_I^μ where[22]

$$\sigma_I^\mu=\frac{1}{4\pi}\int d^2z\,\chi_{\bar z}{}^+(z)\psi_+^\mu(z)\omega_I(z) . \tag{3.186}$$

Thus we find, using Eq. (3.182) and momentum conservation,

$$\begin{aligned}\mathcal{K}^0&=\mathcal{L}_+^0+\mathcal{L}_-^0+2\pi\sum_{ij}k_i^\mu k_j^\mu\mathrm{Im}\int_P^{z_i}\omega_I(\mathrm{Im}\Omega)^{-1}_{IJ}\mathrm{Im}\int_P^{z_j}\omega_J ,\\ \mathcal{K}&=\mathcal{L}_++\mathcal{L}_--4\pi i\,\mathrm{Im}\sigma_I^\mu(\mathrm{Im}\Omega)^{-1}_{IJ}\sum_i k_i^\mu\mathrm{Im}\int_P^{z_i}\omega_J ,\\ \mathcal{K}'&=\mathcal{L}'_++\mathcal{L}'_--2\pi\,\mathrm{Im}\sigma_I^\mu(\mathrm{Im}\Omega)^{-1}_{IJ}\mathrm{Im}\sigma_J^\mu ,\end{aligned} \tag{3.187}$$

where P is an arbitrary point on the worldsheet. The combinations $\mathcal{L}_+^0$, $\mathcal{L}_+$, and $\mathcal{L}'_+$ depend analytically on the z_i and on Ω_{IJ} and involve only the chiral fields ψ_+^μ,

$$\mathcal{L}_+^0=\sum_{i<j}k_i^\mu k_j^\mu\ln E(z_i,z_j) \tag{3.188}$$

and

$$\begin{aligned}\mathcal{L}_+&=\frac{i}{4\pi}\sum_{i=1}^n k_i^\mu\int d^2z\,\chi_{\bar z}{}^+(z)\psi_+^\mu(z)\partial_z\ln E(z,z_i) ,\\ \mathcal{L}'_+&=-\frac{1}{32\pi^2}\int d^2z\int d^2w\,\chi_{\bar z}{}^+\psi_+(z)\chi_{\bar w}{}^+\psi_+(w)\\&\quad\times\partial_z\partial_w\ln E(z,w) .\end{aligned} \tag{3.189}$$

In practice, the expression $\exp(\mathcal{L}_+^0+\mathcal{L}_++\mathcal{L}'_+)$ can be viewed as resulting from contractions of an effectively chiral field $x_+(z)$ with effective propagator

$$\langle x_+(z)x_+(w)\rangle=-\ln E(z,w) , \tag{3.190}$$

so that

$$\exp(\mathcal{L}_+^0+\mathcal{L}_++\mathcal{L}'_+)=\langle e^{-I_m^1+ik_i^\mu x_+^\mu(z_i)}\rangle . \tag{3.191}$$

Finally, $\mathcal{L}_-^0$, $\mathcal{L}_-$, and $\mathcal{L}'_-$ are the complex conjugates of $\mathcal{L}_+^0$, $\mathcal{L}_+$, and $\mathcal{L}'_+$ with the understanding that k_i^μ is taken to be purely imaginary.

Returning to Eq. (3.184), the amplitude $\mathcal{A}_x$ can be rewritten as

[22]In the remainder of this section, the lower index on $\chi_{\bar z}{}^+$ is now an Einstein index, and repeated $I,J,\ldots$ indices are summed over.

$$\mathcal{A}_x=(2\pi)^{10}\delta(k)\left[\frac{8\pi^2\det'\Delta}{\int_M d^2\xi\sqrt{g}\,\det \mathrm{Im}\Omega}\right]^{-5}\times\exp(\mathcal{L}_+^0+\mathcal{L}_-^0+\mathcal{L}_++\mathcal{L}'_++\mathcal{L}_-+\mathcal{L}'_-)\mathcal{A}'_x\ , \tag{3.192}$$

and the remaining amplitude $\mathcal{A}'_x$ is gotten by collecting the pieces that are not yet manifestly chirally split. For later convenience we have rearranged a factor of $\det \mathrm{Im}\Omega$. Thus $\mathcal{A}'_x$ is given by

$$\mathcal{A}'_x=(\det \mathrm{Im}\Omega)^{-5}\exp\left[-2\pi\left[\mathrm{Im}\sigma_I^\mu+i\sum_{i=1}^n k_i^\mu \mathrm{Im}\int_P^{z_i}\omega_I\right](\mathrm{Im}\Omega)^{-1}_{IJ}\left[\mathrm{Im}\sigma_J^\mu+i\sum_{j=1}^n k_j^\mu \mathrm{Im}\int_P^{z_j}\omega_J\right]\right] . \tag{3.193}$$

We previously indicated a good reason for taking the external momenta purely imaginary. We now see that if k_i^μ are all imaginary, $\mathcal{A}'_x$ admits a remarkable representation generalizing the one encountered in Sec. II.G:

$$\mathcal{A}'_x=\int_{\mathfrak{J}}dp_I^\mu\left|\exp\left[i\pi p_I^\mu\Omega_{IJ}p_J^\mu+2\pi p_I^\mu\left[\sigma_I^\mu+i\sum_{i=1}^n k_i^\mu\int_P^{z_i}\omega_I\right]\right]\right|^2 . \tag{3.194}$$

Here p_I^μ represent the internal loop momenta, and for consistency they have been analytically continued to imaginary values as well—this has been indicated by the subscript $\mathfrak{J}$ to the integral. Of course, the integral would not be convergent, so it should be symbolically understood: the absolute value square is taken with p_I^μ imaginary, but to evaluate the integral one must analytically continue to real p_I^μ.

The combination involving $\det'\Delta$ admits a splitting in terms of left- and right-movers on the Riemann surface as well (up to an anomaly that will ultimately be cancelled, as explained in Secs. VII.A and VII.D),

$$\frac{8\pi^2\det'\Delta}{\int_M d^2\xi\sqrt{g}\,\det \mathrm{Im}\Omega}=|Z_\Delta(\Omega)|^4 . \tag{3.195}$$

Taking this into account, it becomes transparent that the full amplitude—for fixed internal, imaginary momenta p_I^μ—has been split (or factorized) as a product of an expression involving chiral operators ψ_+ and holomorphic $\mathbf{z}_i=(z_i,\theta_i)$ times its complex conjugate.

$$\mathcal{A}_x=(2\pi)^{10}\delta(k)\int_{\mathfrak{J}}dp_I^\mu\mathcal{F}_\nu(\mathbf{z}_i,\psi_+,\Omega,\chi;p_I^\mu)\mathcal{F}_\nu(\bar{\mathbf{z}}_i,\psi_-,\bar\Omega,\bar\chi;p_I^\mu) , \tag{3.196}$$

where the operator $\mathcal{F}_\nu$ only depends on ψ_+, $\mathbf{z}_i$, and Ω and not on ψ_-, $\bar{\mathbf{z}}_i$, or $\bar\Omega$,

$$\mathcal{F}_\nu(\mathbf{z}_i,\psi_+,\Omega,\chi;p_I^\mu)=[Z_\Delta(\Omega)]^{-10}\prod_{i<j}E(z_i,z_j)^{k_i\cdot k_j}e^{\mathcal{L}_++\mathcal{L}'_+}\exp\left[i\pi p_I^\mu\Omega_{IJ}p_J^\mu+2\pi p_I^\mu\left[\sigma_I^\mu+i\sum_i k_i^\mu\int_P^{z_i}\omega_I\right]\right] . \tag{3.197}$$

In formulating the type-II superstring, it was necessary to sum separately over the spin structures of left and right chiralities. This can now be easily achieved by evaluating the expectation value for the ψ_+ and ψ_- fields separately on each chiral component, each with its own spin structure. The two halves may then be brought back together for the same value of p_I^μ and the p_I^μ integral carried out. Thus the amplitude for different left- and right-spin structures ν and $\bar\nu$ is a simple generalization of Eq. (3.196),

$$\mathcal{A}_x=(2\pi)^{10}\delta(k)\int_{\mathfrak{J}}dp_I^\mu\mathcal{F}_\nu(\mathbf{z}_i,\psi_+,\Omega,\chi;p_I^\mu)\mathcal{F}_{\bar\nu}(\bar{\mathbf{z}}_i,\psi_-,\bar\Omega,\bar\chi;p_I^\mu) . \tag{3.198}$$

This entirely defines the matter contribution to the type-II superstring amplitudes involving only exponential insertions. The contributions of higher vertex operator insertions (containing in addition derivatives of x) can be similarly evaluated, and one arrives at an expression like (3.196), with $\mathcal{F}$ still chiral, but now also dependent on the derivative insertions. We shall work out the amplitudes for the scattering of massless particles for tree level in Sec. III.L and one-loop level in Sec. III.M.

Next, we must evaluate the amplitude for the full matter contribution, gotten by integrating out the Dirac fermion fields ψ_+ and ψ_-,

$$\mathcal{A}_m=\int D\psi_+D\psi_-\mathcal{A}_x e^{-I_\psi}\prod_{i=1}^n\exp[ik_i^\mu\theta_i\psi_+^\mu(z_i)+ik_i^\mu\bar\theta_i\psi_-^\mu(z_i)] , \tag{3.199}$$

where it is understood that ψ_+^μ and ψ_-^μ are endowed with spin structures ν and $\bar\nu$, respectively. With the help of Eq. (3.198) we may rewrite this expression,

$$\mathcal{A}_m=(2\pi)^{10}\delta(k)\int_{\mathfrak{J}}dp_I^\mu\mathcal{C}_\nu(\mathbf{z}_i,\Omega,\chi;p_I^\mu)\mathcal{C}_{\bar\nu}(\bar{\mathbf{z}}_i,\bar\Omega,\bar\chi;p_I^\mu) , \tag{3.200}$$

where

$$\mathcal{C}_\nu(\mathbf{z}_i,\Omega,\chi;p_I^\mu)=\mathcal{C}'_\nu[Z_\Delta(\Omega)]^{-10}\prod_{i<j}E(z_i,z_j)^{k_i\cdot k_j}\exp\left[i\pi p_I^\mu\Omega_{IJ}p_J^\mu+2\pi i p_I^\mu\sum_{i=1}^n k_i^\mu\int_P^{z_i}\omega_I\right],$$
$$\mathcal{C}'_\nu=\int D\psi_+^\mu e^{-I_{\psi_+}+\mathcal{L}_++\mathcal{L}'_+}\prod_{i=1}^n e^{ik_i^\mu\theta_i\psi_+^\mu(z_i)}e^{2\pi p_I^\mu\sigma_I^\mu}. \tag{3.201}$$

When the spin structure is even, there are generically no zero modes to the Dirac operator, and the Dirac propagator is given by the Szegö kernel (Secs. VI.F and VII.C),

$$S_\nu(z,w)=-\langle\psi_+(z)\psi_+(w)\rangle_\nu=\frac{\vartheta[\nu]\left[\int_w^z\omega_I,\Omega\right]}{E(z,w)\vartheta[\nu](0,\Omega)}, \tag{3.202}$$

which is meromorphic in z and w, and analytic in Ω. As a consequence, the reduced amplitudes $\mathcal{C}_\nu$ of Eq. (3.201) are analytic functions of $\mathbf{z}_i,\Omega_{IJ}$ and they depend only on $\chi_{\bar z}{}^+$.

When the spin structure is odd, there is generically one zero mode $h_\nu(z)$ to the Dirac operator, and the Dirac propagator is not uniquely defined. One choice is to take the propagator orthogonal to the zero mode, which can be achieved by demanding

$$\nabla^z S'_\nu(z,w)=2\pi\delta^2(z,w)-2\pi\frac{\overline{h_\nu(z)}h_\nu(w)}{\langle h_\nu|h_\nu\rangle}. \tag{3.203}$$

Since $h_\nu(z)$ depends holomorphically on Ω, $\tilde S_\nu$ itself will not be holomorphic in Ω. One can define an analytic propagator, at the expense of letting it transform with the wrong weight, and depend on an arbitrary point y on the Riemann surface,

$$S_\nu(z,w)=\frac{1}{E(z,w)}\frac{\sum_I\partial_I\vartheta[\nu]\left[\int_w^z\omega,\Omega\right]\omega_I(y)}{\sum_I\partial_I\vartheta[\nu](0,\Omega)\omega_I(y)}. \tag{3.204}$$

This propagator obeys

$$\nabla^z S_\nu(z,w)=2\pi\delta^2(z,w).$$

Actually, S'_ν can be represented in terms of S,

$$S'_\nu(z,w)=S_\nu(z,w)+\frac{h_\nu(z)h_\nu(w)}{\langle h_\nu|h_\nu\rangle^2}\int d^2P\int d^2Q\,\overline{h_\nu(P)h_\nu(Q)}S_\nu(P,Q)$$
$$-\frac{h_\nu(z)}{\langle h_\nu|h_\nu\rangle}\int d^2P\,\overline{h_\nu(P)}S_\nu(P,w)+\frac{h_\nu(w)}{\langle h_\nu|h_\nu\rangle}\int d^2P\,\overline{h_\nu(P)}S_\nu(P,z), \tag{3.205}$$

and does not depend on the extra point y any longer. S' is antisymmetric in z and w, as expected, and orthogonal to the zero mode. It is thus appropriate to write

$$S'_\nu(z,w)=-\langle\psi'_+(z)\psi'_+(w)\rangle, \tag{3.206}$$

where the prime on the fields stands for the fact that ψ_+ is considered in the space orthogonal to the zero mode.

Whereas for even-spin structure it was straightforward to show the holomorphicity of $\mathcal{C}_\nu$ in $\mathbf{z}_i$, Ω, and χ, for odd-spin structures there are several obstacles. First, the Dirac determinant with zero modes removed is no longer the absolute value square of a holomorphic function of Ω. Second, the Dirac propagator S' orthogonal to zero modes must be used to contract ψ', and it contains nonholomorphic dependences. We now show that a careful treatment actually produces a fully holomorphic amplitude $\mathcal{C}_\nu$ for odd-spin structure ν.

It is convenient to recast the contribution $\mathcal{L}_+^0+\mathcal{L}_++\mathcal{L}'_+$ in terms of a contraction over the chiral Bose field x_+, as shown in Eq. (3.191). Thus the amplitude $\mathcal{C}_\nu$ becomes

$$\mathcal{C}_\nu=Z_\Delta(\Omega)^{-10}\exp\left[i\pi p_I^\mu\Omega_{IJ}p_J^\mu+2\pi i p_I^\mu k_i^\mu\int_P^{z_i}\omega_I\right]\times\left\langle\prod_{i=1}^n e^{ik_i\cdot x_+(z_i)}\mathcal{R}_\nu\right\rangle, \tag{3.207}$$

where the reduced amplitude $\mathcal{R}_\nu$ is given by

$$\mathcal{R}_\nu=\int D\psi_+e^{-I_{\psi_+}-I_m^1}e^{2\pi p_I^\mu\sigma_I^\mu}\prod_{i=1}^n e^{ik_i^\mu\theta_i\psi_+^\mu(z_i)}$$
$$=\int D\psi_+\exp\left[-I_{\psi_+}+\int d^2z\,\eta^\mu(z)\psi_+^\mu(z)\right] \tag{3.208}$$

and the source $\eta^\mu(z)$ is independent of ψ_+,

$$\eta^\mu(z)=-\frac{1}{4\pi}\chi_{\bar z}{}^+[\partial_z x_+^\mu(z)-2\pi p_I^\mu\omega_I(z)]+i\sum_{i=1}^n k_i^\mu\theta_i\delta(z-z_i). \tag{3.209}$$

We now isolate the zero mode h_ν of ψ_+,

$$\psi_+(z)=\hat{h}_\nu(z)\psi_+^0+\psi'_+(z),\quad \hat{h}_\nu(z)=\frac{h_\nu(z)}{\langle h_\nu \mid h_\nu\rangle^{1/2}},$$

and ψ'_+ is understood to be orthogonal to h_ν, so that the functional integral simply splits,

$$\mathcal{R}_\nu=\int d\psi_+^0 e^{\langle \eta \mid \hat{h}_\nu\rangle \psi_+^0}\int D\psi'_+ e^{-I_{\psi'_+}+\langle \eta \mid \psi'_+\rangle}$$
$$=\prod_\mu \langle \eta^\mu \mid \hat{h}_\nu\rangle(\det{}'\not{D})^5\exp\left[\tfrac{1}{2}\int\int \eta S'_\nu\eta\right].$$

The difference between S'_ν and S_ν consists of terms proportional either to $h_\nu(z)$ or to $h_\nu(w)$. In view of the prefactor resulting from the zero-mode integration, such terms cancel. Furthermore, multilinearity of the same prefactor allows us to rearrange the normalization factor of h_ν,

$$\mathcal{R}_\nu=\prod_\mu \langle \eta^\mu \mid h_\nu\rangle\left[\frac{\det{}'\not{D}}{\langle h_\nu \mid h_\nu\rangle}\right]^5\exp\left[\tfrac{1}{2}\int\int \eta S_\nu\eta\right]. \tag{3.210}$$

As we shall see in Sec. VII.A, the determinant factor now precisely contains the correct zero-mode normalization to make it the absolute value square of a holomorphic function of Ω, and S_ν itself was of course holomorphic. Thus we have established full holomorphic splitting of the amplitudes with exponential insertions for even- and odd-spin structures.

What happens for full-fledged scattering amplitudes—say, of massless particles? There are further obstacles in principle to chiral splitting. Foremost among these is the fact that the superderivatives that enter the vertex operator construction themselves involve fields of both chiralities. This can be seen directly from Eq. (3.66), and is actually already familiar from the study of the superstring action which involves the chirality-violating term $\chi\bar{\chi}\psi_+\psi_-$. Thus the extension to higher vertex operators of the property of chiral splitting is nontrivial. In the case of massless external particles, we have checked that chiral splitting holds in exactly the same way as for simple exponential insertions, with the additional property that if ζ is the source term to $\mathcal{D}_+X$ and $\bar{\zeta}$ to $\mathcal{D}_-X$, then there will be holomorphic dependence on ζ as well. We shall not reproduce these calculations here, but postpone to the one-loop case the treatment of amplitudes of massless bosons and the proof of their chiral and holomorphic splitting properties. A general proof of these properties will be given elsewhere (D'Hoker and Phong, 1988a).

2. Spin structure versus space-time parity

It is interesting to examine the space-time character of the various amplitudes we have evaluated. Clearly, we have not directly dealt with physical external particles, but only with exponential insertions, but the observations listed below in fact easily extend to the case of any type of massless external particles, as we shall see more explicitly in the case of one loop in Sec. III.M.

From inspection of Eqs. (3.200) and (3.201), it is clear that the space-time amplitude corresponding to the *chiral half* $\mathcal{C}_\nu$ with ν even, is space-time parity conserving. External momenta and polarization tensors are contracted only with the metric tensor of space-time—the Minkowski or Euclidean metric in this case.

On the other hand, from inspection of Eqs. (3.207)–(3.210), we see that to the *chiral half* $\mathcal{C}_\nu$ with ν odd there corresponds an amplitude invariably containing a ten-dimensional space-time ϵ or completely antisymmetric tensor. It arises directly from the integration over the Dirac zero modes, which produces the product of the ten components of a Grassmann-valued space-time vector,

$$\prod_\mu \langle \eta^\mu \mid h_\nu\rangle=\frac{1}{10!}\varepsilon^{\mu_1\mu_2\cdots\mu_{10}}\eta_{\mu_1}\eta_{\mu_2}\cdots\eta_{\mu_{10}},$$

with $\eta_\mu=\langle \eta_\mu \mid h_\nu\rangle$. All remaining contractions of space-time indices are done with the ten-dimensional metric tensor. Thus the chiral amplitude $\mathcal{C}_\nu$ for ν odd is space-time parity violating—actually parity odd.

This means that the full amplitudes for the type-II superstring will be parity conserving if left and right worldsheet chiralities are endowed with either both even-spin or both odd-spin structure, and will be parity violating if the spin structure parities are opposite. Of course this reasoning has assumed that the vertex operators themselves do not involve the ε symbol, as is indeed always the case for low enough mass level $(m^2<12)$; if it is present, the assignments should of course be reversed.

L. Tree-level amplitudes for the type-II superstring

In this section we present a reasonably complete discussion of the tree-level calculation of superstring amplitudes. To remain specific, we shall deal with the tree-level case of the type-II superstring, determine the measure, factor out the superconformal Killing vector fields, and evaluate the three and four massless boson scattering amplitudes.

For $h=0$, there are six real conformal Killing vectors, four conformal Killing spinors, and no supermoduli parameters. The measure must thus be modified to

$$DE_M^A D\Omega_M \delta(T)=(\text{sdet}'\mathcal{P}_1^+\mathcal{P}_1)^{1/2}D'V^M D\Sigma DL, \tag{3.211}$$

where the prime on $D'V^M$ denotes the fact that it is restricted to the complement of the $\text{Ker}\mathcal{P}_1$. As in the bosonic case, a super Weyl transformation Σ brings out the following dependence:

$$DE_M{}^A D\Omega_M\delta(T)=(\text{sdet}'\hat{\mathcal{P}}_1^\dagger\hat{\mathcal{P}}_1)^{1/2}\frac{1}{\text{Vol}(\text{Ker}\hat{\mathcal{P}}_1)}\times e^{-5S_{SL}(\Sigma)}D\Sigma DV^M DL. \tag{3.212}$$

Assuming that the correct procedure is to divide by the factor of $s\mathcal{N}=\text{Vol(sDiff)}\times\text{Vol(sWeyl)}\times\text{Vol(sU(1))}$, one obtains the formula for the tree-level scattering amplitudes

$$\langle V_1(k_1)\cdots V_n(k_n)\rangle = ce^{-2\lambda}\langle\langle V_1(k_1)\cdots V_n(k_n)\rangle\rangle \times\frac{1}{\mathrm{Vol}(\mathrm{Ker}\hat{\mathcal{P}}_1)}\ , \qquad (3.213)$$

where the symbol $\langle\langle\ \rangle\rangle$ denotes the fact that the functional integral over X alone was performed. The determinants of $\hat{\mathcal{D}}_+\hat{\mathcal{D}}_-^{(0)}$ and $\hat{\mathcal{P}}_1^+\hat{\mathcal{P}}_1$ are constants, since there are no supermoduli, and we denote their effect by c.

1. Superconformal transformations

The next issue we must settle is the volume of $\mathrm{Ker}\hat{\mathcal{P}}_1$. To analyze this, we must write down the invariant volume element on this space. The superconformal invariance group is isomorphic to complexified OSp(1,1)—the superconformal extension of PSL(2,C) defined in Eq. (2.106). To see this, we start with homogeneous coordinates ($v\ w\ \psi$), where latin (greek) variables describe (anti-commuting) commuting variables. On this triplet, we have a natural action of GL(2 | 1) T: $W\to TW$,

$$W=\begin{bmatrix} v\\ w\\ \psi\end{bmatrix},\quad T=\begin{bmatrix} a & b & \alpha\\ c & d & \beta\\ \gamma & \delta & A\end{bmatrix}\ . \qquad (3.214)$$

To make contact with $N=1$ superspace, we introduce the projective coordinates

$$z=\frac{v}{w},\quad \theta=\frac{\psi}{w}\ ,$$

on which GL(2 | 1) acts by super Möbius transformation:

$$z\to\frac{az+b+\alpha\theta}{cz+d+\beta\theta},\quad \theta\to\frac{\gamma z+\delta+A\theta}{cz+d+\beta\theta}\ . \qquad (3.215)$$

To obtain a *superconformal* transformation T, we must transform the line element $d\mathbf{z}=dz+\theta\, d\theta$ into itself up to a conformal scaling. Equivalently, "the quadratic form"

$$z_{12}=z_1-z_2-\theta_1\theta_2=\frac{v_1w_2-v_2w_1-\psi_1\psi_2}{w_1w_2} \qquad (3.216)$$

should transform into itself up to a conformal scaling. This is uniquely achieved when the orthosymplectic form

$$K=\begin{bmatrix} 0 & +1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad (3.217)$$

is left invariant under T:

$$T^TKT=K\ . \qquad (3.218)$$

Note that the transpose of a matrix T is defined by

$$T^T=\begin{bmatrix} a & c & \gamma\\ b & d & \delta\\ -\alpha & -\beta & A\end{bmatrix}\ ,$$

so that $(TW)^T=W^TT^T$, and $\mathrm{sdet}T^T=\mathrm{sdet}T$. Thus the transformations (3.215) with T satisfying (3.218) are superconformal. The weight under which the difference transforms is easily derived, and we have

$$T: z_{12}\to\tilde{z}_{12}=\frac{z_{12}}{(cz_1+d+\beta\theta_1)(cz_2+d+\beta\theta_2)}\ . \qquad (3.219)$$

Similarly the line element transforms as

$$d\mathbf{z}\to d\tilde{\mathbf{z}}=\frac{d\mathbf{z}}{(cz+d+\beta\theta)^2} \qquad (3.220)$$

and the volume element as

$$d\mathbf{z}\wedge d\theta\to\frac{d\mathbf{z}\wedge d\theta}{cz+d+\beta\theta}\ . \qquad (3.221)$$

Elements in OSp(1,1) are in unique correspondence with a triplet of points in the superplane $(z_1,\theta_1),(z_2,\theta_2),(z_3,\theta_3)$ obeying one single (Grassmann-valued) constraint. The counting works out because OSp(1,1) has three commuting and two anticommuting parameters. The constraint is an Osp(1,1)-invariant Grassmann-valued function, dependent on three points (Aoki, 1988), given by

$$\Delta=\frac{z_{12}\theta_3+z_{31}\theta_2+z_{23}\theta_1+\theta_1\theta_2\theta_3}{(z_{12}z_{23}z_{31})^{1/2}}\ . \qquad (3.222)$$

The natural value for Δ is of course 0, which implies that one θ is dependent. With this value for Δ, it is easy to see that there is a unique correspondence between triplets of points satisfying $\Delta=0$ and elements of OSp(1,1), so that the latter may be accordingly parametrized.

In particular, the volume element on OSp(1,1) may be calculated in this fashion. We already know from Eqs. (3.219) and (3.220) that the six-dimensional volume element

$$\frac{dz_1dz_2dz_3d\theta_1d\theta_2d\theta_3}{(z_{12}z_{23}z_{31})^{1/2}} \qquad (3.223)$$

is invariant under OSp(1,1). The invariant volume element induced on OSp(1,1) is obtained by multiplying it by the δ function of the constraint $\delta(\Delta)=\Delta$:

$$d\mu=\frac{dz_1dz_2dz_3d\theta_1d\theta_2d\theta_3}{(z_{12}z_{23}z_{31})^{1/2}}\Delta\ . \qquad (3.224)$$

2. Evaluation of correlation functions

To calculate the correlation functions of a sequence of vertex operators, we would need the Green's function for the super Laplacian on the sphere. However, the Weyl invariance of the measure and the correlation functions, as well as the conservation of momentum, imply that one may instead work on the superplane after a stereographic projection, exactly as in the bosonic case. Here, the propagator is very simple,

$$G(\mathbf{z},\mathbf{z}')=-\ln(|z-z'-\theta\theta'|^2+\varepsilon^2)\ , \qquad (3.225)$$

and ε is understood to be infinitesimal. The vertex operators will be described extensively in Sec. VIII. Here we shall provide an example involving the simplest possible physical vertex operator: the one for bosonic particles at zero-mass level $k^2=0$,

$$V(\varepsilon,k)=g\varepsilon_{\mu;\bar{\mu}}\int d^2\mathbf{z}\,E\mathcal{D}_+X^\mu\mathcal{D}_-X^{\bar{\mu}}e^{ik\cdot X}\;, \qquad (3.226)$$

describing the graviton, the antisymmetric tensor field, and the dilaton. The polarization tensor $\varepsilon_{\mu;\bar{\mu}}$ is understood to be transverse in k, and the vertex is effectively normal ordered. To compute correlation functions of several of these vertices, it is useful to recall a trick known from the bosonic string. It consists of introducing a source for both X and its derivatives, and then isolating the correct expansion coefficient when developing in powers of the source. The key observation is that we may *formally* write $\varepsilon_{\mu;\bar{\mu}}=\zeta_\mu\bar{\zeta}_{\bar{\mu}}$, where ζ_μ and $\bar{\zeta}_{\bar{\mu}}$ are Grassmann-valued vectors. By linearity of any amplitude in the $\varepsilon_{\mu;\bar{\mu}}$'s, clearly any $\varepsilon_{\mu;\bar{\mu}}$ can be written as a formal sum, but we shall not explicitly need this construction. Once this has been done, we may introduce a generalized vertex

$$V^*(\zeta,\bar{\zeta},k)=g\int d^2\mathbf{z}\,E e^{ik\cdot X+\zeta\cdot\mathcal{D}_+X+\bar{\zeta}\cdot\mathcal{D}_-X}\;, \qquad (3.227)$$

whose $\zeta\bar{\zeta}$ coefficient is precisely $V(\varepsilon,k)$. Thus we shall perform our calculations on V^*, introducing a different set of $\zeta\bar{\zeta}$'s for every ε of V and selecting the correct term in the expansion in ζ's.

We thus calculate the n vertex correlation function starting from the V^* operators,

$$\langle V^*(\zeta_1,\bar{\zeta}_1,k_1)\cdots V^*(\zeta_n,\bar{\zeta}_n,k_n)\rangle =g^n\int d^2\mathbf{z}_1\cdots d^2\mathbf{z}_n\left\langle \exp\int d^2\mathbf{z}\,J^\mu(\mathbf{z})X_\mu(\mathbf{z})\right\rangle\;, \qquad (3.228)$$

where the source can be read off from the definition of V^*:

$$J^\mu(\mathbf{z})=\sum_{i=1}^{n}(ik_i^\mu+\zeta_i^\mu\mathcal{D}_+^i+\bar{\zeta}_i^\mu\mathcal{D}_-^i)\delta^2(\mathbf{z},\mathbf{z}_i)\;. \qquad (3.229)$$

By completing the square in the expectation value, we get

$$\left\langle\exp\left[\int d^2\mathbf{z}\,J^\mu(\mathbf{z})X_\mu(\mathbf{z})\right]\right\rangle=(2\pi)^{10}\delta(k)\exp\left[\mathcal{G}_n-\tfrac{1}{2}\sum_{i\neq j=1}^{n}k_i\cdot k_jG(\mathbf{z}_i,\mathbf{z}_j)\right]\;. \qquad (3.230)$$

Here the terms with $i=j$ are independent of momenta k and of the coordinates $\mathbf{z}_i$. Their contribution is absorbed into an overall normalization factor for each vertex, which will be omitted here:

$$\mathcal{G}_n=\sum_{i\neq j=1}^{n}(-ik_i\cdot\zeta_j\mathcal{D}_+^j-ik_i\cdot\bar{\zeta}_j\mathcal{D}_-^j-\tfrac{1}{2}\zeta_i\cdot\zeta_j\mathcal{D}_+^i\mathcal{D}_+^j-\tfrac{1}{2}\bar{\zeta}_i\cdot\bar{\zeta}_j\mathcal{D}_-^i\mathcal{D}_-^j-\tfrac{1}{2}\zeta_i\cdot\bar{\zeta}_j\mathcal{D}_+^i\mathcal{D}_-^j-\tfrac{1}{2}\bar{\zeta}_i\cdot\zeta_j\mathcal{D}_-^i\mathcal{D}_+^j)G(\mathbf{z}_i,\mathbf{z}_j)\;. \qquad (3.231)$$

For tree-level amplitudes, we work on the superplane and we use the Green's function of Eq. (3.225). Thus we have (with $\theta_{ij}=\theta_i-\theta_j$)

$$\begin{aligned}&\mathcal{D}_+^jG(\mathbf{z}_i,\mathbf{z}_j)=-\frac{\theta_{ij}}{z_{ij}},\quad \mathcal{D}_-^jG(\mathbf{z}_i,\mathbf{z}_j)=-\frac{\bar{\theta}_{ij}}{\bar{z}_{ij}}\;,\\ &\mathcal{D}_+^i\mathcal{D}_+^jG(\mathbf{z}_i,\mathbf{z}_j)=-\frac{1}{z_{ij}},\quad \mathcal{D}_-^i\mathcal{D}_-^jG(\mathbf{z}_i,\mathbf{z}_j)=-\frac{1}{\bar{z}_{ij}}\;,\\ &\mathcal{D}_+^i\mathcal{D}_-^jG(\mathbf{z}_i,\mathbf{z}_j)=0,\quad \mathcal{D}_-^i\mathcal{D}_+^jG(\mathbf{z}_i,\mathbf{z}_j)=0\;.\end{aligned} \qquad (3.232)$$

Actual calculations of the above from Eq. (3.225) would yield additional $\delta(\mathbf{z}_i,\mathbf{z}_j)$ functions, which in the tree-amplitude calculations disappear in view of analyticity in the external momenta.[23] Thus we are effectively left with

$$\mathcal{G}_n=\sum_{i\neq j=1}^{n}\left[+ik_i\cdot\zeta_j\frac{\theta_{ij}}{z_{ij}}+ik_i\cdot\bar{\zeta}_j\frac{\bar{\theta}_{ij}}{\bar{z}_{ij}}+\tfrac{1}{2}\zeta_i\cdot\zeta_j\frac{1}{z_{ij}}+\tfrac{1}{2}\bar{\zeta}_i\cdot\bar{\zeta}_j\frac{1}{\bar{z}_{ij}}\right]\;. \qquad (3.233)$$

We now work out the three-point amplitude first and separate $\mathcal{G}_n$ as a function of ζ's and $\bar{\zeta}$ ($\mathcal{G}_n=\mathcal{G}_n^\zeta+\mathcal{G}_n^{\bar{\zeta}}$):

$$\mathcal{G}_3^\zeta=i\left[+k_1\cdot\zeta_2\frac{\theta_{12}}{z_{12}}+k_2\cdot\zeta_1\frac{\theta_{12}}{z_{12}}+k_1\cdot\zeta_3\frac{\theta_{13}}{z_{13}}+k_3\cdot\zeta_1\frac{\theta_{13}}{z_{13}}+k_2\cdot\zeta_3\frac{\theta_{23}}{z_{23}}+k_3\cdot\zeta_2\frac{\theta_{23}}{z_{23}}-i\zeta_1\cdot\zeta_2\frac{1}{z_{12}}-i\zeta_1\cdot\zeta_3\frac{1}{z_{13}}-i\zeta_2\cdot\zeta_3\frac{1}{z_{23}}\right]\;. \qquad (2.234)$$

In evaluating $\exp(\mathcal{G}_n^\zeta)$, one retains terms proportional to $\zeta_1\zeta_2\zeta_3$; however, the term with three θ's vanishes because

[23]This is equivalent to the old argument of the "cancelled propagator."

$\theta_{12}\theta_{23}\theta_{31}=0$. Thus one is left with

$$\exp(\mathcal{G}_3^\zeta)\sim i\left[+\frac{\zeta_1\cdot\zeta_2}{z_{12}}\left[k_1\cdot\zeta_3\frac{\theta_{13}}{z_{13}}+k_2\cdot\zeta_3\frac{\theta_{23}}{z_{23}}\right]+\frac{\zeta_2\cdot\zeta_3}{z_{23}}\left[k_2\cdot\zeta_1\frac{\theta_{12}}{z_{12}}+k_3\cdot\zeta_1\frac{\theta_{13}}{z_{13}}\right]+\frac{\zeta_1\cdot\zeta_3}{z_{13}}\left[k_1\cdot\zeta_2\frac{\theta_{12}}{z_{12}}+k_3\cdot\zeta_2\frac{\theta_{23}}{z_{23}}\right]\right]. \tag{3.235}$$

Using transversality and momentum conservation, we have $k_2\cdot\zeta_3=-k_1\cdot\zeta_3$, etc., so that

$$\exp(\mathcal{G}_3^\zeta)\sim -i\frac{1}{z_{12}z_{23}z_{31}}[\zeta_1\cdot\zeta_2\, k_1\cdot\zeta_3(z_{23}\theta_{13}+z_{31}\theta_{23})+\text{cyclic perm.}]\ . \tag{3.236}$$

Now there is a remarkable identity:

$$z_{23}\theta_{13}+z_{31}\theta_{23}=z_{23}\theta_1+z_{31}\theta_2+z_{12}\theta_3+\theta_1\theta_2\theta_3=(z_{12}z_{23}z_{31})^{1/2}\Delta\ , \tag{3.237}$$

where Δ was the OSp(1,1)-invariant function introduced in Eq. (3.222). Thus

$$\langle\langle V(\varepsilon_1,k_1)V(\varepsilon_2,k_2)V(\varepsilon_3,k_3)\rangle\rangle=4(2\pi)^{10}\delta(k)\int\frac{d^2\mathbf{z}_1d^2\mathbf{z}_2d^2\mathbf{z}_3}{|z_{12}z_{23}z_{31}|}\Delta\bar{\Delta}\zeta_1^{\mu_1}\zeta_2^{\mu_2}\zeta_3^{\mu_3}\bar{\zeta}_1^{\bar{\mu}_1}\bar{\zeta}_2^{\bar{\mu}_2}\bar{\zeta}_3^{\bar{\mu}_3}K_{\mu_1\mu_2\mu_3}K_{\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3}\ , \tag{3.238}$$

where

$$\begin{aligned}&K_{\mu_1\mu_2\mu_3}=\eta_{\mu_1\mu_2}k_{1\mu_3}+\eta_{\mu_2\mu_3}k_{2\mu_1}+\eta_{\mu_3\mu_1}k_{3\mu_2}\ ,\\&\langle V(\varepsilon_1,k_1)V(\varepsilon_2k_2)V(\varepsilon_3k_3)\rangle=4(2\pi)^{10}\delta(k)\varepsilon_1^{\mu_1\bar{\mu}_1}\varepsilon_2^{\mu_2\bar{\mu}_2}\varepsilon_3^{\mu_3\bar{\mu}_3}K_{\mu_1\mu_2\mu_3}K_{\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3}\ .\end{aligned} \tag{3.239}$$

The factors Δ and $\bar{\Delta}$ appeared rather magically in the course of the above calculation. Actually, one never needs to isolate Δ or $\bar{\Delta}$ explicitly, provided one makes the following choice for the gauge fixing of the superconformal group:

$$z_1=0,\quad z_2=1,\quad z_3=\infty,\quad \theta_1,\quad \theta_2=0,\quad \theta_3=0\ .$$

The variable Δ in this gauge takes on the expression $\Delta=-\theta_1$, so that fixing the superconformal gauge is performed upon removal of

$$\frac{d^2z_1d^2z_2d^2\theta_2d^2z_3d^2\theta_3}{|z_{12}z_{23}z_{31}|}\ ,$$

the factor of Δ being taken care of automatically by the θ_1 integration.

To compute the four-point amplitude, we shall make use of the above gauge from the outset. We choose $z=z_1$, $z_2=0$, $z_3=1$, $z_4=\infty$, $\theta_1,\theta_2,\theta_3=\theta_4=0$, and then have

$$\begin{aligned}\mathcal{G}_4^\zeta=i\Bigg[&+k_1\cdot\zeta_2\frac{\theta_{12}}{z_{12}}+k_2\cdot\zeta_1\frac{\theta_{21}}{z_{21}}+k_1\cdot\zeta_3\frac{\theta_1}{z_{13}}+k_3\cdot\zeta_1\frac{\theta_1}{z_{13}}+k_1\cdot\zeta_4\frac{\theta_1}{z_{14}}+k_4\cdot\zeta_1\frac{\theta_1}{z_{14}}+k_2\cdot\zeta_3\frac{\theta_2}{z_{23}}+k_3\cdot\zeta_2\frac{\theta_2}{z_{23}}\\&+k_2\cdot\zeta_4\frac{\theta_2}{z_{24}}+k_4\cdot\zeta_2\frac{\theta_2}{z_{24}}+i\zeta_1\cdot\zeta_2\frac{1}{z_{12}}+i\zeta_1\cdot\zeta_3\frac{1}{z_{13}}+i\zeta_1\cdot\zeta_4\frac{1}{z_{14}}+i\zeta_2\cdot\zeta_3\frac{1}{z_{23}}+i\zeta_2\cdot\zeta_4\frac{1}{z_{24}}+i\zeta_3\cdot\zeta_4\frac{1}{z_{34}}\Bigg]\ .\end{aligned} \tag{3.240}$$

It is easy to see that $\exp(\mathcal{G}_4^\zeta)$ contains no terms with $4k$'s because there are only two θ's. Thus

$$\begin{aligned}\exp(\mathcal{G}_4^\zeta)=&\left[\zeta_1\cdot\zeta_2\zeta_3\cdot\zeta_4\frac{1}{z_{12}z_{34}}+\zeta_1\cdot\zeta_3\zeta_2\cdot\zeta_4\frac{1}{z_{13}z_{24}}+\zeta_1\cdot\zeta_4\zeta_2\cdot\zeta_3\frac{1}{z_{14}z_{23}}\right]\\&+\left[\zeta_1\cdot\zeta_2\left[k_1\cdot\zeta_3k_2\cdot\zeta_4\frac{\theta_2\theta_1}{z_{12}z_{13}z_{24}}+k_1\cdot\zeta_4k_2\cdot\zeta_3\frac{\theta_2\theta_1}{z_{12}z_{14}z_{23}}\right]+\text{perm.}\right]\ .\end{aligned} \tag{3.241}$$

In principle, one should now multiply this whole expression by the one involving the $\bar{\zeta}$'s, perform the integrals over z and θ, and regroup terms, clearly a feudal task. The calculation is enormously simplified by the factorization properties of the Veneziano integrals.

Recall that we have the ordinary integrals

$$\int\frac{d^2z}{\pi}z^A\bar{z}^{\tilde{A}}(1-z)^B(1-\bar{z})^{\tilde{B}}=\frac{\Gamma(-1-\tilde{A}-\tilde{B})}{\Gamma(-\tilde{A})\Gamma(-\tilde{B})}\frac{\Gamma(1+A)\Gamma(1+B)}{\Gamma(A+B+2)} \tag{3.242}$$

provided $A-\tilde{A}$ and $B-\tilde{B}$ are integers, which is always the case in string theory. Using the reciprocity formula for Γ

functions,

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}\ , \tag{3.243}$$

and the fact that $A-\tilde{A}$ and $B-\tilde{B}$ are integers, we see that this expression is actually symmetric under $(A,B)\leftrightarrow(\tilde{A},\tilde{B})$, as one might expect from complex conjugation. More importantly, the answer factorizes into a product of factors, each dependent only on the parameters for either the z or $\bar{z}$ coordinates. This product property implies that one need only consider, say, the z coordinates to find the full amplitude, which by the same token will also completely factorize as a function of ζ's and $\bar{\zeta}$'s. An analogous formula is derived for the superintegrals we need:

$$\int\frac{d^2\mathbf{z}_1}{\pi}d^2\theta_2[\theta_1\theta_2]^a[\bar{\theta}_1\bar{\theta}_2]^{\tilde{a}}z_{12}^{A}\bar{z}_{12}^{\tilde{A}}(1-z_1)^B(1-\bar{z}_1)^{\tilde{B}}=(-2i)^{1-a}(+2i)^{1-\tilde{a}}\frac{\Gamma(-\tilde{a}-\tilde{A}-\tilde{B})}{\Gamma(-\tilde{A})\Gamma(-\tilde{B})}\frac{\Gamma(1+A)\Gamma(1+B)}{\Gamma(A+B+1+a)} \tag{3.244}$$

Here a and $\tilde{a}$ are either 0 or 1, and the integrals are symmetric under $(aAB)\leftrightarrow(\tilde{a}\tilde{A}\tilde{B})$ using Eq. (3.243) and the fact that $A-B$ and $\tilde{A}-\tilde{B}$ are integers. With the help of Eq. (3.244), it is now straightforward to evaluate the four-point function,

$$\begin{aligned}\langle V(\varepsilon_1,k_1)V(\varepsilon_2,k_2)V(\varepsilon_3,k_3)V(\varepsilon_4,k_4)\rangle&=(2\pi)^{10}\delta(k)g^4\int d^2\mathbf{z}_1d^2\theta_2\,|z_{12}|^{-s}|z_1-1|^{-u}e^{\mathcal{G}_4^{\zeta}+\mathcal{G}_4^{\bar{\zeta}}}\\&=\pi(2\pi)^{10}\delta(k)g^4\frac{\Gamma(-s/2)\Gamma(-t/2)\Gamma(-u/2)}{\Gamma(1+s/2)\Gamma(1+t/2)\Gamma(1+u/2)}\varepsilon^{1\bar{1}}\varepsilon^{2\bar{2}}\varepsilon^{3\bar{3}}\varepsilon^{4\bar{4}}K_{1234}K_{\bar{1}\bar{2}\bar{3}\bar{4}}\ .\end{aligned} \tag{3.245}$$

Using the abbreviation i for μ_i to save some writing we have $K_{\mu_1\mu_2\mu_3\mu_4}=K_{1234}$, and K is then given by

$$\begin{aligned}K_{1234}=&(st\eta_{13}\eta_{24}-su\eta_{14}\eta_{23}-tu\eta_{12}\eta_{34})-s(k_1^4k_3^2\eta_{24}+k_2^3k_4^1\eta_{13}-k_1^3k_4^2\eta_{23}-k_2^4k_3^1\eta_{14})\\&+t(k_2^1k_4^3\eta_{13}+k_3^4k_1^2\eta_{24}-k_2^4k_1^3\eta_{34}-k_3^1k_4^2\eta_{12})-u(k_1^2k_4^3\eta_{23}+k_3^4k_2^1\eta_{14}-k_1^4k_2^3\eta_{34}-k_3^2k_4^1\eta_{12})\end{aligned} \tag{3.246}$$

We conclude this subsection by remarking that by superconformal invariance, the zero-, one-, and two-point functions of the superstring all vanish. The fastest way of obtaining this result is by remarking that $SL(2,C)$ is a subgroup of the superconformal group, and that the respective subgroups leaving 0, 1, or 2 points fixed all have infinite volume, so that the amplitudes vanish.

M. One-loop amplitudes for the type-II superstring

To deal with one-loop amplitudes, it is convenient to return to the component formulation of Sec. III.K. On the torus, there are four spin structures, one odd corresponding to periodic×periodic boundary conditions for all worldsheet spinors, and three even-spin structures, containing at least one antiperiodic boundary condition. For even-spin structure, there is one complex modulus and one complex conformal Killing vector. For odd-spin structure, there is in addition an odd modulus and a complex conformal Killing spinor. It will be convenient to represent a spin structure by its corresponding characteristics $\nu=(a,b)$. Here a and b take the value 0 or 1 according to whether the boundary conditions are antiperiodic or periodic, respectively, about A and B cycles. Left and right chiralities will be endowed with separate spin structures ν and $\bar{\nu}$. Thus it is appropriate to decompose the one-loop amplitude as follows:

$$\langle V_1\cdots V_n\rangle=\sum_{\nu\bar{\nu}}C'_{\nu\bar{\nu}}\langle V_1\cdots V_n\rangle_{\nu\bar{\nu}}\ . \tag{3.247}$$

The presence of conformal Killing vectors and spinors requires the insertion of the ghost c and the superghost $\delta(\gamma_0)$ where γ_0 is the zero mode (for odd-spin structure). Thus

$$\langle V_1\cdots V_n\rangle_{\nu\bar{\nu}}=\int_{s\mathcal{M}_1}dm_K\int D(x\psi bc\beta\gamma)\mathcal{I}_\nu\bar{\mathcal{I}}_{\bar{\nu}}V_1\cdots V_ne^{-I}\ . \tag{3.248}$$

When $\nu=(1,1)$ is odd, we have

$$\mathcal{I}_\nu=bc\,\delta(\beta_0)\delta(\gamma_0)\ , \tag{3.249}$$

where β_0 and γ_0 are the zero modes of the corresponding fields. If ν is even, on the other hand, the β_0 and γ_0 modes are absent and we have

$$\mathcal{I}_\nu=bc \tag{3.250}$$

and $\bar{\mathcal{I}}_{\bar{\nu}}$ is the complex conjugate of $\mathcal{I}_\nu$, considered for spin structure $\bar{\nu}$.

We shall now evaluate this expression for the case of bosonic vertex operators. In this case, the vertex operators are independent of the ghosts, and this integral may be performed separately. Both ghost chiralities may be integrated over independently, and one recovers the formulas derived earlier. For even-spin structure ν

$$\begin{aligned}\mathcal{A}_{\rm sgh}&=\int_\nu D(bc\beta\gamma)bce^{-I_{\rm sgh}}\\&=\left[\frac{1}{A}\det{}'\nabla_{-1}^z\right](\det\nabla_{-1/2}^z)_\nu^{-1}\ ,\end{aligned} \tag{3.251}$$

whereas for odd-spin structure

$$\mathcal{A}_{\text{sgh}}=\int_{(1,1)} D(bc\beta\gamma)bc\,\delta(\beta_0)\delta(\gamma_0)e^{-I_{\text{sgh}}} =\det{}'\nabla^z_{-1}(\det{}'\nabla^z_{-1/2})^{-1}_{(1,1)}=1 \ . \tag{3.252}$$

Here A is a normalization factor for conformal Killing vectors and spinors, and is given by the area of the worldsheet: $A=2\tau_2$. Unity results in Eq. (3.252) because the operators ∇^z_{-1} and $\nabla^z_{-1/2}$ are identical on the torus (with Euclidean metric) when both have periodic boundary conditions.

It is straightforward to evaluate [for these and the matter determinants (3.257) below, we refer the reader to Sec. V.A]

$$\frac{1}{A}\det{}'\nabla^z_{-1}=\tfrac{1}{2}\eta(\tau)^2 \ ,$$

and for even-spin structure ν

$$(\det\nabla^z_{-1/2})_\nu=\frac{\vartheta[\nu](0,\tau)}{\eta(\tau)} \ . \tag{3.253}$$

Notice that the superghost part of the amplitude is independent of the supermodulus χ. Recall indeed that the Faddeev-Popov operator could be separated into P_1 and $P_{1/2}$ without cross terms (see Sec. III.E).

1. Exponential insertions

Next, we evaluate the matter contribution, and again use the results of Sec. III.K. Recall that in principle all vertex insertions for bosonic external particles could be obtained from the insertion of (unintegrated) exponential factors. Thus it is best to evaluate these first, since they are simplest. Consider the amplitude

$$\mathcal{A}_m=\int D(x\psi)\prod_{i=1}^n e^{ik_i^\mu X^\mu(z_i,\theta_i)}e^{-I_m} \ , \tag{3.254}$$

where $X^\mu=x^\mu+\theta\psi^\mu_+ +\bar\theta\psi^\mu_- +i\theta\bar\theta F^\mu$ and I_m is the matter action in components. It is implicit that left- and right-spin structures are fixed to be ν and $\bar\nu$. Using the results of Eqs. (3.196) and (3.197), we have

$$\mathcal{A}_m=(2\pi)^{10}\delta(k)\left[\frac{4\pi^2\det{}'\Delta}{(\text{Im}\tau)^2}\right]^{-5}\left[\frac{\det{}'\not{D}_+}{\langle h\,|\,h\rangle}\right]^5_\nu \times\left[\frac{\det{}'\not{D}_-}{\langle h\,|\,h\rangle}\right]^5_{\bar\nu} \times\int_{\mathfrak{F}} dp^\mu \mathcal{F}'_\nu(z_i,\theta_i,\tau;p^\mu)\overline{\mathcal{F}'_{\bar\nu}(z_i,\theta_i,\tau;p^\mu)} \ , \tag{3.255}$$

where the reduced chiral amplitude $\mathcal{F}'_\nu$ is given by

$$\mathcal{F}'_\nu=\prod_{i<j}E(z_i,z_j)^{k_i\cdot k_j}\exp(i\pi p^\mu\tau p^\mu+i2\pi p^\mu k_i^\mu z_i) \times\left\langle e^{\mathcal{L}_+ +2\pi p^\mu\sigma^\mu}\prod_{i=1}^n\exp[ik_i^\mu\theta_i\psi^\mu_+(z_i)]\right\rangle \ . \tag{3.256}$$

The contribution of $\mathcal{L}'_+$, present in Eq. (3.196), vanishes for the torus, since there is only one χ. Further, when there are Dirac zero modes (for odd-spin structure), the last expectation value involves an integral over all of them. Finally, the determinants of the Dirac operators are understood to be primed with the zero mode h factored out when the spin structure is odd, and to have no such modification when the spin structure is even.

We have the explicit formulas

$$\frac{\det{}'\Delta}{(\text{Im}\tau)^2}=|\eta(\tau)|^4 \ ,$$
$$(\det\not{D}_+)_\nu=\frac{\vartheta[\nu](0,\tau)}{\eta(\tau)}, \quad \nu\neq(1,1) \ , \tag{3.257}$$
$$\left[\frac{\det{}'\not{D}_+}{\langle h\,|\,h\rangle}\right]_{(1,1)}=\eta(\tau)^2 \ .$$

Considerable simplification occurs upon putting the matter $\mathcal{A}_m$ and ghost $\mathcal{A}_{\text{sgh}}$ parts together to obtain the full amplitude $\mathcal{A}=\mathcal{A}_m\times\mathcal{A}_{\text{sgh}}\times\overline{\mathcal{A}_{\text{sgh}}}$:

$$\mathcal{A}=\int D(x\psi bc\beta\gamma)\prod_{i=1}^n e^{ik_i^\mu X^\mu(z_i,\theta_i)}\mathcal{J}_\nu\bar{\mathcal{J}}_{\bar\nu}e^{-I} =(2\pi)^{10}\delta(k)M_\nu\bar{M}_{\bar\nu}\int_{\mathfrak{F}} dp^\mu\mathcal{F}_\nu\bar{F}_{\bar\nu} \ . \tag{3.258}$$

For even-spin structure, we have

$$M_\nu=\frac{\vartheta[\nu](0,\tau)^4}{\eta(\tau)^{12}} \ , \tag{3.259a}$$

whereas for odd-spin structure

$$M_{(1,1)}=1 \ . \tag{3.259b}$$

It remains to evaluate $\mathcal{F}_\nu$. Here again, we distinguish between even- and odd-spin structure.

For even-spin structure, $\mathcal{L}_+$ and σ^μ vanish, and

$$\left\langle\prod_{i=1}^n e^{ik_i^\mu\theta_i\psi_+(z_i)}\right\rangle_{\psi_+}=\exp\left[\tfrac{1}{2}\sum_{ij}k_i\cdot k_j\theta_i\theta_j S_\nu(z_i,z_j)\right] \ , \tag{3.260}$$

where $S_\nu(z,w)$ is the Dirac propagator, given by the Szegö kernel

$$S_\nu(z,w)=\frac{\vartheta[\nu](z-w,\tau)\vartheta_1'(0,\tau)}{\vartheta[\nu](0,\tau)\vartheta_1(z-w,\tau)} \ . \tag{3.261}$$

It may also be useful to recall that the prime form E takes on a simple form for the torus,

$$E(z,w)=\frac{\vartheta_1(z-w,\tau)}{\vartheta_1'(0,\tau)} \ . \tag{3.262}$$

Clearly, it is advantageous to define the "chiral X propagator" in analogy with Eq. (3.190),

$$G_\nu(\mathbf{z},\mathbf{w})=G_1(z,w)+\theta_z\theta_w S_\nu(z,w) \ , \qquad G_1(z,w)=-\ln E(z,w) \ , \tag{3.263}$$

so that

$$\mathcal{F}_\nu = \exp\left[ip^\mu \tau p^\mu + i2\pi p^\mu \sum_i k_i^\mu z_i - \sum_{i<j} k_i \cdot k_j G_\nu(\mathbf{z}_i,\mathbf{z}_j)\right] . \tag{3.264}$$

Note, however, that $E(z,w)$ is multivalued around B cycles on the surfaces. The full propagator for the X field is simply related to G_ν:

$$G_{\nu\bar\nu}(\mathbf{z},\mathbf{w}) = \langle X(\mathbf{z})X(\mathbf{w})\rangle = G_\nu(\mathbf{z},\mathbf{w}) + \overline{G_{\bar\nu}(\mathbf{z},\mathbf{w})} - \frac{\pi}{2\tau_2}(z-w-\bar z+\bar w)^2 , \tag{3.265}$$

and it is well defined on the surface, though no longer meromorphic. The last term arises because of the x zero mode. No analogous terms arise for the Dirac propagator because for the even-spin structure there are no Dirac zero modes. Notice also that since the auxiliary field F^μ in X^μ has been set to zero from the outset, we do not pick up a δ-function contribution to the propagator. Analyticity in the external momenta justifies dropping such terms, as long as the propagator is evaluated between vertex operators, as will always be the case here.

For odd-spin structure, $\mathcal{L}_+$ and σ^μ do contribute; however, since they are linear in χ, each of them can only be contracted with the exponential insertion. The ψ_+ propagator $\tilde S_\nu$, orthogonal to the constant-zero mode of $\mathcal{D}$, is given by

$$\tilde S_0(z,w) = S_0(z,w) - \frac{\pi}{\tau_2}(z-w-\bar z+\bar w) , \qquad S_0(z,w) = \frac{\vartheta_1'(z-w,\tau)}{\vartheta_1(z-w,\tau)} . \tag{3.266}$$

Here we have abbreviated the odd-spin structure by $0=(1,1)$. It is easy to see that this is a well-defined function on the torus. The full propagator for odd-spin structure is then given by

$$G_{(1,1)}(\mathbf{z},\mathbf{w}) = \langle X(\mathbf{z})X(\mathbf{w})\rangle = G_0(\mathbf{z},\mathbf{w}) + \overline{G_0(\mathbf{z},\mathbf{w})} - \frac{\pi}{2\tau_2}(z-w-\bar z+\bar w-\theta_z\theta_w+\bar\theta_z\bar\theta_w)^2 ,$$

where

$$G_0(\mathbf{z},\mathbf{w}) = G_1(z,w) + \theta_z\theta_w S_0(z,w) .$$

2. Modular invariance

We now discuss the coefficients $C_{\nu\bar\nu}$ occurring in the summation over spin structures. $G_{\nu\bar\nu}$ is manifestly modular covariant, as may be seen by using Eq. (E5): the only effect of a modular transformation on $G_{\nu\bar\nu}$ is to permute the spin structure according to the modular group,

$$G_{\nu\bar\nu}(z-z',\theta\theta';\tau+1) = G_{\nu_1\bar\nu_1}(z-z',\theta\theta';\tau) , \qquad G_{\nu\bar\nu}\left[\frac{z-z'}{\tau},\frac{\theta\theta'}{\tau},-\frac{1}{\tau}\right] = G_{\nu_\tau\bar\nu_\tau}(z-z',\theta\theta';\tau) , \tag{3.267}$$

where

$$\nu_1=(a,b+a+1),\quad \bar\nu_1=(\bar a,\bar b+\bar a+1) ,$$
$$\nu_\tau=(b,a),\quad \bar\nu_\tau=(\bar b,\bar a) \pmod 2 .$$

Note that the odd-spin structure is transformed into itself. This at once implies that the vertex operator contractions $\langle\langle V_1(k_1)\cdots V_n(k_n)\rangle\rangle_{\nu\bar\nu}$ are also modular invariant in this sense. Modular invariance of the full amplitude $\langle V_1(k_1)\cdots V_n(k_n)\rangle$ will be achieved provided a choice for $C_{\nu\bar\nu}$ is made that is consistent with modular invariance. It is easily checked that the measure in Eq. (3.258) transforms correctly under modular transformations, except perhaps for a constant phase:

$$\langle V_1(k_1)\cdots V_n(k_n)\rangle_{\nu\bar\nu}(\tau+1) = (-1)^{a+\bar a}\langle V_1(k_1)\cdots V_n(k_n)\rangle_{\nu_1\bar\nu_1}(\tau) ,$$
$$\langle V_1(k_1)\cdots V_n(k_n)\rangle_{\nu\bar\nu}\left[-\frac{1}{\tau}\right] = \langle V_1(k_1)\cdots v_n(k_n)\rangle_{\nu_\tau\bar\nu_\tau}(\tau) . \tag{3.268}$$

Hence modular invariance of the full amplitude requires the following choice for the constants $C_{\nu\bar\nu}$:

$$\tau\to-\frac{1}{\tau},\quad C_{(1,0)\bar\nu}=C_{(0,1)\bar\nu},\quad C_{\nu(1,0)}=C_{\nu(0,1)} ,$$
$$\tau\to\tau+1,\quad C_{(0,1)\bar\nu}=-C_{(0,0)\bar\nu},\quad C_{\nu(0,1)}=-C_{\nu(0,0)} , \tag{3.269}$$

and this should hold for all ν and $\bar\nu$. Note that, since the odd-spin structure (1,1) transforms as a singlet under the modular group, the relative magnitude with even-spin structures is not fixed by modular invariance. It should be determined by factorization, in the limit where the torus degenerates to the sphere.

3. Three- and four-point amplitudes for massless bosons

Though the prescriptions given above are complete and explicit, it may be instructive to work things out for an example. Let us consider scattering amplitudes with massless external particles only (the graviton, dilaton, and antisymmetric tensor field). Such operators are produced by the generating vertex $V^*(\zeta,\bar\zeta;k)$ introduced in Eq. (3.227). As in the case of tree level, the amplitude (3.228) is expressed through Eqs. (3.230) and (3.231), but the propagator is now understood to be $G_{\nu\bar\nu}$, of Eq. (3.265).

For even-spin structure, we consider the chirality-conserving form first and then split it to obtain the chiral amplitude. The relevant superderivatives are

$$\mathcal{D}_+^j G_{\nu\bar\nu}(\mathbf{z}_i,\mathbf{z}_j) = \mathcal{D}_+^j G_\nu(\mathbf{z}_i,\mathbf{z}_j) + \frac{\pi}{\tau_2}\theta_j(z_i-z_j-\bar z_i+\bar z_j) ,$$
$$\mathcal{D}_+^i\mathcal{D}_+^j G_{\nu\bar\nu}(\mathbf{z}_i,\mathbf{z}_j) = \mathcal{D}_+^i\mathcal{D}_+^j G_\nu(\mathbf{z}_i,\mathbf{z}_j) + \frac{\pi}{\tau_2}\theta_i\theta_j , \tag{3.270}$$
$$\mathcal{D}_+^i\mathcal{D}_-^j G_{\nu\bar\nu}(\mathbf{z}_i,\mathbf{z}_j) = -\frac{\pi}{\tau_2}\theta_i\bar\theta_j .$$

Again, we have neglected all $\delta(z_i,z_j)$'s, because they do not contribute to the amplitude in view of analyticity in the external momenta. Thus we may separate $\mathcal{G}_n$ of Eq. (3.231) into two chiral parts expressed only in terms of the chiral propagator G_ν (and its complex conjugate) and a mixed part, which we shall call $\hat{\mathcal{G}}_n$:

$$\mathcal{G}_n-\tfrac{1}{2}\sum_{i\neq j}k_i\cdot k_j G_{\nu\bar\nu}(\mathbf{z}_i,\mathbf{z}_j) = \mathcal{G}_n^{\zeta}+\mathcal{G}_n^{\bar\zeta}+\hat{\mathcal{G}}_n -\tfrac{1}{2}\sum_{i\neq j}k_i\cdot k_j[G_\nu(\mathbf{z}_i,\mathbf{z}_j)+\overline{G_{\bar\nu}(\mathbf{z}_i,\mathbf{z}_j)}] , \quad (3.271)$$

where the chiral part is given by

$$\mathcal{G}_n^{\zeta}=\sum_{i\neq j}[-ik_i\cdot\zeta_j\mathcal{D}_+^j G_\nu(\mathbf{z}_i,\mathbf{z}_j)-\tfrac{1}{2}\zeta_i\cdot\zeta_j\mathcal{D}_+^i\mathcal{D}_+^j G_\nu(\mathbf{z}_i,\mathbf{z}_j)] \quad (3.272)$$

and $\mathcal{G}_n^{\bar\zeta}$ is its complex conjugate (for imaginary k_i^μ). The mixed part can be simplified with the use of rearrangements familiar from Sec. III.K:

$$\hat{\mathcal{G}}_n=-\frac{2\pi}{\tau_2}\left[\sum_i[-\mathrm{Im}(\zeta_i^\mu\theta_i)+ik_i^\mu\mathrm{Im}z_i]\right]^2 .$$

Following the derivation of Sec. III.K, we may introduce the loop momenta p^μ and write

$$e^{\hat{\mathcal{G}}_n}=(\tau_2)^5\int_{\mathfrak{F}}dp^\mu\,|\exp[i\pi p^\mu\tau p^\mu +2\pi p^\mu(-\zeta_i^\mu\theta_i+ik_i^\mu z_i)]|^2 . \quad (3.273)$$

Thus the full amplitude (still for even-spin structure) may be recast in a familiar form,

$$\langle V_1^*\cdots V_n^*\rangle_{\nu\bar\nu}=(2\pi)^{10}\delta(k)M_\nu\overline{M}_{\bar\nu}\int_{\mathfrak{F}}dp^\mu\mathcal{F}_\nu\overline{\mathcal{F}}_{\bar\nu} , \quad (3.274)$$

where

$$\mathcal{F}_\nu(z_i,k,\zeta,\tau;p^\mu) =\exp\left[i\pi p^2\tau+2\pi p^\mu(-\zeta_i^\mu\theta_i+ik_i^\mu z_i) -\tfrac{1}{2}\sum_{i\neq j}k_i\cdot k_jG_\nu(z_i,z_j)+\mathcal{G}_n^{\zeta}\right] . \quad (3.275)$$

Of course this amplitude should now be integrated over moduli space.

To evaluate the zero-, one-, two-, three-, four-, and five-point amplitudes, the above is in fact enough, for only the even-spin structures contribute to their amplitudes. Indeed, for the odd-spin structure the Dirac operator has one (chiral) zero mode for each dimension of space-time $d=10$; there is thus a total of ten zero modes. Inserting, for example, a massless vertex eats up two zero modes. However, one fermion mode is also eaten up by fixing a conformal spinor gauge for the supersymmetry operator. One more is produced by the presence of the supermoduli parameters. All zero modes must of course be killed, so naively the lowest number of vertex operator insertions necessary to make the amplitude nonzero is five. However, overall momentum conservation implies that this amplitude also vanishes, and one has to go to six external particles to obtain a nonzero contribution from the odd-spin structure.

We first show that the zero-, one-, two-, and three-point functions vanish identically. This fact is based upon two fundamental observations. For three or fewer external massless particles, one always has $k_i\cdot k_j=0$ for all i and j, so that $\mathcal{F}_\nu$ only involves $\mathcal{G}_n^{\zeta}$, which depends on the derivatives of G_ν only. These derivatives are given by

$$\mathcal{D}_+^i G_\nu(\mathbf{z}_i,\mathbf{z}_j;\tau)=\theta_jS_\nu(z_i,z_j)+\theta_i\partial_iG_1(z_i,z_j) ,$$
$$\mathcal{D}_+^i\mathcal{D}_+^j G_\nu(\mathbf{z}_i,\mathbf{z}_j;\tau)=-S_\nu(z_i,z_j)+\theta_i\theta_j\partial_i\partial_jG_1(z_i,z_j) . \quad (3.276)$$

The partition function and the one- and two-particle amplitudes all vanish simply by the use of the famous Jacobi identity of (E11) and the assignments of the coefficients $C_{\nu\bar\nu}$:

$$\sum_\nu C_{\nu\bar\nu}\vartheta_{ab}(0,\tau)^4=0,\quad \nu=(a,b),\quad \bar\nu=(\bar a,\bar b) . \quad (3.277)$$

For the three-point function one uses, in addition to the above, the facts that

$$\sum_\nu C_{\nu\bar\nu}\vartheta_{ab}(0,\tau)^4\mathcal{D}_+^1G_\nu(1,2)\mathcal{D}_+^2G_\nu(2,3)\mathcal{D}_+^3G_\nu(3,1)=0 ,$$
$$\sum_\nu C_{\nu\bar\nu}\vartheta_{ab}(0,\tau)^4\mathcal{D}_+^1G_\nu(1,2)\mathcal{D}_+^2G_\nu(2,1)\mathcal{D}_+^3G_\nu(3,1)=0 , \quad (3.278)$$

which are equivalent—in component language—to the equations

$$\sum_\nu C_{\nu\bar\nu}\vartheta_{ab}(0,\tau)^4S_\nu(z_1-z_2)S_\nu(z_2-z_3)S_\nu(z_3-z_1)=0 ,$$
$$\sum_\nu C_{\nu\bar\nu}\vartheta_{ab}(0,\tau)^4S_\nu(z_1-z_2)S_\nu(z_2-z_1)=0 . \quad (3.279)$$

All these identities are easily proven with the help of Eqs. (3.277) and (E7′).

The calculation of the four-point function is more involved, and ϑ-function identities are heavily used. There are three types of terms: those with four factors of k, those with two factors of k, and those without explicit k's at all contracted onto the polarization tensors. Our first task is to show that the terms with four factors of k cancel after summation over all spin structures. One needs the following Riemann-type identity:

$$\sum_\nu C_{\nu\bar\nu}\vartheta_{ab}(0,\tau)^4\mathcal{D}_+^1G_\nu(1,i_1)\mathcal{D}_+^2G_\nu(2,i_2)\mathcal{D}_+^3G_\nu(3,i_3)\mathcal{D}_+^4G_\nu(4,i_4) =\theta_{i_1}\theta_{i_2}\theta_{i_3}\theta_{i_4}\sum_\nu C_{\nu\bar\nu}\vartheta_{ab}(0,\tau)^4S_\nu(1,i_1)S_\nu(2,i_2)S_\nu(3,i_3)S_\nu(4,i_4) . \quad (3.280)$$

To establish this, use the representation (3.276) for the derivatives: terms with four G_1's cancel because of (3.277), terms with three G_1's due to transversality, and terms with two or one due to (3.279). Permutations $(1,2,3,4)\to(i_1,i_2,i_3,i_4)$ which leave one or more points fixed need not be considered, as their contribution cancels due to transversality of the polarization tensors. The remaining nine permutations cancel in view of the Riemann identity (E7′).

Now we calculate the terms with two momenta k; it is useful to take an example. Consider terms arising as the coefficient of

$$\zeta_1\cdot\zeta_2\zeta_3\cdot k_{i_3}\zeta_4\cdot k_{i_4} ,$$

where i_3 and i_4 are different from three and four, respectively. The spin structure sum is then again simplified, with the help of the Riemann identities, and one finds

$$\sum_\nu C_{\nu\bar\nu}\vartheta_\nu^4(0,\tau)\mathcal{D}_+^1\mathcal{D}_+^2G_\nu(1,2)\mathcal{D}_+^3G_\nu(3,i_3)\mathcal{D}_+^4G_\nu(4,i_4)\prod_{i<j}[1+k_i\cdot k_j\theta_i\theta_jS_\nu(i,j)]$$
$$=\theta_{i_3}\theta_{i_4}\sum_\nu C_{\nu\bar\nu}\vartheta_\nu(0,\tau)^4S_\nu(1,2)S_\nu(3,i_3)S_\nu(4,i_4)\prod_{i<j}(\ ) , \quad (3.281)$$

where the last factor arises from the expansion of the superspace Green's function. Since we must end up with four θ's, the product $\prod_{i<j}$ produces only terms with two θ's, so that the answer will be a linear function of s, t, and u. With some further use of the Riemann identities, one can evaluate it rather easily, and one finds

$$\zeta_1\cdot\zeta_2(t\zeta_3\cdot k_1\zeta_4\cdot k_2+u\zeta_3\cdot k_2\zeta_4\cdot k_1) . \quad (3.282)$$

Upon inspection, one notices that this result is reminiscent of the tree-level answer obtained in Eq. (3.246). One can now easily complete the analysis by checking that the other terms also have the same form as the tree-level answer. Thus our final expression for the one-loop four-point function in the type-II superstring is

$$\langle V(\varepsilon_1,k_1)\cdots V(\varepsilon_4,k_4)\rangle=g^4\delta(k)\mathcal{A}_1\varepsilon^{1\bar1}\varepsilon^{2\bar2}\varepsilon^{3\bar3}\varepsilon^{4\bar4}\times K_{1234}K_{\bar1\bar2\bar3\bar4} , \quad (3.283)$$

where the reduced amplitude is given by

$$\mathcal{A}_1=\int_{\mathcal{M}_1}\frac{d^2\tau}{2\tau_2^2}\frac{1}{(\tau_2)^4}\int d^2z_1d^2z_2d^2z_3d^2z_4 \times|F_{12}F_{34}|^{-s/2}|F_{23}F_{14}|^{-t/2}\times|F_{13}F_{24}|^{-u/2} . \quad (3.284)$$

We have used identity (E9) and we have abbreviated $F_{ij}=F(z_i,z_j)$, where the function F was defined in Eq. (2.91) or (3.183). Overall translation invariance on the torus allows us to integrate over one of the four positions, so that we may set $z_4=0$ and

$$\mathcal{A}_1=\frac{1}{2}\int_{\mathcal{M}_1}\frac{d^2\tau}{(\tau_2)^5}\int d^2z_1d^2z_2d^2z_3\left|\frac{F_{12}F_{34}}{F_{13}F_{24}}\right|^{-s/2}\times\left|\frac{F_{23}F_{14}}{F_{13}F_{24}}\right|^{-t/2} , \quad (3.285)$$

which agrees with the classic formula derived in the operator formalism.

Several remarks are in order here. First, it is remarkable that the kinematical form for the one-loop amplitude coincides with that for the tree-level amplitude. Second, our calculation of the one-loop four-point amplitude is perhaps more involved than when it is performed in the light-cone operator formalism. However, it has to be recalled that the corresponding calculation in the light-cone formulation was simple only for graphs with very few external legs, ultimately becoming unwieldy for graphs with more than six legs. In our covariant RNS formulation, the difficulty increases, but only slightly so.

4. Higher-point amplitudes and odd-spin structure

Let us now come back to the case of odd-spin structure and derive explicit formulas for scattering amplitudes of massless particles. There are three additional complications as compared to the even-spin structure case. First, we have an odd modulus to integrate over (constant χ), second there is a (constant) Dirac zero mode, third since there is a Dirac zero mode, the chiral amplitude analogous to $\mathcal{F}$ (but now with massless vertex insertions) is no longer holomorphic in τ and χ, but there are mixed terms. We shall tackle these issues by evaluating the matter contribution of the path integral with generating functions for massless operators inserted at points z_i,θ_i, which we do not integrate over.

We begin with the nonchiral amplitude for odd-spin structure

$$\mathcal{A}=\mathcal{A}_{\rm sgh}\int D(x\psi)\prod_{i=1}^n\exp(ik_i^\mu X^\mu+\zeta_i^\mu\mathcal{D}_+X^\mu+\bar\zeta_i^\mu\mathcal{D}_-X^\mu)e^{-I_m} . \quad (3.286)$$

Recall that the superghost contribution was unity for odd-spin structure: $\mathcal{A}_{\rm sgh}=1$.

Care has to be taken to include the full superderivatives in this expression, since the χ field does not vanish now. To be specific, if $X^\mu=x^\mu+\theta\psi_+^\mu+\theta\psi_-^\mu+i\theta\bar\theta F^\mu$, we get

$$\mathcal{D}_+X=\psi_++i\bar\theta F+\theta(\partial_zx+\tfrac{1}{2}\chi_z{}^-\psi_-)+\theta\bar\theta(-\tfrac{1}{4}\chi_{\bar z}{}^+\chi_z{}^-\psi_++\tfrac{1}{2}\chi_z{}^-\partial_{\bar z}x+D_z\psi_+) \quad (3.287)$$

and $\mathcal{D}_- X$ is its complex conjugate.

Integration over the x field is performed as before, and we find a complicated expression due to the presence of several contributions from the vertex. However, there is a remarkable partial cancellation with the ψ_+- and ψ_--dependent terms in the vertex, which considerably simplifies the final answer. Some further partial contractions of fermionic insertions ultimately lead to

$$\mathcal{A}=(2\pi)^{10}\delta(k)\left[\frac{4\pi^2\det'\Delta}{(\tau_2)^2}\right]^{-5} e^{\mathcal{L}^0_+ + \mathcal{L}^0_- + \mathcal{L}^1_+ + \mathcal{L}^1_-}(\mathrm{Im}\tau)^{-5}$$
$$\times\int D\psi_+ D\psi_- e^{\mathcal{L}^2_+ + \mathcal{L}^2_-}$$
$$\times\exp\left[-\frac{2\pi}{\tau_2}(\mathrm{Im}\sigma^\mu + ik_i^\mu \mathrm{Im} z_i)^2\right] . \tag{3.288}$$

We have also used the following abbreviations:

$$\mathcal{L}^0_+ = \sum_{i<j} k_i\cdot k_j \ln E(z_i,z_j) ,$$
$$\mathcal{L}^1_+ = \sum_{i<j} [\, \theta_i\theta_j \zeta_i\cdot\zeta_j \partial_{z_i}\partial_{z_j}\ln E(z_i,z_j)$$
$$-2ik_i\cdot\zeta_j\theta_j\partial_{z_j}\ln E(z_i,z_j)] , \tag{3.289}$$
$$\mathcal{L}^2_+ = \sum_i \Bigg[ik_i^\mu\theta_i\psi_+^\mu(z_i) - \frac{1}{4\pi}\zeta_i^\mu\theta_i\int d^2w\, \chi_{\bar z}{}^+\psi_+^\mu(w)\partial_{z_i}\partial_w \ln E(z_i,w)$$
$$+\zeta_i^\mu\psi_+^\mu(z_i) - \frac{1}{4\pi}k_i^\mu\int d^2w\,\chi_{\bar z}{}^+\psi_+^\mu(w)\partial_w\ln E(z_i,w)\Bigg] .$$

Note that $\mathcal{L}^0_+$ and $\mathcal{L}^1_+$ are independent of the fermion field, whereas $\mathcal{L}^2_+$ is chiral in the sense used throughout. We have also defined

$$\sigma^\mu = \frac{1}{4\pi}\int d^2z\,\chi_{\bar z}{}^+\psi_+^\mu(z) - \sum_i \zeta_i^\mu\theta_i . \tag{3.290}$$

Since $\chi_{\bar z}{}^+$ is a constant, only the zero mode ψ^0_+ of ψ_+ contributes to σ^μ. Notice that the amplitude $\mathcal{A}$ is chirally split in ψ_+ and ψ_-, except for its zero modes. Thus it is necessary to isolate these zero modes explicitly, which is achieved by splitting $\psi_\pm = \psi'_\pm + \psi^0_\pm$. The contractions of the nonzero modes must then be performed with the propagator $\tilde S_0$ of Eq. (3.266), which is indeed orthogonal to constants. One readily finds that

$$\mathcal{A}=(2\pi)^{10}\delta(k)(\mathrm{Im}\tau)^{-5} e^{\mathcal{L}^0_+ + \mathcal{L}^0_- + \mathcal{L}^1_+ + \mathcal{L}^1_- + \mathcal{L}^3_+ + \mathcal{L}^3_-}$$
$$\times\int d\psi_+^{0\mu} d\psi_-^{0\mu} e^{Z}\prod_{i=1}^n e^{v_i^\mu\psi_+^{0\mu} + \bar v_i^\mu\psi_-^{0\mu}} , \tag{3.291}$$

where we use the abbreviation $v_i^\mu = ik_i^\mu\theta_i + \zeta_i^\mu\theta_i$,

$$Z = -\frac{2\pi}{\tau_2}\sum_i [-\mathrm{Im}(\zeta_i^\mu\theta_i) + ik_i^\mu \mathrm{Im} z_i$$
$$+(v_i^\mu\chi_{\bar z}{}^+ + \bar v_i^\mu\chi_z{}^-)\mathrm{Im} z_i]^2 . \tag{3.292}$$

Contraction of the ψ_+-dependent terms $\mathcal{L}^2_+$ produces also a chiral part $\mathcal{L}^3_+ = \frac{1}{2}\langle\mathcal{L}^2_+\mathcal{L}^2_+\rangle$, where all the ψ contractions have been carried out with the propagator S_0 instead of $\tilde S_0$. This function is explicitly given by

$$\mathcal{L}^3_+ = \sum_{ij}\Bigg[-\tfrac{1}{2}v_i^\mu v_j^\mu S_0(z_i,z_j) + \frac{1}{4\pi}v_i^\mu\zeta_j^\mu\theta_j\chi_{\bar z}{}^+\int d^2w\,\partial_{z_j}\partial_w\ln E(z_i,w)S_0(z_i,w)$$
$$+\frac{1}{4\pi}v_i^\mu k_j^\mu\chi_{\bar z}{}^+\int d^2w\,\partial_w \ln E(z_j,w)S_0(z_i,w)\Bigg] . \tag{3.293}$$

The nonmanifestly chiral terms arising in the full contraction have been lumped into Z. Now we see that the amplitude again splits when we introduce the internal momenta p^μ. Putting all these together, we find

$$\mathcal{A}=(2\pi)^{10}\delta(k)\int_{\Im} dp^\mu \int d\psi_+^{0\mu} d\psi_-^{0\mu}\mathcal{F}_0\bar{\mathcal{F}}_0 , \tag{3.294}$$

where the reduced chiral amplitude $\mathcal{F}_0$ is given by

$$\mathcal{F}_0 = \exp\Bigg[\mathcal{L}^0_+ + \mathcal{L}^1_+ + \mathcal{L}^3_+ + i\pi\tau p^2 + 2\pi p^\mu\zeta_p^\mu + \psi_+^{0\mu}\sum_i v_i^\mu\Bigg] , \tag{3.295}$$

where

$$\zeta_p^\mu = \sum_i(-\zeta_i^\mu\theta_i + ik_i^\mu z_i - \chi_{\bar z}{}^+ v_i^\mu z_i) . \tag{3.296}$$

Note that the presence of the zero-mode integral ensures that $\sum_i v_i^\mu$ vanishes at all intermediate steps in the derivation of those formulas. Furthermore, it guarantees that ζ_p^μ be invariant under overall translations in z_i, as it should.

This answer looks rather complicated, but in fact the combination of $\mathcal{L}^0_+$, $\mathcal{L}^1_+$, and $\mathcal{L}^3_+$ can be obtained from a very simple recipe. Start with the functional integral (3.286), but instead of using full superderivatives $\mathcal{D}_+$ and $\mathcal{D}_-$, rather use the flat superderivatives ∂_+ and ∂_- alone, and use the propagators G_0 and S_0 instead of the full propagators. Also ignore all possible complications that could arise because of zero modes to the various fields. Thus we can symbolically write

$$\exp(\mathcal{L}_+^0+\mathcal{L}_+^1+\mathcal{L}_+^3)=\left\langle e^{-I_m^1}\prod_{i=1}^n e^{ik_i^\mu X_+^\mu+\zeta_i^\mu \partial_+ X_+^\mu}\right\rangle , \tag{3.297}$$

where all the fields and propagators are now "chiral"

$$\begin{aligned} &X_+^\mu = x_+^\mu + \theta\psi_+^\mu , \\ &\partial_+ X_+^\mu = \psi_+^\mu + \theta\partial_z x_+^\mu , \\ &\langle x_+(z)x_+(w)\rangle = -\ln E(z,w) = G_1(z,w) , \\ &\langle \psi_+(z)\psi_+(w)\rangle = \partial_z \ln E(z,w) = S_0(z,w) . \end{aligned} \tag{3.298}$$

In fact, one may also introduce a full chiral superfield propagator, including the effects of the supermodulus

$$\begin{aligned} G_0(\mathbf{z},\mathbf{w}) &= \langle X_+(\mathbf{z})X_+(\mathbf{w})\rangle_{\text{full}} \\ &= -\ln\frac{\vartheta_1(z-w-\theta_z\theta_w,\tilde{\tau})}{\vartheta_1'(0,\tilde{\tau})} , \end{aligned} \tag{3.299}$$

where $\tilde{\tau}=\tau-\chi_{\bar{z}}{}^+(\theta_z+\theta_w)$. The amplitude is then given by

$$\begin{aligned} \mathcal{L}_+^0+\mathcal{L}_+^1+\mathcal{L}_+^3 = &-\tfrac{1}{2}(ik_i^\mu+\zeta_i^\mu\partial_+^i)(ik_j^\mu+\zeta_j^\mu\partial_+^j) \\ &\times G_0(\mathbf{z}_i,\mathbf{z}_j) . \end{aligned} \tag{3.300}$$

One-loop amplitudes for four-graviton scattering have been computed in the operator formalism by Green and Schwarz (1982) and Schwarz (1982) for the type-II string. Space-time supersymmetry breaking to one-loop order was investigated by Rohm (1984). For the heterotic string, one-loop four-point functions were calculated by Gross *et al.* (1986) and Yashikozawa (1986, 1987) for gauge bosons, Sakai and Tanii (1987) for gravitons, and Cai and Nunez (1987) for gravitons, gauge bosons, and antisymmetric tensor fields. The first two works rely on the operator method, the third on path integrals. Our present method based on path integrals is more complicated than the operator method for a small number of external states (up to six), but it remains tractable as that number increases.

Issues of modular invariance are addressed by Witten (1984), Arnaudon *et al.* (1987), Gliozzi (1987), and Parkes (1987). Generating functions for anomalies as modular forms are introduced in Schellekens and Warner (1986, 1987), Pilch, Schellekens, and Warner (1987), and Witten (1987). Nonrenormalization theorems were stated in Martinec (1986) and shown explicitly to apply in the one-loop case by Tanii (1985, 1986), and Namazie, Narain, and Sarmidi (1986). The hexagon anomaly was shown to vanish to one loop in the heterotic string for gauge groups Spin(32)/Z_2 and $E_8\times E_8$ by Gross and Mende (1987a). The "supertheta" function of Eq. (3.299) also occurs in Freund and Rabin (1988).

Open-string amplitudes to one loop are discussed in the report of Schwarz (1982) and more recently in Frampton, Moxhay, and Ng (1985), Clavelli (1986), Frampton, Kikuchi, and Ng (1986), Burgess (1987), and Kostelecky, Lechtenfeld, and Samuel (1987).

N. Heterotic strings

The heterotic string was constructed by Gross, Harvey, Martinec, and Rohm (1985a, 1985b, 1986) as a hybrid of one chiral half of the type-II string (say left chirality) and one half of the closed bosonic string, compactified on a 16-dimensional torus T^{16}. As a string theory, it lives in ten space-time dimensions, and we may alternatively regard it as a theory of ten bosonic degrees of freedom x^μ, ten Majorana-Weyl worldsheet spinors ψ_+^μ (left chirality), and a number of fields representing the internal degrees of freedom. These could be 16 bosonic (right-chirality) x^a or, when fermionized, 32 right-chirality Majorana-Weyl spinors ψ_-^a. It is in terms of the latter that we had written the heterotic string worldsheet action of Eq. (3.3). We shall repeat it here for convenience:

$$\begin{aligned} I_m &= I_H + I_i , \\ I_H &= \frac{1}{4\pi}\int_M d^2\xi\sqrt{g}\,(D_z x^\mu D_{\bar{z}} x^\mu - \psi_+^\mu D_{\bar{z}}\psi_+^\mu \\ &\qquad + \chi_{\bar{z}}{}^+\psi_+^\mu D_z x^\mu) , \end{aligned} \tag{3.301}$$

where I_i is the action for the internal degrees of freedom,

$$I_i = \frac{1}{4\pi}\int_M d^2\xi\sqrt{g}\,(-\psi_-^a D_z\psi_-^a), \quad a=1,\ldots,32 \tag{3.302a}$$

when written in fermionic representation, and

$$I_i = \frac{1}{4\pi}\int_M d^2\xi\sqrt{g}\,D_z x^a D_{\bar{z}} x^a, \quad a=1,\ldots,16 \tag{3.302b}$$

when written in bosonic representation, and it is understood that only the chiral halves of the bosonic contributions are kept. This action exhibits $N=\frac{1}{2}$ local supersymmetry invariance and may be quantized as a supergravity theory in its own right. In Sec. III.N.1 we shall give a brief account of this approach, without entering into any details. Instead we shall rather study the heterotic string as a cross breeding of half a type-II string and half a (partially) compactified bosonic string. An advantage of the latter approach is that we can gain direct information about the torus T^{16} or, equivalently, about the lattice[24] Λ out of which the torus is constructed: $T^{16}=R^{16}/\Lambda$. This second approach will be discussed more extensively in Sec. III.N.2. Incidentally, it has already been stressed, when discussing super Weyl anomalies, that in a worldsheet chirality-nonconserving theory, super Weyl invariance must cancel for both left and right chiralities separately. This is equivalent to cancellation of super Weyl and local U(1) anomalies of the whole theory. Clearly, this requires that 16 internal bosons x^a or 32 internal Majorana-Weyl fermions ψ_-^a be present, as discussed before, when the critical dimension is $d=10$. The structure of the lattice Λ is at this point

[24]We shall always assume that Λ is indeed 16 dimensional.

left open, and will be narrowed down—through insistence on modular invariance—to the root lattice of $E_8 \times E_8$ or of $\mathrm{Spin}(32)/Z_2$.

1. $N=\frac{1}{2}$ supergeometry

$N=\frac{1}{2}$ superspace is parametrized by two commuting coordinates ξ and $\bar{\xi}$ and one anticommuting θ, collected into a supercoordinate $z^M=(\xi,\bar{\xi},\theta)$. The U(1) frame is similarly reduced to $A=(z,\bar{z},+)$. Covariant derivatives, torsion, and curvature are defined as in Eqs. (3.7) and (3.10), but the torsion constraints (3.11) are now restricted to the $A=(z,\bar{z},+)$. Using the Bianchi identities, one then has

$$T_{++}{}^{+}=T_{\bar{z}+}{}^{A}=T_{z+}{}^{+}=T_{++}{}^{\bar{z}}=0, \quad T_{++}{}^{z}=2 ,$$
$$T_{z\bar{z}}{}^{+}=-\frac{i}{2}R_{\bar{z}+}, \quad R_{z\bar{z}}=-\mathcal{D}_{+}R_{\bar{z}+}, \quad R_{++}=R_{z+}=0 , \tag{3.303}$$

so that all components of the torsion and curvature are expressible in terms of $R_{\bar{z}+}$. The transformation laws of these fields under super-reparametrizations, super Weyl transformations, and local U(1) transformations may be readily obtained by restriction of the $N=1$ case, and we shall not rewrite them here.

A difficult feature of the $N=\frac{1}{2}$ supergeometry is that the supercurvature field $R_{\bar{z}+}$ now has U(1) weight-$\frac{1}{2}$ and is anticommuting, so that there is no sense to setting it to a constant other than zero. In view of the super Gauss-Bonnet formula analogous to that for $N=1$ supergeometry, $R_{\bar{z}+}$ should not vanish whenever $\chi(M)\neq 0$. Asking $R_{\bar{z}+}$ to be covariantly constant now leads to nontrivial differential equations. Thus it is not clear in the case of heterotic geometry how the geometric ideas discussed in the case of $N=1$ supergeometry can be implemented; as a matter of fact, it is not clear that they can be.

The superspace action for the heterotic string is

$$I=\frac{1}{4\pi}\int d^2\xi\, d\theta(\mathrm{sdet}E_M{}^{A})(\mathcal{D}_{+}X^{\mu}\mathcal{D}_{\bar{z}}X^{\mu}+\Psi^{a}\mathcal{D}_{+}\Psi^{a}) , \tag{3.304}$$

where X^{μ} is the even superfield $X^{\mu}(\xi,\bar{\xi};\theta)=x^{\mu}+\theta\psi^{\mu}_{+}$ and Ψ^a is the odd superfield $\Psi^a=\psi^a_{-}+\bar{\theta}F^a$, with ψ^{μ}_{+} the space-time fermions, ψ^a_{-} the internal fermions, and F^a an auxiliary field.

$N=\frac{1}{2}$ supergeometry was investigated by Hull and Witten (1985), Brooks, Muhammad and Gates (1986), Gates, Brooks, and Muhammad (1987), Nelson and Moore (1986), and Evans and Ovrut (1986a, 1986b, 1987).

2. Heterosis

The fundamental idea behind heterosis is that the left- and right-moving degrees of freedom on the worldsheet are described by independent degrees of freedom, sharing only their common overall momentum. The notion of left- and right-movers may be understood on a compact surface with a metric of Euclidean signature as analytic and antianalytic, or for fermionic degrees of freedom of course as left and right chirality. Unfortunately, the notions of left- and right-movers or analytic and antianalytic are defined only when the fields satisfy their equations of motion. They do not *a priori* make sense in a functional integral formulation where all fields are to be integrated over. This is especially a problem for the bosonic fields x^{μ} or x^{a} which are real.

In our discussion of the type-II string, we have already had to separate left- and right-chirality components in order to endow them with separate spin structures. We have actually achieved much more. When loop momenta p_I^{μ} are fixed, and for a fixed point in supermoduli space, the integrand splits as a function that is analytic in the period matrix Ω_{IJ}, analytic in the positions of the vertex insertions z_i, and dependent only on $\chi_{\bar{z}}{}^{+}$, times its complex conjugate.[25] This chiral splitting at fixed internal momenta will be reconsidered in much more detail in Sec. VII and identified there with holomorphic splitting at fixed internal momenta on supermoduli space. The holomorphic structure of supermoduli space is that introduced in Sec. III.G, and it will be shown in Sec. VII that Ω_{IJ} and $\chi_{\bar{z}}{}^{+}$ are holomorphic coordinates for supermoduli. This holomorphic splitting points to a way of identifying the contributions of the right-movers in the bosonic x^{μ}. In fact, the closed bosonic string amplitudes could be split in a similar fashion, even though x^{μ} is not a chiral field. Again, at fixed internal momenta, the integrand is the absolute-value square of a function analytic in Ω_{IJ} and in the positions of the vertex operator insertions z_i. (Of course we will have to check that this kind of splitting continues to hold when the closed bosonic string is compactified on a torus T^{16}.) The right-movers' contributions can now be taken to be the antiholomorphic factor. The vertex operators (at fixed positions) for the heterotic string are similarly constructed of half a type-II vertex and half a bosonic vertex. Actually, this is not quite so, because each contains pieces of both chirality. However, in the end, all pieces can be put together and split when the internal momenta on the string are kept fixed.

Thus the recipe for heterosis will be to take the left chiral half of the type-II string and the right chiral half of the bosonic string at the *same internal momenta* and to multiply them together and integrate over the internal momenta.

That this prescription is the correct one is confirmed by the fact that it alone will reproduce quantized $N=\frac{1}{2}$

[25]Recall that the complex conjugate is in general evaluated for a different spin structure. Also recall that all momenta—internal p_I^{μ} and external k_i^{μ}—have been analytically continued to imaginary values.

supergravity from the chirally split type-II superstring. Indeed, the amplitudes for the heterotic geometry may be gotten by setting $\chi_{\bar z}{}^{+}=0$, so that we can read off from Eq. (3.196) that an exponential insertion would give

$$\mathcal{A}_H=(2\pi)^{10}\delta(k)\int_{\mathfrak{F}} dp_I^\mu \mathcal{F}_\nu(z_i,\psi_+,\Omega,\chi;p_I^\mu)\times\overline{\mathcal{B}_{10}(z_i,\Omega;p_I^\mu)}\,\overline{\mathcal{B}_{16}(z_i,\Omega)}\ , \qquad (3.305)$$

where $\mathcal{F}_\nu$ was defined in Eq. (3.197) and the ten-dimensional chiral bosonic amplitude is given by

$$\mathcal{B}_{10}(z_i,\Omega;p_I^\mu)=Z_\Delta(\Omega)^{-10}\prod_{i<j}E(z_i,z_j)^{k_i\cdot k_j}\times\exp\left[i\pi p_I^\mu\Omega_{IJ}p_J^\mu+2\pi i p_I^\mu k_i^\mu\int_P^{z_i}\omega_I\right]. \qquad (3.306)$$

The symbols are the same as in the case of the type-II string analysis of Sec. III.K.

We shall now derive an expression for the contribution to the amplitude $\mathcal{B}_{16}$ of the internal degrees of freedom. We begin with the fermionic representation, described by the action (3.302a). For convenience, we shall consider its complex conjugate, so as to obtain $\mathcal{B}_{16}$ directly. We shall also restrict ourselves to considering only insertions of ψ^a and not its derivatives, which is enough for the case of vertex operators for massless particles. Furthermore, all 32 fermions ψ^a are decoupled from one another, so we shall evaluate the contributions of a single one first, endowed with spin structure $\bar\nu$. Actually, the 32 fermions were understood to be Majorana-Weyl, which is not realizable on a worldsheet with Euclidean signature. Thus we shall pair them two by two and endow these with the same spin structure. We then have

$$\mathcal{B}_{\bar\nu}^1=\int_{\bar\nu}D\psi\prod_{i=1}^n e^{\eta_i\psi(z_i)}e^{-I_\psi}\ . \qquad (3.307)$$

For even-spin structure, this integral has no zero modes, and we get

$$\mathcal{B}_{\bar\nu}^1=(\det\bar{\partial}\!\!\!/_+)_{\bar\nu}\exp\left[+\tfrac{1}{2}\sum_{ij}\eta_i\eta_j S_{\bar\nu}(z_i,z_j)\right], \qquad (3.308)$$

where $S_{\bar\nu}$ is the Dirac propagator already encountered in Eq. (3.202) and given by the Szegö kernel. The Dirac determinant will be evaluated using bosonization methods in Sec. VII, and we just quote here the answer from Eq. (7.61):

$$(\det\bar{\partial}\!\!\!/_+)_{\bar\nu}=Z_\Delta(\Omega)^{-1}\vartheta[\bar\nu](0,\Omega)\ , \qquad (3.309)$$

very much in analogy with the one-loop formula of Eq. (3.253).

For odd-spin structure, there is generically one zero-mode $h_{\bar\nu}$, and the chiral Dirac propagator is given by Eq. (3.206). Thus

$$\mathcal{B}_\nu^1=\int d\psi^0\int D\psi' e^{-I_\psi}\exp\left[\sum_i\eta_i\psi^0\right]\prod_{i=1}^n e^{\eta_i\psi'(z_i)}=\left[\sum_i\eta_i h_{\bar\nu}(z_i)\right](\det{}'\bar{\partial}\!\!\!/_+)_{\bar\nu}\times\exp\left[+\tfrac{1}{2}\sum_{ij}\eta_i\eta_j\tilde S_{\bar\nu}(z_i,z_j)\right], \qquad (3.310)$$

where $(\det'\bar{\partial}\!\!\!/_+)_{\bar\nu}$ is the chiral half of $(\det'\bar{\partial}\!\!\!/\,/\langle h_\nu\mid h_\nu\rangle)_{\bar\nu}$. Due to the overall factor linear in η, $\tilde S_{\bar\nu}$ in the exponential is equivalent to $S_{\bar\nu}(z,w)$ of Eq. (3.204) in view of Eq. (3.205), so that Eq. (3.310) is analytic in z_i, η_i, and Ω, but also well defined on the surface.

Now that we have evaluated the contribution of a single (complex) fermion, it remains to put the 16 copies together. This must be done in a modular-invariant fashion. Recall that in the type-II string one had to sum independently over the spin structures assigned to left and right chirality. Each chirality sector was responsible for a space-time supersymmetry, all by itself, so that the theory exhibits $N=2$ supersymmetry. In the heterotic string, left and right chiralities are very different objects, and one could sum separately over the spin structures of left and right chirality, where right chirality now encompasses the internal degrees of freedom. One might also imagine linking the spin structure sum for left and right chirality. In the latter case, it should be expected that space-time supersymmetry would be destroyed. This leaves open a vast class of possibilities, which is narrowed down by the requirement of modular invariance and spin statistics. Seiberg and Witten (1986) have argued that modular invariance requires the fermions ψ^a to have the same spin structure in groups of eight (or four of our complexified ones). This eight is familiar from the modular transformation properties of the ϑ function, which always involves an eighth root of unity. This indeed occurs when the ψ's all carry a space-time index. However, in that case, they describe both bosonic and fermionic space-time degrees of freedom. Since internal ψ^a's should describe only space-time bosonic degrees of freedom, the ψ^a's should actually have the same spin structure in groups of 16 (or eight of our complexified ones). Hence the internal degrees of freedom must exhibit a symmetry that contains SO(16)×SO(16).

When the spin structure of left and right chirality are intertwined in a nontrivial fashion, one will in fact obtain an SO(16)×SO(16) string that is modular invariant (at least to one loop) but not supersymmetric. This type of string theory was investigated by Dixon and Harvey (1986), Seiberg and Witten (1986), and Alvarez-Gaumé *et al.* (1986). Its compactifications were explored by Ginsparg and Vafa (1987).

On the other hand, if spin structures for left and right chirality are summed over independently, then $N=1$ supersymmetry is maintained. The general expression for the internal amplitude is

$$\mathcal{B}_{16}=\sum_{\bar\nu_1\bar\nu_2}C_{\bar\nu_1\bar\nu_2}(\mathcal{B}_{\bar\nu_1}^1)^8(\mathcal{B}_{\bar\nu_2}^1)^8\ . \qquad (3.311)$$

Under a modular transformation

$$M=\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

even- and odd-spin structures are mapped into themselves, so we may limit our discussion to the even case. Hence

$$M(\mathcal{B}_{16})=[\det(C\Omega+D)]^{-8}\sum_{\bar v_1\bar v_2} C_{\bar v_1\bar v_2}(\mathcal{B}^1_{M\bar v_1})^8(\mathcal{B}^1_{M\bar v_2})^8$$
$$=[\det(C\Omega+D)]^{-8}\sum_{\bar v_1\bar v_2} C_{M^{-1}\bar v_1 M^{-1}\bar v_2}(\mathcal{B}^1_{\bar v_1})^8(\mathcal{B}^1_{\bar v_2})^8 \quad (3.312)$$

and

$$C_{\bar v_1\bar v_2}=C_{M^{-1}\bar v_1 M^{-1}\bar v_2} \quad (3.313)$$

for all M. If $\bar v_1\neq\bar v_2$, then let M fix $\bar v_1$. This reduces the modular group from Sp$(2h,\mathbf{Z})$ to Sp$(2h-2,\mathbf{Z})$. This is enough for us to see that

$$C_{\bar v_1\bar v_2}=C_{\bar v_1\bar v_2'} \quad \text{if } \bar v_2\neq\bar v_1\neq\bar v_2' .$$

Since C is symmetric, all off-diagonal elements in C must be equal. On the other hand, taking $\bar v_1=\bar v_2$, we see that all on-diagonal elements must be equal as well. Thus there are two independent solutions. All $C_{\bar v_1\bar v_2}$ are equal for all $\bar v_1$ and $\bar v_2$, and

$$\mathcal{B}_{16}=\left[\sum_{\bar v}(\mathcal{B}^1_{\bar v})^8\right]^2 \quad (3.314\text{a})$$

or all off-diagonal elements of C vanish, so that

$$\mathcal{B}'_{16}=\sum_{\bar v}(\mathcal{B}^1_{\bar v})^{16} . \quad (3.314\text{b})$$

In the latter case, we see that all ψ^a's are endowed with the same spin structure, thus exhibiting Spin(32) symmetry.

Now let us consider the one-loop partition function only, and evaluate the above partial amplitudes:

$$\sum_{\bar v}(\mathcal{B}^1_{\bar v})^8=\vartheta^8_{00}(0,\tau)+\vartheta^8_{01}(0,\tau)+\vartheta^8_{10}(0,\tau) . \quad (3.315)$$

This is a modular form of weight 4. With the help of Jacobi's theorem on the number $r_4(n)$ of representations of an integer n as a sum of four squares $r_4(n)=8\sigma_1(n)$ one easily finds that the above sum of three theta functions equals

$$1+240\sum_n \sigma_3(n)e^{i\pi n\tau}, \quad \sigma_\alpha(n)=\sum_{d\mid n} d^\alpha , \quad (3.316)$$

which is the theta function for the root lattice of E_8. Hence $\mathcal{B}_{16}$ is the amplitude for the group $E_8\times E_8$, and $\mathcal{B}'_{16}$ for Spin(32)/Z_2.

Next we derive an expression for the contribution of internal degrees of freedom to the same amplitude $\mathcal{B}_{16}$ in terms of the bosonic variable x^a. When we compactify the closed bosonic string on a torus T^{16}, the $x^a(z)$ field is no longer single valued, but is shifted by a lattice vector of Λ as z moves around a homology cycle,

$$x^a(\gamma z)=x^a(z)+T^a_\gamma, \quad T^a_\gamma\in\Lambda .$$

We may interpolate T^a_γ with the use of a harmonic function $T^a_\gamma(z)$ and introduce a single-valued field y^a,

$$x^a(z)=y^a(z)+T^a_\gamma(z) .$$

Hence we can represent the differential dx^a as

$$dx^a(z)=dy^a(z)+\sum_I[m^a_I h^A_I(z)+n^a_I h^B_I(z)] , \quad (3.317)$$

where h^A and h^B are harmonic (real) one-forms, normalized to

$$\oint_{A_I}h^A_J=\oint_{B_I}h^B_J=\delta_{IJ}, \quad \oint_{A_I}h^B_J=\oint_{B_I}h^A_J=0 .$$

The vectors m^a_I and n^a_I belong to Λ and determine the winding number of the Riemann surface in T^{16}. The action in a given topological sector is now easily computed, and one finds

$$I_x(x)=I_x(y)+\frac{\pi}{2}(n^a_I-m^a_K\bar\Omega_{KI})(\mathrm{Im}\Omega)^{-1}_{IJ}(n^a_J-\Omega_{JL}m^a_L) . \quad (3.318)$$

We are now going to make the following assumptions concerning m_I and n_I and the lattice they lie on. First, we assume that m_I and n_I run throughout the full lattice. Hence, if λ_α are the 16 basis vectors generating Λ, then

$$m^a_I=m^\alpha_I\lambda^a_\alpha, \quad n^a_I=n^\alpha_I\lambda^a_\alpha ,$$

where m^α_I and n^α_I run over all integers. We shall denote the lattice metric by $g_{\alpha\beta}=\lambda_\alpha\cdot\lambda_\beta$, and furthermore restrict ourselves to lattices for which the volume of the unit cell is one: $\det g_{\alpha\beta}=1$. Finally, we assume that the entries of $g_{\alpha\beta}$ are integers; since $\det g_{\alpha\beta}=1$, this means that $g^{\alpha\beta}$ also has integer entries. When all the above requirements are met, then the amplitude

$$\mathcal{A}=\int Dx^a\prod_{i=1}^n e^{iK^a_i x^a(z_i)}e^{-I_x} \quad (3.319)$$

will be Weyl invariant, provided the external momenta satisfy $K_i^2=2$ so that the lattice must be even. The lattice metric $g_{\alpha\beta}$ can now be viewed as the Cartan matrix of a Lie algebra, and since $g_{\alpha\beta}$ is symmetric, the possible Lie algebras are SO$(2n)$ and E^8 or products thereof. The amplitude $\mathcal{A}$ is easily worked out:[26]

$$\mathcal{A}=(2\pi)^{16}\delta(K)\int dP^a_I\sum_{a_I,b_I}\left|\mathcal{B}\begin{bmatrix}\delta'_I\\ \delta''_I\end{bmatrix}(z_i,\Omega,P^\mu_I)\right|^2 . \quad (3.320)$$

Here δ'^α_I and δ''^α_I are half-order characteristics and take values 0 and $\frac{1}{2}$. The reduced amplitude is given by

[26]This formula is a special case of toroidal compactifications considered in collaboration with V. Periwal.

$$\mathcal{B}\begin{bmatrix}\delta'\\ \delta''\end{bmatrix}(z_i,\Omega;P_I^\mu)=Z_\Delta(\Omega)^{-16}\prod_{i<j}E(z_i,z_j)^{K_i\cdot K_j}\exp\left[i\pi P_I^a\Omega_{IJ}P_J^a+2\pi i P_I\cdot K_i\int_P^{z_i}\omega_I\right]\vartheta_\Lambda\begin{bmatrix}\delta'\\ \delta''\end{bmatrix}\left[K_i^a\int_P^{z_i}\omega_I,\Omega\right]. \quad (3.321)$$

Here the theta function for the lattice Λ with characteristics $\delta_I'^{\alpha}$ and $\delta_I''^{\alpha}$ is defined by

$$\vartheta_\Lambda\begin{bmatrix}\delta'\\ \delta''\end{bmatrix}(z_I^\alpha,\Omega)=\sum_{m_I^\alpha}\exp[i\pi(m_I^\alpha+\delta_I'^\alpha)\Omega_{IJ}g_{\alpha\beta}(m_J^\beta+\delta_J'^\beta)+2\pi i g_{\alpha\beta}(m_I^\beta+\delta_I'^\beta)(\delta_I''^\alpha+z_I^\alpha)]\,. \quad (3.322)$$

Under a modular transformation, we have

$$\vartheta_\Lambda\begin{bmatrix}D\delta'-C\delta''\\ -B\delta'+A\delta''\end{bmatrix}((C\Omega+D)^{-1}z,\Omega')=\zeta[\det(C\Omega+D)]^8 e^{i\pi z(C\Omega+D)^{-1}Cz}\vartheta_{\Lambda'}\begin{bmatrix}\delta'\\ \delta''\end{bmatrix}(z,\Omega)\,,$$

where the image of Ω under the modular transformation Ω' is given by

$$\Omega'=(A\Omega+B)(C\Omega+D)^{-1}\,.$$

Moreover, Λ' is the lattice dual to Λ and

$$\zeta=\exp(-i\pi\delta'^{\mathrm{T}}B^{\mathrm{T}}D\delta'-i\pi\delta''^{\mathrm{T}}A^{\mathrm{T}}C\delta''+2\pi i\delta'^{\mathrm{T}}B^{\mathrm{T}}C\delta'')\,.$$

To have modular invariance, one clearly requires that the lattice be self-dual, $\Lambda'=\Lambda$, and that $\zeta=1$. The latter must hold for all modular transformations. The lattice must also be even, and this restricts the choice to $E_8\times E_8$ and Spin(32)/$Z(2)$ and causes the characteristics to vanish: $\delta'=\delta''=0$.

O. Inverse heterosis and general structure of amplitudes

Although the heterotic string was originally defined as the hybrid between a chiral half of the type-II string and another chiral half of the bosonic string, we have repeatedly witnessed the emergence of simplicity when working with the heterotic string directly. When calculations are initiated with the heterotic string action I_H of Eq. (3.301), using vertex operator insertions of the heterotic string for the relevant chirality, the final amplitude could be directly recast as an integral over internal momenta of the known ten-dimensional bosonic right-chirality part, times the left chirality of the heterotic string—alias the type-II string. Thus it appears that in practice the simplest way to compute in the heterotic or in the type-II superstring is to begin with Eq. (3.301) and, for fixed internal momenta, to decompose the amplitude into left- and right-chirality components.

Scattering amplitudes at fixed internal momenta, fixed supermoduli, and fixed spin structure are easily defined by inserting the constraints fixing the various internal momenta as in Eq. (2.97) for the bosonic string:

$$\langle V_1\cdots V_n\rangle(m_K,\overline{m}_K;p_I^\mu)=\int D(x\psi bc\beta\gamma)\prod_k|\langle\mu_k|B\rangle|^2\prod_b|\delta(\langle\mu_b|B\rangle)|^2V_1\cdots V_ne^{-I}\prod_{\mu,I}\delta\left[\oint_{A_I}\frac{dz}{2\pi}\partial_zx^\mu-p_I^\mu\right]. \quad (3.323)$$

This amplitude is Weyl invariant, local U(1) invariant, local reparametrization invariant, and local supersymmetry invariant. However, it is not modular invariant because we have picked out a preferred homology basis. Modular invariance is, however, recovered for the full amplitude after integrating over p_I^μ,

$$\langle V_1\cdots V_n\rangle=\int_{s\mathcal{M}_h}d^2m_K\int_{\mathfrak{F}}dp_I^\mu\langle V_1\cdots V_n\rangle(m_K,\overline{m}_K;p_I^\mu)\,. \quad (3.324)$$

The key feature of amplitudes at fixed momenta is that they will factorize as

$$\langle V_1\cdots V_n\rangle(m_K,\overline{m}_K;p_I^\mu)=(2\pi)^{10}\delta(k)|\mathcal{F}_\nu(m_K,z_i,\zeta;p_I^\mu)|^2\,, \quad (3.325)$$

where $\mathcal{F}_\nu$ is holomorphic in moduli Ω_{IJ}, odd moduli $\chi_{\bar z}{}^+$, positions of vertex operators z_i, and parameters of the vertex operators ζ_i, and depends on left-chirality spin structure ν. When dealing with the heterotic string, there is an analogous statement,

$$\langle V_1\cdots V_n\rangle(m_K,\overline{m}_k;p_I^\mu)=\int D(x\psi bc\beta\gamma)\prod_k|\langle\mu_k|B\rangle|^2\prod_b\delta(\langle\mu_b|B\rangle)V_1\cdots V_ne^{-I}\prod_{\mu,I}\delta\left[\oint_{A_I}\frac{dz}{2\pi}\partial_zx^\mu-p_I^\mu\right]. \quad (3.326)$$

Here

$$\langle V_1\cdots V_n\rangle(m_K,\overline{m}_k;p_I^\mu)=(2\pi)^{10}\delta(k)\mathcal{F}_\nu(m_K,z_i,\zeta;p_I^\mu)\mathcal{B}(\overline{m}_k,\overline{z}_i,\overline{\zeta};p_I^\mu)\,, \quad (3.327)$$

where $\mathcal{F}_\nu$ is the same as in Eq. (3.325). Since $\mathcal{B}$ is a known quantity, independent of supermoduli, we see that all the information of the type-II or heterotic strings is contained in the heterotic amplitudes.

P. Picture-changing formalism

Since all the information for the type-II and heterotic strings can be extracted from the study of the heterotic string (at fixed internal momenta), we shall restrict our attention to the heterotic string, keeping in mind that we may always fix the internal momenta preserving all symmetries but modular invariance. The formula for the amplitudes is then given in Eq. (3.178), where the heterotic string action is

$$I = I_H + I_{\mathrm{sgh}} \;, \tag{3.328}$$

with I_H given in Eq. (3.301) and I_{sgh} that of Eq. (3.175) but with $\bar\beta = \bar\gamma = 0$. It will be convenient to express it as

$$I = I_0 - \frac{1}{2\pi} \int d^2\xi \sqrt{g}\, \chi_{\bar z}{}^{+} S \tag{3.329}$$

with

$$I_0 = \frac{1}{2\pi} \int d^2\xi \sqrt{g}\, \left(\tfrac{1}{2} D_z x^\mu D_{\bar z} x^\mu - \tfrac{1}{2}\psi_+^\mu D_{\bar z} \psi_+^\mu + b D_{\bar z} c + \bar b D_z \bar c + \beta D_{\bar z} \gamma \right) \tag{3.330}$$

and the full supercurrent

$$S = -\tfrac{1}{2}\psi_+^\mu D_z x^\mu - (D_z \beta) c - \tfrac{3}{2}\beta D_z c + \tfrac{1}{2} b \gamma \;. \tag{3.331}$$

It is always understood that anomalies are appropriately canceled by the presence of the internal degrees of freedom that we suppress here.

There is a BRST invariance inherited from the type-II string, and obtained by restricting Eq. (3.149) to left chirality only, using the bosonic BRST for the right components. In particular,

$$\delta_{\mathrm{BRST}} \beta = -\lambda S \;, \tag{3.332}$$

so that the full supercurrent is BRST invariant.

We shall now follow the treatment of Verlinde and Verlinde (1987a, 1987b) in order to make contact with the formulation of conformal field theory, usually expressed in terms of picture-changing operators. For the time being, our considerations will be local on moduli space; we shall need a better understanding of moduli space, and of its connection to supermoduli space, before being able to attack the global issues in Sec. VII. Locally, we can choose a slice for supermoduli space in which the super Beltrami differentials are of a special type. We assume that

$$\chi_{\bar z}{}^{+} = \sum_{b=1}^{2h-2} m_b \mu_b^0 \;, \tag{3.333}$$

where m_b are the odd moduli and μ_b^0 are super Beltrami differentials independent of m_b. We can take the metric independent of m_b, so that $\mu_b^1 = 0$. This choice is justified by considerable simplifications in the superghost insertions in the ghost functional integral. We now have

$$\langle \mu_b \mid B \rangle = \langle \mu_b^0 \mid \beta \rangle \;. \tag{3.334}$$

By construction, the only thing that depends on odd moduli is $\chi_{\bar z}{}^{+}$ in the action, so that the odd moduli are easily integrated out. One finds

$$\langle V_1 \cdots V_n \rangle (m_K) = \int D(x\psi bc\beta\gamma) \times \prod_k |\langle \mu_k \mid B \rangle|^2 \times \prod_b \delta(\langle \mu_b^0 \mid \beta \rangle) \langle \mu_b^0 \mid S \rangle \times V_1 \cdots V_n e^{-I_0} \;. \tag{3.335}$$

The product

$$Y_{\mu_b} = \delta(\langle \mu_b^0 \mid B \rangle) \langle \mu_b^0 \mid S \rangle \tag{3.336}$$

is a BRST invariant, of ghost number 1, and can formally be thought of as the BRST transform of the step function

$$Y_{\mu_b} = \delta_{\mathrm{BRST}} H(\langle \mu_b^0 \mid \beta \rangle) \;, \tag{3.337}$$

though of course the step function is not well defined. From these quantum-number considerations, and the BRST invariance of Y, it is natural to guess that it is a generalization of the picture-changing operator. Indeed, for general μ_b, the operator Y is nonlocal, but if we choose

$$\mu_b(z) = \delta(z - z_b) \tag{3.338}$$

then it becomes local. Since z_b are points that could depend on moduli, they should properly be viewed as sections of the universal Teichmüller curve of Sec. IV.H. This bundle has no global smooth sections, which is another reason why these considerations should be considered as local on small patches of moduli space. Bosonization of superghost arguments, which we shall not present here (see, however, a brief discussion in Sec. VIII.E), give further evidence that this local version precisely coincides with the picture-changing operator of conformal field theory,

$$Y(z_b) = \delta(\beta(z_b)) S(z_b) = e^{-i\sigma(z_b)} S(z_b) \;, \tag{3.339}$$

where $\sigma(z)$ is the bosonic field of Eq. (8.42). The expression for the general amplitude becomes

$$\langle V_1 \cdots V_n \rangle (m_K) = \int D(x\psi bc\beta\gamma) \times \prod_k |\langle \mu_k \mid B \rangle|^2 \times \prod_b Y(z_b) V_1 \cdots V_n e^{-I_0} \;. \tag{3.340}$$

This amplitude is formally BRST invariant. There are two issues that should, however, be investigated. The first is the usual possible contributions from the boundary of moduli space. The second is what happens when we make a different choice of insertions $\tilde{z}_b$, as will eventually be required by the topology of the universal Teichmüller curve. Using BRST arguments, Verlinde and Verlinde (1987) have argued that the difference will then be a total derivative on the patch where both z_b and $\tilde{z}_b$ are well defined. The effects of such total derivatives will be discussed in Sec. VII.G.

The prescription for the superstring multiloop measure based on BRST invariance and picture-changing operators is presented by Friedan, Martinec, and Shenker (1986) and Martinec (1987). In the path-integral formulation, that the ghost insertions can be recast as picture-changing operators [cf. Eq. (3.336)] was recognized by Witten (1986) in a superstring field theory context and later by Verlinde and Verlinde (1987). The last authors also provide key formulas for the conformal field theory of the superghosts and their bosonization.

IV. PARAMETRIZATIONS OF MODULI SPACE

In the previous sections we have considered the string partition function and scattering amplitudes as integrals over the moduli space $\mathcal{M}_h$ of compact Riemann surfaces or over the moduli space $\mathcal{M}_{h,n}$ of Riemann surfaces with n punctures as in Sec. II.J. These finite-dimensional spaces so far have been given abstract definitions as coset spaces, and it is imperative to describe them in a concrete manner, i.e., to provide some insight into their coordinates, curvatures, and function theory. A diverse choice of such parametrizations is available, and we shall here only describe some of those that have been used for the description of closed-string theory.

First, Riemann surfaces of constant curvature may be uniformized by the round sphere, the Euclidean plane, or the upper half plane. The natural geometry induced on $\mathcal{M}_h$ by such representation is by the Weil-Petersson metric. One of the remarkable aspects of the Weil-Petersson geometry is the abundance of completely explicit formulas, especially concerning the available coordinates and curvature formulas. In particular, we shall see how the Fenchel-Nielsen coordinates provide an elegant parametrization of Teichmüller space and yield explicit formulas for the Weil-Petersson geometry, though it is hard to identify moduli space, i.e., a fundamental domain for the mapping class group. If one is willing to formulate string theory on surfaces with at least one puncture, then the recently developed Penner decomposition (Penner, 1987a, 1987b) provides interesting formulas for the Weil-Petersson geometry on moduli space directly, identifying the boundaries of $\mathcal{M}_{h,n}$ as well.

The parametrization of moduli space with at least two punctures by Mandelstam diagrams has been discovered through string theory. In many ways, this is the parametrization diametrically opposed to constant curvature, since the curvature of Mandelstam diagrams vanishes everywhere except at some isolated interaction points where it is a Dirac δ function.

Another parametrization of the moduli space of surfaces with at least one boundary component, or in general for open strings, is provided by the open-string field theory of Witten (1986a, 1986b); in the mathematics literature it has been discussed independently in the work of Thurston (1980) and Bowditch and Epstein (1988). In a sense, it is analogous to the Penner decomposition, though it does not require constant curvature. We shall not discuss it further here, and instead refer the reader to the work of Witten (1986) and Giddings, Martinec, and Witten (1986), where these constructions were discussed.

Finally, Riemann surfaces may be parametrized by their $h\times h$ complex symmetric period matrix with the advantage of making the dependence on the complex structure of moduli space manifest. Every $h\times h$ complex symmetric matrix is not, however, the period matrix of a Riemann surface. The issue of which matrices do arise as period matrices of a Riemann surface (the so-called Schottky problem) still raises difficult questions. Actually, we shall deal with the complex structure of moduli space in its entirety in a completely separate section, VI.

A. Uniformization for constant-curvature geometry

Given a compact Riemann surface M of genus h and a metric of constant curvature R (recall that any metric is Weyl equivalent to a constant-curvature metric), it follows from the Gauss-Bonnet formula for the Euler number of the Riemann surface that R must be positive for $h=0$, zero for $h=1$, and negative for $h\geq 2$. For definiteness, we shall normalize the metric so that R is 1, 0, or -1, respectively, for the three cases.

The uniformization theorem states that M is isometric to a coset $\tilde{M}/T$, where $\tilde{M}$ is the simply connected covering of M, and Γ is a discrete subgroup of the isometry group of $\tilde{M}$, isomorphic to the first homotopy group of M:

$$\Gamma\sim\pi_1(M)\ .$$

Furthermore, the corresponding simply connected surfaces are unique for $h=0$, $h=1$ and $h\geq 2$, respectively, and are given by

the sphere $\mathbf{C}\cup\{\infty\}$, $ds^2=4(1+|z|^2)^{-2}dz\,d\bar{z}$, with $R=1$, isometry group: SU(2);

the plane $\mathbf{C}$, $ds^2=2\,dz\,d\bar{z}$, with $R=0$, isometry group: $z\mapsto az+b$, $|a|=1$;

the upper half plane, $\mathbf{H}=\{z=x+iy, y>0\}$, $ds^2=2y^{-2}dz\,d\bar{z}$, with $R=-1$, isometry group: SL(2, $\mathbf{R}$).

Actually, for compact surfaces, Γ should have no fixed points inside $\tilde{M}$ (note that Γ may have fixed points on the real line if $\tilde{M}=\mathbf{H}$). Thus, for $h=0$, Γ must be trivial, so for genus 0 there is only one sphere. We shall now study the cases for $h=1$ and $h\geq 2$ separately in the following sections.

B. The genus-1 and genus-2 cases and hyperelliptic surfaces

In the case of genus 1, moduli space and the Weil-Petersson metric can be identified easily. In fact it is readily seen that the only discrete fixed-point free subgroups of the isometry group of the plane are those generated by two translations, in two different directions if the quotient space is to be compact. Choosing two generators, we may characterize the complex structure by their ratio τ, which may be assumed to satisfy $\mathrm{Im}\tau>0$. Different choices of generators lead to changes in τ which are generated by the transformations $\tau\mapsto -1/\tau$ and $\tau\mapsto\tau+1$. Thus the Riemann surface M can be identified with a parallelogram of sides 1 and τ, with opposite sides identified, Teichmüller space is just the upper half-space $\mathbf{H}=\{\tau_1+i\tau_2;\ \tau_2>0\}$, and the mapping class group is $\mathrm{SL}(2,\mathbf{Z})$. Up to a factor of $\frac{1}{2}$[cf. Eq. (2.123)], moduli space corresponds to a fundamental domain for $\mathrm{SL}(2,\mathbf{Z})$ within $\mathbf{H}$, which can be taken as (see Fig. 7)

$$\mathcal{M}_1=\{\ |\,\mathrm{Re}\tau\,|\le\tfrac{1}{2},\ |\,\tau\,|\ge 1\}\ . \tag{4.1}$$

Note that SL(2,Z) admits fixed points, which will be crucial to the determination of phases of chiral determinants and space-time supersymmetry.

A quadratic differential on M must then be of the form $2\,\mathrm{Re}(\delta\kappa\, d\bar z^{\,2})$, with $\delta\kappa$ a complex constant, and the induced deformation of complex structures is

$$|\,dz\,|^2\mapsto|\,dz\,|^2+2\,\mathrm{Re}(\delta\kappa\, d\bar z^{\,2})=|\,dz+\delta\kappa\, d\bar z\,|^2+O(\delta\kappa^2)\ . \tag{4.2}$$

This means that the complex coordinate z has undergone a "quasiconformal" deformation,

$$z\mapsto z+\delta\kappa\bar z\ , \tag{4.3}$$

and hence the new Riemann surface should be represented by a parallelogram of sides $1+\delta\kappa$, $\tau+\delta\kappa\bar\tau$, and the new ratio is $(\tau+\delta\kappa\bar\tau)/(1+\delta\kappa)$. In particular $|\,\delta\tau\,|^2=4\,|\,\delta\kappa\,|^2\tau_2^2$, and since $\|2\,\mathrm{Re}(\delta\kappa\, d\bar z^{\,2})\|_{\mathrm{WP}}^2=8\,|\,\delta\kappa\,|^2$, it follows that

$$ds^2=2\,|\,\delta\tau\,|^2/\tau_2^2\ , \tag{4.4}$$

which is invariant under the mapping class group SL(2,Z).

At higher genus, such a simple parametrization is generally not available. However, when a surface can be represented as a double covering of the sphere—and is so-called *hyperelliptic*—then of course we have a polynomial equation for it, of the form

$$y^2=P(z)\ , \tag{4.5}$$

where $P(z)$ is a polynomial of degree $2h+2$ (or $2h+1$ if one point is sent to ∞). For the torus, this provides a representation familiar from the theory of elliptic functions.

Spin structures have a simple classification for hyperelliptic surfaces. Such surfaces have $2h+2$ branch points. Each partition of the branch points into two sets of $h+1+2k$ and $h+1-2k$ points correspond to a spin structure whose parity is that of k. The spin structures corresponding to $k=0$ and 1 are the nonsingular ones, where the Dirac operator has exactly no zero modes and one zero mode. More generally, k is the order of vanishing of the ϑ function with characteristics at $z=0$.

The set of hyperelliptic surfaces is a subvariety of moduli space of dimension $2h-1$ and hence is of measure zero for genus $h\ge 3$. At genus 2, however, every surface is hyperelliptic, and Eq. (4.5) is a good representation of a generic $h=2$ surface. Thus we have for genus 2

$$y^2=(z-a_1)(z-a_2)(z-a_3)(z-a_4)(z-a_5)(z-a_6)\ , \tag{4.6}$$

but of course, it should be realized that three distinct points can be fixed at will, so that we may set $a_4=0$, $a_5=1$, $a_6=\infty$. There then remain three complex coordinates: the three moduli of $\mathcal{M}_2$. The cut sphere is represented in Fig. 13, and the ramification points are exactly labeled by the a_i's. Actually, each distinct geometry of the cut plane provides a different Riemann surface, i.e., a different point in $\mathcal{M}_2$, and if all a_1, a_2, and a_3 run throughout $\mathbf{C}^3$, $\mathcal{M}_2$ is covered $6!=720$ times.

Holomorphic and meromorphic differentials for $h=2$ are completely explicit in this representation, and we have

- two holomorphic Abelian differentials

$$\omega_1=\frac{dz}{y},\quad \omega_2=\frac{z\,dz}{y}\ , \tag{4.7}$$

- three holomorphic quadratic differentials

$$\phi_j=\frac{z^{j-1}(dz)^2}{y^2},\quad j=1,2,3\ , \tag{4.8}$$

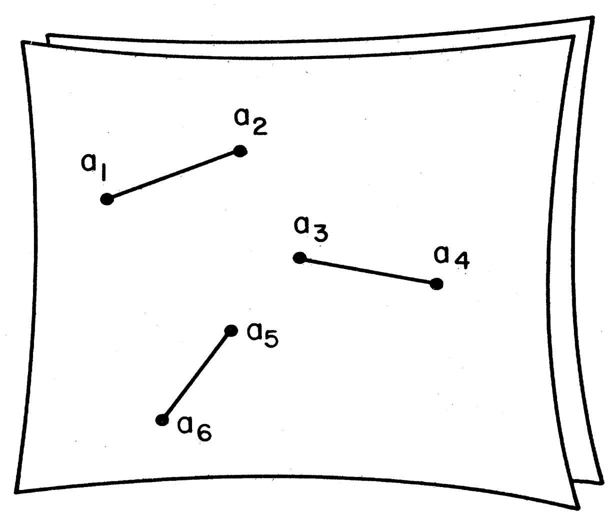


FIG. 13. Representation of a genus-2 surface as a square-root branched covering of the sphere (stereographically projected onto the complex plane).

● six holomorphic $\frac{1}{2}$ differentials, each corresponding to an odd-spin structure

$$v_i=\sqrt{(z-a_i)/y}\,(dz)^{1/2}, \quad i=1,\ldots,6 \tag{4.9}$$

with single zeros at $z=a_i$,
● no holomorphic $\frac{1}{2}$ differentials with even-spin structure,
● two holomorphic $\frac{3}{2}$ differentials for each spin structure. An odd-spin structure is just a selection of a branch point a_i, and the two holomorphic $\frac{3}{2}$ differentials are

$$(z-a_i)^{1/2}\left[\frac{dz}{y}\right]^{3/2}, \quad z(z-a_i)^{1/2}\left[\frac{dz}{y}\right]^{3/2}.$$

Even-spin structures are partitions of the six branch points into two sets A and B of three elements each, and the holomorphic $\frac{3}{2}$ differentials are then

$$\prod_{a_i\in A}(z-a_i)^{1/2}\left[\frac{dz}{y}\right]^{3/2},$$

$$\prod_{a_i\in B}(z-a_i)^{1/2}\left[\frac{dz}{y}\right]^{3/2}.$$

A natural metric on the surface is obtained by using the holomorphic $\frac{1}{2}$ differentials

$$ds^2=|v_i v_{i'}|^2, \tag{4.10}$$

which has a double zero at a_i and $a_{i'}$ or a fourth-order zero at a_i if $i=i'$.

The above techniques extend to the case of hyperelliptic surfaces at higher genus as well, in a straightforward fashion, but we shall not discuss these here.

Work on explicit formulas for two-loop amplitudes includes that of Belavin *et al.* (1986), Kato, Matsuo, and Odake (1986), Moore (1986), and Lebedev and Morozov (1987). Conformal field theory on hyperelliptic surfaces has been dealt with by Zamolodchikov (1987). Two-loop studies were also carried out for the fermionic strings by Atick, Rabin, and Sen (1987), Atick and Sen (1987a), Moore and Morozov (1987), Morozov (1987), Parkes (1987), Lechtenfeld and Parkes (1988).

C. The higher-genus case

For $h\geq 2$, Γ is a subgroup of PSL(2,**R**) and its elements γ act on $z\in H$ by

$$z\to\gamma z=\frac{az+b}{cz+d}, \quad \gamma=\begin{bmatrix}a & b\\ c & d\end{bmatrix}, \quad ad-bc=1. \tag{4.11}$$

All γ's (except for the identity) act without fixed points, which requires that γ be *hyperbolic*, i.e.,

$$|\mathrm{tr}\gamma|>2 \quad \text{for all } \gamma\in\Gamma-\{1\}. \tag{4.12}$$

Such groups are called *Fuchsian groups of the first kind.* We always choose representatives γ with $\mathrm{tr}\gamma>0$, without loss of generality. The Poincaré metric on **H** may be integrated, and we obtain the hyperbolic distance, given by

$$d(z,z')\geq 0, \quad \cosh d(z,z')=1-2\frac{(z-z')(\bar z-\bar z')}{(z-\bar z)(z'-\bar z')}. \tag{4.13}$$

Now since $\mathrm{tr}\gamma>2$, γ is conjugate within SL(2,**R**) to a pure *dilation*:

$$z\to e^{l_\gamma}z$$

whose fixed points are the origin and infinity, neither of which belongs to **H**. The scaling parameter l_γ is actually the shortest distance between any $z\in\mathbf{H}$ and its image γz:

$$l_\gamma=\min_{z\in H} d(z,\gamma z). \tag{4.14}$$

On the surface M, the points z and γz are identified under the action of Γ, so the geodesic from z to γz is closed on M, and it belongs to the homotopy class of $\pi_1(M)$ corresponding to γ. Thus l_γ is the length of the shortest closed geodesic within the homotopy class of γ.

The group Γ is generated by its Fuchsian transformations around a canonical homology basis, say of A cycles and B cycles, so let us denote the corresponding generators by γ_{A_I} and γ_{B_I} for $I=1,2,\ldots,h$. Actually, these generators are only determined up to an overall isometry $\mathfrak{J}$ of **H**,

$$\gamma_{A_I}\to\mathfrak{J}^{-1}\gamma_{A_I}\mathfrak{J}, \quad \gamma_{B_I}\to\mathfrak{J}^{-1}\gamma_{B_I}\mathfrak{J},$$

and the product of their commutators over all A and B cycles is the identity [the corresponding homotopy class of $\pi_1(M)$ is trivial]

$$\prod_{I=1}^{h}\gamma_{A_I}\gamma_{B_I}\gamma_{A_I}^{-1}\gamma_{B_I}^{-1}=1. \tag{4.15}$$

Since each generator depends on three real parameters (say a, b, and c), there are in total $6h$ real parameters, minus 3 for the global isometry and 3 more for the constraint, so altogether we are left with $6h-6$ parameters. Not surprisingly, this is exactly the dimension of Teichmüller and moduli space.

Does this means that we have an explicit parametrization of Teichmüller or moduli space? Suppose we pick γ_{A_I}'s and γ_{B_I}'s hyperbolic and satisfying Eq. (4.15); this can rather easily be done and implies some restrictions on the parameters of these matrices. Still, products of the γ_A and γ_B's need no longer be hyperbolic, leading to further restrictions on the parameter space for these matrices. The result for the $6h-6$ dimensional parameter space is a region of $\mathbf{R}^{6h-6}$ with a highly dented boundary. The corresponding coordinates are the so-called Fricke-Klein (1926) coordinates.

More on Fricke-Klein coordinates can be found in McKean (1972), Harvey (1978), Bers (1981), and Bers and Gardiner (1986).

D. Normal coordinates in the higher-genus case

For Riemannian manifolds there is a natural way of parametrizing a local neighborhood of a given point by tangent vectors at that point. In fact, to a tangent vector corresponds simply the point a unit amount of time away on the geodesic tangent to that vector. These are usually called normal coordinates and can be constructed as follows in the case of moduli space. Consider a fixed complex structure which will be identified with a Fuchsian group of the first kind. A tangent vector to moduli is a quadratic differential ϕ (equivalently a harmonic Beltrami differential $\bar{\mu}=\phi y^2$ with $y=\mathrm{Im} z$), the geodesic with initial velocity ϕ a one-parameter family of Fuchsian groups Γ_ε. The Γ_ε's can be obtained by solving the Beltrami equation

$$\partial_{\bar{z}} w = \varepsilon \mu \partial_z w \tag{4.16}$$

for a mapping w sending $\mathbf{H}$ into itself, and setting $\Gamma_\varepsilon = w^{-1}\Gamma w$. Here we have extended μ by $\mu(z)=\mu(\bar{z})$ for z in the lower half-space. A related construction putting the complex structure of moduli better in evidence is based on extending μ to be 0 instead. The resulting w will then no longer preserve the real axis, so that $\mathbf{H}$ will be deformed into a quasi-half-space and Γ into "quasi-Fuchsian groups." Despite this difference in emphasis, the two ways lead to the same deformation, since the w's obtained either way differ only by a holomorphic mapping. Note that w is not conformal, and that this construction is the natural generalization of that described in Eq. (4.3) for the torus. Choosing now an orthonormal system of quadratic differentials ϕ_α, $\alpha=1,\ldots,3h-3$, and repeating the construction for $\phi=\sum_{\alpha=1}^{3h-3} t_\alpha \phi_\alpha$ we can parametrize a neighborhood of Γ by $(t_\alpha)\in \mathbf{C}^{3h-3}$. In this coordinate system the Weil-Petersson metric will satisfy $\partial_\gamma g_{\alpha\bar{\beta}}\,|\,_\Gamma = \partial_{\bar{\gamma}} g_{\alpha\bar{\beta}}\,|\,_\Gamma = 0$, which implies in particular that it is Kählerian. We shall be especially interested in the Kähler form

$$\omega_{\mathrm{WP}} = g_{\alpha\bar{\beta}}\, dt^\alpha \wedge dt^{\bar{\beta}} , \tag{4.17}$$

since its $(3h-3)$ power is the desired volume form.

Normal coordinates for the Weil-Petersson metric were introduced by Ahlfors (1966). A modern account including a detailed analysis of second variations of the area element of the surface under quasiconformal deformations (4.16) is that of Wolpert (1986). Normal coordinates for general Riemannian manifolds are also useful in background field calculations. See, for example, Alvarez-Gaumé, Freedman, and Mukhi (1981).

E. Fenchel-Nielsen coordinates

We now describe briefly Fenchel-Nielsen coordinates which are real coordinates for Teichmüller space. Although the complex structure is not evident in this system, they have the advantage of presenting Teichmüller space as $(\mathbf{R}\times\mathbf{R}^+)^{3h-3}$ and of providing a particularly simple formula for the Weil-Petersson Kähler form. To define these coordinates, one makes use of the following construction. Consider the maximal set of closed, nonintersecting geodesics on a given surface M. It is clear that $3h-3$ nonintersecting closed geodesics may always be drawn on a surface of genus h (see Fig. 14 for an example). It is not hard to see that any additional closed geodesic has to intersect at least one of the $3h-3$ initial ones. Thus the maximal number is $3h-3$; let us call their lengths l_i, $i=1,2,\ldots,3h-3$. Along each of these geodesics, we may now cut the surface apart and reglue it after a relative twist by an angle θ_i. For the range

$$0<l_i<\infty,\quad i=1,2,\ldots,3h-3\ , \qquad -\infty<\theta_i<\infty\ , \tag{4.18}$$

these parametrize precisely one copy of Teichmüller space. Surfaces with nodes arise at the boundary of this domain when one of the l_i's goes to zero. Notice that the $3h-3$ closed geodesics divide M into $2h-2$ surfaces of genus 0 with three discs removed, which are called *pants*.

Now it is not difficult to see that the hyperbolic structure on a pant can be characterized by the lengths of the three boundaries. In fact, each pant can be viewed as built out of two copies of right hexagons (whose corners have 90° angles), and hexagons are characterized by three alternate sides. The gluing of these pants involves the relative twist of an angle θ_i along each geodesic.

The Weil-Petersson Kähler form has been shown to take the simple form

$$\omega_{\mathrm{WP}} = \sum_{j=1}^{3h-3} l_j\, dl_j \wedge d\theta_j\ , \tag{4.19}$$

so that the Weil-Petersson measure is completely explicit and given by

$$d(\mathrm{WP}) = \prod_{j=1}^{3h-3} l_j\, dl_j\, d\theta_j\ . \tag{4.20}$$

This very geometric viewpoint is also natural to path-integral quantization. In particular, D'Hoker and Phong (1986c) have exhibited string determinants in terms of these coordinates and Green's functions on pants. These in principle can be built of prime forms on hyperelliptic surfaces and may ultimately lead to rules for string amplitudes more closely analogous to the usual Feynman rules of field theory. The main difficulty of this approach is that the action of the mapping class group on the Fenchel-Nielsen coordinates and on the pant decomposition is extremely complicated, and unless some more direct way of representing this action can be found, their usefulness as a characterization of moduli space is unclear.

This decomposition is reminiscent of the division into

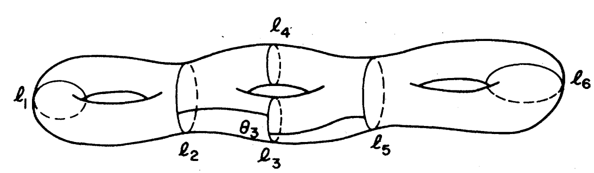


FIG. 14. Decomposition of a surface of genus 3 into four "pants" and corresponding Fenchel-Nielsen coordinates.

primitive graphs briefly discussed in the days of dual models by Gross and Schwarz (1970). In fact, it is quite tempting to construct a *fieldlike theory* this way, without propagators, however. Indeed, take the pant and now sew enough pants together so as to reconstruct the desired string diagram. Unfortunately, this approach does not seem to work, for essentially the same reasons as are given above: though one formally obtains the measure and the integrand, naive sewing will also yield an infinite factor in front, which is basically the cardinality of the mapping class group. One would have to factor out the proper (infinite) combinations right away, and this seems rather hopeless. The latter idea was explored by D'Hoker and Gross; similar attempts at string field theory based on pants may be found in Tseytlin (1986).

Equation (4.19), which shows that Fenchel-Nielsen coordinates are canonical coordinates with respect to the symplectic structure defined by Weil-Petersson Kähler form, is due to Wolpert (1982, 1983). These references also contain a great deal more on the interplay between the symplectic geometry of Teichmüller space and the hyperbolic geometry of the surface.

F. Penner decomposition

Recently, another description of the Teichmüller space $\mathcal{T}_{h,n}$ of Riemann surfaces endowed with constant-curvature metrics has emerged. However, this construction works only when there is at least one puncture, so that $n \geq 1$. The Penner decomposition exhibits a simple behavior under the action of the mapping class group, so that it may be used to describe the corresponding moduli space $\mathcal{M}_{h,n}$ of punctured surfaces. In addition, the Weil-Petersson Kähler form admits an explicit representation, and there is a reasonable description of the boundary of the fundamental domain of moduli space. In practice, we shall here restrict ourselves to Riemann surfaces with only one puncture, the generalization to the case with more punctures being straightforward.

One starts by representing a two-dimensional hyperbolic geometry by one copy $\mathcal{H}$ of the two-sheeted hyperboloid in $\mathbf{R}^3$, endowed with the Minkowski inner product $\xi\cdot\xi' = \xi^1\xi'^1 + \xi^2\xi'^2 - \xi^3\xi'^3$ for $\xi = (\xi^1,\xi^2,\xi^3)$:

$$\mathcal{H} = \{\xi \in \mathbf{R}^3; \xi\cdot\xi = -1,\ \xi^3 > 0\}\ . \tag{4.21}$$

The component connected to the identity $SO^+(2,1)$ of SO(2,1) leaves $\mathcal{H}$ invariant and acts isometrically. The metric induced on $\mathcal{H}$ by the flat Minkowski metric has curvature -1. Actually, there is a simple correspondence between the complex upper half-plane $\mathbf{H} = \{z = x + iy \in \mathbf{C}, y > 0\}$ and $\mathcal{H}$, given by

$$\xi^1 = \frac{x}{y},\quad \xi^2 = \frac{\Lambda^2 - x^2 - y^2}{2\Lambda y},\quad \xi^3 = \frac{\Lambda^2 + x^2 + y^2}{2\Lambda y}\ , \tag{4.22}$$

where Λ is an arbitrary constant > 0 and the hyperbolic metric on $\mathbf{H}$ is linked to the inner product on $\mathcal{H}$ by $\cosh d = -\xi\cdot\xi'$. The geodesics of $\mathbf{H}$, half-circles centered on the real line and arbitrary radius, are mapped onto the geodesics of $\mathcal{H}$, hyperbolas lying in planes that pass through $\xi = 0$; see Fig. 15.

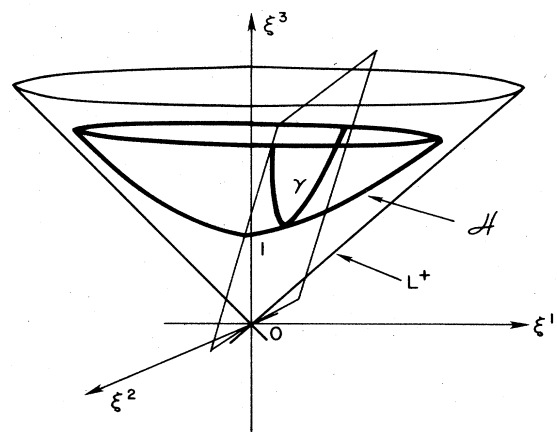


FIG. 15. Hyperbolic geometry as constructed from the three-dimensional hyperboloid $\mathcal{H}$. A geodesic γ is the intersection of $\mathcal{H}$ with a plane through the origin.

For every element of $\mathcal{M}_{h,1}$, we have a representation of $\pi_1(M)$ into $SO^+(2,1)$ by a Fuchsian group Γ, and the Riemann surface is represented by $M = \mathcal{H}/\Gamma$. The positive light cone is given by

$$L^+ = \{\xi \in \mathbf{R}^3,\ \xi\cdot\xi = 0,\ \xi^3 > 0\}\ . \tag{4.23}$$

Now consider a Riemann surface with one puncture P, so that Γ has a parabolic[27] generator $\gamma_P \in SO^+(2,1)$, corresponding to that puncture. A parabolic element $\neq 1$ is characterized by the light ray in L^+ it leaves invariant, and we may pick a particular point z in L^+ on this ray to represent γ_P.

A geodesic of $\mathcal{H}$ that starts at P and returns to P is called an *ideal arc*. The total length of an ideal arc is clearly infinite, but we shall now give a natural regularization. We draw a small circle around the puncture, orthogonal to all geodesics emanating from P; this circle, denoted by h, is called a *horocycle* and is characterized by its length (see Fig. 16 for the torus). If we had several punctures, each of them would inherit a horocycle. The horocycle defines a small disc with puncture P, and this disc may be viewed as the coset of $\mathcal{H}$ by the cyclic group generated by γ_P. The hyperbolic length of a geodesic c starting at P and returning to it, as measured from its intersections with the horocycle h, is now finite and denoted by $d_h(c)$. As the radius of the horocycle tends to zero ($h \to P$), this length again diverges, but the difference between the lengths of two geodesics c_1 and c_2 converges,

$$\lim_{h\to P} \exp[d_h(c_1) - d_h(c_2)] = \left[\frac{\lambda(c_1)}{\lambda(c_2)}\right]^2 . \tag{4.24}$$

[27]An element $\gamma \neq 1$ of $SO^+(2,1)$ is hyperbolic, parabolic, or elliptic provided the eigenvector with eigenvalue 1 lies outside L^+, on L^+, or inside L^+.

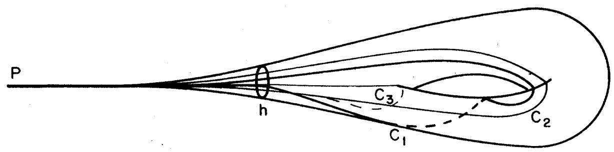


FIG. 16. An ideal triangulation of the once-punctured torus. The ideal arcs are C_1, C_2, and C_3. The horocycle h is also indicated.

If $\gamma(c)$ denotes the element of $SO^+(2,1)$ describing the ideal arc c starting at P and returning to it, and ξ is the element of L^+ fixed by γ_P, then one can show that the λ lengths are given by

$$\lambda^2(c) = -\xi\cdot[\gamma(c)\xi] \ . \tag{4.25}$$

Each length $\lambda(c)=\lambda_\Gamma(c)$ depends on the Fuchsian group, and the lengths are natural coordinates in the sense that elements φ of the mapping class group act simply on their ratios,

$$\frac{\lambda_{\varphi^*\Gamma}(c_1)}{\lambda_{\varphi^*\Gamma}(c_2)} = \frac{\lambda_\Gamma(\varphi^{-1}c_1)}{\lambda_\Gamma(\varphi^{-1}c_2)} \ . \tag{4.26}$$

It is the purpose of this construction to use the λ lengths as coordinates for moduli space.

1. Ideal triangulations

One obtains an *ideal triangulation* Δ of the Riemann surface $M_{h,1}$ by considering a maximal family of disjointly embedded simple ideal arcs Δ, so that no component of $M_{h,1}-\Delta$ is a mono-gon or bi-gon.

Thus an ideal triangulation is a decomposition of $M_{h,1}$ by ideal arcs into regions whose double is a sphere with three punctures. It is easy to see that $6h-3$ ideal arcs is the maximal number, and they divide the surface into $4h-2$ triangles, whose corners are all identified with the puncture P (see Fig. 16). Now given $M_{h,1}$ with its horocycle around the puncture P, the λ lengths of an ideal triangulation provide a one-to-one and onto map between $\mathcal{T}_{h,1}$ (together with its horocycle length) and $\mathbf{R}_+^q$, under which the ideal arcs $c_1, c_2, \ldots, c_q$ of Δ are sent into $\lambda(c_1), \lambda(c_2), \ldots \lambda(c_q)$. Here $q=6h-3$, but recall that only ratios of λ lengths had a geometrical meaning, so that actually only $6h-4$ independent values survive, exactly the number of moduli of $\mathcal{M}_{h,1}$. (If we had n punctures, there would be $6h-6+3n$ ideal arcs, dividing the surface into $4h-4+2n$ triangles.)

2. Ideal cell decompositions

We now define a slightly refined decomposition of $M_{h,1}$ into triangles, which will naturally induce a mapping-class-group-invariant cell decomposition on $\mathcal{T}_{h,1}$.

Consider the orbit Γz for $z\in L^+$ and z invariant under the parabolic element γ_P, and consider its convex hull K, which is Γ invariant. The boundary ∂K consists of a countable set of faces, each of which is the convex hull of a finite number of points. Hyperbolicity of Γ guarantees that any face of ∂K intersects L^+ in an ellipse. Each edge of a face determines a geodesic in $\mathcal{H}$ and hence an ideal arc in $M_{h,1}$. The collection of ideal arcs in $M_{h,1}$ arising from all edges of faces of ∂K is a disjointly embedded collection Δ of arcs in $M_{h,1}$ connecting punctures, so that $M_{h,1}-\Delta$ is simply connected. The homotopy class of such a family of arcs is called an *ideal cell decomposition.* Thus with each $\Gamma\in\mathcal{T}_{h,1}$ we have associated an ideal cell decomposition $\Delta(\Gamma)$ of $M_{h,1}$.

A cell decomposition $\mathcal{C}$ of $\mathcal{T}_{h,1}$ is obtained by considering the collection of

$$\mathcal{C}(\Delta) = \{\Gamma\in\mathcal{T}_{h,1} \text{ such that } \Delta(\Gamma) \text{ belongs to the class } \Delta\} \tag{4.27}$$

and Δ ranges over all ideal cell decompositions of $M_{h,1}$. Each $\mathcal{C}(\Delta)$ is contractible, and $\mathcal{C}$ is a mapping-class-invariant decomposition.

The action of the mapping class group on Δ is generated by the following operation on two consecutive triangles with edges $\{a,b,e\}$ and $\{c,d,e\}$ of Δ, where the side e is common. Consider the quadrangle $\{a,b,c,d,e\}$ with one diagonal e. Remove e from the quadrangle and replace it by the other diagonal f; this yields a new triangulation Δ'. The corresponding λ lengths are polynomially related:

$$\lambda(e)\lambda(f) = \lambda(a)\lambda(c)+\lambda(b)\lambda(d) \ . \tag{4.28}$$

When Teichmüller space is described in terms of the λ coordinates, a cell $\mathcal{C}(\Delta)$ is described by the following inequalities. Let $\{a,b,e\}$ and $\{c,d,e\}$ be two consecutive triangles as before and let $\{\alpha,\beta,\varepsilon\}$ and $\{\gamma,\delta,\varepsilon\}$ be the λ lengths of their sides; then we have

$$\alpha+\beta>\gamma, \quad \alpha+\gamma>\beta, \quad \beta+\gamma>\alpha \ , \tag{4.29}$$

as well as

$$\begin{aligned} 0 &\le \operatorname{sgn}(\alpha^2+\beta^2-\varepsilon^2)K(\alpha,\beta,\varepsilon) \\ &\quad + \operatorname{sgn}(\gamma^2+\delta^2-\varepsilon^2)K(\gamma,\delta,\varepsilon) \\ K(\alpha,\beta,\gamma) &= [(\alpha+\beta-\gamma)(\alpha+\gamma-\beta) \\ &\quad \times(\beta+\gamma-\alpha)(\alpha+\beta+\gamma)]^{1/2} \ . \end{aligned} \tag{4.30}$$

With the help of this cell decomposition and the action of the mapping class group, Penner has succeeded in computing the orbifold Euler number of moduli space $\mathcal{M}_h$ and rederived the well-known formula of Harer and Zagier (1986),

$$\chi(\mathcal{M}_{h,1}) = \zeta(1-2h) \ , \tag{4.31}$$

where $\zeta(z)$ is the Riemann zeta function.

3. Integration over moduli space

There also exists a method for integrating forms invariant under the action of the mapping class group, in terms of integrations over cells whose edges satisfy the nonlinear inequalities above. The triangles are assembled into "fat graphs" G, which describe concisely the homotopy classes Δ and thus the cells $\mathcal{C}(\Delta)$ of $\mathcal{T}_{h,1}$. The formula for the integration of a top-dimensional form of ω of $\mathcal{M}_{h,1}$ is particularly elegant:

$$\int_{\mathcal{M}_{h,1}} \omega = \sum_{[G]} \frac{2^{\varepsilon[G]}}{\#\Gamma(G)} \int_{D_G} \phi^* \omega \ , \tag{4.32}$$

where D_G is the region of integration described by the nonlinear inequalities for the fat graph G, and the sum is over all G with trivalent vertices only. $\varepsilon[G]=1$ when G has two vertices and is hyperelliptic, and $\varepsilon[G]=0$ otherwise. $\Gamma(G)$ is the isotropy group of the cell corresponding to the fat graph G (i.e., the combinatorial factor familiar from Feynman diagrams).

The Weil-Petersson measure can be obtained from the Weil-Petersson Kähler form in the standard fashion, and the latter is given in terms of the real (positive) λ lengths $\{\alpha,\beta,\gamma\}$ assigned to each vertex $\{a,b,c\}$ of G by

$$\omega_{\mathrm{WP}} = -2\sum (d\ln\alpha \wedge d\ln\beta + d\ln\beta \wedge d\ln\gamma + d\ln\gamma \wedge d\ln\alpha) \ , \tag{4.33}$$

where the sum runs over all vertices of G. This formula results from Wolpert's explicit expression for the Weil-Petersson Kähler form as a function of geodesic lengths.

For the mathematical literature, we refer the reader to Epstein and Penner (1988), and Penner (1987a, 1987b). The original proof of Eq. (4.31) is in Harer and Zagier (1986). A recent survey is that of Harer (1982, 1985).

To conclude, we remark that the Penner decomposition method has been used in string theory applications in only one instance so far, even though it allows for some of the most explicit and calculable formulations of the integration measure on moduli space (see Gross and Periwal, 1988).

G. Mandelstam diagrams

The formulation of string theory in the light-cone gauge by Goddard, Goldstone, Rebbi, and Thorn (1973) has led Mandelstam (1973a, 1973b, 1974a, 1974b) to introduce the *interacting string picture.* In this picture, freely moving strings propagate as cylinders and split and join at definite light-cone interaction times τ_a, and the interaction vertex (at least for the bosonic string) is just the overlap integral between initial and final strings. In the light-cone picture, the radius of an intermediate cylinder corresponds to the p^+ component of the momentum of that string, so that the sum of all radii remains conserved as a function of light-cone time. We shall, however, abstract the diagram from the momentum conservation issue and simply keep the same geometry. Such diagrams will be referred to as *Mandelstam diagrams,* and a typical example is presented in Fig. 17.

As a Riemann surface, the number of internal slits corresponds to the genus h, whereas the number of cylinders running off the ∞ corresponds to the number of punctures n. Clearly, then, the Mandelstam diagram can only describe surfaces with at least two punctures. In view of the results in Sec. II.L, however, the Polyakov integrals may be reformulated on surfaces with punctures.

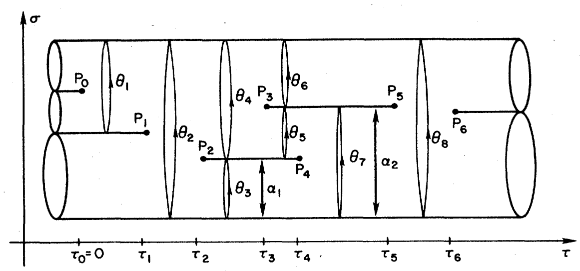


FIG. 17. A Mandelstam diagram for a surface with $n=5$ punctures, genus $h=2$, with the corresponding coordinates.

The coordinates labeling the diagram for genus h and n punctures are (see again Fig. 17)

$$\begin{aligned} &\tau_a, \quad a=1,2,\ldots,2h+n-3, \quad \text{interaction times} \ , \\ &\theta_\alpha, \quad \alpha=1,2,\ldots,3h+n-3, \quad \text{twist angles} \ , \\ &\alpha_I, \quad I=1,2,\ldots,h, \quad \text{internal } \textit{momenta} \ , \\ &\alpha_p, \quad p=1,2,\ldots,n, \quad \text{external } \textit{momenta} \ . \end{aligned} \tag{4.34}$$

The remarkable fact, implicit in the work of Mandelstam (1986a, 1986b) and proven by Giddings and Wolpert (1987), is that natural ranges for τ_a, θ_α, and α_I, as well as fixed α_p's, provide a single cover for the moduli space $\mathcal{M}_{h,n}$ of surfaces of genus h and with n punctures, with *residues* α_p prescribed at the punctures Q_i. The proof proceeds along the following lines.[28]

The quintessential property of a Mandelstam diagram (as is of course the case for the light-cone gauge in general) is that it admits a globally defined light-cone time τ, and so the differential $d\tau$ is exact. The σ direction, on the other hand, is not global, first because on each cylinder σ is not single valued, and second because there is a separate σ variable for each cylinder. However, the differential $d\sigma$ is well defined on each cylinder separately and is single valued there. It is thus natural to introduce the *light cone coordinate* $w=\tau+i\sigma$, whose differential is well defined,

$$\omega = dw = d\tau + i\,d\sigma \ . \tag{4.35}$$

Of course, w does not define a smooth coordinate in the neighborhood of an interaction point w_0. Smooth holomorphic coordinates z may be introduced in the neighborhood of w_0 by mapping the region into a planar region

$$w - w_0 = (z - z_0)^2 \ . \tag{4.36}$$

[28]Arguments along somewhat different lines were presented by Taylor (1987).

Actually, if N interaction times coincide,[29] one will have to introduce z through

$$w-w_0=(z-z_0)^N .$$

In any case, it is clear that ω vanishes at the interaction points and is nonvanishing everywhere else on the surface. Since the radii of the outgoing cylinders are α_p, we also have

$$\frac{1}{2\pi i}\oint_{\text{around } Q_p}\omega=\alpha_p, \quad \sum_{p=1}^{n}\alpha_p=0 , \tag{4.37}$$

so we may view ω as having a simple pole at the puncture Q_p with residue α_p.

Since ω is a (1,0) form, $\bar\omega\omega$ is a metric on the surface. It is flat Euclidean everywhere, except at the interaction points, where all the curvature of the diagram is concentrated:

$$\sqrt{g}\,R(z)=-2\pi\sum_{a=0}^{2h-3+n}\delta(z-P_a) , \tag{4.38}$$

where $g_{z\bar z}=\omega_z\bar\omega_z$.

Now notice that ω has purely imaginary periods around any homology cycle, since τ is single valued. Thus for every Mandelstam diagram there exists a unique holomorphic Abelian differential ω, with purely imaginary periods, and with residues α_p at the puncture Q_p. Conversely, every such holomorphic differntial ω determines a metric $\bar\omega\omega$ and hence a unique Riemann surface.

For a general Riemann surface (not necessarily viewed in the Mandelstam picture) there exists an analogous holomorphic differential. It is constructed out of the meromorphic Abelian differentials of the third kind $\omega_{PQ}(z)$ with simple poles at P and Q of residues 1 and -1, respectively. Such a differential is defined only up to holomorphic differentials ω_I, but a unique $\omega_{PQ}(z)$ emerges if one demands that all its periods be purely imaginary:

$$\mathrm{Re}\oint_{A_I}\omega_{PQ}=\mathrm{Re}\oint_{B_I}\omega_{PQ}=0 . \tag{4.39}$$

Then ω is given by

$$\omega(z)=\sum_{p=1}^{n-1}\beta_p\,\omega_{Q_pQ_{p+1}}(z) , \tag{4.40}$$

with the coefficients β_p expressed in terms of the residues α_p:

$$\beta_p-\beta_{p-1}=\alpha_p, \quad p=1,\dots,n$$

and

$$\beta_0=\beta_n=0 .$$

As a differential on the surface with punctures $Q_1,\dots,Q_n$, ω is of course holomorphic, and by construction all its periods are imaginary.

Thus on every Riemann surface there exists a unique holomorphic differential ω with residues α_p at the punctures Q_p and with purely imaginary periods. Conversely, such a differential defines a metric and hence specifies the Riemann surface uniquely.

The proof is completed by setting the two unique differentials ω equal to one another.

Actually, this proof also informs us immediately about the natural range of parameters mentioned above. The differential ω will be completely specified once the "geometry" of the Mandelstam diagram is given. But there are redundancies in the parametrization of the geometry which yield the same geometrical configuration of the diagram. Clearly, these are the only restrictions on the range of the parameters.

Another remarkable property of the Mandelstam diagram representation is that quadratic and $\frac{3}{2}$ holomorphic differentials admit an explicit representation in terms of the canonical differential ω and h holomorphic first-Abelian differentials ω_I, $I=1,\dots,h$.

- Holomorphic quadratic differentials are given by

$$\begin{aligned}&\phi_I=\omega\omega_I, \quad I=1,\dots,h ,\\ &\phi_a=\omega\omega_{P_0P_a}, \quad a=1,\dots,2h+n-3 ,\end{aligned} \tag{4.41}$$

where $P_0,P_1,\dots,P_{2h+n-3}$ are the interaction points of the diagram. The poles of the meromorphic (third) Abelian differentials ω_{PQ} are precisely canceled by the zeros of ω, so that ϕ_a and ϕ_I are holomorphic on the n-punctured surface.

- Holomorphic $\frac{3}{2}$ differentials for even-spin structure ν and a generic point on $\mathcal{M}_{h,n}$ are constructed as follows. There are no holomorphic $\frac{1}{2}$ differentials on the underlying compact surface (the analogs of ω_I), and there is a unique meromorphic $\frac{1}{2}$ differential [the so-called Szegö kernel of Eq. (3.202)] $\kappa_P(z)=S_\nu(z,P)$ with a single simple pole at P. We obtain the $2h+n-3$ holomorphic $\frac{3}{2}$ differentials as

$$\rho_a=\omega\kappa_{P_a}, \quad a=0,1,\dots,2h+n-3 , \tag{4.42}$$

where again the pole of κ is canceled by the zeros of ω.

- Holomorphic $\frac{3}{2}$ differentials for odd-spin structure ν and a generic point in $\mathcal{M}_{h,n}$ are obtained as follows. There is now a unique holomorphic $\frac{1}{2}$ differential h_ν on the underlying compact surface (the analog of ω_I), and there is a unique meromorphic $\frac{1}{2}$ differential $\kappa_{PQ}(z)=S_\nu(z;P,Q)$, with simple poles of opposite residue $1,-1$ at P and Q (the analog of ω_{PQ}). We again obtain $2h+n-3$ holomorphic $\frac{3}{2}$ differentials as

$$\begin{aligned}&\rho_a=\omega\kappa_{P_0P_a}, \quad a=1,\dots,2h+n-3 ,\\ &\rho_{2h+n-3}=\omega h_\nu .\end{aligned} \tag{4.43}$$

In Sec. V.G, we shall make use of these constructions to exhibit certain simple relations between ghost and matter determinants on Mandelstam diagrams.

[29]Only τ's coincide here; the interaction points may well be distinct.

H. Universal Teichmüller curve and compactification of moduli space

1. Teichmüller curve

We conclude this section by discussing the geometry of the (universal) Teichmüller curve. This is the fiber bundle over moduli space whose fiber above a given point in moduli space (a given complex structure m) is just the Riemann surface with this complex structure m. (See Fig. 18.) Its interest to us stems from the close connection between its curvature and the curvature of determinant line bundles over moduli space (Sec. VII) and, even more importantly, from the fact that it provides the proper setting for certain gauge-fixing procedures in the superstring.

The formal construction of the universal Teichmüller curve is the following. Let $\mathcal{M}$ be the space of all metrics g on a fixed topological surface M and consider the fibration

$$\begin{array}{ccc} (\xi,g) & \in & M\times\mathcal{M} \\ \downarrow & & \downarrow \\ g & \in & \mathcal{M} \end{array} \quad . \tag{4.44}$$

In the product space $M\times\mathcal{M}$ we shall view the pairs (ξ,g) and (ξ',g') as equivalent if there is a reparametrization of M sending simultaneously $\xi\to\xi'$ and $g\to g'$, or if ξ equals ξ' and g and g' differ only by a Weyl scaling. Denoting this equivalence by $\mathrm{Diff}(M)\times\mathrm{Weyl}(M)$, we can now define the universal Teichmüller curve as the fiber bundle

$$\begin{array}{ccc} \mathcal{C}_h & = & (M\times\mathcal{M})/\mathrm{Diff}(M)\times\mathrm{Weyl}(M) \\ \downarrow & & \downarrow \\ \mathcal{M}_h & = & \mathcal{M}/\mathrm{Diff}(M)\times\mathrm{Weyl}(M) \end{array} \quad . \tag{4.45}$$

We note that the original bundle (4.44) is trivial, but it follows, for example, from curvature computations (4.47) below that bundle (4.45) is not. In fact, the universal curve does not admit any global continuous sections.[30] Nevertheless, local sections exist and are important: for example, in a basis of $\frac{3}{2}$ differentials $\mu_a(z)=\delta(z-z_a)$, the points z_a should be reviewed as local sections of the Teichmüller curve. Another useful property that emerges out of the construction (4.44) and (4.45) is that vector fields on moduli space have a lifting to vector fields on the Teichmüller curve: the natural lift $g\to(\xi,g)$ of Eq. (4.44) is invariant under $\mathrm{Diff}(M)\times\mathrm{Weyl}(M)$, and hence makes sense as a lifting from $\mathcal{M}_h$ to $\mathcal{C}_h$. Liftings are needed to have a proper notion of derivatives of points of the surface M with respect to moduli parameters.

In the above construction the holomorphic structure of the Teichmüller curve is not manifest. However, if we

[30]This follows from results of Johnson (1980), as pointed out by E. Miller.

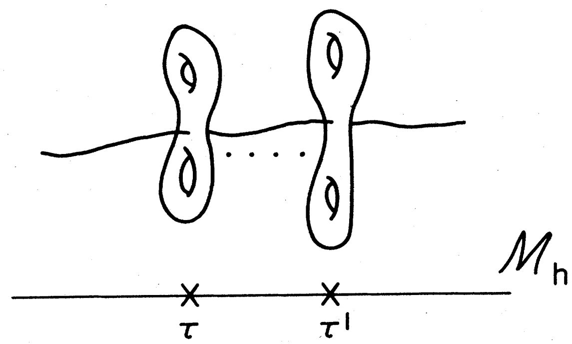


FIG. 18. The Teichmüller curve and its sections for the moduli space of surfaces of genus h.

represent a Riemann surface by a domain D_0 in the upper half plane $\mathbf{H}$ and extend the Beltrami differentials $\epsilon\mu$ to be zero in the lower half plane, then D_0 will be deformed "holomorphically" to another domain $D_{\epsilon\mu}=w(D_0)$ by quasiconformal solutions w of the Beltrami equation (4.16). As μ vary over a small neighborhood of 0 in $\mathbf{C}^{3h-3}$, this provides an embedding of a small neighborhood of the fiber corresponding to D_0 in the Teichmüller curve into $\mathbf{C}^{3h-3}\times\mathbf{C}$. We have just presented a very rough description of the Bers embedding, which endows the Teichmüller curve with a holomorphic structure. It is instructive to realize the lifting discussed in the preceding paragraph in the Bers embedding. If μ is a tangent vector to moduli space at $M=\mathbf{H}/\Gamma$, let $w_{\epsilon\mu}$ again be the solution to Eq. (4.16), which fixes 0, 1, and ∞. The vector field

$$\left[\frac{d}{d\epsilon}w_{\epsilon\mu}\bigg|_{\epsilon=0}\right]\frac{\partial}{\partial z}$$

is defined on the upper half plane, but cannot be viewed as a vector field on the surface M, since the arbitrary choices $0,1,\infty$ prevent it from transforming equivariantly under the group Γ. On the other hand, at each point z of M the vector field on the universal Teichmüller curve

$$\tau_\mu=\left[\frac{d}{d\epsilon}w_{\epsilon\mu}\bigg|_{\epsilon=0}\right]\frac{\partial}{\partial z}+\mu=\dot{w}_\mu\frac{\partial}{\partial z}+\mu \tag{4.46}$$

will be equivariant and hence well defined. The correspondence $\mu\to\tau_\mu$ is the lifting we are looking for. This means that in the Bers realization, as we deform the complex structure along μ, the fundamental domain will be deformed as well. Each point in the fundamental domain describes, then, a path in $\mathbf{C}^{3h-3}\times\mathbf{C}$, whose direction is the vector field τ_μ of Eq. (4.46). We observe that τ_μ is a smooth vector field on the universal curve, while the related quasiconformal vector field V^z of Sec. II.I is a vector field on the surface M which cannot be smooth if μ is a nontrivial deformation.

It is evident that any choice of metrics on Riemann surfaces can be viewed as a metric on the vertical bundle above the Teichmüller curve, i.e., the bundle of tangent vectors to the fibers of the Teichmüller curve. In general a metric and a holomorphic structure will then determine a unique connection by the requirements of unitarity, hermiticity, and that its (0,1) component agree with the $\bar{\partial}$ operator (cf. Sec. VI.A). If we choose constant-curvature metrics on the surface M, the curvature of the corresponding connection on the Teichmüller curve can be computed explicitly. Since the curvature can be viewed as a Hermitian 2-form, it can be described by its values on pairs of vielbein vectors for the Teichmüller curve. Since a vielbein for moduli space is provided by a basis of Beltrami differentials μ_j, $j=1,\ldots,3h-3$, which lifts to vectors

$$\tau_j = \dot{w}_{\mu_j}\frac{\partial}{\partial z}+\frac{\partial}{\partial t_j}$$

on the universal Teichmüller curve, a vielbein for the latter will consist of $\{\tau_j\}$ and $\partial/\partial z$, this last vector field being viewed as a vector field along each fiber. The curvature Ω is now given by

$$\Omega\left[\frac{\partial}{\partial \bar{z}},\frac{\partial}{\partial z}\right]=-\frac{2}{(z-\bar{z})^2},\quad \Omega\left[\frac{\partial}{\partial \bar{z}},\tau_j\right]=0\,,$$
$$\Omega(\bar{\tau}_j,\tau_k)=2(\Delta+2)^{-1}(\bar{\tau}_j\cdot\tau_k)\,. \tag{4.47}$$

Here Δ is the Laplacian on scalars. From this it is easy to deduce the higher powers of the curvature tensor and the characteristic classes

$$\kappa_n=\int_{\text{fiber}}\left[\frac{i\Omega}{2\pi}\right]^{n+1} \tag{4.48}$$

of the Teichmüller curve. In particular, one readily finds the first Chern class of the vertical line bundle

$$\kappa_1=-\frac{1}{4\pi^2}\int_{\text{fiber}}\Omega^2=\frac{1}{2\pi^2}\omega_{\text{WP}}\,, \tag{4.49}$$

where ω_{WP} is the Weil-Petersson Kähler form encountered in Eq. (4.17). This result will help us later (cf. Sec. VII.E) to identify the precise form of the holomorphic anomaly from the determinant line bundles formalism.

The Teichmüller curve is the natural setting for a careful treatment of derivatives of differentials with respect to moduli parameters. Indeed such differentials can be viewed as sections of tensor powers of the vertical line bundle over the Teichmüller curve. If metrics are chosen to represent conformal classes (as is usually done in gauge-fixing the superstring), these bundles will be endowed with a metric as well, and hence with a connection in the presence of a holomorphic structure. If we wanted to differentiate in the moduli direction, μ, it would suffice to again lift μ to a vector field along the fiber of the Teichmüller curve by the natural lift, and differentiate along that vector field using the connection we just discussed. This procedure can be applied, for example, to the $\frac{3}{2}$ differentials χ_a needed to absorb the superconformal ghost zero modes.

We also note that the Teichmüller curve can be viewed as the moduli space of surfaces with one puncture, already encountered in Secs. II.L and IV.F. Formulating string amplitudes as integrals over the Teichmüller curve and its generalizations is quite convenient in many respects.

2. Compactification of moduli space

Finally, we come to the issue of the boundary of moduli space, of which mention has been made in connection with BRST invariance (Secs. III.J–III.P), and whose role will emerge more clearly in connection with finiteness of string amplitudes and supersymmetry breaking.

That moduli space $\mathcal{M}_h$ is not a space without boundary is not evident from the definition we adopted in Eq. (2.33). However, for genus $h=1$, we have an explicit representation of $\mathcal{M}_1$ as a fundamental domain for $SL(2,\mathbf{Z})$ within the upper half space, which admits a natural one-point compactification. For higher genus $h\geq 2$, the Fenchel-Nielsen coordinates of Sec. IV.E for Teichmüller space provide a simple explanation: the boundary of Teichmüller space consists of the surfaces where one of the $3h-3$ geodesics has been pinched to a point (see Fig. 19). This is the basic geometric principle underlying the Deligne-Mumford compactification of moduli space, where one adjoins to the regular Riemann surfaces the divisor Δ of Riemann surfaces with nodes. A Riemann surface M_0 with nodes is a surface with special points p_i called nodes, around each of which the surface is conformally equivalent to two discs with their centers identified. If the coordinate of the node is 0, such a neighborhood of the node can be given by

$$\mathcal{U}_0=\{(z,w);\ zw=0,\ |z|,|w|<1\}\,. \tag{4.50}$$

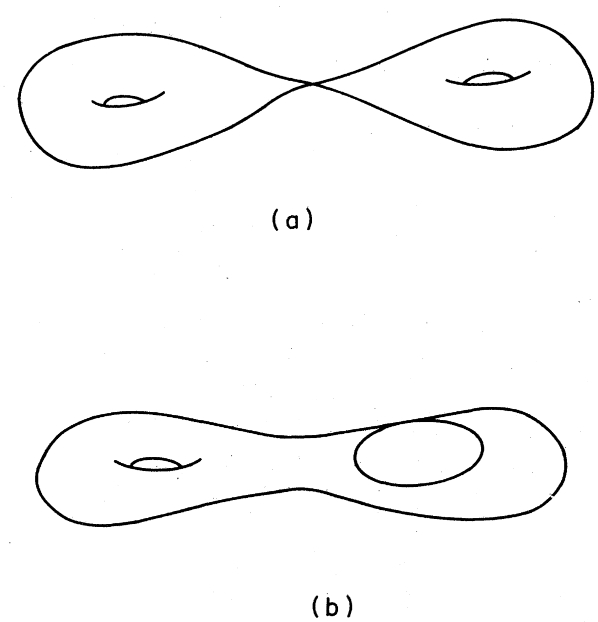


FIG. 19. The boundary of moduli space—surfaces with nodes: (a) pinching a cycle homologous to zero: (b) pinching a cycle not homologous to zero.

This neighborhood can be viewed as the end product of a degeneration of regular holomorphic neighborhoods indexed by a parameter t, $|t|<1$ tending to 0:

$$\mathcal{U}_t=\{(z,w);\ zw=t,\ |t|<|z|<1,\ \ |t|<|w|<1\}\ . \tag{4.51}$$

For each fixed t, $\mathcal{U}_t$ can be viewed as an annulus, or equivalently a cylinder. The principle of the "plumbing fixture" is that the degeneracy family $\mathcal{U}_t$ of cylinders can be fitted in a family of regular Riemann surfaces M_t, which tend to M_0 as $t\to 0$. More precisely, let M_1 and M_2 be two regular Riemann surfaces of genus i and $h-i$ and p_1 and p_2 be points on M_1 and M_2, respectively. If z_i are holomorphic coordinates around p_i and D_i are the discs $\{|z_i|<1\}$, we can remove the smaller discs $\{|z_i|\le|t|^{1/2}\}$ and join the remaining points of M_1 and M_2 by attaching them both to the plumbing fixture $\mathcal{U}_t$ in the following manner:

z_1 in D_1 is identified with $(z=z_1,\ w=t/z_1)$ in $\mathcal{U}_t$ if $|t|^{1/2}<|z_1|<1$;

z_2 in D_2 is identified with $(z=t/z_2,\ w=z_2)$ in $\mathcal{U}_t$ if $|t|^{1/2}<|z_2|<1$.

This gives us a family of regular Riemann surfaces M_t of genus h which tend to a Riemann surface M_0 with node $p=p_1,p_2$. This type of degeneration corresponds to pinching to a point a cycle homologous to zero [Fig. 19(a)]. Holomorphic parameters for moduli space near such a moduli boundary point M_0 are the moduli parameters for M_1 and M_2, the points p_1 and p_2, and the parameter t characterizing the annulus $\mathcal{U}_t$. Of these, t should viewed as the parameter transversal to the boundary of moduli space, while the others parametrize the boundary itself. If we choose instead to pinch a cycle that is not homologous to zero [Fig. 19(b)], then we can repeat essentially the same plumbing fixture construction, starting this time with a regular Riemann surface M of genus $h-1$ with two marked points p_1 and p_2. Again coordinate discs $D_i=\{|z_i|<1\}$ around p_i are introduced, smaller discs $\{|z_i|<|t|^{1/2}\}$ are removed, and the above construction yields a "bridge" between the remaining parts of D_i. The resulting surfaces M_t now have genus h and tend to a surface M_0 with node $p=p_1,p_2$. Holomorphic coordinates for moduli space are the moduli parameters for M, the points p_1 and p_2, and the parameter t, which is again the transversal coordinate. The two types of degenerations can equivalently be distinguished by whether removal of the node at the end disconnects the surface or not. In either case, the plumbing fixture construction shows explicitly that the behavior of M_t outside of $\mathcal{U}_t$ is independent of the degeneration taking place within $\mathcal{U}_t$.

Viewing Δ as arising from regular Riemann surfaces by pinching closed curves to a point, it should be evident that there are nevertheless restrictions as to which and how many curves can be pinched simultaneously. For $h\ge 2$, in the Fenchel-Nielsen picture, it is any number of $3h-3$ defining geodesics. More formally, one requires that the surfaces with nodes have at most as many conformal Killing vectors as the regular surfaces that converge to them. The compactification $\overline{\mathcal{M}}_h$ of $\mathcal{M}_h$ obtained in this manner is called the moduli space of stable curves.

The compactification locus $\Delta=\overline{\mathcal{M}}_h-\mathcal{M}_h$ is a divisor with normal crossings. It is reducible and can be written as

$$\Delta=\Delta_0\cup\cdots\cup\Delta_{[h/2]}\ , \tag{4.52}$$

where the generic surface M in Δ_i has exactly one node separating it into two components of genus i and $h-i$, while Δ_0 consists of degenerations that do not disconnect the surface. The divisors Δ_k define cohomology classes in $H_{6h-8}(\overline{\mathcal{M}}_h)$. If $[\varpi_{\mathrm{WP}}]$ is the cohomology class obtained by Poincaré duality from the Weil-Petersson Kähler form, it is a remarkable theorem of Harer (1985) and Wolpert (1987) that $(\Delta_0,\ldots,\Delta_{[h/2]},[\varpi_{\mathrm{WP}}])$ is actually a basis for $H_{6h-8}(\overline{\mathcal{M}}_h)$.

A last fundamental feature of degenerations is that the universal Teichmüller curve extends to a holomorphic fibration over the compactified moduli space $\overline{\mathcal{M}}_h$ if the fibers above the locus Δ are the corresponding surfaces with nodes. That the total space of the fibration has no singularities (by opposition to the fiber) can be intuitively seen from the degeneration picture provided by Eq. (4.51): the total space there can be viewed as the perfectly regular two-dimensional complex variety

$$\{(z,w,t);\ zw=t,\ \ |z|,|w|,|t|<1\}\ , \tag{4.53}$$

whose projection by $(z,w,t)\to t$ just ceases to be a submersion at $t=0$. The compactified universal Teichmüller curve has been used to investigate the asymptotic behavior near the boundary of moduli space of the string integrand. It is likely that its potential applications to string theory have not been exhausted.

Differential geometric constructions of the universal Teichmüller curve seem to have started with Earle and Eells (1969). The Bers embedding is described in Bers (1973). The curvature of the (uncompactified) universal curve given in Eq. (4.47) is due to Wolpert (1986). A more recent treatment extending to the curvature of surfaces with nodes over the Diligne-Mumford compactification (see Sec. IV.H) is that of Wolpert (1988). The role of the universal Teichmüller curve in Grothendieck-Riemann-Roch constructions is explained in detail by Nelson (1987a). Although we do not need them here, it may be worth reporting that the curvature of moduli space with respect to the Weil-Petersson metric is completely known. Different methods of calculation are given in Royden (1985), Siu (1986), Tromba (1986), Wolpert (1986), and Wolf (1986).

V. EVALUATION OF DETERMINANTS

Determinants of Laplacians and $\bar\partial$ operators for the torus can be expressed in terms of the Dedekind eta function $\eta(\tau)$ and special values of the theta function $\vartheta(z,\tau)$. For higher loops, this can be generalized in a number of ways. If we choose to represent the conformal class by a constant-curvature metric, then the natural function is the Selberg zeta function $Z(s)$. We shall show in this section that all the determinants of Laplacians needed for quantization are given by special values of $Z(s)$. This will allow a simple analysis of the asymptotic behavior of the string integrand near the boundary of moduli space, confirming the presence of a tachyon in the bosonic string spectrum, and clarifying the respective roles of worldsheet and space-time supersymmetry in finiteness issues. It is difficult to extract the determinants of chiral $\bar\partial$ operators in this approach, if only because hyperbolic geometry and Selberg zeta functions are defined by real quantities. Actually, even an appropriate definition of a chiral determinant is problematic. The proper resolution of these issues requires a study of holomorphic anomalies and bosonization, and will lead to expressions for chiral determinants in terms of Riemann theta functions. A full account will be provided in Sec. VII.

Mandelstam diagrams are another convenient way of representing (punctured) Riemann surfaces. Although we shall not evaluate the determinants for Mandelstam diagrams individually, we shall show that remarkable relations hold between determinants of different U(1) weights. Such relations are usually required to relate the light-cone gauge to the Polyakov formulation.

A. One-loop formulas

We begin with the simpler case of one loop, which will serve as an introduction to the more complicated formulas required for multiloops. Recall that a complex structure for a torus is characterized by a lattice $\mathbf{Z}+\tau\mathbf{Z}$ in the complex plane, and that moduli space $\mathcal{M}_1$ is $\mathbf{H}/PSL(2,\mathbf{Z})$, with $\mathbf{H}=\{\tau=\tau_1+i\tau_2;\tau_2>0\}$. The key forms on moduli space are the Dedekind eta and the theta functions defined in Appendix E, which transform according to the Jacobi rule of Eqs. (E4) and (E10). For fermion determinants, a spin structure has to be prescribed. There are four spin structures $\nu=(\nu_1,\nu_2)$, $\nu_{1,2}=0,1$ corresponding to the boundary conditions

$$\begin{aligned}\varphi(\xi_1+1,\xi_2)&=-\varphi(\xi_1,\xi_2)e^{\pi i\nu_1}\,,\\ \varphi(\xi_1+\tau_1,\xi_2+\tau_2)&=-\varphi(\xi_1,\xi_2)e^{\pi i\nu_2}\,.\end{aligned}\tag{5.1}$$

The relevant operators become

$$\begin{aligned}&\Delta=-2\partial\bar\partial,\quad P_1^\dagger P_1=2\Delta\,,\\ &\not{D}=\begin{bmatrix}0&\bar\partial\\-\partial&0\end{bmatrix},\quad P_{1/2}=\not{D}\,.\end{aligned}\tag{5.2}$$

Introducing the basis

$$\varphi_{n_1,n_2}=\exp\left[2\pi i\left[(n_1+\tfrac12-\tfrac12\nu_1)\xi^1+\frac{1}{\tau_2}[n_2-(n_1+\tfrac12-\tfrac12\nu_1)\tau_1+\tfrac12-\tfrac12\nu_2]\xi^2\right]\right],$$

we find the eigenvalues of $\bar\partial$,

$$\lambda_{n_1n_2}=\frac{2\pi}{\tau_2}[(n_1+\tfrac12-\tfrac12\nu_1)\tau-(n_2+\tfrac12-\tfrac12\nu_2)]\,,$$

and zeta-function regularization[31] produces

$$\begin{aligned}&\ln\det{}_\nu\partial\bar\partial=-\zeta'(0)+2\ln\left[\frac{2\pi}{\tau_2}\right]\zeta(0)\,,\\ &\zeta(s)=\sum_{n_1,n_2}\{(n_1+\tfrac12-\tfrac12\nu_1)^2\tau_2^2\\ &\qquad+[(n_1+\tfrac12-\tfrac12\nu_1)\tau_1-n_2-\tfrac12+\tfrac12\nu_2]^2\}^{-s}\,,\end{aligned}\tag{5.3}$$

which is absolutely convergent for $\mathrm{Re}(s)>1$. When $\nu=(1,1)$, it is understood that the summation does not include $n_1=n_2=0$. The n_2 sum may be represented by a contour integral

$$\begin{aligned}\zeta(s)=\sum_{n_1}\oint_C dz\frac{1}{1-e^{-2\pi iz}}\{&(n_1+\tfrac12-\tfrac12\nu_1)^2\tau_2^2\\ &+[(n_1+\tfrac12-\tfrac12\nu_1)\tau_1\\ &-z-\tfrac12+\tfrac12\nu_2]^2\}^{-s}\,,\end{aligned}\tag{5.4}$$

where the contour surrounds the real axis once in the counterclockwise direction. The contour may be restricted to the line infinitesimally above the real axis, the other contribution being related to the complex conjugate. In turn, this contour can be deformed into an integration along both sides of a vertical cut in the upper half plane, starting at $(n_1+\frac12-\frac12\nu_1)\tau+\frac12\nu_2-\frac12$. When $\nu\neq(1,1)$ it is straightforward to see that $\zeta(0)=0$, whereas for $\nu=(1,1)$ we have $\zeta(0)=-1$. Furthermore

$$\zeta'(0)=\sum_{n_1}2\ln\{\exp[2\pi i(n_1+\tfrac12-\tfrac12\nu_1)\tau+\pi i(\nu_2-1)]-1\}+\text{c.c.}\,,$$

with the appropriate subtraction when $\nu=(1,1)$. Using the product representation (E6) from Appendix E for the theta function, and (E10) for the eta function, we find

$$\begin{aligned}&\det{}_\nu(-\bar\partial\partial)=\left|\frac{\vartheta_{\nu_1\nu_2}(0,\tau)}{\eta(\tau)}\right|^2,\quad\nu\neq(1,1)\,,\\ &\det{}'_\nu(-\bar\partial\partial)=\tau_2^2\,|\eta(\tau)|^4\,,\quad\nu=(1,1)\,,\end{aligned}$$

where the τ_2 factor above comes from the term proportional to $\zeta(0)$, which contributes only when there are zero modes. We observe that since the left-hand side is reparametrization invariant by construction, this result can be used to derive the usual transformation law for the η function under modular transformations. Separating out the holomorphic factors where appropriate gives

$$\begin{aligned}&\det{}'\Delta=\tau_2^2\,|\eta(\tau)|^4\,,\\ &\det{}'_\nu\bar\partial=\tau_2/2\pi\vartheta_1'(0,\tau)/\eta(\tau),\quad\nu=(1,1)\,,\\ &\det{}_\nu\bar\partial=\vartheta_{\nu_1\nu_2}(0,\tau)/\eta(\tau),\quad\nu\neq(1,1)\,,\end{aligned}\tag{5.5}$$

[31]See the opening remark of Sec. V.E.

The contributions of left-movers can be identified with those of $\bar{\partial}$ operators. The above formulas actually determine the functional determinants only up to global phases independent of the moduli parameter τ. Recall that spin structures can be divided into two groups, characterized by the parity of the number of zero modes of the Dirac operator. The mapping class group will permute all of them and hence determine the phases within each group. The relative phases of the two groups should be determined by factorization requirements. For one loop, the odd-spin structure $\nu=(1,1)$ does not contribute to the partition function because the zero mode of the Dirac operator decouples from the supermoduli modes, and modular invariance forces a combination of phases for the remaining three even-spin structures which produces 0 by the Jacobi identity. This vanishing of the cosmological constant can be viewed as a consequence of space-time supersymmetry (for details see Sec. III.M).

One-loop determinants for the bosonic string were evaluated by Polchinski (1986) and for the fermionic string by D'Hoker and Phong (1986b) and Namazie, Narain, and Sarmidi (1986).

B. Multiloop formulas and Selberg zeta functions

In this case a complex structure on the worldsheet M can be represented by a metric of constant negative curvature -1, and an analog of the theta function is provided by the Selberg zeta functions

$$Z(s)=\prod_{\gamma \text{ primitive}}\prod_{k=1}^{\infty}[1-\nu(\gamma)^n e^{-(s+k)l_\gamma}]\ ,\qquad n=0,1\ . \tag{5.6}$$

Here the primitive γ denotes simple closed geodesics on M, l_γ is the length of γ in the hyperbolic metric, and $\nu(\gamma)\in\{\pm 1\}$ is determined by the spin structure. In more algebraic terms, we view the worldsheet as $\mathbf{H}/\Gamma$, with Γ a Fuchsian group with compact quotient. A primitive γ is then an element of Γ that cannot be written as a power ≥ 2 of any element, l_γ is equal to $\cosh^{-1}(\mathrm{tr}\gamma/2)$, and $\nu(\gamma)$ is a $\mathbf{Z}_2$-valued character of the group $\tilde{\Gamma}\subset SL(2,\mathbf{R})$ which projects onto $\Gamma\subset PSL(2,\mathbf{R})$. If we recall that the complex parameter τ for the torus is just $\varphi+i/l$ in Fenchel-Nielsen coordinates, there is evidently a close similarity between Eqs. (5.10) and (5.6), with the difference, however, that $Z(s)$ is real and that there are many more geodesics on a hyperbolic surface. Although we use the same symbol $Z(s)$ for the various Selberg zeta functions, it should be clear that the definition (5.6) with $n=0$ is to be taken when dealing with bosons, while $n=1$ corresponds to fermions.

The function $Z(s)$ will converge for $\mathrm{Re}\, s>1$, admit a functional equation similar to the Riemann zeta function,

$$Z(1-s)=x(s)Z(s)\ ,$$
$$x(s)=\exp\left[4\pi(h-1)\int_0^{s-1/2}dy\, y\, tg\pi y\right]\ ,$$

and extend to an entire function in the s complex plane. In terms of $Z(s)$ the functional determinants appearing in the quantum superstring measure were evaluated by D'Hoker and Phong (1986d), Fried (1986b), and Sarnak (1987);

$$\begin{aligned}&\det\Delta_g=e^{-c_0\chi}Z'(1)\ ,\\&\det P_1^\dagger P_1=e^{-c_1\chi}Z(2)\ ,\\&\det P_{1/2}^\dagger P_{1/2}=e^{-c_{1/2}\chi}Z(\tfrac{3}{2})\ ,\\&\det\bar{\not D}\not D=e^{-c_{-1/2}\chi}Z^{(2N)}(\tfrac{1}{2})/(2N)!\ ,\end{aligned} \tag{5.7}$$

with

$$c_n=\sum_{0\leq m<n-1/2}(2n-2m-1)\ln(2n-m)-(n+\tfrac{1}{2})^2+2(n-[n])(n+\tfrac{1}{2})\ln 2\pi+2\zeta'(-1)\ .$$

Here N is the number of zero modes of the chiral Dirac operator. In general it depends on the spin structure as well as on the complex structure. This will be discussed at length in Sec. VI.F.

The Selberg zeta function was introduced by Selberg (1956) and appeared in evaluation of determinants in the work of Ray and Singer (1971, 1973). A good review of its properties can be found in Hejhal (1976a, 1976b). Special cases of Eq. (5.7) have been obtained by Baranov and Schwartz (1985), D'Hoker and Phong (1986a), Fried (1986a), Gilbert (1986), Kierlanczyk (1986), and Namazie and Rajeev (1986). Various relations between Selberg zeta functions were discussed by Beilinson and Manin (1986) and by Voros (1987). The case of worldsheets with boundary is treated in Blau and Clements (1987). Superdeterminants on super Riemann surfaces of constant supercurvature were related to super Selberg zeta functions by Aoki (1988) and by Baranov, Manin, Frolov, and Schwartz (1987). Character-valued generalizations of the Selberg function in which $\nu(\gamma)$ is the character of an Abelian group have been related to determinants on Riemann surfaces with an Abelian orbifold as target space-time by Periwal (1987). Finally, Mandelstam (1986a, 1986b) also used Selberg trace formula techniques to relate determinants to the partition function of the old dual models. The superstring case is discussed by Martinec (1987).

C. Hyperbolic geometry on a Riemann surface

In the remaining subsections of Sec. V we shall discuss the mathematical ingredients necessary to an understanding of Eq. (5.7). Some fundamental facts about Fuchsian groups $\Gamma\subset PSL(2,\mathbf{R})$ with compact quotient are the following. Elements γ of Γ all have traces $|\mathrm{Tr}\gamma|$ greater than 2 and are conjugate within $SL(2,\mathbf{R})$ to a dilation $z\mapsto e^{l_\gamma}z$. The dilation coefficient follows from the hyperbolic distance $d(z,z')$ defined in Eq. (4.13):

$$\min_z d(z,\gamma z)=l_\gamma\ . \tag{5.8}$$

The number of simple closed geodesics of length smaller than l is asymptotically given by $l^{-1}e^{l}[1+O(1)]$ as $l\to\infty$. For a fixed Γ, the set of lengths is bounded from below by some smallest length $l_0>0$.

Next a tensor $f(z)dz^n d\bar{z}^m$ on $H=\mathbf{H}/\Gamma$ may be identified with a function $f(z)$ on $\mathbf{H}$ transforming under Γ as

$$f(\gamma z)=(cz+d)^{2n}(c\bar{z}+d)^{2m}f(z) \quad \text{for } \gamma=\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in \mathrm{PSL}(2,\mathbf{R}) . \tag{5.9}$$

There is no ambiguity for $n+m$ integer, but for n half-integer (which corresponds to spinor fields) the sign of the trace of γ matters, and we must introduce a multiplier $\nu(\tilde{\gamma})\in\{\pm 1\}$ for $\tilde{\gamma}\subset \mathrm{SL}(2,\mathbf{R})$ projecting onto Γ. The condition (5.9) is then replaced by

$$f(\tilde{\gamma}z)=\nu(\tilde{\gamma})^{2(n+m)}(cz+d)^{2n}(c\bar{z}+d)^{2m}f(z) , \quad \tilde{\gamma}=\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in\tilde{\Gamma} . \tag{5.10}$$

In the number theory literature, functions $f(z)$ satisfying Eq. (5.10) are called automorphic forms. Their spaces are denoted by $\mathbf{T}^{n,m}$ and carry the natural inner product

$$\|f\|^2=\int_{H/\Gamma} d^2z\,|f|^2 y^{-2+n+m} , \tag{5.11}$$

which is just an inner product of the form (2.21) or (2.45) in terms of the constant-curvature metric $ds^2=2y^{-2}dz\,d\bar{z}$. Similarly, from Eqs. (2.42)–(2.44), the covariant complex derivatives $\nabla^n:\mathbf{T}^n\to\mathbf{T}^{n+1}$, $\nabla_n:\mathbf{T}^n\to\mathbf{T}^{n-1}$ become operators on automorphic forms:

$$\begin{aligned} &\nabla_n=y^2\bar{\partial}, \quad \nabla^n=y^{-2n}\partial(y^{2n}) , \\ &\Delta_n^{(+)}=\nabla_n^\dagger\nabla_n, \quad \Delta_n^{(-)}=(\nabla^n)^\dagger\nabla^n . \end{aligned} \tag{5.12}$$

It will usually be simpler to work with the space $S(n)=\mathbf{T}^{n/2,-n/2}$ and the Maass operators K_n and L_n defined by

$$\begin{aligned} &K_n:S(n)\to S(n+1), \quad K_n=(z-\bar{z})\partial/\partial z+n , \\ &L_n:S(n)\to S(n-1), \quad L_n=(z-\bar{z})\partial/\partial\bar{z}-n , \end{aligned} \tag{5.13}$$

which are isomorphic to $\mathbf{T}^n$, ∇^n, and ∇_n through the isometry $\mathbf{T}^n\ni f\mapsto y^{n/2}f\in S(n)$. The Laplacians $\Delta_n^{(\pm)}$ on $\mathbf{T}^n$ reduce to operators on $S(n)$,

$$\Delta_n^{(\pm)}=-D_{-n}+n(n\pm 1)$$

with

$$D_n=y^2\partial\bar{\partial}-2iny(\partial+\bar{\partial}) . \tag{5.14}$$

Note that D_n is real: $\bar{D}_n=D_{-n}$.

D. Poincaré series, heat kernels, Selberg trace formulas

To construct automorphic forms we rely on the method of images. More precisely, with any function $h(z)$ on Γ with suitable decrease at infinity we can associate the Poincaré series

$$\theta_h(z)=\sum_{\tilde{\gamma}\in\tilde{\Gamma}}[\nu(\tilde{\gamma})]^{2(p+q)}(cz+d)^{2p}(c\bar{z}+d)^{2q}h(\gamma z) , \tag{5.15}$$

which will then be an automorphic form of weight (p,q). In particular, we may construct the heat kernels for operators on $\mathbf{H}/\Gamma$ from heat kernels for the corresponding operators on $\mathbf{H}$. In view of Eq. (5.14), the key ingredient is then the heat kernel $g_n{}^t(z,z')$ for $-D_{-n}$ on $\mathbf{H}$, which has actually been computed by McKean (1972), Hejhal (1976b), and Fay (1977):[32]

$$g_n{}^t(z,z')=\frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}}\int_d^\infty db\frac{be^{-b^2/4t}}{\sqrt{\cosh b-\cosh d}} \times T_{2n}\left[\frac{\cosh b/2}{\cosh d/2}\right] , \tag{5.16}$$

where $T_{2n}(t)$ is the $2n$th Chebyshev polynomial and d is the hyperbolic distance between z and z'. As a consequence the heat kernel for $\Delta_n^{(\pm)}$ on $S(n)$ is given by $e^{-tn(n\pm1)}K_n{}^t(z,z')$ with

$$K_n{}^t(z,z')=\sum_{\gamma\in\tilde{\Gamma}}\nu(\gamma)^{2n}\left[\frac{c\bar{z}'+d}{cz'+d}\right]^n\left[\frac{z-\gamma\bar{z}'}{\gamma z'-\bar{z}}\right]^n \times g_n{}^t(z,\gamma z') . \tag{5.17}$$

(Observe that K_n^t is of weight n in z', but of weight $-n$ in z.)

The trace of the heat kernel is given by

$$\mathrm{Tr}(e^{-t\Delta_n^{(\pm)}})=e^{-tn(n\pm1)}\int_{\mathbf{H}/\Gamma}dx\,dy\,y^{-2}K_n^t(z,z) \tag{5.18}$$

and can be computed through Selberg trace formula techniques. The method roughly goes as follows. The heat kernel $K_n^t(z,z)$ is a sum over elements of Γ, which can be classified into conjugacy classes and elements in their centralizers. If Γ_β is the conjugacy class of β, then the integral over H/Γ of the sum over elements $k\in\Gamma/\Gamma_\beta$ can be viewed as the integral over a fundamental domain of Γ_β. But Γ_β is cyclic and of the form $\{\gamma^p, p\in Z\}$ for some primitive element γ. Since the heat kernel in the upper half space is invariant under SL(2,**R**), we may now conjugate γ within SL(2,**R**) to a dilation, choose a fundamental domain for Γ_γ to be of the form $\{-\infty<x<\infty, 1\le y<e^l\}$, and carry out the integrals explicitly. Using the generating function for Chebyshev polynomials, we obtain

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[32]John Fay has kindly pointed out to us that the discrete series that occurs in his expression for g^t is erroneous and should be deleted, so that one indeed obtains Eq. (5.16).

$$\mathrm{Tr}(e^{-t\Delta_n^{(\pm)}})=e^{-tn(n\pm1)}I^n(t)+e^{-tn(n\pm1)}I_e^n(t)\ , \tag{5.19}$$

$$\begin{aligned} I^n(t)&=\sum_{\gamma\ \mathrm{primitive}}\sum_{p=1}^{\infty}\nu(\gamma)^{2np}\frac{1}{\sinh pl/2}\frac{e^{-t/4}}{4\sqrt{\pi t}}e^{-p^2l^2/4t}\ ,\\ I_e^n(t)&=|\chi(M)|\sum_{0\le m<|n|-1/2}(2|n|-2m-1)e^{(|n|-m)(|n|-m-1)t}\\ &\quad+|\chi(M)|\frac{e^{-t/4}}{2\sqrt{\pi}t^{3/2}}\int_0^\infty db\frac{be^{-b^2/4t}}{\sinh b/2}\cosh(|n|-[|n|])b\ . \end{aligned} \tag{5.20}$$

Here γ and γ^{-1} are counted as distinct primitive elements. We have singled out the contribution $I_e^n(t)$ of the identity element in the Poincaré series, which encodes all the short-time information of the heat kernel. Note that $I^n(t)$ depends only on whether the field is a tensor or a spinor, and otherwise not on its weight n.

From Eq. (5.20) we can obtain the number $N_n^{\pm}[=\lim_{t\to\infty}\mathrm{Tr}(e^{-t\Delta_n^{(\pm)}})]$ of zero modes of $\Delta_n^{(\pm)}$. Since $N_0^{\pm}$ is just 1 and $\mathrm{Tr}(e^{-t\Delta_{1/2}^{(-)}})$ is certainly bounded as $t\to\infty$, it follows that $I^0(t)\to1$ and $|I^{1/2}(t)|\le O(e^{-t/4})$ as $t\to\infty$. W can now combine this with asymptotics for $I_e^n(t)$ to deduce that $N_n^+=0$ for $n\ge\frac{1}{2}$, $N_1^-=2h$, $N_n^-=(2n-1)|\chi(M)|$ for $n\ge\frac{3}{2}$. The number $N_{1/2}^-=N_{-1/2}^+$ of zero modes of the Dirac operator satisfies no such simple formula, since it depends in general on both the spin and the complex structure [cf. Eq. (2.51) and Sec. VI.F].

Automorphic forms are discussed by Ford (1951). Fay's formula for the heat kernel appeared in Fay (1977). The Selberg trace formula was introduced by Selberg (1956) and applied in McKean (1972) and McKean and Singer (1967) for the scalar Laplacian. An extensive discussion is in Hejhal (1976b). The above generalization based on Maass operators and Fay's formula (5.16) appeared in D'Hoker and Phong (1986d). The generalization to the case of the superstring is discussed by Aoki (1988).

E. Zeta-function regularization

The determinants will be evaluated through zeta-function regularization,[33]

$$\begin{aligned} \det{}'\Delta_n^{(\pm)}&=\exp[-\zeta_n^{\pm\prime}(0)]\ ,\\ \zeta_n^{\pm}(s)&=\mathrm{Tr}'(\Delta_n^{(\pm)})^{-s}\\ &=\frac{1}{\Gamma(s)}\int_0^\infty dt\ t^{s-1}[\mathrm{Tr}(e^{-t\Delta_n^{\pm}})-N_n^{\pm}]\ . \end{aligned} \tag{5.21}$$

With the above formulas for the heat kernel, it is simplest to adapt to our context the elegant method of Fried (1986a). Set

$$\begin{aligned} M_n^{\pm}(s)&=\int_0^\infty dt\ t^{s-1}e^{-tn(n\pm1)}I(t)\ ,\\ M_{n,e}^{\pm}(s)&=\int_0^\infty dt\ t^{s-1}e^{-tn(n\pm1)}I_e^n(t)\ , \end{aligned} \tag{5.22}$$

and choose $\alpha_n^{\pm}$ so that $M_{n,e}^{\pm}(s)-\alpha_n^{\pm}\Gamma(s)$ will be holomorphic at $s=0$. From Eq. (5.20) we see readily that α_n^+ should be taken as the constant term in the short-time expansion of $e^{-t(n\pm1)n}I_e^n(t)$ for $n\ge-\frac{1}{2}$, while α_n^- should be the constant term in the short-time expansion of $e^{-tn(n-1)}I_e^n(t)-(2n-1)|\chi(M)|$ for $n\ge1$. [This distinction is based on the asymptotic behavior for large t of $e^{-t(n\pm1)n}I_e^n(t)$, which is at most $\sim t^{-1/2}$ in the first case while it is $(2n-1)|\chi(M)|$ in the second case.] The exact value at $s=0$ of $M_{n,e}^{\pm}-\alpha_n^{\pm}\Gamma(s)\equiv c_n^{\pm}|\chi(M)|$ can actually be computed to be

$$c_n^+=c_n,\quad c_n^-=c_{n-1}\ , \tag{5.23}$$

with c_n as in Eq. (5.7). Returning to $\zeta_n^{\pm\prime}(0)$ we decompose it as

$$\begin{aligned} \zeta_n^{\pm\prime}(0)&=\lim_{s\to0^-}[M_n^{\pm}(s)-\Gamma(s)\zeta_n^{\pm}(0)+\alpha_n^{\pm}\Gamma(s)]\\ &\quad+\lim_{s\to0^-}[M_n^{\pm}(s)-\alpha_n^{\pm}\Gamma(s)]\ , \end{aligned} \tag{5.24}$$

where the zeta function is known to be holomorphic near $s=0$ by the general theory of functional determinants. To express it in terms of number-theoretical zeta functions, rewrite t^{s-1} in Eq. (5.22) in terms of its Mellin transform, change the order of integration, and evaluate the integrals explicitly to obtain

$$\begin{aligned} M_n^{\pm}(s)&=\frac{1}{\Gamma(1-s)}\int_0^\infty d\lambda[\lambda(\lambda+2|n\pm\tfrac{1}{2}|)]^{-s}\sum_{\gamma,p}\nu(\gamma)^{2np}\frac{l}{2\sinh pl/2}e^{-pl(\lambda+|n\pm1/2|)}\\ &=\frac{1}{\Gamma(1-s)}\int_0^\infty d\lambda[\lambda(\lambda+2|n\pm\tfrac{1}{2}|)]^{-s}\frac{Z'(\lambda+|n\pm\frac{1}{2}|+\frac{1}{2})}{Z(\lambda+|n\pm\frac{1}{2}|+\frac{1}{2})}\ . \end{aligned} \tag{5.25}$$

[33]The result from zeta-function regularization differs from that of small-time cutoff by harmless factors involving the area and the Euler characteristic of the worldsheet, and the number of zero modes.

This is the key relation linking the heat kernel and Selberg zeta functions, allowing us to determine the poles at $s=0$ of $M_n^{\pm}(s)$ in terms of the order of vanishing $r_n^{\pm}$ of $Z(s)$ at $s=|n\pm\frac{1}{2}|+\frac{1}{2}$. In fact, for small $\varepsilon>0$ we can split the integral representing $M_n^{\pm}(s)$ into an integral over $\lambda>\varepsilon$ and an integral over $\lambda<\varepsilon$. The first is holomorphic at $s=0$ and behaves like $-r_n^{\pm}/s+r_n^{\pm}\ln|n\pm\frac{1}{2}|+r_n^{\pm}\ln\varepsilon+O(s)$ for $n\pm\frac{1}{2}\neq 0$, and like $-r_n^{\pm}/2s+r_n^{\pm}\ln\varepsilon$ for $n\pm\frac{1}{2}=0$. Since the pole of the second integral must cancel that of $\Gamma(s)[\zeta_n^{\pm}(0)-\alpha_n^{\pm}]$, it follows that $r_n^{\pm}=\alpha_n^{\pm}-\zeta_n^{\pm}(0)$ for $n\pm\frac{1}{2}\neq 0$, $r_n^{\pm}=2[\alpha_n^{\pm}-\zeta_n^{\pm}(0)]$ for $n\pm\frac{1}{2}=0$. Now it is easy to see that $\zeta_n^{\pm}(0)$ equals the difference between the constant term in the short-time expansion of $\mathrm{Tr}(e^{-t\Delta_n^{\pm}})$ and the number of zero modes $N_n^{\pm}$. Recalling the definition of $\alpha_n^{\pm}$ and the formulas for $N_n^{\pm}$ we obtain at once

$$
\begin{aligned}
&r_{-1/2}^{+}=2N_{-1/2}^{+},\quad r_{1/2}^{-}=2N_{1/2}^{-}\ ,\\
&r_0^{+}=1,\quad r_1^{-}=1\ , \qquad\qquad (5.26)\\
&r_n^{+}=0\quad\text{for } n\geq\tfrac{1}{2},\quad r_n^{-}=0\quad\text{for } n\geq\tfrac{3}{2}\ .
\end{aligned}
$$

This cancellation of the poles leaves us with

$$
M_n^{\pm}(s)-\Gamma(s)\zeta_n^{\pm}(0)+\alpha_n^{\pm}\Gamma(s)\,|_{s=0}
=-\ln\left[\frac{1}{r_n^{\pm}!}Z^{(r_n^{\pm})}(|n\pm\tfrac{1}{2}|+\tfrac{1}{2})\right] \qquad (5.27)
$$

and thus

$$
\det{}'\Delta_n^{(\pm)}=e^{-c_n^{\pm}\chi(M)}\frac{1}{(r_n^{\pm})!}Z^{(r_n^{\pm})}(|n\pm\tfrac{1}{2}|+\tfrac{1}{2})\ . \qquad (5.28)
$$

This formula includes Eqs. (5.7) as special cases.

In the mathematical literature, zeta-function regularization of determinants goes back to Ray and Singer (1971). The above techniques were used to evaluate determinants by Fried (1986a, 1986b), D'Hoker and Phong (1986a, 1986d), and Sarnak (1987).

F. Asymptotics for determinants

Physical quantities are given in string theory by integrals over moduli space. The integrands have no singularity inside, so the only possible divergences must come from their asymptotic behavior near the boundary of moduli. The importance of boundary contributions has emerged before in Sec. II.K in our discussion of BRST invariance. As explained in Sec. IV.H, this boundary corresponds to the length of some closed geodesic on the worldsheet tending to 0, and our first task is to study the behavior of the determinants (5.7) in such a limit. As expected, the partition function for the bosonic string will diverge. For the fermionic string the evidence suggests that the contribution from some spin structures will diverge as well, so that finiteness of superstring amplitudes (if true) must result from delicate cancellations between various spin structures.

In view of Eqs. (5.7) the asymptotic behavior of the absolute values of the determinants of string theory reduces to that of special values of Selberg zeta functions. Let l_0 be the length of the closed geodesic γ_0 that is being pinched. We begin by noting that the asymptotics of its contribution to $Z(s)$ for $\nu(\gamma_0)=1$ can be determined from the Jacobi identity,

$$
\prod_{k=1}^{\infty}(1-e^{-(s+k)l_0})\sim l_0^{-s+1/2}\exp(-\pi^2/6l_0)\ . \qquad (5.29)
$$

For values of $\mathrm{Re}(s)>1$ this actually gives already the asymptotics for the full $Z(s)$. The key to understanding this lies in the collar phenomenon when a geodesic tends to 0. As l_0 tends to 0, a collar, i.e., a region diffeomorphic to a cylinder, around the geodesic will stretch out with its length of the size of $\ln 1/l_0$. Outside the collar the area and diameters remain bounded independently of l_0 (see Fig. 20). (Note that this is consistent with the fact that the area must remain constant for hyperbolic metrics of fixed negative curvature.) It is also indicated by the plumbing fixture constructions of Sec. IV.H. This suggests dividing the remaining geodesics in the infinite product defining $Z(s)$ into two groups. The first group consists of the geodesics not intersecting γ_0. Their contributions will tend to the special value of the Selberg zeta function of the punctured surface obtained at $l_0=0$. These are well behaved and will merely contribute an asymptotic constant. The second group consists of geodesics intersecting γ_0. Since these geodesics must cross the collar, their lengths must go to ∞, and hence their contribution will tend to 1. (Strictly speaking, before these lengths increase they may first decrease due to the fact that they may wrap around γ_0 a large number of times.) Thus recalling that γ_0 and γ_0^{-1} are counted as distinct primitive geodesics in the Selberg zeta function we deduce that

$$
Z(s)\sim l_0^{-2s+1}\exp(-\pi^2/3l_0),\quad \mathrm{Re}(s)>1\ . \qquad (5.30)
$$

The cases of $d^kZ(s)/ds^k$ for $\mathrm{Re}(s)\leq 1$ are more difficult, since analytic continuation is needed to define these values. It is a good heuristic principle, however, that up to smaller factors the asymptotics of $d^kZ(s)/ds^k$ (k is the first integer for which the derivative does not vanish) are given by the same formula (5.29) coming from the asymptotics of the terms in $Z(s)$ involving the pinching geodesics. This heuristic principle is justified by the precise

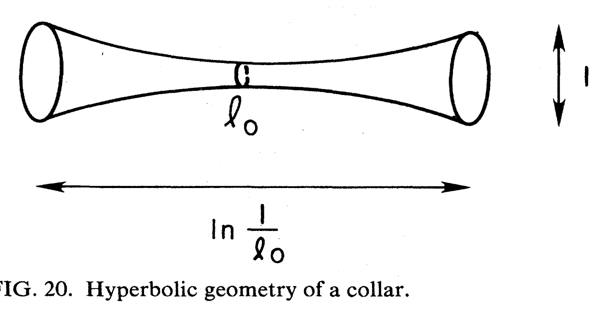

FIG. 20. Hyperbolic geometry of a collar.

formula for $Z'(1)$ obtained recently by Gava *et al.* (1986), Wolpert (1986), and Hejhal (1987):

$$Z'(1) \sim l_0^{-1} \exp(-\pi^2/3l_0) \prod_{0<\lambda_n<1/4} \lambda_n . \qquad (5.31)$$

Here λ_n are the eigenvalues of the Laplacian on scalars on the Riemann surface. It is known from work by Schoen *et al.* (1980) and Dodziuk *et al.* (1986) that there are at most $4h-2$ eigenvalues that are less than $\frac{1}{4}$, and that the lowest eigenvalues λ_n are of the size of the sum of lengths of closed geodesics disconnecting the surface into $(n+1)$ components. Thus for a closed geodesic γ_0 of the type in Fig. 19, λ_1 is of the order of l_0. (Observe that this does not contradict the fact that the diameter of the surface is of the order of $\ln 1/l_0$. The reason is that, in the hyperbolic metric, the area element of a cylinder grows exponentially. More precisely, in the energy integral the contribution from the complement of the collar remains bounded, while by conformal invariance the collar contributions are the same as an energy integral for a Euclidean cylinder of radius 1 and length $\sim 1/l_0$.) Thus the additional factors involving λ_n in Eq. (5.31) are of lower order than the main terms.

In the presence of a spin structure $\nu(\gamma)$ with $\nu(\gamma_0)=-1$, the asymptotics of the contributions from γ_0 to $Z(s)$ become

$$\prod_{k=1}^{\infty} (1+e^{-(s+k)l_0}) \sim e^{\pi^2/12 l_0} , \qquad (5.32)$$

and the above arguments apply when $\mathrm{Re}(s)>1$. No precise asymptotics such as those of Eq. (5.31) have been established rigorously for $d^{2N_{1/2}^-} Z(s)/ds^{2N_{1/2}^-}|_{s=1/2}$ at the present time. There is, nevertheless, a general method that should give good information in principle on any $Z^{(k)}(\rho)$ for $\mathrm{Re}(\rho)\leq 1$. This method, based on functional equations, goes back to Lavrik and was suggested in this context by Goldfeld. Roughly speaking $Z^{(k)}(\rho)$ can be obtained by integrating $Z(\rho+s)/s^k$ on a vertical line far to the right in the s plane, and shifting the line of integration far to the left, picking up the only pole in $Z(\rho+s)/s^k$ at $s=0$. The functional equation for Z allows us to rewrite the integral on the far left as an integral on the far right, where the infinite products for Z_ν converge absolutely and collar arguments are valid. Asymptotics follow in principle by expanding the zeta functions into Dirichlet series in l. Applied, for example, to $Z'(1)$, this method gives back Eq. (5.31) with the precise factor $\prod \lambda_n$ replaced by $O(l^{-\varepsilon})$ for any $\varepsilon>0$. For $Z^{(2N)}(\frac{1}{2})$ we expect it to confirm the heuristic principle stated above again, with a possible uncertainty of $O(l_0^{-\varepsilon})$.

In the above we have written down formulas for the pinching of only one geodesic. It should, however, be evident that they can be extended to the case of several pinching geodesics, and that the maximum number of geodesics that can be pinched independently is $3h-3$.

Before returning to string partition functions, we will need one more ingredient, namely, the asymptotic behavior of the Weil-Petersson measure. Recalling the correspondence between the complex coordinate t defining the divisor Δ of Riemann surfaces with nodes and the length of a pinching geodesic,

$$|t| \sim \exp(-2\pi^2/l_0) , \qquad (5.33)$$

we note that the Weil-Petersson metric is described near Δ by Masur (1976),

$$d(\mathrm{WP}) \sim \prod_j |dt_j^2| / |t_j|^2 (\ln 1/|t_j|)^3 . \qquad (5.34)$$

For the bosonic string, the partition function is the integral over moduli of

$$\exp[c\chi(M)] Z'(1)^{-13} Z(2) d(\mathrm{WP}) , \qquad (5.35)$$

which in view of Eqs. (5.30)–(5.35) behaves up to smaller factors as

$$\prod_j |dt_j^2| \, |t_j|^{-4} . \qquad (5.36)$$

This is the double-pole behavior obtained by Belavin and Knizhnik (1986) using essentially holomorphicity and characteristic classes arguments. A rigorous treatment along Selberg zeta-function lines as above may be found in Wolpert (1987).

Assuming the heuristic principle stated in the previous paragraph for asymptotics of Selberg zeta functions beyond $\mathrm{Re}(s)>1$ and neglecting the factors arising from supermoduli in the superstring functional integrals, we can derive similar asymptotics as well for fermionic partition functions. The importance of spin structures then becomes manifest, since the asymptotic behavior of fermionic determinants can change drastically if the sign of parallel transport around γ_0 is flipped. Thus for "wrong-spin structures" we cannot have cancellation between bosonic and fermionic determinants and must hope instead for a cancellation between the various spin structures. In physical terms, this means that finiteness of superstring theories must come here from space-time supersymmetry rather than worldsheet supersymmetry. Some of these issued have also been addressed by Iengo (1987) and Bonini and Iengo (1987a, 1987b).

G. Determinants on Mandelstam diagrams and unitarity

We have shown in Sec. II.L that Polyakov string amplitudes may be obtained in two different ways, yielding, however, the same answer. In the first approach, reparametrization-invariant vertex operators that satisfy certain Weyl invariance conditions are inserted on a compact surface, and their positions are integrated over. To obtain the full amplitude, one sums over all (inequivalent) compact surfaces of a given number of handles. In the second approach, one sums instead over surfaces of a given number of handles, with each vertex operator replaced by a puncture on the surface. Thus one sums over all surfaces of given genus h and given number of punctures n, corresponding to n vertex operators.

Now it must have become clear from Secs. IV.F and IV.G that very nice parametrizations are available for surfaces and their moduli spaces as soon as one allows for at least one puncture. Here, we shall only consider the case of Mandelstam diagrams, for which it was proven in Sec. IV.G that with their natural ranges of parameters, the Mandelstam diagrams parametrize the moduli space $\mathcal{M}_{h,n}$ of surfaces with n punctures precisely once.

This important result would be even more useful in string theory if the measure and the integrand for the scattering amplitudes were to assume a relatively simple and explicit form. We shall not analyze this question in full here, but restrict ourselves to showing the following important simplifications that occur when evaluating ghost determinants on the Mandelstam diagrams. The determinant for the spin-1 ghost, together with its finite-dimensional determinant involving holomorphic quadratic differentials, is simply given in terms of the determinant on spin-0 scalars. Similarly, the determinant for the spin-$\frac{1}{2}$ superghost is simply related to the Dirac determinant.

1. The spin-1 ghost determinant

Recall the quadratic holomorphic differentials on the Mandelstam diagram that were produced in Sec. IV.G:

$$\begin{aligned}\phi_I &= \omega\omega_I, \quad I=1,\ldots,h \ ,\\ \phi_a &= \omega\omega_{P_0P_a}, \quad a=1,\ldots,2h+n-3 \ ,\end{aligned} \tag{5.37}$$

where ω, ω_I, and $\omega_{P_0P_a}$ are the canonical differentials, Abelian of first and third kind, respectively. Using the Riemann bilinear relations of Appendix D, it is easy to compute the corresponding inner products,

$$\begin{aligned}\langle \phi_I \mid \phi_J\rangle &= 4\,\mathrm{Im}\Omega_{IJ}, \quad \langle \phi_I \mid \phi_a\rangle = 0 \ ,\\ \langle \phi_a \mid \phi_b\rangle &= 2\,\mathrm{Re}\int_{P_0}^{P_b}\omega_{P_0P_a} \equiv 2G_{P_0}(P_a,P_b) \ ,\end{aligned} \tag{5.38}$$

where Ω is the period matrix and G is symmetric in P_a and P_b. Actually, the function $G_{P_0}(P_a,P_b)$ is a Green's function for the scalar Laplacian on the Riemann surface, as can be seen by considering

$$\begin{aligned}\Delta_P G_{P_0}(P,Q) &= -2\frac{\partial}{\partial \bar P}\omega_{P_0Q}(P)\\ &= 2\pi[\delta(P,Q)-\delta(P,P_0)] \ .\end{aligned} \tag{5.39}$$

Having the inner products between ϕ's, it is straightforward to compute the finite-dimensional determinants

$$(\det\langle \phi_\alpha \mid \phi_\beta\rangle)^{1/2} = 2^h \det(\mathrm{Im}\Omega)\det G_{P_0}(P_a,P_b) \ , \tag{5.40}$$

where the latter determinant is taken of a $(2h+n-3)\times(2h+n-3)$ matrix. Similarly, the finite-dimensional determinant involving the Beltrami differentials μ_α can be evaluated by using the quasiconformal vector fields listed in Eq. (2.59). A somewhat lengthy argument, given in D'Hoker and Giddings (1987), allows one to find that

$$\det\langle \mu_\alpha \mid \phi_\beta\rangle = (8\pi)^h(4\pi)^{2h+n-3}\det(\mathrm{Im}\Omega) \ . \tag{5.41}$$

Next we evaluate the infinite-dimensional Faddeev-Popov ghost determinant $\det{}^* P_1^\dagger P_1$, considered on those reparametrization vector fields V^z and $V^{\bar z}$ that vanish at the punctures and are regular anywhere else. With the help of the canonical differential $\omega = \omega_z dz$, such vector fields may be rewritten in terms of scalar fields ϕ,

$$V^z = \frac{1}{\omega_z}\phi, \quad V^{\bar z} = \frac{1}{\omega_{\bar z}}\bar\phi \ , \tag{5.42}$$

provided the scalar field ϕ vanishes at the interaction points P_a where ω_z also vanishes. Provided ϕ is continuous at the punctures, V^z will automatically vanish there. Away from interaction points and punctures, the operators $P_1^\dagger P_1$ and Δ coincide, since the metric is Euclidean. Actually, the only reason they differ is that they act on vectors and scalars, respectively. But we have established above a correspondence between these two, and so the determinant of $P_1^\dagger P_1$ on vector fields vanishing at punctures equals the determinant of Δ vanishing at the interaction points (again indicated by an asterisk),

$$(\det{}^* P_1^\dagger P_1)^{1/2} = \det{}^*\Delta \ . \tag{5.43}$$

This is easily evaluated with the functional integral representation by inserting delta functions of the scalar field at the interaction points:

$$\begin{aligned}(\det{}^*\Delta)^{-1/2} &= \int \mathcal{D}\phi\, e^{-\langle \bar\phi \mid \Delta\phi\rangle/8\pi}\delta(\phi(P_0))\\ &\quad\times\delta(\phi(P_1))\cdots\delta(\phi(P_{2h+n-3})) \ .\end{aligned} \tag{5.44}$$

Delta functions can be represented by their Fourier transform, the constant ϕ mode can be integrated out, and the remaining Gaussian integral (with source at P_a) evaluated in terms of the Green's function $G_{P_0}(P_a,P_b)$ satisfying Eq. (5.39), and one finds

$$\begin{aligned}(\det{}^* P_1^\dagger P_1)^{1/2} &= \det{}^*\Delta\\ &= \left[\frac{8\pi^2}{\int_M d^2\xi\sqrt{g}}\det{}'\Delta\right]\det G_{P_0}(P_a,P_b) \ .\end{aligned} \tag{5.45}$$

It is instructive to combine this answer with that of the finite-dimensional determinants in Eq. (5.40):

$$\left[\frac{\det{}^* P_1^\dagger P_1}{\det\langle \phi_\alpha \mid \phi_\beta\rangle}\right]^{1/2} = 2^{-h}\frac{8\pi^2\det{}'\Delta}{\int_M d^2\xi\sqrt{g}\,\det\mathrm{Im}\Omega} \ . \tag{5.46}$$

The equivalence of the above determinants, together with the equivalence of the formulations with vertex operators and with punctures was obtained by D'Hoker and Giddings (1987) and was used to establish equivalence between the Polyakov approach and the interacting string picture. Since the latter is (formally) unitary by construction—recall it has a tachyon—this establishes the unitarity of the Polyakov approach. A direct comparison was also made by Sonoda (1987b).

2. The superghost determinant for even-spin structure

For even-spin structure and a generic point in moduli space, there is a unique meromorphic $\frac{1}{2}$ differential[34]—the Szegö kernel—with a single pole at P,

$$S_\nu(z,P)=\kappa_P(z)\sim\frac{(dz)^{1/2}}{z-P}\ ,$$

and on a Mandelstam diagram with canonical differential ω, the holomorphic $\frac{3}{2}$ forms are given by

$$\rho_a=\omega\kappa_{P_a},\quad a=1,2,\ldots,2h+n-2\ , \tag{5.47}$$

as was shown in Sec. IV.G. The matrix of inner products of $\omega\kappa_P$ with P not necessarily at an interaction point is closely related to the Dirac Green's function

$$S(P,Q)=\langle\omega\kappa_Q\,|\,\omega\kappa_P\rangle=\int(\omega\bar\omega)^{1/2}\kappa_P\overline{\kappa_Q}\ . \tag{5.48}$$

In conformal coordinates the Dirac operator is given by

$$\not{D}=\nabla^z_{1/2}\oplus\nabla_z^{-1/2}\ , \tag{5.49}$$

with

$$\begin{aligned}\nabla^z_{1/2}&=g^{z\bar z}\frac{\partial}{\partial\bar P}\ ,\\ (\nabla^z_{1/2})^\dagger&=-(g_{z\bar z})^{-1/2}\frac{\partial}{\partial P}(g_{z\bar z})^{1/2}\ ,\end{aligned} \tag{5.50}$$

so that

$$\begin{aligned}\not{D}\not{D}_PS(P,Q)&=-(\omega\bar\omega)^{-1/2}(P)\frac{\partial}{\partial P}(\omega\bar\omega)^{-1/2}(P)\\ &\quad\times\int(\omega\bar\omega)^{1/2}(z)\frac{\partial\kappa_P(z)}{\partial\bar P}\overline{\kappa_Q(z)}\\ &=2\pi(\omega\bar\omega)^{-1/2}(P)\frac{\partial}{\partial P}\overline{\kappa_Q(P)}\\ &=4\pi^2\delta^2(P,Q)\ .\end{aligned} \tag{5.51}$$

Thus we obtain

$$\det\langle\rho_a\,|\,\rho_b\rangle=(2\pi)^{2h+n-2}\det S(P_a,P_b)\ , \tag{5.52}$$

a formula very similar to Eq. (5.40).

The infinite-dimensional determinant for the superghost $\det{}^*P^\dagger_{1/2}P_{1/2}$, considered on spinor fields that vanish at the punctures, is also easily computed, since it can be related to the Dirac determinant $\det{}^*\not{D}$ over spinor fields that vanish at the interaction points. We shall not reproduce this calculation here and only quote the answer,

$$(\det{}^*P^\dagger_{1/2}P_{1/2})^{1/2}=(\det\not{D})\det S(P_a,P_b)\ . \tag{5.53}$$

[34]In Sec. VI.F we shall write down this differential explicitly in terms of ϑ functions.

Combining it with the expression for the finite-dimensional determinants, we find

$$\left[\frac{\det{}^*P^\dagger_{1/2}P_{1/2}}{\det\langle\rho_a\,|\,\rho_b\rangle}\right]^{1/2}=\det\not{D}\ . \tag{5.54}$$

3. The superghost determinant for odd-spin structure

For odd-spin structure δ and a generic point in moduli space, there is one holomorphic $\frac{1}{2}$ differential h_δ and a unique meromorphic $\frac{1}{2}$ differential[35] κ_{PQ} with simple poles at P and Q and unit residue at P. Holomorphic $\frac{3}{2}$ differentials on the Mandelstam diagram are given by (see Sec. IV.G)

$$\begin{aligned}&\rho_a=\omega\kappa_{P_aP_{a+1}},\quad a=1,2,\ldots,2h+n-3\ ,\\ &\rho_{2h-n-2}=\omega h_\delta\ .\end{aligned} \tag{5.55}$$

Actually, the meromorphic differential $\kappa_{PP'}(z)$ depends both on an auxiliary point R, where it has a zero, and on the spin structure. We shall denote such a differential (in particular the one exhibited in Sec. VI.F) by $S_\delta(z,P,R,P')$, and reserve for κ_{PQ} the one that is orthogonal to h_δ:

$$\kappa_{PP'}(z)=S_\delta(z,P,R,P')-A(P,R,P')h_\delta(z)$$

and

$$A(P,R,P')=\frac{\int(\omega\bar\omega)^{1/2}(z)\overline{h_\delta(z)}S_\delta(z,P,R,P')}{\langle h_\delta\,|\,h_\delta\rangle}\ . \tag{5.56}$$

Note that the normalization $\langle h_\delta\,|\,h_\delta\rangle$ is independent of P,R,P'. Now κ_{PQ} satisfies two essential equations (they can be established using the results of Sec. VI.F),

$$\begin{aligned}\frac{1}{2\pi}\frac{\partial}{\partial\bar z}\kappa_{PP'}(z)&=\delta^2(z-P)-\delta^2(z-P')\frac{h_\delta(P)}{h_\delta(P')}\ ,\\ \frac{1}{2\pi}\frac{\partial}{\partial\bar P}\kappa_{PP'}(z)&=-\delta^2(z-P)+\frac{h_\delta(z)\overline{h_\delta(P)}}{\langle h_\delta\,|\,h_\delta\rangle}(\omega\bar\omega)^{1/2}(P)\ .\end{aligned} \tag{5.57}$$

It is now easy to show that the inner product between $\frac{3}{2}$ differentials produces a propagator for the square of the Dirac operator:

$$\begin{aligned}S'(P,Q)&=\langle\omega\kappa_{QQ'}\,|\,\omega\kappa_{PP'}\rangle\\ &=\int(\omega\bar\omega)^{1/2}(z)\overline{\kappa_{QQ'}(z)}\kappa_{PP'}(z)\ .\end{aligned} \tag{5.58}$$

This is seen by applying a derivative in $\bar P$:

$$\frac{\partial}{\partial\bar P}S'(P,Q)=\int(\omega\bar\omega)^{1/2}(z)\overline{\kappa_{QQ'}(z)}\frac{\partial}{\partial\bar P}\kappa_{PP'}(z)\ . \tag{5.59}$$

Using the second equation in (5.57) and the orthogonality of $\kappa_{QQ'}$ and h_δ, we get

[35]In Sec. VI.F we shall give explicit formulas for both these differentials in terms of ϑ functions.

$$\frac{\partial}{\partial \bar{P}} S'(P,Q) = -2\pi(\omega\bar{\omega})^{1/2}(P)\overline{\kappa_{QQ'}(P)} \ . \quad (5.60)$$

Applying now the Dirac operator as in Eq. (5.51), we find

$$\not{D}\not{D}_P S'(P,Q) = 2\pi(\omega\bar{\omega})^{-1/2}(P)\frac{\partial}{\partial P}\overline{\kappa_{QQ'}(P)} \ , \quad (5.61)$$

and using the first equation in (5.57), we get

$$\not{D}\not{D}_P S'(P,Q) = 4\pi^2\delta^2(P,Q) - 4\pi^2\delta^2(P,Q')\frac{h_\delta(Q)}{h_\delta(Q')} \ . \quad (5.62)$$

Thus the matrix of inner products of all holomorphic $\frac{3}{2}$ differentials is given by

$$\det\langle \rho_a \mid \rho_b \rangle = \langle h_\delta \mid h_\delta \rangle \det S'(P_a,P_b) \ , \quad (5.63)$$

where the last determinant is over a $(2h+n-3)\times(2h+n-3)$ matrix. It should be noted that the Green's function S' depends on the auxiliary points P' and Q' and that one has the property

$$S'(P',Q) = S'(P,Q') = 0 \quad (5.64)$$

for fixed P' and Q'. Computing the infinite-dimensional determinant

$$(\det{}^* P_{1/2}^\dagger P_{1/2})^{1/2} = \det{}^* \not{D} \quad (5.65)$$

is done by functional integral methods again, and it is a matter of patiently sorting out the zero-mode contribution and using Eq. (5.64) to obtain

$$(\det{}^* P_{1/2}^\dagger P_{1/2})^{1/2} = h_\delta(P')\overline{h_\delta(Q')}(\det{}'\not{D}) \times \det S'(P_a,P_b) \ . \quad (5.66)$$

Combining this result with the finite-dimensional determinant of the holomorphic $\frac{3}{2}$ differentials, we obtain the remarkable relation

$$\left[\frac{\det{}^* P_{1/2}^\dagger P_{1/2}}{\det\langle \rho_a \mid \rho_b \rangle}\right]^{1/2} = \frac{\det{}'\not{D}}{\langle h_\delta \mid h_\delta \rangle} h_\delta(P')\overline{h_\delta(Q')} \ . \quad (5.67)$$

A more detailed treatment with appropriate regularizations is in D'Hoker and Phong (1988b). Equations (5.54) and (5.67) are crucial ingredients in a unitarity proof of the fermionic string in component language (D'Hoker and Phong, 1988).

VI. COMPLEX GEOMETRY OF MODULI SPACE

In this section, we present the necessary mathematical background for the study in Sec. VII of the holomorphic structure of strings. A key ingredient is the topology and geometry of line bundles, so we begin with a short survey of their formalism. Accessible full treatments of the theory of line bundles over Riemann surfaces are provided by Gunning (1966, 1967), Hirzebruch (1966), and Griffiths and Harris (1978). The basic mathematical references for Secs. VI.D–VI.F are Fay (1973) and Mumford (1975, 1983).

A. Line bundles, Chern classes, and curvature

Let M be a smooth manifold. A line bundle L on M is an assignment of a one-dimensional complex vector space L_z to each point z of M. Sections of L are then functions assigning an element of L_z to each z. The vector spaces L_z should fit together smoothly, and we enforce this in two stages. First locally, i.e., for all z in small coordinate charts $\{B_\alpha\}$ for M, the set $\{L_z\}_{z\in B_\alpha}$ should just become isomorphic to a product $\mathbf{C}\times B_\alpha$, so that a section f of L on B_α should reduce to a smooth $\mathbf{C}$-valued function f_α on B_α. Second, there should exist smooth nowhere-vanishing complex functions $\phi_{\alpha\beta}$ defined on $B_\alpha \cap B_\beta$ so that the f_α's arising from a section f of L are characterized by the condition

$$f_\alpha = \phi_{\alpha\beta} f_\beta \quad \text{on } B_\alpha \cap B_\beta \ . \quad (6.1)$$

Clearly the $\phi_{\alpha\beta}$ themselves must satisfy the consistency condition

$$\phi_{\alpha\beta}\phi_{\beta\gamma} = \phi_{\alpha\gamma} \ . \quad (6.2)$$

They are called *transition functions* and describe L completely. Examples of line bundles are the space $\mathbf{T}^n$ on Riemann surfaces M, encountered earlier in Sec. II.E, where L_z is the $(-n)$th power of the tangent space to M of z, and transition functions are $(\partial z_\alpha/\partial z_\beta)^{-n}$ with z_α, z_β coordinate systems for the patches B_α and B_β. A more sophisticated example relevant to anomalies is that of determinant bundles. Here the manifold M is, for example, the space of metrics on a surface, and the vector space L_g at a metric g is

$$(\overset{\max}{\wedge} \operatorname{Ker}\bar{\partial}_n)^{-1} \otimes (\overset{\max}{\wedge} \operatorname{Ker}\bar{\partial}_n^\dagger) \ .$$

For chiral anomalies, M is instead the space of vector potentials A_μ, and L_{A_μ} at A_μ is similarly built out of zero modes for the Dirac operator coupled to A_μ and its adjoint. In these situations we note that the number of zero modes may jump, and it is a subtle issue to define properly the transition functions. This fact has important ramifications that will be discussed at length in Sec. VII.E.

Our next task is to introduce a topological classification of line bundles over a manifold. For this write the transition functions as $\phi_{\alpha\beta} = \exp(\psi_{\alpha\beta})$; then Eq. (6.2) is equivalent to $\psi_{\alpha\beta} + \psi_{\beta\gamma} - \psi_{\alpha\gamma} = 2\pi i(c_{\alpha\beta\gamma})$. The integers $c_{\alpha\beta\gamma}$ satisfy

$$c_{\alpha\beta\gamma} - c_{\beta\gamma\delta} + c_{\gamma\delta\alpha} - c_{\delta\alpha\beta} = 0 \ . \quad (6.3)$$

Since the $\psi_{\alpha\beta}$ are defined only modulo $2\pi i n_{\alpha\beta}$ with $n_{\alpha\beta}$ integers, we should identify two sets $c_{\alpha\beta\gamma}$ and $c'_{\alpha\beta\gamma}$ differing by elements of the form

$$n_{\alpha\beta} + n_{\beta\gamma} - n_{\alpha\gamma} \ . \quad (6.4)$$

Coefficients $(c_{\alpha\beta\gamma})$ satisfying (6.3) are called *closed cocycles,* while those that can be written in the form (6.4) are called *exact cocycles.* The space of closed cocycles modulo the exact ones is the second (Čech) cohomology group $H^2(M,\mathbf{Z})$ (with coefficients in $\mathbf{Z}$), and our discussion has shown that to each line bundle L on M corresponds an element of $H^2(M,\mathbf{Z})$, usually denoted $c_1(L)$ and called the first Chern class (or topological charge) of L.

The line bundles we need usually have more structure, whether it be under the form of a metric, a holomorphic structure, or a connection. A metric on L is a set of positive functions g_α satisfying $g_\alpha = |\phi_{\alpha\beta}|^{-2} g_\beta$. Thus a metric on L is a metric on each fiber L_z varying smoothly with z and does not involve any metric on the manifold M itself. Given a section f of L, $g_\alpha f_\alpha \bar{f}_\alpha$ is then a scalar on M which represents the modulus squared of f at each point. It will sometimes be denoted by $\|f\|^2$. The line bundle M is said to be a holomorphic line bundle if the manifold M is a complex manifold, and the transition functions $\phi_{\alpha\beta}$ are holomorphic functions on M. A connection is simply a U(1) gauge field on M, i.e., a collection $A_{\mu,\alpha}$ transforming as $A_{\mu,\alpha} = A_{\mu,\beta} - \partial_\mu \ln\phi_{\alpha\beta}$ under change of coordinate patches. There are of course many connections, but in the presence of a metric and a holomorphic structure on M there is a unique connection compatible with them both. To see this, let z^j be holomorphic coordinates on M and observe that, for any section f of L, $(\partial f_\alpha / \partial \bar{z}^j)$ satisfies Eq. (6.2), since $\phi_{\alpha\beta}$ is holomorphic. Thus

$$(\nabla_{\bar{j}} f)_\alpha = \frac{\partial}{\partial \bar{z}^j} f_\alpha \tag{6.5}$$

makes a well-defined section of L. The covariant derivatives $\nabla_j f$ are determined next by the requirement that

$$\frac{\partial}{\partial z^j}(g_\alpha f_\alpha \bar{f}'_\alpha) = g_\alpha \nabla_j f_\alpha \bar{f}'_\alpha + g_\alpha f_\alpha \overline{\nabla_{\bar{j}} f'_\alpha} \tag{6.6}$$

for any sections f and f', which implies that

$$\nabla_j f_\alpha = \left[\frac{\partial}{\partial z^j} + \frac{\partial}{\partial z^j}\ln g_\alpha\right] f_\alpha \ . \tag{6.7}$$

We have actually seen this process at work before when dealing with the spaces $\mathbf{T}^n$. They are holomorphic line bundles over the Riemann surface M, and Eqs. (6.5) and (6.7) are just extensions to this more general case of the constructions of covariant derivatives in Sec. II.E.

Since the connection is Abelian, the curvature $F_{\mu\nu}$ is a 2-form on the manifold M. It is immediate that the only nonvanishing components are

$$F_{j\bar{k}} = \partial^2(\ln g_\alpha)/\partial z_j \partial \bar{z}_{\bar{k}} \ . \tag{6.8}$$

A convenient way of phrasing this in completely intrinsic terms is the following: let f be any nonvanishing holomorphic section of L, i.e., a section for which the f_α's are holomorphic functions. Then the curvature of L is the (1,1) form given by

$$F = \partial\bar{\partial} \ln \|f\|^2 \ . \tag{6.9}$$

Here $d = \partial + \bar{\partial}$ is the splitting of the exterior derivative in the presence of a complex structure, and it is obvious that Eq. (6.9) is independent of the choice of f.

Finally, the Gauss-Bonnet theorem and the anomalous fermion number currents of Sec. II.I have taught us that there should be a direct link between topology and integrals of curvature. The proper generalization to the present context can be based on the DeRham theorem and formulated as follows. Recall that the kth DeRham cohomology group $H^k_{\mathrm{DR}}(M)$ is defined by

$$H^k_{\mathrm{DR}}(M) = \{\text{closed } k\text{-forms}\}/\{\text{exact } k\text{-forms}\} \ , \tag{6.10}$$

where a form ϕ is closed if $d\phi = 0$ and exact if $\phi = d\psi$ for some globally defined $(k-1)$ form ψ. The DeRham theorem asserts identity between these groups and real Čech cohomology groups $H^k(M,R)$ defined by real cocycles $c_{\alpha_1 \cdots \alpha_{k+1}}$ with conditions generalizing Eqs. (6.3) and (6.4). Since we shall need only the case $k=2$, we shall restrict ourselves to this case to simplify the discussion.

Let $[F]$ be an element of $H^2_{\mathrm{DR}}(M)$ with representative a closed 2-form F. On small patches B_α we can write $F = dA_\alpha$ for some one-forms A_α. On $B_\alpha \cap B_\beta$, $A_\alpha - A_\beta$ can in turn be written as

$$A_\alpha - A_\beta = d\lambda_{\alpha\beta} \tag{6.11}$$

for some functions $\lambda_{\alpha\beta}$. The class of the real cocycle $c_{\alpha\beta\gamma} = \lambda_{\alpha\beta} + \lambda_{\beta\gamma} - \lambda_{\alpha\gamma}$ can be checked to depend only on $[F]$ and defines an element of $H^2(M,R)$. According to the DeRham theorem this correspondence $[F] \leftrightarrow [c_{\alpha\beta\gamma}]$ is an isomorphism $H^2_{\mathrm{DR}}(M) \to H^2(M,R)$.

Returning now to any U(1) connections $A_{\mu,\alpha}$ on L [not necessarily the one singled out by metric and complex structures as in Eqs. (6.5) and (6.6)], we see that the curvature form $F_{\mu\nu}$ is clearly closed and thus defines a DeRham cohomology class $[(i/2\pi)F]$ in $H^2_{\mathrm{DR}}(M)$. Retracing the above steps, we see that the A_α in this case can be taken to be the connection forms $(i/2\pi)A_{\mu,\alpha}dx^\mu$, the $\psi_{\alpha\beta}$ become $(1/2\pi i)\ln\phi_{\alpha\beta}$ in view of Eq. (6.11), and thus the cohomology class $[(i/2\pi)F]$ actually coincides with the first Chern class $c_1(L)$.

To make contact with Gauss-Bonnet theorems, we observe that cohomology classes $[F]$ in $H^2_{\mathrm{DR}}(M)$ are characterized by their integrals $\int_C F$ over two-dimensional cycles C. When the dimension of the manifold M is two and M is compact and connected, M itself is the unique such cycle, and the integrals give topological numbers and hence multiples of the Euler characteristic,

$$c_1(L) = \frac{i}{2\pi}\int_M F \ . \tag{6.12}$$

That was essentially the content of equations such as (2.2), (2.27), and (2.50). In particular, we see that the Chern class of the canonical bundle K is $c_1(K) = 2h-2$, and more generally $c_1(\mathbf{T}^n) = n(2h-2)$.

B. The Jacobian variety of a Riemann surface

In this section we specialize to the case where the base manifold M is a Riemann surface and provide a classification of holomorphic line bundles on M.

Recall from Sec. VI.A that line bundles on M are distinguished already by their first Chern classes, which are elements of $H^2(M,\mathbf{Z})$. For compact two-dimensional surfaces, $H^2(M,\mathbf{Z})=\mathbf{Z}$, so that bundles are first indexed by integers. Next, bundles with the same Chern class may be topologically but not necessarily holomorphically equivalent, i.e., the smooth sections are in correspondence but not the holomorphic ones. Thus we introduce the *Picard varieties*

$$\mathrm{Pic}_d=\{\text{line bundles } L \text{ on } M \text{ with } c_1(L)=d\} . \tag{6.13}$$

It will turn out that the Picard varieties for various values of d are very similar, so we shall often concentrate on Pic_0, which is usually called the *Jacobian variety* of M and is denoted by $J(M)$. There are several ways of describing the Jacobian, each suitable for a different purpose, so we give them in turn.

First we observe that the space of all holomorphic line bundles on M can be conveniently viewed as a Čech cohomology group, albeit with coefficients that are not integers. More precisely, given a holomorphic line bundle L, recall that its transition functions $\phi_{\alpha\beta}$ satisfy Eq. (6.2) and note that bundles L' whose functions $\phi'_{\alpha\beta}$ are of the form

$$\phi'_{\alpha\beta}=\phi_{\alpha\beta}h_\alpha h_\beta^{-1} \tag{6.14}$$

for holomorphic nonvanishing functions h_α have their holomorphic sections f'_α in one-to-one correspondence with those of L: $f'_\alpha=h_\alpha f_\alpha$. We shall not distinguish between such bundles L and L'. If we introduce the Čech cohomology group $H^1(M,O^*)$ "with coefficients in O^*"[36] as the class of (multiplicative) holomorphic cocyles $\phi_{\alpha\beta}$ satisfying Eq. (6.2) modulo the exact ones $h_\alpha h_\beta^{-1}$, we see that $H^1(M,O^*)$ is just the space of holomorphic line bundles on M.

To single out the Jacobian variety from within $H^1(M,O^*)$, we begin by defining the first Čech cohomology group $H^1(M,O)$ with coefficients in[37] O in analogy with the previous discussion: $H^1(M,O)$ is the space of (additive) holomorphic cocycles $\psi_{\alpha\beta}$ satisfying

$$\psi_{\alpha\beta}+\psi_{\beta\gamma}-\psi_{\alpha\gamma}=0 \tag{6.15}$$

modulo exact cocycles, i.e., those of the form $\Theta_\alpha-\Theta_\beta$ for holomorphic $\Theta_\alpha,\Theta_\beta$.

There is then a natural mapping $H^1(M,O)\to H^1(M,O^*)$ given by $\phi_{\alpha\beta}=\exp(\psi_{\alpha\beta})$. In view of Eq. (6.15), the Chern class of bundles in $H^1(M,O^*)$ arising from this map must be 0. Furthermore the kernel of the map is evidently given by integer-valued cocycles $\psi_{\alpha\beta}$, naturally called the first Čech cohomology group $H^1(M,\mathbf{Z})$. The net outcome is the fundamental equation

$$J(M)=H^1(M,O)/H^1(M,\mathbf{Z}) . \tag{6.16}$$

[36] O^* usually stands for "germs of holomorphic nonvanishing functions."

[37] O denotes the space of germs of holomorphic functions.

In the mathematical literature, the above arguments are summarized by saying that the short exact sequence of sheaves

$$0\to 2\pi i Z\to O\xrightarrow{\exp} O^*\to 0 \tag{6.17}$$

leads to a long exact sequence in cohomology,

$$\cdots \to H^1(M,\mathbf{Z})\to H^1(M,O)\to H^1(M,O^*)\xrightarrow{c_1} H^2(M,\mathbf{Z})\to H^2(M,O)\to\cdots$$

which terminates, since $H^2(M,O)=0$, as we shall see below. Equation (6.16) follows at once.

Our next description of the Jacobian variety is based on rewriting the Čech cohomology group $H^1(M,O)$ as the dual of the space of holomorphic one-forms on M. The key tool is the Dolbeault theorem, which is the version of the DeRham theorem that applies to the $\bar\partial$ operator instead of the exterior derivative. Let

$$H^{0,q}(M)=\frac{\{(0,q)\text{ forms } \omega \text{ on } M \text{ with } \bar\partial\omega=0\}}{\{\text{exact forms } \bar\partial s\}}$$

be the $(0,q)$ Dolbeault cohomology group. When $q=1$, we can associate with an element $[\Theta]$ of $H^{0,1}(M)$ an element of $H^1(M,O)$, very much as in the earlier DeRham discussion: on patches B_α write $\Theta=\bar\partial s_\alpha$. Clearly $\psi_{\alpha\beta}=s_\alpha-s_\beta$ is a holomorphic function on $B_\alpha\cap B_\beta$ satisfying the additive cocycle condition (6.15). Moreover, $\psi_{\alpha\beta}$ is an exact cocycle if and only if $\psi_{\alpha\beta}=h_\alpha-h_\beta$ for some holomorphic h_α, so that $s_\alpha-h_\alpha$ is then a globally defined function s on M still satisfying $\bar\partial s=\Theta$, which means that Θ is $\bar\partial$ exact. Thus we have a correspondence $H^{0,1}(M)\to H^1(M,O)$, which is an isomorphism by Dolbeault's theorem. Similarly the cohomology groups $H^{0,q}(M)$ and $H^q(M,O)$ can be shown to be isomorphic. Since for $q=2$ and M a Riemann surface there are no (0,2) forms, we conclude that $H^2(M,O)=0$, which is the statement we made in the previous paragraph.

It is actually more convenient to view $H^{0,1}(M)$ as the dual of $H^0(M,K)$, which is defined to be the space of holomorphic sections of the canonical bundle K, in other words, the space of Abelian differentials on M. This duality arises from the natural nondegenerate pairing

$$\begin{aligned} &H^{0,1}(M)\times H^0(M,K)\to \mathbf{C} ,\\ &(\Theta,\omega)\to \int_M \Theta\wedge\omega , \end{aligned} \tag{6.18}$$

where the right-hand side makes sense intrinsically since $\Theta\wedge\omega$ is a (1,1) form. Thus another formula for the Jacobian of M is

$$J(M)=H^0(M,K)^+/H^1(M,\mathbf{Z}) . \tag{6.19}$$

We now pass to a description of the Jacobian in terms of curvature and holonomy. The key observation here is that a line bundle L has zero Chern class if and only if it admits a metric whose holomorphic connection has identically vanishing curvature. That the existence of such a metric implies that $c_1(L)=0$ is an immediate consequence of the Gauss-Bonnet formula (6.12). Conversely let $\hat{g}$ be any metric on L, and look for a factor $e^{2\sigma}$ so that the curvature of $g=e^{2\sigma}\hat{g}$ will be 0. This means that $2\partial\bar{\partial}\sigma=-$(curvature of $\hat{g}$), an equation that can be solved, since the right-hand side is orthogonal to constants again by the Gauss-Bonnet formula. We also note that a metric with zero curvature is unique up to constants. This follows from the simple fact that the ratio of two metrics with the same curvature must be the exponential of a harmonic function.

Now let L be a line bundle with $c_1(L)=0$, equipped with the unique flat metric as above. Let $A_\mu dx^\mu = A_z dz$ be the corresponding connection. Flatness is a local statement, and all such bundles L are locally the same. Globally, however, there are holonomy issues, and it is the values of parallel transport around closed cycles in M that completely determine the complex structure of L. More precisely, we introduce the following version of the familiar Wilson loop observable of gauge theories:

$$W(\gamma)=\frac{i}{2\pi}\oint_\gamma A_z dz \pmod{\mathbf{Z}} . \tag{6.20}$$

The function $W(\gamma)$ is real and can be interpreted as a phase shift. Indeed, if we parallel transport a vector around γ, it will return with a phase shift of $2\pi i W(\gamma)$. Since the curvature of $A_z dz$ is zero, it follows from Green's formula that $W(\gamma)$ depends only on the homology class of the cycle γ. In other words, W should be viewed as a real cohomology class, an element of $H^1(M;R)/H^1(M,\mathbf{Z})$.

We have thus associated an element W of $H^1(M;R)/H^1(M,\mathbf{Z})$ with each line bundle L with $c_1(L)=0$. Conversely, given W, we can construct L by taking the line bundle with constant transition functions $\exp[-2\pi i W(A_i)]$, $\exp[-2\pi i W(B_i)]$ across the cuts. Since W is trivial as a cohomology class if and only if L is trivial as a line bundle (if W is trivial, we can construct a covariantly constant section of L by parallel transport on the cut surface; the triviality of W guarantees that this section has no jumps across the cuts; the reverse statement is obvious), we have a third description of the Jacobian variety,

$$J(M)=H^1(M;R)/H^1(M,\mathbf{Z}) . \tag{6.21}$$

Another useful characterization of the Jacobian is in terms of *divisors*. The basic construction is the following. Given a point w of the surface M, we let z be a holomorphic coordinate centered at w, B_0 a small disk around w, and $B_\infty = M\setminus\{w\}$. A line bundle $[w]$ can now be defined by taking z as a transition function between B_0 and B_∞. Thus a holomorphic section f of $[w]$ is just a pair f_0, f_∞ of holomorphic functions on B_0, B_∞ satisfying $f_0 = z f_\infty$. In particular, the constant holomorphic function 1 on $M\setminus\{w\}$ gives rise to the holomorphic section $1_{[w]}$ defined by $f_\infty=1$ and $f_0=z$. Note that this section has a simple zero at w. Furthermore, the first Chern class $c_1[w]$ is equal to 1. This corresponds to the simple fact that the logarithm of the transition function is multiple valued in $B_0\cap B_\infty$ and changes by $(2\pi i)\times 1$ as we go around a small circle $|z|=$const in $B_0\cap B_\infty$. This argument can easily be made rigorous by taking a refinement of the covering $\{B_0, B_\infty\}$ and retracing the definition of Chern classes of line bundles.

More generally, given a formal expression of the form

$$D=\sum_{i=1}^{N} n_i w_i \quad \text{positive or negative } n_i \text{ integers} , \tag{6.22}$$

we can take holomorphic coordinates z_i centered at w_i, small disjoint disks B_i around w_i, and set $B_\infty = M\setminus\{w_1,\dots,w_N\}$. The line bundle $[\sum_{i=1}^N n_i w_i]$ is defined by the covering $\{B_1,\dots,B_N,B_\infty\}$ and the transition functions $z_i^{n_i}$ on the overlap $B_\infty\cap B_i$. Holomorphic sections of $[\Sigma n_i w_i]$ are now holomorphic functions $f_1,\dots,f_N,f_\infty$ on $B_1,\dots B_N,B_\infty$, respectively, satisfying $f_i=z_i^{n_i}f_\infty$. The holomorphic function 1 on $M\setminus\{w_1,\dots,w_N\}$ thus extends to a meromorphic section of $[\sum_{i=1}^N n_i w_i]$ that has a pole of order n_i at w_i if n_i is negative. The multiple valuedness of the transition functions $z_i^{n_i}$ adds up to a net value of $\sum_{i=1}^N n_i$ for the Chern class of $[\sum_{i=1}^N n_i w_i]$.

The above construction will yield a trivial line bundle provided the expression $\sum_{i=1}^N n_i w_i$ is the set of zeros and poles of a meromorphic function ϕ, counted with their multiplicities. Indeed $\phi^{-1}1_{[\Sigma n_i w_i]}$ is then a holomorphic nowhere-vanishing section of $[\Sigma n_i w_i]$, and a line bundle with a nowhere-vanishing global section is evidently trivial. Similarly the line bundles arising from two formal expressions D and D' differing by the zeros and poles of a meromorphic function will be isomorphic. Thus we define a divisor $[D]$ to be a class of expressions (6.22) modulo such zeros and poles, and actually have a correspondence between divisors and line bundles. Every line bundle L does arise in this manner, since with L we can associate the divisor of zeros and poles of one of its meromorphic sections. It will be shown later from index theorems that such sections do exist, and it does not matter which one we choose, since the divisors of two sections will differ only by the zeros and poles of their quotient, which is a meromorphic function.

In this way we obtain another description of the Jacobian:

$$J(M)=\{\text{divisors } [\Sigma n_i w_i] \text{ with } \Sigma n_i=0\} . \tag{6.23}$$

As a by-product of the above discussion we have the useful fact that the difference between the number of zeros and the number of poles of a meromorphic function must be 0, and more generally, for a section of a line bundle L,

$$\#(\text{zeros})-\#(\text{poles})=c_1(L) . \tag{6.24}$$

C. Index and Riemann-Roch theorems

The basic operator on a line bundle L on a Riemann surface M is the Cauchy-Riemann operator $\bar{\partial}_L{:}L$

$\to L\otimes\bar{K}$. It will be important to determine the number of its zero modes, i.e., the number of holomorphic sections of L. When L is a bundle $\mathbf{T}^n$ of spinors, these are the zero modes of the Dirac operator (coupled to various vector potentials), and we determined them through index theorems and heat kernels in Secs. II.E and V.D. Here we discuss the version that applies for general L.

In the presence of a metric on L, $\bar\partial_L$ has an adjoint $\bar\partial_L^\dagger: L\otimes\bar{K}\to L$ which is just $\bar\partial_L^\dagger f=-\nabla_z f$ where ∇ is the holomorphic connection determined by the metric on L [cf. Eq. (6.7)]. The index theorem familiar from the study of chiral anomalies suggests that the index of $\bar\partial_L$ should be the integral over M of a polynomial in the curvature of L and M. Since the dimension of M is 2 and curvatures are 2-forms, we must have a linear function of $c_1(L)$ and $\chi(M)$. Comparing with Eq. (2.50) and recalling that $c_1(\mathbf{T}^n)=-n\chi(M)$, we arrive at

$$\dim \mathrm{Ker}\bar\partial_L-\dim\mathrm{Ker}\bar\partial_L^\dagger=c_1(L)+\tfrac{1}{2}\chi(M)\ . \tag{6.25}$$

It will be useful to reformulate this result independently of any metric g on L and just in terms of $\bar\partial$ operators. For this we appeal again to a duality statement known as Serre duality,

$$\begin{aligned}&(\mathrm{Ker}\bar\partial_L^\dagger)\times(\mathrm{Ker}\bar\partial_{L^{-1}\otimes K})\to\mathbf{C}\ ,\\&(f\,d\bar z,e\,dz)\to\int fe\,d\bar z\,dz\ .\end{aligned}\tag{6.26}$$

The right-hand side is well defined for f and e in L and L^{-1}. This pairing is nondegenerate, since the vanishing of (6.26) means that $e\,dz$ is orthogonal in the Hilbert space sense to $h\bar f\,dz$. Since $f\,d\bar z$ is in the kernel of $\nabla_{L\otimes\bar K}$ if and only if $h\bar f\,dz$ is in the kernel of $\bar\nabla_{L^{-1}\otimes K}=\bar\partial_{L^{-1}\otimes K}$, our assertion follows. The index theorem (6.25) becomes

$$\dim\mathrm{Ker}\bar\partial_L-\dim\mathrm{Ker}\bar\partial_{L^{-1}\otimes K}=c_1(L)+\tfrac{1}{2}\chi(M) \tag{6.27}$$

and as such is known as the Riemann-Roch theorem.

We now illustrate the use of the Riemann-Roch theorem by deriving the existence of meromorphic sections of various line bundles. First recall that we claimed in Sec. VI.B that any holomorphic line bundle L admitted meromorphic sections. To see this, apply Eq. (6.27) with L replaced by $L\otimes[nw]$ where w is some fixed point and n is an integer taken so large that the right-hand side of Eq. (6.27) becomes positive. In particular, $L\otimes[nw]$ admits some holomorphic section s. But then $s1_{[-nw]}$ is a meromorphic section of L.

Next we investigate meromorphic differentials on M, i.e., sections of K. The case of Abelian (i.e., holomorphic) differentials has already been considered several times (Secs. II.E and V.D) and follows from Eq. (6.27) with $L=$trivial bundle. There are h Abelian differentials $\omega_1,\ldots,\omega_h$. Turning to the meromorphic ones, we shall establish the existence of meromorphic differentials with simple poles at exactly any two given points w_1 and w_2, or a double pole at any given w. We apply Eq. (6.27) with $L=[-w_1-w_2]$. Since L has Chern class $=-2$ and admits no holomorphic sections, it follows that $\dim\bar\partial_{[w_1+w_2]\otimes K}=h+1$. There must exist some section f of $[w_1+w_2]\otimes K$ that will complete $1_{[w_1+w_2]}\omega_1,\ldots,1_{[w_1+w_2]}\omega_h$ into a basis for $\mathrm{Ker}\bar\partial_{[w_1+w_2]\otimes K}$. Evidently $1_{[-w_1-w_2]}f$ is then a section of K, with at most simple poles at w_1 and w_2, and in fact exactly simple poles at both points (if it had a pole at only one point the residue there would vanish, since we can integrate the differential on a closed contour and deform it away). It is now not difficult to see that by scaling and adding a suitable combination of Abelian differentials, we can produce a unique $\omega_{w_1w_2}$ having simple poles at w_1,w_2 with residues ±1, and vanishing A periods. These $\omega_{w_1w_1}$ are the normalized differential of the third kind, encountered already in Sec. IV.G. Finally the above arguments can be modified to produce a meromorphic differential ω_w with exactly one double pole at w. These are called differentials of the second kind. It is not, however, true that there exist differentials with poles of any order at a given point. Certain orders $n_1,\ldots,n_k$ $(k\le h-1)$, which are called Weierstrass gaps, may be missing.

D. Period matrix and Abel map

The above concepts take a very concrete form if we choose a homology basis (A_J,B_J) satisfying the canonical relations (3.5). To a choice of homology basis corresponds a choice of basis $\omega_1,\ldots,\omega_h$ of Abelian differentials, defined unambiguously by the requirement

$$\oint_{A_J}\omega_K=\delta_{JK}\ . \tag{6.28}$$

The period matrix Ω is then the $h\times h$ complex matrix with entries

$$\oint_{B_J}\omega_K=\Omega_{JK}\ . \tag{6.29}$$

Two crucial properties of period matrices are the bilinear relations of Riemann (see Appendix D for a proof), which in particular imply that

$$\begin{aligned}&\Omega_{JK}=\Omega_{KJ}\ ,\\&\mathrm{Im}\Omega\text{ is positive definite}\ .\end{aligned}\tag{6.30}$$

The space of all $h\times h$ matrices satisfying these conditions is called the Siegel upper half space. We note that it has complex dimension $h(h+1)/2$, while the subspace of all period matrices of Riemann surfaces has dimension at most $3h-3$ (for $h\ge2$), in fact exactly $3h-3$, as period matrices actually characterize complex structures.

Next recall that the Jacobian variety of M can be viewed as the coset space $H^0(M,K)^+/H^1(M,\mathbf{Z})$. Since we have chosen a basis $\omega_1,\ldots,\omega_h$ of Abelian differentials, a cycle C in $H^1(M,\mathbf{Z})$ can be identified with its vector of periods $(\int_C\omega_1,\ldots,\int_C\omega_h)$ and hence with a point on the lattice $\mathbf{Z}^h+\Omega\mathbf{Z}^h$. Thus the Jacobian variety becomes

$$J(M)=\mathbf{C}^h/(\mathbf{Z}^h+\Omega\mathbf{Z}^h)\ , \tag{6.31}$$

which is evidently a complex torus of dimension h.

Observe that a change of homology bases preserving the intersection numbers (3.5) is effected by a modular transformation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

in $\mathrm{Sp}(2h,\mathbf{Z})$. Under such a transformation the period matrix changes as

$$\Omega' = (A\Omega + B)(C\Omega + D)^{-1} \ . \tag{6.32}$$

It is evident that the lattice $\mathbf{Z}^h + \Omega\mathbf{Z}^h$ is then unchanged, which confirms the intrinsic meaning of the Jacobian (6.31).

We can now construct explicitly the correspondence between the divisors and elements of the Jacobian variety, and in particular embed the surface M itself within $J(M)$ (see Fig. 21). Since $J(M)$ is a torus, its function theory can be built on modular forms, and this fundamental embedding will allow us to study function theory on M through modular forms.

Fix a point z_0. Then for d points $z_1, \ldots, z_d$ in M, the *Abel map* is defined by

$$I(z_1 + \cdots + z_d) = \int_{z_0}^{z_1} \omega + \cdots + \int_{z_0}^{z_d} \omega \ , \tag{6.33}$$

where the addition signs in the argument of I are understood in the divisor sense. The right-hand side represents an h-dimensional vector, with ω denoting the h vector of Abelian differentials $(\omega_1, \ldots, \omega_h)$. Evidently there is an arbitrariness in the choice of integration paths, but this leads only to an ambiguity of the form of a lattice point in $\mathbf{Z}^h + \Omega\mathbf{Z}^h$, so that the Abel map I is single valued in the Jacobian variety. Actually I is naturally defined on divisor classes in the sense that $I(\Sigma z_j - \Sigma w_l) \equiv 0$ if and only if z_j, w_l are the zeros and poles of a meromorphic function. This statement is usually known as Abel's theorem, for which we refer the reader to Appendix D. The Abel map I viewed as a map from divisor classes to the Jacobian variety becomes one-to-one and onto when restricted to the space of divisors with zero Chern class. This is the explicit correspondence between such divisors and the elements of a complex torus that we are looking for, although, strictly speaking, we have not as yet checked that under this identification $-I(D)$ does go over to the line bundle admitting D for divisor. This will follow most easily from theta-function constructions to be outlined in the next section.

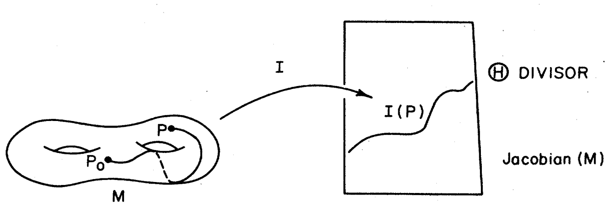

FIG. 21. The embedding of the Riemann surface into the Jacobian by the Abel map.

We note that the Abel map I can be viewed as well as a map from the space of the bundles, since with each line bundle L we can associate $I(D)$ where D is the divisor class of L. We shall use indiscriminately both symbols $I(L)$ and $I(D)$ to denote this same point on the Jacobian variety.

If we restrict I to single points $I(z) = \int_{z_0}^{z} \omega$, we obtain a one-to-one map of the surface M. This embedding is not completely free of any choices, since it depends on the base point z_0. This results in an arbitrary translation within $J(M)$, which may serve as justification for some formulas we shall encounter later.

Finally, it is not difficult to establish the following useful formula for the variation of the period matrix as we deform the complex structure by a Beltrami differential:

$$\delta\Omega_{IJ} = -i \int d^2z \, \mu_{\bar{z}}{}^{z} \omega_I \omega_J \ . \tag{6.34}$$

E. Theta functions

The Jacobi theta function is defined by

$$\vartheta(\zeta, \Omega) = \sum_{n \in \mathbf{Z}^h} \exp(\pi i n^t \Omega n + 2\pi i n^t \zeta), \quad \zeta \in \mathbf{C}^h \ . \tag{6.35}$$

It satisfies the heat equation

$$\left[4\pi i \frac{\partial}{\partial \Omega_{IJ}} + \frac{\partial^2}{\partial \zeta_I \partial \zeta_J} \right] \vartheta(\zeta, \Omega) = 0$$

and the key transformation laws

$$\vartheta(\zeta + M + \Omega N, \Omega) = \exp(-\pi i N^t \Omega N - 2\pi i N^t \zeta) \vartheta(\zeta, \Omega) \tag{6.36}$$

for M and N vectors of integers. This periodicity up to a factor with respect to the lattice $Z^h + \Omega Z^h$ shows that $\vartheta(\zeta, \Omega)$ should be viewed as a holomorphic section of a line bundle over the Jacobian variety, a line bundle whose holonomy around the cycles of $J(M)$ is defined by the factors in Eq. (6.36). It can be shown that this bundle—called the ϑ line bundle—admits in fact (up to multiplicative constants) only one holomorphic section, represented by the theta function.

Although theta functions are strictly speaking sections of line bundles, we can easily manufacture meromorphic functions on $J(M)$ out of them. For example, it is easy to check that

$$\frac{\prod_{i=1}^{n} \vartheta(\zeta + a_i, \Omega)}{\prod_{i=1}^{n} \vartheta(\zeta + b_i, \Omega)} \quad \text{with} \ \sum_i a_i - \sum b_i \equiv 0 \ (\text{mod} \ \mathbf{Z}^h) \ ,$$

$$\frac{\partial}{\partial \zeta^I} \ln \frac{\vartheta(\zeta + a, \Omega)}{\vartheta(\zeta + b, \Omega)} \ ,$$

$$\frac{\partial^2}{\partial \zeta^I \partial \zeta^J} \ln \vartheta(\zeta, \Omega) \ ,$$

are periodic and hence functions on $J(M)$. To go further we need a detailed knowledge of the zero set of $\vartheta(\zeta,\Omega)$, and more precisely of its intersection with the image of the Abel map. Such information is provided by the *Riemann vanishing theorem.*

Let Δ, the "vector of Riemann constants," be defined by

$$\Delta_J=\tfrac{1}{2}-\tfrac{1}{2}\Omega_{JJ}+\sum_{K\neq J}\oint_{A_K}\omega_K(z)\int_{z_0}^{z}\omega_J \ . \qquad (6.37)$$

Then

- $2\Delta=I(K), K=$canonical bundle ;
- $\vartheta(\zeta,\Omega)=0$ if and only if $\zeta=\Delta-I(z_1+\cdots+z_{h-1})$ for any $h-1$ points $z_1,\ldots,z_{h-1}$ in M;
- $\vartheta(\zeta+I(z),\Omega)$ either vanishes identically as a function of z, or else has exactly h zeros $z_1,\ldots,z_h$ characterized by the equation

$$I(z_1+\cdots+z_h)=-\zeta+\Delta \ . \qquad (6.38)$$

The zero set of $\vartheta(\zeta,\Omega)$ is called the Θ divisor. Note that it is well defined as a subset of the Jacobian, thanks to its periodicity.

We pause to discuss briefly some ramifications of the Riemann vanishing theorem. The points ζ for which $\vartheta(\zeta+I(z),\Omega)=0$ as a function of z are rather special, and can be shown to coincide with points of the form

$$\zeta=\Delta-I(z_1+\cdots+z_h) \ ,$$

where $z_1+\cdots+z_h$ is a so-called *special divisor,* i.e., must contain all the poles of some nonconstant meromorphic function. A general divisor $w_1+\cdots+w_h$ in general will not satisfy this property, and the set of special ζ's above (6.38) is a strict analytic subvariety of the Jacobian. Points w for which hw is special are called the *Weierstrass points.* From the Riemann-Roch theorem, w is a Weierstrass point if and only if there exists a holomorphic Abelian differential vanishing to order h at w. Weierstrass points carry a lot of information about the complex structure of M. It is known that there are none in genus $h\leq 1$ and exactly $2h+2$ when M is hyperelliptic. In this case they can be viewed as the branch points of M, when represented as a double covering of the sphere. More generally a theorem of Hurwitz asserts that the number of Weierstrass points is between $2h+2$ and $h(h^2-1)$.

The last statement in the Riemann vanishing theorem provides an explicit answer to a question of Jacobi, namely, given ζ in $J(M)$, find h points $z_1,\ldots,z_h$ so that

$$I(z_1+\cdots+z_h)=\zeta \ . \qquad (6.39)$$

For generic ζ the desired points $z_1,\ldots,z_h$ are obtained simply by translating by $-\zeta+\Delta$ the image by the Abel map I of the Riemann surface M, and taking its intersections with the zero set Θ of the theta function. This invertibility of Eq. (6.39) for generic ζ is usually referred to as the Jacobi inversion theorem and will play a key role in the study of Bose-Fermi correspondence in Sec. VII.

Finally, the equation $2\Delta=I(K)$ suggests that Δ is intimately linked with the square root of K, in other words, bundles of spinors. We shall discuss this aspect in greater detail in the next section.

It is now easy to see why functions such as

$$\vartheta\left[\zeta+\int_w^z\omega,\Omega\right] \qquad (6.40)$$

will be the main ingredient in the construction of propagators. Indeed, if ζ is in the zero set of $\vartheta(\zeta,\Omega)$, then w must be among the h zeros $z_1,\ldots,z_h$ of this function. If, say, $w=z_h$, Eq. (6.38) will reduce to

$$\sum_{i=1}^{h-1}\int_{z_0}^{z_i}\omega_J\equiv-\zeta_J+\Delta_J \ , \qquad (6.41)$$

which shows that the points $z_1,\ldots,z_{h-1}$ are actually independent of w and depend on ζ alone. Since the function $\vartheta(\zeta,\Omega)$ is even, we can interchange the roles of z and w and conclude that there exist points $z_1,\ldots,z_{h-1},w_1,\ldots,w_{h-1}$ depending on ζ such that

$$\vartheta\left[\zeta+\int_w^z\omega,\Omega\right]=0\leftrightarrow\begin{cases}z=w,\\ \text{or } z \text{ is among } z_1,\ldots,z_{h-1},\\ \text{or } w \text{ is among } w_1,\ldots,w_{h-1}.\end{cases} \qquad (6.42)$$

Thus $\vartheta(\zeta+\int_w^z\omega,\Omega)$ has the key property of essentially vanishing only along the diagonal. It is still multiple valued as a function of z and w, but this difficulty can often be overcome as before, by taking suitable ratios.

As a simple illustration we can produce explicitly the meromorphic function with given divisor $z_1+\cdots+z_d-(w_1+\cdots+w_d)$ under the condition of Abel's theorem, i.e., that $I(z_1+\cdots+z_d)=I(w_1+\cdots+w_d)$. A candidate is

$$f(z)=\frac{\prod_{i=1}^{d}\vartheta\left[\zeta+\int_{z_i}^{z}\omega,\Omega\right]}{\prod_{i=1}^{d}\vartheta\left[\zeta+\int_{w_i}^{z}\omega,\Omega\right]} \ , \qquad (6.43)$$

where ζ is chosen so as not to have the functions involved vanish identically, $\vartheta(\zeta)=0$, and the paths of integration are yet to be described. A natural way of prescribing the paths is to choose one same path from z_0 to z, and link it to fixed paths α_i and β_i from z_i to z_0, and from w_i to z_0, respectively. Under changes of the path from z_0 to z the transformation laws (6.36) show that $f(z)$ may change by integral powers of

$$\exp\left[-2\pi i\sum_{i=1}^{d}\left[\int_{\alpha_i}\omega_J-\int_{\beta_i}\omega_J\right]\right] \ . \qquad (6.44)$$

By hypothesis the expression in the exponential is a lattice point in $\mathbf{Z}^h+\Omega\mathbf{Z}^h$ for any paths α_i,β_i from z_i and z_i to z_0. By adding, if necessary, appropriate multiples of the homology basis cycles A_J,B_J, we may make sure that it actually vanishes. Thus $f(z)$ is a single-valued meromorphic function on M and has exactly the desired zeros and poles.

Next, we should like to construct explicitly sections of any line bundle in the Jacobian of M. For this it is most convenient to introduce the *theta function with characteristics,*

$$\vartheta[\delta](\zeta,\Omega)=\sum_{n\in\mathbf{Z}^h}\exp[\pi i(n+\delta')\Omega(n+\delta')+2\pi i(n+\delta')(\zeta+\delta'')]$$
$$=\exp[\pi i\delta'\Omega\delta'+2\pi i\delta'(\zeta+\delta'')]\times\vartheta(\zeta+\Omega\delta'+\delta'',\Omega) \quad (6.45)$$

for any *characteristics* $\delta=[^{\delta'}_{\delta''}]$ in $(0,1)^{2h}$. It is readily seen that the transformation laws are

$$\vartheta[\delta](\zeta+M+\Omega N,\Omega)=\exp[-\pi i N\Omega N-2\pi i N(\zeta+\delta'')+2\pi i\delta' M]\vartheta[\delta](\zeta,\Omega)\ ,$$
$$\vartheta\begin{bmatrix}\delta'+M\\ \delta''+N\end{bmatrix}(\zeta,\Omega)=\exp[2\pi i\delta' N]\vartheta\begin{bmatrix}\delta'\\ \delta''\end{bmatrix}(\zeta,\Omega)\ . \quad (6.46)$$

Given $[\delta]$, we can now construct three different objects that describe in different ways the same line bundle with zero Chern class: $-(\delta''+\Omega\delta')\in\mathbf{C}^h/\mathbf{Z}^h+\Omega\mathbf{Z}^h$; sections f defined by holonomy conditions $(\delta',\delta'')\in H^1(M,\mathbf{R})/H^1(M,\mathbf{Z})$ around the A_I,B_I cycles

$$f\left[z+\int_{A_I}\right]=e^{2\pi i\delta'_I}f(z)\ ,$$
$$f\left[z+\int_{B_I}\right]=e^{-2\pi i\delta''_I}f(z)\ ; \quad (6.47)$$

and

$$\frac{\vartheta[\delta]\left[\zeta+\int_{z_0}^{z}\omega,\Omega\right]}{\vartheta[0]\left[\zeta+\int_{z_0}^{z}\omega,\Omega\right]}\ . \quad (6.48)$$

To see that they indeed correspond to the same line bundle, it suffices to observe that the expressions in Eq. (6.48) transform as (6.47) when z is transported around each cycle A_I or B_I, so that (6.48) is a section of the line bundle with holonomy (δ',δ''). Furthermore, its divisor can be read off from the Riemann vanishing theorem: the zeros $z_1,\dots,z_h$ and poles $w_1,\dots,w_h$ must satisfy

$$I(z_1+\cdots+z_h)=-\zeta-(\delta''+\Omega\delta')+\Delta\ ,$$
$$I(w_1+\cdots+w_h)=-\zeta+\Delta\ , \quad (6.49)$$

and hence

$$I(z_1+\cdots+z_h-w_1-\cdots-w_h)=-(\delta''+\Omega\delta')$$

as predicted.

In Sec. VI.B, we gave several equivalent descriptions of the Jacobian variety of M as a *set*, but it was not so clear exactly how to pass from one description to another for a specific element L of $J(M)$. Equations (6.46)–(6.48) provide a satisfying answer to this question, and allow us as well to characterize a line bundle in $J(M)$ by its characteristics $[\delta]$.

F. Spin structures, Dirac zero modes, and the prime form

The previous sections have provided a thorough investigation of line bundles with zero Chern class. Fermionic strings, however, involve spinors on the worldsheet M, i.e., sections of square roots of the canonical bundle K. Such square roots are called spin bundles and must have Chern class

$$c_1=\tfrac{1}{2}c_1(K)=h-1\ .$$

We have argued elsewhere (Sec. III.A) that there are 2^{2h} distinct spin bundles. They form a finite set inside the $(2h)$-dimensional Picard variety Pic_{h-1} of line bundles of Chern class $h-1$.

Now the Picard varieties Pic_d for different values of d are very similar in structure, but there is no natural correspondence between them without making some choices. One way of obtaining a correspondence is to single out a specific element within Pic_d, so that other elements of Pic_d can be identified by their differences from the chosen element. Since these differences must have vanishing Chern classes, this provides us with an isomorphism between Pic_d and the Jacobian variety.

It is remarkable that once a homology basis A_I, B_I has been chosen, we have in fact a particular spin structure S_0 determined by the basis. The key to this phenomenon lies in the fact that there is a natural correspondence between spin bundles and symmetric translates of the Θ divisor:

$$S\ \text{spin bundle}\leftrightarrow\text{translate of }\Theta\text{ divisor by }I(S)-\Delta\ ,$$
$$\{\text{spin bundles}\}\leftrightarrow\{\text{symmetric translates of }\Theta\}\ . \quad (6.50)$$

Here by a symmetric subset of the Jacobian variety, we mean a subset invariant under $\zeta\to-\zeta$. To establish Eq. (6.50), we begin by noting that a line bundle admits holomorphic sections if its divisor is positive and, in particular for line bundles L of Chern class $h-1$, if its divisor is of the form $z_1+\cdots+z_{h-1}$. In view of the Riemann-Roch theorem, L will admit holomorphic sections if and only if $L^{-1}\otimes K$ does. In other words,

$$[L]=z_1+\cdots+z_{h-1}\Longleftrightarrow[L^{-1}\otimes K]=w_1+\cdots+w_{h-1}\ .$$

In particular, for each $z_1,\dots,z_{h-1}$, there exist $w_1,\dots,w_{h-1}$ so that

$$I(K)-I(z_1+\cdots+z_{h-1})=I(w_1+\cdots+w_{h-1})\ .$$

For a spin bundle S, $I(K)=2I(S)$, so this equation becomes

$$I(S)-I(z_1+\cdots+z_{h-1})=-[I(S)-I(w_1+\cdots+w_{h-1})]\ ,$$

which just means that $\Theta+I(S)-\Delta$ is symmetric. Since $\vartheta(\zeta,\Omega)$ is an even function, the Θ divisor itself is symmetric. Furthermore, it is not difficult to show that the only way of obtaining a symmetric translate of Θ is actually to translate it by half-lattice points, i.e., points of the form $-(\delta''+\Omega\delta')$ where δ',δ'' are half-integers. There are thus exactly 2^{2h} symmetric translates, so the above correspondence is one-to-one and onto. In particular, to the Θ divisor itself must correspond some specific spin bundle S_0, and this is the one we are looking for. Note that it satisfies $I(S_0)=\Delta$, but depends only on the homology basis, not on the choice of base point P_0.

With the choice of the spin bundle S_0 we can identify the Jacobian variety and the Picard variety Pic_{h-1} via

$$\begin{bmatrix}\delta'\\ \delta''\end{bmatrix}\in J(M)\leftrightarrow S_0\otimes\begin{bmatrix}\delta'\\ \delta''\end{bmatrix}\in \mathrm{Pic}_{h-1}\ ,$$

while spin bundles $S_0\otimes[{\delta' \atop \delta''}]$ within the Jacobian are given by $(S_0\otimes[{\delta' \atop \delta''}])^2=K$. This means that $[{\delta' \atop \delta''}]$ must be a half-integer point, in agreement with the discussion based on symmetric translates of the Θ divisor. Observe that, for spin bundles, the theta function with characteristics satisfies

$$\vartheta[\delta](-\zeta,\Omega)=(-1)^{4\delta'\delta''}\vartheta[\delta](\zeta,\Omega)\ , \tag{6.51}$$

which shows that spin structures can be divided into two groups, the even and the odd ones, depending on whether $4\delta'\delta''$ is an even or an odd integer. Simple counting yields $2^{h-1}(2^h+1)$ even-spin and $2^{h-1}(2^h-1)$ odd-spin structures. This parity will reflect itself in the number of zero modes in the Dirac operator.

We can now characterize within Pic_{h-1} those line bundles which admit holomorphic sections. Of course, when we have a spin bundle, the Dirac operator reduces to the $\bar\partial$ operator, and holomorphic sections are simply Dirac zero modes. The description is actually very simple: a line bundle $S_0\otimes[\delta]$ in Pic_{h-1} admits zero modes if and only if its divisor is of the form $z_1+\cdots+z_{h-1}$ for some points $z_1,\ldots,z_{h-1}$. Taking the Abel map and recalling that $I(S_0)=\Delta$, $I([\delta])=-(\delta''+\Omega\delta')$ [cf. Eqs. (6.47) and (6.48)], we obtain

$$-(\delta''+\Omega\delta')+\Delta-I(z_1+\cdots+z_{h-1})=0\ .$$

In other words, $\delta''+\Omega\delta'$ is in the Θ divisor,

$$\vartheta[\delta](0,\Omega)=0\ ,$$

in view of the Riemann vanishing theorem.

This characterization suggests strongly that the number of zero modes is just the order of vanishing of the theta function. This is, for example, in the same spirit as Selberg zeta-function-type formulas derived earlier for regularized determinants and can actually be proved with further work. In particular, for spin bundles it confirms that the parity of the number of zero modes is the same as the parity of the spin structure, and that generically there is no zero mode for the even-spin structures and exactly one for the odd ones.

We turn next to the remaining fundamental ingredient in the construction of chiral fields on a Riemann surface, namely the prime form.

Let $[\delta]$ be an odd-spin structure and assume that we are in the generic case where $\vartheta[\delta](\zeta,\Omega)$ vanishes exactly to first order at $\zeta=0$. This means that the Dirac operator has exactly one zero mode, which we can construct explicitly. For this consider the holomorphic Abelian differential

$$\omega_\delta(w)=\sum_{I=1}^{h}\frac{\partial\vartheta}{\partial\zeta^I}[\delta](0,\Omega)\omega_I(w)\ . \tag{6.52}$$

We claim that it vanishes to second order at $(h-1)$ points $z_1,\ldots,z_{h-1}$ and that these points are determined by $I(z_1+\cdots+z_{h-1})=\Delta-\delta$. To see this, let $w,z_1,\ldots,z_{h-1}$ be the h zeros of the function $\vartheta[\delta](\int_w^z\omega,\Omega)$. Thus Riemann's theorem implies that the stated relation holds, and in particular $z_1,\ldots,z_{h-1}$ are independent of w. Taking the differential of $\vartheta[\delta](\int_w^z\omega,\Omega)$ with respect to w at $z=w$ yields $\omega_\delta(w)$, which must then vanish at $z_1,\ldots,z_{h-1}$. Since ω_δ is an Abelian differential, its divisor is the divisor of the canonical bundle. This fact together with Eq. (6.24) readily implies that the missing $(h-1)$ zeros of ω_δ are again $z_1,\ldots,z_{h-1}$, which is the desired statement. Since the zeros of ω_δ are double, the spinor

$$h_\delta(w)=\sqrt{\omega_\delta(w)} \tag{6.53}$$

is well defined and holomorphic, and in fact is a section of the spin bundle corresponding to δ. We may now generalize the construction (6.40) to obtain the prime form

$$E(z,w)=\frac{\vartheta[\delta]\left[\int_w^z\omega,\Omega\right]}{h_\delta(z)h_\delta(w)}\ . \tag{6.54}$$

The prime form $E(z,w)$ can be viewed as a $(-\frac{1}{2},0)$ form in each variable on the universal covering of the surface M, whose transformation laws can be easily read off from Eq. (6.46). The introduction of the factors $h_\delta(z)h_\delta(w)$ in Eq. (6.54) has several beneficial effects: $E(z,w)$ has the correct U(1) weight for inverses of fermion propagators, is actually independent of the spin structure δ we selected originally, and vanishes only when $z=w$.

When the point z is moved around an A_I cycle once, E is left invariant up to a $\pm$ sign, whereas when it is moved around a B_I cycle one, it transforms as

$$E(z,w)\to -\exp\left[-i\pi\Omega_{II}-2\pi i\int_z^w\omega_I\right]E(z,w)\ . \tag{6.55a}$$

It is important to note that the prime form depends on the choice of homology basis and will transform under modular transformations as

$$E(z,w)\to\exp\left[\pi i\int_z^w\omega(C\Omega+D)^{-1}C\int_z^w\omega\right]E(z,w)\ . \tag{6.55b}$$

Meromorphic differentials, whose existence was established in Sec. VI.C through indirect index theorems arguments, can be written very simply in terms of the prime form. In fact,

$$\omega_{w_1 w_2}(z)=d_z \ln\frac{E(z,w_1)}{E(z,w_2)} \tag{6.56}$$

is a differential of the third kind with zero A periods and residues ± 1 at w_1 and w_2, while

$$\omega_{w_1}(z)=d_z \frac{\partial}{\partial w}\bigg|_{w=w_1} \ln E(z,w) \tag{6.57}$$

is a differential of the second kind with zero A periods and a double pole at w_1. Similarly, propagators can be constructed out of the prime form, but we shall return to this issue later.

All variations with respect to moduli parameters can be deduced from the following variational formulas for the Abelian differentials and the prime form:

$$\begin{aligned}&\frac{4\pi}{\sqrt{g}}\frac{\delta}{\delta g^{ww}}\omega_I(z)=\omega_I(w)\partial_z\partial_w \ln E(z,w)\ ,\\ &\frac{4\pi}{\sqrt{g}}\frac{\delta}{\delta g^{ww}}E(z,x)=-\tfrac{1}{2}[\omega_{zx}(w)]^2\ .\end{aligned} \tag{6.58}$$

Finally, we discuss degenerations of Riemann surfaces in terms of plumbing fixtures (see Sec. IV.H), period matrices, and prime forms. Recall that there are two cases, distinguished by whether the plumbing fixture $\mathcal{U}_t$ used to model the degeneration process disconnects the Riemann surfaces M_t or not. In the case where it does, let M_1' and M_2' be the components of the complement in M_t of the plumbing fixture $\mathcal{U}_t$, and let M_1 and M_2 be the components of M_0 in the degeneration limit. Then the normalized basis of Abelian differentials $\omega_I^t(z)$ will approach the combined bases of Abelian differentials $\omega_{I_1}^t(z_1)$, $1\leq I_1 \leq i$, $\omega_{I_2}(z_2)$, $i+1\leq I_2\leq h$, of the surfaces M_1 and M_2. More precisely, we have

$$\omega_{I_1}^t(z)=\begin{cases}\omega_{I_1}(z)+\frac{1}{4}t\omega_{I_1}(p_1)\omega_{p_1}^1(z)+O(t^2) & \text{for } z\in M_1',\\ \frac{1}{4}t\omega_{I_1}(p_1)\omega_{p_2}^2(z)+O(t^2) & \text{for } z\in M_2',\end{cases} \tag{6.59}$$

and similarly for $\omega_{I_2}(z,t)$ with the roles of M_1 and M_2 interchanged. Here $\omega_w^i(z)$ are the Abelian differentials of the second kind on M_i, with double pole at w [cf. Eq. (6.57)]. The terms $O(t^2)$ are holomorphic differentials whose limits $t^{-2}O(t^2)$ may have a pole of order at most 4 at p_1 and p_2. Integrating over basis cycles gives the asymptotic behavior of the period matrix of M_t,

$$\Omega(t)=\begin{bmatrix}\Omega_1 & 0\\ 0 & \Omega_2\end{bmatrix}+O(t)\ , \tag{6.60}$$

where Ω_i are the period matrices of M_i. Next, if D_i is a divisor on M_t with zero Chern class, which decomposes into $D_t=D_1+D_2+D_{\mathcal{U}}$ with D_1, D_2, and $D_{\mathcal{U}}$ divisors on M_1', M_2', and $\mathcal{U}_t$ of degrees d_1, d_2, and $d_{\mathcal{U}}$, respectively, then the theta function will factorize as

$$\vartheta(D_t,\Omega)\rightarrow\vartheta(D_1-d_1p_1,\Omega_1)\vartheta(D_2-d_2p_2,\Omega_2)\ . \tag{6.61}$$

In particular, the Riemann class factorizes as $\Delta(t)\rightarrow\Delta_1+\Delta_2+p_{1,2}$. As for the prime form $E(z,w)$, it will behave as $z-w$ when both z and w are in the plumbing fixture, and otherwise

$$\begin{aligned}&E(z_1,w)\rightarrow E_1(z_1,p_1)wt^{-3/4}\ ,\\ &E(z_2,w)\rightarrow E_2(z_2,p_2)t^{-1/4}\ ,\\ &E(z_1,z_2)\rightarrow E_1(z_1,p_1)E_2(p_2,z_2)t^{-1/2}\ ,\end{aligned} \tag{6.62}$$

for $z_i\in M_i'$ and w in the plumbing fixture. Here of course the E_i's denote the prime forms of the surfaces M_i.

In the case in which the complement M_t' of the fixture remains connected [Fig. 19(b)], the normalized basis of h Abelian differentials $\omega_I^t(z)$ for the degenerating surface M_t will approach the normalized basis ω_I, $I=1,\ldots,h-1$ for the limiting surface M, while $\omega_h^t(z)$ will tend to the Abelian differential of the third kind $\omega_{p_1p_2}(z)$ [Eq. (6.56)], with poles at $z=p_1$ and p_2. On M_t' one can give precise asymptotes,

$$\begin{aligned}\omega_I^t(z)&=\omega_I(z)+\tfrac{1}{4}t[\omega_I(p_1)-\omega_I(p_2)]\\ &\qquad\times[\omega_{p_1}^1(z)-\omega_{p_2}^2(z)]+O(t^2)\ ,\\ &\qquad\qquad I=1,\ldots,h-1\ ,\\ \omega_h^t(z)&=\omega_{p_1p_2}(z)+t\tilde{\omega}_h(z)+O(t^2)\ ,\end{aligned} \tag{6.63}$$

where $\tilde{\omega}_h$ is a meromorphic differential with poles at p_1 and p_2 of order three. It follows that the period matrix of M_t can be written as

$$\Omega_t=\begin{bmatrix}\Omega_{IJ}(M) & \int_{p_1}^{p_2}\omega_I\\ \int_{p_1}^{p_2}\omega_J & \ln t+\text{const}\end{bmatrix}+O(t)\ , \tag{6.64}$$

with $1\leq I,J\leq h-1$. The asymptotics of theta functions and the prime form can now be derived in analogy with the previous case.

A detailed discussion of theta functions, the prime form, and their degeneration is to be found in Fay (1973). In the physics literature, the characterization of line bundles with holomorphic sections by the theta divisor appears in Alvarez-Gaumé, Moore, and Vafa (1986).

VII. HOLOMORPHIC STRUCTURE OF STRINGS

A fundamental principle underlying theories of closed oriented strings is that massless fields in two dimensions decompose into independent left- and right-movers. The independence is maintained at the interacting level, since the action is still the free action, and the presence of interactions is only indicated by the topology of the worldsheet. This principle is crucial in the construction of fermionic strings: in the type-II string we have to separate the contributions of left- and right-movers to assign them independent spin structures, while in the heterotic string we have to amalgamate the left-movers of the fermionic string with the right-movers of the bosonic string. For bosonic strings, separation of the left- and right-movers is not required, but it should remain a useful property of the partition function. A careful treatment of this chiral splitting and of the related issue of internal loop momenta has been provided in Secs. III.K and III.O.

A key observation due to Belavin and Knizhnik (1986) is that separation of left- and right-movers on the worldsheet can be translated into holomorphicity of the string integrand on moduli space. In fact, if Z is the partition function of a conformal field theory with respect to a background metric $ds^2=\rho\, dz\, d\bar{z}$ and we deform the background metric by a Beltrami differential μ, then

$$\delta_\mu \delta_{\bar{\mu}} \ln Z = \left[\frac{1}{4\pi}\right]^2 \int d^2z \sqrt{g}\, g^{z\bar{z}} \times \int d^2w \sqrt{g}\, g^{w\bar{w}} \mu_{\bar{z}}{}^{z} \bar{\mu}_{w}{}^{\bar{w}} \langle T_{zz} T_{\bar{w}\bar{w}} \rangle_{\text{conn}} .$$

Thus the vanishing of $\langle T_{zz} T_{\bar{w}\bar{w}} \rangle_{\text{conn}}$ would imply that Z is the absolute value squared of a holomorphic function on moduli space.

Anomalies in principle could spoil this picture. Recall (Secs. II.I and II.J) that the bosonic string is built out of the conformal systems of the matter fields x^μ, $\mu=1,\ldots,d$ and ghost fields b,c. If we consider, say, the x^μ fields alone, reparametrization invariance and separation of left- and right-movers (in Euclidean signature, holomorphic and antiholomorphic) cannot be achieved simultaneously, the obstruction being the nonvanishing central charge in the Virasoro algebra. This means that $\langle T_{zz} T_{\bar{w}\bar{w}} \rangle_{\text{conn}}$ develops a Schwinger term that prevents the vanishing of $\delta\bar{\delta}\ln Z$. The same is true for the isolated b,c system. For the combined x^μ,b,c system, however, the anomalies should cancel in $d=26$, and it is after cancellation that the string partition function should split into a holomorphic factor times its antiholomorphic conjugate on moduli space. Earlier expressions for the bosonic string such as (2.145) should be understood in this sense.

A complete analysis of the second variation $\delta\bar{\delta}\ln Z$ was carried out by Belavin and Knizhnik (1986). Besides justifying the above principles, their results also provide a basis for investigating the holomorphic structure on moduli space of the conformal field theories encountered earlier in Secs. II.I and II.J. In particular, they can be a starting point for a detailed study on higher-genus surfaces of the Bose-Fermi correspondence of two-dimensional field theory.

In Sec. VII.A we provide an exposition of the holomorphic anomaly formula of Belavin and Knizhnik, based on heat-kernel regularization. This leads to their characterization of the string partition function as the unique (up to constants) holomorphic nonvanishing section of a line bundle over moduli space. Sections VII.C and VII.D are devoted to bosonization, following Verlinde and Verlinde. They culminate in complete expressions for correlation functions of bosons and chiral fermions in terms of the prime form. In Sec. VII.E a geometric interpretation of the holomorphic anomaly is given in terms of curvature of determinant line bundles. This refines the Atiyah-Singer (1984) interpretation of chiral anomalies as nontriviality of these bundles. There the bundle was that of Dirac operators over the space of vector potentials modulo gauge transformations. Here it is the bundle of $\bar{\partial}$ operators over moduli space. The key new feature is the existence of a new metric built out of regularized determinants, the Quillen (1984) metric, so that nontriviality of the line bundle can be measured at the level of differential forms (rather than Chern classes) by the curvature and holonomy of its connection.

The determinants for Dirac and gauge-fixing operators obtained this way in terms of theta functions encode nicely their dependence on spin structures. They also allow a simple study of degeneration behavior. However, the resulting expressions for string scattering amplitudes are still somewhat formal, since they require a convenient parametrization of period matrices within the Siegel upper half space. It is possible that recent solutions of the Schottky problem based on the KP hierarchy may be helpful in this context, but the issue has not been fully explored as yet.

In Sec. VII.F we investigate the superholomorphic structure of superstrings, following D'Hoker and Phong (1987b). There is indeed a superholomorphic anomaly, which cancels in the critical dimension $d=10$ and for the heterotic string with rank 16 gauge groups. Thus we may hope that the superholomorphic structure of supermoduli space will impose powerful constraints on the superstring. The full consequences will require a better understanding of superalgebraic geometry, which is being developed by many authors. Finally, in Sec. VII.G we provide a detailed comparison of chiral splitting, holomorphic splitting, and holomorphic splitting at fixed internal momenta. A crucial ingredient in this comparison is a supersymmetric extension $\hat{\Omega}$ of the period matrix Ω. One of the major difficulties encountered in multiloop amplitudes has been the fact that supermoduli space does not seem to have a natural projection onto moduli space. The existence of $\hat{\Omega}$ indicates that such a projection exists if we represent supermoduli by $\{\hat{\Omega},\chi\}$ and moduli by $\{\hat{\Omega}\}$, though it need not coincide with the standard idea of split supermanifolds. The matrix $\hat{\Omega}$ may ultimately be the way to express superstring amplitudes in terms of modular forms.

A. Holomorphic anomalies

There is a simple way of viewing the holomorphic anomalies we shall discuss in this section as chiral anomalies. In fact, if we wish to consider the chiral version of the fermionic theories $b(dz)^n$, $c(dz)^{1-n}$ of Sec. II.J, quantization will demand a suitable notion of a determinant for the chiral operator ∇_n^z. Now determinants of chiral operators make sense only as sections of line bundles, as we shall see in Sec. VII.E. To obtain a scalar we could try instead to construct an appropriate square root for the nonchiral determinant of $\nabla_n^{z\dagger}\nabla_n^z=\Delta_n^{(-)}$. The phases of such square roots are arbitrary, however, and can only be determined by requiring further that the dependence on moduli of $\det\nabla_n^z$ mimic that of ∇_n^z itself. To understand this dependence, let

$$ds^2=\rho\mid dz+\mu\, d\bar z\mid^2 \tag{7.1}$$

parametrize deformations of a fixed conformal structure $ds^2=\rho\mid dz\mid^2$. The corresponding deformation of ∇_n^z is

$$\delta\nabla_n^z=\mu_{\bar z}{}^z\nabla^{\bar z}+n\,\nabla^{\bar z}\mu_{\bar z}{}^z\ . \tag{7.2}$$

Since $\mu_{\bar z}{}^z$ constitutes the holomorphic coordinates for moduli space near $\rho\mid dz\mid^2$, the fact that $\delta\nabla_n^z$ depends only on $\mu_{\bar z}{}^z$ and not $\bar\mu_z{}^{\bar z}$ means that ∇_n^z depends holomorphically on moduli parameters. Thus a chiral theory of b,c fermions requires a reparametrization-invariant, holomorphic square root of $\det'\Delta_n^{(-)}$, with suitable modifications necessitated by absorption of zero modes.

If we choose to maintain manifest reparametrization invariance, say by a heat-kernel regularization, we shall see that we cannot extract a holomorphic square root on moduli space, as may naively have been expected from the previous discussion. A local "holomorphic anomaly" is measured by

$$\delta_\mu\delta_{\bar\mu}\ln\left[\frac{\det'\Delta_n^{(\pm)}}{\det\langle\phi_a\mid\phi_b\rangle\det\langle\psi_a\mid\psi_b\rangle}\right] \tag{7.3}$$

and has a very similar structure to the conformal anomaly. We turn now to its evaluation. We shall work with Lorentz-covariant derivatives D_z^n, $D_{\bar z}^n$ on tensors of weight n,

$$D_z^n=e_z^m(\partial_m+in\,\omega_m)\ ,$$
$$\Delta_n^{(+)}=-2D_{\bar z}^{n+1}D_z^n\ ,$$
$$\Delta_n^{(-)}=-2D_z^{n-1}D_{\bar z}^n\ ,$$

instead of the covariant derivatives ∇_n^z, since this setup is more convenient for the generalization to superholomorphic anomalies in Sec. VII.F. Recall that determinants are defined by

$$\ln\det'\Delta_n^{(+)}=-\int_\varepsilon^\infty\frac{dt}{t}(\mathrm{tr}e^{-t\Delta_n^{(+)}}-N_n^+)\ , \tag{7.4}$$

and a change with respect to μ produces

$$\delta_\mu\ln\det'\Delta_n^{(+)}=\int_\epsilon^\infty dt\ \mathrm{tr}(\delta_\mu\Delta_n^{(+)}e^{-t\Delta_n^{(+)}})\ . \tag{7.5}$$

Now the operators $\Delta_n^{(+)}$ and $\Delta_{n+1}^{(-)}$ are not in general invertible on the entire function spaces of rank n and $n+1$ tensors, so it is appropriate to single out their kernels by introducing the projection operators

$$\Pi_n^+\equiv 1+2D_{\bar z}^{n+1}(\Delta_{n+1}^{(-)})^{-1}D_z^n\ \ \text{onto}\ \mathrm{Ker}D_z^n\ ,$$
$$\Pi_{n+1}^-\equiv 1+2D_z^n(\Delta_n^{(+)})^{-1}D_{\bar z}^{n+1}\ \ \text{onto}\ \mathrm{Ker}D_{\bar z}^{n+1}\ . \tag{7.6}$$

Hence we decompose the trace as follows:[38]

$$\begin{aligned}\mathrm{tr}\delta_\mu\Delta_n^{(+)}e^{-t\Delta_n^{(+)}}&=-2\,\mathrm{tr}\delta_\mu D_{\bar z}D_ze^{-t\Delta_n^{(+)}}\\&\quad-2\,\mathrm{tr}\delta_\mu D_zD_{\bar z}e^{-t\Delta_{n+1}^{(-)}}\\&=-2\,\mathrm{tr}\delta_\mu D_{\bar z}D_ze^{-t\Delta_n^{(+)}}(1-\Pi_n^+)\\&\quad-2\,\mathrm{tr}\delta_\mu D_zD_{\bar z}e^{-t\Delta_{n+1}^{(-)}}(1-\Pi_{n+1}^-)\ .\end{aligned} \tag{7.7}$$

Since, on the space complementary to their kernel, the Laplace operators are now invertible, this equals

$$\begin{aligned}&2\frac{\partial}{\partial t}\mathrm{tr}\delta_\mu D_{\bar z}D_z(\Delta_n^{(+)})^{-1}e^{-t\Delta_n^{(+)}}(1-\Pi_n^+)\\&\quad+2\frac{\partial}{\partial t}\mathrm{tr}\delta_\mu D_zD_{\bar z}(\Delta_{n+1}^{(-)})^{-1}e^{-t\Delta_{n+1}^{(-)}}(1-\Pi_{n+1}^-)\ .\end{aligned} \tag{7.8}$$

Thus

$$\begin{aligned}\delta_\mu\ln\det'\Delta_n^{(+)}&=-2\,\mathrm{tr}\delta_\mu D_{\bar z}D_z(\Delta_n^{(+)})^{-1}\\&\quad\times e^{-\varepsilon\Delta_n^{(+)}}(1-\Pi_n^+)\\&\quad-2\,\mathrm{tr}\delta_\mu D_zD_{\bar z}(\Delta_{n+1}^{(-)})^{-1}\\&\quad\times e^{-\varepsilon\Delta_{n+1}^{(-)}}(1-\Pi_{n+1}^-)\ .\end{aligned} \tag{7.9}$$

Then we evaluate the second derivative with respect to $\bar\mu$. Two formulas come in handy:

$$\delta_{\bar\mu}(1-\Pi_n^+)=-2D_{\bar z}(\Delta_{n+1}^{(-)})^{-1}\delta_{\bar\mu}D_z\Pi_n^++O(\delta_{\bar\mu}D_{\bar z})\ , \tag{7.10}$$

$$\delta_{\bar\mu}(1-\Pi_{n+1}^-)=-2\Pi_{n+1}^-\delta_{\bar\mu}D_z(\Delta_n^{(+)})^{-1}D_{\bar z}+O(\delta_{\bar\mu}D_{\bar z})\ ,$$

as well as

$$\delta e^A=\int_0^1du\ e^{uA}\delta A e^{(1-u)A}\ . \tag{7.11}$$

With the help of these and some straightforward algebra, one finds

$$\begin{aligned}\delta_{\bar\mu}\delta_\mu\ln\det'\Delta_n^{(+)}&=-2\,\mathrm{tr}(\delta_{\bar\mu}D_{\bar z})D_z(\Delta_n^{(+)})^{-1}e^{-\varepsilon\Delta_n^{(+)}}(1-\Pi_n^+)-2\,\mathrm{tr}(\delta_{\bar\mu}\delta_\mu D_z)D_{\bar z}(\Delta_{n+1}^{(-)})^{-1}e^{-\varepsilon\Delta_{n+1}^{(-)}}(1-\Pi_{n+1}^-)\\&\quad-2\,\mathrm{tr}\delta_\mu D_{\bar z}\Pi_{n+1}^-\delta_{\bar\mu}D_z(\Delta_n^{(+)})^{-1}(1-\Pi_n^+)-2\,\mathrm{tr}\delta_\mu D_{\bar z}(1-\Pi_{n+1}^-)(\Delta_{n+1}^{(-)})^{-1}\delta_{\bar\mu}D_z\Pi_n^+\\&\quad+2\varepsilon\int_0^1du\ \mathrm{tr}\delta_\mu D_{\bar z}e^{-\varepsilon u\Delta_{n+1}^{(-)}}\delta_{\bar\mu}D_ze^{-\varepsilon(1-u)\Delta_n^{(+)}}\ .\end{aligned} \tag{7.12}$$

[38]For brevity, we shall denote $D_z=D_z^n$ and $D_{\bar z}=D_{\bar z}^{n+1}$.

Here we have made use of the fact that within finite-dimensional traces, the heat kernel for short time ε reduces to the identity operator. The presence of nonlocal contributions in the first four terms on the right-hand side reminds us of the fact that it is natural to work with determinants divided by normalizations of zero modes, as in the case of the Weyl anomaly. The changes of the finite-dimensional determinants under δ_μ and $\delta_{\bar\mu}$ are obtained as follows. Let ϕ_j span a basis for $\mathrm{Ker} D_{\bar z}^{n+1}$ and ψ_a a basis for $\mathrm{Ker} D_z^n$. It is easy to show that

$$\delta_{\bar\mu}\delta_\mu \ln\det\langle \phi_j \mid \phi_k\rangle = 2\langle \delta_{\bar\mu}\delta_\mu\phi_j \mid \phi_j\rangle + \langle \delta_{\bar\mu}\phi_j \mid (1-\Pi_{n+1}^-) \mid \delta_\mu\phi_j\rangle .$$

From $D_{\bar z}^{n+1}\phi_j = 0$, we deduce

$$\delta_\mu D_{\bar z}^{n+1}\phi_j + D_{\bar z}^{n+1}\delta_\mu\phi_j = 0 \tag{7.13}$$

and hence

$$(1-\Pi_{n+1}^-)\mid \delta_\mu\phi_j\rangle = 2D_z(\Delta_n^{(+)})^{-1}\delta_\mu D_{\bar z}^{n+1}\mid \phi_j\rangle , \tag{7.14}$$

so that

$$\delta_{\bar\mu}\delta_\mu \ln\det\langle \phi_j \mid \phi_k\rangle = 2\langle \delta_{\bar\mu}\delta_\mu\phi_j \mid \phi_j\rangle - 2\,\mathrm{tr}\Pi_{n+1}^-\delta_{\bar\mu}D_z^n(1-\Pi_n^+)(\Delta_n^{(+)})^{-1}\delta_\mu D_{\bar z}^{n+1} \tag{7.15}$$

and similarly

$$\delta_{\bar\mu}\delta_\mu \ln\det\langle \psi_a \mid \psi_b\rangle = 2\langle \delta_{\bar\mu}\delta_\mu\psi_a \mid \psi_a\rangle - 2\,\mathrm{tr}\Pi_n^+\delta_\mu D_{\bar z}^{n+1}(1-\Pi_{n+1}^-)(\Delta_{n+1}^{(-)})^{-1}\delta_{\bar\mu}D_z^n . \tag{7.16}$$

We may recast Eq. (7.12) in the form

$$\begin{aligned}\delta_{\bar\mu}\delta_\mu \ln\frac{\det'\Delta_n^{(+)}}{\det\langle \phi_j \mid \phi_k\rangle\det\langle \psi_a \mid \psi_b\rangle} = &-2\,\mathrm{tr}(\delta_{\bar\mu}\delta_\mu D_{\bar z}^{n+1})D_z^n(\Delta_n^{(+)})^{-1}e^{-\varepsilon\Delta_n^{(+)}}(1-\Pi_n^+)\\ &-2\,\mathrm{tr}(\delta_{\bar\mu}\delta_\mu D_z^n)D_{\bar z}^{n+1}(\Delta_{n+1}^{(-)})^{-1}e^{-\varepsilon\Delta_{n+1}^{(-)}}(1-\Pi_{n+1}^-)\\ &-2\langle \delta_{\bar\mu}\delta_\mu\phi_j \mid \phi_j\rangle - 2\langle \delta_{\bar\mu}\delta_\mu\psi_a \mid \psi_a\rangle\\ &+2\varepsilon\int_0^1 du\,\mathrm{tr}\delta_\mu D_{\bar z}^{n+1}e^{-\varepsilon u\Delta_{n+1}^{(-)}}\delta_{\bar\mu}D_z^n e^{-\varepsilon(1-u)\Delta_n^{(+)}} .\end{aligned} \tag{7.17}$$

The next crucial observation is that the only way the operators and zero modes can depend on μ and $\bar\mu$ simultaneously is through a conformal change, as we indicated when we first wrote down the corresponding differential operators. Denoting this Weyl scaling by $\delta\sigma$, we have

$$\begin{aligned}&\delta_{\bar\mu}\delta_\mu D_{\bar z}^{n+1} = -(n+2)\delta\sigma D_{\bar z}^{n+1} + (n+1)D_{\bar z}^{n+1}\delta\sigma ,\\ &\delta_{\bar\mu}\delta_\mu D_z^n = (n-1)\delta\sigma D_z^n - nD_z^n\delta\sigma ,\end{aligned} \tag{7.18}$$

and correspondingly

$$\begin{aligned}&\delta_{\bar\mu}\delta_\mu\phi_j = -(n+1)\delta\sigma\phi_j ,\\ &\delta_{\bar\mu}\delta_\mu\psi_a = n\,\delta\sigma\psi_a .\end{aligned} \tag{7.19}$$

With the help of these, we see that the first four terms on the right-hand side of Eq. (7.17) reduce to

$$\begin{aligned}n\,\mathrm{tr}\delta\sigma e^{-\varepsilon\Delta_{n+1}^{(-)}} - (n+1)\mathrm{tr}\delta\sigma e^{-\varepsilon\Delta_n^{(+)}} &= -\frac{1}{4\pi\varepsilon}\int d^2\xi\sqrt{g}\,\delta\sigma\\ &\quad -\frac{6n^2+6n+1}{12\pi}\int d^2\xi\sqrt{g}R\,\delta\sigma + O(\varepsilon) ,\end{aligned} \tag{7.20}$$

which is precisely the effect of the conformal anomaly on the determinant of $\Delta_n^{(+)}$ [see Eq. (2.69)]. For a deformation of the form (7.1) $\delta\sigma = \bar\mu\mu$. Thus the only term that remains is the integral over u in Eq. (7.17), and this term is both local on the worldsheet and, as can be seen from its definition, reparametrization invariant. Thus we may evaluate it locally on the Riemann surface, its reparametrization invariance guaranteeing that these local contributions will fit together consistently. Since we work only up to a Weyl anomaly, we can in fact work around flat space, and the calculation is then easily performed. Putting all together, one finds

$$\begin{aligned}\delta_{\bar\mu}\delta_\mu \ln&\frac{\det'\Delta_n^{(\pm)}}{\det\langle \phi_j \mid \phi_k\rangle\det\langle \psi_a \mid \psi_b\rangle}\\ &= -\frac{6n^2\pm 6n+1}{12\pi}\int d^2\xi\sqrt{g}\,(\nabla_z\bar\mu\nabla^z\mu + 2R\mu\bar\mu) .\end{aligned} \tag{7.21}$$

We note that this second variation corresponds to the particular choice of variations $\rho\mid dz\mid^2 \to \rho\mid dz+\mu\,d\bar z\mid^2$. Clearly we can accompany this variation with any additional Weyl scaling without changing the complex structure, so strictly speaking the holomorphic anomaly is not intrinsic and must be considered modulo the conformal anomaly.

We can now give a complex analytic characterization of the bosonic string partition function. In view of Eq. (7.21), the function

$$F=\left[\frac{\det'\Delta_0}{\int d^2\xi\sqrt{g}\,\det\langle\omega_I\mid\omega_J\rangle}\right]^{-13}\left[\frac{\det'\Delta_2^{(-)}}{\det\langle\phi_a\mid\phi_b\rangle}\right] \tag{7.22a}$$

is the square modulus of a holomorphic function on moduli space, as long as the Abelian differentials ω_I and quadratic differentials ϕ_a are chosen to depend holomorphically on moduli parameters. Stated as generally as that, such choices are not possible globally on moduli space. A weaker choice is, however, possible, which is dictated by the structure of Eq. (7.22) and suffices for our purposes. Let K and Λ be the maximum wedge powers of the spaces of quadratic differentials and Abelian differentials, respectively. Since these spaces vary holomorphically with moduli, they should be viewed as making two holomorphic line bundles over moduli space. Given a holomorphic section s of $K\otimes\Lambda^{-13}$, we can write it locally as

$$s=(\phi_1\wedge\cdots\wedge\phi_{3h-3})\otimes(\omega_1\wedge\cdots\wedge\omega_h)^{-13}\ , \tag{7.22b}$$

and the function F in Eq. (7.22a) depends only on s and not on the particular factorization into ϕ_a and ω_I. We can now apply a theorem of Mumford (1977) which guarantees the existence of the weaker choice we referred to earlier, namely, that of a global nowhere-vanishing holomorphic section s of $K\otimes\Lambda^{-13}$. In other words, neither line bundle K nor Λ is trivial over moduli space, but $K\otimes\Lambda^{-13}$ is. If s is a global section of $K\otimes\Lambda^{-13}$, the function F will be globally defined on moduli space and hence must be constant. Writing s as in Eq. (7.22b), we note that $\det\langle\omega_I\mid\omega_J\rangle^{-13}\ \phi_1\wedge\cdots\wedge\bar{\phi}_{3h-3}$ is now a well-defined global $(6h-6)$ volume form over moduli space, which coincides in local coordinates with the measure $[dm]\det\langle\mu_j\mid\phi_k\rangle\det\langle\omega_I\mid\omega_J\rangle^{-13}$ of Sec. II.G. Thus, up to a multiplicative constant c, the bosonic string partition function can be rewritten as

$$Z=c\int_{\mathcal{M}_h}\phi_1\wedge\cdots\wedge\bar{\phi}_{3h-3}\det\langle\omega_I\mid\omega_J\rangle^{-13}\ , \tag{7.23}$$

a formula that is manifestly conformally invariant.

The line bundles Λ and K are usually called, respectively, the Hodge bundle and the canonical bundle of moduli space.

In the Deligne-Mumford compactification $\overline{\mathcal{M}}_h$, one adjoins to moduli space the divisor $[\Delta]$ of Riemann surfaces with nodes. Both the canonical bundle K and the Hodge bundle admit natural extensions to $\overline{\mathcal{M}}_h$, the first as the canonical bundle of $\overline{\mathcal{M}}_h$, and the second as the line bundle of dualizing differentials. A characteristic class computation then shows that $K\otimes\Lambda^{-13}$ over $\overline{\mathcal{M}}_h$ is actually not trivial and admits $[-2\Delta]$ as divisor. Since the components of Δ are independent and F is nowhere vanishing on the interior of $\mathcal{M}_h$, it follows that F must have a second-order pole along Δ. Physically, this pole corresponds to the presence in the string mass spectrum of the tachyon.

Holomorphic anomalies in string theory were discovered by Belavin and Knizhnik (1986). In retrospect, related issues had occurred earlier in the work of Schwinger (1951), Coleman, Gross, and Jackiw (1969), and Quillen (1984) on two-dimensional Dirac operators coupled to vector potentials. Equation (7.23), which makes no reference to regularized determinants, appeared in Belavin and Knizhnik (1986) and also in Bost and Jolicoeur (1986) and Catenacci *et al.* (1986). It is the starting point for several expressions of the string partition function in terms of modular forms and theta functions, e.g., Beilinson and Manin (1986), Belavin *et al.* (1986), Manin (1986), Moore (1986), Dugan (1987), Morozov (1987a, 1987b). Other expressions in terms of theta functions can be derived from chiral bosonization formulas below, as indicated in Sec. VII.D. Applications to chiral determinants are considered in Knizhnik (1986a, 1986b, 1987). A careful discussion of the extensions of the Hodge and canonical bundles to the compactified moduli space $\overline{\mathcal{M}}_h$ is provided in the review of Nelson (1987a).

B. The free scalar field

We now begin a detailed study of the conformal fields introduced in Sec. II.J. The simplest field is a free scalar boson x, with action

$$I_x(x)=\frac{1}{4\pi}\int d^2z\,\partial_z x\partial_{\bar z}x\ .$$

Its two-point function $G(z,w)=\langle x(z)x(w)\rangle$ is familiar from Sec. II.G. Recall that it is not Weyl invariant, so that $x(z)$ does not have a well-defined conformal dimension. However, both $\partial_z x$ and the vertex operator

$$V_q(z)=\rho^{q^2/2}e^{iqx(z)}$$

are well-behaved conformal fields in view of Eqs. (2.87) and (2.90), and have conformal dimensions (1,0) and $(q^2/2,q^2/2)$, respectively.

Finally, we can now address the issue of chiral scalar fields. In the presence of the holomorphic anomaly discussed in Sec. VII.A, the partition function of x is not the absolute value squared of a holomorphic function on moduli space. Nevertheless we can define the partition function Z_Δ^{-1} of a chiral scalar field by

$$\int Dx\,\exp\left[-\frac{1}{4\pi}\int d^2z\,\partial_z x\partial_{\bar z}x\right]=|Z_\Delta^{-1}|^2e^{S_L(\rho)}(\det\mathrm{Im}\Omega)^{-1/2}\ , \tag{7.24}$$

where $S_L(\rho)$ is the Liouville action in conformal gauge with $ds^2=\rho\,dz\,d\bar z$,

$$S_L(\rho)=\frac{1}{48\pi}\int d^2z\,\partial_z\ln\rho\partial_{\bar z}\ln\rho\ .$$

In this way, Z_Δ will be holomorphic on moduli space, although it has both local and global gravitational anomalies, as indicated by the presence of the Liouville action and det $\mathrm{Im}\Omega$.

It is easy to determine the variation of Z_Δ with respect to moduli, since it reduces to the expectation value of the chiral stress tensor T_{zz} of Eq. (2.178),

$$\frac{4\pi}{\sqrt{g}}\frac{\delta}{\delta g^{zz}}\ln Z_\Delta=-T_{zz}\ . \tag{7.25}$$

Equation (2.90) for the propagator and chiral renormalization procedure yields

$$T_{zz}=-\tfrac{1}{2}\lim_{z\to w}\left[\partial_z\partial_w \ln E(z,w)-\frac{1}{(z-w)^2}\right]\ . \tag{7.26}$$

This means that the stress tensor is the third Taylor expansion coefficient of the expansion of E for z near w,

$$E(z,w)=z-w+(z-w)^3T_{zz}+O(z-w)^5)\ . \tag{7.27}$$

C. Spin-$\frac{1}{2}$ bosonization

In this section and those that follow, we shall solve completely the theory of circle-valued bosonic fields. The formulas we shall derive for correlation functions will be explicit enough to allow us to identify them with the corresponding correlation functions for chiral fermions. We begin with the simplest case of no background charge Q, where the action reduces to

$$I_x(\varphi)=\frac{1}{4\pi}\int d^2z\ \partial_z\varphi\partial_{\bar z}\varphi\ . \tag{7.28}$$

The first task is a suitable indexing of the soliton sector. Recall that φ is to be thought of as circle valued, i.e., $d\varphi$ is a closed 1-form that is not necessarily exact [note, however, that the action $I_x(\varphi)$ in Eq. (7.28) is unambiguous, since it can as well be written as the integral of the (1,1) form $\partial\varphi\wedge\bar\partial\varphi$, after splitting $d\varphi$ as $\partial\varphi+\bar\partial\varphi$]. Up to exact forms, a closed 1-form is characterized by its winding numbers along cycles of the homology basis

$$2\pi n_I=\oint_{A_I}d\varphi,\quad 2\pi m_I=\oint_{B_I}d\varphi\ .$$

If we fix once and for all a set of closed 1-forms ϕ_{mn} with precisely winding numbers m_I and n_I, $d\varphi$ can be unambiguously written as

$$d\varphi=\phi_{mn}+dx\ , \tag{7.29}$$

where x is a genuine single-valued scalar, completely determined by the familiar normalization requirement $\int d^2z\sqrt{g}\,x=0$, needed to remove the zero mode of the scalar Laplacian. Since the action $I_x(\varphi)$ then splits completely as

$$I_x(\varphi)=I_x(\phi_{mn})+I_x(x)=I_{mn}+I_x(x)\ ,$$

the path integral becomes

$$\sum_{m,n}e^{-I_{mn}}\int Dx\ e^{-I_x(x)}\ ,$$

which is tractable. With the canonical basis choice for Abelian differentials, $\omega_1,\dots,\omega_h$, it is easy to write down such forms ϕ_{mn}:

$$\phi_{mn}=-i\pi(m+\bar\Omega n)_I(\mathrm{Im}\Omega)^{-1}_{IJ}\omega_J+\text{c.c.}\ , \tag{7.30}$$

and the soliton contribution to the action is

$$I_{mn}=\frac{\pi}{2}(m+\bar\Omega n)(\mathrm{Im}\Omega)^{-1}(m+\Omega n)\ . \tag{7.31}$$

We shall illustrate the procedure with an explicit calculation of the partition function. In this case the contribution to the Dx integral is $(8\pi^2\det{}'\Delta_g/\int d^2z\sqrt{g}\,)^{-1/2}$, while the sum over soliton sectors produces the factor

$$\sum_{m,n}\exp\left[-\frac{\pi}{2}(m+\bar\Omega n)(\mathrm{Im}\Omega)^{-1}(m+\Omega n)\right]\ . \tag{7.32}$$

This actually is a sum over all spin structures of theta functions evaluated at 0. To see this, we rewrite the summation index m as $2(k+\delta'')$ with δ'' half-integer valued and apply the Poisson summation formula to get

$$(\det\mathrm{Im}\Omega)^{1/2}\sum_{\delta''}\sum_{n,k}e^{-\pi n(\mathrm{Im}\Omega)n/2}e^{-\pi k(\mathrm{Im}\Omega)k/2}\times e^{i(2\pi\delta''+\pi n\,\mathrm{Re}\Omega)k}\ . \tag{7.33}$$

If we now rewrite the summation over n,k as the summation over integers p,q and half-integers δ' with $p+q+2\delta'=n$, $p-q=k$, we recognize the sum over n,k as

$$\sum_{\delta'}|\vartheta[\delta](0,\Omega)|^2\ .$$

Thus the final formula for the circle-valued bosonic field with vanishing background charge is

$$Z_B=\sum_\delta Z_B^\delta\ ,$$

where

$$Z_B^\delta=\left[\frac{8\pi^2\det{}'\Delta_g}{\int d^2z\sqrt{g}\,\det\mathrm{Im}\Omega}\right]^{-1/2}|\vartheta[\delta](0,\Omega)|^2\ . \tag{7.34}$$

We can compare this expression with the partition function of the chirally symmetric fermion theory with spin $\frac{1}{2}$ (cf. Sec. II.J):

$$Z_F^\delta=\int e^{-I(b,c)+\text{c.c.}}\ .$$

Clearly, both Z_B^δ and Z_F^δ vanish when there is a Dirac zero mode, so we discuss only generic even-spin structures, in which case

$$Z_F^\delta=(\det\Delta_{1/2}^{(-)})\ . \tag{7.35}$$

To compare Eq. (7.34) with (7.35) it suffices to compare their variations with respect to the background metric. Since determinants are regularized by heat kernels, they are manifestly reparametrization invariant. Modular anomalies could come from changing the basis of Abelian differentials and hence changing Ω, but this is compensated by the theta factors, which render Eq. (7.34) modular invariant. Finally, the conformal and holomorphic anomalies of both expressions have the same coefficient $c_{1/2}=-\frac{1}{2}c_0=-\frac{1}{2}$. This means that the two expressions differ by a multiplicative constant depending only on the genus h. When $h=1$, explicit calculation of the fermionic determinants shows that the constant is one, and the general case can be determined by letting both sides degenerate.

The bosonic theory thus corresponds to a sum over spin structures of the fermionic ones. Such a sum should be expected, since it is hard to imagine a particular spin structure being preferred by the bosonic theory.

Correlation functions can be evaluated in the same way. Let the metric in conformal gauge be $ds^2=\rho\, dz\, d\bar{z}$, and set

$$Z_B(z_1,\ldots,w_M)=\int D\varphi\, e^{-I_x(\varphi)}\times\prod_1^M \rho^{1/2}(z_i)e^{i\varphi(z_i)}\times\prod_1^M\rho^{1/2}(w_i)e^{-i\varphi(w_i)} .$$

In each soliton sector (m,n) we can replace $\varphi(z_i)-\varphi(w_i)$ by

$$\varphi(z_i)-\varphi(w_i)=\int_{w_i}^{z_i}\phi_{mn}+x(z_i)-x(w_i) ,$$

so that the full functional integral over $D\varphi$ becomes

$$\sum_{m,n} e^{-I_{mn}}\exp[-\pi i(m+\bar{\Omega}n)(\mathrm{Im}\Omega)^{-1}I(\Sigma z_i-\Sigma w_i)+\mathrm{c.c}]\int Dx\, e^{-I_x(x)}\prod_1^M\rho^{1/2}(z_i)e^{ix(z_i)}\prod_1^M\rho^{1/2}(w_i)e^{-ix(w_i)} ,$$

with $I(\Sigma z_i-\Sigma w_i)$ the Abel map defined in Eq. (6.33). The contributions of the scalar field x have been calculated before and can be written in terms of the prime form, while a similar Poisson summation argument can be applied to the sum over the soliton sector. The result is

$$Z_B(z_1,\ldots,w_M)=\left[\frac{8\pi^2\det{}'\Delta_g}{\int d^2z\sqrt{g}\,\det\mathrm{Im}\Omega}\right]^{-1/2}\times\sum_\delta|\vartheta[\delta](\Sigma z_i-\Sigma w_i,\Omega)|^2\times\left|\frac{\prod_{i<j}E(z_i,z_j)\prod_{i<j}E(w_i,w_j)}{\prod_{i,j}E(z_i,w_j)}\right|^2 , \tag{7.36}$$

where we have followed common practice in writing $\sum z_i-\sum w_j$ for $I(\Sigma z_i-\Sigma w_i)$ when no confusion is possible.

We should like to arrive at a chiral form of bosonization. If δ is a generic even-spin structure, $\vartheta[\delta](0,\Omega)$ will not vanish, and the contribution of δ to the above sum can be rewritten as

$$Z_B^\delta\,|A_B^\delta(z_1,\ldots,w_M)|^2 ,$$

where we have introduced the "normalized" amplitudes

$$A_B^\delta(z_1,\ldots,w_M)=\frac{\vartheta[\delta]\left[\sum z_i-\sum w_i,\Omega\right]}{\vartheta[\delta](0,\Omega)}\times\frac{\prod_{i<j}E(z_i,z_j)\prod_{i<j}E(w_i,w_j)}{\prod_{i,j}E(z_i,w_j)} . \tag{7.37}$$

This expression transforms for each z_i as a section of the spin bundle corresponding to δ. As a section of z_1, say, it has "physical" zeros and poles at $z_2,\ldots,z_M$ and $w_1,\ldots,w_M$, respectively. Since the Chern class of a spin bundle is $h-1$, it must have unphysical zeros $p_1,\ldots,p_h$ as well, as determined by the divisor equation

$$I\left[\sum_2^M z_i-\sum_1^M w_i+\sum_1^h p_k\right]=I(\text{divisor of }\delta) . \tag{7.38}$$

By the Jacobi inversion theorem, Sec. VI.E, the p_k's are completely determined by the z_i's and w_i's.

Let us now examine the structure of correlation functions of the theory of chiral fermions b, c of rank $\frac{1}{2},-\frac{1}{2}$, with spin structure also δ. The operator product expansions of Eq. (2.166) simply say that

$$\left\langle\prod_{i=1}^M b(z_i)\prod_{i=1}^M c(w_i)\right\rangle \tag{7.39}$$

also transforms in z_1 as a section of the spin bundle δ, and it also has zeros at $z_2,\ldots,z_M$, poles at $w_1,\ldots,w_M$. The unphysical zeros must coincide with those dictated by Eq. (7.38). Thus the fermionic correlation function agrees with the bosonic one given by Eq. (7.37).

Bosonization will yield theta-function identities if correlation functions for the fermionic system can be evaluated independently. For the spin-$\frac{1}{2}$ system we can produce explicit propagators using the prime form. If δ is a generic even-spin structure, there is no zero mode. The fermionic propagator is given by the so-called Szegö kernel

$$S_\delta(z,w)=\langle b(z)c(w)\rangle=\frac{1}{E(z,w)}\frac{\vartheta[\delta](z-w,\Omega)}{\vartheta[\delta](0,\Omega)} , \tag{7.40}$$

and the correlation functions (7.39) become

$$\det S_\delta(z_i,w_j) . \tag{7.41}$$

Taking $M=2$ and comparing with Eq. (7.37) gives Fay's trisecant formula,

$$\vartheta[\delta](z_1+z_2-w_1-w_2,\Omega)\vartheta[\delta](0,\Omega)E(z_1,z_2)E(w_1,w_2)=\vartheta[\delta](z_1-w_1,\Omega)\vartheta[\delta](z_2-w_2,\Omega)E(z_2,w_1)\times E(z_1,w_2)-(z_1\to z_2)\ . \tag{7.42}$$

This is known to be a rather remarkable identity, since it does not hold for theta functions defined out of arbitrary matrices Ω in the Siegel upper half space. It relies heavily on the fact that Ω is the period matrix of a Riemann surface.

For odd-spin structures δ, we have to modify Eq. (7.37) as a candidate for normalized amplitudes, since the theta function will vanish at 0. In this case we replace $1/\vartheta[\delta](0,\Omega)$ in Eq. (7.37) by $(\det \mathrm{Im}\Omega)^{-1/4}$. Generally there will be exactly one Dirac zero mode $h_\delta(z)$, which we actually constructed in Sec. VI.F. To evaluate the fermionic correlation functions, we normally have to project out this zero mode. As in the case of the scalar Green's function [see Eq. (7.25)], this will spoil the meromorphicity of the propagator. In practice it is more convenient to work with the following propagator, which was already encountered in Eq. (3.204):

$$S_\delta(z,w)=\frac{1}{E(z,w)}\frac{\sum \partial_I\vartheta[\delta](z-w,\Omega)\omega_I(y)}{\sum \partial_I\vartheta[\delta](0,\Omega)\omega_I(y)}\ , \tag{7.43}$$

where y is an arbitrary point on the surface M where h_δ does not vanish. This propagator is meromorphic with a simple pole at $z=w$. Its drawback is that it is multiple valued and strictly speaking should be viewed as defined on the universal covering of M. This multiple valuedness, however, will disappear from the correlation functions

$$\left\langle \prod_{i=1}^{M} b(z_i)\prod_{i=1}^{M} c(w_i)\right\rangle=\sum_{k,l}(-1)^{k+l}h_\delta(z_k)h_\delta(w_l)\times\det_{\substack{i\neq k\\ j\neq l}} S_\delta(z_i,w_j)\ . \tag{7.44}$$

These can now be checked to coincide with the above prescription for the bosonic amplitudes.

Bosonization in field theory goes back to Skyrme (1961, 1962), Coleman (1975), Mandelstam (1975), and Witten (1984). For the ghost system, it was considered by Marnelius (1983). As discussed earlier in Sec. II.J, the equivalence of the fermionic b,c and bosonic φ systems was suggested by Friedan, Martinec, and Shenker (1986) based on current algebra. The proof for spin $\frac{1}{2}$ was given by Alvarez-Gaumé, Moore, and Vafa (1986) and Bost and Nelson (1986) using the Belavin-Knizhnik theorem. A different argument based on explicit computation of the expectation values of the stress tensor of both theories can be found in Sonoda (1987c). Chiral bosonization as presented here and in the next section is due to Verlinde and Verlinde (1987a). These formulas for chiral amplitudes are among the most powerful tools in the study of string amplitudes available today.

D. Higher-spin bosonization

We now come to the general circle-valued Bose field coupled to a nonvanishing background charge Q. An especially lucid treatment has been given by Verlinde and Verlinde (1987a). We shall follow them and limit ourselves to some clarifying comments on their work. Recall that a naive action leading to the right current algebra has been written down in Eq. (2.180). It suffers, however, from ambiguities, since φ is multiple valued. The prescription is to replace it by

$$I_Q(\varphi)=\frac{1}{4\pi}\int d^2z\,\partial_z\varphi\partial_{\bar z}\varphi -iQ\left[\frac{1}{4\pi}\int_{M_{\rm cut}}d^2z\sqrt{g}R\varphi+\frac{1}{2\pi}\int_{\partial M_{\rm cut}}ds\,k\varphi+\frac{1}{2\pi}\sum_{k=1}^{4h}\Lambda_k\varphi(p_k)\right]\ , \tag{7.45}$$

where we have fixed a base point P_0 and cut the Riemann surface along homology cycles to a polygon $M_{\rm cut}$ with $4h$ boundary curves as in Appendix D (see Fig. 11). The p_k's are the corners of $M_{\rm cut}$, the Λ_k's are given by $\Lambda_k=\pi/2-\vartheta_k-2\pi l_k$ where the ϑ_k's are the inner angles at the corners p_k's, and the l_k's are integers chosen to satisfy $\sum l_k=h-1$. Finally, k is the geodesic curvature.

Since the geodesic curvature transforms as $k=e^{-\sigma}\partial_n\sigma+e^{-\sigma}\hat k$ under Weyl scalings $g=e^{2\sigma}\hat g$ (here ∂_n is the normal derivative), and Weyl scalings preserve angles, $I_Q(\varphi)$ is easily seen to be invariant under scalings. If we let ϑ denote the angle that the boundary curves make with a fixed given direction, and ds denote arc length, then the form $d\vartheta-k\,ds$, which is defined originally only along boundary curves, extends to a 1-form in a two-dimensional neighborhood of each curve. Furthermore, its exterior derivative is just the curvature form.

We can now verify that the action $I_Q(\varphi)$ is also invariant under changes of the base point and of the curves a_i, b_i chosen to cut open the surface M. This follows by integrating the form $(d\vartheta-k\,ds)\varphi$ along the closed cycles $a_i-\tilde a_i, b_i-\tilde b_i$ if $\tilde a_i,\tilde b_i$ is another choice. Finally, it will emerge as the result of explicit calculations below that the correlation functions of the theory will be independent as well of the choice of integer l_k's (see Fig. 22).

With the proper definition for the action, we can now proceed very much as in the study of spin-$\frac{1}{2}$ bozonization. On a typical soliton configuration $\varphi(z)=\int_P^z\omega_i$, the contribution to the action of the background charge Q can be evaluated to be

$$-Q\int_{(h-1)P}^{\Delta}\omega_i\ . \tag{7.46}$$

This follows by observing that, since the action is Weyl invariant, its value can be calculated using the (singular) metric $ds^2=|s_0|^4dz\,d\bar z$ where s_0 is a meromorphic section of the spin bundle S_0, determined by the Riemann class Δ. The curvature reduces to a combination of Dirac measures, and the Gauss-Bonnet theorem will

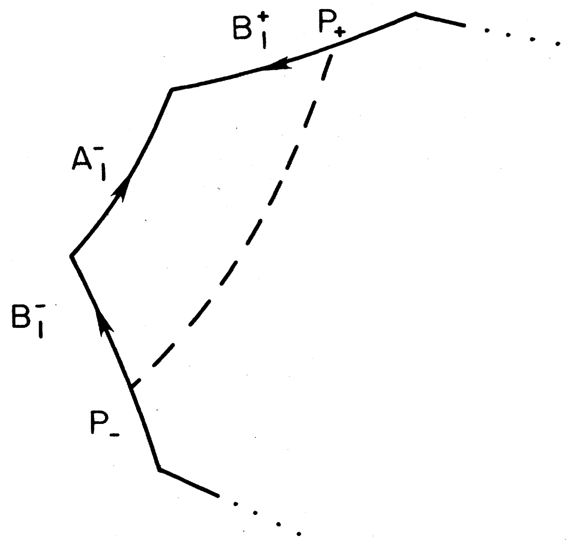

FIG. 22. Integrating an Abelian differential gives a function with jump discontinuities at the cuts.

yield Eq. (7.46) with the right choice of integers l_k's. If we consider a correlation function of the form

$$Z_B(z_1,\ldots,w_m)=\left\langle \prod_{i=1}^{M+\Upsilon} \rho^n(z_i)e^{i\varphi(z_i)} \times \prod_{j=1}^{M} \rho^{1-n}(w_j)e^{-i\varphi(w_j)}\right\rangle \tag{7.47}$$

with $\Upsilon=(2n-1)(h-1)$ and decompose a soliton configuration as $d\varphi=\phi_{mn}+dx$, ϕ_{mn} as in Eq. (7.29), then the contribution of the multiple-valued piece of φ is

$$\sum_{m,n} e^{-I_{mn}}\exp\left[(m+\bar{\Omega}n)(\mathrm{Im}\Omega)^{-1} \times I\left[\sum z_i-\sum w_j-Q\Delta\right]\right].$$

Applying the Poisson summation formula as in Sec. VII.C we can rewrite this as

$$(\det \mathrm{Im}\Omega)^{1/2}\sum_{\delta} |\vartheta[\delta](z)|^2\exp[-2\pi\,\mathrm{Im}z(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z]\ ,$$

where

$$z=I\left[\sum z_i-\sum w_j-Q\Delta\right].$$

Turning to the contributions of the scalar field x, we begin by noting that the boundary and corner terms in the action $I_Q(x)$ cancel for x single valued. The coupling to the background Q results only in the term $Q\sqrt{g}Rx$ in the action, and path integrals can be evaluated as in Sec. VII.B. The result is

$$\left[\frac{8\pi^2\det'\Delta_0}{\int d^2\xi\sqrt{g}}\right]^{-1/2}\frac{\prod_{i<j}F(z_i,z_j)\prod_{i<j}F(w_i,w_j)}{\prod F(z_i,w_j)} \times\frac{\prod H^Q(z_i)}{\prod H^Q(w_j)}e^{-3Q^2U(g)}\ , \tag{7.48}$$

where $H(z)$ is the coupling of the vertex operator to the background charge

$$H(z)=\rho^{1/2}(z)\exp\left[\frac{1}{4\pi}\int d^2y\sqrt{g}R(y)\ln F(z,y)\right] \tag{7.49}$$

and $U(g)$ is the general form of the Liouville action

$$U(g)=-\frac{1}{96\pi^2}\int d^2x\,d^2y\sqrt{g}R(x)\sqrt{g}R(y)\ln F(x,y)\ . \tag{7.50}$$

Combining Eqs. (7.47) and (7.48) then gives a completely explicit form for the correlation functions of the Bose theory. To establish the desired Bose-Fermi correspondence, it suffices as before to compare the anomaly structures of both theories, as well as the zeros and poles of the chiral correlation functions.

Now the chirally symmetric correlation functions of the fermionic theory

$$Z_F^{\alpha,\mathrm{sym}}(z_1,\ldots,w_M)=\int D(b\bar{b}c\bar{c}) \times\prod_1^{M+\Upsilon} b(z_i)\bar{b}(z_i) \times\prod_1^{M} c(w_j)\bar{c}(w_j) \times e^{-I_n(b,c)+\mathrm{c.c.}} \tag{7.51}$$

can be exhibited as the square of Eq. (2.160) in Sec. II.J. This form is manifestly reparametrization invariant and carries conformal and holomorphic anomalies with central charge $c_n=6n^2-6n+1$. This means, as explained in Sec. II.J, that there is a conflict between holomorphicity and covariance, so that the chiral fermionic amplitudes $Z_F^\alpha(z_1,\ldots,w_M)$ we shall produce must have a gravitational anomaly. More specifically, they will be defined by

$$Z_F^{\alpha,\mathrm{sym}}(z_1,\ldots,w_M)=|Z_F^\alpha(z_1,\ldots,w_M)|^2e^{-2c_nS_L(\rho)}\ , \tag{7.52}$$

where $S_L(\rho)$ is the Liouville action in conformal gauge,

$$S_L(\rho)=\frac{1}{48\pi}\int d^2z\,\partial_z\ln\rho\,\partial_{\bar{z}}\ln\rho\ , \tag{7.53}$$

and Z_F^α will be holomorphic with zeros in z_1 at $z_2,\ldots,z_{M+\Upsilon}$ and poles at $w_1,\ldots,w_M$.

Thus we must rewrite the bosonic amplitudes arising from Eqs. (7.48) and (7.47) under the form (7.52). First we note that conformal and holomorphic anomalies arise from the regularized determinant of Δ_0 and the factors $H(z)\exp[-3/(h-1)U(g)]$ for each insertion. Using the Polyakov and Belavin-Knizhnik formula for $\det'\Delta_0$ and standard variational formulas for the curvature, we find that the central charges for both anomalies add up to $-\frac{1}{2}+\frac{3}{2}\Upsilon Q/(h-1)=-\frac{1}{2}+3Q^2/2=6n^2-6n+1$. This suggests extracting out the Liouville action for each insertion,

$$H(z)\exp\left[-\frac{3}{h-1}U(g)\right]=\exp\left[-\frac{3}{h-1}S_L(\rho)+\frac{2\pi}{h-1}\mathrm{Im}\int_{(h-1)z}^{\Delta}\omega_I(\mathrm{Im}\Omega)^{-1}_{IJ}\mathrm{Im}\int_{(h-1)z}^{\Delta}\omega_J\right]|\sigma(z)|^2 . \tag{7.54}$$

Here it is $\sigma(z)$ that is invariant under Weyl scalings, holomorphic with respect to both z and moduli parameters, but not single-valued on the surface M. It is multivalued around B cycles,

$$\sigma(z')=\sigma(z)\exp\left[+\pi i(h-1)\Omega_{II}-2\pi i\int_{(h-1)z}^{\Delta}\omega_I\right], \tag{7.55}$$

and transforms as a tensor of rank $h/2$. The ratio $\sigma(z)/\sigma(w)$ can be written in terms of the prime form

$$\frac{\sigma(z)}{\sigma(w)}=\frac{\vartheta\left[z-\sum q_i+\Delta\right]}{\vartheta\left[w-\sum q_i+\Delta\right]}\prod_i\frac{E(w,q_i)}{E(z,q_i)}, \tag{7.56}$$

where $q_1,\ldots,q_h$ are arbitrary points on M. It is useful to know the variations of $\sigma(z)$ with respect to the moduli parameters,

$$\frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{ww}}\ln\sigma(z)=\frac{1}{2(h-1)}\{[\partial_w\ln\psi(w,z)]^2-\partial_w^2\ln\psi(w,z)\},$$
$$\psi(w,z)=\sigma(w)E(w,z)^{h-1}. \tag{7.57}$$

With Eq. (7.54) we can now express the bosonic amplitudes (7.47) in the desired form:

$$Z_B(z_1,\ldots,w_M)=\sum_\delta|Z_\Delta(\Omega)|^{-2}|A_B^\delta(z_1,\ldots,w_M)|^2\times e^{-2c_nS_L(\rho)}, \tag{7.58}$$

where $Z_\Delta(\Omega)^{-1}$ can be interpreted as the partition function of a chiral scalar field,

$$|Z_\Delta(\Omega)^{-1}|^2=\left[\frac{8\pi^2\det'\Delta_0}{\int d^2\xi\sqrt{g}\det\mathrm{Im}\Omega}\right]^{-1/2}e^{-S_L(\rho)}, \tag{7.59}$$

and A_B^δ are chiral amplitudes:

$$A_B^\delta(z_1,\ldots,w_M)=\vartheta[\delta](z,\Omega)\frac{\prod_{i<j}E(z_i,z_j)\prod_{i<j}E(w_i,w_j)}{\prod E(z_i,w_j)}\times\left[\frac{\prod\sigma(z_i)}{\prod\sigma(w_j)}\right]^Q, \tag{7.60}$$

$$z=\sum_i z_i-\sum_j w_j-Q\Delta .$$

Direct inspection of Eq. (7.60) shows that the right-hand side is a form of rank n in each z_i, and $1-n$ in each w_i. Furthermore, as a form in z_1 it has zeros at $z_2,\ldots,z_{M+\Upsilon}$, poles at $w_1,\ldots,w_M$, and additional unphysical zeros, $p_1,\ldots,p_h$, fixed by the Jacobi inversion theorem,

$$I(p_1+\cdots+p_h)=I\left[\sum_2^{M+\Upsilon}z_i-\sum_1^M w_j\right]-Q\Delta+(\text{divisor of }\delta).$$

Those are exactly the zeros and poles of the chiral fermionic correlation functions in Eq. (7.52). We conclude that

$$Z_F^\delta(z_1,..,w_M)=Z_\Delta(\Omega)^{-1}\vartheta[\delta](z,\Omega)\times\frac{\prod_{i<j}E(z_i,z_j)\prod_{i<j}E(w_i,w_j)}{\prod_{i,j}E(z_i,w_j)}\times\left[\frac{\prod\sigma(z_i)}{\prod\sigma(w_j)}\right]^Q \tag{7.61}$$

with z given in Eq. (7.60). When n is an integer we can evidently ignore spin structures and set $\delta=0$. Finally, an explicit and manifestly holomorphic expression for $Z_\Delta(\Omega)$ can be found by noting that, when $n=1$, $Z_F^{n=1}(z_1,\ldots,z_h,w)$ can be expressed easily in terms of $Z_\Delta(\Omega)$,

$$Z_F^{n=1}(z_1,\ldots,z_h,w)=\det\omega_i(z_j)Z_\Delta(\Omega)^2 . \tag{7.62}$$

Applying Eq. (7.61) gives an equation from which we can deduce $Z_\Delta(\Omega)$. Thus Eqs. (7.61) and (7.62) provide a complete solution to the chiral fermion system $b(dz)^n$, $c(dz)^{1-n}$.

It is now simple to write down a holomorphic square root for the Polyakov bosonic string partition function in the critical dimension $d=26$. Recall that the partition function is given by Eq. (2.142) and becomes, in terms of the chiral fields discussed above,

$$Z=\int(\det\mathrm{Im}\Omega)^{-13}F\wedge\bar{F},$$

$$F=\phi_1\wedge\cdots\wedge\phi_{3h-3}\frac{Z_F^{n=2}(z_1,\ldots,z_{3h-3})}{\det\phi_j(z_k)}Z_\Delta(\Omega)^{-26}.$$

Substituting in Eqs. (7.61) and (7.62) yields an expression for Z entirely in terms of the prime form.

As we have seen in Secs. III.J and III.K, the superghosts β and γ arising from gauge-fixing local supersymmetry are commuting fields. Still, a useful bosonization procedure has been proposed by Friedan, Martinec, and

Shenker (1986), which expresses β and γ in terms of anticommuting fields ξ and η and a commuting field σ,

$$\beta = e^{i\sigma}\partial_z \xi, \quad \gamma = e^{-i\sigma}\eta .$$

As pointed out by Verlinde and Verlinde (1987b), operator product expansions indicate that

$$\delta(\beta(z)) = e^{-i\sigma(z)}, \quad \delta(\gamma(z)) = e^{i\sigma(z)} ,$$

$$\xi(z) = H(\beta(z)), \quad \eta(z) = \partial_z \gamma(z)\delta(\gamma(z)) .$$

General correlation functions for these fields are then given by

$$\left\langle \prod_i^{n+1} \xi(x_i) \prod_j^{n} \eta(y_j) \prod_k e^{-iq_k\sigma(z_k)} \right\rangle = Z_\Delta \frac{\prod_{j=1}^{n} \vartheta[\delta](-y_j + \Sigma x - \Sigma y + \Sigma qz - 2\Delta)}{\prod_{i=1}^{n+1} \vartheta[\delta](-x_i + \Sigma x - \Sigma y + \Sigma qz - 2\Delta)} \times \frac{\prod_{i<i'} E(x_i, x_{i'}) \prod_{j<j'} E(y_j, y_{j'})}{\prod_{i,j} E(x_i, y_j) \prod_{k<l} E(z_k, z_l)^{q_k q_l} \prod_k \sigma(z_k)^{2q_k}} .$$

The reference here, as well as for Sec. VII.C, is Verlinde and Verlinde (1987a, 1987b). A different proof of bosonization based on Arakelov metrics and Quillen geometry is given in Alvarez-Gaumé, Moore, and Vafa (1986), and Alvarez-Gaumé *et al.* (1987), Sonoda (1987c) Dugan and Sonoda (1987), and Fay (1987). Chirally symmetric correlation functions are obtained there in terms of Arakelov Green's functions.

E. Determinant line bundles and Quillen's metric

Holomorphic anomalies have an especially attractive geometric interpretation, which we shall discuss in this section.

In general, an effective action is scalar. In chiral theories, this scalar is to be extracted from the determinant of a chiral Dirac operator. Since chiral Dirac operators reverse chiralities, its determinant can only be defined after choices of bases in each space S_+, S_- of spinors of definite chirality. In other words, it is not a scalar but an element of $(\overset{\max}{\wedge} S_+) \otimes (\overset{\max}{\wedge} S_-)^{-1}$. These one-dimensional spaces form a line bundle over the space of background gauge fields, modulo gauge transformations. Zero modes are sources of anomalies and could cause this bundle to be twisted, so that its sections cannot be equated with scalars. Topological obstructions are given by characteristic classes, which can be evaluated by the index density formula. This is the geometric approach to chiral anomalies pioneered by Atiyah and Singer (1984). The gravitational anomalies of Alvarez-Gaumé and Witten (1983) can also be viewed in the same light (see Alvarez, Singer, and Zumino, 1984).

In our first treatment of holomorphic anomalies in Sec. VII.A, we bypassed this issue by attempting to construct a reparametrization-invariant, holomorphic square root of the chirally symmetric determinant (suitably normalized with zero modes), that was a scalar. We shall now come to the geometric point of view, and determine whether the determinant line bundle can be trivialized by investigating directly the existence of a covariantly constant section. This requires a notion of connection. We shall see that determinant line bundles carry an intrinsic metric, the Quillen metric. In the presence of holomorphic structures, this determines a connection whose curvature and holonomy give exactly the perturbative and global anomalies of the theory. Such interpretations of anomalies had been suggested by Witten (1985a).

We begin with a brief review of the setup for determinant line bundles. Let ∇ be a family of operators indexed by a parameter τ varying over some parameter space B. We shall adopt the terminology of the case $\nabla = \nabla_n^z$, τ = metric on the surface M, although the setup will be obviously quite general. Exactly as the chiral Dirac operator interchanges spinors of positive and negative chiralities, the operator ∇_n^z sends $\mathbf{T}^n$ to a different space, $\mathbf{T}^{n-1}$, and thus its determinant can only make sense after a choice of bases within each space. Since these bases are infinite dimensional, we introduce the finite-dimensional approximations

$$T^n_{\alpha,-} = \oplus \{\text{eigenspaces with eigenvalues} < \alpha \text{ of } \Delta_n^-\} ,$$
$$(7.63)$$
$$T^{n-1}_{\alpha,+} = \oplus \{\text{eigenspaces with eigenvalues} < \alpha \text{ of } \Delta_{n-1}^+\} .$$

Now the key observation familiar from index theory is that $\varphi \to \nabla_n^z \varphi$, $\psi \to (\nabla_n^z)^\dagger \psi$ is a one-to-one correspondence between eigenspaces of Δ_n^- and Δ_{n-1}^+ as long as the eigenvalues are strictly positive. In particular, ∇_n^z restricts to an operator from $T^n_{\alpha,-}$ to $T^{n-1}_{\alpha,+}$. If these spaces have different dimensions (in other words, if the index of ∇_n^z does not vanish), then we just define the determinant to be 0. Otherwise, let $\phi_1, \ldots, \phi_{M_\alpha}$ be a base for $T^n_{\alpha,-}$, and define $(\det \nabla_n^z)_\alpha$ to be

$$(\det \nabla_n^z)_\alpha = \frac{\nabla_n^z \phi_1 \wedge \cdots \wedge \nabla_n^z \phi_{M_\alpha}}{\phi_1 \wedge \cdots \wedge \phi_{M_\alpha}} . \tag{7.64}$$

This should be viewed as an element of the one-dimensional space

$$(\overset{\max}{\wedge} T^n_{\alpha,-})^{-1} \otimes (\overset{\max}{\wedge} T^{n-1}_{\alpha,+}) \tag{7.65}$$

and does not depend on the bases. If we had chosen a different eigenvalue cutoff β, say $\beta > \alpha$, then the two determinants would satisfy a relation of the form

$$s_\beta = s_\alpha \frac{\nabla^z_n \phi_{M_\alpha+1} \wedge \cdots \wedge \nabla^z_n \phi_{M_\beta}}{\phi_{M_\alpha+1} \wedge \cdots \wedge \phi_{M_\beta}} \tag{7.66}$$

with $s_\alpha = (\det \nabla^z_n)_\alpha$ and $s_\beta = (\det \nabla^z_n)_\beta$. Equation (7.66) is actually a way of identifying the spaces $(\overset{\max}{\wedge} T^n_{\alpha,-})^{-1} \otimes (\overset{\max}{\wedge} T^{n-1}_{\alpha,+})$ for different values of α, so we shall view them all as identical. In particular, they can be viewed as

$$(\overset{\max}{\wedge} \mathrm{Ker} \nabla^z_n)^{-1} \otimes (\overset{\max}{\wedge} \mathrm{Ker}(\nabla^z_n)^+) \ , \tag{7.67}$$

and we shall henceforth use for them this last notation. It is important, however, to keep in mind the interpretation of Eq. (7.67) as (7.65) for α positive, since the dimensions of $\mathrm{Ker}\nabla^z_n$ and $\mathrm{Ker}(\nabla^z_n)^\dagger$ may jump, while the parameter space B can always be covered by small coordinate patches B_α over which $T^n_{\alpha,-}$ and $T^{n-1}_{\alpha,+}$ have constant dimensions and in fact vary smoothly with respect to parameters. Thus Eq. (7.65) forms a smooth line bundle over B_α for each α, and patching these line bundles together over overlaps $B_\alpha \cap B_\beta$, using the transition rule (7.66), we obtain a smooth line bundle over the full parameter space B. This is the determinant line bundle, which will be denoted alternatively by Eq. (7.67) or just $\mathrm{DET}(\nabla^z_n)$. We recall that it has a natural global section, namely the determinant of ∇^z_n, which is identically 0 when the index does not vanish and is otherwise given by Eq. (7.64). In the latter case the determinant section is 0 when the operator ∇^z_n has a zero mode.

We turn next to the construction of a metric on $\mathrm{DET}(\nabla^z_n)$. Now over a coordinate patch B_α, a section s of $\mathrm{DET}(\nabla^z_n)$ can be written as

$$s = (\phi_1 \wedge \cdots \wedge \phi_{M_\alpha})^{-1} \otimes (\psi_1 \wedge \cdots \wedge \psi_{N_\alpha}) \ ,$$

where $\phi_1, \ldots, \phi_{M_\alpha}$ and $\psi_1, \ldots, \psi_{N_\alpha}$ are bases for $T^n_{\alpha,-}$ and $T^{n-1}_{\alpha,+}$. Using the worldsheet metric indexed by τ, we could introduce a metric

$$\|s\|^2_\alpha = \det{}^{-1}\langle \phi_a \mid \phi_b \rangle \det \langle \psi_a \mid \psi_b \rangle \tag{7.68}$$

above B_α. The transition laws (7.66), however, require that the $\|s\|^2_\alpha$ arising from a global section satisfy

$$\|s\|^2_\beta = \|s\|^2_\alpha \prod_{\alpha < \lambda < \beta} \lambda \ , \tag{7.69}$$

where the λ's are the positive eigenvalues of Δ^-_n. This condition is not satisfied by Eq. (7.68). The modification proposed by Quillen is

$$\|s\|^2_Q = \det{}^{-1}\langle \phi_a \mid \phi_b \rangle \det \langle \psi_a \mid \psi_b \rangle \prod_{\lambda > \alpha} \lambda \ . \tag{7.70}$$

After zeta regularization of the infinite-dimensional product on the right-hand side, this does lead to a smooth metric on $\mathrm{DET}(\nabla^n_z)$, which is called the Quillen metric.

So far, our discussion has been quite general, although we have used the terminology relevant to ∇^z_n. Specializing to ∇^z_n proper, we note that for $n = \frac{1}{2}$ the index is 0 and the determinant section is a section that vanishes exactly when the theta function vanishes. This incidentally is the case where the number of zero modes does jump. In genus $h \geq 2$, the index does not vanish for any weight $n \neq \frac{1}{2}$, so the determinant section in these cases is trivially identically zero. Although there is no natural global section, local sections are quite important and have appeared implicitly before. To see this we observe that the duality explained in Eq. (6.26) allows us to rewrite $\mathrm{DET}(\nabla^z_n)$ as

$$\mathrm{DET}(\nabla^z_n) = (\overset{\max}{\wedge} \mathrm{Ker} \nabla^z_n)^{-1} \otimes (\overset{\max}{\wedge} \mathrm{Ker}(\nabla^z_{1-n}))^{-1} \ . \tag{7.71}$$

Taking α sufficiently small but positive, we can then represent a section s of $\mathrm{DET}(\nabla^z_n)$ over B_α as

$$s = (\phi_1 \wedge \cdots \wedge \phi_M)^{-1} \otimes (\psi_1 \wedge \cdots \wedge \psi_N)^{-1} \tag{7.72}$$

with $\phi_1, \ldots, \phi_M$ and $\psi_1, \ldots, \psi_N$ zero modes of ∇^z_n and ∇^z_{1-n}. Note that the ranks M and N are constant when the weight n is not $\frac{1}{2}$. The Quillen metric then takes the form

$$\|s\|^2_Q = \frac{\det' \Delta^-_n}{\det\langle \phi_a \mid \phi_b \rangle \det \langle \psi_a \mid \psi_b \rangle} \ , \tag{7.73}$$

familiar from our earlier evaluations of conformal and holomorphic anomalies. In general, it is not possible to choose bases of zero modes invariant under large reparametrizations, so the sections s we just discussed are defined only locally.

Equation (7.71) for $\mathrm{DET}(\nabla^z_n)$ also shows that $\mathrm{DET}(\nabla^z_n)$ is a holomorphic line bundle above moduli space, since we have seen in Sec. VII.A that the operators ∇^z_n depend holomorphically on moduli parameters. It should be pointed out that we now view the base space of parameters as moduli space instead of the space of metrics. This is essentially possible because our constructions are manifestly reparametrization invariant. As for the Weyl symmetry, we can restrict our discussion to a specific conformal gauge (e.g., metrics of constant curvature) or only to combinations of determinants that are ultimately Weyl invariant. This harmless arbitrariness of conformal gauge is the same as was encountered earlier in the derivation of the Belavin-Knizhnik formula.

If we choose bases of zero modes depending holomorphically on moduli, the section s of Eq. (7.72) is a holomorphic local section of the line bundle $\mathrm{DET}(\nabla^z_n)$, and hence the curvature form of $\mathrm{DET}(\nabla^z_n)$ is just given by

$$\partial\bar\partial \ln \|s\|^2_Q = \partial\bar\partial \ln \frac{\det' \Delta_g}{\det\langle \phi_a \mid \phi_b \rangle \det \langle \psi_a \mid \psi_b \rangle} \ , \tag{7.74}$$

as we saw in Sec. VI.A. Thus the Belavin-Knizhnik formula gives exactly the curvature of the determinant line bundle with respect to the Quillen metric.

The cases relevant to the bosonic string are $\mathrm{DET}(\nabla_2^z)$ and $\mathrm{DET}(\nabla_0^z)$. The spaces $\mathrm{Ker}\nabla_2^z$ and $\mathrm{Ker}\nabla_1^z$ are the usual quadratic and Abelian differentials, while $\mathrm{Ker}\nabla_0^z$ represents the constants, and $\mathrm{Ker}\nabla_{-1}^z$ consists of conformal Killing vectors that all vanish for genus $h \geq 2$. As a consequence

$$\mathrm{DET}(\nabla_2^z)=K^{-1}, \quad \mathrm{DET}(\nabla_0^z)=\Lambda^{-1},$$

where K and Λ are the canonical and Hodge bundles and we have ignored the trivial bundle of constants over moduli space. The Belavin-Knizhnik formula now asserts that the bundle $\mathrm{DET}(\nabla_2^z)^{-1}\otimes[\mathrm{DET}(\nabla_0^z)]^{13} = K\otimes\Lambda^{-13}$ is flat. By covariant transport we can then construct a global nonvanishing section over Teichmüller space. To really obtain a section over moduli space we need to investigate the holonomy of the connection. Such investigations were initiated by Witten (1985b) and worked out in detail there for the heterotic string. In particular, the holonomy around a loop is expressed as an adiabatic limit of the Atiyah-Patodi-Singer (1975) eta invariant of the fibration above the loop.

In the above setting the determinant line bundles carried a holomorphic structure and the connection was determined by the metric. Such was also the case in the setting originally considered by Quillen, namely, determinant line bundles over Jacobian varieties. In the more general situation of chiral Dirac operators indexed by parameters τ, varying over a space B, Bismut and Freed (1986) have extended Quillen's construction to construct a connection separately. Such a connection depends on a choice of horizontal spaces in the "universal curve," i.e., the fiber bundle over B whose fiber above τ is the spin manifold on which $\not{D}_\tau$ is defined. For the operators ∇_n^z this fiber bundle is the usual Teichmüller universal curve, and there are natural horizontal spaces, namely, the horizontal spaces discussed in Sec. IV.H. The main steps in the construction of the connection are then as follows: with the Riemannian metric in the fiber and the choice of horizontal subspace, one can produce a unitary connection on the infinite-dimensional bundle on B whose fibers are spaces of spinors. This connection projects in turn to a connection on eigenspaces of $\not{D}_\tau{}^*\not{D}_\tau$ and $\not{D}_\tau\not{D}_\tau{}^*$ and hence to a connection on determinant spaces of the form (7.63), (7.65) for each eigenvalue cutoff α. Again these connections do not match, as the eigenvalue cutoff varies. A correction factor with regularization yields a well-defined global connection, whose curvature is given by the index density formula

$$\left\{2\pi\int_{\text{fiber}}\hat{A}\left[\frac{\Omega}{2\pi}\right]\mathrm{Tr}\left[\exp\left[-\frac{F}{2\pi i}\right]\right]\right\}_{(2)} . \qquad (7.75)$$

Here $\hat{A}$ is the standard $\hat{A}$ polynomial, the index (2) indicates retention of only the 2-form terms, and Ω is the curvature of a connection ω^{Vert} on the universal curve, which can be described as follows: Choose a metric on B and consider the metric on the full universal curve obtained by requiring that the fibers and horizontal spaces be orthogonal and the horizontal spaces be isometric to the tangent spaces to the base. Then ω^{Vert} is the projection onto the tangent space to the fiber of the Levi-Cività connection on the universal curve. It is in fact independent of the choice of metrics on the base. As for F, it is as usual the curvature of external gauge couplings. In the presence of a compatible holomorphic structure, this construction gives back the holomorphic connection in the determinant line bundle, and the $\hat{A}$ genus in Eq. (7.75) should be replaced by the Todd polynomial for $\bar{\partial}$ operators. Finally, Bismut and Freed also evaluate the holonomy of $\mathrm{DET}(\not{D})$, generalizing the Witten (1985a) formula for global anomalies.

We shall now illustrate Eq. (7.75) by rederiving the Belavin-Knizhnik formula. We parametrize moduli by constant-curvature metrics and take the horizontal subspaces as defined by the lifts $\mu\rightarrow\dot{\omega}(\mu)\partial/\partial z+\mu$ of Sec. IV.H. The connection ω^{Vert} described above coincides with the connection on the Teichmüller curve determined by the metric in the fiber and the holomorphic structure. In particular, the curvature Ω of Eq. (7.75) coincides with the curvature of the Teichmüller curve in Sec. IV.H. Since the curvature form $iF/2\pi$ of the cotangent bundle is $-\Omega/2\pi$, the operator $\bar{\partial}_n = g_{z\bar{z}}\nabla_n^z$ is just the $\bar{\partial}$ operator coupled to the nth power of the holomorphic cotangent bundle, and the expression for the Todd polynomial is Todd $x = 1+x/2+x^2/12+\cdots$. We obtain with $x=\Omega/2\pi$

$$\text{curvature of } \mathrm{DET}(\bar{\partial}_n)=2\pi\left[\int_{\text{fiber}}\left[1+\frac{x}{2}+\frac{x^2}{12}\right]\left[1-nx+\frac{n^2x^2}{2}\right]\right]_{(2)}$$

$$=\frac{6n^2-6n+1}{24\pi}\int_{\text{fiber}}\Omega^2=-\frac{6n^2-6n+1}{12\pi}\omega_{\text{WP}} .$$

In the last equation we have used Eq. (4.49) to express the final result in terms of the Weil-Petersson Kähler form. This agrees with Eq. (7.21) for constant-curvature metrics.

The original chiral anomaly of Adler (1969), Bell and Jackiw (1969), Bardeen (1970), and Gross and Jackiw (1972) emerged in fermion triangle diagrams, where it was found that one could not conserve vector currents without violating the conservation of the axial-vector current. For a recent review see Jackiw, Witten, and Zumino (1984). The fact that chiral determinants are not scalars is stressed in Alvarez-Gaumé and Witten (1983) and in Atiyah and Singer (1984), where the basic geometric setting is also introduced. Although we lacked space to discuss it properly, the material of this section is heavily influenced by the family's index theorem of Atiyah and Singer (1968). That the conformal anomaly can be derived from the index theorem is first noted by Alvarez (1986). The Quillen metric is defined in Quillen (1984). Mathematical treatments of adiabatic limits and holonomy are in Cheeger (1987) and Lee, Miller, and Weintraub (1987). Further applications to global anomalies and torsion issues are developed in Freed (1986).

F. Superholomorphic anomalies

One of the most remarkable features of the previous discussion is that the bosonic string partition function is almost completely characterized (up to a constant) by its being a nowhere-vanishing holomorphic section of a line bundle on moduli space. As remarkable as it may be, one can discuss the bosonic string theory and its scattering amplitudes without really ever using this fact, however, as Secs. II and IV illustrate.

For the superstring case, the situation is rather different. The very construction of type-II and heterotic strings in the Ramond-Neveu-Schwarz formulation requires a summation over spin structures for left- and right-moving worldsheet fermions *separately*. Thus we always make use of the chiral components of the string as exhibited in Sec. III.K. One resorts to considering a nonchiral Lagrangian and nonchiral scattering amplitudes and then uses the principle of chiral splitting discussed in Sec. III.K to *split* the string into its two chiral modes. Such splitting will in general be unique only up to a phase, but further physical principles, including unitarity and modular invariance, may then be used to fix this remaining phase. We achieved a splitting in terms of the chiral components of the fields in Sec. III.K and in terms of holomorphic square roots of determinants dependent only on the period matrix Ω and $\chi_{\bar z}{}^{+}$, and not on $\bar\Omega$ and $\chi_{z}{}^{-}$. In view of the complex structure of supermoduli themselves—as explained in Sec. III.G—it is natural to seek a superholomorphic splitting of superdeterminants and amplitudes. We shall establish in this section that superdeterminants indeed split holomorphically, with respect to this complex structure on supermoduli, except for a superholomorphic anomaly which cancels in $d=10$. In Sec. VII.G we shall show that superholomorphic splitting coincides with the chiral splitting in terms of Ω and $\chi_{\bar z}{}^{+}$.

It is likely that the algebraic-geometric considerations addressed in Secs. VII.A and VII.E for the bosonic string can be generalized to the case of supermoduli space. In particular, the generalization of the Mumford form of Eq. (7.22b) can be deduced from the form of the superholomorphic anomaly equation to be presented in Eq. (7.109) below and will be given by

$$s=(\phi_1\wedge\cdots\wedge\phi_{5h-5})\otimes(\omega_1\wedge\cdots\wedge\omega_h)^{-5}$$

for even-spin structure and

$$s=(\phi_1\wedge\cdots\wedge\phi_{5h-5})\otimes(\omega_1\wedge\cdots\wedge\omega_{h+1})^{-5}$$

for odd-spin structure. ϕ_J are the holomorphic superquadratic differentials of Sec. III.E and ω_I are the super-Abelian differentials. The Mumford form thus arises as a global section of $K\otimes\Lambda^{-5}$ with K the generalization of the canonical line bundle of Λ of the Hodge bundle in the above sense. Our results will show that this bundle is indeed flat.

1. Holomorphic coordinates for supermoduli space

Recall that to each supergeometry satisfying the torsion constraints (3.11) corresponds a supercomplex structure $J_M{}^N$ given by (3.23) which characterizes the superconformal class of the supergeometry. Thus supermoduli space may be viewed as

$$s\mathcal{M}_h=\{\text{supercomplex structures } J_M{}^N\}/s\text{Diff}\,, \qquad (7.76)$$

where it is understood that $J_M{}^N$ arises from an $N=1$ supergeometry that satisfies the torsion constraints. This allows us to introduce superholomorphic coordinates on $s\mathcal{M}_h$ through the complex structure $\mathcal{J}$ of supermoduli space as explained in Sec. III.G. The complex structure $J_M{}^N$ itself may be used as a complex coordinate for $s\mathcal{M}_h$, and holomorphic and antiholomorphic directions are, respectively, solutions to

$$\begin{aligned}0=\Gamma_M{}^N&=dJ_M{}^N-i\mathcal{J}(dJ_M{}^N)=dJ_M{}^N-iJ_M{}^P dJ_P{}^N\,,\\ 0=\bar\Gamma_M{}^N&=dJ_M{}^N+i\mathcal{J}(dJ_M{}^N)=dJ_M{}^N+iJ_M{}^P dJ_P{}^N\,.\end{aligned} \qquad (7.77)$$

As was shown in Sec. III.G, this system is integrable and thus defines complex coordinates.

We now wish to parametrize the complex structure $J_M{}^N$ with the help of the $N=1$ supergeometry. We shall first do this locally and then discuss global issues in Sec. VII.G. It follows from the torsion constraints that all components of $H_A{}^B$ can be expressed in terms of $H_-{}^z=H$, $H_-{}^-$, and $H_-{}^{\bar z}$ and their complex conjugates. The latter two are eliminated by superreparametrizations δV^α, and local U(1) and super Weyl rescalings, and we may set them to zero. Supermoduli space may now be parametrized in terms of the super Beltrami differential $\mu=H_-{}^z$. With the help of Eq. (3.22), we get[39]

$$\begin{aligned}&H_-{}^-=H_-{}^{\bar z}=H_{\bar z}{}^{\bar z}=0\,,\\ &H_{\bar z}{}^z=D_-H,\quad H_-{}^+=-\tfrac12 D_+H\,,\\ &H_{\bar z}{}^+=-\tfrac12 D_-D_+H-\frac{i}{2}R_{+-}H,\quad H_{\bar z}{}^-=-\tfrac12 D_+^2H\,,\\ &\delta\Omega_-=iD_zH+\tfrac12 D_+H\Omega_+-H\Omega_z\,.\end{aligned} \qquad (7.78)$$

Thus we see that to first order all components of $H_A{}^B$ de-

[39]We shall denote superderivatives by D instead of by $\mathcal{D}$ at this point, reserving $\mathcal{D}$ for later use.

pend either on H or on $\bar{H}$ separately, and H provides holomorphic coordinates on supermoduli space. We conclude that the operator $\mathcal{D}_-$ varies holomorphically with respect to supermoduli parameters.

The situation to second order is more subtle. In the case of moduli space, there are no constraints on specifying zweibeins, and second-order terms in deformations can be chosen so that $e_{\bar{z}}{}^m$ depends holomorphically on the Beltrami differential up to second order. For $N=1$ supergeometries, however, even $E_-{}^M$ alone cannot be specified arbitrarily due to the torsion constraints, as may be seen by counting degrees of freedom. Indeed, upon specifying $E_-{}^M$, we have also right away its complex conjugate $E_+{}^M$, or eight real superfields all together. But the supergeometry depends on 16 $E_M{}^A$'s and 4 Ω_M's minus 14 (real) torsion constraints. Thus giving all the components of $E_-{}^M$ overspecifies the system. This phenomenon may also be understood as a manifestation of the fact that the structure group of the connection is reduced to O(2). Thus the space of $E_-{}^M$ satisfying Eq. (3.11) is a nontrivial curved manifold, unlike the space of metrics $\{g_{mn}\}$, which is a contractible cone.

Satisfying the torsion constraints to second order requires modifications of Eq. (7.78), so that the supergeometry to second order is specified by

$$\mathcal{E}_-{}^M = E_-{}^M + \tfrac{1}{2}(D_+H + D_-M)E_+{}^M - HE_z{}^M , \tag{7.79}$$

$$\mathcal{E}_{\bar{z}}{}^M = E_{\bar{z}}{}^M - D_-HE_z{}^M + \tfrac{1}{2}(D_-D_+H + D_{\bar{z}}M)E_+{}^M + \tfrac{1}{2}(D_zH + D_-K)E_-{}^M ,$$

where

$$M = \bar{H}D_-H - \tfrac{1}{2}D_-\bar{H}H . \tag{7.80}$$

The U(1) connection is given by

$$\begin{aligned}\delta\Omega_- &= iD_zH + iD_-K + \tfrac{1}{2}D_+H\Omega_+ - H\Omega_z \\ &\quad - \tfrac{1}{2}R_{+-}M + \tfrac{1}{2}D_-M\Omega_+ , \\ K &= \tfrac{1}{2}D_+D_-\bar{H}H + \bar{H}D_+D_-H - \frac{3i}{2}R_{+-}\bar{H}H .\end{aligned} \tag{7.81}$$

Here we have kept only the $H\bar{H}$ part of the quadratic terms in H, since it can be checked that H^2 terms will not upset the superholomorphic dependence found to first order in H. To linear order, the quantities $\mathcal{E}_-{}^M$, Ω_-, and hence $\mathcal{D}_-$ depend only on H. However, to second order, new dependence arises through K and M, which involve $\bar{H}$ as well. To understand this phenomenon, we need to introduce true complex coordinates on supermoduli space, and these can be built out of the $J_M{}^N$ through Eq. (7.77) and the complex structure $\mathcal{J}$.

It is convenient to express $J_M{}^N$ in terms of U(1) indices as seen from the geometry $E_M{}^A$:

$$\begin{aligned}J_M{}^N &= \mathcal{E}_M{}^a\varepsilon_a{}^b\mathcal{E}_b{}^N + \mathcal{E}_M{}^\alpha(\gamma_5)_\alpha{}^\beta\mathcal{E}_\beta{}^N \\ &= E_M{}^AJ_A{}^BE_B{}^N .\end{aligned} \tag{7.82}$$

The results are

$$\begin{aligned}&J_-{}^z = 2iH, \quad J_{\bar{z}}{}^z = D_-J_-{}^z , \\ &J_-{}^+ = -i(D_+H + D_-M), \quad J_{\bar{z}}{}^+ = D_-J_-{}^+ , \\ &J_-{}^{\bar{z}} = i\bar{H}D_+H - 2iHD_+\bar{H} , \\ &J_{\bar{z}}{}^{\bar{z}} = -i(1 + 2D_-HD_+\bar{H} - D_-D_+H\bar{H}) , \\ &J_-{}^- = -i(1 - HD_+D_-\bar{H} + D_+HD_-\bar{H}) , \\ &J_{\bar{z}}{}^- = -\frac{i}{2}D_-D_+HD_-\bar{H} + iD_-HD_+D_-\bar{H} .\end{aligned} \tag{7.83}$$

The holomorphic components can be read off from Eq. (7.77), and it is clear that $J_-{}^z$, $J_-{}^+$, $J_{\bar{z}}{}^z$, and $J_{\bar{z}}{}^+$ are holomorphic. Their complex conjugates are antiholomorphic, and the remaining eight components (like $J_+{}^+, \ldots$) are neither holomorphic nor antiholomorphic, and should be considered as auxiliary fields arising from the overdetermination characteristic of $J_M{}^N$. Note that the number of truly independent holomorphic components ($J_-{}^z$ and $J_-{}^+$) is exactly what you would have expected from a naive extension of ordinary geometry (functions of z and functions of θ).

The U(1) connection involves yet another complication. One can easily show by rearranging terms that

$$\begin{aligned}\delta\Omega_- &= -D_+J_-{}^+ + \frac{i}{2}J_-{}^+\Omega_+ + \frac{i}{2}J_-{}^z\Omega_z \\ &\quad + iD_-(K + D_+M) ,\end{aligned} \tag{7.84}$$

so that Ω_- becomes holomorphic except for a total D_- derivative. We may now recast all components of the supergeometry in terms of linear functions of the components of the complex structure, except for the D_- derivative in Ω_-:

$$\begin{aligned}\mathcal{E}_-{}^M &= E_-{}^M + \frac{i}{2}J_-{}^+E_+{}^M + \frac{i}{2}J_-{}^zE_z{}^M , \\ \delta\Omega_- &= -D_+J_-{}^+ + \frac{i}{2}J_-{}^+\Omega_+ + \frac{i}{2}J_-{}^z\Omega_z \\ &\quad + iD_-(K + D_+M) , \\ \mathcal{D}_-^{(n)} &= D_-^{(n)} + \frac{i}{2}J_-{}^+D_+^{(n)} + \frac{i}{2}J_-{}^zD_z^{(n)} \\ &\quad - inD_+J_-{}^+ - nD_-(K + D_+M) .\end{aligned} \tag{7.85}$$

The total derivative, however, is simply a local U(1) and super Weyl rescaling, which may be done away with by a conjugation,

$$\hat{\mathcal{D}}_-^n = e^{-n(K + D_+M)}\mathcal{D}_-^n e^{n(K + D_+M)} ,$$

analogously to the bosonic case, where only the derivatives $\bar{\partial}_n$ are holomorphic, and not ∇^z or $D_{\bar{z}}$. This means that the parametrization (7.79) is holomorphic to second order and can be used to calculate the holomorphic anomaly.

2. Variational derivatives of superdeterminants

For the calculation of variational derivatives of the determinants, it will be useful to introduce projection operators onto the null spaces of the superderivatives. We define

$$\Pi_+^n \equiv 1-\mathcal{D}_-^{n+1/2}\frac{1}{\Box_{n+1/2}^{(-)}}\mathcal{D}_+^n \quad \text{onto } \operatorname{Ker}\mathcal{D}_+^n=\operatorname{Ker}\Box_n^{(+)} \quad \text{for } n\neq 0\,,$$

$$\Pi_-^n \equiv 1-\mathcal{D}_+^{n-1/2}\frac{1}{\Box_{n-1/2}^{(+)}}\mathcal{D}_-^n \quad \text{onto } \operatorname{Ker}\mathcal{D}_-^n=\operatorname{Ker}\Box_n^{(-)} \quad \text{for } n\neq 0\,, \tag{7.86}$$

$$\Pi_\pm^0 \equiv 1-\frac{1}{\Box_0^{(\pm)}}\Box_0^{(\pm)} \quad \text{onto } \operatorname{Ker}\Box_0^{(\pm)}\,.$$

To show that these operators indeed project onto the above-mentioned null space, one relies on the properties derived in Sec. III.F. Take, for instance, the case of Π_+^n. When $n\le -1$, $\operatorname{Ker}\mathcal{D}_-^{n+1/2}=\operatorname{Ker}\Box_{n+1/2}^{(-)}=0$ according to Eq. (3.72). Hence

$$\operatorname{Ker}(1-\Pi_+^n)=\operatorname{Ker}\mathcal{D}_-^{n+1/2}\frac{1}{\Box_{n+1/2}^{(-)}}\mathcal{D}_+^n=\operatorname{Ker}\mathcal{D}_+^n\,. \tag{7.87}$$

When $n\ge \frac{1}{2}$, we have rather $\operatorname{Ker}\mathcal{D}_+^n=\operatorname{Ker}\Box_n^{(+)}=0$ according to Eq. (3.72), which implies that $(\operatorname{Range}\mathcal{D}_-^{n+1/2})^\perp=0$. Hence every element to which Π_+^n is applied is in $\operatorname{Range}\mathcal{D}_-^{n+1/2}$, and it follows readily that $\Pi_+^n=0$, which coincides with $\operatorname{Ker}\mathcal{D}_+^n$. For $n=-\frac{1}{2}$, $\operatorname{Ker}\mathcal{D}_-^0=\{\text{const}\}$, so that

$$\operatorname{Ker}(1-\Pi_+^{-1/2})=\left\{V \text{ such that } \frac{1}{\Box_0^{(-)}}\mathcal{D}_+^{-1/2}V=\text{const}\right\}.$$

The Green's function on scalars being chosen to have zero integral over the surface (as one has in the bosonic case), it is clear that the constant must vanish. This in turn implies that $\mathcal{D}_+^{-1/2}V=0$ by arguments similar to those that led to Eq. (3.80). For $n=0$, the property is straightforward.

It follows that for all n, Φ_j span $\operatorname{Ker}(1-\Pi_-^n)$ and Ψ_α span $\operatorname{Ker}(1-\Pi_+^{n-1/2})$.

We now compute the variational derivatives of the superdeterminants with respect to H, viewed as independent coordinates on $s\mathcal{M}_h$. Given a supergeometry $E_M{}^A$ we shall deform it to another supergeometry $\mathcal{E}_M{}^A$ under the change[40] of $\mathcal{H}$ as described by Eqs. (7.79)–(7.81). However, it appears profitable to consider, as well, rescalings of both $E_M{}^A$ and $\mathcal{E}_M{}^A$ by the same super Weyl transformation $-\Sigma$ and local U(1) transformation $-iL$. These new geometries will be denoted by $\hat{E}_M{}^A$ and $\hat{\mathcal{E}}_M{}^A$ and the corresponding deformation by H. Thus the supergeometries $\hat{E}_M{}^A$ and E_M^A should be viewed as fixed, and the deformation H induces the geometries $\hat{\mathcal{E}}_M{}^A$ and $\mathcal{E}_M{}^A$, which can be represented in terms of the following diagram:

$$\begin{array}{ccc} E_M{}^A & \xrightarrow{\ \mathcal{H}\ } & \mathcal{E}_M{}^A \\ \uparrow\ \Sigma+iL & & \uparrow\ \Sigma+iL\,. \\ \hat{E}_M{}^A & \xrightarrow{\ H\ } & \hat{\mathcal{E}}_M{}^A \end{array} \tag{7.88}$$

Here $\mathcal{H}$ is the corresponding scaling of H, given (for $L=0$) by

$$\mathcal{H}_+{}^a=e^{\Sigma/2}H_+{}^a\,,$$
$$\mathcal{H}_+{}^\alpha=H_+{}^\alpha+(\gamma_a)^{\alpha\beta}D_\beta\Sigma H_+{}^a\,, \tag{7.89}$$

and the deformation $\hat{E}_M{}^A\to E_M{}^A$ is given by Eqs. (3.17)–(3.19). To perform the calculation of the $\mathcal{H}$ deformation of the superdeterminants, we shall first pull $\mathcal{H}$ back by the local U(1) and super Weyl transformation $\Sigma+iL$, then calculate the H deformation of $\hat{E}_M{}^A$, and finally reassemble the answer in terms of $E_M{}^A$ and $\mathcal{H}$. The fact that this can be done provides a check that we preserved super-reparametrization invariance.

Superdeterminants are again defined through the heat-kernel short-time cutoff method employed in Sec. III.G, and we shall restrict our attention to the case $h\ge 2$. We need the expression for the super Weyl rescaling Eq. (3.114), which we here repeat for convenience.[41] Without loss of generality, we can restrict to $\Box_n^{(-)}$:

$$\ln\frac{\operatorname{sdet}'\Box_n^{(-)}}{\operatorname{sdet}\langle\Phi_j\,|\,\Phi_k\rangle\langle\Psi_\alpha\,|\,\Psi_\beta\rangle}=\ln\frac{\operatorname{sdet}'\hat{\Box}_n^{(-)}}{\operatorname{sdet}\langle\hat{\Phi}_j\,|\,\hat{\Phi}_k\rangle\langle\hat{\Psi}_\alpha\,|\,\hat{\Psi}_\beta\rangle}-\frac{1-4n}{4\pi}(-)^{2n}\int d^2z\,\hat{\mathcal{E}}(\hat{\mathcal{D}}_+\Sigma\hat{\mathcal{D}}_-\Sigma-i\hat{R}_{+-}\Sigma)\,. \tag{7.90}$$

Here, $\Phi_j\in\operatorname{Ker}\mathcal{D}_-^n$ except when $n=0$ where $\Phi_j\in\operatorname{Ker}\Box_0^{(-)}$, and $\Psi_\alpha\in\operatorname{Ker}\mathcal{D}_+^{n-1/2}$, except when $n=\frac{1}{2}$ where $\Psi_\alpha\in\operatorname{Ker}\Box_0^{(+)}$.

Mixed variational derivatives with respect to supermoduli are given by

$$\delta_{\mathcal{H}}\delta_{\overline{\mathcal{H}}}\ln\frac{\operatorname{sdet}'\Box_n^{(-)}}{\operatorname{sdet}\langle\Phi_j\,|\,\Phi_k\rangle\langle\Psi_\alpha\,|\,\Psi_\beta\rangle}=\delta_H\delta_{\overline{H}}\ln\frac{\operatorname{sdet}'\hat{\Box}_n^{(-)}}{\operatorname{sdet}\langle\hat{\Phi}_j\,|\,\hat{\Phi}_k\rangle\langle\hat{\Psi}_\alpha\,|\,\hat{\Psi}_\beta\rangle}-\frac{1-4n}{4\pi}(-)^{2n}\delta_H\delta_{\overline{H}}\int d^3z\,\hat{\mathcal{E}}(\hat{\mathcal{D}}_+\Sigma\hat{\mathcal{D}}_-\Sigma-i\hat{R}_{+-}\Sigma)\,. \tag{7.91}$$

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[40] We have denoted the change H of Eq. (7.79) by $\mathcal{H}$ here, reserving H for later use.

[41] Since we are dealing with nonchiral determinants, recall that there is no local U(1) anomaly.

The first term is calculated using the heat-kernel short-time cutoff method, as for the case of the super Weyl anomaly. One begins by introducing

$$\ln\hat{\delta}_n^{(-)}(s)=\ln \mathrm{sdet}[(\hat{\Box}_n^{(-)})^2+s]=-\int_\varepsilon^\infty \frac{dt}{t}e^{-ts}\mathrm{str}e^{-t(\hat{\Box}_n^{(-)})^2} . \tag{7.92}$$

Variation with respect to $\overline{H}$ yields

$$\delta_{\overline{H}}\ln\hat{\delta}_n^{(-)}(s)=2\int_\varepsilon^\infty dt\, e^{-ts}\mathrm{str}\delta_{\overline{H}}\hat{\Box}_n^{(-)}\hat{\Box}_n^{(-)}e^{-t(\hat{\Box}_n^{(-)})^2} . \tag{7.93}$$

Elements of $\mathrm{Ker}\hat{\Box}_n^{(-)}$ will not contribute to the supertrace in the above formula, and it is useful to insert (redundantly) the projection operator $\hat{\Pi}_-^n$ onto $\mathrm{Ker}\hat{\Box}_n^{(-)}$, so that $\hat{\Box}_n^{(-)}$ becomes invertible on this restricted function space,

$$\delta_{\overline{H}}\ln\hat{\delta}_n^{(-)}(s)=-2\int_\varepsilon^\infty dt\, e^{-ts}\frac{\partial}{\partial t}\mathrm{str}\delta_{\overline{H}}\hat{\Box}_n^{(-)}\frac{1}{\hat{\Box}_n^{(-)}}e^{-t(\hat{\Box}_n^{(-)})^2}(1-\hat{\Pi}_-^n) . \tag{7.94}$$

We use the arguments of analytic continuation in s familiar from the super Weyl case, and since there are no zero modes, we find in the limit $s\to 0$ that

$$\delta_{\overline{H}}\ln \mathrm{sdet}'\hat{\Box}_n^{(-)}=\mathrm{str}\delta_{\overline{H}}\hat{\Box}_n^{(-)}\frac{1}{\hat{\Box}_n^{(-)}}e^{-\varepsilon(\hat{\Box}_n^{(-)})^2}(1-\hat{\Pi}_-^n) . \tag{7.95}$$

Second variations yield

$$\begin{aligned}\delta_H\delta_{\overline{H}}\ln \mathrm{sdet}'\hat{\Box}_n^{(-)}=&+\mathrm{str}\delta_H\delta_{\overline{H}}\hat{\Box}_n^{(-)}\frac{1}{\hat{\Box}_n^{(-)}}e^{-\varepsilon(\hat{\Box}_n^{(-)})^2}(1-\hat{\Pi}_-^n)-\mathrm{str}\delta_{\overline{H}}\hat{\Box}_n^{(-)}\frac{1}{\hat{\Box}_n^{(-)}}\delta_H\hat{\Box}_n^{(-)}\frac{1}{\hat{\Box}_n^{(-)}}e^{-\varepsilon(\hat{\Box}_n^{(-)})^2}(1-\hat{\Pi}_-^n)\\&+\mathrm{str}\delta_{\overline{H}}\hat{\Box}_n^{(-)}\frac{1}{\hat{\Box}_n^{(-)}}e^{-\varepsilon(\hat{\Box}_n^{(-)})^2}\delta_H(1-\hat{\Pi}_-^n)\\&-\varepsilon\int_0^1 du\ \mathrm{str}\delta_{\overline{H}}\hat{\Box}_n^{(-)}\frac{1}{\hat{\Box}_n^{(-)}}e^{-\varepsilon u(\hat{\Box}_n^{(-)})^2}(\hat{\Box}_n^{(-)}\delta_H\hat{\Box}_n^{(-)}+\delta_H\hat{\Box}_n^{(-)}\hat{\Box}_n^{(-)})e^{-\varepsilon(1-u)(\hat{\Box}_n^{(-)})^2}(1-\hat{\Pi}_-^n) .\end{aligned} \tag{7.96}$$

This expression may be considerably simplified by noticing that, to order $H\overline{H}$, $\hat{\mathcal{D}}_+$ and $\hat{\mathcal{D}}_-$ depend only on (anti)holomorphic coordinates.

For the change in the projection operator we have ($n\neq 0$)

$$\delta_H(1-\hat{\Pi}_-^n)=\hat{\mathcal{D}}_+\frac{1}{\hat{\Box}_n^{(-)}}\delta_H\hat{\mathcal{D}}_-\hat{\Pi}_-^n , \tag{7.97}$$

and for $n=0$

$$\delta_H(1-\hat{\Pi}_-^0)=\frac{1}{\Box_0^{(+)}}\hat{\mathcal{D}}_+\delta_H\hat{\mathcal{D}}_-\hat{\Pi}_-^0 . \tag{7.98}$$

Using also the fact that ε may be set to 0 in finite-dimensional traces, we may continue the above calculations to obtain

$$\begin{aligned}\delta_H\delta_{\overline{H}}\ln \mathrm{sdet}'\hat{\Box}_n^{(-)}=&+\mathrm{str}(\delta_H\delta_{\overline{H}}\hat{\mathcal{D}}_+\hat{\mathcal{D}}_-+\hat{\mathcal{D}}_+\delta_H\delta_{\overline{H}}\hat{\mathcal{D}}_-)\frac{1}{\hat{\Box}_n^{(-)}}e^{-\varepsilon(\hat{\Box}_n^{(-)})^2}(1-\hat{\Pi}_-^n)\\&+\mathrm{str}\delta_{\overline{H}}\hat{\mathcal{D}}_+\hat{\Pi}_+^{n-1/2}\delta_H\hat{\mathcal{D}}_-\frac{1}{\hat{\Box}_n^{(-)}}(1-\hat{\Pi}_-^n)+\mathrm{str}\delta_{\overline{H}}\hat{\mathcal{D}}_+\frac{1}{\hat{\Box}_n^{(-)}}(1-\hat{\Pi}_+^{n-1/2})\delta_H\hat{\mathcal{D}}_-\hat{\Pi}_-^n\\&-\varepsilon\int_0^1 du\ \mathrm{str}\delta_{\overline{H}}\hat{\mathcal{D}}_+\hat{\mathcal{D}}_-e^{-\varepsilon u(\hat{\Box}_n^{(-)})^2}\hat{\mathcal{D}}_+\delta_H\hat{\mathcal{D}}_-e^{-\varepsilon(1-u)(\hat{\Box}_n^{(-)})^2}\\&+\varepsilon\int_0^1 du\ \mathrm{str}\hat{\mathcal{D}}_-\delta_{\overline{H}}\hat{\mathcal{D}}_+e^{-\varepsilon u(\hat{\Box}_{n-1/2}^{(+)})^2}\delta_H\hat{\mathcal{D}}_-\hat{\mathcal{D}}_+e^{-\varepsilon(1-u)(\hat{\Box}_{n-1/2}^{(+)})^2} .\end{aligned} \tag{7.99}$$

The second and third terms can be linked to the changes in the finite-dimensional determinants of zero modes. Using the fact that the operator $\hat{\Pi}_-^n$ projects onto the $\hat{\Phi}_j$'s, we have

$$\delta_H\delta_{\overline{H}}\ln \mathrm{sdet}\langle\hat{\Phi}_j\,|\,\hat{\Phi}_k\rangle=2\langle\delta_{\overline{H}}\delta_H\hat{\Phi}_j\,|\,\hat{\Phi}_J\rangle+\langle\delta_H\hat{\Phi}_j\,|\,(1-\hat{\Pi}_-^n)\,|\,\delta_H\hat{\Phi}_j\rangle . \tag{7.100}$$

Now the variations in the zero modes can be determined from a differential equation that follows from their definition,

$$\hat{\mathcal{D}}_-\hat{\Phi}_j=0\Longrightarrow\delta_H\hat{\mathcal{D}}_-(\hat{\Phi}_j)+\hat{\mathcal{D}}_-(\delta_H\hat{\Phi}_j)=0 . \tag{7.101}$$

Fortunately, we do not need the full change of the zero modes in Eq. (7.100), but only the projection onto the complement to the $\hat{\Phi}_j$'s:

$$\hat{\mathcal{D}}_+ \frac{1}{\hat{\Box}^{(+)}_{n-1/2}} (\delta_H \hat{\mathcal{D}}_-)\hat{\Phi}_j + (1-\hat{\Pi}^{n}_{-})\delta_H \hat{\Phi}_j = 0 \ , \tag{7.102}$$

so that

$$\langle \delta_H \hat{\Phi}_j \mid (1-\hat{\Pi}^{n}_{-}) \mid \delta_H \hat{\Phi}_j \rangle = \mathrm{str}\,\delta_H \hat{\mathcal{D}}_+ \frac{1}{\hat{\Box}^{(+)}_{n-1/2}} (1-\hat{\Pi}^{n-1/2}_{+})\delta_H \hat{\mathcal{D}}_- \hat{\Pi}^{n}_{-} \ . \tag{7.103}$$

Analogously, we have

$$\delta_H \delta_{\bar{H}} \ln \mathrm{sdet}\langle \hat{\Psi}_\alpha \mid \hat{\Psi}_\beta \rangle = 2\langle \delta_{\bar{H}} \delta_H \hat{\Psi}_\alpha \mid \hat{\Psi}_\alpha \rangle + \langle \delta_{\bar{H}} \hat{\Psi}_\alpha \mid (1-\hat{\Pi}^{n-1/2}_{+}) \mid \delta_{\bar{H}} \hat{\Psi}_\alpha \rangle \tag{7.104}$$

and

$$\langle \delta_{\bar{H}} \hat{\Psi}_\alpha \mid (1-\hat{\Pi}^{n-1/2}_{+}) \mid \delta_{\bar{H}} \hat{\Psi}_\beta \rangle = \mathrm{str}\,\delta_{\bar{H}} \hat{\mathcal{D}}_+ \hat{\Pi}^{n-1/2}_{+} \delta_H \hat{\mathcal{D}}_- \frac{1}{\hat{\Box}^{(-)}_{n}} (1-\hat{\Pi}^{n}_{-}) \ . \tag{7.105}$$

The two terms in the first line of Eq. (7.99), combined with the first terms on the right-hand side of Eqs. (7.100) and (7.102), are essentially super Weyl anomalies. Exactly as in the bosonic case, our calculation of the superholomorphic anomaly is consistent only modulo super Weyl transformations, so that the above effects due to the super Weyl anomaly may be ignored. The last two terms can now be easily evaluated with the help of the heat kernel flat superspace given in Appendix C, as well as the dependences of the superderivatives $\hat{\mathcal{D}}_+$ and $\hat{\mathcal{D}}_-$ on H and $\bar{H}$, respectively, as given by Eq. (7.85):

$$\delta_H \delta_{\bar{H}} \ln \frac{\mathrm{sdet}\hat{\Box}^{(-)}_{n}}{\mathrm{sdet}\langle \hat{\Phi}_j \mid \hat{\Phi}_k \rangle \langle \hat{\Psi}_\alpha \mid \hat{\Psi}_\beta \rangle} = \frac{1-4n}{\pi}(-)^{2n} \int d^2\mathbf{z}\, \hat{E}\, \hat{D}_z H \hat{D}_{\bar{z}} \bar{H} \ . \tag{7.106}$$

Next, we need the change in the super Weyl anomaly of Eq. (7.91). We may now pick a convenient slice for $\hat{E}_M{}^A$ by using the local U(1) and super Weyl rescalings to make $\Omega_\pm = 0$ in a small patch. As a consequence the supergeometry is flat, $R_{+-}=0$. Super-reparametrization invariance guarantees that we can put such patches together, as long as the expressions are covariant. The kinetic term yields

$$\begin{aligned}\hat{\mathcal{E}}\hat{\mathcal{D}}_+\Sigma\hat{\mathcal{D}}_-\Sigma = \hat{E}(&\hat{D}_+\Sigma\hat{D}_-\Sigma - \bar{H}\hat{D}_{\bar{z}}\Sigma\hat{D}_-\Sigma + H\hat{D}_+\Sigma\hat{D}_z\Sigma + \bar{H}H\hat{D}_{\bar{z}}\Sigma\hat{D}_z\Sigma \\ &+\tfrac{1}{2}\{\hat{D}_-\bar{H}H\hat{D}_-\Sigma\hat{D}_z\Sigma - \bar{H}\hat{D}_+H\hat{D}_{\bar{z}}\Sigma\hat{D}_+\Sigma\} + \hat{D}_+\Sigma\hat{D}_-\Sigma\{\tfrac{1}{2}\hat{D}_+\hat{D}_-\bar{H}H + \tfrac{1}{2}\hat{D}_-\hat{D}_+H\bar{H} - \hat{D}_-H\hat{D}_+\bar{H}\}) \ ,\end{aligned} \tag{7.107}$$

whereas the change in the curvature is given by

$$\begin{aligned}\hat{\mathcal{E}}\hat{\mathcal{R}}_{+-} = \hat{E}\Bigg[&\hat{R}_{+-} + i\hat{D}_+{}^3H - i\hat{D}_-{}^3\bar{H} + i\hat{D}_+\hat{D}_-(K+\bar{K}) \\ &-\frac{i}{2}\hat{D}_-\{-2\bar{H}\hat{D}_-\hat{D}_z H + \hat{D}_-\bar{H}\hat{D}_z H\} + \frac{i}{2}\hat{D}_+\{-2H\hat{D}_+\hat{D}_{\bar{z}}\bar{H} + \hat{D}_+ H\hat{D}_{\bar{z}}\bar{H}\}\Bigg] \ .\end{aligned} \tag{7.108}$$

Notice that the function M has completely disappeared from the final result of the calculation and that the factor K enters only as a super Weyl transformation would. Thus the net effect of the second-order terms that were needed to perform the variation (7.91) with respect to superholomorphic coordinates of $s\mathcal{M}_h$ is only a super Weyl transformation. Putting together Eqs. (7.91) and (7.106)–(7.108), we can actually reassemble all terms and rewrite them in function of the supergeometry $\mathcal{E}_M{}^A$ and the deformations $\mathcal{H}$ only. This provides us with a powerful check on the covariance of the result, and we obtain

$$\begin{aligned}\delta_{\mathcal{H}}\delta_{\bar{\mathcal{H}}}\ln\frac{\mathrm{sdet}'\Box^{(-)}_n}{\mathrm{sdet}\langle\Phi_j\mid\Phi_k\rangle\langle\Psi_\alpha\mid\Psi_\beta\rangle} = \frac{1-4n}{2\pi}(-)^{2n}\int d^2\mathbf{z}\,E\Bigg[&-\tfrac{1}{2}R^2\mathcal{H}\bar{\mathcal{H}} + D_z\mathcal{H}D_{\bar{z}}\bar{\mathcal{H}} \\ &+\frac{i}{2}R(4D_+\bar{\mathcal{H}}D_-\mathcal{H} + \mathcal{H}D_+D_-\bar{\mathcal{H}} + \bar{\mathcal{H}}D_-D_+\mathcal{H})\Bigg] \ ,\end{aligned} \tag{7.109}$$

which is indeed completely covariant. We have used Eq. (7.89), relating H and $\mathcal{H}$, and set $R_{+-}=R$.

As in the bosonic string, this expression simplifies considerably if we represent superconformal classes by constant-curvature geometries and restrict $\mathcal{H}$ to elements in $\mathrm{Ker}\mathcal{P}_1^\dagger$. Recalling that the super Weil-Petersson metric on supermoduli is given by Eq. (3.44) with constant R_{+-}, we obtain at once that

$$\delta_{\mathcal{H}}\delta_{\overline{\mathcal{H}}}\ln\frac{\mathrm{sdet}'\Box_n^{(-)}}{\mathrm{sdet}\langle\Phi_j\mid\Phi_k\rangle\langle\Psi_\alpha\mid\Psi_\beta\rangle} = -7\frac{4n-1}{4\pi}(-)^{2n}\|\mathcal{H}\|^2_{\mathrm{sWP}} . \quad (7.110)$$

The above holomorphic coordinates for supermoduli space, second variations of superdeterminants, and holomorphic splitting of the superstring measure appear in D'Hoker and Phong (1987a). The approach taken there is in the supergeometry formalism. A different argument in Wess-Zumino gauge, also leading to holomorphic splitting, was provided later by Sonoda (1987d) and Bershadsky (1988). That Howe's solutions (3.32) provide holomorphic coordinates for supermoduli space in the sense of Eqs. (7.77) and (7.78), when the zweibein depends holomorphically on moduli, was verified by Nelson (1987b). Independent approaches to ratios of superdeterminants and superholomorphic splitting are discussed by Baranov and Schwarz (1987).

G. Global issues for the superstring

In this section we shall tie together the various properties of string amplitudes uncovered so far and propose a solution for a number of contradictions and ambiguities that have seemed to affect superstrings.

1. Chiral and superholomorphic splitting

We have discussed a number of different approaches to the splitting of string amplitudes as a function of left and right chirality degrees of freedom in Secs. III.K and III.M–III.O, or as a function of holomorphic and antiholomorphic dependence on moduli space in Secs. VII.A and VII.C–VII.E, or finally as a function of superholomorphic and antisuperholomorphic dependence on supermoduli space in Sec. VII.G—the previous section. The question thus arises whether all such approaches are the same or, if they are different, which one is correct. To discuss this, we shall make a finer distinction.

(a) Holomorphic splitting over moduli space

$$\text{amplitude}=\int_{\mathcal{M}_h} dm_k d\overline{m}_k \mid \mathcal{F}(m_k)\mid^2 .$$

(b) Holomorphic splitting on moduli space at fixed internal momenta

$$\text{amplitude}=\int_{\mathfrak{F}} dp_I^\mu \int_{\mathcal{M}_h} dm_k d\overline{m}_k \mid \mathcal{F}(m_k,p_I^\mu)\mid^2 .$$

(c) Chiral splitting at fixed internal momenta (and for odd-spin structure at fixed Dirac zero modes $\psi_\pm^{0\mu}$)

$$\text{amplitude}=\int_{\mathfrak{F}} dp_I^\mu \int_{s\mathcal{M}_h} d^2m_K \mid \mathcal{F}(\Omega,\chi_{\bar z}{}^+;p_I^\mu)\mid^2 \quad \text{even} ,$$

$$\text{amplitude}=\int_{\mathfrak{F}} dp_I^\mu \int d\psi_\pm^{0\mu} \int_{s\mathcal{M}_h} d^2m_K \mid \mathcal{F}(\Omega,\chi_{\bar z}{}^+;\psi_\pm^{0\mu};p_I^\mu)\mid^2 \quad \text{odd} .$$

(d) Superholomorphic splitting on supermoduli space

$$\text{amplitude}=\int_{s\mathcal{M}_h} d^2m_K \mid \mathcal{F}(m_K)\mid^2 .$$

(e) Superholomorphic splitting on supermoduli space at fixed internal momenta (and fixed Dirac zero modes for odd-spin structure)

$$\text{amplitude}=\int_{\mathfrak{F}} dp_I^\mu \left[\int d\psi_\pm^{0\mu}\right] \times \int_{s\mathcal{M}_h} d^2m_K \mid \mathcal{F}(m_K(;\psi_\pm^0);p_I^\mu)\mid^2 .$$

Of course it is understood that the absolute value square is taken for the nonchiral theory.

We now discuss the validity and interrelation of the various possibilities.

(a) Holds for the partition function of the bosonic string only, provided the measure is defined to include the factor $\det(\mathrm{Im}\Omega)$. It does not hold for nontrivial scattering amplitudes of the bosonic string. It also does not hold for type-II or heterotic strings.

(b) Holds for any scattering amplitude in the bosonic string. In a modified form that will be explained below it holds for type-II or heterotic strings after odd moduli have been integrated out.

(c) Holds for type-II or heterotic strings, as was shown in Sec. III.K for exponential insertions. In Sec. III.M it was also shown to hold in detail for the one-loop case. We can argue that it holds for all amplitudes. Note that we do not assume here that Ω and $\chi_{\bar z}{}^+$ are complex coordinates for $s\mathcal{M}_h$.

(d) Holds for the partition function of type-II or heterotic strings, as was shown in Sec. VII.F, on the condition that a factor $\mathrm{sdet}(\mathrm{Im}\hat{\Omega})$ be included in the measure. In fact, it follows from (c) as we shall show below.

(e) Holds provided we can argue—as we will indeed—that Ω and $\chi_{\bar z}{}^+$ of (c) are complex coordinates for $s\mathcal{M}_h$. In that case it is equivalent to (c), and valid for all amplitudes in type-II or heterotic strings. Thus it is the properties (c) and (e) which provide the correct framework for superstring perturbation theory.

In the remainder of this section, we shall show that (c) implies (e) and finally see how odd moduli can be integrated out to obtain a result of type (b) for the superstring.

2. Supersymmetric period matrix

Our starting point is an arbitrary scattering amplitude at fixed momenta, encountered already in Sec. III.K, and more explicitly in Eq. (3.323) for type-II and Eq. (3.326) for heterotic strings. We shall mostly be interested in the matter part,

$$\mathcal{A}_m(\Omega,\overline{\Omega};\chi_{\bar z}{}^+;\chi_z{}^-;\zeta_i,\bar\zeta_i;p_I^\mu)=\int D(x\psi)\prod_{\mu,I}\delta\left[\oint_{A_I}dz\,\partial_z x^\mu-p_I^\mu\right]V_1(\zeta_1,\bar\zeta_1)\cdots V_n(\zeta_n,\bar\zeta_n)e^{-I_m}\ , \tag{7.111}$$

where the emission vertices $V_1\cdots V_n$ are physical and independent of the ghost fields. For simplicity, we shall consider only the case of even-spin structure, and we shall list the modifications resulting from odd-spin structure at the end.

It is easy to see that $\mathcal{A}_m$, defined above and for all internal momenta, is invariant under local reparametrizations (connected to the identity) and local supersymmetry and has the standard Weyl and U(1) anomalies, which should be thought of as compensated by the ghost fields. $\mathcal{A}_m$ fails to be modular invariant because we picked a canonical homology basis. Thus it may be expected to transform "covariantly" under a modular transformation, provided χ is transformed appropriately. It is also invariant under any large diffeomorphism that preserves the homology basis and hence is invariant under the Torelli group. The most important thing here is that it is reparametrization and supersymmetry invariant.

Now the chiral splitting established in Sec. III.K implies that this is the norm square of a function $\mathcal{C}_\nu$ dependent only on Ω, $\chi_{\bar z}{}^+$, ζ_i, and p_I^μ:

$$\mathcal{A}_m(\Omega,\overline{\Omega};\chi_{\bar z}{}^+,\chi_z{}^-;\zeta_i,\bar\zeta_i;p_I^\mu)=(2\pi)^{10}\delta(k)\,|\,\mathcal{C}_\nu(\Omega,\chi_{\bar z}{}^+,\zeta_i;p_I^\mu)\,|^2\ . \tag{7.112}$$

Here $\mathcal{C}_\nu$ inherits[42] the symmetries of $\mathcal{A}_m$. But we know most of the p dependence on $\mathcal{C}_\nu$: it is a Gaussian in p. A particularly interesting quantity is the variance of the Gaussian:

$$\delta^{\mu\nu}\hat\Omega_{IJ}=\frac{1}{2\pi i}\frac{\partial^2}{\partial p_I^\mu\partial p_J^\nu}\ln\mathcal{C}_\nu(\Omega,\chi_{\bar z}{}^+,\zeta_i;p_I^\mu)\ . \tag{7.113}$$

Using the functional integral representation of Eq. (3.201), we see that it is independent of the external momenta k, and we get

$$\hat\Omega_{IJ}=\Omega_{IJ}+\frac{1}{2\pi i d}\frac{\partial^2}{\partial p_I^\mu\partial p_J^\mu}\ln\int D\psi_+^\mu e^{-I_{\psi_+}+\mathcal{L}'_++2\pi p_I^\mu\sigma_I^\mu} \tag{7.114}$$

with the field σ given as before,

$$\sigma_I^\mu=\frac{1}{4\pi}\int d^2z\,\chi_{\bar z}{}^+(z)\psi_+^\mu(z)\omega_I(z)\ .$$

Since the ψ_+ integral is again Gaussian in p, $\hat\Omega$ is actually independent of p as well and depends only on the supermoduli. The term $\mathcal{L}'_+$ introduces the coupling to the Dirac field of a nonlocal potential (since we have already integrated out the x field), as can be seen from Eq. (3.189) directly. Thus it is appropriate to introduce a full Dirac propagator $\hat S_\nu(z,w)$ for the combinations of I_{ψ_+} and $\mathcal{L}'_+$:

$$\partial_{\bar z}\hat S_\nu(z,w)+\frac{1}{8\pi}\chi_{\bar z}{}^+(z)\int d^2z'\chi_{\bar z}{}^+(z')\partial_z\partial_{z'}\ln E(z,z')\times\hat S_\nu(z',w)=2\pi\delta^2(z,w)\ . \tag{7.115}$$

With the help of this propagator, we have

$$\hat\Omega_{IJ}=\Omega_{IJ}-2\pi i\langle\sigma_I\sigma_J\rangle_{\hat S} \tag{7.116}$$

or

$$\hat\Omega_{IJ}=\Omega_{IJ}-\frac{i}{8\pi}\int d^2z\int d^2w\,\omega_I(z)\chi_{\bar z}{}^+(z)\hat S_\nu(z,w)\times\chi_{\bar z}{}^+(w)\omega_J(w)\ . \tag{7.117}$$

Reparametrizations, Weyl, and local U(1) invariance of $\hat\Omega$ are manifest, but since $\mathcal{C}_\nu$ was also supersymmetric, we conclude that $\hat\Omega$ must be supersymmetric.[43]

Thus chiral splitting has led to a supersymmetric extension of the period matrix—exactly the type of thing we were looking for, as will become clear shortly. The imaginary part of this *supersymmetric period matrix* was already encountered in the discussion of the superholomorphic splitting of the superdeterminant on scalar superfields (not surprisingly) in Sec. VII.F for $n=0$,

$$\langle\Phi_I\,|\,\Phi_J\rangle=\mathrm{Im}\hat\Omega_{IJ}\ , \tag{7.118}$$

with $\Phi_{I,J}\in\mathrm{Ker}\mathcal{D}_-^{1/2}$. Recall that, since we are dealing with even-spin structure only, there are precisely h such super Abelian differentials. For odd-spin structure, there is (generically) one more.

In fact, this shows us right away, when we integrate over the internal momenta, that the correct normalization of the matter functional integrals is sdet(Im$\hat\Omega$), as

[42]An arbitrary (real) phase that could in principle come in the definition of $\mathcal{C}_\nu$ cannot depend on the complex variables Ω, $\chi_{\bar z}{}^+$, and ζ_i, and can thus be neglected—in particular in symmetry considerations. From one spin structure to another it is determined by the action of the modular group.

[43]Since the propagator $\hat S$ has a perturbative series in $\chi_{\bar z}{}^+$ that ends after h terms, Eq. (7.117) is the full answer for $\hat S$ replaced by S when $h=2$. It is straightforward to show directly in that case that $\hat\Omega$ is supersymmetric.

opposed to det(ImΩ), recovering (for the amplitudes with no vertex insertions) that

$$\frac{\text{sdet}\mathcal{D}_+\mathcal{D}_-^{(0)}}{\int d^2\mathbf{z}\, E\ \text{sdet}\,\text{Im}\hat{\Omega}}$$

indeed factors into the absolute value square of a function dependent only on Ω and $\chi_{\bar{z}}{}^+$. This very strongly suggests that Ω and $\chi_{\bar{z}}{}^+$ should be good holomorphic coordinates for supermoduli space. Actually, this may be seen very directly from the fact that $H_-{}^z$ of Eq. (7.83) was a good complex coordinate for supermoduli space and that its expression in Wess-Zumino gauge [Eq. (3.129)] indicates that $e_{\bar{z}}{}^m\delta e_m{}^z$ and $\delta\chi_{\bar{z}}{}^+$ are both good complex coordinates. Hence we have shown that chiral splitting of (c) implies the superholomorphic splitting of (d) for the partition function as well as (e), since now Ω and $\chi_{\bar{z}}{}^+$ are good coordinates for $s\mathcal{M}_h$. As indicated before, this means that (c) and (e) are in fact equivalent.

Actually, the above construction of the supersymmetric period matrix is equivalent to a generalization of the usual construction of the period matrix in terms of line integrals of Abelian differentials. To see this, recall that a holomorphic super Abelian differential $\varpi=\varpi_0+\theta\hat{\omega}+(i/4)\theta\bar{\theta}A\varpi_0$ satisfies the following set of differential equations:

$$\begin{aligned} &D_{\bar{z}}\varpi_0+\tfrac{1}{2}\chi_{\bar{z}}{}^+\hat{\omega}=0\ ,\\ &D_{\bar{z}}\hat{\omega}+\tfrac{1}{2}D_z(\chi_{\bar{z}}{}^+\varpi_0)=0\ . \end{aligned} \tag{7.119}$$

The general solution to the second equation is given in terms of h complex integration constants c_I,

$$\hat{\omega}(z)=\sum_I c_I\omega_I(z) -\frac{i}{4\pi}\int d^2w\ \partial_z\partial_w G(z,w)\chi_{\bar{w}}{}^+\varpi_0(w)\ , \tag{7.120}$$

where G is the scalar Green's function. Using the fact that

$$\partial_z\partial_w G(z,w)=-\partial_z\partial_w \ln E(z,w) +\pi\omega_I(z)(\text{Im}\Omega)^{-1}_{IJ}\omega_J(w)\ ,$$

we see that the latter contribution can be lumped together with the integration constants c_I. In view of the fact that the prime form is single valued around A cycles, it is then clear that the differentials

$$\hat{\omega}_I(z)=\omega_I(z)-\frac{1}{16\pi^2}\int d^2y\int d^2w\ \partial_z\partial_w\ln E(z,w) \times\chi_{\bar{w}}{}^+S_\nu(w,y)\chi_{\bar{y}}{}^+\hat{\omega}_I(y) \tag{7.121}$$

are canonically normalized around A cycles,

$$\oint_{A_K}d\mathbf{z}\,\varpi_J=\oint_{A_K}dz\,\hat{\omega}_J=\delta_{JK}\ .$$

Here $S_\nu(z,w)$ denotes the meromorphic Dirac propagator [the Szegö kernel for even-spin structure and the propagator (3.204) for odd-spin structure]. The integral of these normalized differentials around B cycles reproduces precisely the supersymmetric period matrix defined above:

$$\begin{aligned}\hat{\Omega}_{JK}&=\oint_{B_K}\varpi_J\\ &=\Omega_{JK}-\frac{i}{8\pi}\int d^2w\int d^2y\ \omega_K(w)\chi_{\bar{w}}{}^+S_\nu(w,y)\\ &\qquad\times\chi_{\bar{y}}{}^+\hat{\omega}_J(y)\ ,\end{aligned} \tag{7.122}$$

as is easily seen, order by order, in an expansion in powers of χ.

The supersymmetric period matrix in the context of Eq. (7.118) was first encountered in the general formulation of amplitudes in terms of two-dimensional supergeometry in Sec. III.I, and in D'Hoker and Phong (1987a). The construction in terms of line integrals around closed contours of Abelian differentials for even spin structure is due to Bershadsky (1988) and Sonoda (1987d, 1987e). Its supersymmetry was also checked explicitly in Sonoda (1987e). Generalizations to the case of odd-spin structures are given in D'Hoker and Phong (1988a).

3. Splitting of supermoduli space over "moduli"

It remains to work out how the split expressions of (c) and (e) can be reduced to the holomorphic splitting in the sense of (b). In short, we should integrate out the odd moduli.

If $s\mathcal{M}_h$ were a vector bundle above $\mathcal{M}_h$, with the odd moduli as fibers and transition functions that depend only on $\mathcal{M}_h$, then such integration would be straightforward. However, supermoduli space rather emerges as a coset space of two-dimensional supergeometries by reparametrizations, supersymmetry, local U(1), and Weyl transformations. Especially supersymmetry is very tricky, since its action on the two-dimensional metric is not only along the reparametrization and Weyl directions, but also along moduli. Thus changes in $\chi_{\bar{z}}{}^+$ viewed as supersymmetry transformations can be undone only at the expense of a simultaneous motion on moduli space. If one indeed wants to exhibit a projection from any slice of supergeometry for supermoduli space, taken as some specific choice of $e_m{}^a,\chi_{\bar{z}}{}^+$, one has to confront the problem that supersymmetry acts by mixed transformations on both the zweibein and the gravitino field. In practice it has not appeared to be possible in general to disentangle this action and decompose it onto reparametrizations and Weyl transformations without affecting moduli. This observation may lead one to believe that no natural projection of supermoduli onto moduli exists in general.

When this is the case, the formulas for the amplitudes of superstring scattering processes seem ambiguous because, without a preferred projection, the answer for physical amplitudes should be independent of the projection. Actual calculations show that this is not the case: a difference in projection produces a shift in the even coordinates that depends on the odd moduli and, upon integrating out the odd moduli, results in a total derivative term on moduli space.

In fact the emergence of total derivative terms is directly observed when performing a change of slice for the super Beltrami differentials in Eq. (3.335). It was argued by Verlinde and Verlinde (1987b) that a change in $\chi_{\bar{z}}{}^{+}$ induces a BRST change that may be pulled out of the integral and as usual produces a total derivative on moduli space. Actually, their argument is only local on moduli space, so that a change in $\chi_{\bar{z}}{}^{+}$ produces a total derivative on moduli within the open patch one is considering, and the question arises how to put such patches together. To be more precise, the super Beltrami differentials are characterized by points z_a, which should move independently of moduli if the proposed formula (3.340) in terms of picture-changing operators is to hold. Thus the issue is whether one can have points moving quasiconformally on moduli space in a global way. Reformulated in terms of the Teichmüller universal curve, it is a question of whether there are any covariantly constant global sections of this fiber bundle. Certainly this bundle is not flat, since we evaluated its nonzero characteristic class c_1. It also has no global sections. It would thus appear that the fermionic string integral is intrinsically ambiguous.

One is faced very much with a problem in Čech cohomology, as it appears perhaps most simply in the problem of the magnetic monopole inside a sphere. One has an object (say the field strength) that is a total derivative (say of the vector potential) in an open patch. However, if one is dealing with an underlying topologically nontrivial manifold, the integral can still be nonzero because one can never cover that manifold with just one patch. Correct expressions must also include the Wu-Yang-type corrections that take the effects of patch changing into account.

Such a treatment was proposed by Verlinde (1987) and independently by the authors, and it leads to a well-defined expression for the full amplitudes, with no further total derivative ambiguities. Thus in general there are additional contributions coming from the boundary terms in a cell decomposition of moduli space, which may be evaluated explicitly. It is tempting to propose that such a treatment could be obtained directly from an argument based on the preservation of worldsheet supersymmetry, but we shall not explore this possibility further here.

To make contact with the discussion given above, we could, for example, consider the chiral amplitude $\mathcal{C}_\nu$. It depends on Ω and $\chi_{\bar{z}}{}^{+}$, which were argued to be good complex coordinates for supermoduli space. However, they exhibit the same problem mentioned above: $\chi_{\bar{z}}{}^{+}$ transforms simply under a local supersymmetry, but Ω_{IJ} also transforms. Thus it seems that we cannot expect to integrate out $\chi_{\bar{z}}{}^{+}$ and be left with a sensible theory on moduli space in terms of Ω_{IJ}, which is not supersymmetry.

Here, however, we are saved by the existence of $\hat{\Omega}$, which is supersymmetric. Indeed, it is clear that the amplitude $\mathcal{C}_\nu$ can be expressed as a function of $\hat{\Omega}$ and $\chi_{\bar{z}}{}^{+}$ instead of Ω and $\chi_{\bar{z}}{}^{+}$:

$$\hat{\mathcal{C}}_\nu(\hat{\Omega},\chi_{\bar{z}}{}^{+},\zeta_i;p_i^\mu)=\mathcal{C}_\nu(\Omega,\chi_{\bar{z}}{}^{+},\zeta_i;p_I^\mu) .$$

Since $\hat{C}_\nu$ is itself supersymmetric in the sense that polarization tensor and position of vertex operators transform covariantly under supersymmetry, it is clear that we are no longer concerned by the fact that no global slice can be chosen for the super Beltrami differentials.

In fact, for all practical purposes, $(\hat{\Omega},\chi)$ admits a natural projection to $\hat{\Omega}$ trivially defined by omitting χ. Thus, in order to integrate out the odd moduli in a supersymmetric fashion, one should keep $\hat{\Omega}$ fixed and integrate the remaining independent variable χ. This is not to say that χ suddenly admits global sections above moduli space, but rather that a change of section (a gauge transformation—in this case a supersymmetry) acts in a tensorial fashion, so that, upon transition from one patch to the next, quantities transform in a tensorial way, and no boundary problems occur between patches. In this way we obtain a well-defined measure on "moduli space," viewed as the space of matrices $\hat{\Omega}$.

4. Modular invariance

We can now argue that the superstring measure in terms of $\hat{\Omega}$, as prescribed above with the odd moduli integrated out, is modular invariant. It may be convenient to review here the points of the previous discussion that we shall need in our arguments. The first important fact is that

(a) $\hat{\Omega}$ transforms under modular transformations

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

exactly the same way as Ω. This is most easily seen from the description (7.122) of $\hat{\Omega}$ in terms of line integrals of super Abelian differentials over homology cycles, since a modular transformation is just a change of homology basis. The theta characteristics $[\delta]$ of the $\frac{3}{2}$ differentials χ_a change accordingly,

$$[\delta]\to\begin{bmatrix} D & -C \\ -B & A \end{bmatrix}[\delta]+\frac{1}{2}\begin{bmatrix}\mathrm{diag}\,CD^t \\ \mathrm{diag}\,AB^t\end{bmatrix} . \qquad (7.123)$$

The next outcome of our earlier discussions is that

(b) the superstring measure on $\hat{\Omega}$ resulting from integrating out odd moduli is invariant under small changes of the $2h-2$ $\frac{3}{2}$ differentials χ_a which leave $\hat{\Omega}$ fixed. In fact the chirally symmetric superdeterminants are regularized in a manifestly super-reparametrization-invariant way, and although chiral splitting of each of them would lead to anomalies, the anomalies cancel in the full gauge-fixed superstring, as we saw in Sec. VII.F. This means that there is no local supersymmetry anomaly, and our assertion follows from the fact that a change in χ_a with $\hat{\Omega}$ fixed is just a supersymmetry transformation. The final point we wish to make is that

(c) modular transformations can be used to pull back or push forward measures on $[\hat{\Omega},\chi]$ without ambiguities. To see why, we note that measures written in terms of $(\hat{\Omega},\chi)$ involve combinations of chiral Dirac determinants, correlation functions of spinors, and correlation functions of scalars. There are no difficulties with spinor correlation functions, but Dirac determinants and chiral scalars in general cannot be defined individually in a modular-invariant way. In our situation, however, we know that modular anomalies will cancel for the combination of Dirac determinants alone, as can be checked from the explicit bosonization formulas (7.61), or by invoking Witten's (1985b) result on global anomalies. As for chiral scalars, the fixed internal momenta splitting prescription applies, which transforms under modular transformations as it should.

We can now see that there is no ambiguity in the superstring measure. More precisely, we can cover the space of $\{\hat{\Omega}\}$ by patches $\{B_\alpha\}$ over each of which a choice of $2h-2$ $\frac{3}{2}$ differentials $\{\chi_{a,\alpha}\}$ is made, and the superstring measure is obtained by expressing χ as $\chi=\sum m_a\chi_{a,\alpha}$ and integrating with respect to $\prod_{a=1}^{2h-2}dm_a$. Over overlaps $B_\alpha\cap B_\beta$ there is no ambiguity in view of observation (b) above. Now let B_α and B_γ be patches for which there exists a modular transformation M sending B_α into MB_α with a nonempty intersection with B_γ. For the superstring measure to be consistent, we need to know that the superstring measure on B_α is pushed to a measure on MB_α that agrees with the measure on B_γ chosen independently at the outset. Under a modular transformation, the measure on B_α is pushed by (c) to a measure of the same functional form, with the only difference that the χ_a from the push forward in general will not agree with the $\chi_{a,\gamma}$ on B_γ. In view of observation (b), this leaves the measure unchanged, and we have shown the absence of modular anomalies.

We can now trace easily the origin of the ambiguities discussed by Verlinde (1987), Verlinde and Verlinde (1987b), Atick, Rabin, and Sen (1988), and Moore and Morozov (1988). These ambiguities seem to be inherent in a choice of slice in which the zweibein ${e_m}^a$ is independent of odd moduli. In this case a change of χ_a keeping ${e_m}^a$ fixed is not a supersymmetry transformation, and the difference in the χ_a results in a total derivative defined only on intersections of small patches on moduli space. The argument we just gave above then fails, since the measure pushed forward on MB_α differs from that on B_γ by a local total derivative. This is why Wu-Yang terms have to be introduced by hand to lead to a well-defined cosmological constant. We also note that, if there existed global sections of the universal Teichmüller curve, so that the $\{\chi_a\}$ could be chosen globally to be invariant under modular transformations, then the above argument would apply trivially. Indeed the measure pushed forward on MB_α would clearly agree with the one on B_γ since they would both come from the same choice of χ_a. However, the Teichmüller curve has no global sections, and Wu-Yang terms will be needed. They are usually difficult to evaluate explicitly.

We observe that the issue of modular invariance, which is a global issue, has been reduced to local considerations by the above arguments. The reason for this is that we already know how to cancel modular anomalies in the chiral Dirac determinants and how to define chiral scalars, using internal loop momenta. The main problem at this point is really the problem of making small changes in the χ_a, which is solved by using the supersymmetric period matrix. In particular, we make no assumption about global choices of χ_a's through the $\hat{\Omega}$ space and just use small covering patches.

As we just noted, slices $[\hat{\Omega},\chi]$ correspond to zweibeins depending usually on odd moduli m_a. This means that the terms $\partial/\partial m_a$ arising from the $\prod dm_a$ integration cannot be dropped. In principle we should expand the contractions in χ, which will stop after $h-1$ terms.

The supersymmetric period matrix $\hat{\Omega}$ will be an element of Siegel space of $h\times h$ symmetric matrices with even Grassmann values. Such a space will have dimensions $\frac{1}{2}h(h+1)$, evidently larger than the dimension $3h-3$ of the space of superperiod matrices. It is obviously an important issue in the present approach to solve the corresponding Schottky problem of characterizing the supersymmetric period matrices arising from super-Riemann surfaces.

5. The cosmological constant to two loops

That these ideas make sense is easily seen by reconsidering some of the calculations performed in the literature. In Morozov and Perelomov (1987) and Atick, Rabin, and Sen (1988) it was argued that, in order to make the string measure well behaved, the insertions of the picture-changing operators in the case of genus 2 should be taken at special points. It is now easy to see why by examining the difference between $\hat{\Omega}$ and Ω in Eq. (7.117). When $h=2$, we can replace $\hat{S}_\nu$ by S_ν, which is the Szegö kernel for genus 2. Furthermore, ${\chi_{\bar{z}}}^+$ is given by

$$ {\chi_{\bar{z}}}^+(z)=\alpha_1\delta(z-z_1)+\alpha_2\delta(z-z_2)\ , \tag{7.124}$$

where z_1 and z_2 are two arbitrary points and α_1 and α_2 are the two odd moduli. Substitution into Eq. (7.117) shows that only the term in $\alpha_1\alpha_2$ survives, so that the Szegö kernel is evaluated between z_1 and z_2,

$$\hat{\Omega}_{IJ}=\Omega_{IJ}-\frac{i}{4\pi}\alpha_1\alpha_2\omega_I(z_1)S_\nu(z_1,z_2)\omega_J(z_2)\ . \tag{7.125}$$

An explicit formula for the Szegö kernel in terms of branch points was given by Fay (1977):

$$S_\nu(x,y)=\frac{1}{2}\left[\left[\frac{\psi(x)}{\psi(y)}\right]^{1/4}+\left[\frac{\psi(y)}{\psi(x)}\right]^{1/4}\right]\times\frac{1}{z(x)-z(y)}\left[\frac{\partial z(x)}{\partial x}\frac{\partial z(y)}{\partial y}\right]^{1/2} \quad (7.126)$$

with

$$\psi(x)=\frac{\prod_{a_i\in A}[z(x)-a_i]}{\prod_{a_i\in B}[z(x)-a_i]} \quad (7.127)$$

in the notation of Sec. IV.B, so that a_i, $i=1,\ldots,6$ are the branch points and $A\cup B$ is the partition of the branch points into two groups of 3 corresponding to the spin structure ν. This formula shows that the divisor in y of $S_\nu(a_1,y)$ is

$$\sum_A a_i - 2a_1$$

and similarly if $a_1\in B$, so that one is left with $\hat{\Omega}=\Omega$ when z_1 and z_2 are two branch points within either A or B.

Thus it is to be expected that the correct superstring measure can be written in terms of the usual measure in Ω when the support of χ is located at the branch points. This also includes the calculation of the cosmological constant to two-loop order by Moore and Morozov (1987), also performed with the insertions at the branch points.

We have, however, the general supersymmetry covariant formula available, and hence we can insert the picture-changing operators anywhere. When they are not inserted at the branch points, Ω will not equal $\hat{\Omega}$, and the difference is an odd-moduli-dependent shift, which according to general integration formulas for Grassmann variables will produce a total derivative term on moduli space. The difference here, however, is that this contribution was defined tensorially throughout, so that only a term coming from the boundary of moduli space and not from the boundary of individual cells is obtained. We shall report elsewhere on more explicit verifications of these ideas.

The importance of a modular-invariant choice of insertions z_a is stressed in Verlinde and Verlinde (1987b). Ambiguities caused by total derivatives on local patches of moduli space were investigated by Verlinde (1987), Atick, Rabin, and Sen (1988), and Moore and Morozov (1988). Appropriate corrections to the cosmological constant dictated by Čech cohomology considerations (as in Wu-Yang terms for a particle in a gauge field; see Alvarez, 1985) were introduced by Verlinde (1987) and in unpublished work of D'Hoker and Phong (1987). A different approach assuming the existence of global sections of the universal Teichmüller curve is to be found in Atick, Moore, and Sen (1988a, 1988b) where a discussion of advantages and disadvantages of various choices of slices in the literature is also given. The possibility that additional contributions from the boundary of moduli space may be required in the covariant or light-cone gauge superstring in view of supersymmetry has been suggested by Green and Seiberg (1987) and Greensite and Klinkhamer (1987). Detailed analyses of contributions from the boundary of moduli space are presented by Atick and Sen (1987a, 1987b), who show that two-loop string-theoretic calculations of Fayet-Iliopoulos D terms agree with the effective field considerations of Dine, Ichinose, and Seiberg (1987), and Dine, Seiberg, and Witten (1988); Atick, Moore, and Sen (1988b) also address ambiguities of n-point functions.

VIII. VERTEX OPERATORS FOR ON-SHELL PHYSICAL PARTICLES

One of the remarkable features of string theories is that correlation functions of certain local operators—the vertex operators—on the worldsheet give scattering amplitudes of physical particles in space-time. The spectrum of the space-time theory as well as its gauge invariances are thus dictated by the structure of those vertex operators. The general rules for the construction of vertex operators have been partially known from the days of dual models. A key requirement is that they have conformal dimension 1 for open strings and (1,1) for closed strings. In this section we shall present the complete rules for vertex operators in the functional formulation for the closed bosonic, type-II, and heterotic string theories. Essentially, bosonic vertex operators must be consistent with all the symmetries of the corresponding worldsheet theory, after inclusion of all possible anomalies. The condition of conformal dimension (1,1) just guarantees the integrability of a vertex operator on the worldsheet. Of particular importance is the Weyl anomaly. It is in fact the anomalous dimension of vertex operators that is responsible for the appearance of massless spin-2 particles in the string spectrum. Vertex operators also give a simple explanation of gauge invariances in space-time as modifications by total derivatives on the worldsheet, since these should not change the scattering amplitudes. We shall discuss in some detail the example of the gauge symmetry of the graviton and antisymmetric tensor field.

Vertex operators for emission of fermions are more complicated. Some of the difficulties can already be gathered from the fact that we must manufacture space-time spinors when the fundamental fields on the worldsheet are space-time vectors. Moreover, insertion of a fermion emission vertex operator should change the spin structure on the worldsheet. In Sec. VIII.E we shall present the fermion vertex construction due to Friedan, Shenker, and Martinec (1985; Friedan, Martinec, and Shenker, 1986) and Knizhnik (1985) based on bosonization and coupling to the ghosts. The space-time supersymmetry charge is then easily obtained from the fermion vertex operator, and some basic consequences of supersymmetry will be discussed.

A. Covariance properties of vertex operators

Vertex operators for on-shell physical states of given momentum k must obey the following covariance properties.

(i) Space-time translation invariance requires that all x^μ dependence occur through a factor of $\exp(ik\cdot x)$. The remaining factors depend only on the derivatives of x^μ.

(ii) Space-time Lorentz invariance requires that space-time indices $(\mu,\nu,\ldots)$ on all fields be contracted with a polarization tensor $\varepsilon_{\mu\nu}(k)$ which transforms under a real representation of the little group of k_μ.

(iii) Worldsheet reparametrization invariance is ensured when Einstein indices are contracted with the zweibein to yield U(1) indices. A factor $\sqrt{g}=\det e_m{}^a$ is required for the volume element.

(iv) Worldsheet local U(1) invariance requires that derivatives be covariant, all U(1) indices properly contracted, and all U(1) anomalies canceled.

(v) Weyl invariance requires that the vertex be invariant under Weyl rescalings after inclusion of all anomalies.

Fermionic strings require, in addition to the above, the following.

(vi) Local worldsheet supersymmetry. Vertices must be invariant under arbitrary reparametrizations of superspace ($N=1$ for the type-II superstring, $N=\frac{1}{2}$ for heterotic strings). With superfields, all super Einstein indices must be contracted with the superzweibein, only local U(1) covariant derivatives should be used, and a factor $E=\mathrm{sdet}E_M{}^A$ should be included instead of $\sqrt{g}$.

(vii) Super Weyl invariance must be preserved after inclusion of anomalies.

Now requirements (i)–(iii) and (vi) are easily enforced by use of U(1) covariant (super)derivatives, while (vii) will follow from (v) and (vi). Further, there will be no U(1) anomaly if Weyl anomalies cancel separately for left- and right-movers. Thus (super)Weyl invariance is the key property that distinguishes physical states from ghost states.

B. The bosonic string and space-time gauge invariance

1. The bosonic string vertex operators

The general vertex operator consistent with the requirements of Sec. VIII.A is given by

$$V(\varepsilon,k)=\int_M d^3\xi\sqrt{g}\,U(\varepsilon,Dx,R)e^{ik\cdot x}\ . \tag{8.1}$$

Here U is a polynomial scalar expression in the U(1) covariant derivatives of x^μ and the two-dimensional curvature R.[44] Using the Heisenberg equations of motion for the x^μ field ($D_zD_{\bar z}x^\mu=0$) under the time-ordering symbol, we see that vertex operators involving $D_zD_{\bar z}x^\mu$ must be omitted, and on a given x^μ only D_z or $D_{\bar z}$ derivatives are applied. We turn then to the Weyl transformation laws of U(1) covariant derivatives. If $e_m{}^a$ is a zweibein, the connection and curvature are

$$\omega_m=-e_m{}^c\varepsilon^{ab}e_a{}^n e_b{}^p\partial_n e_p{}^c,\quad R=\varepsilon^{mn}\partial_m\omega_n\ .$$

The covariant derivatives on tensors of U(1) weight n are given by

$$D_z^n=e_z{}^m\nabla_m^n,\quad D_{\bar z}^n=e_{\bar z}{}^m\nabla_m^n,\quad [D_z,D_{\bar z}]^n=nR\ . \tag{8.2}$$

Under Weyl transformations ($e_m{}^a=e^\sigma\hat e_m{}^a$) we have

$$\begin{aligned} D_z^n&=e^{(n-1)\sigma}\hat D_z^n e^{-n\sigma}\ ,\\ D_{\bar z}^n&=e^{-(n+1)\sigma}\hat D_{\bar z}^n e^{n\sigma}\ ,\\ R&=e^{-2\sigma}(\hat R-2\hat D_z\hat D_{\bar z}\sigma)\ . \end{aligned} \tag{8.3}$$

U(1) invariance of $U(\varepsilon,Dx,R)$ implies that the total number of derivatives—independently of how they are distributed—must satisfy

$$\#D_z=\#D_{\bar z}\ . \tag{8.4}$$

On the other hand, the possible sources of Weyl anomalies are the following.

(a) Contractions within $\exp(ik\cdot x)$. Under constant Weyl rescalings we have

$$e^{ik\cdot x}\mapsto e^{-\sigma k\cdot k}e^{ik\cdot x}\ , \tag{8.5}$$

so that in view of the Weyl scalings of the derivative factors we find

$$m^2=-k\cdot k=2(N-1),\quad N=0,1,\ldots\ . \tag{8.6}$$

Thus, at the lowest mass level, $N=0$, we have a tachyon whose presence has manifested itself in the asymptotic behavior of the string partition function (cf. Secs. II.H, V.F, and VII.A). At the next mass level, $N=1$, we have massless particles. Under the Lorentz group they decompose into the graviton, the dilaton, and the antisymmetric tensor field. It is a remarkable property of string theory that the graviton must invariably be present. (At least in the critical dimension.)

(b) Contractions of D_z derivatives with each other or with $\exp(ik\cdot x)$. The first type produces an anomaly proportional to $\eta_{\mu\nu}$, while the second produces an anomaly proportional to k_μ. Such anomalies disappear when the polarization tensor is made to satisfy

$$k^\mu\varepsilon_{\mu\nu}\ldots=k^\nu\varepsilon_{\mu\nu}\ldots=0\ . \tag{8.7}$$

These polarization tensors do not make up the complete list, however, since cancellation of anomalies could occur by combining different terms. The same considerations apply to contractions between $D_{\bar z}$ derivatives.

(c) Contractions between D_z and $D_{\bar z}$ derivatives. These also lead to anomalous terms. However, these mixed

[44]To obtain similarity with the case of fermionic strings, where derivatives are taken U(1) covariant, we have adopted this same strategy for the bosonic case. The translation to the covariant derivatives ∇ introduced previously is straightforward.

contractions always require curvature counterterms, and conversely, since curvature terms by themselves are not Weyl invariant, they can only compensate for mixed derivatives in view of their tensor structure. If we introduce "normal ordering" conventions so that no mixed contractions are to be performed, no curvature terms are required and none will ever appear. We shall throughout assume that such normal ordering has been performed.

Thus the complete classification of vertex operators is contingent upon the evaluation of all contractions of derivatives of x^μ at coincident points. Since the action is quadratic we need only consider bilinear composites in x^μ. The use of the Heisenberg equations of motion under the time-ordered product and the commutation relations of (8.2) allow us to restrict ourselves to the case of no mixed derivatives in z and $\bar z$ on a given x^μ. Thus the only contractions of interest are

$$\langle D_z{}^m x(z) D_z{}^n x(z)\rangle \; . \tag{8.8}$$

We shall regularize the ultraviolet behavior of composites by the heat kernel and a finite-time cutoff, which is reparametrization invariant. The results agree with those obtained from reparametrization-invariant (but not translation-invariant) Pauli-Villars regulators to the order we have checked. Dimensional regularization would yield different expressions, which we believe are inconsistent with reparametrization invariance; moreover, this method is well known to have problems with infrared behavior. With any of these methods, Leibnitz's rule is satisfied:

$$\partial\langle \partial^p x(z)\partial^q x(z)\rangle = \langle \partial^{p+1}x(z)\partial^q x(z)\rangle + \langle \partial^p x(z)\partial^{q+1}x(z)\rangle \; . \tag{8.9}$$

Here $\partial=\partial/\partial z$ with z a local conformal coordinate. Thus, in view of Leibnitz's rule, we need only compute the contractions $\langle x(z)\partial^m x(z)\rangle$.

In Appendix A we calculate the general expression for such contractions, and we find

$$\langle x(z)\partial^m x(z)\rangle = \sum_{p=1}^{m} \sum_{m_i=1,2,\ldots}^{m_1+\cdots+m_p=m} A_{m_1,\ldots,m_p} e^{2m\sigma}(\partial^{m_1}e^{-2\sigma})\cdots(\partial^{m_p}e^{-2\sigma}) \; , \tag{8.10}$$

where the A's are finite (rational) coefficients given by

$$A_{m_1,\ldots,m_p} = \frac{1}{m}\int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{p-1}} dt_p \frac{\partial^p}{\partial\tau_1\cdots\tau_p}[\tau_0^{m_1}(\tau_0+\tau_1)^{m_2}\cdots(\tau_0+\cdots+\tau_{p-1})^{m_p}(\tau_0+\cdots+\tau_p)^{-m}] \; . \tag{8.11}$$

The rule here is to differentiate with respect to the τ's first and then to set $\tau_0=1-t_1$, $\tau_1=t_1-t_2,\ldots,\tau_{p-1}=t_{p-1}-t_p$, and $\tau_p=t_p$. By performing only the last two differentiations with respect to τ and only the last integration, we may easily obtain a recursion formula ($m=m_1+\cdots+m_p$):

$$A_{m_1m_2\cdots m_p} = -A_{m_1\cdots m_{p-2}(m_{p-1}+m_p)} + \frac{m+1}{m_p+1}A_{m_1\cdots m_{p-2}(m_{p-1}+m_p+1)} - \frac{m-m_p}{m_p+1}A_{m_1\cdots m_{p-2}m_{p-1}} \; .$$

Special cases are

$$A_m = -\frac{1}{m+1} \; ,$$

$$A_{m_1m_2} = \frac{1}{m_1+m_2+1} - \frac{m_1+m_2+1}{m_2+1}\frac{1}{m_1+m_2+2} + \frac{m_1}{m_2+1}\frac{1}{m_1+1} \; .$$

Some low-order terms are easily obtained. In function of covariant derivatives this leads to the following formulas for contractions:

$$\begin{aligned} &\langle xx\rangle = -\frac{2}{\varepsilon}+2\sigma, \quad \langle D_z xx\rangle = D_z\sigma \; , \\ &\langle D_z{}^2 xx\rangle = \tfrac{1}{3}[2D_z{}^2\sigma-(D_z\sigma)^2] \; , \\ &\langle D_z{}^3 xx\rangle = \tfrac{1}{2}(D_z{}^3\sigma - 2D_z{}^2\sigma D_z\sigma) \; . \end{aligned} \tag{8.12}$$

Thus contractions of covariant derivatives are polynomials in covariant derivatives of the conformal factor.

Finally, we provide a general argument for equivalence between vertices in the Polyakov formalism and vertices in the operator langauge. Polyakov vertices must be Weyl and reparametrization invariant after anomalies have been taken into account. For the operator vertices, one requires instead that they have conformal weight 1, so that physical states are annihilated by the Virasoro generators L_n for $n\geq 1$. When we put the Polyakov vertex in the conformal gauge, the worldsheet metric is Euclidean and the residual invariance is just the Virasoro invariance. Conversely, every vertex of conformal weight 1 can be lifted to a Weyl-invariant vertex by adding appropriate contractions. This can also be checked explicitly by inspection of the commutators of vertices with the components of the stress tensor.

2. Space-time gauge invariance and examples

The vertex operators constructed by the above methods may not produce truly distinct particle states, since they may differ by a total derivative on the worldsheet providing zero upon integration. In space-time this would correspond to a gauge transformation. As an example consider the $m^2=0$ vertex

$$V_0 = \int d^2\xi\sqrt{g}\,\varepsilon_{\mu\bar\mu}D_z x^\mu D_{\bar z}x^{\bar\mu}e^{ik\cdot x} \; . \tag{8.13}$$

We may add to V_0 expressions of the following form, which reduce to zero through integrations by parts and the (Heisenberg) equations of motion:

$$\varepsilon_{\mu\bar{\mu}} \mapsto \varepsilon_{\mu\bar{\mu}} + \theta^1_\mu k_{\bar{\mu}} + \theta^2_{\bar{\mu}} k_\mu + \theta^3 k_\mu k_{\bar{\mu}} . \tag{8.14}$$

Clearly, in terms of ε, this precisely corresponds to the gauge transformations associated with the graviton and the antisymmetric tensor field. For massless particles, Eq. (8.14) generates all gauge transformations. To obtain gauge transformations for higher-mass particles it suffices to write down all worldsheet derivatives with the appropriate U(1) structure. A somewhat more complicated example may be given for mass level $m^2=2$, where

$$\begin{aligned} V_1 = \int d^2\xi\sqrt{g}\,(&\varepsilon_{\mu\nu,\bar{\mu}\bar{\nu}} D_z x^\mu D_z x^\nu D_{\bar{z}} x^{\bar{\mu}} D_{\bar{z}} x^{\bar{\nu}} \\ &+\varepsilon_{\mu\nu,\bar{\mu}} D_z x^\mu D_z x^\nu D_{\bar{z}}{}^2 x^{\bar{\mu}} \\ &+\varepsilon_{\mu,\bar{\mu}\bar{\nu}} D_z{}^2 x^\mu D_{\bar{z}} x^{\bar{\mu}} D_{\bar{z}} x^{\bar{\nu}} \\ &+\varepsilon_{\mu,\bar{\mu}} D_z{}^2 x^\mu D_{\bar{z}}{}^2 x^{\bar{\mu}}) e^{ik\cdot x} . \end{aligned} \tag{8.15}$$

The conditions for Weyl invariance are easily obtained with the rules for contractions given above (recall that our vertex was implicitly normal ordered with respect to mixed contractions, so that no curvature terms occur),

$$\begin{aligned} (k^2\eta^{\mu\nu}-6k^\mu k^\nu)\varepsilon_{\mu\nu,\bar{\mu}\bar{\nu}} &= ik^2 k^\mu \varepsilon_{\mu,\bar{\mu}\bar{\nu}} , \\ (k^2\eta^{\mu\nu}-6k^\mu k^\nu)\varepsilon_{\mu\nu,\bar{\mu}} &= ik^2 k^\mu \varepsilon_{\mu,\bar{\mu}} , \end{aligned} \tag{8.16}$$

and the analogous conditions with $\mu \to \bar{\mu}$, etc. These constraints are trivially satisfied when the ε's are transverse and traceless, but there are more solutions.

In particular, there is a gauge invariance obtained by adding total derivatives of the type

$$\begin{aligned} 0 = \int d^2\xi\sqrt{g}\,[&\theta^1_{\mu,\bar{\mu}\bar{\nu}} D_z (D_z x^\mu D_{\bar{z}} x^{\bar{\mu}} D_{\bar{z}} x^{\bar{\nu}} e^{ik\cdot x}) \\ &+\theta^2_{\mu\nu,\bar{\mu}} D_{\bar{z}} (D_z x^\mu D_z x^\nu D_{\bar{z}} x^{\bar{\nu}} e^{ik\cdot x})] , \end{aligned}$$

inducing the gauge transformations

$$\begin{aligned} &\varepsilon_{\mu\nu,\bar{\mu}\bar{\nu}} \to \varepsilon_{\mu\nu,\bar{\mu}\bar{\nu}} + \frac{i}{2}(k_\mu \theta^1_{\nu,\bar{\mu}\bar{\nu}} + k_\nu \theta^1_{\mu,\bar{\mu}\bar{\nu}} + k_{\bar{\mu}} \theta^2_{\mu\nu,\bar{\nu}} \\ &\qquad\qquad + k_{\bar{\nu}} \theta^2_{\mu\nu,\bar{\mu}}) , \\ &\varepsilon_{\mu\nu,\bar{\mu}} \to \varepsilon_{\mu\nu,\bar{\mu}} + \theta^2_{\mu\nu,\bar{\mu}}, \quad \varepsilon_{\mu,\bar{\mu}\bar{\nu}} \to \varepsilon_{\mu,\bar{\mu}\bar{\nu}} + \theta^1_{\mu,\bar{\mu}\bar{\nu}} . \end{aligned} \tag{8.17}$$

More details about these examples, as well as the structure of the cancellation of the curvature terms, may be found in D'Hoker and Phong (1987b).

The general rules for vertex operators above were formulated by Weinberg (1985). The contraction $\langle D_x x D_{\bar{z}} x \rangle$ was shown by de Alwis (1986) to lead to the correct dilaton vertex of Fradkin and Tseytlin (1985a, 1985b). Anomalous Weyl scalings of general composites were calculated in D'Hoker and Phong (1987b). Weyl rescalings based on dimensional regularization are to be found in Tanii and Watabiki (1986). Weyl anomalies when several emission points get close to one another are evaluated in Seiberg (1987), Sen (1987), and Watabiki (1987). Vertices from operator considerations are derived in Sasaki and Yamanaka (1985) and Ichinose and Sakita (1986). Indirect methods for obtaining vertex operators from tachyon amplitudes are discussed in Aldazabal *et al.* (1987). Interpretations of gauge invariances as total derivatives are given in Callan and Gan (1986), Cohen *et al.* (1987) and D'Hoker and Phong (1987b).

C. The type-II superstring

Vertex operators for the type-II superstring must be of the form

$$V(\varepsilon,k) = \int d^2\mathbf{z}\, E U(\varepsilon, E_M{}^A, \mathcal{D}X) e^{ik\cdot X} , \tag{8.18}$$

where U is a polynomial scalar expression in the U(1) covariant superderivatives of X^μ and of the supercurvature R_{+-}. Super-reparametrization and local U(1) invariance require U to be a scalar under these transformations. Here we use again the Heisenberg equations of motion $\mathcal{D}_+\mathcal{D}_- X^\mu = 0$ under the time-ordering symbol to eliminate mixed derivatives on a single X^μ. Thus the building blocks of U are

(i) $\mathcal{D}_+{}^p X^\mu, \mathcal{D}_-{}^q X^\nu, \quad p \ge 1,\ q \ge 1$,

(ii) R_{+-} and covariant derivatives thereof .

Local U(1) invariance requires the total[45] number of $\mathcal{D}_+$ and the total number of $\mathcal{D}_-$ derivatives to be equal:

$$\#\mathcal{D}_+ = \#\mathcal{D}_- . \tag{8.19}$$

Constant super Weyl transformations, including the anomaly of the exponential, require

$$\tfrac{1}{2}\#\mathcal{D}_+ + \tfrac{1}{2}\#\mathcal{D}_- + \#R_{+-} = 1 - k_\mu k^\mu . \tag{8.20}$$

Finally, to perform the Gliozzi-Scherk-Olive projection in order to obtain supersymmetry, one must also require that left and right fermion numbers be separately conserved. Including the factors of $d\theta\, d\bar{\theta}$, one gets

$$\#\mathcal{D}_+ + \#R_{+-} = \#\mathcal{D}_- + \#R_{+-} = \text{odd} . \tag{8.21}$$

Combining Eqs. (8.19)–(8.21) gives the mass spectrum

$$m^2 = -k_\mu k^\mu = 2N, \quad N = 0,1,2,\ldots , \tag{8.22}$$

which is quite familiar from the operator formulation.

Finally, we must insist on local super Weyl invariance, independently for left- and right-movers to ensure local U(1) invariance as well. Contractions of $\mathcal{D}_+{}^n X^\mu$ with $\exp(ik\cdot X)$ and with $\mathcal{D}_+{}^m X^\nu$ will produce factors of covariant derivatives of Σ. Contractions of $\mathcal{D}_+{}^m X^\mu$ with $\mathcal{D}_-{}^n X^\nu$ will always produce curvature terms and covariant derivatives thereof. Assuming that a normal ordering convention has been adopted, so that no mixed derivatives are contracted, there will be no curvature

[45]This total number counts those $\mathcal{D}$'s applied to X^μ or R_{+-} equally.

terms either.

We can now outline the procedure for finding all vertex operators at a given mass level N.

Determine all contractions $\langle \mathcal{D}_+{}^n X \mathcal{D}_+{}^m X \rangle$ with the total number of $\mathcal{D}_+$'s less than or equal to $2N+1$. Consider all expressions of the form (8.18) for the vertex without any curvature terms. Take the polarization tensor to be (anti)symmetric when two powers, say m and n, are equal and (odd) even. Group the terms in Eq. (8.18) into those having an even and those having an odd number of space-time indices. These groups do not mix and can be treated separately. In, say, the even terms, take the tensor of highest weight and separate its traces. Using the results for the contractions between $\mathcal{D}_+$ derivatives, determine successively conditions on the lower weight tensors, so that they combine with the anomalies of the trace of the higher tensors to produce super-Weyl-invariant expressions.

Clearly, the most difficult step in the above procedure is the calculation of the anomalous contractions. As a regulator we again use heat-kernel, short-time cutoff methods, which are guaranteed to be super-reparametrization invariant. The calculation of the contractions can be performed with the help of the super heat kernel constructed in Appendix B (restricted to the very simple case of $n=0$), and we shall just quote the results here. Furthermore, some algebraic relations exist among the different p,q exponents. This comes about because the anomalous contractions satisfy the "derivative property"

$$\mathcal{D}_+\langle \mathcal{D}_+{}^p X \mathcal{D}_+{}^q X\rangle = \langle \mathcal{D}_+{}^{p+1} X \mathcal{D}_+{}^q X\rangle + (-1)^p \langle \mathcal{D}_+{}^p X \mathcal{D}_+{}^{q+1} X\rangle \ , \quad (8.23)$$

so that it suffices to compute $\langle \mathcal{D}_+{}^p XX\rangle$, the other cases being deduced from it using Eq. (8.23).

Though an explicit formula with known coefficients is not available for the type-II superstring, in contrast to the bosonic string, the calculations are sufficiently tractable to low order, and we get

$$\begin{aligned}
&\langle \mathcal{D}_+ XX\rangle = \mathcal{D}_+\Sigma \ ,\\
&\langle \mathcal{D}_+{}^2 XX\rangle = \mathcal{D}_+{}^2\Sigma \ ,\\
&\langle \mathcal{D}_+{}^3 XX\rangle = \tfrac{1}{2}(\mathcal{D}_+{}^3\Sigma - \mathcal{D}_+\Sigma\mathcal{D}_+{}^2\Sigma) \ ,\\
&\langle \mathcal{D}_+{}^4 XX\rangle = \tfrac{1}{3}(\mathcal{D}_+{}^4\Sigma + \mathcal{D}_+{}^2\Sigma\mathcal{D}_+{}^2\Sigma + \mathcal{D}_+\Sigma\mathcal{D}_+{}^3\Sigma) \ ,\\
&\langle \mathcal{D}_+{}^5 XX\rangle = \tfrac{1}{3}(\mathcal{D}_+{}^5\Sigma - 2\mathcal{D}_+{}^4\Sigma\mathcal{D}_+\Sigma) \ .
\end{aligned} \quad (8.24)$$

Applying the above rules, we can easily derive the U functions for the lowest mass levels. Certain symmetrization properties that automatically arise in this construction can be usefully represented in terms of Young tableaux of the representations of the target space-time Lorentz group corresponding to the particles of the vertex operator. Thus we have the following.

$m^2=0$:

$$U = \varepsilon_{\mu;\nu}\mathcal{D}_+ X^\mu D_- X^\nu \ ,$$

$$k^\mu \varepsilon_{\mu;\nu} = k^\nu \varepsilon_{\mu;\nu} = 0$$

($\varepsilon_{\mu;\nu}$ is *not* traceless, however). Symmetric traceless $\varepsilon_{\mu;\nu}$ corresponds to the graviton; antisymmetric $\varepsilon_{\mu;\nu}$ corresponds to the antisymmetric tensor, and the trace part of $\varepsilon_{\mu;\nu}$ is the dilation (it would require an R_{+-} term if contractions had not already been performed).

$m^2=2$:

$$\begin{aligned}
U &= \varepsilon_{\mu\nu\kappa;\lambda\rho\sigma}\mathcal{D}_+X^\mu\mathcal{D}_+X^\nu\mathcal{D}_+X^\kappa\mathcal{D}_-X^\lambda\mathcal{D}_-X^\rho\mathcal{D}_-X^\sigma\\
&\quad + \varepsilon_{\mu\nu;\kappa\lambda}\mathcal{D}_+X^{\{\mu}\mathcal{D}_+{}^2X^{\nu\}}\mathcal{D}_-X^{\{\kappa}\mathcal{D}_-{}^2X^{\lambda\}}\\
&\quad + \varepsilon_{\mu\nu\kappa;\lambda\sigma}\mathcal{D}_+X^\mu\mathcal{D}_+X^\nu\mathcal{D}_+X^\kappa\mathcal{D}_-X^{\{\lambda}\mathcal{D}_-{}^2X^{\sigma\}} \ .
\end{aligned}$$

It is not hard to extend this list to higher-mass levels.

D. The heterotic string

The general vertex operator for the heterotic string is of the form

$$V = \int d^2\xi\, d\theta (\mathrm{sdet} E_M{}^A) U^+ e^{ik\cdot x} \ . \quad (8.25)$$

Since $d\theta(\mathrm{sdet}E_M{}^A)$ is a spinor superfield of weight $\frac{1}{2}$, super-reparametrization and U(1) invariance require that U^+ be a spinor superfield of weight $\frac{1}{2}$, built out of $\mathcal{D}_+{}^pX^\mu$, $\mathcal{D}_{\bar z}{}^qX^\mu$, and $\Psi^I\mathcal{D}_{\bar z}{}^r\Psi^I$, as well as factors of $R_{+\bar z}$ and its covariant derivatives (mixed derivatives on a single factor again have been eliminated though equations of motion). Simple and important examples are the Yang-Mills vertex,

$$V = \varepsilon_\mu^{IJ}\int d^2\xi\, d\theta\, E\mathcal{D}_+X^\mu\Psi^I\Psi^J e^{ik\cdot x} \ , \quad (8.26)$$

and the gravity multiplet vertex,

$$V = \varepsilon_{\mu\nu}\int d^2\xi\, d\theta\, E\mathcal{D}_+X^\mu\mathcal{D}_{\bar z}X^\nu e^{ik\cdot x} \ . \quad (8.27)$$

Turning now to the general U^+ we observe that invariance under U(1) and constant Weyl transformations imply, respectively, that

$$\begin{aligned}
&-\tfrac{1}{2}\#\mathcal{D}_+ + \#\mathcal{D}_{\bar z} + \tfrac{1}{2}\#R_{+\bar z} + \tfrac{1}{2}\#\Psi = \tfrac{1}{2} \ ,\\
&\tfrac{1}{2}\#\mathcal{D}_+ + \#\mathcal{D}_{\bar z} + \tfrac{3}{2}\#R_{+\bar z} + \tfrac{1}{2}\#\Psi = \tfrac{3}{2} - k\cdot k \ .
\end{aligned} \quad (8.28)$$

On the other hand, the vertex must have even worldsheet fermion number, that is, $\#\Psi$ must be even. It follows that

$$m^2 = -k\cdot k = 2N, \ N=0,1,2,\ldots \ . \quad (8.29)$$

The rules of construction from the principle of super Weyl invariance are now completely analogous to those stated for the type-II superstring once we have identified the potential anomalous contractions. There are new anomalies coming from the contractions $\langle (\mathcal{D}_{\bar z})^p\Psi\Psi\rangle$, while the anomalous contractions for X^μ are of the form $\langle (\mathcal{D}_{\bar z})^pXX\rangle$, $\langle (\mathcal{D}_+)^qXX\rangle$, and $\langle (\mathcal{D}_{\bar z})^pX(\mathcal{D}_+)^qX\rangle$. The first two types of terms involving X^μ, however, are essentially given by those of the bosonic string involving $\mathcal{D}_{\bar z}$ derivatives and those of the type-II superstring involving

$\mathcal{D}_+$ derivatives. To see this, we regulate the theory by the heat-kernel, short-time cutoff method. The natural operator for X^μ in the heterotic string is $\mathcal{D}_+\mathcal{D}_{\bar z}$, which is not a U(1) scalar, so we use instead $-\mathcal{D}_+{}^2\mathcal{D}_{\bar z}$, which is a positive operator transforming as

$$\mathcal{D}_+{}^2\mathcal{D}_{\bar z}=e^{-2\Sigma}(\hat{\mathcal{D}}_+{}^2\hat{\mathcal{D}}_{\bar z}-\hat{\mathcal{D}}_+\Sigma\hat{\mathcal{D}}_+\hat{\mathcal{D}}_{\bar z}) \tag{8.30}$$

under super Weyl scalings. When computing $\langle(\mathcal{D}_+)^pXX\rangle$ we can infer from dimensional analysis and U(1) covariance that p $\mathcal{D}_+$ derivatives acting on Σ will appear in the answer. Omitting then any reference to $\mathcal{D}_-$ in the type-II superstring calculation reduces the operators in the heat-kernel regularization. On the other hand, in an anomaly computation involving only $\mathcal{D}_{\bar z}$ derivatives on Σ, the $\mathcal{D}_+$ derivatives may be effectively omitted, and we recover the result from the bosonic string. Finally, the anomalies in mixed derivatives are polynomials in $R_{+\bar z}$ and its superderivatives. However, as in the bosonic or type-II superstring, such contractions are precisely compensated by the curvature terms, and every curvature term is present only to compensate for the anomalous contractions. Consequently, the normal ordering convention will be adopted in which no mixed contractions are allowed, and thus no curvature terms should appear.

We now discuss anomalies of the spinor superfields Ψ in order to complete our analysis. The basic object is the propagator

$$(\mathcal{D}_+)^{-1}(\mathbf{z},\mathbf{z}')=\langle\Psi(\mathbf{z})\Psi(\mathbf{z}')\rangle\ , \tag{8.31}$$

which in superconformal gauge is related to the flat superspace propagator by

$$(\mathcal{D}_+)^{-1}(\mathbf{z},\mathbf{z}')=e^{-\Sigma(\mathbf{z})/2}(\hat{\mathcal{D}}_+)^{-1}(\mathbf{z},\mathbf{z}')e^{-\Sigma(\mathbf{z}')/2}\ , \tag{8.32}$$

with

$$\hat{\mathcal{D}}_+^{-1}(\mathbf{z},\mathbf{z}')=\frac{z-z'}{|z-z'|^2+\varepsilon^2}+\theta\theta'\delta^2(z-z')\ . \tag{8.33}$$

The natural heat kernel is

$$\mathcal{K}^t_\Sigma(\mathbf{z},\mathbf{z}')=\langle\mathbf{z}\,|\,e^{t2\mathcal{D}_+{}^2\mathcal{D}_{\bar z}}\,|\,\mathbf{z}'\rangle\theta(t)\ , \tag{8.34}$$

which will be given by the perturbative expansion

$$e^{\Sigma(z)/2}\mathcal{K}^t_\Sigma(\mathbf{z},\mathbf{z}')e^{\Sigma(z')/2}=\hat{\mathcal{K}}^t(\mathbf{z},\mathbf{z}')+\int_0^t dt_1\int d^2\mathbf{z}_1\hat{\mathcal{K}}^{t-t_1}(\mathbf{z},\mathbf{z}_1)W(\mathbf{z}_1)\hat{\mathcal{K}}^{t_1}(\mathbf{z},\mathbf{z}')+\cdots\ , \tag{8.35}$$

where

$$\hat{\mathcal{K}}^t(\mathbf{z},\mathbf{z}')=\frac{1}{4\pi t}\exp\left[-\frac{|z-z'|^2}{2t}\right](\theta'-\theta)\ ,$$
$$W(\mathbf{z})=(e^{-2\Sigma(\mathbf{z})}-1)\frac{\partial}{\partial t}+2(\partial_{\bar z}e^{-2\Sigma(\mathbf{z})})\partial_{\bar z}\ . \tag{8.36}$$

The contractions can be obtained from

$$\langle\Psi\partial_z\Psi\rangle=\partial^m_{z'}\left[e^{-\Sigma(z')/2}\int d^2\mathbf{w}\,\mathcal{K}^\varepsilon_\Sigma(\mathbf{z},\mathbf{w})e^{\Sigma(\mathbf{w})/2}\hat{\mathcal{D}}_+^{-1}(\mathbf{w},\mathbf{z}')\right]\bigg|_{\mathbf{z}=\mathbf{z}'}\ . \tag{8.37}$$

As an example we get

$$\langle\Psi\mathcal{D}_{\bar z}\Psi\rangle=\frac{i}{2}\mathcal{D}_{\bar z}{}^2\Sigma+\frac{i}{3}(\mathcal{D}_{\bar z}\Sigma)^2\ . \tag{8.38}$$

It is clear that all other contractions could be derived in an analogous fashion, albeit by rather lengthy calculations.

Vertices in the operator language for the heterotic string may be found in Gross *et al.* (1986). The above formulas for the type-II superstring and heterotic strings, as well as the evaluation of super Weyl anomalies, are in D'Hoker and Phong (1987b).

E. The covariant fermion emission vertex operator and space-time supersymmetry

1. Covariant fermion emission vertex operator

Two related problems arise in the construction of fermion emission vertex operators, which together point to a solution. The first is to manufacture space-time spinors out of the worldsheet fermions ψ^μ which transform rather as an SO(10) vector. The second is that inserting a fermion emission vertex operator at a point z on the worldsheet must introduce a branch cut originating at z. To see this, we recall that in canonical quantization, free strings propagate along cylinders, and fermions and bosons correspond, respectively, to states in which ψ^μ are periodic $\psi^\mu(\sigma+\pi)=\psi^\mu(\sigma)$ (Ramond sector), and states in which ψ^μ are antiperiodic $\psi^\mu(\sigma+\pi)=-\psi^\mu(\sigma)$ (Neveu-Schwarz sector). Thus a fermion emission vertex must switch boundary conditions in order to preserve spin statistics. The way to achieve this is to introduce a cut originating at the insertion that contributes a factor of -1 when we cross it (see Fig. 23). Now a cut must originate at one point and end at some other point on the worldsheet. This means that we should look for operators S such that correlations of two S's with the ψ^μ's will be defined on the double cover of the worldsheet with branch points at the S insertions. Such operators can be obtained by bosonizing the worldsheet fermions. We

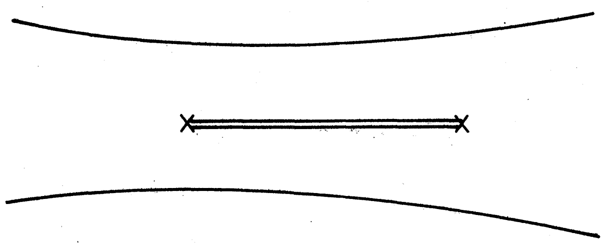

FIG. 23. Quadratic branch cut introduced by a fermion emission vertex.

shall see that they can also be combined into spinor representations of SO(10).

If we group the fields ψ^μ into

$$\psi_+^a=\psi^a+i\psi^{a+5}, \quad \psi_-^a=\psi^a-i\psi^{a+5},$$

they can be represented by scalar bosons ϕ^a,

$$\psi_\pm^a=e^{\pm i\phi^a}. \tag{8.39}$$

The spin fields S_α can next be defined by

$$\begin{aligned} &S_{\pm a}=e^{\pm i\phi^a/2}, \\ &S_\alpha=e^{\pm i\phi^1/2}\cdots e^{\pm i\phi^5/2}, \\ &\alpha=(\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2}). \end{aligned} \tag{8.40}$$

Now a key observation is that the integrals of the currents

$$J_a=-i\partial_z\phi_a \tag{8.41}$$

for the bosonized theories can be viewed as generating a Cartan subalgebra of SO(10), with the weights of S_α given by $\alpha=(\pm\frac{1}{2},\ldots,\pm\frac{1}{2})$. These are precisely the weights of the spinor representation of SO(10).

Since the ψ^μ's have conformal dimension $\frac{1}{2}$, it follows from Sec. II.J that the spin fields $S_{\pm a}$ have dimensions $\frac{1}{8}$, and hence the operators S_α have dimension $\frac{5}{8}$. Recall that a physical vertex operator must have conformal dimension 1, so that it may be integrated on the worldsheet. This appearance of the dimension $\frac{5}{8}$ instead of 1 was one of the major difficulties of fermion emission amplitudes already encountered in dual models, where a number of cures involving either projections onto physical states or a light-cone gauge approach were proposed. A natural resolution in the covariant formalism is in terms of the superconformal ghosts. Indeed, when a fermion vertex is inserted, it changes the boundary conditions of the worldsheet fermions and consequently the boundary conditions of the gravitino field χ, since χ couples to ψ^μ. In particular, the Grassmann parameter for supersymmetry transformations must be double valued around the insertions. In terms of the superconformal ghosts, this means that we must introduce a cut in the β,γ fields as well.

To achieve this, we need, as before, to bosonize the ghosts. There is an added complication here, since the superconformal ghosts are bosons, and $e^{\pm i\sigma}$ obey Fermi statistics. A way out is given by Friedan, Shenker, and Martinec (1985; Friedan, Martinec, and Shenker, 1986), who introduced fermion fields ξ and η, so that

$$\beta(z)=\partial_z\xi(z)e^{i\sigma(z)}, \quad \gamma(z)=\eta(z)e^{-i\sigma(z)} \tag{8.42}$$

are consistent with spin statistics. Spin fields $\Sigma_{\pm 1/2}$ in the fields of the superconformal ghosts can now be introduced by

$$\Sigma_{\pm 1/2}=e^{\pm i\sigma(z)/2}. \tag{8.43}$$

Since the β,γ have conformal dimensions $\frac{3}{2}$ and $-\frac{1}{2}$, the bosonized theory is coupled to a background charge $Q=-2$, the minus sign being due to the fact that the operator product expansion $\beta(z)\gamma(w)$ is now $-1/(z-w)$ instead of $1/(z-w)$. Similarly the conformal dimension of $e^{iq\sigma}$ in Eq. (2.184) becomes now $-q(q+Q)/2$, so that the conformal dimensions of $\Sigma_{+1/2}$ and $\Sigma_{-1/2}$ are $\frac{3}{8}$ and $-\frac{5}{8}$, respectively. This suggests that the fermion emission vertex is given by

$$V_{-1/2}^F(u^\alpha,k^\mu)=\Sigma_{1/2}(u^\alpha S_\alpha)e^{ik_\mu x^\mu}, \tag{8.44}$$

where $u^\alpha(k)$ is a space-time spinor. The correct conformal dimension is achieved for $k^\mu\Gamma_\mu u=k^2=0$, so that the space-time fermion emitted is massless. (Here Γ denote the ten-dimensional gamma matrices.)

It is useful to know the operator product expansions of the S_α's. In ten dimensions, the charge-conjugation matrix C is antisymmetric and interchanges chiralities. Left and right spinors correspond to an even or odd number of $+\frac{1}{2}$ in α. We should have then

$$\begin{aligned} S_\alpha(z)S^\beta(w)&\sim\frac{-\delta_\alpha{}^\beta}{(z-w)^{5/4}}+\frac{(\Gamma_\mu)_\alpha{}^\beta\psi^\mu}{(z-w)^{3/4}} \\ &\quad+\frac{(\Gamma_{\mu\nu})_\alpha{}^\beta\psi^\mu\psi^\nu}{(z-w)^{1/4}}+\cdots, \\ \psi^\mu(\xi)S_\alpha(z)S_\beta(w)&\sim\frac{(\Gamma^\mu)_{\alpha\beta}}{(\xi-z)^{3/4}}+\cdots, \\ J^{\mu\nu}(\xi)S_\alpha(w)&\sim\frac{(\Gamma^{\mu\nu})_\alpha{}^\beta S_\beta(z)}{\xi-w}+\cdots. \end{aligned} \tag{8.45}$$

To compute tree-level fermion scattering amplitudes for four or fewer fermions, we note that the amplitudes decompose into separate functions for the vertex operators e^{ikx}, for the bosonized superghosts, and for the spin operators. The first two types of correlation functions are familiar by now, and we can use Eq. (8.45) and ten-dimensional spinor algebra to arrive at projectively invariant expressions consistent with SO(10) transformation properties. The results reproduce those obtained by the earlier methods.

In general we also need another fermion emission vertex $V_{1/2}^F$ with opposite ghost charge. It is natural to make use of $\Sigma_{-1/2}$, which, combined with S_α, will however produce an operator of dimension 0. We have then to introduce a dimension-1 bosonic vertex. The vertex

$V^F_{1/2}$ is obtained by choosing the massless boson emission vertex

$$V^F_{1/2}=u^{\alpha}(k)\Sigma_{-1/2}(\partial x^{\mu}+k_{\nu}\psi^{\nu}\psi^{\mu})\Gamma^{\alpha\beta}_{\mu}S_{\beta}e^{ikx} . \quad (8.46)$$

The vertices $V^F_{-1/2}$ and $V^F_{1/2}$ are just two of an infinite number of versions of the fermion emission vertex, which are actually related to one another by

$$V^F_{1/2}=[Q_{\mathrm{BRST}},\xi V^F_{-1/2}] ,$$
$$V^F_{3/2}=[Q_{\mathrm{BRST}},\xi V^F_{1/2}] . \quad (8.47)$$

The operators on the right-hand side are BRST invariant (as physical vertices must be), but not spurious despite the fact that they appear as BRST transforms. The reason is that the irreducible representation of the current algebra is built out of $\phi,\eta,\partial\xi$, but not of ξ itself. These many versions are caused by the necessity of prescribing an arbitrary Bose sea level for the superconformal ghosts. The Hilbert space of states must include all representations corresponding to various choices of sea levels. The vertex V^F_{λ} is the BRST-invariant vertex operator with Bose sea level λ. These ubiquitous "picture-changing" phenomena are explained in detail in Friedan, Martinec, and Shenker (1986). They are crucial in the construction of the ☆ operation of Witten's superstring field theory (1986b) and, as shown by Verlinde and Verlinde (1987b), in the gauge-fixed superstring of Sec. III.P.

Finally, we note that N-fermion emission for $N\geq 6$ is already more complicated even at tree level. In fact the above discussion shows that the worldsheet has to be viewed as a sphere with N punctures, and such a surface admits $(N-4)/2$ supermoduli parameters. Proper treatment of integration over moduli parameters presents some of the problems encountered earlier in multiloop amplitudes.

2. Space-time supersymmetry

The supersymmetry charge in the covariant formalism can now be obtained as a contour integral,

$$Q_{\alpha}=\oint\frac{dz}{2\pi i}V^F_{\alpha}(k=0) , \quad (8.48)$$

where $V^F_{\alpha}(k=0)=\Sigma_{1/2}S_{\alpha}$ is the fermion emission vertex at zero momentum. From operator product expansions we can check that Q_{α} transforms massless fermion vertices into massless boson vertices and vice versa,

$$\{Q_{\alpha},V^F(u,k)\}=V^B(\zeta^{\mu}=u\Gamma^{\mu},k) ,$$
$$[Q_{\alpha},V^B(\zeta,k)]=V^F(u=ik^{\mu}\Gamma_{\mu\nu}\zeta^{\nu},k) . \quad (8.49)$$

In fact, the full supersymmetry algebra

$$[P^{\mu},P^{\nu}]=[P^{\mu},Q_{\alpha}]=0 ,$$
$$\{Q_{\alpha},Q_{\beta}\}=(\Gamma^{\mu})_{\alpha\beta}P^{\mu} , \quad (8.50)$$

is obtained by taking P^{μ} to be the contour integral of the vector emission vertex at zero momentum: ∂x^{μ}. Powerful nonrenormalization theorems can in principle be deduced from this setup. For example, since vertices for massless bosons and fermions can be obtained from one another by a contour integral of the supersymmetry current [see Eq. (8.49)], boson propagator corrections follow from fermion propagator corrections by deformation of contours. Thus there will be no mass renormalization for massless bosons if the fermions are chiral.

It should be noted that this discussion is only local. For worldsheets of nontrivial topology, the supersymmetry current develops unphysical poles, which must be taken into account before any firm conclusion can be drawn. Furthermore, there may be contributions from the boundary of moduli space.

Some classical papers on the fermion emission vertex are those of Schwarz and Wu (1971), Thorn (1971), Corrigan and Olive (1972), Brink *et al.* (1973), and Mandelstam (1974a), the last paper being based on path-integral methods in the light-cone gauge. The covariant fermion vertex operator was constructed by Friedan, Shenker, and Martinec (1985; Friedan, Martinec, and Shenker, 1986) and Knizhnik (1985, 1986a, 1986b). That the ghosts should contribute was suggested by Goddard and Olive. A derivation of quantum numbers for the vertex from the Polyakov integral on surfaces with punctures was proposed by Knizhnik (1986a, 1986b). Explicit calculations of fermion emissions in the covariant formalism are given in Cohn *et al.* (1986), Knizhnik (1986a, 1986b), and Kostelecky, Lechtenfeld, and Samuel (1987). Nonrenormalization theorems are in Martinec (1986). Picture-changing phenomena were uncovered by Friedan, Martinec, and Shenker (1986). Their role in superstring field theory is discussed by Witten (1986b), and in the gauge-fixed multiloop partition function by Friedan, Martinec, and Shenker (1986) and Verlinde and Verlinde (1987b), as we saw earlier in Sec. III.P. Verlinde and Verlinde discovered the unphysical poles in the supersymmetry current and argued that their residues must be total derivatives on moduli space. They also provided expressions for the correlation functions of the bosonized superconformal ghosts in terms of the prime form. Unphysical poles as well as contributions from the boundary of moduli space in the two-loop case are treated by Atick and Sen (1987a, 1987b, 1987c).

IX. CONCLUSION AND OUTLOOK

In this paper we have reviewed some of the most recent developments in string perturbation theory. We shall now give a brief survey of the main objectives achieved so far, as well as of the questions that remain. We shall also take the opportunity to mention developments in other directions and include some references that have not occurred earlier in the text.

The structure of strings is amazingly rich, and in many ways quite rigid. Progress in the study of the bosonic string has been spectacular thanks to the concerted efforts of many authors, and we have now a very good understanding of scattering amplitudes, order by order in perturbation theory. The fermionic strings, on the other hand, have revealed themselves to be much more profound and fraught with dangerous subtleties. Their investigation has forced us to come to grips with some of the deepest questions in geometry. Nevertheless we have reached a stage now where the required machinery is in place, and the proposals we described in Secs. III.K, III.O, and VII.G point to a consistent formulation of superstrings. More specifically, with internal loop momenta, we have a way of separating left from right chiralities, which does reproduce the heterotic string from the chirally split RNS string. This way is also precisely the one agreeing with holomorphic splitting on supermoduli space, allowing us to integrate out the odd moduli, and holomorphic splitting on supermoduli will then reduce to holomorphic splitting on "moduli," if "moduli" space is viewed as the $(3h-3)$-dimensional space of supersymmetry period matrices $\hat{\Omega}$. The formulation can then be argued to lead to modular-invariant amplitudes, even taking into account the fact that no global section over moduli space of $\frac{3}{2}$ differentials can be chosen to gauge-fix the superstring. It also offers a way out of the apparent ambiguities of the picture-changing formalism discussed by Atick, Rabin, and Sen (1987), Moore and Morozov (1987) and Verlinde (1987). These ambiguities, for example, could have led to a nonvanishing cosmological constant at two-loop order if not treated properly.

A more explicit implementation with the required technology of the above program is the natural next step. Here we are encouraged by the rapid progress in the understanding of two-dimensional supergeometry and supermoduli space. Difficulties with indefinite metrics have been resolved (Secs. III.F and III.H), a complex structure of supermoduli space has been introduced (Secs. III.G and VII.F), and foundations of superalgebraic geometry are on the way with the super Abelian differentials and supersymmetric period matrix (Secs. VII.F and VII.G; Sonoda, 1987b). Line bundles over super Riemann surfaces have been investigated by Giddings and Nelson (1987). It is perhaps timely to formulate and solve a Schottky problem for supersymmetric period matrices. From the component point of view, we now have at our disposal the chiral bosonization formulas for ghosts and superghosts of Verlinde and Verlinde (1987a, 1987b), as well as a good understanding of Mandelstam diagrams (Sec. IV.G; Giddings and Wolpert, 1987) and of relations between their determinants (Sec. V.G). That the present formulation is an efficient tool for practical calculations is illustrated to one loop in Sec. III.M. All this is grounds for believing that we shall shortly have explicit confirmation of consistency and unitarity of superstrings, together with simple rules for calculating scattering amplitudes.

In this paper we have discussed only briefly the picture-changing formalism and the necessary Wu-Yang correction terms, and we have not pursued it further. This is clearly an important issue, since it is intimately connected with manifest BRST invariance. A detailed discussion of this topic and of whether supermoduli space splits over moduli is to be found in Verlinde (1987). Other options have been suggested by Atick, Moore, and Sen (1988a, 1988b).

Perhaps after mastering the subtleties of string perturbation theory we may find a mechanism for breaking supersymmetry while maintaining a vanishing cosmological constant. A proposal based on modular forms to one loop has been presented by Moore (1987).

In a different direction, the string ground state should be determined by physics at the Planck scale, and formally perturbative amplitudes may be used to probe the higher-energy (limit as the Planck mass tends to zero) behavior of string theory. Such investigations have been initiated by Gross and Mende (1987, 1988) and Gross (1988), who argue that in the $T\to 0$ limit, contributions from surfaces with discrete symmetry dominate, and an infinite number of relations then hold between scattering amplitudes. This suggests the presence of a huge spontaneously broken symmetry.

At the other end, in the low-energy limit ($T\to\infty$), string theory should reduce to an effective field theory, whose equations of motion are given by the requirement of conformal invariance. Thus a vacuum configuration corresponds to a conformal field theory. Of particular interest are vacuum configurations in which space-time splits into four-dimensional Minkowski space-time times a six-dimensional internal space M_6. Vanishing of the beta functions as well as unbroken $N=1$ supersymmetry restricts M_6 to be essentially a Calabi-Yau (i.e., Ricci flat and Kähler) manifold. This was argued by Callan, Friedan, Martinec, and Perry (1985), Candelas, Horowitz, Strominger, and Witten (1985), Green, Schwarz, and West (1985), Sen (1985, 1986a), Grisaru, Van de Ven, and Zanon (1986), Howe, Papadopoulos, and Stelle (1986), and Witten (1986a, 1986b). Other conformal field theories are provided by orbifolds, introduced by Dixon, Harvey, Vafa, and Witten (1985, 1986), toroidal compactifications (Narain, 1986; Ginsparg and Vafa, 1987; Narain and Sarmidi, 1987; Narain, Sarmidi, and Vafa, 1987; Narain, Sarmidi, and Witten, 1987), quasicrystalline orbifolds (Harvey, Moore, and Vafa, 1988), and group manifolds (Jain, Shankar, and Wadia, 1985; Gepner and Witten, 1986; Jain, Mandal, and Wadia, 1987). The moduli space of conformal field theories and renormalization-group equations are considered, respectively, in Seiberg (1987) and Banks and Martinec (1987). Four-dimensional theories from the $d=10$ type-II theories with chiral asymmetry are constructed by Antoniadis *et al.* (1986), Bluhm, Dolan, and Goddard (1987), Dixon, Kaplunovsky, and Vafa (1987), and Kawai, Lewellen, and Tye (1987).

The large number of candidate vacua will require a better understanding of nonperturbative effects, for example, of stringy instantons. Very early on in string theory, attempts were made to derive string perturbation theory from a string field theory, in the hope that string field theory might be consistently interpolated off-shell. Some of the earliest works are those of Mandelstam (1973a, 1973b), Cremmer and Gervais (1974), and Kaku and Kikkawa (1974). More recently, superstring fields in the light-cone gauge have been formulated by Green and Schwarz (1983, 1984), Green, Schwarz, and Brink (1983), and Gross and Periwal (1988), although in a background-dependent way. Covariant formulations requiring an unphysical length parameter are presented by Kazama *et al.* (1986), Hata *et al.* (1987), and Neveu and West (1987). String fields based on BRST invariance have been developed by Friedan (1985), Siegel (1985), Siegel and Zwiebach (1986), Banks and Peskin (1986), and Witten (1986a, 1986b). Witten's theory is based on a remarkable interaction on the worldsheet. Its bosonic version has been shown to reproduce the correct (open-string) amplitudes by Giddings (1986), Giddings and Martinec (1986), Giddings, Martinec, and Witten (1986), and Thorn (1987). Background-independent formulations for it have been proposed by Horowitz *et al.* (1986), as well as closed-string versions by Strominger (1987). Operator formulations have been worked out by Gross and Jevicki (1987).

More radical proposals for the study of nonperturbative effects have been put forth by Friedan and Shenker (1986, 1987) and by Bowick and Rajeev (1987). Friedan and Shenker use factorization requirements to lump moduli spaces of all genera, including surfaces with nodes, into a universal moduli space. An abstract string theory corresponds to a holomorphic vector bundle together with a flat connection on the universal moduli space. Nonperturbative effects correspond to a completion of the universal moduli space, which must then include some classes of surfaces of infinite genus. This approach has been extended to the case of the superstring by Cohn (1988). On the other hand, Bowick and Rajeev (1987) view conformal invariance as invariance under $\mathrm{Diff}S^1/S^1$, so that the key requirement becomes flatness and trivial holonomy of parallel transport along $\mathrm{Diff}S^1/S^1$. Now $\mathrm{Diff}S^1/S^1$ is a Kähler manifold with nonvanishing Ricci curvature, and an acceptable theory can be viewed as a vector bundle on this space, whose curvature cancels that of the tangent bundle. This would be an analog of the anomaly cancellation between matter and ghost parts in the Polyakov string. Related ideas have been developed by Witten (1987).

A natural setup in which Riemann surfaces (more precisely, with a puncture and a local coordinate system) of all genera appear on the same footing is provided by Sato's universal Grassmannian, which has been at the center of great developments in connection with integrable systems. It is similar in many ways to moduli space and is already known to provide an operator proof of Bose-Fermi equivalence (Sato, Jimbo, and Miwa, 1977, 1978, 1979; Date *et al.* 1983; Segal and Wilson, 1985). Possibilities of string theory formulations in terms of Grassmannians are investigated by Ishibashi, Matsuo, and Ooguri (1986), Alvarez-Gaumé, Gomez, and Reina (1987), Vafa (1987), and Witten (1988a, 1988b). Grassmannians and the homology of the mapping class group are studied by Arbarello *et al.* (1987). Of course, for the fermionic string we would need an analog of this theory based on super Riemann surfaces.

Finally, several authors have also suggested considering string theories over number fields different from the complex. This may help to solve some theories, as well as to uncover any arithmetic structure that may make the theory even more rigid.

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APPENDIX A: CONVENTIONS

1. Differential geometry

We use the following conventions for covariant derivatives and connections:

$$\begin{aligned}\nabla_m V_n &= \partial_m V_n - \Gamma^p_{mn} V_p \ ,\\ \nabla_m V^n &= \partial_m V^n + \Gamma^n_{mp} V^p \ .\end{aligned} \tag{A1}$$

Generalizations to tensors of arbitrary rank may be deduced by applying covariant derivatives to tensor products and using Leibnitz's rule.

The Riemann curvature tensor is given by

$$R^l{}_{mnk}=\frac{\partial\Gamma^l_{mn}}{\partial x^k}-\frac{\partial\Gamma^l_{mk}}{\partial x^n}+\Gamma^p_{mn}\Gamma^l_{kp}-\Gamma^p_{mk}\Gamma^l_{np} \tag{A2}$$

or

$$[\nabla_m,\nabla_n]V_k=-R^l{}_{kmn}V_l\ . \tag{A3}$$

One has the symmetry properties

$$\begin{aligned}&R_{lmnk}=R_{nklm}\ ,\\ &R_{lmnk}=R_{mlkn}=-R_{mlnk}=-R_{lmkn}\ ,\\ &R_{lmnk}+R_{lkmn}+R_{lnkm}=0\ .\end{aligned} \tag{A4}$$

In Riemannian geometry, the connection Γ^p_{mn} is symmetric (i.e., has zero torsion) and the metric is covariantly constant,

$$\nabla_k g_{mn}=0\ .$$

Hence the Levi-Cività connection is given by

$$\Gamma^p_{mn}=\tfrac{1}{2}g^{pq}(\partial_m g_{nq}+\partial_n g_{mq}-\partial_q g_{mn})\ . \tag{A5}$$

One also defines the Ricci curvature tensor

$$R_{mn}=R^l{}_{mln}$$

and the (Gaussian) scalar curvature

$$R=-\tfrac{1}{2}g^{mn}R_{mn}\quad\text{or}\quad R_{mn}=-g_{mn}R\ . \tag{A6}$$

A change in the Levi-Cività connection is a tensor, given by

$$\delta\Gamma^p_{mn}=\tfrac{1}{2}g^{pq}(\nabla_m\delta g_{nq}+\nabla_n\delta g_{mq}-\nabla_q\delta g_{mn})\ , \tag{A7}$$

generating a change in the Riemann curvature,

$$\delta R^l{}_{mnk}=\nabla_k\delta\Gamma^l_{mn}-\nabla_n\delta\Gamma^l_{mk}\ ,$$

and in the Gaussian curvature,

$$\begin{aligned}\delta R=&-\tfrac{1}{2}Rg^{mn}\delta g_{mn}-\tfrac{1}{2}\nabla_p\nabla^p(g^{mn}\delta g_{mn})\\ &+\tfrac{1}{2}\nabla^m\nabla^n\delta g_{mn}\ ,\end{aligned} \tag{A8}$$

which is Eq. (2.34). It is also useful to record the changes in the covariant derivatives,

$$\delta\nabla^z=-\delta\sigma\nabla^z+\tfrac{1}{2}\delta g^{zz}\nabla_z+\frac{n}{2}\nabla_z(\delta g^{zz})\ ,$$

$$\delta\nabla_z=-n\,\partial_z\delta\sigma-\tfrac{1}{2}\delta g_{zz}\nabla^z+\frac{n}{2}\nabla^z(\delta g_{zz})$$

with $\delta g^{zz}=-(g^{z\bar z})^2\delta g_{\bar z\bar z}$.

General two-dimensional coordinates are denoted by ξ^1 and ξ^2 or $\xi=(1/\sqrt{2})(\xi^1+i\xi^2)$. The metric is then $ds^2=g_{mn}d\xi^m d\xi^n$. Locally conformally flat coordinates are denoted by z (or w, etc.) and the metric is then $ds^2=2g_{z\bar z}dz\,d\bar z$. With these conventions, one has the following metrics and curvatures, $g_{z\bar z}R=-\partial_z\partial_{\bar z}\ln g_{z\bar z}$, for the sphere, plane, and upper half plane, respectively:

$$ds^2=\frac{4\,dz\,d\bar z}{(1+|z|^2)^2}\ ,\quad R=1\ ,$$

$$ds^2=2\,dz\,d\bar z\ ,\quad R=0\ ,$$

$$ds^2=-\frac{dz\,d\bar z}{(z-\bar z)^2}\ ,\quad R=-1\ .$$

The sphere has area 4π. Furthermore, the covariant derivatives in locally conformally flat coordinates are

$$\nabla_z V_{\underbrace{zz\cdots z}_{n}}=(\partial_z-n\Gamma^z_{zz})V_{\underbrace{zz\cdots z}_{n}}\ , \tag{A9}$$

encountered in Eqs. (2.41)–(2.44), with $\Gamma^z_{zz}=\partial_z\ln g_{z\bar z}$. We use the notation for flat metric $g_{z\bar z}=g_{\bar z z}=\sqrt{g}=1$:

$$d^2z\sqrt{g}=dx\,dy=i\,dz\wedge d\bar z\ . \tag{A10}$$

2. Spinors, Dirac matrices

U(1) vector indices are denoted by $a,b,\ldots$ and take on the values z and $\bar z$; spinor indices are denoted by $\alpha,\beta,\ldots$ and take values $+$ and $-$. We use the same notations z and $\bar z$ for conformally flat coordinates and U(1) indices because in conformally flat coordinate systems they may be identified, so no confusion should arise. Dirac matrices satisfy

$$\{\gamma^a,\gamma^b\}=-\delta^{ab},\quad[\gamma^a,\gamma^b]=-\varepsilon^{ab}\gamma_5\ . \tag{A11}$$

We take the convenient representation of this Clifford algebra,

$$\begin{aligned}&(\gamma^z)_{++}=(\gamma^{\bar z})_{--}=-(\gamma^z)_+{}^-=(\gamma^{\bar z})_-{}^+=1\ ,\\ &(\gamma^a)_{\alpha\beta}=0,\quad\alpha\neq\beta\ ,\\ &(\gamma_5)_{+-}=(\gamma_5)_{-+}=(\gamma_5)_+{}^+=-(\gamma_5)_-{}^-=i\ ,\\ &(\gamma_5)_{\pm\pm}=0\ .\end{aligned} \tag{A12}$$

It is also useful to have the following formulas at hand:

$$\psi_+=-\psi^-,\quad\psi_-=\psi^+\ ,$$

and the following conventions for the antisymmetric tensor:

$$\varepsilon^{\bar z z}=-\varepsilon^{z\bar z}=\varepsilon_{z\bar z}=-\varepsilon_{\bar z z}=i,\quad\varepsilon_{zz}=\varepsilon_{z\bar z}=0\ . \tag{A13}$$

Contractions without indices written explicitly are understood as

$$\omega\psi=\omega^\alpha\psi_\alpha=-\omega_\alpha\psi^\alpha=\omega^+\psi_++\omega^-\psi_-\ . \tag{A14}$$

It is useful to have

$$\theta_\alpha\theta^\beta=\delta_\alpha{}^\beta\theta\bar\theta\ .$$

3. Covariant derivatives on U(1) tensors

For Weyl spinors of U(1) weight $\pm\frac{1}{2}$ we have

$$D_a\psi_\pm=e_a{}^m\left[\partial_m\pm\frac{i}{2}\omega_m\right]\psi_\pm\ .$$

On a Dirac spinor we have

$$D_a\psi_\alpha = e_a{}^m[\partial_m\psi_\alpha + \tfrac{1}{2}(\gamma_5)_\alpha{}^\beta\omega_m\psi_\beta] .$$

On a general tensor-spinor of weight n, we have

$$D_a\psi_{(n)} = e_a{}^m(\partial_m\psi_{(n)} + in\omega_m\psi_{(n)}) .$$

Here the spin connection is given by

$$\omega_m = -e_m{}^a\varepsilon^{pq}\partial_p e_q{}^b\delta_{ab} ,$$

where

$$\varepsilon^{pq} = e_a{}^p e_b{}^q\varepsilon^{ab} .$$

The Gaussian curvature is expressed in terms of the spin connection

$$R = \varepsilon^{mn}\partial_m\omega_n ,$$

which is Eq. (3.37).

4. Dirac singularity

By definition

$$\int d^2w\, \delta(z-w)f(w) = f(z) .$$

We shall also use the covariant Dirac delta function

$$\delta(z,w) = \frac{1}{\sqrt{g}}\delta(z-w) .$$

Notice the minus sign versus the comma. It may be viewed as the limit in the sense of distributions of certain functions as $\varepsilon\to 0$,

$$\frac{1}{2\pi}\frac{\varepsilon^2}{(|z-w|^2+\varepsilon^2)^2} \to \delta(z-w)$$

and

$$\frac{1}{4\pi\varepsilon}\exp\left[-\frac{1}{2\varepsilon}|z-w|^2\right] \to \delta(z-w) .$$

In particular, we have

$$\partial_{\bar z}\frac{1}{z-w} = 2\pi\delta(z-w) .$$

Notice the unusual factor of 2 in this convention.

APPENDIX B: SHORT-TIME EXPANSIONS OF THE HEAT KERNEL

The heat kernel for the operator $\Delta_n^{(-)}$ satisfies the equation

$$\left[\frac{\partial}{\partial t} + \Delta_n^{(-)}\right] K_n^t(z,z') = \delta^2(z,z')\delta(t) \tag{B1}$$

with the solution

$$K_n^t = \theta(t)e^{-t\Delta_n^{(-)}} .$$

We wish to calculate elements on or close to the diagonal $z=z'$ for short times t. As the issues involved are exclusively local in z, we may locally perform a reparametrization rendering the metric conformal to the Euclidean metric: $g_{mn}(z) = e^{2\sigma(z)}\delta_{mn}$. From Eqs. (2.42)–(2.44) and (2.47), we readily find the σ dependence of $\Delta_n^{(-)}$:

$$\Delta_n^{(-)} = e^{-2\sigma}\Delta + 4n\partial_z\sigma e^{-2\sigma}\partial_{\bar z} , \tag{B2}$$

where Δ is the flat-space Laplacian $\Delta = -2\partial_z\partial_{\bar z}$. Combining Eqs. (B1) and (B2) and the scaling of $\delta^2(z,z')$ under constant Weyl transformations σ_0, we find that

$$K_n^t(z,z') = e^{-2\sigma_0}K_n^{\hat t}(z,z') \quad \text{with } \hat t = e^{-2\sigma_0}t . \tag{B3}$$

Thus, without loss of generality, we may assume that $\sigma=0$ at the point of interest z'. We now rewrite (B1) as

$$\left[\frac{\partial}{\partial t} + \Delta - V_n\right] K_n^t(z,z') = \delta^2(z,z')\delta(t) \tag{B4}$$

with

$$V_n = (1-e^{-2\sigma})\Delta - 4n\partial_z\sigma e^{-2\sigma}\partial_{\bar z} .$$

The flat-space heat kernel satisfies the equation

$$\left[\frac{\partial}{\partial t} + \Delta\right] K^t(z,z') = \delta^2(z,z')\delta(t)$$

with explicit solution

$$K^t(z,z') = \frac{1}{4\pi t}e^{-|z-z'|^2/2t}\theta(t) . \tag{B5}$$

With the flat-space heat kernel, we can derive an integral equation for K_n^t,

$$K_n^t = K^t + \int dt' K^{t-t'}V_n K_n^{t'} ,$$

where pairwise integrations over z coordinates are understood, and which is solved by the following formal infinite series:

$$K_n^t = K^t + \int dt' K^{t-t'}V_n K^{t'} + \int dt' \int dt'' K^{t-t'}V_n K^{t'-t''}V_n K^{t''} + \cdots . \tag{B6}$$

1. The diagonal of the heat kernel

On the diagonal $z=z'$, K^t is of order $1/t$. V_n must involve at least one derivative on σ, and two-dimensional rotational invariance requires equal numbers of z and $\bar z$ derivatives, so that Eq. (B6) is easily seen to be an expansion in increasing powers of t, starting with $1/t$. Thus, in the short-time limit, we shall be interested in contributions with 0 derivatives coming from the first term (of order $1/t$) and with one z and one $\bar z$ derivative coming from the second term (of order t^0). Actually, the terms proportional to $\partial_z\sigma\partial_{\bar z}\sigma$ cancel between the second and third terms, as can be seen by a simple calculation not reproduced here. It remains to obtain the terms in $\partial_z\partial_{\bar z}\sigma$, which arise solely from the second term in (B6):

$$K_n^t(z,z)=\frac{1}{4\pi t}+\int_0^t dt'\int d^2z' K^{t-t'}(z,z')\bar{V}_n(z') \times K^{t'}(z',z)+O(t)\ , \qquad \text{(B7)}$$

where the contribution of V_n proportional to $\partial_z\partial_{\bar z}\sigma$ is denoted by $\bar{V}_n$ and is given by

$$\bar{V}_n(z')=2\,|\,z'-z\,|^2(\partial_z\partial_{\bar z}\sigma)\Delta_{z'} -4n(\bar z'-\bar z)(\partial_z\partial_{\bar z}\sigma)\partial_{\bar z'}\ . \qquad \text{(B8)}$$

It is straightforward to evaluate the necessary z integrals:

$$\int d^2z' K^{t-t'}(z,z')\,|\,z'-z\,|^2\Delta_{z'}K^{t'}(z',z) =\frac{1}{2\pi t^3}(t-t')(2t'-t)$$

and

$$\int d^2z' K^{t-t'}(z,z')(\bar z-\bar z')\partial_{\bar z'}K^{t'}(z',z)=\frac{1}{4\pi t^3}(t-t')\ .$$

Putting all together, one finds

$$K_n^t(z,z)=\frac{1}{4\pi t}+\frac{1-3n}{12\pi}\Delta\sigma\ , \qquad \text{(B9)}$$

and taking into account the additional constant Weyl rescalings as given in Eq. (B3) one finds with the help of Eq. (2.31) that

$$K_n^t(z,z)=\frac{1}{4\pi t}+\frac{1-3n}{12\pi}R\ , \qquad \text{(B10)}$$

whence Eq. (2.68).

2. The anomaly in the ghost number current

The ghost number current $j_z=c^z b_{zz}$ is naively analytic, but suffers an anomaly, which we shall now calculate. Observe that the ghost propagator

$$G(z,w)=\langle c^z b_{ww}\rangle \qquad \text{(B11)}$$

is Weyl invariant off the diagonal. The regularized ghost number current may be defined in a reparametrization-invariant way with the help of the heat kernel and a short-time cutoff:

$$\langle j_z\rangle=\int d^2w\sqrt{g(w)}G(z,w)K_2^{\varepsilon}(w,z)\ , \qquad \text{(B12)}$$

where K_2^t is the heat kernel defined above. The anomaly in $\nabla^z j_z$ is a local scalar function of dimension 2, dependent only on the metric and its derivatives. Thus it must be proportional to the curvature. The coefficient may be gotten by calculating the term proportional to $\partial_z\sigma$ is $\langle j_z\rangle$ in the limit where $\varepsilon\to 0$, and then taking the $\partial_{\bar z}$ derivative. To calculate $\langle j_z\rangle$ we use again the expansion (B6), but this time away from the diagonal. As one is interested in a contribution linear in σ, one need only retain the first two terms in Eq. (B6), and the relevant part of V_2 of Eq. (B4) is $\tilde V_2$:

$$\langle j_z\rangle=\int d^2w\, e^{2\sigma(w)}\frac{1}{z-w}\left[K^t(w,z)+\int_0^t dt'\int d^2z' e^{2\sigma(z')}K^{t-t'}(w,z')\tilde V_2(z')K^{t'}(z',z)\right]\ , \qquad \text{(B13)}$$

where

$$\tilde V_2(z')=2(\partial_z\sigma)(z'-z)\Delta'-4n\,\partial_z\sigma\,\partial_{\bar z'}\ .$$

To leading order in t, one finds after some calculation

$$\partial_{\bar z}\langle j_z\rangle=-\tfrac{3}{2}R\ . \qquad \text{(B14)}$$

3. Anomalous contractions

Since Weyl anomalies are purely local, we may work in local isothermal coordinates. The propagator may be regularized at short distances by convolution with the heat kernel, evaluated after a short time ε. The propagator at distinct points is Weyl invariant and given by

$$\Delta^{-1}(z,z')=-\frac{1}{4\pi}\ln|\,z-z'\,|^2,\quad z\neq z'\ . \qquad \text{(B15)}$$

The regularized propagator instead is given by

$$\Delta_t^{-1}(z,z')=\int d^2w\,\Delta^{-1}(z,w)K_0^t(w,z')\ , \qquad \text{(B16)}$$

where K_0^t may be thought of as given by the expansion (B6), but this time evaluated at distinct points w and z'. Contractions will be performed at some fixed point, say $z=0$. Fixing the overall scale so that $\sigma(0)=0$, we may expand the potential W about 0 to yield

$$W=\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{1}{m!(m-k)!}(\partial^m\bar\partial^{k-m}e^{-2\sigma})(-\Delta)\Big|_{z=0} \times z^m\bar z^{k-m}\ . \qquad \text{(B17)}$$

It is now simple algebra to substitute this into Eq. (B6) and evaluate perturbatively. Note that we need only deal with derivatives of the propagator, which simplifies matters. Thus we have

$$\langle x(0)\partial^m x(0)\rangle=4\pi\int d^2z\,K_0^t(0,z;\varepsilon)\partial_{z'}^m\Delta^{-1}(z,z')\Big|_{z'=0} =\int d^2z\,K_0^t(0,z;\varepsilon)(m-1)!z^{-m}\ . \qquad \text{(B18)}$$

If $\langle x(0)\partial^m x(0)\rangle_p$ denotes the contribution to the above expression of the terms in (B6) with p interaction factors $W_{z_1},\ldots,W_{z_p}$, each given by a formula such as (B17) with exponents k_i and m_i, we must have from invariance under $z\to e^{i\theta}z$ and $z_i\to e^{i\theta}z_i$,

$$\sum_{i=1}^{p}(2m_i-k_i)=m\ .$$

On the other hand, simple power counting shows that

$$\sum_{i=1}^{p} k_i \le m \ ,$$

otherwise the limit as $\varepsilon \to 0$ will vanish. Thus $k_i = m_i$ for all i, and $\sum_{i=1}^{p} m_i = m$. This means that only ∂ derivatives on σ will appear in the final answer, and no $\bar\partial$. Furthermore, the interaction $W_{z_i} = (1-e^{-2\sigma})\Delta_{z_i}$ applied to $K^{t_i - t_{i+1}}(z_i, z_{i+1})$ produces a term

$$(e^{-2\sigma}-1)\frac{\partial}{\partial \tau_i} K^{\tau_i}(z_i, z_{i+1}) \ ,$$

where one sets $\tau_i = t_i - t_{i+1}$ after differentiation has been performed. Substituting in (B6) and carrying out the Gaussian integrals yields Eq. (8.11).

APPENDIX C: THE DIAGONAL OF THE SUPER HEAT KERNEL

We shall now compute the heat kernel for the superspace Laplacian $(\square_n^-)^2$, satisfying the equation

$$\left[\frac{\partial}{\partial t} + (\square_n^-)^2\right] \mathcal{K}_n^t(z,z';\theta,\theta') = \delta^2(z,z')\delta^2(\theta,\theta')\delta(t) \tag{C1}$$

with solution

$$\mathcal{K}_n^t = \theta(t) e^{-t(\square_n^-)^2} \ .$$

We are specially interested in calculating elements on the diagonal: $z=z'$, $\theta=\theta'$ for short times t. This problem is entirely local, so we may perform a super-reparametrization and local U(1) transformation to render the geometry superconformally flat. From Eq. (3.101) we find the super Weyl dependence of $(\square_n^-)^2$,

$$\begin{aligned}\square_n^- V &= \mathcal{D}_+^{n-1/2}\mathcal{D}_-^n V \\ &= e^{(n-1)\Sigma}\hat{\mathcal{D}}_+^{n-1/2} e^{-2n\Sigma}\hat{\mathcal{D}}_-^n e^{n\Sigma} V \ .\end{aligned} \tag{C2}$$

Since the quantities with hats are taken with respect to flat supergeometry, we may replace $\hat{\mathcal{D}}_\pm^n = \partial_\pm$, so that

$$\begin{aligned}\square_n^- V &= e^{(n-1)\Sigma}\partial_+[e^{-n\Sigma}(\partial_- V + n\partial_-\Sigma V)] \\ &= e^{-\Sigma}(\partial_+\partial_- V + n\partial_+\partial_-\Sigma V - n\partial_-\Sigma\partial_+ V \\ &\quad - n\partial_+\Sigma\partial_- V - n^2\partial_+\Sigma\partial_-\Sigma V)\end{aligned}$$

and its square is a very lengthy expression which can be worked out in a straightforward manner. Actually, we shall be interested only in the contribution that has at most two superderivatives on Σ fields in total, the other contributions tending to 0 at $t \to 0$. The two derivatives, moreover, must be one ∂_+ and one ∂_- in order to get a nonzero answer. With this restriction, we get

$$\begin{aligned}(\square_n^-)^2|_{\text{restr.}} V = e^{-2\Sigma}[&\partial_+\partial_-\partial_+\partial_- V - n\partial_-\Sigma\partial_+\partial_-\partial_+ V - (2n-1)\partial_-\partial_+\Sigma\partial_+\partial_- V + \partial_+\Sigma\partial_+\partial_-^2 V \\ &- n^2\partial_+\Sigma\partial_-\Sigma\partial_+\partial_- V - (n-1)\partial_-\Sigma\partial_+^2\partial_- V + n(n-1)\partial_-\Sigma\partial_+\Sigma\partial_+\partial_- V \\ &- n(n+1)\partial_+\Sigma\partial_-\Sigma\partial_-\partial_+ V - (n^2-1)\partial_+\Sigma\partial_-\Sigma\partial_+\partial_- V] \ .\end{aligned} \tag{C3}$$

The contributions from the terms quadratic in Σ are easily seen to vanish in computing the kernel at coincident points, so that we are effectively left with

$$\begin{aligned}(\square_n^-)^2 V = e^{-2\Sigma}[&-\partial_+^2\partial_-^2 V + \partial_-\Sigma\partial_-\partial_+^2 V + \partial_+\Sigma\partial_+\partial_-^2 V \\ &+ (2n-1)\partial_+\partial_-\Sigma\partial_+\partial_- V] \ .\end{aligned} \tag{C4}$$

As in the bosonic case, we may omit the scalings by constant Σ and easily restore them at the end. Thus, without loss of generality, we may assume that $\Sigma=0$ at the point of interest (z',θ'). Using the fact that $-\partial_+^2\partial_-^2 = \frac{1}{2}\Delta$, where the flat-space Laplacian is given by $\Delta = -2\partial_z\partial_{\bar z}$, we rewrite the equation defining the heat kernel as

$$\left[\frac{\partial}{\partial t} + \Delta - \mathcal{V}_n\right] \mathcal{K}_n^t(z,z';\theta,\theta') = \delta^2(z,z')\delta^2(\theta,\theta')\delta(t) \ . \tag{C5}$$

The part $\bar{\mathcal{V}}_n$ relevant to the diagonal of $\mathcal{K}_n^t$ in $\mathcal{V}_n$ is given by

$$\begin{aligned}\bar{\mathcal{V}}_n = (1-e^{-2\Sigma})\Delta - e^{-2\Sigma}[&\partial_-\Sigma\partial_+^2\partial_- - \partial_+\Sigma\partial_+\partial_-^2 \\ &-(2n-1)\partial_+\partial_-\Sigma\partial_+\partial_-] \ .\end{aligned} \tag{C6}$$

The flat-space heat kernel satisfies

$$\left[\frac{\partial}{\partial t} + \Delta\right] \mathcal{K}^t(z,z';\theta,\theta') = \delta^2(z,z')\delta^2(\theta,\theta')\delta(t) \ ,$$

which is solved by

$$\mathcal{K}^t(z,z';\theta,\theta') = K^t(z,z')\delta^2(\theta,\theta')$$

where the usual flat-space heat kernel K^t is given by Eq. (B5). The major distinction from the bosonic case is that $\mathcal{K}^t$ vanishes on the diagonal because $\delta^2(\theta,\theta)=0$, which implies that there is *no* term in t^{-1} in the expansion of $\mathcal{K}_n^t$ in terms of small t. In analogy with Eqs. (B6) and (B7) we readily find that the relevant contributions are

$$\begin{aligned}\mathcal{K}_n^t(z,z;\theta,\theta) = \int_0^t dt' \int d^2z' d^2\theta' \mathcal{K}^{t-t'}(z,z';\theta,\theta') \\ \times \bar{\mathcal{V}}_n(z',\theta')\mathcal{K}^{t'}(z',z;\theta',\theta) \\ + O(t) \ .\end{aligned}$$

It is convenient first to work out the θ' integral:

$$\begin{aligned}\int d^2\theta' \delta^2(\theta,\theta')\bar{\mathcal{V}}_n(z',\theta')\delta^2(\theta',\theta) &= \bar{\mathcal{V}}_n(z',\theta')\delta^2(\theta',\theta)|_{\theta'=\theta} \\ &= -(2n-1)\partial_+\partial_-\Sigma\end{aligned} \tag{C7}$$

so that

$$\mathcal{K}_n^t(z,z;\theta,\theta) = -(2n-1)\partial_+\partial_-\Sigma \times \int_0^t dt' \int d^2z' K^{t-t'}(z,z')K^{t'}(z',z) = -\frac{2n-1}{2\pi}\partial_+\partial_-\Sigma \ . \quad \text{(C8)}$$

Restoring the factor of constant Σ scalings and using Eq. (3.20) for R_{+-}, we find

$$\mathcal{K}_n^t(z,z;\theta,\theta) = -i\frac{2n-1}{4\pi}R_{+-} + O(t) \ . \quad \text{(C9)}$$

An analogous calculation for $\Box_n^+$ will give a coefficient $2n+1$ in front of R_{+-} instead of $2n-1$.

APPENDIX D: RIEMANN VANISHING AND ABEL THEOREMS

In this appendix we shall present some of the methods of the theory of Riemann surfaces, and in particular provide proofs for some of the properties of the period matrix and theta functions used in the text. The key tool is Green's theorem on a cut Riemann surface. Recall that we can choose a homology basis satisfying the intersection pairings (3.5), and that representatives of the cycles A_I, B_I, $I=1,\ldots,h$ in the basis may be chosen as in Fig. 10. It is not difficult to see that the surface M can then be cut along these cycles in a $4h$ polygonal region (see Fig. 11).

Here we have labeled by $+$ and $-$ the oriented edges of each cycle, and the oriented boundary of the cut Riemann surface M_{cut} is

$$\partial M_{\text{cut}} = -\sum A_I^+ - \sum B_I^+ + \sum A_I^- + \sum B_I^- \ .$$

The advantage of working with a cut surface is that any holomorphic differential ω can be integrated, $\omega = dg$, where g is a holomorphic function on M_{cut} with, however, different values on the $+,-$ edges of each cycle. If P_+ and P_- are the corresponding points on the $+,-$ edges of, say, the cycle B, then we may join them by the dotted path as in Fig. 22. Since this path can be deformed to A_1, we obtain the important identity

$$g(P_-) - g(P_+) = \int_{A_1} \omega \ . \quad \text{(D1)}$$

We conclude these preliminaries by observing that a holomorphic differential ω is automatically closed as a 1-form, i.e., $d\omega$ must be 0.

It is now easy to derive Riemann's bilinear relations. Let ω_J, ω_K be two elements of the homology basis, write $\omega_J = dg_J$ on the cut Riemann surface, and apply Green's theorem. The result is

$$\begin{aligned} 0 = \int \omega_J \wedge \omega_K = \int d(g_J\omega_K) &= \sum_{I=1}^h \int_{-B_I^+ \cup B_I^-} \int g_J\omega_K + \int_{-A_I^+ \cup A_I^-} g_J\omega_K \\ &= \sum_{I=1}^h \oint_{A_I}\omega_J \oint_{B_I}\omega_K - \oint_{B_I}\omega_J \oint_{A_I}\omega_K = \Omega_{JK} - \Omega_{KJ} \ , \end{aligned} \quad \text{(D2)}$$

showing that Ω is indeed symmetric. Next let $\omega = \sum_{J=1}^h c_J\omega_J$ by any holomorphic differential, which we again write as $\omega = dg$. The same arguments yield

$$\begin{aligned} 0 < \frac{1}{2i}\int \bar{\omega} \wedge \omega = \frac{1}{2i}\int d(\bar{g}\omega) &= \sum_{I=1}^h \int_{-B_I^+ \cup B_I^-} \bar{g}\omega + \int_{-A_I^+ \cup A_I^-} g\omega \\ &= \sum_{I=1}^h \oint_{A_I}\bar{\omega} \oint_{B_I}\omega - \oint_{B_I}\bar{\omega} \oint_{A_I}\omega = \text{Im} \sum_{1 \le J,K \le h} \Omega_{JK}c_J\bar{c}_K \ , \end{aligned} \quad \text{(D3)}$$

and the second Riemann bilinear relation is established.

We now provide a proof of Abel's theorem characterizing the divisors of meromorphic functions as the kernel of Abel's map. Let f be a meromorphic function on M, and let $z_1,..,z_k, w_1,\ldots,w_k$ be its zeros and poles. Then df/f is an Abelian differential of the third kind with simple poles and residues ± 1 at these points, and thus can be expressed as

$$df/f = \sum_{i=1}^n \omega_{z_i w_i} + \sum_{J=1}^h c_J\omega_J \ . \quad \text{(D4)}$$

Here ω_{z_i,w_i} are the normalized meromorphic differentials introduced in Sec. VI.F, and c_J are some complex scalar coefficients. Since the integral of df/f over any closed cycle must be a multiple of $2\pi i$, we deduce that

$$\begin{aligned} 2\pi i n_K &= \oint_{A_K} df/f = c_K \ , \\ 2\pi i m_K &= \oint_{B_K} df/f = 2\pi i \sum_{j=1}^m \int_{w_j}^{z_j}\omega_K + \sum_{J=1}^h c_J\Omega_{JK} \ , \end{aligned} \quad \text{(D5)}$$

for some integers n_K and m_K. This just means that $I(\sum_{j=1}^n z_j - \sum_{j=1}^n w_j)$ belongs to the lattice $\mathbf{Z}^h + \Omega\mathbf{Z}^h$. The converse has already been established via theta-function formulas, but we can also obtain it easily at this point by reversing the above arguments. Indeed if $I(\sum_{j=1}^n z_j - \sum_{j=1}^n w_j) \equiv 0$ then Eq. (D5) defines integers (n_K, m_K) out of which a differential ω can be constructed as in Eq. (D4) with periods multiples of $2\pi i$. In particular, $f(z) = \exp(\int_{z_0}^z \omega)$ is well defined on M and has the desired zeros and poles.

Finally we come to the zeros of the theta functions $\vartheta(\zeta,\Omega)$. Set

$$f(z)=\vartheta\left[\int_{z_0}^{z}\omega+\zeta,\Omega\right] . \tag{D6}$$

If f does not vanish identically as a function of z, df/f will be holomorphic away from the zeros of f, and hence we may apply Green's theorem to the cut Riemann surface with tiny disks S_i around the zeros of f removed:

$$\begin{aligned}0&=\int_{M_{\rm cut}\setminus US_i}d(df/f)\\&=-\sum\int_{\partial S_i}df/f+\sum_{J=1}^{h}\int_{-A_J^{+}+A_J^{-}}df/f\\&\quad+\int_{-B_J^{+}\cup B_J^{-}}df/f .\end{aligned} \tag{D7}$$

The last term on the right-hand side is 0, since f is invariant under A periods. Under the B_K period df/f changes by $-2\pi i\omega_K$. Since the integrals over ∂S_i just produce $2\pi i$ (# of zeros of f), it follows that f has exactly h zeros.

To establish Eq. (6.37) we again use the cut Riemann surface with base point P (see Fig. 11) to write the Abelian differentials ω_K as $\omega_K=dg_K$. The jumps of g_K across A_L and B_L are then $-\Omega_{KL}$ and δ_{KL}, respectively. Green's theorem implies

$$\begin{aligned}0&=\int_{M\setminus US_i}d(g_K df/f)\\&=-\sum\int_{\partial S_i}g_K df/f+\sum_{L=1}^{h}\int_{(-B_L^{+})\cup B_L^{-}}g_K df/f+\sum_{L=1}^{h}\int_{(-A_L^{+})\cup A_L^{-}}g_K df/f\\&=-2\pi i\sum g(z_i)+\sum_{L=1}^{h}\delta_{KL}\int_{B_L}df/f-\sum_{L=1}^{h}\Omega_{KL}\int_{A_L}df/f+\sum 2\pi i\int_{A_L}g_K\omega_L+2\pi i\Omega_{KL}\int_{A_L}\omega_L\\&\equiv-2\pi i\sum g(z_i)+\left[-\pi i\Omega_{KK}-2\pi i\int_{z_0}^{P}\omega_K-2\pi i\zeta_K\right]+2\pi i\int_{A_L}g_K\omega_L\end{aligned} \tag{D8}$$

up to lattice points on $Z^h+\Omega Z^h$. Here P is the common point to all the basis cycles. Taking $P=z_0$, which is no loss of information, we recognize this relation as the desired relation, with the right definition of Δ as in Eq. (6.37).

It is now easy to deduce the zero set of $\vartheta(\zeta,\Omega)$ itself if we assume the characterization of those ζ for which f vanishes identically to be of the form $\zeta=I(w_1+\cdots+w_h)-\Delta$, where $w_1,\ldots,w_h$ is the set of poles of a nonconstant meromorphic function g. In fact, if ζ falls within this case, we may arrange for g to vanish at z_0. By Abel's theorem $I(\text{zeros of } g)\equiv I(w_1+\cdots+w_h)$. Substituting in the formula for ζ and noting that g must have zeros, one of which is z_0, gives $\zeta=I(z_1+\cdots+z_{h-1})-\Delta$. When z does not fall in this case, the first part of Riemann's vanishing theorem expresses ζ as $I(z_1+\cdots+z_h)-\Delta$, and again z_0 must be among these points.

APPENDIX E: THETA FUNCTIONS FOR THE TORUS

Ordinary theta functions, together with their properties, will be listed here. All four theta functions can be expressed in terms of a single one as translations thereof by a half-period:

$$\begin{aligned}&\vartheta_{00}(z,\tau)=\vartheta_3(z,\tau)=\vartheta(z,\tau) ,\\&\vartheta_{01}(z,\tau)=\vartheta_4(z,\tau)=\vartheta(z+\tfrac{1}{2},\tau) ,\\&\vartheta_{10}(z,\tau)=\vartheta_2(z,\tau)=e^{\pi i\tau/4+\pi iz}\vartheta(z+\tfrac{1}{2},\tau) ,\\&\vartheta_{11}(z,\tau)=\vartheta_1(z,\tau)=e^{\pi i\tau/4+\pi iz}\vartheta\left[z+\tfrac{1}{2}+\frac{\tau}{2},\tau\right] ,\end{aligned} \tag{E1}$$

where ϑ may be defined through

$$\vartheta(z,\tau)=\sum_{n=-\infty}^{\infty}e^{i\pi n^2+2\pi niz} . \tag{E2}$$

This series is absolutely convergent for $\mathrm{Im}\tau>0$, and $\theta(z,\tau)$ is holomorphic in z. Under shifts by the periods, we have

$$\begin{aligned}&\vartheta_{ab}(z+1,\tau)=(-1)^a\vartheta_{ab}(z,\tau) ,\\&\vartheta_{ab}(z+\tau,\tau)=(-)^b e^{-\pi i\tau-2\pi iz}\vartheta_{ab}(z,\tau) ,\end{aligned} \tag{E3}$$

whereas shifts under half-periods produce

$$\begin{aligned}&\vartheta_{ab}(z+\tfrac{1}{2},\tau)=(-)^{ab}\vartheta_{a(b+a)}(z,\tau) ,\\&\vartheta_{ab}(z+\tfrac{1}{2}\tau,\tau)=(-i)^b e^{-\pi i\tau/4-\pi iz}\vartheta_{(a+b)b}(z,\tau) ,\end{aligned} \tag{E4}$$

where addition of a and b is understood modulo 2. Their next fundamental property is their behavior under modular transformations:

$$\begin{aligned}&\vartheta_{ab}(z,\tau+1)=e^{\pi ia/4}\vartheta_{a(b+a+1)}(z,\tau) ,\\&\vartheta_{ab}\left[\frac{z}{\tau},-\frac{1}{\tau}\right]=(-)^{ab}\sqrt{-i\tau}\,e^{\pi iz^2/\tau}\vartheta_{ba}(z,\tau) .\end{aligned} \tag{E5}$$

There also exist famous infinite product representations for these functions:

$$\vartheta_{0b}(z,\tau)=\prod_{n=1}^{\infty}(1-e^{2\pi in\tau})\left\{1-\exp\left[2\pi i\left[n\tau+z-\frac{1-b}{2}-\frac{\tau}{2}\right]\right]\right\}\left\{1-\exp\left[2\pi i\left[n\tau-z-\frac{1-b}{2}-\frac{\tau}{2}\right]\right]\right\},$$

$$\vartheta_{1b}(z,\tau)=i^b e^{\pi i\tau/4}[e^{i\pi z}+(-)^b e^{-i\pi z}] \times\prod_{n=1}^{\infty}(1-e^{2\pi in\tau})\left\{1-\exp\left[2\pi i\left[n\tau+z-\frac{1-b}{2}\right]\right]\right\}\left\{1-\exp\left[2\pi i\left[n\tau-z-\frac{1-b}{2}\right]\right]\right\}. \tag{E6}$$

Let us also mention an example of a Riemann identity:

$$\vartheta_{00}(x)\vartheta_{01}(y)\vartheta_{10}(u)\vartheta_{11}(v)-\vartheta_{00}(y)\vartheta_{01}(x)\vartheta_{10}(v)\vartheta_{11}(u)-\vartheta_{00}(u)\vartheta_{01}(v)\vartheta_{10}(x)\vartheta_{11}(y) +\vartheta_{00}(v)\vartheta_{01}(u)\vartheta_{10}(y)\vartheta_{11}(x)=2\vartheta_{00}(x_1)\vartheta_{01}(y_1)\vartheta_{10}(u_1)\vartheta_{11}(v_1)\ , \tag{E7}$$

with

$$x_1=\tfrac{1}{2}(x+y+u+v),\quad y_1=\tfrac{1}{2}(x+y-u-v)\ ,$$
$$u_1=\tfrac{1}{2}(x-y+u-v),\quad v_1=\tfrac{1}{2}(x-y-u+v)\ .$$

It may be reexpressed as

$$\sum_{a,b}(-1)^{a+b}\vartheta_{ab}(x)\vartheta_{ab}(y)\vartheta_{ab}(u)\vartheta_{ab}(v) =2\vartheta_{11}(x_1)\vartheta_{11}(y_1)\vartheta_{11}(u_1)\vartheta_{11}(v_1)\ . \tag{E7'}$$

Here we have suppressed the common τ dependence. In particular, upon setting $y=u=v=0$ one gets

$$\vartheta_{11}(x)\vartheta_{00}(o)\vartheta_{01}(o)\vartheta_{10}(o) =2\vartheta_{11}(\tfrac{1}{2}x)\vartheta_{00}(\tfrac{1}{2}x)\vartheta_{01}(\tfrac{1}{2}x)\vartheta_{10}(\tfrac{1}{2}x)\ . \tag{E8}$$

For a list of Riemann identities, see Mumford (1983), Lecture I, pp. 16–23.

Also of importance are the "theta constants," obtained by setting $z=0$ in the above. As modular forms, they have profound significance in number theory. Especially well known is

$$\vartheta'_{11}(o,\tau)=-2\pi\eta(\tau)^3\ , \tag{E9}$$

where the Dedekind eta function is given by

$$\eta(\tau)=e^{i\pi\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi in\tau})\ . \tag{E10}$$

Under a modular transformation, $\eta(\tau)$ transforms according to the Jacobi rule with $\varepsilon^4=1$,

$$\eta\left[\frac{a\tau+b}{c\tau+d}\right]=\varepsilon(cz+d)^{1/2}\eta(\tau)\ .$$

In addition one has the Jacobi relations

$$\vartheta'_{11}(o,\tau)=-\pi\vartheta_{00}(o,\tau)\vartheta_{01}(o,\tau)\vartheta_{10}(o,\tau)\ ,$$
$$\vartheta^4_{00}(o,\tau)=\vartheta^4_{01}(o,\tau)+\vartheta^4_{10}(o,\tau)\ . \tag{E11}$$

We take the opportunity to recall the Poisson resummation formula,

$$\sum_{n=-\infty}^{\infty}e^{-\lambda 2\pi^2 n^2}=\left[\frac{1}{2\pi\lambda}\right]^{1/2}\sum_{n=-\infty}^{\infty}e^{-n^2/2\lambda}\ ,$$

or more generally, if $\hat{f}$ is the Fourier transform of f,

$$\hat{f}(n)=\int_{-\infty}^{\infty}dx\, e^{-inx}f(x)\ ,$$

then

$$\sum_{n\in\mathbf{Z}}f(2\pi n)=\frac{1}{2\pi}\sum_{n\in\mathbf{Z}}\hat{f}(n)\ .$$

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