Astrophysical blastwaves

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The authors present a general discussion of spherical, nonrelativistic blastwaves in an astrophysical context. A variety of effects has been included: expansion of the ambient medium, gravitation, and an embedded fluid of clouds capable of exchanging mass, energy, or momentum with the medium. The authors also consider cases of energy injection due either to a central source or to detonations. Cosmological solutions are extensively treated. Most attention is devoted to problems in which it is permissible to assume self-similarity, as in the prototype Sedov-Taylor blastwave. A general virial theorem for blastwaves is derived. For self-similar blastwaves, the radius varies as a power of the time, $R_s \propto t^{\eta}$. The integral properties of the solution are completely specified by two dimensionless numbers measuring the relative importance of thermal and kinetic energy. The authors find certain exact kinematical relations and a variety of analytic approximations to determine these numbers with varying degrees of accuracy. The approximations may be based on assumptions about the internal density distributions (e.g., shell-like), pressure distribution, or velocity distribution. In many cases exact conditions from, for example, boundary conditions or other constraints may be used to determine unspecified parameters. One new set of exact integral constraints has been derived. The various approximation schemes are tested with known solutions. The authors find that for blastwaves in which the flow extends to the origin, the assumption that the internal velocity is linear with radius is reasonably accurate. For blastwaves in which an interior vacuum develops, the equally simple approximation of constant interior velocity is accurate. These lowest-order approximations are shown to give numerical coefficients in the relation $R = \text{const} \times t^{\eta}$ which are accurate to about 1–2 %. The higher-order approximations show an accuracy that in some cases equals that obtained, to date, by direct numerical integration. In addition to the new methods presented, the authors have obtained new results for evaporative blastwaves, impeded blastwaves, blastwaves with cloud crushing, bubbles, cosmological blastwaves (self-similar and non-self-similar, radiative and nonradiative), blastwaves in a wind, and detonations. Some of the new results found are exact. Included are the radiative, cosmological self-similar solution, appropriate to the universe ($z > 10$) when inverse Compton cooling is efficient [$\ln R = \text{const} + (\ln t)(15+\sqrt{17})/24$], and certain properties of the solutions mentioned above. In a series of appendixes several related issues are treated: energy conservation for multicomponent fluid in an expanding universe; central and edge derivatives of physical quantities in self-similar adiabatic blastwaves; shock jump conditions including energy input (detonations), and a variety of other matters.

CONTENTS

LIST OF SYMBOLS[1]

a	cloud radius [Eq. (5.30)]; a constant [Eq. (9.87)].
a_x	coefficient in two-power approximation for x [Eq. (4.9)].
b	a constant [Eqs. (4.31) and (9.88)].
c	a constant [Eq. (9.89)]; speed of light (Secs. IX and X).
C, C_1	isothermal sound speed [Eq. (2.1)], behind shock [Eq. (10.1)].
C_d	isothermal sound speed of dark matter [Eq. (9.5)].
C_R	integration constant [Eq. (9.93)].
D	$(b^2-4ac)^{1/2}$ [Eq. (9.91)].
E	total energy of the intercloud gas inside the blastwave, including the energy of the swept-up ambient medium: $E=E_a+E_b$ [Eqs. (3.14) and (A1)]. For a cold, stationary medium $E=E_b$.
$\dot{E}$	dE/dT [e.g., Eq. (A2)].
E_t, E'	total energy of all components in blastwave [Eqs. (A16) and (9.74)], of components other than intercloud gas, $E'=E_t-E$ [Eq. (A16)].
E_0, E_{51}, E_{61}	initial energy of blastwave, $E_0=E_b(t=0)$, [Eq. (A21)]; in units of 10^{51} erg [Eq. (6.63)], 10^{61} erg [Eq. (9.39)].
E_a	energy of ambient intercloud gas in absence of blastwave [Eqs. (3.14) and (A18)].
E_b	energy due to blastwave [Eqs. (3.31) and (A19)].
E_{det}	detonation energy [Eq. (A21)].
E_{in}	energy injected by central source in the time interval $(0,t)$ [Eq. (7.1)].
E_K	kinetic energy in blastwave [Eq. (3.16)].
E_{rad}	energy radiated by blastwave [see (A5)].
E_{th}	thermal energy in blastwave [Eq. (3.18)].
f	cloud filling factor = fraction of volume occupied by clouds [Eq. (2.1)]; may depend on position.
f_0	value of f just outside shock [Eq. (2.6)].
f_i, f_i'	mean values of f inside blastwave [Eqs. (2.10) and (D6)].
F	$(\gamma+1)K_{11}/K_{02}$ [Eq. (4.77a)].
g, G	gravitational acceleration [Eq. (2.2)], constant [Eq. (2.12)].
h	parameter characterizing cosmological expansion [Eq. (9.1)]; $h=1$ $(\Omega\ll 1)$, $h=\frac{2}{3}$ $(\Omega=1)$.

[1]Equations in or near which symbol is defined are given in square brackets.

h_* — $H_0/(100\ \mathrm{km\,s^{-1}\,Mpc^{-1}})$ [Eq. (9.32)].

H, H_0 — Hubble constant at arbitrary time [Eq. (9.3)], now [Eq. (9.4)].

I_m — form factor for blastwave luminosity [Eq. (4.95)].

j_x — for arbitrary hydrodynamic variable x, the power law for $x(\tilde{M})$: $j_x \equiv l_x/l_M$ [Eq. (4.54)]. Special value: $j_\lambda = 1/l_M$ [Eq. (4.52)].

k_x — for arbitrary variable x characterizing blastwave; $x \propto R^{-k_x}$ [Eq. (3.7)]. Relation to time dependence t^{η_x}: $k_x = -\eta_x/\eta$ [Eq. (7.4)], except $\rho_0(r,t) \propto r^{-k_{\rho_0}} t^{\eta_{\rho_0}}$ [Eq. (3.25)]. Special cases: $k_v = \eta^{-1} - 1$ [Eq. (3.13)], $k_P = k_E + 3$, $k_M = k_\rho - 3$ [Eq. (3.29)].

$k_{\rho\,\mathrm{crit}}$ — for $k_\rho = k_{\rho\,\mathrm{crit}} \equiv (7-\gamma)/(\gamma+1)$ and $k_E = 0$, linear velocity approximation exact; central vacuum forms for $k_\rho > k_{\rho\,\mathrm{crit}}$ [Eq. (4.18)].

K_{mn} — dimensionless moment of $r^m v^n$ [Eq. (2.9)].

$K_m(\mathrm{LVA})$ — $K_{m,2-m}$ in the linear velocity approximation [Eq. (4.16)].

$K_{mn}(\mathrm{OPA})$ — K_{mn} in one-power approximation [Eq. (4.24)].

K_m — $K_{m,n}$ in the shell approximation [Eq. (4.43)].

$K_{n,0}^{\mathrm{ev}}$ — moment of evaporated gas [Eq. (C4)].

K_P — dimensionless moment of pressure [Eq. (D7)].

l — power law in cooling function: $\Lambda \propto \bar{\rho}^{\,l}$ [Eq. (6.1)].

l_x — for arbitrary variable x, approximate $x \propto (r/R_s) l_x$ inside blastwave [Eq. (4.8)].

l_{x2} — power-law index in Kahn approximation [Eq. (4.34)].

l_{xi} — power law for asymptotic behavior at inner edge of hollow blastwave [Eq. (B37)].

L, L_{rad} — radiative luminosity of blastwave [Eq. (4.93)], of outer radiative shock [Eq. (7.14)].

L_{in} — rate of energy injection by central source [Eq. (7.2)].

L_1 — a constant proportional to the rate of energy injection: $L_{\mathrm{in}} = L_1 t^{\eta_{\mathrm{in}}-1}$ [Eq. (7.9)].

m — power law in cooling function: $\Lambda \propto T^m$ [Eq. (4.94)].

$\dot{m}$ — rate of mass flow from a cloud to the intercloud medium [Eq. (2.1)].

M — mass of intercloud gas inside blastwave [Eq. (2.1)].

M_a, M_{ev} — swept-up [Eq. (3.27)], evaporated [Eq. (C7)] mass inside blastwave.

M_1 — integration constant [Eq. (6.27)].

M_H — characteristic mass for blastwave in a wind [Eq. (8.5)].

M_t, M' — total mass of all components in blastwave [Eq. (A8)], of all but intercloud gas, $M' = M_t - M$ [Eq. (A8)].

M_i — mass of ith component in blastwave: $i=1$, intercloud gas ($M = M_1$); $i=2$, matter other than intercloud gas, assumed to have same spatial distribution; $i=3$, central mass ($M' = M_2 + M_3$) [Eq. (A22)].

$\dot{M}_{\mathrm{in}}$ — rate at which mass is injected into blastwave from central source [Eqs. (2.15) and (7.23)].

$\dot{M}_w$ — mass loss rate in wind prior to blastwave [Eq. (4.81)].

$\dot{M}_{\mathrm{ev}}$ — rate at which mass is injected into blastwave by cloud evaporation [Eq. (C3)].

$\mathcal{M}$, $\mathcal{M}_s$ — isothermal Mach number of shock v_s/C_0 [Eq. (6.20)], in saturated evaporation [Eq. (5.34)].

$\mathcal{M}_a$, $\mathcal{M}_{a1}$ — adiabatic Mach number of shock [Eq. (E7)], of postshock flow [Eq. (E8)].

n — density of hydrogen nuclei ρ/μ_H [Eq. (4.93)].

$\bar{n}$ — mean density of hydrogen nuclei in blastwave $M/\mu_H V$ [Eq. (4.94)].

p_1 — momentum of radiative blastwave [Eqs. (6.5a) and (8.15)].

P, P_0, P_1 — pressure [Eq. (2.1)], ambient pressure [Eq. (2.1)], post-shock pressure [Eq. (2.1)].

$\bar{P}$, $\bar{P}_c'$ — mean pressure in blastwave [Eq. (2.11)], just after radiative cooling leads to shell formation [Eq. (6.18)].

P^*, P_1^* — logarithmic derivative of pressure [Eq. (B13)], just inside shock [Eq. (B12)].

P_{cr} — cosmic-ray pressure [Eq. (4.88)].

Q — parameter describing cloud evaporation rate per unit volume [Eq. (5.15a)].

Q_{av} — average value of Q [Eq. (5.15b)].

r — radial coordinate from center of blastwave [Eq. (2.1)].

$R_s(t)$ — blastwave radius at time t [Eq. (2.6)].

$R_s(1)$ — blastwave radius at fiducial epoch [Eq. (3.7)].

R — normalized blastwave radius $R_s/R_s(1)$ [Eq. (3.7)].

R_{ev} — for $R_s < R_{\mathrm{ev}}$, evaporated mass is important [Eq. (5.25)].

R_{sat} — for $R_s < R_{\mathrm{sat}}$, evaporation rate saturates [Eq. (5.36)].

R_c — at $R_s = R_c$, radiative losses from the shocked intercloud gas become impor-

	tant [Eqs. (6.16) and (6.63)].
R_c^*	$R_c/\lambda_{\rm cl}$ [Eq. (6.25)].
R_d, R_g	radius of outer caustic in dark matter, of gas shock, in a combined dark matter/gas blastwave [Eq. (9.72)].
$R_{\rm eq}$	at $R_s=R_{\rm eq}$, $\bar{P}$ equals ambient pressure, [Eq. (6.18)].
$R_{\rm stop}$	maximum radius during overshoot of radiative solution [Eq. (6.21)].
$R_{\rm im}$	for $R_s>R_{\rm im}$, blastwave impeded by clouds [Eq. (6.39)].
$R_{\rm eq,im}$	for $R_s=R_{\rm eq,im}$, $\bar{P}=P_0$ for impeded blastwave [Eq. (6.48)].
$R_{\rm cc}$	for $R_s>R_{\rm cc}$, clouds completely crushed [Eq. (6.53)].
R_{cw}	for $R_s>R_{cw}$, cloud-crushing losses are significant [Eq. (6.61)].
$R_{{\rm eq},cw}$	at $R_s=R_{{\rm eq},cw}$, $\bar{P}=P_0$ for cloud-crushing solution [Eq. (6.62)].
s	entropy P/ρ^γ [Eq. (4.12)].
t	age of blastwave [Eq. (2.1)].
t_H	for blastwave in a wind, time at which injected energy is comparable to energy of swept-up wind [Eq. (8.8)].
T, $\bar{T}$	temperature [Eq. (4.93)], mass-weighted mean temperature in blastwave [Eq. (5.1)].
u	internal energy (Appendix A) [Eq. (A2)]; shock parameter (Appendix E) [Eq. (E15a)].
v	velocity of intercloud gas [Eq. (2.1)].
v_s, v_1	shock velocity [Eq. (2.1)], post-shock fluid velocity [Eq. (2.1)].
v_H, v_w	velocity of ambient medium [Eq. (2.1)], for a particular case of wind [Eq. (4.81)].
$v_{\rm in}$	velocity of mass injected from central source [Eqs. (2.15) and (7.2)].
$v_{\rm cl}$	velocity of cloud-crushing shock [Eq. (6.52)].
v^*, v_1^*	logarithmic derivative of v [Eq. (B11)], just inside shock [Eq. (B16)].
v_λ	velocity of the self-similar coordinate, $v_\lambda \equiv v_\lambda s$ [Eq. (B5)].
Δv	shock velocity relative to Hubble flow [Eq. (9.15)].
V	volume [Eq. (2.6)].
w, w'	gravitational energy coefficients [Eqs. (7.16) and (A34)].
w_{ij}, w'_{ij}	gravitational interaction energy coefficients for components i,j [Eqs. (A25) and (A35)].
$w_{\rm cr}$	fraction of post-shock pressure in cosmic rays [Eq (4.86)].
W	gravitational energy of intercloud gas [Eqs. (2.12) and (A9)].
$W_{\rm int}$, $W_{\rm self}$	gravitational interaction energy [Eq. (A8b)], self-energy [Eq. (A8a)]
ΔW	gravitational energy transferred out of gas [Eq. (A12)].
x	age of blastwave/age of universe [Eq. (9.40)].
x, x_1	arbitrary variable [Eq. (3.7)], just behind shock [Eq. (B1)].
$x(1)$	arbitrary variable at fiducial radius $R_s(1)$ [Eq. (3.7)].
$\tilde{x}$, x^*	normalized x [Eq. (4.6)], logarithmic derivative of x [Eq. (4.10)].
$x^\dagger$	$(1-\nu)x^*$ [Eq. (B29)].
x_i	arbitrary variable evaluated at inner edge of hollow blastwave [Eq. (4.48)].
y	normalized relative shock velocity in Hubble flow [Eq. (9.85)].
y_1, y_2	defined in [Eq. (9.91)]; $y_1>y_2$.
z	Redshift [Eq. (10.18)].
α	$v_s/\bar{C}=(\bar{\rho}v_s^2/\bar{P})^{1/2}$ [Eq. (3.17)].
α_1, α_2	constants [Eqs. (4.59b) and (4.59c)].
β	detonation velocity/c [Eq. (10.2)].
β_P	effective pressure compressing clouds/$\bar{P}$ [Eq. (6.54)].
γ, γ_i	ratio of specific heats at shock [Eq. (3.22)], in interior of blastwave [Eq. (3.14)].
$\gamma_{\rm eff}$	effective γ just behind shock, allowing for cosmic rays [Eq. (4.86)].
Γ	energy in bubble/energy injected [Eq. (7.6)]; for fully adiabatic bubble ($\Gamma=1$), for bubble with radiative outer shock [Eq. (7.16)]
$\Gamma(x)$	gamma function [Eq. (6.51)]
δ	shell thickness in shell approximation is $R_s\delta$ [Eq. (9.21)].
ζ	$K_{20}^{\rm ev}=(1+\zeta)^{-1}$ [Eq. (C18)].
ε	εc^2 is the energy/g released in detonation wave [Eq. (10.1)].
ε^*	total energy density of gas [Eq. (B48)].
η, $\bar{\eta}$	$R_s \propto t^\eta$ for power-law behavior [Eq. (3.1)], $\bar{\eta}\equiv v_s t/R$ in general [Eq. (9.105)].
η_x	for arbitrary variable x characterizing blastwave: $x\propto t^{\eta_x}$ [Eq. (7.3)]. Relation to space dependence R^{-k_x}: $\eta_x=-\eta k_x$ [Eq. (7.4)]. Exception: $R_s\propto t^{\eta_r}$ in radiative phase [Eq. (6.17)]. Special cases: $E_{\rm in}\propto t^{\eta_{\rm in}}$ in bubbles [Eq. (7.1)]; $\rho_0(r,t)\propto r^{-k_{\rho_0}}t^{\eta_{\rho_0}}$ [Eq. (3.25)]. For cosmological blastwaves with $\Omega_g<\Omega$, η_{E_0} is defined by [Eq. (9.102)].
θ	normalized temperature $kT/\mu v_s^2$ [Eq. (E6)].
κ_ρ, κ_T	power-law index for density, temperature dependence of evaporation rate per unit volume [Eq. (5.15)].

λ	r/R_s [Eq. (4.7)].
λ_{cl}	cloud mean free path $(\omega_{cl}\sigma_{cl})^{-1}$ [Eq. (2.1)].
Λ	the radiative cooling rate per unit volume is $n^2\Lambda$ [Eq. (4.93)].
μ, μ_H	mean mass per particle [Eq. (5.1)], per hydrogen nucleus [Eq. (4.94)].
ν, ν_1, ν_H	$v/\lambda v_s$ [Eq. (B5)], v_1/v_s [Eq. (3.15)], v_H/v_s [Eq. (3.21)].
ξ	dimensionless numerical coefficient in general expression for R_s [Eq. (3.6)].
ξ_{ST}	value of ξ for Sedov-Taylor blastwave [Eq. (4.79)].
ξ_a, ξ_r	ξ for adiabatic, radiative blastwaves [Eq. (6.17)].
ξ_c	effective ξ for combined dark matter/gas blastwaves [Eq. (9.75)].
ξ_0, ξ_e, $\bar{\xi}_e$	for stage 1 of blastwaves in a $\Omega_g<\Omega$ universe: ξ in the precosmological stage, at late times and at arbitrary time [Eq. (9.103)].
Π	normalized momentum flux through shock; $=1$ for strong shocks and strong detonations [Eq. (E10)].
ρ, ρ_0, ρ_1	intercloud density [Eq. (2.1)], ambient intercloud density [Eq. (2.3)], postshock intercloud density [Eq. (2.3)].
$\bar{\rho}$	mean intercloud density in blastwave [Eq. (3.4)].
ρ_{cl}	density in a cloud [Eq. (6.52)].
ρ_{crit}	critical cosmological density [Eq. (9.4b)].
ρ_t	total mass density, including intercloud gas, clouds, stars, neutrinos, etc. [Eq. (9.4b)].
ρ_u	constant characterizing ambient density in cosmological blastwaves [Eq. (9.2)].
ρ^*, ρ_1^*	logarithmic derivative of ρ [Eq. (B1)] just inside shock [Eq. (B17)].
σ	E_b/Mv_s^2
σ_0	saturation parameter [Eq. (5.34)].
σ_d, σ_g	in combined dark matter/gas blastwaves, E_b/Mv_s^2 evaluated for dark matter and gas separately [Eq. (9.73)].
σ_{cl}	cloud cross section πa^2 [Eq. (2.1)].
Σ, Σ_{pc}	cloud evaporation parameter, measured in pc^2 [Eq. (5.30)].
τ, τ_b	cosmic time [Eq. (9.1)], at start of blastwave [above Eq. (9.5)].
τ_b'	integration constant approximately equal to τ_b [Eq. (9.93)].
τ_c	cosmic time at which radiative cooling becomes important in blastwave [Eq. (9.52)].
φ	used to describe pressure gradient [Eq. (4.68)]; to measure heat flux [Eq. (5.30)].
Φ	normalized energy flux through shock [Eq. (E10b)]; $=1$ for strong shocks, but not for detonations [Eq. (E22)].
χ_1	shock compression [Eq. (4.2)].
Ψ	momentum per steradian of intercloud gas [Eq. (D2)].
ω_{cl}	number of clouds per unit volume [Eq. (2.1)].
Ω	cosmological density parameter [Eq. (9.3)].
Ω_g, Ω_{g0}	cosmological density parameter based on intercloud gas only [Eq. (9.4b)], at present epoch [Eq. (9.4a)].

I. INTRODUCTION

There is growing evidence that explosions are critically important to the evolution of astrophysical systems. In addition to classic point explosions, continuous energy input of one kind or another is often important. The treatment in this paper on blastwaves is sufficiently general that its methods can be applied to winds from early-type stars, *D* critical ionization fronts, detonations, or any situation in which an expanding spherical disturbance is preceded by a shock front. Quite apart from the nature or origin of the explosive process itself, the expanding blast wave will shock, heat, and accelerate the surrounding ambient medium. If there are many explosions, the multiple interacting blastwaves may dominate over other physical processes to the extent that they determine the overall density and temperature of the medium. In our galaxy, shocks from supernovae are thought to be important in the acceleration, collapse, and perhaps formation of interstellar clouds. Shocks may also be important for accelerating cosmic rays (Axford, Leer, and Skadron, 1977; Bell, 1978; Blandford and Ostriker, 1979) and for determining the intercloud temperature and pressure as well. In the first paper in this series (hereafter denoted as paper I) (McKee and Ostriker, 1977), we attempted to construct a general theory for an interstellar medium dominated by shocks. In other work (Schwarz, Ostriker, and Yahil, 1975; Blandford and McKee, 1977; Ostriker and Cowie, 1981; Vishniac *et al.*, 1985) we have studied the effect of shocks on the intergalactic medium, where it may be as important for galaxy formation and the confinement of intergalactic clouds as are shocks for star formation and confinement of interstellar clouds within the galaxy.

The astronomical paradigm of a blastwave is the supernova remnant (SNR). Some stars undergo a cataclysmic explosion, a supernova, at the end of their lives (see, for example, Chevalier, 1981): low-mass stars (i.e., stars of mass comparable to that of the sun, $1M_\odot$) in close binary systems can explode either when mass transfer from one star to the other drives the latter over the Chandrasekhar limit, or when the two stars coalesce, again exceeding the Chandrasekhar limit. Stars of mass $M \gtrsim 7M_\odot$ cannot shed enough mass in their lifetimes to end up below the

Chandrasekhar limit, and they explode when the nuclear fuel in their cores is exhausted. Remarkably enough, these very different beginnings both lead to an explosion with an energy $E_b \sim 10^{51}$ erg, or $6\times10^{-4}M_{\odot}c^2$. Initially, the energy is largely thermal, but as the supernova expands, adiabatic expansion converts the thermal energy to kinetic energy. The mean expansion velocity of the matter ejected by the explosion is $v_{\rm ej}=10^9[E_{51}/(M_{\rm ej}/M_{\odot})]^{1/2}$ cm s^{-1}. Not all the matter in the pre-supernova star is ejected: for massive progenitors, a neutron star or black hole may be left behind.

As the remnant of the supernova expands, it goes through several well-defined stages: (1) In the first stage the ejected mass exceeds the swept-up ambient mass, so that to lowest order the ejecta undergo free expansion. The dynamics in this stage are affected by both the density distribution of the pre-supernova star and that of the ambient material. (2) After the blastwave expands to the point that the swept-up, shocked mass of gas exceeds the ejected mass, the SNR enters the second stage, the adiabatic (or Sedov-Taylor) stage, and the dynamics becomes simple. For a uniform, homogeneous ambient medium, the distributions of density, velocity, and pressure in this stage depend only upon two-dimensional parameters, the energy of the explosion E_b and the ambient density ρ_0. Even if the initial explosion is aspherical, it approaches sphericity in this stage. SNR's in this stage of evolution have been observed at radio, infrared, optical, and x-ray wavelengths. X-ray observations, particularly by the *Einstein* satellite, best reflect the distribution of the hot, shocked gas inside the SNR, and are approximately consistent with theoretical expectations in terms of the degree of brightening at the edge and other properties. With the exception of young, pulsar-powered remnants such as the Crab Nebula, the optical emission from SNR's is primarily emission-line radiation from shocked clouds, many of which have a filamentary appearance. (3) Eventually the radiative losses from the SNR interior become significant, and the remnant enters the third, radiative stage of its evolution, when a shell-like structure is expected to occur. Much of the interstellar medium is at too low a density for the SNR to be observable in this stage. (4) Finally the SNR expands so far that its interior pressure drops to the point that the SNR merges with the ambient interstellar medium.

The theoretical paradigm for the simple blastwave is the Sedov-Taylor (ST) self-similar solution (Sedov, 1946, 1959; Taylor, 1950) for a point explosion in a homogeneous medium with zero pressure and a fixed ratio of specific heats. This solution thus applies to the second of the four stages of SNR evolution outlined above, when the stable spherical shock propagates, at a decelerating rate, into the unperturbed medium. Since the solution depends only on the blastwave energy E_b and the ambient density ρ_0, there is no characteristic radius or time in the problem, and the blastwave radius must be of the form $R_s=(\xi E_b t^2/\rho_0)^{1/5}$, where ξ is a numerical constant, with the interior structure maintaining an invariant form. The dependent variables describing the flow—the velocity $v(r,t)$, the density $\rho(r,t)$, and the pressure $P(r,t)$—must all be of the form $x(r,t)=x_1(t)\tilde{x}(r/R_s)$, where $x_1(t)$ is the value of $x(r,t)$ just behind the shock and $\tilde{x}$ is dimensionless; for example, $v_1(t)\propto v_s(t)\equiv dR_s/dt$, $\rho_1(t)\propto\rho_0(R_s,t)$, and $P_1(t)\propto\rho_0(R_s,t)v_s(t)^2$. Because the structure of the blastwave at any time is related to that at any other time by simple scaling, the solution is said to be self-similar. All dimensionless quantities approach fixed values in this phase with, for example, the thermal energy fixed at about 70% of the total energy. It is often possible to obtain explicit analytic solutions to self-similar problems because the partial differential equations of hydrodynamics reduce to ordinary differential equations for the dimensionless dependent variables $\tilde{x}(r/R_s)$. In the interior, the density and velocity approximate power laws, with the mass concentrated towards the edge ($\rho\propto r^9$) but the velocity nearly linear throughout the interior.

Sedov (1959), in a classic work, has presented an extensive analysis of solutions of this type, which can be obtained through dimensional analysis. The example of the evolution of a supernova remnant shows the power of this technique: in reality, the ejected mass that dominates the first stage of the evolution and the radiation that dominates the third stage are never completely absent, so the Sedov-Taylor solution represents an *intermediate asymptotic* solution (Barenblatt and Zel'dovich, 1972; Barenblatt, 1979), which is valid in the limit $R_{\rm ej}\ll R_s\ll R_c$, where $R_{\rm ej}$ is the radius at which the swept-up mass equals the ejected loss and R_c is the radius at which radiative cooling begins to determine the dynamics. The Sedov-Taylor solution is exact only in the limits $R_{\rm ej}/R_s\to0$, $R_s/R_c\to0$, but it is a good approximation whenever the radiative losses are not dominant, and it thus has wide applicability.

There is a second type of self-similar solution, in which dimensional analysis is inadequate (Barenblatt and Zel'dovich, 1972; Barenblatt, 1979); in particular, the values of the parameter η in the expression $R_s\propto t^\eta$ is indeterminate through dimensional analysis. For blastwaves, self-similar solutions of the second type can occur when energy is not conserved. If the energy in the blastwave $E_b=E_0(R_c/R_s)^{k_E}$ for $R_s\gg R_c$, then for $R_s\gg R_c$ the solution will be self-similar with $R_s=[\xi E_0(R_c/R_s)^{k_E}t^2/\rho_0]^{1/5}$, or $R_s=(\xi E_0 R_c^{k_E}t^2/\rho_0)^{1/(5+k_E)}$. In this case the solution does not depend separately on the two length scales $(E_0t^2/\rho_0)^{1/5}$ and R_c but only on a combination of the two. If the combination is not specified in advance—in other words, if k_E is an unknown to be solved for—then the solution is of the second type. Radiative blastwaves (Sec. VI) and cosmological blastwaves with $\Omega_g<\Omega$ (Sec. IX) fall into this category.

In recent years the literature concerning blastwaves in an astronomical context has become quite voluminous as more and more complexities have been added to the theoretical model (e.g., Chernyi, 1957; Parker, 1963; Dryer, 1974; Cavaliere and Messina, 1976; Gaffet, 1978; Chieze and Lazareff, 1981). Most improvements involve relaxing the simplifying assumptions of the ST solution and allowing for conduction, radiative losses, external pressure, gravity, relativity, or symmetric spatial or temporal changes in the ambient gas. These variations still permit the explosion to be described as spherically symmetrical, which is, of course, a great convenience to the theoretician. It is thus reassuring that many of the *Einstein* images of supernova remnants show a fair degree of circular symmetry. However, we know that the interstellar medium is far from uniform, so it is useful to see if an approximate treatment can be found which allows for this irregularity. To the extent that much of the mass of the interstellar medium is confined to a small fraction of the volume and that the high-density component is mostly in irregularities (henceforth called "clouds") small in scale compared to the radii of the blastwaves studied, we can consider the medium to be made up of two fluids. There is a background medium with mean density and isothermal sound speed (ρ_0, C_0) and an embedded fluid of clouds. The blastwave propagates in the low-density medium, with clouds treated as "impurities" that can add or subtract mass, energy, or momentum from the local flow. An approximate treatment of this problem was given in paper I of this series (McKee and Ostriker, 1977); a detailed numerical discussion was presented (hereafter denoted as paper III; Cowie, McKee, and Ostriker, 1981), and the converse problem of the effect on the cloud fluid of a blastwave propagating through was discussed earlier (hereafter denoted as paper II; McKee, Cowie, and Ostriker, 1978).

To treat either the effects of inhomogeneity or the other mentioned physical effects requires detailed numerical solutions of the hydrodynamic equations. However, in many cases it is possible to obtain analytic solutions for the shock radius R_s versus time which are exact to within constant numerical factors.

In previous papers by the authors in various combinations and with others, a variety of solutions has been found for the expansion of spherical blastwaves. Various numerical and semianalytical techniques have been used, and the results published in various notations and to varying degrees of accuracy. A preliminary and elementary summary of the present method was presented by McKee (1982). The intention of this paper is to present on a unified basis a new analytic treatment of our previous results, and to extend the method to other cases in which it may provide a simple and accurate model for computing evolution in other problems of interest, such as stellar wind bubbles or explosions in cosmologically expanding media. We have found that the relatively simple analytical treatment presented here (based primarily on the virial theorem) is remarkably accurate and provides much greater insight than that obtainable from more complex numerical simulations.

A brief review of our organizational plan and primary results may be useful as a guide to the reader. Sections II and III should be read by those interested in any of the specific problems addressed subsequently, since they establish the notation and derive the fundamental dynamical equations to be used thereafter. A general time-dependent virial theorem is derived which includes surface terms, gravity, and interaction with another coextensive fluid (such as a matrix of clouds). The terms in this equation retain their intuitive basis, such as relations to commonly understood thermal, kinetic, and gravitational energies. The derived equation also allows for expansion of the underlying fluid as would be appropriate for cosmological explosions or for supernovae occurring in stars surrounded by a stellar wind. In the limit where the internal fluid becomes compressed into a thin shell, a particularly simple equation results. Both the general and the limited form of these equations can be used for an arbitrary spherical blastwave. They are ordinary nonlinear differential equations well suited for exploratory numerical calculations in cases in which the full complexity of numerical hydrodynamics is not required. Section III treats the broad and important class of self-similar flows, showing how the exponents of the power laws in the time development can be found exactly and how most of the integral properties of the solution can be summarized by two numbers: σ, the ratio of total blastwave energy to Mv_s^2, and α^2, the ratio of $\bar{\rho}v_s^2$ to the mean internal pressure. With these determined, one can find, in addition to the kinetic and thermal energies, the value of the shock radius versus time and other quantities of interest. The remainder of the paper can be seen as primarily an exercise in determining these two numbers (α,σ) in a variety of contexts by simple but approximate means. Section IV outlines several different useful approximations, the simplest being one that is reasonably accurate for the Sedov-Taylor case, assuming that the internal radial velocity is linear in radius. The most useful approximation is termed the pressure-gradient approximation (PGA), and is an extension of an approximation introduced by Gaffet (1981a). Sections V, VI, VII, and VIII apply the theory developed so far to a variety of contexts that appear in galactic astronomy. For example, mass input from evaporating clouds, which can be more important than mass input at the outer shock, is treated in Sec. V. Radiative losses, or the losses due to PdV work done on internal clouds, are treated in Sec. VI. All of these complex and interacting processes are important for galactic supernova remnants propagating in the inhomogeneous interstellar medium. We attempt to summarize the overall situation in Fig. 1. Section VII treats the bubbles that form around early-type stars having strong stellar winds, and Sec. VIII discusses explosions in a wind, as might be appropriate for either the stellar case or for a galaxy developing an active galactic nucleus in an environment where there was a preexisting galactic wind (M82 may represent an example of such a phenomenon).

In Sec. IX we turn to cosmology, where, although the previously established principles are still valid, gravity and the underlying expansion combine to change the picture quantitatively. Very thin shells with low rates of energy loss are the rule. In a real sense cosmological blastwaves can be thought of as solitonlike rearrangements of the Hubble flow via a nonlinear wave. This section can be read by those interested primarily in cosmology without Secs. V–VIII. Section X treats detonation waves, including cosmological detonations and stressing the simple fact that they always propagate at a constant velocity regardless of the density and velocity structure of the underlying fluid.

The appendixes treat a variety of subjects used in the main body of the text normally with greater rigor and generality. Since many of the results are derived here or collected for the first time they may be useful to readers with other agendas. For example, Appendixes B, C, and E, would be useful for those performing detailed numerical integrations for the treatment of edge derivatives or exact integral constraints. We hope that these solutions will facilitate a comparison between theory and observation and thus contribute to an understanding of the energetics of the interstellar and intergalactic media.

II. THE VIRIAL THEOREM FOR BLASTWAVES

In this derivation we shall assume that spherical symmetry holds. Magnetic fields will be neglected except insofar as they contribute an isotropic pressure. The shock front at the perimeter of the blastwave may be broadened by thermal conduction, a magnetic precursor (Draine, 1980), or accelerating cosmic rays (Drury and Volk, 1981), but its thickness is assumed to be small compared to the radius of the blastwave. The shock is assumed to satisfy the usual jump conditions with some effective value of γ, the ratio of specific heats, in most of our work. The virial theorem for blastwaves differs from the standard virial theorem in that it does not apply to a fixed mass. In addition, we explicitly allow for two complications that are important in some applications: either (1) the ambient medium may be expanding or (2) it may consist of an intercloud medium with embedded clouds. The theorem we derive applies to the intercloud medium. For simplicity, we assume that the clouds have negligible velocities. Mass exchange from the clouds to the intercloud medium due to evaporation (Cowie and McKee, 1977; Balbus and McKee, 1982) or turbulent stripping is permitted; since the clouds are essentially at rest, no momentum is transferred by this process. Mass exchange in the reverse direction via condensation is also permitted, but here we neglect the associated momentum transfer. Finally, we assume that the clouds exert a drag on the intercloud medium of $-\rho v^2/\lambda_{\rm cl}$, where $\lambda_{\rm cl}$ is the cloud mean free path.[2] No further assumptions are made, and, given these assumptions, the treatment is exact.

We begin by defining the quantities needed in our derivation. Let r be the radius of an arbitrary point, and let R_s be the radius of the shock front, idealized as an infinitesimally thin boundary. Let v be the fluid velocity at r, measured in the laboratory frame; $v_H(r,t)$ and $v_1(t)$ are the values of v ahead of and just behind the shock, respectively. The velocity of the shock itself is $v_s(t)$. The density ρ and pressure P have the values $\rho_0(r,t)$, $P_0(r,t)$ ahead of the shock and $\rho_1(t)$, $P_1(t)$ just behind the shock; the isothermal sound speed is given by $C=(P/\rho)^{1/2}$. Let M be the mass of intercloud fluid inside R_s. To describe the clouds, we introduce $f(r,t)$ as the volume filling factor, $\omega_{\rm cl}$ as the number of clouds per unit volume, $\sigma_{\rm cl}$ as the cloud cross section, and $\lambda_{\rm cl}\equiv(\omega_{\rm cl}\sigma_{\rm cl})^{-1}$ as the mean free path. The rate of mass exchange with the intercloud medium per cloud is $\dot m$, and g is the gravitational acceleration.

A. General virial theorem

The basic relations we need are the equations of continuity and motion for the intercloud fluid (Shu *et al.*, 1972; Cowie *et al.*, 1981), with the simplifying assumption that the cloud velocity is negligible:

$$\frac{\partial}{\partial t}[\rho(1-f)]+\frac{1}{r^2}\frac{\partial}{\partial r}r^2\rho(1-f)v=\omega_{\rm cl}\dot m\ , \qquad (2.1)$$

$$\rho\frac{dv}{dt}=-\frac{\partial P}{\partial r}-\frac{\omega_{\rm cl}\dot m v}{1-f}-\frac{\rho v^2}{\lambda_{\rm cl}}+\rho g\ . \qquad (2.2)$$

Our assumption that the shock front is thin means that the cloud filling factor f is constant across the shock, so that the usual shock jump conditions apply (Appendix E). Mass and momentum conservation in our notation are

$$\rho_1(v_s-v_1)=\rho_0(v_s-v_H)\ , \qquad (2.3)$$

$$P_1+\rho_1(v_s-v_1)^2=P_0+\rho_0(v_s-v_H)^2\ , \qquad (2.4)$$

which imply

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[2]The expression $\rho v^2/\lambda_{\rm cl}$ for the typically unimportant frictional drag in Eq. (2.2) is fairly uncertain. Calculations of an interstellar cloud overtaken by a shock in the absence of any magnetic field (Woodward, 1976) indicate a drag about half this large (Woodward, 1983); in the presence of a magnetic field the drag would be greater. Note that $\lambda_{\rm cl}$ may not be constant for small clouds: as they are reduced and $\lambda_{\rm cl}$ is increased. The suggestion that the drag on an interstellar cloud is only about $0.1\rho v^2/\lambda_{\rm cl}$ (Yuan and Wang, 1982) is based on an incorrect interpretation of Woodward's (1976) results.

$$P_1=P_0+\rho_0(v_s-v_H)(v_1-v_H)\ . \tag{2.5}$$

As is usual in derivations of the virial theorem, we begin with the identity

$$\frac{d}{dt}\int_0^{R_s} r\rho(1-f)v\,dV=4\pi R_s^3\rho_1(1-f_0)v_s v_1+\int_0^{R_s} r\frac{\partial}{\partial t}[\rho(1-f)v]dV\ . \tag{2.6}$$

The partial time derivative may be evaluated with the aid of Eq. (2.1),

$$\frac{\partial}{\partial t}[\rho(1-f)v]=\rho(1-f)\frac{dv}{dt}+\omega_{\rm cl}\dot{m}v-\frac{1}{r^2}\frac{\partial}{\partial r}r^2[\rho(1-f)v^2]\ . \tag{2.7}$$

Substituting into Eq. (2.6) with the aid of Eq. (2.2) and integrating by parts gives

$$\frac{d}{dt}\int_0^{R_s} r\rho(1-f)v\,dV=4\pi R_s^3\rho_1(1-f_0)v_1(v_s-v_1)+\int_0^{R_s}\rho(1-f)v^2dV+\int_0^{R_s} r(1-f)\left[-\frac{\partial P}{\partial r}-\frac{\rho v^2}{\lambda_{\rm cl}}+\rho g\right]dV\ . \tag{2.8}$$

We simplify this expression by introducing the dimensionless moments of the radius and velocity,

$$K_{mn}\equiv\int_0^{R_s}\left[\frac{r}{R_s}\right]^m\left[\frac{v}{v_1}\right]^n\frac{4\pi r^2\rho(1-f)dr}{M}\ . \tag{2.9}$$

We define the weighted mean interior value of the cloud filling factor f_i by

$$f_i\int_0^{R_s} r\frac{\partial P}{\partial r}dV\equiv\int_0^{R_s} fr\frac{\partial P}{\partial r}dV\ . \tag{2.10}$$

Typically $\partial P/\partial r$ increases strongly near the edge of the blastwave, so that f_i is weighted toward the edge. Then an integration by parts gives

$$\int_0^{R_s} r(1-f)\frac{\partial P}{\partial r}dV=(1-f_i)4\pi R_s^3(P_1-\bar{P})\ , \tag{2.11}$$

where $\bar{P}$ is the mean pressure of the intercloud gas, defined as $R_s^3\bar{P}=3\int r^2P\,dr$. Next, we define the gravitational virial energy term

$$W\equiv\int_0^{R_s}\rho g r(1-f)dV\ , \tag{2.12}$$

which is evaluated in Appendix A. Note that $g=-GM_t(r)/r^2$ is the gravitational acceleration due to the total mass inside r, including the clouds, nongaseous material such as stars, neutrinos, etc., and any possible central mass. As shown in Appendix A, W is the same as the gravitational energy of the gas.

Finally, we use Eq. (2.3) to rewrite the first term on the right-hand side of Eq. (2.8) as

$$4\pi R_s^3\rho_1(1-f_0)v_1(v_s-v_1)=4\pi R_s^3(1-f_0)v_1\rho_0(v_s-v_H)\ . \tag{2.13}$$

Then our final result for the virial theorem for blast waves arises from inserting Eqs. (2.5) and (2.9)–(2.13) into (2.8):

$$\frac{d}{dt}(K_{11}MR_sv_1)=4\pi R_s^3(1-f_i)(\bar{P}-P_0)+4\pi R_s^3\rho_0(v_s-v_H)[(1-f_i)v_H-(f_0-f_i)v_1]+\left[K_{02}-K_{12}\frac{R_s}{\lambda_{\rm cl}}\right]Mv_1^2+W\ , \tag{2.14}$$

where we have assumed $\lambda_{\rm cl}=$const.

In this form, the virial theorem is quite general and applies to any nonrelativistic, spherically symmetric, inviscid, nonmagnetic flow. For example, expansion into a vacuum is covered by setting $P_0=\rho_0=0$. In our applications, we shall assume that a shock front is located at R_s. Since the derivation is independent of the energy jump condition, the virial theorem allows for thermal conduction of heat from the hot interior to the shock front (Sec. V) and for detonations in which energy is injected at the shock front (Sec. X).

One feature of this equation that makes it particularly useful is that it is possible to derive reasonably accurate analytic expressions for the moments K_{mn}; this is accomplished in Appendix C.

B. The thin-shell approximation

As an alternative to the virial theorem approach described above, it is possible to derive an equation of motion for the blastwave (Appendix D). We know from similarity solutions and from numerical calculations that most of the mass in the blastwave is concentrated near R_s. In the *shell approximation* all the gas is assumed to be concentrated in a shell at $r\sim R_s$ and to be moving at velocity v_1. In the *thin-shell approximation*, the shell is assumed to have zero thickness and to be at $r=R_s$; hence its velocity is v_s. In this latter case, the equation of motion (D8) reduces to

$$\frac{d}{dt}(Mv_s)=4\pi R_s^2(1-f)[(\bar{P}-P_0)+\rho_0 v_H(v_s-v_H)]$$
$$-Mv_s^2/\lambda_{\rm cl}+W/R_s+\dot{M}_{\rm in}v_{\rm in} \ , \qquad (2.15)$$

where $\dot{M}_{\rm in}v_{\rm in}$ is the rate at which momentum is injected at the origin. Zel'dovich and Raizer (1966) have discussed this approximation for the simpler case $f=W=\lambda_{\rm cl}^{-1}=v_H=0$. It is often easier to apply Eq. (2.15) than (2.14), but it is less accurate. It can be shown that usually the two approaches are equivalent if, in evaluating the moments required for the virial theorem, it is assumed that v is a linear function of radius.

III. SELF-SIMILAR BLASTWAVES ($P_0=0$)

We now develop the formalism to describe the evolution of a spherical blastwave under the assumption that it is self-similar. Since the inclusion of an ambient pressure P_0 usually destroys the self-similarity, we set $P_0=0$ here.

As discussed in the Introduction, self-similar blastwaves are characterized by a power-law dependence of the blastwave radius R_s on time,

$$R_s \propto t^{\eta} \ . \qquad (3.1)$$

The velocity of the strong shock at the periphery of the blastwave is then

$$v_s=\eta R_s/t \ . \qquad (3.2)$$

Let E_b be the total energy of the intercloud gas due to the blastwave, including kinetic, thermal, and gravitational terms, and let M be the total mass of the intercloud gas in the blastwave. Then the dimensionless quantity

$$\sigma \equiv E_b/Mv_s^2 \qquad (3.3)$$

is generally constant for self-similar solutions, even though E, M, and v_s may all depend on time (this requirement is relaxed in Sec. VI.C). The mean intercloud density $\bar{\rho}(R_s)$ is defined by

$$M\equiv\bar{\rho}(R_s)V(1-f_i) \ , \qquad (3.4)$$

where $V=4\pi R_s^3/3$ is the volume and f_i is given by Eq. (2.10). The solution of Eqs. (3.2)–(3.4) is

$$R_s=\left[\frac{\xi E_b(R_s)}{\bar{\rho}(R_s)}\right]^{1/5} t^{2/5} \ , \qquad (3.5)$$

where

$$\xi\equiv\frac{3}{4\pi\eta^2\sigma(1-f_i)} \ . \qquad (3.6)$$

Thus the radius of a self-similar blastwave is always related to the current energy and mean interior density by a Sedov-Taylor-like relation, even when E_b and $\bar{\rho}$ are functions of time and/or radius. This simple result is a direct consequence of dimensional analysis.

In order to obtain an explicit solution for R_s, we must unravel the dependence of E_b and $\bar{\rho}$ on R_s. Let R be the shock radius R_s normalized to some fiducial radius, denoted $R_s(1)$. In general the requirement of self-similarity implies that a hydrodynamic variable x which is a function of the blastwave radius can be written

$$x=x(1)R^{-k_x} \ , \qquad (3.7)$$

where $x(1)\equiv x(R=1)$ and k_x is constant. In particular, we have $R_s=R_s(1)R$, consistent with our notation for the fiducial radius $R_s(1)$; furthermore,

$$E_b=E_b(1)R^{-k_E} \ , \qquad (3.8)$$

$$\bar{\rho}=\bar{\rho}(1)R^{-k_\rho} \ . \qquad (3.9)$$

Equation (3.5) then becomes

$$R_s=R_s(1)\left[\frac{\xi E_b(1)}{\bar{\rho}(1)R_s(1)^5}\right]^{\eta/2} t^{\eta} \ , \qquad (3.10)$$

with

$$\eta=\frac{2}{5+k_E-k_\rho} \ . \qquad (3.11)$$

It is sometimes useful to have explicit expressions for the age t and velocity v_s of the blastwave in terms of R; Eqs. (3.2) and (3.10) give

$$t=t(1)R^{1/\eta}=\left[\frac{\bar{\rho}(1)R_s(1)^5}{\xi E_b(1)}\right]^{1/2} R^{1/\eta} \ , \qquad (3.12)$$

$$v_s=v_s(1)R^{1-1/\eta}=\left[\frac{\eta^2\xi E_b(1)}{\bar{\rho}(1)R_s(1)^3}\right]^{1/2} R^{1-1/\eta} \ . \qquad (3.13)$$

The determination of ξ, or equivalently $\sigma\equiv E_b/Mv_s^2$, completes the solution for the dynamics of the blastwave. The two basic equations we use to constrain these parameters are the energy equation and the virial theorem. The total energy E of the intercloud gas is the sum of the kinetic, thermal, and gravitational energy of the gas inside the blastwave. It consists of two parts: the energy due to the blastwave E_b plus the energy E_a the ambient intercloud gas would have had in the absence of the blastwave:

$$E=\int \tfrac{1}{2}\rho v^2(1-f)dV+\left[\frac{1-f_i}{\gamma_i-1}\right]\bar{P}V+W$$
$$\equiv E_a+E_b \ , \qquad (3.14)$$

where we have made the approximation $\int fP\,dV\simeq f_i\bar{P}V$, with f_i defined in Eq. (2.10). The effective specific-heat ratio for the interior of the blastwave is γ_i. The gravitational energy term W includes the self-energy of the intercloud gas, together with the interaction energy due to other mass inside the blastwave; it is discussed in Appendix A. For blastwaves in cold, stationary media, the energy of the ambient medium E_a vanishes, and for such blastwaves we shall generally drop the distinction between E and E_b.

Defining

$$\nu_1 \equiv v_1/v_s \tag{3.15}$$

permits the kinetic energy E_K to be simplified to

$$\frac{E_K}{E_b} = \frac{\nu_1^2 K_{02}}{2\sigma} . \tag{3.16}$$

The thermal energy $E_{\rm th}$ can be expressed in terms of an effective mean Mach number α given by

$$\alpha^2 \equiv \frac{\bar{\rho} v_s^2}{\bar{P}} = \frac{\bar{\rho}(1) v_s^2(1)}{\bar{P}(1)} , \tag{3.17}$$

which is generally constant for self-similar blastwaves; we obtain

$$\frac{E_{\rm th}}{E_b} = \frac{1}{\alpha^2 \sigma (\gamma_i - 1)} \tag{3.18}$$

with the aid of Eqs. (3.3) and (3.4). Equations (3.14), (3.16), and (3.18) then yield the energy equation

$$\sigma = \tfrac{1}{2}\nu_1^2 K_{02} + \frac{1}{\alpha^2(\gamma_i - 1)} + \frac{W}{Mv_s^2} - \frac{E_a}{Mv_s^2} . \tag{3.19}$$

The virial theorem (2.14) provides the second basic relation for determining the dynamics of the blastwave. For self-similar blastwaves, the moment K_{11} is constant and the factor $MRv_1 \propto R^{(5-k_\rho-k_E)/2}$ from Eqs. (3.4), (3.9), (3.11), and (3.13). Hence, the virial theorem for self-similar blastwaves can be expressed as

$$\tfrac{1}{2}\nu_1(5-k_\rho-k_E)K_{11} = \frac{3}{\alpha^2} + \frac{3\rho_0}{\bar{\rho}}(1-\nu_H)\left[\nu_H - \left[\frac{f_0 - f_i}{1-f_i}\right]\nu_1\right] + \nu_1^2\left[K_{02} - K_{12}\frac{R_s}{\lambda_{\rm cl}}\right] + \frac{W}{Mv_s^2} , \tag{3.20}$$

where

$$\nu_H \equiv v_H/v_s . \tag{3.21}$$

In order for the solution to be self-similar each of the terms in Eq. (3.20) must be constant; in particular, self-similar blastwaves in expanding media have a constant ratio ν_H of expansion speed to shock speed. [In the case of cloud drag it is possible to relax this requirement that each term in (3.20) be constant—see Sec. VI. C.]

The energy equation (3.19) and the virial theorem (3.20) thus determine σ (and α^2) in terms of various moments of the velocity distribution K_{mn}, and the parameters ν_1, ν_H, and W. We defer discussion of the last two to Sec. IX on cosmological blastwaves. The normalized post-shock velocity ν_1 is determined by the shock jump conditions (Appendix E); in the absence of energy injection at the shock front it is

$$\nu_1 = \frac{2+(\gamma-1)\nu_H}{\gamma+1} , \tag{3.22}$$

which reduces to $2/(\gamma+1)$ for $\nu_H = 0$; here γ is the ratio of specific heats just behind the shock front. Our problem thus reduces to determining the moments K_{mn}, and in succeeding sections we develop simple approximations for doing this.

Before focusing on particular cases, however, we note some general relations that apply to all self-similar blastwaves. The mean pressure $\bar{P}$ may be inferred from Eqs. (3.3), (3.4), (3.6), and (3.17) to be

$$\bar{P} = \frac{E_b}{\alpha^2 \sigma (1-f_i) V} = \frac{\eta^2 \xi E_b}{\alpha^2 R_s^3} . \tag{3.23}$$

The pressure just behind the shock P_1 is given by the jump condition (2.5), so the ratio $\bar{P}/P_1$ is

$$\frac{\bar{P}}{P_1} = \frac{(\bar{\rho}/\rho_0)}{\alpha^2 (1-\nu_H)(\nu_1 - \nu_H)} . \tag{3.24}$$

The evaluation of the mean density is complicated by the possibility that the ambient density may be a function of time and by the possibility of mass injection by evaporation. In general, the ambient density $\rho_0(r,t)$ may depend explicitly on both radius and time:

$$\rho_0(r,t) \propto r^{-k_{\rho_0}} t^{\eta_{\rho_0}} . \tag{3.25}$$

At the shock, this simplifies to $\rho_0(R_s) \propto R_s^{-k_\rho}$ because $\rho_0(R_s)/\bar{\rho}$ must be constant in a self-similar blastwave and $\bar{\rho} \propto R^{-k_\rho}$ by Eq. (3.9). Since $R_s \propto t^\eta$, this implies

$$k_\rho = k_{\rho_0} - \eta_{\rho_0}/\eta . \tag{3.26}$$

The mass of intercloud gas in the blastwave is comprised of swept-up ambient gas M_a and evaporated gas $M_{\rm ev}$. The mass of the swept-up gas is

$$M_a = \frac{4\pi R_s^3}{3-k_{\rho_0}} \rho_0(R_s)(1-f_0) \equiv \frac{4\pi}{3}\bar{\rho}_a(R_s) R_s^3 (1-f_i) , \tag{3.27}$$

so that

$$\frac{\bar{\rho}_a}{\rho_0} = \frac{3}{3-k_{\rho_0}} \left[\frac{1-f_0}{1-f_i}\right] . \tag{3.28}$$

In the absence of evaporation $\bar{\rho}_a = \bar{\rho}$ and $M_a = M$. Note that it is k_{ρ_0}, not k_ρ, which appears in Eqs. (3.27) and (3.28).

Another relation between k_ρ and k_{ρ_0}, which brings out the connection to the expansion or contraction of the ambient medium, can be obtained by equating two expressions for dM_a/dt. The requirement of self-similarity implies that $M_a \propto R^{3-k_\rho} \propto t^{\eta(3-k_\rho)}$. On the other hand, the blastwave sweeps up mass at a rate $\rho_0(v_s - v_H)(1-f_0)$, so that

$$\frac{dM_a}{dt}(3-k_\rho)\frac{\eta M_a}{t}=4\pi R_s^2\rho_0 v_s(1-f_0)(1-\nu_H) \ . \qquad (3.29)$$

With the aid of Eq. (3.27) this reduces to

$$\frac{3-k_\rho}{3-k_{\rho_0}}=1-\nu_H \ ; \qquad (3.30)$$

hence k_ρ differs from k_{ρ_0} only if the ambient medium is in motion ($\nu_H \neq 0$). Equations (3.26) and (3.30) together yield

$$\eta\nu_H=-\eta_{\rho_0}/(3-k_{\rho_0}) \ , \qquad (3.31)$$

as a kinematic constraint on self-similar blastwaves in such a case.

IV. APPROXIMATIONS FOR ADIABATIC, SELF-SIMILAR BLASTWAVES IN STATIONARY, HOMOGENEOUS MEDIA ($\nu_H=W=\lambda_{cl}^{-1}=f_0-f_i=\gamma-\gamma_i=\dot{m}=\varepsilon=P_0=0$)

A. General results

Most analyses of blastwaves have focused on adiabatic blastwaves in stationary, homogeneous media. Here we shall allow for the effects of embedded clouds only insofar as they occupy a fixed fraction $f_0=f_i$ of the volume; cloud drag, cloud evaporation, and cloud crushing are all neglected. In an adiabatic blastwave, the entropy per particle is conserved, but it is not necessary for the total energy to be constant. Since we treat gravity only in the cosmological case ($v_H \neq 0$), the assumption of a stationary ambient medium allows us to set $W=0$. Finally reserving a discussion of detonations to Sec. X, we find that the shock jump conditions [Eq. (E22)] give

$$\nu_1 \equiv v_1/v_s=2/(\gamma+1) \ , \qquad (4.1)$$

$$\chi_1 \equiv \rho_1/\rho_0=(\gamma+1)/(\gamma-1) \ , \qquad (4.2)$$

for the post-shock velocity [from Eq. (3.22)] and the compression [from Eq. (E22)], respectively. In view of the fact that the ambient medium is stationary, it is not necessary to distinguish between k_ρ and k_{ρ_0} [Eq. (3.30)].

With these simplifications, the energy equation (3.19) and the virial theorem (3.20) for self-similar blastwaves can be solved for the parameters α^2 and σ:

$$\alpha^2=\frac{3(\gamma+1)^2}{(5-k_\rho-k_E)(\gamma+1)K_{11}-4K_{02}} \ , \qquad (4.3)$$

$$\sigma=\frac{2K_{02}(3\gamma_i-5)+(5-k_\rho-k_E)(\gamma+1)K_{11}}{3(\gamma_i-1)(\gamma+1)^2} \ . \qquad (4.4)$$

The distinction between γ and γ_i in Eq. (4.4) has been maintained for generality; in adiabatic blastwaves they are equal.

In order to complete the solution, the moments K_{11} and K_{02} must be obtained. Of course the complete solution would require specification of the hydrodynamic variables as functions of λ. However, to obtain all the *integral* quantities of interest and the expansion rate, only two numbers are required, and straightforward methods can be found that will provide very accurate values for these numbers. There is one general relation among the moments which follows from the continuity equation and is exact. As demonstrated in Appendix C [Eq. (C10)], for the case at hand ($v_H=\dot{m}=\varepsilon=0$), this *kinematic moment relation* is

$$K_{n0}=\frac{(3-k_\rho)+\dfrac{2n}{\gamma+1}K_{n-1,1}}{3-k_\rho+n} \ . \qquad (4.5)$$

This relation does not reduce the number of unknown moments in the equations for α^2 and σ, since it introduces a new moment, K_{20}, but it does play an important role in constructing approximations, since it relates density and velocity moments.

In a self-similar blastwave, a hydrodynamic variable $x(r,t)$, such as the density ρ, the velocity v, or the pressure P, can be written

$$x(r,t)=x_1(t)\tilde{x}(\lambda) \ , \qquad (4.6)$$

with

$$\lambda \equiv r/R_s \ , \qquad (4.7)$$

$x_1(t)$ being the post-shock value and $\tilde{x}(1)=1$. The hydrodynamic equations for self-similar, adiabatic blastwaves are given in Appendix B. To construct approximations for the moments we shall adopt a one-power approximation,

$$\tilde{x}(\lambda) \simeq \lambda^{l_x} \ , \qquad (4.8)$$

or two-power approximation,

$$\tilde{x}(\lambda) \simeq a_x\lambda^{l_{x1}}+(1-a_x)\lambda^{l_{x2}} \ , \qquad (4.9)$$

for one or more of the hydrodynamic variables. The parameters l_x, a_x, etc., are generally fixed by an integral constraint on $\tilde{x}(\lambda)$, or by requiring the logarithmic derivative,

$$x^*(\lambda) \equiv \frac{\partial \ln x}{\partial \ln r}=\frac{d\ln\tilde{x}}{d\ln\lambda} \ , \qquad (4.10)$$

to match the correct values (cf. Appendix B) at the boundaries.

Conservation of mass provides an integral constraint on $\tilde{\rho}(\lambda)$:

$$\int_0^1 \tilde{\rho}(\lambda)\lambda^2 d\lambda=\frac{1}{3-k_\rho}\left[\frac{\rho_0}{\rho_1}\right]=\frac{\gamma-1}{(3-k_\rho)(\gamma+1)} \ . \qquad (4.11)$$

This implies that $K_{00}=1$, as it must from the general condition (4.5). For adiabatic blastwaves, a second constraint is provided by entropy conservation. The quantity

$$s \equiv P/\rho^\gamma \qquad (4.12)$$

is a function of the entropy per unit mass and hence is a constant in comoving coordinates. As shown in Appendix B.4, this implies

$$\tilde{s}(\lambda)=\tilde{M}(\lambda)^{(\gamma k_\rho - k_p)/(3-k_\rho)} . \tag{4.13}$$

Evaluation of $\int \tilde{s}\, d\tilde{M}$ then yields

$$\int_0^1 \tilde{\rho}\tilde{s}\lambda^2 d\lambda = \frac{\gamma-1}{(\gamma+1)[(\gamma-1)k_\rho - k_E]} . \tag{4.14}$$

The utility of this result is limited, however, since the integral on the left-hand side can be evaluated analytically only for simple approximations for the pressure and density.

In the following, we first develop approximations for the density and velocity structure of "filled" blastwaves, in which the shocked intercloud gas extends from the origin to the edge of the blastwave, as in the case of the Sedov-Taylor blastwave. We then treat "hollow" blastwaves, in which the intercloud gas is confined to a shell between $\lambda=\lambda_i>0$ and $\lambda=1$. The central cavity can either be a vacuum, as in the case of Sedov blastwaves in sufficiently steep density gradients (and cosmological blastwaves), or it can be filled with very hot gas, as in the case of bubbles. Approximations for the pressure distribution, which provide the most accurate, simple results for the blastwave structure, are discussed in Sec. IV.D. The results are then applied to the known exact solutions for Sedov blastwaves; the effects of cosmic-ray acceleration at the shock are also briefly considered.

In subsequent sections we shall find that some of the approximations developed here are useful for nonadiabatic and nonstationary blastwaves as well. Care must be exercised in extending these results to those cases, however, since Eq. (4.11) does not apply to evaporative blastwaves, and the relation $k_\rho = k_{\rho_0}$ does not apply to cosmological blastwaves.

B. Density and velocity structure of filled blastwaves ($\lambda_i = k_E = 0$, $k_\rho \le k_{\rho\,\mathrm{crit}}$)

Since central energy injection leads to hollow blastwaves and we are deferring a discussion of detonations, we can assume that the energy is constant in a filled, adiabatic blastwave ($k_E=0$). A central vacuum forms if the density gradient exceeds a critical value $k_{\rho\,\mathrm{crit}}$ [Eq. (4.18)]; here we assume $k_\rho \le k_{\rho\,\mathrm{crit}}$. We consider several approximations, in order of increasing accuracy.

1. Linear velocity approximation (LVA)

The simplest approximation for the velocity structure of a filled blastwave is that the velocity is a linear function of radius, $v\propto r$, $\tilde{v}=\lambda^{l_v}$ with $l_v=1$. As will be shown below, if one assumes that both the density and the velocity are simple power laws, then the kinematic moment relation (4.5) implies that the velocity power law l_v must be unity; hence, the LVA is the most natural simple approximation for the velocity structure. The corresponding power law l_ρ for the density is determined by the mass constraint, Eq. (4.11):

$$l_\rho = \frac{6-(\gamma+1)k_\rho}{\gamma-1} . \tag{4.15}$$

The same result is obtained from the equation of continuity (B6) with $v^*=1$.

In the LVA we have

$$K_{mn} = K_{m+n,0} \equiv K_{m+n}(\mathrm{LVA}) , \tag{4.16}$$

which, with the kinematic moment relation (4.5), determines all the moments

$$K_n(\mathrm{LVA}) = \frac{1}{1+\dfrac{n(\gamma-1)}{(3-k_\rho)(\gamma+1)}} . \tag{4.17}$$

This result, when inserted into Eqs. (4.3) and (4.4) to give α^2 and σ, completes the solution.

Although the LVA is generally approximate, there are values of γ and k_ρ for which it becomes exact (Sedov, 1959). The self-similar hydrodynamic equation (B11) for v^* (where the asterisk denotes logarithmic derivatives) is consistent with setting $v^*=1$ only if the normalized temperature $\theta \propto T/\lambda^2$ is constant, so that $\theta^*=0$. Since $\theta^* = P^* - \rho^* - 2$, one can use Eqs. (B12) and (B13) to determine the condition that θ^* vanish. Setting $\theta=\theta_1$, the post-shock value, from Eq. (E27), we find $v\propto r$ provides an exact solution to the hydrodynamic equations for

$$k_\rho = k_{\rho\,\mathrm{crit}} \equiv \frac{7-\gamma}{\gamma+1} , \tag{4.18}$$

and $k_E=0$. In this case Eqs. (4.3), (4.4), and (4.17) give the exact results $K_2=\frac{2}{3}$,

$$\alpha^2 = \frac{3(\gamma+1)^2}{4(\gamma-1)} , \tag{4.19a}$$

$$\sigma = \frac{8}{3(\gamma+1)^2} \quad (k_\rho = k_{\rho\,\mathrm{crit}}) , \tag{4.19b}$$

since $\gamma=\gamma_i$ in adiabatic blastwaves. In astrophysical applications the density distribution $\rho\propto r^{-2}$ commonly occurs (as in winds or galactic halos). For this case the LVA is exact when $\gamma=\frac{5}{3}$ and is a good approximation for nearby values of γ.

2. One-power approximation (OPA) for velocity

The next level of sophistication in approximating the moments is to allow the velocity power law l_v to differ from unity while maintaining the density power law l_ρ at the value given by the mass constraint, Eq. (4.15). With power laws for both $\tilde{\rho}$ and $\tilde{v}$, the moments become

$$K_{mn} = \frac{3+l_\rho}{3+l_\rho+m+nl_v} . \tag{4.20}$$

Hence, in this approximation, K_{11} is the harmonic mean of K_{20} and K_{02}:

$$\frac{1}{K_{11}}=\frac{1}{2}\left[\frac{1}{K_{20}}+\frac{1}{K_{02}}\right] . \tag{4.21}$$

Note that if K_{20} and K_{11} are known, then one can solve Eq. (4.20) for l_ρ and l_v:

$$l_\rho=\frac{5K_{20}-3}{1-K_{20}} , \tag{4.22a}$$

$$l_v=\frac{2K_{20}-K_{11}(1+K_{20})}{(1-K_{20})K_{11}} . \tag{4.22b}$$

If we require that Eq. (4.20) for K_{mn} satisfy the kinematic moment relation (4.5) for all m, then we recover the LVA. Similarly, if we set l_ρ equal to the value in Eq. (4.15) and require the moment relation to be satisfied for some n, we again recover the LVA. However, a more accurate estimate of the moments is obtained by setting $l_v=v_1^*$, the exact edge value for the logarithmic velocity gradient. In terms of the critical density gradient $k_{\rho\,\mathrm{crit}}=(7-\gamma)/(\gamma+1)$ defined in Eq. (4.18), Eq. (B19) becomes

$$v_1^*=1+\tfrac{1}{2}(k_{\rho\,\mathrm{crit}}-k_\rho) , \tag{4.23}$$

which together with Eq. (4.15) yields

$$K_{mn}(\mathrm{OPA})=\frac{1}{1+\dfrac{(\gamma-1)[m+n+\frac{1}{2}n(k_{\rho\,\mathrm{crit}}-k_\rho)]}{(\gamma+1)(3-k_\rho)}} . \tag{4.24}$$

This result shows that the OPA value for K_{mn} reduces to the exact LVA value when $k_\rho=k_{\rho\,\mathrm{crit}}$. The increase in accuracy afforded by the OPA over the LVA is shown in Table I, which compares several approximations with the exact results for Sedov-Taylor blastwaves ($k_E=k_\rho=0$) at $\gamma=\frac{5}{3},\frac{7}{5}$, and $\frac{4}{3}$.

A variation of the OPA that affords greater accuracy can be obtained by noting that the most accurate moment in the OPA is K_{02}, and then calculating K_{11} from the geometric mean approximation [Eq. (C13b)]. The result for K_{11} is

$$K_{11}=\frac{(3-k_\rho)(\gamma+1)}{\{4+2(5-k_\rho)[2\gamma^2+7\gamma-3-\gamma(\gamma+1)k_\rho]\}^{1/2}-2} . \tag{4.25}$$

This reduces to the LVA result for $k_\rho=k_{\rho\,\mathrm{crit}}$ and is more accurate than the OPA result for $k_\rho<k_{\rho\,\mathrm{crit}}$. When combined with the K_{02} result from Eq. (4.24), this yields values of α^2 and σ [Eqs. (4.3) and (4.4)] of comparable or somewhat better accuracy than the pressure-gradient approximation in Sec. IV.D below.

3. Two-power approximation (TPA) for velocity

A significant improvement in the accuracy of the approximation is obtained by assuming that the density and velocity can each be represented as the sum of two power laws, as in Eq. (4.9). Requiring that the velocity have the correct derivatives at the origin and the edge [see Appendix B, Eqs. (B19) and (B27)] leads to

TABLE I. Sedov-Taylor blastwaves in a uniform medium ($k_E=k_\rho=0$).

	LVA	OPA	TPA	PGA/$\bar{K}$	Exact
		$\gamma=\frac{5}{3}$			
K_{20}	0.8571	0.8571	0.8367		0.835 67
K_{11}	0.8571	0.8000	0.7889		0.785 57
K_{02}	0.8571	0.7500	0.7445		0.740 42
α^2	2.6667	2.7826	2.8293	2.8571	2.839 66
σ	0.8036	0.7500	0.7396	0.7500	0.736 47
$(1-f)\xi$	1.8568	1.9894	2.0175	1.9894	2.025 97
		$\gamma=\frac{7}{5}$			
K_{20}	0.9000	0.9000	0.8797		0.879 37
K_{11}	0.9000	0.8504	0.8391		0.838 19
K_{02}	0.9000	0.8060	0.8009		0.799 93
α^2	2.4000	2.4753	2.5168	2.5400	2.519 87
σ	1.3542	1.2898	1.2714	1.2795	1.269 87
$(1-f)\xi$	1.1018	1.1568	1.1736	1.1661	1.174 99
		$\gamma=\frac{4}{3}$			
K_{20}	0.9130	0.9130	0.8936		0.893 44
K_{11}	0.9130	0.8673	0.8563		0.855 85
K_{02}	0.9130	0.8258	0.8210		0.820 77
α^2	2.3333	2.3968	2.4355	2.4565	2.437 15
σ	1.6211	1.5550	1.5334	1.5398	1.532 45
$(1-f)\xi$	0.9204	0.9595	0.9731	0.9690	0.973 65

$$\tilde{v}(\lambda)\simeq\frac{(\gamma+1)\lambda+(\gamma-1)\lambda^{l_{v_2}}}{2\gamma}\ , \tag{4.26a}$$

with

$$l_{v_2}=1+\frac{\gamma}{\gamma-1}(k_{\rho\,\mathrm{crit}}-k_\rho)\ ; \tag{4.26b}$$

for $k_\rho=k_{\rho\,\mathrm{crit}}$, the exact LVA result $\tilde{v}(\lambda)=\lambda$ is recovered.

The TPA result for the density is obtained by requiring $\rho^*(\lambda)$ to agree with the exact values in Appendix B [Eqs. (B20) and (B28)] and by imposing the mass constraint (4.11):

$$\tilde{\rho}(\lambda)=a_\rho\lambda^{l_{\rho_1}}+(1-a_\rho)\lambda^{l_{\rho_2}} \tag{4.27}$$

with

$$l_{\rho_1}\equiv\frac{3-\gamma k_\rho}{\gamma-1},\quad a_\rho\equiv\frac{\gamma(k_{\rho\,\mathrm{crit}}-k_\rho)}{10-\gamma-(\gamma+2)k_\rho}\ , \tag{4.28a}$$

and

$$l_{\rho_2}\equiv\frac{6-(\gamma+1)(2k_\rho-k_{\rho\,\mathrm{crit}})}{\gamma-1}\geq(1,l_{\rho_1})\ . \tag{4.28b}$$

The approximation represented in Eq. (4.27) is valid only when the denominator of a_ρ in Eq. (4.28a) is >0, which, in turn, is assured (for $k_\rho<k_{\rho\,\mathrm{crit}}$) so long as $\gamma>1$. Again, when $k_\rho=k_{\rho\,\mathrm{crit}}$, this expression reduces to the exact LVA result. For $k_\rho=0$, this expression agrees with the result obtained by Gaffet (1981a), which when $\gamma=\frac{5}{3}$ gives $(l_{\rho_1},l_{\rho_2})=(\frac{9}{2},14)$.

The moments K_{mn} can be evaluated by direct integration with approximations (4.25) and (4.27) for $\tilde{v}(\lambda)$ and $\tilde{\rho}(\lambda)$. This procedure is quite accurate at $\gamma=\frac{5}{3}$, but becomes less so for smaller γ as the exponents l_{v_2} and l_{ρ_1}, l_{ρ_2} become large. The TPA and the kinematic moment relation can be used to show that

$$K_{11}=\frac{\gamma+1+(\gamma-1+2\gamma b)K_{l_{v_2}+1,0}}{2\gamma(1+2b)}\ , \tag{4.29}$$

$$K_{02}=\frac{\gamma+1}{2\gamma}K_{11}+\frac{\gamma^2-1}{4\gamma(l_{v_2}+1)}\times[(4-k_\rho+l_{v_2})K_{l_{v_2}+1,0}-(3-k_\rho)]\ , \tag{4.30}$$

where

$$b\equiv(\gamma-1)/[\gamma(3-k_\rho)]\ , \tag{4.31}$$

$$K_{l_{v_2}+1,0}=\frac{\gamma+1}{\gamma b}\left[\frac{a_\rho}{4+l_{v_2}+l_{\rho_1}}+\frac{1-a_\rho}{4+l_{v_2}+l_{\rho_2}}\right]\ , \tag{4.32}$$

and the coefficients a_ρ and l_{ρ_1} are given by Eq. (4.28a). In common with the LVA and OPA discussed above, this approximation reduces to the exact LVA when $k_\rho=k_{\rho\,\mathrm{crit}}$ and to the thin-shell limit $K_{mn}=1$ as $\gamma\to1$. As shown in Table I, the accuracy is typically better than 1%.

4. Kahn's approximation for mass distribution

Kahn (1975) has developed an approximation for the internal structure of blastwaves which is more accurate than those considered above, although it does not readily lend itself to calculation of the moments. The approximation has been extended and applied by Cox and Franco (1981) and Cox and Anderson (1982). Rather than approximating the density, Kahn approximates the logarithmic derivative of the mass by the sum of two powers: for $k_\rho<k_{\rho\,\mathrm{crit}}$, the approximation is

$$\frac{M^*}{M_1^*}=\frac{\gamma+\lambda^{l_{x_2}}}{\gamma+1}\ , \tag{4.33}$$

where

$$l_{x_2}=\chi_1(k_{\rho\,\mathrm{crit}}-k_\rho)\ , \tag{4.34}$$

and M_1^* is $\partial\ln M(r)/\partial\ln r$ evaluated at $\lambda=1$; this approximation has the correct values and logarithmic derivatives at the boundaries, just as does the TPA. Integration gives

$$\tilde{M}(\lambda)=\lambda^{\gamma(3-k_\rho)/(\gamma-1)}\exp\left[\frac{(3-k_\rho)(\lambda^{l_{x_2}}-1)}{(\gamma+1)(k_{\rho\,\mathrm{crit}}-k_\rho)}\right]\ . \tag{4.35}$$

Now

$$d\tilde{M}=(3-k_\rho)\chi_1\tilde{\rho}(\lambda)\lambda^2d\lambda\ , \tag{4.36}$$

so that

$$M_1^*=(3-k_\rho)\chi_1 \tag{4.37}$$

and

$$\frac{M^*}{M_1^*}=\frac{\tilde{\rho}\lambda^3}{\tilde{M}}\ . \tag{4.38}$$

Combining Eqs. (4.33) and (4.38) gives

$$\tilde{\rho}(\lambda)=\frac{(\gamma+\lambda^{l_{x_2}})}{(\gamma+1)}\frac{\tilde{M}(\lambda)}{\lambda^3}\ , \tag{4.39}$$

with $\tilde{M}(\lambda)$ given by Eq. (4.35).

One reason for the accuracy of this approximation is that

$$\frac{M^*}{M_1^*}=\frac{1-\nu_1}{1-\nu} \tag{4.40}$$

for self-similar blastwaves, according to Eqs. (4.38) and (B47); since $(1-\nu)$ changes by only a small factor from the center to the edge of the blastwave, the quantity being approximated is nearly constant. Solving Eq. (4.40) for $\tilde{v}=\lambda v/v_1$ yields

$$\tilde{v}=\lambda\left[\frac{1+\lambda^{l_{x_2}}}{\gamma+\lambda^{l_{x_2}}}\right]\left[\frac{\gamma+1}{2}\right] . \tag{4.41}$$

This result explicitly demonstrates the linearity of v near the origin. Because $l_{x_2}=0$ at $k_{\rho\,\mathrm{crit}}$, the LVA is recovered when it is exact. Finally, the pressure can be obtained from the adiabatic condition derived in Appendix B, Eq. (B56).

For Sedov-Taylor blastwaves ($k_\rho=k_E=0$) with $\gamma=\frac{5}{3}$, Kahn's approximation for the density is accurate locally to 4% (Cox and Franco, 1981), whereas the TPA is accurate only to 17%. However, the presence of the exponential in Kahn's approximation renders it unsuitable for analytic calculation of integral quantities, whereas the accuracy of such quantities calculated with the TPA is generally much better than 1%.

C. Density and velocity structure of hollow blastwaves ($\lambda_i>0$)

Blastwaves in steep density gradients ($k_\rho>k_{\rho\,\mathrm{crit}}$), blastwaves with central energy injection (Sec. VII), and cosmological blastwaves (Sec. IX) all develop central cavities, so that the swept-up mass is confined to a shell, $0<\lambda_i\le\lambda\le 1$. Fully radiative, momentum-conserving blastwaves are also hollow, with the swept-up mass in a thin shell at $\lambda\simeq 1$. We now consider several approximations for hollow blastwaves in order of increasing sophistication.

1. Thin-shell approximation

In this approximation all the gas is concentrated at $r=R_s$ so that

$$K_{mn}=1 , \tag{4.42}$$

except for radiative bubbles (Sec. VII.B), in which the kinetic energy of the injected fluid makes $K_{02}>1$. This approximation is appropriate for most radiative blastwaves; an exception is radiative *evaporative* blastwaves, in which the radiative shell is formed in the interior rather than at the edge (Cowie, McKee, and Ostriker, 1981). The thin-shell approximation is also appropriate for adiabatic blastwaves with a soft equation of state, $\gamma\to 1$; such blastwaves have the same moments K_{mn} as do radiative blastwaves, but they have α^2 finite and $\sigma\to\infty$ rather than the values $\alpha^2\to\infty$ and σ finite appropriate for momentum-conserving blastwaves (Sec. VI).

Historically, the thin-shell approximation has been used in treating the equation of motion for nonradiative blastwaves [Eq. (2.15)] as well as for radiative blastwaves. However, there is an unfortunate kinematic inconsistency in this approach: If all the gas is in a thin shell at R_s, then that shell must move at the shock velocity v_s, rather than at the shocked fluid velocity v_1, and the two differ unless $\gamma=1$. We shall use the thin-shell approximation only for $\gamma\simeq 1$. Note that the approximations discussed above (LVA, OPA, and TPA) all reduce to the thin-shell approximation in the limit $\gamma\to 1$.

2. Shell approximation

The simplest generalization of the thin-shell approximation that removes the kinematic inconsistency of the latter is to assume the flow velocity inside the blastwave is fixed at $v=v_1$. In this case the moments satisfy the relation

$$K_{mn}=K_{m0}\equiv K_m \quad (\text{shell}) . \tag{4.43}$$

The kinematic moment relation (4.5) becomes

$$(3-k_\rho+m)K_m(\text{shell})=3-k_\rho+\frac{2m}{\gamma+1}K_{m-1}(\text{shell}) . \tag{4.44}$$

Since $K_0(\text{shell})=K_{00}=1$, this relation determines the moments:

$$K_{11}\simeq K_1(\text{shell})=\frac{(3-k_\rho)(\gamma+1)+2}{(\gamma+1)(4-k_\rho)} , \tag{4.45}$$

$$K_{20}\simeq K_2(\text{shell})=\frac{(4-k_\rho)(3-k_\rho)(\gamma+1)^2+4(3-k_\rho)(\gamma+1)+8}{(\gamma+1)^2(4-k_\rho)(5-k_\rho)} . \tag{4.46}$$

In the shell approximation, the density can be obtained by integrating the equation of continuity (2.1), with the result

$$\tilde{\rho}(\lambda)=\frac{1}{\lambda^2}\left[\frac{\lambda-\nu_1}{1-\nu_1}\right]^{2-k_\rho} . \tag{4.47}$$

Note that $\lambda_i=\nu_1$ in this approximation. The accuracy of the approximation is best when the shell is thin, which requires $\nu_1\simeq 1$. Below, we shall use this approximation both for bubbles and, after suitable generalization, for cosmological blastwaves.

3. Shell one-power approximation ($k_E=0$)

A considerable improvement in the accuracy of the approximation for hollow blastwaves can be attained by allowing a power-law variation of velocity behind the shock. The simple one-power approximation for filled blastwaves, $\tilde{x}=\lambda^{l_x}$, becomes

$$\tilde{x}=\tilde{x}_i+(1-\tilde{x}_i)\left[\frac{\lambda-\lambda_i}{1-\lambda_i}\right]^{l_x} \tag{4.48}$$

for hollow blastwaves, where $\tilde{x}_i\equiv\tilde{x}(\lambda_i)$ and λ is restricted to the range $\lambda_i\le\lambda\le 1$.

We now demonstrate that the velocity is a linear function of position in this approximation; in other words, the shell OPA is equivalent to the shell LVA. The logarithmic derivative of $\tilde{x}$ is

$$x^*=l_x\frac{\lambda}{\tilde{x}}\frac{(1-\tilde{x}_i)}{(1-\lambda_i)}\left[\frac{\lambda-\lambda_i}{1-\lambda_i}\right]^{l_x-1} . \tag{4.49}$$

Hence, if $\tilde{x}_i\neq 0$, then $x_i^*\equiv x^*(\lambda_i)$ is a finite number unequal to zero only if $l_x=1$. As shown in Appendix B.3, v_i^* is finite and nonzero; hence $l_v=1$ and the velocity is linear:

$$\tilde{v}=\tilde{v}_i+(1-\tilde{v}_i)\left[\frac{\lambda-\lambda_i}{1-\lambda_i}\right] . \tag{4.50}$$

The value of $\tilde{v}_i$ is determined by the requirement that the gas at λ_i remain there, so that $v_i=\lambda_i v_s$, or

$$\tilde{v}_i=\lambda_i/\nu_1, \quad \nu_i=1 . \tag{4.51}$$

It is convenient to express this approximation in terms of the mass fraction $\tilde{M}(\lambda)\equiv M(r)/M$. Since $\tilde{M}_i=0$, we have, from Eq. (4.48),

$$\frac{\lambda-\lambda_i}{1-\lambda_i}=\tilde{M}^{1/l_M}\equiv\tilde{M}^{j_\lambda} , \tag{4.52}$$

so that

$$\frac{\tilde{x}-\tilde{x}_i}{1-\tilde{x}_i}\equiv\tilde{M}^{j_x} \tag{4.53}$$

with

$$j_x\equiv l_x/l_M=l_x j_\lambda . \tag{4.54}$$

The logarithmic derivatives x^* and M^* are related [from Eqs. (4.48), (4.49), and (4.52)] by

$$x^*=j_x\left[\frac{\tilde{x}-\tilde{x}_i}{\tilde{x}}\right]M^* . \tag{4.55}$$

In particular, for $\tilde{x}=\lambda$ we have

$$j_\lambda=\frac{\lambda}{(\lambda-\lambda_i)M^*} . \tag{4.56}$$

The moments are

$$K_{mn}=\int\lambda^m\tilde{v}^n d\tilde{M} , \tag{4.57}$$

so that

$$K_{11}=\frac{2j_\lambda^2\lambda_i\tilde{v}_i+j_\lambda(\lambda_i+\tilde{v}_i)+j_\lambda+1}{2j_\lambda^2+3j_\lambda+1} , \tag{4.58a}$$

$$K_{02}=\frac{2j_\lambda^2\tilde{v}_i^2+2j_\lambda\tilde{v}_i+j_\lambda+1}{2j_\lambda^2+3j_\lambda+1} , \tag{4.58b}$$

and

$$K_{20}=\frac{2j_\lambda^2\lambda_i^2+2j_\lambda\lambda_i+j_\lambda+1}{2j_\lambda^2+3j_\lambda+1} . \tag{4.58c}$$

To this point the discussion of the shell OPA has been general, and applies to bubbles and cosmological blastwaves as well as Sedov blastwaves. In order to obtain values for λ_i and j_λ we can substitute Eqs. (4.58a) and (4.58c) into the exact moment relation (C9) and use the value of j_λ obtained later in this section. However, if we restrict the treatment to Sedov blastwaves ($k_E=\nu_H=0$), it is possible to use the exact, explicit expression for λ_i obtained by Sedov (1959):

$$\lambda_i=\left[\frac{2}{\gamma+1}\right]^{\eta}\frac{1}{(\gamma+1)^{\alpha_2}}\left[\frac{k_\rho-k_{\rho\,\mathrm{crit}}}{3\gamma+k_\rho-6}\right]^{\alpha_1} , \tag{4.59a}$$

where

$$\alpha_2\equiv-\frac{(\gamma-1)}{2\gamma+1-\gamma k_\rho} , \tag{4.59b}$$

$$\alpha_1\equiv\frac{\gamma(5-k_\rho)}{3\gamma-1}\left[\frac{2(6-3\gamma-k_\rho)}{\gamma(5-k_\rho)^2}-\alpha_2\right] . \tag{4.59c}$$

Note that $\lambda_i=0$ for $k_\rho=k_{\rho\,\mathrm{crit}}$ and $\lambda_i\to 1$ for $k_\rho\to\infty$.

To complete the approximation, we must select a value for j_λ. Let j_{λ_i} be the exact value of j_λ at λ_i, and let j_{λ_1} be the exact value at $\lambda=1$. Near λ_i, we have $\tilde{M}\propto(\lambda-\lambda_i)\tilde{\rho}$ so that

$$j_{\lambda_i}=\frac{1}{1+l_{\rho_i}} \tag{4.60}$$

$$=\frac{k_\rho+3\gamma-6}{\gamma(3-k_\rho)} , \tag{4.61}$$

where we have used Eq. (B42) to obtain l_{ρ_i}. On the other hand, Eq. (4.37) for M_1^* gives

$$j_{\lambda_1}=\frac{1}{(3-k_\rho)X_1(1-\lambda_i)} , \tag{4.62}$$

from Eq. (4.56). We then adopt the weighted mean

$$j_\lambda=\frac{j_{\lambda_1}+\lambda_i j_{\lambda_i}}{1+\lambda_i} , \tag{4.63}$$

which has the desirable attributes of reducing to the exact LVA for $k_\rho=k_{\rho\,\mathrm{crit}}$ and $\lambda_i=0$, and giving approximately equal weight to the inner and outer edges of the shell for $\lambda_i\to 1$. Inserting Eqs. (4.61) and (4.62) into this expression, we find

$$j_\lambda = \frac{1}{(3-k_\rho)(1+\lambda_i)}\left[\frac{\gamma-1}{(\gamma+1)(1-\lambda_i)} + \frac{\lambda_i(k_\rho+3\gamma-6)}{\gamma}\right] . \quad (4.64)$$

The result of using this expression in the approximation (4.58) for the moments is shown in Table II. When used to calculate σ [Eq. (4.4)] and ξ [Eq. (3.5)], these results are within 2% of the exact values; the value of α^2 is less accurate.

D. The pressure-gradient approximation (PGA)

A complementary approach to approximating the blastwave structure is to consider the pressure rather than the density and velocity. We adopt a power-law form for the pressure gradient,

$$\frac{d\tilde{P}}{d\lambda} = P_1^* \lambda^{l_{P_2}-1} . \quad (4.65)$$

Strictly speaking, one should give a separate analysis for hollow blastwaves ($k_\rho > k_{\rho\,\mathrm{crit}}$) without central energy injection, since such blastwaves have zero pressure in the central cavity, but the above approximation leads to a satisfactory result for the mean pressure. Integration of Eq. (4.65) yields

$$\tilde{P} = \tilde{P}(0) + \frac{P_1^*}{l_{P_2}} \lambda^{l_{P_2}} , \quad (4.66)$$

where the central pressure is

$$\tilde{P}(0) = 1 - \frac{P_1^*}{l_{P_2}} . \quad (4.67)$$

The PGA is thus a two-power approximation for the pressure.

The edge derivative P_1^* is evaluated in Appendix B, so it remains to determine l_{P_2}. For self-similar blastwaves, the equation of motion (B7) and the jump condition (E22) for θ imply

$$\frac{d\tilde{P}}{d\lambda} = P_1^* \tilde{\rho} \tilde{v} \tilde{\varphi} , \quad (4.68)$$

where

$$\tilde{\varphi} \equiv \frac{(1-\nu)v^* + (3+k_E-k_\rho)}{(1-\nu_1)v_1^* + (3+k_E-k_\rho)} . \quad (4.69)$$

The quantity $\tilde{\varphi}$ is unity at $\lambda=1$ and is usually close to unity for the entire range $0 \le \lambda \le 1$; hence we ignore it in estimating l_{P_2}. Adopting power-law approximations $\tilde{\rho} = \lambda^{l_\rho}$ and $\tilde{v} = \lambda^{l_v}$, we have $l_{P_2} - 1 = l_\rho + l_v$. The mass constraint (4.11) is equivalent to $l_\rho = l_M - 3$ with $l_M = (3-k_\rho)\chi_1$, so that

$$l_{P_2} = l_M + l_v - 2 . \quad (4.70)$$

For a first approximation, one might ignore the factor $l_v - 2$ in comparison with l_M and obtain

$$\tilde{P} = \tilde{P}(0) + P_1^* M / l_M \quad \text{(LPA)} . \quad (4.71)$$

Since the pressure is a linear function of the mass frac-

TABLE II. Hollow Sedov blastwaves ($3 > k_\rho > k_{\rho\,\mathrm{crit}}, k_E = 0$) for $\gamma = \frac{5}{3}$.

	Shell	PGA/GM	Shell OPA	Exact
		$k_\rho = 2.2$		
K_{20}	0.7470	0.6296	0.6126	0.637 07
K_{11}	0.8611	0.6420	0.6461	0.655 86
K_{02}	1.0000	0.6546	0.6833	0.676 23
α^2	8.7805	9.8077	10.2025	9.731 68
σ	0.4521	0.3370	0.3392	0.344 32
$(1-f)\xi$	1.0350	1.3883	1.3795	1.358 94
		$k_\rho = 2.5$		
K_{20}	0.7000	0.5556	0.5809	0.591 09
K_{11}	0.8333	0.5926	0.6575	0.651 82
K_{02}	1.0000	0.6321	0.7500	0.724 48
α^2	13.714	15.000	15.417	14.738
σ	0.3906	0.2778	0.3082	0.305 54
$(1-f)\xi$	0.9549	1.3429	1.2102	1.220 85
		$k_\rho = 2.8$		
K_{20}	0.6307	0.4444	0.5384	0.540 29
K_{11}	0.7917	0.5185	0.6655	0.659 10
K_{02}	1.0000	0.6049	0.8288	0.811 66
α^2	33.103	34.286	36.218	34.404
σ	0.3266	0.2139	0.2745	0.271 88
$(1-f)\xi$	0.8846	1.3505	1.0523	1.062 48

tion $\tilde{M}$, this is termed the linear pressure approximation (LPA); Gaffet (1981) has presented an extensive analysis of this approximation for astrophysical blastwaves and has extended it to higher order. Earlier, Laumbach and Probstein (1969) developed an approximation for blastwaves in exponential density gradients which reduces to the linear pressure approximation in the limit of infinite scale height. Morita and Sakashita (1978) have also considered this approximation.

To obtain the PGA, or pressure-gradient approximation, we include the term (l_v-2), approximating it as v_1^*-2. The results of Appendix B.2 then give

$$l_{P_2}=\frac{3\gamma^2+20\gamma+1-(\gamma+1)[(3\gamma+1)k_\rho-3(\gamma-1)k_E]}{2(\gamma^2-1)}, \quad (4.72)$$

$$\tilde{P}(0)=\frac{(\gamma+1)^2(k_{\rho\,\mathrm{crit}}-k_\rho-k_E)}{3\gamma^2+20\gamma+1-(\gamma+1)[(3\gamma+1)k_\rho-3(\gamma-1)k_E]}. \quad (4.73)$$

The volume average of the pressure (4.66) is

$$\bar{\bar{P}}=1-P_1^*/(3+l_{P_2}). \quad (4.74)$$

Equation (3.24) relates this average pressure to α^2:

$$\alpha^2=\frac{3(\gamma+1)(3-l_{P_2})}{2(3-k_\rho)(3+l_{P_2}-P_1^*)}. \quad (4.75)$$

With the aid of Eqs. (B21) and (4.72), this becomes

$$\alpha^2=\frac{3}{2}\,\frac{9\gamma^2+20\gamma-5-(\gamma+1)[(3\gamma+1)k_\rho-3(\gamma-1)k_E]}{(3-k_\rho)[5\gamma+1-(\gamma+1)(k_\rho+k_E)]}\quad(\mathrm{PGA}). \quad (4.76)$$

This is the basic result of the PGA. It is remarkably accurate over a wide range of γ and k_ρ; its accuracy for $k_E\neq0$ has not been checked. It is generally more accurate than the linear pressure approximation, and, in contrast to the latter, it reduces to the exact LVA for $k_\rho=k_{\rho\,\mathrm{crit}}$ and $k_E=0$. It also becomes exact in the limit $\gamma\to1$. Of the approximations considered in Table I, only the two-power approximation is more accurate. Although the derivation focused on the case $\lambda_i=0$, Eq. (4.76) is within 2% of the values of α^2 for blastwaves with $\lambda_i>0$ in Table II; indeed, it is substantially more accurate than the shell OPA.

In order to obtain the value of σ and complete the solution, a further approximation is necessary. We present two versions of different complexity and accuracy, depending on different assumptions for the ratio of the moments (K_{11}/K_{02}). Defining

$$F\equiv(\gamma+1)K_{11}/K_{02}, \quad (4.77a)$$

we can combine Eqs. (4.3) and (4.4) to give

$$\sigma=\frac{1}{\alpha^2(\gamma-1)}\left[\frac{(5-k_\rho-k_E)F-2(5-3\gamma)}{(5-k_\rho-k_E)F-4}\right]\quad(\mathrm{exact}), \quad (4.77b)$$

with α^2 given by Eq. (4.76).

1. The $\bar{K}$ approximation (PGA/$\bar{K}$)

The simplest approximation is to take $K_{11}=K_{02}$, which we shall call the $\bar{K}$ approximation. This is not the LVA; to recover that approximation, it is necessary to set $K_{11}=K_{02}$ in evaluating α^2 also. Equations (4.76) and (4.77b) then yield

$$\sigma=\frac{2(3-k_\rho)[11\gamma-5-(\gamma+1)(k_\rho+k_E)]}{3(\gamma-1)\{9\gamma^2+20\gamma-5-(\gamma+1)[(3\gamma+1)k_\rho-3(\gamma-1)k_E]\}}\quad(\mathrm{PGA}/\bar{K}). \quad (4.78a)$$

As shown in Table I, the resulting values for σ for filled blastwaves agree with the exact ones to within 2% for the cases shown. The $\bar{K}$ approximation is exact at $k_\rho=k_{\rho\,\mathrm{crit}}$, but becomes very poor for hollow blastwaves $(k_\rho>k_{\rho\,\mathrm{crit}})$.

2. The geometric mean approximation (PGA/GM)

With α^2 determined by the PGA, Eq. (4.3) and the kinematic moment relation (C9) provide two of the three relations needed to determine all three second-order moments. To close the system, we adopt the geometric mean approximation, $K_{11}^2=K_{20}K_{02}$, which is satisfied to within 1% for the cases in Tables I and II. Equations (4.3), (4.77a), and (C13b) then imply (independently of the PGA)

$$\alpha^2=\frac{3F}{3-k_\rho}\left[\frac{(5-k_\rho)F-4}{(5-k_\rho-k_E)F-4}\right]\quad(\mathrm{GM}). \quad (4.78b)$$

For energy-conserving blastwaves $(k_E=0)$, this immediately implies

$$F=\tfrac{1}{3}(3-k_\rho)\alpha^2\quad(\mathrm{GM},\ k_E=0). \quad (4.78c)$$

Equations (4.78b), (4.78c), together with (4.76) determine

F in the PGA and consequently fix the moment ratios via (4.77a), and σ through (4.77b). The resulting values of σ are somewhat less accurate than those from the $\bar{K}$ approximation for the filled blastwaves illustrated in Table I, despite the fact that the underlying approximation in the latter case is less accurate. However, for hollow blastwaves the values of σ from the PGA/GM approximation are substantially better than those from the $\bar{K}$ approximation (Table II), which suggests that the PGA/GM is a robust approximation. The accuracy of α^2 is substantially better than that of the moments and σ in this case. As for the $\bar{K}$ approximation, this approximation becomes exact at $k_\rho = k_{\rho\,\mathrm{crit}}$, with F reducing to $(\gamma+1)$ as it should.

For the more general case in which $k_E \neq 0$, Eq. (4.78b) gives

$$F = \frac{B + [B^2 - 48(3-k_\rho)(5-k_\rho)\alpha^2]^{1/2}}{6(5-k_\rho)} \quad \text{(PGA/GM)}\,, \tag{4.78d}$$

where

TABLE III. Sedov-Taylor blastwaves in media with power-law density variations ($k_E = v_H = W = \lambda_{\mathrm{cl}}^{-1} = f_0 - f_i = \gamma - \gamma_i = \dot{m} = \varepsilon = P_0 = 0$), $\rho_0(R) \equiv \rho_0(1)[R_s/R_s(1)]^{-k_\rho}$.

Result	$\gamma = \frac{5}{3}$	Note
$R_s = R_s(1)\left[\frac{\xi E_b t^2}{\bar{\rho}(1) R_s(1)^5}\right]^{1/(5-k_\rho)}$		Exact[a]
$= \left[\frac{1.52\times 10^{-3}(1-\frac{1}{3}k_\rho)\xi E_{51}}{n_0(1)}\left[\frac{t}{1\ \mathrm{y}}\right]^2\right]^{1/(5-k_\rho)}$ pc		
$t = 25.6\left[\frac{n_0(1)}{(1-\frac{1}{3}k_\rho)\xi E_{51}}\right]^{1/2}\left[\frac{R_s}{1\ \mathrm{pc}}\right]^{(5-k_\rho)/2}$ y		Exact[a]
$v_s = 1.53\times 10^4\left[\frac{(1-\frac{1}{3}k_\rho)\xi E_{51}}{(1-\frac{1}{5}k_\rho)^2 n_0(1)}\right]^{1/2}\left[\frac{R_s}{1\ \mathrm{pc}}\right]^{-(3-k_\rho)/2}\frac{\mathrm{km}}{\mathrm{s}}$		Exact[a]
$\sigma \equiv \frac{E_b}{M v_s^2}$		PGA[b]
$\simeq \frac{2(3-k_\rho)[11\gamma - 5 - (\gamma+1)k_\rho]}{3(\gamma-1)[9\gamma^2+20\gamma-5-(\gamma+1)(3\gamma+1)k_\rho]}$	$\frac{(3-k_\rho)(5-k_\rho)}{2(10-3k_\rho)}$	$+\bar{K}$ ($k_\rho < k_{\rho\,\mathrm{crit}}$)
$\xi \equiv \frac{3}{4\pi\eta^2\sigma(1-f)}$		PGA[b]
$\simeq \frac{9(5-k_\rho)^2(\gamma-1)[9\gamma^2+20\gamma-5-(\gamma+1)(3\gamma+1)k_\rho]}{32\pi(1-f)(3-k_\rho)[11\gamma-5-(\gamma+1)k_\rho]}$	$\frac{3(5-k_\rho)(10-3k_\rho)}{8\pi(1-f)(3-k_\rho)}$	$+\bar{K}$
$\alpha^2 \equiv \frac{\bar{\rho} v_s^2}{\bar{P}} \simeq \frac{3[9\gamma^2+20\gamma-5-(\gamma+1)(3\gamma+1)k_\rho]}{2(3-k_\rho)[5\gamma+1-(\gamma+1)k_\rho]}$	$\frac{6(10-3k_\rho)}{(3-k_\rho)(7-2k_\rho)}$	PGA[b]
$\frac{E_{\mathrm{th}}}{E} = \frac{1}{\alpha^2\sigma(\gamma-1)} \simeq \frac{5\gamma+1-(\gamma+1)k_\rho}{11\gamma-5-(\gamma+1)k_\rho}$	$\frac{7-2k_\rho}{2(5-k_\rho)}$	PGA[b]
$\frac{\bar{P}}{P_1} \simeq \frac{(\gamma+1)[5\gamma+1-(\gamma+1)k_\rho]}{9\gamma^2+20\gamma-5-(\gamma+1)(3\gamma+1)k_\rho}$	$\frac{2(7-2k_\rho)}{3(10-3k_\rho)}$	PGA[b]
$\frac{\bar{\rho}}{\rho_1} = \frac{3(\gamma-1)}{(3-k_\rho)(\gamma+1)}$	$\frac{3}{4(3-k_\rho)}$	Exact
$\frac{\bar{T}}{T_1} \equiv \frac{\bar{P}/P_1}{\bar{\rho}/\rho_1} \simeq \frac{(\gamma+1)^2(3-k_\rho)[5\gamma+1-(\gamma+1)k_\rho]}{3(\gamma-1)[9\gamma^2+20\gamma-5-(\gamma+1)(3\gamma+1)k_\rho]}$	$\frac{8(3-k_\rho)(7-2k_\rho)}{9(10-3k_\rho)}$	PGA[b] (mass-weighted average)

[a] 1 pc $= 3.086\times 10^{18}$ cm; $E_{51} \equiv E_b/(10^{51}$ erg); the conversion from mass density ρ_0 to hydrogen number density n_0 is based on cosmic abundances, with $\rho_0/n_0 = 2.34\times 10^{-24}$ g.
[b] See Secs. IV.D and IV.E for details of derivation and applicability.

$$B \equiv (3-k_\rho)(5-k_\rho-k_E)\alpha^2+12 \; . \qquad (4.78e)$$

For $\gamma=1$, Eqs. (4.75) and (4.78d) imply $F=2$, corresponding to $K_{11}=K_{02}$, as it should in this case.

E. Application to Sedov blastwaves ($k_E=0$)

1. Dependence on density distribution (k_ρ)

Sedov (1959) was the first to obtain the solution for blastwaves in media with a power-law variation of the density, $\rho_0 \propto R^{-k_\rho}$. The methods discussed above allow us to test our approximations for determination of constants such as σ and α^2 which can otherwise be obtained only through numerical integration; the results are summarized in Table III. The nature of the blastwave changes as k_ρ increases.

$k_\rho<0$. *Blastwave in a preexisting cavity* (Cox and Franco, 1981). As k_ρ becomes large and negative, the blastwave decelerates rapidly ($v_s \propto R^{-(3-k_\rho)/2}$). The pressure becomes uniform except for a sharp rise at the edge of the blastwave [$l_{P_2} \propto -k_\rho$ in Eq. (4.72)], which is required to effect the deceleration.

$k_\rho=0$. *Sedov-Taylor blastwave.* The classic problem of the dynamics of a strong explosion in a uniform medium (Sedov, 1946, 1959; Taylor, 1950; von Neumann, 1947) has the solution

$$R_s=(\xi_{\rm ST}E/\rho_0)^{1/5}t^{2/5} \qquad (4.79)$$

with

$$\xi_{\rm ST}=2.026\ldots/(1-f) \qquad (4.80)$$

for $\gamma=\frac{5}{3}$; approximate values of ξ for $\gamma\neq\frac{5}{3}$ can be obtained from Table III. The accuracy of the approximations developed above is portrayed in Table I for several values of γ. The two-power approximation (TPA) and the pressure-gradient approximation (PGA) are both quite accurate (better than about 0.4% and 2%, respectively, in Table I), and both can be used to treat the large part of the three-parameter space (γ, k_ρ, k_E) for which numerical solutions are not available.

$0<k_\rho\le k_{\rho\,crit}=(7-\gamma)/(\gamma+1)$. *Filled blastwaves in a decreasing density gradient.* For $k_\rho<k_{\rho\,\rm crit}$ the pressure is finite at the origin; the case $k_\rho=k_{\rho\,\rm crit}$ will be discussed further below. The most important astrophysical case of a power-law density gradient is $\rho_0 \propto r^{-2}$, which corresponds to an explosion in a preexisting supersonic wind. The condition $k_\rho \le k_{\rho\,\rm crit}$ imposes the restriction $\gamma\le\frac{5}{3}$; at $\gamma=\frac{5}{3}$, the velocity is linear. It is useful to define a (possibly fictional) mass flow $\dot M_w$ and wind velocity v_w in a hypothetical preexisting wind such that

$$R_s^2\bar\rho(R_s)=3R_s^2\rho_0(R_s)=3\dot M_w/4\pi v_w \; . \qquad (4.81)$$

Inserting this result into Eq. (3.10) for $R_s(t)$ gives

$$R_s(t)=\left[\frac{4\pi\xi v_w E t^2}{3\dot M_w}\right]^{1/3}, \quad k_\rho=2 \; . \qquad (4.82)$$

In this case the PGA expressions for the constants are

$$\alpha^2=\frac{3(3\gamma^2+12\gamma-7)}{2(3\gamma-1)} \, ,$$

$$\sigma=\frac{2(9\gamma-7)}{3(\gamma-1)(3\gamma^2+12\gamma-7)} \, ,$$

$$\text{PGA (exact for } \gamma=\tfrac{5}{3}, k_\rho=2) \, , \qquad (4.83)$$

$$\xi=\frac{81(\gamma-1)(3\gamma^2+12\gamma-7)}{32\pi(1-f)(9\gamma-7)} \, ,$$

from Table III.

Finally, for $\gamma=\frac{5}{3}$ the PGA is exact, and Eq. (4.83) yields $\alpha^2=8$, $\sigma=\frac{3}{8}$, and $4\pi\xi/3=6/(1-f)$; Eq. (3.18) shows that exactly half of the energy is thermal. Our results for this particular case ($k_\rho=2$, $\gamma=\frac{5}{3}$) agree with those of previous workers (Sedov, 1959; Chevalier, 1982).

$k_\rho=k_{\rho\,crit}$. *Homologous blastwaves.* In this case the velocity field is a linear function of position,

$$v(r,t)=v_1\lambda v_s=v_1 r v_s/R_s \; , \qquad (4.84)$$

so that the linear velocity approximation (as well as the OPA, TPA, and PGA) is exact. This solution is sometimes referred to as the Primakoff solution because Primakoff (cited in Courant and Friedrichs, 1948) studied it in connection with underwater explosions, for which $\gamma\simeq7$ and $k_\rho\simeq k_{\rho\,\rm crit}=0$. The power-law indexes for the density and pressure are

$$l_\rho=1, \quad l_P=3 \; , \qquad (4.85)$$

independent of γ, from Eqs. (4.15) and (4.72). Astrophysically, the most important example of a homologous blastwave is a blastwave in a wind ($k_\rho=2$) with $\gamma=\frac{5}{3}$, discussed above.

$k_{\rho\,crit}<k_\rho<3$. *Hollow blastwaves with finite mass.* In sufficiently steep density gradients, a vacuum forms at the center and the shocked intercloud gas is confined to $\lambda_i\le\lambda\le1$; for $k_\rho<3$, the total mass in the blastwave is finite. Approximations for this case were developed in Sec. IV.C and are compared with the exact solution in Table II. As shown in Eq. (4.51), the velocity at λ_i is $v_i=\lambda_i v_s$. The temperature and pressure both vanish at λ_i [see Eqs. (B30) and (B43)]. The density there vanishes if $k_\rho<6/(\gamma+1)$ and becomes infinite if $k_\rho>6/(\gamma+1)$, from Eq. (B42).

$3<k_\rho<5$. *Accelerating blastwaves.* Blastwaves in this range of density gradients are also hollow, but they formally have infinite mass and energy; hence the similarity solution applies only to an undetermined region near the shock front, and our integral methods cannot be applied. Since $k_\rho>3$, the exponent $\eta=2/(5-k_\rho)$ exceeds unity and the blastwave accelerates. Provided $k_\rho<5$, the velocity and acceleration remain finite.

2. Blastwaves with cosmic rays ($\gamma \neq \gamma_i$)

As an application of these results for a case in which $\gamma \neq \frac{5}{3}$, consider a blastwave in which cosmic rays are accelerated at the shock (e.g., Blandford and Ostriker, 1978). Chevalier (1983) has found a similarity solution for this problem under the assumptions that the cosmic rays and thermal gas have the same mean velocity and that they each expand adiabatically, with specific-heat ratios of $\frac{4}{3}$ and $\frac{5}{3}$, respectively. If the fraction of the postshock pressure in cosmic rays is denoted by $w_{\rm cr}$, then the effective value of γ at the shock is determined by the energy jump condition to be

$$\frac{\gamma_{\rm eff}}{\gamma_{\rm eff}-1}=\tfrac{5}{2}(1-w_{\rm cr})+4w_{\rm cr} , \tag{4.86}$$

so that (Chevalier, 1983)

$$\gamma_{\rm eff}=\frac{5+3w_{\rm cr}}{3(1+w_{\rm cr})} . \tag{4.87}$$

Note that only the *relativistic* cosmic rays contribute to $w_{\rm cr}$. The value of α^2 can be determined from Table III, with γ replaced by $\gamma_{\rm eff}$.

In general, the internal ratio of specific heats γ_i that enters into σ [see Eq. (4.4)] is less than $\gamma_{\rm eff}$, since the expansion of the gas behind the shock reduces the pressure of the thermal gas more rapidly than that of the cosmic rays. The quantity γ_i enters in Eq. (3.14) for the energy, whence

$$\frac{\bar{P}}{\gamma_i-1}=\tfrac{3}{2}\bar{P}_{\rm th}+3\bar{P}_{\rm cr} , \tag{4.88}$$

where $\bar{P}_{\rm th}$ and $\bar{P}_{\rm cr}$ are the mean pressures of the thermal gas and cosmic rays, respectively. The mean pressure of a component k of the gas in the blastwave can be calculated with the aid of Eq. (B56) for the pressure; in the one-power approximation for the density with $l_\rho = l_M - 3 = \chi_1(3-k_\rho)-3$ [cf. Eq. (4.37)], we find

$$\bar{\bar{P}}_k \equiv \bar{P}_k / P_{k_1} = \frac{1}{(\chi_1-1)(\gamma_k-1)} . \tag{4.89}$$

[Note that the compression χ_1 is $(\gamma_{\rm eff}+1)/(\gamma_{\rm eff}-1)$ and does not explicitly depend on γ_k.] Using Eqs. (4.88), (4.89), and the definition $w_{\rm cr}\equiv P_{\rm cr,1}/(P_{\rm th,1}+P_{\rm cr,1})$, with $\bar{P}=\bar{P}_{\rm th}+\bar{P}_{\rm cr}$, we find

$$\gamma_i-1=\frac{2}{3}\left[\frac{1+w_{\rm cr}}{1+3w_{\rm cr}}\right] . \tag{4.90}$$

The values of σ (and hence ξ) can be obtained from α^2 in the $\bar{K}$ approximation from Eqs. (4.3) and (4.4),

$$\sigma=\frac{6(\gamma_i-1)+5\gamma+1-(\gamma+1)k_\rho}{(\gamma_i-1)[5\gamma+1-(\gamma+1)k_\rho]\alpha^2} , \tag{4.91}$$

which becomes, on taking α^2 from the PGA [Eq. (4.76)],

$$\sigma=\frac{(3-k_\rho)\{4(1+w_{\rm cr})+(1+3w_{\rm cr})[5\gamma_{\rm eff}+1-(\gamma_{\rm eff}+1)k_\rho]\}}{(1+w_{\rm cr})[9\gamma_{\rm eff}^2+20\gamma_{\rm eff}-5-(\gamma_{\rm eff}+1)(3\gamma_{\rm eff}+1)k_\rho]} . \tag{4.92}$$

For a uniform ambient medium ($k_\rho=0$), this is within 2% of Chevalier's numerical results. As the value of $w_{\rm cr}$ goes from 0 to 1, the corresponding value of ξ drops by a factor of about 2: the pressure is lower for a given energy at $\gamma=\frac{4}{3}$ than at $\gamma=\frac{5}{3}$, so the blastwave expands more slowly.

3. Blastwave structure and emissivity ($k_\rho \le k_{\rho\,\rm crit}$)

Next, consider the internal structure of the blastwaves. We focus on filled blastwaves ($k_\rho \le k_{\rho\,\rm crit}$) for simplicity. The most accurate expressions for $\tilde{\rho}$, $\tilde{P}$, and $\tilde{v}$ are given by Kahn's approximation (Sec. IV. B.4). The two-power approximation for $\tilde{\rho}$ and $\tilde{v}$ is given in Eqs. (4.25)–(4.28), and that for $\tilde{P}$ is given by Eqs. (4.66), (4.67), (4.72), and (B21). In some cases, the yet simpler one-power approximation $\tilde{x}=\lambda^{l_x}$ is required. The choice of such an approximation is necessarily somewhat arbitrary. If values near the edge of the blastwave are needed, the edge derivatives (B19)–(B21) can be used for the exponents l_x. Alternatively, an integral constraint can be used: For the density, the value of l_ρ which ensures $M=\int \rho\, dV$ is given by Eq. (4.15); it is related to l_M by $l_\rho = l_M - 3$, and it differs from ρ_1^* for $k_\rho \neq k_{\rho\,\rm crit}$.

A quantity of considerable importance in astrophysical blastwaves is the total luminosity L. In low-density gas in which the emission varies as the square of the particle density n, the luminosity L can be expressed as

$$L=\int n^2\Lambda(T)(1-f)4\pi r^2 dr \tag{4.93}$$

in the absence of absorption, where $n^2\Lambda$ is the emissivity per unit volume. Since astrophysical plasmas are predominantly hydrogen, we adopt the convention that n is the density of hydrogen nuclei; hence $n=\rho/\mu_H$, where μ_H is the mass per hydrogen nucleus. If Λ has an approximate power-law dependence on temperature, $\Lambda \propto T^m$, this becomes (Cox and Franco, 1981)

$$\begin{aligned} L&=\frac{4\pi}{3}R_s^3(1-f)\bar{n}n_0\Lambda(T_1)I_m \\ &=\left[\frac{M}{\mu_H}\right]n_0\Lambda(T_1)I_m \end{aligned} \tag{4.94}$$

with

$$I_m \equiv \chi_1^2(3-k_\rho)\int \tilde{\rho}^2\tilde{T}^m\lambda^2 d\lambda . \tag{4.95}$$

The effective density of the particles radiating at a temperature T_1 is thus $I_m n_0$. [For evaporative blastwaves $\bar{n}$ is much larger than n_0, and one defines $L \propto \bar{n}(n_0+\bar{n}_{ev})I_m$, where $\bar{n}_{ev}$ is the mean density of evaporated gas.]

To evaluate I_m for adiabatic blastwaves, we express the integrand in terms of $\tilde{P}$ and the entropy variable $\tilde{s}=\tilde{P}/\tilde{\rho}^{\gamma}$, which is given exactly in terms of $\tilde{M}$ by Eq. (4.13):

$$I_m = \chi_1 \int \tilde{P}^{a_1} \tilde{s}^{-(1-m)/\gamma} d\tilde{M} \tag{4.96}$$

with

$$a_1 \equiv [1+m(\gamma-1)]/\gamma \ . \tag{4.97}$$

The pressure can be written in terms of $\tilde{M}$ by expressing the pressure-gradient approximation (4.66) in terms of M [using $\lambda \simeq \tilde{M}^{1/(3-k_\rho)\chi_1}$ from Eq. (4.62)]:

$$\tilde{P} \simeq \tilde{P}(0)+[1-\tilde{P}(0)]\tilde{M}^{l_{P_2}/(3-k_\rho)\chi_1} \ . \tag{4.98}$$

This procedure, which is reasonably accurate for filled blastwaves ($k_\rho < k_{\rho\,\rm crit}$) is exact for $k_\rho = k_{\rho\,\rm crit}$, but becomes increasingly inaccurate as k_ρ increases above $k_{\rho\,\rm crit}$ (hollow blastwaves). Finally, we approximate the resulting integral for I_m as a weighted mean of its values at $\tilde{P}(0)=0$ and $\tilde{P}(0)=1$ (a procedure that is exact for $a_1=0$ or 1), obtaining from Eq. (4.96)

$$I_m = \frac{\chi_1^2(3-k_\rho)}{l_{P_2}} \left[\frac{1-\tilde{P}(0)}{a_1+a_2} + \frac{\tilde{P}(0)}{a_2} \right] , \tag{4.99}$$

where

$$a_2 = \frac{\chi_1}{\gamma l_{P_2}} [3(\gamma+1)-2\gamma k_\rho + k_E - m(3+k_E-\gamma k_\rho)] , \tag{4.100}$$

and l_{P_2} and $\tilde{P}(0)$ are given by Eqs. (4.72) and (4.73). This result is exact for $k_\rho = k_{\rho\,\rm crit}$ and $k_E=0$, since then the PGA is exact and $\tilde{P}(0)=0$.

The cases of greatest astrophysical interest are $m=\frac{1}{2}$, corresponding to bremsstrahlung cooling, and $m=-\frac{1}{2}$, corresponding to line cooling in a plasma of cosmic abundances with $10^5 \lesssim T \lesssim 10^{7.5}$ K (Kahn, 1976). For $\gamma=\frac{5}{3}$ and $k_E=0$, we find

$$I_{1/2} = \frac{40}{9} \frac{(3-k_\rho)}{(8-3k_\rho)} \left[\frac{16-5k_\rho}{21-8k_\rho} + \frac{12(2-k_\rho)}{39-15k_\rho} \right] , \tag{4.101}$$

$$I_{-1/2} = \frac{80}{3} \frac{(3-k_\rho)}{(8-3k_\rho)} \left[\frac{16-5k_\rho}{138-59k_\rho} + \frac{2(2-k_\rho)}{57-25k_\rho} \right] , \tag{4.102}$$

which are within 1.5% of the numerical results of Cox and Franco (1981) for $0 \geq k_\rho \geq -4$; they are exact for $k_\rho = k_{\rho\,\rm crit} = 2$. Cox (1986) has given an alternative, somewhat simpler approximation for (4.101).

As shown in the examples (4.101) and (4.102), the radiative form factor I_m diverges for sufficiently large k_ρ. This divergence is not simply due to the fact that the PGA is less accurate for hollow blastwaves, but rather to the fact that the radiative losses for such blastwaves are concentrated toward the center rather than toward the edge as they are for ST blastwaves. For example, the cooling time, which varies as $nkT/n^2\Lambda \propto T^{1-m}/n$, decreases inward at the edge if

$$k_\rho > \frac{-2\gamma^2+3\gamma+29+2m(\gamma^2+\gamma-8)}{(\gamma+1)(\gamma+4-2m)} , \tag{4.103}$$

which lies between 1.8 and 2 for $-\frac{1}{2} < m \leq \frac{1}{2}$. Blastwaves in steeper density gradients will form dense radiative shells in the interior rather than at the edge.

V. ENERGY-CONSERVING BLASTWAVES WITH THERMAL CONDUCTION ($k_E = v_H = W = \lambda_{\rm cl}^{-1} = f_0 - f_i = \gamma - \gamma_i = P_0 = 0$)

Thermal conduction can alter the evolution of a blastwave in two ways: in a homogeneous medium, it reduces the temperature gradients and eliminates the temperature singularity that occurs at the origin in the ST solution; in an inhomogeneous medium, it leads to evaporation of the embedded clouds. The simplest way to treat conduction in a homogeneous medium is to assume that it is so efficient that it renders the temperature constant throughout the remnant. Such isothermal blastwaves were first analyzed by Korobeinikov (1956) and Korobeinikov, Melnikova, and Ryazanov (1962); Solinger, Rappaport, and Buff (1975) considered the implications of such blastwaves for supernova remnants. The blastwave radius differs only slightly from the Sedov value, but the density jump at the shock differs significantly from the adiabatic value $\chi_1=(\gamma+1)/(\gamma-1)$ due to conduction of energy to the shock front from the hot interior (for a blastwave in a steep density gradient, the heat flow is reversed). More recently, Naidu, Rao, and Yadav (1983) have developed approximate analytic solutions using the technique of Laumbach and Probstein (1969), and they have generalized the treatment to allow for energy injection. Although the assumption of isothermality is convenient and approximately accurate, it is physically inconsistent when the conduction is due to electrons in a fully ionized plasma (Lerche and Vasyliunas, 1976): if the electrons are hot enough to transport heat efficiently across the blastwave, they are too hot to maintain equipartition between the electron and ion temperatures. Numerical solutions (Cowie, 1977), including the saturation of the heat flux, which occurs when the temperature-gradient length is comparable to or larger than the blastwave radius, show that the electrons are nearly isothermal and the ions nearly adiabatic at early times; at intermediate times, thermal conduction and Coulomb collisions between the electrons and ions result in a structure somewhat similar to the idealized isothermal blastwave. Cox and Edgar (1983) have obtained the similarity solution for the early phase in which the ions are adiabatic and the electrons isothermal. An analytic model for the evolution of such a blastwave toward the adiabatic Sedov structure is given by Edgar and Cox (1984).

The evolution of a blastwave in an inhomogeneous medium is quite different from the Sedov case in which cloud evaporation is important (McKee and Ostriker, 1977; see also Cowie, 1976). The injection of mass into the interior retards the blastwave, but since the mass injection is generally less at later times when the temperature is lower, the mean density declines with time and $R_s \propto t^\eta$ with $\eta = 2/(5-k_\rho) > \frac{2}{5}$, the ST value. Chieze and Lazareff (1981) have found the similarity solution for this problem when the conductivity $\propto T^{5/2}$. Numerical solutions of SNR evolution including cloud evaporation are also available (Cowie, McKee, and Ostriker, 1981).

A. Isothermal blast waves ($\dot{m}=0$)

Although truly isothermal blastwaves with $T_e = T_i$ are an idealization unlikely to be encountered in any astrophysical setting, they do provide a useful model for assessing the maximum effects of thermal conduction in a homogeneous medium. We shall use the pressure-gradient approximation, and the $\bar{K}$ approximation (Sec. IV.D) to obtain a simple analytic solution for such blastwaves.

The assumption of isothermality allows us to relate α^2 directly to the post-shock compression χ_1. In terms of the mass-weighted average of the temperature

$$k\bar{T}/\mu \equiv \bar{P}/\bar{\rho}\ , \tag{5.1}$$

where μ is the mean mass per particle, the definition (3.17) gives

$$\alpha^2 = \mu v_s^2 / k\bar{T} = \theta_1^{-1}\ , \tag{5.2}$$

since $\bar{T} = T_1$, where θ_1 is the normalized temperature just behind the shock [Eq. (E6)]. Now the shock jump conditions imply

$$v_1 = (\chi_1 - 1)/\chi_1\ , \tag{5.3}$$

$$\theta_1 = (\chi_1 - 1)/\chi_1^2\ , \tag{5.4}$$

for a strong shock in a stationary medium [Eq. (E24)], independent of whether energy is added to or taken away from the shocked gas by thermal conduction. We conclude that

$$\alpha^2 = \chi_1^2/(\chi_1 - 1) \tag{5.5}$$

for isothermal blastwaves. Because the flux of kinetic and internal energy is not conserved across the shock in this case, however, the compression χ_1 is not known *a priori*.

The mean pressure for an isothermal blastwave follows from inserting Eqs. (5.3) and (5.5) into Eq. (3.24):

$$\frac{\bar{P}}{P_1} = \frac{3}{(3-k_{\rho_0})\chi_1}\ . \tag{5.6}$$

In the PGA, the mean pressure is

$$\frac{\bar{P}}{P_1} = 1 - \frac{P_1^*}{(3-k_{\rho_0})\chi_1 + v_1^* + 1} \tag{5.7}$$

from Eqs. (4.70) and (4.74). The edge derivatives P_1^* and v_1^* must be evaluated directly from Eqs. (B6) and (B7), since the adiabatic condition (B8) no longer holds. Since T is constant we have $P^* = \rho^*$, so that with $k_E = 0$

$$v_1^* = \frac{4-(1-k_{\rho_0})\chi_1}{2(\chi_1 - 2)}\ , \tag{5.8}$$

$$P_1^* = \frac{k_{\rho_0}\chi_1(3-\chi_1) + (3\chi_1 - 4)(\chi_1 - 1)}{2(\chi_1 - 2)}\ . \tag{5.9}$$

(Note that the mean density $\bar{\rho}$ is proportional to the external density ρ_0, so that $k_\rho = k_{\rho_0}$; to distinguish this case from that of evaporative blastwaves to follow, we consistently use k_{ρ_0} in this section to emphasize that the dynamics are governed by the external medium.) Combining Eqs. (5.6)–(5.9) then yields the solution for the shock compression:

$$\chi_1 = \frac{5 - k_{\rho_0} + (4 + k_{\rho_0} + k_{\rho_0}^2)^{1/2}}{3 - k_{\rho_0}} \quad \text{(PGA)}\ . \tag{5.10}$$

For $k_{\rho_0} = 0$, this is within 2% of the exact result.

To complete the solution, it is necessary to evaluate σ, which we do using the PGA/GM approximation developed in Sec. IV.D. Generalizing the definition of F in Eq. (4.77a) to $(2/v_1)K_{11}/K_{02}$, one finds that the basic relations (3.19) and (3.20) leave Eqs. (4.77b) and (4.78c) unchanged (note that $k_E = 0$). These last two equations together with Eq. (5.5) yield

$$\sigma = \frac{\chi_1 - 1}{(\gamma - 1)\chi_1^2}\left[\frac{(5-k_\rho)(3-k_\rho)\chi_1^2 - 6(5-3\gamma)(\chi_1 - 1)}{(5-k_\rho)(3-k_\rho)\chi_1^2 - 12(\chi_1 - 1)}\right] \quad \text{(PGA/GM)}\ . \tag{5.11}$$

For the case of greatest astrophysical interest, $\gamma = \frac{5}{3}$ and

$$\sigma = \frac{3}{2}\left[\frac{(5-k_\rho)(3-k_\rho)(\chi_1 - 1)}{(5-k_\rho)(3-k_\rho)\chi_1^2 - 12(\chi_1 - 1)}\right] \quad (\gamma = \tfrac{5}{3})\ . \tag{5.12}$$

For $k_{\rho_0} = 0$, this is within 1% of the exact value (Solinger *et al.*, 1975). The exact value of σ leads to $\xi = 3.243$, so that the blastwave radius is 10% greater than in the ST case.

Finally, note that the thermal energy, which has a relatively simple form in the adopted approximation, is

$$\frac{E_{\rm th}}{E}=\frac{(5-k_\rho)(3-k_\rho)\mathcal{X}_1^2-12(\mathcal{X}_1-1)}{(5-k_\rho)(3-k_\rho)\mathcal{X}_1^2-6(5-3\gamma)(\mathcal{X}_1-1)} \quad \text{(PGA/GM)}, \qquad (5.13)$$

which also exceeds the ST value by 10% for $(k_{\rho_0},\gamma)=(0,\frac{5}{3})$.

B. Evaporative blastwaves

Diffuse astrophysical gases are often inhomogeneous, with most of the mass concentrated in clouds that occupy a small fraction of the volume. A blastwave in such a medium propagates primarily in the lowest-density phase, engulfing the clouds in hot gas. Thermal conduction between the hot gas and the clouds results in cloud evaporation, and it has been proposed that this mechanism is responsible for regulating the structure of the hot gas in the interstellar medium (McKee and Ostriker, 1977). The injection of mass into the interior of the blastwave implies that the mean density varies with radius ($k_\rho \neq 0$). In the limit in which the evaporated mass dominates the swept-up mass, which we shall focus on, the density variation of the ambient medium ($\rho_0 \propto R^{-k_{\rho_0}}$) is irrelevant. In a less extreme situation, a self-similar solution is possible only if $k_\rho = k_{\rho_0}$ (Chieze and Lazareff, 1981).

The internal structure of an evaporative blastwave differs dramatically from that of other blastwaves in that the post-shock pressure P_1 is of order $\rho_0 v_s^2$, whereas the mean internal pressure $\bar{P}$ is of order $\bar{\rho} v_s^2 \gg \rho_0 v_s^2$. The large negative pressure gradient behind the shock accelerates the evaporated gas from the clouds (which are assumed to be stationary) up to the mean flow velocity.

Because the pressure is not a monotonic function of position, the pressure-gradient approximation is inappropriate; instead we adopt the linear velocity approximation. Some of the details of the analysis, including the modification of the kinematic moment relation by the evaporated mass, are deferred to Appendix C.

We assume that the blastwave is nonradiative and cloud crushing is unimportant, so that the total contained energy is constant. Then the solution is of the form given by Eq. (3.10) with $k_E=0$. We must first find the mean internal density $\bar{\rho}(R)$. By hypothesis all of the mass added is from evaporation of interior clouds with assumed spatial density $\omega_{\rm cl}$, so that

$$\frac{dM}{dR_s}=\frac{\langle \omega_{\rm cl}\dot{m}\rangle}{v_s}\frac{4\pi}{3}R_s^3 . \qquad (5.14)$$

We do *not* assume that the blastwave is isothermal. At any point inside the blastwave, the evaporation rate per unit volume $\omega_{\rm cl}\dot{m}$ can be parametrized as

$$\omega_{\rm cl}\dot{m}=Q(r)\left[\frac{kT(r)}{\mu}\right]^{\kappa_T}[\rho(r)]^{\kappa_\rho} , \qquad (5.15a)$$

where the coefficients $Q(r)$, κ_T, and κ_ρ depend on the nature of the evaporation (e.g., saturated or classical evaporation—see Cowie and McKee, 1977); in addition, $Q(r)$ depends on the properties of the clouds. What is needed in Eq. (5.14), however, is the volume-averaged evaporation rate $\langle \omega_{\rm cl}\dot{m}\rangle$. Hence we define the quantity $Q_{\rm av}(R)$ such that

$$\langle \omega_{\rm cl}\dot{m}\rangle = Q_{\rm av}(R)\left[\frac{k\bar{T}(R)}{\mu}\right]^{\kappa_T}[\bar{\rho}(R)]^{\kappa_\rho} . \qquad (5.15b)$$

Recall that $k\bar{T}/\mu=\bar{P}/\bar{\rho}$ [Eq. (5.1)], so that $\bar{T}$ differs somewhat from the volume-averaged temperature $\langle T\rangle$. For simplicity we shall assume that $Q(r)$ and $Q_{\rm av}(R)$ have the same radial dependence, as would be the case if Q were unaffected by the passage of the blastwave; then we write $Q\propto(r/R_s)^{-k_Q}$ and $Q_{\rm av}\propto R^{-k_Q}$. We further assume that $\rho(r)$, $T(r)$, and $P(r)$ can be expressed as power laws, $\rho\propto r^{l_\rho}$, etc. The similarity solution of Chieze and Lazareff (1981) for classical evaporation shows that the power-law approximation is good for T but crude for ρ and P; below, we therefore use integral constraints from the kinematic moment relation rather than the edge derivatives to evaluate the power laws. With these assumptions, we readily find

$$\frac{Q_{\rm av}(R)}{Q(R_s)}=\frac{3^{1-\kappa_\rho}(3+l_\rho+l_T)^{\kappa_T}}{(3-k_Q+\kappa_T l_T+\kappa_\rho l_\rho)(3+l_\rho)^{\kappa_T-\kappa_\rho}} , \qquad (5.16)$$

where we made use of the fact that $l_P=l_\rho+l_T$.

Equations (3.13) and (5.2) allow one to rewrite (5.14) in the form

$$\frac{dM}{dR}=\frac{4\pi}{3\alpha^2\kappa_T}Q_{\rm av}(1)[v_s(1)]^{2\kappa_T-1}\bar{\rho}(1)^{\kappa_\rho}R_s(1)^4R^{-k_{\rm ev}} , \qquad (5.17)$$

where

$$k_{\rm ev}\equiv -3+k_Q+\kappa_\rho k_\rho+(2\kappa_T-1)(1-\eta)/\eta . \qquad (5.18)$$

Then by differentiating the scaling relation for mass $[M=M(1)R^{-k_\rho+3}]$ and using Eq. (5.17), one finds that $k_\rho=2+k_{\rm ev}$, or, with (3.11),

$$k_\rho=\frac{k_Q+3\kappa_T-\frac{5}{2}+k_E(\kappa_T-\frac{1}{2})}{\kappa_T-\kappa_\rho+\frac{1}{2}} . \qquad (5.19)$$

Integrating Eq. (5.17) and substituting from Eqs. (3.4) and (3.13) then yields

$$\bar{\rho}(1)^{\kappa_T-\kappa_\rho+1/2}=\left[\frac{Q_{\rm av}(1)}{\alpha(3-k_\rho)(1-f)}\right]R_s(1)^{5/2-3\kappa_T}\times\left[\frac{\eta^2\xi E(1)}{\alpha^2}\right]^{\kappa_T-1/2} , \qquad (5.20)$$

which completes the determination of $\bar{\rho}(R)=\bar{\rho}(1)R^{-k_\rho}$. We have retained the dependence on k_E for future reference, but we set $k_E=0$ for the remainder of this section.

As we have seen in Eq. (5.16), the analysis of evaporative blastwaves is complicated by the necessity of determining l_ρ and l_T in addition to the usual structure parameters. Indeed, for evaporative blastwaves, the kinematic relation among the moments [Eq. (C10)] also depends on l_ρ and l_T, through Eq. (C14). Even with the linear velocity approximation the analysis is somewhat tedious and is given in Appendix C.3. Here we note a simple relation for α^2, which follows immediately from the assumption that $P(r)$ and $\rho(r)$ are power laws:

$$\alpha^2 \equiv \frac{\bar{\rho} v_s^2}{\bar{P}} = \left[\frac{3+l_\rho+l_T}{3+l_\rho}\right] \frac{\rho_1 v_s^2}{P_1} = \left[\frac{3+l_\rho+l_T}{3+l_\rho}\right] \frac{(\gamma+1)^2}{2(\gamma-1)} , \quad (5.21)$$

where the final step follows the jump condition (E22).

1. Classical evaporation

First we consider the case of classical evaporation, in which the thermal conduction has the value given by Spitzer (1962), so that $(\kappa_T,\kappa_\rho)=(\frac{5}{2},0)$. This form of the conductivity is valid only for a completely ionized nonrelativistic gas; hence $\gamma_i=\frac{5}{3}$. From Eqs. (5.19) and (3.11) we find

$$k_\rho = \frac{5+k_Q}{3} , \quad (5.22a)$$

$$\eta = \frac{6}{10-k_Q} , \quad (5.22b)$$

which reduces to the familiar result $(k_\rho,\eta)=(\frac{5}{3},\frac{3}{5})$ for a uniform medium. From Eqs. (5.20) and (3.10)

$$R_s(t) = \left[\frac{(4-k_Q)(10-k_Q)^4(1-f)\xi\alpha^5 E}{3888 Q_{\rm av}(1) R_s(1)^{k_Q}}\right]^{1/(10-k_Q)} t^{6/(10-k_Q)} , \quad (5.23)$$

$$\bar{\rho}(R_s) = \left[\frac{3888}{(4-k_Q)(10-k_Q)^4} \frac{R_s(1)^{k_Q} Q_{\rm av}(1)\xi^2 E^2}{\alpha^5(1-f)}\right]^{1/3} R_s^{-(5+k_Q)/3} . \quad (5.24)$$

In order for this evaporation-dominated solution to hold, the interior density $\bar{\rho}(R)=\bar{\rho}(1)R^{-k_\rho}$ must exceed the ambient density ρ_0, which implies

$$R_s < \left[\frac{3888 Q_{\rm av}(R_{\rm ev})\xi^2 E^2}{\alpha^5(10-k_Q)^4(4-k_Q)(1-f)\rho_0^3}\right]^{1/5} \equiv R_{\rm ev} . \quad (5.25)$$

Thereafer the evolution approaches the ST result.

To complete the solution, we must evaluate the moments K_{20}, etc., the powers l_ρ and l_T, and the parameters α^2 and ξ. The solution, in the linear velocity approximation, is given in Eqs. (C12)–(C20). The moment K_{20} is determined by the evaporative moment $K_{20}^{\rm ev} \equiv (1+\zeta)^{-1}$. In the case of classical evaporation (with $\gamma=\gamma_i=\frac{5}{3}$), Eqs. (C15), (C16), (4.4), and (3.6) give

$$K_2^{\rm LVA} = K_{20} = K_{11} = K_{02} = \frac{2(4-k_Q)}{(11-2k_Q)(1+\zeta)} ,$$

$$\alpha^2 = \frac{24(1+\zeta)}{4-k_Q}, \quad \sigma = \frac{(4-k_Q)(10-k_Q)}{8(11-2k_Q)(1+\zeta)} , \quad (5.26)$$

$$\xi = \frac{1}{6\pi} \frac{(10-k_Q)(11-2k_Q)(1+\zeta)}{(1-f)(4-k_Q)} .$$

The exponents l_ρ and l_T follow from Eqs. (4.22a) and (C17):

$$l_\rho = -\left[\frac{3(11-2k_Q)\zeta+4k_Q-7}{(11-2k_Q)\zeta+3}\right] ,$$

$$l_T = \frac{2(9\zeta+1+2k_Q)}{(11-2k_Q)\zeta+3} . \quad (5.27)$$

First we consider the case of a uniform medium with $f \ll 1$ and $k_Q=0$, as treated in paper I (McKee and Ostriker, 1977). Equation (5.22) yields

$$\eta=\tfrac{3}{5}, \quad k_\rho=\tfrac{3}{5} \quad \text{(classical, } \gamma=\tfrac{5}{3},\ k_Q=0) . \quad (5.28)$$

Solution of Eq. (C19) gives $\zeta=\frac{1}{3}$, so that

$$K_{20}=K_{11}=K_{02}=\tfrac{6}{11}, \quad \alpha^2=8 ,$$

$$\sigma=\tfrac{15}{44}, \quad \xi=55/9\pi=1.945 ,$$

$$l_\rho=-\tfrac{3}{5}, \quad l_T=\tfrac{6}{5} , \quad (5.29)$$

$$Q_{\rm av}/Q=1.38 ,$$

$$R_s(t)=2.20\left[\frac{E}{Q}\right]^{1/10} t^{3/5} < 0.31 \left[\frac{QE^2}{\rho_0^3}\right]^{1/5} .$$

These results agree remarkably well with the exact similarity solution obtained by Chieze and Lazareff (1981). In the limit of vanishing external density, they found $\alpha=2.89$ (2% higher than our result) and $R_s=2.22$ $(E/Q)^{1/10}t^{3/5}$ (1% higher); their results correspond to $\xi=2.06$ and $Q_{\rm av}/Q=1.44$.

In paper I (McKee and Ostriker, 1977), the efficiency of the evaporation was measured by $\Sigma^{-1} \equiv 3f\varphi/\alpha a^2$, where $\varphi \leq 1$ is the ratio of the actual conductivity to the Spitzer value and a is the cloud radius. If a is measured in pc and Σ in pc^2, then

$$Q = 3.49 \times 10^{-73} \frac{\alpha}{\Sigma_{pc}} \text{ g s}^{-1} \text{cm}^{-3} \text{ K}^{-5/2} . \quad (5.30)$$

In paper I we estimated $\Sigma_{pc} \simeq 50$ pc^2 in the local interstellar medium. Using Eqs. (5.24) and (5.30) together with the exact Chieze and Lazareff (1981) solution, we obtain the following numerical results for classical evaporative blastwaves:

$$R_s = 0.045(E_{51}\Sigma_{pc})^{1/10}(t/1 \text{ y})^{3/5} \text{ pc} , \quad (5.31)$$

$$\bar{n} = 80.6(E_{51}^2/\Sigma_{pc})^{1/3}(R_s/1 \text{ pc})^{-5/3} \text{ cm}^{-3} \quad (f = k_Q = 0) , \quad (5.32)$$

where $\bar{n} = \bar{\rho}/\mu_H$ is the mean number density of hydrogen nuclei in a gas of cosmic abundances. These differ slightly from the results of paper I because of different values for α and ξ.

The case $k_Q = 2$ ($\omega_{cl} \propto r^{-2}$ initially) is of interest for a supernova explosion occurring in a star surrounded by an inhomogeneous wind, or a galactic explosion occurring in a galaxy about which the cloud density falls as r^{-2}. In this situation Eq. (5.22) implies

$$\eta = \tfrac{3}{4}, \quad k_\rho = \tfrac{7}{3} \quad (\text{classical}, \ \gamma = \tfrac{5}{3}, \ k_Q = 2) . \quad (5.33)$$

Solution of Eq. (C19) gives $\zeta = 3/13$, and Eqs. (5.26) and (5.27) then yield $K_{20} = \frac{13}{28}$, $l_\rho = -\frac{19}{15}$, $l_T = \frac{46}{15}$, and $\xi = 1.828$.

2. Saturated evaporation

When the mean free path in the hot intercloud gas becomes comparable to the radius of the embedded clouds, the heat flux is no longer directly proportional to the temperature gradient, and it is said to saturate or to be flux limited. An approximate theory of cloud evaporation in this case has been developed by Cowie and McKee (1977). It applies when the saturation parameter σ_0, which is proportional to the ratio of the mean free path to the cloud radius, is in the range 1–100; in astrophysical units, $\sigma_0 = (T/1.54 \times 10^7 \text{ K})^2/na_{pc}$, where n and T are the density and temperature of the intercloud gas far from the cloud. Their results imply

$$\kappa_\rho = 5/(6 + \mathcal{M}_s^2), \quad \kappa_T = \kappa_\rho(1 + \tfrac{1}{2}\mathcal{M}_s^2) , \quad (5.34)$$

where $\mathcal{M}_s$, the Mach number of the evaporative outflow in the region of saturated heat flux, is a constant that depends on the uncertain magnitude of this heat flow (see Cowie and McKee, 1977). For a typical value $\mathcal{M}_s = \sqrt{2}$, Eqs. (5.19) and (3.11) yield.

$$k_\rho = \tfrac{10}{9}, \quad \eta = \tfrac{18}{35} \quad [\text{saturated } (\mathcal{M}_s^2 = 2), \ \gamma = \tfrac{5}{3}, \ k_Q = 0] . \quad (5.35)$$

With the linear velocity approximation, Eq. (C19) then gives $\zeta = 0.671$, Eq. (C15) gives $K_{20} = K_{11} = 0.473$, and Eqs. (4.22) and (5.16) imply $l_\rho = -1.203$, $l_T = 0.587$, and $Q_{av}/Q = 1.040$. Note that this solution applies only for $\sigma_0 > 1$, or radii less than R_{sat}, where

$$R_{sat} = 0.67(1-f)(\Sigma_{pc}/a_{pc}) \text{ pc} . \quad (5.36)$$

This result is independent of the energy of the explosion and is nearly independent of $\mathcal{M}_s$.

VI. APPLICATION TO BLASTWAVES WITH MOMENTUM AND ENERGY LOSS ($v_H = W = \dot{m} = \varepsilon = 0$)

A. Radiative blastwaves in homogeneous media ($\lambda_{cl}^{-1} = f = 0$)

In all of the cases treated so far it was assumed that, although the hot gas might lose or gain energy by doing $\int P\,dV$ work, the total energy of the blast was constant. However, as is well known, after long times the interior will cool and, under most circumstances, cooling will occur first in a thin, thermally unstable shell. At that time, in the numerical simulations (Chevalier, 1974; Mansfield and Salpeter, 1974), approximately one-half of the interior mass is compressed into the thin shell, which continues to move into the unperturbed medium. The shock is now nearly isothermal, $\gamma = 1$, so subsequently the shocked gas is simply added to the shell. This phase is correspondingly called the radiative, snowplow, Oort, or momentum-limited phase, and as we shall see gives $R \propto t^{2/7}$ [for $\gamma = \frac{5}{3}$, cf. Eq. (6.14)] or $t^{1/4}$ depending on whether or not the rarefied interior gas has cooled or not; if cooling is very efficient ($\eta = \frac{1}{4}$) then the total energy, all in the form of kinetic energy of the shell, scales as $E \propto t^{-3/4} \propto R^{-3}$.

Under some circumstances cooling may occur fairly uniformly throughout the interior (e.g., when catalyzed by interior clouds), and then solutions of the type described in Sec. IV may exist. In general, however, cooling flows will either not be self-similar or will be self-similar only in the thin-shell limit. Before turning to that case let us check on the applicability of general self-similar flows. Let the average cooling function in the interior be

$$\bar{\Lambda} = \bar{\Lambda}(1) \left[\frac{\bar{\rho}}{\bar{\rho}(1)} \right]^l \left[\frac{\bar{T}}{\bar{T}(1)} \right]^m \quad (6.1)$$

(where $l = 0$ or -1 and $-2 \lesssim m \lesssim +2$, depending on temperature and composition), so that

$$\frac{dE}{dt} = -\tfrac{4}{3}\pi R_s^3 \bar{\Lambda}(\bar{\rho}, \bar{T})\bar{n}^2 , \quad (6.2)$$

where n is the particle density. Substituting the dependences assumed for the self-similar solution in Eq. (3.7) gives

$$k_E+\eta^{-1}=k_\rho(l+2)+mk_T-3\ , \tag{6.3a}$$

which can be solved for k_E using Eq. (3.11) and the relation $k_T=k_P-k_\rho=3+k_E-k_\rho$:

$$k_E=\frac{-11+6m+k_\rho(5+2l-2m)}{3-2m}\ . \tag{6.3b}$$

For almost all circumstances this gives $k_E<0$, energy increasing with radius, which is an impossible result for a radiating blastwave in the absence of energy injection. An example of a case with $k_E>0$ is a somewhat relativistic bremsstrahlung in an inverse-square density distribution ($k_\rho=2$); then $l=0$, $\frac{1}{2}<m<1$, and $k_E=(2m-1)/(3-2m)>0$. With the exception of unusual cases such as this, however, uniformly cooling blastwaves are not self-similar.

We now apply our analysis to the case in which the cooling occurs just behind the shock, so that a dense shell forms. We make the approximation that all of the mass inside the blastwave has been concentrated into the thin shell at R_s. (The flow of hot interior gas into the shell as the gas cools has been studied analytically by Gaffet, 1983.) The linear velocity approximation then holds; indeed, the moments K_{mn} are all unity. The virial theorem approach is equivalent to the radiative thin-shell equation of motion [Eq. (D13)] in this case, since $\gamma=1$,

$$\frac{d(Mv_s)}{dR_s}=4\pi R_s^2(\bar{P}-P_0)/v_s\ . \tag{6.4}$$

Let us first use this to rederive the familiar Oort (1951) solution in the limit $(\bar{P},P_0)=0$. Setting Mv_s to the constant radial momentum p_1,

$$p_1\equiv Mv_s=\text{const}\ , \tag{6.5a}$$

we have, in the absence of internal mass injection ($\dot{m}=0$), the parametric solution

$$R_s\int_0^{R_s}R_s'^2\rho_0(R_s')dR_s'-\int_0^{R_s}R_s'^3\rho_0(R_s')dR_s'=p_1t/4\pi\ , \tag{6.5b}$$

for any $\rho_0(R_s)$. For power-law distributions

$$\rho_0(R_s)\equiv\rho_0(1)R^{-k_{\rho_0}} \tag{6.6a}$$

$$=(3-k_{\rho_0})\bar{\rho}(R_s)/3\quad(\dot{m}=0)\ , \tag{6.6b}$$

this reduces to the form

$$R_s=\left[\frac{3(4-k_{\rho_0})}{4\pi}\right]^{1/4}\left[\frac{p_1t}{\bar{\rho}}\right]^{1/4}\ . \tag{6.7}$$

We designate this generalized Oort solution as the momentum-conserving snowplow (MCS). It gives the familiar result for $k_{\rho_0}=0$ and it applies to inverse-Compton-cooled explosions for which the interior cooling is very efficient even when the density is low.

Of greater relevance for the normal interstellar medium is the pressure-driven radiative shell. If the interior gas has negligible mass but significant pressure which suffers no radiation losses then, due to adiabatic decompression $\bar{P}=\bar{P}(1)R^{-3\gamma_i}$, and we may seek self-similar solutions. The fact that all of the energy of the swept-up gas is radiated away implies that the blastwave energy decreases in the thin-shell limit as

$$\frac{dE}{dt}=-4\pi R_s^2\tfrac{1}{2}(\rho_0v_s^3)\ . \tag{6.8}$$

Since the blastwave is assumed to be self-similar (the non-self-similar case has been discussed by Cox, 1972), this can be integrated to give

$$E\equiv E(1)R^{-k_E}=\frac{2\pi}{3}\frac{3-k_{\rho_0}}{k_E}\bar{\rho}(1)v_s(1)^2R_s(1)^3R^{-k_E}\ , \tag{6.9}$$

provided $k_E>0$. With the aid of Eqs. (3.6) and (3.13), this becomes an equation for k_E:

$$k_E=\frac{3-k_{\rho_0}}{2\sigma}\ . \tag{6.10}$$

In the thin-shell approximation, the moments are $K_{mn}=1$; since $\gamma=1$ for a radiative shock and $k_\rho=k_{\rho_0}$ for $\dot{m}=0$, Eq. (4.4) becomes

$$\sigma=\frac{3\gamma_i-k_E-k_{\rho_0}}{6(\gamma_i-1)}\ . \tag{6.11}$$

Simultaneous solution of Eqs. (6.10) and (6.11) gives a quadratic, the two roots for k_E being

$$k_E=\begin{cases}3-k_{\rho_0} & \text{(MCS)}\\ 3(\gamma_i-1) & \text{(PDS)}\ .\end{cases} \tag{6.12}$$

The first root is the already derived (MCS) solution for which, in the notation of Sec. III,

$$\sigma=\tfrac{1}{2},\quad \eta=\frac{1}{4-k_{\rho_0}},\quad \xi=\frac{3}{2\pi}(4-k_{\rho_0})^2\quad\text{(MCS)}\ . \tag{6.13}$$

The second root is the pressure-driven snowplow (PDS):

$$\begin{aligned}&\sigma=(3-k_{\rho_0})/6(\gamma_i-1)\ ,\\ &\eta=2/(2+3\gamma_i-k_{\rho_0})\ ,\\ &\xi=\frac{9(\gamma_i-1)(2+3\gamma_i-k_{\rho_0})^2}{8\pi(3-k_{\rho_0})}\ ,\\ &\alpha^2=1/[1-(k_{\rho_0}+3\gamma_i)/6]\quad\text{(PDS)}\ .\end{aligned} \tag{6.14}$$

For the simplest case of explosions in a uniform medium ($k_{\rho_0}=0$) with $\gamma_i=\frac{5}{3}$, Eq. (6.14) gives $\eta\equiv d\ln R/d\ln t=\frac{2}{7}\simeq 0.29$, slightly less than the value $\eta\simeq 0.31$ found by Chevalier (1974) in numerical integrations. The small difference is due to the "memory" of the Sedov-Taylor phase in the numerical solution, which leads to a greater internal pressure than in the self-similar PDS solution (Cioffi, McKee, and Bertschinger, 1988).

Even if the pressure at the beginning of the radiative phase is large, there will be no PDS-type similarity solution if the density decreases too rapidly. From its definition [Eq. (3.3)], we know that in the thin-shell case $\sigma=\frac{1}{2}+E_{\rm thermal}/2E_{\rm kinetic}>\frac{1}{2}$. Comparing this requirement with (6.14), we see that

$$\sigma>\tfrac{1}{2},\quad k_{\rho_0}<3(2-\gamma_i)\quad \text{(PDS)} \tag{6.15}$$

is required. If this is not satisfied then, however large the initial pressure, the solution after some transition period becomes of the MCS type. Furthermore, it is likely that most, if not all, solutions of the PDS type are unstable to nonradial perturbations due to the instability found by Vishniac (1983). It is not clear what, if any, effect such small-scale instabilities will have on the gross properties of the blastwave.

In either the MCS or PDS case, $R_s(t)$ is given by Eq. (3.10). We make use of the freedom in choosing $R_s(1)$ to identify it as the radius at which $E_b(R)$ would have been the initial energy E_0, labeling quantities at this point as R_c, etc. Then the solution for a radiative blastwave in a homogeneous medium is

$$R_s(t)=R_c\left[\frac{\xi E_0}{\bar\rho_c R_c^5}\right]^{\eta/2} t^{\eta}\,, \tag{6.16}$$

where ξ and η are given in Eqs. (6.13) and (6.14) for the two cases. Numerically R_c can be estimated as the point at which half the energy of the initial blastwave has been radiated away.

It is sometimes useful to have a single expression that applies approximately to both adiabatic and radiative blastwaves. Let ξ_a be the value of ξ in the adiabatic phase (Table III); let (ξ_r,η_r) be the values of (ξ,η) in the radiative phase [Eqs. (6.13) and (6.14)]. In the radiative phase we have $E=E_0(t/t_c)^{-\eta_r k_E}$. Inserting an approximate expression for ξE which has the correct limiting values into Eq. (3.5), we obtain

$$R_s(t)\simeq\left\{\frac{E_0t^2}{\bar\rho(R_s)}\left[\frac{1}{\xi_a}+\frac{1}{\xi_r}\left[\frac{t}{t_c}\right]^{\eta_r k_E}\right]^{-1}\right\}^{1/5}. \tag{6.17}$$

An alternative method for providing smooth analytic fits to blastwaves as they pass from the adiabatic to the PDS and MCS stages has been given by Cioffi *et al.* (1988).

Next let us find the solution for the case of finite external pressure ($P_0\neq 0$) with the internal pressure decreasing adiabatically as $P\propto R^{-3\gamma_i}$. The solution is no longer self-similar. We define an equilibrium radius $R_{\rm eq}$ at which the interior pressure equals P_0:

$$R_{\rm eq}\equiv R_c(P_0/\bar P_c')^{-1/3\gamma_i},\quad \bar P=P_0(R_s/R_{\rm eq})^{-3\gamma_i}\,, \tag{6.18}$$

where we continue to use $R_s(1)=R_c$, the radius at which the shell forms, and where $\bar P_c'$ is the interior pressure just after shell formation. Numerical calculations (e.g., Chevalier, 1974) suggest that P_c' is about half the pressure an adiabatic blastwave would have had at R_c. Integration of Eq. (6.4) with $(f,\lambda_{\rm cl}^{-1})=0$ yields

$$v_s^2=v_c^2\left[\frac{R_c}{R_s}\right]^{2(3-k_{\rho_0})}+\frac{6}{3(2-\gamma_i)-k_{\rho_0}}\left[\frac{P_0}{\bar\rho_c}\right]\left[\frac{R_s}{R_c}\right]^{k_{\rho_0}}\times\left\{\left[\frac{R_{\rm eq}}{R_s}\right]^{3\gamma_i}\left[1-\left[\frac{R_c}{R_s}\right]^{3(2-\gamma_i)-k_{\rho_0}}\right]-\frac{3(2-\gamma_i)-k_{\rho_0}}{6-k_{\rho_0}}\left[1-\left[\frac{R_c}{R_s}\right]^{6-k_{\rho_0}}\right]\right\}. \tag{6.19}$$

Since $k_{\rho_0}<3(2-\gamma_i)$ [Eq. (6.15)], this reduces to the PDS solution [Eqs. (6.14) and (6.16)] for $R_{\rm eq}\gg R_s\gg R_c$. At the equilibrium radius $R_{\rm eq}$, the isothermal Mach number of the shock is for $R_{\rm eq}\gg R_c$:

$$\mathcal{M}^2(R_{\rm eq})=\frac{v_s^2}{P_0/\rho_0(R_{\rm eq})}=\left[\frac{3-k_{\rho_0}}{6-k_{\rho_0}}\right]\frac{6\gamma_i}{6-3\gamma_i-k_{\rho_0}}\,; \tag{6.20}$$

for example, if $\gamma_i=\frac{5}{3}$ and $k_\rho=0$, then $\mathcal{M}^2(R_{\rm eq})=5$. The remnant will continue to expand until

$$R_{\rm stop}=R_{\rm eq}\left[\frac{6-k_{\rho_0}}{6-k_{\rho_0}-3\gamma_i}\right]^{1/(3\gamma_i)}\to 1.43R_{\rm eq}\,, \tag{6.21}$$

according to Eq. (6.19), where the numerical evaluation is for the case $(\gamma_i,k_{\rho_0})=(\frac{5}{3},0)$. At $R_{\rm stop}$, the interior is underpressured by a factor $(6-k_{\rho_0})/(6-k_{\rho_0}-3\gamma_i)\to 6$, compared to the ambient medium.

B. Radiative blastwaves in inhomogeneous media ($P_0=f_0-f_i=0$)

If the mass per unit area in the shell is small compared to that in typical clouds, the shell will lose mass as it accretes onto clouds that are passed. For simplicity, we shall focus on the case in which the clouds are uniformly distributed ($\lambda_{\rm cl}$=const), leaving the more general case for Sec. VI.C.

The solution for the momentum-conserving snowplow is trivial: since the pressure is everywhere negligible, each element of the shell expands according to Eq. (6.7) until it strikes a cloud and is absorbed. At any radius R_s the fraction of the shell that survives is $\exp(-R_s/\lambda_{cl})$.

In the case of the pressure-driven snowplow, we assume that the internal pressure causes the shock, and hence the shell, to reform after passing a cloud. For a stationary ambient medium ($v_H=0$) with negligible pressure ($P_0=0$), and in the absence of cloud crushing ($f_0=f_i=f$), the equation of motion for the shell [Eq. (D13)] becomes

$$\frac{d}{dR_s}(Mv_s)=\frac{4\pi R_s^2(1-f)\bar{P}(1)R^{-k_P}}{v_s}-\frac{Mv_s}{\lambda_{cl}} . \tag{6.22}$$

Since we are deferring discussion of blastwaves with energy input to Sec. VII, we may set $k_P=3\gamma_i$. The integral of this equation is then

$$(Mv_s)^2=8\pi\lambda_{cl}^3(1-f)\bar{P}(1)e^{-2R}\int_{R_c^*}^{R}M(R')R'^{(2-3\gamma_i)}e^{2R'}dR'+[M(R_c)v_s(R_c)]^2e^{-2(R-R_c^*)} , \tag{6.23}$$

where $M(R_c)v_s(R_c)$ is the radial momentum at the point at which the shell forms. We have adopted

$$R_s(1)=\lambda_{cl} \tag{6.24}$$

and have introduced the notation

$$R_c^*\equiv R(R_s=R_c)=R_c/\lambda_{cl} . \tag{6.25}$$

The mass in the shell obeys the equation

$$\frac{dM}{dR_s}=4\pi R_s^2\rho_0(1-f)-\frac{M}{\lambda_{cl}} , \tag{6.26}$$

for $R_s>R_c$. For a uniform ambient medium ($\rho_0=$const) this may be solved to give

$$M(R)=4\pi\lambda_{cl}^3\rho_0(1-f)(R^2-2R+2)+M_1e^{-R} , \tag{6.27}$$

where M_1 is an integration constant. Equations (6.23) and (6.27) can be integrated exactly to determine the velocity and radius of the shell. In the limit where $(R_s/R_c, R_s/\lambda_{cl})\gg 1$, we have

$$v_s\simeq[(\bar{P}(1)/\rho_0]^{1/2}R^{-3\gamma_i/2} , \tag{6.28}$$

$$R_s\simeq\lambda_{cl}\left[(1+\tfrac{3}{2}\gamma_i)\left[\frac{\bar{P}(1)}{\rho_0}\right]^{1/2}\frac{t}{\lambda_{cl}}\right]^{1/(1+3\gamma_i/2)} , \quad k_{\rho_0}=0 . \tag{6.29}$$

In this limit the solution is self-similar and can be treated by a generalization of the methods of Sec. III (see below). Note that the exponent in $R_s(t)$ is identical to that for the PDS in a homogeneous medium [Eq. (6.14)]. In the opposite limit $R_s\ll\lambda_{cl}$, but maintaining $R_s>R_c$, the solution (6.23) reduces to the non-self-similar, homogeneous PDS analyzed by Cox (1972).

C. Self-similar impeded blastwaves ($P_0=\dot{m}=0$)

We now turn to the case in which the blastwave is slowed by frictional interaction with embedded clouds. Since the frictional heat, like the entropy generated by the shock itself, is deposited in the interior where it is available to drive the shock further, energy-conserving evolution is possible. Initially in this treatment we will consider both energy input and loss, later specializing to the adiabatic and radiative cases. Examining the terms in the virial theorem [Eq. (2.14) or (3.20)], we see that no truly self-similar solution is possible because the frictional term varies as R_s/λ_{cl}. In the limit in which R_s/λ_{cl} is large, however, examination of (3.20) shows that the problem reduces to that of the "porous plug," with the pressure gradient in the remnant balanced by the viscous drag on the clouds. Both inertial and kinetic energy terms are unimportant, and an approximately self-similar solution is obtained with

$$\alpha^2\simeq\frac{3(\gamma+1)^2}{4K_{12}}\left[\frac{\lambda_{cl}}{R_s}\right] \tag{6.30a}$$

and

$$\sigma\simeq\frac{1}{\alpha^2(\gamma_i-1)} . \tag{6.30b}$$

Two cases might be considered: First, following Cox (1979), we might assume that thermal conduction can maintain an approximately constant temperature in the interior; then the pressure will be roughly constant and the expansion will be confined to the outer layer of the blastwave, $\Delta r\sim\lambda_{cl}$. In this case the moments K_{mn} are of order $\lambda_{cl}/R_s\ll 1$ for $n\geq 1$. The dynamics of such a blastwave are qualitatively similar to those of an ST blastwave (Cox, 1979). Our neglect of conduction at the shock front and our assumption that the moments are constant prevent us from treating this case.

In the more generally applicable second case, appropriate to sufficiently large blastwaves that conduction is unimportant, the interior is approximately adiabatic. We also require of course that the density be low enough or the radius small enough that interior radiative losses are not important. The drag due to the clouds prevents mass from piling up at the outer edge, so a large pressure drop develops between the interior and the edge of the blastwave ($\bar{P}\gg P_1$). This forces the gas to expand outward through a large fraction of the interior, so that the moments K_{mn} are not small and variable, as in the first case. In order to analyze this second case—an impeded blastwave with an adiabatic interior—we must generalize the treatment of Sec. III to allow for variable α^2 and σ as indicated in Eq. (6.30). In general, let

$$\alpha \equiv \alpha(1) R^{-k_\alpha}, \quad \sigma \equiv \sigma(1) R^{-k_\sigma} , \tag{6.31}$$

so that Eqs. (3.3), (3.4), (3.6), and (3.13) imply

$$\begin{aligned} k_E &= k_\sigma + 2(1-\eta)/\eta + k_\rho - 3 , \\ \xi &= \xi(1) R^{+k_\sigma} . \end{aligned} \tag{6.32}$$

Then the dynamical results [Eqs. (3.10), (3.12), and (3.13)] remain valid if ξ is replaced by $\xi(1)$; Eq. (3.11) becomes

$$\eta = \frac{2}{5+k_E-k_\sigma-k_\rho} , \tag{6.33}$$

and Eqs. (3.5) and (3.6) remain unchanged.

We further generalize our discussion by allowing for a power-law variation in the cloud mean free path,

$$\lambda_{\rm cl} \equiv \lambda_{\rm cl}(1) R^{-k_\lambda} . \tag{6.34}$$

The moment K_{12} in the cloud drag term in the virial theorem (3.20) becomes $K_{1+k_\lambda,2}$. For the particular case of an impeded blastwave with an adiabatic interior, Eq. (6.30) implies $\sigma \propto \alpha^{-2} \propto R/\lambda_{\rm cl}$, so that

$$k_\sigma = -2k_\alpha = -(1+k_\lambda) , \tag{6.35}$$

$$\sigma(1) = \frac{4K_{1+k_\lambda,2}}{3(\gamma+1)^2(\gamma_i-1)} . \tag{6.36}$$

Here we have taken

$$R_s(1) = \lambda_{\rm cl}(1) , \tag{6.37}$$

which is possible provided $k_\lambda \neq 1$ (i.e., $\lambda_{\rm cl}$ does not vary linearly with R_s); the case $k_\lambda = -1$ has σ=const and does not require any generalization of the methods of Sec. III. Equations (3.6), (3.10), and (6.33)–(6.36) then yield

$$R_s(t) = \lambda_{\rm cl}(1) \left[\frac{\xi(1)E(1)}{\bar{\rho}(1)\lambda_{\rm cl}(1)^5} \right]^{\eta/2} t^\eta , \tag{6.38a}$$

$$\xi(1) = \frac{9(\gamma_i-1)(\gamma+1)^2(6+k_E+k_\lambda-k_\rho)^2}{64\pi K_{1+k_\lambda,2}(1-f_i)} , \tag{6.38b}$$

$$\eta = \frac{2}{6+k_E+k_\lambda-k_\rho} . \tag{6.38c}$$

The transition from a Sedov-Taylor blastwave to an impeded blastwave occurs at about the point at which their respective radii are equal. Using Eq. (3.5), we find that this occurs when $\xi = \xi_{\rm ST}$, or at a radius $R_{\rm im}$,

$$\frac{R_{\rm im}}{\lambda_{\rm cl}(1)} \equiv \left[\frac{\xi(1)}{\xi_{\rm ST}} \right]^{1/(k_\lambda+1)} . \tag{6.39}$$

For $k_\lambda > -1$, the blastwave is impeded beyond this radius; for $k_\lambda < -1$, it is impeded at smaller radii. If we evaluate (6.38b) and earlier formulas for the simplest case ($\gamma = \gamma_i = \frac{5}{3}$), ($k_E = k_\rho = k_\lambda = 0$) in the linear velocity approximation, we find that adiabatic blastwaves will be impeded for $R > R_{\rm im} = (45/8)\lambda_{\rm cl}$.

In an impeded blastwave with an adiabatic interior, the mean pressure is much greater than the pressure at the shock front as the hot gas tries to force its way past the clouds. Allowing for the possibility that f_i is less than f_0 because of cloud crushing (see below), we use Eqs. (3.24) and (3.28) to obtain

$$\frac{\bar{P}}{P_1} = \frac{2K_{1+k_\lambda,2}}{(3-k_{\rho_0})(\gamma+1)} \left[\frac{1-f_0}{1-f_i} \right] \frac{R_s}{\lambda_{\rm cl}} , \tag{6.40}$$

which is very large in this case ($R_s \gg \lambda_{\rm cl}$).

As one example, consider an impeded blastwave with no radiative losses ($k_E = 0$) with $(\gamma, \gamma_i, k_\rho, k_\lambda) = (\frac{5}{3}, \frac{5}{3}, 0, 0)$ and $f_0 = f_i = f$. Evaluating K_{12} in the linear velocity approximation, which is of uncertain accuracy in this case, gives $K_{12} = \frac{4}{5}$ and $\xi = 30/\pi(1-f) = 9.55/(1-f)$. Since $\xi_{\rm ST} = 2.03/(1-f)$ (Table I), the transition from the adiabatic ST solution to the impeded blastwave occurs at $R_{\rm im} = 4.7\lambda_{\rm cl}$; thereafter $R = 4.7\lambda_{\rm cl}(t/t_{\rm im})^{1/3}$, where $t_{\rm im} = 33.9\,[\rho\lambda_{\rm cl}^5(1-f)/E]^{1/2}$.

D. Impeded blastwaves: pressure-driven snowplow ($f_0 - f_i = 0$)

Radiative blastwaves in inhomogeneous media, which were analyzed in Sec. VI.B above, can be treated with this formalism also, provided they have expanded to large radii ($R_s \gg \lambda_{\rm cl}, R_c$) so that $M \propto R^{3-k_\rho}$. We reserve the discussion of cloud crushing to Sec. VI.E, so we here set $f_0 = f_i = f$. For $R_s \gg \lambda_{\rm cl}$, the solution of Eq. (6.26) for the mass in the shell is simply

$$M = 4\pi R_s^2 \rho_0 (1-f) \lambda_{\rm cl} , \tag{6.41}$$

so that in this case

$$k_\rho = 1 + k_{\rho_0} + k_\lambda , \tag{6.42}$$

$$\bar{\rho}(1) = 3\rho_0(1) , \tag{6.43}$$

where we used Eq. (6.37) in obtaining the last result. For an impeded pressure-driven snowplow with all the mass in a thin shell, we have $k_E = 3(\gamma_i - 1)$ from Eq. (6.12) (no cloud crushing) and $K_{12} = 1$. As in the energy-conserving case in Sec. VI.C above, the kinetic energy is negligible. Then, since $\gamma = 1$, Eqs. (6.38b) and (6.38c) become

$$\xi(1) = \frac{9(\gamma_i-1)(2+3\gamma_i-k_{\rho_0})^2}{16\pi(1-f)} , \tag{6.44}$$

$$\eta = \frac{2}{2+3\gamma_i-k_{\rho_0}} . \tag{6.45}$$

To express the radius in terms of the conditions at the cooling radius R_c, note that $ER_s^{k_E} = E_0 R_c^{k_E}$ and $\rho_0 R_s^{k_{\rho_0}} = \rho_0(R_c) R_c^{k_{\rho_0}}$ for $R_s \geq R_c$, where $E(R_c) \equiv E$. [However, note that if $\lambda_{\rm cl}(R_c) > R_c$ then the blastwave goes through a homogeneous PDS phase (Sec. VI.A) before entering the impeded PDS; in contrast to ρ_0 and $E, \bar{\rho}$ does not have the same power-law dependence in the two phases, so that $\bar{\rho}R_s^{k_\rho} \neq \bar{\rho}(R_c)R_c^{k_\rho}$.] Inserting these results into Eq. (6.38a) and using Eq. (6.43) we then obtain

$$R_s(t) = \left[\frac{\xi(1)E_0 R_c^{3(\gamma_i-1)-k_{\rho_0}}}{3\rho_0(R_c)}\right]^{\eta/2} t^\eta \ . \tag{6.46}$$

Equation (6.29) for a uniform ambient medium can be recovered from Eq. (6.46) by setting $k_{\rho_0}=0$ and

$$E_0 R_c^{3(\gamma_i-1)} = 4\pi(1-f)\lambda_{\rm cl}(1)^{3\gamma_i}\bar{P}(1)/3(\gamma_i-1) \ .$$

As another example, an impeded PDS in a wind ($k_{\rho_0}=2$) with $\gamma_i=\frac{5}{3}$ has $\eta=\frac{2}{5}$, just as the adiabatic Sedov-Taylor blastwave.

Recall that pressure-driven snowplows can occur in homogeneous media only if the density gradient is not too steep, $k_{\rho_0} < 3(2-\gamma_i)$ [see Eq. (6.15)]. This restriction does not occur for an impeded PDS: since σ increases with R (for $k_\lambda > -1$), it always exceeds $\frac{1}{2}$ at sufficiently large R. In fact, Eq. (6.35) shows that at $R_s = \lambda_{\rm cl}(1)$ we have $\sigma \ge \frac{1}{2}$ for $\gamma_i \le \frac{5}{3}$ (since $K_{12}=1$ for a snowplow). However, in order for η to be positive, we do require

$$k_{\rho_0} < 2+3\gamma_i \ . \tag{6.47}$$

The solution derived in this section is of course valid only so long as the internal pressure $\bar{P}$ greatly exceeds P_0. When $(\bar{P}/P_0)$ approaches 1, a stopping solution analogous to Eq. (6.19) can be derived. It will of course not be self-similar. Since the pressure in the impeded PDS solution at the fiducial epoch when $R_s = R_s(1) = \lambda_{\rm cl}$ is $\bar{P}(1)$, an equilibrium radius can be defined analogous to that in Eq. (6.18):

$$R_{\rm eq,im} \equiv \lambda_{\rm cl}[P_0/\bar{P}(1)]^{-1/(3\gamma_i)} \ , \tag{6.48}$$

in terms of which Eqs. (6.4), (6.22), and (6.27) can be combined to give (for $R \gg 1$, $k_{\rho_0}=k_\lambda=0$)

$$\frac{d}{dR}\tfrac{1}{2}(v_s R^2)^2 + (v_s R^2)^2 = C_0^2\left[\left[\frac{\lambda_{\rm cl}}{R_{\rm eq,im}}\right]^{-3\gamma_i} R^{-3\gamma_i+4} - R^4\right] , \tag{6.49}$$

where $C_0^2 \equiv P_0/\rho_0$. In the same limit, this has the solution

$$v_s = \frac{dR_s}{dt} = C_0\left[\left[\frac{R_s}{R_{\rm eq,im}}\right]^{-3\gamma_i} - 1\right]^{1/2} . \tag{6.50}$$

The shock radius approaches $R_{\rm eq,im}$ in a finite time $t_{\rm eq}$, where

$$t_{\rm eq} = \frac{R_{\rm eq,im}}{C_0}\frac{\pi^{1/2}}{3\gamma_i}\frac{\Gamma(\frac{1}{2}+1/3\gamma_i)}{\Gamma(1+1/3\gamma_i)} = 0.501\frac{R_{\rm eq,im}}{C_0} \quad (\text{for } \gamma_i=\tfrac{5}{3}) \ . \tag{6.51}$$

E. Cloud crushing ($k_\rho = k_Q = k_\lambda = 0$)

The previous solution may be applicable during very late stages of evolution, when the pressure difference between the intercloud medium $(\bar{P},\bar{\rho})$ and the clouds is not too large. At earlier stages the energy loss due to $\int P\,dV$ work done on interior clouds can be great, as stressed by Cox (1979). This effect, which was found to produce about half of the energy loss in the numerical simulations of paper III (Cowie, McKee, and Ostriker, 1981), is treated in this section as the only energy-loss process. The treatment neglects the external pressure P_0 and the approximately equal pressure in the clouds $P_{\rm cl}$ and is thus valid only for $R < R_{\rm eq}$ [Eq. (6.18)].

When the expanding blastwave engulfs a cloud, the high pressure inside the blastwave drives a strong shock into the cloud at a velocity of about

$$v_{\rm cl} \simeq (\rho_0/\rho_{\rm cl})^{1/2} v_s \ , \tag{6.52}$$

where $\rho_{\rm cl}$ is the initial gas density in the cloud (McKee and Cowie, 1975). For $v_{\rm cl} \ge$ a few hundred km s^{-1} the cloud shock is nonradiative and the $P\,dV$ work done by the intercloud gas on the cloud is stored as internal energy in the cloud to be released back into the intercloud medium as the blastwave expands. When the cloud shock becomes radiative, however, the work done on the cloud is lost from the system. Here we assume the cloud shocks are radiative.

Let R_{cc} be the radius of the blastwave at which the clouds can be completely crushed, defined by equating the shock crossing time to the age of the blastwave: $a/v_{\rm cl} = \eta R_{cc}/v_s$, or

$$R_{\rm cc} = \left[\frac{\rho_{\rm cl}}{\rho_0}\right]^{1/2}\frac{a}{\eta} \ , \tag{6.53a}$$

where a is the initial cloud radius. In terms of the cloud filling factor $f_0 = (4\pi/3)\omega_{\rm cl}a^3$, we have $a = (3f_0/4)\lambda_{\rm cl}$, so that

$$R_{\rm cc} = \frac{3f_0}{4\eta}\left[\frac{\rho_{\rm cl}}{\rho_0}\right]^{1/2}\lambda_{\rm cl} \ . \tag{6.53b}$$

For simplicity, we set $k_\rho = k_Q = k_\lambda = 0$ so that $R_{\rm cc}$, f_0, and $\lambda_{\rm cl}$ are constant.

For $R_s < R_{\rm cc}$, the clouds are only partially crushed. Let $\beta_P P$ be the effective pressure compressing the clouds; for $R_s \lesssim R_{\rm cc}$, the clouds are being crushed throughout the volume of the blastwave, and we expect $\beta_P \simeq 1$. If the clouds are compressed spherically, the compressional energy loss is

$$\frac{dE}{dt} = -\frac{4\pi}{3}R_s^2\omega_{\rm cl}\beta_P\bar{P}4\pi a^2 v_{\rm cl} \ . \tag{6.54}$$

Integration yields (for the case where the energy is almost entirely thermal)

$$E(R_s)=E_0\exp-\left[\frac{4(\gamma_i-1)\beta_P}{1-f_i}\left[\frac{\rho_0}{\rho_{\rm cl}}\right]^{1/2}\frac{R_s}{\lambda_{\rm cl}}\right], \quad (6.55)$$

with $R_s(t)$ given by Eq. (3.5). Thus

$$E(R_{\rm cc})=E_0\exp[-3(\gamma_i-1)\beta_P f_0], \quad (6.56)$$

where $f_i \ll 1$ at $R_s=R_{\rm cc}$. Hence a significant fraction of the initial energy E_0 remains at $R_s=R_{\rm cc}$.

For $R_s>R_{\rm cc}$ the clouds are completely crushed, so the energy loss is

$$\frac{dE}{dt}=-4\pi R_s^2 v_s\beta_P\bar{P}\omega_{\rm cl}4\pi a^3/3=-4\pi R_s^2 v_s\beta_P\bar{P}f_0\ . \quad (6.57)$$

The compression is anisotropic, with most of the compression occurring in the radial direction, so that the crushed cloud has the shape of a distorted pancake (see Woodward, 1976). Then Eqs. (6.39) and (6.53b) show that the blastwave is impeded unless $f_0 \ll 1$. For an impeded blastwave the energy is entirely thermal, so that (6.57) becomes

$$\frac{dE}{dR_s}=-\frac{3\beta_P f_0(\gamma_i-1)E}{R_s}\ . \quad (6.58)$$

In this case the interior filling factor f_i is nearly zero and, since P may significantly exceed the pressure P_1 just behind the blastwave shock [Eq. (6.41)], the factor β_P may be significantly less than unity. Equation (6.58) implies

$$k_E=3\beta_P f_0(\gamma_i-1)\ , \quad (6.59)$$

and from Eqs. (6.36) and (6.38) we find

$$R_s=\left[\frac{\xi(1)E(R_{\rm cc})R_{\rm cc}^{k_E}\lambda_{\rm cl}t^2}{\bar{\rho}}\right]^{1/(6+k_E)}\ . \quad (6.60)$$

This result reflects the balance between the pressure gradient $dP/dr\propto E/R^4$ and the cloud drag $\bar{\rho}v^2/\lambda\propto\bar{\rho}R^2/t^2\lambda$, which gives $R^6\propto\lambda Et^2/\bar{\rho}$ (Cox, 1983). Note that the pressure gradient extends throughout the blastwave because the clouds impede the flow even after they are crushed. Also, note that in most applications $k_E \ll 1$, so that $R_s\propto t^{1/3}$. We may define two radii: R_{cw} is the radius at which the energy has been reduced by a factor of $1/e$ due to $\int P\,dV$ work on crushing clouds, and $R_{{\rm eq},cw}$ is the radius at which the pressure has been reduced to the ambient and equilibrium results:

$$R_{cw}\equiv R_{\rm cc}\exp(1/k_E-1)\ , \quad (6.61)$$

$$R_{{\rm eq},cw}\equiv R_{\rm cc}(2\pi P_0R_{\rm cc}^3e^{k_E}/E_0)^{-1/(3+k_E)}\ . \quad (6.62)$$

F. Summary of blastwaves in uniform inhomogeneous media ($k_{\rho_0}=k_\lambda=0,\gamma_i=\frac{5}{3}$)

Before proceeding to discuss the additional complexities introduced by energy injection, gravity, and an expanding substrate, it may be useful to review the variety of solutions already derived and to indicate the domain of validity of each type of solution. We assume a uniform ambient medium ($k_{\rho_0}=k_\lambda=0$) and $\gamma_i=\frac{5}{3}$. Several important characteristic radii have been introduced in previous sections: $R_{\rm ev}$ [Eq. (5.25)], the radius at which classical evaporative solutions switch to the Sedov-Taylor solution; $R_{\rm sat}$ [Eq. (5.30)], which marks the transition from saturated to classical evaporation; R_c, the radius at which half the energy has been radiated away by the intercloud gas; $R_{\rm im}\sim 5\lambda_{\rm cl}$ [Eq. (6.39)], the radius at which the motion of the blastwave is impeded by clouds; $R_{\rm eq}$ [Eq. (6.18)] and $R_{\rm eq,im}$ [Eq. (6.48)], the radii at which the internal pressure equals the ambient pressure; $R_{\rm cc}$ [Eq. (6.53)], the radius beyond which clouds are completely crushed; R_{cw} and $R_{{\rm eq},cw}$ [Eqs. (6.61) and (6.62)], which measure the energy losses of cloud crushing; and $R_{\rm stop}>R_{\rm eq}$ [Eq. (6.21)], where the pressure-driven snowplow finally comes to a halt. A given solution corresponds to a particular ordering of these parameters and the blastwave radius. For example, the Sedov-Taylor blastwave is characterized by $R_{\rm ev}<R_s<(R_c,R_{\rm im},R_{\rm eq})$. The values of the characteristic radii also depend on the solution; e.g., the value of $R_{\rm ev}$ given in Eq. (5.25) is based on the assumption that cloud drag, cooling, and ambient pressure are all negligible ($R_{\rm ev}<R_{\rm im},R_c,R_{\rm eq}$).

The cooling radius R_c generally determines the volume of the ambient medium heated by the blastwave. We approximate the cooling function for a gas of cosmic abundances (Raymond, Cox, and Smith, 1976) as $\Lambda=1.6\times10^{-19}T^{-1/2}$ erg cm^3 s^{-1} for 10^5 K $\lesssim T\lesssim 10^{7.5}$ K, and let $\beta>1$ be the enhancement of the radiative losses due to nonequilibrium ionization. Assuming the blastwave is neither impeded nor near pressure equilibrium and normalizing to the characteristic 10^{51} erg energy of a supernova remnant, we find that half the energy has been radiated by the intercloud gas at

$$R_c=\left[\frac{1.34\times10^{10}(7-3k_\rho)\eta^2\xi E_{51}^2}{(3-k_\rho)\beta(1-f)\bar{n}(1)^3I_{-1/2}}\right]^{1/(7-3k_\rho)}\ {\rm pc}\ , \quad (6.63a)$$

where $\bar{n}(1)=\bar{\rho}(1)/(2.34\times10^{-24}$ g) is the density of hydrogen nuclei at $R_s=1$ pc and $I_{-1/2}$ is given in Eq. (4.102). This expression reduces to

$$R_c=\begin{cases}26[E_{51}^2(1-f)^2n_0^3]^{1/7}\ {\rm pc}\ , & (6.63b)\\ 30\Sigma_{\rm pc}^{1/2}(1-f)^{1/2}\ {\rm pc}\ , & (6.63c)\end{cases}$$

for ST and evaporative blastwaves, respectively. For the ST case we set $\beta=1$ and $I_{-1/2}=1.88$ (Cox and Franco, 1981); for the evaporative case, nonequilibrium ionization effects are likely to be important so we set $\beta I_{-1/2}=10$. Note that, in the latter case, R_c is independent of the blastwave energy. For $R_s>R_c$ a dense shell forms at the periphery of the blastwave provided evaporation is unim-

TABLE IV. Catalog of solutions for uniform external medium ($\gamma_i=\frac{5}{3}$, $v_H=W=\varepsilon=0$).

(I) Adiabatic solutions	
$\mathcal{A}1$: $R_{\rm ev}<R_s<(R_s,R_{\rm im},R_{cw},R_{\rm eq})$ Sedov-Taylor, Eq. (3.5)	$\eta=\frac{2}{5}$
$\mathcal{A}2$: $(R_{\rm ev},R_{\rm im})<R_s<(R_c,R_{cw},R_{\rm eq})$ impeded adiabatic, Eq. (6.38)	$\eta=\frac{1}{3}$
(II) Evaporative solutions	
$\mathcal{E}1$: $R_s<(R_{\rm sat},R_{\rm ev},R_{cw},R_{\rm eq},R_c,R_{\rm im})$ saturated evaporation, Eq. (5.35)	$\eta=\frac{18}{35}$
$\mathcal{E}2$: $R_{\rm sat}<R_s<(R_{\rm ev},R_c,R_{\rm im},R_{cw},R_{\rm eq})$ classical evaporation, Eq. (5.23)	$\eta=\frac{3}{5}$
(III) Cloud-crushing solutions (internal $\int P\,dV$ losses)	
$\mathcal{C}1$: $R_{\rm ev}<R_s<(R_{cw},R_c,R_{\rm eq})$, partial cloud crushing, Eqs. (6.55) and (3.5)	Not self-similar
$\mathcal{C}2$: $(R_{\rm ev},R_{cw},R_{\rm im})<R_s<(R_c,R_{\rm eq})$, total cloud crushing, Eq. (6.60)	$\eta=\frac{1}{3}$
(IV) Radiative solutions ($R_{\rm ev}\ll R_c$)	
$\mathcal{R}1$: $R_c<R_s<(R_{\rm eq},R_{\rm im},R_{cw})$, pressure-driven snowplow, Eq. (6.14)	$\eta=\frac{2}{7}$
$\mathcal{R}2$: $(R_c,R_{\rm im})<R_s<(R_{\rm eq,im},R_{\rm cc})$, radiative impeded snowplow, Eqs. (6.29) and (6.46)	$\eta=\frac{2}{7}$
$\mathcal{R}3$: $(R_c,R_{\rm eq})<R_s<(R_{\rm stop},R_{\rm im},R_{\rm cc})$, stopping solution, Eq. (6.19)	Not self-similar
$\mathcal{R}4$: $(R_c,R_{\rm eq,im},R_{\rm im})<R_s<(R_{\rm stop},R_{\rm cc})$ impeded stopping solution, Eqs. (6.50) and (6.51)	Not self-similar

portant ($R_c\gg R_{\rm ev}$); solutions have been discussed in Secs. VI.A, VI.B, and VI.D. However, numerical calculations by Cowie, McKee, and Ostriker (1981) show that cooling occurs in the interior of evaporative blastwaves rather than at the edge. We have not discussed this case here.

Based on the theoretical treatment of the local interstellar medium in paper I (McKee and Ostriker, 1977), the ordering (for $E_{51}=1$) and approximate value of the radii is $R_{\rm sat}<R_{cw}<R_{\rm im}<R_c\sim R_{\rm eq}\sim R_{\rm ev}<R_{\rm stop}$. But many of these inequalities are strongly dependent on conditions that will vary from place to place within the galaxy and from galaxy to galaxy by large amounts (Habe, Ikeuchi, and Tanaka, 1981; Cox, 1986). To clarify this confusing situation we present a catalog of relevant solutions for the simplest case, i.e., no preexisting motion, nor gravity, nor density gradients with γ_i fixed to $\frac{5}{3}$. The catalog, presented as Table IV, divides the solutions into adiabatic $\mathcal{A}$, evaporative $\mathcal{E}$, cloud crushing $\mathcal{C}$, and radiative $\mathcal{R}$, and gives the range of validity, solution name, relevant equation numbers, and exponent $\eta(R\propto t^\eta)$ for each.

For a given explosion in a definite medium the dimensional radii are all determined and, depending on the ordering of these radii, the solution will go through various phases in a definite order. We can schematically indicate this order if we know all the dimensionless ratios of radii and, in fact, two numbers ($R_{\rm im}/R_c$) and ($R_{\rm ev}/R_c$) allow one to predict most of the details of the evolution.

Thus, as a further aid to the reader, we present in Fig. 1 just such a schematic breakdown of the solution space. Script capital letters indicate the solutions labeled in the Table IV, with arrows pointing to the direction of evolution. Since more than two dimensionless numbers would be required to completely specify the evolution, we indicate within each area the different possible paths, in the manner used in block diagrams. Certain solutions in which two effects must be included—for example, evaporation and cloud crushing—have not been calculated in the text but are required in certain parts of the Fig. 1. We enclose those in parentheses and invite the reader to calculate them by the methods outlined. To simplify the presentation we have assumed that $R_{\rm eq}>R_c$ and $R_{\rm eq,im}>R_{cw}$, both of which hold under most circumstances. We have further neglected the solution $\mathcal{C}1$, since the evolution is normally adiabatic during the interval of partial cloud crushing and, finally, neglected cloud crushing during phases when radiative losses are important. In all cases the solutions described begin when the matter evaporated from clouds exceeds that directly from the exploding star, since none of the solutions developed in this paper are valid prior to that time.

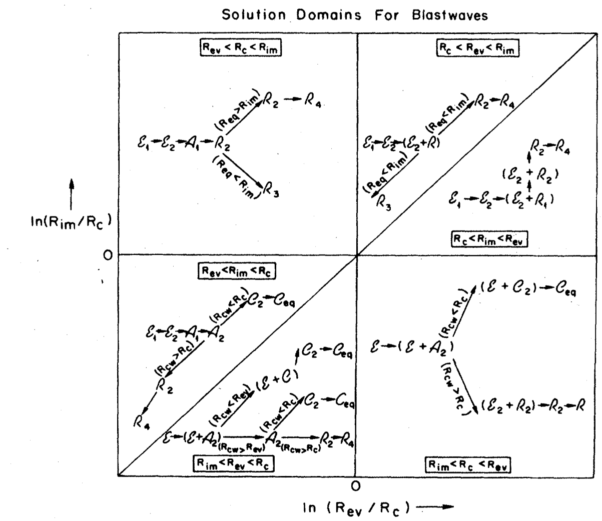

FIG. 1. *Solution Domains for Blastwaves:* For a specific ambient medium and explosion energy, the dimensional radii are fixed (see List of Symbols), which determines the domain in Fig. 1 within which the solution lies. In each domain above we give the possible solutions through which a blastwave passes, labeling solutions by script letters defined in Table III ($\mathcal{R}3$ is the stopping solution [Eq. (6.19)] and $\mathcal{R}4$ is the impeded stopping solution [Eq. (6.50)]). Since more than two dimensions are required to define the parameter space, we must allow for forks within each domain. To further simplify matters we have assumed (1) that $R_{cw} > R_{\rm im}$, $R_{\rm eq} > R_c$, and $R_{\rm eq,im} > R_{cw}$ (usually true), (2) that we may neglect cloud crushing in $\mathcal{R}$, and (3) that the end of the ejecta-dominated phase (not treated in this paper) occurs while evaporation is still important. Also $\mathcal{C}_{\rm eq}$ corresponds to an approach towards equilibrium radius $R_{{\rm eq},cw}$ and solutions enclosed in parentheses, e.g., $(\mathcal{E}+\mathcal{A})$ correspond to those not worked out in the text in which two effects are important.

VII. APPLICATION TO SELF-SIMILAR BLASTWAVES WITH ENERGY INJECTION ($v_H = W = \lambda_{\rm cl}^{-1} = f_0 - f_i = \varepsilon = 0$)

Up to this point we have assumed that the energy in the blastwave is injected impulsively. However, there are a number of astrophysical situations in which the energy is injected over a period of time. The resulting blastwave is self-similar provided the total injected energy $E_{\rm in}$ (or the rate of energy injection $L_{\rm in}$) is a power law in time,

$$E_{\rm in} = E_{\rm in}(1)[t/t(1)]^{\eta_{\rm in}} , \tag{7.1}$$

$$L_{\rm in} \equiv L_{\rm in}(1)[t/t(1)]^{\eta_{\rm in}-1} = \dot{E}_{\rm in} = \eta_{\rm in} E_{\rm in}/t , \tag{7.2}$$

and provided the injected mass is negligible. [Self-similar flows with finite injected mass have been analyzed by Simon and Axford (1966) for blastwaves in the solar wind and by Chevalier (1982) for the initial phase of the expansion of a supernova remnant. We shall not consider these cases here.]

There are two prototypes of a blastwave with energy injection. The first is the stellar wind from a massive star (Pikel'ner and Shcheglov, 1969; Avedisova, 1972; Castor, McCray, and Weaver, 1975; Steigman, Strittmatter, and Williams, 1975; Weaver *et al.*, 1977; McKee, Van Buren, and Lazareff, 1984). Mass flows out from the star at high velocity $v_{\rm in}$ and undergoes a shock when the dynamic pressure $\rho v_{\rm in}^2$ drops to the thermal pressure of the surrounding gas. The shocked wind gas drives a shock into the ambient interstellar medium, with the shocked interstellar gas and the shocked wind separated by a contact discontinuity. Since much of the volume of the blastwave is filled with low-density stellar wind, this structure has been termed an interstellar bubble by Castor, McCray, and Weaver (1975).

The second prototype of a blastwave with energy injection is the detonation wave, in which the passage of the shock releases additional energy from the gas. This problem will be treated in Sec. X.

The methods we have developed can be readily extended to treat the expansion of bubbles and detonation waves. The total energy of the bubble will be a power law in time and in radius, with

$$E \equiv E(1)[t/t(1)]^{\eta_E} \equiv E(1)R^{-k_E} , \tag{7.3}$$

which defines η_E and k_E. Since $R \propto t^{\eta}$, this implies

$$\eta_E = -\eta k_E . \tag{7.4}$$

Solving this equation together with the result (3.11) for η yields (see Parker, 1963)

$$k_E = -\eta_E \left[\frac{5-k_\rho}{2+\eta_E}\right], \quad \eta = \frac{2+\eta_E}{5-k_\rho} . \tag{7.5}$$

To proceed further, it is necessary to specify the relation between the blastwave energy $E \propto t^{\eta_E}$ and the injected energy $E_{\rm in} \propto t^{\eta_{\rm in}}$. We consider two general cases for bubbles: first, if the injected fluid is nonradiative, then $E \propto E_{\rm in}$ and $\eta_E = \eta_{\rm in}$; second, if the injected fluid and the shocked ambient gas are radiative, then a momentum-conserving solution is obtained.

A. Bubbles: nonradiative injection

If the injected fluid does not radiate away its energy, then the energy in the bubble is some fraction Γ of the injected energy

$$E = \Gamma E_{\rm in} , \tag{7.6}$$

which implies $\eta_E = \eta_{\rm in}$. With the aid of Eqs. (3.12) and (7.2) we can express the fiducial energy $E(1)$ in terms of the fiducial luminosity $L_{\rm in}(1)$:

$$E(1) = \left[\frac{\Gamma L_{\rm in}(1)}{\eta_{\rm in}}\right]^{2/3} \left[\frac{\bar{\rho}(1)R_s(1)^5}{\xi}\right]^{1/3} . \tag{7.7}$$

The radius of the outer shock of the bubble [Eq. (3.10)] then becomes

$$R_s(t)=R_s(1)\left[\frac{\xi\Gamma L_{\rm in}(1)}{\eta_{\rm in}\bar{\rho}(1)R_s(1)^5}\right]^{\eta/3}t^{\eta}\ , \qquad (7.8a)$$

or $R=[t/t(1)]^{\eta}$ with

$$t(1)\equiv\left[\frac{\eta_{\rm in}\bar{\rho}(1)R_s(1)^5}{\xi\Gamma L_{\rm in}(1)}\right]^{1/3}\ , \qquad (7.8b)$$

where

$$\eta=(2+\eta_{\rm in})/(5-k_\rho)\ , \qquad (7.8c)$$

from Eq. (7.5). If $L_{\rm in}$ is given in the form $L_{\rm in}=L_1 t^{\eta_{\rm in}-1}$, then $L_{\rm in}(1)=L_1 t(1)^{\eta_{\rm in}-1}$ is given by

$$L_{\rm in}(1)=L_1^{3/(2+\eta_{\rm in})}\left[\frac{\eta_{\rm in}\bar{\rho}(1)R_s(1)^5}{\xi\Gamma}\right]^{(\eta_{\rm in}-1)/(2+\eta_{\rm in})}\ . \qquad (7.9)$$

We now turn to several particular cases.

1. Adiabatic bubble

In the absence of any energy losses from the injected fluid or the shocked ambient medium $\Gamma=1$; the bubble is fully adiabatic. The mass in the bubble consists almost entirely of swept-up ambient gas, which is confined to a relatively thin shell. The most accurate explicit approximation for this case is the pressure-gradient approximation in Sec. IV.D. Results for α^2 and σ can be obtained by inserting Eq. (7.5), with $\eta_E=\eta_{\rm in}$, into Eqs. (4.76), (4.77), and (4.78). The general results are not very illuminating, but particular cases of astrophysical interest are as follows.

For $\gamma=\frac{5}{3}$:

$$\alpha^2=\frac{6[20-6k_\rho+(5-2k_\rho)\eta_{\rm in}]}{(3-k_\rho)[14-4k_\rho+(17-4k_\rho)\eta_{\rm in}]}\quad ({\rm PGA})\ , \qquad (7.10a)$$

$$\sigma=\frac{(3-k_\rho)(5-k_\rho)(1+\eta_{\rm in})}{20-6k_\rho+(5-2k_\rho)\eta_{\rm in}}\quad ({\rm PGA}/\bar{K})\ . \qquad (7.10b)$$

For $\eta_{\rm in}=1$:

$$\alpha^2=\frac{9[2\gamma(\gamma+5)-(\gamma+1)^2k_\rho]}{4(3-k_\rho)[5\gamma+2-(\gamma+1)k_\rho]}\quad ({\rm PGA})\ , \qquad (7.11a)$$

$$\sigma=\frac{2(3-k_\rho)[19\gamma-5-2(\gamma+1)k_\rho]}{9(\gamma-1)[2\gamma(\gamma+5)-(\gamma+1)^2k_\rho]}\quad ({\rm PGA}/\bar{K})\ . \qquad (7.11b)$$

For the particular case $\gamma=\frac{5}{3}$, $\eta_{\rm in}=1$, $k_\rho=0$, this gives $\alpha^2=\frac{50}{31}$, $\sigma=\frac{6}{5}$, and $\xi=125/72\pi(1-f)$, which corresponds to $\xi^{1/5}=0.888$ for $f=0$. The more accurate PGA/GM treatment [Eqs. (4.76), (4.77b), (4.78d), and (4.78e)] gives $\sigma=1.2372$ for this case, or $\xi^{1/5}=0.8828$. By comparison, the detailed numerical solution of Weaver *et al.* (1977) for this case gives $\xi^{1/5}=0.88$, while the analytic approximation of Cavaliere and Messina (1976) gives $\xi^{1/5}=0.901$, with all treatments of course giving $\eta=\frac{3}{5}$. Thus the simple and general (PGA/$\bar{K}$) treatment is accurate to approximately 1% in the $R(t)$ relation, with the somewhat more complicated (PGA/GM) approach of higher but unknown accuracy. The radiative form factor [cf. Eq. (4.95)] in this case is

$$I_m=\frac{160}{19}\left[\frac{13}{209+6m}+\frac{11}{2(19-4m)}\right] \qquad (7.12)$$

from Sec. IV.E, somewhat greater than when $\eta_{\rm in}=0$.

As remarked at the end of Sec. IV.D, the accuracy of the PGA is compromised for steep density gradients. A necessary condition for the accuracy of the approximation is that $P(0)$ be finite, which implies

$$k_\rho<\frac{3\gamma^2+20\gamma+1+2\eta_{\rm in}(-3\gamma^2+5\gamma+4)}{(\gamma+1)(3\gamma+1+2\eta_{\rm in})} \qquad (7.13)$$

from Eq. (4.73). For $\gamma=\frac{5}{3}$ this reduces to $k_\rho<(16+3\eta_{\rm in})/2(3+\eta_{\rm in})$.

2. Radiative outer shock

Since we have assumed that the ambient medium is much denser than the injected fluid, it is quite possible for the outer shock to be radiative and the injected fluid to be adiabatic. Interstellar bubbles typically spend most of their existence in this stage.

The luminosity radiated by the outer shock is

$$L_{\rm rad}=4\pi R_s^2(\tfrac{1}{2}\rho_0 v_s^3) \qquad (7.14a)$$

$$=\frac{2\pi(3-k_\rho)\xi\eta^3\Gamma L_{\rm in}(t)}{3\eta_{\rm in}}\ , \qquad (7.14b)$$

using Eqs. (7.8) and (7.9) with $\eta_E=\eta_{\rm in}$. From Eq. (7.6) we obtain

$$\dot{E}=\Gamma L_{\rm in}=L_{\rm in}-L_{\rm rad}\ , \qquad (7.15)$$

so that

$$\Gamma=\left[1+\frac{2\pi(3-k_\rho)\xi\eta^3}{3\eta_{\rm in}}\right]^{-1}\ . \qquad (7.16)$$

Note that $k_\rho\leq 3$ to ensure $\Gamma\leq 1$. Since the outer shock is radiative, all the swept-up gas is in a thin shell and σ is given by Eq. (6.11). Then the parameter ξ can be evaluated with the aid of Eqs. (3.6) and (7.5):

$$\xi=\frac{9(\gamma_i-1)(5-k_\rho)^2}{2\pi(1-f_i)(2+\eta_{\rm in})[3\gamma_i(2+\eta_{\rm in})+5\eta_{\rm in}-2(1+\eta_{\rm in})k_\rho]}\ . \qquad (7.17)$$

As a check of the general result we can specialize to the case $\eta_{in}=1$, $\gamma_i=\frac{5}{3}$, $k_\rho=f_i=0$ considered by Weaver *et al.* (1977), and obtain $\eta=\frac{3}{5}$, $\xi=5/4\pi$, and $\Gamma=\frac{50}{77}$, in agreement with their results.

3. Evaporative bubble ($\Gamma=1$)

If the bubble expands into a low-density medium with embedded clouds, and if cloud evaporation is efficient, then the structure of the bubble is substantially modified. The swept-up ambient gas has negligible mass; instead, the mass of the bubble is dominated by the evaporated gas in the shocked wind. We assume that radiative losses are negligible, so $\Gamma=1$. This problem has been considered previously by Königl (1983) and Van Buren, Lazareff, and McKee (1987). Castor, McCray, and Weaver (1975) and Weaver *et al.* (1977) considered the effect of evaporating gas from the shell of swept-up ambient gas into the region of the shocked wind. However, since the mass of this evaporated gas represents a negligible fraction of the gas in the shell, the dynamics of the expansion are unaffected by the evaporation and are given by the results of Secs. VII.A.1 and VII.A.2.

Simultaneous solution of Eqs. (5.19) for k_ρ and Eq. (7.5) for k_E yield

$$k_\rho=\frac{6\kappa_T-5+2k_Q-\eta_{in}(2\kappa_T-k_Q)}{2\kappa_T-2\kappa_\rho+1+\eta_{in}(1-\kappa_\rho)}\ , \tag{7.18}$$

$$\eta=\frac{2\kappa_T-2\kappa_\rho+1+\eta_{in}(1-\kappa_\rho)}{2\kappa_T+5(1-\kappa_\rho)-k_Q}\ , \tag{7.19}$$

with η given by Eq. (7.4); $k_E=-\eta_{in}/\eta$ with $\eta_E=\eta_{in}$. The mean density $\bar{\rho}(1)$ that enters into $R_s(t)$ must be determined self-consistently for evaporative bubbles; Eqs. (5.20) and (7.7) give

$$\bar{\rho}(1)^{2\kappa_T-3\kappa_\rho+2}=\left[\frac{Q_{av}(1)}{\alpha^{2\kappa_T}(3-k_\rho)(1-f)}\right]^3 R_s(1)^{5-4\kappa_T}\left[\frac{\eta^3\xi L_{in}(1)}{\eta_{in}}\right]^{2\kappa_T-1}\ . \tag{7.20}$$

When substituted into Eq. (7.8a) this yields the bubble radius

$$\frac{R_s}{R_s(1)}=\left[\frac{\alpha^{2\kappa_T}(3-k_\rho)(1-f)}{\eta^{2\kappa_T-1}Q_{av}(1)R_s(1)^{2\kappa_T+5(1-\kappa_\rho)}}\left[\frac{\xi L_{in}(1)}{\eta_{in}}\right]^{1-\kappa_\rho}\right]^{\eta/(2\kappa_T-3\kappa_\rho+2)} t^\eta\ . \tag{7.21}$$

As an example, consider the special case of uniform ($k_Q=0$) classical evaporation ($\kappa_T=\frac{5}{2}$, $\kappa_\rho=0$) in a constant-luminosity bubble ($\eta_{in}=1$). Then $\eta=\frac{7}{10}$ (Königl, 1983), $k_\rho=\frac{5}{7}$, and Eq. (7.21) reduces to

$$R_s(t)=1.253[\alpha^5\xi(1-f)L_{in}/Q_{av}]^{1/10}t^{7/10}\ . \tag{7.22}$$

The parameters α^2 and ξ have been evaluated by Van Buren, Lazareff, and McKee (1987), and for this particular case they find $(\alpha^2,\xi)=[3.48,0.47/(1-f)]$. Their similarity solution for this problem shows that, since the energy is injected at the origin, the velocity, temperature, and density all generally decrease outward.

B. Radiative bubbles

The bubbles analyzed above contain a significant fraction of the total injected energy E_{in} in the hot, shocked injected fluid. If this energy is radiated away, as well as that in the shocked ambient medium, then a momentum-conserving bubble results (Avedisova, 1972; Steigman, Strittmatter, and Williams, 1975). Evaporation of embedded clouds may enhance the radiation from the shocked injection fluid and thereby facilitate the formation of a radiative bubble. To simplify the discussion, we ignore this possibility here.

Let $\dot{M}_{in}$ be the mass injection rate and v_{in} the injection velocity, so that $L_{in}=\frac{1}{2}\dot{M}_{in}v_{in}^2$. For simplicity, we assume that v_{in} is constant; then $\dot{M}_{in}\propto L_{in}\propto t^{\eta_{in}-1}$. In contrast to the bubbles with nonradiative interiors, here $\eta_{in}\neq\eta_E$, since most of the energy is radiated away. Momentum conservation gives

$$Mv_s=\int \dot{M}_{in}v_{in}dt=2\frac{E_{in}}{v_{in}}=2\frac{E_{in}(1)}{v_{in}}R^{\eta_{in}/\eta}\ . \tag{7.23}$$

Recall that we are assuming that the injected fluid has negligible mass ($M\gg M_{in}$), so that $v_{in}\gg v_s$. Equating exponents in Eq. (7.23) yields

$$k_E=-\frac{\eta_{in}(5-k_\rho)-(3-k_\rho)}{1+\eta_{in}}\ ,$$

$$\eta_E=\frac{\eta_{in}(5-k_\rho)-(3-k_\rho)}{4-k_\rho}\ , \tag{7.24}$$

$$\eta=(1+\eta_{in})/(4-k_\rho)\ ,$$

where Eqs. (3.4), (3.9), (3.13), and (7.5) were used. The expression for $R_s(t)$ may be obtained directly from Eq. (7.23) with the aid of Eq. (7.2):

$$R_s(t)=R_s(1)\left[\frac{3L_{in}(1)}{2\pi\eta\eta_{in}\bar{\rho}(1)(1-f_i)v_{in}R_s^4(1)}\right]^{\eta/2}t^\eta\ . \tag{7.25}$$

Avedisova (1972) and Steigman, Strittmatter, Williams (1975) obtained a result of this form for the case $\eta_{in}=1$, $k_\rho=0$, but Avedisova found a different numerical coefficient.

In common with the radiative blastwaves considered in Sec. V, the moments K_{20} and K_{11} are unity. However, a significant fraction of the total energy in the bubble is contained in the unshocked wind, and one can show that

$$K_{02}=1+\eta_{in}/\eta \; . \tag{7.26}$$

This ensures that $\alpha^2\to\infty$ [see Eq. (3.20)], as it must for a blastwave with no internal energy. Furthermore, for such blastwaves one has $\sigma=K_{02}/2$ from Eq. (3.19).

VIII. BLASTWAVE IN AN EXPANDING WIND ($f=\lambda_{cl}^{-1}=\eta_{in}=\dot{m}=P_0=0$)

Far from its source, a steady spherical wind obeys an R^{-2} density law. The initial stage of evolution of a blastwave in a wind is, as usual, dominated by the ejected mass, which acts as a piston driving a forward shock into the ambient medium while a reverse shock propagates backward into the ejecta. Such a system has been analyzed for both flare-driven blastwaves in the solar wind (Simon and Axford, 1966) and for supernova-driven blastwaves in the interstellar medium (McKee, 1974; Chevalier, 1982). During the later stages of evolution of the blastwave a number of effects occur: the swept-up mass dominates the ejected mass, the blastwave velocity v_s becomes of the order of the wind velocity v_w or smaller, the ambient pressure becomes important ($v_s\sim C_0$), and radiative losses may become significant.

Up to this point, our analysis of blastwaves in winds (Secs. IV.E and VI) has ignored the expansion velocity of the wind—i.e., we have assumed $v_s\gg v_w$. Here we shall determine the effects of the expansion on the blastwave evolution for adiabatic and radiative blastwaves. We shall assume that there is no energy injection ($\eta_{in}=0$); Chevalier (1984) has considered the case of an expanding bubble in a wind. As we did in all of Sec. VIII, we assume the medium is homogeneous ($f=\lambda_{cl}^{-1}=\dot{m}=0$) and we neglect the ambient pressure ($v_s\gg C_0$). In addition we ignore self-gravity. Finally we assume that the swept-up mass is dominant: not only is the ejected mass negligible, but, if the wind continues after the explosion (as it does in the case of a solar flare), then this assumption requires that this wind not interact with the blastwave. Hence, the mass in the blastwave at time t is the mass it would have had if the wind had continued to maintain an R^{-2} density, minus the mass injected after $t=0$:

$$M=4\pi R_s^3\rho_0(R_s)-\dot{M}_w t=\frac{\dot{M}_w}{v_w}(R_s-v_w t)\; , \tag{8.1}$$

where the wind mass-loss rate is given by Eq. (4.81). It is important to verify that the swept-up mass is indeed dominant before applying the results of this section.

A. Adiabatic blastwave

We model the blastwave as a cold, thin shell. This should be a good approximation at late times, when the adiabatic expansion losses due to the expanding wind refrigerate the gas; at early times we can compare our result with the exact solution, Eq. (4.82). The energy of the blastwave E_b is then

$$E_b=\tfrac{1}{2}M(v_1^2-v_w^2)\; , \tag{8.2}$$

where the post-shock velocity v_1 may be related to $\Delta v=v_s-v_w$ with Eq. (2.4),

$$v_1=v_w+\frac{2}{\gamma+1}\Delta v\; . \tag{8.3}$$

We have replaced v_H by v_w to emphasize the difference from the cosmological problems treated below. Solving Eqs. (8.2) and (8.3) for Δv, we find

$$\frac{\Delta v}{v_w}=\frac{\gamma+1}{2}\left[\left[1+\frac{M_H}{M}\right]^{1/2}-1\right]\; , \tag{8.4}$$

where the fiducial mass M_H is given by

$$E_b\equiv\tfrac{1}{2}M_H v_w^2\; . \tag{8.5}$$

When the energy-conservation result (8.4) is supplemented with the equation of continuity,

$$\frac{dM}{dt}=4\pi R_s^2\rho_0\Delta v=\dot{M}_w\left[\frac{\Delta v}{v_w}\right]\; , \tag{8.6}$$

the resulting system can be integrated to give

$$\frac{t}{t_H}=\frac{1}{\gamma+1}\left\{\left[\frac{1}{2}+\frac{M}{M_H}\right]\left[\frac{M}{M_H}+\frac{M^2}{M_H^2}\right]^{1/2}+\frac{M^2}{M_H^2}\right.$$
$$\left.-\tfrac{1}{4}\ln\left[1+\frac{2M}{M_H}+2\left[\frac{M}{M_H}+\frac{M^2}{M_H^2}\right]^{1/2}\right]\right\}\; , \tag{8.7}$$

where

$$t_H\equiv M_H/\dot{M}_w=2E_b/\dot{M}_w v_w^2 \tag{8.8}$$

is the characteristic time at which v_s is of order v_w. This parametric solution in terms of M/M_H can be completed with Eq. (8.1):

$$R_s=v_w t_H\left[\frac{M}{M_H}+\frac{t}{t_H}\right]\; . \tag{8.9}$$

These results simplify considerably at late times, and one finds for $t\gg t_H$:

$$R_s\to v_w t\; , \tag{8.10}$$

$$\frac{M}{M_H}\to\left[\left[\frac{\gamma+1}{2}\right]\frac{t}{t_H}\right]^{1/2}\; , \tag{8.11}$$

$$\frac{\Delta v}{v_w} \to \left[\left[\frac{\gamma+1}{8} \right] \frac{t_H}{t} \right]^{1/2} . \tag{8.12}$$

In contrast to the $\Omega=0$ cosmological solution discussed below [Eq. (9.32)], in this case the mass swept-up by the blastwave grows without limit.

At early times ($t \ll t_H$) expansion of Eq. (8.7) shows that this solution approaches that given in Eq. (4.82) for an adiabatic blastwave in a wind, except that the factor $4\pi\xi/3$ is replaced by $\frac{9}{8}(\gamma+1)^2$. This factor is too large by $\frac{4}{3}$ for $\gamma=\frac{5}{3}$ and $\frac{35}{18}$ for $\gamma=\frac{4}{3}$: the neglect of the internal energy at the starting point [Eq. (8.2)] leads to a progressively greater error as γ becomes smaller. An approximation for $R_s(t)$ which has the correct limit for $t \ll t_H$ [Eq. (4.82)] and agrees with Eqs. (8.7) and (8.9) for $t \gg t_H$ is

$$R_s \simeq v_w t_H \left[\left[\frac{2\pi\xi}{3} \right]^{1/2} \frac{t}{t_H} + \left[\frac{t}{t_H} \right]^{3/2} \right]^{2/3} . \tag{8.13}$$

If ξ is taken to be the LVA value in Eq. (4.83), then for $t > t_H$ and $\frac{4}{3} \le \gamma \le \frac{5}{3}$ the agreement with Eqs. (8.7) and (8.9) is within 11%. Similarly, one can show that the expression

$$\frac{\Delta v}{v_w} \simeq \left[\frac{81t/t_H}{16\pi\xi} + \left[\frac{8t/t_H}{\gamma+1} \right]^{3/2} \right]^{-1/3} \tag{8.14}$$

has the correct limits and agrees with Eq. (8.4) to within 5% for $t > t_H$ and for $\frac{4}{3} \le \gamma \le \frac{5}{3}$.

B. Radiative blastwave in a wind

If the shock is radiative, then the momentum associated with the increment in velocity Δv is conserved:

$$p_1 \equiv M \Delta v = \text{const} . \tag{8.15}$$

The energy in the blastwave over and above the initial wind energy is

$$\Delta E = \tfrac{1}{2} M (v_s^2 - v_w^2) = p_1 v_w + \frac{p_1^2}{2M} , \tag{8.16}$$

since $v_1 = v_s$ for a radiative shock. Assume that the shock becomes radiative when $M = M_c$; then at M_c, $\Delta E = E_b$ and Eq. (8.16) implies

$$p_1 = \frac{2E_b}{v_w} \left[1 + \left[1 + \frac{M_H}{M_c} \right]^{1/2} \right]^{-1} . \tag{8.17}$$

The equation of continuity (8.6) may then be integrated with the aid of Eq. (8.15) to give

$$\frac{M}{M_H} = \left[\frac{2t}{t_H} \right]^{1/2} \left[1 + \left[1 + \frac{M_H}{M_c} \right]^{1/2} \right]^{-1/2} . \tag{8.18}$$

Note that this agrees with the exact result (8.11) for our model of an adiabatic blastwave when $M > M_c \gg M_H$, since then Δv is small and the radiative losses are negligible. For any value of M_c/M_H, the energy in the blastwave approaches a nonzero constant $p_1 v_w$ at late times [Eq. (8.16)], so the fraction of the energy that is radiated is

$$\frac{E_{\text{rad}}}{E_b} = \frac{E_b - p_1 v_w}{E_b} = \frac{p_1^2}{2M_c E_b} = \frac{M_H/M_c}{[1+(1+M_H/M_c)^{1/2}]^2} . \tag{8.19}$$

For $M_c \ll M_H$ the medium is effectively stationary when cooling sets in and nearly all the energy is radiated away; for $M_c \gg M_H$, however, only a fraction $M_H/4M_c$ is radiated away. Eventually, of course, the wind itself terminates in a shock, and then the remainder of the energy is radiated.

IX. COSMOLOGICAL BLASTWAVES ($f = \lambda_{\text{cl}}^{-1} = \gamma - \gamma_i = \dot{m} = P_0 = 0$)

Blastwaves in an expanding universe have been analyzed by Schwarz, Ostriker, and Yahil (1975), Ozernoi and Chernomordik (1978), Bertschinger (1983, 1985a), Ikeuchi, Tomisaka, and Ostriker (1983), and Fillmore and Goldreich (1984). Maeda and Sato (1983a, 1983b, 1983c) have numerically treated the fully relativistic problem and have independently derived some of the analytic results we present in this section. Numerical treatments of a closely related problem, the development of cosmological voids, by Peebles (1982), Hausman *et al.* (1983), Hoffman *et al.* (1983), and Lake and Pim (1985) are also relevant. The methods we have developed here are particularly useful in obtaining analytic solutions to this problem. Using these methods, we have developed a somewhat more general treatment of adiabatic cosmological blastwaves and obtained new results for blastwaves with radiative losses, with energy injection, or in universes with dark matter (McKee and Ostriker, 1987).

For simplicity, we restrict ourselves to a matter-dominated universe and to the two cosmologically important cases in which the density is a power law in cosmic time τ: the just closed $\Omega=1$ case, in which the gravitational energy balances the kinetic energy of the expansion, and the $\Omega=0$ case, in which gravity is negligible. These cases can be described by an ambient expansion velocity and density

$$v_H = hR/\tau , \tag{9.1}$$

$$\rho_0 = \bar{\rho} = \rho_u \tau^{-3h} , \tag{9.2}$$

$$\Omega = \rho_t (8\pi G/3H^2) = 2GM_t/Rv_H^2 , \tag{9.3}$$

where for $\Omega \ll 1$

$$h = 1, \rho_u = 3\Omega_{g0}/8\pi G H_0 , \tag{9.4a}$$

and for $\Omega = 1$

$$h = \tfrac{2}{3}, \quad \rho_u = \Omega_g/6\pi G, \quad \rho_{\text{crit}} \equiv 1/(6\pi G\tau^2) . \tag{9.4b}$$

Here Ω_0 and H_0 are the present ($z=0$) values of Ω and the Hubble constant, respectively, and $\Omega_g=(\rho/\rho_t)\Omega$ is the density parameter for the gas. For the $\Omega=1$ case, we allow for the case in which a small fraction of the mass is in gas and the remainder (stars, neutrinos, etc.—"dark matter") interacts with the gas only gravitationally by distinguishing between (ρ,M,Ω_g) and (ρ_t,M_t,Ω). The dark matter is classified as cold, warm, or hot depending on its velocity dispersion at the time the baryons decoupled from the radiation, since that determines the length scale over which initial density perturbations are erased by free-streaming (Bond and Szalay, 1983). An analogous classification can be made for blastwaves, in which the ratio of the velocity dispersion of the dark matter to the shock velocity is the governing parameter. If C_d is the isothermal sound speed of the dark matter, then $C_d \ll v_s$ corresponds to "cold" dark matter and $C_d \gg v_s$ corresponds to "hot" dark matter. Note that since this classification is made at the red shift of the blastwave and not at decoupling, it need not correspond to that used for galaxy formation. Our restriction to matter-dominated universes eliminates consideration of "very hot" dark matter with $C_d^2 \gtrsim \Omega_g c^2$. Further discussion of blastwaves with $\Omega_g<\Omega$ is given in Secs. IX.A.4 and IX.B.2. In all cases we neglect Compton drag by the microwave background.

Our discussion is divided into two parts. In Sec. IX.A we consider self-similar blastwaves, show that in all cases the shocked gas is confined to a thin shell, and find certain general results. In Sec. IX.B we drop the assumption of self-similarity. These latter results are of greater utility.

A. Self-similar cosmological blastwaves

If the blastwave begins its expansion at cosmic time τ_b, it will become self-similar only at $\tau \gg \tau_b$. At these late times, we have $R_s \propto \tau^\eta$ and $v_s=\eta R_s/\tau$. Hence we may define a dimensionless constant ν_H, analogous to ν_1 [Eq. (3.15)],

$$\nu_H \equiv v_H/v_s = h/\eta = \text{const} \le 1 \ . \tag{9.5}$$

That is, self-similarity requires a constant ratio of shock velocity to Hubble velocity at the shock. The general self-similar solution for radial expansion [Eq. (3.5)] still applies,

$$R_s=\left[\frac{\xi E_b}{\bar\rho}\right]^{1/5}\tau^{2/5}=\left[\frac{\xi E_b}{\rho_u}\right]^{1/5}\tau^{(2+3h)/5} , \tag{9.6}$$

where $E_b=\sigma M v_s^2$ is the energy of the blastwave at any time ($=E_0$ in the absence of losses), $\xi=3/(4\pi\eta^2\sigma)$, and where we used Eq. (9.2) for $\bar\rho$. The definitions of the constants (σ,ξ) are identical to those proposed for the general case in Sec. III [Eqs. (3.3) and (3.6)].

Due to the energetics of the Hubble flow and the possibility that the gas represents only a fraction of the gravitating matter, we must be quite careful in our definition of the energy. Recall that the blastwave energy E_b is defined as the *difference* between the energy of the gas inside the volume, bounded by R_s, in the actual case, and the energy E_a in an equivalent unperturbed part of the (Hubble) flow, bounded by a sphere of the same radius [see Eq. (3.14)]. Now, the total gravitational energy of all the matter inside R_s is

$$-\int_0^{R_s}\frac{GM_t dM_t}{r}=-\int_0^{R_s}\frac{GM_t dM}{r}-\int_0^{R_s}\frac{GM_t dM_d}{r} , \tag{9.7}$$

where $M_t(r)$ is the total mass inside r and $M_d(r)$ is the mass of nongaseous, or "dark," matter inside R. The first term represents the energy required to assemble the gas in the presence of the dark matter; it is identical to the gravitational virial term W in Eqs. (2.12) and (2.14). The second term represents the energy required to assemble the dark matter in the presence of the gas. Only the first term contributes to the total energy of the gas, so that

$$E_b=(E_K+E_{\rm th}+W)-E_a \ . \tag{9.8}$$

The rate of change of E_b due to radiative losses, energy injection at the origin (bubbles) or edge of the blast wave (detonations), and gravitational interaction with the dark matter is given in Appendix A. The magnitude of W is proportional to $\Omega M v_H^2$, so it is convenient to introduce the coefficient w defined by

$$W \equiv -\tfrac{1}{2} w \Omega M v_H^2 \ . \tag{9.9}$$

For the unperturbed Hubble flow $w=\frac{3}{5}$ [see Appendix A, Eq. (A28)], whence

$$E_a=\tfrac{3}{10}(1-\Omega)Mv_H^2 \ . \tag{9.10a}$$

The total energy in the gas is $E=E_b+E_a$, so that

$$\frac{E}{E_b}=1+\left[\frac{3}{10}\right]\frac{(1-\Omega)\nu_H^2}{\sigma} \ . \tag{9.10b}$$

With Eq. (3.16) for E_K and (3.18) for $E_{\rm th}$, Eqs. (9.8)–(9.10) yield

$$\sigma \equiv \frac{E_b}{Mv_s^2}=\tfrac{1}{2}\nu_1^2 K_{02}+\frac{1}{\alpha^2(\gamma-1)}-\tfrac{3}{10}\nu_H^2[1-(1-\tfrac{5}{3}w)\Omega] \ . \tag{9.11}$$

We can further generalize the treatment to allow $E_b \propto t^{\eta_E}$ in the cases of energy injection or radiative losses. Note that Eq. (9.2) implies that $\rho_0 \propto R^{-k_\rho}$ with

$$k_\rho=3h/\eta=3\nu_H \tag{9.12}$$

[on the other hand, $\rho_0(r,t)$ is independent of r so that $k_{\rho_0}=0$]. Since $k_E=-\eta_E/\eta$ from Eq. (7.4), the general expression (3.11) for η becomes

$$\eta=(2+3h+\eta_E)/5 \ . \tag{9.13}$$

The Hubble velocity parameter $\nu_H \equiv h/\eta$ is then, from Eqs. (9.6) and (9.13),

$$\nu_H \equiv v_H/v_s = 5h/(2+3h+\eta_E) , \qquad (9.14)$$

so that the velocity of the shock relative to the Hubble flow is

$$\frac{\Delta v}{v_s} \equiv \frac{v_s - v_H}{v_s} \equiv 1-\nu_H = 1 - \left[\frac{h}{\eta}\right] = \frac{2(1-h)+\eta_E}{2+3h+\eta_E} . \qquad (9.15)$$

The normalized post-shock velocity is given by Eq. (3.22) as

$$\nu_1 \equiv \frac{v_1}{v_s} = \frac{2+(\gamma-1)\nu_H}{(\gamma+1)} . \qquad (9.16)$$

The remaining general and exact relation needed to determine the evolution of the blastwave is the virial theorem [Eq. (3.20)]. Since $\bar{\rho}=\rho_0$, Eqs. (9.8) and (9.13) enable us to write it as

$$\frac{3}{\alpha^2} = \left[\frac{1+\eta_E}{\eta}\right]\nu_1 K_{11} - 3\nu_H(1-\nu_H) - K_{02}\nu_1^2 + \tfrac{1}{2}w\Omega\nu_H^2 , \qquad (9.17)$$

where α^2 is defined in Eq. (3.17).

In order to evaluate the constants ξ and σ we must assume a specific dynamical solution. Numerical integrations (Schwarz *et al.*, 1975; Bertschinger, 1983; Ikeuchi *et al.*, 1983) have shown that, in all cases studied, an extremely thin shell develops after long times, although the reasons for this behavior are different in the different instances. For an adiabatic blastwave in a low-density universe, the pressure in the shocked gas declines because of the decompression due to the expansion of the universe; as a result, there is no resistance to an element of gas overtaking another element that is shocked later to a lower velocity, and the gas piles up just behind the shock. For a blastwave in an $\Omega_g=1$ universe, the innermost gas, which is not decelerated by gravity, tends to plow into the outermost gas, which is decelerated; again, a very thin shell results. In both cases the mean density of the shocked gas can substantially exceed the value $(\gamma+1)/(\gamma-1)$ attained just behind the shock. The velocity behind the shock is nearly constant (actually increasing slightly with decreasing radius), so our "shell approximation" (Appendix C.4) should be excellent. Thus, with $v=v_1$, the moments appearing in the virial theorem are given by Eq. (C22). The analysis of Ikeuchi *et al.* (1983) suggests that the shell approximation is exact for a constant-energy blastwave in a low-density universe ($\Omega=\eta_E=0$). More generally, the shell approximation will be accurate when the internal energy is small ($\alpha^{-2} \equiv \bar{P}/\bar{\rho}v_s^2 \ll 1$).

In view of the fundamental importance of the shell structure of cosmological blastwaves, it is worthwhile to estimate α^{-2} for energy-conserving blastwaves and verify that is is self-consistent to regard it as small. Equations (3.24) and (9.16) give the relation between α^{-2} and the mean pressure as

$$\alpha^{-2} = \frac{2(1-\nu_H)^2}{\gamma+1}\frac{\bar{P}}{P_1} . \qquad (9.18)$$

In the shell approximation, the normalized thickness of the shell is $1-\lambda_i = 1-\nu_1$ [see below Eq. (4.47)]. We estimate $\bar{P}$ by taking $P(\lambda)=P_1$, a constant. Then, since the average extends over the entire sphere, whereas the pressure is nonzero only in the shell, we have $\bar{P}/P_1 = 3(1-\nu_1)$ and

$$\alpha^{-2} \simeq \frac{6(\gamma-1)(1-\nu_H)^3}{(\gamma+1)^2} , \qquad (9.19)$$

where we have used Eq. (9.16) again. For the case $\gamma=\frac{5}{3}$, $\eta_E=0$, $\Omega=\Omega_g=1$, this rough estimate gives $\alpha^2=384$ in comparison with the exact value 408.4. When $\eta_E=0$, Eqs. (9.4) and (9.15) show that $1-\nu_H \le \frac{1}{6}$, so that α^{-2} is indeed very small. [This estimate of α^{-2} breaks down for adiabatic blastwaves with a soft equation of state ($\gamma \to 1$), since α^{-2} does not in fact vanish for such blastwaves—see Eq. (9.36).] The thermal pressure in energy-conserving cosmological blastwaves is small because the velocity difference at the shock is small compared to the Hubble velocity, and thus the gas piles up in a thin shell as the blastwave decelerates. Kinematically, this results in a negative velocity gradient at the shock front (see Appendix B) and the natural formation of a void or cavity.

Since all moments of the density are determined in the shell approximation, we can derive the gravitational energy to any desired accuracy. The velocity moment $K_{02}=1$ and, for the others, it suffices for our purposes here to treat the matter as if in a thin homogeneous shell of thickness δR_s. We also assume that if any dark matter is present, the force it exerts on the shell is the same as if the flow of dark matter were unperturbed by the blastwave (the opposite case is taken up in Sec. IX.A.4 below). Then, from Eqs. (A30) and (A35), the gravitational coefficient is

$$w = (1-\delta) - \tfrac{1}{2}(1-\tfrac{7}{3}\delta)\frac{\Omega_g}{\Omega} . \qquad (9.20)$$

Here δ is determined in terms of the moments K_n [cf. Eq. (C23)]; we adopt the $n=1$ relation to lowest order in δ,

$$\delta = 2(1-K_1) . \qquad (9.21)$$

Note that δ is the *effective* shell thickness for a homogeneous shell; numerical solutions (Bertschinger, 1983; Ikeuchi *et al.*, 1983) show that the density in the shell is far from constant, and the actual shell thickness generally differs from δ. Equations (9.11), (9.14), (9.17)–(9.19), and (C22) let us determine α^2 and σ to complete the solution of cosmological blastwaves. We now focus separately on the cases $\Omega \ll 1$ and $\Omega = 1$.

$\Omega \ll 1$. Since $h=1$, Eqs. (9.13)–(9.16) reduce to

$$\eta = 1 + \tfrac{1}{5}\eta_E, \quad \frac{\Delta v}{v_s} = \frac{\eta_E}{5+\eta_E}, \tag{9.22a}$$

$$v_1 = \frac{5(\gamma+1)+2\eta_E}{(\gamma+1)(5+\eta_E)}, \quad v_H = \frac{5}{5+\eta_E} \quad (\Omega \ll 1). \tag{9.22b}$$

Hence the solution becomes degenerate (the shock comoving with the Hubble flow) in the limit $\eta_E \to 0$. The relevant moments are obtained from Eq. (C22):

$$K_1 \equiv K_{10} = K_{11} = 1 - \frac{\eta_E}{4\eta_E+5}\left[\frac{\gamma-1}{\gamma+1}\right], \tag{9.23a}$$

$$K_2 \equiv K_{20} = 1 - \left[\frac{2}{\gamma+1}\right] \frac{\eta_E(\gamma-1)+(2\eta_E+5\gamma+5)(1-K_1)}{5(\eta_E+2)} \quad (\Omega \ll 1). \tag{9.23b}$$

For all of these solutions there is a simple relation between the swept-up mass and the energy of the explosion derivable from Eqs. (9.7), (9.2), and (9.4),

$$M(\tau) = \left[\frac{\Omega_{g0}}{2GH_0}\right]^{2/5} \left[\frac{4\pi\xi E_b(\tau)}{3}\right]^{3/5}. \tag{9.24}$$

$\Omega = 1$. Here Eqs. (9.13)–(9.16) give

$$\eta = \tfrac{1}{5}(4+\eta_E), \quad \frac{\Delta v}{v_s} = \frac{2+3\eta_E}{3(4+\eta_E)}, \tag{9.25}$$

$$v_1 = \frac{2(7+5\gamma+3\eta_E)}{3(4+\eta_E)(\gamma+1)}, \quad v_H = \frac{10}{3(4+\eta_E)} \quad (\Omega = 1), \tag{9.26}$$

so that the shell decelerates for $\eta_E < 1$. In the just closed case the moments are

$$K_1 \equiv K_{10} = K_{11} = 1 - \frac{3\eta_E+2}{6(2\eta_E+3)}\left[\frac{\gamma-1}{\gamma+1}\right] \tag{9.27}$$

$$K_2 \equiv K_{20} = 1 - \left[\frac{2}{\gamma+1}\right] \frac{(12+3\eta_E-10K_1)(\gamma-1)+6(4+\eta_E)(1-K_1)}{15(\eta_E+2)} \quad (\Omega = 1). \tag{9.28}$$

We now consider the adiabatic blastwave and the nonradiative bubble in greater detail.

1. Energy-conserving blastwaves ($\eta_E = 0$, $\Omega = \Omega_g$)

$\Omega \ll 1$. In a low-density universe, Eqs. (9.20) and (9.21) imply that $\eta = 1$, $\Delta v = 0$, and $v_1 = v_H = 1$. An adiabatic blastwave comoves with the Hubble expansion at late times. Equations (9.22) and (9.23) give $K_1 = K_2 = 1$: all matter is in an infinitely thin shell, and our solution is exact. The internal energy produced at the shock is proportional to Δv^2, which vanishes; thus we expect $\alpha^{-2} = \bar{P}/\bar{\rho} v_s^2$ to be zero, as is indeed the case for $\Omega \to 0$ and all values of γ [cf. Eq. (9.18)]. Finally Eq. (9.11) gives

$$\sigma = \tfrac{1}{5} \quad (\Omega \ll 1) \tag{9.29}$$

or

$$E_b = M v_s^2/5, \quad \xi = 15/4\pi \quad (\Omega \ll 1) \tag{9.30}$$

independent of γ, as it must be, since the internal energy vanishes in this case. The total energy is $E = 2.5 E_b$ from Eq. (9.10b), so that $E = \tfrac{1}{2} M v_s^2$ as expected for a thin, comoving shell in the absence of gravity. The radius of the blastwave is

$$R_s = \left[\frac{10 G H_0 E_b}{\Omega_{g0}}\right]^{1/5} \tau \quad (\Omega \ll 1) \tag{9.31}$$

and the mass enclosed approaches a constant:

$$M = (5E_b)^{3/5} \left[\frac{\Omega_{g0}}{2GH_0}\right]^{2/5} = 2.9 \times 10^{13} E_{b61}^{3/5} (\Omega_{g0}/h_*)^{2/5} M_\odot \quad (\Omega \ll 1) \tag{9.32}$$

where $h_* = H_0/(100 \text{ km s}^{-1} \text{Mpc}^{-1})$ and E_{b61} is in units of 10^{61} erg. These results become exact in the limit $\tau/\tau_b \to \infty$.

$\Omega = 1$. In a marginally closed universe, Eqs. (9.25) and (9.26) yield

$$\eta = \tfrac{4}{5}, \quad \Delta v = v_s/6 = v_H/5, \quad v_1 = (5\gamma+7)/6(\gamma+1), \quad v_H = \tfrac{5}{6}. \tag{9.33}$$

The gravitational deceleration of the gas ahead of the blastwave allows it to overtake more and more gas, so it never becomes comoving as in the $\Omega=0$ case. The moments are determined [cf. Eqs. (9.20), (9.21), (9.27), and (9.28)] to be

$$K_1=\frac{2}{9}\left[\frac{4\gamma+5}{\gamma+1}\right],\quad K_2=\frac{107\gamma^2+266\gamma+167}{135(\gamma+1)^2}, \tag{9.34}$$

$$\delta=\frac{2}{9}\left[\frac{\gamma-1}{\gamma+1}\right],\quad w=(29\gamma+25)/54(\gamma+1)\quad (\Omega=1)\ . \tag{9.35}$$

Now we can determine α^2 using Eqs. (9.17) and (9.33)–(9.35), finding

$$\alpha^2=\frac{11\,664(\gamma+1)^2}{5\gamma^2+90\gamma+13}\rightarrow 468.9\quad (\Omega=1)\ , \tag{9.36}$$

where the numerical values assume $\gamma=\frac{5}{3}$ as before. These relations then allow us to determine σ using Eqs. (9.11) and (9.33)–(9.36),

$$\sigma=\frac{1875\gamma^3+5420\gamma^2-1137\gamma-6050}{11\,664(\gamma+1)^2(\gamma-1)}\rightarrow 0.2856\quad (\Omega=1) \tag{9.37}$$

so that [Eq. (9.6)]

$$R_s=\left[\frac{225GE_b}{32\sigma}\right]^{1/5}\tau^{4/5}$$
$$\rightarrow 3(360/3553)^{1/5}(GE_b)^{1/5}\tau^{4/5}\quad (\Omega=1)\ , \tag{9.38}$$

where the numerical values assume $\gamma=\frac{5}{3}$, as before. We can compare the solution ($\Omega_g=\Omega=1$) with that obtained by Bertschinger (1983). The coefficient in our answer, $R_s=1.898(E_bG)^{1/5}\tau^{4/5}$, differs from the result of his numerical integration by 0.56%, demonstrating, in this case, the accuracy of the shell approximation. The result for α^2, although obtained from the virial theorem (9.17), which involves the difference between quantities of comparable magnitude, is also reasonably accurate: it is 9% high at $\gamma=\frac{5}{3}$ and 2% high at $\gamma=\frac{4}{3}$. Bertschinger was unable to find a similarity solution for $\gamma\le\frac{7}{6}$ because the density and pressure had divergent exponents at the inner edge of the shell [see Eqs. (B42) and (B43)]. Hence, our solution does not apply for $\gamma\le\frac{7}{6}$. The total energy E and the blastwave energy E_b are the same in this case, since the ambient gravitational and kinetic energy exactly cancel in an $\Omega=1$ flow [Eq. (9.10)]. The mass increases slowly with time. Taking $\gamma=\frac{5}{3}$, we have

$$M(\tau)=6\left(\tfrac{360}{3553}\right)^{3/5}\frac{E_b^{3/5}\tau^{2/5}}{G^{2/5}} \tag{9.39a}$$

$$=2.3\times10^{13}E_{b61}^{3/5}\tau_{10}^{2/5}M_\odot\quad (\Omega=1) \tag{9.39b}$$

where τ_{10} is measured in units of 10^{10} yr.

Joint ST solution. The results obtained so far apply only at late times $\tau\gg\tau_b$. For $t\equiv\tau-\tau_b$ small compared to τ_b, we expect the standard Sedov-Taylor (ST) solution to be valid. Because the ST and cosmological solutions have the same form, it is straightforward to write an analytic expression that smoothly connects the two and that should be relatively accurate. Introducing the parameter x which lies between 0 and 1,

$$x\equiv t/\tau,\quad 1-x=\tau_b/\tau\ , \tag{9.40}$$

we can write

$$R_s=[\xi_{\rm ST}(1-x)+\xi x]^{1/5}\left[\frac{E_b t^2\tau^{3h}}{\rho_u}\right]^{1/5}. \tag{9.41}$$

Inserting the exact numerical value of $\xi_{\rm ST}$ for $\gamma=\frac{5}{3}$ from Table I, we obtain

$$R_s=\begin{cases}33.2(1-0.41x)^{1/5}(h_*E_{b61}/\Omega_{g0})^{1/5}t_8^{2/5}\tau_8^{3/5}\ \text{kpc} & (\Omega\ll1),\\ 97.8(1-0.36x)^{1/5}E_{b61}^{1/5}t_8^{2/5}\tau_8^{2/5}\ \text{kpc} & (\Omega_g=\Omega=1).\end{cases} \tag{9.42}$$

where t_8 and τ_8 are measured in units of 10^8 yr. These results are more accurate than those of Ikeuchi *et al.* (1983).

2. Nonradiative cosmological bubbles ($\Omega=\Omega_g$)

Radio galaxies, quasars, and perhaps young galaxies with high supernova rates can inject more than 10^{61} ergs into the surrounding medium in the form of kinetic energy. If energy is injected into the blastwave at a rate

$$L_{\rm in}=L_1t^{\eta_{\rm in}-1}\ , \tag{9.43}$$

for $\tau>\tau_b$, then the evolution of the resulting bubble at late times ($t=\tau-\tau_b\rightarrow\tau$) is described by Eq. (9.6) with

$$E_b=\frac{L_{\rm in}\tau}{\eta_{\rm in}}=\frac{L_1\tau^{\eta_{\rm in}}}{\eta_{\rm in}}\ . \tag{9.44}$$

The result is

$$R_s=\left[\frac{\xi L_1}{\eta_{\rm in}\rho_u}\right]^{1/5}\tau^{(2+3h+\eta_{\rm in})/5}\ . \tag{9.45}$$

The value of η (the exponent of τ) is given by Eq. (9.13), since $\eta_E=\eta_{\rm in}$ for nonradiative bubbles.

The constant ξ is determined by Eq. (3.6) in terms of σ, which is given by Eq. (9.11). For $\gamma=\frac{5}{3}$, the parameters ν_1 and σ are

$$\nu_1=\begin{cases}\dfrac{20+3\eta_{\rm in}}{4(5+\eta_{\rm in})} & (\Omega\ll 1),\\[2mm] \dfrac{46+9\eta_{\rm in}}{12(4+\eta_{\rm in})} & (\Omega=1).\end{cases} \tag{9.46a}$$

$$\sigma=\begin{cases}\dfrac{5(45\eta_{\rm in}^3+213\eta_{\rm in}^2+328\eta_{\rm in}+160)}{32(4\eta_{\rm in}+5)(5+\eta_{\rm in})^2} & (\Omega\ll 1),\\[2mm] \dfrac{5(1215\eta_{\rm in}^3+5859\eta_{\rm in}^2+9272\eta_{\rm in}+14\,212/3)}{1728(4+\eta_{\rm in})^2(2\eta_{\rm in}+3)} & (\Omega=1).\end{cases} \tag{9.46b}$$

Specializing further to the case $\eta_{\rm in}=1$ (steady energy injection), we find

$$\sigma=\begin{cases}\frac{1865}{5184}\simeq 0.360 & (\Omega\ll 1),\\ \frac{1265}{2592}\simeq 0.488 & (\Omega=1).\end{cases} \tag{9.47}$$

The parameter α^2 [Eq. (9.17)] in this case is

$$\alpha^2=\begin{cases}\frac{15552}{1129}\simeq 13.8 & (\Omega\ll 1),\\ \frac{486}{59}\simeq 8.24 & (\Omega=1).\end{cases} \tag{9.48}$$

Finally, the radius of the bubble is, for a constant rate of energy injection L,

$$R_s=\begin{cases}\left[\dfrac{1440GH_0L}{373\Omega_{g0}}\right]^{1/5}\tau^{6/5} & (\Omega\ll 1),\\[2mm] \left[\dfrac{11\,664GL}{1265}\right]^{1/5}\tau & (\Omega=1).\end{cases} \tag{9.49}$$

For the $\Omega=1$ case the velocity of expansion is constant at $v_s=770\,(L/10^{11}L_\odot)^{1/5}$ km/s.

3. Radiative cosmological blastwaves ($\eta_{\rm in}=0$)

The non-self-similar case will be taken up in Sec. IX.B below. The shell approximation adopted in the present section is exact for the radiative case, since the shell collapses to zero thickness and our equations become very simple with $\nu_1=K_n=1$, $\delta=0$, and $w=1-\Omega_g/2\Omega$. (Radiative cosmological bubbles, with $\eta_{\rm in}\neq 0$ and $K_{02}\neq 1$, have been discussed by Ostriker *et al.*, 1986.) In this case we need not restrict ourselves to the case $\Omega=\Omega_g$. Since we are assuming $\gamma=\gamma_i$, a radiative blastwave must be a momentum-conserving snowplow and not a pressure-driven snowplow; hence $\alpha^{-1}=0$. Equations (9.13) and (9.17) then imply

$$0=\frac{5\eta-1-3h}{\eta}-\frac{3h}{\eta}\left[1-\frac{h}{\eta}\right]-1+\frac{w\Omega h^2}{2\eta^2}, \tag{9.50}$$

which can be solved for η to give

$$\eta=\tfrac{1}{8}\{1+6h+[1+12h-4(3+2w\Omega)h^2]^{1/2}\}. \tag{9.51}$$

We have discarded the negative root because it gives $\eta<h$, in violation of Eq. (9.6).

The radius of the radiative blastwave is given by Eq. (3.10). If we define R_c as the blastwave radius at which $E_b=E_0$, the energy prior to any losses, and set $R_s(1)=R_c$, then we have

$$R_s=R_c\left[\frac{\xi E_0\tau_c^{3h}}{\rho_u R_c^5}\right]^{\eta/2}\tau^\eta, \tag{9.52}$$

where $\tau_c\equiv\tau(R_s=R_c)$. If the blastwave becomes radiative in the pre-cosmological phase ($t_c\lesssim\tau_b$), then R_c and τ_c have only formal significance and one should use the non-self-similar results below [Eqs. (9.97) and (9.100)]. Evaluation of Eq. (9.52) at τ_c implies

$$R_c=(\xi E_0/\rho_u)^{1/5}\tau_c^{(2+3h)/5}, \tag{9.53}$$

as expected from Eq. (9.6). Inserting Eq. (9.53) into (9.52) then yields the final expressions for the radius of a radiative, cosmological blastwave,

$$R_s=\left[\frac{\xi E_0}{\rho_u R_c^{\eta_E/\eta}}\right]^{\eta/(3h+2)}\tau^\eta=\left[\frac{\xi E_0\tau_c^2}{\rho_c}\right]^{1/5}\left[\frac{\tau}{\tau_c}\right]^\eta, \tag{9.54}$$

in terms of R_c and τ_c, respectively; here $\rho_c\equiv\rho(\tau_c)$. The value of $\xi=3/(4\pi\eta^2\sigma)$ is determined by the energy equation (9.11), which for radiative blastwaves reduces to

$$\sigma=\frac{1}{2}-\frac{3}{10}\left[\frac{h}{\eta}\right]^2(1+\tfrac{2}{3}\Omega-\tfrac{5}{6}\Omega_g). \tag{9.55}$$

In a low-density universe ($\Omega\ll 1$, $h=1$), Eq. (9.51) gives $\eta=1$, which is identical to the comoving solution found for the adiabatic case above. Using Eqs. (9.11) and (9.13), we can thus summarize a radiative blastwave in a low-density cosmology as follows:

$$\eta=1,\quad \eta_E=0,\quad \sigma=\tfrac{1}{5},\quad \xi=\frac{15}{4\pi}\quad(\Omega\ll 1), \tag{9.56}$$

so that the radius is given by Eq. (9.31). For $\Omega\ll 1$, the radiative and nonradiative blastwaves are completely equivalent: since the blastwave is comoving ($\Delta v=0$), no radiation is possible and the energy is conserved ($\eta_E=0$). If the blastwave becomes radiative in the pre-cosmological phase, then the non-self-similar solution derived in Sec. IX.B below is necessary to relate E_0 to the actual initial energy.

The just closed case ($\Omega=1$, $h=\frac{2}{3}$) is more interesting. Equations (9.51) and (9.55) give

$$\eta=[15+(1+16\Omega_g)^{1/2}]/24, \tag{9.57a}$$

$$\eta_E=[-21+5(1+16\Omega_g)^{1/2}]/24\quad(\Omega=1), \tag{9.57b}$$

$$\sigma=\frac{1}{2}-\frac{128(1-\frac{1}{2}\Omega_g)}{[15+(1+16\Omega_g)^{1/2}]^2}. \tag{9.58}$$

For $\Omega_g=1$, these equations, together with Eqs. (9.13) and (9.15), yield

$$\begin{aligned}&\eta=(15+\sqrt{17})/24=0.7968\ ,\\&\eta_E=-(21-5\sqrt{17})/24=-0.0160\ ,\\&\sigma=0.3250,\quad \xi=1.1567\ ,\\&\Delta v=0.1633v_s=0.1952v_H\quad(\Omega_g=1)\ .\end{aligned}\tag{9.59}$$

In this solution, like the adiabatic blastwave, the shock propagates about 20% faster than the Hubble velocity. The blastwave radius [Eq. (9.54)] becomes

$$R_s=(21.80GE_0R_c^{0.0201})^{0.1992}\tau^{0.7968}\quad(\Omega_g=\Omega=1)\ .\tag{9.60}$$

In the opposite limit $\Omega_g \ll \Omega=1$, we obtain

$$\begin{aligned}&\eta=(2+\Omega_g)/3,\quad \eta_E=(-2+5\Omega_g)/3\ ,\\&\sigma=\tfrac{3}{4}\Omega_g,\quad \xi=\frac{9}{4\pi\Omega_g}\ ,\\&\Delta v=\tfrac{1}{2}\Omega_g v_s=\tfrac{1}{2}\Omega_g v_H\quad(\Omega_g\ll\Omega=1)\ .\end{aligned}\tag{9.61}$$

In the limit $\Omega_g\to 0$, this leads to the comoving solution with

$$M\to\left[\frac{4E_b^3\tau^2}{3G^2\Omega_g}\right]^{1/5}\quad(\Omega_g\to 0)\ ,\tag{9.62}$$

where $E_b^3\tau^2$ is constant in this limit. Clearly to reach this comoving state requires non-self-similar evolution, which must be treated separately. During this non-self-similar evolution most of the blastwave energy is transferred to the dark matter, so E_b is considerably less than the explosion energy. The energetics of blastwaves with $\Omega_g<\Omega$ is discussed below.

4. Blastwaves with $\Omega_g<\Omega=1$

If only a fraction of the matter is gaseous, then the gravitational interaction between the gas and the dark matter dramatically alters the evolution of the blastwave. For "cold" dark matter (see the discussion at the beginning of Sec. IX) two stages in the evolution may be identified: In the first stage, energy is drained from the gas ($\eta_E<0$) and added to the dark matter as the shell of gas sweeps over the dark matter; in the case of the radiative cosmological blastwaves treated above, the gravitational energy loss far exceeds the radiative losses by the factor $(1-\Omega_g)(v_H/\Delta v)^2$ [see Eq. (9.69)]. This energy is not truly lost, but rather transferred to the dark component. In the second stage of evolution, the accelerated dark matter has overtaken the gas and is also in a thin shell; for $\Omega_g\ll\Omega=1$, the blastwave will closely follow Bertschinger's (1985a) solution for a collisionless compensated hole (a blastwave in which the initial energy is purely gravitational but the mass inside the blastwave is unchanged from that in the Hubble flow). In this case the energy in the gas is $E\simeq\Omega_g E_0$, and the total energy in all the matter is $E_t\simeq E_0$. The first stage does not have a truly self-similar character, since the time required to evolve into the self-similar configuration is comparable to the time required for the dark matter to begin overtaking the gas shell. The second stage will become self-similar because, in the absence of thermal energy in the Hubble flow, it lasts indefinitely.

It is also possible to apply our analysis to "hot" dark matter, in which the velocity dispersion is large compared to the shock velocity (but not so large that the universe is pressure dominated). In this case the hole in the gas distribution due to the blastwave induces a density perturbation in the dark matter no greater than of the order of $\Omega_g(v_s/C_d)^2$, which is negligible for $v_s\ll C_d$. The first stage still exists, but since the blastwave is subsonic with respect to the dark matter, a shock can never develop in the dark matter and the second stage will not occur. If the dark matter dominates the mass, it must be nonrelativistic in the matter-dominated era of the Hubble expansion, so that $C_d\propto\tau^{-2/3}$ for $\Omega=1$. By contrast, the Hubble velocity at the shock declines more slowly than $\tau^{-1/3}$, and hence C_d/v_s decreases with time and "hot" dark matter eventually becomes "warm." Our analysis does not apply to "warm" dark matter ($C_d\sim v_s$), since the density perturbation in the dark matter can be comparable to the gas density.

Stage I: Dark matter acceleration. During this phase there is no truly self-similar evolution, so we generalize the treatment of radiative blastwaves to determine the asymptotic value of η. Since $\eta_E=5\eta-4$ for $\Omega=1$ [Eq. (9.13)], we have

$$\delta=2(1-K_1)=\frac{3\eta-2}{3(2\eta-1)X_1}\tag{9.63}$$

from Eqs. (9.21) and (9.27), so that

$$w=\frac{9X_1(2\eta-1)(2-\Omega_g)-(3\eta-2)(6-7\Omega_g)}{18X_1(2\eta-1)}\tag{9.64}$$

from Eq. (9.20). Since the solution is not strictly self-similar, the virial theorem in the form we have derived it is only approximately valid; however, using it [Eq. (9.17)] and approximation (9.19) for α^{-2} we find

$$\begin{aligned}\frac{6(\gamma-1)(3\eta-2)^3}{\eta(\gamma+1)^2}&=\frac{(5\eta-3)(3\eta+\gamma-1)[(9\gamma+15)\eta-4\gamma-8]}{(2\eta-1)(\gamma+1)^2}\\&\quad-6(3\eta-2)-\frac{4(3\eta+\gamma-1)^2}{(\gamma+1)^2}+2w\ .\end{aligned}\tag{9.65}$$

Equation (9.65) generalizes the result for radiative blastwaves ($\gamma=1$, $\alpha^{-2}=0$), Eq. (9.50). An approximate solution to Eqs. (9.64) and (9.65) is

$$\eta\simeq\frac{4}{3}\left[\frac{1+2\Omega_g}{2+3\Omega_g}\right],\tag{9.66a}$$

$$\eta_E \simeq -\frac{4}{3}\left[\frac{1-\Omega_g}{2+3\Omega_g}\right] . \quad (9.66b)$$

The expression for η is accurate to within 1.5% for $\frac{4}{3} \le \gamma \le \frac{5}{3}$; that for η_E is accurate to within 5% over the same range of γ. For $\Omega_g \ll 1$, it is straightforward to show that $\eta \simeq (2+\Omega_g)/3$, just as for radiative blastwaves. As $\Omega_g \to 0$, self-similar test particles comove with the Hubble expansion. In the opposite limit, $\Omega_g = 1$, we derive from Eq. (9.65), $\eta = (0.8008, 0.7992, 0.7968)$ for $\gamma = (\frac{5}{3}, \frac{4}{3}, 1)$, respectively, in excellent agreement with the correct value for energy-conserving blastwaves, $\eta = \frac{4}{5}$. Knowing η and hence K_1 and w, it is possible to find σ and ξ, but in view of the approximate nature of the treatment we shall not pursue this here. The results for radiative blastwaves, which are more accurate, have been given in Sec. IX.A.3.

The energetics of a blastwave in a partially gaseous Hubble flow can be treated exactly in the limit in which the gas is in a thin shell; the discussion will thus be approximate for energy-conserving blastwaves, which have a finite thickness, but exact for the radiative blastwaves considered above. For the gas, the energy balance is (see Appendix A)

$$E_K + E_{\rm th} + W + E_{\rm rad} + \Delta W = E_0 , \quad (9.67)$$

where W is the gravitational potential energy of the gas in the presence of the dark matter, $E_{\rm rad}$ is the energy radiated by the blastwave, and ΔW is the gravitational energy transferred to the dark matter from the gas. Since $\Omega = 1$, the energy of the ambient gas E_a vanishes [Eq. (9.10a)]. The rates of energy loss from the gas are

$$\frac{dE_{\rm rad}}{d\tau} = \tfrac{1}{2}\dot{M}(\Delta v)^2 , \quad (9.68a)$$

$$\frac{d\Delta W}{d\tau} = 4\pi R_s^2 \rho_d \Delta v \frac{GM}{R_s} = \tfrac{1}{2}(1-\Omega_g)\dot{M} v_H^2 , \quad (9.68b)$$

where $\dot{M}$ is the rate at which gas is swept up by the blastwave. These results have two important consequences. First, the radiative losses are small compared to the gravitational losses,

$$\frac{E_{\rm rad}}{\Delta W} = \frac{(\Delta v)^2}{(1-\Omega_g)v_H^2} = \frac{1}{1-\Omega_g}(\tfrac{3}{2}\eta - 1)^2 , \quad (9.69)$$

unless $\Omega_g \simeq 1$. Second, both losses are largest at early times (since $E \propto \tau^{\eta_E - 1}$), prior to the onset of self-similar flow. The energy balance for the dark matter is

$$E_{K,d} + E_{{\rm th},d} + W_d = \Delta W , \quad (9.70)$$

where W_d is the potential energy of the dark matter as defined by the second term of Eq. (9.7); since the gas is confined to a shell outside the dark matter, the interaction part vanishes. The sum of Eqs. (9.67) and (9.70) gives the overall conservation relation

$$E_{K,t} + E_{{\rm th},t} + W_t + E_{\rm rad} = E_0 , \quad (9.71)$$

where $E_{K,t} = E_K + E_{K,d}$, etc., as it should. These results hold so long as the dark matter does not overtake the gas; when it does, energy is transferred back to the gas, and the flow makes a transition to the second stage.

Stage II. Combined dark matter/gas blastwave (cold dark matter only). Even if the blast begins initially in the baryonic component, after long times the dark matter, if cold, will also have been swept up into a thin shell. That material will, to an excellent approximation (for $\Omega_g \ll \Omega$), follow the self-similar solution for a collisionless compensated hole found by Bertschinger (1985a), so both components will be in thin overlapping shells expanding as $\tau^{4/5}$. Alternatively, if the perturbation begins as simultaneous and proportional positive energy perturbations in the combined dark matter plus baryonic fluids, due to some combination of initial negative density perturbations or positive (super-Hubble) velocity perturbations, both fluids will approach a self-similar $R \propto \tau^{4/5}$ solution together.

In either case most of the energy will be in the dark matter. The simplest, and quite accurate, way of treating this problem is to take the combined blastwave to be the superposition of the blastwave in the gas and that in the dark matter. Since the latter is collisionless, the velocity dispersion it acquires is entirely one dimensional; hence, we treat it as a gas with $\gamma = 3$. We then infer a shell thickness for the dark matter (subscript d) of $1-\lambda_{id} = 1-\nu_{1d} = (1-\nu_H)/\chi_{id} = \frac{1}{12}$ for an energy-conserving blastwave, compared to the exact value 0.0996 (Bertschinger, 1985a), and $\sigma_d = 0.2410$ from Eq. (9.37), compared to the exact 0.2506. Because the collisionless shell thickness exceeds that of the gas, the blastwave radii R_g and R_d differ for the two components; R_g is the radius of the gas shock and R_d is that of the outer caustic in the collisionless gas. We adopt the simple approximation that the gas shell is centered in the shell of dark matter, so that

$$\frac{1+\nu_{1g}}{2} R_g = \frac{1+\nu_{1d}}{2} R_d , \quad (9.72)$$

where ν_{1g} is the value of ν_1 for the gas [Eq. (9.16)]; $\nu_{1g} R_g$ is the inner radius of the shell in the shell approximation. Since $\gamma = 3$ for the dark matter, we find, using Eqs. (9.72) and (E23),

$$\frac{R_d}{R_g} = \frac{2[\gamma + 3 + (\gamma-1)\nu_H]}{(\gamma+1)(3+\nu_H)} . \quad (9.73)$$

The energies in the two components are $E_d \equiv \sigma_d M_d v_{sd}^2$ and $E_g \equiv \sigma_g M_g v_{sg}^2$, where we have added the subscript g on E and M for clarity. The superposition approximation implies that σ_g has the same value that it would have in the absence of the dark matter. Since the dark matter extends beyond the gas, we have

$$M_d v_{sd}^2 / M_g v_{sg}^2 = (1-\Omega_g) R_d^5 / (\Omega_g R_g^5)$$

and

$$\frac{E_t}{E_g}=\frac{E_g+E_d}{E_g}=1+\frac{\sigma_d(1-\Omega_g)}{\sigma_g\Omega_g}\left[\frac{R_d}{R_g}\right]^5 . \tag{9.74}$$

Let us define the effective value of ξ_c for the combined blastwave to be

$$\frac{\xi_g E_g}{\rho_g}\equiv\frac{\xi_c E_t}{\rho_t} , \tag{9.75}$$

so that

$$\xi_c=\frac{3E_g}{4\pi\eta^2\sigma_g\Omega_g E_t} . \tag{9.76}$$

The blastwave radius in the gas is then

$$R_g=\left[\frac{\xi_g E_g\tau^4}{\rho_u}\right]^{1/5}=(6\pi\xi_c G E_t\tau^4)^{1/5} \tag{9.77}$$

from Eqs. (9.4b), (9.6), and (9.75). These results allow one to determine the radii R_g and R_d for arbitrary Ω_g and γ, and thus provide a useful extension to Bertschinger's (1985a) results. Adopting Bertschinger's value $\sigma_d=0.2506$ ensures that R_d is exact in the limit $\Omega_g=0$; we then find

$$R_g/(6\pi G E_t\tau^4)^{1/5}=(1.060,1.050)$$

for $\gamma=(\frac{5}{3},\frac{4}{3})$ and $\Omega_g=0$, compared to the exact values (1.052, 1.048). For $\Omega_g=1$, the results reduce to the $\Omega_g=1$ results obtained above.

Remarkably enough, radiative blastwaves differ only slightly from energy-conserving ones if Ω_g is small, both because the losses scale as $(\Delta v)^2$ and because most of the energy is in the dark matter:

$$\dot E_t=-\tfrac{1}{2}\dot M_g(\Delta v)^2=-\tfrac{3}{2}\eta\frac{M_g v_{sg}^2}{\tau}\left[1-\frac{2}{3\eta}\right]^3$$

$$\equiv E_t\frac{\eta_E}{\tau}=\frac{E_t(5\eta-4)}{\tau} . \tag{9.78}$$

Since, in the limit $\Omega_g\ll\Omega$, η will be very close to $\frac{4}{5}$, we solve the above equation in that limit to obtain

$$\eta-\tfrac{4}{5}=-\tfrac{1}{900}\frac{M_g v_{sg}^2}{E_t}=-\frac{1}{900}\frac{\Omega_g}{\sigma_c} , \tag{9.79}$$

where $\sigma_c=3/(4\pi\eta^2\xi_c)=\sigma_g\Omega_g E_t/E_g$ from Eq. (9.76). For $\gamma=1$, Eq. (9.74) gives $\sigma_c=0.3100$ in this limit, so that

$$\eta=\tfrac{4}{5}-0.0036\Omega_g,\quad \eta_E=-0.018\Omega_g . \tag{9.80}$$

Note that although this solution was obtained in the limit $\Omega_g\ll1$, it is close to the result (9.59) if $\Omega_g=1$ and so should be expected to be a reasonable approximation for the whole range $0<\Omega_g<1$. Greater accuracy can be attained by inserting $\sigma_g=0.3250$ [from Eq. (9.59)] into Eq. (9.76) and solving (9.78) directly.

If, as would be the case in certain currently popular cosmologies, $\Omega=1$ and $\Omega_g\approx0.1$, then disturbances in the combined-matter distribution will be able to propagate through the epoch in which inverse Compton cooling is efficient ($z>7$) as growing self-similar shells (Vishniac, Ostriker, and Bertschinger, 1985). It is remarkable that, even though the thermal cooling time is very small compared to the age of the universe, $\tau_{\rm cool}/\tau\approx[7/(1+z)]^{5/2}\ll1$, during these epochs, the propagating blast (or "void" if that term is to be preferred) will only lose energy at the very much slower rate of $\tau_{\rm loss}/\tau=1/\eta_E\approx500\gg1$.

We note that in both this case and the nonradiative case, the gaseous component is so cold that it is likely to be unstable to nonradial perturbations. Both the initial analysis of Ostriker and Cowie (1981) and subsequent work by Bertschinger (1983) and Vishniac (1983) have found that the shells are likely to fragment into pieces with mass of order $10^{-3}M_s$ after the self-similar state is reached (i.e., for $\tau\gg\tau_b$).

B. Non-self-similar cosmological blastwaves

1. Solution from the equation of motion ($\eta_{\rm in}=0$)

Blast waves in expanding media are self-similar only in the limit $\tau\gg\tau_b$, where τ_b marks the onset of the blastwave. We have obtained approximate expressions for the behavior of the blastwave by meshing the cosmological similarity solutions with the Sedov-Taylor similarity solution appropriate at early times [$t=\tau-\tau_b\ll\tau_b$; cf. Eq. (9.42)]. An alternate approach is to attempt to solve the equation of motion for the blastwave directly. To do this we make two approximations: we assume that all the gas is swept up into a thin shell; and we assume that $\bar P=\bar\rho v_s^2/\alpha^2=\rho_0 v_s^2/\alpha^2$ with α^2 constant, just as for self-similar blastwaves. For radiative blastwaves, both approximations are exact. For adiabatic blastwaves at late times, the first approximation is exact for $\Omega=0$ and quite good for $\Omega=1$, as discussed above, and the second approximation becomes exact for both cases. If $\Omega_g<\Omega=1$, we assume that no dark matter has overtaken the gas shell—i.e., we treat only the first stage of the blastwave. With these approximations we can verify that the self-similar solutions of Sec. IX.A do obtain in the limit $\tau\gg\tau_b$, and we can obtain the cosmological generalization of the Oort snowplow for radiative blastwaves.

For the case at hand, the equation of motion [Eq. (D8)] in the shell approximation ($K_{01}=1$) with the pressure confined to a thin shell [$K_P=\frac{2}{3}$ from Eq. (D7)] reduces to

$$\frac{d}{d\tau}(Mv_1)=4\pi R_s^2(\tfrac{2}{3}\bar P+\rho_0 v_H\Delta v)+\int g\,dM \tag{9.81}$$

$$=\frac{3M}{R_s}\left[\frac{2}{3}\frac{v_s^2}{\alpha^2}+v_H\Delta v-\tfrac{1}{6}w'\Omega v_H^2\right] , \tag{9.82}$$

where $\int g\,dM$ and w' are given in Appendix A [Eqs. (A34)–(A36)]:

$$w'=1-\tfrac{1}{2}\delta-\tfrac{1}{2}(1-\tfrac{5}{3}\delta)\frac{\Omega_g}{\Omega} \ . \tag{9.83}$$

The velocity v_1 is given by Eq. (9.16), and the mass in the shell increases according to

$$\frac{dM}{d\tau}=4\pi R_s^2\rho_0\Delta v \ . \tag{9.84}$$

Defining

$$y\equiv\frac{v_s-v_H}{v_H}\equiv\frac{\Delta v}{v_H} \ , \tag{9.85}$$

so that $v_s=(1+y)v_H$, we can rewrite Eq. (9.81) as

$$\frac{dy}{ay^2+by+c}\equiv\frac{dy}{a(y-y_1)(y-y_2)}=-d\ln\tau \ , \tag{9.86}$$

where

$$a\equiv4h\left[1-\frac{1}{4}\frac{(\gamma+1)}{\alpha^2}\right] \ , \tag{9.87}$$

$$b\equiv h\left[\frac{\gamma+3}{2}-\frac{2(\gamma+1)}{\alpha^2}\right]-1 \ , \tag{9.88}$$

$$c\equiv\frac{\gamma+1}{2}\left[h\left[1+\frac{1}{2}w'\Omega-\frac{2}{\alpha^2}\right]-1\right] \tag{9.89a}$$

$$=-\frac{(\gamma+1)h}{8}[8\alpha^{-2}+\Omega\delta+(1-\tfrac{5}{3}\delta)\Omega_g] \ , \tag{9.89b}$$

and y_1, y_2 are the roots of the quadratic with $y_1>y_2$. In deriving (9.89b), we have used the relation $\Omega=2(1-h)/h$, which is valid for $\Omega=0,1$, and Eq. (9.83). The solution of Eq. (9.86) is

$$y=\frac{y_1-y_2(\tau_b'/\tau)^D}{1-(\tau_b'/\tau)^D} \ , \tag{9.90}$$

where

$$D\equiv(b^2-4ac)^{1/2}, \quad y_{1,2}\equiv\frac{\pm D-b}{2a} \ , \tag{9.91}$$

and τ_b' is an integration constant that would equal τ_b if the solution were valid at $\tau\to\tau_b$. Since

$$\frac{dR_s}{d\tau}=v_s=v_H(1+y) \ , \tag{9.92}$$

an integration yields

$$R_s=C_R\tau^{h(1+y_1)}[1-(\tau_b'/\tau)^D]^{h/a} \ , \tag{9.93}$$

where C_R is another integration constant. For $\tau\gg\tau_b'$, we have $R_s\propto\tau^\eta$ with

$$\eta=h(1+y_1)=\frac{D+h(13-\gamma)/2+1}{8-2(\gamma+1)\alpha^{-2}} \ , \tag{9.94}$$

where D is given by Eq. (9.95). On the other hand, in the pre-cosmological phase ($\tau\to\tau_b$), Eq. (9.93) yields

$$R_s\to C_R\tau_b^{h(1+y_1-a^{-1})}D^{h/a}t^{h/a} \quad (t\ll\tau_b) \ , \tag{9.95}$$

if we approximate $\tau_b'=\tau_b$. A more convenient form for R_s, which is exact in the limits $t\ll\tau_b$ and $\tau\gg\tau_b$ and accurate to within about 5% for intermediate values of τ, is

$$R_s=\frac{C_R t^{h/a}\tau^{\eta-h/a}}{\left[1+\left[\frac{1}{D}-1\right](1-x)^D\right]^{h/a}} \ , \tag{9.96}$$

where η is given by Eq. (9.94). These results can be applied to both nonradiative and radiative blastwaves.

a. Nonradiative blastwaves: approach to self-similar flow

Equation (9.93) confirms the power-law behavior found for $R_s(t)$ in the similarity solutions of Sec. IX.A. We adopt the values of α^2 found there in order to ensure the correct asymptotic behavior. For $\Omega=0$, we have $\alpha^{-2}=0$, $c=0$, and $D=(\gamma+1)/2$; hence Eq. (9.94) gives $\eta=1$ for $\tau\gg\tau_b'$, in agreement with Eq. (9.31). The integration (9.93) determines the comoving mass.

For $\Omega=1$, there are two cases. If $\Omega_g=1$, then α^2 is given by Eq. (9.36) and δ by (9.35). For $\gamma=\frac{5}{3}$, Eqs. (9.83), (9.87)–(9.89), (9.91), and (9.94) then yield $\eta=0.8007$; the similarity solution has $\eta=\frac{4}{5}$, in agreement to $O(\delta^2)$. In the other limit, $\Omega_g\ll1$, we evaluate α^2 and δ in the self-similar limit, finding $\alpha^{-2}=O(\Omega_g^3)\simeq0$ and δ given by Eq. (9.64). As a result, we obtain $\eta=(2+\Omega_g)/3$, in agreement with the similarity solution [Eq. (9.66a)].

The limiting expression for R_s at early times given by Eq. (9.95) is approximately $R_s\propto t^{1/4}$ in all three cases, if the cosmological values of α^2 are used; this differs significantly from the correct ST behavior $R_s\propto t^{2/5}$. On the other hand, if the LVA expression for α^2 is used [$\alpha^2=\gamma+1$, Eqs. (4.3) and (4.17)], then Eq. (9.95) gives the correct $t^{2/5}$ behavior. We conclude that Eq. (9.95) is not an accurate way to handle the transition from ST to cosmological blastwaves because α^2 changes during the transition. Equation (9.42) is the preferred general result. For the same reason, it is impossible to evaluate the integration constant C_R in terms of the energy in the pre-cosmological phase; this problem is addressed in Sec. IX.B.2 below.

b. Radiative blastwaves

Radiative blastwaves are characterized by $\gamma=1$. We further assume that the blastwaves have negligible internal pressure ($\alpha^{-2}=0$). This is appropriate to cosmological applications at early epochs ($z\gtrsim7$), when inverse Compton losses effectively cool the shock-heated interior electrons. Equations (9.87)–(9.89), (9.91), and (9.93) then yield

$$R_s=C_R\tau^\eta[1-(\tau_b'/\tau)^D]^{1/4} \ , \tag{9.97}$$

$$D=[1+12h-4h^2(3+2w'\Omega)]^{1/2} \ , \tag{9.98}$$

where η is given by Eq. (9.94). Comparison with Eq.

(9.51) demonstrates that this value of η is identical to that found for self-similar radiative blastwaves. If the blastwave makes the transition from adiabatic to radiative at late time ($\tau \gg \tau_b$), and if $\Omega = \Omega_g$, then the coefficient C_R is given by Eq. (9.54) as

$$C_R = (\xi E_b / \rho_u R_c^{\eta_E/\eta})^{\eta/(3h+2)} \quad (\tau_c \gg \tau_b) . \tag{9.99}$$

On the other hand, if the blastwave becomes radiative in the pre-cosmological phase ($t_c \ll \tau_b$), then at these early times R_s is given by Eq. (9.95) with $R_s \propto t^{h/a} = t^{1/4}$. This time dependence conforms with that for the momentum-conserving snowplow [Eqs. (6.13) and (6.16)], which allows one to determine C_R:

$$C_R = \left[\frac{24 E_b R_c^3}{\pi D^2 \rho_u \tau_b^{D+3h-1}} \right]^{1/8} \quad (t_c \ll \tau_b) . \tag{9.100}$$

The cases of high and low density ($\Omega_g = \Omega = 1,0$) have already been discussed in Sec. IX.A.3 on self-similar radiative blastwaves, and do not require further elaboration. It is worth noting that, in the low-density case ($\Omega = 0$) in which gravitational effects are negligible, we have $(D, y_1, y_2) = (1, 0, -\frac{1}{4})$, so that Eqs. (9.84), (9.85), and (9.90) imply

$$M(v_s - \tfrac{3}{4} v_H) = \text{const} \quad (\Omega = 0) . \tag{9.101}$$

This verifies directly that this solution is the generalized momentum-conserving snowplow: the difference between the total and swept-up momentum remains constant in time.

2. Model for blastwaves with $\Omega_g < \Omega$: dark matter acceleration

If only a fraction of the matter is gaseous, then, as emphasized in Sec. IX.A.4 above, the dynamics of a blastwave is strongly affected by the gravitational interaction between the gas and the "dark" matter. In the first stage, energy is drained from the gas as the dark matter is accelerated. This stage is not truly self-similar because the time it takes for the blastwave to evolve into a self-similar configuration is comparable to the time it takes for the accelerated dark matter to overtake the gas shell; furthermore, the similarity methods we have used cannot treat the evolution at $t \sim t_b$, when a significant energy loss can occur. Solution of the equation of motion (Sec. IX.B.1) provides an accurate means for treating blastwaves that become radiative in the pre-cosmological era ($t_{\text{cool}} \ll t_b$), but it is inaccurate for nonradiative blastwaves, in which the fraction of the energy that is thermal drops precipitously as the blastwave evolves from the ST stage to the self-similar cosmological stage. Here we develop a simple analytic approximation for the non-self-similar evolution of cosmological blastwaves under the assumption that the accelerated dark matter does not overtake the gas shell (Stage I in the nomenclature of Sec. IX.A.4).

All the blastwaves we have discussed obey a relation of the form (3.5), $R_s = (\xi E_b t^2/\bar{\rho})^{1/5}$. In the problem at hand the energy E_b is affected by the gravitational energy loss to the dark matter and must be determined. We adopt the ansatz

$$\xi E_b = \bar{\xi}_e E_0 \left[\frac{t}{t_c} \right]^{\eta_{E_0}} \left[\frac{\tau}{\tau_b} \right]^{\eta_E - \eta_{E_0}} , \tag{9.102}$$

with

$$\bar{\xi}_e \equiv \xi_0 (1-x) + \xi_e x . \tag{9.103}$$

Here E_0 is the initial blastwave energy; we allow for radiative energy losses on the blastwave time scale $t = \tau - \tau_b \equiv x\tau$ and for gravitational losses on the cosmic time scale τ. If there are radiative losses ($\eta_{E_0} > 0$), then Eq. (9.102) is valid only for $t > t_c$. The constant ξ_0 is known from the pre-cosmological solution (e.g., the ST solution), and the constant ξ_e remains to be determined. It should be emphasized that $\bar{\xi}_e$ may differ substantially from ξ, so that E_b in turn may differ from $E_0 (t/t_c)^{\eta_{E_0}} (\tau/\tau_b)^{\eta_E - \eta_{E_0}}$. This ansatz will be reasonably accurate only if ξ_e is of order ξ_0, so that $\bar{\xi}_e$ is approximately constant. For radiative blastwaves in a universe with $\Omega_g \ll \Omega = 1$ we found $\xi = 9/(4\pi\Omega_g) \gg \xi_{\text{ST}} = \xi_0$ [Eq. (9.61)], so this accuracy requirement will be satisfied only if $\xi_e \ll \xi$ in this case. Inserting this ansatz into the general relation (3.5) yields

$$R_s = \left[\frac{\bar{\xi}_e E_0 \tau^{3h+\eta_E - \eta_{E_0}} t^{2+\eta_{E_0}}}{\rho_u t_c^{\eta_{E_0}} \tau_b^{\eta_E - \eta_{E_0}}} \right]^{1/5} , \tag{9.104}$$

where Eq. (9.2) has been used for $\bar{\rho}$. The blastwave velocity is then

$$v_s \equiv \bar{\eta} \frac{R}{t} , \tag{9.105}$$

with

$$\bar{\eta} = \frac{1}{5} \left[\frac{(\xi_e - \xi_0) x (1-x)}{\bar{\xi}_e} + (3h + \eta_E - \eta_{E_0}) x + 2 + \eta_{E_0} \right] . \tag{9.106}$$

To evaluate the constant ξ_e, we assume that the energy-loss processes would eventually reduce E_b to zero if the dark matter did not over take the gas. If we divide the total blastwave energy E_b into two parts, the gravitational interaction $-\Delta W$ and everything else,

$$E_b \equiv -\Delta W + \Delta E \tag{9.107}$$

[see Eq. (A21)], then the rate of change of E_b is

$$\frac{dE_b}{dt}=-4\pi R_s^2\rho_d(v_1-v_H)\frac{GM}{R_s}+\Delta\dot{E}\ , \qquad (9.108)$$

where we have replaced the thin-shell approximation for $d\Delta W/dt$ in Eq. (9.68b) with the more accurate shell approximation (v_1-v_H in place of v_s-v_H). Noting that

$$t=\frac{x\tau_b}{1-x}\ , \qquad (9.109)$$

$$\frac{dt}{dx}=\frac{\tau_b}{(1-x)^2}\ , \qquad (9.110)$$

we can use Eq. (9.104) to reexpress the energy-loss rate as

$$\frac{dE_b}{dx}=-2\pi E_0 h^2(1-\Omega_g)\left[\frac{1-x_c}{x_c}\right]^{\eta_{E_0}}\frac{\bar{\xi}_e(\nu_1\bar{\eta}-hx)x^{\eta_{E_0}+1}}{(1-x)^{\eta_E+1}}+\frac{\tau_b\Delta\dot{E}}{(1-x)^2}\ . \qquad (9.111)$$

The explicit x dependence of the gravitational energy-loss term can be found with the aid of Eqs. (9.103), (9.106), and (E21):

$$\bar{\xi}_e(\nu_1\bar{\eta}-hx)=\frac{2}{5(\gamma+1)}\{\xi_0(1-x)[2+\eta_{E_0}-(2h-\eta_E+\eta_{E_0}+1)x]+\xi_e x[3+\eta_{E_0}-(2h-\eta_E+\eta_{E_0}+1)x]\}\ . \qquad (9.112)$$

(We note that for detonations the factor $\frac{2}{5}$ should be replaced by $\frac{1}{5}$.) The value of ξ_e is then obtained by integrating Eq. (9.111) and requiring $E_b(x=1)=0$. The integral in the gravitational term is

$$\int_0^1\frac{\bar{\xi}_e(\nu_1\bar{\eta}-hx)x^{\eta_{E_0}+1}}{(1-x)^{\eta_E+1}}dx=\frac{4(1-h)\Gamma(2+\eta_{E_0})\Gamma(-\eta_E)(2+\eta_{E_0})}{5(\gamma+1)\Gamma(4+\eta_{E_0}-\eta_E)}[-\eta_E\xi_0+(3+\eta_{E_0})\xi_e]\ , \qquad (9.113)$$

where Γ denotes the gamma function.

As an example, consider the simple case in which the only energy losses are gravitational ($\Delta E=0$). Then $h=\frac{2}{3}$, since these losses are not important in an empty universe; $\eta_{E_0}=0$, since the losses occur on the cosmic time scale τ; and $\xi_0=\xi_{\rm ST}=2.026$. The integral of Eq. (9.111) then gives

$$-E_0=\frac{-64\pi E_0(1-\Omega_g)\Gamma(-\eta_E)}{135(\gamma+1)\Gamma(4-\eta_E)}(-\eta_E\xi_{\rm ST}+3\xi_e)\ , \qquad (9.114)$$

so that

$$\xi_e=\tfrac{1}{3}\eta_E\xi_{\rm ST}+\frac{45(\gamma+1)(3-\eta_E)(2-\eta_E)(1-\eta_E)(-\eta_E)}{64\pi(1-\Omega_g)}\ . \qquad (9.115)$$

The value of η_E is given by Eq. (9.66b). The accuracy of this method may be gauged by taking the limit $\Omega_g\to 1$, corresponding to the adiabatic blastwaves discussed in Sec. IX.A.1. Since $\eta_E\to -4(1-\Omega_g)/15$ in this case, we find $\xi_e=9(\gamma+1)/8\pi$; for $\gamma=\frac{5}{3}$, this becomes $\xi_e=3/\pi=0.955$, which is lower than the exact value (Bertschinger, 1983) by a factor 1.33 (corresponding to R's being too low by 1.06). This is remarkably accurate considering that the rate of the gravitational energy losses, which was used to infer ξ_e, goes to zero in this limit. In the other limit $\Omega_g\to 0$, $\eta_E\to\frac{2}{3}$ and we find $\xi_e=6.03$.

The blastwave energy at late times can be found by integrating Eq. (9.111) with $x\simeq 1$. For $\Delta\dot{E}=0$ and $\Omega=1$, we obtain

$$\frac{E_b}{E_0}=\frac{8\pi\xi_e\Omega_g}{9(\gamma+1)}\left[\frac{\tau}{\tau_b}\right]^{\eta_E}\quad(\tau\gtrsim\tau_b/\Omega_g)\ , \qquad (9.116)$$

where we have used Eq. (9.66b) for η_E. For $\Omega_g=1$, extrapolation back to $\tau=\tau_b$ gives $E_b(x=0)=E_0$; on the other hand, for $\Omega_g\ll 1$, extrapolation gives $E_b(x=0)=6.31\Omega_g E_0$, so that the rate of energy loss in the pre-self-similar stage can differ significantly from that in the self-similar stage. The onset of the self-similar stage for $\Omega_g\ll 1$ can be estimated from Eq. (9.112): terms of order $1-x$ can be neglected, as they were in obtaining Eq. (9.116), only if $1-x=\tau_b/\tau\ll\Omega_g$, so that the onset of the self-similar stage occurs at $\tau\sim\tau_b/\Omega_g$.

X. DETONATION WAVES ($f=\lambda_{\rm cl}^{-1}=\gamma-\gamma_i=\dot{m}=P_0=0$)

A. General results

Energy can be provided to blastwaves over a period of time either from the interior, producing a "bubble" (Secs. VII and IX), or at the shock front. The latter case is termed a detonation wave. Astrophysical examples include nuclear explosions in stars (Hoyle and Fowler, 1960), self-propagating star formation (Mueller and Arnett, 1976; Seiden and Gerola, 1979; Cowie and Rybicki, 1982), and self-propagating galaxy formation (Ikeuchi, 1981; Ostriker and Cowie, 1981).

In a detonation, the energy released is generally proportional to the swept-up mass, so that $E_b\propto M$ (in this section we set $\dot{M}=0$ so that $M_a=M$). In a self-similar blastwave, the energy is also proportional to Mv_s^2 [Eq. (3.3)]; hence the shock velocity v_s is constant, so that $R\propto t$ and $\eta=1$. Since the thermal energy is also proportional to the mass, we have $E/R^3 \propto \bar{P} \propto M/R^3 \propto \rho$, which implies $k_E+3=k_P=k_\rho$, Eq. (3.11) then confirms that $\eta=1$. To summarize, self-similar detonations with $E_b \propto M$ satisfy the general relations

$$\eta=1, \quad k_E=k_\rho-3, \quad \eta_E=3-k_\rho, \quad k_P=k_\rho \ . \tag{10.1}$$

Detonation waves differ from the blastwaves considered in previous sections of this paper because the shock jump conditions across the shock are altered by the energy injection (see Landau and Lifschitz, 1959). Let εc^2 be the energy per gram released in the detonation wave, so that the energy jump condition for a strong shock becomes

$$\tfrac{1}{2}(v_s-v_1)^2+\frac{\gamma}{\gamma-1}C_1^2=\tfrac{1}{2}(v_s-v_H)^2+\varepsilon c^2 \ , \tag{10.2a}$$

where C_1 is the isothermal sound speed behind the shock. Define the parameters u and β by

$$(v_s-v_H)^2 \equiv 2\left[\frac{\gamma^2-1}{1-u^2}\right]\varepsilon c^2 \equiv \beta^2 c^2 \ ; \tag{10.2b}$$

the more general definition of u, allowing for finite preshock temperature, is given in Eq. (E15). Solution of the jump conditions in the strong shock limit then yields (Appendix E)

$$\chi_1 \equiv \frac{\rho_1}{\rho_0}=\frac{\gamma+1}{\gamma-u} \ , \tag{10.3a}$$

$$\nu_1 \equiv \frac{v_1}{v_s}=1-(1-\nu_H)\left[\frac{\gamma-u}{\gamma+1}\right] , \tag{10.3b}$$

$$\frac{v_s-v_1}{C_1}=\frac{1-\nu_1}{\theta_1^{1/2}}=\left[\frac{\gamma-u}{1+u}\right]^{1/2} . \tag{10.3c}$$

For ordinary shocks with $\varepsilon=0$ we have $u\rightarrow 1$ and Eqs. (10.3) reduce to the standard results. In the case $u=0$ we see from (10.3c) that the flow behind the detonation wave moves at the adiabatic sound speed relative to the wave. This condition, called the *Chapman-Jouguet condition,* is shown in Zel'dovich and Kompanyets (1960) to be the normal steady state for a detonation wave and obtains, for instance, in the spherically symmetrical self-similar solution (cf. Landau and Lifschitz, 1959), although only for a limited range of k_ρ (Sedov, 1959). We shall henceforward take $u=0$ (see Appendix E for a brief discussion of the general case). Then Eq. (10.2b) implies that in the frame of the unshocked gas a steady detonation wave will move at a velocity βc:

$$\beta=[2(\gamma^2-1)\varepsilon]^{1/2} \ , \tag{10.4a}$$

$$v_s=v_H+\beta c, \quad v_1=v_H+\frac{\beta c}{(\gamma+1)} \tag{10.4b}$$

($u=0$, Chapman-Jouguet conditon). The density jump is smaller in a detonation wave than in a normal shock, with (ρ_1/ρ_0) equal to $(\frac{8}{5})$ rather than (4) in the usual case of $\gamma=\frac{5}{3}$.

B. Stationary media [$v_H=u=0$, $k_\rho<2\gamma/(\gamma+1)$]

Detonation waves in a stationary medium can be self-similar if the density is distributed like a power law, $\rho=\rho(1)R^{-k_\rho}$. In our terminology the requirement that $E=\sigma M v_s^2=\varepsilon M c^2$ implies that

$$v_s=(\varepsilon/\sigma)^{1/2}c=\text{const} \ . \tag{10.5}$$

For a Chapman-Jouguet detonation, $v_s=\beta c$ from Eq. (10.4a), so that $k_v=0$,

$$\sigma=\frac{1}{2(\gamma^2-1)}\rightarrow\frac{9}{32}, \quad \nu_1=\frac{v_1}{v_s}=\frac{1}{\gamma+1}\rightarrow\frac{3}{8} \ , \tag{10.6}$$

where the numerical values are for $\gamma=\frac{5}{3}$. These results, which are exact and independent of k_ρ, can be used with the virial theorem (modified to take into account the changed jump conditions) to obtain the properties of self-similar detonation waves in stationary media without resorting to approximate treatments such as the linear velocity approximation. Equations (10.5) and (10.6), together with the results of Sec. III, then give

$$\xi=\frac{3\beta^2}{4\pi\varepsilon}=\frac{3(\gamma^2-1)}{2\pi} \ . \tag{10.7}$$

Sedov (1959) shows that the Chapman-Jouguet condition can be satisfied only for $k_\rho<2\gamma/(\gamma+1)$. Reference to Eqs. (B16)–(B18) shows that the edge derivatives, which are infinite for Chapman-Jouguet detonations, would change sign at $k_\rho=2\gamma/(\gamma+1)$. For $k_\rho<2\gamma/(\gamma+1)$, the edge derivatives equal $+\infty$, which is consistent with a rarefaction just behind the shock front. However, it is impossible for the edge derivatives to equal $-\infty$, since that occurs in the shock, not behind it. Hence we restrict our discussion to the case $k_\rho<2\gamma/(\gamma+1)$, where the Chapman-Jouguet condition is satisfied.

In order to proceed further, a discussion of the integrals is necessary. The general relation among the moments for nonevaporative blastwaves, Eq. (C9), together with Eq. (10.6), gives

$$K_{11}=\frac{\gamma+1}{2}[K_{20}(5-k_\rho)-(3-k_\rho)] \ , \tag{10.8}$$

which replaces (4.5); note that $k_{\rho_0}=k_\rho$ for nonevaporative blastwaves in a stationary medium. One can show that the virial theorem for detonation waves is given by Eq. (3.20), with ν_1 from Eq. (10.3b):

$$\frac{K_{11}(4-k_\rho)}{\gamma+1}=\frac{K_{02}}{(\gamma+1)^2}+\frac{3}{\alpha^2} \ . \tag{10.9}$$

The energy equation (3.19) applies to blastwaves with arbitrary jump conditions and implies

$$\sigma=\frac{1}{2(\gamma+1)^2}K_{02}+\frac{1}{(\gamma-1)\alpha^2} \ . \tag{10.10}$$

Now, equating the values for σ in Eqs. (10.6) and (10.10) gives

$$\alpha^2=\frac{2(\gamma+1)^2}{(\gamma+1)-K_{02}(\gamma-1)}\ , \tag{10.11}$$

which with the aid of the virial theorem (10.9) allows one to write

$$\tfrac{1}{2}K_{02}(3\gamma-5)+K_{11}(4-k_\rho)(\gamma+1)=\tfrac{3}{2}(\gamma+1)\ . \tag{10.12}$$

Equations (10.8) and (10.11) give two exact relations among the three moments K_{02}, K_{20}, and K_{11}. If we adopt for the third relation the harmonic mean relation among the moments [Eq. (C13a)], we can derive a quadratic equation for K_{11}:

$$2K_{11}^2[20\gamma^2+21\gamma+9-(\gamma+1)(9\gamma+5)k_\rho+(\gamma+1)^2k_\rho^2]-2(\gamma+1)K_{11}[18(2\gamma+1)-(17\gamma+13)k_\rho+2(\gamma+1)k_\rho^2]+6(3-k_\rho)(\gamma+1)^2=0\ . \tag{10.13}$$

For the case of greatest interest ($\gamma=\frac{5}{3}$), we can solve for K_{11} and K_{20} without using the approximate equation (10.13). We find the exact results,

$$K_{11}=\frac{3}{2(4-k_\rho)}\rightarrow\frac{3}{8}\ , \tag{10.14a}$$

$$K_{30}=\frac{105-56k_\rho+8k_\rho^2}{8(4-k_\rho)(5-k_\rho)}\rightarrow\frac{105}{160}\quad(\gamma=\tfrac{5}{3}), \tag{10.14b}$$

where the numerical values are for $k_\rho=0$. The harmonic mean relation (C13a) then yields the approximate values

$$K_{02}=\frac{3K_{20}}{4(4-k_\rho)K_{20}-3}\rightarrow\frac{21}{80}\ , \tag{10.15a}$$

$$\alpha^2=\frac{2(\gamma+1)}{1-\left[\frac{\gamma-1}{\gamma+1}\right]\frac{3K_{20}}{4(4-k_\rho)K_{20}-3}}\rightarrow\frac{5120}{897}\simeq 5.71\ , \tag{10.15b}$$

with K_{20} given by Eq. (10.14b). The accuracy of the harmonic mean relation in this case is not known, but a rigorous lower bound on K_{02} is provided by Eq. (C12), $K_{02}\geq\frac{3}{14}$. The values of l_ρ and l_v, the power laws for the density and velocity, are obtained from Eq. (4.22a) and (4.22b) as 0.818 and 5.36, respectively, for $k_\rho=0$. The large value of l_v is characteristic of detonations: the exact solution has $v=0$ inside a critical radius and $dv/dr\rightarrow\infty$ as $r\rightarrow R_s$ (Landau and Lifschitz, 1959).

It is interesting and somewhat surprising that the relative kinetic energy σ is determined without need for numerical integration and independent of whether or not there is a density gradient (k_ρ); the relative thermal energy α^2 is weakly dependent on k_ρ and γ and always close to the value $2(\gamma+1)$. Our relatively simple analytical treatment can be used for arbitrary other values of (k_ρ,γ) for which numerical solutions do not exist.

C. Hubble flow

The solutions for cosmological detonations are clearly not self-similar in general, but their features can be obtained immediately from Eq. (10.4b). Bertschinger (1985b) and Kazhdan (1986) have analyzed (with slightly different results) the $\Omega=1$ case. Here we follow the analytic treatment in McKee and Ostriker (1987). Essentially, the Chapman-Jouguet condition requires that

$$\frac{dR_s}{d\tau}=h\frac{R_s}{\tau}+\beta c\ , \tag{10.16a}$$

where the cosmological parameter h is defined by Eqs. (9.1)–(9.5) and τ is the cosmic time. For detonations that begin at $\tau=\tau_b$, the general solution for (10.16a) is

$$R_s=\beta c\tau^h\int_{\tau_b}^{\tau}\tau'^{-h}d\tau'\ , \tag{10.16b}$$

which is trivially integrated for the cosmologically interesting cases of $h=(1,\frac{2}{3})$ to give

$$R_s(\tau)=\beta c\tau\begin{cases}\ln(\tau/\tau_b), \\ 3[1-(\tau_b/\tau)^{1/3}]\ ,\end{cases}\quad\Omega=\begin{cases}0,\\1.\end{cases} \tag{10.17}$$

It is interesting to express the result as the present radius of a detonation that propagated from $z=z_b$ to $z=z_1$ and that thereafter satisfied the equation $\ddot{R}_s=-h(1-h)R_s\tau^{-2}$. We find

$$R_s(z=0)=\beta\frac{c}{H_0}\begin{cases}\ln\left[\frac{1+z_b}{1+z_1}\right]+\frac{z_1}{1+z_1}\\ \frac{2}{3}\left[4-\left[\frac{1+z_b}{1+z_1}\right]^{1/2}-\frac{1}{(1+z_1)}\right]\end{cases}\ \text{for}\ \Omega=\begin{cases}0,\\1.\end{cases} \tag{10.18}$$

The numerical coefficient in Eq. (10.18) is 57 Mpc for $\gamma=\frac{5}{3}$, $\varepsilon=10^{-4}$, and $H_0=100$ km/s/Mpc. The corresponding masses are of the order of $10^{17.5}\Omega M_\odot$. This treatment supersedes the more approximate calculation in Ostriker and Cowie (1981) but confirms the conclusion presented there that detonations propagating over cosmological epochs can process volumes corresponding to spheres of radius ~ 50 Mpc for explosions of efficiency $\varepsilon\approx 10^{-4}$.

D. Self-similar detonations in a Hubble flow ($\Omega=\Omega_g=1$)

As in the case of cosmological blastwaves, solutions will become self-similar only in the limit that the age of the wave t approaches the age of the universe τ. We will assume that is the case. The results of Sec. IX show that the requirement of self-similarity can only be satisfied for the just closed (with $\Omega_g=\Omega=1$) universe [see Eq. (10.17)]. In this case the total energy of the flow is the same as the detonation energy [see Eq. (9.10)]. The energy in the blastwave $E_b=M\varepsilon c^2\propto\rho_0R_s^3\propto\tau^{-3h+3\eta}$. However, by definition $E_b\propto\tau^{\eta_E}$. Equating these expressions and using the relation among (η,η_E,h) given by Eq. (9.13) gives us the simple result

$$\eta=\eta_E=1,\quad k_E=-1\quad(\Omega_g=\Omega=1)\ . \tag{10.19}$$

Since v_s is constant, it follows from Eqs. (10.4a) and (10.4b) that v_1 and v_H must be constant as well, and, since $v_s/v_H=\eta/h$, we find $v_s=(\frac{3}{2})v_H$, which leads to the simple exact results

$$v_s=3\beta c,\quad v_H=2\beta c,\quad \Delta v=v_s/3=v_H/2\ , \tag{10.20a}$$

$$v_1=\left[\frac{2\gamma+3}{\gamma+1}\right]\beta c\rightarrow\tfrac{19}{8}\beta c\ ,$$
$$\nu_1\equiv v_1/v_s=\left[\frac{2\gamma+3}{3\gamma+3}\right]\rightarrow\tfrac{19}{24}\beta c\ , \tag{10.20b}$$

where the numerical values are for $\gamma=\frac{5}{3}$. The $R_s(t)$ relation is given by Eq. (10.17): $R_s(\tau)=3\beta c\tau$. The dimensionless constants σ and ξ are also obtained exactly:

$$\sigma\equiv\frac{E_b}{Mv_s^2}=\frac{\varepsilon c^2}{v_s^2}=\frac{1}{18(\gamma^2-1)}\rightarrow\frac{1}{32}\ , \tag{10.21a}$$

giving

$$\xi=\frac{27(\gamma^2-1)}{2\pi}\rightarrow\frac{24}{\pi}\ . \tag{10.21b}$$

To obtain information about the other integral properties we must proceed as we did in Sec. X.A.

First, we note the general kinematic result for nonevaporative self-similar flows given by Eq. (C9). For cosmological detonation waves we use (10.20b) for ν_1 and set $k_{\rho_0}=0$, obtaining

$$K_{11}=\frac{3(\gamma+1)(3K_{20}-1)}{2(2\gamma+3)}\ . \tag{10.22}$$

Next, we use the virial theorem (3.20), which in this context becomes

$$\frac{3}{\alpha^2}=\frac{2(2\gamma+3)}{3(\gamma+1)}K_{11}-\frac{2}{3}-\frac{(2\gamma+3)^2}{9(\gamma+1)^2}K_{02}+\tfrac{2}{9}w\ , \tag{10.23}$$

with the aid of Eqs. (9.9) and (10.20b). The energy equation (3.19) yields

$$\sigma=\frac{(2\gamma+3)^2}{18(\gamma+1)^2}K_{02}+\frac{1}{(\gamma-1)\alpha^2}-\tfrac{2}{9}w\ , \tag{10.24}$$

and since we know σ from Eq. (10.21) we obtain

$$\alpha^2=\frac{18(\gamma+1)}{1-\frac{(2\gamma+3)^2}{(\gamma+1)}(\gamma-1)K_{02}+4(\gamma^2-1)w}\ . \tag{10.25}$$

Inserting this result into the virial theorem (10.23) provides a second relation among the moments,

$$12(\gamma+1)(2\gamma+3)K_{11}-(5-3\gamma)(2\gamma+3)^2K_{02}-4(\gamma+1)^2(3\gamma-4)w-3(\gamma+1)(4\gamma+5)=0\ . \tag{10.26}$$

Equations (10.22) and (10.26) give two exact relations among the three moments K_{20}, K_{11}, and K_{02}. To proceed further we introduce the harmonic mean approximation, as before, and Eq. (C13a) becomes the third relation among the moments. The gravitational factor $w=w_{11}$ is given by Eq. (A28); using Eq. (4.22a) to eliminate l_ρ in favor of K_{20}, we find

$$w=\frac{2K_{20}}{5K_{20}-1}\ . \tag{10.27}$$

Substituting Eqs. (10.22), (10.27), and (C13a) into Eq. (10.26) yields a cubic equation for K_{20}, which completes the solution. That equation, which is general but not particularly illuminating, will not be presented, but we shall pass on to the most important special case.

As in Sec. X.A above, the problem simplifies considerably for $\gamma=\frac{5}{3}$, since the coefficient of K_{02} in Eq. (10.26) vanishes in this case, so Eqs. (10.22), (10.26), and (10.27) determine K_{11}, K_{20}, and w. The remaining quantities of interest can then be found with the aid of the harmonic mean relation [Eq. (C13a)] and (10.27). The results are as follows.

For $\Omega_g=\Omega=1$ ($\gamma=\frac{5}{3}$):

$$K_{11}=0.5434,\quad K_{20}=0.6201,\quad w=0.5904\ ,$$
$$K_{02}\simeq0.4836,\quad \alpha^2\simeq137.3\ , \tag{10.28}$$
$$l_\rho\simeq0.2648,\quad l_v\simeq1.743\ .$$

In comparison with these results, those of Bertschinger (1985b) are $w=0.552$, $K_{02}=0.452$, and $\alpha^2=124.6$. About half the energy is in thermal form [cf. Eq. (3.18)]. The cosmological expansion has reduced l_v considerably below the value for a detonation in a stationary medium, which is quoted below Eq. (10.15b).

As in all detonations, the temperature is independent of time. From Eqs. (3.23) and (10.5) we have

$$\frac{\bar{P}}{\rho_0}=\frac{4\pi}{3}\frac{\varepsilon c^2\xi}{\alpha^2}\ . \tag{10.29}$$

Defining the mean temperature as $\bar{P}/\rho_0=\bar{P}/\rho\equiv k\bar{T}/\mu$, we find with (10.21b)

$$\frac{k\overline{T}}{\mu}=18\varepsilon c^2(\gamma^2-1)/\alpha^2=(0.36,0.23)\varepsilon c^2 \qquad (10.30)$$

for $(\Omega_g=0,1)$ and $\gamma=\frac{5}{3}$. Adopting parameters appropriate for the intergalactic medium ($\varepsilon=10^{-4.5}$, $\mu=10^{-24.0}$ g), we compute temperatures of approximately 10^8 K.

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APPENDIX A: ENERGY CONSERVATION AND THE GRAVITATIONAL ENERGY

1. Energy conservation for a multicomponent fluid ($f_0-f_i=\lambda_{\rm cl}^{-1}=\dot{m}=0$)

We wish to derive the energy conservation law for one component of a spherically symmetric, multicomponent fluid. This component is assumed to interact with the other components only through gravitation. For the problems analyzed in the text, the component of interest is the intercloud gas, and the remaining components could be clouds, stars, dark matter, a central mass, etc. The assumption that its interaction is purely gravitational rules out cloud crushing (so that the cloud filling factor f is constant), cloud drag ($\lambda_{\rm cl}^{-1}=0$), and cloud evaporation ($\dot{m}=0$). Since f is constant, we shall ignore it here, but it can be recovered by replacing the volume element dV by $(1-f)dV$.

Let E be the total energy of the intercloud gas. It is comprised of three terms: the kinetic energy E_K, the thermal energy $E_{\rm th}$, and the gravitational energy W:

$$E\equiv E_K+E_{\rm th}+W\ . \qquad (A1)$$

Operationally, we seek a definition for W such that dE/dt has a clear significance in terms of energy gains and losses from the intercloud gas.

The differential form of energy conservation is

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho v^2+u)+\frac{1}{r^2}\frac{\partial}{\partial r}r^2 v(\tfrac{1}{2}\rho v^2+u+P)=-n^2\Lambda+\dot{E}_{\rm in}\delta(V)+\rho v g\ , \qquad (A2)$$

where u is the internal energy density; $-n^2\Lambda$ is the radiative cooling rate per unit volume, which in general is a function of both n and T and will be positive if the radiation heats the gas; $\dot{E}_{\rm in}\delta(V)$ is the rate of central energy injection; and g is the gravitational acceleration due to all the matter, not just the gas. The sum of the kinetic and thermal energies inside a sphere of radius $R_s(t)$ and volume $V_s(t)$ is

$$E_K+E_{\rm th}=\int_0^{V_s}(\tfrac{1}{2}\rho v^2+u)dV\ . \qquad (A3)$$

With the aid of Eq. (A2), we can show that the rate of change of $(E_K+E_{\rm th})$ is

$$\frac{d}{dt}(E_K+E_{\rm th})=4\pi R_s^2[(\tfrac{1}{2}\rho_1 v_1^2+u_1)v_s-(\tfrac{1}{2}\rho_1 v_1^2+u_1+P_1)v_1]-\dot{E}_{\rm rad}+\dot{E}_{\rm in}+\int_0^{V_s}\rho v g\, dV\ , \qquad (A4)$$

with $E_{\rm rad}$ is the radiation energy emitted by the gas and

$$\dot{E}_{\rm rad}=\int_0^{V_s}n^2\Lambda\, dV\ . \qquad (A5)$$

The shock jump conditions (E1)–(E3) imply that the term in brackets on the right-hand side of Eq. (A4) is

$$(\tfrac{1}{2}\rho_0 v_H^2+u_0+\rho_0\varepsilon c^2)(v_s-v_H)-v_H P_0\ .$$

If $E_{\rm det}$ is the energy released by a detonation, then the rate of change of $E_{\rm det}$ is

$$\dot{E}_{\rm det}=4\pi R_s^2\rho_0\varepsilon c^2(v_s-v_H) \qquad (A6)$$

and Eq. (A4) becomes

$$\frac{d}{dt}(E_K+E_{\rm th})=4\pi R_s^2[(\tfrac{1}{2}\rho_0 v_H^2+u_0)(v_s-v_H)-P_0 v_H]+\dot{E}_{\rm det}-\dot{E}_{\rm rad}+\int vg\, dM\ . \qquad (A7)$$

Now consider the gravitational term. Let $M_t(r)$ be the total mass inside r and $M'(r)\equiv M_t(r)-M(r)$ be the mass of the nongaseous components. The self-energy of the gas is

$$W_{\rm self}\equiv-\int\frac{GM\, dM}{r}\ , \qquad (A8a)$$

and the interaction energy of the gas in the field of the other components is

$$W_{\rm int}=-\int\frac{GM'dM}{r}\ . \qquad (A8b)$$

Observe that the sum of these two energies is just the gravitational virial term introduced in Eq. (2.12):

$$W_{\rm self}+W_{\rm int}=-\int\frac{GM_t dM}{r}=\int\rho g r\, dV\equiv W\ . \qquad (A9)$$

Let ρ' and v' be the mean density and velocity of the nongaseous components, and let g' be the acceleration due to M'. Using the equation of continuity and the jump condition (E1), one finds

$$\frac{d}{dt}W_{\rm int} = -4\pi R_s^2\left[\frac{GM'}{R_s}\right]\rho_0(v_s - v_H) - G\int 4\pi r^2\rho'(v - v')\frac{dM}{r} - \int v g' dM \ . \quad (A10)$$

The rate of change of $W_{\rm self}$ can then be obtained by replacing (ρ', v', g') by (ρ, v, g). Summing the rates of change of $W_{\rm int}$ and $W_{\rm self}$ then yields

$$\frac{dW}{dt} = -4\pi R_s^2\left[\frac{GM_t}{R_s}\right]\rho_0(v_s - v_H) - \frac{d\Delta W}{dt} - \int v g\, dM \ , \quad (A11)$$

where

$$\frac{d\Delta W}{dt} = G\int 4\pi r^2\rho'(v - v')\frac{dM}{r} \quad (A12)$$

is the rate at which gravitational energy is transferred from the gas to the other components. To gain further insight into the gravitational energy transfer ΔW, we rewrite Eq. (A12) as

$$\frac{d}{dt}\Delta W = G\int \frac{\partial M'}{\partial r}\frac{\partial M}{\partial r}\frac{(v - v')}{r}dr \ , \quad (A13)$$

which demonstrates that

$$\Delta W = -\Delta W' \ ; \quad (A14)$$

the gas's loss is the other components' gain. For the particular case in which the gas is concentrated in a thin spherical shell, moving at a velocity $\Delta v = (v - v')$ relative to the other components, the rate of gravitational energy transfer is just

$$\frac{d}{dt}\Delta W = 4\pi R_s^2\rho'\Delta v\left[\frac{GM}{R_s}\right] . \quad (A15)$$

Thus, as the shell expands past a mass $dM' = 4\pi R_s^2\rho'\Delta v\, dt$, that mass is no longer in the field of the shell and has its gravitational energy increased by $GM\, dM'/R_s$. This increase in energy results in an acceleration of the mass dM'.

With the gravitational contribution to the energy determined, we can obtain a clear statement of energy conservation by inserting Eqs. (A1) and (A11) into (A7):

$$\frac{dE}{dt} = 4\pi R_s^2\left[\left[\tfrac{1}{2}\rho_0 v_H^2 + u_0 - \rho_0\frac{GM_t}{R_s}\right](v_s - v_H) - v_H P_0\right] + \dot{E}_{\rm in} + \dot{E}_{\rm det} - \dot{E}_{\rm rad} - \frac{d\Delta W}{dt} \ . \quad (A16)$$

Thus the energy of the intercloud gas inside R_s changes due to the energy content of the swept-up matter (the term proportional to $v_s - v_H$), the work done by the gas inside R_s on that outside ($\propto v_H P_0$), the energy gains due to injection at the origin or to a detonation, and the losses due to radiation and gravitational transfer to the other components. Note that an energy conservation relation identical to that in Eq. (A16) can be written for the other components, but with all quantities primed. In particular, $W'_{\rm int}$ is the energy of the other components in the field of the gas. When the two relations are summed, one obtains the energy conservation for the entire system, with $E_t = E + E' = E_{K,t} + E_{{\rm th},t} + W_t$ and $W_t = \int GM_t dM_t/r$ the total gravitational energy of the system.

Equation (A16) is the basic statement of energy conservation for a spherically symmetric, multicomponent fluid, and it demonstrates that in this case the gravitational virial term W can be interpreted as the gravitational energy of the gas.

2. Energy conservation for a blastwave

If we now apply the energy conservation relation (A16) to a blastwave, it is convenient to distinguish the energy due to the blastwave E_b, from that which would have been present in the absence of the blastwave E_a. Assume that $\dot{E}_{\rm rad}$ vanishes in the absence of the blastwave. Then, since the shock jump conditions apply even in the absence of a shock, Eq. (A16) applies to any spherically symmetric volume of radius $R_s(t)$, whether or not a shock is present, and we have

$$\frac{dE_a}{dt} = 4\pi R_s^2\left[\left[\tfrac{1}{2}\rho_0 v_H^2 + u_0 - \rho_0\frac{GM_t}{R_s}\right](v_s - v_H) - v_H P_0\right] \quad (A17)$$

for the rate of change of the energy of an equivalent volume of ambient medium. The energy E_a itself is given by Eqs. (A1), (A3), and (A9) evaluated in the absence of the blastwave; for example, for the cosmological case we have $\Omega v_H^2/2 = GM_t/R$ and

$$E_a = \tfrac{3}{10}Mv_H^2(1 - \Omega) + E_{{\rm th},a} \ . \quad (A18)$$

The blastwave energy E_b is defined as

$$E_b \equiv E - E_a \ . \quad (A19)$$

When Eq. (A17) is inserted into (A16), we obtain

$$\frac{dE_b}{dt} = \dot{E}_{\rm in} + \dot{E}_{\rm det} - \dot{E}_{\rm rad} - \frac{d\Delta W}{dt} \ , \quad (A20)$$

which may be immediately integrated to give

$$E_b = E_0 + E_{\rm in} + E_{\rm det} - E_{\rm rad} - \Delta W \ . \quad (A21)$$

The blastwave energy is the sum of the initial energy of the explosion E_0 and the energy added by central injection $E_{\rm in}$ and detonation $E_{\rm det}$ minus the losses due to radiation $E_{\rm rad}$ and gravitational transfer to other components ΔW.

3. Evaluation of the gravitational terms

Let $M_i(r)$ be the mass of component i inside r: $M_1(r)$—denoted above and in the text as $M(r)$—is the mass of intercloud gas, $M_2(r)$ is the mass interior to r of nongaseous distributed material which interacts only gravitationally with the intercloud gas (drag-free clouds, stars, neutrinos, etc.), and M_3 is the mass of any central object. In the virial theorem, the effects of gravity enter through the term [cf. Eq. (2.12)]

$$W=\int_0^{R_s}\rho g r(1-f)dV \tag{A22}$$

$$=-\int_0^{R_s}\frac{GM_t(r)dM_1(r)}{r}\ , \tag{A23}$$

where the total mass is

$$M_t(r)=\sum_{i=1}^{3}M_i(r)\ . \tag{A24}$$

We now introduce the dimensionless coefficients w_{ij}:

$$w_{ij}\frac{M_i(R_s)M_j(R_s)}{R_s}\equiv\int_0^{R_s}\frac{M_i(r)dM_j(r)}{r}\ , \tag{A25}$$

so that

$$W=-\frac{GM_1(R_s)}{R_s}\sum_{i=1}^{3}w_{i1}M_i(R_s)\ . \tag{A26}$$

To evaluate these coefficients we assume that the density of the distributed nonintercloud material is proportional to the preexplosion intercloud density, $\rho_2\propto r^{-k_{\rho_0}}$. This suffices to determine all the coefficients except w_{11} in terms of the moments defined in Eq. (2.9):

$$w_{21}=K_{2-k_{\rho_0},0}\ ,$$
$$w_{12}=\left[\frac{3-k_{\rho_0}}{2-k_{\rho_0}}\right](1-K_{2-k_{\rho_0},0})\ , \tag{A27}$$
$$w_{31}=K_{-1,0},\quad w_{13}=w_{23}=0\ .$$

We have considered two separate approximations to evaluate w_{11} and the other coefficients. The first is a power law for the intercloud density, $\rho_1\propto r^{l_\rho}$. Equation (4.20) for K_{mn} in this case then implies

$$w_{11}=\frac{3+l_\rho}{5+2l_\rho},\quad w_{12}=\frac{3-k_{\rho_0}}{5+l_\rho-k_{\rho_0}}\ ,$$
$$w_{21}=\frac{3+l_\rho}{5+l_\rho-k_{\rho_0}},\quad w_{22}=\frac{3-k_{\rho_0}}{5-2k_{\rho_0}}\ , \tag{A28}$$
$$w_{31}=\frac{3+l_\rho}{2+l_\rho},\quad w_{32}=\frac{3-k_{\rho_0}}{2-k_{\rho_0}}\ .$$

Since these coefficients are relatively weak functions of l_ρ, we conclude that the gravitational term W is insensitive to the details of the internal density distribution. In the limit $l_\rho\to\infty$ we recover the thin-shell approximation discussed in Appendix D: $w_{11}=\frac{1}{2}$ and $w_{21}=w_{31}=1$ so that

$$W=-\frac{1}{2}\frac{GM_1^2(R_s)}{R_s}\left[1+\frac{2M'}{M_1(R_s)}\right]\quad(l_\rho\to\infty)\ , \tag{A29}$$

where $M'=M_2(R_s)+M_3$.

The second approximation is a variant of the shell approximation in which the density rather than the velocity is assumed independent of position behind the shock; this is equivalent to the shell approximation to first order in the shell thickness, which is adequate for our purpose. If we assume the intercloud density is constant in the region $(1-\delta)R_s<r<R_s$ and zero inside $(1-\delta)R_s$, we find for $k_{\rho_0}=0$,

$$w_{11}=\frac{1-5\delta/3+\delta^2-\delta^3/5}{2(1-\delta+\delta^2/3)^2}\to\tfrac{1}{2}(1+\delta/3)\ ,$$
$$w_{21}=\frac{1-2\delta+2\delta^2-\delta^3+\delta^4/5}{1-\delta+\delta^2/3}\to1-\delta\ ,$$
$$w_{12}=\left[\frac{3\delta}{2}\right]\frac{(1-5\delta/3+\delta^2-\delta^3/5)}{1-\delta+\delta^2/3}\to\tfrac{3}{2}\delta\ , \tag{A30}$$
$$w_{31}=\frac{1-\delta/2}{1-\delta+\delta^2/3}\to1+\delta/2\ ,$$

where the results for a thin shell (small δ) have also been indicated.

In the text we have considered gravitational effects only in the context of cosmology (Sec. IX). The mass is parametrized in terms of the density parameter Ω_i and the Hubble velocity v_H,

$$\frac{GM_i(R_s)}{R_s}=\tfrac{1}{2}\Omega_i v_H^2\ . \tag{A31}$$

Then if we write the gravitational integral in terms of the dimensionless quantity w,

$$W\equiv-\tfrac{1}{2}w\Omega M_1(R_s)v_H^2\ , \tag{A32}$$

we find

$$w=w_{11}\frac{\Omega_1}{\Omega}+w_{21}\frac{\Omega_2}{\Omega}\ , \tag{A33}$$

where we have set $M_3=0$.

In certain cases [see Appendix D, Eq. (D8)] a different integral is required. If w' is defined by

$$\int g\,dM\equiv-\tfrac{1}{2}w'\Omega M_1(R_s)v_H^2/R_s\ , \tag{A34}$$

then

$$w'=w'_{11}\frac{\Omega_1}{\Omega}+w'_{21}\frac{\Omega_2}{\Omega} \tag{A35}$$

with

$$w'_{11}=\tfrac{1}{2}(1+\tfrac{2}{3}\delta),\quad w'_{21}=(1-\tfrac{1}{2}\delta)\ . \tag{A36}$$

APPENDIX B: INTERNAL STRUCTURE OF SELF-SIMILAR, ADIABATIC BLASTWAVES ($\lambda_{\rm cl}^{-1}=f_0-f_i=\gamma-\gamma_i=\dot m=0$)

1. General results

The assumption of self-similarity reduces the spherically symmetric hydrodynamic equations to a set of three ordinary differential equations. Any hydrodynamic variable x can be expressed

$$x=x_1(t)\tilde{x}(\lambda)\ , \tag{B1}$$

where

$$\lambda\equiv r/R_s\ , \tag{B2}$$

$x_1(t)$ is the post-shock value, and $\tilde{x}(1)=1$. The logarithmic derivative of x is denoted

$$x^*\equiv\frac{\partial\ln x}{\partial\ln r}=\frac{d\ln\tilde{x}}{d\ln\lambda}\ . \tag{B3}$$

Since $x_1(t)\propto R_s^{-k_x}\propto t^{-\eta k_x}$, the partial derivatives of x are

$$\frac{\partial x}{\partial r}=\frac{\eta x x^*}{\lambda v_s t}\ , \tag{B4}$$

$$\frac{\partial x}{\partial t}=-\frac{\eta x}{t}(k_x+x^*)\ . \tag{B5}$$

Define $\nu\equiv v/v_\lambda$ to be the fluid velocity normalized to the velocity of the self-similar coordinates $v_\lambda\equiv\lambda v_s$. Then the mass [Eq. (2.1)], momentum [Eq. (2.2)], and entropy $[d(P/\rho^\gamma)/dt=0]$ equations yield

$$\nu v^*=\rho^*(1-\nu)+k_\rho-2\nu\ , \tag{B6}$$

$$(1-\nu)v^*=\frac{\theta P^*}{\nu}+\tfrac{1}{2}(k_\rho-k_P)+\frac{\Omega\nu_H^2\tilde{g}}{2\nu\lambda^3}\ , \tag{B7}$$

$$(1-\nu)P^*=\gamma(1-\nu)\rho^*+\gamma k_\rho-k_P\ , \tag{B8}$$

where

$$\theta\equiv\frac{P}{\rho v_\lambda^2}=\frac{kT}{\mu v_\lambda^2}=\frac{C^2}{v_\lambda^2} \tag{B9}$$

may be viewed as a normalized temperature or as an inverse Mach number squared. We have written the gravitational acceleration as

$$g=g_1(t)\tilde{g}(\lambda)=-\frac{GM_{\rm tot}}{R_s^2}\tilde{g}(\lambda)=-\frac{\Omega v_H^2}{2R_s}\tilde{g}(\lambda)\ , \tag{B10}$$

and set $\nu_H\equiv v_H/v_s$.

Solution of Eqs. (B6)–(B8) gives the logarithmic derivatives at an arbitrary point in the flow:

$$v^*=\frac{2\theta(k_P-2\gamma\nu)+(1-\nu)[\nu(k_P-k_\rho)-\Omega\nu_H^2\tilde{g}\lambda^{-3}]}{2\nu[\gamma\theta-(1-\nu)^2]}\ , \tag{B11}$$

$$\rho^*=\frac{2\theta(k_P-\gamma k_\rho)/(1-\nu)-(3\nu-2)k_\rho+\nu k_P-4\nu(1-\nu)-\Omega\nu_H^2\tilde{g}\lambda^{-3}}{2[\gamma\theta-(1-\nu)^2]}\ , \tag{B12}$$

$$P^*=\frac{-\gamma\nu k_\rho+[\gamma\nu+2(1-\nu)]k_P-4\gamma\nu(1-\nu)-\gamma\Omega\nu_H^2\tilde{g}\lambda^{-3}}{2[\gamma\theta-(1-\nu)^2]}\ . \tag{B13}$$

These equations apply to any self-similar, spherically symmetric flow in which the entropy per unit mass is conserved. No assumption has been made about the external velocity v_H, density ρ_0, or pressure P_0, other than that they be consistent with self-similarity. Bubbles (Sec. VII) and detonations (Sec. X) are both included.

In the absence of gravity ($\Omega=0$), these equations can be reduced to a single equation in terms of ν and θ by noting that $v^*=1+\nu^*$ and $\theta^*=P^*-\rho^*-2$. For the case in which the energy is conserved ($k_E=0$, so that $k_P=k_E+3=3$), Sedov (1959) has given an explicit analytic solution with $\eta\nu$ as the independent variable.

2. Edge derivatives x_1^* for strong shocks ($P_0=0$)

The values of the derivatives at the edge of the blastwave

$$x_1^*\equiv x^*|_{\lambda=1} \tag{B14}$$

are necessary for analytic approximations to the internal structure as well as to provide starting points for numerical integrations. For strong shocks, Eqs. (E12) and (E13) imply

$$\theta_1=(\chi_1-1)(1-\nu_1)^2\ . \tag{B15}$$

Then, setting $\lambda=1$ and $\tilde{g}(1)=1$, we find that Eqs. (B11)–B13) yield

$$v_1^*=\frac{-\nu_1k_\rho+[2\chi_1(1-\nu_1)+3\nu_1-2]k_P-4\gamma\nu_1(1-\nu_1)(\chi_1-1)-\Omega\nu_H^2}{2\nu_1(1-\nu_1)[\gamma(\chi_1-1)-1]}\ , \tag{B16}$$

$$\rho_1^* = \frac{-[2\gamma(\chi_1-1)(1-\nu_1)+3\nu_1-2]k_\rho + [2\chi_1(1-\nu_1)+3\nu_1-2]k_P - 4\nu_1(1-\nu_1) - \Omega\nu_H^2}{2(1-\nu_1)^2[\gamma(\chi_1-1)-1]} , \quad \text{(B17)}$$

$$P_1^* = \frac{-\gamma\nu_1 k_\rho + [\gamma\nu_1 + 2(1-\nu_1)]k_P - 4\gamma\nu_1(1-\nu_1) - \gamma\Omega\nu_H^2}{2(1-\nu_1)^2[\gamma(\chi_1-1)-1]} . \quad \text{(B18)}$$

We now consider several special cases. For blastwaves in stationary media ($v_H=0$) with no energy injection at the shock (i.e., not detonations), the jump conditions (E22) yield $\nu_1 = 2/(\gamma+1)$ and $\chi_1 = (\gamma+1)/(\gamma-1)$. Since $k_P = k_E + 3$, Eqs. (B16)–B18) simplify to

$$v_1^* = \frac{(3k_E - k_\rho)(\gamma+1) + \gamma + 9}{2(\gamma+1)} , \quad \text{(B19)}$$

$$\rho_1^* = \frac{[3k_E - (\gamma+2)k_\rho](\gamma+1) + 5\gamma + 13}{\gamma^2 - 1} , \quad \text{(B20)}$$

$$P_1^* = \frac{[(2\gamma-1)k_E - \gamma k_\rho](\gamma+1) + 2\gamma^2 + 7\gamma - 3}{\gamma^2 - 1} \quad (v_H = 0,\ \varepsilon = 0) . \quad \text{(B21)}$$

If we further specialize to the case of no central energy injection and no radiative losses ($k_E = 0$), then Eq. (B19) shows that $v_1^* = 1$ for $k_\rho = (7-\gamma)/(\gamma+1)$. As shown by Sedov (1959), the LVA is satisfied exactly at this point (and not merely at the edge), and for steeper density gradients (larger k_ρ) a vacuum develops at the center. For $v_1^* \lesssim 0$ the central vacuum is large and the shell approximation (Appendix C.4) is better than the LVA.

For cosmological blastwaves (Sec. IX) we distinguish the low- and high-density cases: for $\Omega = 0$, we have $k_\rho = 15/(5+\eta_E)$ and $k_P = 3 - \eta_E/\eta = (15 - 2\eta_E)/(5+\eta_E)$, so that

$$v_1^* = -\frac{2\eta_E(7\gamma+3) + 25(\gamma^2-1)}{(\gamma+1)[2\eta_E + 5(\gamma+1)]} , \quad \text{(B22)}$$

which is always negative: the gas piles up just behind the shock. For $\Omega = 1$, we have $k_\rho = 10/(4+\eta_E)$ and $k_P = 2(6-\eta_E)/(4+\eta_E)$, so that

$$v_1^* = -\frac{9\eta_E^2(7\gamma+3) + 3\eta_E(25\gamma^2 + 13\gamma - 28) + 2(25\gamma^2 - \gamma - 34)}{(\gamma+1)(2+3\eta_E)(3\eta_E + 5\gamma + 7)} . \quad \text{(B23)}$$

This is also negative, provided $\eta_E \geq 0$ and $\gamma > 1.19$. In both cases the shell approximation is appropriate.

Finally, for Chapman-Jouguet detonations, the compression is $\chi_1 = (\gamma+1)/\gamma$, so that the denominators in Eqs. (B16)–(B18) all vanish; the slopes of ρ, v, and P diverge at the shock (Zel'dovich and Kompanyets, 1960), and this remains true for an expanding medium.

3. Central derivatives (no central energy injection: $L_{in} = 0$)

We now evaluate the derivatives at the origin for filled blastwaves, and at the inner edge of the mass distribution for hollow blastwaves. First consider *filled blastwaves,* which have $\tilde{\rho} > 0$ for $\lambda > 0$; in addition we assume a nonzero central pressure, $\tilde{P}(0) > 0$. This rules out bubbles, which have central energy injection, but not blastwaves with energy gains or losses at the periphery. Since the pressure gradient must be finite at the origin, it follows that

$$P^*(0) = \left[\frac{\lambda}{\tilde{P}(0)}\right]\left[\frac{d\tilde{P}}{d\lambda}\right]_{\lambda=0} = 0 . \quad \text{(B24)}$$

Eliminating ρ^* between Eqs. (B6) and (B8) then yields

$$\nu(0) = \frac{k_P}{\gamma[2 + v^*(0)]} . \quad \text{(B25)}$$

This quantity is finite because the possibility that $v^*(0) = -2$ is ruled out by the requirement that $v(0)$ be finite. Since $v \propto \lambda\nu$, we have $v^* = 1 + \nu^*$; a generalization of the argument leading to (B24) then implies either that $\nu(0) \propto k_P \neq 0$, $\nu^*(0) = 0$, and $v^*(0) = 1$, or that $\nu(0) \propto k_P = 0$. In either case, we conclude that

$$\nu(0) = \frac{k_P}{3\gamma} = \frac{3 + k_E}{3\gamma} . \quad \text{(B26)}$$

Note from Sec. X that the condition $k_P = 0$ is satisfied by a detonation in a constant-density medium: the gas near the origin of such a detonation is stationary. The normalized velocity implied by Eqs. (B26) and (E22) is

$$\tilde{v} = \lambda\nu/\nu_1 \xrightarrow[\lambda\to 0]{} \lambda(\gamma+1)(3+k_E)/6\gamma \quad \text{(B27)}$$

for $\nu_H = 0$. The density gradient at the origin is then readily found from Eq. (B8),

$$\rho^*(0) = \frac{3(k_P - \gamma k_\rho)}{3\gamma - k_P} = \frac{3(3 + k_E - \gamma k_\rho)}{3(\gamma-1) - k_E} . \quad \text{(B28)}$$

For $k_E = 0$, these results reduce to the results obtained by

Sedov (1959), $v^*(0)=1$, $\nu(0)=1/\gamma$, $\tilde{v}\to(\gamma+1)\lambda/2\gamma$, and $\rho^*(0)=(3-\gamma k_\rho)/(\gamma-1)$.

Next, consider *hollow blastwaves.* As before, we assume no central energy injection, so that there is a vacuum for $\lambda<\lambda_i$. For cosmological blastwaves, we assume that all the matter is gaseous ($\Omega=\Omega_g$), so that again there is no matter inside λ_i. Since λ_i is a constant, it follows that the velocity at λ_i is $\lambda_i v_s$, whence $\nu_i=v_i/\lambda v_s=1$. Reference to the entropy equation (B8) then indicates that ρ^* and/or P^* must diverge at λ_i in order for the equation to be satisfied there. We define

$$\rho^\dagger\equiv(1-\nu)\rho^*,\quad P^\dagger\equiv(1-\nu)P^* \ . \tag{B29}$$

Now, Eq. (B52) below shows that as $\lambda\to\lambda_i$ from above (so that the enclosed mass fraction $\tilde{M}_{\rm tot}\to 0$), the normalized temperature becomes

$$\theta=\frac{(\gamma-1)(1-\nu)\nu^2}{2(\gamma\nu-1)} \ . \tag{B30}$$

Then the self-similar hydrodynamic equations (B6)–(B8) become in this limit

$$\nu v^*=\rho^\dagger+k_\rho-2\nu \ , \tag{B31}$$

$$(1-\nu)v^*=\frac{(\gamma-1)\nu P^\dagger}{2(\gamma\nu-1)}+\tfrac{1}{2}(k_\rho-k_P) \ , \tag{B32}$$

$$P^\dagger=\gamma\rho^\dagger+\gamma k_\rho-k_P \ . \tag{B33}$$

Solving these equations at $\lambda=\lambda_i$, where $\nu=1$, yields

$$P_i^\dagger=3+k_E-k_\rho \ , \tag{B34}$$

$$\rho_i^\dagger=\frac{6+2k_E-(\gamma+1)k_\rho}{\gamma} \ , \tag{B35}$$

$$v_i^*=\frac{6+2k_E-k_\rho-2\gamma}{\gamma}\quad(\Omega=\Omega_g) \ . \tag{B36}$$

For $k_E=0$ these expressions agree with results obtained from the exact solution (Sedov, 1959).

In order to interpret $\rho^\dagger$ and $P^\dagger$, consider a hydrodynamic variable x with the asymptotic behavior

$$x\propto(\lambda-\lambda_i)^{l_{xi}} \tag{B37}$$

as $\lambda\to\lambda_i$, so that

$$x^*\to\frac{\lambda l_{xi}}{\lambda-\lambda_i} \ . \tag{B38}$$

In particular, we have

$$(1-\nu)^*=-\frac{\nu\nu^*}{1-\nu}=\frac{\lambda l_{(1-\nu)_i}}{\lambda-\lambda_i} \ . \tag{B39}$$

However,

$$\nu_i^*=v_i^*-1=-(k_\rho+3\gamma-6-2k_E)/\gamma \tag{B40}$$

is a finite, nonzero constant, which is consistent with Eq. (B39) only if $(1-\nu)\propto(\lambda-\lambda_i)$, so that $l_{(1-\nu)i}=1$. Combining Eqs. (B38) and (B39) then yields

$$l_{xi}=-x_i^\dagger/\nu_i^* \ , \tag{B41}$$

so that

$$l_{\rho i}=\frac{2(3+k_E)-(\gamma+1)k_\rho}{k_\rho+3\gamma-2(3+k_E)} \ , \tag{B42}$$

$$l_{Pi}=\frac{\gamma(3+k_E-k_\rho)}{k_\rho+3\gamma-2(3+k_E)} \ , \tag{B43}$$

from (B34), (B35), and (B40).

4. Integrals ($\dot{M}_{\rm in}=0$)

The conservation laws for mass, energy, and entropy yield exact integrals for self-similar flow (Sedov, 1959). The mass inside a radius r is

$$M(r)\equiv M\tilde{M}(\lambda)=(1-f)\int_0^r 4\pi r^2\rho\,dr \ . \tag{B44}$$

(Note the convention that M without an argument is the total gas mass inside the blastwave.) For a self-similar blastwave, $\tilde{M}(\lambda)$ is independent of time for fixed λ, and hence

$$\tilde{M}(\lambda)\frac{dM}{dt}=\left[4\pi r^2\rho(r)\lambda v_s+\int_0^r 4\pi r^2\frac{\partial\rho}{\partial t}dr\right](1-f) \ . \tag{B45}$$

Then assuming no mass injection inside the blast wave ($\dot{m}=\dot{M}_{\rm in}=0$), we have

$$\frac{dM}{dt}=4\pi R_s^2\rho_0(1-f)(v_s-v_H) \ . \tag{B46}$$

In combination with the continuity equation (2.1) and the jump condition (2.3), this gives

$$\tilde{\rho}(\lambda)=\frac{(1-\nu_1)}{(1-\nu)}\frac{\tilde{M}(\lambda)}{\lambda^3} \ , \tag{B47}$$

which is an exact relation for the normalized density $\tilde{\rho}(\lambda)$.

Next consider the energy integral. Let

$$\varepsilon^*(r)=\tfrac{1}{2}\rho v^2+\frac{1}{\gamma-1}P-\frac{GM(r)\rho}{r} \tag{B48}$$

be the energy density of the gas at r, where we assume all the matter is gaseous ($\Omega=\Omega_g$); ε^* is not to be confused with ε, the energy release in a detonation (Sec. X). The energy inside r is then

$$E(r)\equiv E\tilde{E}(\lambda)=(1-f)\int_0^r\varepsilon^*(r)4\pi r^2dr \ . \tag{B49}$$

Allowing for energy injection at the origin, we have for the equation of energy conservation

$$\frac{\partial\varepsilon^*}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}r^2v(\varepsilon^*+P)=L_{\rm in}\delta\left[\frac{4\pi}{3}r^3(1-f)\right] \ , \tag{B50}$$

where $\delta(x)$ is the Dirac δ function. Now note that

$$\dot{E}=\eta_E E/t, \quad L_{\rm in}=\eta_{\rm in}E_{\rm in}/t \ , \tag{B51a}$$

$$GM(r)=\tfrac{1}{2}\Omega R v_H^2 \tilde{M}(\lambda) \ , \tag{B51b}$$

$$E=\sigma M v_s^2=4\pi\sigma\rho_0(1-f)R_s^3 v_s^2/(3-k_{\rho_0}) \ . \tag{B51c}$$

The requirement of self-similarity is that $\dot{E}(\lambda)$ be constant for fixed λ. Following a procedure analogous to that which led to Eq. (B47), we use Eqs. (B48)–(B51) to find an exact relation for the normalized temperature:

$$\theta=\frac{\gamma-1}{\gamma\nu-1}\left[\tfrac{1}{2}(1-\nu)\left[\nu^2-\frac{\Omega\nu_H^2\tilde{M}}{\lambda^3}\right] + \frac{\sigma\eta_E}{(3-k_{\rho_0})\eta\chi_1\tilde{\rho}\lambda^5}\left[\frac{\eta_{\rm in}E_{\rm in}}{\eta_E E}-\tilde{E}\right]\right] . \tag{B52}$$

A similar result has been given by Sedov (1959); for $\Omega=0$, by Gaffet (1981); and for $\eta_E=0$, by Bertschinger (1983). In the absence of gravity and for constant energy ($\eta_E=0$), this result reduces to Eq. (B30).

Finally, consider the entropy. The quantity

$$s\equiv P/\rho^\gamma \tag{B53}$$

is a function of the entropy per unit mass and is constant in comoving coordinates since the blastwave is adiabatic; hence we have $s=s(\tilde{M})$. Since P, ρ, and M are all power laws in R_s, it follows that

$$s=s_1\tilde{M}^{j_s} \ , \tag{B54}$$

where

$$j_s=\frac{\gamma k_\rho-3-k_E}{3-k_\rho} \ . \tag{B55}$$

Because $s/s_1=\tilde{P}/\tilde{\rho}^\gamma$, Eq. (B54) can be rewritten as

$$\tilde{s}\equiv\tilde{P}/\tilde{\rho}^\gamma=\tilde{M}^{j_s} \ , \tag{B56}$$

which is exact for self-similar, adiabatic blastwaves.

APPENDIX C: KINEMATIC RELATIONS AMONG THE MOMENTS

1. General relations between $K_{n,0}$ and $K_{n-1,1}$ for self-similar blastwaves

In terms of the moment $K_{m,n}$ defined in Eq. (2.9),

$$K_{m,n}\equiv\int_0^{R_s}\left[\frac{r}{R_s}\right]^m\left[\frac{v}{v_1}\right]^n\left[\frac{dM}{M}\right] , \tag{C1}$$

we have

$$K_{n,0}MR_s^n=\int_0^{R_s}4\pi r^{n+2}\rho(1-f)dr \ . \tag{C2}$$

Taking the time derivative of both sides gives

$$K_{n,0}[\dot{M}_{\rm ev}R_s^n+4\pi R_s^{2+n}\rho_0(1-f_0)(v_s-v_H)+nMR_s^{n-1}v_s]=\int_0^{R_s}4\pi r^{n+2}\frac{\partial}{\partial t}[\rho(1-f)]dr+4\pi R_s^{2+n}\rho_1(1-f)v_s \ , \tag{C3}$$

since $K_{n,0}$ is constant for self-similar blastwaves. The right-hand side can be simplified by using the equation of continuity (2.1), the jump condition (2.3), and the moment

$$K_{n,0}^{\rm ev}\equiv\frac{1}{\dot{M}_{\rm ev}}\int\left[\frac{r}{R_s}\right]^n\omega_{\rm cl}\dot{m}\,dV \ , \tag{C4}$$

which depends on the internal structure of the blastwave. Then the right-hand side of Eq. (C3) becomes

$$K_{n,0}^{\rm ev}R_s^n\dot{M}_{\rm ev}+4\pi R_s^{n+2}\rho_0(1-f_0)(v_s-v_H)+nK_{n-1,1}MR_s^{n-1}v_1 \ . \tag{C5}$$

The mass in the blastwave M is comprised of the swept-up ambient gas M_a and the evaporated mass $M_{\rm ev}$. More generally, $M_{\rm ev}$ can be due to any mass exchange with the clouds and may be positive or negative. One can readily show that [Eq. (3.27)]

$$M_a=\frac{4\pi}{3-k_{\rho_0}}\rho_0(1-f_0)R_s^3 \ , \tag{C6}$$

and for evaporation-dominated blastwaves

$$M_{\rm ev}=\frac{\dot{M}_{\rm ev}R_s}{(3-k_\rho)v_s} \ . \tag{C7}$$

In general we write $v_1\equiv\nu_1 v_s$ and $v_H\equiv\nu_H v_s$ (Sec. III). Inserting Eqs. (C4)–(C6) into Eq. (C2) then gives the general kinematic result

$$K_{n,0}=\frac{(3-k_{\rho_0})(1-\nu_H)\left[\frac{M_a}{M}\right]+(3-k_\rho)K_{n,0}^{\rm ev}\left[\frac{M_{\rm ev}}{M}\right]+nK_{n-1,1}\nu_1}{(3-k_{\rho_0})(1-\nu_H)\left[\frac{M_a}{M}\right]+(3-k_\rho)\left[\frac{M_{\rm ev}}{M}\right]+n} \ . \tag{C8}$$

Our derivation of the virial theorem allows for either a nonstationary ambient medium or evaporation, but not both, since we assumed the clouds to be stationary. Hence we can simplify this result for the relevant cases of (i) nonevaporative blastwaves and (ii) evaporative blastwaves in a stationary ambient medium.

a. Nonevaporative blastwaves ($M_{\rm ev}=0$, $M_a=M$)

We find

$$K_{n,0}=\frac{(3-k_{\rho_0})(1-\nu_H)+nK_{n-1,1}\nu_1}{(3-k_{\rho_0})(1-\nu_H)+n} . \quad \text{(C9)}$$

For a strong shock in a stationary ambient medium ($\nu_H=0$), this reduces to Eq. (4.5) in the absence of energy injection at the shock [Eq. (E27), $\nu_1=2/(\gamma+1)$], and to Eq. (10.9) for a Chapman-Jouguet detonation [Eq. (E26), $\nu_1=1/(\gamma+1)$].

b. Evaporative blastwaves in a stationary medium ($\nu_H=0$)

We require $k_\rho=k_{\rho_0}$ for self-similarity (Chieze and Lazareff, 1981). The general result is given by Eq. (C8). Focusing on the case in which the evaporation is dominant ($M_a=0$, $M=M_{\rm ev}$) and there is no energy injection at the shock, we obtain

$$K_{n,0}=\frac{(3-k_\rho)K_{n,0}^{\rm ev}+2nK_{n-1,1}/(\gamma+1)}{3-k_\rho+n} . \quad \text{(C10)}$$

2. The harmonic mean and geometric mean approximations

From the definition of the moments K_{mn}, it is apparent that K_{11} is in some sense intermediate between K_{20} and K_{02}. The moments may be regarded as the scalar product of $(r/R_s)^m$ and $(v/v_1)^n$ with a weighting factor, to which one can apply the Schwarz inequality and obtain

$$K_{mn}^2\le K_{2m,0}K_{0,2n} . \quad \text{(C11)}$$

In particular, for $m=n=1$ we have

$$K_{11}^2\le K_{20}K_{02} . \quad \text{(C12)}$$

Thus K_{11} is no greater than the geometric mean of K_{20} and K_{02}. Since K_{20} and K_{02} are both positive, one can show that the harmonic mean is also no greater than the geometric mean (which in turn is no greater than the arithmetic mean). This crude argument suggests that we may approximate K_{11} as the harmonic mean of K_{20} and K_{02}:

$$\frac{1}{K_{11}}\simeq\frac{1}{2}\left[\frac{1}{K_{20}}+\frac{1}{K_{02}}\right] . \quad \text{(C13a)}$$

The numerical results in Tables I and II show that this relation is satisfied to a remarkable degree for Sedov blastwaves: for the cases given, the agreement with the approximation is better than 0.1% except for the case $k_\rho=2.8$ in Table II, where the agreement drops to 1%. Furthermore, the one-power approximation (OPA) for the density and velocity (see Sec. IX.B.2) satisfies this relation exactly.

The geometric mean approximation $K_{11}^2=K_{20}K_{02}$ is almost as accurate as the harmonic mean approximation (C13a); in fact, K_{11} is between the geometric and harmonic means in 5 of the 6 cases listed in Tables I and II. Together with the kinematic moment relation derived above, the geometric mean approximation can be used to determine the deviations from the LVA ($K_{20}=K_{11}=K_{02}$) and the $\bar{K}$ approximation ($K_{11}=K_{02}$). For the particular case of nonevaporative blastwaves in a stationary medium, the geometric mean approximation and the moment relation (C9) imply

$$\frac{K_{11}}{K_{02}}=\frac{1}{5-k_{\rho_0}}\left[\frac{3-k_{\rho_0}}{K_{11}}+\frac{4}{\gamma+1}\right] \quad (\text{GMA},\ \nu_H=0) . \quad \text{(C13b)}$$

3. Moments for evaporative blastwaves ($k_E=0$, $\nu_H=0$)

For evaporative blastwaves, the kinematic relation among the moments (C10) depends on the evaporative moment $K_{n,0}^{\rm ev}$. We assume that inside the blastwave we have $T(r)\propto r^{l_T}$, $\rho(r)\propto r^{l_\rho}$, and $\omega_{\rm cl}\dot{m}\propto r^{-k_Q}T'^{\kappa_T}\rho^{\kappa_\rho}$, which is consistent with Eq. (5.15), so that

$$K_{n,0}^{\rm ev}=\left[1+\frac{n}{3-k_Q+\kappa_T l_T+\kappa_\rho l_\rho}\right]^{-1} . \quad \text{(C14)}$$

In addition, l_ρ is given in terms of K_{20} by Eq. (4.22). Adopting the linear velocity approximation, we have $K_{20}=K_{11}$, so that Eq. (C10) reduces to

$$[(5-k_\rho)(\gamma+1)-4]K_{20}=(3-k_\rho)(\gamma+1)K_{20}^{\rm ev} . \quad \text{(C15)}$$

Equation (4.3) for α^2, derived from the virial theorem and energy equation, remains valid for evaporative blastwaves. The LVA simplifies this equation to

$$\alpha^2=\left[\frac{3(\gamma+1)^2}{(5-k_\rho)(\gamma+1)-4}\right]\frac{1}{K_{20}} . \quad \text{(C16)}$$

Solving this equation together with Eq. (5.21) then yields a relation among l_ρ, l_T, and K_{20}:

$$\frac{3+l_\rho+l_T}{3+l_\rho}=\left[\frac{6(\gamma-1)}{(5-k_\rho)(\gamma+1)-4}\right]\frac{1}{K_{20}} . \quad \text{(C17)}$$

Combining Eqs. (4.22), (C14), (C15), and (C17), we find a quadratic equation for

$$\zeta\equiv(K_{20}^{\rm ev})^{-1}-1 , \quad \text{(C18)}$$

namely,

$$A\zeta^2+B\zeta+C=0 \,, \tag{C19}$$

where

$$A\equiv\tfrac{1}{2}(3-k_Q-3\kappa_\rho)[(5-k_\rho)(\gamma+1)-4]+6\kappa_T(\gamma-1) \,,$$

$$B\equiv(3-k_Q)(\gamma-1)+4-(5-k_\rho)(\gamma+1)$$
$$+\kappa_T[3(\gamma-3)+k_\rho(\gamma+1)]+\kappa_\rho[6-(\gamma+1)k_\rho] \,, \tag{C20}$$

$$C\equiv-2(\gamma-1) \,.$$

The moment K_{20} is then given by Eq. (C15). Applications of this result are given in Sec. V.B.

4. The shell approximation

In a variety of circumstances the velocity interior to the shock is constant or increases inwards (cosmological explosions, radiative cooling, propagation into a steeply declining density) and a vacuum develops in the interior. As discussed in Sec. IV.C, the moments may be approximated in this case by assuming $v(r)=v_1=\text{const}$:

$$K_{mn}=K_{m0}\equiv K_m \,, \tag{C21}$$

with, of course, $K_0=1$. The general kinematic relation (C8) (neglecting evaporation and detonations, which are important only if shells do not form) becomes a recursion relation. For a stationary ambient medium, this relation yields Eq. (4.44); in the cosmological case, it gives

$$K_n=\frac{3(1-h/\eta)+\dfrac{n}{\gamma+1}[2+(\gamma-1)h/\eta]K_{n-1}}{3(1-h/\eta)+n}$$
$$(\text{cosmology}, \dot{M}_{\text{ev}}=0) \,, \tag{C22}$$

where we have used the cosmological relations $(v_H/v_s)=(h/\eta)$ and $k_{\rho_0}=0$ from Sec. IX. Note that for $\gamma=1$ Eq. (C22) gives all $K_n=1$; i.e., we recover the thin-shell approximation for radiative (isothermal) blastwaves.

We note in passing that, if all of the matter is restricted to a homogeneous shell in the region $(1-\delta)R_s\rightarrow R_s$ then

$$K_n=\left[\frac{3}{n+3}\right]\frac{1-(1-\delta)^{n+3}}{1-(1-\delta)^3} \,. \tag{C23}$$

APPENDIX D: EQUATION OF MOTION AND THE THIN-SHELL APPROXIMATION

The virial theorem derived in Sec. II is exact, but the moments entering into it are unknown. As an alternative it is possible to develop an equation for the evolution of a blastwave directly from the equation of motion. We begin with the equation of motion for an element of intercloud fluid of mass $\delta M=\rho(1-f)\delta V$ (Cowie *et al.*, 1981):

$$\frac{d}{dt}v\,\delta M=-(1-f)\frac{\partial P}{\partial r}\delta V-\frac{v^2\delta M\,\text{sgn}(v)}{\lambda_{\text{cl}}}+g\,\delta M \,, \tag{D1}$$

where we have assumed spherical symmetry and stationary clouds; a gravitational term has been added. Let

$$\Psi(R)\equiv\frac{1}{4\pi}\int_0^R v\,dM \,, \tag{D2}$$

be the momentum per steradian of the intercloud gas inside R. Summing Eq. (D1) over all the mass elements inside a radius R_0 that is comoving with the external local gas gives

$$4\pi\frac{d}{dt}\Psi(R_0)=-\int_0^{R_0}(1-f)\frac{\partial P}{\partial r}dV$$
$$-\frac{1}{\lambda_{\text{cl}}}\int v^2 dM+\int g\,dM+\dot{M}_{\text{in}}v_{\text{in}} \,, \tag{D3}$$

where we set $\lambda_{\text{cl}}=\text{const}$ and $\text{sgn}(v)=1$. The last term allows for injection of momentum at the origin; $\dot{M}_{\text{in}}$ is the mass injection rate and v_{in} the injection velocity (see Sec. VII). If we choose R_0 to be just outside the shock radius R_s, we can derive from (D2) the kinematic relation

$$\frac{d}{dt}\Psi(R_s)=\frac{d}{dt}\Psi(R_0)+\rho_0R_s^2v_H(v_s-v_H)(1-f_0) \,. \tag{D4}$$

With the aid of the definitions of the K_{mn} factors in Eq. (2.9), we obtain

$$4\pi\Psi(R_s)=K_{01}Mv_1 \,. \tag{D5}$$

Let

$$f_i'\int_0^{R_s}\frac{\partial P}{\partial r}dV\equiv\int_0^{R_s}f\frac{\partial P}{\partial r}dV \,, \tag{D6}$$

which differs by a weighting factor r from f_i in Eq. (2.10), and let

$$K_P\equiv2\int_0^{R_s}\left[\frac{P}{\bar{P}}\right]\frac{r}{R_s}\frac{dr}{R_s} \,. \tag{D7}$$

Inserting Eqs. (D4)–(D7) into (D3) and integrating the pressure term by parts then yields the equation of motion for the blastwave:

$$\frac{d}{dt}K_{01}Mv_1=4\pi R_s^2(1-f_i')(K_P\bar{P}-P_0)+4\pi R_s^2\rho_0(v_s-v_H)[(1-f_i')v_H-(f_0-f_i')v_1]-\frac{K_{02}Mv_1^2}{\lambda_{\text{cl}}}+\int g\,dM+\dot{M}_{\text{in}}v_{\text{in}} \,. \tag{D8}$$

This equation is equivalent to the virial theorem, Eq. (2.14).

The pressure moment K_P has been defined so that it is unity if the pressure is uniform inside the blastwave, as is approximately true for a filled blastwave in which the mass is concentrated at the edge; this approximation becomes exact as $\gamma \to 1$. However, for cosmological blastwaves the pressure is nonzero only in a thin shell; in the limit in which the pressure is a δ function of position, Eq. (D7) gives $K_P = \frac{2}{3}$.

For self-similar flow with $P_0 = \lambda_{\rm cl}^{-1} = \dot{M}_{\rm in} = f_0 - f_i' = 0$, the equation of motion implies

$$\frac{3K_p}{\alpha^2} = \left[\frac{1+\eta_E}{\eta}\right] \nu_1 K_{01} - 3\nu_H(1-\nu_H) - K_{01}\nu_1 + \tfrac{1}{2} w'\Omega \nu_H^2 , \tag{D9}$$

which is similar in form to the virial theorem for self-similar flows in an expanding, gravitating medium, Eq. (9.17). Here we have assumed $\rho_0 \nu_H / \bar{\rho} = \nu_H$, since that is true for all the cases considered in the text, and we have used Eq. (A34) to replace $\int g\, dM$.

Equation (D9) may be used to evaluate the moment K_{01}, which determined the total momentum in the blastwave [Eq. (D5)]. Assuming that $\nu_H = W = \gamma - \gamma_i = \dot{m} = \varepsilon = 0$ in addition to the assumptions stated above Eq. (D9), we apply the PGA expression for the pressure [Eq. (4.66)] and find

$$K_P = \left[\frac{(\gamma+1)(3-k_\rho-k_E)}{(5\gamma+1)-(\gamma+1)(k_\rho+k_E)}\right] \frac{9\gamma^2+20\gamma-5-(\gamma+1)[(3\gamma+1)k_\rho-3(\gamma-1)k_E]}{7\gamma^2+20\gamma-3-(\gamma+1)[(3\gamma+1)k_\rho-3(\gamma-1)k_E]} . \tag{D10a}$$

For radiative blastwaves ($\gamma = 1$), this collapses to $K_P = 1$. This result together with Eq. (4.76) for α^2 and the virial theorem (D9) then yields the PGA result for K_{01}:

$$K_{01} = \frac{2(\gamma+1)^2(3-k_\rho)}{(7\gamma-1)(\gamma+3)-(\gamma+1)[(3\gamma+1)k_\rho-3(\gamma-1)k_E]} . \tag{D10b}$$

As expected, this moment also reduces to unity for radiative blastwaves. The PGA result for the moment K_{10} follows from the general moment relation (C9).

We now return to the general case and develop the *shell approximation,* which takes advantage of the fact that, since most of the mass in a blastwave (except for detonations) is concentrated in a shell near R_s, the velocity of the shocked intercloud gas is approximately constant. In the shell approximation we take $v = v_1$, so that $K_{01} = K_{02} = 1$. For simplicity, we assume that the cloud compression in this shell is negligible, so that $f_i' = f_0 = f$. The blastwave equation of motion in the shell approximation is then

$$\frac{d}{dt} M v_1 = 4\pi R_s^2 (1-f)[(K_P \bar{P} - P_0) + \rho_0 v_H (v_s - v_H)] - M v_1^2 / \lambda_{\rm cl} + \int g\, dM + \dot{M}_{\rm in} v_{\rm in} . \tag{D11}$$

We consider two special cases, neither of which allows for detonations. First, if the ambient medium is stationary ($v_H = 0$) and the shock is strong ($v_s^2 \gg C_0^2$), this equation can be simplified by writing $v_1 = 2v_s/(\gamma+1)$ and $dR_s = v_s dt$:

$$\frac{d}{dR_s} M v_s = 2\pi(\gamma+1)\frac{R_s^2}{v_s}(1-f)(K_P\bar{P}-P_0) - \frac{2Mv_s}{(\gamma+1)\lambda_{\rm cl}} - \frac{(\gamma+1)GM^2}{4v_s R_s^2}\left[1+\frac{2M'}{M}\right] + \frac{\gamma+1}{2}\dot{M}_{\rm in}\frac{v_{\rm in}}{v_s} . \tag{D12}$$

We have used Eq. (A29) for the gravitational term, which is a good approximation if the matter is in a thin shell; here M' is the mass of the matter other than the intercloud gas. If K_P is approximated as unity, then for $f = P_0 = G = \lambda_{\rm cl}^{-1} = \dot{M}_{\rm in} = 0$, this reduces to the approximation discussed by Zel'dovich and Raizer (1966), which in turn was based on the expansion developed by Chernyi (1957).

Second, if the shock is radiative, then the compression will be large ($\rho_1 \gg \rho_0$), ($v_s - v_1$) will be small [Eq. (2.3)], and the effective γ at the shock will be close to unity, $\gamma \simeq 1$ [see Eq. (E15)]. Such blastwaves satisfy a simple equation even if the ambient medium is expanding or contracting ($v_H \neq 0$) or if the shock is not strong. The high compression implies $v_1 = v_s$, so that, in this *thin-shell approximation,* Eq. (D11) becomes

$$\frac{d}{dR_s} M v_s = \frac{4\pi R_s^2}{v_s}(1-f)[(K_P\bar{P}-P_0)+\rho_0 v_H(v_s - v_H)] - \frac{Mv_s}{\lambda_{\rm cl}} - \frac{1}{2}\frac{GM^2}{v_s R_s^2}\left[1+\frac{2M'}{M}\right] + \dot{M}_{\rm in}\frac{v_{\rm in}}{v_s} . \tag{D13}$$

For filled blastwaves, in which the central pressure is nonzero, the moment $K_P = 1$ in this approximation; for hollow blastwaves, including cosmological blastwaves, $K_P = \frac{2}{3}$ in this approximation.

APPENDIX E: SHOCK JUMP CONDITIONS

1. General solution

For a plane, nonrelativistic shock propagating at velocity v_s through an ideal gas, which is itself moving at velocity v_H, the fluxes of mass, momentum, and energy are constant in the shock frame, which yields the shock jump conditions (or Rankine-Hugoniot conditions):

$$\rho_1(v_s - v_1) = \rho_0(v_s - v_H) , \tag{E1}$$

$$P_1 + \rho_1(v_s - v_1)^2 = P_0 + \rho_0(v_s - v_H)^2 , \tag{E2}$$

$$\tfrac{1}{2}(v_s - v_1)^2 + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1} = \tfrac{1}{2}(v_s - v_H)^2 + \frac{\gamma_0}{\gamma_0 - 1}\frac{P_0}{\rho_0} + \varepsilon c^2 . \tag{E3}$$

Here the conditions ahead of the shock are denoted by the subscript 0 and those behind by 1; all velocities are measured in the frame in which the shock velocity is v_s. We have allowed for the injection of an energy εc^2 per gram at the shock front, due to the release of chemical, nuclear, or gravitational energy in a detonation, or to conduction of energy from the shocked gas to the shock front. For chemical or nuclear energy, ε is determined by the change in composition across the shock: If n_j is the density of species j, and I_j is the binding energy per particle, then $\varepsilon c^2 \equiv [(n_{j0}/\rho_0) - (n_j/\rho)]I_j$. If energy is absorbed due to chemical or nuclear dissociation, then ε will be negative. We have also allowed for the possibility that the ratio of specific heats ahead of the shock, γ_0, differs from the value, γ, behind the shock; if $\gamma \neq \gamma_0$, then ε will generally be nonzero, as in the case of dissociation of molecular hydrogen. However, these equations do not apply to magnetized gas, in which the magnetic field and the gas have different effective values of γ.

We define the compression χ_1, the normalized velocities ν_1 and ν_H, and the normalized temperatures θ_1 and θ_0 by

$$\chi_1 \equiv \rho_1/\rho_0 , \tag{E4}$$

$$\nu_1 \equiv v_1/v_s , \quad \nu_H \equiv v_H/v_s , \tag{E5}$$

$$\theta_1 \equiv \frac{P_1}{\rho_1 v_s^2} \equiv \frac{C_1^2}{v_s^2} \equiv \frac{kT_1}{\mu v_s^2} , \quad \theta_0 \equiv \frac{P_0}{\rho_0 v_s^2} \equiv \frac{C_0^2}{v_s^2} \equiv \frac{kT_0}{\mu_0 v_s^2} , \tag{E6}$$

where (μ, μ_0) is the mean mass per particle (behind, ahead of) the shock. The adiabatic Mach number $\mathcal{M}_a$ of the shock is then given by

$$\mathcal{M}_a^2 \equiv \frac{(v_s - v_H)^2}{\gamma_0 C_0^2} \equiv \frac{(1-\nu_H)^2}{\gamma_0 \theta_0} , \tag{E7}$$

whereas the adiabatic Mach number of the post-shock flow is

$$\mathcal{M}_{a1}^2 \equiv \frac{(v_s - v_1)^2}{\gamma C_1^2} = \frac{(1-\nu_1)^2}{\gamma \theta_1} . \tag{E8}$$

We can express the fluxes of momentum and energy across the shock in terms of the parameters Π and Φ defined as follows:

$$\Pi \rho_0 (v_s - v_H)^2 \equiv P_0 + \rho_0 (v_s - v_H)^2 , \tag{E9a}$$

$$\tfrac{1}{2}\Phi(v_s - v_H)^2 \equiv \tfrac{1}{2}(v_s - v_H)^2 + \frac{\gamma_0}{\gamma_0 - 1}\frac{P_0}{\rho_0} + \varepsilon c^2 , \tag{E9b}$$

so that

$$\Pi = 1 + \frac{\theta_0}{(1-\nu_H)^2} , \tag{E10a}$$

$$\Phi = 1 + \frac{2}{(1-\nu_H)^2}\left[\frac{\gamma_0}{\gamma_0 - 1}\theta_0 + \frac{\varepsilon c^2}{v_s^2}\right] . \tag{E10b}$$

Note that Π and Φ reduce to unity for strong shocks $[P_0 \ll \rho_0(v_s - v_H)^2$ or $\theta_0 \ll (1-\nu_H)^2]$ in the absence of energy injection at the shock front ($\varepsilon = 0$). In terms of the adiabatic Mach number $\mathcal{M}_a$, these relations become

$$\Pi = 1 + \frac{1}{\gamma_0 \mathcal{M}_a^2} , \tag{E11a}$$

$$\Phi = 1 + \left[\frac{1}{\gamma_0 - 1} + \frac{\varepsilon c^2}{\gamma_0 C_0^2}\right]\frac{2}{\mathcal{M}_a^2} . \tag{E11b}$$

With these definitions, the jump conditions (E1)–(E3) become

$$\chi_1(1-\nu_1) = 1 - \nu_H , \tag{E12}$$

$$\chi_1[\theta_1 + (1-\nu_1)^2] = \Pi(1-\nu_H)^2 , \tag{E13}$$

$$(1-\nu_1)^2 + \frac{2\gamma}{\gamma - 1}\theta_1 = \Phi(1-\nu_H)^2 . \tag{E14}$$

The solution of these equations is conveniently expressed in terms of the parameter

$$u \equiv \pm[\gamma^2 - (\gamma^2 - 1)\Phi/\Pi^2]^{1/2} , \tag{E15a}$$

which may be rewritten as

$$u^2 = \frac{\gamma_0^2 \mathcal{M}_a^4 - 2\gamma_0\left[\dfrac{\gamma^2 - \gamma_0}{\gamma_0 - 1} + \dfrac{(\gamma^2 - 1)\varepsilon c^2}{C_0^2}\right]\mathcal{M}_a^2 + \gamma^2}{(\gamma_0 \mathcal{M}_a^2 + 1)^2} . \tag{E15b}$$

It is straightforward to show that $u^2 \leq 1$ if $\varepsilon \geq 0$, $\mathcal{M}_a^2 > 1$, and $\gamma \geq 1$. Even if ε is negative, it follows from the definition [Eq. (E15a)] that $|u| \leq \gamma$. Note that radiative shocks have $\varepsilon < 0$; alternatively, such shocks can be approximated as having $\varepsilon = 0$ and an effective γ close to unity. In terms of u, the solution to the jump conditions (E12)–(E14) is

$$\chi_1=\frac{\gamma+1}{\Pi(\gamma-u)}\ , \tag{E16a}$$

$$\nu_1=1-\Pi(1-\nu_H)\left[\frac{\gamma-u}{\gamma+1}\right]\ , \tag{E16b}$$

$$\theta_1=\left[\frac{1+u}{\gamma-u}\right](1-\nu_1)^2$$

$$=\Pi^2(1-\nu_H)^2\frac{(1+u)(\gamma-u)}{(\gamma+1)^2}\ , \tag{E16c}$$

$$\frac{P_1}{\rho_0 v_s^2}=\chi_1\theta_1=\Pi(1-\nu_H)^2\frac{1+u}{\gamma+1}\ . \tag{E16d}$$

The adiabatic Mach number of the post-shock flow is given by

$$\mathcal{M}_{a1}^2=\frac{\gamma-u}{\gamma(1+u)}\ , \tag{E17}$$

so that the point $u=0$ corresponds to $\mathcal{M}_{a1}^2=1$; for $u>0$, the post-shock flow is subsonic, whereas for $u<0$ it is supersonic.

In some cases it is useful to express the solution implicitly in terms of the compression χ_1. Equations (E12) and (E13) give

$$\nu_1=\frac{\chi_1-(1-\nu_H)}{\chi_1}\ , \tag{E18a}$$

$$\theta_1=\frac{(\Pi\chi_1-1)(1-\nu_H)^2}{\chi_1^2}\ ; \tag{E18b}$$

Eq. (E16a) can be solved for $u(\chi_1)$.

The jump conditions (E16a)–(E16c) apply to an arbitrary discontinuity in the fluid flow. In discussing their significance, we shall assume that the ratio of specific heats is constant across the shock ($\gamma=\gamma_0$), thereby eliminating the possibility of rarefaction shocks, in which $\chi_1<1$ (Zel'dovich and Raizer, 1966). For shock waves, in which there is no energy injection ($\varepsilon=0$), the requirement that the entropy increase across the shock implies

$$u>0,\quad \mathcal{M}_a>1,\quad \mathcal{M}_{a1}<1\quad \text{(shock)} \tag{E19}$$

for a nontrivial solution (Landau and Lifshitz, 1959). The shock is said to be strong if $\mathcal{M}_a\gg1$ and weak if $\mathcal{M}_a\simeq1$; the corresponding values of u are

$$u\to\begin{cases}1, & \mathcal{M}_a\to\infty\ ,\\ 0, & \mathcal{M}_a\to1\ .\end{cases} \tag{E20}$$

If there is an energy release at the discontinuity ($\varepsilon>0$), then the Mach number $\mathcal{M}_a$ need not exceed unity: if $\mathcal{M}_a<1$, the flow is termed a deflagration, whereas if $\mathcal{M}_a>1$ it is termed a detonation. Detonations are discussed by Landau and Lifshitz (1959), Stanyukovich (1960), and Zel'dovich and Raizer (1966). For $\gamma=\gamma_0$ and $\varepsilon>0$ one can show that the supersonic case is always compressive ($\chi_1>1$). Both positive and negative values of u are possible, with $u>0$ corresponding to a strong detonation and $u<0$ to a weak detonation; $u=0$ is termed a Chapman-Jouguet detonation:

$$\mathcal{M}_a>1,\ \chi_1>1\begin{cases}u>0, & \mathcal{M}_{a1}<1\ \text{(strong detonation)}\ ,\\ u=0, & \mathcal{M}_{a1}=1\ \text{(Chapman-Jouguet)}\ ,\\ u<0, & \mathcal{M}_{a1}>1\ \text{(weak detonation)}\ .\end{cases} \tag{E21}$$

Strong detonations and Chapman-Jouguet detonations are preceded by shocks, with the energy relase occurring behind the shock front. The jump conditions apply at every point behind the shock front, with the energy release increasing from 0 to ε, and u decreasing from the value just behind the shock front ($\simeq1$ for strong shocks) to its final value ≥0 [Eq. (E15b)]; a weak detonation cannot occur in this case. Note that $P\propto\chi\theta$ is a linear function of u, as is χ_1^{-1}; hence, in the P-χ^{-1} plane, the evolution of the gas behind the shock front lies on a straight line. If the detonation is driven by a piston, or if it is propagating in a sufficiently steep density gradient (Sedov, 1959), so that the shocked gas acts like a piston, then the detonation is strong ($u>0$); in the absence of such effects, however, a rarefaction wave follows the detonation, so that $\mathcal{M}_{a1}=1$ and $u=0$: the Chapman-Jouguet condition is satisfied. A weak detonation can be preceded by a shock only if (1) energy is absorbed or radiated behind the region where it is relased and (2) $u=0$ at the point where the energy injection changes sign. Otherwise, weak detonations do not have associated shock fronts and can occur only if the release of energy is triggered by some supersonic mechanism, such as radiation.

An astrophysically important example of a discontinuity with $\varepsilon>0$ is the ionization front (see, for example, Kahn, 1969), in which ionizing radiation propagates upstream and deposits energy in the neutral gas flowing across the front. (The deposition of momentum as well as energy is considered by Max and McKee, 1977.) In ionization fronts, the gas loses energy to radiation, so the value of ε is not fixed. The final temperature θ_1 is determined by balancing radiative cooling with photoionization heating, and since θ_1 is known, the mass and momentum jump conditions suffice to determine the conditions behind a front of given velocity. The velocity of the ionization front itself is determined by the criterion that the flux of ionizing photons equal the flux of neutrals across the front plus the number of recombinations per unit area. Supersonic ionization fronts are termed *R* fronts; subsonic ones are termed *D* fronts. Strong *R* fronts have subsonic flow and a relatively high compression behind the front; weak *R* fronts have supersonic flow and a small compression behind the front. If the ionizing flux at the front is sufficiently large (photon flux $>2n_0C_1$), the ionization front races through the gas without much dynamical effect (a weak *R* front). If the ionizing flux reaching the front is in the range $2n_0C_1>$ photon flux $>n_0C_0^2/2C_1$, then the flow generally relaxes to a shock front followed by a "*D*-critical" front with $\mathcal{M}_1=1$, analogous to the Chapman-Jouguet condition.

2. Strong shocks ($\theta_0 \to 0$, $\mathcal{M}_a \to \infty$)

We now focus on strong shocks, the case of greatest interest in the text. In this case we have

$$\Pi = 1, \quad \Phi = 1 + \frac{2\varepsilon c^2}{v_s^2(1-\nu_H)^2} ,$$
$$u^2 = 1 - \frac{2(\gamma^2-1)\varepsilon c^2}{v_s^2(1-\nu_H)^2}, \quad \chi_1 = \frac{\gamma+1}{\gamma-u} ,$$
$$\nu_1 = 1-(1-\nu_H)\left[\frac{\gamma-u}{\gamma+1}\right] , \tag{E22}$$
$$\theta_1 = (1-\nu_H)^2 \frac{(1+u)(\gamma-u)}{(\gamma+1)^2} ,$$
$$\frac{P_1}{\rho_0 v_s^2} = (1-\nu_H)^2 \frac{(1+u)}{(\gamma+1)}$$

(arbitrary strong shock, $\theta_0 = 0$). We now consider several special cases.

a. No energy injection at the shock ($\varepsilon = 0$)

We have $\Pi = \Phi = u = 1$ and

$$\chi_1 = \frac{\gamma+1}{\gamma-1} ,$$
$$\nu_1 = \frac{2+(\gamma-1)\nu_H}{\gamma+1} , \tag{E23}$$
$$\theta_1 = \frac{2(\gamma-1)}{(\gamma+1)^2}(1-\nu_H)^2 ,$$
$$\frac{P_1}{\rho_0 v_s^2} = \frac{2}{\gamma+1}(1-\nu_H)^2 \quad (\theta_0 = 0, \ \varepsilon = 0) .$$

The compression χ_1 is independent of the ambient velocity ν_H in this case. For $\nu_H = v_H / v_s = 0$, these jump conditions reduce to the usual ones for a strong gas dynamic shock [see, for example, Zel'dovich and Raizer, 1966, and Eq. (E27) below].

b. Stationary ambient medium

Alternatively, for a strong shock in a stationary medium with arbitrary energy injection at the shock front, we have $\nu_H = 0$ and $\Pi = 1$, but ε (and hence Φ and u) arbitrary; equations (E12) and (E18b) then yield

$$\chi_1(1-\nu_1) = 1 ,$$
$$\theta_1 = \frac{\chi_1 - 1}{\chi_1^2} = \frac{\nu_1}{\chi_1} = \nu_1(1-\nu_1) \quad (\theta_0 = 0, \ \nu_H = 0) . \tag{E24}$$

The parameters u and ε, which describe the energy injection at the shock, can then be obtained from Eq. (E22):

$$u = \gamma - \frac{\gamma+1}{\chi_1} = \nu_1(\gamma+1) - 1 ,$$
$$\frac{\varepsilon c^2}{v_s^2} = \frac{\chi_1 - 1}{2\chi_1^2}\left[\frac{\gamma+1}{\gamma-1} - \chi_1\right] = \frac{\nu_1[2-(\gamma+1)\nu_1]}{2(\gamma-1)} \tag{E25}$$
$$(\theta_0 = 0, \ \nu_H = 0) .$$

These relations allow one to express the jump conditions implicitly in terms of either χ_1 or ν_1. Note that in the absence of energy injection at the shock ($\varepsilon = 0$, $u = 1$), Eq. (E25) gives the usual strong shock jump conditions for χ_1 and ν_1.

c. Chapman-Jouguet detonation ($u = 0$)

In this case Eq. (E22) yields

$$\chi_1 = \frac{\gamma+1}{\gamma} ,$$
$$\nu_1 = \frac{\gamma\nu_H + 1}{\gamma+1} ,$$
$$\theta_1 = \frac{\gamma(1-\nu_H)^2}{(\gamma+1)^2} , \tag{E26}$$
$$\frac{P_1}{\rho_0 v_s^2} = \frac{(1-\nu_H)^2}{\gamma+1} ,$$
$$v_s^2 = \frac{2(\gamma^2-1)\varepsilon c^2}{(1-\nu_H)^2} \quad (\theta_0 = 0, \ u = 0) .$$

d. Stationary ambient medium, no energy injection ($\nu_H = \varepsilon = 0$)

In this case we recover the standard jump conditions for a strong shock from Eq. (E23):

$$\chi_1 = \frac{\gamma+1}{\gamma-1}, \quad \theta_0 = 0 ,$$
$$\nu_1 = \frac{2}{\gamma+1} = \frac{P_1}{\rho_0 v_s^2}, \quad \nu_H = 0 , \tag{E27}$$
$$\theta_1 = \frac{2(\gamma-1)}{(\gamma+1)^2}, \quad \varepsilon = 0 .$$

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