Brownian Motion as a Natural Limit to all Measuring Processes

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TABLE OF CONTENTS

§1. Introduction

It is the purpose of this paper to review the literature showing that Nature has set a definite limit to the ultimate sensitivity of measuring instruments beyond which we cannot advance, and that this limit is determined by the Brownian motion or one of a group of related effects discussed below.

1A. Historical

In 1827, the naturalist, Robert Brown, while examining suspensions of various pollens with the aid of one of the then newly constructed achromatic objectives, discovered that the individual particles were constantly in a very animated state of motion. After his announcement of this fact, there followed a deluge of experiments and theories which sought to arrive at the true nature and cause of this motion. The explanation was first looked for in the possibility that the particles were alive. This theory was quickly disproved however, for very soon particles of glass, minerals, petrified wood, pollen known to be over 100 years old, and even stone dust from the Egyptian Sphinx were shown to behave similarly to Brown's original pollen particles. All such possibilities as convection currents in the solution, internal motion due to uneven evaporation, hygroscopic or capillary action, mutual forces between particles, formation of small bubbles of gas, temperature effect of the illumination, etc., were carefully investigated. None of these having proved to be the true cause, the search continued. Wiener[1] in the year 1863 published a lengthy paper in which he made the suggestion that the molecular structure of the solution might be the cause of the irregular motion. Convection currents whose cross sections were small compared to the diameter of the particles were considered. Cantoni was led to believe that the specific heats of the particles and the solutions were connected with the effect, while Jevons thought that the source was one of an electric nature. In 1877 Delsaulx[2] published a paper in which for the first time an explanation was offered which is still considered to be the correct one. In this paper he showed, that in a given small interval of time, the impulses which a particle would receive from the molecules of the surrounding liquid would not always be equal in all directions, but would very often have a resultant in some one direction. Subject to these resultant impulses the particles would exhibit a random zigzag motion. Thirion pointed out in 1880 that Carbonelle had previously in 1874 considered and discussed this same theory, and so to Carbonelle goes the priority for the true explanation of the Brownian motion. To Gouy,[3] however, goes the credit for having really prepared the way for our present point of view, since his experiments established conclusively the fact that the source of the impulses really was in the molecules of the surrounding liquid and not in the particles themselves. In spite of these papers however, many authors disagreed with this view and ascribed still other causes to the effect, and it was not until the papers of Einstein,[4] v. Smoluchowski[5] and Perrin[6] had appeared that the issue was finally settled.

Let us assume that we have a particle which is constrained to move backward and forward along a straight line, and that the motion takes place in a random manner by jumps of equal length l. We can calculate the probability W that out of n such jumps $(n/2-b)$ will be to the left and $(n/2+b)$ will be to the right, and obtain

$$W=\frac{1}{2^n}\cdot\frac{n!}{(\frac{1}{2}n-b)!(\frac{1}{2}n+b)!}. \tag{1}$$

Using now the Stirling formula, we can state that the probability for a given value of b is

$$W=(2/\pi n)^{\frac{1}{2}}e^{-2b^2/n}. \tag{1'}$$

Obviously, the net displacement of the particle is given by $2bl=a$. The probability that the displacement will be between a and $a+da$ is

$$W=(1/l)(1/2\pi n)^{\frac{1}{2}}e^{-a^2/2nl^2}da, \tag{2}$$

from which we see that the mean square of the distance covered is

$$\overline{S^2}=\frac{1}{l(2\pi n)^{\frac{1}{2}}}\int_{-\infty}^{+\infty}a^2e^{-a^2/2nl^2}da=nl^2 \tag{3}$$

and

$$W=(2\pi\overline{S^2})^{-1}\cdot e^{-a^2/2\overline{S^2}}da. \tag{4}$$

In a similar manner we can show that if the particles are able to move in every direction, the mean square displacement in a given direction will be

$$\overline{S^2}=\tfrac{1}{3}nl^2. \tag{5}$$

Now let us assume, as in the Brownian movement, a random zigzag motion of the particles and consider the projection of the motion of some one particle onto a straight line. If $S_1, S_2\cdots S_n$ are these projections during equal time intervals τ, the projected displacement is $S=S_1+S_2\cdots S_n$. We assume that successive jumps are absolutely independent, then $\overline{S_kS_l}=0$, and therefore

$$\overline{S^2}=\overline{S_1^2}+\overline{S_2^2}+\cdots+\overline{S_n^2}, \tag{5a}$$

or

$$\overline{S^2}=n\overline{S_1^2}.$$

The mean square displacement is then proportional to the number of jumps, n, and to the total time $n\tau=t$. Since the S's are only the projections of the individual displacements, nothing has been said regarding the shape of the path of the particle during the time intervals τ. This path may be quite complex. It is easily seen that if S_x, S_y and S_z refer to the projections on the axes x, y, z, and S_v and S_r refer to the projections on the plane xy and in space, respectively, Eq. (5a) holds for $\overline{S_x^2}$, $\overline{S_y^2}$, $\overline{S_z^2}$. As a result of this, $\overline{S_v^2}=2n\overline{S_1^2}$ and $\overline{S_r^2}=3n\overline{S_1^2}$.

These results show that $\overline{S^2}$ is a function of the time t, $\overline{S^2}=Qt$, but say little concerning the coefficient Q. Since this coefficient gives the time rate of increase of the mean square displacement it must obviously be a coefficient of diffusion of the particles. It remained for Einstein first to evaluate Q. Let us assume that the particles are under the influence of a force which is proportional to some potential Φ. One can then investigate the manner in which the condition of a system remains stationary due to the fact that the changes in this condition caused by the Brownian motion and this assumed force equalize each other. If Δ is the displacement of the coordinate α, which is chosen to describe the motion of the particle, the result is

$$\overline{\Delta^2} = (2R/N)BTt. \qquad (6)$$

For spherical particles, one may use Stokes' formula and so replace B by $(1/6\pi\zeta a)$. The final result in this case is that the mean square displacement is given by

$$\overline{S^2} = \overline{\Delta^2} = (RT/N)(1/3\pi\zeta a)t, \qquad (7)$$

where a is the radius of the particles, and ζ is the coefficient of friction of the surrounding medium, and the other constants have their usual meanings. The potential Φ is missing from the formula since the restoring force quite naturally depends upon it.

After the derivation of this formula, it was obvious that in order to test the validity of this molecular interpretation of the Brownian motion one needs only to observe the mean square displacement of a number of particles for a time t, the sizes of the particles being accurately known. Perrin and his co-workers succeeded in preparing suspensions of very uniform particle size and in a "camera lucida" traced the positions of a given particle from half-minute to half-minute. Since the microscope was vertical the horizontal displacements S_v of the motions were observed. If cross-section paper be used the projections onto two rectangular axes can be read off immediately. This is not necessary however for the sum of the squares of these projections is equal to the sum of the squares of the separate tracks, and to obtain the mean square of the projection upon one axis it is only necessary to measure these tracks one by one, square them and divide by 2. Fig. 1 is reproduced from Perrin's book *Brownian Movement and Molecular Reality*. By evaluating such traces as the one shown, Perrin was not only able to verify Einstein's formula, but also to obtain from his experimental values of $\overline{S^2}$ a very accurate value for N, and so to establish definitely the Brownian movement as a molecular phenomenon.

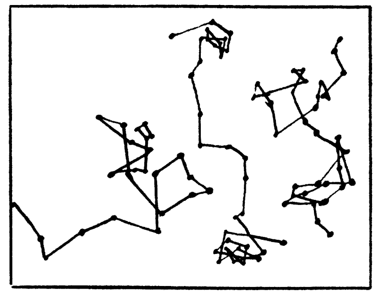


FIG. 1. Brownian motion of a suspended particle. (After Perrin, *Brownian Movement and Molecular Reality*.)

The coordinate α used by Einstein to describe the motion of the particles can represent any degree of freedom of the system and is not at all restricted to translation. If this α refers to the rotation of the particles one finds that the mean square angular displacement $\overline{\theta^2}$ is given by

$$\overline{\theta^2} = \overline{\Delta^2} = (RT/N)(1/4\pi\zeta a^3)t. \qquad (8)$$

This formula was also beautifully checked by Perrin.

von Smoluchowski, also in 1906, published the results of his calculations in which, without the assumption of an outside force, he arrived at the expression

$$\overline{S^2} = \overline{\Delta^2} = (64/81)(RT/N)(1/\pi\zeta a)t \qquad (9)$$

for the mean square displacement in any one direction. This, as will be seen at once, differs from the result of Einstein only in the numerical factor. In 1908 Langevin published a discussion of the problem and derived by different methods of calculation a formula identical with that of Einstein.

Following these calculations many others appeared dealing with various phases of the subject. In 1913 Frau G. L. de Haas-Lorentz[4a] published a small volume in which the complete history and status of the question of the Brownian motion was given in detail, and in which the above-mentioned investigations were discussed. In spite of the completeness and thoroughness of this booklet, very little attention was paid by physicists as a whole to the subject for quite some years. The seven cases, where Brownian movement plays an important role, discussed by Frau de Haas-Lorentz, in which unfortunately no numerical values were given, were never considered further until 1926, at which time G. Ising[7] published a paper in which for the first time the relationship of Brownian motion to an experimental problem in physics was shown numerically.

In 1925 Moll and Burger,[8] with their now well-known "thermorelay" magnified the deflections of a Moll galvanometer about 100 times, and recorded the deflections photographically as shown in Fig. 2. It will be noticed that the zero line thus obtained is not at all straight but shows small irregularities. Their galvanometers were of course well shielded against thermal, electric, magnetic, and mechanical disturbances, and so they attributed this residual unsteadiness to disturbances of a micro-seismic nature. Ising, in the paper mentioned above, made a careful study of the nature of these small zero deflections, and from them calculated the mean square deviation and showed conclusively that they were for the most part purely of Brownian motion origin. v. Smoluchowski had previously discussed the fluctuations of a suspended system. These results will be discussed in detail in the section devoted to the Brownian motion of galvanometers.

Occasionally one finds, in the literature, work of others who had used very sensitive instruments such as radiometers, micro-radiometers, galvanometers, etc., in which in spite of all precautions and shields, zero disturbances similar to those just mentioned are recorded. The Brownian motion of these instruments had thus for years been observed but never understood. Ising's paper in pointing out the true nature of these disturbances may be considered a classical contribution to the subject.

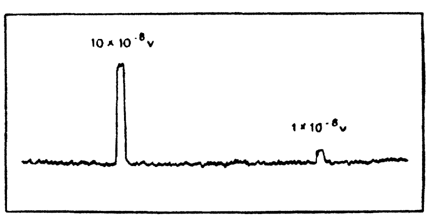


FIG. 2. Fluctuations of galvanometer zero. (From Moll and Burger.)

1B. Statistical nature of Brownian motion

The molecular nature of matter and the interpretation of heat as the energy of chaotic motions of the molecules are fundamental principles in physics today. According to the principles of classical statistical mechanics we have to describe the state of a given system in terms of coordinates, $q_1q_2\cdots$ and conjugate momenta, $p_1p_2\cdots$ the number of each being equal to the number of degrees of freedom of the system. If $H(p_iq_i)$ is the Hamiltonian function of the system, it may be shown that the relative probability of finding the state of the system in the range $dq_1\cdots dp_N$, $dp_1\cdots dp_N$ is given by

$$fd\tau = Ce^{-H(p_iq_i)/kt}dq_1dp_1\cdots dq_Ndp_N. \quad (10)$$

The coefficient C must obviously be chosen so that $\int fd\tau = 1$, when the integration extends over all possible values of the variables. This supposes the system to be in thermodynamic equilibrium at the temperature T.

Since there is a continual interchange of energy between the system and its surroundings, the total energy content H at a given temperature does not have a definite value. In calculating resulting averages for the system it is convenient to introduce the partition function Z defined by

$$Z = \int e^{-H/kT}dq_1dp_1\cdots dq_Ndp_N.$$

The average value of any function of the p's and q's, $f(p_iq_i)$, is then given by

$$\overline{f(p_iq_i)} = (1/Z)\int fe^{-H/kT}d\tau. \quad (11)$$

In particular the average total energy is given by

$$\bar{H} = (1/Z)\int He^{-H/kT}d\tau = \partial \log Z/\partial\beta, \quad (12)$$

where $\beta = 1/kT$ and the average value of any coordinate is

$$\overline{q_i} = (1/Z)\int q_i e^{-H/kT} d\tau. \quad (13)$$

Suppose the coordinates are so chosen that the kinetic energy is expressed as a sum of squares of the momenta with constant coefficients

$$T(p_i) = \sum p_i^2/2m_i \quad (14)$$

and the Hamiltonian is thus

$$H = \sum p_i^2/2m_i + V(q_i); \quad (15)$$

then in calculating Z or the average of any function of the coordinates alone, the integrals occurring are the product of integrals over the position and momenta coordinates respectively. For example in getting $\overline{H}$ we have

$$\frac{1}{2m_i}\overline{p_i^2} = \frac{\frac{1}{2m_i}\int p_i^2 e^{-p_i^2/2mkT} dp_i \int [\text{over other coordinates}]}{\int e^{-p_i^2/2mkT} dp_i \int [\text{over other coordinates}]}. \quad (16)$$

Now

$$\int_{-\infty}^{+\infty} e^{-ap^2} dp = (\pi/a)^{\frac{1}{2}}, \quad \text{so} \quad (\partial/\partial a)\int_{-\infty}^{+\infty} e^{-ap^2} dp = \int_{-\infty}^{+\infty} p^2 e^{-ap^2} dp = (1/2a)(\pi/a)^{\frac{1}{2}}.$$

The ratio is $1/2a$. Hence the mean value of $\overline{p_i^2}/2m_i$ is given by

$$\overline{p_i^2}/2m_i = (1/2m_i)(2m_i kT/2) = \tfrac{1}{2}kT, \quad (17)$$

and so the mean value of the kinetic energy in any coordinate is $\frac{1}{2}kT$. The same is evidently true of potential energy terms which are simply of the form $\frac{1}{2}K_i q_i^2$ assuming that q_i does not enter the Hamiltonian in any other way.

If, however, the coordinate q_i, say, enters the Hamiltonian solely as $(1/2m_1)p_1^2 + \frac{1}{2}K_1 q_1^2$, the particle cannot come into thermal equilibrium with the rest of the system: there must be some kind of interaction between it and the rest of the system. This is generally provided by some mechanism such as collisions of gas molecules with the mass particle which constitutes the oscillator. Supposing the interaction energy depends solely on the coordinates, not on the momenta, then the result for the mean kinetic energy of each degree of freedom is rigorously given by statistical mechanics.

Associated with each degree of freedom is a mean kinetic energy of $\frac{1}{2}kT$. One can, then, calculate the Brownian movement fluctuations due to this $\frac{1}{2}kT$ energy in a system in perfect generality from the laws of statistical mechanics. This is possible since the average energy of these random motions will be exactly the same for all systems at the same temperature (so long as they are each in thermodynamic equilibrium with their surroundings) entirely independent of the nature of the systems and the mechanism which produces them. The energy distribution, however, will be a function of the particular system in question, and it is the purpose of the main body of this paper to develop in detail the characteristics of the Brownian motion in some of the more common measuring devices.

Since the fluctuations of a given system which are due to Brownian motion are statistical in nature, we may at once calculate the probability of finding our system in the region x to $x+dx$, the result being the well-known Gauss error curve, given by

$$W(x)dx = [2\pi\overline{x^2}]^{-1} e^{-x^2/2\overline{x^2}} dx. \quad (18)$$

In using our system for the purpose of measuring we must read off by means of the position of an indicator of some type the energy of the system after the quantity of energy to be measured has been impressed upon said system. We must know whether the excess energy is due to the quantity

TABLE I.

Coordinate values	Energy values $a(\frac{1}{2}kT)$	Probability $W(x)$
$\lvert x\rvert \geq 1(\overline{x^2})^{\frac{1}{2}}$	E_{pot} $1(\frac{1}{2}kT)$	0.317
$\lvert x\rvert \geq 2(\overline{x^2})^{\frac{1}{2}}$	$4(\frac{1}{2}kT)$	0.045
$\lvert x\rvert \geq 3(\overline{x^2})^{\frac{1}{2}}$	$9(\frac{1}{2}kT)$	0.003
$\lvert x\rvert \geq 4(\overline{x^2})^{\frac{1}{2}}$	$16(\frac{1}{2}kT)$	0.00006

being measured or simply to the statistical fluctuations of the system. From the above-mentioned probability equation, (18), we can, following Czerny,[9] draw up Table I in the following manner.

If, then, we find the energy of our system equal to a $\frac{1}{2}kT$ as shown by the coordinate x of our indicator, we can immediately state the probability that this constitutes a real measurement. The factor a is clearly arbitrary, and depends only upon the degree of certainty desired.

The statistical laws governing the behavior of our system have nothing to say regarding the detailed mechanism of the energy fluctuations. It is therefore very unsatisfactory to attempt in a particular case to say that the source of the existing variations is this or that effect. Furthermore we may not hope by removing one of these supposed causes, let us say by evacuating a galvanometer and thus preventing bombardment by air molecules, to decrease the magnitude of these fluctuations. There must be, regardless of what the system and the cause may be, a mean kinetic energy of exactly $\frac{1}{2}kT$ associated with each degree of freedom of the system.

Since x, the coordinate of our indicator, may be any coordinate which describes the position of our system, many common measuring devices are at once seen to be subject to these slight deflections of amount $(\overline{x^2})^{\frac{1}{2}}$. These deflections in most cases are relatively easy to predict, and when expressed in terms of the constants of the instrument in question give at once the theoretical limit of that system. Let the sensitivity of any instrument be given by

$$\sigma = \delta x/\delta q,$$

where δx is the change in the quantity to be measured and δq the corresponding deflection produced by this change. This may be written as

$$\sigma = \delta x/\delta q = B/A, \tag{19}$$

where A is the directional force depending upon the moving system, and B may be thought of as a deviation factor. If δx is a steady deflection, then $B\delta q$ is the turning moment produced by δq. If we now substitute for δx the value $\bar{x} = (2\epsilon/A)^{\frac{1}{2}}$ we obtain

$$\overline{\delta q} = (A/B)\bar{x} = (2\epsilon A)^{\frac{1}{2}}/B, \quad \text{where} \quad \epsilon = \tfrac{1}{2}kT \tag{20}$$

as the average error in a single measurement of the quantity q. In general there is a relation between A and B which is characteristic of every measuring instrument, and this fact together with the validity of the Einstein equation results in a natural limit of sensitivity for all single measurements of q.

It is well to mention that in this paper we are interested primarily in such measurements. One might take many measurements of the zero of any instrument, and so accurately determine the zero and the most probable deviations from this zero. We would thus be able to draw our zero very accurately as in Fig. 3a. We might now apply a

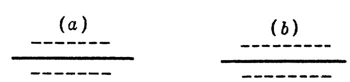


FIG. 3.

quantity smaller than δq to the instrument and again make a great many readings of the new position, obtaining the value indicated in Fig. 3b. The difference between the two positions would give us a measure of a quantity much smaller than δq, the individual deflections for which would have been entirely indiscernible. One must, however, remember that in such a case the amount of energy which has been delivered to our instrument during the course of this measurement is greater than $\frac{1}{2}kT$.

§2. NON-ELECTRIC SYSTEMS

2A. Suspended mirror

Let M be a very light mirror suspended upon a fine quartz fiber of torsion constant A. The motion of this system may be characterized by the one coordinate ϕ, where ϕ is the angle through which the system has rotated from its position of equilibrium (Fig. 4). Since it is a motion of one degree of freedom, we may expect that the system will oscillate back and forth with a Brownian motion of such magnitude that

$$\tfrac{1}{2}A\overline{\phi^2}=\epsilon=\tfrac{1}{2}kT. \qquad (21)$$

An example will make clear the order of magnitude of these oscillations. At 18°C, $\epsilon=\frac{1}{2}kT=2\times10^{-14}$ ergs. Assuming A to be 10^{-6} dyne cm/radian^2, as for a thin quartz fiber, we find $\overline{\delta\phi}=2\times10^{-4}$ radian.

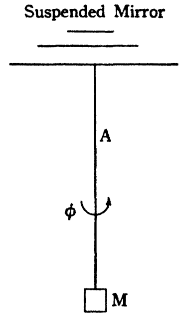


FIG. 4.

Gerlach[10] allowed a beam of light to be reflected from such a mirror onto a distant scale and then made a study of the inherent zero unsteadiness of the system. With a quartz fiber a few tenths of a μ in diameter and a few centimeters long, a mirror 0.8×1.6 mm and a scale at 1.5 m, Brownian movements of several cm were observed. A system with a fiber sufficiently thin showed at 1 m a Brownian movement of over a meter. The mean square deflection, $\overline{\phi^2}$, was successfully measured with an accuracy of 7 percent, and found to agree very closely with the value predicted.

Kappler,[11] continuing this work, investigated the possibility of using the Brownian movement of such a suspended system to obtain a more accurate value of Avogadro's number. A photographic record of the Brownian movement of a system, and the equation $\frac{1}{2}A\overline{\phi^2}=\frac{1}{2}kT$ and measurements of A and $\overline{\phi^2}$ permit one to obtain a value of k, the Boltzmann constant. Having this, N is obtained from the relation

$$N=R/k.$$

The constant of the fiber, A, was determined carefully to ±0.2 percent. From 101 hours of registrations at 287°K, $\overline{\phi^2}$ was found to be $4.178\cdot10^{-6}$ radian$^2\pm0.4$ percent. From these values N was found to be $6.059\cdot10^{23}$ which is probably accurate to ±1 percent. One important result must be emphasized: the fact that only the character of the Brownian movement was dependent upon the pressure, while the average energy of course was not.

In order to determine whether the value of $\overline{\phi^2}$ had been made too large by some outside impulses, such for example as mechanical vibrations, curves were taken at various pressures ranging from 1 atmosphere to 10^{-4} mm of mercury. As is clearly seen in the curves shown in Fig. 5, which were recorded at the two extremes of the pressure range, the pressure dependency of the form of the Brownian movement is shown. It is quite striking that at low pressures the motion approaches the sinusoidal natural mode of oscillation of the system and tends to lose its random character. But for slight outside disturbances which make themselves felt only at the lowest pressures, all of the curves, in spite of the difference in their forms, yield for the Brownian movement of the system identical values of $\overline{\phi^2}$.

In a second paper Kappler studied the influence exerted by these outside mechanical disturbances in order to see if the 1 percent error in the previous determination of N could be reduced. By studying the Brownian movement for long periods of time with respect to its dependence upon the original conditions (original velocities, etc.) one can determine what parts of the $\overline{\phi^2}$ are due to spurious effects. In addition one obtains the mean square velocities, $\overline{u^2}$, and from them can make a quite independent determination of k, the Boltzmann constant, since $\frac{1}{2}m\overline{u^2}=\frac{1}{2}kT$.

Uhlenbeck and Goudsmit[12] investigated theoretically the dependence of the form of the Brownian movement upon the pressure. They developed the displacement of the small mirror, for a time interval long compared with the free period of the system, into a Fourier series. The

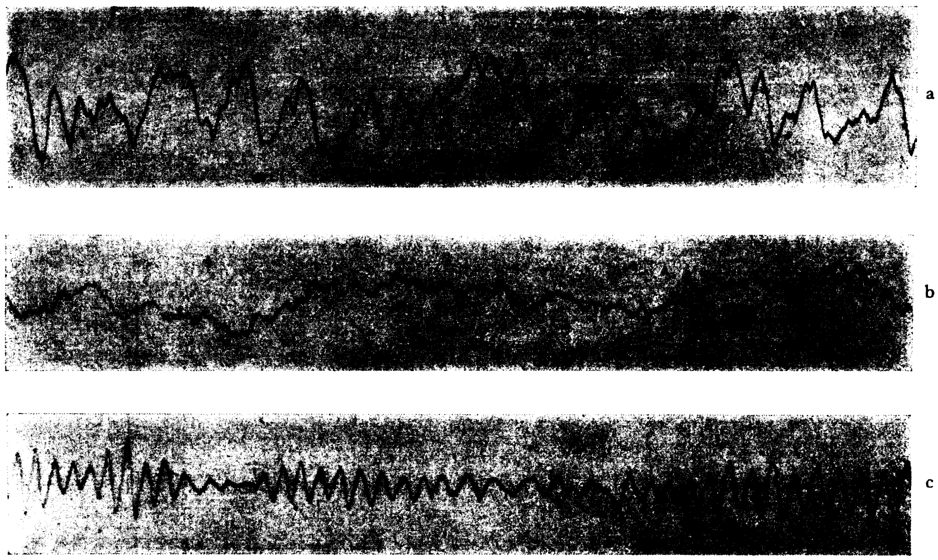


FIG. 5. Brownian fluctuation of a suspended mirror. (From Kappler.)

(a) Restoring force 2.66×10^{-9} abs. units. $I=6.1\times10^{-6}$. Camera distance 86.5 cm. Time 30 sec. equivalent to 2 mm. Pressure 4×10^{-3} mm.
(b) Restoring force 9.428×10^{-9} abs. units. $I=10^{-7}$. Camera distance 72.1 cm. Time 30 sec. equivalent to 1 mm. Atmospheric pressure.
(c) Same system as curve 5b, except that the pressure is 10^{-4} mm.

squares of the amplitudes of each Fourier component were found to be given by

$$\overline{\phi^2{}_K}=\frac{\pi^{\frac{1}{2}}m^{\frac{1}{2}}(8kT)^{\frac{3}{2}}}{\rho IT}\frac{1}{\pi kT(\omega^2-\omega^2{}_K)^2+32p^2\rho^2\omega^2{}_K}, \quad (22)$$

where m = mass of gas molecules, ρ = mass of mirror per cm². They are, then, explicit functions of the pressure and the molecular weight of the surrounding gas molecules as well as the absolute temperature. The sums of these quantities, however, just as must be expected are entirely independent of the pressure and the molecular weight, being functions only of T and yielding the result predicted from the equipartition theorem,

$$\tfrac{1}{2}A\overline{\phi^2}=\tfrac{1}{2}A\sum_{K=0}^{\infty}\overline{\phi^2{}_K}=\tfrac{1}{2}kT. \quad (23)$$

The above discussion shows that such an instrument as a Nichols radiometer, which consists of a tube evacuated to about 0.01 mm in which a pair of light vanes and a small mirror are suspended by a quartz fiber as shown in Fig. 6, must always have an unsteady zero. Experimentally such instruments, in spite of all possible mechanical precautions, actually show these residual deflections around their zeros.

For any suspended system with a fixed torsion constant A, there is then a definite limit set by the Brownian motion. However, as may readily be seen, the sensitivity of the system may be increased by diminishing A. For, whereas the Brownian movement fluctuation is represented by $A\overline{\phi^2}$, the constant turning moment produced by the quantity being measured is represented by $A\bar{\phi}$. The Brownian motion is then proportional to $1/A^{\frac{1}{2}}$, while the real deflection produced by our given energy quantity is proportional to $1/A$. Our net gain is therefore inversely proportional to $A^{\frac{1}{2}}$. Mechanical stability and the patience necessary to read long period instruments are the

FIG. 6.

Suspended Double Mirror

FIG. 7.

limiting factors of sensitivity, for an ideal instrument.

2B. Compound torsion pendulum

The kind of calculation involved in the theory of the Brownian movement for a system of more than one degree of freedom is illustrated by the double torsion pendulum worked out for this report by Professor Condon. Suppose by attaching a second mirror half way up the fiber as in Fig. 7 we cause the system to have a motion characterized by two degrees of freedom. We now require the two angles θ and ϕ to describe the complete motion. According to our statement of the validity of the equipartition theory, the system should now have a mean kinetic and potential energy equal to kT. Let us proceed to investigate the motion of such a system.

$$\left.\begin{aligned} T&=\text{K.E.}=\tfrac{1}{2}\{I_1\dot{\theta}^2+I_2\dot{\phi}^2\},\\ V&=\text{P.E.}=\tfrac{1}{2}\{k_1\theta^2+k_2(\phi-\theta)^2\},\end{aligned}\right\}\quad(24\text{a})$$

or if we let $\Theta=I_1^{\frac{1}{2}}\theta$, and $\Phi=(I_2)^{\frac{1}{2}}\phi$,

$$\left.\begin{aligned} T&=\tfrac{1}{2}(\dot{\Theta}^2+\dot{\Phi}^2),\\ V&=\tfrac{1}{2}(\omega_1^2\Theta^2+\omega_2\Phi^2-2\lambda\Theta\Phi),\end{aligned}\right\}\quad(24\text{b})$$

where $\omega_1^2=(k_1+k_2)/I_1$, $\omega_2^2=k_2/I_2$, $\lambda^2=k_2/(I_1I_2)^{\frac{1}{2}}$. ω_1 and ω_2 are, respectively, the natural frequency of oscillation of the upper mirror with the lower one fixed and of the lower one with the upper one fixed. Now if we write down the equations of motion,

$$d^2\Theta/dt^2+\omega_1^2\Theta-\lambda^2\Phi=0, \qquad (25)$$

$$d^2\Phi/dt^2+\omega_2^2\Phi-\lambda^2\Theta=0, \qquad (26)$$

and put $\Theta=Ae^{i\omega t}$ and $\Phi=Be^{i\omega t}$ the equation for ω is

$$(\omega^2-\omega_1^2)(\omega^2-\omega_2^2)-\lambda^4=0, \qquad (27\text{a})$$

which has the two roots

$$\omega_\pm^2=\omega_1^2+\omega_2^2/2\pm[\lambda^4+(\omega_1^2-\omega_2^2/2)^2]^{\frac{1}{2}}. \qquad (27\text{b})$$

These are the frequencies of the normal modes of vibration. If we now let x and y be the normal coordinates we get

$$\left.\begin{aligned} \Theta&=x\cos\alpha-y\sin\alpha,\\ \Phi&=x\sin\alpha+y\cos\alpha,\end{aligned}\right\}\quad(28)$$

$$\left.\begin{aligned} T&=\tfrac{1}{2}(\dot{x}^2+\dot{y}^2),\\ 2V&=(\omega_1^2\cos^2\alpha+\omega_2^2\sin^2\alpha-2\lambda^2\sin\alpha\cos\alpha)x^2\\ &\quad+(\omega_1^2\sin^2\alpha+\omega_2^2\cos^2\alpha+2\lambda^2\sin\alpha\cos\alpha)y^2\\ &\quad+[(\omega_2^2-\omega_1^2)\sin 2\alpha-2\lambda^2\cos 2\alpha]xy.\end{aligned}\right\}\quad(29)$$

But since x and y are normal coordinates, the term in xy must vanish, so for α we have

$$2\tan 2\alpha=\lambda^2/(\omega_2^2-\omega_1^2).$$

Using this value for α the coefficient of x^2 in $2V$ turns out to be equal to ω_-^2. That for y^2 is equal to ω_+^2, so the potential energy function is

$$V=\tfrac{1}{2}[\omega_-^2x^2+\omega_+^2y^2]. \qquad (30)$$

Now since the Hamiltonian is composed additively of a function of x and of y, the two coordinates will be independent statistically. So, we shall have $\overline{xy}=0$. Also by the equipartition theorem

$$\tfrac{1}{2}\omega_-^2\overline{x^2}=\tfrac{1}{2}kT \quad\text{and}\quad \tfrac{1}{2}\omega_+^2\overline{y^2}=\tfrac{1}{2}kT,$$

so

$$\overline{x^2}=kT/\omega_-^2 \quad\text{and}\quad \overline{y^2}=kT/\omega_+^2.$$

The motion of Θ is given by

$$\begin{aligned}\overline{\Theta^2}&=\overline{x^2}\cos^2\alpha+\overline{y^2}\sin^2\alpha\\ &=\tfrac{1}{2}(\overline{x^2}+\overline{y^2})+\tfrac{1}{2}(\overline{x^2}-\overline{y^2})\cos 2\alpha.\end{aligned} \qquad (31\text{a})$$

Similarly the motion of Φ is given by

$$\begin{aligned}\overline{\Phi^2}&=\overline{x^2}\sin^2\alpha+\overline{y^2}\cos^2\alpha\\ &=\tfrac{1}{2}(\overline{x^2}+\overline{y^2})-\tfrac{1}{2}(\overline{x^2}-\overline{y^2})\cos 2\alpha.\end{aligned} \qquad (31\text{b})$$

Substituting the values for $\overline{x^2}$, $\overline{y^2}$ and α from the preceding results we have

$$\overline{\Theta^2}=kT\frac{\omega_2^2}{(\omega_1^2\omega_2^2-\lambda^4)},\qquad(32a)$$

$$\overline{\Phi^2}=kT\frac{\omega_1^2}{(\omega_1^2\omega_2^2-\lambda^4)}.\qquad(32b)$$

Substituting for

$$\overline{\Theta^2}=I_1\overline{\theta^2},$$

we have

$$I_1\overline{\theta^2}=kT\frac{k_2/I_2}{(k_1+k_2/I_1)\cdot(k_2/I_2)-(k_2^2/I_1I_2)}=kTI_1/k_1$$

or

$$\tfrac{1}{2}k_1\overline{\theta^2}=\tfrac{1}{2}kT.\qquad(33)$$

Accordingly the upper mirror fluctuates just as if the lower body were not present. Similarly substituting for $\overline{\Phi^2}$ we get

$$\overline{\phi^2}=kT/k_2+kT/k_1.\qquad(34)$$

The lower body therefore fluctuates due to Brownian movement with its own fluctuations, as it would if the upper body were fixed, $\frac{1}{2}k_2\overline{\phi^2}=\frac{1}{2}kT$, plus those of the upper body $\frac{1}{2}k_1\overline{\theta^2}=\frac{1}{2}kT$. This motion of the lower mirror is exactly the same as that which it would have if the upper mirror were not present. The torsion constant K would then be given by $1/K=1/k_1+1/k_2$.

If we calculate the quantity

$$\overline{\Theta\Phi}=\overline{x^2}\sin\alpha\cos\alpha-\overline{y^2}\sin\alpha\cos\alpha$$

we obtain

$$\overline{\Theta\Phi}=kT(I_1I_2)^{\frac{1}{2}}/k_1=(I_1I_2)^{\frac{1}{2}}\overline{\theta\varphi},$$

which gives us the degree of correlation of the two motions.

$$\overline{\theta\phi}=kT/k_1\qquad(35)$$

or the mean value of $\overline{\theta\phi}$ is the same as $\overline{\theta^2}$.

$$\begin{aligned}\overline{(\phi-\theta)^2}&=\overline{\phi^2}-2\overline{\theta\phi}+\overline{\theta^2}\\&=kT(1/k_2+1/k_1-2/k_1+1/k_1)\qquad(36)\\&=kT/k_2,\end{aligned}$$

i.e., the fluctuations in twist of the second mirror relative to the first are the same as if the upper mirror were fixed.

Obviously, this two-mirror treatment may be extended to the case of n mirrors.

2C. Elastic rods and strings

Smoluchowski derived the relationship for a string fastened at one end:

$$\overline{\Lambda^2}=(2RT/N)(1/\pi a^2\rho g),\qquad(37)$$

where a and ρ are the radius and density of the string, respectively. Przibram while studying the Brownian movements of long chains of bacteria was able to prove the accuracy of this equation, when one of the ends of several of the chains was fastened to the cover glass. Houdijk, later, observed the Brownian movement of the lower ends of thin fibers which were fastened at their upper ends and allowed to hang vertically. In collaboration with Zeeman he studied filaments of 1μ platinum and 2μ quartz, and they were able to verify the above theoretical prediction. Van Lear and Uhlenbeck[13] have discussed the problem of a string fastened at both ends following the method of calculation first worked out by Ornstein, in which one calculates the mean square deviations when the surrounding medium is air. By improving upon all previous calculations of these quantities, one is able to calculate the mean square deviation of a given point x at any time t, after having started at $t=t_0$ with some given initial deviation C of that point. For the string it is found that

$$\overline{S^2}=\frac{kT}{\rho}F(x)+\left[C^2-\frac{kT}{\rho}F(x)\right]e^{-\beta t}\times\left[\frac{1}{F(x)}\frac{\partial G}{\partial t}+\frac{\beta}{2F(x)}G\right]^2,\qquad(38)$$

where

$$G(x,t)=\sum_{n=1}^{\infty}x_n^2\sin\omega_nt/\omega_n\lambda_n,$$

$$F(x)=\sum_{n=1}^{\infty}x_n^2(x)/\lambda_n,$$

and ρ and f are, respectively, the density and friction coefficient of the string, while $\beta=f/\rho$. For an elastic rod,

$$\overline{S^2}(L)=kT\frac{4L^3}{3\pi r^4E+0.98\pi r^3dgL^3},\qquad(39)$$

where $E=$ Young's modulus, $r=$ radius of the rod, $d=$ volume density and $L=$ the length. This formula is practically identical with that given by Houdijk, the only difference being the factor 0.98 instead of 1. Houdijk, using his own formula and experimental data, was able to determine N to within 5 percent of the accepted value.

If we are interested in the midpoint of a string fastened at each end (Fig. 8), the calculations become very much simplified. Let $u(x, t)$ be the transverse displacement at the position x and time t. The whole length of the string is

$$\int_0^l [1+(\partial u/\partial x)^2]^{\frac{1}{2}}dx \sim l+\tfrac{1}{2}\int_0^l (\partial u/\partial x)^2 dx.$$

Assuming that the tension is practically constant along the length, the work done against it in putting the string into the distorted shape is

$$\tfrac{1}{2}F\int_0^l (\partial u/\partial x)^2 dx$$

over and above the potential it has (Fl) when $u=0$. The normal modes of vibration are of the general form

$$U_n = a_n \sin(n\pi x/l).$$

The Brownian movement of the system we know will be such that

$$\frac{1}{2}\left(\frac{n\pi}{l}\right)^2 T\int_0^l \overline{a_n^2}\cos^2\left(\frac{n\pi x}{l}\right)dx = \tfrac{1}{2}kT, \tag{40}$$

$$\frac{n^2\pi^2}{l^2}\frac{1}{2}F\frac{l}{2}\overline{a_n^2} = \tfrac{1}{2}kT,$$

$$\overline{a_n^2} = \frac{2kT}{n^2\pi^2 F}l, \tag{41}$$

$$[\overline{a_n a_m} = 0 \text{ when } n \neq m].$$

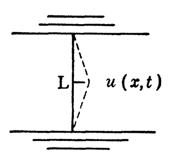
$$u = \sum a_n \sin\frac{n\pi x}{l}, \quad \overline{u^2} = \sum \overline{a_n^2}\sin^2\frac{n\pi x}{l}.$$

At the midpoint

$$\sin^2\frac{n\pi x}{l} = \begin{cases} 0 \text{ even } n \\ 1 \text{ odd } n \end{cases};$$

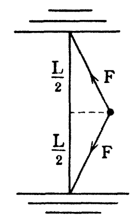
hence $\overline{u^2} = (\overline{a_1^2}+\overline{a_3^2}+\cdots)$

$$= \frac{2l}{\pi^2 F}kT\left(\frac{1}{1}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right).$$



Elastic String

FIG. 8.



Weightless String

FIG. 9.

Since the sum of this series is $\pi^2/8$

$$\overline{u^2} = kTl/4F \quad \text{or} \quad (4F/l)\overline{u^2} = kT. \tag{42}$$

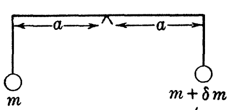
This result shows us that the Brownian movement fluctuations of the midpoint of such a string are the same as those of a single mass particle whose force constant is $4F/l$. It is interesting to note further that if all the mass of the string were concentrated at the midpoint the restoring force would then be $4Fu/l$ which states that the midpoint of a stretched string fluctuates just as it would if the rest of the string were without mass (Fig. 9).

From this general discussion of strings and rods it is clear that any instrument in which they form part of the system will be subject also to an ultimate limit of sensitivity as a result of Brownian movements. Einthoven, using a string galvanometer with a string 18 mm long and 0.2μ thick, with a 2000-fold magnification found that the system was always in motion. Just as Kappler had found with a suspended mirror, at pressures above 100 mm of mercury this motion is quite irregular while at low pressures it appears to be the same as the principal mode of vibration of the fiber. Tinbergen had shown with a typical instrument of this kind that measurements of direct currents less than 10^{-12} ampere were quite impossible due to these Brownian motion deflections. The agreement between predictions and observed values was again very good.

2D. Vibrating membranes

Let us assume that in a given instrument a membrane is to be used, for example for the measurement of a pressure, p, or a change of pressure δp. This membrane, with its individual boundary conditions and characterizing constants, will possess a position of equilibrium, and its motion can be described by one coordinate. Again, by applying the equipartition theorem we know that this membrane will vibrate constantly and that its average potential and kinetic energies will each be equal to $\frac{1}{2}kT$. As in the above cases the sensitivity as an instrument will be given by $\sigma=\delta x/\delta p$, where δx is the deflection produced by the pressure δp. The fluctuations about the position x_0 are therefore equivalent to slight changes of the pressure, and so every measurement will be in error by an amount δp. To find the actual value of this δp, one needs only to set up the potential energy equation of the deformed membrane and the equation for the free period and then to proceed to evaluate $\delta x/\delta p$ in terms of the constants of the instrument. There is, then, a minimum pressure which can be measured with an instrument such as an aneroid barometer.

An interesting question may be raised regarding the human ear. Is the intensity threshold, measured in units of M(inimum) A(udible) P(ressure), determined by the above-mentioned δp? The ear-drum is vibrating constantly as if the incident pressure were fluctuating by δp. This should produce a faint background noise beneath which no sounds of smaller intensity may be distinguished. If the ear were infinitely sensitive in other words, there would be, due to statistical fluctuations of the pressure, an ever present "noise."



Chemical Balance

FIG. 10.

2E. Chemical balance

Recalling our equation that the average error in making a measurement of a quantity q is $\overline{\delta q}=(2\epsilon A/B)^{\frac{1}{2}}$, it is only necessary to determine the constants A and B for a chemical balance in order to arrive at its theoretical sensitivity limit (Fig. 10). Ising[14] found this $\overline{\delta m}$ to be of the order of 10^{-9} g. Let $2a$ = length of the balance arms, $k_0=m_0a^2$ = moment of inertia, m and $m+\delta m$ the loads on the ends of the arms, A the restoring force of the acceleration of gravity and x the angle through which the balance arms turn. For a steady deflection $A\,\delta x=ag\delta m$ and as a result the sensitivity is given by

$$\delta x/\delta m=ag/A.$$

The undamped period of the balance T_0 is given by

$$T_0=2\pi a(m_0+2m/A)^{\frac{1}{2}}$$

or

$$A^{\frac{1}{2}}/ag=(2\pi/T_0g)(m_0+2m)^{\frac{1}{2}}. \tag{43}$$

Again $\delta x=\bar{x}=(2\epsilon/A)^{\frac{1}{2}}$. Substituting this and eliminating $A^{\frac{1}{2}}/a$ we have $\overline{\delta m}=(2\pi/T_0g)(2\epsilon(m_0+2m))^{\frac{1}{2}}$ for the average error in measuring the mass m. The average downward motion of one of the scale pans $\overline{\delta z}$ when carrying a load $\overline{\delta m}$, is

$$\overline{\delta z}=a\bar{x}=a\left(\frac{2\epsilon}{A}\right)^{\frac{1}{2}}=\frac{T_0}{2\pi}\left(\frac{2\epsilon}{m_0+2m}\right)^{\frac{1}{2}}. \tag{44}$$

This can be calculated from the equation

$$g\overline{\delta m}\;\overline{\delta z}=2\epsilon.$$

If we neglect m_0, the mass of the scale arms

$$\overline{\delta m}=(4\pi/T_0g)(\epsilon m)^{\frac{1}{2}}.$$

If we now put $\epsilon=2\times10^{-14}$ and $g=981$, we obtain the final formulas

$$\overline{\delta m}=1.81\cdot10^{-9}\,m^{\frac{1}{2}}/T_0\text{ gram}, \tag{45a}$$

$$\overline{\delta z}=2.25\cdot10^{-8}\,T_0/m^{\frac{1}{2}}\text{ cm}. \tag{45b}$$

If for example $m=1000$ g, $T_0=10$ sec., then $\overline{\delta m}=5.72\cdot10^{-9}$ g; $\overline{\delta z}=7.12\times10^{-9}$ cm.

Needless to say this particular limit $\overline{\delta m}=10^{-9}$ g has not yet been approached experimentally, as the present limit is about 10^{-5} g.

2F. Spring balance

In the same manner one can show that the sensitivity of a spring balance is limited by a zero unsteadiness $\overline{\delta x}$ such that again $\frac{1}{2}A\,\overline{\delta x^2}=\epsilon$. Again the sensitivity $\sigma=\delta x/\delta m=B/A$. For the deflection produced by δm we have the expression $g\delta m=-Adx$, where $-A=(m_1x_1/g)=(Mx/g)$, which gives

$$\sigma=\delta x/\partial m=g/A.$$

The free period of such a system is

$$T_0 = 2\pi(M_0+m/-A)^{\frac{1}{2}}.$$

From this

$$A/g = (2\pi/gT_0)A^{\frac{1}{2}}(M_0+m)^{\frac{1}{2}}. \tag{46}$$

Substituting this in the equation for the sensitivity, we get

$$\overline{\delta m} = \overline{\delta x}\frac{2\pi}{gT_0}A^{\frac{1}{2}}(M_0+m)^{\frac{1}{2}}$$

and by inserting the value $\overline{\delta x} = (2\epsilon/A)^{\frac{1}{2}}$,

$$\overline{\delta m} = (2\pi, gT_0)(2\epsilon)^{\frac{1}{2}}(M_0+m)^{\frac{1}{2}}. \tag{47}$$

Neglecting the weight of the spring and pan M_0 and substituting the numerical values for ϵ and g we have the final formula

$$\overline{\delta m} = 0.905\times 10^{-9}(m^{\frac{1}{2}}/T_0)\ \mathrm{g}. \tag{48}$$

The result then of Brownian movement in the case of either type of balance is an unsteady zero and, therefore, limited sensitivity. This unsteadiness is the same as would be produced if the mass m on the balance were fluctuating by an amount $\overline{\delta m}$.

2G. Other mechanical cases

In addition to the cases thus far discussed, a few other simple ones may be mentioned very briefly. The atoms which make up a steel meter stick are constantly in motion and so the length of the stick is, in the final analysis, a quantity which fluctuates statistically. Accepting this, it is clear that all measurements of lengths with such an instrument are also subject to uncertainties.

For similar reasons any measuring process relating to the phenomenon of expansion, whether the expansion be in the quantity to be measured or in some part of the measuring device, will necessarily be limited in its ultimate accuracy.

If it is desired to determine a length by means of some form of interferometer, one can see at once that here, too, a limit must be reached as the mirror supports will fluctuate with respect to their relative positions.

The last three examples have their explanations in the fact that a massive steel bar is quite analogous to a spring balance. Here again the length of the system, according to the laws discussed above, must fluctuate so that the mean potential energy is $=\frac{1}{2}kT$.

Due to such fluctuations a simple pendulum must have a fluctuating period due to the changes in its length. Neglecting this fact, however, the pendulum still does not give absolutely accurate values of time intervals, since as a result of having one degree of freedom, it must have a random mean kinetic energy equal to $\frac{1}{2}kT$.

Such phenomena as ionization, scattering of electrons, etc., determination of optical constants, mass-spectroscopic work, photography and many others need only be mentioned in passing. These clearly are statistical in nature and so the sensitivity of measurements of them or by them must be limited.

§3. Electric Systems

3A. Simple electric systems

In the booklet of de Haas-Lorentz[6a] discussed above, six such systems are treated. These deserve mention at this point.

1. A single conductor. Given a single electrical conductor with a self inductance L, resistance r, carrying a current i, a time of observation t, and F the e.m.f. produced by the thermal agitation of the molecules, one sees that

$$ri + L\,di/dt = F. \tag{49}$$

Letting $\lambda = 1 - rt/L$ and $X = \int F dt$ one gets

$$\overline{i^2} = \frac{\overline{x^2}}{L^2}\frac{1}{(1-\lambda)^2} = \frac{\overline{x^2}}{2rLt}. \tag{50}$$

If now one degree of freedom is assumed, $\frac{1}{2}L\overline{i^2}$ must be set equal to $\frac{1}{2}kT$, and so

$$\overline{x^2} = 2rkTt. \tag{51}$$

This shows the mean magnitude of the e.m.f. which exists in such a conductor as a result of statistical fluctuations. (This result, arrived at in 1913, is quite similar to that of the Johnson effect discussed below.)

2. Two conductors. If now the system consists of two electrical conductors which are coupled, one can show that it has an average magnetic energy given by

$$\overline{\tfrac{1}{2}L_1i_1^2+Mi_1i_2+\tfrac{1}{2}L_2i_2^2}=kT. \qquad (52)$$

This mean energy is equal to kT and not $\frac{1}{2}kT$, because the system is characterized by two degrees of freedom.

3. Fluctuations in charge of system. In addition to discussing the fluctuations in the quantity $\overline{i^2}$, which corresponds to the mean square velocity $\overline{u^2}$ of a particle, one might ask about the electrical analog of the mean square distance, $\overline{x^2}$, which the particle travels due to Brownian Movement. In other words, if e is the amount of electricity which passes a given cross section in a time interval t, one wishes to calculate $\overline{e^2}$. The result is obtained at once from the Einstein formula (6) if for α, the coordinate which describes the motion, the quantity of electricity which has passed since the beginning of the time interval is taken. Δ then becomes equal to e. Since $\dot{\alpha}$ is the current, and the force connected with α is the e.m.f. of the conductor, one must understand by B the current which is caused to flow by a steady e.m.f. of value 1. Therefore, if r = the resistance, $B=1/r$ and the result is at once seen to be

$$\overline{e^2}=(2kT/r)t. \qquad (53)$$

4. Tangent galvanometer. If the influence of the earth's field is neglected entirely, one can calculate the extent to which the needle of a tangent galvanometer must be constantly in motion due to the spontaneous currents in the coil. If Q is the moment of inertia and ω the angular velocity of the needle the final equation is

$$\overline{\omega^2}=kT/Q. \qquad (54)$$

This is obviously equivalent to $\frac{1}{2}Q\overline{\omega^2}=\frac{1}{2}kT$ = mean kinetic energy of the needle corresponding to its one degree of freedom.

5. Circuit containing capacity. Let C = capacity of a condenser, r and L equal the resistance and self-inductance of the circuit, ϕ the potential difference between the two condenser plates, and $+e$ and $-e$ the charges on these plates. In such a system

$$\tfrac{1}{2}L\overline{i^2}=\tfrac{1}{2}\overline{e^2}/C=\overline{x^2}/4rt. \qquad (55)$$

Here, exactly as must be expected, one finds that the electric and magnetic energies are equal and each equal to $\frac{1}{2}kT$. $[\overline{x^2}/4rt=\frac{1}{2}kT$ since in Case 1 (No. 1) (51), $\overline{x^2}=2rkTt.]$

6. Fluctuations of temperature of two parts of a conductor. If P and Q, two bodies of heat capacities c_1 and c_2, are connected by a metal conductor D, one wishes to know whether the temperatures of P and Q will vary due to statistical fluctuations in the system. Letting T be the equilibrium temperature and θ_1 and θ_2 be the deviations of P and Q from T, one finds that

$$\overline{\theta^2}=kT^2/c. \qquad (56)$$

7. System containing a thermocouple. If now P and Q represent the junctions of a thermocouple, which are connected by two wires D and D', it will be found that due to the spontaneous temperature differences of P and Q, a fluctuating electric current will flow through the system. One finds, in fact, that,

$$\overline{i^2}=kT/L, \quad \overline{\theta^2}=kT^2/c,$$

where

$$c=c_1c_2/(c_1+c_2). \qquad (57)$$

Here in this complicated system the mean magnetic energy is exactly the same as that found in the simple system No. 1 where $\frac{1}{2}L\overline{i^2}=\frac{1}{2}kT$. Also, the mean temperature differences are the same as those found in No. 6 where $\overline{\theta^2}=kT^2/c$. This demonstrates a very important result, namely the fact that the mean value of the spontaneous electric current is not increased by the accidental temperature differences of the two junctions of the thermocouple. Also the mean values of the temperature differences are not increased by the Peltier effects which must accompany the electric currents. As was shown in §1 there is associated with one degree of freedom of any system a mean energy of $\frac{1}{2}kT$.

3B. Electrometer

The natural sensitivity of a Hoffmann duant electrometer (Fig. 11) has been treated theoretically by Engel.[15] Again the starting point is the Einstein equation. Due to Brownian movement the potential energy of the system will be equal on the average to $\frac{1}{2}kT$, or for the electrometer (at 18°C)

FIG. 11.

$$\tfrac{1}{2}R\overline{\phi^2}=\tfrac{1}{2}kT=2\cdot 10^{-14}\text{ erg.} \qquad (58)$$

If R, the restoring force is expressed in volts² cm and ϕ is the angular displacement, then $\frac{1}{2}R\overline{\phi^2}=2\cdot 10^{-14}(300)^2$. R is given by

$$R=R_0/n=R_0-(b/2-a^2/C_{33})V^2, \qquad (59)$$

where the c_{ik} are the capacity coefficients and a and b are constants of the apparatus. A charge of Δe_3 is necessary to cause a deflection of $(\overline{\phi^2})^{1/2}$, where

$$\Delta e_3=C_{33}\frac{R}{aV}(\overline{\phi^2})^{1/2}=C_{33}\frac{2\cdot 10^{-7}\cdot 300}{aV}R^{1/2}.$$

With the help of the above equation for R, V can be eliminated, and one then gets

$$\Delta e_3=\frac{300\cdot 2\cdot 10^{-7}}{a(n-1)^{1/2}}C_{33}\left(\frac{b}{2}-\frac{a^2}{C_{33}}\right)^{1/2}\text{ volt cm}$$
$$=\frac{300\cdot 2\cdot 10^{-1}\cdot 7.06}{a(n-1)^{1/2}}C_{33}\left(\frac{b}{2}-\frac{a^2}{C_{33}}\right)^{1/2}\text{ “electrical quanta.”} \qquad (60)$$

For example, if $R_0=5.6$ volts² cm

$U=30.56$ volts, $(b/2-a^2/C_{33})=0.0043$ cm,

$C_{33}=5.45$ cm, then n will be 4.65,

and $\Delta e_3=200$ el. qu. This means of course that 200 el. qu. will produce a deflection equal to the mean square Brownian movement deflection, and therefore no charge smaller than several times this amount could be measured accurately in a simple measurement.

In a later paper Eggers[16] showed experimentally that such a limit exists. By substituting in the equation $R\overline{\phi^2}=kT$, either the value of R when measuring volts, $R_V=R_0-\frac{1}{2}bV^2$, or the value when measuring charge, $R_{Ch}=R_0-(b/2-a^2/C_{33})V^2$, one can state that the **average** fluctuations are

$$(\overline{\phi^2}_{\text{volt}})^{1/2}=\frac{8\cdot 10^{-4}}{(R_V)^{1/2}}\,\frac{\text{millimeters}}{\text{meter}}, \qquad (61a)$$

$$(\overline{\phi^2}_{Ch})^{1/2}=\frac{8\cdot 10^{-4}}{(R_{Ch})^{1/2}}\,\frac{\text{millimeters}}{\text{meter}}. \qquad (61b)$$

[In Egger's instrument $R_0=0.89$ volt² cm, $a=0.033$ cm, $b=0.014$ cm, $C_{33}=2.90$ cm.] These fluctuations are proportional to V^2 where V is the voltage applied to the duants. By measuring them carefully he determined k, $(k=R\overline{\phi^2}/T)$, to be $1.425\cdot 10^{-16}$ erg/degree with an accuracy of 5 percent.

Substituting the constants of this instrument in the formula of Engel a theoretical limit of sensitivity of 900 el. qu./mm was found, the theoretical limit being defined as that sensitivity at which the disturbances $(\overline{\phi^2})^{1/2}$ were equal to 1 mm/m. At an actual sensitivity of 850 el. qu./mm $(\overline{\phi^2})^{1/2}$ was $=1.00$ mm/m corresponding almost exactly to the predicted limit. The practical limit however was found to lie about 1500 el. qu./mm where measurements could be made with an accuracy of 5 percent. The time necessary to reach maximum deflection was 20 seconds.

3C. Galvanometer

(a) Probably one of the most interesting and important cases where the Brownian movement limits the sensitivity of a measuring instrument is that of the galvanometer. Since the motion of this instrument may also be described by one coordinate, the angular rotation ϕ, one knows in advance that it must be subject to zero fluctuations such that $\frac{1}{2}A\overline{\phi^2}=\frac{1}{2}kT$. Following the general procedure outlined above, its unsteadiness can be expressed in terms of apparent fluctuations in the current or voltage. Let the sensitivity of the instrument be defined by $\sigma=$deviation per unit current$=\delta x/\delta i$. Let us assume further from experimental experience that the smallest deflection which will be significant (i.e., which can be reproduced with sufficient accuracy to be of any value to the observer, see §1B) is given by

$$\delta x_{\min}=4\overline{\delta x}=4(\overline{\phi^2})^{1/2}.$$

Although we are selecting our smallest reliable deflection as $\delta x = a\overline{\delta x}$, instead of using the root mean square, $(\overline{\delta x^2})^{\frac{1}{2}}$, the correlation of the two is nearly unity; so it is clear then that for $a=4$, the least energy value we are willing to consider as certain is $16\cdot\frac{1}{2}kT$, and the probability that it is obtained by a stray kick rather than by our impressed force is quite small indeed. It is evident that if a smaller value of a be selected the uncertainty will be greater, requiring more readings to attain the same accuracy. The net result is that the total energy must add up to be the same.

Now let us write $\delta x/\delta i = C/A$ where $C\delta i$ is the deflecting moment produced by a current change δi. Therefore $C = dN/dx$ where N denotes the flux of magnetic induction through the galvanometer coil. Let us restrict ourselves to the usual case in which our instrument is critically damped; the mechanical damping is then negligible in comparison with the electromagnetic damping, and the calculations are simplified.

Let $p\,dx/dt$ = moment of the frictional forces,

$2\pi/\omega = \tau$ = period of oscillation,

for $p=0$ we have τ_0 = period of undamped oscillation $= 2\pi/\omega$,

$\lambda = p/2K$ = the damping constant,

K = moment of inertia of the system.

The differential equation of the motion of the system is then

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + (\lambda^2+\omega^2)x = 0. \tag{62}$$

Remembering that $\omega_0 = (A/K)^{\frac{1}{2}}$, we have $\lambda^2+\omega^2 = \omega_0^2$. Also if R be the resistance in the circuit

$$\lambda = \lambda_0 + \lambda_{el} = \lambda_0 + C^2/2KR, \tag{63}$$

where λ_{el} is the electromagnetic damping and λ_0 is the air, or any other damping. At the limit of aperiodicity, or at critical damping, $\lambda = \omega_0$ and $\lambda_0 + C^2/2KR = \omega_0$ from Eq. (63), and this determines C when the other quantities are known. Let us set $\lambda_0 = 0$ as this will make the change of flux, and hence C, a maximum. Then $\omega_0 = C^2/2KR$ and from above $\sigma = \delta x/\delta i = C/A$. Eliminating C **and K** we have the general formulae:

$$\text{current sensibility, } \sigma_i = \frac{\delta x}{\delta i} = \left(\frac{2R}{A\omega_0}\right)^{\frac{1}{2}}, \tag{64a}$$

$$\text{voltage sensibility, } \sigma_v = \frac{\delta x}{\delta v} = \frac{\sigma_i}{R} = \left(\frac{2}{AR\omega_0}\right)^{\frac{1}{2}} \tag{64b}$$

From Eqs. (19) and (20) we can calculate the disturbances set up by the Brownian motion, for we have $\overline{\delta x} = (2\epsilon/A)^{\frac{1}{2}}$. Hence we find from (64a) and (64b)

$$\overline{\delta i} = (\epsilon\omega_0/R)^{\frac{1}{2}} = (\pi kT/R\tau)^{\frac{1}{2}} \quad \text{the current equivalent} \tag{65a}$$

and

$$\overline{\delta v} = (\epsilon\omega_0 R)^{\frac{1}{2}} = (\pi kTR/\tau)^{\frac{1}{2}} \quad \text{the voltage equivalent} \tag{65b}$$

of the Brownian fluctuations, respectively.

Introducing $\omega_0 = 2\pi/\tau_0$ and $\epsilon = 2.0\times10^{-14}$ erg, we can reduce these last two equations to practical units giving

$$\overline{\delta i'} = 1.12\cdot10^{-10}(1/RT_0)^{\frac{1}{2}} \text{ amperes,} \tag{66a}$$

$$\overline{\delta v'} = 1.12\cdot10^{-10}(R/T_0)^{\frac{1}{2}} \text{ volts.} \tag{66b}$$

And since we have $\delta x_{\min} = 4\overline{\delta x}$, the smallest accurately observable quantities will be four times as great.

Upon reducing the deflections obtained by Moll and Burger (see §1A) to their voltage equivalence, Ising found an observed $\overline{\delta v'} = 9.22\times10^{-10}$ volt, compared with a value of 6.3×10^{-10} volt as calculated from (66b) for their particular circuit. This result shows that any single measurement of current or voltage will be uncertain at least to the extent just stated.

(b) It was pointed out by Zernike[17] that the same results could be obtained by considering the galvanometer circuit purely from an electrical standpoint. This means then, if Brownian fluctuation is inherent in the electrical circuit independent of mechanical effects and of equal magnitude, that such precautions as evacuation or cooling of the galvanometer alone are useless; for these steps cannot reduce the net effect of the random disturbances if the remainder of the circuit be at room temperature.

Let us assume a circuit of resistance R, self-induction L, kept for simplicity at a uniform temperature T. As a constant current is without significance, we may assume $\bar{i}=0$. From the equipartition theory, we have that the energy of spontaneous current fluctuations is

$$\tfrac{1}{2}L\overline{i^2}=\tfrac{1}{2}kT,$$

as was shown in §3A. These kicks will occur far too rapidly to permit their instantaneous measurement by any ordinary current recorder.

It is now useful to imagine these currents to be set up by a haphazard e.m.f., E. It is then possible to write down as the differential equation of our circuit

$$Ldi/dt+Ri=E,$$

which, through integration, yields for the value of the current at any instant, t

$$i_t=\frac{1}{L}e^{-Rt/L}\int_0^t Ee^{Rt/L}dt+i_0e^{-Rt/L}. \quad (67)$$

To obtain the effect of one disturbance upon a later one, we multiply Eq. (67) by i_0, and since i_0 is independent of later e.m.f.'s, we obtain

$$\overline{i_ti_0}=\overline{i_0{}^2}e^{-Rt/L}. \quad (68)$$

So, two kicks i_0 and i_t, separated by a time interval t are correlated, the factor of correlation being $e^{-Rt/L}$. This relationship becomes vanishingly small as $t\gg R/L$.

It is now possible to calculate the average current strength j over a time τ

$$j=\frac{1}{\tau}\int_0^\tau idt$$

and

$$\overline{j^2}=\frac{2}{\tau^2}\int_0^\tau dt\int_t^\tau i_ti_t{}'dt'$$

$$=\frac{2\overline{i^2}}{\tau^2}\int_0^\tau dt\int_0^{\tau-t}e^{-Rx/L}dx. \quad (69)$$

Again, if $\tau\gg R/L$ we can substitute ∞ for the values of the upper limit of the integral of the last integrand, without introducing any appreciable error. We can then write Eq. (69) as

$$\overline{j^2}=(2\overline{i^2}/\tau^2)\tau L/R=2kT/R\tau, \quad (70)$$

which is in agreement with Ising's value of Eq. (65) to within a numerical factor. This agreement may be made complete if we select a galvanometer which is critically damped, as was done in the mechanical example above (to simplify the results, and also because this is the usual laboratory type). Such an apparatus averages out the current to a mean value "u," rather than gives the momentary current strength i. It can be represented by an equation such as

$$d^2u/dt^2+2\lambda du/dt+\lambda^2u=\lambda^2i, \quad (71)$$

where conditions and units are so chosen that for steady currents $u=i$. Integrating we obtain for a circuit which has been closed a long while

$$u=\lambda^2\int_0^\infty i(-t)te^{-\lambda t}dt \quad (72)$$

and for the mean square

$$\overline{u^2}=2\lambda^4\int_0^\infty t^2e^{-2\lambda t}dt\int_0^\infty i(-t)i(-t+x)dx. \quad (73)$$

Utilizing the correlation factor of Eq. (68) and proceeding as we did above in evaluating $\overline{j^2}$ in (69), we have the result that,

$$\overline{u^2}=2\lambda^4\cdot\frac{1}{4\lambda^3}\cdot\frac{\overline{i^2}L}{R}=\lambda\frac{L\overline{i^2}}{2R}=\frac{\pi kT}{R\tau} \quad (74)$$

upon introducing $\tau=2\pi/\lambda$. This is identical with the result Eq. (65) obtained on the mechanical basis.

(c) The above treatments which are due to Ising and Zernike, and which lead to the same result for the Brownian motion of a galvanometer coil, show that this theoretical value agrees qualitatively with the experimental results obtained by Moll and Berger. A direct proof that these fluctuations are truly due to Brownian motion has been offered by Ornstein[17a] and his coworkers. The classical theory shows the dependence of the motion upon temperature. It is necessary, then, only to carry this general theory over to the specific case of the galvanometer, and to compare the calculations with the experimental results obtained by observing the change of the inherent fluctuations of a galvanometer with change of temperature. Practically, it is hard to change the temperature of the galvanometer itself; it is easier to insert a resistance coil whose temperature may be varied at will.

This requires the development of a formula to cover the case of a system of two components which are not at the same temperature.

For a system at one temperature, subject to a varying e.m.f., E, we can write as our equation

$$Ldi/dt+Ri=E, \tag{75}$$

where we have again, since E is perfectly random, $\bar{E}=0$.

For our system, which is no longer at one uniform temperature, we consider two inductances L_1 and L_2, and two resistances R_1 and R_2 which are at temperatures T_1 and T_2. Now it is known that if $F(z)$ be the spontaneous Brownian force at a time z on a particle and $F(z+\psi)$ be the force at time $z+\psi$ then $\overline{F(z)F(z+\psi)}\neq 0$. The correlation will of course be greatest when ψ is small. It can be shown very simply that $\int F(z)F(z+\psi)d\psi$ is not dependent upon z or ψ and is a constant. This value of the integral which we may designate by $\overline{FF}$, is $2\beta kT$. On the same lines we find for the case of an electrical circuit $\overline{EE}=2RkT$.

For a two-coil system at the same temperature, the accidental E_1 for the first coil would have the values $\overline{E_1}=0$ and $\overline{E_1E_1}=2R_1\cdot kT$, and for the second coil $\overline{E_2}=0$ and $\overline{E_2E_2}=2R_2\cdot kT$. We shall assume for our "two-temperature" system that the same relations hold true, with the additional hypothesis that $\overline{E_1E_2}=0$, as the Brownian fluctuation in one coil will be completely independent of the motion in the other.

The equation for our circuit becomes

$$(L_2+L_2)(di/dt)+(R_1+R_2)i=E_1+E_2. \tag{76}$$

Upon integrating we obtain for i

$$i=\exp\left[-\frac{(R_1+R_2)t}{L_1+L_2}\right]\left\{i_0+\int_0^t (E_1(z)+E_2(z))\cdot\exp\left[\frac{R_1+R_2}{L_1+L_2}z\right]\cdot dz\right\} \tag{77}$$

and for the value of $\overline{i^2}$, neglecting terms with i_0 as they are very small for appreciable values of t, we obtain

$$\overline{i^2}=(L_1+L_2)^{-2}\cdot\exp\left[-\frac{2(R_1+R_2)t}{L_1+L_2}\right]\cdot\int_0^t\int_0^\tau \overline{\{E_1(z)+E_2(z)\}\{E_1(x)-E_2(x)\}}\cdot\exp\left[\frac{R_1+R_2}{L_1+L_2}(z+x)\right]\cdot dzdx$$

$$=\frac{1}{L_1+L_2}\cdot\frac{1}{R_1+R_2}(2kR_1T_1+2kR_2T_2) \tag{78}$$

using our results that $\overline{E_nE_m}=\begin{cases}0 & \text{for } n\neq m\\ 2kRT & \text{for } n=m\end{cases}$. Therefore

$$\tfrac{1}{2}(L_1+L_2)\overline{i^2}=\tfrac{1}{2}k(R_1T_1+R_2T_2)/(R_1+R_2). \tag{79}$$

For $T_1=T_2$ we get the usual equipartition value; otherwise, we obtain a weighted value in which the coil of higher resistance-temperature product plays the dominant rôle.

We now are able to estimate the Brownian motion of the galvanometer system. Let R_1 be the resistance of the galvanometer coil, which is at temperature T_1; R_2 and T_2 be the resistance and temperature respectively of the external coil. Neglecting the inductance L, the equation of motion and the electrical motion will be, respectively,

$$Kd^2\theta/dt^2+\beta d\theta/dt+\alpha\theta+qi=F,\qquad (R_1+R_2)i-qd\theta/dt=E_1+E_2.$$

Eliminating i, we obtain as our galvanometric equation

$$K\frac{d^2\theta}{dt^2}+\left(\beta+\frac{q^2}{R_1+R_2}\right)\frac{d\theta}{dt}+\alpha\theta=F-\frac{q}{R_1+R_2}(E_1+E_2). \tag{80}$$

If we assume as above

$$\overline{FF}=2\beta kT_1,\qquad \overline{FE_1}=0,$$
$$\overline{E_1E_1}=2R_1kT_1,\qquad \overline{FE_2}=0,$$
$$\overline{E_2E_2}=2R_2kT_2,\qquad \overline{E_1E_2}=0,$$

we obtain

$$\alpha\overline{\theta^2}=\left[\alpha\left(\beta+\frac{q^2}{R_1+R_2}\right)\right]^{-1}\left\{\overline{FF}+\left(\frac{q}{R_1+R_2}\right)^2(\overline{E_1E_1}+\overline{E_2E_2})\right\} \tag{81}$$

$$=k\left[\beta+\frac{q^2}{R_1+R_2}\right]^{-1}\left\{\beta T_1+\left(\frac{q}{R_1+R_2}\right)^2(R_1T_1+R_2T_2)\right\}. \tag{82}$$

For the case where the air damping β is very small in comparison with the electromagnetic damping q, so that $\beta<q^2/R_1+R_2$, we obtain

$$\tfrac{1}{2}\alpha\overline{\theta^2}=\tfrac{1}{2}k(R_1T_1+R_2T_2/R_1+R_2). \tag{83}$$

This states that the Brownian fluctuation as measured in potential energy of rotation is exactly equivalent to the energy of the current fluctuation (79). The results can be checked experimentally by varying T_2, and observing the corresponding change in the disturbance of the zero. Ornstein used a coil of 400 ohms resistance for R_2, and found that the ratio of the amplitudes of his zero fluctuation at liquid air temperature and room temperature was 0.59; the theoretical value was 0.61. This is excellent verification that the Brownian movement is the inherent cause of the galvanometer's uncertainty.

§4. Effects Related to Brownian Motion

Thus far we have centered our attention upon those cases in which the measuring instrument itself exhibited fluctuations. In this section it is proposed to discuss a few cases in which the fluctuations occur in the quantities which are to be measured. Although some of the phenomena discussed are not strictly of a Brownian nature, they are considered to be of sufficiently allied interest to be included in this review.

4A. Shot effect

The thermionic current in a vacuum tube is not a smooth flow of electricity, but is subject to rapid and irregular changes in magnitude. These fluctuations were discovered by Schottky[18, 19] and were called by him the Schrot or small-shot effect. They are caused by random emission from the cathode, and show their presence by voltage or current fluctuations in any circuit in which the tube is connected. When sufficient amplification is used to make these variations audible in a telephone receiver they cause a continual sound of indefinite frequency; so that a continual background of noise is heard.

A mathematical treatment of the problem is possible if we assume that the passage of any one electron from cathode to anode is entirely independent of the passage of any other electron. We can then interpret this condition by the simple laws of probability of an unordered distribution of similar events. Let the average number of outgoing particles per second be N (the average to be taken over a long interval of time), and let the number of particles which happen to be emitted in any particular interval of t seconds be n_t. In general n_t will not equal $N\cdot t$, the average number of particles for t seconds. The deviation from the mean value for the interval is given by

$$\delta_t=n_t-Nt. \tag{84}$$

If a large number of particles is contained in this assemblage, and if our observational period extends over t seconds, we have from statistical theory that

$$\overline{\delta_t^2}=Nt. \tag{85}$$

Let $i_0 = e \cdot N$ be the mean value of the current taken over a long time, and let i_t be the average value of the current over a single interval of t seconds. Then the average current fluctuation for this interval is given by

$$\Delta i = i_t - i_0 = e\delta_t / t \tag{86}$$

and for the mean square variation we obtain

$$\overline{\Delta i^2} = (e^2/t^2)\overline{\delta_t^2} = (e^2/t^2)Nt = i_0(e/t). \tag{87}$$

Results in good agreement with this theory have been obtained by Rajewsky[20] for the case of a photo-cell. Here he observed the mean relative fluctuation $f = \delta_t / N$. For an average rate of emission of 42.4 electrons per minute he obtained the results given in Table II.

TABLE II.

t	$f_{\text{experimental}}$	$f_{\text{calculated}}$
2 sec.	0.179	0.109
6	.061	.063
10	.042	.048
20	.019	.034

However, difficulties in the experimental verification with a discharge or vacuum tube arise as it is not possible in these cases to measure i directly. Schottky and Johnson[21] have shown that the difficulty may be overcome by making a Fourier analysis of the shot effect current fluctuations in a discharge tube over a long interval of time $T \gg t$. One can then show that the fluctuations will affect a resonance circuit, of eigenfrequency $1/t$, in parallel with the discharge tube as if there were a pure sinusoidal alternating current in the tube of equal period t and effective amplitude $a = (\pi e i_0 / t)^{\frac{1}{2}}$.

To measure a from the energy which is absorbed in the resonator it is necessary to know the damping of the oscillating circuit. The amplitude a and the eigenfrequency $1/t$ can be found from the constants (capacity, self-inductance and ohmic resistance) of the resonance circuit, and i_0 can be measured directly. This gives a method of determining the value of e, the charge on the electron. This was first attempted by Hartmann,[22, 23] but his results gave only qualitative agreement with the value as determined by Millikan. Improved methods by Hull and Williams[24] led to a result of $e = 4.756 \times 10^{-10}$ e.s.u. which deviates from the oil-drop value by about 1/3 percent.

Johnson[25] found that under certain conditions the voltage fluctuations did not obey the theory of the shot effect. These deviations occurred chiefly at low frequencies especially with oxide-coated filaments under operating conditions such that space charge effects were not noticeable. Johnson attributed these abnormalities to a fluctuating surface change, and this was substantiated by Schottky,[26] who showed that this Flicker effect could be established as being dependent upon the life-times of foreign atoms upon the cathode surface. In general, the magnitude of the disturbances in an amplifying tube in ordinary use is distorted either by the Flicker effect or by space charge effects so that its value cannot be predicted by measurements on the true shot effect. In addition, in a well-designed amplifier, Brownian fluctuations set up in the grid circuit by thermal agitation may be the predominant source of noise.

4B. Johnson effect

The problem of thermal agitation of electricity in conductors has been studied experimentally by Johnson,[27] and theoretically by Nyquist.[28] Let us consider the simple circuit shown below in Fig. 12 which consists of two equal resistances

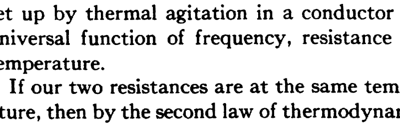


FIG. 12.

separated by a long non-dissipative transmission line, having an inductance L and a capacity per unit length of C, so chosen that $R = (L/C)^{\frac{1}{2}}$. Now, it may be shown quite easily that an e.m.f. set up by thermal agitation in a conductor is a universal function of frequency, resistance and temperature.

If our two resistances are at the same temperature, then by the second law of thermodynamics they must also be in equilibrium as regards total power transfer. This must also be true for any particular frequency; for if we still **have** temperature equilibrium and if we assume that one conductor delivers more energy at a frequency range $\Delta\nu$ than it receives, we may connect a non-dissipative network such as a resonator between the two conductors in such a manner as to interfere more with the transfer of energy in the range $\Delta\nu$ than in any other. Since there is equilibrium of power transfer before insertion of this network, there will be an unequal flow after it is in position. But our conductors are still at the same temperature, and we should violate the second law. We have, therefore, equilibrium of power transfer at any frequency.

Under the conditions of our ideal circuit above, our lines possess the characteristic impedance R, and there is no reflection at either end of the line. Let the length of the line be l, and the velocity of propagation be v, and let the absolute temperature after equilibrium is established be T. Then we have two trains of energy traversing this transmission line, one being the power delivered from conductor I and absorbed by conductor II, and the other which runs in the opposite direction. Now, let the line be isolated at any instant after equilibrium has been reached. We shall then have complete reflection at the two ends, and the energy which was on the line at the time the circuit was broken is trapped. This permits a description of our system in terms of its vibrations at its natural frequencies, for we have established the conditions required for standing waves. The lowest frequency of the voltage wave will be $v/2l$ and the higher ones will be $mv/2l$ where m is an integer.

Now consider a frequency range of width $d\nu$. The number of modes of vibration (or degrees of freedom) lying within this range is equal to $2ld\nu/v$, assuming that l is large enough to make the expression a large number. If this be so, we can assign to each degree of freedom the property that its average energy is a definite quantity, which on the average will be kT, by the law of equipartition. The total energy of the vibrations within the frequency range is then $2lkTd\nu/v$. As our line is non-reflecting, this is the energy within the given frequency range which is transferred from the two conductors to the line during the time of transit l/v, and the average power transferred from each conductor is $kTd\nu$. The e.m.f. due to thermal agitation in either of the conductors produces a current which is obtained by dividing the e.m.f. by $2R$, and the power transferred to the opposite conductor is obtained by the usual I^2R product. If we call $E^2d\nu$ the square of the e.m.f. within the range $d\nu$, we have

$$E^2d\nu = 4RkTd\nu. \tag{88}$$

We are now able to extend these results for conductors which are pure resistances to a network built of impedance members as in the following Fig. 13. Here we have a pure resistance

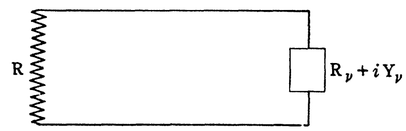


FIG. 13.

R connected to an impedance $R_\nu + iY_\nu$ which is at the same temperature T, where R_ν and Y_ν are some functions of the frequency. By reasoning similar to that above, it may be deduced that the power transfer from conductor to impedance network is equal to that from the impedance to the conductor. By simple current theory the first is:

$$E^2R_\nu d\nu/[(R_\nu+R)^2+Y_\nu^2]$$

and the second is:

$$E_\nu^2Rd\nu/[(R_\nu+R)^2+Y_\nu^2],$$

where E_ν^2 is the square of the voltage at the frequency ν. It follows from the equality of these two expressions that the relationship of Eq. (88) holds for any particular frequency interval and

$$E_\nu^2d\nu = 4R_\nu kTd\nu. \tag{89}$$

To set this relationship in terms of experimentally observable quantities let $M(\omega)$ be the ratio of the output current to the applied input voltage at frequency ν, i.e., the transfer admittance of the network from the member in which the e.m.f. originates, to the member in which the resulting current is measured. Setting $\omega = 2\pi\nu$ we can rewrite R_ν as $R(\omega)$, the resistance in which the e.m.f. is generated. The square of the measured current within the range $d\nu$ is then

$$\overline{i^2}d\nu = E_\nu^2|M(\omega)|^2 d\nu = \frac{2}{\pi}kTR(\omega)|M(\omega)|^2 d\omega \quad (90)$$

and the square of the total current is obtained by integration from 0 to ∞ so

$$\overline{i^2} = (2/\pi)kT\int_0^\infty R(\omega)|M(\omega)|^2 d\omega. \quad (91)$$

Johnson[27] has determined the value of k from Eq. (91), and his results are in qualitative agreement with the theory.

This theory of thermal agitation in conductors makes three suggestions regarding the reduction of noise in amplifiers. The first is the use of a low input resistance; this cannot always be done, however, due to limitations set by the apparatus which is generating the voltage phenomena that are being investigated. Secondly, the temperature may be lowered. Thirdly, the frequency range may be made very narrow by using a sharply tuned circuit.[29]

A practical example of the effect of thermal agitation is the case of a good amplifier in which the noise due to this agitation is greater than that of either the Shot or Flicker effects. The latter can usually be suppressed in a tube which is working so that there is full space charge limitation of the current. The amount of disturbance which is present with grid connected directly to the filament may be determined roughly in terms of a resistance R_g which, connected between grid and filament, would cause an equal amount of noise due to thermal agitation. Then $\overline{V^2} = W(R+R_g)$ where R is the resistance of the conductor in the circuit, where $\overline{V^2}$ is the mean squared voltage fluctuation and W is the power produced thereby.

In the region of voice frequency, 60 to 5000 cycles, we find for a given tube $R_g = 1.5\times 10^5 i_p/\mu$ experimentally, where μ is the effective amplification and i_p is the space current in milliamperes. For a tube in which $\mu = 15$, and $i_p = 0.5$ milliampere, the minimum noise corresponds to the thermal agitation of $R_g = 5000$ ohms. For a set-up of constant amplification in a region ν_2 to ν_1 and a pure resistance circuit we can rewrite Eq. (91) as

$$\overline{i^2} = 4kTR(\nu_2 - \nu_1)M^2 \quad (92)$$

and at the limit where the output current is just at the noise level

$$\overline{V^2} = 4kTR(\nu_2 - \nu_1) = WR. \quad (93)$$

For the vocal range $\nu_2 - \nu_1 = 5000$, and $W = 0.8 \times 10^{-16}$ watt. The corresponding voltage is then of the order of magnitude of 10^{-6} volt.

Hafstad[30] gives an excellent account of the limitations imposed upon a more general circuit in which the input-impedance varies with frequency. He finds fluctuations in the grid circuit of a sensitive FP-54 tube in agreement with those to be expected from either the Shot or Johnson effects. (Cf. Fig. 14.)

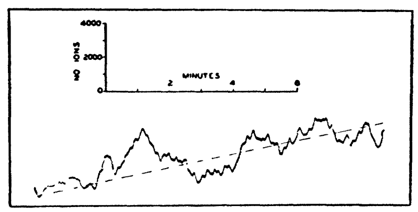


FIG. 14. Fluctuations in grid circuit of FP-54 tube. Scale 250,000 mm per volt. (From Hafstad.[30])

In passing, it is worth commenting upon the use of a bolometer in the grid circuit of a vacuum tube, as a radiometric device. Inasmuch as the bolometer element is ordinarily of high resistance, it produces a very high Brownian kick in the usual galvanometer circuit which is used—generally a Wheatstone bridge. To obtain small heat capacity and consequently a large temperature rise to produce the desired large resistance change, one must necessarily reduce the amount of material in the element, and the resistance increases accordingly. It has been often suggested that a sufficiently high resistance bolometer be placed in the grid circuit of a vacuum tube. However, from Eq. (92), which is a modified form of the general equation, we see that thermal agitation sets up a voltage fluctuation which is dependent upon $R^{1/2}$, precisely as for a simple galvanometer. There is then no advantage to be gained from the standpoint of reduction of Brownian fluctuations by using a vacuum tube amplifier in place of a galvanometer as a measuring device. From the standpoint of construction it is rather remarkable that thus far the experimental efficiencies of the thermocouple and bolometer considered as heat engines are the same: for one is an instrument which develops power by means of thermoelectric currents, and the other varies the amount of power transfer by a pure resistance change.

4C. Shot effect of photons

Suppose the eye were to be used as an instrument to detect the presence of weak sources of light. The question may be asked whether or not there would be a natural limit to this measuring process. The answer is, of course, yes. Quite analogously to the Shot effect just discussed, the emission of photons is quantized and so must be subject to statistical fluctuations. These fluctuations will mean, obviously, that whatever instrument is employed to measure the absolute intensity of a source of light or to detect a weak source will be limited in its performance.

One of us, together with M. Czerny,[31] investigated experimentally the possibility of seeing these fluctuations with the unaided eye. A strong source of light having a spectral distribution very nearly the same as that corresponding to the sensitivity of the darkness adapted eye was set up and its intensity measured as exactly as possible. The light from this source was then greatly reduced in intensity and allowed to fall upon a metal screen containing about 50 small holes. With the head upon a rest about 50 cm away from this screen the number of quanta entering the eye per second from each of the weak sources could be calculated. These sources were so weak that they could be seen only after the eyes had been 15 or 30 minutes in total darkness. The field of small sources was seen to flicker and twinkle much the same as a field of stars.

The authors realized the complexity of effects entering into an experiment of this type. Could the eye detect such a small number of quanta that the statistical fluctuations would form an appreciable percent of the absolute intensity? Could the eye then detect these percentage changes of intensity? Would the physical condition of the observer play a great rôle? How many physiological effects could produce such a twinkling effect under the conditions used? All of these questions and more had to be considered. As a result however of threshold determinations and calculations of the expected fluctuations the authors stated in their concluding remarks that "One can then say, that the eye is developed right up to that limit, where the Shot effect of photons makes itself noticeable."

§5. Conclusion

It has been shown above that practically every process which might be used to make physical measurements is in some way limited by Brownian motion. In conclusion one might state simply that matter and energy are divided into small units (atoms, molecules, electrons, photons, etc.) and that the natural limit of sensitivity is reached by any method of measuring as soon as that method begins to detect the individual effects of these small units.

§6. Appendix

6A. Special cases of galvanometer circuits

Lately there have been several galvanometric circuits devised to obtain high sensitivity, for although the limit of sensitivity is clearly defined by Brownian fluctuations, in practice there are disturbances present of order of magnitude larger than that set by molecular motion. For example, the thermorelay under proper shielding with no source of e.m.f. but only a fixed resistance in the circuit of the first galvanometer will be quite steady to external disturbances. The introduction of a sensitive radiometric device such as a good thermocouple in the circuit of the first galvanometer, however, invariably so magnifies the problem of shielding from electromagnetic and thermal radiations as to make the instrument less sensitive than is theoretically attainable. Ising[32] has built a very ingenious galvanometer which is so small that it lies upon the stage of a microscope, and the deflections may be observed to the Brownian limit directly without use of any other magnifying apparatus. This scheme is still subject to the difficulties of the relay as regards external troubles. A few years ago a very significant suggestion was set forth by Pfund,[33] who proposed a system of underdamped galvanometers to be set into resonance by an alternating current. The proposed advantage was the well-known lack of sensitivity of a resonating system to mechanical disturbances, in addition to its insensibility to any current disturbances of period different from that of the impressed current. The problem was worked out under Pfund by Hardy,[34] and the instrument—the resonance radiometer—was used very successfully in mapping infrared spectra. The device consists of two identical galvanometers which are highly underdamped. The radiation falling upon the thermocouple in the first circuit is interrupted periodically so that the thermocouple is illuminated for one-half of a period, and darkened the other half. This produces a current whose strength rises from zero to a maximum with the proper timing to set the galvanometer in resonance. The deflections of the first galvanometer are magnified in the second by a very neat adaptation of the Moll and Burger relay. (For complete details see Hardy's paper.) With no unusual shielding against temperature drifts or stray electric fields, it was found possible to record the swinging amplitudes at magnifications 200 times as high as those possible with the critically damped instrument. Mechanical vibrations due to motor trucks and the usual building tremors made very little impression upon the instrument, despite the fact that it was mounted directly upon the wall of a basement room, and no vibration-free supports were used. Hardy arrived at a sensitivity of some 4×10^{-11} volt (later corrected to about 4×10^{-10} by Van Lear) and stated that the instrument had lowered the Brownian fluctuation limit some 200-fold. This was, of course a misstatement, as was pointed out by Czerny,[9] who showed that in fact the resonance system was theoretically less sensitive than a critically damped system, and that the great sensitivity was due only to the large time of response: it required 40 seconds in place of the usual 6 or 7 to attain maximum amplitude. Czerny pointed out that the instrument magnified all disturbances which were not mutually independent. Subsequently Firestone[35] carried Hardy's calculations further to fit the case of an instrument of rapid response—about 6 seconds. He arrived at the conclusion that the Pfund resonance radiometer reduced drifts by a factor of several hundred, since the actual drift involved in reading must occur during one swing—i.e., a half-period—as amplitudes to both sides of the zero are read, while the total time of response is some 10 to 11 times larger; but a Moll relay of equal time of response had only one-half the Brownian error. Firestone remarked that the Brownian motion of a swinging system affects the amplitude rather slowly, so that it will appear to be much more constant over a short interval, say 3 complete periods, than it will over a longer interval of some 10 complete periods. Curves bearing this out have recently been published by Hardy.[36] Firestone described a very clever arrangement of short period which is drift-free. He built a periodic radiometer using two critically damped galvanometers, such that the total response time was only 6 seconds. To reduce drifts, two condensers are placed in series with the amplifying circuit which effectually stops all drifts while passing the periodic deflection. The Brownian error is of course again greater than that of a Moll relay, but the experimental error due to drifts is eliminated. To approach the resonance radiometer in sensitivity, it is necessary only to take a sufficient number of readings to make the actual time of reading large. The specific problem of the resonance radiometer has been worked in complete detail by Van Lear.[37] Here he finds that the actual sensitivity of the resonance radiometer is 1/3 that of a critically damped instrument, and that of Firestone's instrument about 1/7.

6B. Observational methods

(a). It remained for Zernike[38] to work out the general problem of the statistical influence of observational methods upon the measured Brownian error. This leads immediately to the estimation of the error for different types of damping. The problem of the effect of damping upon the Brownian motion has been treated somewhat more specially by Czerny[9] and by Ising,[39] but they introduced simplifications which did not permit as complete a statement of the problem, and which in some instances yielded misleading information.

The problem of observational method is concerned primarily with a determination of the correlation of the Brownian fluctuations of successions of readings taken on a galvanometer.

A knowledge of this correlation permits an expression of the accuracy of different combinations of sets of readings, and the best method of selection of readings for different damping cases.

(b). Let us consider a system subject to random fluctuations. Then the mean square value of the simple case of two Brownian readings X_1 and X_2, which are made in an interval of time $0 \rightarrow t$ seconds, is given by

$$\overline{(X_1+X_2)^2}=\overline{X_1^2}+\overline{X_2^2}+2\overline{X_1X_2}=2\overline{X^2}(1+\rho), \tag{94}$$

where $\rho=\overline{X_1X_2}/\overline{X^2}$ is the correlation coefficient which determines the influence of X_1 upon X_2. We wish to calculate ρ as a function of t so that we can determine the influence of one "kick" upon a subsequent one. Let us begin by writing the equation of motion of the system where as usual $F(t)$ is the varying Brownian force:

$$K\ddot{X}+\beta\dot{X}+\alpha X=F(t). \tag{95}$$

By introduction of the undamped frequency $\omega=(\alpha/K)^{\frac{1}{2}}$, the damping constant for the critical case $\beta_0=2(K\alpha)^{\frac{1}{2}}$ and $n=\beta/\beta_0$; our equation becomes

$$\ddot{X}+2n\omega\dot{X}+\omega^2X=F(t)/K. \tag{96}$$

This can be integrated by the method of variation of constants, and for the initial conditions X_0, and $\dot{X}_0$, one obtains the solution

$$x(t)=\frac{X_0}{\lambda_2-\lambda_1}(\lambda_2e^{-\lambda_1t}-\lambda_1e^{-\lambda_2t})+\frac{\dot{X}_0}{\lambda_2-\lambda_1}(e^{-\lambda_1t}-e^{-\lambda_2t})+\frac{1}{K(\lambda_2-\lambda_1)}\int_0^t\{e^{-\lambda_1(t-S)}-e^{-\lambda_2(t-S)}\}F(S)dS, \tag{97}$$

where the λ's are the roots of the equation

$$\lambda^2-2n\omega\lambda+\omega^2=0. \tag{98}$$

For a system that has been left to itself for a long time (i.e., one for which the lower limit of integration is now $-\infty$ rather than 0) Eq. (97) becomes

$$x(t)=\frac{1}{K(\lambda_2-\lambda_1)}\int_{-\infty}^t\{e^{-\lambda_1(t-S)}-e^{-\lambda_2(t-S)}\}F(S)dS. \tag{99}$$

To obtain the average value for many observations of $x(t)$ from Eq. (99) it is necessary to know the statistical properties of the unknown force $F(t)$. The initial conditions are imposed that $\overline{F(t)}=0$ at X_0 and $\dot{X}_0=0$. We have immediately then that at X_0 (99) reduces to

$$\overline{X_t}=X_0(\lambda_2e^{-\lambda_1t}-\lambda_1e^{-\lambda_2t})/(\lambda_2-\lambda_1). \tag{100}$$

This states that the influence of a Brownian kick on this mean value diminishes according to the same periodic decrement as is observed for a macroscopic oscillation. To find our general mean of two readings we multiply by X_0:

$$\overline{X_0X_t}=X_0^2(\lambda_2e^{-\lambda_1t}-\lambda_1e^{-\lambda_2t})/(\lambda_2-\lambda_1) \tag{101}$$

and from its definition in (94) the correlation factor ρ is

$$\rho=(\lambda_2e^{-\lambda_1t}-\lambda_1e^{-\lambda_2t})/(\lambda_2-\lambda_1).$$

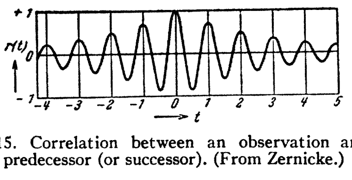


FIG. 15. Correlation between an observation and its predecessor (or successor). (From Zernicke.)

The calculations for negative values of t are done likewise and it is seen immediately that $\overline{X_{-t}}=\overline{X_t}$; which states that the correlation factor ρ is the same for past as for future events. $\rho=1$ for $t=0$, and falls off to zero for large values of t, according to the following curve; (Fig. 15).

Next, let us calculate the Brownian error for the case in which we determine the position of equilibrium of our system by integrating the random vibrations over a time interval S. We wish to find the accuracy of the quantity $1/s\int_0^S xdt$, which is to say, our mean square error is given by

$$\epsilon^2{}_S=\epsilon^2{}_{\text{integration}}=\overline{\left[\frac{1}{S}\int_0^S xdt\right]^2}=\frac{1}{S^2}\int\int_0^S \overline{X(S)X(S')dSdS'}$$

$$=\frac{2x^2}{S}\int_0^\infty \rho(S-S')d(S-S'), \qquad (102)$$

assuming that S is large compared with the "periodic" time of ρ. This is the usual experimental condition. From Eqs. (102) and (100) we have as our expression for the error

$$\epsilon_S{}^2=\frac{2\overline{X^2}}{(\lambda_2-\lambda_1)}\left(\frac{\lambda_2}{\lambda_1}-\frac{\lambda_1}{\lambda_2}\right)=\frac{4n}{\omega}\overline{X^2}=\frac{2nt}{\pi}\overline{X^2}. \qquad (103)$$

The factor $2nt/\pi$ is the accuracy factor of the integration since $\overline{X^2}$ is the mean square error of a single observation.

As $\overline{X^2}/\epsilon^2$ is proportional to the time of observation S, we can write $\overline{X^2}/\epsilon^2=\eta$ a constant, which we may term the efficiency of the particular observational method. Physically, η represents the number of independent single observations which must be taken per second to attain the same accuracy. For the above case of the time-integral, from (103),

$$\eta_s=\pi/2nt.$$

Of special interest is an instrument for which the damping constant n is very small, and for which the deflection sensitivity is independent of n (as in the case of the balance, treated elsewhere in this paper), for then there is no natural sensitivity limit, if we integrate over a very long period of time. Turning to the case of the galvanometer, we can replace our general coordinate X by the current i. Then our Brownian fluctuation as derived above is given by

$$\overline{i^2}=\pi kT/Rtn \qquad (104)$$

and

$$\epsilon^2{}_S=\overline{i^2}_S=\frac{2nt}{\pi S}\overline{i^2}=\frac{2kT}{RS}. \qquad (105)$$

It will be noted that in this expression neither the damping nor the free period of the galvanometer plays any role. An instrument of this type is the electrometer, with which one measures the total charge $e=\int idt$. This latter satisfies the differential equation

$$Ld^2e/dt^2+Rde/dt=E(t), \qquad (106)$$

where $E(t)$ is the Brownian e.m.f. As shown elsewhere in this paper (since this is merely the well known Langevin formula for an emulsion particle), this yields for our mean square fluctuation in charge (cf. Eq. (53))

$$\overline{e^2}=2kTt/R \qquad (107)$$

and this is identical with Eq. (105) inasmuch as $e=i\cdot t$. Accordingly, the integration method of observation gives the same limit of accuracy as an electrometer measurement of equal time of reading. Therefore, the galvanometric system is most sensitive when used as an electrometer for then it measures total charge; therefore, no other observational method will equal integration in exactness. This latter method will serve as an excellent basis of comparison, and we can define as the relative efficiency of any observational method $H=\eta/\eta_s=\epsilon_s^2/\epsilon^2$ which we may henceforth call the efficiency of the system.[40]

One very important fact was mentioned by Czerny—namely, that from thermodynamics, if one considers the usual thermocouple-galvanometer system as a heat engine, working at temperature difference ΔT between irradiated and non-radiated portions, the best possible thermodynamic efficiency is $\Delta T/T$ by the second law. As ΔT varies with the intensity of radiation, decreasing with smaller intensity, it is evident that under the most stringent conditions (for e.m.f.'s approaching the Brownian limit) the efficiency becomes alarmingly low. Assuming a galvanometer system for which the smallest readable voltage experimentally is 10^{-10} volt, the corresponding temperature rise in the hot junction will be about 10^{-7}°C, so that the thermodynamic efficiency is about $10^{-7}/300=3\times10^{-10}$. This agrees rather well with estimates made that the least detectable energy for the usual evacuated thermocouple is around 5×10^{-6} erg. Considering, as in Pfund's resonance radiometer, that the observation time is 40 sec., and that the Brownian limit is $\frac{1}{2}kT=2\times10^{-14}$ erg ($\sim$20°C), our thermodynamic efficiency is $2\times10^{-14}/40\times5\times10^{-6}=10^{-10}$. This factor gives only the efficiency as determined from heat engine considerations, and usual experimental sensitivity limits are chosen, with no attempt made to select the best possible case.

(c) In practice we are unable to use the method of integration, unless we are interested in electrometer measurements. For ordinary current measurements, one must replace integration by a series of individual readings. We shall now calculate the Brownian fluctuation for this type of measurement. Let us make N observations, on a given position, representing the Brownian fluctuations by $X_0, X_1, \cdots X_{n-1}$, taken at times $0, s, 2s, \cdots (N-1)s$, respectively. The mean square error of the arithmetic mean is

$$\epsilon^2{}_\Sigma=\frac{\overline{(X_0+X_1+\cdots X_{n-1})^2}}{N}=\overline{X^2}+\frac{2\overline{X^2}}{N}\sum_{z=1}^{\infty}\rho(zs) \tag{108}$$

and proceeding as we did in deriving Eq. (102), by neglecting vanishing terms, we obtain

$$\frac{N\epsilon^2{}_\Sigma}{\overline{X^2}}=1+\frac{2}{\lambda_2-\lambda_1}\left(\frac{\lambda^2e^{-\lambda_1 s}}{1-e^{-\lambda_1 s}}-\frac{\lambda_1e^{-\lambda_2 s}}{1-e^{-\lambda_2 s}}\right). \tag{109}$$

Recalling from Eq. (100) that $\lambda^2-2n\omega\lambda+\omega^2=0$ defines the λ's we have for the periodic underdamped case; i.e., $n<1$,

$$\frac{N\epsilon^2{}_\Sigma}{\overline{X^2}}=\frac{\sinh n\omega s+(n\omega/\omega')\sin\omega' s}{\cosh n\omega s-\cos\omega' s},\quad\text{where}\quad \omega'=\omega(1-n^2)^{\frac{1}{2}}, \tag{110}$$

and for the overdamped case of $n>1$, we replace the hyperbolic by the ordinary trigonometric functions, and obtain

$$\frac{N\epsilon^2{}_\Sigma}{\overline{X^2}}=\frac{\sin n\omega s+(n\omega/\omega'')\sin\omega'' s,}{\cos n\omega s-\cos\omega'' s},\quad\text{where}\quad \omega''=(n^2-1)^{\frac{1}{2}}. \tag{111}$$

We find for values of $s\rightarrow0$,

$$N\epsilon^2{}_\Sigma/\overline{X^2}=4n/\omega S. \tag{112}$$

Calling our observation time $Ns=S$, we have for this limiting case

$$\lim_{S\to 0} \epsilon^2{}_\Sigma = \overline{X^2}\cdot 4n/\omega. \tag{113}$$

This is in exact accord with the expression derived in Eq. (105) for the integration method.

$\epsilon^2{}_\Sigma$ may be evaluated for various values of s. If this is done the efficiency $H=\epsilon_s{}^2/\epsilon^2{}_\Sigma$ may be plotted against s, as in the following curve of Fig. 16 which is taken from Zernike.

It is seen that regardless of damping the summation method remains practically on a par with integration as long as $s \lessdot T/2$. This result is quite remarkable, for it illustrates that high accuracy may be obtained in this manner even for very weak damping, although such an instrument is very poor for a single zero reading. For times of observation separated by as long as one period, however, H drops to 0.6 even for a critically damped instrument.

The more usual laboratory use of a weakly damped system, however, involves an observation time of the same order of magnitude as the period. We shall next consider, then, the case of such an instrument when S, the total observation time, is not large.

The general integral as above gives the squared error

$$\begin{aligned}\epsilon^2{}_\Sigma &= \frac{2\overline{X^2}}{S^2}\int_0^s dt\int_0^t \rho ds \\ &= \frac{2\overline{X^2}}{(\lambda_2-\lambda_1)}\left\{\frac{\lambda_2}{\lambda_1}-\frac{\lambda_1}{\lambda_2}-\frac{\lambda_2}{\lambda_1}\left(\frac{1-e^{-\lambda_1 s}}{\lambda_1 s}\right)+\frac{\lambda_1}{\lambda_2}\left(\frac{1-e^{-\lambda_2 s}}{\lambda_2}\right)\right\},\end{aligned} \tag{114}$$

which reduces, for small values of n, to

$$\epsilon^2{}_\Sigma = \frac{2\overline{X^2}}{S\omega}\left\{2n+\frac{1-4n^2}{\omega S}[1-e^{-\omega Sn}\cos S\omega']\right\}. \tag{115}$$

This function is periodic and the efficiency $H=\epsilon_s{}^2/\epsilon^2{}_\Sigma$ shows sharp maxima at values of S corresponding to whole periods, for at these points $\cos S\omega'=1$ is a maximum so that $\epsilon^2{}_\Sigma$ is a minimum. Correspondingly, H is a minimum at the half-periods where $\cos\omega' S=-1$ and $\epsilon^2{}_\Sigma$ a maximum. As $S\to\infty$, H approaches a stationary value. A graph of H for the value of $n=0.02$ is presented below, following Zernike in Fig. 17.

The effect of this changing effectiveness of a swinging system is immediately apparent if one considers the accuracy of even a small number of readings. Here we have a short time interval in which $S=Ns$ where N may be as small as 2. The error is:

$$\epsilon^2{}_N=\epsilon_2{}^2=\tfrac{1}{4}\overline{(X_1+X_2)^2}=\tfrac{1}{2}\overline{X^2}(1+\rho(s)). \tag{116}$$

We wish then to make $\rho(s)$ as small as possible. For the aperiodic case we saw that ρ continually diminished with time, so it would be advantageous to take the two readings as widely separated as possible. However, for a vibrating instrument the value of ρ oscillates and is a minimum when our interval $S=\pi/\omega'$, as seen from the preceding Eq. (115). For small values of n, the exact formula becomes

$$\begin{aligned}\epsilon^2{}_{\min} &= \tfrac{1}{2}\overline{X^2}[1-\exp(-\pi n\omega/\omega')] \\ &= \tfrac{1}{2}\pi n\overline{X^2}\quad \text{when expanded.}\end{aligned} \tag{117}$$

Therefore the best possible case is the one for which the periodic error ϵ for two readings is $\frac{1}{2}\pi n$ as large as the error for a single reading; for a system in which n is very small, the gain in efficiency may be made to be very large. For example, for $n=0.3$, $\epsilon_2=0.5\epsilon_1$, or two readings have only one-half the error of a single observation.

The advantage gained by thus taking a small number of readings at the proper intervals is further borne out if we consider the general case of a set of m observations taken at uniformly spaced intervals during the time of one period. That is to say $S=2\pi/m\omega'$.

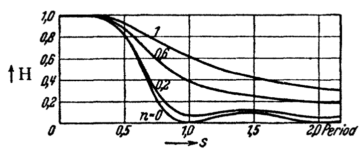


FIG. 16. Efficiency of the mean of many observations, taken at fixed time interval S, for various damping constants. (From Zernicke.)

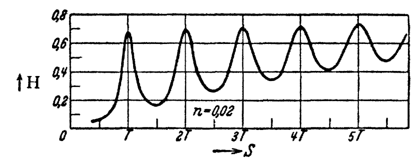


FIG. 17. Efficiency of time-integral observation as a function of the elapsed time S, for a weakly damped galvanometer. (From Zernicke.)

For very weak damping

$$\epsilon^2_{\min}=\frac{1}{m}\frac{(4\pi/m)+\sin(2\pi/m)}{1-\cos(2\pi/m)}n\overline{X^2}. \quad (118)$$

Evaluating this for various values of m gives the values for H (see Table III). The gain in

TABLE III.

m	2	3	4	5	6	∞
H	0.41	0.57	0.62	0.64	0.65	0.667

accuracy is quite appreciable when as few as 4 readings per period are taken; increasing m thereafter does little good.

(*d*) *Observational accuracy for the measurement of the difference between equilibrium positions.* Thus far we have confined ourselves only to the determination of the accuracy with which a single point may be known. In practice it is always necessary to know at least 2 points: first the zero or reference point, and secondly, the position of equilibrium or throw with steady current flowing in the galvanometer circuit. Essentially we measure the difference between two positions, and the accuracy with which the difference is known is at heart the problem with which we are concerned. The answer in the two cases of time-integral and a discrete-manifold set is found instantly. Since the long observation time for the zero must be followed by an equally long deflection time, the correlation factor will be exceedingly small. The mean square error of the difference of the two integrals will be $2\epsilon_s^2$; in addition, the time is also doubled so that, all told, our effectiveness is reduced to 1/4 the value for a single point. As remarked above, usually one is compelled to use a relatively short series of readings (for reasons of fatigue and expediency). It is worth while then to leave our optimum case to calculate the error of the difference of two readings where observations on each point are not numerous.

Let $X_1, X_2\cdots X_{n-1}$ be the various readings. For the simplest case of one zero and one deflection reading, taken t seconds apart, we have:

$$\epsilon^2_{\text{difference}}=2\overline{X^2}(1-\rho(t)). \quad (119)$$

We want, then, to make ρ as large as possible; that is, we desire to make $\rho\rightarrow 1$.

For an underdamped system, ρ attains its maximum value of $e^{-2\pi n}$ when $t=2\pi/\omega'$, or, at the end of a whole period. Now, the galvanometer throw is a maximum at the half-period. It is, therefore, best to wait a half-period after the zero reading before closing the circuit to deflect the galvanometer. In addition, the return swing carries almost as far to the negative side of the zero, and the double throw is $1+e^{-\pi n}$ times the first maximum. Our time of course is doubled. To calculate the efficiency, we have

$$H\,(\text{difference})=\frac{(\omega'/2\pi)(1+e^{-\pi n})^2}{(\omega/2\pi)(1-e^{-2\pi n})}\ \text{and passing to the limit} \quad (120)$$

$$\lim_{n\rightarrow 0} H\,(\text{difference})=2/\pi^2=\tfrac{1}{4}(0.81).$$

This means then that taking into account that the efficiency of all methods for differences are at best only 1/4 as high as those for a single reading, the poorly damped galvanometer, read with isolated readings under optimum conditions can be about 81 percent as good as that obtainable under the best possible method. This high accuracy results from the fact that the Brownian correlation is so strong at the end of one period that the fluctuation which it has produced has changed very little; so that it very nearly cancels out in taking differences in readings.

It is true that at the end of the return swing the galvanometer is at a maximum and is not yet ready to be used, and one might well inquire whether or not this would reduce the efficiency of the instrument. In truth the effectiveness H would be reduced proportionally to the time for a given ϵ^2, but ϵ^2 is also proportional to $1/t$, so that the value of H remains unaltered. This may be demonstrated very readily. For, if a second zero reading be taken a whole period after the first deflection reading, or half-period after the current is cut off, we may write

$$\epsilon'^2_{\text{difference}} = \overline{(-\tfrac{1}{2}X_0 + X_1 - \tfrac{1}{2}X_2)^2} = \tfrac{3}{2}\overline{X^2} - \overline{X^2}(2\rho(t) - \tfrac{1}{2}\rho(2t)). \quad (121)$$

For n quite small $\rho(t) = 1 - 2\pi n$, $\rho(2t) = 1 - 4\pi n$, and $\epsilon'^2_{\text{difference}} = 2\pi n\overline{X^2} = \frac{1}{2}\epsilon^2_{\text{difference}}$ when we substitute $\rho(2t)$ in Eq. (119) for $\epsilon^2_{\text{difference}}$.

The total lapse of time is 2 periods, so the efficiency remains the same.

This line of reasoning may be readily adapted to a calculation of the sensitivity of the resonance radiometer, and the results of Czerny[9] and Firestone[35] are found to be qualitatively correct. As stated earlier in this paper, the resonant state is attained for this underdamped instrument if the galvanometer be excited during half of the cycle. For a system of damping constant n, the amplitude attained after a long time is $1/(1-e^{-\pi n})$ that attained by a single impulse. Also, as we read both ends of the swing, there is no zero recording to be made. This gives us an unfavorable correlation factor since it is a minimum at values of t equal to a half-period interval; and it will be recalled that for a periodic case we wish the correlation to be a maximum. From (101) we find:

$$\epsilon^2 = 2\overline{X^2}(1 + e^{-\pi n}). \quad (122)$$

In addition the time of observation is much longer; to reach a final amplitude which is 99 percent of the limiting value, the time $S = 4.6/n\omega$.

For small values of n, then

$$H_{\text{resonance}} = \left[\frac{(4n/\omega)}{2(1+e^{-\pi n})S}\right]\left[\frac{4}{(1-e^{-\pi n})^2}\right] = \frac{4}{4.6\pi^2} = \frac{1}{4}(0.35), \quad (123)$$

so that the resonance radiometer is only 35 percent as effective as the integration method as regards observational method. It will be noticed that this result does not depend upon damping; but if the time S be reduced to $1.53\ n/\omega$ then the final amplitude will be 78 percent of the limiting value, and $H = 65$ percent instead of 35 percent. This will be more efficient in lowering the Brownian limit as compared to a single reading, but is still short of being as good as two readings taken under best conditions; for here we can attain a value of $H = 81$ percent.

There is one further disadvantage of the resonance radiometer. In addition to yielding a poor result as regards observational method, it yields a poor result for the average current disturbance itself. This limit, set by general conditions, is $\overline{i^2} = \pi kT/Rtn$ and as n is very small compared to the value of 1 for a critically damped galvanometer, the resonance radiometer suffers again in the ratio of $1/n$. Van Lear has worked out the problem of the resonance radiometer in complete detail.[37] He considered the first galvanometer to be a harmonic oscillator coupled with an electromagnetic circuit. The fluctuations in the first system are broken up into their Fourier components, and thence the fluctuations in the second circuit are calculated. Van Lear found that the resonance radiometer's Brownian limit was about three times as high as that of a critically damped instrument of equal resistance and equal time of response. This is in agreement with the results found above by Zernike. In addition, Van Lear found that no advantage is to be gained by departing from conditions of equal damping for the two galvanometers; and in particular, the periodic radiometer of Firestone which uses two critically damped galvanometers is only half as good as the underdamped ones.

(e). *The aperiodic case.* The calculations for the difference of two readings are precisely similar to those for the periodic case. Here $n > 1$, and $\rho(t)$ diminishes continuously with time. Further from Eq. (119) $\epsilon^2_{\text{diff.}} = 2\overline{X^2}(1-\rho(t))$, so that the error increases with time. The deflection X also increases with time, proportionally to the function $1-\rho(t)$; so that the relative square error is proportional to $1/(1-\rho(t))$.

It follows then that

$$H(t) = 2n(1-\rho(t))/\omega t. \quad (124)$$

TABLE IV.

t/τ	$n=1$ $1-\rho(t)$	$H(t)$	t/τ	$n=10$ $1-\rho(t)$	$H(t)$
0.05	0.94	0.26	0.25	0.07	.93
.15	0.24	.52	0.50	0.15	.93
.25	0.47	.59	0.75	0.21	.89
0.4	0.72	.57	1.0	0.27	.86
0.6	0.89	.47	2.5	0.54	.69
1.0	0.97	.31	10.0	0.96	.30

τ is the period of the instrument.

Table IV which gives values for $n=1$ and $n=10$ is taken from Zernike. For the critically damped galvanometer it will be observed that the efficiency is a maximum (59 percent) when the deflection is about 50 percent of its final value, and when $t/\tau=25$ percent. Moreover, one must also wait for a time about 0.6τ before the instrument is ready for use again. To calculate the efficiency on this basis of total elapsed time $t+0.6\tau$, we find our efficiency is about 20 percent.

The very heavily overdamped galvanometer is actually one without a torsion constant. n is very large, and the coil creeps forward slowly. Its equation of motion is simply

$$m\ddot{X}+\beta\dot{X}=F(t), \qquad (125)$$

which is the Langevin formula, for which the solution is

$$\overline{X^2}=(2kT/\beta)t. \qquad (126)$$

For a steady current $\beta\dot{X}=Bi$ and if the initial conditions are $X=0$ for $t=0$

$$X(t)=(B/\beta)it. \qquad (127)$$

The Brownian motion will create an error in the difference of two readings taken at an interval of t seconds which may be expressed in terms of current by

$$\overline{i^2}=\frac{\beta^2}{B^2t^2}\overline{X^2}=\frac{2kT}{t}\frac{\beta}{B^2}. \qquad (128)$$

Now the electromagnetic damping constant is known to be equal to B^2/R so $\overline{i^2}=2kT/Rt$, which is the expression obtained for the electrometer. This is equivalent to observing a time integral and our relative efficiency is the best obtainable.

In conclusion, the best theoretical galvanometer is one which is used as an electrometer; in practice, the resonance radiometer despite its inefficient method of observation remains most free from macroscopic disturbances, and for poorly protected mountings is most trustworthy.

BIBLIOGRAPHY

1. Chr. Wiener, Ann. d. Phys. u. Chem. **118**, 79 (1863).
2. Delsaulx, *Revue des questions scientifiques* (Louvain) **2**, 319 (1877).
3. Gouy, J. de Physique **7**, 561 (1888).
4. A. Einstein, Ann. d. Physik **19**, 371 (1906); cf. also *The Brownian Movement*, Methuen (1926).
5. v. Smoluchowski, Ann. d. Physik **21**, 756 (1906).
6. M. Jean Perrin, Annales de Chimie et de Physique, 8th. ser, 1909.

6a. G. L. de Haas-Lorentz, *Die Brownsche Bewegung*, Vieweg (1913).

7. G. Ising, Phil. Mag. **1**, 827 (1926).
8. Moll and Burger, Phil. Mag. **1**, 626 (1925).
9. M. Czerny, Ann. d. Physik **12**, 993 (1932).
10. W. Gerlach and E. Lehrer, Naturwiss. **15**, 15 (1927).
11. E. Kappler, Ann. d. Physik **11**, 233 (1931).
12. G. E. Uhlenbeck and S. Goudsmit, Phys. Rev. **34**, 145 (1929).
13. G. A. Van Lear and G. E. Uhlenbeck, Phys. Rev. **38**, 1583 (1931).
14. G. Ising, Ann. d. Physik **8**, 905 (1931).
15. K. Engel, Ann. d. Physik **4**, 575 (1930).
16. W. Eggers, Ann. d. Physik **7**, 833 (1930).
17. F. Zernike, Zeits. f. Physik **40**, 628 (1926).

17a. Ornstein, Burger and Taylor, Proc. Roy. Soc. **A115**, 391 (1927).

18. W. Schottky, Ann. d. Physik **57**, 541 (1918).
19. W. Schottky, Ann. d. Physik **68**, 157 (1922).
20. B. Rajewsky, Phys. Zeits. **32**, 121 (1931).
21. Handbuch der Experimental Physik **13**, Part 2, 270 (gives an excellent resumé of the work).
22. Hartmann, Ann. d. Physik **65**, 51 (1921).
23. Hartmann, Phys. Zeits. **23**, 436 (1922).
24. Hull and Williams, Phys. Rev. **25**, 147 (1925).
25. J. B. Johnson, Phys. Rev. **26**, 71 (1925).
26. W. Schottky, Phys. Rev. **28**, 74 (1926).
27. J. B. Johnson, Phys. Rev. **32**, 97 (1928).
28. H. Nyquist, Phys. Rev. **32**, 110 (1928).
29. E. A. Johnson and C. Neitzert, Rev. Sci. Inst. **5**, 196 (1934).
30. L. Hafstad, Phys. Rev. **44**, 201 (1933).
31. R. Bowling Barnes and M. Czerny, Zeits. f. Physik **79**, 436 (1932).
32. G. Ising, Ann. d. Physik **14**, 755 (1932).
33. A. H. Pfund, Science **69**, 11 (1929).
34. J. D. Hardy, Rev. Sci. Inst. **1**, 429 (1930).
35. F. A. Firestone, Rev. Sci. Inst. **3**, 161 (1932).
36. J. D. Hardy, Rev. Sci. Inst. **5**, 120 (1934).
37. G. A. Van Lear, Jr., Rev. Sci. Inst. **4**, 21 (1933).
38. F. Zernike, Zeits. f. Physik **79**, 516 (1932).
39. G. Ising, Ann. d. Physik **8**, 911 (1931).
40. Czerny[9] was the first to point out that the efficiency of a galvanometer was the underlying factor in its usefulness. His calculations of efficiency were based entirely upon the consideration of the system as a heat engine. He neglected to take account of the effect of observational method upon the observed Brownian error, so that his results are of limited value.

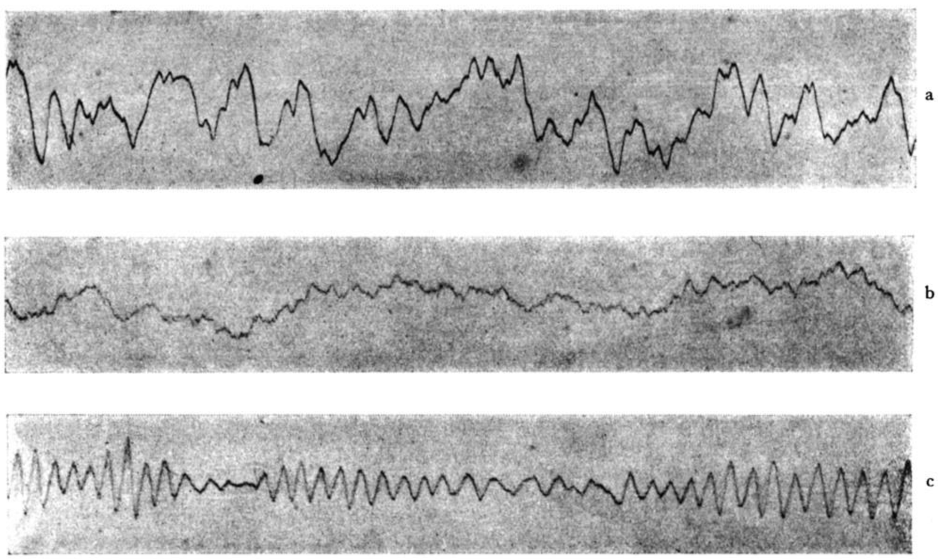


FIG. 5. Brownian fluctuation of a suspended mirror. (From Kappler.)

(a) Restoring force 2.66×10^{-9} abs. units. $I=6.1\times10^{-6}$. Camera distance 86.5 cm. Time 30 sec. equivalent to 2 mm. Pressure 4×10^{-3} mm.

(b) Restoring force 9.428×10^{-9} abs. units. $I=10^{-7}$. Camera distance 72.1 cm. Time 30 sec. equivalent to 1 mm. Atmospheric pressure.

(c) Same system as curve 5b, except that the pressure is 10^{-4} mm.