Vortex oscillations and hydrodynamics of rotating superfluids

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This review covers the progress in the study of vortex oscillations in rotating superfluids. The paper deals with the theory as its principal concern, but the experiments that one can compare with the theory considered are also discussed. Attention is focused mainly on the effects of crystalline order in the vortex lattice (the Tkachenko waves especially) and on the boundary problems arising in studies of vortex oscillations in finite containers. The approach is based mostly on the continuum hydrodynamic theory dealing with dense vortex arrays, and considerable attention is devoted to discussion of this theory in order to understand better the principles upon which the obtained results rest. The theory is traced from the simple description of a rotating classical fluid with continuous vorticity, through that of a perfect fluid with quantized vorticity in the form of an array of vortex lines, then the two-fluid theory of an isotropic superfluid, and finally the theory of rotating anisotropic superfluids such as ^{3}He-A. Applications of the theory to He II, the superfluid phases of ^{3}He, and the superfluid neutron matter in pulsars are discussed.

CONTENTS

I. INTRODUCTION

The motion of vortices has already been studied for more than a century. During the classical period of vortex dynamics, which began in the late 1800s, many interesting properties of vortices were discovered, beginning with the notable Kelvin waves propagating on an isolated vortex line (Thompson, 1880). The main subject of theoretical studies at that time was a dissipationless perfect fluid (Lamb, 1945). It was difficult to make contact with theory experimentally, since any classical fluid exhibits viscous effects. The situation changed after the works of Onsager (1949) and Feynman (1955), who revealed that rotating superfluids are threaded by an array of vortex lines with quantized circulation. With this discovery, the quantum period of vortex dynamics began. The rotating superfluid ^{4}He provided the testing ground for the theories of vortex motion developed for the perfect fluid. Hall (1958) and Andronikashvili and Tsakadze (1958) were the first to study experimentally the elastic properties of vortex lines. It was possible to observe resonances on Kelvin waves with the spectrum modified by interaction between vortices. The properties of the vortex waves were to a large degree well understandable within the confines of the theory of an inviscid perfect fluid. But some effects, damping of vortex waves in particular, required an extension of the theory to include two-fluid effects, so the quantum period of vortex science was marked by the progress of vortex dynamics in the frame of the two-fluid theory. The first step in this direction was taken by Hall and Vinen (1956), who studied mutual friction between vortices and the normal part of the superfluid and derived the law of vortex motion in two-fluid hydrodynamics.

The second important theoretical framework, invented to describe vortex motion in rotating superfluids, was the so-called macroscopic hydrodynamics. It relied on hydrodynamical equations averaged over scales much larger than the intervortex spacing. Such a hydrodynamic theory was formulated by Hall (1960) and Bekarevich and Khalatnikov (1961). It was a continuum theory similar to the elasticity theory, including, however, only bending deformations of vortex lines and ignoring the crystalline order of the vortex array. The theory was successful in explaining a variety of experiments.

In the late 1960s attention was attracted to phenomena connected with crystalline order in the vortex array. The first to show that vortex lines in a rotating superfluid form a stable triangular lattice, as in a type-II superconductor (Abrikosov, 1957), was Tkachenko (1965). He predicted (Tkachenko, 1966) that the vortex lattice sustains collective elastic waves, or "Tkachenko modes," in which vortex lines undergo dispacements homogeneous along the vortex lines and transverse to the wave vector. Such a wave is a transverse-sound mode of the vortex lattice and is derived from the elasticity theory of the two-dimensional vortex lattice when the wavelength is much larger than the distance between vortices (Tkachenko, 1969). Tkachenko modes were not describable within the Bekarevich-Khalatnikov hydrodynamics, but later the continuum theory was developed, which incorporated the effects of vortex-lattice rigidity (Tkachenko rigidity) and of vortex-line bending (Sonin, 1976; Williams and Fetter, 1977). This theory predicted a mixed mode of vortex oscillations involving the Kelvin mode and the Tkachenko mode as particular cases when the vortex wave propagates along, and normal to, the rotation axis.

Observation of the regular vortex lattice turned out to be a much more difficult experimental problem in rotating He II than in type-II superconductors because vortices in He II do not create magnetic or electric fields facilitating their identification. Only 13 years after the paper of Tkachenko (1965), the existence of the regular vortex lattice in rotating He II was proven experimentally. The first evidence was obtained by S. J. Tsakadze (1978), who deduced a value of the vortex-lattice shear rigidity by observing a mixed mode of vortex oscillations (the Tkachenko mode modified by vortex bending) in a free-spinning cell with He II. Shortly afterwards direct photographs of vortex arrays in He II were obtained showing that vortices form stable, regular arrays similar to the predictions of theory (Gordon *et al.*, 1978; Yarmichuk *et al.*, 1979).

Recently new problems have challenged vortex dynamics research as a result of the discovery of superfluid phases of ^{3}He. The A phase turned out to be especially unusual. It possesses a remarkable and hitherto unknown property: the rotating A phase, while remaining a superfluid, sustains a continuous vorticity that is not homogeneous in space, as in a rotating classical fluid, but forms a two-dimensional periodic texture with more intricate symmetry properties than the simple hexagonal symmetry of the triangular vortex array in He II. The study of such periodic textures originated with the work of Volovik and Kopnin (1977). A continuous vorticity in the rotating A phase has been detected by Hakonen *et al.* (1982) with use of NMR techniques. But investigations of the dynamical behavior of vortex textures in the rotating A phase are still in an early state. The first theoretical work was done by Kopnin (1978), who derived the law of motion for a continuous axisymmetric vortex in the A phase.

At first, the B phase seemed less intriguing than the A phase, because only "common" singular vortices were expected to exist in it. But the NMR experiments on the rotating B phase were a surprise for experimentalists and theoreticians alike (Ikkala *et al.*, 1982). They showed that properties of vortices in the B phase are quite nontrivial, and that the vortex texture undergoes a phase transition. The spontaneous magnetic moment of the vortices was also discovered (Hakonen *et al.*, 1983). All these interesting phenomena have been explained by a complicated structure and symmetry of the vortex core in the B phase (see the recent paper by Salomaa and Volovik, 1985). One may expect that core effects play an important role in vortex dynamics too, as can be judged from the only experiment on vortex dynamics in the B phase, dealing with vortex-induced mutual friction in rotating ^{3}He-B (Hall *et al.*, 1984).

Up until now only two superfluids have been available for laboratory research: ^{4}He and ^{3}He (putting aside the superconductive electron fluid in metals). However, these two superfluids do not exhaust all the possible applications of superfluid vortex dynamics. Long ago it was proposed that the interior matter of neutron stars is in a superfluid state (Migdal, 1959) and is threaded by quantized vortices because of rotation (Ginzburg and Kirzhnitz, 1964). A rich variety of phenomena in pulsars are interpreted by the use of this vortex conception. In the past astrophysical applications strongly stimulated the study of vortex dynamics. For example, experimental studies of Tkachenko modes in Tbilisi (Tsakadze and Tsakadze, 1973) were prompted by the theory of Ruderman (1970) associating variations observed in the pulse period of pulsars with Tkachenko waves.

The study of vortex dynamics in rotating superfluids is advantageous because rotation creates vortices with well-controlled form and density. But vortices are generated by uniform superflow too, and they are responsible for the critical velocity of the superflow (see the recent review by Sonin, 1982). Vortices are present even in undisturbed ^{4}He (Awschalom and Schwarz, 1984). Results obtained by studying vortices in rotating superfluids may be used for other, more intricate vortex structures. An example of such an extension is the theory of Vinen (1957, 1961) describing the behavior of the vortex tangle arising at large superflows. Recent developments in this theory are discussed by Tough (1982; see also Nemirovskii and Lebedev, 1983).

The importance of vortex dynamics is not restricted to superfluid applications. There are number of linear singularities (linear defects) in other ordered condensed media that are similar in some aspects to superfluid vortices. Besides vortices in superconductors these include dislocations in solids, Bloch lines in magnetic materials, disclinations in liquid crystals, and some others. The general methods of their topological classification and of stability analysis have been developed [see the review of Mineev (1980) and the book of Kléman (1983) for introductions to the literature]. But the dynamics of linear defects also has much in common with that of superfluid vortices. In particular, the motion of the Bloch line is described by an equation including a gyrotropic force similar to the Magnus force. Gyrotropy of the equation of motion always results in elliptical or circular polarization of linear-defect oscillations, as occurs in the case of the Kelvin wave. Such oscillations of Bloch lines, or magnetic vortices, have been seen experimentally and studied in theory (Argyle *et al.*, 1984; Dedukh *et al.*, 1985).

Theoretical and experimental achievements in vortex dynamics have been treated in a number of comprehensive reviews and books (Hall, 1960, 1963; Andronikashvili *et al.*, 1961, 1978; Andronikashvili and Mamaladze, 1966, 1967; Khalatnikov, 1971; Putterman, 1974), but phenomena associated with crystalline order in the vortex lattice were considered there only fragmentarily. Our intention is to fill this gap. The theory is reviewed and found capable of describing all oscillation modes of a regular vortex lattice. The experiments that have been performed or may be done to prove this theory are also considered, though from the position of a theorist: results, but not techniques, are discussed. The experiments deal mostly with effects produced by a large number of vortices, so our theoretical approach is based on macroscopic hydrodynamics referring to the infinite vortex lattice. Even though a great deal of attention is devoted to boundary problems for finite vortex arrays (because of their importance for contacts between theory and experiment on vortex oscillations), they are assumed to be large enough and are treated using the hydrodynamical theory. In order to make the review self-contained, it was necessary to limit it essentially to problems of superfluid hydrodynamics as a whole, keeping in mind that vortex-oscillation modes can mix with other hydrodynamical modes and sometimes it is difficult to distinguish between them in theory and experiment. But we deal with general problems of the hydrodynamics of rotating superfluids only to the extent necessary for understanding vortex motion.

As for the style of the present review, the principle "from particular to general" is preferred to the principle "from general to particular." Though the latter makes the paper more compact and helps to avoid repetition, the former is more convenient for readers who have no intention of entering deeply into the theory and who want to stop at some level. Following the same principle, we discuss an experimental result immediately after the theory has been presented adequately for its discussion. It is quite remarkable that so much of the theory and experiment on vortex oscillations can be understood within the framework of the model of the perfect fluid, without referring to the more complicated two-fluid theory. We exploit this circumstance as far as possible.

Discussion of the perfect fluid begins with a very simple model of a classical fluid with continuous homogeneous vorticity (Sec. II). It is well known that superfluid vorticity is concentrated along singular vortex lines with quantized circulation. It is in this quantization of circulation that the quantum fluid differs from a common classical perfect fluid. But sometimes the circulation quantum may be considered to be small compared to other relevant parameters and it is even possible to forget the quantum nature of the superfluid for a while.[1] In Secs. III and IV the laws of vortex motion and oscillation modes in an unbound rotating superfluid with quantized vorticity are considered. Boundary problems for the perfect fluid are treated in Secs. V and VI. Only after analysis of the boundary problems are we able to discuss experiments on vortex oscillations in finite containers with a superfluid, and this is done is Sec. VI.

The theory is extended to include two-fluid effects for an unbound fluid in Sec. VII and for a fluid in finite containers in Sec. VIII. Section IX is devoted to the proper-

[1] So, like Putterman (1974), we shall heed Uhlenbeck's appeal: "one must watch like a hawk to see how Planck's constant comes into hydrodynamics" (cited in Preface to Putterman's book).

ties of anisotropic superfluids such as the A phase of ^{3}He or the neutron matter in the $^3P_2$ superfluid phase. This section gives a preliminary outlook on new problems that we encounter in studying vortex dynamics in anisotropic superfluids. The review is concluded by Sec. X, briefly discussing the mutual friction problem arising at determination of the vortex velocity in the two-fluid hydrodynamics.

Experiments discussed in the review were performed mostly on He II, which has been up until now the main field of application of the presented theory. As for superfluid ^{3}He, we are compelled to discuss more often future experiments, since experimental research on vortex dynamics in superfluid ^{3}He is in its early stages. In our paper we touch also on astrophysical applications of the theory.

II. WAVES IN A PERFECT FLUID WITH CONTINUOUS VORTICITY

We begin our analysis of vortex oscillations within the theory of a rotating classical perfect fluid.

Equations governing the motion of a perfect fluid are the continuity equation,

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho \mathbf{v})=0 , \tag{2.1}$$

and the Euler equation

$$\frac{\partial \mathbf{v}}{\partial t}+(\nabla\cdot\mathbf{v})\mathbf{v}=-\nabla P/\rho . \tag{2.2}$$

Here $\mathbf{v}$ is the fluid velocity, ρ is the density, and P is the pressure. The pressure obeys the differential Gibbs-Duhem relation

$$dP=\rho\, d\mu+S\, dT , \tag{2.3}$$

where μ is the chemical potential, S is the entropy per unit volume, and T is the temperature. Formally one may describe a superfluid within the perfect-fluid theory only at $T=0$ when $S=0$ too. Then $\nabla\mu$ appears instead of $\nabla P/\rho$ in the right-hand side of Eq. (2.2), as usual for the superfluid Euler equation. But the perfect-fluid theory may be applied at finite temperatures until the effects associated with the normal part of the fluid are not very important.

One can prove directly that Eqs. (2.1) and (2.2) obey the momentum conservation law

$$\frac{\partial j_i}{\partial t}+\nabla_j \Pi_{ij}=0 , \tag{2.4}$$

where $\mathbf{j}=\rho\mathbf{v}$ is the momentum density or the mass flow, and Π_{ij} is the momentum-flux tensor given by

$$\Pi_{ij}=P\delta_{ij}+\rho v_i v_j . \tag{2.5}$$

After transformation of the second term on the left-hand side of Eq. (2.2) it takes the form

$$\frac{\partial \mathbf{v}}{\partial t}+\tilde{\boldsymbol{\omega}}\times\mathbf{v}=-\nabla P/\rho-\nabla(\tfrac{1}{2}v^2) . \tag{2.6}$$

Here $\tilde{\boldsymbol{\omega}}=\nabla\times\mathbf{v}$ is the vorticity.

Suppose that the fluid rotates with the angular velocity $\boldsymbol{\Omega}$. Transformation to a rotating coordinate frame does not change the continuity equation (2.1), but the Euler equation (2.2) becomes

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}+2\boldsymbol{\Omega}\times\mathbf{v}=-\nabla P/\rho+\nabla(\tfrac{1}{2}\Omega^2 r^2) . \tag{2.7}$$

Here r is the distance from the rotation axis. Equation (2.7) differs from Eq. (2.2) written for the inertial frame by the Coriolis force $2\boldsymbol{\Omega}\times\mathbf{v}$ and the centrifugal force $\nabla(\frac{1}{2}\Omega r^2)$. Further, the centrifugal force will be ignored as being of minor importance for the problems under consideration. Indeed, this force becomes essential when the linear velocity due to rotation approaches the sound velocity and the fluid density begins to depend or r. Such fast rotations are not available in present-day laboratory experiments on superfluids.

The transformation to the rotating frame changes the vorticity too, so the velocity field, originally irrotational in the inertial frame, possesses after the transformation the vorticity $\nabla\times\mathbf{v}=-2\boldsymbol{\Omega}$. Further, it is convenient to deal with the absolute vorticity, always determined in the inertial frame. Then the vorticity is not touched upon by the transformation to the rotating frame, but is connected with the fluid velocity in the rotating frame by the relation

$$\tilde{\boldsymbol{\omega}}=2\boldsymbol{\Omega}+\nabla\times\mathbf{v} . \tag{2.8}$$

When $\tilde{\boldsymbol{\omega}}$ denotes the absolute vorticity, the Euler equation in the form of Eq. (2.6) is invariant with respect to the transformation to the rotating frame.

Now let us consider waves of small amplitude. Deviations of the pressure and the density are connected by the linear relation $P'=c^2\rho'$, where c is the sound velocity. In the linear theory, the velocity $\mathbf{v}$ in the rotating frame is small, and the vorticity takes its equilibrium value $2\boldsymbol{\Omega}$, so the linearized equations in the rotating frame are given by

$$\frac{\partial \rho'}{\partial t}+\rho\nabla\cdot\mathbf{v}=0 , \tag{2.9}$$

$$\frac{\partial \mathbf{v}}{\partial t}+2\boldsymbol{\Omega}\times\mathbf{v}=-\frac{c^2}{\rho}\nabla\rho' . \tag{2.10}$$

Suppose that a plane wave propagates in the fluid. Then

$$\begin{aligned}\rho'&=\rho(\mathbf{Q})\exp(i\mathbf{Q}\cdot\mathbf{R}-i\omega t)\\&=\rho(\mathbf{Q})\exp(ipz+i\mathbf{q}\cdot\mathbf{r}-i\omega t) ,\\ \mathbf{v}&=\mathbf{v}(\mathbf{Q})\exp(i\mathbf{Q}\cdot\mathbf{R}-i\omega t)\\&=\mathbf{v}(\mathbf{Q})\exp(ipz+i\mathbf{q}\cdot\mathbf{r}-i\omega t) .\end{aligned} \tag{2.11}$$

Here and throughout the paper $\mathbf{R}$ is a three-dimensional position vector, the z axis is directed along the rotation axis, a two-dimensional vector $\mathbf{r}$ is the component of $\mathbf{R}$ in the xy plane normal to the rotation axis, a wave number p and a two-dimensional wave vector $\mathbf{q}$ are components of the three-dimensional wave vector $\mathbf{Q}$ on the z axis and in the xy plane, respectively. Three components of the velocity $\mathbf{v}(\mathbf{Q})$ will be introduced as follows: the component v_z along the z axis, the component v_q along the wave vector $\mathbf{q}$, and the the component v_t along the axis normal to both $\mathbf{q}$ and the rotation axis.

We begin by solving the continuity equation (2.9) and the equation for v_z [one of three scalar equations given by the vector Eq. (2.10)] and find the expressions connecting ρ' and v_z with v_q:

$$\rho' = \frac{\omega q}{\omega^2 - c^2 p^2} \rho v_q , \quad v_z = \frac{c^2 p q}{\omega^2 - c^2 p^2} v_q . \tag{2.12}$$

Substituting these expressions into the remaining two scalar equations given by Eq. (2.10) we obtain equations to determine v_q and v_t,

$$\begin{aligned} &-i\omega v_q - \frac{\omega^2 - c^2 p^2}{\omega^2 - c^2 Q^2} 2\Omega v_t = 0 , \\ &-i\omega v_t + 2\Omega v_q = 0 . \end{aligned} \tag{2.13}$$

The dispersion law for this system of linear equations is

$$\omega^2 = 4\Omega^2 \frac{\omega^2 - c^2 p^2}{\omega^2 - c^2 Q^2} . \tag{2.14}$$

Let us assume that the fluid is incompressible, taking a limit $c \to \infty$. Then Eq. (2.14) yields

$$\omega^2 = 4\Omega^2 \frac{p^2}{Q^2} = 4\Omega^2 \frac{p^2}{p^2 + q^2} . \tag{2.15}$$

Equations (2.12) and (2.13) give the following relations between velocity components in the wave:

$$v_q = \frac{i\omega}{2\Omega} v_t , \quad v_z = -\frac{i\omega}{2\Omega} \frac{q}{p} v_t . \tag{2.16}$$

Thus we have an elliptically polarized wave, well known in the hydrodynamics of a rotating classical fluid as the inertial wave. This wave plays an important role in meteorology and geophysics (Greenspan, 1968). Properties of the inertial wave are very peculiar. For example, the group velocity is directed normal to the wave vector. The inertial wave is the sole linear mode of motion in an incompressible inviscid rotating fluid. Looking at the dispersion law, Eq. (2.15), one arrives at a result known as the Taylor-Proudman theorem: any slow motion in the rotating fluid is two-dimensional and homogeneous along the rotation axis. This statement follows from the property that small frequencies $\omega \ll 2\Omega$ correspond to small values of the ratio p/Q. Thus the slow motion of the fluid is columnar. Suppose that a solid disk is towed across the bottom of the rotating tank with a fluid. The entire vertical pillar of fluid above the disk moves as a unit. This pillar is called a "Taylor column." A number of convincing experimental demonstrations of this striking phenomenon are described in the book of Greenspan (1968). Slow columnar motion, known as a geostrophic mode, is especially important for problems considered in the present paper.

The spectrum of the inertial wave, Eq. (2.15), was proved by observation of resonances on standing waves in a fluid contained in a cylindrical region with finite height (Fultz, 1959). One can find further discussion and additional references in Sec. 2.15 of Greenspan (1968) and in Sec. 7.6 of Batchelor (1970).

Now let us discuss the dispersion law for oscillations in a compressible fluid. Solving Eq. (2.14) as an equation for ω^2 one obtains the expression

$$\omega^2 = \tfrac{1}{2}(4\Omega^2 + c^2 Q^2) \pm [\tfrac{1}{4}(4\Omega^2 + c^2 Q^2)^2 - (2\Omega c p)^2]^{1/2} . \tag{2.17}$$

Comparing this spectrum with the spectrum of oscillations in a fluid at rest ($\Omega = 0$, $\omega_1 = cQ$, $\omega_2 = 0$), we see that rotation adds a second mode with a finite frequency. It occurs due to the Coriolis force: rotation makes a fluid rigid in the direction normal to the rotation axis. The dispersion law, Eq. (2.17), shows that natural units of the frequency and the wave vector in a rotating fluid are Ω and Ω/c, respectively. Thus the expansion in $\mathbf{Q}$ is an expansion in the dimensionless parameter $c\mathbf{Q}/\Omega$, and the limit of an incompressible fluid $c \to \infty$ is approached at large wave vectors when $cQ/\Omega \to \infty$. Then one branch of the spectrum, Eq. (2.17), yields the inertial wave with the spectrum Eq. (2.15), and another branch is a sound wave modified by rotation. Frequencies of the latter are given by

$$\omega^2 = (cQ)^2 + (2\Omega q/Q)^2 . \tag{2.18}$$

In the inverse limit of small wave vectors $cQ/\Omega \ll 1$, expansion of Eq. (2.17) in cQ/Ω yields two frequencies,

$$\omega = \begin{cases} 4\Omega^2 + c^2 Q^2 , \\ c^2 p^2 . \end{cases} \tag{2.19}$$

This means that the model of the incompressible fluid is invalid when the oscillation wavelength is large enough. This important conclusion, obtained in the continuous-vorticity model, holds as well in a fluid with quantized vorticity. Therefore one should be careful with the term "hydrodynamics of an incompressible fluid." On the one hand, the hydrodynamical theory should describe the long-wavelength behavior of a fluid; on the other hand, it is just the long-wavelength limit that the hydrodynamics of an incompressible fluid fails. However, this problem is academic to a certain extent, because the space scale c/Ω at which the incompressible-fluid hydrodynamics becomes invalid is extremely large (of order hundreds of meters) and is not relevant to any real laboratory experiment. Probably such a large space scale is relevant in some astrophysical applications. But one should remember that at the distance c/Ω from the rotation axis the fluid velocity approaches the sound velocity c, and the centrifugal force may not be ignored. Then our theory should be modified.

III. MOTION OF VORTEX LINES IN A PERFECT FLUID

A. Vortex lines in a perfect fluid. Energy of vortex lines

The concept of a vortex line appeared in classical hydrodynamics many years ago (see Lamb, 1945). This is a line whose direction is everywhere that of the vorticity vector $\nabla\times\mathbf{v}$. Now suppose that a bunch of vortex lines forms a tube, and motion of the fluid is irrotational (curl-free) everywhere except for space within the tube. This is known as a "vortex tube" or a "vortex filament" or simply a "vortex." Circulation around a vortex filament is a measure of its "strength":

$$\kappa=\oint \mathbf{v}\,dl\ . \tag{3.1}$$

Here one integrates over any closed path around the filament. Suppose that at a given circulation the diameter of the vortex filament decreases and becomes much smaller than any relevant hydrodynamical scale (it may be the curvature radius of the vortex filament or the distance from other filaments). Such an infinitely thin vortex filament is widely known as a "vortex line." But in contrast with the vortex line introduced in a fluid with continuously distributed vorticity, we are now dealing with a singular vortex line along which all vorticity is concentrated. A fluid may contain an arbitrary set of singular vortex lines; outside of them motion of the fluid is curl-free. If in addition the fluid is incompressible, then in a multiply connected region around the vortex lines the velocity field satisfies the conditions

$$\nabla\cdot\mathbf{v}=0,\ \ \nabla\times\mathbf{v}=0\ . \tag{3.2}$$

At given circulations around all vortex lines we come to a standard classical-field problem. Any vortex line induces a velocity field given by the Biot-Savart formula (in complete analogy with a magnetic field around a filament with an electrical current.) The net velocity field at the point $\mathbf{R}$ is equal to a sum of contributions of all vortex lines:

$$\mathbf{v}(\mathbf{R})=\sum_j \frac{\kappa_j}{4\pi}\int \frac{d\mathbf{R}_j\times(\mathbf{R}-\mathbf{R}_j)}{|\mathbf{R}-\mathbf{R}_j|^3}\ . \tag{3.3}$$

Here κ_j is the circulation around the jth vortex line, and $\mathbf{R}_j$ is the position vector of the point on the same line. Integration is performed over the whole length of any vortex line. The velocity field induced by one rectilinear vortex line is especially simple. In this case integration over the line length yields a field

$$\mathbf{v}_v(\mathbf{r})=\frac{\boldsymbol{\kappa}\times\mathbf{r}}{2\pi r^2}\ . \tag{3.4}$$

Here $\boldsymbol{\kappa}$ is the circulation vector of magnitude κ directed parallel to the vortex line in accordance with the right-hand rule. The two-dimensional position vector $\mathbf{r}$ lies in the plane normal to the vortex line, and the origin of $\mathbf{r}$ is on the vortex line. The velocity field Eq. (3.4) refers not only to a straight vortex line, but to a curved one as well, until the distance r from the vortex line is small compared to the curvature radius.

Though the concept of the singular vortex line was invented in classical hydrodynamics, it was there thought of as being very distant from a real fluid, since viscosity let to diffusion of vorticity, initially concentrated along singular lines, over the entire bulk of the fluid (Lin, 1963, p. 105). But in the superfluid, vorticity is not compatible with the existence of a scalar complex order parameter; therefore the superfluid endeavors to contract the region of vorticity, providing stability of the vortex lines.

The energy of vortex lines is equal to the kinetic energy of the velocity field induced by them. For a straight vortex line [the velocity field is given by Eq. (3.4)] the energy per unit length is

$$\varepsilon=\rho\int d\mathbf{r}\tfrac{1}{2}v_v(\mathbf{r})^2=\frac{\rho\kappa^2}{4\pi}\int\frac{dr}{r}=\frac{\rho\kappa^2}{4\pi}\ln\frac{r_m}{r_c}\ . \tag{3.5}$$

The upper cutoff of the logarithmically divergent integral depends on the particular hydrodynamical problem under consideration. It is a distance from the vortex line at which the velocity begins to decrease faster than $1/r$. For example, when one deals with a vortex ring formed by a vortex line, the cutoff r_m is of the order of the ring radius.

The lower cutoff r_c is a core radius. The vortex core is a region around the vortex line where the hydrodynamics of an incompressible perfect fluid fails. One can approximately determine the core radius as a distance r at which the velocity $\mathbf{v}_v$ given by Eq. (3.4) is of the order of the sound velocity c. This means that $r_c\sim\kappa/c$. Exact determination of r_c requires an analysis of the vortex core structure beyond the hydrodynamical approach, based on the concept of infinitely thin vortex lines. In classical hydrodynamics a number of models were proposed to deal with the vortex core, for example, a hollow core or a solid core with uniform distribution of vorticity in it. In quantum hydrodynamics the core radius $r_c\sim\kappa/c$ is of order of the coherence length. Studying a quantum fluid on such a space scale is far from easy in general, except for cases when one may apply some local field theory describing the order parameter inside of the core by differential equations. The Ginzburg-Pitaevskii theory is such a theory, and within its framework the first calculation of the superfluid vortex core was carried out (Ginzburg and Pitaevskii, 1958) that provided an exact value of the cutoff r_c in Eq. (3.5).

The phenomenological Ginzburg-Pitaevskii theory is a mean-field theory similar to the well-known Ginzburg-Landau theory describing superconductors near the critical point. But such a theory is not suitable for applications to He II, where the effect of critical fluctuations is much stronger than in superconductors. Therefore a modified version of the Ginzburg-Pitaevskii theory has been proposed, matching the scaling laws and experimental data in the critical region for He II (Ginzburg and Sobyanin, 1976, 1982). This is the phenomenological Ψ theory, which will be used in Sec. X.C in connection with the mutual friction problem. The calculation of the vortex line energy in the Ψ theory is similar to that in the Ginzburg-Pitaevskii theory (Ginzburg and Sobyanin, 1976). Similar calculations of the vortex core structure and its effect on the energy of the vortex line have been carried out using the Gross-Pitaevskii theory for the weakly interacting Bose gas (Gross, 1961; Pitaevskii, 1961; Fetter, 1965). This theory is derived from the Schrödinger equation for bosons written in the second-quantization formalism, but also results in the nonlinear Schrödinger equation for the condensate wave function, or the order parameter, like that in the Ginzburg-Pitaevskii theory. Properties of vortex lines in terms of the Gross-Pitaevskii and Ginzburg-Pitaevskii theory were considered by Vinen (1966).

The structure of the vortex core has also been studied within the healing theory of Hills and Roberts (1977b, 1978a, 1978b), who generalized the traditional two-fluid theory in a way that allows the superfluid density to be an independent thermodynamic variable. One can find a discussion of this and other models for the vortex core in the review paper of Barenghi *et al.* (1983).

By performing calculations of the vortex core structure we can obtain an exact number factor in the expression for the lower cutoff r_c in Eq. (3.5). But this factor is not very important for our purposes, since the logarithm in Eq. (3.5) is large as a rule (of order ten). Furthermore, we shall be using the "logarithmic approximation" neglecting number factors in the argument of the logarithm in the expression of the energy of vortex lines.

Let us find now the energy of an arbitrary pattern of vortex lines in an incompressible fluid. The energy is approximately equal to the product of the energy per unit length given by Eq. (3.5) and the total length of vortex lines. A more involved approach is to transform by means of partial integration the volume integral for the kinetic energy,

$$\mathscr{E}=\tfrac{1}{2}\rho \int d\mathbf{R}\, v(\mathbf{R})^2 \,, \tag{3.6}$$

into the sum of double linear integrals over vortex lines

$$\mathscr{E}=\frac{\rho}{4\pi}\sum_{ij}\kappa_i\kappa_j \int\int \frac{d\mathbf{R}_i\cdot d\mathbf{R}_j}{|\mathbf{R}_i-\mathbf{R}_j|} \,. \tag{3.7}$$

Performing this transformation we used Eq. (3.3) for the velocity field and conditions given by Eq. (3.2). Equation (3.7) is identical with the expression for the magnetic energy of electrical currents flowing along thin filaments and for the electrostatic energy of charged filaments.

Any term $i\neq j$ in the sum Eq. (3.7) is the energy of interaction between two vortex lines. The self-action term $i=j$ contains the logarithmically divergent integral that we have encountered in estimating the energy of one vortex line. It is therefore cut off by the core radius r_c.

B. The Magnus force and the Helmholtz theorem

In order to describe the dynamical behavior of a prefect fluid with singular vortex lines we solve the hydrodynamical equations for the curl-free perfect fluid in a multiply connected region around the vortex lines. These equations should be supplemented by equations governing the motion of vortex lines. Now we are going to discuss such equations.

It is well known that vortex lines in a perfect fluid move with the fluid, or are "frozen into" the fluid. This is stated by the Helmholtz theorem, which holds for singular vortex lines too, but in the latter case some points deserve discussion: (i) the hydrodynamical theory becomes invalid in the approach to a singular vortex line, (ii) in connection with it the question arises how to determine correctly the fluid velocity of points on vortex lines. In order to make these points clear we review the derivation of the Magnus relation connecting the vortex line velocity $\mathbf{v}_L$ and the external force $\mathbf{F}$ per unit length applied to the vortex line.

Suppose that we have an isolated vortex line in an incompressible fluid. The line induces the velocity field $\mathbf{v}_v(r)$ given by Eq. (3.4), and there is a fluid current past the vortex line with a constant velocity $\mathbf{v}_0$. Then the net velocity field around the line is

$$\mathbf{v}(\mathbf{r})=\mathbf{v}_v(\mathbf{r})+\mathbf{v}_0 \,. \tag{3.8}$$

Equation (3.8) yields the sole velocity field in an incompressible perfect fluid satisfying conditions (3.2) together with the condition imposed by the given circulation of the vortex line. The pressure around the vortex line can be found from the Euler equation (2.2), in which the temporal derivative is

$$\frac{\partial \mathbf{v}}{\partial t}=-(\mathbf{v}_L\cdot\nabla)\mathbf{v} \,.$$

Then we obtain the Bernoulli law for a stationary state of the fluid in the frame of reference connected with the vortex:

$$P=\text{const}-\tfrac{1}{2}\rho[\mathbf{v}(\mathbf{r})-\mathbf{v}_L]^2 \,. \tag{3.9}$$

Next we write an equation of momentum balance for a cylindrical region of a radius r_0 around the vortex line. The momentum conservation law requires that an external force $\mathbf{F}$ exerted on the fluid in the balance region be equal to the momentum flux through the entire boundary of the balance region. The momentum-flux tensor is given by Eq. (2.5), in which the velocity $\mathbf{v}$ should be replaced by the relative velocity $\mathbf{v}-\mathbf{v}_L$. Using Eqs. (3.4), (3.8), and (3.9) and integrating the momentum-flux tensor over the cylindrical region of the radius r_0, we obtain the total momentum flux, which should be equal to the external force. It yields the following relation:

$$\mathbf{F}=\rho\boldsymbol{\kappa}\times(\mathbf{v}_0-\mathbf{v}_L) \,. \tag{3.10}$$

On the right-hand side of Eq. (3.10) we see the Magnus force.

We dwell on this simple derivation of the Magnus relation, which can be found in many books on hydrodynamics, in order to discuss the rather wide range of its validity. During the derivation we referred to the hydrodynamical equations only at large distance r_0 from the vortex line. It may have appeared as if the behavior of the fluid in the vortex core did not matter at all. However, we made an implicit assumption concerning the fluid in the vortex core. It was assumed that the momentum and the momentum flux were well-defined quantities everywhere, even inside of the vortex core where the hydrodynamical theory fails. This assumption does not arouse suspicion in the case of hydrodynamical vortices. But sometimes the assumption does not hold, as in the case of magnetic vortices in the easy-plane (planar) ferromagnet. The magnetic vortex is a linear defect of an ordered material like the superfluid vortex. On the close path around the magnetic vortex, the magnetic moment rotates through the angle 2π, remaining in the easy plane. The dynamics of the easy-plane ferromagnet is governed by the phenomenological Landau-Lifshitz theory. In this theory there is a momentum conservation law derived from the Lagrangian by means of the Noether theorem. But the momentum and its flux are divergent on the axis of a magnetic vortex, and this should be allowed for when one derives the equation of motion for the magnetic vortex from the momentum balance. The balance equation must include not only the momentum flux through the cylindrical surface remote from the vortex, but also the momentum flux through the surface on an infinitely small radius surrounding the axis of the vortex (Nikiforov and Sonin, 1983).

The Magnus relation may be extended to cases in which the vortex line is not straight and in which there are other vortex lines. Then the velocity field near the vortex line is a more general function of the position vector than that given by Eq. (3.8). But one can expand this function close to some point on the vortex line, and the first two terms of such an expansion should give the field, Eq. (3.8) (a singular term $\sim 1/r$ and a constant vector). In the case of a curved vortex the expansion also contains a logarithmically divergent term $\sim \ln r$, but it is cut off by the core radius r_c and gives a contribution $\sim \ln r_c$ to the constant velocity $\mathbf{v}_0$. Other terms of the expansion are important if the position vector $\mathbf{r}$ is the same order of magnitude as the distance from other vortex lines or the curvature radius of the vortex line itself. Our derivation of the Magnus relation, Eq. (3.10), is valid if one chooses the radius r_0 of the balance region to be larger than the radius r_c of the vortex core but smaller than the distance from the vortex line, where higher-order terms in the expansion of the velocity field become important. Thus the condition that restricts the validity of Eq. (3.10) is that the core radius be smaller than any spatial scale relevant for the hydrodynamical problem under consideration. One can apply the Magnus relation to a compressible fluid, too. In this case the relevant scale is a sound wavelength, and it should be larger than the core radius. Then, in deriving the Magnus relation, one can ignore the spatial variation of the density ρ and the fluid velocity $\mathbf{v}$ produced by the sound wave.

When we derived the Magnus relation we considered a vortex in steady motion at a constant velocity $\mathbf{v}_L$. If the velocity $\mathbf{v}_L$ varies in time, the inertial force of the fluid inside the balance region can contribute to the momentum balance. It is clear, however, that not the whole inertial force of the fluid contributes, but only the part of this force associated with the presence of the vortex (the difference between the inertial force of the fluid with and without the vortex). The rest of the inertial force is canceled by nonstationary corrections to the momentum flux. As a crude estimate for the inertial force that can be added to the Magnus relation we take the quantity $\sim \rho r_c^2 (d\mathbf{v}_L/dt)$ or $\sim \rho r_c^2 \omega \mathbf{v}_L$, where ω is a frequency. The inertial force is smaller than the Magnus force $\sim \rho\kappa \mathbf{v}_L$ provided

$$\omega < \kappa/r_c^2 \sim c^2/\kappa \ .$$

This inequality is violated only at quite large frequencies. For a sound wave it is the frequency at which the wavelength is of the same order as the vortex core radius. Under such conditions the hydrodynamical framework on which our derivation of the motion equation rests, completely fails. Therefore we adhere to the opinion of Baym and Chandler (1983), who studied the effect of vortex inertia on the dynamics, that a calculation of vortex inertia on the basis of hydrodynamics is questionable. Throughout this paper we assume that the Magnus relation [Eq. (3.10)] without the inertial force is exact enough.

The external force in the Magnus relation [Eq. (3.10)] may be any force localized over distances from the vortex line smaller than other relevant hydrodynamical scales. An example of such a force would be the electrical force on ions captured by the vortex core. But if the whole fluid is in some external force field, then the force in Eq. (3.10) includes that part of the total force on the fluid which is associated with the presence of the vortex. For example, when the fluid is in a gravitational field the Archimedes force acts upon on the vortex. It is proportional to the mass difference of the fluid with and without the vortex. The Archimedes force was introduced by Muslimov and Tsygan (1985) to explain the expulsion of vortices from the superconducting interiors of neutron stars.

When the external force $\mathbf{F}$ is absent, the vortex moves with the velocity $\mathbf{v}_0$ of the fluid current past the vortex. Thus we arrive at the Helmholtz theorem again, but now it is clear what the fluid velocity of the point on the vortex line is. The velocity $\mathbf{v}_0$ in Eq. (3.10) is the first term of the expansion for the velocity field near a point on a vortex line, regularized by subtracting the divergent term $\sim 1/r$ and by cutting off the logarithm term ($\ln r \rightarrow \ln r_c$). From here on we shall drop the subscript 0, assuming that the regularization of the velocity field is always performed.

In the Magnus relation we encounter a noteworthy feature of vortex dynamics: the resultant of forces on the vortex is balanced not by the inertia force, proportional to an acceleration (as in Newton's second law), but the gyrotropic Magnus force, proportional to a velocity. Any force acting upon the vortex can be described by its contribution to the net vortex velocity, and vice versa, any contribution to the vortex velocity may be presented as some force acting upon the vortex. Particularly, one can rewrite the Magnus relation, Eq. (3.10), in the following form:

$$-\rho\boldsymbol{\kappa}\times\mathbf{v}_L=\mathbf{F}+\mathbf{F}_0\ , \tag{3.11}$$

where $\mathbf{F}_0=-\rho\boldsymbol{\kappa}\times\mathbf{v}_0$ is the force exerted on the vortex by the fluid current. Such a "force versus velocity" relation is widely exploited in vortex dynamics.

Since the force on the vortex fixes its velocity, the latter cannot be an independent variable determined by initial conditions as in the case of a particle. So the particle in two-dimensional space has twice the degrees of freedom of a rectilinear vortex performing two-dimensional motion. As a result, the dynamical behavior of the vortex and of the particle is essentially different. Suppose both are located in a two-dimensional well. The particle in such a well would have two linearly polarized oscillation modes, but the vortex would have one elliptically polarized mode reducing to a circularly polarized mode when the well is axisymmetric. The Kelvin mode considered in Secs. III.D and III.E gives an example of such a mode for which the bending energy of the vortex line plays the role of the potential energy forming the well.

C. Canonical equations of motion of vortex lines in an incompressible perfect fluid

The Helmholtz theorem forms the basis of the theory of vortex motion in an incompressible perfect fluid, a theory which was developed in the past century. Suppose there is a set of vortex lines inducing the velocity field, Eq. (3.3). One obtains the velocity of the point $\mathbf{R}_i$ on the ith vortex line by taking a limit $\mathbf{R}\to\mathbf{R}_i$ in Eq. (3.3) and subtracting the singularity $\propto 1/|\mathbf{R}-\mathbf{R}_i|$ in the self-induction term $i=j$, in accordance with the procedure of regularization described in Sec. III.B. The velocity of the point $\mathbf{R}_i$ is

$$\frac{d\mathbf{R}_i}{dt}=\sum_j\frac{\kappa_j}{4\pi}\int\frac{d\mathbf{R}_j\times(\mathbf{R}_i-\mathbf{R}_j)}{|\mathbf{R}_i-\mathbf{R}_j|^3}\ . \tag{3.12}$$

For the set of parallel rectilinear vortex lines, integration in Eq. (3.12) is readily performed and yields

$$\frac{d\mathbf{r}_i}{dt}=\sum_{j\neq i}\frac{\boldsymbol{\kappa}_j\times(\mathbf{r}_i-\mathbf{r}_j)}{2\pi|\mathbf{r}_i-\mathbf{r}_j|^2}\ . \tag{3.13}$$

Here $\mathbf{r}_i$ is the two-dimensional position vector of the ith vortex line. The self-induction term $i=j$ drops out after regularization.

Let us rewrite Eq. (3.12) in the following form:

$$-\rho\boldsymbol{\kappa}_i(\mathbf{R}_i)\times\frac{d\mathbf{R}_i}{dt}=-\frac{\delta\mathscr{E}}{\delta\mathbf{R}_i}\ . \tag{3.14}$$

Here the vector $\boldsymbol{\kappa}_i(\mathbf{R}_i)$ is a tangent to the ith vortex line in the point $\mathbf{R}_i$, and the magnitude of this vector is the circulation κ_i. One can readily prove the identity of Eqs. (3.12) and (3.14) by calculating the functional derivative of the energy $\mathscr{E}$ given by Eq. (3.7):

$$\frac{\delta\mathscr{E}}{\delta\mathbf{R}_i}=\rho\sum_j\frac{\kappa_j}{4\pi}\left[\int\frac{d\mathbf{R}_j}{|\mathbf{R}_i-\mathbf{R}_j|^3}[\boldsymbol{\kappa}_i\cdot(\mathbf{R}_i-\mathbf{R}_j)]-\int\frac{d\mathbf{R}_j\cdot\boldsymbol{\kappa}_i}{|\mathbf{R}_i-\mathbf{r}_j|^3}(\mathbf{R}_i-\mathbf{R}_j)\right]$$
$$=\rho\sum_j\frac{\kappa_j}{4\pi}\int\frac{\boldsymbol{\kappa}_i\times[d\mathbf{R}_j\times(\mathbf{R}_i-\mathbf{R}_j)]}{|\mathbf{R}_i-\mathbf{R}_j|^3}\ . \tag{3.15}$$

On the right-hand side of Eq. (3.14) we see the generalized force acting upon the ith vortex line in the point $\mathbf{R}_i$. This force is balanced by the Magnus force, in accordance with the basic law of vortex motion. Terms in the expression for the generalized force [see Eq. (3.15)] correspond to forces exerted by all vortex lines. Each of them has its counterpart in Eq. (3.12) for the velocity.

Equations of motion given by Eq. (3.14) form a closed system describing the dynamical behavior of vortex lines, as well as that of the fluid as a whole. The Euler equation is necessary only if one wants to find the distribution of the chemical potential or the pressure over the fluid. Let us see what form the Euler equation takes in the model of singular vortex lines. The vorticity may be presented as a sum of contributions due to all vortex lines:

$$\tilde{\boldsymbol{\omega}}(\mathbf{R})=\sum_j\kappa_j\int d\mathbf{R}_j\delta(\mathbf{R}-\mathbf{R}_j)\ . \tag{3.16}$$

Here $\delta(\mathbf{R}-\mathbf{R}_j)$ is a three-dimensional δ function and $\int d\mathbf{R}_j$ is an integral over the jth vortex line. Substituting Eq. (3.16) into the Euler equation (2.6) and using the Gibbs-Duhem relation, Eq. (2.3), at $T=0$ to replace $\nabla P/\rho$ by $\nabla\mu$, we obtain

$$\frac{\partial\mathbf{v}}{\partial t}+\sum_j\kappa_j\int[d\mathbf{R}_j\times\mathbf{v}(\mathbf{R})]\delta(\mathbf{R}-\mathbf{R}_j)=-\nabla(\mu+\tfrac{1}{2}\mathbf{v}^2)\ . \tag{3.17}$$

Singular terms in Eq. (3.17) play the role of pseudopotentials, allowing us to extend the Euler equation to points of vortex lines where the hydrodynamical theory is invalid in a strict sense.

The Euler equation is a field equation, and Eq. (3.14) is an equation of motion of "charges." In electrodynamics the field possesses its own degrees of freedom only when the retardation of the interaction is important, but may be eliminated from the nonrelativistic dynamical theory. Likewise one can eliminate the Euler equation, describing an incompressible fluid by equations of motion for vortex lines only.

In the rest of this review the most attention will be devoted to small oscillations of an array of rectilinear vortices parallel to the z axis. One can describe their motion by a set of two-dimensional vectors of displacement $\mathbf{u}_i(z)$ in the xy plane which depend on the coordinate z and the index i of the vortex. The original three-dimensional position vector $\mathbf{R}_i$ is connected with $\mathbf{u}_i(z)$ by

$$\mathbf{R}_i=z_i\hat{\mathbf{z}}+\mathbf{r}_i+\mathbf{u}_i(z)\ . \tag{3.18}$$

Here $\hat{\mathbf{z}}$ is the unit vector along the z axis and $\mathbf{r}_i$ is the two-dimensional equilibrium position vector of the ith vortex line. Then we can rewrite Eq. (3.14) as follows (Fetter, 1967):

$$\frac{\partial \mathbf{u}_i(z)}{\partial t} = -\frac{1}{\rho\kappa_i}\hat{\mathbf{z}}\times\frac{\delta\mathscr{E}}{\delta\mathbf{u}_i(z)} . \tag{3.19}$$

These are Hamiltonian equations for pairs of conjugate variables (x_i,y_i) that are components of the displacement $\mathbf{u}_i$. If one assumes x_i to be a coordinate, then $\rho\kappa y_i$ is a canonical momentum. The particle moving in the xy plane would have two pairs of conjugate variables: (x,p_x) and (y,p_y).

D. Kelvin oscillations of an isolated vortex line

Here we consider circularly polarized waves propagating on an isolated vortex line. This problem was solved by Lord Kelvin a hundred years ago (Thompson, 1880). Using the concept of linear tension ε given by Eq. (3.5), we estimate the energy of the vortex line as $\mathscr{E}=\varepsilon L$, where L is the length of a slightly bent vortex line:

$$L=\int dz\left[1+\left[\frac{d\mathbf{u}(z)}{dz}\right]^2\right]^{1/2} \approx L_0+\frac{1}{2}\int dz\left[\frac{d\mathbf{u}(z)}{dz}\right]^2 . \tag{3.20}$$

We drop a subscript of the displacement $\mathbf{u}$, since only one vortex is retained in our analysis. Taking the functional derivative of the energy $\mathscr{E}$ and substituting it into Eq. (3.19), we obtain the equation governing small oscillations of the vortex line:

$$-\rho\kappa\hat{\mathbf{z}}\times\frac{d\mathbf{u}}{dt}=\varepsilon\frac{d^2\mathbf{u}(z)}{dz^2} . \tag{3.21}$$

The force of linear tension on the right-hand side of Eq. (3.21) is balanced by the Magnus force.

Equation (3.21) admits plane-wave solutions $\propto \exp(ipz-i\omega t)$ with the dispersion law

$$\omega=\pm v_s p^2 , \tag{3.22}$$

where rigidity v_s is

$$v_s=\frac{\varepsilon}{\rho\kappa}=\frac{\kappa}{4\pi}\ln\frac{r_m}{r_c} . \tag{3.23}$$

As an upper cutoff r_m one should choose a distance where perturbations produced by oscillations penetrate. In an unbound fluid, such a distance is expected to be of order of the wavelength $\sim 1/p$. This means that our approach based on the differential equation (3.21) is not quite rigorous, since the coefficient of this equation depends on the wave number p. But it is possible to develop a more rigorous approach.

Let us look for a general solution of the problem as a superposition of normal-mode solutions, each being a propagating plane wave. We perform the Fourier transformation

$$\mathbf{u}(z,t)=\frac{1}{2\pi}\int dp\, \mathbf{u}(p,t)\exp(ipz) . \tag{3.24}$$

The expression for the energy in the Fourier representation is obtained from Eq. (3.7) retaining only the self-action term for one vortex line. Using Eqs. (3.18) and (3.24), we have

$$\begin{aligned}\mathscr{E}&=\frac{\rho\kappa^2}{4\pi}\int\int dz_1dz_2\frac{1+\dfrac{d\mathbf{u}(z_1)}{dz}\cdot\dfrac{d\mathbf{u}(z_2)}{dz}}{[(z_1-z_2)^2+|\mathbf{u}(z_1)-\mathbf{u}(z_2)|^2]^{1/2}}\\ &\approx\mathscr{E}_0+\frac{\rho\kappa^2}{4\pi}\int\int dz_1dz_2\left[\frac{\dfrac{d\mathbf{u}(z_1)}{dz}\cdot\dfrac{d\mathbf{u}(z_2)}{dz}}{|z_1-z_2|}-\frac{1}{2}\frac{|\mathbf{u}(z_1)-\mathbf{u}(z_2)|^2}{|z_1-z_2|^3}\right]\\ &=\mathscr{E}_0+L_0\frac{\rho\kappa^2}{8\pi^2}\int dp\, \mathbf{u}(p)^*\mathbf{u}(p)K(p) .\end{aligned} \tag{3.25}$$

Here $\mathscr{E}_0$ and L_0 are the energy and the length of the straight vortex line and

$$K(p)=\int_{-\infty}^{\infty}\frac{dz}{|z|}\left[p^2e^{ipz}-\frac{1}{z^2}(1-e^{ipz})\right] . \tag{3.26}$$

Divergence at small z is cut off by the core radius r_c. Then by partial integration one obtains

$$K(p)=p^2\int_{r_c}^{\infty}\frac{dz}{z}\cos pz=p^2\ln(1/pr_c) . \tag{3.27}$$

The equation of motion (3.19) after Fourier transformation takes the form

$$\frac{d\mathbf{u}(p)}{dt}=-\frac{2\pi}{\rho\kappa L_0}\left[\hat{z}\times\frac{\delta\mathscr{E}}{\delta\mathbf{u}(p)^*}\right]=-\frac{\kappa K(p)}{4\pi}\hat{z}\times\mathbf{u}(p) . \tag{3.28}$$

This equation describes the Kelvin oscillations with the frequency of Eq. (3.22), but now the upper cutoff r_m in

Eq. (3.23) is rigorously determined.

Oscillations of the single vortex line have also been considered within the frame work of the Gross-Pitaevskii theory for a weakly interacting Bose gas (Gross, 1961; Pitaevskii, 1961; Fetter, 1965). Outside of the vortex core the equations of the Gross-Pitaevskii theory coincide with the hydrodynamical equations. (A justification of hydrodynamics is thus provided by this microscopic model.) But the Gross-Pitaevskii theory is able to provide an exact value of the lower cutoff r_c in the expression for rigidity ν_s given by Eq. (3.23).

E. Experimental observation of the Kelvin mode

The Kelvin modes have been observed in experiments on torsion oscillations of a pile of disks in rotating He II. But in these experiments frequencies were rather low, and the Kelvin oscillations were strongly modified by collective effects due to long-range interaction of vortices. Pile-of-disks experiments will be discussed later, in Sec. VI, after the analysis of boundary problems that is necessary for their interpretation.

Some time ago Ashton and Glaberson (1979) tested the dispersion law [Eq. (3.22)] at high frequencies where collective effects are not important; in this experiment they dealt with pure Kelvin waves in isolated vortex lines. Ashton and Glaberson investigated the motions of ions along vortex lines in rotating He II in the presence of a rf electric field, transverse to the vortex lines. It had been suggested earlier (Halley and Cheung, 1968; Halley and Ostermeier, 1977) that such a field would strongly couple to vortex waves under suitable conditions. Resonant generation of vortex waves occurs when the following conditions are satisfied:

$$\omega_{\rm rf}=\omega(p)-v_{\rm ion}p \ , \tag{3.29}$$

$$v_{\rm ion}=\frac{d\omega(p)}{dp} \ . \tag{3.30}$$

Here $\omega_{\rm rf}$ is a rf frequency of the field, which was 10^7 Hz in the experiment, $v_{\rm ion}$ is an ion velocity. The first condition means that the vortex wave frequency in the frame of reference of the moving ion is the same as the frequency $\omega_{\rm rf}$ of the field. The second condition ensures that the ion, pumping the energy into the vortex wave, remains in the vicinity of the vortex wave packet moving with the group velocity $d\omega/dp$. In addition to these two conditions, the sense of circular polarization of the rf field should be the same as that of the vortex wave in the frame of reference of the ion. Ashton and Glaberson (1979) measured the ion velocity as a function of the dc electric field, driving ions along the vortex line. They observed anomalies on the plot when all conditions of the resonance were satisfied. Despite a small discrepancy with theory, explained by Ashton and Glaberson in terms of the field inhomogeneity, the experiment provides rather convincing evidence of the existence of propagating Kelvin wave at high frequencies.

IV. OSCILLATIONS OF QUANTIZED VORTICES IN A ROTATING PERFECT FLUID

A. Hydrodynamics of rotating superfluids

The equations of motion (3.12) for vortex lines yield a complete description of the dynamical behavior of an incompressible perfect fluid with quantized vorticity. However, analytic solution of these equations is a tractable problem only when one deals with a few vortices or with regular arrays of straight parallel vortices. In the latter case the apparatus of complex functions can be applied (Milne-Thomson, 1960). In the hydrodynamics of superfluids the most remarkable analytic result belongs to Tkachenko (1966), who has solved exactly and completely the problem of small oscillations for the infinite two-dimensional regular array of rectilinear vortices using the theory of elliptic functions. According to Dyson (1971, p. 51), Tkachenko's solution was "a *tour de force* of powerful mathematics." However, this analytic technique cannot be extended to three-dimensional problems when vortex flexure occurs. Moreover, most experiments deal with fluids involving a large number of vortices constituting a very dense array, and only averaged parameters are available for experimental observation. Thus we follow the approach of the continuum elasticity theory of solids and approximate equations for discrete vortex lines by equations for continuous fields of such averaged parameters as the vortex density, deformations of the vortex array, and the vortex velocity. It is assumed that these parameters slowly vary over the distance between vortices, and one may use an expansion in gradients or in wave vectors in the Fourier representation. The initial equations governing motion of the fluid with singular vortex lines play the role of microscopic equations for atoms in a solid. This is why the term "microscopic" is sometimes used to refer to equations in terms of quantized vortex lines and "macroscopic" to refer to equations of the continuum model (Baym and Chandler, 1983). But in fact "microscopic" equations are formulated within the scope of the phenomenological hydrodynamical theory and have nothing to do with truly microscopic equations of the fluid.

Macroscopic hydrodynamical equations can be derived from microscopic hydrodynamics by means of a coarse-graining procedure, or they may be formulated on a phenomenological basis using conservation laws and requirements imposed by symmetry. Macroscopic hydrodynamics was derived and applied to the description of rotating superfluids beginning with the pioneering work of Hall (1958), Mamaladze and Matinyan (1960), and Bekarevich and Khalatnikov (1961). Bekarevich and Khalatnikov developed the first general nonlinear phenomenological description of rotating superfluids. However, their hydrodynamics neglected the energy increase produced by shearing of the vortex lattice. Therefore Tkachenko modes could not be obtained in such a hydrodynamics. A continuum hydrodynamical theory allowing for the crystalline order in the vortex lattice and its shearing rigidity was developed by Tkachenko (1969), but his theory did not consider possible flexure of vortices. The extension of the hydrodynamical continuum theory to include effects of shearing rigidity and vortex flexure was carried out by Sonin (1976) and Williams and Fetter (1977). In both papers linear equations for quantized vorticity were derived in the Fourier representation, then expanded in the wave vector, retaining only first terms of the expansion. This procedure yielded the long-wavelength equations of motion within the continuum theory. An equivalent coarse-graining procedure directly in the coordinate space was suggested by Baym and Chandler (1983). They also restricted themselves to a linear theory. Volovik and Dotsenko (1980) have derived hydrodynamical equations for the vortex lattice using Poisson bracket techniques. A general nonlinear hydrodynamics of a rotating superfluid involving the effects of both vortex tension and Tkachenko shearing rigidity of the vortex lattice was formulated by Andreev and Kagan (1984).

Though nearly all calculations concerning vortex oscillations remain within the scope of the linear theory, we prefer to begin with formulation of the general nonlinear macroscopic hydrodynamics in order to have an outlook on the problem as a whole. Following the plan of the present review we restrict ourselves in this section to the one-fluid hydrodynamics of a perfect fluid. Resultant equations differ from those of Andreev and Kagan (1984) by another choice of variables characterizing the form of the vortex lattice. We shall use displacements and deformations instead of the variables connected with the metric tensor of the vortex lattice used in the paper of Andreev and Kagan.

First let us see what form the Euler equation takes in macroscopic hydrodynamics. In the initial microscopic hydrodynamics of the perfect fluid with singular vortex lines it is given by Eq. (3.17). From here on we assume that any vortex line bears one quantum of circulation, $\kappa = h/M$, where M is the mass of bosons or Cooper pairs of fermions. We shall average Eq. (3.17) over a vortex lattice cell. Because of the singular character of the vorticity field, the velocity of the fluid in the vector product is not affected by the procedure of averaging; it remains a regularized local velocity of the point on the vortex line. This velocity will be denoted as $\mathbf{v}_L$. Thus the averaged equation is

$$\frac{\partial \mathbf{v}}{\partial t} + \tilde{\boldsymbol{\omega}} \times \mathbf{v}_L = -\nabla(\mu + \tfrac{1}{2}v^2) \ . \tag{4.1}$$

Here $\mathbf{v}$ and $\tilde{\boldsymbol{\omega}}$ are the averaged velocity and vorticity, respectively, the latter equal to the number density of vortex lines per unit area multiplied by the circulation quantum κ. Symbols denoting averages are dropped. We write the gradient term on the right-hand side of Eq. (4.1) in the same form as in Eq. (3.17). But μ in Eq. (4.1) now denotes some scalar function, which differs from the chemical potential μ in Eq. (3.17). We shall not explore the relation between old and new μ because our intention is to formulate the hydrodynamics phenomenologically, appealing only to conservation laws and symmetry. But later it will be clear that μ in Eq. (4.1) is the chemical potential in macroscopic hydrodynamics, as required by the energy conservation law.

The Euler equation (4.1) is closely connected with the continuity equation for vorticity. The latter is obtained by taking a curl of both parts of Eq. (4.1),

$$\frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t} + \mathbf{v} \times (\tilde{\boldsymbol{\omega}} \times \mathbf{v}_L) = 0 \tag{4.2}$$

or

$$\frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t} + (\mathbf{v}_L \cdot \nabla)\tilde{\boldsymbol{\omega}} + [\tilde{\boldsymbol{\omega}}(\nabla \cdot \mathbf{v}_L) - (\tilde{\boldsymbol{\omega}} \cdot \nabla)\mathbf{v}_L] = 0 \ . \tag{4.3}$$

Because $\tilde{\boldsymbol{\omega}}$ is proportional to the density of vortex lines, Eq. (4.3) is a differential conservation law for vortex lines, but its validity is not restricted to the model of singular vortex lines; it holds even when vorticity is distributed continuously over the entire space. Therefore one can arrive at the Euler equation [Eq. (4.1)] starting from the evident continuity equation for vorticity in the form of Eq. (4.2) or (4.3). Integration yields Eq. (4.1), but with an indeterminate scalar function $\mu + v^2/2$.

Another insight into the physical meaning of Eq. (4.1) is achieved by recalling the connection between the velocity $\mathbf{v}$ and the phase φ of the order parameter of the superfluid: $\mathbf{v} = (\hbar/M)\nabla\varphi$. A phase difference along the path around the vortex line is equal to 2π. Suppose we integrate Eq. (4.1) along a path between some points 1 and 2. This yields an expression for the temporal derivative of the phase difference:

$$\begin{aligned}\frac{d(\varphi_2 - \varphi_1)}{dt} &= \frac{M}{\hbar}\int_1^2 d\mathbf{l} \cdot \frac{\partial \mathbf{v}}{\partial t} \\ &= -\frac{M}{\hbar}\left[\left[\mu + \frac{v^2}{2}\right]_2 - \left[\mu + \frac{v^2}{2}\right]_1\right] \\ &\quad - \frac{M}{\hbar}\int_1^2 d\mathbf{l} \cdot (\tilde{\boldsymbol{\omega}} \times \mathbf{v}_L) \ .\end{aligned}$$

The term originating with the vector product in Eq. (4.1) gives a contribution to the phase difference variation due to a flow of vortex lines across the path between points 1 and 2. Indeed, any passage of a vortex line across the path produces a change 2π in the phase difference. This is a "phase slippage," a concept invented to explain Josephson-type phenomena in superconductors and superfluids (Anderson, 1966).

The next step is to rewrite Eq. (4.1),

$$\frac{\partial \mathbf{v}}{\partial t} + \tilde{\boldsymbol{\omega}} \times \mathbf{v} + \nabla(\mu + \tfrac{1}{2}v^2) = \mathbf{f}/\rho \ , \tag{4.4}$$

introducing a force

$$\mathbf{f} = -\rho\tilde{\boldsymbol{\omega}} \times (\mathbf{v}_L - \mathbf{v}) \ . \tag{4.5}$$

Comparing Eq. (4.5) with Eq. (3.10) and recalling that $\tilde{\boldsymbol{\omega}}$ is equal to the circulation quantum κ multiplied by the two-dimensional density of vortex lines, we see that $\mathbf{f}$ is a force acting upon vortex lines in unit volume of the fluid moving with velocity $\mathbf{v}$. Thus it should be connected with a variation of the energy due to vortex displacements. The total energy density is given by

$$E = E_0 + \tfrac{1}{2}\rho v^2, \quad E_0 = E_{\rm in}(\rho) + E_v(\mathbf{u}) \; . \tag{4.6}$$

Here E_0 is the energy density in the frame of reference moving with the averaged velocity $\mathbf{v}$. It includes the internal energy density $E_{\rm in}(\rho)$, depending on the fluid density ρ, and the vortex energy density $E_v(\mathbf{u})$, which is a functional of the vortex line displacements $\mathbf{u}$. Displacements are determined referring to some arbitrarily chosen equilibrium pattern of vortex lines. Here we consider a general nonlinear theory, so displacements $\mathbf{u}$ are arbitrary three-dimensional vectors and do not necessarily lie in the xy plane as assumed in Sec. III.C. In fact, $E_v(\mathbf{u})$ is the density of the total kinetic energy of the fluid after subtraction of the kinetic energy of the averaged flow with density $\rho v^2/2$. Of course, $E_v(\mathbf{u})$ depends only on gradients of $\mathbf{u}$, not on their absolute values, so the Gibbs thermodynamic relation is

$$\begin{aligned} dE_0 &= \mu\, d\rho + \frac{\partial E_0}{\partial \nabla_i u_j} d(\nabla_i u_j) \\ &= \mu\, d\rho + \frac{\delta E_0}{\delta u_i} du_i + \nabla_i \left[\frac{\partial E_0}{\partial \nabla_i u_j} du_j \right] , \end{aligned} \tag{4.7}$$

or, for the density of the total energy,

$$\begin{aligned} dE &= (\mu + \tfrac{1}{2} v^2) d\rho + \rho \mathbf{v} \cdot d\mathbf{v} + \frac{\delta E}{\delta \mathbf{u}} \cdot d\mathbf{u} \\ &\quad + \nabla_i \left[\frac{\partial E}{\partial \nabla_i \mathbf{u}} \cdot d\mathbf{u} \right] . \end{aligned} \tag{4.8}$$

Here the functional derivative is determined by the usual expression:

$$\frac{\delta E}{\delta \mathbf{u}} = \frac{\partial E}{\partial \mathbf{u}} - \nabla_j \left[\frac{\partial E}{\partial \nabla_j \mathbf{u}} \right] = -\nabla_j \left[\frac{\partial E}{\partial \nabla_j \mathbf{u}} \right] . \tag{4.9}$$

However, it is not the derivative that determines the force $\mathbf{f}$ in Eq. (4.4). Let us consider a small variation of positions of the vortex lines. There are two ways to define a small variation of displacements $\mathbf{u}$. The first definition refers to a fixed point on a vortex line, and the variation is defined as the difference between two positions of this point. It is a small variation $D\mathbf{u}$ in the Lagrange representation of hydrodynamics, and the force $\mathbf{f}$ should be connected with the energy variation with respect to the Lagrange variation. However, the displacement $\mathbf{u}(\mathbf{R},t)$ in Eqs. (4.6)–(4.9) is a field variable referring to a fixed point $\mathbf{R}$ of the coordinate space, i.e., a variable in the Euler representation of hydrodynamics. In this representation a small variation $d\mathbf{u}$ is the difference between displacements of those two points on vortex lines which were located at point $\mathbf{R}$ after and before variation of positions of all vortex lines. The functional variation in Eqs. (4.7)–(4.9) is determined with respect to the Euler variation $d\mathbf{u}$. The relation between the Lagrange and the Euler variations is well known:

$$D\mathbf{u} = d\mathbf{u} + (D\mathbf{u} \cdot \nabla)\mathbf{u} \; . \tag{4.10}$$

Now we can find the variation of the total energy in the Lagrange sense. At first we write the energy variation in the Euler sense:

$$\delta \mathscr{E} = \int d\mathbf{R} \left[\frac{\partial E}{\partial \mathbf{u}} d\mathbf{u} + \frac{\partial E}{\partial \nabla_j \mathbf{u}} d\nabla_j \mathbf{u} \right] . \tag{4.11}$$

Then we eliminate $d\mathbf{u}$ with the help of Eq. (4.10),

$$\begin{aligned} \delta \mathscr{E} = \int d\mathbf{R} \Bigg[& \frac{\partial E}{\partial \mathbf{u}} [D\mathbf{u} - (D\mathbf{u} \cdot \nabla)\mathbf{u}] \\ & + \frac{\partial E}{\partial \nabla_j \mathbf{u}} \nabla_j [D\mathbf{u} - (D\mathbf{u} \cdot \nabla)\mathbf{u}] \Bigg] . \end{aligned} \tag{4.12}$$

Integrating by parts and allowing for $\partial E / \partial \mathbf{u} = 0$, we obtain

$$\begin{aligned} \delta \mathscr{E} = \int d\mathbf{R} \Bigg[& -\nabla_j \left[\frac{\partial E}{\partial \nabla_j \mathbf{u}} \right] \\ & + \nabla_j \left[\frac{\partial E}{\partial \nabla_j u_i} \right] \nabla u_i \Bigg] D\mathbf{u} \; . \end{aligned} \tag{4.13}$$

The quantity in brackets is the functional derivative in the Lagrange sense determining the force $\mathbf{f}$:

$$\mathbf{f} = \nabla_j \left[\frac{\partial E}{\partial \nabla_j \mathbf{u}} \right] - \nabla_j \left[\frac{\partial E}{\partial \nabla_j u_i} \right] \nabla u_i \; . \tag{4.14}$$

Equations (4.4), (4.5), (4.14), and the continuity equation (2.1) at given dependence of the energy on all variables constitute a closed system of hydrodynamical equations. It is invariant with respect to the Galilean transformation and also to the transformation to the rotating coordinate frame, provided that $\tilde{\omega}$ is an absolute vorticity in the inertial frame of reference and centrifugal forces are ignored. We can prove that our system of equations obeys the conservation laws for the momentum,

$$\frac{\partial j_i}{\partial t} + \nabla_j \Pi_{ij} = 0 \; , \tag{4.15}$$

and for the energy,

$$\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{J} = 0 \; . \tag{4.16}$$

The momentum-flux tensor is

$$\Pi_{ij} = P\delta_{ij} + \rho v_i v_j + \sigma_{ij} \; , \tag{4.17}$$

where the elastic stress tensor of the vortex lattice is given by

$$\sigma_{ij} = -\frac{\partial E}{\partial \nabla_j u_i} + \nabla_i u_k \frac{\partial E}{\partial \nabla_j u_k} \; . \tag{4.18}$$

For the energy flow one has

$$\mathbf{J} = (\mu + \tfrac{1}{2} v^2)\mathbf{j} - \frac{\partial E}{\partial \nabla u_i} [v_{Li} - (\mathbf{v}_L \cdot \nabla) u_i] \; . \tag{4.19}$$

The pressure

$$P=-E_0+\mu\rho \tag{4.20}$$

satisfies the differential Gibbs-Duhem relation,

$$dP=\rho\, d\mu-\frac{\partial E}{\partial \nabla_j u_i} d\nabla_j u_i \ . \tag{4.21}$$

The energy conservation law is obeyed only provided that μ in the Euler equation Eq. (4.1) is the true thermodynamical chemical potential.

The hydrodynamics of Bekarevich and Khalatnikov follows from the present theory when one ignores the dependence of the energy on shearing deformation of the vortex array, i.e., the crystalline order in the vortex array. Then the vortex energy density $E_v(\tilde{\omega})$ depends only on the vortex density, and the variation of the energy produced by variation of the pattern of vortex lines is

$$\begin{aligned}\delta\mathscr{E}&=\int d\mathbf{R}\frac{\partial E}{\partial\tilde{\omega}}d\tilde{\omega}\\ &=\int d\mathbf{R}\frac{\partial E}{\partial\tilde{\omega}}\cdot\frac{1}{\tilde{\omega}}(\tilde{\boldsymbol{\omega}}\cdot d\tilde{\boldsymbol{\omega}}) \ .\end{aligned} \tag{4.22}$$

The variation $d\tilde{\boldsymbol{\omega}}$ is connected with the Lagrange variation of the displacement $D\mathbf{u}$, following from purely kinematical arguments, by the formula [compare with Eq. (4.3)]

$$d\tilde{\boldsymbol{\omega}}=-(D\mathbf{u}\cdot\nabla)\tilde{\boldsymbol{\omega}}-[\tilde{\boldsymbol{\omega}}(\nabla\cdot D\mathbf{u})-(\tilde{\boldsymbol{\omega}}\cdot\nabla)D\mathbf{u}] \ . \tag{4.23}$$

We eliminate $d\tilde{\boldsymbol{\omega}}$ from Eq. (4.22) by substitution of Eq. (4.23) and compare the obtained expression with Eq. (4.13). After some integration by parts, this yields the following formula connecting derivatives with respect to $\tilde{\omega}$ and to gradients of the displacement $\mathbf{u}$:

$$-\frac{\partial E}{\partial\tilde{\omega}}\left[\tilde{\omega}\delta_{ij}-\frac{\tilde{\omega}_i\tilde{\omega}_j}{\tilde{\omega}}\right]=\frac{\partial E}{\partial\nabla_i u_j}-\nabla_j u_k\frac{\partial E}{\partial\nabla_j u_k} \ . \tag{4.24}$$

With the help of this formula it is not difficult to prove that our hydrodynamical equations are identical with those of Bekarevich and Khalatnikov (1961) at $T=0$.

B. Equations of motion in linear hydrodynamics

Now we descend from the general theory to its particular cases, which permit a quantitative analysis. Let us derive the expression for the vortex energy density E_v in the harmonic approximation. The vortex array constitutes a triangular lattice possessing hexagonal symmetry. Its energy in terms of displacement gradients is elastic. One can find a general expression for the elastic energy for the lattice with hexagonal symmetry in the book of Landau and Lifshitz (1965). This expression may be simplified, since in the harmonic approximation only displacements in the xy plane normal to the rotation axis increase the energy. Therefore all terms containing components of the deformation tensor

$$u_{ij}=\tfrac{1}{2}(\nabla_i u_j+\nabla_j u_i) \tag{4.25}$$

with $i=j=z$ drop out. As a result, the density of the elastic energy is determined by three elasticity moduli,

$$\begin{aligned}E_v=&4C_1(u_{xz}^2+u_{yz}^2)+C_2(u_{xx}+u_{yy})^2\\ &+C_3[(u_{xx}-u_{yy})^2+4u_{xy}^2] \ ,\end{aligned} \tag{4.26}$$

or in terms of displacements $\mathbf{u}$, which are two-dimensional vectors in the xy plane, henceforth

$$\begin{aligned}E_v=&C_1\left[\frac{d\mathbf{u}}{dz}\right]^2+C_2(\nabla\cdot\mathbf{u})^2\\ &+C_3\left[\left[\frac{\partial u_x}{\partial x}-\frac{\partial u_y}{\partial x}\right]^2+\left[\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right]^2\right] \ .\end{aligned} \tag{4.27}$$

The first two terms in Eq. (4.27) are not connected with the crystalline order and shear elasticity. Therefore they can be derived from the Bekarevich-Khalatnikov theory. The density of the vortex energy in this theory is equal to the energy ε of one vortex line per unit length [see Eq. (3.5)] multiplied by the two-dimensional density of vortex lines $\tilde{\omega}/\kappa$:

$$E_v=\frac{\rho\kappa}{4\pi}\tilde{\omega}\ln\frac{r_m}{r_c}=\frac{\rho\kappa}{8\pi}\tilde{\omega}\ln\frac{\omega_c}{\tilde{\omega}} \ . \tag{4.28}$$

Here we chose the intervortex distance on the triangular lattice,

$$r_v=\left[\frac{2\kappa}{\sqrt{3}\tilde{\omega}}\right]^{1/2} \tag{4.29}$$

as the upper cutoff r_m in the logarithm argument. This means that $\omega_c=2\kappa/\sqrt{3}r_c^2$ in Eq. (4.28).

In the harmonic approximation, the vortex energy E_v may be expanded in terms of a small deviation

$$\boldsymbol{\omega}'=\tilde{\boldsymbol{\omega}}-2\boldsymbol{\Omega} \tag{4.30}$$

of vorticity $\tilde{\boldsymbol{\omega}}$ from the equilibrium value 2Ω. The expansion includes terms of first order in $\boldsymbol{\omega}'$. But the variation of vorticity is not independent, being coupled with the variation of the velocity of the fluid. In the correct theory, first-order terms in $\boldsymbol{\omega}'$ should be canceled out by other first-order terms, so we retain in the expansion of the Bekarevich-Khalatnikov vortex energy, Eq. (4.28), only terms of the second order in $\boldsymbol{\omega}'$:

$$\begin{aligned}E_v&\simeq E_{v0}+\frac{1}{2}\frac{\partial^2 E_v}{\partial\tilde{\omega}_i\partial\tilde{\omega}_j}\omega_i'\omega_j'\\ &=E_{v0}+\frac{1}{4\Omega}\frac{\partial E_v}{\partial\tilde{\omega}}\left[\omega'^2-\frac{(\boldsymbol{\omega}'\cdot\boldsymbol{\Omega})^2}{\Omega^2}\right]\\ &\quad+\frac{1}{2\Omega^2}\frac{\partial^2 E_v}{\partial\tilde{\omega}^2}(\boldsymbol{\omega}'\cdot\boldsymbol{\Omega})^2 \ .\end{aligned} \tag{4.31}$$

The deviation of vorticity $\boldsymbol{\omega}'$ is connected with small displacements of vortex lines by a relation similar to the linearized version of Eq. (4.23) connecting $d\tilde{\boldsymbol{\omega}}$ and $D\mathbf{u}$:

$$\boldsymbol{\omega}' = -2\Omega(\boldsymbol{\nabla}\cdot\mathbf{u}) + 2(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\mathbf{u} \ . \tag{4.32}$$

Now we can calculate derivatives $\partial E_v/\partial\tilde{\omega}$ and $\partial^2 E_v/\partial\tilde{\omega}^2$ using Eq. (4.28), substitute them together with $\boldsymbol{\omega}'$ given by Eq. (4.32) into Eq. (4.31), and compare the obtained expression with the first two terms in Eq. (4.27). This enables us to deduce values of two elastic moduli:

$$C_1 = \rho \nu_s \Omega, \quad C_2 = -\frac{\rho\kappa\Omega}{8\pi} \ , \qquad \nu_s = \frac{\kappa}{8\pi}\ln\frac{\omega_c}{2\Omega} \ . \tag{4.33}$$

The negative elastic constant C_2 does not lead to instability, since, as we have already mentioned, the displacements are not completely independent variables: the longitudinal part of the displacement field is connected with the variation of vorticity and therefore with the fluid velocity (Baym and Chandler, 1983).

Such a simple calculation of two elastic moduli in the Bekarevich-Khalatnikov theory was possible because we used the logarithmic approximation, neglecting numbers comparable to the large logarithm. But it is these very numbers, which are different for different types of vortex lattice, that determine shear rigidity. Therefore determination of the shear elastic modulus C_3 involves more ingenious calculations, like those performed by Tkachenko. The reader is referred to the original papers of Tkachenko (1965, 1966) on the subject and to Appendix B in the paper of Baym and Chandler (1983). The only conclusion we can draw without detailed calculations is that C_3 should be of the same order as C_2. The exact calculation yields

$$C_3 = \frac{\rho c_T^2}{2} = \frac{\rho\kappa\Omega}{16\pi} \ . \tag{4.34}$$

Here $c_T = (\kappa\Omega/8\pi)^{1/2}$ is the Tkachenko wave velocity. The same relation between the shear elastic modulus and the transverse sound velocity holds in atomic crystals.

Having calculated the values of elastic constants in the density of the vortex energy [Eq. (4.27)], we can rewrite Eq. (4.18) for the elastic stress tensor as follows:

$$\sigma_{ij} = -2\Omega\rho\nu_s \frac{\partial u_i}{\partial z}\delta_{jz} + \rho c_T^2[3(\boldsymbol{\nabla}\cdot\mathbf{u})\delta_{ij} - (\nabla_i u_j + \nabla_j u_i)] \times (1-\delta_{iz})(1-\delta_{jz}) \ . \tag{4.35}$$

To close this section, we present the complete system of linearized equations of motion for macroscopic hydrodynamics:

$$\frac{\partial\rho'}{\partial t} + \rho(\boldsymbol{\nabla}\cdot\mathbf{v}) = 0 \ , \tag{4.36}$$

$$\frac{\partial\mathbf{v}}{\partial t} + 2\Omega\times\mathbf{v}_L = -\boldsymbol{\nabla}P'/\rho \ , \tag{4.37}$$

$$-\rho\kappa[\hat{\mathbf{z}}\times(\mathbf{v}_L - \mathbf{v})] = +\rho\kappa\nu_s\frac{\partial^2\mathbf{u}}{\partial z^2} - \frac{\rho\kappa c_T^2}{2\Omega}[2\boldsymbol{\nabla}_\perp(\boldsymbol{\nabla}\cdot\mathbf{u}) - \nabla_\perp^2\mathbf{u}] \ . \tag{4.38}$$

Here $\boldsymbol{\nabla}_\perp(\nabla_x,\nabla_y)$ is the two-dimensional vector of the gradient in the xy plane, and $\boldsymbol{\nabla}P' = c^2\boldsymbol{\nabla}\rho'$ where $c = (\partial P/\partial\rho)^{1/2}$ is the sound velocity.

Equations are written in the rotating coordinate frame in which velocities $\mathbf{v}$ and $\mathbf{v}_L$ are small. The right-hand side of Eq. (4.38) is the force on the vortex line. It originates entirely due to quantization of vorticity and vanishes if $\kappa\to 0$. In this limit Eqs. (4.36)–(4.38) transform into Eqs. (2.9) and (2.10) in the continuous-vorticity model. Then the velocity of vortex lines does not differ from the averaged velocity of the fluid, i.e., $\mathbf{v} = \mathbf{v}_L$.

C. Spectrum of oscillations in an incompressible fluid

In an incompressible fluid ($c\to\infty$) the density does not vary, $\rho' = 0$, and the velocity field is divergence-free, $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$. Any vector field $\mathbf{A}(\mathbf{R})$ may be divided into a longitudinal and a transverse part, $\mathbf{A}(\mathbf{R}) = \mathbf{A}_{\parallel}(\mathbf{R}) + \mathbf{A}_\perp(\mathbf{R})$, then $\boldsymbol{\nabla}\times\mathbf{A}_{\parallel}(\mathbf{R}) = 0$ and $\boldsymbol{\nabla}\cdot\mathbf{A}_\perp(\mathbf{R}) = 0$. The gradient term on the right-hand side of Eq. (4.37), which is longitudinal, cancels exactly with the longitudinal part of the vector-product term on the left-hand side if the fluid is incompressible. Thus we can rewrite Eq. (4.37)

$$\frac{\partial\mathbf{v}}{\partial t} + (2\boldsymbol{\Omega}\times\mathbf{v}_L)_\perp = 0 \ . \tag{4.39}$$

Separation of longitudinal and transverse parts of a vector field involves an intricate integration-differentiation operation in the coordinate space, but it is easily performed in the Fourier representation. Suppose that

$$\mathbf{A}(\mathbf{R},t) = \int d\mathbf{Q}\cdot d\omega\ \mathbf{A}(\mathbf{Q},\omega)\exp(ipz + i\mathbf{q}\cdot\mathbf{r} - i\omega t) \ , \tag{4.40}$$

where p and $\mathbf{q}$ are the components of the wave vector $\mathbf{Q}$ on the z axis and in the xy plane. Then we have for $\mathbf{A} = \mathbf{A}(\mathbf{Q},\omega)$

$$\mathbf{A}_{\parallel} = \frac{\mathbf{A}\cdot\mathbf{Q}}{Q^2}\mathbf{Q}, \quad \mathbf{A}_\perp = \mathbf{A} - \frac{\mathbf{A}\cdot\mathbf{Q}}{Q^2}\mathbf{Q} \ . \tag{4.41}$$

So in the Fourier representation Eqs. (4.39) and (4.38) look like

$$-i\omega\mathbf{v} + 2\boldsymbol{\Omega}\times\mathbf{v}_L - [\mathbf{Q}\cdot(2\Omega\times\mathbf{v}_L)]\mathbf{Q}/Q^2 = 0 \ , \tag{4.42}$$

$$\hat{\mathbf{z}}\times(\mathbf{v}_L - \mathbf{v}) = \nu_s p^2\mathbf{u} + \frac{c_T^2}{2\Omega}[q^2\mathbf{u} - 2(\mathbf{q}\cdot\mathbf{u})\mathbf{q}] \ . \tag{4.43}$$

Equations (4.42) and (4.43) look like three-dimensional vector equations; they are, however, effectively two dimensional. Indeed, vectors $\mathbf{u}$ and $\mathbf{v}_L = -i\omega\mathbf{u}$ have components only in the xy plane, and the z component of $\mathbf{v}$, v_z, can be eliminated with the help of the incompressibility condition $\mathbf{Q}\cdot\mathbf{v} = 0$. As in the continuous-vorticity model (Sec. II), we choose axes in the xy plane parallel and normal to the vector $\mathbf{q}$ and denote corresponding components by subscripts q and t. In terms of q and t components, Eqs. (4.42) and (4.43) take the form

$$-i\omega v_q - 2\Omega\frac{p^2}{Q^2}v_{Lt}=0 \ ,$$
$$-i\omega v_t + 2\Omega v_{Lq}=0 \ , \tag{4.44}$$

$$-i\omega(v_{Lt}-v_t)=-\nu_s p^2 v_{Lq}+\frac{c_T^2 q^2}{2\Omega}v_{Lq} \ ,$$
$$+i\omega(v_{Lq}-v_q)=-\nu_s p^2 v_{Lt}-\frac{c_T^2 q^2}{2\Omega}v_{Lt} \ . \tag{4.45}$$

These equations have a solution when the following dispersion law holds:

$$\omega^2=(2\Omega+\nu_s p^2)\left[2\Omega\frac{p^2}{Q^2}+\nu_s p^2+\frac{c_T^2 q^2}{2\Omega}\right] . \tag{4.46}$$

In deriving the dispersion law we neglected the term $\propto c_T^2 q^2$ in the first multiplier in Eq. (4.46) as being of higher order than our approximation, but retained a similar term in the second multiplier. This is because this term is always unimportant compared to 2Ω, but is not small compared to $2\Omega(p/Q)^2$ when the ratio p/q is small enough. As for terms $\propto \nu_s p^2$, they contain a large logarithm in ν_s and are retained in both multipliers.

Further analysis of oscillation modes in an incompressible fluid will be carried out for different particular cases separately.

D. Axial modes of vortex oscillations

We shall call waves axial when their wave vectors are directed along the z axis ($q=0$). According to Eq. (4.46) the dispersion law for axial modes is given by

$$\omega^2=(2\Omega+\nu_s p^2)^2 \ . \tag{4.47}$$

These are circularly polarized torsion vortex waves, studied extensively from the 1950s onward. The waves involve motion only within the xy plane, and all displacements and velocities are two dimensional, so it is convenient to use the complex representation for two-dimensional vector, widely applied in classical hydrodynamics (Milne-Thomson, 1960). In this representation any two-component vector is presented by a complex number. The components of our vectors, however, are already complex in the Fourier representation as a result of the presence of $i\omega$ in the equations. In order to distinguish between complexity due to the Fourier transformation and that connected with the representation of two-dimensional vectors, we introduce a new imaginary unit j, assuming $j^2=-1$ as usual. Any vector in the xy plane is presented by a complex number,

$$\widetilde{A}=A_q+jA_t \ . \tag{4.48}$$

Separation of a j-complex number $\widetilde{A}$ into its real and imaginary parts yields q and t components of a vector $\mathbf{A}$:

$$A_q=\mathrm{Re}_j(\widetilde{A}), \ \ A_t=\mathrm{Im}_j(\widetilde{A}) \ . \tag{4.49}$$

In performing these operations, one should treat the other complex unit i as real. That is why we have introduced the second imaginary unit j.

In the j-complex representation the vector product $\hat{\mathbf{z}}\times\mathbf{v}$ is $j\widetilde{v}$, and our hydrodynamical equations (4.44) and (4.45) for axial modes become simple and compact:

$$-i\omega\widetilde{v}+2\Omega j\widetilde{v}_L=0 \ , \tag{4.50}$$

$$\widetilde{v}_L=\widetilde{v}+\frac{\nu_s p^2}{i\omega}j\widetilde{v}_L=0 \ . \tag{4.51}$$

Eigenfrequencies of axial modes correspond to zeros of the complex determinant of Eqs. (4.50) and (4.51):

$$D(j)=i\omega-j(2\Omega+\nu_s p^2) \ . \tag{4.52}$$

Then

$$\omega=-ij(2\Omega+\nu_s p^2) \ , \tag{4.53}$$

and relations between velocity components are

$$v_t=-jv_q, \ \ v_{Lt}=-jv_{Lq} \ . \tag{4.54}$$

We arrive at explicit formulas for axial modes by replacing j by $\pm i$. Then Eq. (4.53) agrees with Eq. (4.47). The two signs correspond to two possible senses of the circular polarization.

Probably the j-complex representation for two-dimensional vectors looks too artificial for the simple problem under consideration. But it will turn out to be convenient for more complicated problems dealing with axial modes in two-fluid hydrodynamics and in boundary problems.

The spectrum of axial modes in macroscopic hydrodynamics differs from the spectrum of the pure Kelvin modes (Sec. III.D) by the gap 2Ω and by another choice of the upper cutoff r_m in Eq. (3.23) for ν_s (the intervortex distance r_v instead of $r_m=1/p$ for the pure Kelvin mode). The gap arises as a result of long-range interaction between vortices, in analogy with the gap in the plasma-oscillation spectrum as a result of Coulomb interaction. The axial vortex wave in macroscopic hydrodynamics may be considered to be the collective Kelvin mode. One may not follow the transition from the collective to the pure Kelvin mode and still remain within the scope of macroscopic hydrodynamics, because the presence of the wave number p in the argument of the logarithm function in the dispersion law of the pure Kelvin mode, is incompatible with the second-order differential equations of the hydrodynamical theory, as was pointed out in Sec. III.D. These equations provide a rigorous approach until $pr_v \ll 1$. A more general theory, capable of treating both collective and pure Kelvin modes, deals with the system of equations (3.19) in the Fourier representation and is restricted by a weaker condition $pr_c \ll 1$ (Sonin, 1976; Williams and Fetter, 1977). According to this theory, whether $pr_v \ll 1$ or $pr_v \gg 1$, the dispersion law [Eq. (4.47)] holds with r_v or $1/p$ as the upper cutoff of the logarithm. But according to Rajagopal (1964), in the region $pr_v \gg 1$ the shift of the Kelvin wave frequency due to rotation is Ω instead of 2Ω in Eq. (4.47). The aforementioned theory disproved this result based on some simplifying assumptions.

E. Mixed modes. Slow motion of an incompressible perfect fluid

The general dispersion law for an incompressible fluid, Eq. (4.46), shows that low-frequency oscillations $\omega \ll \Omega$ are possible only if $p \ll q$. This means that the Taylor-Proudman theorem (Sec. II) holds in a perfect fluid with quantized vorticity too: the fluid in a state of slow motion is homogeneous along the rotation axis. Considering the slow motion one can neglect the vortex tension (the term $\propto \nu_s$ in the dispersion law) and obtain from Eq. (4.46) (Fetter, 1975; Sonin, 1976)

$$\omega^2 = 4\Omega^2 \frac{p^2}{Q^2} + c_T^2 q^2 \ . \tag{4.55}$$

The first term on the right-hand side is of classical origin and yields the frequency of the inertial wave. The second term is due to quantization of vorticity and is responsible for the Tkachenko waves. We shall call the oscillation mode with the dispersion law Eq. (4.55) the mixed mode.[2] It has some noteworthy features pointed out by Williams and Fetter (1977). The frequency ω as a function of q at given nonzero p has a minimum. The values of q and of ω in the minimum at $pr_v \ll 1$ are given by the expressions

$$\begin{aligned} q_m^2 &= \frac{2\Omega p}{c_T} = 4p\sqrt{2}\pi\Omega/\kappa \ , \\ \omega_m^2 &= 4\Omega c_T p \ . \end{aligned} \tag{4.56}$$

As usual, the minimum on the dispersion curve ω vs q should correspond to a peak of the density of states. This peak will be discussed later (Sec. VI.E) in connection with the interpretation of the experiments of Glaberson's group.

Let us find the relations between velocity components in the mixed wave. The solutions of Eqs. (4.44) and (4.45), together with the incompressibility condition $pv_z + qv_q = 0$, yield

$$\begin{aligned} & v_t \simeq v_{Lt}, \quad \frac{v_{Lq}}{v_{Lt}} = \frac{i\omega}{2\Omega} \ , \\ & \frac{v_q}{v_t} = -\frac{2\Omega}{i\omega}\frac{p^2}{Q^2} = -\frac{\omega^2 - c_T^2 q^2}{i\omega 2\Omega} \ , \\ & v_z = -\frac{q}{p} v_q = \frac{2\Omega}{i\omega}\frac{p}{q} v_t = \frac{(\omega^2 - c_T^2 q^2)^{1/2}}{i\omega} v_t \ . \end{aligned} \tag{4.57}$$

When the quantum Tkachenko contribution $c_T^2 q^2$ increases from zero, the oscillatory motion of the fluid transforms from the circularly polarized motion $v_z \approx i v_t$, as in the classical inertial wave, into the motion with transverse linear polarization corresponding to the Tkachenko wave ($v_t \gg v_q, v_z$). As for vortices, they move in the xy plane on elliptical paths with their major axes perpendicular to $\mathbf{q}$. The ratio of the axes of the ellipse v_{Lq}/v_{Lt} is small at $\omega \ll \Omega$, so one can neglect small longitudinal components v_q and v_{Lq} and consider the slow motion in the xy plane to be transverse with coinciding vortex and averaged fluid velocities $v_t = v_{Lt}$. Eliminating from Eqs. (4.44) and (4.45) all velocity components except for v_{Lt}, we obtain

$$\left[\omega^2 - 4\Omega^2 \frac{p^2}{Q^2} - c_T^2 q^2\right] v_{Lt} = 0 \ . \tag{4.58}$$

The inverse Fourier transformation of this equation yields the following equation for the vortex velocity $\mathbf{v}_L$ lying always in the xy plane ($p \ll q$):

$$\frac{\partial^2 \mathbf{v}_L}{\partial t^2} = -4\Omega^2 \frac{1}{\Delta_\perp}\frac{\partial^2 v_L}{\partial z^2} + c_T^2 \Delta_\perp \mathbf{v}_L \ . \tag{4.59}$$

Here $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator in the xy plane. The inverse Laplace operator $1/\Delta_\perp$ is an integral operator determined by the Green's function of the Laplace equation. Applying the Laplace operator $\Delta_\perp$ to both sides of Eq. (4.59), we can transform it into the differential equation

$$\Delta_\perp \frac{\partial^2 \mathbf{v}_L}{\partial t^2} = -4\Omega^2 \frac{\partial^2 \mathbf{v}_L}{\partial z^2} + c_T^2 \Delta_\perp^2 \mathbf{v}_L \ . \tag{4.60}$$

This equation follows after some simplification from the more general equation obtained by Williams and Fetter [(1977); see Eq. (29) in their paper].

Neglecting the longitudinal velocity component v_{Lq} means that vortices behave as an incompressible fluid and $\nabla \cdot \mathbf{v}_L = 0$. This condition together with Eq. (4.59) or (4.60) constitutes the hydrodynamical theory of slow motion. Although the longitudinal part of the vortex velocity is small, it plays an important role in the dispersion law and in the derivation of the basic equation of motion Eq. (4.59) and cannot be ignored until this equation is obtained. Just the small longitudinal component v_q of the fluid velocity provides the possibility of fluid motion along the z axis, as follows from the fluid incompressibility condition $pv_z + qv_q = 0$. Sometimes geometry does not allow such motion; then both the fluid and the vortices move together only in the xy plane. Variation of velocities along the z axis, which is slow in accordance with the Taylor-Proudman theorem, may not be ignored either, since the small derivative $\partial^2 \mathbf{v}_L/\partial z^2$ in Eq. (4.59) is multiplied by the large factor $4\Omega^2$. Indeed, this derivative is responsible for the gap in the oscillation spectrum at $q \to 0$. This gap is important for observation of Tkachenko waves in finite vessels and will be discussed later (Secs. VI.D and E).

F. Tkachenko waves. Elasticity theory of a two-dimensional vortex crystal

When $p = 0$ the mixed mode becomes the Tkachenko wave with the gapless dispersion law

[2]It was named a transverse vortex wave before (Sonin, 1976), because the wave vector $\mathbf{Q}$ for this wave is nearly transverse to the rotation axis. But in the present review the word "transverse" is widely used to refer to divergence-free velocity fields, not to the mixed node.

$$\omega = c_T q \ . \tag{4.61}$$

Though vortices in the Tkachenko wave move on elliptical paths, the axis of the ellipse parallel to $\mathbf{q}$ is small ($v_{Lq} \ll v_{Lt}$), and motion of vortices is nearly linear and transverse with respect to $\mathbf{q}$. Thus it is fairly accurate to consider the Tkachenko wave to be a transverse sound wave in the two-dimensional lattice of rectilinear vortices (Tkachenko, 1969). In order to see this better, let us rewrite Eq. (4.38) omitting $\partial \mathbf{u}/\partial z$:

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{v}_L = \mathbf{v} - \frac{c_T^2}{2\Omega}[\hat{\mathbf{z}} \times (2\nabla\nabla\cdot\mathbf{u} - \nabla\mathbf{u})] \ . \tag{4.62}$$

Dividing the field of vortex displacements $\mathbf{u}$ into longitudinal and transverse parts, $\mathbf{u} = \mathbf{u}_\parallel + \mathbf{u}_\perp$ ($\nabla\cdot\mathbf{u}_\perp = 0$, $\nabla\times\mathbf{u}_\parallel = 0$), and using Eq. (4.39) for elimination of the fluid velocity $\mathbf{v}$, we obtain for displacements in the long-wavelength limit

$$\frac{\partial \mathbf{u}_\parallel}{\partial t} = + \frac{c_T^2}{2\Omega}\hat{\mathbf{z}} \times \Delta \mathbf{u}_\perp \ , \tag{4.63}$$

$$\frac{\partial \mathbf{u}_\perp}{\partial t} = -2\mathbf{\Omega} \times \mathbf{u}_\parallel \ . \tag{4.64}$$

Exclusion of the small longitudinal displacement $\mathbf{u}_\parallel$ yields an equation similar to that for the transverse sound in conventional elasticity theory:

$$\frac{\partial^2 \mathbf{u}_\perp}{\partial t^2} - c_T^2 \Delta \mathbf{u}_\perp = 0 \ . \tag{4.65}$$

This follows from the more general equation of slow motion, Eq. (4.59), when the fluid is uniform along the z axis. Let us rewrite Eq. (4.65) in the form, also widely used in elasticity theory (the subscript $\perp$ is omitted),

$$\rho \frac{\partial^2 u_i}{\partial t^2} = -\nabla_j \sigma_{ij} \ . \tag{4.66}$$

Here

$$\sigma_{ij} = -\rho c_T^2 (\nabla_i u_j + \nabla_j u_j) \tag{4.67}$$

is the elastic stress tensor, as follows from Eq. (4.35) assuming that $\partial \mathbf{u}/\partial z = 0$ and $\nabla\cdot\mathbf{u} = 0$. Subscripts i and j take only two values corresponding to the two axes in the xy plane.

We see that the elasticity theory of the vortex crystal contains a single elastic modulus, the shear modulus $\mu = \rho c_T^2$. Formally one arrives at such a version of the elasticity theory (Landau and Lifshitz, 1965) assuming hexagonal symmetry and taking the limit of the infinite compression modulus that rules out longitudinal displacements (Ignatiev and Sonin, 1981). More exactly, longitudinal displacements are not ruled out, but excluded from the equations. However small, they remain finite and are not independent variables, as in the atomic crystal. This is one of the peculiar features of vortex dynamics discussed earlier. Even small longitudinal displacements of vortices generate the flow of the fluid with the averaged velocity $\mathbf{v}$, because they are coupled with the vorticity field. The motion of the fluid is responsible for the inertia force in the dynamical equation (4.66), in which massless vortices provide the elastic force. Such an interpretation of Eq. (4.66) is possible because the fluid velocity $\mathbf{v}$ is approximately equal to the vortex velocity $\mathbf{v}_L$.

In the stationary case longitudinal displacements are exactly ruled out by Eq. (4.64).

G. Vortex oscillations in a compressible perfect fluid

For a discussion of the effect of compressibility on vortex oscillations we need to return to the general linear equations of motion, Eqs. (4.36)–(4.38). Performing the Fourier transformation, we obtain

$$-i\omega\rho' + \rho\mathbf{Q}\cdot\mathbf{v} = 0 \ , \tag{4.68}$$

$$-i\omega\mathbf{v} + 2\mathbf{\Omega}\times\mathbf{v}_L + \frac{c^2}{\rho} i\mathbf{Q}\rho' = 0 \ , \tag{4.69}$$

and the third equation of vortex motion does not differ from that in the incompressible fluid, Eq. (4.43) or (4.45).

As we do for the fluid with continuous vorticity (Sec. II), we first solve the continuity equation, Eq. (4.68), and the equation for v_z given by Eq. (4.69). These equations do not differ from those in the continuous-vorticity model, since the vector product $\mathbf{\Omega}\times\mathbf{v}_L$ has no z component, so we obtain again Eq. (2.12) connecting ρ' and v_z with v_q. Eliminating ρ' and v_z from equations for v_q and v_t given by Eq. (4.69) yields two equations,

$$\begin{aligned} &-i\omega v_q - 2\Omega \frac{\omega^2 - c^2 p^2}{\omega^2 - c^2 Q^2} v_{Lt} = 0 \ , \\ &-i\omega v_t + 2\Omega v_{Lq} = 0 \ . \end{aligned} \tag{4.70}$$

These equations transform into the equations of motion in the continuous-vorticity model, Eq. (2.13), when $\mathbf{v}_L = \mathbf{v}$, and into equations of motion for an incompressible fluid, Eq. (4.44), when the sound velocity $c \to \infty$.

Solving Eq. (4.70) together with Eq. (4.45), we obtain the dispersion equation for oscillations in a compressible fluid,

$$\omega^2 = (2\Omega + \nu_s p^2)\left[2\Omega \frac{\omega^2 - c^2 p^2}{\omega^2 - c^2 Q^2} + \nu_s p^2 + \frac{c_T^2 q^2}{2\Omega} \right] . \tag{4.71}$$

This equation has two solutions for ω^2 at any given wave vector $\mathbf{Q}$. We restrict ourselves to the case $p = 0$ when the wave vector $\mathbf{Q}$ lies in the xy plane (in-plane modes) and use the inequality $c \gg c_T$. Then the first solution of Eq. (4.71),

$$\omega^2 = 4\Omega^2 + c^2 q^2 \ , \tag{4.72}$$

corresponds to the usual sound wave modified by rotation [cf. Eq. (2.18)]. The second solution,

$$\omega^2 = \frac{c^2 c_T^2 q^4}{c^2 q^2 + 4\Omega^2} \ , \tag{4.73}$$

yields the Tkachenko wave in the limit $c \to \infty$. But we see that compressibility strongly alters the spectrum of this wave at small $q \ll 2\Omega/c$, making it parabolic:

$$\omega = \frac{c_T c}{2\Omega} q^2 . \quad (4.74)$$

We have already encountered the drastic effect of compressibility on the inertial wave in the long-wavelength limit when we studied the continuous-vorticity model in Sec. II. This effect is important for our conception of the hydrodynamics of an incompressible fluid, as was remarked after Eq. (2.19). That comment is also relevant for a fluid with quantized vorticity.

The strong effect of compressibility on the Tkachenko wave spectrum in the long-wavelength limit was discovered by Reatto (1968). But he obtained a dispersion law different from Eq. (4.73), giving complex frequencies despite there being no physical source of dissipation in a perfect fluid. This result is due to imperfections of the model used by Reatto (Sonin, 1976). The theory of a compressible fluid developed above allows a rigorous analysis of the problem without referring to the assumptions of Reatto's model.

V. OSCILLATIONS OF FINITE VORTEX ARRAYS. TWO-DIMENSIONAL BOUNDARY PROBLEMS

A. Introductory comments

For most experiments dealing with vortex oscillations, finite dimensions of containers play an important role. The theory may make contact with such experiments only after analysis of the boundary problem. This problem involves formulation of boundary conditions for the equations of macroscopic hydrodynamics treated in the previous section and solution of the equations for the geometry of a particular experiment. Such a program implies that all effects of the boundary are taken into account by a proper choice of boundary conditions. The symmetry and local properties of the finite array are assumed to be the same as those of the infinite array. This approach is common for theories of continuum media, such as the elasticity theory, for example.

We restrict ourselves in the present section to two-dimensional problems in which vortices move in the xy plane remaining rectilinear and parallel to the rotation axis (the z axis). It is difficult to find experimental situations to which such a theory may be applied. As we shall see (Sec. VI) even weak pinning of vortices strongly modifies their oscillation spectra, and bending of vortices may not be ignored. Therefore only numerical experiments modeling two-dimensional motion are at our disposal for comparison with the theory. Extensive numerical calculations of a large number of finite two-dimensional vortex patterns and their oscillation modes have been carried out by Campbell and Ziff (1979) and Campbell (1981a). In addition, some important general relations between eigenfrequencies of finite vortex patterns have been analytically derived and oscillation modes classified (Campbell, 1981a).

In spite of the limited applicability of the two-dimensional theory, a continuous and steady interest in it is completely justified. The theory allows us to obtain elegant exact results, the Tkachenko theory being an impressive example. Besides its esthetic and pedagogical value for further study of three-dimensional problems, the two-dimensional theory enables us to solve some key problems of vortex dynamics. Of utmost importance for us is the question: what reliance can be placed on the continuum-hydrodynamics approach developed for the infinite-vortex lattice when dealing with a finite array? Recently such an approach was called into doubt on the grounds that boundary effects strongly affect the structure of the vortex lattice, even deep within the interior of the vortex array (Campbell, 1981a, 1981b). Therefore our analysis of finite-vortex-array dynamics is preceded by a discussion of the equilibrium properties of a finite vortex array. We shall see that distortions of the finite vortex array produced by the boundary are stronger than in atomic crystals, as numerical calculations have shown (Campbell and Ziff, 1979). This is explained by the long-range interaction of the vortices. But further comparison of the eigenmodes of finite vortex patterns, obtained from the continuum theory and numerically, convinces us that the distortions are not strong enough to discredit the results of the dynamic continuum theory relying on the infinite-lattice properties inside a finite vortex pattern.

Within the scope of the two-dimensional theory we also treat surface waves propagating along the boundary of the vortex pattern. The study of surface waves is in its early stages, and it is difficult as yet to draw any conclusions concerning their observability. But they are interesting in principle and deserve discussion.

B. An equilibrium finite vortex array. Distortion produced by a boundary

It is well known now that a superfluid in a rotating container imitates solid-body rotation, despite the fact that the velocity field remains curl-free nearly everywhere in the bulk. This is possible on a macroscopic scale if the superfluid contains vortex lines parallel to the rotation axis and uniformly spaced at a density n_v per unit area given by the formula of Feynman (1955),

$$n_v = \frac{2\Omega}{\kappa} . \quad (5.1)$$

The formula is obtained by minimization of the fluid energy in the rotating coordinate frame,

$$\mathscr{E} = \int d\mathbf{r}[\tfrac{1}{2}\rho v^2 - \rho \mathbf{v}\cdot(\mathbf{\Omega}\times\mathbf{r}) + E_v] , \quad (5.2)$$

where E_v is the energy density of the vortices given by Eq. (4.28) and depends on the magnitude of vorticity $\tilde{\omega} = \kappa n_v$, so shear rigidity of the vortex lattice is not taken into account. Then the Feynman formula is exactly true even for finite vortex arrays. If it were not true, vortex lines would rotate with a velocity different from that of the containing vessel, and mutual friction between vortex lines and the normal part of the fluid would lead to energy dissipation, but this is impossible for the equilibrium state.[3]

On the other hand, it was noticed by Hall (1960) that vortices do not fill the vessel completely, and a so-called irrotational region free from vortices should exist near the wall of the cylindrical vessel. Later it became clear that the vortex-free region is formed near any solid surface bounding a superfluid and parallel to the rotation axis (Bendt and Oliphant, 1961; Kemoklidze and Khalatnikov, 1964; Stauffer and Fetter, 1968). When the curvature radius of the solid surface is large compared with the intervortex distance, the width of the vortex-free region is (Stauffer and Fetter, 1968)[4]

$$d=\left[\frac{\kappa}{4\pi\Omega}\ln\frac{r_v}{r_c}\right]^{1/2} . \tag{5.3}$$

The width d differs by the factor $\sim\sqrt{\ln(r_v/r_c)}$, large in the logarithmic approximation, from the intervortex distance r_v. It justifies determination of d within the scope of the continuum theory, though this factor is not quite so large in practice. The existance of the vortex-free region was experimentally proven by Tsakadze (1964), though quantitative discrepancy with theory was considerable (see also the discussions by Andronikashvili and Mamaladze, 1966, and Andronikashvili *et al.*, 1978).

The width of the vortex-free region can be different from its equilibrium value given by Eq. (5.3). Suppose that the rotation speed of the container with superfluid changes its value. Then the number of vortices in the container has to change too. But generation of new vortices and their annihilation at the boundary are much slower than other relaxation processes (see Chap. IX of Andronikashvili *et al.*, 1978). Therefore one can consider a state of restricted equilibrium at a fixed number of vortices. If the number of vortices is smaller than the equilibrium value, then the width d of the vortex-free region is larger than that given by Eq. (5.3). When d is infinitely large, the problem corresponds to the case of a finite number of vortices in unbound fluid. Vortices, however, "know" the rotation speed of the vessel, since we look for their equilibrium distribution in the reference frame rotating together with the vessel. Vortices always congregate around the rotation axis forming a finite pattern of cylindrical shape with the density given by Eq. (5.1). This behavior may be considered as a result of two competing effects. First, vortices effectively repel each other, trying to stay apart. Second, the rotation velocity produces a force (recall that in vortex dynamics any velocity is equivalent to a force; see Sec. III.B) attracting vortices to the rotation axis and trying to make the vortex pattern round and compact. The latter effect may be interpreted in terms of the surface tension of the vortex-pattern boundary. But unlike the surface tension of common fluids and crystals, the surface tension of the vortex pattern is proportional to its volume, and not to the surface area. In order to show this, let us calculate the energy associated with a small distortion of the boundary of the finite vortex array occupying a cylindrical region of radius R_0. In the ground state the velocity field $\mathbf{v}_0$ is determined in the cylindrical system of coordinates as follows:

$$v_{0r}=0,\quad v_{0\varphi}=\begin{cases}\Omega r, & r<R_0\ ,\\ \dfrac{\Omega R_0^2}{r}, & r>R_0\ .\end{cases} \tag{5.4}$$

As a result of a small distortion, the distance of the vortex-pattern edge from the rotation axis is $R_0+\delta u$; we choose the small radial displacement δu in the form of the nth cylindrical wave

$$\delta u=u_0e^{in\varphi}\ . \tag{5.5}$$

Here φ is an angle of a point on the vortex-pattern edge and n is an integer. Due to distortion the velocity field changes and becomes $\mathbf{v}_0+\mathbf{v}'$. The vorticity $\widetilde{\omega}=\nabla\times\mathbf{v}$ remains $2\boldsymbol{\Omega}$ everywhere inside the vortex pattern with the distorted edge. Then $\widetilde{\omega}$ changes only near the edge. Since the displacement δu is small, we can write that the vorticity deviation due to distortion is

$$\widetilde{\omega}'=\nabla\times\mathbf{v}'=2\Omega u_0\delta(r-R_0)e^{in\varphi}\ . \tag{5.6}$$

Suppose that our vortex pattern is inside of a cylindrical vessel of radius R. Then the distortion velocity field $\mathbf{v}'$ has no radial component at $r=R$ and satisfies the conditions $\nabla\cdot\mathbf{v}'=0$ and $\nabla\times\mathbf{v}'=0$ except at the edge $r=R_0$. It is given by the expressions for $r<R_0$

$$\begin{aligned}v_r'&=iw_+\left[\frac{r}{R_0}\right]^{n-1}\left[1-\left[\frac{R_0}{R}\right]^{2n}\right]e^{in\varphi}\ ,\\ v_\varphi'&=-w_+\left[\frac{r}{R_0}\right]^{n-1}\left[1-\left[\frac{R_0}{R}\right]^{2n}\right]e^{in\varphi}\ ,\end{aligned} \tag{5.7}$$

and for $R_0<r<R$

$$\begin{aligned}v_r'&=iw_-\left[\left[\frac{R_0}{r}\right]^{n+1}-\left[\frac{r}{R_0}\right]^{n-1}\left[\frac{R_0}{R}\right]^{2n}\right]e^{in\varphi}\ ,\\ v_\varphi'&=w_-\left[\left[\frac{R_0}{r}\right]^{n+1}+\left[\frac{r}{r_0}\right]^{n-1}\left[\frac{R_0}{R}\right]^{2n}\right]e^{in\varphi}\ .\end{aligned} \tag{5.8}$$

[3]According to Baym and Chandler (1983), there is a difference of order h/mR^2 between the angular velocity of a container of radius R and of the vortex array inside of the container. But this is the result of the incorrect energy minimization procedure.

[4]Kemoklidze and Khalatnikov (1964; see also Khalatnikov, 1971) gave a formula that differed by the factor $\sqrt{\pi}$. I have repeated their calculations and have not revealed this factor, which probably results from a misprint or an arithmetical error. Stauffer and Fetter explained the disagreement by a discontinuity of the velocity on the boundary of the vortex pattern allowed by Kemoklidze and Khalatnikov. But this discontinuity is not important within the approximation adopted both by Stauffer and Fetter and by Kemoklidze and Khalatnikov.

The radial component v_r' should be continuous at the edge; this means that $w_+ = w_-$. But the azimuthal component v_φ' is discontinuous, and the magnitude of discontinuity follows from Eq. (5.6):

$$w_-\left[1+\left[\frac{R_0}{R}\right]^{2n}\right]+w_+\left[1-\left[\frac{R_0}{R}\right]^{2n}\right]=2w_+ =2w_-=2\Omega u_0 \; . \tag{5.9}$$

Substituting the real part of the determined distortion velocity field into the expression for the energy, Eq. (5.2), we find that distortion of the edge of the vortex pattern increases the energy by an amount

$$\delta\mathscr{E}_d(n)=\pi R_0^2\rho\Omega^2\frac{u_0^2}{n}\left[1-\left[\frac{R_0}{R}\right]^{2n}\right] . \tag{5.10}$$

We see that the energy of distortion is proportional to a two-dimensional volume πR_0^2, but not a surface area $2\pi R_0$ at $R\to\infty$.

But the form of the finite vortex pattern is strictly cylindrical only in the continuous-velocity model. Now let us take into account that vortices form a triangular lattice. Because of the lack of correspondence between the circular symmetry of the vortex-pattern edge and the hexagonal symmetry of the infinite-vortex lattice, the edge should deviate from a cylindrical shape. Indeed, if we cut out a cylindrical region from the infinite-vortex lattice, its boundary vortices cannot be located on one circumference; some of them will be at a distance of the order of the vortex spacing. Thus we can say that there is a small distortion of the edge of the vortex pattern and, as a result of it, the velocity deviates from solid-body-rotation velocity inside the vortex pattern. Just this velocity deviation was found by numerical calculations and was called the destabilizing velocity (Campbell and Ziff, 1979). The net destabilizing velocity is yielded by a sum over distortion harmonics labeled by an integer n, each given by Eq. (5.7). The sum should include $n=6,12,18,\ldots$ allowed by hexagonal symmetry. Deep in the interior of the vortex pattern the contribution of the fundamental harmonic $n=6$ becomes most important, and the destabilizing velocity is proportional to r^5, in complete agreement with the numerical calculations of Campbell and Ziff (1979).

The destabilizing velocity makes the vortex pattern with the structure of the infinite-vortex lattice unstable and deforms it. Deformation tends to decrease distortion of the edge and the energy associated with distortion. But at the same time the elastic energy of the vortex lattice increases. The competition between these energies determines the equilibrium structure of the vortex pattern. In the equilibrium state the solid-body rotation of vortices must be restored, since lattice deformation contributes to the vortex velocity, as one can see from Eq. (4.62), and this contribution exactly cancels the destabilizing velocity. Calculations supporting such a picture were carried out by Ignatiev and Sonin (1981). It was shown that elastic deformation strongly diminishes distortion of the vortex-pattern edge and the energy of this distortion. But at the same time distortion of the regular vortex lattice in the bulk arises. Deep in the interior of the vortex pattern displacements of vortices from sites of the regular triangular lattice fall as r^5 in agreement with the numerical calculations of Campbell and Ziff (1979).

Thus numerical calculations, as well as the simple continuum theory of Ignatiev and Sonin (1981), show that the boundary of a vortex crystal distorts the lattice more than the boundary of an atomic crystal. According to Campbell and Ziff (1979), the ratio of the surface energy of the vortex pattern (the difference between its energy and the energy of the same number of vortices in the infinite lattice) to its total energy decreases too slowly when the number of vortices in the pattern increases and probably does not approach zero, i.e., there is no thermodynamical limit for vortex crystals. The effect of finiteness of the vortex pattern on its dynamic behavior will be discussed in the following section. In conclusion we point out that the numerical calculations mentioned above were carried out for vortex patterns in an unbound fluid. But the walls of the vessel strongly reduce boundary effects because images of periphery vortices decrease their long-range velocity field. We see from Eq. (5.7) that distortion velocity inside of the vortex pattern is a factor $[1-(R_0/R)^{2n}]\approx 2nd/R$ smaller than the same velocity in an unbound fluid ($R\to\infty$). The same small factor enters the expression for the distortion energy [Eq. (5.10)]. This should affect our conclusions concerning the thermodynamical limit for vortex patterns.

C. Axisymmetric Tkachenko modes in a finite vortex pattern. Comparison of the continuum theory and numerical experiments

In considering axisymmetric modes we need not derive detailed boundary conditions on velocity fields (this will be done later) because axisymmetric oscillations involve variation of the angular momentum of the fluid. Any correct boundary condition should provide that the net angular momentum be conserved, so in deriving the dispersion law for axisymmetric modes we may refer directly to the conservation law for the angular momentum. Such an approach was taken by Ruderman (1970) in studying Tkachenko waves in a cylindric vessel of a finite radius. Let us refer to the elasticity theory of the two-dimensional vortex crystal (Sec. IV.F). The field of transverse displacements $\mathbf{u}$ may be determined by a vector potential $\boldsymbol{\Psi}=\Psi\hat{\mathbf{z}}$:

$$\mathbf{u}=\nabla\times\boldsymbol{\Psi}=-\hat{\mathbf{z}}\times\nabla\Psi \; . \tag{5.11}$$

The potential Ψ must satisfy the wave equation

$$\frac{\partial^2\Psi}{\partial t^2}-c_T^2\Delta\Psi=0 \; . \tag{5.12}$$

Axisymmetric modes correspond to a cylindrical wave

$$\Psi=\Psi_0 J_0(qr)e^{-i\omega t}\,,$$
$$u_r=0,\quad u_\varphi=-\frac{\partial\Psi}{\partial r}=\Psi_0 q J_1(qr)e^{-i\omega t} \tag{5.13}$$

with the soundlike spectrum $\omega=c_T q$.

Suppose that no external force acts upon the fluid. Then eigenfrequencies are defined by the condition that the total angular momentum not vary (recall that in the Tkachenko wave the fluid and vortices move together with nearly the same velocity):

$$\begin{aligned}M&=2\pi\rho\int_0^R v_\varphi r^2 dr\\&=-2\pi\rho i\omega\int_0^R u_\varphi r^2 dr\\&=-2\pi\rho i\omega\Psi_0 R^2 J_2(qR)e^{-i\omega t}=0\;.\end{aligned} \tag{5.14}$$

Here we do not distinguish between the radius R of the vessel and the radius R_0 of the vortex pattern. Equation (5.14) yields Ruderman's eigenfrequencies,

$$\omega_R(s)=j_{2,s}c_T/R\;. \tag{5.15}$$

Here $j_{n,s}$ denotes the sth zero of the Bessel function $J_n(z)$. For the fundamental frequency $j_{2,1}=5.14$.

Another simple condition for determination of eigenfrequencies of axisymmetric modes was proposed by Williams and Fetter (1977) (WF):

$$\omega_{\mathrm{WF}}(s)=j_{1,s}c_T/R\;. \tag{5.16}$$

This follows from the more general condition given by Eq. (40) of Williams and Fetter, for the case when the motion of the fluid is two dimensional as considered here. Equation (5.16) supposes that the vortices and the fluid are at rest near the wall of the vessel, i.e., $u_\varphi \propto J_1(qR)=0$.

Now we consider the more general condition imposed on the axisymmetric modes, which reduces to the conditions of Ruderman and of Williams and Fetter in two opposite limits. It is assumed that some external force is applied to the fluid boundary, restoring it to its initial state and proportional to the azimuthal displacement at the boundary. Then the following equation for the angular momentum balance holds:

$$2\pi R\sigma_{\varphi r}(R)=-2\pi Rku_\varphi(R)\;, \tag{5.17}$$

where k is the ratio of the force to the displacement. The right-hand side of this equation is the torque exerted on the fluid, and the left-hand side is the flux of the angular momentum through the circular boundary. Here $\sigma_{\varphi r}$ is the component of the stress tensor, given by Eq. (4.67), in the cylindric coordinate frame:

$$\sigma_{\varphi r}=-\rho c_T^2\left[\frac{\partial u_\varphi}{\partial r}-\frac{u_\varphi}{r}+\frac{1}{r}\frac{\partial u_r}{\partial\varphi}\right]. \tag{5.18}$$

For axisymmetric modes $u_r=0$, and substitution of Eq. (5.18) in Eq. (5.17) yields

$$\alpha\left[\frac{\partial u_\varphi(R)}{\partial r}-\frac{u_\varphi(R)}{R}\right]+u_\varphi(R)=0\;. \tag{5.19}$$

Equation (5.19), together with Eq. (5.13), gives the condition for determination of eigenfrequencies $\omega=c_T q$ of axisymmetric modes:

$$\alpha q J_2(qR)-J_1(qR)=0\;. \tag{5.20}$$

Here $\alpha=\rho c_T^2/k$.

The force sticking vortices to the wall (it is assumed in this section to be at rest in the rotating frame) arises due to mutual friction between the vortices and the normal part of the fluid, which sticks to the wall. So the parameter α may be determined only within two-fluid hydrodynamics, as done in Sec. VII.D. We shall see that coupling between vortices and the wall is considerable, even at rather low temperatures, and αq is quite small. At zero temperature vortices can interact with the wall because any real wall is not smooth. Effective friction arises as a result of the averaging of vortex motion over the irregular relief of the wall, as the residual resistance for electrons in dirty solids.

The dispersion equation (5.20) yields Ruderman's spectrum [Eq. (5.15)] when $\alpha\to\infty$ and the spectrum of Williams and Fetter when $\alpha=0$. Let us consider now the variation of α from ∞ to 0. Eigenfrequencies of axisymmetric modes increase as shown in Fig. 1 by arrows. Two lower eigenfrequencies map the two intervals on the frequency scale indicated by solid lines. It is important that the fundamental frequency of Ruderman, $\omega_R(1)=5.14c_T/R$, is not the lowest one. The spectrum also includes the zero frequency corresponding to the Goldstone mode due to invariance with respect to rotations. This mode is of no physical interest. But when rotational invariance is broken, the zero-frequency mode becomes an observable mode with finite frequency.

Such a character of the spectrum of axisymmetric modes should be taken into account when comparing predictions of the continuum theory with results of numerical calculations for finite vortex patterns. In order to simulate the boundary condition $u_\varphi=0$ used by Williams and Fetter, Campbell (1981a, 1981b) numerically calculated eigenfrequencies of the vortex pattern when the outermost ring of vortices was constrained to be strictly fixed. The calculated frequencies turned out to be a factor of 2 smaller than the frequencies given by Eq. (5.16) and considerably smaller than the frequencies of Ruderman's spectrum [Eq. (5.15)]. Campbell considered them to be frequencies of soft oscillation modes that could not be predicted by the continuum theory based on the perfect-

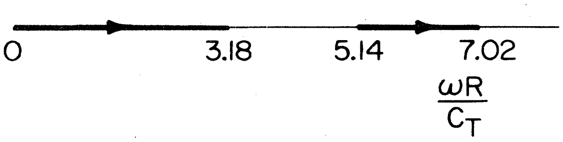

FIG. 1. Eigenfrequencies ω of the axisymmetric Tkachenko modes for a finite vortex pattern of radius R subject to action of the surface restoring force. The frequencies in units of c_T/R are shown for the two lowest modes. The thick solid lines with arrows show how the eigenfrequencies increase when the force increases from zero to infinity.

triangular-lattice properties. Another interpretation of these numerical results may be proposed, however, which is not so discreditable for the continuum theory of the triangular lattice. All vortices in the pattern except those on the fixed outermost ring constitute their own inner pattern. Interaction between inner vortices and the outermost ring of vortices produces an elastic force applied to the inner pattern, so the latter can sustain oscillation modes with frequencies corresponding to some finite α in Eq. (5.20) and falling somewhere in the first interval of the scale of eigenfrequencies shown in Fig. 1. This would explain why numerically calculated eigenfrequencies can be much smaller than the lowest frequency of Williams and Fetter for the whole pattern, the outermost ring included. In order to give a more exact quantitative interpretation of soft modes in a vortex pattern with a fixed outer ring, in terms of the continuum theory, one should determine somehow the force on the inner pattern.

Endeavoring to find a numerical result most appropriate for comparison with the continuum theory, we referred to the first pattern (left and uppermost) shown in Fig. 14(n) of the paper of Campbell (1981a). The pattern contains $N=19$ vortices. The displayed field of displacements for Campbell's parameter $\lambda=0.1586095$ resembles that for Ruderman's fundamental mode. The given value of λ corresponds to the frequency $\omega=\sqrt{\lambda(2-\lambda)}\Omega$ $=0.54\Omega$. On the other hand, the fundamental Ruderman frequency for the pattern of $N=19$ vortices is $\omega=j_{2,1}c_T/R=(j_{2,1}/2\sqrt{N})\Omega=0.589\Omega$. The 10% discrepancy is not too significant, especially if one takes into account that the ratio ω/Ω is not small enough for the long-wavelength continuum theory to be accurate. This comparison convinces us that the continuum theory relying on the perfect triangular lattice is not so poor an approximation for dynamics of large enough vortex patterns, despite strong distortions of the triangular lattice produced by the boundary.

D. Edge waves in the continuous-vorticity model

It is well known that semi-infinite crystals sustain surface waves, called Rayleigh waves. It would be interesting to find surface modes localized near the boundary of the vortex array as well. Such modes involving motion of vortices only close to the boundary have already been found in the numerical calculations and called edge waves (Campbell, 1981a, 1981b). Campbell and Krasnov (1981) have developed a theory of edge waves in the frame of the continuous-vorticity model, taking into account mutual friction. In classical hydrodynamics, edge waves were already known to Kelvin (Thompson, 1880), who discovered them while studying the stability of the columnar vortex tube in an unbound fluid [see Sec. 158 in Lamb (1945) and Sec. 7.3 in Batchelor (1970)]. Here we present a theory of edge waves in the continuous-vorticity model.

Suppose that an edge wave propagates along the edge of a vortex pattern of radius R_0 in a cylindrical vessel of radius R. This leads to distortion of the edge and generation of a distortion velocity field inside of the vortex pattern and in the vortex-free region. The displacements and velocities for the nth cylindrical edge wave differ from those given by Eqs. (5.5)–(5.8) by the time-dependent factor $\exp(-i\omega t)$. Equation (5.9) also holds. In the continuous-vorticity model, the velocities of the vortices and of the fluid coincide, and the edge of the vortex pattern should displace with the radial velocity and

$$\frac{d\delta u}{dt}=v_r' \ . \tag{5.21}$$

Substitution of δu and v_r' readily yields the dispersion law of the edge wave in a rotating reference frame (Campbell and Krasnov, 1981):

$$\omega=-\Omega\left[1-\left[\frac{R_0}{R}\right]^{2n}\right]$$

$$=-2\Omega\{1+\coth[n\ln(R/R_0)]\}^{-1} \ . \tag{5.22}$$

But in the laboratory reference frame

$$\omega=\Omega\left[n-1+\left[\frac{R_0}{R}\right]^{2n}\right] \ . \tag{5.23}$$

The edge wave $n=1$ involves translation of the pattern as a whole and is a displacement mode in the classification of Campbell (1981a). Its frequency approaches 0 in the laboratory frame when $R\to\infty$ because in an unbound fluid translation of the vortex pattern does not change the energy.

If the width of the vortex-free region $d=R-R_0$ is much smaller than R, we can rewrite the dispersion law, introducing the wave number $k=n/R$:

$$\omega=-2\Omega(1+\coth kd)^{-1} \ . \tag{5.24}$$

In the long-wavelength limit $kd\ll 1$

$$\omega=-c_E k, \quad c_E=2\Omega d \ . \tag{5.25}$$

If d takes its equilibrium value from Eq. (5.3), the velocity of the edge waves is equal to

$$c_E=\left[\frac{\kappa\Omega}{\pi}\ln\frac{r_v}{r_c}\right]^{1/2} \ . \tag{5.26}$$

Since this velocity is larger than the Tkachenko wave velocity $c_T=(\kappa\Omega/8\pi)^{1/2}$, the edge wave can emit volume Tkachenko waves and lose energy. But we shall see in the following section that attenuation due to emission of Tkachenko waves is quite weak.

It is interesting to note that the edge wave is unidirectional and can propagate only in the direction opposite to the velocity of solid-body rotation. Therefore reflection of edge waves is impossible.

E. Edge waves and Tkachenko waves in the continuum model of quantized vorticity

Now we shall go beyond the scope of the continuous-vorticity model and include in our considerations the effects of crystalline order in the lattices of quantized vortices. The closed system of equations governing fluid behavior inside a vortex pattern consists of the Euler equation (4.37) and equation of motion of vortices, (4.62), in which we neglect the longitudinal part of the displacement field. For convenience we rewrite both equations together:

$$\frac{\partial \mathbf{v}}{\partial t}+2\boldsymbol{\Omega}+\mathbf{v}_L+\nabla P/\rho=0\ , \tag{5.27}$$

$$\frac{\partial \mathbf{u}}{\partial t}=\mathbf{v}_L=\mathbf{v}+\frac{c_T^2}{2\Omega}\hat{\mathbf{z}}\times\Delta\mathbf{u}\ . \tag{5.28}$$

In the vortex-free region there is only the Euler equation,

$$\frac{\partial \mathbf{v}}{\partial t}+\frac{\nabla P}{\rho}=0\ . \tag{5.29}$$

In the linear momentum-flux tensor,

$$\Pi_{ij}=P\delta_{ij}+\sigma_{ij}\ , \tag{5.30}$$

the stress tensor σ_{ij} given by Eq. (4.67) vanishes in the vortex-free region.

We consider oscillations near the edge of the vortex array when wavelengths are small compared with the curvature radius of the edge and the edge is treated as a plane. The x axis is along the edge and the y axis is directed inside the vortex pattern; the solid surface is located at $y=-d$. Oscillations near the edge are described by a superposition of three solutions of the hydrodynamical equations. The first one includes fields due to distortion of the edge, while the other two describe Tkachenko waves falling onto and reflected from the edge. The distortion velocity field is obtained from Eqs. (5.7) and (5.8) if we add the time-dependent factor $\exp(-i\omega t)$, use the inequality $d=R-R_0\ll R$, and carry out a transformation of coordinates, $x=R_0\varphi$, $y=R_0-r$. The distortion pressure field is determined from the Euler equation, Eq. (5.27) or (5.29). We obtain (a) inside the vortex array $(y>0)$

$$\begin{aligned} v_x^D&=w_+(1-e^{-2kd})\exp(-ky+ikx-i\omega t)\ ,\\ v_y^D&=iw_+(1-e^{-2kd})\exp(-ky+ik+ikx-i\omega t)\ ,\\ P^D&=\rho w_+\frac{2\Omega+\omega}{k}(1-e^{-2kd})\exp(-ky+ikx-i\omega t)\ ;\end{aligned} \tag{5.31}$$

(b) in the vortex-free region $(0>y>-d)$

$$\begin{aligned} v_x^D&=-2w_-\cosh[k(y+d)]\\ &\quad\times\exp(-kd+ikx-i\omega t)\ ,\\ v_y^D&=2iw_-\sinh[k(y+d)]\\ &\quad\times\exp(-kd+ikx-i\omega t)\ ,\\ P^D&=-2\rho w_-\frac{\omega}{k}\cosh[k(y+d)]\\ &\quad\times\exp(-kd+ikx-i\omega t)\ .\end{aligned} \tag{5.32}$$

The wave number k is connected with n in Eqs. (5.7) and (5.8) by the relation $k=n/R$.

The distortion velocity field is divergence-free $(\nabla\cdot\mathbf{v}^D=0)$ and curl-free $(\nabla\times\mathbf{v}^D=0)$, so the elastic term $\propto c_T^2$ drops out of Eq. (5.28) and the vortices move together with the fluid:

$$\mathbf{v}_L^D=-i\omega\mathbf{u}^D=\mathbf{v}^D\ . \tag{5.33}$$

The distortion field contributes, however, to the elastic tensor σ_{ij}, Eq. (4.67), and the components necessary for us are

$$\begin{aligned}\sigma_{yy}^D&=-\frac{2\rho c_T^2 k}{\omega}w_+(1-e^{-2kd})\\ &\quad\times\exp(-ky+ikx-i\omega t)\ ,\end{aligned} \tag{5.34}$$

$$\begin{aligned}\sigma_{xy}^D&=\frac{2i\rho c_T^2 k}{\omega}w_+(1-e^{-2kd})\\ &\quad\times\exp(-ky+ikx-i\omega t)\ .\end{aligned} \tag{5.35}$$

The fields generated by Tkachenko waves are determined by the potential $\boldsymbol{\Psi}=\Psi\hat{\mathbf{z}}$ (see Sec. V.C),

$$\Psi=(\Psi_+e^{iqy}+\Psi_-e^{-iqy})\exp(ikx-i\omega t)\ , \tag{5.36}$$

according to

$$\begin{aligned}&\mathbf{u}^T=\nabla\Psi\times\hat{\mathbf{z}},\ \ \mathbf{v}^T=\mathbf{v}_L{}^T=-i\omega\mathbf{u}^T\ ,\\ &P^T=2\Omega i\omega\rho\Psi\ .\end{aligned} \tag{5.37}$$

These fields satisfy Eqs. (5.27) and (5.28) when $\omega^2=c_T^2(k^2+q^2)$ and $\omega\ll\Omega$.

The contribution of Tkachenko waves to the stress tensor is given by

$$\begin{aligned}\sigma_{yy}^T&=-2\rho c_T^2 qk(\Psi_+e^{iqy}-\Psi_-e^{-iqy})\\ &\quad\times\exp(ikx-i\omega t)\ ,\end{aligned} \tag{5.38}$$

$$\begin{aligned}\sigma_{xy}^T&=-\rho c_T^2(k^2-q^2)(\Psi_+e^{iqy}+\Psi_-e^{-iqy})\\ &\quad\times\exp(ikx-i\omega t)\ .\end{aligned} \tag{5.39}$$

Now it is necessary to match fields inside the vortex array and in the vortex-free region using the conditions of continuity of mass flows and momentum at the edge $y=0$.

Continuity of the mass flow is provided by continuity of the normal velocity v_y:

$$2iw_-\sinh kd\,e^{-kd}=iw_+(1-e^{-2kd})-\omega k(\Psi_++\Psi_-)\ . \tag{5.40}$$

The components Π_{yy} and Π_{xy} of the momentum-flux tensor should be continuous also:

$$\begin{aligned}-2\rho w_-\frac{\omega}{k}e^{-kd}\cosh kd&\\ =\rho w_+(1-e^{-2kd})&\left[\frac{2\Omega+\omega}{k}-\frac{2c_T^2k}{\omega}\right]\\ &+2\Omega i\omega\rho(\Psi_++\Psi_-)-2\rho c_T^2 qk(\Psi_+-\Psi_-)\ ,\end{aligned} \tag{5.41}$$

$$0=\frac{2i\rho c_T^2 k}{\omega}w_+(1-e^{-2kd})-\rho c_T^2(k^2-q^2)(\Psi_++\Psi_-)\ . \tag{5.42}$$

Elimination of w_+ and w_- from Eqs. (5.40)–(5.42) yields the condition imposed on Ψ_+ and Ψ_- which governs reflection of Tkachenko waves from the edge of the vortex array. In deriving this condition it is assumed that $\omega \ll \Omega$ and $kd \ll 1$. These inequalities are necessary to justify application of the long-wavelength theory of Tkachenko waves, since the equilibrium value of d given by Eq. (5.3) is nearly of the same order as the intervortex distance. Thus the reflection condition, or the effective boundary condition on Tkachenko waves, is

$$\left[1+\frac{\omega}{c_E k}\right](\Psi_++\Psi_-)+i\gamma(\Psi_+-\Psi_-)=0\ , \tag{5.43}$$

where small γ is of order ω/Ω:

$$\gamma=\frac{2c_T^2 qk^3}{\omega\Omega(k^2+q^2)}=\frac{2c_T^3 k^3(\omega^2-c_T^2k^2)^{1/2}}{\omega^3\Omega}\ . \tag{5.44}$$

If the frequency ω is far from the frequency $-c_E k$ of the edge wave, the small term $\propto\gamma$ may be ignored. Then we obtain a simple reflection condition

$$\Psi_++\Psi_-=0\ . \tag{5.45}$$

According to Eq. (5.37), this means that $u_y=0$, i.e., there is no displacement of vortices normal to the boundary (compare with the condition on the circular boundary discussed in the following section).

The dispersion law for the edge wave is obtained from Eq. (5.43), assuming that the falling Tkachenko wave is absent. Since at positive k the frequency $-c_E k$ is negative, the amplitude of the falling wave is Ψ_+ when $q=(\omega^2-c_T^2k^2)^{1/2}/c_T$ is positive. Equation (5.43) is therefore satisfied when $\Psi_+=0$ and the factor before Ψ_- vanishes, and the dispersion law for the edge wave is given by

$$\omega=-c_E k(1-i\gamma),\quad \gamma=-\frac{2c_T^3 k}{\Omega c_E^3}(c_E^2-c_T^2)^{1/2}\ . \tag{5.46}$$

The small imaginary part of the frequency [γ was obtained from Eq. (5.44) by the substitution $\omega=-c_E k$] describes attenuation of the edge wave due to emission of the Tkachenko wave. It supposes that the irradiated Tkachenko wave is carrying away energy to infinity. But if there is another boundary at a distance smaller than the absorption length of the Tkachenko wave, the wave reflects there and returns to the former boundary, so no attenuation occurs. As an example we consider propagation along the fluid layer between two rotating planes parallel to the rotation axis. The coordinates of planes in the rotating reference frame are $y=0$ and $y=D$. At the first boundary, the boundary condition is Eq. (5.43) as before. The edge wave can propagate around the vortex pattern only against the velocity circulation (see the end of Sec. V.D). When $k>0$ and $\omega<0$ and the edge wave propagates along the edge $y=0$, it cannot propagate in the same direction along $y=D$. Therefore the boundary condition at $y=D$ takes a simple form of the reflection condition $u_y=0$, which may be rewritten with the help of Eqs. (5.36) and (5.37) in a form similar to Eq. (5.45):

$$\Psi_+e^{iqD}+\Psi_-e^{-iqD}=0\ . \tag{5.47}$$

The determinant of the two linear equations (5.43) and (5.47) yields the dispersion law for waves propagating along the fluid layer parallel to the x axis:

$$\left[1+\frac{\omega}{c_E k}\right]\tan(qD)=\gamma\ . \tag{5.48}$$

Because γ is small, in a good approximation the spectrum consists both of Tkachenko branches given by poles of the tangent,

$$\omega(p)=c_T\left[\left[\frac{\pi p}{D}\right]^2+k^2\right]^{1/2}, \tag{5.49}$$

where integer $p\geq 1$, and of the edge-wave branch $\omega=-c_E k$, which crosses all Tkachenko branches. At the crossing point the small γ becomes important, and repulsion of branches occurs. Though the frequency of the Tkachenko branch $p=0$, $\omega(0)=c_T k$ formally satisfies Eq. (5.48), it does not correspond to some motion of the fluid, since $\Psi=0$ everywhere in this case.

We have obtained the effective boundary condition for Tkachenko waves near the plane wall without referring to the relation connecting discontinuity of the tangential velocity on the edge of the vortex pattern with the displacement of the edge. This relation, however, was used to derive the dispersion law of the edge wave in the continuous-vorticity model (Sec. V.D), and Eq. (5.21) relies on it. Beyond the scope of this model, Eq. (5.21) is incompatible with the conditions of continuity of the momentum flux used in the present section. The contradiction would be removed by the inclusion of an additional mode in the superposition of waves near the boundary. Such a mode would be available if dispersion of Tkachenko waves were taken into account because it would increase the order of the dispersion equation and the number of its solutions. The Tkachenko wave vector, however, exhausts all possible real solutions at $\omega\ll\Omega$, so an additional solution for the wave vector cannot be real. This means that an additional mode would attenuate at a distance of the order of the intervortex spacing and would not be tractable within the continuum theory. In view of this, the question arises which condition we should eliminate from the continuum theory where the number of conditions on the edge exceeds the number of modes. It seems that conditions appealing to conservation laws have priority over other ones. The condition connecting the velocity discontinuity on the vortex-pattern edge with the displacement of the edge may be given up on the grounds that the continuum theory is not capable of finding the displacement exactly enough.

It is worthwhile to note that a similar problem of additional boundary conditions arose in the theory of exciton polaritons in solids (Ivchenko, 1982). It was also resolved by preferring the conditions based on the conservation law.

F. General Tkachenko modes in a cylindric container

In Sec. V.C the axisymmetric Tkachenko modes were considered. In order to extend these considerations to general Tkachenko modes we shall derive the effective boundary condition for Tkachenko waves on a circular boundary. Our derivation follows along the same lines as in the previous section, where the plane boundary was discussed. Here we consider a cylindrical Tkachenko wave with n-fold symmetry in a container of radius R with a vortex pattern of radius R_0. The Tkachenko wave distorts the edge of the vortex pattern, and the velocity field of distortion is given by Eqs. (5.7) and (5.8) multiplied by $\exp(-i\omega t)$. The pressure field associated with distortion of the edge is found from the Euler equation—Eq. (5.27) inside the vortex pattern and Eq. (5.29) in the vortex-free region: for $r<R_0$,

$$P^D=-\rho w_+\frac{2\Omega+\omega}{n}R_0\left[\frac{r}{R_0}\right]^n\left[1-\left[\frac{R_0}{R}\right]^{2n}\right]\times\exp(in\varphi-i\omega t)\ , \tag{5.50}$$

and for $R_0<r<R$,

$$P^D=\rho w_-\frac{\omega R_0}{n}\left[\left[\frac{R_0}{r}\right]^n+\left[\frac{r}{R_0}\right]^n\left[\frac{R_0}{R}\right]^{2n}\right]\times\exp(in\varphi-i\omega t)\ . \tag{5.51}$$

Distortion also contributes to the stress tensor of the vortex array. From here on we shall not retain corrections of order ω/Ω responsible for the coupling of edge waves and Tkachenko waves. Then the distortion contributes only to the component $\sigma_{\varphi r}$ in the cylindric coordinate frame given by

$$\sigma^D_{\varphi r}=-2w_+\frac{\rho c_T^2(n-1)}{i\omega R_0}\left[\frac{r}{R_0}\right]^{n-2}\left[1-\left[\frac{R_0}{R}\right]^{2n}\right]\times\exp(in\varphi-i\omega t)\ . \tag{5.52}$$

The fields of displacements, velocities, and pressure generated by the Tkachenko wave are determined from Eq. (5.37) through the potential

$$\Psi=\Psi_n J_n(qr)\exp(in\varphi-i\omega t)\ . \tag{5.53}$$

The contribution of the Tkachenko wave to the stress-tensor component $\sigma_{\varphi r}$ is given by

$$\sigma^T_{\varphi r}=\rho c_T^2\Psi_n\left[q^2J_n''(qr)-\frac{q}{r}J_n'(qr)+\frac{n^2}{r^2}J_n(qr)\right]\times\exp(in\varphi-i\omega t)\ . \tag{5.54}$$

Now we write the continuity conditions on the vortex-pattern edge for the flows of mass and momentum as in Sec. V.E. Continuity of mass flow requires continuity of the radial component v_r of velocity:

$$iw_-\left[1-\left[\frac{R_0}{R}\right]^{2n}\right]=iw_+\left[1-\left[\frac{R_0}{R}\right]^{2n}\right]+\Psi_n\frac{\omega n}{R_0}J_n(qR_0)\ . \tag{5.55}$$

In the continuity condition for the flow of momentum normal to the boundary (the radial momentum) one may retain only the pressure, ignoring the contribution of the stress tensor:

$$\rho w_-\frac{\omega R_0}{n}\left[1+\left[\frac{R_0}{R}\right]^{2n}\right]=\frac{2i\Omega R_0}{n}\left\{iw_+\left[1-\left[\frac{R_0}{R}\right]^{2n}\right]+\Psi_n\frac{\omega n}{R_0}J_n(qR_0)\right\}\ . \tag{5.56}$$

The third continuity condition is imposed on the flow of the transverse momentum given by $\Pi_{\varphi r}=\sigma_{\varphi r}$:

$$0=-\frac{2c_T^2(n-1)\rho}{i\omega R_0}w_+\left[1-\left[\frac{R_0}{R}\right]^{2n}\right]+\rho c_T^2\Psi_n\left[\left[\frac{2n^2}{R_0^2}-q^2\right]J_n(qR_0)-\frac{2q}{R_0}J_n'(qR_0)\right]\ . \tag{5.57}$$

Solving these equations, we obtain the edge-wave modes and Tkachenko modes. Tkachenko modes are given by solutions in which $w_-=0$, i.e., the vortex-free region is not disturbed and the radial velocity component v_r and the pressure on the edge of the vortex pattern [the right-hand sides of Eqs. (5.55) and (5.56)] vanish. Eliminating w_+ and assuming $R_0\simeq R$, we obtain the effective boundary condition, which contains only the Tkachenko-wave amplitude Ψ_n:

$$[J_{n+2}(qR)-nJ_n(qR)]\Psi_n=0\ . \tag{5.58}$$

The zeros $qR=\Lambda_{n,p}$ of the factor before Ψ_n yield the frequencies of the Tkachenko modes:

$$\omega_n(p)=\frac{c_T}{R}(n^2+\Lambda^2_{n,p})^{1/2}\ . \tag{5.59}$$

When $n=0$, Eq. (5.59) coincides with Eq. (5.15) for the Ruderman spectrum of axisymmetric modes, since $\Lambda_{0,s}=j_{2,s}$. At very large n, when the wavelength

$(n/2\pi R)^{-1}$ is small compared with R, the curvature of the boundary becomes unimportant, and the frequencies of the Tkachenko modes approach those obtained from the boundary condition that the normal displacement vanish $[u_r \propto J_n(qR)=0]$. This condition was derived in Sec. V.E for the plane boundary. For Tkachenko waves in a cylinder it was proposed by Williams and Fetter (1977).[5] But in general the condition $u_r=0$ does not hold. One should bear in mind that u_r is not a true radial displacement, but that produced by the Tkachenko wave. The true displacement of the edge of the vortex pattern differs by the contribution of the distortion field [the right-hand side of Eq. (5.55) is the velocity of this displacement].

It is interesting to note that the lowest frequency of the spectrum Eq. (5.59) belongs not to an axisymmetric mode, but to a mode $n=1$. Indeed, the lowest frequency at $n=1$, $\omega_{1,1}=3.21c_T/R$, is smaller than the fundamental Ruderman frequency $\omega_R(1)=\omega_{0,1}=5.14c_T/R$. But we have already pointed out (Sec. V.C) that formally the lowest axisymmetric mode is the Goldstone mode with a zero frequency. This mode, however, is of no physical interest.

VI. VORTEX OSCILLATIONS IN FINITE ROTATING VESSELS. THREE-DIMENSIONAL BOUNDARY PROBLEMS

A. Boundary conditions on a horizontal solid surface. Pinning

In this section vortex patterns will be considered that are finite not only in the horizontal plane (the xy plane), but also along the rotation axis (the z axis). They sustain vortex oscillations having wave vectors with components along and normal to the rotation axis. We begin with formulation of boundary conditions imposed on the perfect fluid at the horizontal solid surface. The first one is trivial: the normal component of the velocity vanishes at the surface,

$$v_z=0 \; . \tag{6.1}$$

Other boundary conditions arise from constraints on the vortex velocity. Such conditions have been proposed by Hall (1958) and in more general form by Bekarevich and Khalatnikov (1961):

$$\mathbf{v}_L-\mathbf{v}_B=-\zeta\frac{1}{\tilde{\omega}^2}\tilde{\boldsymbol{\omega}}\times(\hat{\mathbf{z}}\times\tilde{\boldsymbol{\omega}})+\zeta'\frac{1}{\tilde{\omega}}\hat{\mathbf{z}}\times\tilde{\boldsymbol{\omega}} \; . \tag{6.2}$$

Here $\mathbf{v}_B$ is the velocity of the solid surface in the xy plane.

In linear theory

$$\tilde{\boldsymbol{\omega}}=2\boldsymbol{\Omega}+2\Omega\frac{\partial\mathbf{u}}{\partial z} \; , \tag{6.3}$$

and one can rewrite Eq. (6.2) as

$$\mathbf{v}_L-\mathbf{v}_B=\zeta\frac{\partial\mathbf{u}}{\partial z}+\zeta'\hat{\mathbf{z}}\times\frac{\partial\mathbf{u}}{\partial z} \; . \tag{6.4}$$

The phenomenological condition of Bekarevich and Khalatnikov assumes that a surface force acts upon the vortices. If the surface is ideally "smooth," the force is absent, ζ and ζ' are infinite, and the vortices terminate normal to the surface plane, $\partial\mathbf{u}/\partial z=0$. In experiment, this last condition holds only on the free surface of the fluid. Any real solid surface is uneven. Suppose that the end of the vortex is located on the tip of a protuberance of the surface. Then there is a restoring force on the vortex proportional to its displacement, since the displacement increases the length of the vortex and therefore its energy. When the fluid flow dragging the vortex is too small, the vortex cannot escape from the protuberance, and stationary motion of the vortex along the surface is impossible. This is a phenomenon of vortex pinning well known for type-II superconductors in the mixed state (Tinkham, 1975). Coercivity affecting domain-wall and Bloch-line motion in ferromagnetic materials yields a similar effect. All these effects may be united under the title "dry friction:" motion is allowed only when a driving force exceeds some critical value. Surface pinning of vortices in experiments on stationary counterflow in He II was revealed by Yarmchuk and Glaberson (1978, 1979), and more systematically studied by Hegde and Glaberson (1980).

Formally one may describe the perfect-pinning case by assigning zero values to the phenomenological coefficients ζ and ζ' in the Bekarevich-Khalatnikov condition [Eq. (6.4)]. But physically it is more correct to rewrite the condition for pinned vortices in the stationary problem in terms of displacements

$$\mathbf{u}-\mathbf{u}_B=a\frac{\partial\mathbf{u}}{\partial z}+a'\hat{\mathbf{z}}\times\frac{\partial\mathbf{u}}{\partial z} \; , \tag{6.5}$$

or in the equivalent form

$$\frac{\partial\mathbf{u}}{\partial z}=b(\mathbf{u}-\mathbf{u}_B)-b'\hat{\mathbf{z}}\times(\mathbf{u}-\mathbf{u}_B) \; . \tag{6.6}$$

Here $\mathbf{u}_B$ is the displacement of the solid surface. Pinning coefficients in Eqs. (6.4)–(6.6) are connected in the Fourier representation by a j-complex relation (see Sec. IV.D),

$$b-jb'=\frac{1}{a+ja'}=\frac{-i\omega}{\zeta+j\zeta'} \; . \tag{6.7}$$

[5]Axisymmetric oscillations do not involve radial displacements of the fluid in the long-wavelength limit, as follows from Eq. (5.13), so the condition $u_r=0$ imposes no restriction on the oscillation frequency at $n=0$. But Williams and Fetter retained in their analysis the small terms of order r_v/R that made the radial displacement at $n=0$ finite. The condition that small u_r vanish at the boundary was satisfied when the azimuthal displacement u_φ vanished too. It yielded Eq. (5.16) from which Williams and Fetter determined the eigenfrequencies of the axisymmetric modes.

Explicit relations between the pairs of coefficients are obtained by separation of the real and imaginary parts with respect to the imaginary unit j (the reader is reminded that in performing this operation i is assumed to be "real").

Any linear version of the phenomenological boundary condition on vortices becomes of limited usefulness when the force dragging the vortices reaches the critical value for depinning. Then the pinning site cannot hold the vortex, and it will tend to jump from one surface protuberance to the next, thus producing an irregular and nonlinear form of slip (Hall, 1958). This means that the parameters of the boundary condition depend on the fluid flow in a complicated way, and no linear idealization is satisfactory. Following a similar line of reasoning, Yarmchuk and Glaberson (1979) questioned whether one could rely on the Bekarevich-Khalatnikov condition at all. This caution is entirely justified in the region where the onset of depinning occurs. But when the flow has far exceeded the critical depinning value, vortices move so fast that irregularities of their motion are not too important; then one may introduce an average velocity of the vortex slip along the surface and the Bekarevich-Khalatnikov condition for such a velocity. In the same manner, the problem of pinning in type-II superconductors is dealt with by introducing a linear resistance for the current much larger than the critical current of depinning. The problem of coercivity in ferromagnets is similarly treated using the concept of linear mobility of the domain walls in magnetic fields exceeding the coercivity field. A more ingenious and general approach to dealing with the pinning problem is to incorporate into the boundary condition the dry-friction force and a friction force proportional to the relative velocity of the vortices and the boundary. This was what Adams *et al.* (1985) did when studying the spin-up problem in superfluid ^{4}He. In this case, however, the problem becomes nonlinear; in the present review we shall stay with the linear Bekarevich-Khalatnikov condition.

Another question concerning vortex-boundary interaction was raised by Campbell and Krasnov (1982) and Adams *et al.* (1985): "how the alternative attachment and release (after some stretching) of a vortex line on surface irregularities could result in a dissipative force proportional to the relative velocities." It is natural to suggest that a dissipative force given by the Bekarevich-Khalatnikov condition is of the same origin as residual resistance in dirty metals at $T=0$: it arises as a result of averaging of vortex motion over random irregularities of the solid surface. Such a dissipative force may arise even if there is no direct contact between a rough wall and the vortices, as in the case of a vertical wall parallel to the vortices [see discussion after Eq. (5.20) in Sec. V.C]. In this case, however, the force is expected to be rather weak. The complete solution of the problem of surface pinning would be a theory connecting the force on the vortices with the amplitude and space scale of the surface irregularities. Some steps in this direction were recently taken by Schwartz (1985). However, up until now he has considered only isolated vortices.

Data on the magnitude of empirical coefficients in the Bekarevich-Khalatnikov condition were available from experiments on vortex oscillations that will be discussed later. Unfortunately, these data were not sufficient to determine both coefficients ζ and ζ' simultaneously. Therefore, beginning with Hall (1958), researchers in this field, were forced to assume that $\zeta'=0$. The direct measurement of the sliding coefficient ζ is due to Gamtsemlidze *et al.* (1966). They determined ζ under stationary-flow conditions. A disk on a suspension head was rotated with a velocity slightly greater than the rotation speed of a vessel containing He II in which the disk was immersed normal to the rotation axis. The parameter ζ was deduced from the displacement angle of the disk with respect to the rotating suspension head. The values of Gamtsemlidze *et al.* (1966) and those of Hall (1958) roughly agree and fall in the interval $\zeta=10^{-1}$–10^{-2} cm sec^{-1}. The data are discussed in greater detail by Andronikashvili and Mamaladze (1966). Additional information on pinning coefficients has been obtained from experiments in which the inertial-wave resonance was observed. These will be discussed in Sec. VI.E.

Data on vortex-surface interaction are available also from studies of transient phenomena such as the spin-up process, in which a freely rotating bucket of superfluid is impulsively spun up and allowed to relax back to solid-body rotation [for references see Chap. IX in the book of Andronikashvili *et al.* (1978) and the recent papers of Campbell and Krasnov (1982) and of Adams *et al.* (1985)]. Campbell and Krasnov (1982) analyzed the spin-up experiments of Reppy *et al.* (1960) and of Reppy and Lane (1965) in some two-fluid-hydrodynamics models (two-fluid hydrodynamics of rotating superfluids is discussed in Secs. VII and VIII) using the Bekarevich-Khalatnikov boundary condition identical to Eq. (6.4) with $\zeta'=0$. They introduced a dimensionless pinning parameter ξ connected with ζ in Eq. (6.4) by the relation $\xi=v_s/L\zeta$ (L is the height of the helium). Fitting to experimental data, Campbell and Krasnov obtained values of ξ a few orders of magnitude larger than those deduced from previous experiments and from those given above. But their model relies upon a rather questionable assumption concerning the rate of vortex creation at lateral walls that undermines the reliability of their quantitative conclusions on the pinning force. Adams *et al.* (1985) have performed spin-up experiments with a rather small aspect ratio $L/R=0.21$ and interpreted them using the same model as that of Campbell and Krasnov (1982) except for presence of the dry-friction force in the boundary condition (see discussion above). They arrived at a quantitative estimate of the pinning parameter quite different from that obtained by Campbell and Krasnov. But, as they believed themselves, the model of Campbell and Krasnov is inadequate to describe normal-fluid behavior in their experiment because it supposes that the normal fluid spins up by viscous diffusion of vorticity from lateral walls homogeneous along the rotation axis. For small aspect ratio in the experiment of Adams *et al.* (1985), another mechanism is expected to dominate, similar to that describing spin-up of the classical fluid between infinite horizontal planes (see Sec. 2.4 in the book of Greenspan, 1968). This mechanism involves formation of the Ekman layer near the bottom, through which fluid is pumped radially by centrifugal action. To compensate for the mass flow in the Ekman layer, a small vertical secondary flow is required. Adams *et al.* (1985) suggested that their pinning parameter, obtained by fitting to experimental data, reflected this mechanism to some extent. But this would change the physical meaning of ξ, which would cease to be a pinning parameter in the strict sense. Thus further development of the theory is necessary in order to obtain reliable data on pinning from spin-up experiments.

B. Pile-of-disks oscillations. The vortex-wave resonance versus the inertial-wave resonance

The experimental study of vortex oscillations in the pile-of-disks geometry began in the 1950s (Andronikashvili and Tsakadze, 1958; Hall, 1958). According to Hall (1958), the original aim of these investigations was to study mutual friction effects in rotating He II by a modification of the classic experiment of Andronikashvili (1946). An increase was expected both in period and in damping of the oscillating pile of disks as a result of drag upon the superfluid by mutual friction. The mutual friction effects, however, were found to play a minor part in comparison with the other damping mechanism of dragging connected with vortex pinning and elastic properties of vortex lines.

Remarkable resonance effects were discovered when the disk separation was an odd number of half-wavelengths of the vortex wave propagating along the vortex line (Sec. IV.D). The simple explanation of these resonances (Hall, 1958) is that the ends of the vortices are completely pinned to rough surfaces of oscillating disks, which generate oscillations of vortex lines resembling oscillations of the elastic string. Motion of the vortices is coupled with motion of the fluid as a whole. The latter affects oscillations of the disks. Only resonances with an odd number of half-wavelengths were observable, because the even-number resonances generated fluid motion with velocity vanishing after averaging over the fluid layer.

The resonances were observed by determining the period of oscillation as a function of rotation speed. The pile of disks was suspended by a long torsion fiber in a rotating ^{4}He cryostat. These resonances were extensively studied experimentally and theoretically by Hall and the Tbilisi group. One can find comprehensive discussions of the matter in the previous reviews (Hall, 1960; Andronikashvili *et al.*, 1961, 1978; Andronikashvili and Mamaladze, 1966, 1967). On the whole, theory and experiment were in a good agreement.

But all these early experiments were performed at comparatively low rotation speed. Recently Andereck *et al.* (1980) and Andereck and Glaberson (1982; see also Glaberson, 1982) returned to these experiments and were able, owing to progress in technique, to extend the experiments to considerably higher rotation speeds (from ~ 1 to ~ 10 rad/sec). They discovered that at high rotation speed the observed resonance frequencies were considerably lower than had been predicted by Hall. This was explained by the generation of Tkachenko waves. Later we shall discuss the arguments on which this claim was based (Sec. VI.E). Another interpretation of the results of Andereck *et al.* has been offered in the framework of Hall's original theory, rejecting, however, some approximations traditionally made within this theory (Sonin, 1983). This discussion attracted attention to some new aspects of the old theory and has been included in the present review.

We consider a perfect fluid between two horizontal solid surfaces located at $z=L$ and $z=-L$, so the width of the fluid layer is $2L$. The dimensions of the surfaces are much larger than $2L$. The surfaces perform harmonic torsional oscillations around the z axis. It seems that one can treat this problem as one dimensional: to find the dependence of fluid variables on z at a given velocity $\mathbf{v}_B$ of the solid surfaces and to take into account the slow variation of $\mathbf{v}_B$ in the xy plane afterwards. But because of the singular behavior of the oscillation spectrum it is preferable to take into account from the very first the variation of $\mathbf{v}_B$ in the xy plane, supposing that the solid-surface velocity field is $\mathbf{v}_B \exp(i\mathbf{q}\mathbf{r}-i\omega t)$ and $\mathbf{v}_B$ is normal to $\mathbf{q}$. Later we shall take the limit $\mathbf{q}\rightarrow 0$. The solution of our hydrodynamical problem must be a linear superposition of plane waves

$$\sim \exp(i\mathbf{Q}\mathbf{R}-i\omega t)=\exp(i\mathbf{q}\mathbf{r}+ipz-i\omega t)$$

with fixed $\mathbf{q}$ and ω. Any wave in the superposition corresponds to a value of p satisfying the dispersion equation for waves in an incompressible perfect fluid [Eq. (4.46)]; we neglect the Tkachenko contribution $\propto c_T^2 q^2$ until the following section. This equation has three solutions for p^2. Two of them are much larger than small q^2 and correspond to torsion vortex waves with the spectrum Eq. (4.53),

$$p(j)^2=\frac{-2\Omega-ji\omega}{\nu_s}, \tag{6.8}$$

where the substitution $j=\pm i$ gives values of p for two circularly polarized waves. The third value of p^2,

$$p_I^2=\frac{\omega^2}{4\Omega^2-\omega^2}q^2, \tag{6.9}$$

yields the inertial wave with velocity homogeneous in space, but varying in time in the limit $\mathbf{q}\rightarrow 0$.

The three possible values of p^2 correspond to three standing waves in the wave superposition. For the field of the fluid velocity in the xy plane we have in the j-complex representation

$$\tilde{v}(z,\mathbf{r},t)=\left[\tilde{v}_K(j)\cos p(j)z+v_I\left[\frac{i\omega}{2\Omega}+j\right]\cos p_I z\right]$$
$$\times\exp(i\mathbf{q}\cdot\mathbf{r}-i\omega t)\ . \qquad (6.10)$$

The real v_I (with respect to j) is the amplitude of the inertial wave, and the complex $\tilde{v}_K=v_K+jv_K'$ determines the amplitudes of two torsion vortex waves (Kelvin modes). The explicit expressions for the velocity components v_q and v_t (along and normal to $\mathbf{q}$) are obtained by separation of the real and the imaginary parts:

$$\begin{aligned}v_q&=\mathrm{Re}_j\tilde{v}=\left[\tfrac{1}{2}(v_K+iv_K')\cos p(+)z+\tfrac{1}{2}(v_K-iv_K')\cos p(-)z+\frac{i\omega}{2\Omega}v_I\cos p_I z\right]\exp(i\mathbf{q}\cdot\mathbf{r}-i\omega t)\ ,\\ v_t&=\mathrm{Im}_j\tilde{v}=\left[\frac{1}{2i}(v_K+iv_K')\cos p(+)z-\frac{1}{2i}(v_K-iv_K')\cos p(-)z+v_I\cos p_I z\right]\exp(i\mathbf{q}\cdot\mathbf{r}-i\omega t)\ .\end{aligned} \qquad (6.11)$$

Here $p(\pm)$ are the values of $p(j)$ at $j=\pm i$. We see that the amplitudes of the Kelvin waves are given by v_K+iv_K' and v_K-iv_K'; the relations between the velocity components in the Kelvin waves [Eq. (4.54)] and in the inertial wave [Eq. (4.57)] at $c_Tq=0$ are satisfied by the fluid velocity field [Eq. (6.10)]. The third component of velocity, v_z, is found from the incompressibility condition $pv_z+qv_q=0$:

$$v_z(z,\mathbf{r},t)=-q\left[\mathrm{Re}_j\left[\frac{\tilde{v}_K(j)\sin p(j)z}{p(j)}\right]+\frac{i\omega}{2\Omega}v_I\frac{\sin p_I z}{p_I}\right]\ . \qquad (6.12)$$

Now we take into account that p_I is proportional to q [see Eq. (6.9)] and therefore small, and expand in p_Iz. Using Eqs. (4.54) and (4.57) between velocity components, we can find the vortex velocity $\mathbf{v}_L$ and substitute it into the Bekarevich-Khalatnikov condition, Eq. (6.4) or (6.5). We limit ourselves to the simple case of perfect pinning when the vortex velocity $\mathbf{v}_L$ is equal to the solid-surface velocity and the Bekarevich-Khalatnikov condition at the surfaces $z=\pm L$ reduces to the complex relation involving two boundary conditions,

$$jv_B=\frac{i\omega}{2\Omega j}\tilde{v}_K\cos p(j)L+\left[\frac{i\omega}{2\Omega}+j\right]v_I\ . \qquad (6.13)$$

The third boundary condition [Eq. (6.1)] is imposed on v_z; according to Eq. (6.12) one may write it as

$$0=\mathrm{Re}_j\left[\frac{\tilde{v}_K(j)\sin p(j)L}{p(j)}\right]+\frac{i\omega}{2\Omega}Lv_I\ . \qquad (6.14)$$

Now one can see the reason why we retained the small wave vector $\mathbf{q}$ in the xy plane. If we had not, we would have lost the boundary condition, Eq. (6.14), which follows from Eq. (6.12) when $\mathbf{q}\neq 0$. But there is another way to deduce Eq. (6.14). Suppose that the total fluid flow, integrated over the whole width of the fluid layer, must vanish along some direction in the xy plane. This restriction yields Eq. (6.14) when the direction is that of the vector $\mathbf{q}$. In the pile-of-disks geometry $\mathbf{q}$ should be directed along the radius, since the absence of radial mass flow is required by the incompressibility of the fluid.

The solution of Eqs. (6.13) and (6.14) yields the amplitudes of all waves in the space between disks. The fluid velocity averaged over the fluid layer is of most interest. It has only the t component [the q component vanishes as a result of Eq. (6.14)] and its ratio to the solid-surface velocity is equal to the ratio of the effective density ρ' that moves with disks to the total density ρ. This ratio was introduced into the theory by Hall (1958) and is equal to

$$\begin{aligned}\frac{\rho'}{\rho}&=\frac{1}{2v_BL}\int_{-L}^{L}dz\,v_t(z)\\&=\frac{-\dfrac{4\Omega^2}{\omega^2}\mathrm{Re}_j[Z_K(j)]-\dfrac{2\Omega}{i\omega}\mathrm{Im}_j[Z_K(j)]+\dfrac{4\Omega^2}{\omega^2}Z_K(j)Z_K(-j)}{1-\dfrac{4\Omega^2}{\omega^2}\mathrm{Re}_j\left[\left[1-\dfrac{ji\omega}{2\Omega}\right]Z_K(j)\right]}\ .\end{aligned} \qquad (6.15)$$

In a more explicit form, this ratio coincides with the expression obtained by Hall (1958):

$$\frac{\rho'}{\rho}=\frac{Z_K(+)\dfrac{\Omega(\omega-2\Omega)}{\omega^2}-Z_K(-)\dfrac{\Omega(\omega+2\Omega)}{\omega^2}+\dfrac{4\Omega^2}{\omega^2}Z_K(+)Z_K(-)}{1-\dfrac{\Omega(\omega+2\Omega)}{\omega^2}Z_K(+)+\dfrac{\Omega(\omega-2\Omega)}{\omega^2}Z_K(-)}\ . \qquad (6.16)$$

Here $Z_K(j)=[\tan p(j)L]/p(j)L$ and $Z_K(\pm)$ are $Z_K(j)$ at $j=\pm i$.

We do not give here any formulas connecting ρ'/ρ with the observable variation of the period and damping of the pile-of-disks oscillations because one can find them in the previous reviews; an especially extensive discussion of them is given by Andronikashvili *et al.* (1961).

The poles of ρ'/ρ as a function of ω determine the frequencies of the resonances. Hall (1958) supposed that resonances are possible only when $\omega \gg 2\Omega$ and simplified Eq. (6.16), assuming that $\omega \gg \Omega$:

$$\frac{\rho'}{\rho}=\frac{\Omega}{\omega}\left[\frac{\tan p(+)L}{p(+)L}-\frac{\tan p(-)L}{p(-)L}\right] . \tag{6.17}$$

The poles of ρ'/ρ are poles of $\tan p(+)L$ (only poles on the real axis correspond to physical resonances), and the resonance frequencies are linear functions of the angular velocity Ω:

$$\omega_n=\nu_s\left[\frac{\pi}{2}\frac{2n-1}{L}\right]^2+2\Omega . \tag{6.18}$$

Here the integer n is a number of a branch of the spectrum.

The Hall approximation at $\omega \gg \Omega$ is equivalent to the assumption that the oscillating disks generate only Kelvin modes in the fluid. Thus one can obtain the resultant formulas of this approximation, Eqs. (6.17) and (6.18), by deleting the inertial-wave contribution and the boundary condition Eq. (6.1) imposed on v_z from the very first. But an approximation based on expansion in Ω is questionable when one deals with the low branches of the Hall spectrum [Eq. (6.18)]. In observing the lowest branch of the Hall resonances, Andereck *et al.* (1980) revealed a serious discrepancy with Eq. (6.18). They observed resonances at large ratio Ω/ω when resonances on the Kelvin modes are impossible, since both values $p(j)^2$ are negative,

$$p(j)^2=-k_t^2=-\frac{2\Omega}{\nu_s} . \tag{6.19}$$

In this case the Kelvin vortex waves are evanescent and penetrate into the fluid only to the length $l_s=1/k_t$, which is sometimes called the width of the superfluid Eckman layer (Alpar, 1978). In the limit $\Omega \gg \omega$ Hall derived from Eq. (6.16) a simple formula,

$$\frac{\rho'}{\rho}=1-\frac{\tanh k_t L}{k_t L} , \tag{6.20}$$

which excludes the possibility of resonances.

In order to verify suspicions concerning the lowest branches of the oscillation spectrum, Sonin (1983) calculated the dispersion law numerically, using the general Hall formula, Eq. (6.16). The resonance frequencies were determined by the condition that the denominator in Eq. (6.16) be equal to zero. The results are shown in Fig. 2 by solid lines in dimensionless variables $\bar{\omega}=\omega L^2/\nu_s$ and $\bar{\Omega}=\Omega L^2/\nu_s$. On the same plot the dispersion curves given by Eq. (6.18) are shown by dashed lines. One can see that the numerically calculated curve $n=1$ entirely differs from that defined by Eq. (6.18). The new curve goes into the region $2\Omega > \omega$, where Hall resonances are impossible, and approaches at $\bar{\Omega}\to\infty$ the asymptotic curve

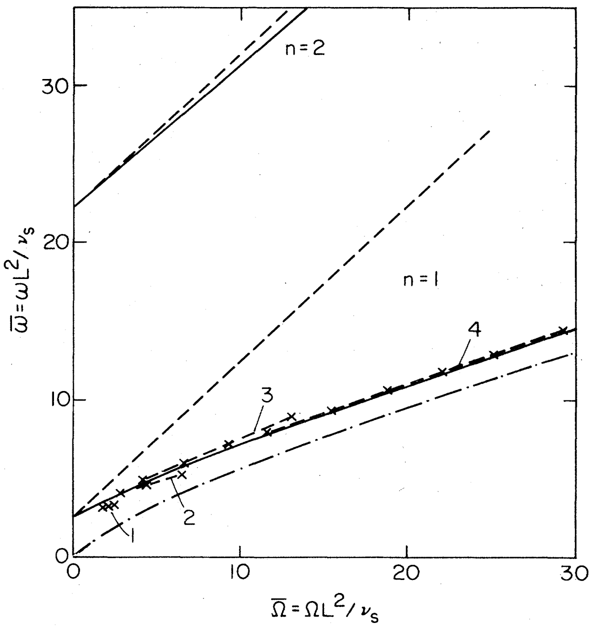

FIG. 2. Low-frequency branches $n=1$ and 2 of the spectrum of oscillations in the superfluid between two horizontal solid surfaces. The solid lines show the eigenfrequencies that are numerically calculated poles of the function ρ'/ρ given by Eq. (6.16). The dashed lines are the eigenfrequencies determined by Hall's approximate formula Eq. (6.18). The dot-dashed line is the asymptotic curve given by Eq. (6.21), which the $n=1$ branch approaches as $\Omega \gg \omega$. The dashed lines with the cross marks are drawn through the experimental points of Andereck and Glaberson (1982) obtained for different distances $d=2L$ between disks: 1, $d=0.0208$ cm; 2, $d=0.0366$ cm; 3, $d=0.0508$ cm; 4, $d=0.0762$ cm. It was assumed that $\nu_s=10^{-3}\ \mathrm{cm^2\,sec^{-1}}$.

$$\bar{\omega}=(2\bar{\Omega})^{3/4}$$

or

$$\omega=\left[\frac{\nu_s}{L^2}\right]^{1/4}(2\Omega)^{3/4}=\left[\frac{\kappa}{4\pi L^2}\ln\frac{r_v}{r_c}\right]^{1/4}(2\Omega)^{3/4} . \tag{6.21}$$

The frequencies on this curve produce a zero denominator in Eq. (6.16) when $\Omega \gg \omega$. It is worth noting that the Hall formula Eq. (6.20) for the region $\Omega \gg \omega$ was criticized in the past and another one proposed by Mamaladze [see Eq. (4.35) in the review paper of Andronikashvili *et al.*, 1961]:

$$\frac{\rho'}{\rho}=\frac{1-(\tanh k_t L)/k_t L}{1-(\omega/2\Omega)^2 k_t L/\tanh k_t L} . \tag{6.22}$$

The zero of the denominator in this formula yields the resonance frequency Eq. (6.21), but Andronikashvili *et al.*

did not discuss such a resonance.

Comparing the dispersion curve $n=2$ from the Hall formula [Eq. (6.18)] with that numerically calculated (both are shown in Fig. 2) does not reveal a serious difference. It does show the peculiar nature of the lowest branch of the oscillation spectrum in the rotating superfluid between two horizontal planes. Unlike the higher branches, the lowest one may be considered to be the Kelvin-mode resonance only at its beginning when $\Omega \simeq 0$. At the growth of Ω the ratio Ω/ω increases, as does the contribution of the inertial wave to the superposition of the waves generated in the fluid bulk. When $\Omega \gg \omega$ the Kelvin oscillations are negligible nearly everywhere in the fluid bulk. Thus we may consider the lowest oscillation branch as corresponding to the inertial-wave resonance. But it should be remembered that even in the limit $\Omega/\omega \to \infty$ one cannot neglect the Kelvin waves completely. Though they penetrate into the fluid only the width of the superfluid Eckman layer $l_s = 1/k_t$, which is small compared with the spacing between disks, the Kelvin modes are vital to the behavior of the inertial wave, since it is governed by the effective boundary condition allowing for the existance of Kelvin waves in the boundary layer. This condition is derived in the following section. Therefore despite the fact that an inertial wave is possible in the classic fluid with continuous vorticity (Sec. II), only the array of quantized vortices can sustain the inertial-wave resonance, which is a *quantum* phenomenon. Indeed, when the circulation quantum κ tends to zero and Kelvin modes disappear, the inertial-wave-resonance frequency given by Eq. (6.21) vanishes also.

C. The effective boundary condition and slow motion of a horizontal layer of rotating fluid

The inertial wave is the classic limit of the mixed wave, which is the only mode that can propagate in a rotating superfluid at low frequencies. Low-frequency oscillations may be described in the frame of slow-motion hydrodynamics developed in Sec. IV.E, but we need a boundary condition for Eq. (4.59) governing slow motion in the horizontal layer of the rotating superfluid. The condition should allow for the existence of Kelvin waves in a superfluid Eckman layer of width l_s. Such an approach, based on the concept of the boundary layer, is widely used in hydrodynamics (Greenspan, 1968).

In order to derive the effective boundary condition for slow motion with $\omega \ll \Omega$, we consider a fluid occupying a semi-infinite space $z>0$. This is the limit of a thick fluid layer $L \gg l_s$ between oscillating disks. The velocity of the solid surface $z=0$ is given by the field $\mathbf{v}_B \exp(i\mathbf{qr}-i\omega t)$, where $\mathbf{v}_B$ is normal to $\mathbf{q}$ and both lie in the xy plane, as was assumed in the previous section. Near the solid surface three waves are generated: the mixed one and the two Kelvin modes, all of them with the same $\mathbf{q}$ and ω. Unlike the previous section, the waves are not standing, but propagating along the z axis. Then the fluid velocity in the xy plane in the j-complex representation is given by

$$\tilde{v}(z,\mathbf{r},t)=\left[\tilde{v}_K(j)\exp[ip(j)z] + v_M\left[-\frac{2\Omega}{i\omega}\frac{p^2}{Q^2}+j\right]\exp(ipz)\right]\times\exp(i\mathbf{qr}-i\omega t)\ . \tag{6.23}$$

Here v_M and p are the amplitude and the z component of the wave vector, respectively, of the mixed wave. This time we do not ignore the Tkachenko contribution $c_T q$ to the frequency of the mixed mode [see Eq. (4.55)], and the latter does not reduce to the classical inertial wave. The amplitudes and wave numbers of the two Kelvin modes are determined by the complex quantities $\tilde{v}_K(j)$ and $p(j)$. The sign of $p(j)$ [see Eq. (6.19)] is chosen so as to provide attenuation of the Kelvin vortex wave far from the boundary.

The vortex velocity $\mathbf{v}_L$ is determined with the help of Eq. (4.50) in the Kelvin vortex wave and with the help of Eq. (4.57) in the mixed wave; the net vortex velocity field $\mathbf{v}_L$ is substituted into the Bekarevich-Khalatnikov condition [Eq. (6.5)] on the surface $z=0$. It yields the j-complex equation

$$-\tilde{a}k_t\frac{i\omega}{2\Omega j}\tilde{v}_K=\frac{i\omega}{2\Omega j}\tilde{v}_K+\left[\frac{i\omega}{2\Omega}+j\right]v_M-jv_B\ . \tag{6.24}$$

Here we used $\tilde{a}=a+ja'$, the relation $\mathbf{v}_L=-i\omega\mathbf{u}$, and Eq. (6.19) for $p(j)^2$, and neglected the contribution of the mixed wave to $\partial\mathbf{u}/\partial z$, since p is small at $\omega \ll \Omega$.

The component v_z of the fluid velocity, found from the incompressibility condition as in the previous section, is substituted into the boundary condition Eq. (6.1). It yields the equation

$$-\frac{q}{ik_t}\mathrm{Re}_j\tilde{v}_K+\frac{2\Omega}{i\omega}\frac{p}{q}v_M=0\ . \tag{6.25}$$

Elimination of the j-complex amplitude $\tilde{v}_K$ of the Kelvin waves from Eqs. (6.24) and (6.25) gives the equation for the mixed-wave amplitude:

$$A\frac{ip}{q^2}v_M+(v_M-v_B)=0\ , \tag{6.26}$$

where

$$A^{-1}=k_t^{-1}\,\mathrm{Re}_j[(1+\tilde{a}k_t)^{-1}]\ , \tag{6.27}$$

or in explicit form [see Eq. (6.7)]

$$A=k_t\frac{(1+ak_t)^2+(a'k_t)^2}{1+ak_t} =k_t\frac{(b+k_t)^2+b'^2}{b(b+k_t)+b'^2}\ . \tag{6.28}$$

For perfect pinning $(b,b'\to\infty)$ $A=k_t$, otherwise $A>k_t$, since the dissipation parameter b is always positive.

We have considered a single-plane mixed wave near the solid surface. A general velocity field in slow-motion hydrodynamics is a superposition of such plane waves. Then Eq. (6.26) is a Fourier transform of the boundary condition we look for, and v_M is a Fourier transform of the vortex velocity $\mathbf{v}_L$ in the fluid bulk. The inverse Fourier transformation of Eq. (6.26) yields the boundary condition imposed on the vortex velocity $\mathbf{v}_L$:

$$A\frac{1}{\Delta_\perp}\frac{\partial \mathbf{v}_L}{\partial z}-(\mathbf{v}_L-\mathbf{v}_B)=0 . \tag{6.29}$$

In deriving Eq. (6.29), the correspondence principle $ip \rightarrow \partial/\partial z$, $q^2 \rightarrow -\Delta_\perp$ was used. The meaning of the inverse Laplace operator $1/\Delta_\perp$ is explained in the paragraph after Eq. (4.59). In a more general form, Eq. (6.29) may be rewritten as

$$A\frac{1}{\Delta_\perp}(\hat{\mathbf{n}}\cdot\nabla)\mathbf{v}_L-(\mathbf{v}_L-\mathbf{v}_B)=0 , \tag{6.30}$$

where $\hat{\mathbf{n}}$ is the unit normal to the solid surface directed inside the fluid.

Now we shall return to the problem of the fluid layer between two oscillating solid surfaces located at $z=\pm L$. In contrast with the previous section, we limit ourselves to conditions of slow motion $\omega \ll \Omega$ and large spacing $L \gg l_s$, but we do take into account the Tkachenko contribution to the mixed-wave frequency given by Eq. (4.55). Since the Taylor-Proudman theorem holds in a fluid with quantized vorticity, we expect the fluid motion to be columnar. This means that the velocity slowly varies along the z axis and may be expanded in the Taylor series:

$$\mathbf{v}_L(z,\mathbf{r},t)=\mathbf{v}_0(\mathbf{r},t)+\tfrac{1}{2}\mathbf{v}_1(\mathbf{r},t)z^2 . \tag{6.31}$$

Substitution of this expansion in the equation of motion (4.59) and the boundary condition Eq. (6.30) yields

$$\frac{\partial^2 \mathbf{v}_0}{\partial t^2}=4\Omega^2\frac{1}{\Delta_\perp}\mathbf{v}_1(\mathbf{r},t)+c_T^2\Delta_\perp\mathbf{v}_0 , \tag{6.32}$$

$$AL\frac{1}{\Delta_\perp}\mathbf{v}_1-(\mathbf{v}_0-\mathbf{v}_B)=0 . \tag{6.33}$$

We neglect the small $\mathbf{v}_1$ everywhere except when it appears under the operator $\partial/\partial z$. According to Eq. (6.33),

$$\mathbf{v}_1=\frac{1}{AL}\Delta_\perp(\mathbf{v}_0-\mathbf{v}_B) , \tag{6.34}$$

and elimination of $\mathbf{v}_1$ yields the following equation for the velocity field homogeneous along the z axis (from here on, the subscript 0 will be omitted):

$$\frac{\partial^2 \mathbf{v}}{\partial t^2}-c_T^2\Delta_\perp\mathbf{v}+\omega_L^2\mathbf{v}=\omega_L^2\mathbf{v}_B , \tag{6.35}$$

where the frequency ω_L is

$$\omega_L=\frac{2\Omega}{\sqrt{AL}} . \tag{6.36}$$

Thus we have reduced the problem of fluid motion between two oscillating horizontal solid surfaces to the $2+1$ partial differential nonhomogeneous equation for the two-dimensional velocity $\mathbf{v}\simeq\mathbf{v}_L$. The eigenfrequencies of free oscillations of the fluid layer are given by the dispersion law

$$\omega^2=\omega_L^2+c_T^2q^2 . \tag{6.37}$$

Though variation of the velocity is slow along the z axis and was taken into account with the perturbation theory, even a weak dependence on z, together with pinning of vortices, is responsible for the gap ω_L in the oscillation spectrum Eq. (6.37). When the Tkachenko contribution is small ($q \rightarrow 0$), the eigenfrequency is equal to ω_L, and the mixed eigenmode is the inertial-wave resonance considered in the previous section. Indeed, in the limit of perfect pinning $A=k_t$, the frequency ω_L given by Eq. (6.36) coincides with the inertial-wave-resonance frequency, Eq. (6.21), obtained for the case $\omega \ll \Omega$.

But the columnar motion as just described is not the only type of motion allowed by slow-motion hydrodynamics in a horizontal fluid layer. Let us consider a more general solution of the equation of motion Eq. (4.59):

$$\mathbf{v}(z,\mathbf{r},t)=\mathbf{v}_0\cos pz\,\exp(i\mathbf{q}\mathbf{r}-i\omega t) . \tag{6.38}$$

The values of p, q, and ω are connected by the dispersion relation, Eq. (4.55), for the mixed mode. We consider free oscillations of the fluid layer when the solid surfaces are at rest, i.e., $\mathbf{v}_B=0$. Then substitution of Eq. (6.38) into the boundary condition Eq. (6.30) yields an equation determining the set of discrete values of p at given $\mathbf{q}$ and ω:

$$\frac{Ap}{q^2}\tan pL-1=0 . \tag{6.39}$$

The columnar motion corresponds to the smallest value of p obtained by expanding the tangent in Eq. (6.39):

$$p_0^2=\frac{q^2}{AL} . \tag{6.40}$$

Substitution of $p=p_0$ into the dispersion law [Eq. (4.55)] readily yields the dispersion law [Eq. (6.37)] for the columnar motion.

Motion of the fluid is columnar when $p_0L \ll 1$, or, according to Eq. (6.40),

$$Lq^2 \ll A . \tag{6.41}$$

We see from Eq. (6.34) that this inequality provides that the z-dependent part of the velocity field is small compared with the z-independent part.

When the inequality Eq. (6.41) holds, other values of p satisfying Eq. (6.39) are approximately equal to those at which the tangent vanishes:

$$p_n=\frac{\pi n}{L},\quad n\geq 1 . \tag{6.42}$$

Substitution of $p=p_n$ into Eq. (4.55) yields the eigenfrequencies

$$\omega_n^2=\left[\frac{2\Omega\pi n}{qL}\right]^2+c_T^2q^2 , \tag{6.43}$$

which, together with the eigenfrequencies of the columnar motion given by Eq. (6.37) (the branch $n=0$), constitute the full oscillation spectrum of the horizontal layer of the rotating fluid. Any branch except the lowest $n=0$ has a minimum of the frequency at a finite value of q. Values of q and ω in the minimum are determined from Eq. (4.56) by the substitution $p=p_n$.

D. Low-frequency oscillations of a finite cylindric container with superfluid

The next step in our analysis of the boundary problem is to consider a fluid bound both along the z axis and in the xy plane. We shall consider a fluid filling a cylindric rotating container of height $2L$ and radius R, but all our results will refer as well to a fluid with a free surface when the height of the fluid column is L. The analysis rests on Eq. (6.35) governing slow columnar motion of a fluid between oscillating horizontal solid surfaces. Later it will become clear that such an approach is valid even for very long cylinders with $L \gg R$. For this equation we need a boundary condition on the lateral walls of the cylinder. Such a condition for axisymmetric modes in an infinitely long cylinder in a state of uniform rotation was discussed in Sec. V.C [Eq. (5.19)]. Its extension to the case of a cylinder with oscillating rotation speed is obvious:

$$\alpha\left[\frac{\partial v_\varphi(R)}{\partial r}-\frac{v_\varphi(R)}{R}\right]+[v_\varphi(R)-v_B(R)]=0 . \quad (6.44)$$

Here we wrote the boundary condition in terms of velocities instead of displacements, using the cylindric coordinate frame. The velocity of the lateral walls and the velocity of the horizontal surfaces bounding the fluid are given by

$$\mathbf{v}_B=\boldsymbol{\Omega}'\times\mathbf{r}e^{-i\omega t},\quad v_B(R)=\Omega' Re^{-i\omega t} . \quad (6.45)$$

Here Ω' is the amplitude of the angular velocity oscillations of the container.

It is not difficult to find the solution of the nonhomogeneous equation (6.35) satisfying the boundary condition Eq. (6.44):

$$v_\varphi(r,t)=\left[\frac{\omega_L^2}{\omega_L^2-\omega^2}\frac{r}{R}-\frac{\omega^2}{\omega_L^2-\omega^2}\frac{J_1(qr)}{J_1(qR)-\alpha q J_2(qR)}\right]\times\Omega' Re^{-i\omega t},\quad v_r(r,t)=0 . \quad (6.46)$$

Here $q=(\omega^2-\omega_L^2)^{1/2}/c_T$. The angular momentum per unit height for such a velocity field is

$$M=2\pi\int_0^R r^2 dr\, v_\varphi(r,t)=2\pi\Omega' R^4 e^{-i\omega t}\left[\frac{1}{4}\frac{\omega_L^2}{\omega_L^2-\omega^2}-\frac{\omega^2}{\omega_L^2-\omega^2}\frac{J_2(qR)}{qR[J_1(qR)-\alpha q J_2(qR)]}\right] . \quad (6.47)$$

Suppose that the axial oscillations of a freely suspended container are considered. Then the equation determining the eigenfrequencies is found from the condition that the net angular momentum of the fluid and of the container itself not vary during the oscillations. This yields the dispersion equation

$$\frac{\omega^2}{\omega^2-\omega_L^2}\left[\frac{1}{4}-\frac{J_2(qR)}{qR[J_1(qR)-\alpha q J_2(qR)]}\right]-\frac{1+\beta}{4}=0 . \quad (6.48)$$

Here β is the ratio of the moment of inertia of the container to the moment of inertia of the fluid, if the latter rotates as a solid together with the container.

If the container is very tall [$L\to\infty$, $\omega_L\to 0$; see Eq. (6.36)] and the moment of inertia of the container is negligible ($\beta=0$), then Eq. (6.48) yields the Ruderman spectrum, Eq. (5.15). The fundamental Ruderman frequency is especially interesting for us:

$$\omega_R=\frac{5.14c_T}{R} . \quad (6.49)$$

The fundamental eigenfrequency from the spectrum given by Eq. (6.48) is well approximated by this value until the inequality $\omega_R \gg \omega_L$ holds.

In the opposite limit $\omega_L \gg \omega_R$ the eigenfrequencies approach ω_L. When $\alpha=0$ and β is not very large, they are given approximately by the formula

$$\omega_n^2=\omega_L^2+x_n^2 c_T^2/R^2=\omega_L^2+(x_n/5.14)^2\omega_R^2 . \quad (6.50)$$

Here x_n are roots of the equation,

$$\frac{1}{4}-\frac{J_2(x)}{xJ_1(x)}=0 . \quad (6.51)$$

We see that the eigenfrequencies do indeed approach the inertial-wave-resonance frequency ω_L, and at the same time the spacing between neighboring frequencies decreases when the ratio ω_L/ω_R increases. This creates the possibility of beats in the oscillations. Equation (6.46) shows also that when $\omega_L/\omega_R\to\infty$ and as a result $\omega\to\omega_L$, the fluid velocity in the bulk far exceeds the velocity of the lateral walls. This distinctive feature of the case $\omega_L \gg \omega_R$ is due to the inertial-wave resonance in the fluid inside the container.

Thus oscillations with the Ruderman frequency [Eq. (6.49)] determined by Tkachenko-wave velocity are possible in a container of finite height only if $\omega_R \gg \omega_L$. This inequality in terms of container dimensions [see Eqs. (6.36) and (6.49)] takes the form

$$\frac{L}{R^2} \gg \frac{3.8}{A} \frac{\Omega}{\kappa} . \tag{6.52}$$

On the other hand, the columnar motion of the fluid persists until the inequality Eq. (6.41) holds. At the inertial-wave-resonance condition $\omega_L \gg \omega_R$ we have $q = 6.8/R$ [the number 6.8 is the smallest root of Eq. (6.51)], and Eq. (6.41) may be rewritten as

$$\frac{L}{R^2} \ll \frac{A}{46} . \tag{6.53}$$

Comparing Eqs. (6.52) and (6.53), we see that when A is small (strong pinning) a transition to the Ruderman regime of oscillations bounded by Eq. (6.52) may occur in a region where the theory of columnar motion is invalid. It is clear, however, that when the container is high enough the Ruderman theory must correctly predict eigenfrequencies. So the theory of columnar motion, which includes the Ruderman theory as a particular case, fails to predict *corrections* to Ruderman frequencies, but not frequencies themselves. It cannot, moreover, predict the value of L/R^2 at which the corrections become unimportant, when this value does not satisfy the inequality Eq. (6.53). Nevertheless, one may assign to the frequency ω_L and the parameter A, which is connected with ω_L by Eq. (6.36), a meaning broader than that within the scope of the columnar-motion theory. They can be treated as characteristic parameters determining the transition to the Ruderman regime of oscillations. But then the relation connecting A with the pinning coefficients, Eq. (6.28), does not hold.

The problem of vortex oscillations in a cylindrical vessel of finite height was solved formerly by a more intricate method based on expansion of the fluid velocity field in cylindrical waves (Sonin, 1976). It was found that the obtained series of Bessel functions allowed summation for not very long vessels when an inequality similar to Eq. (6.53) held. As a result the velocity field given by Eq. (6.46) at $\alpha = 0$ was obtained. Such a simple form of result aroused the suspicion that there existed another, more direct and transparent way to achieve the same result. The theory of columnar motion provides this way. But the expansion method gives the dispersion equation beyond the region of columnar motion, though its solution requires numerical calculations.

E. Experiments on Tkachenko waves. The Tkachenko wave versus the inertial wave

The first attempt to observe a Tkachenko wave was undertaken by Tkachenko himself in the 1960s in a study of torsion oscillations of a light cylinder suspended by a thin fiber (see Tkachenko, 1974). No conclusive data were obtained; nevertheless we shall return to this idea later, in Sec. VIII.D, since its discussion requires a knowledge of the two-fluid theory.

Further efforts to discover Tkachenko waves experimentally were stimulated by Ruderman's theory explaining long-period oscillations of the pulsar rotation velocity. Tsakadze and Tsakadze (1973, 1975) tried to simulate pulsar phenomena; they studied free rotation of buckets of various shapes, cylindrical included, filled with He II, and revealed rotation-period oscillations superimposed on the steady deceleration of rotation. The finding was in qualitative agreement with the Ruderman theory. The oscillations disappeared above the λ point that proved their superfluid nature. But the oscillation frequencies observed for cylindrical vessels were nearly eight times higher than the Ruderman fundamental frequency ω_R. This disagreement was explained by the three-dimensional effects of pinning and bending of vortices (Sonin, 1976). We saw in the previous section that such effects can transform a Tkachenko-wave resonance into an inertial-wave resonance in cylinders of moderate aspect ratio L/R. What Tsakadze and Tsakadze (1973) observed, then, was the inertial-wave resonance with frequency ω_L. This was proven by experimental detection of properties predicted by the theory of the inertial-wave resonance (S. Tsakadze, 1976). First of all, the experimental oscillation frequency depended on the height of helium in the vessel (the length of vortices) and on the smoothness of the bottom, but did not depend on the vessel radius. It agreed with the theoretical expression for the inertial-wave-resonance frequency [see Eqs. (6.36) and (6.28)], but contradicted Eq. (6.49) for the Ruderman frequency. In a number of cases, beats arose in the oscillations, which are possible only in the region of the inertial-wave resonance. One can find more detailed comparison and discussion in the papers of Sonin (1976) and S. Tsakadze (1976; see also J. Tsakadze *et al.*, 1980).

In further experiments, S. Tsakadze (1978) used longer cylindrical vessels in an effort to get the region $\omega_R \gg \omega_L$ where pure Tkachenko waves are possible. He could not do it completely, because it required impractical vessels with too large a ratio L/R [see the inequality Eq. (6.52)], but he managed to come fairly close. In his experiments the Tkachenko contribution to the oscillation frequency is the same order as the pinning contribution, but not more, and S. Tsakadze had to refer to the general theory allowing for both contributions. In Fig. 3, reproduced from the paper of S. Tsakadze (1978), the theoretical dependence of the oscillation frequencies on the frequency ω_L is shown, calculated numerically with the help of Eq. (6.48). In order to draw the experimental points on the same plot, values of the parameter A were necessary. They were obtained by extrapolation of the plot A versus the oscillation period (see Fig. 2 in the paper of S. Tsakadze, 1978). The plot was drawn across the values of A obtained in the region of the inertial-wave resonance, $\omega_L \gg \omega_R$, where A is a single fitting parameter and is readily determined from the experimental frequencies. The experimental points for the longest cylinder with the largest value of L/R^2 are most important for determination of the Tkachenko rigidity of the vortex lattice. It is clear that they follow approximately the theoretical curve for the fundamental frequency and approach the region where this curve becomes parallel to the abscissa axis; i.e.,

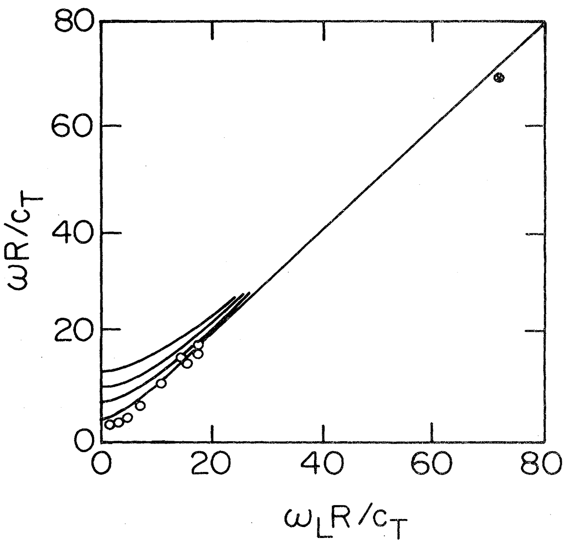

FIG. 3. Dependence of the eigenfrequencies ω of the low-frequency mixed modes on the frequency ω_L characterizing pinning in a cylindric vessel of radius R filled by a superfluid up to height L. ⊗, $R=3.2$ cm, $L=5$ cm; ○, $R=0.75$ cm, $L=7$ cm; ●, $R=0.4$ cm, $L=10$ cm. The solid lines were calculated from Eq. (6.48) at $\alpha=0$ and $\beta=3$. The experimental points and the theoretical curves are taken from S. Tsakadze (1978).

dependence on the frequency ω_L, connected with pinning, vanishes. The discrepancy between experimental points and the theoretical curve is about 30%. This means that the Tkachenko-wave velocity in the theory and in the experiment agree with 30% accuracy. The agreement looks rather satisfactory, especially if we take into consideration the possible sources of error in interpretation. The rotation speed was not constant in the experiment due to deceleration of free rotation. Amplitudes of period oscillations were rather large, but nonlinear effects were not estimated theoretically or experimentally. Extrapolation of the parameter A beyond its measured values was also vulnerable to criticism. These deficiencies, pointed out by Andereck and Glaberson (1982), affect the quantitative, not the qualitative aspect of interpretation. The experiment of S. Tsakadze provided the first experimental evidence of the existence of Tkachenko waves and consequently of cristalline order in the vortex lattice. Regardless of how the parameter A was determined, some experimental points in Fig. 3 for the longest vessel show a tendency to become independent of A, and it was for these very points that the Tkachenko wave velocity was found. To conclude our discussion of the experiment by S. Tsakadze (1978), we note that not all measured values of A satisfy the inequality $A \geq k_t$ following from Eq. (6.28) ($k_t=85$ cm^{-1} for Tsakadze's experiment, but measured A were in the range 20—200 cm^{-1}). One possible explanation of this disagreement is that the inequality Eq. (6.53), necessary for the theory of the columnar motion to be valid, is not satisfied for too small $A < k_t$. This means that these values of A, obtained from experimental oscillation frequencies with the help of formulas of the columnar-motion theory, are not those connected with the pinning coefficients by Eq. (6.28). Nevertheless, as was said in the paragraph after Eq. (6.53), the parameter A may retain its meaning as the parameter characterizing pinning and independent of geometry; that the experimental points for different vessels are approximately on one line favors this interpretation.

In Sec. VI.B we have already mentioned the experiments of Andereck *et al.* (1980) and Andereck and Glaberson (1982), who claimed that they obtained confirmation of the existence of Tkachenko waves. But their experiments are readily explained by the theory of pile-of-disks oscillations, without reference to the Tkachenko rigidity and given in Sec. VI.B. A comparison of this theory with the results of the experiments of Andereck *et al.*, also carried out in pile-of-disks geometry, is presented in Fig. 2. The figure shows the resonance frequencies observed by Andereck *et al.* for different distances $d=2L$ between disks. The experimental frequencies lie quite close to the numerically calculated curve $n=1$, which gives frequencies of the inertial-wave resonance. The experimental data for $d=0.269$ cm, which are distributed beyond the scale of the plot, at large $\overline{\Omega} > 170$, also agree well with theory. Such an agreement allows us to conclude that Andereck and Glaberson (1982) observed the same inertial-wave resonance as that observed by Tsakadze and Tsakadze (1973), though in another geometry and at other frequencies of oscillation and rotation.

But Andereck *et al.* (1980) and Andereck and Glaberson (1982) themselves associated the observed resonances with a peak in the density of state due to a minimum on the dispersion curve of the mixed mode at given p. This minimum was discussed in Sec. IV.E [see Eq. (4.56) there]. Andereck *et al.* took the value $p=\pi/2L$ in Eq. (4.56) that assumed the ends of vortices to be pinned completely to the disk surfaces. Then the frequency of the peak is given by

$$\omega=\frac{1}{2}\left[\frac{\pi\kappa}{L^2}\right]^{1/4}(2\Omega)^{3/4} . \tag{6.54}$$

It is remarkable that this expression differs from that for the inertial-wave-resonance frequency, Eq. (6.21), only by a factor of about 0.7, and it is consequently not surprising that the agreement of Eq. (6.54) with the experiment is not much worse than the theoretical dispersion curve $n=1$ associated with the frequency of the inertial-wave resonance. But the theory used by Andereck *et al.* for interpretation of their experiments left unresolved a serious problem (by admission of the authors themselves; see p. 288 in the paper of Andereck and Glaberson, 1982): how can the oscillations of disks, introducing perturbations with wavelengths of the order of the disk radii, generate Tkachenko waves (more exactly, mixed waves with considerable Tkachenko contributions) whose wavelengths are an order of magnitude smaller than the radius of the disks? The density-of-state peak arguments do not provide an explicit mechanism for generation, but suggest that there should be one. On the other hand, the inertial-wave-resonance interpretation rests on a self-consistent hydrodynamical derivation.

For a further insight into the problem of pile-of-disks oscillations it would be interesting to carry out observations at large rotation speeds of higher branches of the oscillation spectrum, say the branch $n=2$, pictured in Fig. 2. According to the theory of Sec. VI.B, the lowest branch $n=1$ is of a special nature; other branches do not differ essentially from the frequencies given by the Hall expression, Eq. (6.18) (compare the solid and dashed lines $n=2$ in Fig. 2). But the density-of-state peak interpretation does not distinguish between different branches and is extended to the branches $n>1$ assuming that $p=\pi(2n-1)/2L$ in the expression for the frequency given by Eq. (4.56). Therefore it predicts that observable frequencies would be considerably lower than Hall's resonance frequencies.

The spectrum of axial wave numbers $p=\pi(2n-1)/2L$, implied by Andereck *et al.*, follows from the theory of slow motion of a horizontal fluid layer when $Lq^2 \gg A$; these p's are solutions of Eq. (6.39). Since $q \sim 1/R$, the condition $Lq^2 \gg A$ is equivalent to $L/R^2 \gg A$, the inequality opposite to Eq. (6.53) which bounds the region of columnar motion. So the density-of-state peak concept is expected to be relevant only at quite large values of the geometrical parameter L/R^2. In the pile-of-disks geometry L/R^2 is very small as a rule.

In conclusion it is worth noting that the theory of vortex oscillations in finite vessels (Sec. VI.D) helps us to understand some peculiarities of vortex observations by photographic techniques. In the first experiments, Williams and Packard (1974) saw a rather irregular structure with vortex images considerably blurred. The authors explained this by random oscillations of the rotation speed. By modification of technique and geometry they managed later to obtain good photographs of the vortex lattice (Gordon *et al.*, 1978; Yarmchuk *et al.*, 1979). The theory predicts that random oscillations of the rotation speed are especially dangerous in the limit $\omega_L \gg \omega_R$ (small L/R^2) when oscillations approach the inertial-wave resonance. Then, as explained in the paragraph after Eq. (6.51) (Sec. VI.D), the fluid velocity in the bulk far exceeds the velocity of the walls. One can avoid such amplifying of the rotation-speed oscillations by decreasing $\omega_L/\omega_R \propto R/\sqrt{AL}$. In connection with this we note that good photographs of the vortex lattice have been obtained for rather small values of this ratio.

F. Vortex oscillations in pulsars

Many features of pulsar behavior have been explained by hypothesis that the rotating inner matter of pulsars is in the superfluid state and is threaded by vortex lines (Kirzhnitz, 1970). Some of these features are associated with transient phenomena, such as sudden spin-ups (glitches) of pulsars, or relaxation after the glitch, which are beyond the scope of the present paper (see discussion by Alpar, 1978; Anderson *et al.*, 1978; Tsakadze and Tsakadze, 1980). In addition, very slow oscillations of the Crab pulsar's period have been observed (see discussion by Dyson, 1971). As has already been mentioned, Ruderman (1970) has associated this remarkable phenomenon with Tkachenko waves. He considered waves in a cylinder (see Sec. V.C), ignoring the difference between cylindrical and spherical geometry. Inserting into Eq. (6.49) the data for the pulsar in the Crab nebula ($\Omega=200$ rad/sec, $R=10^6$ cm, $\kappa=2\times10^{-3}$ cm^2/sec), Ruderman found that the oscillation period should be

$$T=\frac{2\pi}{\omega_R}=9.73\times10^6 \text{ sec}=3.75 \text{ months},$$

in good agreement with the observed period of ~ 3 months.

But as was shown before, the Ruderman model is too idealized even for very long cylinders when pinning of vortices to the solid surface is important (Sec. VI.D). In pulsars the solid crust confining the neutron matter plays the role of a solid surface. Having no data on vortex pinning to the crust at our disposal, we can, nevertheless, estimate the possible effect of pinning from above (Tsakadze *et al.*, 1980). Pinning can increase the oscillation frequency up to the frequency ω_L of the inertial-wave resonance. Like Ruderman, we ignore the difference between a cylinder and a sphere and put $L=R$ in Eq. (6.36) for ω_L. The frequency ω_L is maximal when pinning is strong and $A=k_t$. So the maximal ω_L is given by

$$\omega_L=\left[\frac{\kappa}{4\pi R^2}\ln\frac{r_v}{r_c}\right]^{1/4}(2\Omega)^{3/4}. \qquad (6.55)$$

Here Eqs. (6.19) and (4.33) for k_t and ν_s were used. According to Dyson (1971, p. 50), the vortex cores occupy a fraction 10^{-20} of the pulsar volume. This means that the ratio of the intervortex spacing to the core radius is $r_v/r_c \sim 10^{10}$, so that Eq. (6.55) yields for the Crab pulsar $\omega_L=1.3\times10^{-2}$ rad/sec. This corresponds to a period of about 8 min. Thus pinning would lead to a strong decrease of the oscillation period.

VII. VORTEX OSCILLATIONS IN TWO-FLUID HYDRODYNAMICS

A. Two-fluid macroscopic hydrodynamics of a rotating superfluid

The two-fluid theory of Landau (Landau, 1941; Khalatnikov, 1971; Putterman, 1974) was formulated for a curl-free superfluid. If one deals with a superfluid threaded by vortex lines, the equations of the Landau theory hold in the multiply connected region outside the vortex lines; they should be supplemented by equations of motion for the vortex lines. Together they constitute the theory governing the behavior of the superfluid at finite temperatures. But, just as in the one-fluid perfect fluid, one can sometimes describe the motion of the vortex lines in terms of the averaged parameters of the vortex array. Such a theory is the macroscopic hydrodynamics defined above (Sec. IV.A), extended to include finite-temperature two-fluid effects. Here we shall formulate a two-fluid macroscopic hydrodynamics omitting all dissipation terms except for mutual friction. The remaining dissipation effects may be introduced into the theory (and will be, as necessary) in the same manner as in the original "microscopic" two-fluid hydrodynamics.

The two-fluid hydrodynamic theory includes continuity equations for mass and entropy, which usually have the form

$$\frac{\partial \rho}{\partial t}+\nabla\cdot\mathbf{j}=0\ , \tag{7.1}$$

$$\frac{\partial S}{\partial t}+\nabla\cdot(S\mathbf{v}_n)=R/T\ . \tag{7.2}$$

Here $\mathbf{j}$ is the net mass current, S is the entropy per unit volume, $\mathbf{v}_n$ is the normal velocity, and R is the dissipation function.

The Euler equation (4.1) retains its former form as deduced from purely kinematic arguments, but now refers to the superfluid part of the fluid and contains the superfluid velocity $\mathbf{v}_s$:

$$\frac{\partial \mathbf{v}_s}{\partial t}+\tilde{\boldsymbol{\omega}}\times\mathbf{v}_L=-\nabla(\mu_0+\tfrac{1}{2}v_s^2)\ . \tag{7.3}$$

To satisfy energy conservation, μ_0 should be the chemical potential at a given current $\mathbf{j}$ in a reference frame moving with velocity $\mathbf{v}_s$

The next equation is provided by the momentum conservation law, which is

$$\frac{\partial j_i}{\partial t}+\nabla_j\Pi_{ij}=0\ . \tag{7.4}$$

Unlike the case of the one-fluid theory, in which the momentum conservation law is derived from the mass continuity equation and the Euler equation, in two-fluid theory momentum conservation must be dealt with in an independent equation.

The system of equations of the two-fluid macroscopic hydrodynamics includes, as well, the thermodynamic identities and expressions for the momentum-flux tensor and the vortex velocity $\mathbf{v}_L$.

The Gibbs relation for the energy density in a reference frame moving with $\mathbf{v}_s$ is

$$dE_0=\mu_0 d\rho+T\,dS+(\mathbf{v}_n-\mathbf{v}_s)\cdot d\mathbf{j}_0+\frac{\partial E}{\partial \nabla_i u_j}d\nabla_i u_j\ . \tag{7.5}$$

Here $\mathbf{u}$ is the displacement of the vortex line, and $\mathbf{j}_0$ is the mass current in the superfluid reference frame,

$$\mathbf{j}_0=\mathbf{j}-\rho\mathbf{v}_s\ . \tag{7.6}$$

For the total energy density in the laboratory frame,

$$E=E_0+\mathbf{j}_0\cdot\mathbf{v}_s/2+\tfrac{1}{2}\rho v_s^2\ , \tag{7.7}$$

the Gibbs relation is

$$dE=\mu\,d\rho+T\,dS+\mathbf{v}_n\cdot d\mathbf{j}+\boldsymbol{\lambda}\cdot d\mathbf{v}_s+\frac{\partial E}{\partial \nabla_i u_j}d\nabla_i u_j\ , \tag{7.8}$$

where

$$\mu=\mu_0-\mathbf{v}_n\cdot\mathbf{v}_s+\tfrac{1}{2}v_s^2 \tag{7.9}$$

is the chemical potential in the laboratory reference frame at a given current $\mathbf{j}$. The current in the reference frame moving with normal velocity $\mathbf{v}_n$ is

$$\boldsymbol{\lambda}=\frac{\delta E}{\delta \mathbf{v}_s}=\mathbf{j}-\rho\mathbf{v}_n\ . \tag{7.10}$$

The velocity $\mathbf{v}_s$ and the vortex displacement $\mathbf{u}$ are not independent variables, since vortex displacements change vorticity. The definition of the derivatives $\delta E/\delta\mathbf{v}_s$ and $\partial E/\partial\nabla_i u_j$ relies on the convention that the kinetic energy of the averaged superflow with velocity $\mathbf{v}_s$ is a function of the velocity and involves the long-range interaction between vortices, but the rest part of the superfluid kinetic energy is a function of the displacement and involves the short-range interaction of vortices.

The pressure is determined by the usual thermodynamic formula:

$$\begin{aligned}P&=-E_0+TS+\mu_0\rho+\mathbf{j}_0\cdot(\mathbf{v}_n-\mathbf{v}_s)\\&=-E+TS+\mu\rho+\mathbf{j}\cdot\mathbf{v}_n\ .\end{aligned} \tag{7.11}$$

Differentiation of Eq. (7.11) yields the Gibbs-Duhem relation:

$$\begin{aligned}dP&=\rho\,d\mu_0+S\,dT+\mathbf{j}_0\cdot d(\mathbf{v}_n-\mathbf{v}_s)-\frac{\partial E}{\partial\nabla_i u_j}d\nabla_i u_j\\&=\rho\,d\mu+S\,dT+\mathbf{j}\cdot d\mathbf{v}_n-\boldsymbol{\lambda}\cdot d\mathbf{v}_s-\frac{\partial E}{\partial\nabla_i u_j}d\nabla_i u_j\ .\end{aligned} \tag{7.12}$$

The momentum-flux tensor is given by

$$\Pi_{ij}=P\delta_{ij}+j_i v_{nj}+v_{si}\lambda_j-\frac{\partial E}{\partial\nabla_j u_i}+\nabla_i u_k\frac{\partial E}{\partial\nabla_j u_k}\ . \tag{7.13}$$

Now we are able to prove the energy conservation law by calculating the time derivative of the energy density with the help of the Gibbs relation (7.8) and the dynamic equations (7.1)—(7.4). These yield

$$\frac{\partial E}{\partial t}+\nabla\cdot\mathbf{J}=0\ , \tag{7.14}$$

where the energy current is given by

$$\begin{aligned}\mathbf{J}&=(\mu_0+\tfrac{1}{2}v_s^2)\mathbf{j}+ST\mathbf{v}_n+(\mathbf{j}_0\cdot\mathbf{v}_n)\mathbf{v}_n\\&\quad-\frac{\partial E_0}{\partial\nabla u_k}[v_{Lk}-(\mathbf{v}_L\cdot\nabla)u_k]\\&=\mu\mathbf{j}+ST\mathbf{v}_n+(\mathbf{j}\cdot\mathbf{v}_n)\mathbf{v}_n+(\mathbf{v}_n\cdot\mathbf{v}_s)\boldsymbol{\lambda}\\&\quad-\frac{\partial E}{\partial\nabla u_k}[v_{Lk}-(\mathbf{v}_L\cdot\nabla)u_k]\ .\end{aligned} \tag{7.15}$$

The energy conservation law [Eq. (7.14)] holds if the dissipation function is equal to

$$R=(\mathbf{v}_L-\mathbf{v}_n)f_\Sigma\ , \tag{7.16}$$

where the force

$$\mathbf{f}_\Sigma=-\tilde{\boldsymbol{\omega}}\times\boldsymbol{\lambda}+\mathbf{f} \tag{7.17}$$

is introduced. Here the elastic force $\mathbf{f}$ is defined the same as in a one-fluid perfect fluid [see Eq. (4.14)].

The requirement that the dissipation function R be a positive definite quadratic form gives the phenomenological equation for the vortex velocity as follows:

$$\mathbf{v}_L=\mathbf{v}_n+\alpha\mathbf{f}_\Sigma+\alpha'\hat{\mathbf{n}}\times\mathbf{f}_\Sigma\ , \tag{7.18}$$

where $\alpha>0$ and $\hat{\mathbf{n}}$ is the unit vector parallel to the vorticity $\tilde{\boldsymbol{\omega}}$.

Our system of equations is closed if the dependence of the energy on all hydrodynamic variables is known. Up to now we have refrained from giving explicit expressions for the energy or currents in order to obtain hydrodynamics in a more general form, allowing extension to an anisotropic superfluid (Sec. IX). Now we shall fill in the formal scheme with specific physical content. As we proceed further it will be advantageous to use the fact that vortex lines perturb a fluid considerably only in their immediate vicinity. Therefore the coarse-graining procedure for deriving macroscopic hydrodynamics yields values of hydrodynamical quantities that differ negligibly from those in the original "microscopic" hydrodynamics, with the exception, of course, of quantities that are completely absent in the original theory (mutual friction parameters and the elastic stress tensor of the vortex lattice). The accuracy of such an approach is determined by a small parameter $\Omega r_c^2/\kappa$ (r_c is the core radius), which is extremely small in He II (see the footnote to Sec. 32 on p. 112 in the Russian edition of the book by Khalatnikov, 1971). Within this approach we have the usual explicit expressions for the currents:

$$\begin{gathered}\mathbf{j}=\rho_s\mathbf{v}_s+\rho_n\mathbf{v}_n,\quad \mathbf{j}_0=\rho_n(\mathbf{v}_n-\mathbf{v}_s)\ ,\\ \boldsymbol{\lambda}=\rho_s(\mathbf{v}_s-\mathbf{v}_n)\ .\end{gathered} \tag{7.19}$$

Here ρ_s and ρ_n are scalar superfluid and normal densities, respectively, taken from microscopic two-fluid hydrodynamics.

Any force in vortex dynamics is connected with some velocity by the Magnus relation. The elastic force $\mathbf{f}$ may be written as

$$\mathbf{f}=-\rho_s\tilde{\boldsymbol{\omega}}\times(\mathbf{v}_{sl}-\mathbf{v}_s)\ . \tag{7.20}$$

Then from Eqs. (7.17) and (7.19) we obtain

$$\mathbf{f}_\Sigma=-\rho_s\tilde{\boldsymbol{\omega}}\times(\mathbf{v}_{sl}-\mathbf{v}_n)\ . \tag{7.21}$$

The velocity $\mathbf{v}_{sl}$ defined by Eq. (7.20) is the local superfluid velocity at a point on the vortex line (in Sec. III.B it was explained what such a local fluid velocity means). It differs from the average superfluid velocity $\mathbf{v}_s$ when the vortex array is deformed and the elastic force $\mathbf{f}$ arises. Comparing Eq. (7.20) with the Magnus relation (4.5) in the perfect fluid, we see ρ_s, $\mathbf{v}_s$, and $\mathbf{v}_{sl}$ instead of ρ, $\mathbf{v}$, and $\mathbf{v}_L$, since vortex lines are singularities of the velocity field of the superfluid part of the fluid. If the elastic force $\mathbf{f}$ were the only force acting upon the vortices, the net velocity of the vortices $\mathbf{v}_L$ would be equal to $\mathbf{v}_{sl}$, which is dependent on the form of the vortex pattern. But the normal part of the fluid produces a drag force on the vortices called the mutual friction force $\mathbf{f}_{\rm fr}$, and this force is responsible for the difference between $\mathbf{v}_{sl}$ and $\mathbf{v}_L$:

$$\mathbf{f}_{\rm fr}=-\rho_s\tilde{\boldsymbol{\omega}}\times(\mathbf{v}_L-\mathbf{v}_{sl})\ . \tag{7.22}$$

The net force on the vortices per unit volume consists of the elastic force and the mutual friction force,

$$\mathbf{f}_s=\mathbf{f}+\mathbf{f}_{\rm fr}=-\rho_s\tilde{\boldsymbol{\omega}}\times(\mathbf{v}_L-\mathbf{v}_s)\ . \tag{7.23}$$

One can introduce the force $\mathbf{f}_s$ into the Euler equation (7.3):

$$\frac{\partial\mathbf{v}_s}{\partial t}+\nabla(\mu_0+\tfrac{1}{2}v_s^2)+\tilde{\boldsymbol{\omega}}\times\mathbf{v}_s=\mathbf{f}_s/\rho_s\ . \tag{7.24}$$

Both the forces $\mathbf{f}$ and $\mathbf{f}_{\rm fr}$ are of quantum origin and vanish in the continuous-vorticity model.

It can be shown that the dissipation function given by Eq. (7.16) contains only the mutual friction force:

$$\begin{aligned}R&=-(\mathbf{v}_L-\mathbf{v}_n)\{\mathbf{f}_{\rm fr}+\rho_s[\tilde{\boldsymbol{\omega}}\times(\mathbf{v}_L-\mathbf{v}_n)]\}\\ &=-(\mathbf{v}_L-\mathbf{v}_n)\mathbf{f}_{\rm fr}=-(\mathbf{v}_{sl}-\mathbf{v}_n)\mathbf{f}_{\rm fr}\ .\end{aligned} \tag{7.25}$$

Here Eqs. (7.17), (7.20), and (7.22) were used. Using Eq. (7.21) one can rewrite the equation of motion of vortices, Eq. (7.18), in the form of the linear relation for velocities $\mathbf{v}_L$, $\mathbf{v}_{sl}$, and $\mathbf{v}_n$:

$$\mathbf{v}_L=\mathbf{v}_{sl}+\frac{\rho_n}{2\rho}B'(\mathbf{v}_n-\mathbf{v}_{sl})+\frac{\rho_n}{2\rho}B\hat{\mathbf{n}}\times(\mathbf{v}_n-\mathbf{v}_{sl})\ . \tag{7.26}$$

Here B and B' are the mutual friction parameters introduced by Hall and Vinen (1956) and connected with α and α' by

$$\alpha=\frac{1}{\rho_s\tilde{\omega}}\frac{\rho_n}{2\rho}B,\quad \alpha'=\frac{1}{\rho_s\tilde{\omega}}\left[1-\frac{\rho_n}{2\rho}B'\right]\ . \tag{7.27}$$

The system of macroscopic hydrodynamic equations formulated above is invariant with respect to the Galilean transformation; the transformation to the rotating reference frame adds the Coriolis force to the momentum conservation law [Eq. (7.4)]. Other equations do not vary when $\tilde{\boldsymbol{\omega}}$ is defined as the absolute vorticity in the inertial reference frame (see Sec. II) and the centrifugal force is neglected.

Various generalizations of this theory have been proposed. Hills and Roberts (1977a) generalized the Bekarevich-Khalatnikov hydrodynamics to include the inertia of the vortex lines. Though the inertia of vortices is usually slight (see Sec. III.B), it may become important when some particles trapped by vortex lines enhance the inertia of the lines considerably. Then the three-fluid theory of Hills and Roberts (1977a) is necessary to describe the wave properties of rotating superfluids.

In the linear theory, the part of the superfluid kinetic energy, which depends on deformations of the vortex array and includes the short-range interaction between vortices, is defined by Eq. (4.27) as in the perfect fluid, but in Eqs. (4.33) and (4.34) for the elastic constants C_1, C_2, and C_3 the superfluid density ρ_s should take the place of ρ. Then the density E_v of this energy is given by

$$E_v=\frac{\rho_s\kappa\Omega}{8\pi}\left\{\ln\frac{\omega_c}{2\Omega}\left[\frac{\partial \mathbf{u}}{\partial z}\right]^2-(\nabla\cdot\mathbf{u})^2+\frac{1}{2}\left[\left[\frac{\partial u_x}{\partial x}-\frac{\partial u_y}{\partial y}\right]^2+\left[\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right]^2\right]\right\}. \quad (7.28)$$

The mutual friction parameters B and B' will be discussed in Sec. X.

Now let us bring together the equations of the linear macroscopic two-fluid hydrodynamics to be used below:

$$\frac{\partial\rho'}{\partial t}+\rho\nabla\cdot\mathbf{v}=0\ , \quad (7.29)$$

$$\frac{\partial S'}{\partial t}+S\nabla\cdot\mathbf{v}_n=0\ , \quad (7.30)$$

$$\frac{\partial \mathbf{v}_s}{\partial t}+\nabla\mu+2\boldsymbol{\Omega}\times\mathbf{v}_L=0\ , \quad (7.31)$$

$$\nabla\mu=\frac{1}{\rho}\nabla P-\frac{S}{\rho}\nabla T\ , \quad (7.32)$$

$$\frac{\partial \mathbf{v}}{\partial t}+\frac{1}{\rho}\nabla P+2\boldsymbol{\Omega}\times\mathbf{v}+\frac{\rho_s}{\rho}2\boldsymbol{\Omega}\times(\mathbf{v}_{sl}-\mathbf{v}_s)=0\ , \quad (7.33)$$

$$\mathbf{v}_{sl}=\mathbf{v}_s+\nu_s\hat{\mathbf{z}}\times\frac{\partial^2\mathbf{u}}{\partial z^2}+\frac{c_T^2}{2\Omega}[\hat{\mathbf{z}}\times\Delta_\perp\mathbf{u}-2\hat{\mathbf{z}}\times\nabla(\nabla\cdot\mathbf{u})]\ . \quad (7.34)$$

Here $\mathbf{v}$ is the center-of-mass fluid velocity,

$$\mathbf{v}=\frac{\rho_s}{\rho}\mathbf{v}_s+\frac{\rho_n}{\rho}\mathbf{v}_n\ , \quad (7.35)$$

and the vortex velocity $\mathbf{v}_L$ is given by Eq. (7.26).

Looking ahead to our further analysis, let us see what oscillation modes are expected in the two-fluid hydrodynamics of a rotating superfluid. We should distinguish between longitudinal and transverse modes involving curl-free and divergence-free motion. Various modes may or may not involve relative motion of the superfluid and normal parts of the fluid (a counterflow). The fluid at rest, without vorticity, has one longitudinal mode, in which it moves as a whole (the first sound). Transverse modes without a relative motion are impossible because of the condition $\nabla\times\mathbf{v}_s=0$. Oscillations involving a counterflow admit one longitudinal mode, the second sound. In addition, the normal part of the fluid sustains two transverse viscous modes. In a rotating superfluid all four modes listed above remain, though with their spectra changed, but they are supplemented by two transverse modes associated with vortex oscillations. Thus in total there are six modes at a given wave vector.

Recently some of these modes (longitudinal and transverse-vortex ones) were independently studied by Chandler and Baym (1986). Their results, though presented in a slightly different form, agree with those given below.

B. Hydrodynamic equations for a completely incompressible fluid

We shall call the fluid completely incompressible when it is incompressible in the mechanical and thermal sense, that is, when the mass density and the entropy density are constants. In practice all results of vortex dynamics dealing with real situations may be obtained within the model of the completely incompressible fluid. Formally this model corresponds to the limit of infinite velocities of the first and second sound ($c_1\to\infty$, $c_2\to\infty$). In this limit both $\mathbf{v}_s$ and $\mathbf{v}_n$ are divergence-free, i.e., $\nabla\cdot\mathbf{v}_s=0$, $\nabla\cdot\mathbf{v}_n=0$. We may transform the equations for the velocities $\mathbf{v}_s$ and $\mathbf{v}$ into equations for the velocities $\mathbf{v}_s$ and $\mathbf{v}_n$, which in the completely incompressible fluid are given by

$$\frac{\partial \mathbf{v}_s}{\partial t}+[2\boldsymbol{\Omega}\times\mathbf{v}_L]_\perp=0\ , \quad (7.36)$$

$$\frac{\partial \mathbf{v}_n}{\partial t}+[2\boldsymbol{\Omega}\times\mathbf{v}_n]_\perp+\frac{\rho_s}{\rho_n}[2\boldsymbol{\Omega}\times(\mathbf{v}_{sl}-\mathbf{v}_L)]_\perp+\nu\nabla\times(\nabla\times\mathbf{v}_n)=0\ . \quad (7.37)$$

As in Sec. IV.C, the subscript $\perp$ indicates that only the transverse part of the corresponding vortex field is retained. The first-viscosity term is included in Eq. (7.37) as important for the problems to be considered later. It is worth mentioning that all dissipation processes except for the first viscosity and mutual friction are impossible in an incompressible fluid, since they involve compression of the superfluid or normal fluid, or both.

Now let us Fourier analyze Eqs. (7.36), (7.37), and (7.34) using the earlier notations for the wave vector $\mathbf{Q}$ and its components p and $\mathbf{q}$ on the z axis and in the xy plane. We present the obtained equations as two components in the xy plane, along and normal to the wave vector $\mathbf{q}$. The components will be denoted by the subscripts q and t, as before:

$$-i\omega v_{sq}-2\Omega\frac{p^2}{Q^2}v_{Lt}=0\ ,\qquad -i\omega v_{st}+2\Omega v_{Lq}=0\ , \quad (7.38)$$

$$-(i\omega-\nu Q^2)v_{nq}-2\Omega\frac{p^2}{Q^2}v_{nt}-\frac{\rho_s}{\rho_n}2\Omega\frac{p^2}{Q^2}(v_{slt}-v_{Lt})=0\ ,$$
$$-(i\omega-\nu Q^2)v_{nt}+2\Omega v_{nq}+\frac{\rho_s}{\rho_n}2\Omega(v_{slq}-v_{Lq})=0\ , \quad (7.39)$$

$$\delta v_{sq} = v_{slq} - v_{sq} = -\frac{\nu_s p^2 + c_T^2 q^2/2\Omega}{i\omega} v_{Lt} \,,$$
$$\delta v_{st} = v_{slt} - v_{st} = \frac{\nu_s p^2 - c_T^2 q^2/2\Omega}{i\omega} v_{Lq} \,. \tag{7.40}$$

The z components v_{sz} and v_{nz} of the superfluid and normal velocities are determined from the incompressibility conditions. Vortex displacements have been excluded with the help of the relation $\mathbf{v}_L = -i\omega \mathbf{u}$. Equations (7.38)–(7.40), together with Eq. (7.26) for the vortex velocity, constitute a closed system of equations governing oscillations in the completely incompressible fluid. Elimination of the components of $\mathbf{v}_s$ from Eqs. (7.38) and (7.40) yields two equations:

$$v_{slq} = -\frac{\Pi}{i\omega} v_{Lt}, \quad v_{slt} = \frac{\Gamma}{i\omega} v_{Lq} \,. \tag{7.41}$$

Here

$$\Pi = 2\Omega \frac{p^2}{Q^2} + \nu_s p^2 + \frac{c_T^2 q^2}{2\Omega} \,,$$
$$\Gamma = 2\Omega + \nu_s p^2 - \frac{c_T^2 q^2}{2\Omega} \simeq 2\Omega + \nu_s p^2 \,. \tag{7.42}$$

The solution of Eqs. (7.26), (7.39), and (7.41) yields all oscillation modes and their dispersion laws in the completely incompressible fluid. But the general dispersion law looks rather intricate, and we prefer to study the most interesting particular modes separately.

C. Axial modes

Axial modes have wave vectors directed along the rotation axis (the z axis); thus in all the equations $q = 0$, $Q = p$. These modes have been comprehensively treated in earlier reviews, but are included in this one in order to make it self-contained.

The axial modes are circularly polarized and involve motion only in the xy plane, so it is convenient to use the j-complex representation for the vectors in the xy plane introduced in Sec. IV.D. Then Eqs. (7.26) and (7.38)–(7.40) take a more compact form:

$$-i\omega \tilde{v}_s + 2\Omega j \tilde{v}_L = 0 \,, \tag{7.43}$$

$$-(i\omega - \nu p^2)\tilde{v}_n + 2\Omega j \tilde{v}_n + \frac{\rho_s}{\rho_n} 2\Omega j (\tilde{v}_{sl} - \tilde{v}_L) = 0 \,, \tag{7.44}$$

$$\tilde{v}_{sl} = \frac{2\Omega + \nu_s p^2}{i\omega} j \tilde{v}_L \,, \tag{7.45}$$

$$\tilde{v}_L = \left[1 - \frac{\rho_n}{2\rho} j\tilde{B}\right] \tilde{v}_{sl} + \frac{\rho_n}{2\rho} j\tilde{B}\, \tilde{v}_n \,. \tag{7.46}$$

Here a tilde means that the quantity is j-complex and $\tilde{B} = B - jB'$, but $\tilde{v}_s = v_{sq} + jv_{st}$, $\tilde{v}_n = v_{nq} + jv_{nt}$, and so on.

Solving Eqs. (7.43)–(7.45), one can express all velocities through $\tilde{v}_s$:

$$\tilde{v}_L = \frac{i\omega}{2\Omega j} \tilde{v}_s \,, \tag{7.47}$$

$$\tilde{v}_{sl} = \frac{2\Omega + \nu_s p^2}{2\Omega} \tilde{v}_s \,, \tag{7.48}$$

$$\tilde{v}_n = \tilde{\gamma}\, \tilde{v}_s, \quad \tilde{\gamma} = -\frac{\rho_s}{\rho_n} \frac{2\Omega + \nu_s p^2 + ji\omega}{2\Omega + j(i\omega - \nu p^2)} \,. \tag{7.49}$$

Substitution of these velocities in Eq. (7.46) yields an equation for $\tilde{v}_L$ that has a solution when the following dispersion equation holds:

$$(i\omega - \nu p^2)\left[i\omega - (2\Omega + \nu_s p^2)\left[j + \frac{\rho_n}{2\rho}\tilde{B}\right]\right] - 2\Omega\left[i\omega\left[j + \frac{\rho_s}{2\rho}\tilde{B}\right] + (2\Omega + \nu_s p^2)(1 - \tfrac{1}{2}j\tilde{B})\right] = 0 \,. \tag{7.50}$$

As explained in Sec. IV.D, the dispersion equation is obtained explicitly after substitution $j = \pm i$, each sign corresponding to one of two possible circular polarizations. The following values of p^2 satisfy the dispersion equation (7.50):

$$p^2 = \frac{1}{2\tilde{\beta}_n}\left\{\frac{i\omega\tilde{\beta}_n - 2\Omega j\tilde{\beta}}{\nu} + \frac{i\omega - 2\Omega j\tilde{\beta}_n}{j\nu_s} \pm \left[\left[\frac{i\omega\tilde{\beta}_n - 2\Omega j\tilde{\beta}}{\nu} - \frac{i\omega - 2\Omega j\tilde{\beta}_n}{\nu_s}\right]^2 - \underline{\frac{\rho_n\rho_s}{\rho^2}\frac{i\omega 2\Omega}{\nu\nu_s}\tilde{B}^2}\right]^{1/2}\right\} \,. \tag{7.51}$$

Here

$$\tilde{\beta} = 1 - \tfrac{1}{2}j\tilde{B}, \quad \tilde{\beta}_n = 1 - \frac{1}{2}\frac{\rho_n}{\rho} j\tilde{B} \,. \tag{7.52}$$

Equation (7.51) gives the same values for the wave number as Eq. (4.7) in the review by Andronikashvili *et al.* (1961), but in a different system of notation.

The underlined term in Eq. (7.51) may sometimes be neglected ($\omega \ll \Omega$ or $\omega \gg \Omega$, small ρ_s/ρ or ρ_n/ρ, small $\tilde{B}$). Then the viscous modes and the Kelvin vortex modes are well separated, and the upper sign before the square brackets yields p^2 for viscous modes,

$$p_v^2 = \frac{1}{\nu}\left[i\omega - 2\Omega j \frac{\tilde{\beta}}{\tilde{\beta}_n}\right]$$
$$= \frac{1}{\nu}\left[i\omega \mp 2\Omega i \frac{1 - B'/2 \mp iB/2}{1 - \frac{\rho_n}{2\rho}B' \mp i\frac{\rho_n}{2\rho}B}\right] \tag{7.53}$$

while the lower sign yields p^2 for Kelvin modes,

$$p_K^2=\frac{1}{\nu_s}\left[\frac{i\omega}{j\tilde{\beta}_n}-2\Omega\right]$$
$$=\frac{1}{\nu_s}\left[\frac{\pm\omega}{1-\frac{\rho_n}{2\rho}B'\mp i\frac{\rho_n}{2\rho}B}-2\Omega\right]. \qquad (7.54)$$

In the limit $\Omega\to 0$, Eq. (7.54) gives p^2 for the pure Kelvin mode of an isolated vortex line, modified by mutual friction. We see that undamped oscillations are possible when

$$1-\frac{\rho_n}{2\rho}B'\gg\frac{\rho_n}{2\rho}B .$$

In the low-frequency limit all axial modes are damped (p^2 are complex) and penetrate into the fluid to a finite depth. The penetration depth of viscous modes $1/\mathrm{Im}(p_v)$ differs from the width of the Eckman layer

$$l_E=\sqrt{\nu/2\Omega} \qquad (7.55)$$

by a factor depending on the mutual friction parameters and deduced from Eq. (7.53). The Kelvin modes penetrate to the width of the superfluid Eckman layer,

$$l_s=k_t^{-1}=\sqrt{\nu_s/2\Omega}\ , \qquad (7.56)$$

which has already appeared in Sec. VI.B.

It is worthwhile at this point to discuss the long-wavelength limit $p\to 0$ for the axial modes. Solving the dispersion equation (7.50) with respect to ω and expanding the solution in p^2, we obtain two pairs of modes, each corresponding to two possible circular polarizations $j=\pm i$. The first pair has oscillation frequencies

$$\omega=-ij\left[2\Omega+\frac{\rho_s}{\rho}\nu_s p^2\right]-i\frac{\rho_n}{\rho}\nu p^2 . \qquad (7.57)$$

These modes do not involve relative motion of the superfluid and normal part of the fluid; therefore mutual friction does not affect them, and they are undamped in the limit $p\to 0$.

The second pair of oscillation modes is damped in the limit $p\to 0$, since they involve a counterflow accompanied by mutual friction. The frequencies are given by

$$\omega=-i\left[2\Omega+\frac{\rho_n}{\rho}\nu_s p^2\right]\left[j\left[1-\frac{B'}{2}\right]+\frac{B}{2}\right]-i\frac{\rho_s}{\rho}\nu p^2 . \qquad (7.58)$$

We see that in the long-wavelength limit the axial modes cannot be labeled as viscous or Kelvin because the vortex tension and viscosity enter the expressions for the frequencies of two pairs of modes under equal conditions.

D. In-plane modes

When the oscillation mode has the wave vector in the xy plane ($p=0$, $Q=q$), we shall call it an in-plane mode. The most trivial of these is the viscous mode, involving the motion of the normal fluid along the z axis. Rotation has no effect on such a mode, and its spectrum is the same as in the fluid at rest:

$$\omega=-\nu q^2 . \qquad (7.59)$$

Other in-plane modes involve motion only in the xy plane. In addition, the velocities $\mathbf{v}_s$ and $\mathbf{v}_n$ have no q components because of the incompressibility conditions. The coefficients Π and Γ connecting the components of $\mathbf{v}_{sl}$ and $\mathbf{v}_L$ in Eq. (7.41) and given by Eq. (7.42) are equal to

$$\Pi=\frac{c_T^2q^2}{2\Omega},\quad \Gamma=2\Omega . \qquad (7.60)$$

We can eliminate from the equation of motion for vortices, Eq. (7.26), all velocities excepting $\mathbf{v}_L$ by using Eqs. (7.38)–(7.40). This yields the following equations for the components of the velocity $\mathbf{v}_L$:

$$v_{Lq}\left[1-\frac{\rho_n}{\rho}\frac{B\Omega}{i\omega}-\frac{\rho_s}{\rho}\frac{B\Omega}{i\omega-\nu q^2}\right]+v_{Lt}\left[1-\frac{\rho_n}{2\rho}B'-\frac{\rho_s}{\rho}\frac{B\Omega}{i\omega-\nu q^2}\right]\frac{\Pi}{i\omega}=0\ ,$$
$$-v_{Lq}\left[\left[1-\frac{\rho_n}{2\rho}B'\right]\frac{2\Omega}{i\omega}-\frac{\rho_s}{\rho}\frac{B'\Omega}{i\omega-\nu q^2}\right]+v_{Lt}\left[1-\frac{\rho_n}{2\rho}B\frac{\Pi}{i\omega}+\frac{\rho_s}{\rho}\frac{B'\Omega}{i\omega-\nu q^2}\frac{\Pi}{i\omega}\right]=0 . \qquad (7.61)$$

The dispersion equation for this system of linear equation is as follows:

$$(i\omega-\nu q^2)\left[1-\frac{\rho_n}{2\rho}B\frac{2\Omega+\Pi}{i\omega}-\frac{2\Omega\Pi}{\omega^2}\left[1-\frac{\rho_n}{\rho}B'+\frac{\rho_n^2}{4\rho^2}(B^2+B'^2)\right]\right]-\frac{\rho_s}{\rho}\Omega\left[B\left[1-\frac{2\Omega\Pi}{\omega^2}\right]-\frac{\rho_n}{2\rho}\frac{\Pi}{i\omega}(B^2+B'^2)\right]=0 . \qquad (7.62)$$

Let us neglect for a while the crystalline order in the vortex lattice. Then $\Pi\propto c_T^2=0$, and the dispersion equation is simplified:

$$\nu q^2=i\omega\frac{i\omega-B\Omega}{i\omega-\frac{\rho_n}{\rho}B\Omega} . \qquad (7.63)$$

This is the dispersion law of the viscous mode modified by rotation. The same law is given by Eq. (4.38) of the review by Andronikashvili *et al.* (1961), though in a more complicated form and in different notation.

When the mutual friction is strong ($B\Omega\to\infty$), one obtains from Eq. (7.63)

$$i\omega = \frac{\rho_n \nu}{\rho} q^2 \; . \tag{7.64}$$

This is a viscous mode in which the superfluid and the normal fluids oscillate simultaneously, coupled by mutual friction.

But one may not ignore the crystalline order of the vortex lattice in the low-frequency limit. Retaining Π in the limit $\omega \to 0$, we obtain from the dispersion equation (7.62) the dispersion law for the viscous mode,

$$\nu q^2 = -\frac{\rho_s}{\rho} \frac{B\Omega}{\left[1 - \frac{\rho_n}{2\rho} B'\right]^2 + \left[\frac{\rho_n}{2\rho} B\right]^2} \; , \tag{7.65}$$

which cannot be obtained from Eq. (7.63). The ratio of the vortex velocity to the normal velocity in the in-plane viscous mode (both have only t components) is small when $\omega \to 0$ and given by

$$\beta_v = \frac{v_{Lt}}{v_{nt}} = \frac{i\omega}{2\Omega} \frac{\rho_n}{\rho_s} \frac{\nu q^2}{\Pi} = i\omega \frac{\rho_n}{\rho_s} \frac{\nu}{c_T^2} = \frac{i\omega}{\Omega} \frac{\rho_n}{\rho_s} \frac{8\pi\nu}{\kappa} \; . \tag{7.66}$$

We see that the in-plane viscous mode in the low-frequency limit has a finite penetration depth of order the Eckman-layer width l_E, unlike the similar mode in the classical rotating viscous fluid. The drastic effect of crystalline order on the low-frequency behavior of the in-plane viscous mode can be used for experimental confirmation of the crystalline order (see Sec. VIII.D).

When the ratio $\Pi/\nu q^2 \sim \kappa/\nu$ increases, the frequency at which the crossover from the dispersion law Eq. (7.63) to the low-frequency dispersion law Eq. (7.65) occurs increases too. But one should remember that when the parameter $qr_v \sim \kappa/\nu$ becomes too large [r_v is the intervortex distance given by Eq. (4.29)], the long-wavelength continuum theory, upon which Eq. (7.65) relies, becomes invalid.

The dispersion equation (7.62) at small $q \to 0$ also yields a Tkachenko wave with the dispersion law

$$\omega = c_t q, \quad c_t = \sqrt{\rho_s/\rho}\, c_T \; , \tag{7.67}$$

where $c_T = (\kappa\Omega/8\pi)^{1/2}$ is the former Tkachenko-wave velocity derived for the one-fluid perfect fluid. Expanding Eq. (7.62) in q^2, it is possible to obtain a small imaginary part of the Tkachenko-wave frequency in the two-fluid theory,[6]

$$\mathrm{Im}\omega = -\frac{\rho_n}{2\rho} q^2 \left[\nu + \frac{\kappa}{8\pi} \frac{(1-B'/2)^2 + B^2/4}{B}\right] . \tag{7.68}$$

The first term in the large parentheses is due to viscous losses, and the second to mutual friction. The mutual friction losses are small because in the low-frequency limit mutual friction strongly couples the superfluid and the normal fluid, and there is no significant counterflow. The same explains the temperature-dependent factor $\sqrt{\rho_s/\rho}$ in the Tkachenko-wave velocity. The fluid as a whole with the total density ρ is involved in the low-frequency Tkachenko oscillations, but the restoring elastic force is decreased by the factor ρ_s/ρ compared with that in the perfect fluid (Tkachenko, 1973).

[6]Equations (7.66) and (7.68) were obtained earlier (Sonin, 1976) for the particular values of B and B' given by Eq. (10.6).

E. Mixed modes in a completely incompressible superfluid

Now we shall consider soft oscillation modes with wave vectors $\mathbf{Q}$ directed at small angles to the xy plane ($p \ll q$). As in the perfect fluid, they will be called mixed modes. It is convenient to study them using equations for the center-of-mass velocity $\mathbf{v}$ and the relative velocity $\mathbf{w} = \mathbf{v}_n - \mathbf{v}_s$. These equations are readily derived from Eqs. (7.38) and (7.39):

$$-i\omega v_q - 2\Omega \frac{p^2}{Q^2}\left[v_t + \frac{\rho_s}{\rho}\delta v_{st}\right] + \frac{\rho_n}{\rho}\nu Q^2 \left[v_q + \frac{\rho_s}{\rho} w_q\right] = 0 \; ,$$

$$-i\omega v_t + 2\Omega\left[v_q + \frac{\rho_s}{\rho}\delta v_{sq}\right] + \frac{\rho_n}{\rho}\nu Q^2\left[v_t + \frac{\rho_s}{\rho} w_t\right] = 0 \; , \tag{7.69}$$

$$\left[-i\omega + \frac{p^2}{Q^2} B\Omega\right] w_q - (2-B')\Omega \frac{p^2}{Q^2} w_t - B\Omega \frac{p^2}{Q^2}\delta v_{sq} + (2-B')\Omega \frac{p^2}{Q^2}\delta v_{st} + \nu Q^2\left[v_q + \frac{\rho_s}{\rho} w_q\right] = 0 \; ,$$

$$(-i\omega + B\Omega) w_t + (2-B')\Omega w_q - B\Omega\delta v_{st} - (2-B')\Omega\delta v_{sq} + \nu Q^2\left[v_t + \frac{\rho_s}{\rho} w_t\right] = 0 \; . \tag{7.70}$$

Substitution of $\mathbf{v}_s = \mathbf{v} - (\rho_n/\rho)\mathbf{w}$ and $\mathbf{v}_n = \mathbf{v} + (\rho_s/\rho)\mathbf{w}$ into Eq. (7.26) yields

$$\mathbf{v}_L = \mathbf{v} + \left[1 - \frac{\rho_n}{2\rho} B'\right](\delta\mathbf{v}_s - \mathbf{w}) - \frac{\rho_n}{2\rho} B\hat{\mathbf{z}} \times (\delta\mathbf{v}_s - \mathbf{w}) \; . \tag{7.71}$$

In Eq. (7.40) for the components of $\delta\mathbf{v}_s = \mathbf{v}_{sl} - \mathbf{v}_s$ one can neglect the vortex tension $\propto \nu_s$; then

$$\delta v_{sq} = -\frac{c_T^2 q^2}{i\omega} v_{Lt}, \quad \delta v_{st} = -\frac{c_T^2 q^2}{i\omega} v_{Lq} \; . \tag{7.72}$$

In the long-wavelength limit the equations for $\mathbf{v}$ and for $\mathbf{w}$ are uncoupled, since $\delta\mathbf{v}_s \to 0$ and $\nu Q^2 \to 0$ in this limit. This means that there is an oscillation mode corresponding to motion of the the fluid as a whole and another mode involving a counterflow without motion of the center of fluid mass. By analogy with the terms "first sound" and "second sound," we shall use the terms "first mixed mode" and "second mixed mode," referring to modes connected with fluid motion as a whole and relative motion of two parts of the fluid.

First we consider the first mixed mode. The transverse velocities of vortices and the fluid, $v_{Lt} \approx v_t$, are larger than the longitudinal velocities, since $v_{Lq} \sim v_q \propto \omega/\Omega$. Conversely, according to Eq. (7.72), the longitudinal component δv_{sq} is larger than the transverse δv_{st}. Thus, neglecting viscosity and coupling with relative motion, we may write Eq. (7.69) in a much simpler form:

$$\begin{aligned} &-i\omega v_q - 2\Omega \frac{p^2}{Q^2} v_t = 0 , \\ &-i\omega v_t + 2\Omega \left[v_q + \frac{\rho_s}{\rho} \delta v_{sq} \right] = 0 , \\ &\delta v_{sq} = -\frac{c_T^2 q^2}{i\omega} v_t . \end{aligned} \tag{7.73}$$

This system of the equations yields the dispersion law

$$\omega^2 = 4\Omega^2 \frac{p^2}{Q^2} + c_t^2 q^2 = 4\Omega^2 \frac{p^2}{Q^2} + \frac{\rho_s}{\rho} c_T^2 q^2 , \tag{7.74}$$

which differs from the dispersion law of the mixed mode in the perfect fluid by the temperature-dependent factor ρ_s/ρ in the Tkachenko contribution.[7] The physical origin of this factor has already been explained in Sec. VII.D.

The small imaginary correction to the frequency Eq. (7.74) is found with the perturbation theory that implies calculation of the small relative velocity $\mathbf{w}$ from Eq. (7.70). As a result we have

$$\mathrm{Im}\omega = -\frac{\rho_n}{2\rho} \left\{ \nu Q^2 \left[1 + \left[\frac{2\Omega}{\omega} \frac{p}{Q} \right]^2 \right] + \frac{\rho_s}{\rho} \frac{(c_T q)^4}{4\Omega\omega^2} \frac{B^2 + (2-B')^2}{B} \right\} . \tag{7.75}$$

When $p=0$, Eqs. (7.74) and (7.75) coincide with Eqs. (7.67) and (7.68) for the pure Tkachenko wave.

The second mixed mode will be considered in the simplest approximation when $\nu q^2 \sim 0$, $\delta \mathbf{v}_s \sim 0$. Then the equations of relative motion given by Eq. (7.70) are

$$\begin{aligned} &\left[-i\omega + \frac{p^2}{Q^2} B\Omega \right] w_q - (2-B')\Omega \frac{p^2}{Q^2} w_t = 0 , \\ &(-i\omega + B\Omega) w_t + (2-B')\Omega w_q = 0 , \end{aligned} \tag{7.76}$$

and the dispersion equation is [cf. Eq. (71) of Chandler and Baym, 1986]

$$(\omega + iB\Omega) \left[\omega + iB\Omega \frac{p^2}{Q^2} \right] = 4\Omega^2 \frac{p^2}{Q^2} \left[1 - \frac{B'}{2} \right]^2 . \tag{7.77}$$

If mutual friction vanishes, $B = B' = 0$, Eq. (7.77) yields the dispersion law of the classical inertial wave, the second inertial wave since it involves only relative motion (a counterflow). But in the low-frequency limit one may not ignore mutual friction, and the second inertial wave is damped:

$$\omega = -i\Omega \frac{p^2}{Q^2} \frac{B^2 + (2-B')^2}{B} . \tag{7.78}$$

F. Oscillations in a clamped regime

In a clamped regime the normal part of the fluid moves together with the solid surfaces bounding the superfluid as a solid body. This is possible, for example, when the viscous penetration depth is much larger than the width of the fluid layer. The fluid layer, nevertheless, may be larger than other relevant hydrodynamic scales. Such a regime of motion is especially easily realized in superfluid ^{3}He because of its high viscosity. The clamped regime has been assumed to exist in neutron stars, where the charged normal part of the fluid is clamped to the solid outer crust of the star by a large magnetic field (Baym *et al.*, 1969).

In the clamped regime one may delete the Navier-Stokes equation from the system of hydrodynamic equations and assume $\mathbf{v}_n = 0$ in the rest equations. More exactly, the normal velocity $\mathbf{v}_n$ is equal to the velocity of the solid surfaces, but it does not matter for the linear theory of free oscillations considered in the present section. Then the number of degrees of freedom reduces to that for the perfect fluid. The difference from the perfect fluid is that superfluid motion involves a smaller mass density and is affected by mutual friction. We shall discuss the oscillation modes in the clamped regime without dwelling on derivations, which are similar to those performed above.

In the clamped regime the first sound transforms into the fourth sound with the velocity (Putterman, 1974, p. 206)

$$c_4 = \sqrt{\rho_s/\rho}\, c_1 . \tag{7.79}$$

The dispersion equation of the Kelvin vortex waves does not change in the clamped regime from that given by Eq. (7.54) because the normal viscosity does not contribute to it.

The incompressible fluid in the clamped regime sustains the mixed mode with the dispersion equation

$$\omega^2 + i\omega \frac{\rho_n}{\rho} B\Omega \left[1 + 2\frac{p^2}{Q^2} \right] - \left[4\Omega^2 \frac{p^2}{Q^2} + c_T^2 q^2 \right] \times \left[\left[\frac{\rho_n}{2\rho} B \right]^2 + \left[1 - \frac{\rho_n}{2\rho} B' \right]^2 \right] = 0 . \tag{7.80}$$

[7]The factor ρ_s/ρ in Eq. (7.74) does not rely on assumptions made for derivation of B and B', as Andereck and Glaberson (1982) supposed. In fact, Eq. (7.74) holds at any temperature if the frequency is low enough, whatever B and B' are. The magnitude of B, however, determines the range of validity of Eq. (7.74).

By analogy with the fourth sound, we shall call it the fourth mixed mode. Like the second mixed mode it is damped at $\omega \to 0$. Without the effects of the vortex-lattice rigidity ($c_T=0$), Eq. (7.80) yields the dispersion law for the fourth inertial wave. At $p=0$ we have the Tkachenko mode with the frequency

$$\omega=-i\frac{\rho_n}{2\rho}B\Omega \pm \left\{ c_T^2 q^2 \left[\left[\frac{\rho_n}{2\rho}B\right]^2 + \left[1-\frac{\rho_n}{2\rho}B'\right]^2 \right] - \Omega^2 \left[\frac{\rho_n}{2\rho}B\right]^2 \right\}^{1/2} . \tag{7.81}$$

This formula agrees with that obtained by Stauffer (1967) and Volovik and Dotsenko (1980).

G. Oscillations in a thermally compressible fluid

In Sec. VII.E we saw that in the completely incompressible fluid the equations of center-of-mass fluid motion and of relative motion of the superfluid and the normal fluid are uncoupled in the long-wavelength limit. When the thermal expansion coefficient is small, as it is assumed to be in He II (Khalatnikov, 1971), the mass flow and the counterflow do not interact in a compressible fluid either. Thus the mechanical and thermal degrees of freedom are uncoupled. The effect of mechanical compressibility on the oscillation modes of the fluid moving as a whole (the first mixed mode, the Tkachenko mode) is similar to that considered in Sec. IV.G for the perfect fluid, and results obtained there remain valid in the two-fluid theory, though with ρ and c_T replaced by ρ_s and the temperature-dependent Tkachenko velocity given by Eq. (7.67) and denoted c_t. For this reason we treat here only the effect of thermal compressibility.

We need the continuity equation for the entropy and the equation for the relative velocity $\mathbf{w}=\mathbf{v}_n-\mathbf{v}_s$:

$$\frac{\partial S'}{\partial t}+\frac{\rho_s}{\rho}S\nabla\cdot\mathbf{w}=-\frac{\lambda}{T}\Delta T' , \tag{7.82}$$

$$\frac{\partial \mathbf{w}}{\partial t}+\frac{S}{\rho_n}\nabla T'+B\Omega\mathbf{w}+(2-B')\boldsymbol{\Omega}\times\mathbf{w}=0 . \tag{7.83}$$

Here we neglect the effects of vortex-lattice rigidity, but include the thermal conductivity $\propto\lambda$. Eliminating w_z and the temperature variation T', as was done with respect to v_z and ρ' in the perfect fluid (Secs. II and IV.G), we obtain the following equations for the in-plane components of the relative velocity:

$$\begin{aligned} &-i\omega\frac{\omega^2-\tilde{c}_2^2Q^2}{\omega^2-\tilde{c}_2^2p^2}w_q+B\Omega w_q-(2-B')\Omega w_t=0 , \\ &-i\omega w_t+B\Omega w_t+(2-B')\Omega w_q=0 . \end{aligned} \tag{7.84}$$

Here

$$\tilde{c}_2=c_2\left[1-\frac{\lambda Q^2}{i\omega\rho C}\right]^{1/2} = \left[\frac{\rho_s}{\rho_n}\frac{\partial T}{\partial(S/\rho)}\left[1-\frac{\lambda Q^2}{i\omega\rho C}\right]\right]^{1/2} \tag{7.85}$$

is an effective second-sound velocity, allowing for the thermal conductivity, and C is the specific heat per unit mass.

These equations yield the dispersion equation for a thermally compressible superfluid,

$$(\omega^2-\tilde{c}_2^2Q^2)(\omega^2+i\omega B\Omega) - (\omega^2-\tilde{c}_2^2p^2)\{\Omega^2[B^2+(2-B')^2]-i\omega B\Omega\}=0 . \tag{7.86}$$

In the limit $\tilde{c}_2\to\infty$ we obtain from Eq. (7.86) the dispersion law [Eq. (7.77)] for the second inertial wave in a completely incompressible superfluid. In a superfluid at rest, Eq. (7.86) gives the second-sound spectrum with dissipation due to thermal conductivity. In the opposite limit of fast rotation, $\omega \ll \Omega$, we expand Eq. (7.86) in ω up to the second-order terms, to obtain

$$\omega=-\frac{i}{2}\left[\frac{\lambda Q^2}{\rho C}+\frac{c_2^2Q^2}{\Omega}\frac{B}{B^2+(2-B')^2}\right] \pm\left[c_2^2p^2-\frac{1}{4}\left[\frac{\lambda Q^2}{\rho C}+\frac{c_2^2Q^2}{\Omega}\frac{B}{B^2+(2-B')^2}\right]^2\right]^{1/2} \tag{7.87}$$

Andreev and Kagan (1984) have derived a similar formula, assuming, however, infinitely strong mutual friction ($B,B'\to\infty$) when its contribution to the frequency drops out. But at the angular velocities available for today's experiments the mutual friction contribution exceeds that of thermal conductivity by many orders. Thermal conductivity becomes important extremely close to the λ point, where c_2 is very small, or probably at very low temperatures.

In the limit $\omega \gg \Omega$ one can derive from Eq. (7.86) the dispersion equation usually used in analyzing measurements of B and B' with the second-sound technique [see Eqs. (67) and (67a) of Chandler and Baym (1986)].

VIII. BOUNDARY PROBLEMS IN TWO-FLUID HYDRODYNAMICS

A. Boundary conditions on a horizontal solid surface

At finite temperatures the number of oscillation modes of the fluid increases, so the number of boundary conditions should increase also. On the horizontal solid surface bounding the perfect fluid along the rotation axis (the z axis), we had the Bekarevich-Khalatnikov condition [Eq. (6.2)] imposed on the vortex velocity $\mathbf{v}_L$ and the condition

[Eq. (6.1)] that the mass flow normal to the solid surface vanish. In the two-fluid theory, Eq. (6.1) is imposed on the center-of-mass velocity,

$$v_z = \frac{\rho_s}{\rho} v_{sz} + \frac{\rho_n}{\rho} v_{nz} = 0 , \tag{8.1}$$

and new conditions should be added: the "stick condition" for the normal velocity $\mathbf{v}_n$ and the thermal boundary condition connecting the normal heat flow and the variation of temperature on the surface.

The stick condition is that the component of $\mathbf{v}_n$ in the horizontal plane coincide with the velocity $\mathbf{v}_B$ of the solid surface:

$$\mathbf{v}_B = \mathbf{v}_n - \hat{\mathbf{z}}(\hat{\mathbf{z}} \cdot \mathbf{v}_n) . \tag{8.2}$$

Sometimes one uses a more general condition involving the effect of a slip of the normal fluid relative to the solid surface (Jensen *et al.*, 1980). The slip effect can be essential when the relevant hydrodynamic scale (the fluid-layer dimension, or the wavelength) is of the same order as the mean free path of quasiparticles. Thus the slip is beyond the scope of the conventional hydrodynamic approach as a rule. But sometimes the slip effect is not weak, even at quite a large ratio of the relevant hydrodynamic scale to the mean free path. By assuming an enlarged slip effect, one can explain the rather low effective viscosity of ^{3}He-B measured at low temperatures. According to Einzel *et al.* (1984), Andreev reflection of quasiparticles is responsible for the increase in the normal-fluid slip. But for the problems treated in the present paper, the slip effect is not very important and can be taken into account without any difficulty if necessary.

As a thermal condition one may assume that there is no heat flux across the solid surface. This means that

$$w_z = v_{nz} - v_{sz} = 0 . \tag{8.3}$$

Together with Eq. (8.1) this tells us that neither velocity, $\mathbf{v}_s$ or $\mathbf{v}_n$, has a z component normal to the surface.

As an alternative to the adiabatic condition, as we shall call Eq. (8.3), the isothermal condition may be assumed: the temperature of the fluid near the solid surface is kept constant. But later on we shall use only the adiabatic thermal condition as more realistic.

B. Pile-of-disks oscillations and the effective boundary condition

Now let us reconsider the problem of fluid motion between two oscillating horizontal solid surfaces in the frame of two-fluid hydrodynamics (see the one-fluid theory in Sec. VI.B). The surfaces are separated by a distance $2L$. Their motion is described by the velocity field $\mathbf{v}_B \exp(i\mathbf{q}\mathbf{r} - i\omega t)$, where $\mathbf{v}_B$ is normal to the wave vector $\mathbf{q}$ in the xy plane. As in Sec. VI.B, we shall take the limit $\mathbf{q} \rightarrow 0$ afterwards, so $\mathbf{q}$ is small and the Tkachenko rigidity can be ignored.

The oscillating surfaces of the disks generate in the fluid a velocity field, which is a superposition of all six oscillation modes possible in two-fluid hydrodynamics: two Kelvin modes, two axial viscous modes, and two inertial waves. Each of these corresponds to a solution of the dispersion equation for p^2 at given q and ω. In the j-complex representation (see Secs. IV.D and VI.B) one can write the field of the superfluid velocity components in the xy plane as

$$\widetilde{v}_s = \left[\widetilde{v}_K(j) \cos p_K(j) z + \widetilde{v}_v(j) \cos p_v(j) z + \left[\frac{i\omega}{2\Omega} + j \right] v_{\mathrm{I}} \cos p_{\mathrm{I}} z - \frac{\rho_n}{\rho} (i\alpha + j) w_{\mathrm{II}} \cos p_{\mathrm{II}} z \right] \times \exp(i\mathbf{q}\cdot\mathbf{r} - i\omega t) . \tag{8.4}$$

Explicit expressions for the components v_{sq} and v_{st} are obtained by separation of the real and imaginary parts with respect to j, the imaginary unit i being treated as "real." Equally, one can write the j-complex expressions for the in-plane components of $\mathbf{v}_n$ and $\mathbf{v}_L$. The axial components v_{sz} and v_{nz} are determined from the incompressibility conditions $\nabla \cdot \mathbf{v}_s = 0$ and $\nabla \cdot \mathbf{v}_n = 0$ as in Sec. VI.B. The j-complex amplitudes $\widetilde{v}_K(j)$ and $\widetilde{v}_v(j)$ and the wave numbers $p_K(j)$ and $p_v(j)$ refer to the Kelvin and axial viscous modes, $p_K(j)$ and $p_v(j)$, given by the lower and upper signs in Eq. (7.51), or the approximate equations (7.53) and (7.54). The amplitudes v_{I} and w_{II} are real in the j sense; they define the transverse t component of the velocities in the first inertial wave ($v_{\mathrm{I}} = v_t = v_{st} = v_{nt} = v_{Lt}$) and the transverse component of the relative velocity in the second inertial wave ($w_{\mathrm{II}} = w_t = v_{nt} - v_{st}$). The relations between velocity components for various oscillations modes were found earlier when they were studied in an infinite fluid. For example, the ratio $v_{sq}/v_{st} = i\omega/2\Omega$ in the first-inertial-wave term $\propto v_{\mathrm{I}}$ in Eq. (8.4) is equal to v_q/v_t in the inertial wave in the perfect fluid [see Eq. (4.57) in the classical limit $c_T \rightarrow 0$]. The ratio

$$i\alpha = \frac{w_q}{w_t} = \frac{i\omega - B\Omega}{\Omega(2 - B')} \tag{8.5}$$

in the second-inertial-wave term $\propto w_{\mathrm{II}}$ follows from Eq. (7.76). In both the first and the second inertial wave the axial components of the wave vector, p_{I} and p_{II}, are proportional to q at given ω, as follows from Eqs. (6.9) and (7.78).

The next step is to substitute the velocity fields into the boundary conditions at $z = \pm L$. The conditions Eqs. (8.1) and (8.3) imposed on the velocities normal to the solid surface yield (see a more detailed derivation in Sec. VI.B)

$$\mathrm{Re}_j \left[\frac{\widetilde{v}_K(j)}{p_K(j)} \left[\frac{\rho_s}{\rho} + \frac{\rho_n}{\rho} \widetilde{\gamma}_K(j) \right] \sin p_K(j) L + \frac{\widetilde{v}_v(j)}{p_v(j)} \left[\frac{\rho_s}{\rho} + \frac{\rho_n}{\rho} \widetilde{\gamma}_v(j) \right] \sin p_v(j) L \right] + \frac{i\omega}{2\Omega} L v_{\mathrm{I}} = 0 , \tag{8.6}$$

$$\mathrm{Re}_j\left[\frac{\widetilde{v}_K(j)}{p_K(j)}[\widetilde{\gamma}_K(j)-1]\sin p_K(j)L+\frac{\widetilde{v}_v(j)}{p_v(j)}[\widetilde{\gamma}_v(j)-1]\sin p_v(j)L\right]+i\alpha L w_{\mathrm{II}}=0\ . \tag{8.7}$$

The boundary conditions are written in the limit $q\rightarrow 0$, in which p_{I} and p_{II} vanish also. The ratio $\widetilde{\gamma}(j)$ between the normal and the superfluid velocity is given by Eq. (7.49) with the subscripts K and v referring to the Kelvin and the viscous modes, respectively.

The stick condition [Eq. (8.2)] after substitution of the normal velocity field is

$$\widetilde{\gamma}_K(j)\widetilde{v}_K(j)\cos p_K(j)L+\widetilde{\gamma}_v(j)\widetilde{v}_v(j)\cos p_v(j)L+\left[\frac{i\omega}{2\Omega}+j\right]v_{\mathrm{I}}+\frac{\rho_s}{\rho}(i\alpha+j)w_{\mathrm{II}}=jv_B\ . \tag{8.8}$$

We limit ourselves to the perfect pinning case, which corresponds to the Bekarevich-Khalatnikov condition [Eq. (6.4)] $\zeta=\zeta'=0$ and consequently $\mathbf{v}_L=\mathbf{v}_B$. Substitution of $\mathbf{v}_L$ in this condition yields in the j-complex representation

$$\frac{i\omega}{2\Omega j}[\widetilde{v}_K(j)\cos p_K(j)L+\widetilde{v}_v(j)\cos p_v(j)L]+\left[\frac{i\omega}{2\Omega}+j\right]v_{\mathrm{I}}-\frac{\rho_n}{\rho}\left[\frac{i\omega}{2\Omega}+tj\right]w_{\mathrm{II}}=jv_B\ . \tag{8.9}$$

Here

$$t=1-\frac{B'}{2}-\frac{B}{2}\frac{i\omega-B/2}{(1-B'/2)2\Omega} \tag{8.10}$$

is the ratio v_{Lt}/v_{st} in the second inertial wave.

Equations (8.6)–(8.9) give a solution of our boundary problem in which the zeros of the determinant correspond to eigenfrequencies. But the general analysis of such a system of equations is quite complicated, and we restrict ourselves to discussion of simpler particular cases. When the frequency is large ($\omega\gg\Omega$), then the eigenfrequencies are given by the resonance condition for Kelvin waves $p_K(+)L=\pi(2n-1)/2$, where $p_K(+)$ is p_K from Eq. (7.54) at $j=i$. This condition yields Hall's resonance frequencies modified by mutual friction [cf. Eq. (6.18)],

$$\omega_n=\left[\nu_s\left[\frac{\pi}{2}\frac{2n-1}{L}\right]^2+2\Omega\right]\left[1-\frac{\rho_n}{2\rho}B'-i\frac{\rho_n}{2\rho}B\right]\ . \tag{8.11}$$

The most interesting case is the low-frequency limit $\omega\ll\Omega$ when an inertial-wave resonance is expected. In this limit $\widetilde{\gamma}_K\rightarrow 0$ (Kelvin modes do not involve motion of the normal fluid), but $\widetilde{\gamma}_v$ remains finite, as do α and t. Then a direct estimate shows that a good approximation at $\omega\ll\Omega$ is to delete the viscous modes and the second inertial wave, together with the stick condition and the thermal boundary condition, Eqs. (8.7) and (8.8). The remaining system of equations contains the same modes and boundary conditions as were in the theory for a perfect fluid (Sec. VI.B). The Kelvin wave number $p_K(j)$ takes as its asymptotic low-frequency value ik_t given by Eq. (6.19). Repeating the derivation of the dispersion equation, we obtain the frequency of the inertial-wave resonance in two-fluid theory:

$$\omega=\left[\frac{\rho_s}{\rho}\frac{1}{k_tL}\right]^{1/2}2\Omega=\left[\frac{\rho_s}{\rho}\right]^{1/2}\left[\frac{\nu_s}{L^2}\right]^{1/4}(2\Omega)^{3/4}\ . \tag{8.12}$$

This differs from the same frequency in a perfect fluid [Eq. (6.21)] by the factor $\sqrt{\rho_s/\rho}$, because the Kelvin modes involve only superfluid motion, and their contribution to the center-of-mass flow at the boundary [see Eq. (8.6)] is diminished by a factor ρ_s/ρ. In contrast, the first inertial wave in the bulk is associated with motion of the fluid as a whole.

In the same way we can reconsider the derivation of the effective boundary condition for a mixed wave in the bulk, which we carried out in Sec. VI.C for a perfect fluid. Again the viscous modes and the second inertial wave are not important, nor are the associated boundary conditions, when the frequency is low. Two-fluid effects result in the factor ρ/ρ_s in the expression for the parameter A in the effective boundary condition, Eq. (6.30) [cf. Eq. (6.28)]:[8]

$$A=\frac{\rho}{\rho_s}k_t\frac{(b+k_t)^2+b'^2}{b(b+k_t)+b'^2}\ . \tag{8.13}$$

C. Pile-of-disks oscillations in a clamped regime

Formally the theory of the clamped regime is the limit of infinite viscosity $\nu\rightarrow\infty$ of the theory developed in the preceding section. But to take such a limit is not a simple procedure, and it is more convenient to derive the theory of pile-of-disks oscillations in the clamped regime anew, referring directly to those modes that are possible in this case: two Kelvin modes and the fourth inertial wave (see Sec. VII.F). Oscillations in the bulk, however, are not free, because a normal fluid clamped by the oscillating disks exerts a driving force on the superfluid by means of mutual friction. Thus the hydrodynamic equations in the bulk are nonhomogeneous and contain the velocity $\mathbf{v}_B$ of the disks. As usual, the general solution of such equa-

[8]Two-fluid corrections to A of the order ω/Ω were also calculated (Sonin, 1976), but not all of them were taken into account, so their total magnitude differs from that presented by the term $\propto\omega/\Omega$ in Eq. (27) of Sonin (1976).

tions consists of some solution of the nonhomogeneous equations and a superposition of solutions for the corresponding homogeneous ones. The latter are the free oscillation modes listed above.

Let us first solve the nonhomogeneous equations, which represent the drag upon the superfluid exerted by mutual friction with the clamped normal fluid. The necessary hydrodynamic equations are Eqs. (7.26) and (7.38), where it is assumed that $\mathbf{v}_n=\mathbf{v}_B$ and $\mathbf{v}_{sl}=\mathbf{v}_s$. The latter condition means that vortex deformation effects on the velocity $\mathbf{v}_L$ are ignored, since the driving velocity field (the velocity $\mathbf{v}_n$ of the clamped normal fluid) varies slowly in space. We are thus considering a case within the continuous-vorticity model, and $p\to 0$ and $q\to 0$, but at the same time $p/q\to 0$ also. As was discussed in Sec. VI.B, in the pile-of-disks geometry the direction of $\mathbf{q}$ is identical to the radial direction. If p/q were not negligible, v_{sq}, i.e., the radial component of the superfluid velocity $\mathbf{v}_s$, would be finite, thus contradicting the incompressibility condition. As a result, Eqs. (7.26) and (7.38) take a quite simple form:

$$-i\omega v_{st}+2\Omega v_{Lq}=0,\quad v_{sq}=0\ , \tag{8.14}$$

$$v_{Lq}=-\frac{\rho_n}{2\rho}B(v_B-v_{st})\ , \tag{8.15}$$

$$v_{Lt}=v_{st}+\frac{\rho_n}{2\rho}B'(v_B-v_{st})\ . \tag{8.16}$$

The solution of these elementary equations (denoted by the superscript D) is

$$v_{st}^D=gv_B\ , \tag{8.17}$$

$$v_{Lq}^D=\frac{i\omega}{2\Omega}gv_B \tag{8.18}$$

$$v_{Lt}^D=\left[1+\frac{i\omega}{2\Omega}g'\right]v_B\ . \tag{8.19}$$

Here

$$g=\frac{(\rho_n/\rho)B\Omega}{-i\omega+(\rho_n/\rho)B\Omega}\ ,\qquad g'=\frac{[2-(\rho_n/\rho)B']\Omega}{-i\omega+(\rho_n/\rho)B\Omega}\ . \tag{8.20}$$

When the vortices do not interact with the disk surfaces (no pinning) Eqs. (8.17)–(8.19) completely describe superfluid motion between oscillating disks (Sonin, 1981a).[9]

In the general solution for velocities, the contributions of free oscillation modes (solutions of the homogeneous equations) should be included. The superfluid velocity field in the j-complex representation is given by

$$\tilde v_s(z,\mathbf{r},t)=[\tilde v_K(j)\cos p_K(j)z+v_4(i\alpha_n+j)+jgv_B]\times\exp(i\mathbf{q}\cdot\mathbf{r}-i\omega t)\ . \tag{8.21}$$

The j-complex expression for the vortex velocity is

$$\tilde v_L(z,\mathbf{r},t)=\left[\frac{i\omega}{2\Omega j}\tilde v_K(j)\cos p_K(j)z+v_4\left[\frac{i\omega}{2\Omega}+t_nj\right]+\left[j+\frac{i\omega}{2\Omega}\tilde g\right]v_B\right]\exp(i\mathbf{q}\cdot\mathbf{r}-i\omega t)\ . \tag{8.22}$$

Here $\tilde g=g+jg'$ and

$$i\alpha_n=\frac{i\omega-(\rho_n/\rho)B\Omega}{2\Omega\left[1-\dfrac{\rho_n}{2\rho}B'\right]}\ ,\qquad t_n=1-\frac{\rho_n}{2\rho}B'-\frac{\rho_n}{2\rho}Bi\alpha_n \tag{8.23}$$

are ratios of the velocity components v_{sq}/v_{st} and v_{Lt}/v_{st} in the fourth inertial wave (see Sec. VII.F). The j-complex amplitude $\tilde v_K(j)$ and the real v_4 define the amplitudes of two Kelvin modes and the fourth inertial wave, respectively.

In the case of perfect pinning, when $\mathbf{v}_L=\mathbf{v}_B$ on the solid surface, substitution of Eq. (8.22) yields

$$\frac{i\omega}{2\Omega j}\tilde v_K(j)\cos p_K(j)L+v_4\left[\frac{i\omega}{2\Omega}+t_nj\right]+v_B\left[j+\frac{i\omega}{2\Omega}\tilde g\right]=jv_B\ . \tag{8.24}$$

The condition that the mass flow across the solid surface vanish is

$$\mathrm{Re}_j\left[\frac{\tilde v_K(j)\sin p_K(j)L}{p_K(j)L}\right]+i\alpha_nLv_4=0\ . \tag{8.25}$$

Solving Eqs. (8.24) and (8.25) for $\tilde v_K(j)$ and v_4, we can find the ratio of the effective superfluid density ρ_s', dragged by the oscillating disks, to the total superfluid density ρ_s:

$$\begin{aligned}\frac{\rho_s'}{\rho_s}&=\frac{1}{2v_BL}\int_{-L}^{L}dz\,v_{st}(z)\\&=g+\frac{v_4}{v_B}+\frac{1}{v_B}\mathrm{Im}_j[\tilde v_K(j)Z_K(j)]\\&=g+\frac{\mathrm{Re}_j[j\tilde gZ_K(j)]-i\alpha_n\mathrm{Im}_j[i\tilde gZ_K(j)]-Z_K(j)Z_K(-j)\mathrm{Re}_j\left[\left[j+t_n\dfrac{2\Omega}{i\omega}\right]\tilde g\right]}{i\alpha_n-\mathrm{Re}_j\left[\left[j-t_n\dfrac{2\Omega}{i\omega}\right]Z_K(j)\right]}\ .\end{aligned} \tag{8.26}$$

[9]The expression for g given in this paper is incorrect.

Here

$$Z_K(j)=[\tan p_K(j)L]/p_K(j)L \ .$$

When mutual friction vanishes and $\rho_s=\rho$, Eq. (8.26) coincides with Eq. (6.15) for ρ'/ρ in the perfect fluid. Let us consider the high-frequency region $\omega \gg \Omega$ where Hall's resonances are expected. There one can expand Eq. (8.26) in Ω:

$$\frac{\rho_s'}{\rho_s} \approx g - \mathrm{Im}_j[j\tilde{g}Z_K(j)]$$
$$\approx g+\frac{\Omega}{\omega}\left[1-\frac{\rho_n}{2\rho}B'-i\frac{\rho_n}{2\rho}B\right]\frac{\tan p_K(+)L}{p_K(+)L} \ . \qquad (8.27)$$

The first term g represents the effect of mutual friction and the second is due to vortex pinning [compare with Eq. (6.17) obtained for the perfect fluid]. Here $p_K(+)$ is the value of $p_K(j)$ at $j=i$. Hall's resonances correspond to poles of the function of tangent in Eq. (8.27). They are observable when

$$1-\frac{\rho_n}{2\rho}B' \gg \frac{\rho_n}{2\rho}B \ .$$

D. Boundary conditions on the vertical solid surface. Axisymmetric oscillations of a cylindric vessel

In Secs. V.C and VI.D we have used the effective boundary condition on the lateral wall, implying that there is a force sticking vortices to the wall, which is parallel to them [Eqs. (5.19) and (6.44)]. Such a force is provided by mutual friction between the vortices and the normal fluid and sticking of the normal fluid to the wall. Now we are going to derive this boundary condition.

Suppose that low-frequency axisymmetric oscillations are excited in a superfluid contained by a cylindric vessel of radius R. The assumption that vortices do not interact directly with the wall means that the component $\sigma_{\varphi r}$ of the stress tensor of the vortex lattice, given by Eq. (5.18) in the cylindric coordinate frame, should vanish on the wall. Then, recalling that $\partial u_r/\partial\varphi=0$ for axisymmetric motion, we have

$$\sigma_{\varphi r}=-\rho c_T^2\left[\frac{\partial u_\varphi(R)}{\partial r}-\frac{u_\varphi(R)}{R}\right]=0 \ . \qquad (8.28)$$

Another boundary condition is the usual stick condition for the normal velocity $\mathbf{v}_n$ at a wall moving with the velocity $\mathbf{v}_B$:

$$v_{n\varphi}(R)=v_B \ . \qquad (8.29)$$

The low-frequency oscillation modes in the superfluid are exhausted by mixed modes which involve motion of both parts of the fluid together with vortices with the same velocity. Obviously, mixed modes cannot satisfy both boundary conditions simultaneously, so we resort to the concept of the boundary layer again, this time supposing that there is a layer in which the moving wall generates the viscous mode and that the width of the boundary layer is the depth to which this damping mode penetrates.

For axisymmetric oscillations the viscous mode in the boundary layer can be approximated by a plane wave with the wave vector $\mathbf{q}_v$ normal to the wall. Then azimuthal velocity components in the boundary layer are

$$\begin{aligned} v_{n\varphi}&=v_\varphi(r)+v_v\exp[iq_v(r-R)] \ , \\ v_{L\varphi}&=v_\varphi(r)+\beta_v v_v\exp[iq_v(r-R)] \ . \end{aligned} \qquad (8.30)$$

Here v_φ is the azimuthal component of the velocity in the mixed wave that propagates in the bulk and β_v is the ratio $v_{L\varphi}/v_{n\varphi}=v_{LT}/v_{nt}$ in the viscous mode with the amplitude v_v of the normal velocity. We consider harmonic oscillations with the frequency ω, but the time-dependent factor $\exp(-i\omega t)$ is omitted.

Substitution of Eq. (8.30) in the boundary conditions (8.28) and (8.29) ($v_{L\varphi}=-i\omega u_\varphi$) yields

$$\frac{\partial v_\varphi(R)}{\partial r}-\frac{v_\varphi(R)}{R}+iq_v\beta_v v_b=0 \ , \qquad (8.31)$$

$$v_\varphi(R)+v_v=v_B \ . \qquad (8.32)$$

Eliminating v_v, we obtain the effective boundary condition, Eq. (6.44), with the parameter α equal to

$$\alpha=-\frac{1}{iq_v\beta_v} \ . \qquad (8.33)$$

Substitution of q_v and β_v given by Eqs. (7.65) and (7.66) for the in-plane viscous mode in the low-frequency limit yields

$$\alpha=-\frac{\kappa}{8\pi\nu}\frac{1}{i\omega}\frac{\rho_s}{\rho_n}\times\left[\frac{\rho}{\rho_s}\nu\Omega\frac{\left[1-\frac{\rho_n}{2\rho}B'\right]^2+\left[\frac{\rho_n}{2\rho}B\right]^2}{B}\right]^{1/2} \ . \qquad (8.34)$$

We see that in the low-frequency limit $\alpha\to\infty$. Thus it would seem that one could not neglect the slip of the superfluid relative to the lateral wall. The magnitude of this slip, however, is determined not by the absolute value of α, but by the dimensionless parameter $q\alpha$, in which q is the in-plane wave vector of the mixed mode in the bulk. Since $q\le\omega/c_t$, we have an inequality,

$$|\alpha q|\le\frac{\rho}{\rho_n}\left[\frac{\kappa}{8\pi\nu}\frac{\left[1-\frac{\rho_n}{2\rho}B'\right]^2+\left[\frac{\rho_n}{2\rho}B\right]^2}{B}\right]^{1/2} \ . \qquad (8.35)$$

One should remember that our derivation of the effective boundary condition relies on the long-wavelength continuum theory and is valid for $q_v r_v \ll 1$, where $r_v\sim\sqrt{\kappa/\Omega}$ is the intervortex distance. According to Eq. (7.65), $q_v r_v$ is small when the parameter $\sqrt{\kappa/8\pi\nu}$ is small, but then the right-hand side of the inequality (8.35) is small too, unless the ratio ρ/ρ_n is too large. So the slip is not very important while we remain within the range of validity of the theory above (except at very low temperatures).

Equation (8.34) for α was obtained earlier (Sonin, 1976) for the particular values of the mutual friction coefficients B and B' given by Eq. (10.6) below.

It is useful to have an expression for the component $\Pi_{\varphi r}$ of the net momentum-flux tensor, since this component determines the force applied by the superfluid to the wall. The net momentum consists of the elastic flux given by the stress tensor of the vortex lattice and the viscous flux given by the viscous tensor. But the former vanishes at the solid surface, according to Eq. (8.28), and $\Pi_{\varphi r}$ is given by

$$\begin{aligned}\Pi_{\varphi r}=\tau_{\varphi r}&=-\rho_n \nu\left[\frac{\partial v_{n\varphi}(R)}{\partial r}-\frac{v_{n\varphi}(R)}{R}\right]\\&=-\rho_n\nu\left[\frac{\partial v_{\varphi}(R)}{\partial \tau}-\frac{v_{\varphi}(R)}{R}+iq_v v_v\right]\\&=-\rho_n \nu i q_v(1-\beta_v)(v_B-v_\varphi)\ .\end{aligned} \tag{8.36}$$

Here Eqs. (8.30)–(8.32) were used. Suppose that the cylindric vessel is an oscillating body of a torsion oscillator. Then the balance of angular momenta provides the following dispersion equation for the eigenfrequencies of the oscillator:

$$\omega^2-\omega_0^2-\frac{2\pi R^3}{I}i\omega\frac{\Pi_{\varphi r}}{v_B}=0\ , \tag{8.37}$$

or, after substitution of Eq. (8.36),

$$\omega^2-\omega_0^2-\frac{2\pi R^3}{I}\rho_n\nu q_v\omega(1-\beta_v)\frac{v_B-v_\varphi(R)}{v_B}\ . \tag{8.38}$$

Here ω_0 is the eigenfrequency of the torsion oscillator without the superfluid, and I is the moment of inertia of the vessel per unit length. When $\omega_0=0$, Eq. (8.38) gives the eigenfrequencies of the freely suspended vessel. Assuming that $v_\varphi(r)=CJ(qr)$, and finding C from the boundary condition (6.44), one can obtain from Eq. (8.38) the dispersion equation (6.48) in the limit of a very long cylinder ($\omega_L=0$).

Let us turn now to the oscillations of a cylinder immersed in a rotating superfluid. The boundary condition for the fluid around the cylinder is Eq. (6.44), as in the previous problem of the fluid inside the cylinder, but $\alpha\propto q_v^{-1}$ has another sign because the choice of the sign of q_v should provide attenuation of the viscous mode deep within the fluid. Equation (8.38) also retains its form. The oscillating cylinder irradiates the Tkachenko wave, which may be approximated by a plane wave $v_\varphi=C\exp[iq(r-R)-i\omega t]$ near the surface of the cylinder when $q=\omega/c_t\gg 1/R$. The amplitude C is determined from the boundary condition equation (6.44). Then substitution of v_φ into Eq. (8.38) yields, in the limit $\omega\ll\Omega$,

$$\omega^2=\omega_0^2-i\omega c_t\frac{2\pi\rho R^3}{I}\ . \tag{8.39}$$

The eigenfrequencies have imaginary parts associated with energy losses due to emission of a Tkachenko wave. The damping $\delta\sim \mathrm{Im}\omega/\omega$ is proportional to $\sqrt{\Omega}/\omega$ when $\omega\ll\Omega$.

Formerly the theory of axisymmetric oscillations of a cylinder in a rotating superfluid did not take into account the Tkachenko-wave effect and assumed that the oscillating cylinder generated only viscous oscillation modes in the surrounding fluid (Andronikashvili *et al.*, 1961, 1978). The equation for the eigenfrequencies in this theory is obtained from Eq. (8.38) by deleting v_φ and β_v and substituting for q_v the value of q given by Eq. (7.63). Then

$$\omega^2=\omega_0^2-\frac{2\pi R^3}{I}\rho_n\omega\left[i\omega\nu\frac{i\omega-B\Omega}{i\omega-(\rho_n/\rho)B\Omega}\right]^{1/2}\ . \tag{8.40}$$

In the region of slow rotation $\Omega\ll\omega$ this formula predicts the linear dependence of the damping δ on Ω [see Eq. (7.3.1) in the review by Andronikashvili and Mamaladze, 1967]. This prediction was confirmed by the experiment of Tsakadze and Chkheidze (1960). Indeed, the Tkachenko-wave effect is not expected to be important at $\Omega\ll\omega$ and cannot be calculated within the continuum theory because the wave vector of the Tkachenko wave is much larger than the inverse intervortex distance in this case. But in the region of fast rotation $\Omega\gg\omega$ the correct formula for eigenfrequencies is Eq. (8.39), allowing for the emission of the Tkachenko wave. Equation (8.40) in this region predicts the dependence $\delta\propto 1/\sqrt{\omega}$, contrary to $\delta\propto\sqrt{\Omega}/\omega$ following from Eq. (8.39). An experimental study of the oscillations of a long cylinder in a rotating superfluid, in the regime of fast rotation, would provide the information on vortex-lattice rigidity that Tkachenko hoped for (Tkachenko, 1974). Therefore extension of the experiment by Tsakadze and Chkheidze (1960) to much higher rotation speeds would be interesting for verification of the vortex-lattice effects.

E. Two-fluid effects in pile-of-disks experiments in He II

Mutual friction is among the most important two-fluid effects and is responsible for attenuation of vortex oscillations. It has been studied in the past by observing Hall's resonances in pile-of-disks experiments (Hall, 1960; Andronikashvili and Mamaladze, 1966, 1967). But the accuracy of the obtained data on B and B' were diminished by the competition of mutual friction in the bulk with pinning and vortex slip on the surface of the disks (see the discussion of the pile-of-disks experiment in ^{3}He in Sec. VIII.F).

The two-fluid theory predicts the temperature-dependence factor $\sqrt{\rho_s/\rho}$ for the frequency of the Tkachenko wave and of the inertial-wave resonance, as can be seen from Eqs. (7.67) and (8.12). The temperature dependence of the resonance frequencies in the pile-of-disks experiment was studied by Andereck and Glaberson (1982) and is shown in Fig. 4, reproduced from Fig. 19 of their paper. The experimental points are compared with the theoretical temperature dependence $\propto\sqrt{\rho_s/\rho}$ following from Eq. (8.12) for the inertial-wave-resonance frequency (the solid line). The proportionality factor of the theoretical curve was chosen to match the low-temperature experimental points. The theoretical and experimental dependence agree quite well, despite the fact that the condition $\omega \ll \Omega$ (more exactly $\omega \ll 2\Omega$) is not well satisfied for the experimental points. A more accurate determination of the theoretical curve, which does not use the assumption $\omega \ll 2\Omega$, requires numerical calculations of the system of equations (8.6)–(8.9).

As discussed above (Sec. VI.E), Andereck and Glaberson adopted another interpretation of their experiments, relating observed resonances with the frequencies of the density-of-state peaks given by Eq. (4.56). In the two-fluid theory Eq. (4.56) remains, but with the temperature-dependent Tkachenko-wave velocity $c_t=\sqrt{\rho_s/\rho}c_T$ instead of c_T. This means that the resonance frequency would be proportional to $(\rho_s/\rho)^{1/4}$. The theoretical curve obtained by Andereck and Glaberson from the density-of-state peak theory is also shown in Fig. 4 by the dashed line. We see that the theory of inertial-wave resonance better explains the experimental temperature dependence.

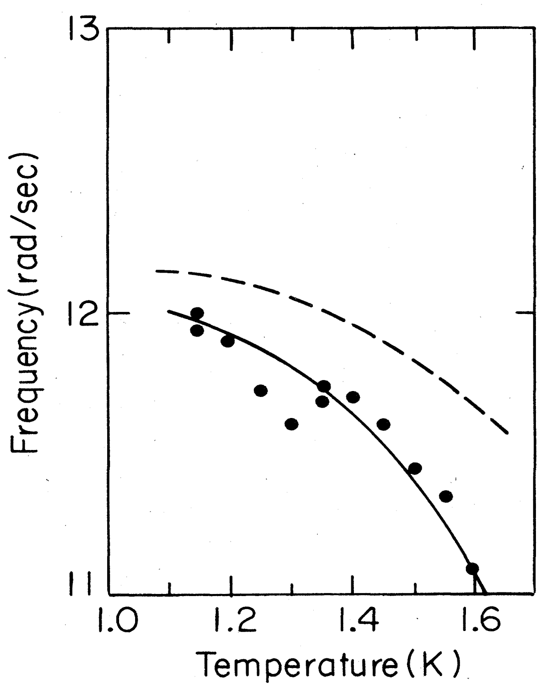

FIG. 4. Dependence of the resonance frequency on the temperature in a pile-of-disks experiment. The data points are for the distance between disks $d=0.051$ cm and the angular velocity $\Omega=10.1$ rad/sec (Andereck and Glaberson, 1982). The solid line shows the dependence $\propto(\rho_s/\rho)^{1/2}$ of the inertial-wave-resonance frequency [Eq. (8.12)]. The dependence was scaled to match the lowest temperature data. The dashed line was calculated by Andereck and Glaberson from the density-of-states peak theory.

F. The clamped regime in superfluid ^{3}He and neutron stars

It has been said above that the clamped regime (in which the normal fluid moves together with the walls of the vessel as a solid body) is readily realized in superfluid ^{3}He. In a pile-of-disks experiment in rotating ^{3}He one may choose a distance between disks much smaller than the viscous penetration length, but still larger than the wavelength scale of the vortex waves. This is possible because the former is given by $\sim\sqrt{\nu/\omega}$ and $\sim\sqrt{\nu/\Omega}$ and the latter is given by $\sim\sqrt{\nu_s/\omega}$ and $\sim\sqrt{\nu_s/\Omega}$ at $\omega \gg \Omega$ and $\omega \ll \Omega$, respectively. At the same time $\nu \gg \nu_s$ in ^{3}He. Therefore a pile-of-disks experiment in superfluid phases of ^{3}He is capable of providing direct data on the mutual friction coefficient B, since mutual friction becomes a primary source of drag on the superfluid under the considered conditions, even in the case of perfect pinning (Sonin, 1981a), (excepting regions of the possible Hall's resonances), as supported by Eqs. (8.26) and (8.27) in Sec. VIII.C. Such an experiment would carry out the original idea of the pile-of-disks experiment as planned for He II (see the first paragraph in Sec. VI.B). Measurements of B for rotating ^{3}He have already been performed by Hall *et al.* (1984), though in a geometry different from that of the pile-of-disks experiment.

But observation of vortex-wave resonances (associated with Kelvin modes or the inertial wave) is expected to be more difficult in ^{3}He than in He II, since the measured values of B in both A and B phases of ^{3}He are quite large (see Hall *et al.*, 1984, Hall and Hook, 1985), and further discussion in Sec. X.E). We saw above (Secs. VIII.B and VIII.C) that vortex-wave resonances are possible when

$$\frac{\rho_n}{2\rho}B \ll 1-\frac{\rho_n}{2\rho}B' .$$

Hence they can probably be observed at low temperatures when large values of B are offset by small ρ_n/ρ.

The clamped regime is assumed also in the two-component model suggested by Baym *et al.* (1969) to describe the behavior of neutron stars. One component is a neutron superfluid. The second consists of charged particles, protons and electrons, clamped to the outer crust of the star by a large magnetic field, so that the charged component rotates rigidly with the crust as a unit. Of the charged particles, only electrons are effective at interaction with vortices threading the neutron superfluid, because the electrons are normal and the protons are superconducting. The superfluid neutron component is coupled with the clamped normal-electron component by mutual friction. Baym *et al.* (1969) describe mutual friction in their model by the relaxation time τ_c determining the mutual-friction torque on the neutron superfluid:

$$K = -\frac{I_c}{\tau_c}(\Omega - \Omega_n) . \tag{8.41}$$

Here Ω and Ω_n are the angular velocities of the neutron superfluid and the crust, while I_c is the moment of inertia of the superfluid. This expression follows from Eqs. (8.14) and (8.15) provided that $v_{st} = \Omega r$, $v_B = \Omega_n r$, and $\tau_c = 1/B\Omega$.

The two-component model of Baym *et al.* (1969) provided an explanation of the long relaxation time, of the order of years, after a starquake. According to Alpar *et al.* (1984a), however, the simple two-component model does not explain adequately a wealth of new data on the postglitch behavior of pulsars. They proposed the vortex creep theory and applied it to the interpretation of the behavior of some pulsars (Alpar *et al.*, 1984b, 1985).

IX. VORTEX OSCILLATIONS AND HYDRODYNAMICS OF A ROTATING ANISOTROPIC SUPERFLUID

A. Vortices and hydrodynamics of an anisotropic superfluid ^{3}He-A

Earlier in this paper we applied the theory developed for He II to the superfluid phases of ^{3}He without any reference to specific features of these phases. Now we are going to examine to what extent one can exploit the theory of the conventional superfluid He II (which remained the single testing ground of the superfluidity theory for about forty years) by considering superfluids with a more complicated order-parameter structure and symmetry. We shall extend the theory to include effects of superfluid anisotropy. The theory relies on phenomenological arguments and may be applied to any anisotropic superfluid, but we have in mind ^{3}He-A as our primary concern, and therefore begin with a short discussion of the "microscopic" hydrodynamics of ^{3}He-A.

The order parameter in the A phase of ^{3}He is specified by the unit orbital vector $\hat{l}$, the unit spin vector $\hat{\mathbf{d}}$, and the phase φ, which is, at the same time, the rotation angle around $\hat{l}$, though it is not well defined globally (see further). The dipole-dipole energy couples vectors $\hat{l}$ and $\hat{\mathbf{d}}$, and we shall consider only the dipole-locked regime, when $\hat{l}$ and $\hat{\mathbf{d}}$ are parallel or antiparallel to each other. This yields a correct description until the fluid variables vary slowly compared with the dipole length $\sim 10^{-3}$ cm (Brinkman and Cross, 1978). Then the spin vector $\hat{\mathbf{d}}$ drops out from the set of independent variables. The remaining order-parameter variables, $\hat{l}$ and φ, are not completely independent. Only the derivatives of the phase φ are well-defined variables; in particular, the superfluid velocity is

$$\mathbf{v}_s = \frac{\hbar}{M}\nabla\varphi , \tag{9.1}$$

where M is the mass of the Cooper pair of ^{3}He atoms. Since φ is not well defined, Eq. (9.1) does not yield the curl-free velocity field. The vorticity is given by the Mermin-Ho equation (Mermin and Ho, 1976)

$$\nabla \times \mathbf{v}_s = \frac{\hbar}{2M}\varepsilon_{ikn} l_i \nabla l_k \times \nabla l_n . \tag{9.2}$$

Once the structure of broken-symmetry variables has been determined, one can apply a standard procedure for deriving the hydrodynamic equations for thermodynamics and the conservation laws (Khalatnikov, 1971), The nonlinear hydrodynamics was formulated by Hu and Saslov (1977). At the same time Khalatnikov and Lebedev (1977) used the canonical Lagrange formalism to derive the general hydrodynamic equations. Since then a considerable number of papers on ^{3}He-A hydrodynamics have been published, but it remains a matter of controversy. For a comprehensive list of references, the reader is referred to the last review by Hall and Hook (1985). Much of the confusion has arisen over the intrinsic angular momentum associated with the orbital vector $\hat{l}$. The simplest idea is that any Cooper pair is in a state with unit orbital momentum and has an angular momentum $\hbar\hat{l}$. Thus the density of the total intrinsic angular momentum is $\hbar\hat{l}$ times the number density of Cooper pairs, i.e., ρ/M at $T=0$ or ρ_s/M at $T>0$. But the situation has turned out to be much more complicated. Some microscopic calculations agreed with such a prediction, but others showed that the intrinsic angular momentum is reduced by a very small factor $(T_c/\varepsilon_F)^2$. This was partly a matter of semantics (Brinkman and Cross, 1978). The intrinsic angular momentum is not a well-defined, one-valued quantity. In principle, different angular momenta may appear in the theory, and it is now thought (Volovik and Mineev, 1981; Hall and Hook, 1985) that different measurements of the intrinsic angular momentum may give different answers. As for the problems considered in this review, the most important is the dynamic angular momentum L, which is directly involved in dynamics and appears in the equation of orbital motion [see Eq. (10.17) below]. It determines orbital inertia as a factor before the term $\partial\hat{l}/\partial t$ [for discussion of the static intrinsic angular momentum see Balatskii and Mineev (1986) and references therein]. The frequency of orbital waves (see the end of Sec. IX.C) and mutual friction coefficients (Sec. X.D) depend on L. It follows from a number of microscopic derivations of orbital hydrodynamics using a quasiclassical expansion in spatial and temporal derivatives of hydrodynamic variables that L is very small due to particle-hole symmetry in the BCS theory (Volovik, 1975; Cross, 1975, 1977; Nagai, 1980). But the solution of the problem does not look definitive. Such a theory gives hydrodynamics that do not satisfy the momentum-conservation law and cannot be formulated within the Lagrange formalism at $T=0$, though derived from the theory with the Lagrangian.

The problem of the intrinsic angular momentum and of the hydrodynamic Lagrangian are closely connected with the problem of the nonlocal term $-(\hbar/M)C_0\hat{l}\,(\hat{l}\cdot(\nabla\times\hat{l})$

in the expression for the supercurrent [see Eq. (9.9) below]). The microscopic theory mentioned above predicted large C_0 at $T=0$ (see Mermin and Muzikar, 1980, and references therein). It is possible to formulate a self-consistent local phenomenological hydrodynamics satisfying the momentum-conservation law in the limit $T\rightarrow 0$, $\rho_n \rightarrow 0$ only if $L\rightarrow \rho\hbar/M$ and $C_0 \rightarrow 0$ in this limit (see the discussion by Sonin, 1984, in which one can find additional references). In order to reconcile the phenomenological hydrodynamics with the microscopic theory, Volovik and Mineev (1981) suggested that the normal density remains finite even at $T\rightarrow 0$ if $\hat{l}$ varies in space (see also Combescot and Dombre, 1986, and references therein). The discussion concerning the magnitude of C_0 is still in progress since McClure and Takagi (1979) showed, using the symmetry of the BCS wave function, that $C_0=0$ at $T=0$. The same result was obtained by Ishikawa *et al.* (1980), who performed calculations with the BCS many-particle wave function. Recently a number of papers have appeared (Volovik, 1985, 1986; Balatskii *et al.*, 1986; Combescot and Dombre, 1986) in which an exact solution was attempted of the quantum problem for boojums (the singular points on the Fermi surface in which the wave vector $\mathbf{k}$ is parallel to $\hat{l}$ or $-\hat{l}$). This is important because near boojums the quasiclassical expansion in gradients is invalid and higher-order terms are essential. The general conclusion was that the nonlocal term $\propto C_0$ is linked to the normal fluid, but not to the condensate. However, a complete solution of the problem has not yet been achieved in this way.

The source of all these ambiguities is the question of how to treat boojums correctly in the microscopic theory. The derivation of the zero-temperature hydrodynamics from the microscopic theory of ^{3}He-A is discussed in the Appendix. It seems that up to this point the theory has not been able to determine the contribution of boojums to L and C_0 unambiguously. But some conjecture linked with the momentum-conservation law makes it possible to derive the self-consistent zero-temperature hydrodynamics with large $L=\rho\hbar/M$ and $C_0=0$.

Another problem of the ^{3}He-A hydrodynamics worthy of mention is the gauge wheel effect invented by Liu and Cross (1979). They showed that there is a term $\hat{l}\cdot(\nabla\times\mathbf{v}_n)$ in the Josephson equation which produces superfluid acceleration. Hu and Saslov's hydrodynamics predict that the gauge wheel effect exists even at $T=0$, $\rho_n=0$, though it depends on $\mathbf{v}_n$. But their hydrodynamic equations are incompatible in this limit (even their linear version). If they are supplemented by some terms that make the theory self-consistent, the gauge wheel effect vanishes at $T\rightarrow 0$, $\rho_n\rightarrow 0$ (Sonin, 1984).

In all, the structure of the hydrodynamical equations for ^{3}He-A is not entirely clear at present. Nevertheless, we may distract ourselves from these difficulties when formulating the "macroscopic" hydrodynamics of the rotating anisotropic superfluid (the next section), which is a cruder theory evolving from the original "microscopic" hydrodynamics with a coarse-graining procedure. However, the problems discussed above become vital again if one attempts to estimate some coefficients of macroscopic hydrodynamics, for example, those for the mutual friction (see Sec. X.D).

The most remarkable feature of ^{3}He-A is continuous vorticity of the superfluid, which is accompanied by an inhomogeneous $\hat{l}$ texture in accordance with the Mermin-Ho relation Eq. (9.2). Volovik and Kopnin (1977) showed that the $\hat{l}$ texture and the vorticity field in a rotating superfluid are doubly periodic functions in the plane normal to the rotation axis (the xy plane). Now a large number of various vortex structures have already been proposed. What vortex structure is in equilibrium depends on the angular velocity and the magnetic field (see the recent papers of Seppälä and Volovik, 1983; Ohmi, 1984; Maki and Zotos, 1985, as the entry points to the literature on the topics).

The cell of the periodical vortex structure may be considered as an elementary superfluid vortex. It can be singular or nonsingular (i.e., with or without the singular line on which the order parameter vanishes), but its circulation is always quantized as in a conventional superfluid. For nonsingular vorticity quantization follows from the Mermin-Ho relation [Eq. (9.2)], or more readily from the integral form of the relation derived by Ho (1978). The Ho circulation theorem is obtained by integration of the z component of the vorticity $\nabla\times\mathbf{v}_s$ over the vortex structure cell in the xy plane. It yields the circulation around the cell, in other words, the circulation around the elementary vortex:

$$\kappa_v=\int_{\text{cell}} dx\,dy\,\nabla\times\mathbf{v}_s$$
$$=\frac{\hbar}{M}\int_{\text{cell}} dx\,dy\,\hat{l}\left[\frac{\partial\hat{l}}{\partial x}\times\frac{\partial\hat{l}}{\partial y}\right]. \tag{9.3}$$

The direction of $\hat{l}$ at any point within the cell is presented by a point on the surface of a unit sphere. Then the $\hat{l}$ texture yields a mapping of the vortex-structure cell on the sphere. Because of the periodical boundary conditions for the cell, it is a mapping of the torus on the sphere surface (Volovik, 1984). The integral in Eq. (9.3) is the area of the mapping, which is equal to the area 4π of the unit-surface sphere surface multiplied by the integer topological charge p. This charge indicates how many times the unit sphere is mapped. As a result

$$\kappa_v=2p\kappa=\frac{2ph}{M}\,, \tag{9.4}$$

so the nonsingular vortex is always characterized by an even number $2p$ of circulation quanta. But the circulation of the singular vortex may be any integer number of quanta, and even a half-integer when the vortex is hybridized with the disclination in the field of the spin vector $\hat{\mathbf{d}}$ (Volovik and Salomaa, 1985).

B. Macroscopic hydrodynamics of a rotating anisotropic superfluid

The macroscopic hydrodynamics of a rotating superfluid is obtained by averaging the original hydrodynamic equations over the vortex cell (the procedure of coarse-graining). Thus its formal structure should not depend on details of vortex structure within the vortex cell, in particular, on whether the vortex is singular or not. As usual, instead of the coarse-graining derivation one can deduce the macroscopic hydrodynamics by referring to thermodynamics, the conservation laws, and the symmetry of the order-parameter variables. It is very important that in the rotating superfluid ^{3}He-A the degeneracy with respect to the direction of $\hat{l}$ is lifted, since $\hat{l}$ is fixed by the vortex structure. Then the vector $\hat{l}$ drops out from the list of independent hydrodynamic variables, and the order-parameter field may be characterized by the same set of variables as in the conventional superfluid: the superfluid velocity $\mathbf{v}_s$ and the deformation tensor of the vortex structure determined by its displacements $\mathbf{u}$. As a result, the formal structure of the macroscopic hydrodynamics, as given by Eqs. (7.1)—(7.17) in Sec. VII.A, remains valid for the anisotropic superfluid too, but the equation of motion for vortices, Eq. (7.18), should be replaced by a more general one allowing a lower symmetry of the superfluid:

$$\mathbf{v}_L=\mathbf{v}_n+\alpha\mathbf{f}_\Sigma+\alpha'\hat{\mathbf{n}}\times\mathbf{f}_\Sigma+\alpha_m\hat{\mathbf{m}}(\hat{\mathbf{m}}\cdot\mathbf{f}_\Sigma)\ . \qquad (9.5)$$

Here the unit vector $\hat{\mathbf{m}}$ determines the direction in the plane normal to the vorticity vector at a given point, and α_m is the third mutual friction parameter in the anisotropic superfluid. The dissipation function R given by Eq. (7.16) is a positive definite quadratic form when

$$\alpha>0,\ \ \alpha+\alpha_m>0\ . \qquad (9.6)$$

A further difference between isotropic and anisotropic superfluids becomes apparent when one turns to the explicit expressions for the energy and the current. Let us write the general expression for that part of the energy density which depends on the order-parameter variables [$\mathbf{v}_s$ and the space gradient $\nabla_i u_j$, or the deformation tensor u_{ij} given by Eq. (4.25)]. In a reference frame moving with normal velocity $\mathbf{v}_n$ we have, in the harmonic approximation ($\mathbf{w}=\mathbf{v}_n-\mathbf{v}_s$),

$$E_B=(\rho_s)_{ij}\frac{w_iw_j}{2}+\gamma_{ijk}w_i\nabla_j u_k+\tfrac{1}{2}\lambda_{iklm}u_{ik}u_{lm}\ . \qquad (9.7)$$

The expression for the current is

$$\lambda_i=-\frac{\partial E_B}{\partial w_i}=-(\rho_s)_{ij}w_j+\gamma_{ijk}\nabla_j u_k\ . \qquad (9.8)$$

Since only displacements normal to the vorticity vector, i.e., to the rotation axis, increase the energy, the tensor γ_{ijk} should vanish when $k=z$, and the tensor λ_{iklm} when $i=k=z$ or $l=m=z$.

We see that the expression for the current includes a term depending on the deformation, which was absent in the conventional superfluid. One can understand its origin by referring to the equation for the current in microscopic hydrodynamics:

$$\lambda_i=-(\rho_s)_{ij}w_j+\frac{\hbar}{M}C\nabla\times\hat{l}-\frac{\hbar}{M}C_0\hat{l}(\hat{l}\cdot\nabla\times\hat{l})\ . \qquad (9.9)$$

By averaging the last term in this equation, one obtains a contribution to the current depending on the deformation of the vortex structure. Just this term in the current was intensively discussed in connection with problems of ^{3}He-A hydrodynamics (see Sec. IX.A and the Appendix). The deformation-dependent current is important in the macroscopic hydrodynamics for the vortex oscillations discussed in the following section.

The deformation-dependent current is the only new term in the hydrodynamic equations for the rotating anisotropic superfluid; other terms have their counterparts in the isotropic superfluid, but differ from them by a more general tensor structure.

It is clear that anisotropic superfluids such as ^{3}He-A on the level of macroscopic hydrodynamics may possess the same symmetry as the isotropic superfluid if the symmetry of the vortex structure is high enough. But the theory, developed above for a superfluid with singular vortex lines, was simplified by the assumption that the vortex lines perturb the superfluid only in their immediate vicinity, which made it possible to use the scalar superfluid density while ignoring lowering of the symmetry due to the vortex lines (see Sec. VII.A). This assumption does not work in a superfluid with nonsingular vorticity, in which the superfluid density along the rotation axis (the z axis) and in the xy plane differ no matter what the symmetry of the vortex structure is. But the oscillation modes that we have considered involved mostly the motion in the xy plane or of the fluid as a whole, when the difference between ρ_s along the z axis and in the xy plane does not matter. In such cases the results of the developed theory can be extended to an anisotropic superfluid with high symmetry of the vortex structure without any modification. Further analysis of the vortex oscillations in an anisotropic superfluid will require symmetry classification of the vortex structures.

C. Symmetry and the effect of axial currents on Kelvin modes

Symmetry classification of the vortex structures in rotating superfluids should follow along the same lines as in solids. It is well known that macroscopic properties of periodic structures, on scales large compared to their period, are entirely determined by the symmetry of point groups, or the "symmetry of directions" (Landau and Lifshitz, 1976). With respect to such symmetry all periodic structures are divided into the crystal classes. The vortex structures are distributions of currents and moments, just like the magnetic structures. Therefore their symmetry should be described by the magnetic crystal classes that supplement the elements of the point groups with time reversal (Landau and Lifshitz, 1982). But two-dimensional periodicity of the vortex structures imposes some restrictions on the possible magnetic classes, and only 21 classes may refer to the vortex structures. The list for two-dimensional crystal systems (syngonies) is given here (see Sonin and Fomin, 1985):

the oblique system: C_1, $\{C_2, S_2 = C_i\}$;

the rectangular system: $\{C_{1v}(C_1), D_1(C_1)\}$, $\{C_{2v}(C_2), D_2(C_2), D_{2d}(S_2)\}$;

the square system: $\{C_4, S_4\}$, $\{C_{4v}(C_4), D_4(C_4), D_{2d}(S_4)\}$;

the hexagonal system: C_3, $\{C_6, S_6\}$, $\{C_{3v}(C_3), D_3(C_3)\}$, $\{C_{6v}(C_6), D_6(C_6), D_{3d}(S_6)\}$.

Here the notations of Landau and Lifshitz (1974, 1976, 1982) are used. The class C_n is determined by the group of the n-fold rotation axis (always the vertical z axis), and S_n corresponds to the n-fold axis of improper rotations (accompanied by reflection in the horizontal plane). In the group C_{nv}, the group C_n is supplemented by n vertical planes of reflection denoted by σ_v. The group D_n contains, besides C_n, n twofold axes U_2 in the horizontal plane. The group D_{nd} is the group D_n supplemented by n vertical reflection planes σ_v. The group denoted by $A(B)$ contains all elements of B, the subgroup of A, and the remaining elements of A, each combined with the time-inversion operation R.

This classification implies that the vortex structure varies only on the xy plane, but currents and moments map a three-dimensional space. When the currents are confined in the xy plane too, and, correspondingly, the moments are along the z axis (as in He II), then the groups united by curly brackets are not distinguishable, since for plane currents the operations σ_v and U_2 are equivalent and the reflection in the horizontal plane is the identity operation. Thus for classification of distributions of plane currents it is enough to retain only the first group in each set of brackets. Then one has the ten crystal classes listed by Kittel (1971).

As an example of a property that appears only in the anisotropic superfluid, we shall consider the effect of deformation-dependent currents on the Kelvin modes, the torsion vortex waves propagating along the z axis. For such currents to exist, the tensor γ_{ijk} in the energy density Eq. (9.7) should include a quantity transforming like a current (changing sign at the inversion of space or time). The symmetry allowing such a current is highest when the current is axial and does not break rotational symmetry. The axial current may be a nonlocal current $\propto \hat{l}(\hat{l}\cdot\nabla\times\hat{l})$ along the rotation axis. Of course, the total current averaged over the cell of the vortex structure vanishes, including if necessary the current $\propto \mathbf{v}_s$. An axial current in equilibrium is forbidden by the symmetry element $\sigma_v R$, so it is possible in the classes C_n and $D_n(C_n)$. For simplicity we shall consider the classes with high rotational symmetry ($n>2$). The clamped regime will also be assumed ($\mathbf{v}_n = 0$, $\mathbf{w} = -\mathbf{v}_s$). Then the energy density equation (9.7) may be rewritten in simpler form, retaining only terms relevant for the problem under consideration:

$$E_B = \tfrac{1}{2}\rho_s v_s^2 + \rho_s \Omega \nu_s \left[\frac{\partial \mathbf{u}}{\partial z}\right]^2 - \gamma \mathbf{v}_s \cdot \frac{\partial \mathbf{u}}{\partial z} - \gamma' \mathbf{v}_s \cdot \left[\hat{\mathbf{z}} \times \frac{\partial \mathbf{u}}{\partial z}\right] . \tag{9.10}$$

The effective force $\mathbf{f}_\Sigma$ determining the vortex velocity $\mathbf{v}_L$ [see Eqs. (7.17) and (7.18)] is given by

$$\begin{aligned} \mathbf{f}_\Sigma &= -2\boldsymbol{\Omega}\times\boldsymbol{\lambda} + \mathbf{f} = -2\boldsymbol{\Omega}\times\frac{\partial E_B}{\delta \mathbf{v}_s} - \frac{\delta E_B}{\delta \mathbf{u}} \\ &= -2\boldsymbol{\Omega}\times\rho_s\mathbf{v}_s + \rho_s 2\Omega\nu_s \frac{\partial^2 \mathbf{u}}{\partial z^2} \\ &\quad + \gamma\left[2\boldsymbol{\Omega}\times\frac{\partial \mathbf{u}}{\partial z} - \frac{\partial \mathbf{v}_s}{\partial z}\right] \\ &\quad - \gamma'\left[2\Omega\frac{\partial \mathbf{u}}{\partial z} - \hat{\mathbf{z}}\times\frac{\partial \mathbf{v}_s}{\partial z}\right] . \end{aligned} \tag{9.11}$$

Substituting Eq. (9.11) in Eq. (7.18) [which is equivalent to Eq. (9.5) when $\alpha_m = 0$ due to rotational symmetry], and using the relations in Eq. (7.27) between α, α' and B, B' yields the equation for the vortex velocity written for the plane Kelvin wave $\mathbf{u} \propto \exp(ipz - i\omega t)$ in the j-complex representation:

$$\begin{aligned} \widetilde{v}_L = &\left[\frac{\rho_n}{2\rho}B + j\left[1 - \frac{\rho_n}{2\rho}B'\right]\right] \\ &\times\left[-j\widetilde{v}_s - \nu_s p^2 \widetilde{u} + \frac{\gamma}{\rho_s}\left[ijp\widetilde{u} - \frac{ip}{2\Omega}\widetilde{v}_s\right]\right. \\ &\left.- \frac{\gamma'}{\rho_s}\left[ip\widetilde{u} - \frac{jip}{2\omega}\widetilde{v}_s\right]\right] . \end{aligned} \tag{9.12}$$

This equation is solved together with the Euler equation, which in the complex representation for vectors in the xy plane is Eq. (7.43). It yields the dispersion law for the Kelvin mode:

$$\begin{aligned} \omega &= -ij\left[1 - \frac{\rho_n}{2\rho}B' - j\frac{\rho_n}{2\rho}B\right]\left[2\Omega + \nu_s p^2 - ij\frac{2\gamma}{\rho_s}p\right] \\ &= \pm\left[1 - \frac{\rho_n}{2\rho}B' \mp i\frac{\rho_n}{2\rho}B\right]\left[2\Omega + \nu_s p^2 \pm \frac{2\gamma}{\rho_s}p\right] . \end{aligned} \tag{9.13}$$

The two values of $j = \pm i$ correspond to two senses of circular polarization. The term $\propto \gamma'$ in the energy cancels and does not affect the eigenfrequency.

Thus the axial current leads to the term linear in p in the expression for the frequency of the Kelvin mode. When one deals with nonsingular vorticity in ^{3}He-A without a magnetic field, the only relevant space scale of the vortex structure is the intervortex distance $r_v \sim \sqrt{\kappa/\Omega}$, and the hydrodynamic expansion in p is in fact an expansion in the dimensionless parameter pr_v. This means that $\gamma \propto \sqrt{\Omega}$. Formally one may apply Eq. (9.13) only to the case $pr_v \ll 1$, where the macroscopic hydrodynamics is rigorous. But the Hall resonances in

He II were observed at small $\Omega \ll \omega$ where $pr_v \gg 1$. In a superfluid with singular vortex lines it is not difficult to extend the theory to wave numbers p larger than $1/r_v$ (but not larger than $1/r_c$, where r_c is the core radius) by deriving it from the dynamics of isolated vortex lines (see Sec. IV.D). No derivation of this sort has been carried out for ^{3}He-A. Nevertheless, assuming that such a theory exists and that it admits an expansion in $(pr_v)^{-1}$ beginning with the term $\propto p^2$, one can apply Eq. (9.13) for the nonsingular vorticity at $pr_v \gg 1$. Then it follows from Eq. (9.13) that the frequency of the Hall resonance in a pile-of-disks experiment would increase as $\gamma \propto \sqrt{\Omega}$ at small Ω, unlike the linear dependence on Ω in Eq. (8.11) for an isotropic superfluid.

Observation of the Hall resonances would provide information on the symmetry of the vortex structure. For example, one could distinguish between the lattices of radial-hyperbolic and circular-hyperbolic vortex pairs, as considered by Seppälä and Volovik (1983) and by Maki and Zotos (1985). These pairs belong to the classes $C_{1v}(C_1)$ and $D_1(C_1)$, respectively. The latter allows axial currents and the former does not.

Observation of effects associated with axial currents would at the same time provide verification of the existence of the nonlocal term $-(\hbar/M)C_0\hat{l}(\hat{l}\cdot\nabla\times\hat{l})$ in Eq. (9.9) for the current.

An axial current may also arise in the core of a vortex in ^{3}He-B. According to the symmetry analysis of Salomaa and Volovik (1985), a spontaneous axial current is possible for w and uvw vortices. But an estimate of its effect on vortex oscillations in ^{3}He-B requires a more involved theory, since simple scaling arguments, used above for nonsingular vortices in ^{3}He-A, are not valid there.

In this section Kelvin waves in ^{3}He-A have been treated in terms of vortex displacements like Kelvin waves along singular vortex lines. But in ^{3}He-A, displacement of the $\hat{l}$ texture associated with a given vortex structure produces variation of $\hat{l}$ at a given point. So Kelvin waves are orbital waves in terms of the original microscopic hydrodynamics of ^{3}He-A, and their properties must strongly depend on the dynamic intrinsic angular momentum L discussed in Sec. IX.A. This dependence shows itself in the equations for mutual friction coefficients B and B' [Eq. (10.31) or Eq. (10.32)] that will be derived later in Sec. X.D.

X. MUTUAL FRICTION

A. Introductory comments

The concept of mutual friction between a superfluid and a normal fluid was invented by Hall and Vinen (1956) to explain the effect of rotation on propagation of the second sound. Since that time it has remained one of the most important and probably most intricate problems of the hydrodynamics of a rotating superfluid. An exhaustive discussion of mutual friction would take up too much room, and this section will be limited to an overview of the subject, mostly its theoretical aspects. The experimental results concerning mutual friction coefficients in He II have been exhaustively reviewed recently by Barenghi *et al.* (1983).

Calculations of mutual friction coefficients inevitably refer to "microscopic" hydrodynamics and sometimes to the really microscopic kinetic theory based on the Boltzmann equation. The calculations follow along different lines, depending on the ratio between various space scales. The pioneer theory of Hall and Vinen was based on the assumption that the free path length of quasiparticles is much larger than the size of the region responsible for mutual friction. Therefore the mutual friction was treated as due to scattering of noninteracting quasiparticles by vortices. This theory will be discussed in Sec. X.B. Because the free path length diminishes when the temperature increases, such a theory is valid only when $T < 1.4$ K (Sonin, 1975). As the temperature rises, however, and approaches the λ point, the radius of the vortex core becomes larger than the free path length of the quasiparticles, and one expects that some sort of phenomenological theory similar to the time-dependent Ginzburg-Landau theory may be valid. This was realized in the theory of mutual friction near the λ point discussed in Sec. X.C. But most experimental data fall in the region of the intermediate temperatures, where neither of the two mentioned theories is capable of providing a satisfactory quantitative picture of mutual friction. Matheiu and Simon (1980) have suggested a model that fits the experimental data well at intermediate temperatures. It will be discussed at the end of Sec. X.B.

In spite of the remaining controversy and disagreement concerning the mutual friction problem in He II, the contours of the theory are well established, and the magnitude of the mutual friction parameters is fairly clear as a result of a considerable number of experimental and theoretical studies. In contrast, investigation of mutual friction in the superfluid phases of ^{3}He is still in an early stage. Both approaches developed for He II (based on the scattering theory of noninteracting quasiparticles and on phenomenological theory near the critical point) are expected to have regions of application for ^{3}He too. But in addition, another possible approach to a quantitative theory of mutual friction has been found in ^{3}He. If nonsingular vorticity arises in ^{3}He-A, the space scale responsible for mutual friction is the scale of the vortex structure itself, which is much larger than the free path length and the coherence length. Then it is possible to study mutual friction within hydrodynamics. This will be discussed in Sec. X.D.

The short Sec. X.E is devoted to mutual friction in pulsars.

B. Mutual friction in He II at low temperatures

The low-temperature theory assumes that mutual friction occurs due to scattering of independent quasiparticles by a potential produced by the nonhomogeneous order parameter around the vortex line. Applying the usual scattering theory for calculating the scattering cross section of quasiparticles, one can find the force that the normal fluid exerts on the vortex:

$$\mathbf{F}_n = -D(\mathbf{v}_{nl}-\mathbf{v}_L) - D'\hat{\mathbf{z}}\times(\mathbf{v}_{nl}-\mathbf{v}_L) . \qquad (10.1)$$

Here $\mathbf{v}_{nl}$ is the local normal velocity in the vicinity of the vortex. There are relations connecting D and D' with the effective cross sections for the transfer of momentum longitudinal and transverse to the relative velocity $\mathbf{v}_{nl}-\mathbf{v}_L$ (Hall and Vinen, 1956; Lifshitz and Pitaevskii, 1957). Like any force acting upon the vortex, the force $\mathbf{F}_n$ should be balanced by the Magnus force:

$$\rho_s \boldsymbol{\kappa}\times(\mathbf{v}_{sl}-\mathbf{v}_L) = -\mathbf{F}_n . \qquad (10.2)$$

We consider here forces per unit length of one vortex instead of per unit volume. Therefore the circulation vector $\boldsymbol{\kappa}$ replaces the vorticity in similar relations in the previous sections. In order to have a relation connecting $\mathbf{v}_{sl}$, $\mathbf{v}_n$, and $\mathbf{v}_L$ like Eq. (7.26), we should find the relation between the local velocity $\mathbf{v}_{nl}$ and the average velocity $\mathbf{v}_n$. The latter may be assumed with high accuracy to be equal to the normal velocity far from the vortex, since the vortex disturbs the fluid only close to itself. The difference $\mathbf{v}_{nl}-\mathbf{v}_n$ is due to viscous drag by the force $\mathbf{F}_n$ (Hall and Vinen, 1956):

$$\mathbf{v}_{nl}-\mathbf{v}_n = \mathbf{F}_n/E, \quad E = \frac{4\pi\rho_n\nu}{\ln(r_u/r_l)} . \qquad (10.3)$$

This equation results from a consideration of the motion of the cylinder through the viscous fluid (the two-dimensional Stokes problem; see Batchelor, 1970). The uniform-velocity motion of the cylinder produces a logarithmically divergent velocity field. The upper cutoff r_u in the logarithm argument is chosen equal to the smallest of four lengths: The viscous penetration depth $\sim\sqrt{\nu/\omega}$, the Eckman-layer width $\sim\sqrt{\nu/\Omega}$, the intervortex distance $\sim\sqrt{\kappa/\Omega}$, or the length $\sim\nu/|\mathbf{v}_{nl}-\mathbf{v}_n|$ (this last is relevant for nonlinear problems). The lower cutoff r_l is usually of the order of the free path length (Hall and Vinen, 1956). A more detailed discussion of r_l is given by Sonin (1975).

From Eqs. (10.1)—(10.3) one can obtain the equation of motion for vortices, Eq. (7.26), with B and B' given by the complex formula

$$B - jB' = \frac{2\rho}{\kappa\rho_n\rho_s}\left[\frac{1}{E}+\frac{1}{D+jD'}-\frac{1}{j\rho_s\kappa}\right]^{-1} . \qquad (10.4)$$

Thus the mutual friction problem is reduced to determination of the scattering cross section of quasiparticles.

Quasiparticles are scattered by the velocity field outside the core and by the nonhomogeneous field of the order parameter inside the core. Only the first of these can be calculated more or less rigorously, and we discuss the theory ignoring the vortex core at first. Scattering of rotons may be described within the quasiclassical scattering theory (Lifshitz and Pitaevskii, 1957), and for phonons the Born approximation has been shown to be accurate enough (Pitaevskii, 1958).

But the scattering theory for quasiparticles in a vortex velocity field turned out to be quite unusual due to the slow decrease of the velocity far from the vortex ($\sim 1/r$). This decrease causes a singularity of the scattering amplitude at small angles. Studying carefully the small-angle behavior of the scattering amplitude for phonons, Iordanskii (1964, 1965) discovered a force transverse to the relative velocity $\mathbf{v}_{nl}-\mathbf{v}_L$, which had been overlooked in the previous Born-approximation calculations. Since then much confusion has arisen about the need for, and the interpretation of, the Iordanskii force. It has been argued that the Iordanskii force should always be added to the transverse force obtained from the cross section. If the cross section is determined by D and D', then one has to write $D'-\kappa\rho_n$ instead of D' in Eq. (10.1) (Barenghi *et al.*, 1983).

The source of disagreement was the contribution of quasiparticles with large impact parameters (responsible for the small-angle scattering) to the momentum balance in the region around the vortex. This contribution was analyzed by Sonin (1975) on the basis of the collisionless Boltzmann equation. It was shown that the effective cross section for the transverse force is an ambiguous concept due to this contribution. In various approaches the cross section does or does not include the term responsible for the Iordanskii force. In particular, it depends on the shape of the region where the momentum balance is considered. Therefore one should be careful not to take the Iordanskii force into account twice, as was done for rotons in earlier papers. In order to avoid ambiguity it is helpful (but this is a matter of taste and convention) to assume that D' in Eq. (10.1) corresponds to the net transverse force. Then the result of Sonin (1975) is as follows. The net transverse force is given by $D'=-\kappa\rho_n$ in Eq. (10.1), independently of whether the quasiparticle gas consists of rotons or of phonons. This statement is in agreement with the transverse force for rotons calculated within the quasiclassical scattering theory of Lifshitz and Pitaevskii (1957; but after correction of a wrong sign of D' in their paper).

The derivation of the mutual friction force from kinetic theory has not put an end to the disagreements. Sometime later Hillel (1981) and Hillel and Vinen (1983) returned to this problem. According to their calculations, $D'=-\frac{3}{2}\kappa\rho_n$ [in their notation $D'=-\frac{1}{2}\kappa\rho_n$, which corresponds to $D'-\kappa\rho_n$ instead of D' in Eq. (10.1)]. Their interpretation of the Iordanskii force may be summarized as follows (Hillel *et al.*, 1974). They emphasize a difference between the momentum density of quasiparticles (rotons), $\rho_n(\mathbf{v}_n-\mathbf{v}_s)$, and the momentum density of the normal fluid, $\rho_n\mathbf{v}_n$. The force due to quasiparticle scattering is the force exerted by the vortex on quasiparticles, but the Magnus force must be balanced by the force on the normal fluid rather than by that on the quasiparticle fluid. So the force-balance equation must take into account that a momentum with density $-\rho_n\mathbf{v}_s$ (the difference between quasiparticle and normal-fluid momentum) is convected past the vortex line with a relative velocity $\mathbf{v}_{nl}-\mathbf{v}_L$. It contributes the Iordanskii force to the force-balance equation. Such an interpretation is illustrative and helps us to understand qualitatively the physical origin of the Iordanskii force, but it does not provide a quantitative solution. The crux of the problem is to trace carefully how and where the convection contribution appears in the calculations. In fact, the calculation of Hillel (1981) suffered from the same deficiency as some previous calculations: a certain contribution is taken into account twice. Hillel calculated the effective scattering cross section by expanding his integration over all impact parameters. At the same time, only those quasiparticles passing by far from the vortex contribute to the Iordanskii force, which he added to the scattering force obtained from the effective cross section. In general, it is dangerous to study mutual friction starting from some intuitive definition of forces acting upon the vortex, however plausible they may seem. A more careful approach is *first* to derive rigorously some balance equation and *then* to label terms that enter this equation as such-and-such a force. This precaution is vital also for the mutual-friction problem in ^{3}He-A, considered in Sec. X.D.

A more detailed discussion of the Iordanskii force requires deeper involvement in the kinetic theory; perhaps this will be presented elsewhere. It is worth mentioning that the mutual friction force has also been calculated for pure type-II superconductors (Gal'perin and Sonin, 1976; Kopnin and Kravtzov, 1976). These calculations were based on the Boltzmann equation, which is derived from microscopic theory (Aronov *et al.*, 1981). When the energy of the quasiparticles is close to the energy gap of the superconductor, the quasiparticle spectrum is identical to the roton spectrum, and the quasiclassical theory should yield the same mutual friction force as that for rotons. Indeed, the above mentioned calculations agree with the results $D'=-\kappa\rho_n$. Later we shall return to these calculations in our discussion of mutual friction in ^{3}He-B (Sec. X.D).

All calculations yield a transverse force that exceeds the longitudinal force due to scattering outside of the core. For phonons the ratio $D/|D'|$ is of order the ratio of the core radius to the phonon wavelength. For rotons the longitudinal force determined in the quasiclassical theory, in the approximation of the large logarithm, is given by

$$D=\frac{\kappa\rho_n}{\pi\sqrt{2\pi}}\frac{\sqrt{\mu kT}}{p_0}\left[\ln\frac{p_0}{\sqrt{\mu kT}}\right]^3 \simeq 2.6\frac{\sqrt{\mu kT}}{p_0}\kappa\rho_n\ . \tag{10.5}$$

Here p_0 is the momentum of the roton minimum and μ is the roton mass. This formula is readily derived from the roton cross section given by Sonin (1975) and Hillel (1981). Lifshitz and Pitaevskii (1957) gave a formula with a factor 1.2 instead of 2.6. Thus for rotons the ratio $D/|D'|$ is $\sim\sqrt{\mu kT/p_0}$ and very small. Relying on these estimates Sonin (1975, 1976) used the approximation $D'=-\kappa\rho_n$, $D\simeq 0$, which yielded after substitution in Eq. (10.4)

$$B-jB'=2\left[j+\frac{\kappa\rho_n\rho_s}{\rho E}\right]^{-1}\ . \tag{10.6}$$

In this approximation it follows from Eqs. (10.1) and (10.2) that the vortex moves with the local center-of-mass velocity

$$\mathbf{v}_L=\frac{\rho_s}{\rho}\mathbf{v}_{sl}+\frac{\rho_n}{\rho}\mathbf{v}_{nl}\ , \tag{10.7}$$

and the entire dissipation is associated with viscous losses.

Comparison of this theory with the experiments is not very conclusive, since the experiments have been carried out at temperatures too high for the theory to give exact quantitative predictions. In Fig. 5 the experimental temperature dependence of the transverse-force parameter D' is reproduced from Fig. 9A of Barenghi *et al.* (1983; D' corresponds to D_t of Barenghi *et al.*). The theoretical values $D'=-\kappa\rho_n$ are shown by the dashed line. They are 2—3 times smaller than the experimental values at the lowest temperatures for which comparison is possible. Probably the agreement would be better at much lower temperatures.

The disagreement between theory and experiment for the longitudinal-force parameter is also considerable (Barenghi *et al.*, 1983; Hillel and Vinen, 1983). This has been attributed to the contribution of the core to mutual friction. It is difficult to estimate this contribution rigorously because of the lack of a reliable theory to describe the vortex core in He II (excepting for a narrow critical region discussed in the following section). In order to describe the core structure, various speculative models have been proposed (see Sec. III.A and the review by Barenghi *et al.*, 1983). The observed longitudinal

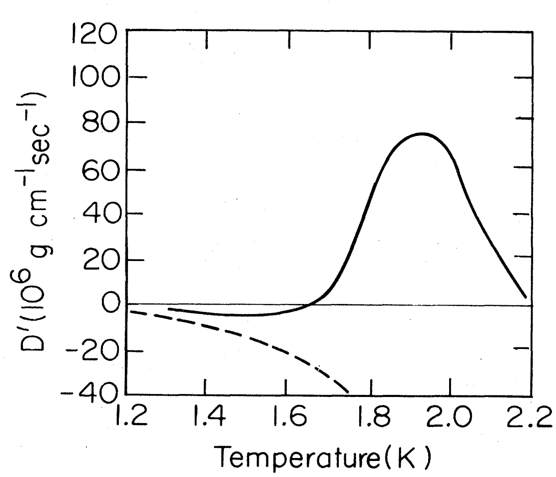

FIG. 5. Dependence of the mutual friction parameter D' on the temperature. The solid line was obtained by Barenghi *et al.* (1983) by fitting the experimental data. The dashed line shows the theoretical values $D'=-\kappa\rho_n$.

force is explained if the core absorbs all rotons falling on it and the collision diameter of the core coincides with its diameter. The concept of the absorbing core was suggested by Lifshitz and Pitaevskii (1957) and was investigated recently by Hillel (1981) and Hillel and Vinen (1983), who found that the core radius, deduced from the experiment according to this concept, is in reasonable agreement with some models of core structure.

Mathieu and Simon (1980) proposed a theoretical model that is in close agreement with experimental data in the temperature range 1.7—2.1 K, just where the other theories discussed here are unsuccessful. Their arguments in a slightly modified and simplified form are as follows. For stationary motion of the vortex in a completely incompressible fluid (ρ=const, S=const) the continuity equations for the mass and the entropy, Eqs. (7.1) and (7.2), yield (dissipation is ignored)

$$\nabla\cdot[\rho_s(\mathbf{v}_s-\mathbf{v}_n)]=0 . \tag{10.8}$$

Mathieu and Simon suggested that the velocity fields are transported by the vortex without deformation; then

$$\nabla\rho_s\cdot(\mathbf{v}_s-\mathbf{v}_n)=0 , \tag{10.9}$$

and, since in the core $\nabla\rho_s\neq 0$, the superfluid and the normal velocity in the core are equal to each other, i.e., $\mathbf{v}_{sl}=\mathbf{v}_{nl}$. Using this relation and Eqs. (10.2)—(10.4) we obtain

$$B=\frac{2\rho E}{\kappa\rho_n\rho_s},\quad B'=0 . \tag{10.10}$$

The model of Mathieu and Simon relies on not quite rigorous conjectures, but its good agreement with experiment is a strong argument in its favor and raises the hope that a more rigorous justification of their model may be found.

Putterman (1974, Sec. 32) proposed a purely hydrodynamic theory of mutual friction that relates the mutual friction dissipation with the second viscosity defined by the coefficient ζ_3. He neglected the spatial variation of the relative velocity $\mathbf{v}_s-\mathbf{v}_n$ and substituted $\nabla\cdot[\rho_s(\mathbf{v}_s-\mathbf{v}_n)]=\nabla\rho_s\cdot(\mathbf{v}_s-\mathbf{v}_n)$ into the second-viscosity term $\propto\zeta_3$ in the dissipation function. But we saw that this term vanishes in a completely incompressible fluid, which is a good approximation for a superfluid far from the critical region. Therefore the first, but not the second, viscosity is responsible for dissipation. In the critical region considered in the following section, thermal compressibility becomes important, and the second viscosity contributes to mutual friction. But there the conjecture of Putterman that $\mathbf{v}_s-\mathbf{v}_n$=const is invalid, and the second-viscosity contribution differs from that calculated by Putterman.

C. Mutual friction in He II near the λ point

The mutual friction coefficients B and B' close to the λ point were measured by Mathieu *et al.* (1976). They investigated the second sound in rotating He II and showed that when T approaches T_λ the coefficients B and B' diverge as $(T_\lambda-T)^{-1/3}$. Such critical behavior was explained by Pitaevskii (1977) on the basis of the dynamic scaling hypothesis, supposing that relaxation of the order parameter modulus was the principle energy-dissipation mechanism. In addition to this, Pitaevskii conjectured that both B and B' diverge at $T\to T_\lambda$ with the same critical exponent. He found that this exponent is close to $-\frac{1}{3}$, in agreement with the experiment. But no conclusions concerning the magnitudes of B and B' were drawn.

Near the λ point the vortex-core radius becomes larger than all other relevant lengths, and one may expect that some phenomenological theory similar to the time-dependent Ginzburg-Landau theory is applicable. Indeed, this theory has already been used for description of vortex motion in type-II superconductors (Gor'kov and Kopnin, 1975). But because of the more important role of critical fluctuations in He II, an analogous theory for He II (the Ginzburg-Pitaevskii theory, see Ginzburg and Pitaevskii, 1958) cannot pretend to a quantitative description. A modification of this theory, however, has been proposed. Its parameters are renormalized to be nonanalytical functions of $T_\lambda-T$ fitted to experimental data and scaling laws. The resultant theory, called the phenomenological Ψ theory, was considered in the reviews of Ginzburg and Sobyanin (1976, 1982). Within the scope of this theory the coefficients B and B' in the critical region have been calculated (Sonin, 1981b). Results of these calculations are better presented in the parameters g and g' connected with B and B' by the complex expression

$$B+jB'=\frac{2}{g+jg'} . \tag{10.11}$$

The calculated g and g' are given by

$$\begin{aligned} g&=\frac{1}{4\Lambda\rho_s}\int_0^\infty\frac{r\,dr}{\rho_s(r)}\left[\left[\frac{\partial\rho_s(r)}{\partial r}\right]^2+\left[\frac{\Lambda\rho_s}{S}\frac{\partial S(r)}{\partial r}\right]^2\right]\\ &\approx\left[\frac{T_\lambda-T}{T_\lambda}\right]^{1/3},\\ g'&=\frac{\Delta C}{\rho_s S}\frac{T_\lambda-T}{T_\lambda}\approx 1.5\left[\frac{T_\lambda-T}{T_\lambda}\right]^{1/3} . \end{aligned} \tag{10.12}$$

Here ΔC is the specific-heat discontinuity on the λ line, while Λ is the relaxation parameter in the Ψ theory. The entropy $S(r)$ and the superfluid density $\rho_s(r)$ in the ground state of the vortex are functions of the distance r from the vortex line, and S and ρ_s are values of $S(r)$ and $\rho_s(r)$ at $r\to\infty$.

The expression for the dissipation parameter g consists of two terms. The first is due to relaxation of the order-parameter modulus. Just this process was considered by Pitaevskii (1977), and his order estimate is in agreement with our quantitative one. The second term in g is connected with relaxation of the order-parameter phase. This relaxation process is responsible for the second viscosity given by the coefficient $\zeta_3=\hbar\Lambda/2M\rho_s$. Thus the second term in g may be interpreted as the contribution of the second viscosity to the mutual friction, proportional to the coefficient ζ_3.

Comparing theoretical g and g' with the experimental values of Mathieu *et al.* (1976), which are

$$g=2.8\left[\frac{T_\lambda-T}{T_\lambda}\right]^{1/3},\quad g'=2\left[\frac{T_\lambda-T}{T_\lambda}\right]^{1/3},\qquad(10.13)$$

shows that the agreement for g' is better than that for g. Indeed, it is harder to calculate g than g'; one must know the relaxation parameter Λ and the spatial distribution of S and ρ_s.

Divergence of B and B' in the critical region means that, according to Eq. (7.26), vortices move much faster than the superfluid and the normal fluid when thermal counterflow takes place. Hence it is expected that vortex pinning strongly affects the counterflow. This is discussed in more detail by Sonin (1981b).

The theory of mutual friction near the λ point has also been developed by Onuki (1983a). He used the phenomenological theory, which is a generalization of Hohenberg and Halperin's (1977) model F. This theory is equivalent to the Ψ theory except that Onuki assumed the relaxation parameter Λ to be complex. In addition, Onuki took into account corrections due to the thermal conductivity, which had been ignored by Sonin (1981b). Without the thermal-conductivity corrections, and for real Λ, the resultant formulas of Onuki coincide with the expressions for g and g' in Eq. (10.12), so his theory agrees with that of Pitaevskii (1977) and Sonin (1981b).[10] The numerical estimate by Onuki,

$$g=0.62\Lambda^{-1},\quad g'=0.58\Lambda^{-1},\qquad(10.14)$$

after substitution of $\Lambda=0.3[(T_\lambda-T)/T]^{-1/3}$ (given by Ginzburg and Sobyanin, 1982), agrees better with experiment for the value of g. The corrections due to thermal conductivity are not large, according to Onuki, so the numerical difference between values of g obtained by Sonin (1981b) and Onuki (1983a) probably arises as a result of a more careful calculation of the distribution of S and ρ_s in the core by Onuki. In all, the theory provides a quite satisfactory explanation of experimental data in the critical region, in particular, the negative sign of B'.

The methods used by Sonin (1981b) and Onuki (1983a) to obtain the equation of motion for vortices are also similar. They are based on the absence of secular terms in the equations describing the superfluid around the moving vortex. This method was widely used to derive the equations of motion for various solitons, including vortices in type-II superconductors (Gor'kov and Kopnin, 1975). The idea of the method will be made clear in the following section, where it is employed to derive the equation of motion of the nonsingular vortex in ^{3}He-A. Another way to derive equations of motion for solitons is based on the variational principle. It was studied in the papers of Kawasaki (1983, 1984) and Ohta *et al.* (1984) for various types of topological defects, including superfluid vortices in the critical region.

Onuki (1983b) has extended the theory of mutual friction in the critical region to ^{3}He-^{4}He mixtures. He found that B and B' do not diverge on the critical line, even at small concentrations of ^{3}He atoms. In conclusion we mention the experiments of Mathieu *et al.* (1982), who measured the coefficient B as a function of pressure and temperature. They found that at high pressures the critical behavior of B is different from that observed at lower pressures and predicted by theory. The experimental critical exponent at $P=25$ bars was -0.11, that is, three times smaller than the theoretical value $-\frac{1}{3}$.

D. Mutual friction in superfluid ^{3}He

The theoretical approach to mutual friction in ^{3}He, like that in He II, depends on the ratio of the free path length of quasiparticles to the size of the region responsible for mutual friction. In ^{3}He-B the vorticity is concentrated along the vortex lines, as a rule at distances small compared to the free path length, so that one can apply the scattering theory of noninteracting quasiparticles. No such calculations have as yet been performed; nevertheless, one is tempted to risk using the results of calculations for pure type-II superconductors already discussed in Sec. B. This looks like a reasonable procedure if mutual friction occurs outside the core and is described by the quasiclassical theory, which is sensitive only to the energy spectrum of quasiparticles, but the energy spectra in ^{3}He-B and in the superconductors do not differ. The values of D and D' calculated for the superconductors (Gal'perin and Sonin, 1976; Kopnin and Kravtzov, 1976) are

$$D'=-\kappa\rho_n\tanh\frac{\Delta(T)}{kT},\qquad(10.15)$$

$$D=\begin{cases}\kappa\rho_n\dfrac{m}{8\pi p_F^2}\left[\dfrac{kT\Delta(T)}{2\pi}\right]^{1/2}\left[\ln\dfrac{\Delta(T)}{kT}\right]^3, & \Delta(T)\gg kT\,,\\[2ex] \kappa\rho_n\dfrac{m}{p_F^2}\dfrac{\Delta(T)^2}{kT}, & \Delta(T)\ll kT\,.\end{cases}\qquad(10.16)$$

Here $\Delta(T)$ is the energy gap and p_F is the Fermi momentum. When $\Delta\gg kT$ these formulas yield the formulas for rotons (see Sec. X.B) provided that the roton mass and momentum, μ and p_0, correspond to $\Delta m^2/p_F^2$ and p_F in superconductors. Near T_c, when $\Delta\ll kT$, a mutual friction occurs mainly inside the core. The value of D in Eq. (10.16) for this region was calculated using crude conjectures on the order-parameter distribution inside the core. The calculated value is much smaller than

[10] Onuki claimed that his entropy production differed from that of Pitaevskii by the factor $\propto(T_\lambda-T)^{2/3}$. But this is puzzling because Eq. (116) of Onuki (1983a) yields the same entropy production as in the paper of Pitaevskii (1977).

$$D \sim \rho_n r_c \frac{p_F}{m} \sim \kappa \rho_n \frac{p_F^2}{m\Delta} ,$$

corresponding to the core's absorbing all quasiparticles falling on it. The calculated value is also too small to explain the experimental value of B measured by Hall *et al.* (1984) in ^{3}He-B. This is not surprising because, for the core, it is difficult to trust the analogy between ^{3}He-B and the superconductor with S pairing. Though completely unsuccessful quantitatively, the discussed theory shows that near T_c the core is a primary contributor to mutual friction, so mutual friction coefficients should depend on the structure of the vortex core.

The theory of mutual friction in ^{3}He-A for nonsingular vortices was developed by Kopnin (1978) on the basis of the Boltzmann equation close to T_c. But in fact, the results of Kopnin are readily obtained within the hydrodynamic theory without referring to the Boltzmann equation, and their validity is not restricted by the Ginzburg-Landau region near T_c. Here we shall give the hydrodynamic derivation of the equation of vortex motion obtained by Kopnin, but generalized to include the effect of orbital inertia (Sonin, 1986a).

Suppose that a nonsingular vortex moves through the superfluid. The latter is assumed incompressible in the mechanical and the thermal sense (ρ=const, S=const), and the normal fluid is not dragged by the vortex because of high viscosity ($\mathbf{v}_n$ is constant). Then we need only the equation of orbital motion and the equation following from complete incompressibility of the superfluid. It is convenient for further analysis to present these in the following form:

$$L\left[\hat{l}\times\left[\frac{\partial\hat{l}}{\partial t}+(\mathbf{v}_n\cdot\nabla)\hat{l}\right]\right]-\mu\left[\frac{\partial\hat{l}}{\partial t}+(\mathbf{v}_n\cdot\nabla)\hat{l}\right]=\frac{\delta E}{\delta\hat{l}} , \tag{10.17}$$

$$0=\frac{\delta E}{\delta\varphi} . \tag{10.18}$$

Here L is the dynamic angular momentum determining the orbital inertia, and μ is the orbital viscosity. The functional derivatives are given by

$$\frac{\delta E}{\delta\varphi}=-\nabla\left[\frac{\partial E}{\partial\nabla\varphi}\right]=-\frac{\hbar}{M}\nabla\left[\frac{\partial E}{\partial\mathbf{v}_s}\right]=-\frac{\hbar}{M}\nabla\cdot\boldsymbol{\lambda} , \tag{10.19}$$

$$\frac{\delta E}{\delta\hat{l}}=\frac{\partial E}{\partial\hat{l}}-\nabla_i\left[\frac{\partial E}{\partial\nabla_i\hat{l}}\right]+\frac{\partial E}{\partial v_{sj}}\frac{\partial v_{sj}}{\partial\hat{l}} . \tag{10.20}$$

Here it is taken into account that $\hat{l}$ and $v_s=(\hbar/M)\nabla\varphi$ are connected by the Mermin-Ho relation, so we have for small perturbations $\mathbf{v}_s'$ and $\mathbf{l}'$

$$\mathbf{v}_{si}'=\frac{\hbar}{M}\nabla_i\varphi'+\frac{\partial v_{si}'}{\partial\hat{l}}\cdot\mathbf{l}' ,$$
$$\frac{\partial v_{si}}{\partial\hat{l}}=\nabla_i\hat{l}\times\hat{l} . \tag{10.21}$$

Using Eqs. (10.19)—(10.21), one immediately proves that Eq. (10.17) is the usual equation of orbital motion, given, for example, by Hu and Saslov (1977), while Eq. (10.18) is the incompressibility condition analogous to Eq. (10.8).

The vortex moves with the constant velocity $\mathbf{v}_L$, so $\partial\hat{l}/\partial t=-(\mathbf{v}_L\cdot\nabla)\hat{l}$. Linearization of Eqs. (10.17) and (10.18) with respect to the small perturbations produced by vortex motion yields

$$L\hat{l}\times[(\mathbf{v}_n-\mathbf{v}_L)\cdot\nabla]\hat{l}-\mu[(\mathbf{v}_n-\mathbf{v}_L)\cdot\nabla]\hat{l}=\frac{\delta^2 E}{\delta\hat{l}^2}\mathbf{l}'+\frac{\delta^2 E}{\delta\hat{l}\cdot\delta\varphi}\varphi' , \tag{10.22}$$

$$0=\frac{\delta^2 E}{\delta\varphi\delta\hat{l}}\mathbf{l}'+\frac{\delta^2 E}{\delta\varphi^2}\varphi' . \tag{10.23}$$

The symbols $\delta^2E/\delta\hat{l}^2$, $\delta^2E/\delta\hat{l}\cdot\delta\varphi$, $\delta^2E/\delta\varphi\delta\hat{l}$, and $\delta^2E/\delta\varphi^2$ denote here the linear differential operators applied to the perturbations $\mathbf{l}'$ and φ'. We write down in an explicit form those applied to φ':

$$\frac{\delta^2 E}{\delta\hat{l}\delta\varphi}\varphi'=\frac{\hbar}{M}\left[\frac{\partial^2 E}{\partial\hat{l}\partial v_{si}}\nabla_i\varphi'-\nabla_i\left[\frac{\partial^2 E}{\partial\nabla_i\hat{l}\partial v_{sj}}\nabla_j\varphi'\right]+\frac{\partial^2 E}{\partial v_{si}\partial v_{sj}}\frac{\partial v_{si}}{\partial\hat{l}}\nabla_j\varphi'\right] ,$$
$$\frac{\delta^2 E}{\delta\varphi^2}\varphi'=-\left[\frac{\hbar}{M}\right]^2\nabla_i\left[\frac{\partial^2 E}{\partial v_{si}\partial v_{sj}}\nabla_j\varphi'\right] . \tag{10.24}$$

We have a system of two nonhomogeneous linear equations, (10.22) and (10.23), for $\mathbf{l}'(\mathbf{r})$ and $\varphi'(\mathbf{r})$. They have a solution only if the left-hand sides (the nonhomogeneous terms) are orthogonal to any solution of the adjoint homogeneous equations (without the left-hand sides); as a result of translational invariance, the solution of the homogeneous equations is obtained by an arbitrary translation of the solution for the vortex at rest. It is denoted by the superscript t:

$$\varphi^t=(\mathbf{t}\cdot\nabla\varphi)=\frac{M}{\hbar}(\mathbf{t}\cdot\mathbf{v}_s),\quad \mathbf{l}^t=(\mathbf{t}\cdot\nabla)\hat{l} ,$$
$$V_{si}^t=\frac{\hbar}{M}\nabla_i\varphi^t+\frac{\partial v_{si}}{\partial\hat{l}}\mathbf{l}^t=\frac{\hbar}{M}\nabla_i\varphi^t+\frac{\hbar}{M}\mathbf{l}^t\cdot[\nabla_i\mathbf{l}\times\mathbf{l}] . \tag{10.25}$$

Here $\mathbf{t}$ is an arbitrary vector of translation. In order to obtain the solvability condition it is necessary to multiply Eq. (10.22) by $\mathbf{l}^t$ and Eq. (10.23) by φ^t, to integrate both equations over the whole plane, and to sum integrals. Because the linear differential operators are self-adjoint, partial integration reduces the volume integrals of the right-hand sides of Eqs. (10.22) and (10.23) to surface integrals. Following Kopnin (1978), we consider the axisymmetric vortex with $\hat{l}$ parallel to the z axis far from the vortex. Then $\mathbf{l}'$ and $\mathbf{l}^t$ vanish at the surface far from the vortex, and the surface integrals contain only φ' and φ^t. In all, the solvability condition is

$$\int d\mathbf{r}(\mathbf{t}\cdot\nabla)\hat{l}\{L\hat{l}\times[(\mathbf{v}_n-\mathbf{v}_L)\cdot\nabla]\hat{l}-\mu[(\mathbf{v}_n-\mathbf{v}_L)\cdot\nabla]\hat{l}\}$$
$$=-\frac{\hbar}{M}\int dS\, n_i[\varphi^t(\mathbf{r})\lambda_i^r(\mathbf{r})-\varphi^r(\mathbf{r})\lambda_i^t(\mathbf{r})] \ . \quad (10.26)$$

Here n_i are components of the unit vector normal to the surface and directed outside. The currents far from the vortex are

$$\lambda^r(\mathbf{r})=\frac{\delta^2 E}{\delta\mathbf{v}_s\delta v_{sj}}v_{sj}^r=\rho_{s\perp}(\mathbf{v}_{sl}-\mathbf{v}_n) \ ,$$
$$\lambda^t(r)=\frac{\delta^2 E}{\delta\mathbf{v}_s\delta v_{sj}}v_{sj}^t=\rho_{s\perp}\frac{\hbar}{M}\nabla\varphi^t \ , \quad (10.27)$$

where $\rho_{s\perp}$ is the superfluid density in the plane normal to $\hat{l}$. The superfluid velocity $\mathbf{v}_{sl}$ is referred to as "local," though it is a velocity far from the vortex, because it is at distances small compared to other hydrodynamic scales—say, the distance from other vortices or from the wall.

If the solvability condition does not hold and the linear equations (10.22) and (10.23) have no solution, stationary motion of the vortex with given constant $\mathbf{v}_L$ is impossible. Then a solution of the equation of orbital motion in a reference frame moving with velocity $\mathbf{v}_L$ should contain terms growing in time, so-called "secular terms." Thus the condition Eq. (10.26) provides for the absence of secular terms. Since it must hold at any translation $\mathbf{t}$, we obtain, after integration,

$$-\frac{ML}{\hbar}\kappa_v\hat{\mathbf{z}}\times(\mathbf{v}_L-\mathbf{v}_n)+\mu\hat{\gamma}(\mathbf{v}_L-\mathbf{v}_n)=-\kappa_v\hat{\mathbf{z}}\times\lambda^r \ . \quad (10.28)$$

The integral in the orbital inertial term has been reduced to the integral Eq. (9.3) for circulation [the first term on the left-hand side of Eq. (10.28)]. The tensor $\hat{\gamma}$ has components

$$\gamma_{ij}=\int d\mathbf{r}\,\nabla_i\hat{l}\cdot\nabla_j\hat{l}_j \quad (10.29)$$

only in the xy plane, since $\nabla_z\hat{l}=0$. For the axisymmetric vortex under consideration it is reduced to a scalar $\gamma\delta_{ij}$ $(i,j\neq z)$ and

$$\gamma=\pi\int_0^\infty r\,dr\left\{\left[\frac{d\beta}{dr}\right]^2+\sin^2\beta\left[\frac{1}{r^2}+\left[\frac{d\alpha}{dr}\right]^2\right]\right\} \quad (10.30)$$

for $\hat{l}$ texture given by $l_x=\sin\beta(r)\cos[\alpha(r)+\varphi]$, $l_y=\sin\beta(r)\sin[\alpha(r)+\varphi]$, $l_z=\cos\beta(r)$, where r and φ are cylindrical coordinates in the plane. Hall (1985a) estimated $\gamma=\pi^3/2$ or $\pi^2/3$ for different models of an Anderson-Toulouse two-quantum nonsingular vortex.

Recently an equation of vortex motion was derived for a moving vortex lattice of arbitrary symmetry (Sonin and Fomin, 1986). It has the same form as Eq. (10.28), in which λ^r signifies the supercurrent averaged over the vortex structure cell.

Comparing Eq. (7.26) with Eq. (10.28) for the antisymmetric vortex, we see that the mutual friction coefficients are given by the complex expression

$$B-jB'=\frac{2\rho}{\rho_{n\perp}}\left[\frac{\kappa_v\rho_{s\perp}}{\mu\gamma-j\kappa_v LM/\hbar}-j\right] \ . \quad (10.31)$$

At $L=0$, Eq. (10.28) yields the equation obtained by Kopnin (1978) for the axisymmetric vortex. Cross (1983) considered vortex motion also neglecting orbital inertia. He estimated the dissipation function for a moving vortex and was able to calculated only the dissipative component of the mutual friction force. He obtained the same value of B as Kopnin.

The explicit expressions for B and B' following from Eq. (10.31) are given by

$$B=\frac{B_K}{1+\left[\frac{\rho_{n\perp}}{2\rho\rho_{s\perp}}\frac{LM}{\hbar}B_K\right]^2} \ ,$$
$$B'=\frac{2\rho}{\rho_{n\perp}}-\frac{\frac{\rho_{n\perp}}{2\rho\rho_{s\perp}}\frac{LM}{\hbar}B_k^2}{1+\left[\frac{\rho_{n\perp}}{2\rho\rho_{s\perp}}\frac{LM}{\hbar}B_K\right]^2} \ . \quad (10.32)$$

Here

$$B_K=\frac{2\rho\rho_{s\perp}}{\rho_{n\perp}}\frac{\kappa_v}{\gamma\mu}\simeq\frac{7}{\gamma}\left[1-\frac{T}{T_c}\right]^{-1/2}$$

is the value of B in the theory of Kopnin. Equation (10.32) shows that the orbital inertia term $\propto L$ contributes to the active and reactive components of the mutual friction force and changes their critical behavior near T_c. Indeed, Kopnin's theory $(L=0)$ predicts that $B\propto(1-T/T_c)^{-1/2}$ and $B'\to 2$, but assuming $L\propto\rho_s$ the critical behavior is $B\propto(1-T/T_c)^{+1/2}$ and $B'\to 2(1-\rho_s\hbar/ML)$. If the absolute value of L is very small, the vicinity of T_c where L is important may be very narrow. Thus a mutual friction measurement may give evidence of the existence of an intrinsic angular momentum, as was pointed out first by Hall (1985a). Hall revealed a discrepancy[11] between values of B measured in ^{3}He-A and those obtained in the theory with $L=0$. In order to explain the effect of the intrinsic momentum on mutual friction, he derived the mutual friction force through the torque due to variation of the intrinsic momentum determined by the

[11]Mineev (1986) estimated this discrepancy differently from Hall (1985a). It is not simple, however, to estimate reliably an absolute value of B in theory because of the texture-dependent factor γ, which is not well known for experimental conditions. It seems, therefore, that there is a clear discrepancy in temperature dependences of theoretical and experimental values B, but not necessarily in their absolute values. In the experiment the growth of B at $T\to T_c$ is slower than Kopnin's theory predicts, and up to now the only explanation for such behavior is due to the intrinsic angular momentum, as discussed further in the text.

conservation law. It is not clear that just this torque is responsible for mutual friction (see further discussion by Liu, 1985, and Hall, 1985b). On the other hand, in the hydrodynamic derivation given above we did not resort to some *a priori* determination of the force. The derivation was based on the absence of secular terms. This condition was intended to point out what weight function one should use in averaging the equation of orbital motion in order to obtain the correct equation of vortex motion. The resultant equation, Eq. (10.28), takes the form of the force-balance equation as usual, but one can see once more why it is dangerous to rely on some *a priori* conception of forces related to the vortex, as was pointed out in the discussion of the Iordanskii force in Sec. X.B. The forces on the left-hand side of Eq. (10.28) are proportional to $\mathbf{v}_L-\mathbf{v}_n$ and look as if they were components of the force exerted on the normal fluid by the vortex (orbital inertia is responsible for the transverse component). They are balanced, however, not by the Magnus force $\propto(\mathbf{v}_L-\mathbf{v}_n)$ as analogous forces in Hall's analysis, but by the force $\propto(\mathbf{v}_s-\mathbf{v}_n)$. It is worthy of mention that a similar force arises in the force-balance equation obtained by averaging of time-dependent equations for the order parameter in the analysis of mutual friction in He II close to the λ point (see Sec. X.C and Sonin, 1981b).

Despite different ways of reasoning in Hall's theory and the theory presented above, both yield similar expressions for B and B' [see Eq. (10.32)], but the angular momentum density L is of different physical origin in the two theories. As was discussed in Sec. IX.A, there is more than one way to introduce the intrinsic angular momentum into the theory. One obtains Hall's result by assuming that in Eq. (10.32) $L=(\lambda-\rho_L+\rho_s)\hbar/M$ in terms of Hall's effective densities ρ_L and λ. The first density determines the angular momentum that Hall and Hook (1985) have introduced into the hydrodynamics by combining some gradient terms in the expression for the supercurrent, and λ determines orbital inertia. Thus the result of the analysis given in this section corresponds to $L=\lambda\hbar/M$ in Hall's notation. Very often orbital inertia has been neglected (as it was by both Kopnin, 1978, and Hall, 1985a) on the grounds that the microscopic theory predicts it to be of rather small magnitude (see discussion in Sec. IX.A and the Appendix). When we neglect orbital inertia, however, our hydrodynamic theory of mutual friction is not able to explain deviations of the measured B from Kopnin's theory. One may consider this fact as a clue that orbital inertia is not so small. But orbital relaxation experiments by Paulson *et al.* (1976) did not reveal any traces of orbital inertia. They claimed the absence of any tendency to oscillation in the process of orbital relaxation, which means $L \ll \mu$. In sum, further efforts are necessary to obtain the answer to this interesting and important experimental problem. Measurements as close as possible to T_c are especially valuable, as well as experimental data on B'. Hall and Hook (1985) suggested that a fourth-sound analog of the second-sound experiment in ^{4}He (Mathieu *et al.*, 1976) be performed. But one may expect that the coefficients B and B' in such experiments would be strongly frequency dependent. In ^{4}He, mutual friction occurs in a small region around the vortex core, and dependence of B and B' on frequency is not significant (aside from a weak logarithmic dependence) until the sound wavelength is large in comparison with the size of the mutual friction region. On the other hand, the size of such a region for nonsingular vorticity in ^{3}He-A is of order of the intervortex distance, and B and B' may depend on frequency beginning from the lowest frequencies. So it is not certain that a fourth-sound experiment would yield values of B and B' like those predicted by the theory of this section.

E. Mutual friction in neutron stars

As was said above (Sec. VIII.F), relaxation after glitches in pulsars has been explained as mutual friction between electrons and vortices in the superfluid neutron matter. Only the dissipative coefficient B is necessary for determination of the relaxation time $\tau_c=(B\Omega)^{-1}$. This time has been calculated by solving the linearized Boltzmann transport equation for noninteracting electrons scattered by vortex cores, an approach similar to that of Hall and Vinen (1956). Feibelman (1971) considered scattering of electrons by magnetic moments of neutron quasiparticles in vortex cores. The neutron superfluid was assumed to be in the S-wave state. He found that the relaxation time depends exponentially on $\Delta^2/\epsilon_F kT$, where Δ is the energy gap in the quasiparticle spectrum and ϵ_F is the Fermi energy. Sauls *et al.* (1982) considered neutron superfluidity in the $^3P_2$ state, where the structure of the vortex core is quite complicated and where the vortices have a spontaneous magnetization. Spontaneous magnetization has also been discovered in some types of vortices in ^{3}He-B (Hakonen *et al.*, 1983). Sauls *et al.* (1982) found electron scattering not only from neutron excitations in the vortex core, as suggested by Feibelman (1971) for S-wave vortices, but also from the vortex-core magnetization. The latter mechanism is especially important for low temperatures, when core-excitation scattering vanishes. But both mechanisms yield quite small values of $B<10^{-4}$ (see Table 1 of Sauls *et al.*, 1982) compared to those experimentally measured in ^{3}He and even in ^{4}He.

XI. CONCLUSION

This paper has reviewed the theory and the latest experimental results on dynamics of the vortex lattice in various rotating superfluids. There now exists a theory capable of describing all vortex-oscillation modes in rotating superfluids, including the effects of sheer rigidity of the lattice and effects of boundaries. The basic concepts of the theory have been tested by experiment, though further efforts at the observation of Tkachenko waves in He II would be desirable. It would be interesting also to have experimental evidence for the existence of surface vortex-oscillation modes, discussed in this review.

The theoretical and experimental study of vortex dynamics in the superfluid phases of ^{3}He is still in its early stages. Up to now vortex structures in rotating ^{3}He have been probed mostly by the rf NMR technique. This was possible due to the remarkable magnetic properties of superfluid ^{3}He. The experimental study of phenomena discussed in the present review requires ultra-low-frequency hydrodynamic measurements successfully used in He II. The application of these methods to new superfluids would be a good supplement to NMR methods and doubtless would provide valuable information on vortex structures, as we endeavored to show in this paper.

The properties of superfluid ^{3}He also permit phenomena based on the interaction of hydrodynamic and magnetic degrees of freedom—say, generation of mechanical oscillation by alternating magnetic fields and vice versa. But discussion of these interesting effects remains beyond the scope of the present review, devoted to purely hydrodynamic analysis.

Note added in proof. Recently the review of Glaberson and Donnelly (1985) became available to me. This review extensively deals with vortex dynamics in He II and is important for studying the subject.

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APPENDIX: THE ZERO-TEMPERATURE HYDRODYNAMICS OF ^{3}HE-A VERSUS THE MICROSCOPIC THEORY (THE DYNAMIC ANGULAR MOMENTUM AND THE SUPERFLUID CURRENT)

The derivation of ^{3}He-A hydrodynamics from the BCS theory (Cross, 1975, 1977; Volovik, 1975) was based on the quasiclassical gradient expansion for the energy gap in the quasiparticle spectrum $\Delta(\mathbf{k})$. We shall intervene in this derivation at the last stage, when one arrives at the linear equations for the complex gap $\Delta(\mathbf{k}) = \widetilde{\Delta}(\mathbf{k})\exp[i\widetilde{\varphi}(\mathbf{k})]$, which is determined on the Fermi surface and depends on the direction of the wave vector $\mathbf{k}(k=k_F)$. Following Cross (1977), one can introduce instead of $\widetilde{\Delta}(\mathbf{k})$ another variable, the density $\widetilde{\rho}(\mathbf{k})$ of particles with given direction $\mathbf{k}$. The total density is the surface integral over the Fermi surface:

$$\rho(\mathbf{R}) = \int_{k=k_F} d\mathbf{k}\,\widetilde{\rho}(\mathbf{R},\mathbf{k}) \ . \tag{A1}$$

For processes slow in space and time, the equations of the linear theory take the form of the Hamiltonian equations for pairs of conjugate $\mathbf{k}$-dependent variables $\widetilde{\rho}(\mathbf{k})$ and $\widetilde{\varphi}(\mathbf{k})$ (Cross, 1977):

$$\begin{aligned} \frac{\hbar}{M}\frac{\partial\widetilde{\varphi}(\mathbf{k})}{\partial t} &= -\frac{\delta H}{\delta\widetilde{\rho}(\mathbf{k})} \ , \\ \frac{\hbar}{M}\frac{\partial\widetilde{\rho}(\mathbf{k})}{\partial t} &= \frac{\delta H}{\delta\widetilde{\varphi}(\mathbf{k})} \ . \end{aligned} \tag{A2}$$

On the right-hand sides of these equations are functional derivatives of the Hamiltonian $H\{\widetilde{\rho}(\mathbf{R},\mathbf{k}),\widetilde{\varphi}(\mathbf{R},\mathbf{k})\}$, which is a functional of $\widetilde{\rho}$ and $\widetilde{\varphi}$, the functions of the space-position vector $\mathbf{R}$ and the wave vector $\mathbf{k}$. These equations can be obtained from the Lagrangian

$$\mathscr{L} = -\frac{\hbar}{M}\int d\mathbf{R}\int_{k=k_F} d\mathbf{k}\,\widetilde{\rho}(\mathbf{k})\frac{\partial\widetilde{\varphi}(\mathbf{k})}{\partial t} - H \ . \tag{A3}$$

Equations (A2) are general enough to be applied not only to the A phase. In the case of the A phase the energy H has a minimum when the $\mathbf{k}$-dependent phase $\widetilde{\varphi}(\mathbf{k})$ is given by

$$\widetilde{\varphi}(\mathbf{k}) = \arctan\frac{\mathbf{\Delta}_2\cdot\mathbf{k}}{\mathbf{\Delta}_1\cdot\mathbf{k}} \ . \tag{A4}$$

Here $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$ are two orthogonal vectors, $\Delta_1=\Delta_2$, which determine hydrodynamical variables: the unit orbital vector $\hat{l}=\mathbf{\Delta}_2\times\mathbf{\Delta}_2/\Delta_1^2$ and the global phase φ. The latter may be defined only in its variations $\delta\varphi$. The variation of the $\mathbf{k}$-dependent phase is given by

$$\delta\widetilde{\varphi}(\mathbf{k}) = \delta\varphi - (\mathbf{k}\cdot\mathbf{l})\frac{\hat{l}\times\mathbf{k}}{(\hat{l}\times\mathbf{k})^2}\delta\hat{l} \ . \tag{A5}$$

In contrast to the global phase φ, the $\mathbf{k}$-dependent phase $\widetilde{\varphi}(\mathbf{k})$ is well defined in the interval $(0,2\pi)$ everywhere on the Fermi surface excepting at boojums (the points in which $\mathbf{k}$ is parallel to $\hat{l}$ or $-\hat{l}$), but it is not single valued. One can see from Eq. (A4) that the boojums are located on the vortex line in $\mathbf{k}$ space, and the change of $\widetilde{\varphi}$ around it is 2π.

One of the possible ways to derive hydrodynamics from the semimicroscopic equations of motion given by Eq. (A2) is to introduce the hydrodynamic variables ρ, φ, and $\hat{l}$ directly into the Lagrangian [Eq. (A3)], assuming that $\widetilde{\rho}(\mathbf{R},\mathbf{k})$ and $\widetilde{\varphi}(\mathbf{R},\mathbf{k})$ depend on $\mathbf{k}$ as in equilibrium (Sonin, 1986b). This yields the hydrodynamic Lagrangian

$$\mathscr{L} = -\frac{\hbar}{M}\int d\mathbf{R}\,\rho(\mathbf{R})\frac{\partial\varphi(\mathbf{R})}{\partial t} - H\{\rho,\nabla\varphi,\hat{l}\} \ . \tag{A6}$$

In deriving hydrodynamic equations from this Lagrangian, one should remember that variation of $\hat{l}$ in the Lagrangian also involves variation of $\partial\varphi/\partial t$ and $\nabla\varphi$ because of the generalized Mermin-Ho relation:

$$\delta_2\delta_1\varphi - \delta_1\delta_2\varphi = \hat{l}\cdot(\delta_2\hat{l}\times\delta_1\hat{l}) \ . \tag{A7}$$

Here δ_1 and δ_2 are arbitrary variations of variables. Assuming $\delta_2 \to \delta$ and $\delta_1 \to \partial/\partial t$ or ∇, one has

$$\delta\frac{\partial\varphi}{\partial t}=\frac{\partial}{\partial t}\delta\varphi+\delta\hat{l}\cdot\left[\frac{\partial\hat{l}}{\partial t}\times\hat{l}\right],$$
$$\delta(\nabla_i\varphi)=\nabla_i\delta\varphi+\delta\hat{l}(\nabla_i\hat{l}\times\hat{l})\,. \tag{A8}$$

Variation of the hydrodynamic Lagrangian with respect to $\hat{l}$ yields

$$\frac{\hbar}{M}\rho\left[\frac{\partial\hat{l}}{\partial t}\times\hat{l}\right]=\frac{\delta H}{\delta\hat{l}} \tag{A9}$$

with

$$\frac{\delta H}{\delta\hat{l}}=\frac{\partial H}{\partial\hat{l}}-\nabla_i\left[\frac{\partial H}{\partial\nabla_i\hat{l}}\right]+\frac{\partial H}{\partial\nabla_i\varphi}(\nabla_i\hat{l}\times\hat{l})\,. \tag{A10}$$

Comparing Eq. (A9) with Eq. (10.17), one can see that Eq. (A9) is the equation of orbital motion at $T=0$ without orbital viscosity ($\mu=0$) and with large dynamic angular momentum $L=\rho\hbar/M$.

But Cross (1977) derived the equation of orbital motion otherwise. We shall trace his derivation with slight modifications to allow generalization to the nonlinear case. It is possible to obtain the relation between the functional derivative, $\delta H/\delta\hat{l}$, given by Eq. (A10) and the time derivatives of hydrodynamic variables using the equations of motion Eq. (A2) and equilibrium relations connecting hydrodynamic and $\mathbf{k}$-dependent variables:

$$\frac{\delta H}{\delta\hat{l}}=\int_{k=k_F}d\mathbf{k}\left[\frac{\partial H}{\partial\tilde{\rho}(\mathbf{k})}\frac{\partial\tilde{\rho}(\mathbf{k})}{\partial\hat{l}}+\frac{\partial H}{\partial\nabla_i\tilde{\varphi}(\mathbf{k})}\frac{\partial\nabla_i\tilde{\varphi}(\mathbf{k})}{\partial\hat{l}}-\nabla_j\left[\frac{\partial H}{\partial\nabla_i\tilde{\varphi}(\mathbf{k})}\frac{\partial\nabla_i\tilde{\varphi}(\mathbf{k})}{\partial\nabla_j\hat{l}}\right]\right]$$
$$=\int_{k=k_F}d\mathbf{k}\left[\frac{\hbar}{M}\left[-\frac{\partial\tilde{\varphi}(k)}{\partial t}\frac{\partial\tilde{\rho}(\mathbf{k})}{\partial\hat{l}}+\frac{\partial\tilde{\rho}(\mathbf{k})}{\partial t}\frac{\partial\tilde{\varphi}(\mathbf{k})}{\partial\hat{l}}\right]+\frac{\partial H}{\partial\nabla_i\tilde{\varphi}(\mathbf{k})}\left[\frac{\partial}{\partial\hat{l}}\nabla_i\tilde{\varphi}(\mathbf{k})-\nabla_i\frac{\partial\tilde{\varphi}(\mathbf{k})}{\partial\hat{l}}\right]\right]. \tag{A11}$$

From here on the following relations are used:

$$\frac{\delta(\partial\tilde{\varphi}/\partial t)}{\delta(\partial\hat{l}/\partial t)}=\frac{\delta\tilde{\varphi}}{\delta\hat{l}},\quad\frac{\delta(\nabla_i\tilde{\varphi})}{\delta(\nabla_j\hat{l})}=\delta_{ij}\frac{\delta\tilde{\varphi}}{\delta\hat{l}}\,. \tag{A12}$$

The last term $\propto\partial H/\partial\nabla_i\tilde{\varphi}$ in this expression is nonlinear and did not arise in the linear theory of Cross. The remaining terms contain $\partial\tilde{\rho}/\partial t$ and $\partial\tilde{\rho}/\partial\hat{l}$, which are small [of order $(T_c/\epsilon_F)^2$] in the BCS theory due to particle-hole symmetry. As a result of this, Eq. (A11) yields after integration over the Fermi surface an equation of orbital motion with extremely small L, of order $(T_c/\epsilon_F)^2$.

Thus two methods of deriving hydrodynamics, usually entirely equivalent, yield essentially different results. In order to understand the discrepancy better, let us transform Eq. (A11):

$$\frac{\delta H}{\delta\hat{l}}=-\frac{\partial}{\partial\hat{l}}\left[\frac{\hbar}{M}\int_{k=k_F}d\mathbf{k}\,\tilde{\rho}(\mathbf{k})\frac{\partial\tilde{\varphi}(\mathbf{k})}{\partial t}\right]+\frac{d}{dt}\frac{\partial}{\partial(\partial\hat{l}/\partial t)}\left[\frac{\hbar}{M}\int_{k=k_F}d\mathbf{k}\,\tilde{\rho}(\mathbf{k})\frac{\partial\tilde{\varphi}(\mathbf{k})}{\partial t}\right]$$
$$+\frac{\hbar}{M}\int_{k=k_F}d\mathbf{k}\,\tilde{\rho}(\mathbf{k})\left[\frac{\partial}{\partial\hat{l}}\frac{\partial\tilde{\varphi}(\mathbf{k})}{\partial t}-\frac{\partial}{\partial t}\frac{\partial\tilde{\varphi}(\mathbf{k})}{\partial\hat{l}}\right]+\int_{k=k_F}d\mathbf{k}\frac{\partial H}{\partial\nabla_i\tilde{\varphi}(\mathbf{k})}\left[\frac{\partial}{\partial\hat{l}}\nabla_i\tilde{\varphi}(\mathbf{k})-\nabla_i\frac{\partial\tilde{\varphi}(\mathbf{k})}{\partial\hat{l}}\right]. \tag{A13}$$

The first two terms are present in the Lagrangian theory (the second one vanishes after integration), but the third and fourth are not. They contain commutators of differentiation operators applied to the phase $\tilde{\varphi}(\mathbf{k})$. They are equal to zero everywhere except in boojums, where the vortex line crosses the Fermi surface. We see that the two methods of derivation of hydrodynamics involve different kinds of extrapolation of the theory onto singular points where the theory fails.

One may obtain the general expression for commutators like those in Eq. (A13) by considering arbitrary successive variations $\delta_2\hat{l}$ and $\delta_1\hat{l}$, resulting in rotations of the vortex in $\mathbf{k}$-space and corresponding variations of the $\mathbf{k}$-dependent phase:

$$\delta_2\delta_1\tilde{\varphi}(\mathbf{k})-\delta_1\delta_2\tilde{\varphi}(\mathbf{k})=2\pi\hat{l}\cdot(\delta_2\hat{l}\times\delta_1\hat{l})[\delta(\mathbf{k}-k\hat{l})+\delta(\mathbf{k}+k\hat{l})]\,. \tag{A14}$$

Here $\delta(\mathbf{k}\pm k\hat{l})$ are δ functions on the Fermi surface.

Direct integration of the commutators in Eq. (A13), assuming that $\tilde{\rho}(\mathbf{k})$ does not depend on $\mathbf{k}$ in the zero-order approximation in T_c/ϵ_F, yields large terms. One of them [the third in Eq. (A13)] cancels the first term in the Lagrangian theory, and it results in an equation of orbital motion with negligible L. But at the same time the fourth nonlinear term $\propto\partial H/\partial\nabla_i\tilde{\varphi}$ in Eq. (A13) cancels the corresponding term $\propto\partial H/\partial\nabla_i\varphi$ in Eq. (A10) for $\delta H/\delta\hat{l}$. So generalization of such a theory to the nonlinear case would mean that the widely adopted condition of equilibrium, $\delta H/\delta\hat{l}=0$, is incorrect. Other conjectures are also possible. Suppose, for example, that the density $\tilde{\rho}(\mathbf{k})$ has some singular contribution and the commutators themselves reduce to a singular function more complicated than the δ function, so that integration of the product of $\tilde{\rho}(\mathbf{k})$ and the commutator yields zero. One may also attempt to redefine $\nabla_i\tilde{\varphi}$ and $\partial\tilde{\varphi}/\partial t$ in boojums by adding some singular terms that cancel contributions of commutators and lead to the Lagrangian theory (Sonin, 1986b).

So the crux of the problem of obtaining the equation of orbital motion is the question whether to take into account the integral of commutators over the Fermi surface. The same question arises in the problem of the nonlocal term in the superfluid current. It should be pointed out that the Lagrangian theory does not avoid the difficulty with the momentum-conservation law without additional restrictions on the Hamiltonian (Sonin, 1986b). It follows from the Noether theorem that the density of momentum in the momentum-conservation law,

$$\mathbf{g}=-\frac{\partial\mathscr{L}}{\partial(\partial\varphi/\partial t)}\nabla\varphi=\frac{\hbar}{M}\rho\nabla\varphi\ , \tag{A15}$$

does not coincide in general with the mass current in the mass-conservation law:

$$\begin{aligned}\mathbf{j}&=-\frac{M}{\hbar}\frac{\partial\mathscr{L}}{\partial\nabla\varphi}\\&=\frac{M}{\hbar}\frac{\partial H}{\partial\nabla\varphi}\\&=\frac{\hbar}{M}\rho\nabla\varphi+\frac{\hbar}{2M}\nabla\times\rho\hat{l}-\frac{\hbar}{M}C_0\hat{l}[\hat{l}\cdot(\nabla\times\hat{l})]\ .\end{aligned} \tag{A16}$$

The difference between $\mathbf{g}$ and $\mathbf{j}$ does not reduce to a divergent-free current (which would lead to no difficulty in the theory) if $C_0\neq 0$. But in the original microscopic theory (before transition to the BCS theory) expressions for both currents are identical; they reduce to the current $\mathbf{j}$ after transition to hydrodynamics. Thus the Noether theorem does not provide the conservation law for the "true" momentum with density $\mathbf{j}$ if the nonlocal term $\propto C_0$ is present in the expression for $\mathbf{j}$. Now let us see how this term arises. It is possible to start from the expression for the current $\mathbf{j}$ in terms of $\mathbf{k}$-dependent variables $\tilde{\rho}(\mathbf{k})$ and $\tilde{\varphi}(\mathbf{k})$ (Volovik and Mineev, 1981),

$$\mathbf{j}=\frac{\hbar}{M}\int_{k=k_F}d\mathbf{k}\,\mathbf{k}\left[\nabla\tilde{\rho}(\mathbf{k})\frac{\partial\tilde{\varphi}(\mathbf{k})}{\partial\mathbf{k}}-\frac{\partial\tilde{\rho}(\mathbf{k})}{\partial\mathbf{k}}\nabla\tilde{\varphi}(\mathbf{k})\right]. \tag{A17}$$

After integration by parts one obtains

$$\begin{aligned}\mathbf{j}=&\frac{\hbar}{M}\int_{k=k_F}d\mathbf{k}\,\tilde{\rho}(\mathbf{k})\nabla\tilde{\varphi}(\mathbf{k})\\&+\frac{\hbar}{M}\nabla_j\int_{k=k_F}d\mathbf{k}\,\mathbf{k}\tilde{\rho}(\mathbf{k})\frac{\partial\tilde{\varphi}(\mathbf{k})}{\partial k_j}\\&+\frac{\hbar}{M}\int_{k=k_F}d\mathbf{k}\,\mathbf{k}\tilde{\rho}(\mathbf{k})\left[\frac{\partial}{\partial\mathbf{k}}\nabla\tilde{\varphi}(\mathbf{k})-\nabla\frac{\partial\tilde{\varphi}(\mathbf{k})}{\partial\mathbf{k}}\right].\end{aligned} \tag{A18}$$

The three terms in Eq. (A18) are in one-to-one correspondence with the three terms in Eq. (A16). The commutator in Eq. (A18) can be reduced to one of those given by Eq. (A14) with the help of the relation $\partial\tilde{\varphi}/\partial\hat{l}=-(\mathbf{k}\cdot\hat{l})\partial\tilde{\varphi}/\partial\mathbf{k}$.

This means that the integral of the commutator is responsible for the nonlocal term in the supercurrent. Adhering to the opinion that this integral vanishes, one may delete it and arrive at a consistent hydrodynamics with the momentum-conservation law satisfied. Thus, having no theory able to treat the vicinity of boojums rigorously, one may expect nevertheless that the correct theory should provide vanishing integrals of commutators in order to satisfy the momentum-conservation law and, as a result, to obtain the correct motion law of the vortex line in $\mathbf{k}$ space (the equation of orbital motion). Similarly, one may obtain the motion law of vortex lines in a perfect fluid (the Helmholtz theorem) by requiring that it satisfy the momentum-conservation law (see Sec. III.B).

Evidently, the conjecture that the integral of the commutator vanishes remains speculative and needs further proof (or disproof). It rests on the belief that hydrodynamics should satisfy the local momentum-conservation law in the form of a differential equation. An alternative would be some complicated form of nonlocal momentum-conservation law that nobody has yet been able to formulate explicitly. What version of hydrodynamics is true may depend on the collisional contribution. Recent developments in attempts to construct a rigorous theory for the vicinity of boojums (Balatskii *et al.*, 1986; Combescot and Dombre, 1986) lead us to believe that the nonlocal supercurrent as a whole is linked to excitation states with zero or negative energy which arise due to $\hat{l}$ texture and have strong analogies with the Landau states of a charged particle in a magnetic field. In a collisionless situation $\omega\tau\gg 1$ these states must remain unoccupied because there is no relaxation process that allows them to come into equilibrium. This convinces us that it is possible to treat orbital waves at $\omega\tau>1$ within the scope of the self-consistent hydrodynamics with $C_0=0$ and $L=\rho\hbar/M$ (though one should be careful with the term "hydrodynamics" in the collisionless situation). The truly hydrodynamic regime $\omega\tau\ll 1$ calls for further analysis because the collision contribution is expected to dominate some reactive parameters including the dynamic angular momentum (Brand *et al.*, 1979). It is well known that collisions strongly change the dynamics of charged particles in a magnetic field when the free path length is smaller than the magnetic length.

A possible experimental check of ^{3}He-A hydrodynamics would be measurement of mutual-friction coefficients near the critical point. This is discussed in Sec. X.D.

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