Magnetic flux effects in disordered conductors

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This paper surveys the magnetic flux effects in multiply connected conductors, both normal metals and superconductors. Discussion of these effects for hopping conduction on the dielectric side of the metal-insulator transition is presented also. The main emphasis in the review is on the modern theoretical picture of these phenomena and comparison of theoretical results with experimental data.

CONTENTS

LIST OF MAIN SYMBOLS

$\mathbf{A}$	magnetic vector potential
D	electron diffusion coefficient
G_1	conductance in one-dimensional case
G	conductance in two-dimensional case
L_φ	phase relaxation diffusion length
l	elastic mean free path of electron
L_T	diffusion length for time $\hbar/2T$
r_H	magnetic length $(\hbar c/2eH)^{1/2}$
T_c	superconducting transition temperature
ϕ	magnetic flux
ϕ_0	magnetic flux quantum hc/e
τ	elastic collision time
τ_φ	phase relaxation time
σ	conductivity
ω_c	cyclotron frequency
μ	Fermi energy
ν	density of states at the Fermi level
τ_s	spin-flip electron scattering time due to spin-spin coupling
τ_{so}	spin-flip electron scattering time due to spin-orbit coupling
Δ	order parameter for superconductor
ξ_0	coherence length at $T=0$ in superconductors

I. INTRODUCTION

Studies of galvanomagnetic properties of disordered conductors have recently revealed a new direction of research that has already yielded reliable results and has a promising potential as well. In the experimental domain it consists in observing magnetoresistance oscillations in topologically different tiny samples at low temperatures. One can draw an analogy between these experiments and the idea of an experiment with electron beams in vacuum (Fig. 1) proposed fairly long ago (Ehrenberg and Siday, 1949; Aharonov and Bohm, 1959) and performed in different versions after Aharonov and Bohm (1959) had pointed out the radical importance and viability of such an experiment. It should also be noted that already in 1939 Franz had pointed out that the phase difference between two electron beams is determined by the magnitude of the magnetic flux enclosed between them (see the comprehensive review by Olariu and Popescu, 1985). Experiments reveal an interference pattern produced on the plate AB by two beams emitted by the source S. The in-

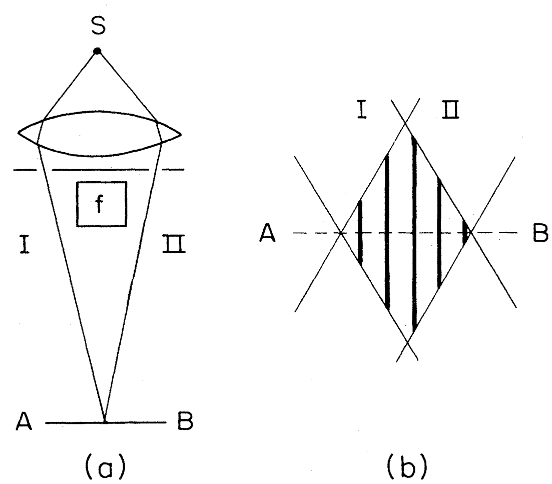

FIG. 1. (a) Schematic of electron-beam interference experiment (Ehrenberg and Siday, 1949); (b) the stationary wave pattern in the beam crossing region of space near the plane AB.

terference bands shift as the magnetic flux in the region f changes, although beams I and II pass through a field-free region, while the outer boundaries of the beam crossing region remain unshifted [Fig. 1(b)]. The change in the phase difference between the beams, which determines the band shift, can be obtained from the expression

$$\Delta\varphi = \frac{e}{\hbar c}\oint \mathbf{A}\,dl = 2\pi\frac{\phi}{\phi_0}\,, \tag{1.1}$$

where the magnetic field vector potential $\mathbf{A}$ is integrated around the contour made by beams I and II, ϕ is the magnetic flux through the region f, and $\phi_0 = hc/e = 4.14 \times 10^{-7}$ G cm^2 is the magnetic flux quantum. Thus the result does not depend on the gauge chosen, in accordance with the requirement that the theory be gauge invariant.

The effect of magnetic flux on the motion of the electrons localized in a region with $H=0$ should obviously manifest itself also in the case of electrons moving in a conductor that encloses a region with $H\neq 0$. Actually, one should observe variations in the sample characteristics providing information on the energy spectrum and mechanisms of the electron scattering.

Consider two simple models with one-dimensional propagation of an electron wave along a given path (Figs. 2 and 3), which illustrate two different approaches to solving thermodynamic and transport problems in systems with magnetic flux.

The first model involves a closed ring of radius R with an electron moving freely in it. The ring encloses a magnetic flux ϕ [Fig. 2(a)]. We can write the electron wave function in the form

$$\psi_n = e^{iny/R}, \quad n = 0, \pm 1, \pm 2, \ldots \tag{1.2a}$$

(y is the electron coordinate on the ring), the electron energy E_n and magnetic moment M being dependent on ϕ [Figs. 2(b) and 2(c), light lines]:

$$E_n = \frac{\hbar^2}{2mR^2}\left[n - \frac{\phi}{\phi_0}\right]^2, \qquad M = \frac{e\hbar}{2mc}\left[n - \frac{\phi}{\phi_0}\right]. \tag{1.2b}$$

If we include now a weak arbitrary potential $V(y)$, a splitting of the electron levels at the points

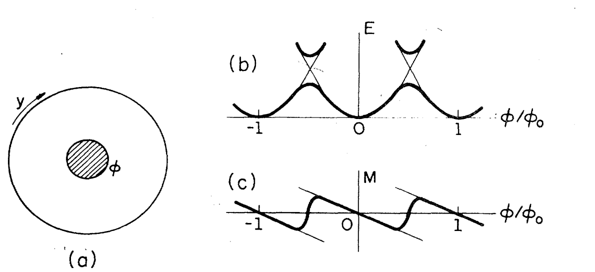

FIG. 2. (a) Schematic of one-dimensional ring confining a magnetic flux; (b) electron energy in the ring reduced magnetic flux ϕ/ϕ_0; (c) magnetic moment vs reduced flux ϕ/ϕ_0.

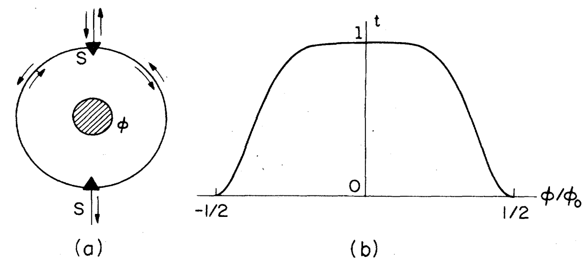

FIG. 3. (a) Schematic of one-dimensional ring with scatterers S and current-carrying contacts by Büttiker *et al.* (1984); (b) relative transmitted wave intensity vs reduced magnetic flux ϕ/ϕ_0.

$\phi = \phi_0(n_1+n_2)/2$ arises where the $E_{n_1}(\phi)$ and $E_{n_2}(\phi)$ parabolas intersect one another. As a result, the electron ground-state energy and magnetic moment and other thermodynamic properties of the model that can be expressed in terms of the electron spectrum turn out to be periodic functions of the magnetic flux with a period ϕ_0 shown by heavy lines in Figs. 2(b) and 2(c) (Büttiker *et al.*, 1983; Landauer and Büttiker, 1985).

We can follow the magnetic flux dependence of the transport properties using the simplest model of a ring connected to two current-carrying leads via elastic scatterers S [Fig. 3(a); Büttiker *et al.*, 1984; Gefen *et al.*, 1984]. For an arbitrary wave vector k_0 of the incident wave, the wave vectors of the waves propagating around the ring in opposite directions will be different for the same energy of the particle and equal to $k_0 \pm (1/R)(\phi/\phi_0)$, if one assumes the tangential component of the vector potential $\mathbf{A}$ to be $\phi/2\pi R$.

The relative intensity of the transmitted wave t representing the transport characteristic of the system will be a periodic function of the flux containing also higher harmonics due to a multiple passage of the waves around the ring. Figure 3(b) illustrates the $t(\phi)$ relationship for the scatterer parameters and the value of k_0R corresponding to the best signal transmission.

By now various magnetic flux interference effects that depend substantially on sample shape have been discovered and predicted. The search for such effects in metals began long after the discovery of oscillations in the resistivity (Schubnikov and de Haas, 1930) and magnetic moment (de Haas and van Alphen, 1930, 1932) of pure bismuth single crystals. These phenomena may be interpreted quasiclassically as size effects in momentum space, where, as the field increases, the quantized areas of electron orbits become successively equal to the extremal cross section area of the Fermi surface of the metal, thus resulting in the onset of oscillations (Onsager, 1952; Lifshitz and Kosevich, 1954). The oscillations have been observed in high magnetic fields where the maximum Larmor radius of the electron trajectories r_L was less than the mean free path l and the sample size. Thus the sample shape was here inessential.

Magnetic moment oscillations related to the geometric size of the samples were first considered by Dingle (1952) for the case in which electrons with an isotropic dispersion relation are specularly reflected from the boundary of a pure cylindrical sample of radius R placed in a weak longitudinal magnetic field. It was assumed that $R \ll r_L, l$, with magnetic field included in the determination of the energy spectrum by a perturbative treatment. The calculated spectrum of the susceptibility oscillations turned out to be very complex; however, it revealed an interesting feature leading to a qualitative understanding of the mechanism by which a magnetic field acts on a metal under the conditions $r_L \gg \min(R,l) \gg \lambda$, where λ is the electron wavelength (including action upon a disordered conductor, to be considered in the following sections).

The spectrum of the Dingle oscillation periods was determined by a set of areas S_i of regular polygons with an arbitrary number of sides inscribed into the circular cross section of a cylinder, the polygons representing closed projections of electron trajectories on the cylinder cross section. For $\lambda \ll R$ the periods were equal to ϕ_0/S_i, which corresponds to a change of the flux through the orbit areas by one quantum, just as in the case shown in Fig. 2. Thus the classical bending of trajectories in a magnetic field turns out to be inessential in this case, the action of the field reducing to that due to the magnetic flux enclosed in the various closed trajectories.

Subsequent studies (Kulik, 1970; Bogachek and Gogadze, 1972; see also Gogadze, 1984, and references therein) drew attention to the fact that specularity in real cylindrical objects should increase for the electrons localized near the surface, thus enhancing the relative contribution to the spectrum due to oscillations with a period close to $\phi_0/\pi R^2$, which should manifest itself for any electron dispersion relation. The authors of these studies suggested also that the magnetoresistance, acoustic absorption, and tunneling current from the cylinder surface be studied instead of the susceptibility oscillations.

Experimental investigation of the magnetoresistance of thin ($R \simeq 10^{-1}$ μm) cylindrical single crystals of pure bismuth (Brandt *et al.*, 1977, 1982) with a large mean free path revealed oscillations in a longitudinal field at $l, r_L \geq R$. The authors separated out from a complex spectrum a period approximately equal to $\phi_0/\pi R^2$. Further studies are required before a comprehensive comparison of experiments of this kind with theoretical calculations will be possible.

There is another class of oscillatory phenomena that can occur in normal disordered metals, namely, oscillations of transport effects in multiply connected samples in the field of a vector potential **A**, whose relative magnitude, rather than decreasing, even grows with decreasing electron mean free path. Oscillations of this kind, with a period of $\phi_0/2$ in magnetic flux were predicted by Altshuler *et al.* (1981). The first observation of these oscillations (Sharvin and Sharvin, 1981) was followed by a number of experimental studies of this phenomenon. These studies are reviewed in Secs. II.A—II.C.2 of the present paper. The search for oscillatory effects in quasi-one-dimensional rings (Umbach *et al.*, 1984; Webb *et al.*, 1984) has led to the observation of a new class of effects in disordered tiny conductors, namely, mesoeffects, which originate from a lack of total averaging of properties over the impurity distribution (Altshuler, 1985; Altshuler and Khmelnitskii, 1985; Lee and Stone, 1985; Stone, 1985; Webb *et al.*, 1985). The resistivity of such multiply connected samples observed as a function of magnetic field reveals fluctuations, their spectrum having a strongly pronounced harmonic of period ϕ_0. These phenomena are discussed in Sec. II.C.3.

Magnetic flux effects manifest themselves not only in normal metals, but also in superconductors. Their importance for the physics of superconductivity and its applications is widely recognized. If concentrated in the form of a flux tube inside a superconductor, the magnetic flux should be quantized.

Before the development of the microscopic theory of superconductivity, London (1948) predicted this phenomenon by suggesting the quantity ϕ_0 to be a flux quantum. In 1961, Deaver and Fairbank (1961) and Doll and Näbauer (1961) measured the magnetic moment of hollow superconducting cylinders and found the quantity $\phi_0/2 = hc/2e$ to be a flux quantum in superconductors. Byers and Yang (1961) and Onsager (1961) attributed this result to electron pairing. The pairing occurs in such a way that the electron single-particle energies are equal at a minimal total energy. Then from Eq. (1.2b) it follows that $n_1 - \phi/\phi_0 = -(n_2 - \phi/\phi_0)$ and, hence, $\phi = \phi_0(n_1 + n_2)/2$ (Schrieffer, 1964). Thus the magnetic flux corresponding to the minima in the total energy of the electron pair is a multiple of the superconducting flux quantum $\phi_0/2$. The phenomenon of magnetic flux quantization in superconductors can be investigated also by studying such a transport quantity as the resistance of a hollow cylinder in the vicinity of the superconducting transition (Little and Parks, 1962; Parks and Little, 1964). All these effects in superconductors are a natural consequence of the existence of a macroscopic wave function that is not destroyed by the electron impurity and electron-photon scattering. These effects are dealt with in Sec. III. Finally, the last section (IV) of this review is devoted to discussion of magnetic flux effects in hopping conduction.

II. RESISTIVITY OSCILLATIONS IN MULTIPLY CONNECTED DISORDERED CONDUCTORS

A. Qualitative considerations

Resistivity oscillations with $\phi_0/2$ flux period in multiply connected disordered conductors are intimately related to the phenomenon of weak localization. Therefore we shall consider first the conductivity correction due to weak localization (Abrahams *et al.*, 1979; Anderson *et al.*, 1979).

In the classical theory of transport phenomena the total probability for a particle to transfer from point P to point Q (Fig. 4) is the sum of probabilities of such a transfer over all possible trajectories. In quantum mechanics this result corresponds to neglecting the interference of scat-

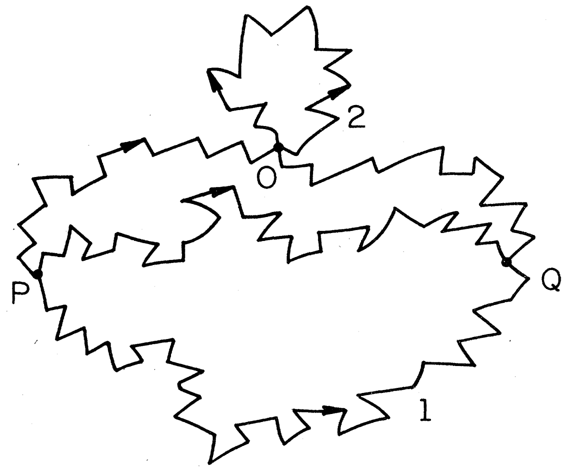

FIG. 4. Different types of quasiclassical particle trajectories connecting P and Q. Point O is the trajectory self-crossing point.

tered electron waves propagating along different paths and having approximately random phases under the quasiclassical condition $\lambda \ll l$. There is, however, a specific class of trajectories, namely, self-crossing trajectories (trajectory 2 in Fig. 4) for which the wave interference turns out to be essential. Indeed, two waves propagating along such trajectories in two opposite directions (conjugated waves) accumulate the same phase difference. Therefore the contribution of these trajectories to the probability of coming to point O (Fig. 4) will be

$$|A_1+A_2|^2=|A_1|^2+|A_2|^2+2\,\mathrm{Re}A_1^*A_2$$
$$=4|A_1|^2\,, \tag{2.1}$$

which is twice the sum of the squared amplitude moduli. A higher probability of returning back to point O means a lower probability of transfer from point P to point Q. Thus the interference of conjugated waves favors particle localization and, hence, results in an increase of resistivity.

Quantitative evaluation of this effect should take into account that the particles return to a given point by diffusion, the interference occurring between conjugated waves propagating along a tube with a cross-section area of λ^2. Note also that the time of their propagation should be less than the time τ_φ during which the conjugated waves remain coherent. The phase relaxation time τ_φ may be related both to inelastic processes and to spin-spin scattering. The relative correction to conductivity $\Delta\sigma/\sigma$ is proportional to the total probability for particles to return in the time τ_φ (Altshuler, Aronov, Khmelnitskii, and Larkin, 1982)

$$\frac{\Delta\sigma}{\sigma}\simeq-\int_\tau^{\tau_\varphi}\frac{v\lambda^2 dt}{(Dt)^{d/2}a^{3-d}}\,, \tag{2.2}$$

where D is the diffusion coefficient, v the particle velocity, σ the metal conductivity, d the effective dimension of the sample for diffusion, and a the small dimension of the sample (see below).

For the lower limit in Eq. (2.2) the elastic collision time τ is chosen, since the diffusion description of particle propagation becomes valid only for longer times, and a particle can turn back only after at least one scattering event. In a film of thickness $a \ll \sqrt{D\tau_\varphi}=L_\varphi$ a particle is capable of passing many times from one wall to another in a time scale τ_φ, the diffusion acquiring two-dimensional character. Similarly, when the wire diameter a is small compared to L_φ, diffusion over times $t>a^2/D$ will be one dimensional.

It follows from Eq. (2.2), in two dimensions the correction to the two-dimensional film conductance $G=a\sigma$ is (Abrahams *et al.*, 1979; Gorkov *et al.*, 1979)

$$\Delta G=-\frac{e^2}{2\pi^2\hbar}\ln\frac{\tau_\varphi}{\tau}\,. \tag{2.3}$$

If the sample is placed in a magnetic field, then the amplitudes of the probability of completing the loop on contour 2 of Fig. 4, clockwise and counterclockwise, acquire additional phase factors

$$\begin{aligned} A_1&\rightarrow A_1\exp\left[\frac{ie}{\hbar c}\oint \mathbf{A}\,dl\right]=A_1\exp\left[i\frac{2\pi\phi}{\phi_0}\right]\,,\\ A_2&\rightarrow A_2\exp\left[-i\frac{2\pi\phi}{\phi_0}\right]\,, \end{aligned} \tag{2.4}$$

where the magnetic flux through the loop is $\phi=HS$, and S is the projection of the loop area on the plane perpendicular to the magnetic field direction. Equation (2.4) implies that the phase difference between the conjugated waves is

$$\Delta\varphi=2\pi\frac{2\phi}{\phi_0}\,. \tag{2.5}$$

Therefore, by producing a phase difference depending on the size of trajectory, the magnetic field destroys the interference, which, in its turn, reduces the probability for the particle to return to the given point and, hence, reduces the resistivity. This is the mechanism that accounts for the negative magnetoresistance phenomenon (Altshuler *et al.*, 1980; Kawabata, 1980a, 1980b).

The characteristic time t_H in which the interference of conjugated waves is destroyed in a magnetic field can be estimated from the condition that in this time the phase difference between the conjugated waves becomes of order unity,

$$\begin{aligned} \Delta\varphi&\simeq\frac{2HDt_H}{\phi_0}\simeq 1\,,\\ t_H&\simeq\frac{\phi_0}{2HD}\simeq\frac{r_H^2}{D}\,, \end{aligned} \tag{2.6}$$

where $r_H=(\hbar c/2eH)^{1/2}$ is the magnetic length for a particle with charge $2e$. The characteristic magnetic fields are determined by the relation $t_H\simeq\tau_\varphi$, i.e.,

$$H \simeq \frac{\hbar c}{e} \frac{1}{D\tau_\varphi}$$

or

$$\omega_c \tau \frac{\mu \tau_\varphi}{\hbar} \simeq 1 , \qquad (2.7)$$

where $\omega_c = eH/mc$ is the cyclotron frequency, μ is the Fermi energy, and m is the effective mass. As follows from Eq. (2.7), the magnetic fields of interest here are less than the classical limit determined by the condition $\omega_c \tau \simeq 1$, by a factor $\mu\tau_\varphi/\hbar \gg 1$.

A detailed exposition of both the theory and the experimental data on anomalous magnetoresistance of singly connected samples can be found in reviews by Altshuler, Aronov, Khmelnitskii, and Larkin (1982) and Bergmann (1983).

The anomalous magnetoresistance of disordered conductors observed in the weak field ($r_L \gg l$) may be considered as a magnetic flux effect caused by the magnetic field confined within electron loop trajectories. The recognition of the fact that these trajectories are actually paths in real space, which may be of macroscopic size, has led to ideas concerning the effect of topological differences in the shape of samples on their magnetoresistance and the possibility of observing interference oscillations in the resistivity of multiply connected samples as a function of magnetic flux enclosed by them (Altshuler, Aronov, and Spivak, 1981).

Consider, for instance, a hollow metal cylinder with a hole that encloses a long solenoid carrying a magnetic flux ϕ so that the magnetic field is zero everywhere outside the solenoid (Fig. 5). However, the vector potential $\mathbf{A}$ within the sample should be nonzero, its tangential component being equal to $\phi/2\pi R$ (the Aharonov-Bohm experiment geometry). We assume that the mean free path l is smaller than the cylinder circumference, so that the size effects of pure metals (see the Introduction) may be neglected. The phase difference accumulated by conjugated waves going around the cylinder is

$$\Delta\varphi = 2\pi \frac{2\phi}{\phi_0} ,$$

i.e., it is the same for all conjugated waves running around the cylinder. As a result, the cylinder resistivity will oscillate with the period $\phi_0/2$.

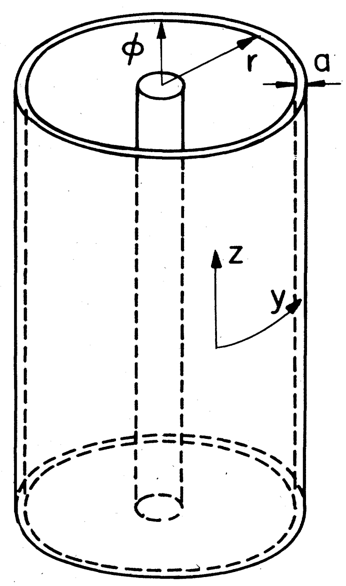

FIG. 5. Schematic of experiment with a solenoid carrying magnetic flux ϕ inside a cylinder with wall thickness a.

B. Theory of resistivity oscillations in multiply connected conductors

As pointed out in the preceding section, the negative magnetoresistance effect is related to quantum corrections to the conductivity originating from the interference of conjugated waves. The main quantum correction to the conductivity can be derived by summing the fan diagrams (Langer and Neal, 1966; Abrahams *et al.*, 1979; Gor'kov *et al.*, 1979; Fig. 6), which determine the function $C_\omega(\mathbf{r},\mathbf{r}')$ or "cooperon." The conductivity correction is related to the cooperon by the expression

$$\frac{\Delta\sigma(\omega)}{\sigma} = -\frac{2}{\pi\nu} C_\omega(\mathbf{r},\mathbf{r}) , \qquad (2.8)$$

where ν is the density of states at the Fermi level and σ is connected with the diffusion coefficient through Einstein's relation

$$\sigma = e^2 D\nu .$$

In the presence of the vector potential $\mathbf{A}$ of magnetic field, the equation for the cooperon takes on the form (Altshuler, Khmelnitskii, Larkin, and Lee, 1980)

$$\hbar \left[D \left[-i\nabla - \frac{2e}{c}\mathbf{A} \right]^2 + i\omega + \frac{1}{\tau_\varphi} \right] C_\omega(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') . \qquad (2.9)$$

Equation (2.9) resembles the Schrödinger equation for a particle of charge $2e$ and mass $\hbar/2D$ with the imaginary energy $i\omega$.

For a thin-walled hollow cylinder with magnetic field in the walls $H = 0$ and vector potential $\mathbf{A} \neq 0$, constant in absolute magnitude and directed tangentially, the solution of Eq. (2.9) with periodic boundary conditions along the coordinate y (i.e., along the cylinder circumference), has the form

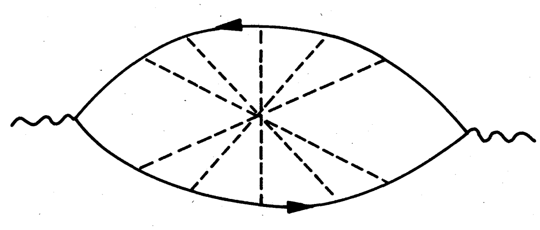

FIG. 6. Fan diagram for the conjugated wave interference.

$$C_\omega(\mathbf{r},\mathbf{r}')=\frac{1}{2\pi\hbar R}\int\frac{d\mathbf{Q}_\perp}{(2\pi)^2}\sum_{l=-\infty}^{\infty}e^{i\mathbf{Q}(\mathbf{r}-\mathbf{r}')}\frac{1}{i\omega+\dfrac{1}{\tau_\varphi}+DQ_\perp^2+D\left[Q_y^l-\dfrac{2e}{c}A\right]^2}\,, \tag{2.10}$$

where $\mathbf{Q}_\perp=(Q_x,Q_z)$, $Q_y^l=l/R$, and R is the cylinder radius. Substituting Eq. (2.10) in Eq. (2.8) yields

$$\Delta\sigma(\omega)=-\frac{2e^2}{\pi\hbar}\frac{1}{2\pi R}\int\frac{d\mathbf{Q}_\perp}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\frac{1}{Q_\perp^2+L_\varphi^{-2}(\omega)+\dfrac{1}{R^2}\left[n-\dfrac{2\phi}{\phi_0}\right]^2}\,, \tag{2.11}$$

where

$$L_\varphi^2(\omega)=\frac{L_\varphi^2}{1+i\omega\tau_\varphi}\,.$$

If the thickness of the cylinder walls is small compared with the length $L_\varphi(\omega)$, then the integration over Q_x should be replaced by a summation with only the term corresponding to $Q_x=0$ retained. If, in addition, the cylinder height is also small compared with $L_\varphi(\omega)$, then the integral over Q_z should be likewise replaced by a sum, with only the term with $Q_z=0$ retained in it (thin ring). This yields for the conductance of a unit length along the circumference of a thin ring $G_1=\sigma ab$ (b is the ring height, a is the ring thickness)

$$\Delta G_1(\omega)=-\frac{e^2}{\pi\hbar}L_\varphi(\omega)\times\sum_{l=-\infty}^{+\infty}\frac{1}{\pi}\frac{R/L_\varphi(\omega)}{\left[\dfrac{R}{L_\varphi(\omega)}\right]^2+\left[l-\dfrac{2\phi}{\phi_0}\right]^2}\,. \tag{2.12}$$

For $\omega=0$, Eq. (2.12) can be presented in the form (Altshuler, Aronov, and Spivak, 1981)

$$\Delta G_1=-\frac{e^2L_\varphi}{\pi\hbar}\frac{\sinh\dfrac{2\pi R}{L_\varphi}}{\cosh\dfrac{2\pi R}{L_\varphi}-\cos 2\pi\dfrac{2\phi}{\phi_0}}\,. \tag{2.13}$$

As can be seen from Eq. (2.12), the magnetic flux dependence of the conductance correction represents a sum of the Lorentzian contours of width $\phi_0R/2L_\varphi$ centered on $\phi=l\phi_0/2$. Such a clearly nonsinusoidal shape of the conductivity oscillations originates from the inclusion in Eq. (2.12) of conjugated waves making multiple turns around the ring. Now if the phase relaxation length $L_\varphi(\omega)$ is small compared with the circumference $2\pi R$, the oscillations become sinusoidal with an exponentially small amplitude.

For a cylindrical sample of radius R and height b, the correction to its longitudinal conductance at $\omega=0$ is (Altshuler, Aronov, and Spivak, 1981)

$$\Delta G=-\frac{e^2}{\pi^2\hbar}\frac{2\pi R}{b}\left[\ln\frac{L_\varphi}{l}+2\sum_{n=1}^{\infty}K_0\left[n\frac{2\pi R}{L_\varphi}\right]\times\cos\left[2\pi n\frac{2\phi}{\phi_0}\right]\right]=-\frac{e^2}{\pi^2\hbar}\frac{2\pi R}{b}Z_\phi(L_\varphi)\,, \tag{2.14}$$

where $K_0(x)$ is the Macdonald function. Just as in the case of the ring, the correction oscillates with a period $\phi_0/2$, becoming exponentially small for $L_\varphi\ll 2\pi R$.

Equations (2.12) and (2.14) do not, however, take into account certain points of importance for a comparison of the theory with experiment.

(1) In the derivation of Eqs. (2.12) and (2.14) no allowance was made for the presence of magnetic field in the sample; in other words, it was assumed that in the sample walls $H=0$, or that the wall thickness was negligible. In the real experiments to be discussed below, the samples were placed in a uniform magnetic field, the field in the cylinder walls being nonzero. As a result, (a) the oscillations decay as the field increases, since the magnetic fluxes enclosed in different trajectories of conjugated waves are different, and (b) a monotonic component appears in the magnetic field dependence of the sample resistivity, which originates from the interference of the conjugated waves, which do not enclose the cylinder axis; i.e., anomalous longitudinal magnetoresistance sets in (Altshuler and Aronov, 1981b). Note also that the magnetic field direction may deviate from the axis of cylindrical samples by an angle θ.

The inclusion of magnetic field results in the phase relaxation length L_φ becoming field dependent. For a cylinder or ring with a wall thickness of a in a field parallel to its axis,

$$\frac{1}{L_\varphi^2(H)}=\frac{1}{D\tau_\varphi}+\frac{1}{3}\left[\frac{aeH}{\hbar c}\right]^2=\frac{1}{L_\varphi^2}+\frac{1}{D\tau_H}\,. \tag{2.15}$$

As pointed out by Altshuler (see review by Sharvin, 1984), if the angle θ between the field and the cylinder axis is not zero but is very small, so that

$$r_H^2 > 4R^2 \sin\theta \ ,$$

then the quantity a in Eq. (2.15) should be replaced by a^*, such that

$$a^{*2} = a^2 \cos^2\theta + 6R^2 \sin^2\theta \ . \tag{2.15a}$$

(2) As demonstrated by Hikami *et al.* (1980), in the presence of spin-orbit scattering the anomalous magnetoresistance in weak fields and at low temperatures reverses its sign. At the same time, in the case of oscillations, the monotonic component undergoes sign reversal, the oscillation phase changing by π. The change in the shape of the curve can be described by replacing the function $Z_\phi(L_\varphi)$ in Eq. (2.14) by

$$\tfrac{3}{2} Z_\phi(\widetilde{L}_\varphi(H)) - \tfrac{1}{2} Z_\phi(L'_\phi(H)) \ , \tag{2.16}$$

where

$$\begin{aligned} \widetilde{L}_\varphi^{-2}(H) &= L_\varphi^{-2}(H) + \frac{1}{D}\left[\frac{2}{3\tau_s} + \frac{4}{3\tau_{so}}\right] , \\ L_\varphi'^{-2}(H) &= L_\varphi^{-2}(H) + \frac{2}{D\tau_s} \ . \end{aligned} \tag{2.17}$$

Here τ_s and τ_{so} are the spin-flip electron scattering times due to spin-spin and spin-orbit coupling, respectively; and $L_\varphi(H)$ is determined by Eq. (2.15). The first term in Eq. (2.16) represents a contribution to the interference which vanishes in the presence of strong spin-orbit coupling (triplet cooperon), and the second term (singlet cooperon) is negative and describes the antilocalization effect created under these conditions.

(3) A substantial contribution to the transport phenomena may come from the interaction of electrons with superconducting fluctuations. This interaction, generally speaking, is present even in normal metals and is essential in superconductors, even at temperatures considerably above T_c, owing primarily to the so-called Maki-Thompson corrections (Larkin, 1980). Since the superconducting pairs are in the singlet state, which is not affected by spin-orbit scattering, the Maki-Thompson correction affects only the singlet part of the cooperon. The factor in front of the second term in Eq. (2.16) can now be written as $\frac{1}{2} + \beta(T/T_c)$. The always positive function $\beta(T/T_c)$ was tabulated by Larkin (1980). For superconducting metals, for $T \to T_c$,

$$\beta \simeq \frac{\pi^2}{4} \frac{1}{\ln \dfrac{T}{T_c}} \tag{2.18}$$

and for $T \gg T_c$

$$\beta \simeq \frac{\pi^2}{6 \ln^2 \dfrac{T}{T_c}} \ . \tag{2.19}$$

Here

$$T_c = \begin{cases} \hbar\omega_D \exp\left[\dfrac{1}{\lambda_c}\right], & \lambda_c < 0 \ , \\ \mu \exp\left[\dfrac{1}{\lambda_c}\right], & \lambda_c > 0 \ , \end{cases}$$

where ω_D is the Debye frequency. Note that for nonsuperconducting materials ($\lambda_c > 0$) the quantity T_c has only a formal meaning, and that $T_c > \mu$. For typical normal metals at helium temperatures $\beta = 10^{-2}$.

All this adds up to the following expression:

$$\Delta G = \frac{e^2}{\pi^2\hbar} \frac{2\pi R}{b}\left[\left(\tfrac{1}{2}+\beta\right) Z_\phi(L'_\varphi(H)) - \tfrac{3}{2} Z_\phi(\widetilde{L}_\varphi(H))\right] . \tag{2.20a}$$

Similarly, for a thin ring,

$$\Delta G_1 = \frac{e^2 R}{\pi^2 \hbar} \sum_{l=-\infty}^{+\infty} \left[\frac{\frac{1}{2}+\beta}{\left[\dfrac{R}{L'_\varphi(H)}\right]^2 + \left[l - \dfrac{2\phi}{\phi_0}\right]^2} - \frac{3}{2} \frac{1}{\left[\dfrac{R}{\widetilde{L}_\varphi(H)}\right]^2 + \left[l - \dfrac{2\phi}{\phi_0}\right]^2} \right] . \tag{2.20b}$$

C. Experimental studies of magnetoresistance oscillations

1. Cylindrical films

The observation of oscillatory magnetoresistance in hollow normal metal cylinders could serve as a rather direct experimental confirmation of the quantum interference concept underlying the weak localization theory.

The first experiments were performed on samples made of magnesium (Sharvin and Sharvin, 1981; Gijs *et al.*, 1984a, 1984b), and lithium (Altshuler *et al.*, 1982; Ladan and Maurer, 1983; see also the review by Sharvin, 1984). In these experiments, cylindrical films 10—100 nm thick were evaporated on the surface of quartz filaments with $d \simeq 1$ μm and placed at liquid-helium temperature in a longitudinal magnetic field H. The sample resistivity R was measured with a dc potentiometer. The tiny size of the samples precluded production of magnetic flux only in the sample channel; the presence of nonzero magnetic field inside the metal accounted for the additional monotonic variation of the resistivity superimposed on the oscillations. The simultaneous measurement of the oscillation amplitude and monotonic component of the magnetoresistance on the same sample provided an additional possibility for comparison with theory, since both phenomena are characterized by the same parameters [see Eqs. (2.14) and (2.15)].

Experiments with magnesium (Fig. 7) demonstrated that the oscillation period did indeed correspond to the variation of the magnetic flux through the sample cross section by an amount $\approx\phi_0/2$; however, the phase of the oscillations (min$R$ for $H=0$) and the monotonic growth of the resistivity with increasing H could not be reconciled with the simplest formula [Eq. (2.14)] suggested above for this phenomenon (see Sec. II.B). On the other hand, the same oscillation phase was observed earlier in the experiments with superconducting materials of Parks and Little (1964) near T_c and of Shablo *et al.* (1974) at $T>T_c$ (for more details see Sec. III). Magnesium was reported to be not superconducting down to 17 mK (Falge *et al.*, 1968), and the positive magnetoresistance in this case can be accounted for by another reason (see below).

The most direct confirmation of the weak localization mechanism, however, from both qualitative and quantitative points of view, was obtained in experiments by Altshuler *et al.*, 1982) and Ladan and Maurer (1983) on lithium films. In the case of lithium the oscillation phase was successfully reversed and a negative sign achieved for the monotonic magnetoresistance (Fig. 8).

The measurement results averaged over four experimental curves are in a good accord with the theoretical curve (dashed line in Fig. 8) drawn by Eqs. (2.14) and (2.15), with the sum in Eq. (2.14) including only the $n=1$ term. The value of the filament diameter accepted in the calculations, $d=1.31$ μm, was found to be fairly close to the value $d=1.30\pm0.30$ μm obtained with a scanning electron microscope. Thus, within this accuracy, the period of the experimental curve in flux through the average sample cross section coincided with $\phi_0/2$. The film thickness $a=127$ nm accepted in the calculations determined at the same time the oscillation decay and the magnitude of the monotonic magnetoresistance component, thus providing an additional test of the theory.

The results of the magnesium film studies shown in Fig. 7 and the more detailed data obtained by Gijs *et al.* (1984a, 1984b) (Fig. 9) may be theoretically explained if one considers spin-orbit interaction under the elastic scattering of electrons [see Sec. II.B, Eqs. (2.20a), (2.16), and (2.17)]. The importance of spin-orbit scattering in the case of magnesium, an element with a small atomic

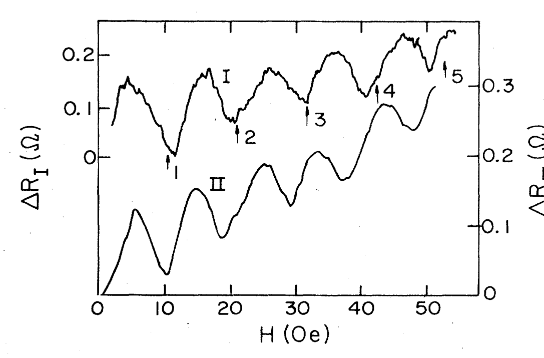

FIG. 7. Longitudinal magnetoresistance $\Delta R(H)$ at $T=1.1$ K for two cylindrical magnesium films on quartz filaments 1 cm long. $R_{4.2\mathrm{I}}=9.2$ kΩ, $R_{4.2\mathrm{II}}=12.3$ kΩ. The ratios $R_{300}/R_{4.2}$ for the two films are 1.39 and 1.25, respectively. Filament diameter of sample I measured with scanning electron microscope is 1.59 ± 0.03 μm. The arrows specify the fields corresponding to integer numbers of magnetic flux quanta $\phi_0/2=hc/2e$ through the filament cross-section area (Sharvin and Sharvin, 1981).

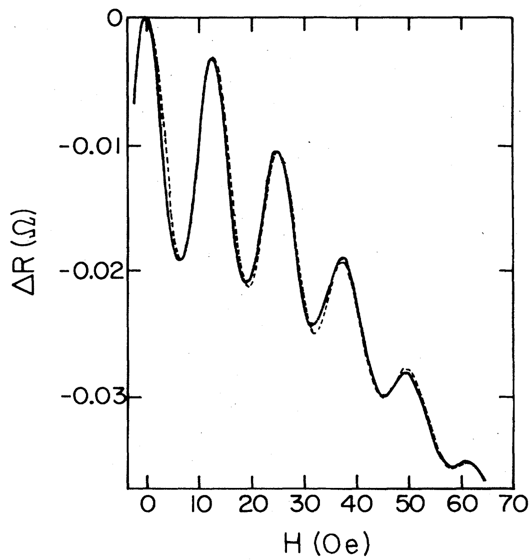

FIG. 8. Longitudinal magnetoresistance $\Delta R(H)$ at $T=1.1$ K for a cylindrical lithium film evaporated onto a 1-cm-long quartz filament. $R_{4.2}=2$ kΩ, $R_{300}/R_{4.2}=2.8$. Solid line: averaged from four experimental curves. Dashed line: calculated for $L_\varphi=2.2$ μm, $\tau_\varphi/\tau_{so}=0$, filament diameter $d=1.31$ μm, film thickness 127 nm. Filament diameter measured with scanning electron microscope yields $d=1.30\pm0.03$ μm (Altshuler *et al.*, 1982; Sharvin, 1984).

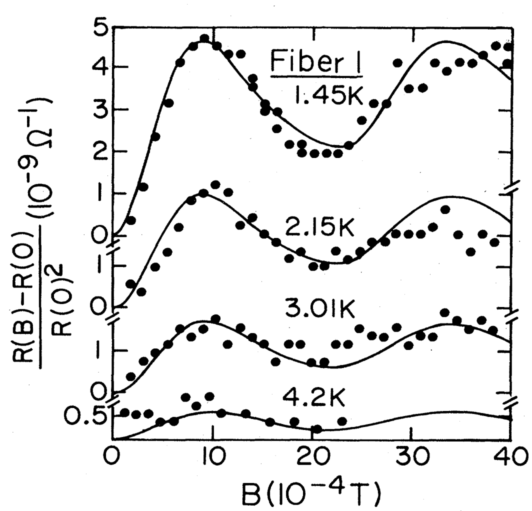

FIG. 9. Longitudinal magnetoresistance $R(H)$ for different temperatures of cylindrical magnesium film on a quartz filament 5.3 mm long and 1.2 ± 0.1 μm in diameter. $R_{4.2}=5056$ Ω. Solid curves show the theory. $L_\varphi=2$ μm, $\tau_\varphi/\tau_{so}=13$ for 1.45 K (Gijs *et al.*, 1984a).

number $Z=12$ is, generally speaking, not obvious, since the relative cross section of spin-orbit interaction in the electron scattering from defects was estimated by Meservey and Tedrow (1978) to scale as $(\alpha Z)^4$, where $\alpha=e^2/\hbar c=\frac{1}{137}$. Thus for magnesium one might expect small values of the ratio $\tau/\tau_{so}\simeq 10^{-4}$. One should, however, bear in mind that the oscillation phase and magnetoresistance sign are determined by the ratio τ_φ/τ_{so}, with τ_φ growing dramatically with decreasing temperature. The curve in Fig. 9 for $T=1.45$ K corresponds to $\tau_\varphi/\tau_{so}=13$, and for the curves in Fig. 7 evaluation yields $\tau_\varphi/\tau_{so}>10$, whereas above helium temperatures Bergmann (1982) observed negative magnetoresistance in planar magnesium films with $\tau_\varphi/\tau_{so}<0.5$.

Turning back to the results for lithium (Fig. 8), which are well described by theory with $\tau_\varphi/\tau_{so}=0$, it should be pointed out that the shape of the curves for small τ_φ/τ_{so} turns out to be relatively insensitive to the magnitude of τ_φ/τ_{so}. Using Eqs. (2.14), (2.17), and (2.20a), one can derive an upper limit for this quantity, $\tau_\varphi/\tau_{so}\leq 0.3$, which corresponds to $\tau_{so}>10^{-8}$ s for the sample used in the experiment of Altshuler *et al.* (1982). The amplitude of the resistance oscillations for this lithium sample could be measured at temperatures up to 2.6 K. These data and the monotonic longitudinal magnetoresistance measured up to 4.2 K were used to derive the value of $L_\varphi(T)$ and the temperature dependence of the phase relaxation length $l_\varphi=v_F\tau_\varphi=3L_\varphi^2/l$ displayed in Fig. 10. This dependence for the quantity $1/l_\varphi$ or $1/\tau_\varphi$ was thus found to be nearly quadratic in T over the temperature range studied, and the value of τ_φ was about one-tenth of the electron transport relaxation time in bulk lithium. For bulk lithium one observes a $\Delta\rho\propto T^2$ law (Krill, 1971; Sinvani *et al.*, 1981) due apparently to the electron-electron interaction in pure metals (Landau, 1956).

For comparison we also show the results of a study into the magnetoresistance of quasi-one-dimensional lithium strips measuring about $10^{-6}\times10^{-6}\times10^{-1}$ cm^3 carried out by Licini *et al.* (1985). The samples were prepared by evaporating lithium onto a substrate with a stencil (Fig. 11) fabricated by "canyon" lithography (Fulton and Dolan, 1983). By using the weak localization theory for the quasi-one-dimensional case, Licini *et al.* (1985) obtained for one of the samples over a broad temperature range $0.1<T<16$ K the relation $L_\varphi=\sqrt{D\tau_\varphi}=1.9T^{-1}$ μm $\pm10\%$, or $\tau_\varphi=1.4\times10^{-10}T^{-2}$ s$\pm10\%$, i.e., the same exponent for the temperature dependence of τ_φ as in Fig. 10 but with an order of magnitude smaller absolute values of τ_φ (assuming by the free-electron model $v_F=1.3\times10^8$ cm/s for lithium). An estimate was obtained also for the temperature-independent quantities $L_s=\sqrt{D\tau_s}=3.1$ μm and $L_{so}=\sqrt{D\tau_{so}}=1.8$ μm. The magnitude of the spin-orbit coupling turned out to be at least an order of magnitude greater than that for a cylindrical sample described earlier, apparently due to the presence of trace amounts of heavy elements, to which lithium is particularly sensitive.

The dependence $\tau_\varphi\propto T^{-2}$ was observed earlier on planar films of other metals as well, e.g., magnesium (Bergmann, 1982). This dependence is obviously not related to the conventional e-e scattering. While a comprehensive discussion of the relaxation processes is beyond the scope of this review (see, for example, Altshuler and Aronov, 1985), it should be noted that a plausible theoretical explanation of the relationship $\tau_\varphi\propto T^{-2}$ has yet to be advanced. Neither electron-electron scattering in dirty metals (Schmid, 1974; Altshuler and Aronov, 1979, 1981a; Altshuler, Aronov, Khmelnitskii, and Larkin, 1982; Abrahams *et al.*, 1982) nor electron-phonon scattering (Schmid, 1973, 1985) are capable of producing such a dependence $\tau_\varphi(T)$.

It is possible that the $\tau_\varphi^{-1}\propto T^2$ dependence is transitional between high-temperature electron-phonon scattering ($\tau_\varphi^{-1}\propto T^3$) and low-temperature electron-electron scattering $\tau_\varphi^{-1}\propto T$ in the two-dimensional case, or $\tau_\varphi^{-1}\propto T^{3/2}$ observed in many three-dimensional cases (see, for example, the review by Altshuler and Aronov, 1985).

2. Multiply connected planar structures

A number of magnetoresistance studies in a transverse field have been carried out on samples shaped as a "mesh" representing a two-dimensional square array, "necklace" samples consisting of square rings strung end to end, or ladders of quasi-one-dimensional strips con-

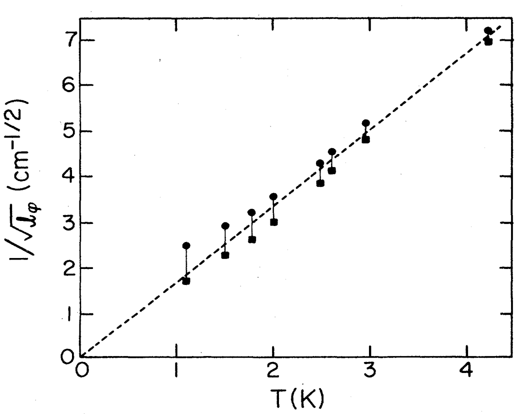

FIG. 10. Temperature dependence of $1/\sqrt{l_\varphi}$, where $l_\varphi=v_F\tau_\varphi$ is the phase relaxation length for the lithium (see Fig. 8). ●, $\tau_\varphi/\tau_{so}=0$; ■, $\tau_\varphi/\tau_{so}=0.2$ for 1.1 K. Dashed line drawn to guide the eye (Altshuler *et al.*, 1982; Sharvin, 1984).

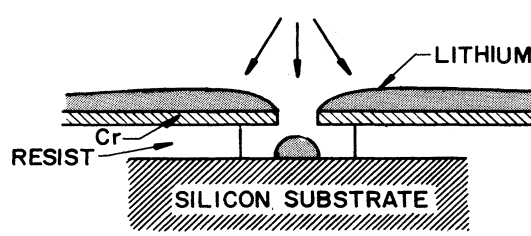

FIG. 11. Silicon substrate for preparation of lithium samples (Bishop *et al.*, 1985) with a stencil fabricated using canyon lithography (Fulton and Dolan, 1983).

nected with one another. Pannetier, Chaussy, Rammal, and Gandit (1984) and Pannetier *et al.* (1985) investigated honeycomb networks obtained by condensation of magnesium (1984), copper, and gold (1985) on microchannel substrates. Bishop *et al.* (1985) and Dolan *et al.* (1986) prepared their samples in the same way (Fig. 11) as they had prepared the above-mentioned singly connected samples for the experiment of Licini *et al.* (1985). A similar technique was used by Petrashov *et al.* (1985) to prepare aluminum arrays.

Some of the results obtained in these studies are illustrated in Figs. 12—14, presenting the curves measured at the lowest temperature reached, 0.13 K. At this temperature the lengths L'_φ and $\widetilde{L}_\varphi$ [see Eq. (2.17)] were the largest, which was conducive to revealing differences between curves for samples of different shapes.

In the reverse case of $L'_\varphi,\widetilde{L}_\varphi \ll 4S$, where $4S$ is the perimeter of a single loop (at temperatures up to 7 K), Dolan *et al.* (1986) obtained for arrays and networks with the same S curves $\Delta R(H)/R$ that closely resemble one another and the curve for a single loop calculated by Eq. (2.20b).

At all the temperatures chosen the separation between the adjoint oscillation peaks corresponded for all samples to a change of the flux through a single loop by $\phi_0/2$. One observes, however, in the curves of Fig. 13 an interesting feature (most clearly pronounced in the upper curve), which is due to a stronger spin-orbit coupling revealed for this group of samples. As a result, the reversed phase of the oscillations was observed in weak fields. As the magnetic field increased, the decrease of the time τ_H in Eq. (2.15) resulted in a decrease of the relative difference between L'_φ and $\widetilde{L}_\varphi$ and a restoration of the original phase at $H \gtrsim 30$ Oe. As can be seen from Eq. (2.20b), curves of this kind can also be obtained with single loops. Ladan and Maurer (1984) have made a similar observation in the case of cylindrical lithium films prepared by slow metal condensation in several steps, which apparently led to an enhanced impurity absorption.

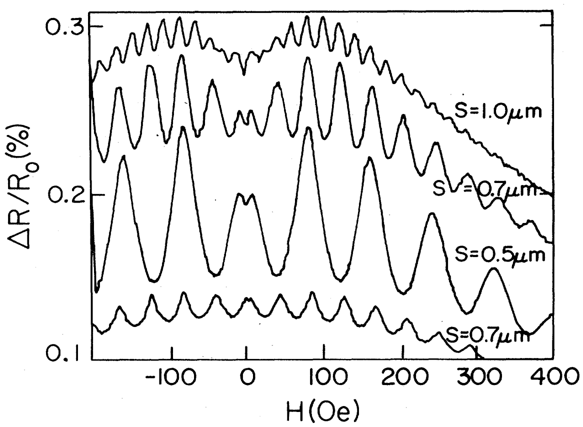

FIG. 13. Transverse magnetoresistance for lithium samples of the same type as for Fig. 12 but with a smaller L_{so}/L_s ratio. The three upper curves are for the necklace geometry; the lower curve is for a mesh (Dolan *et al.*, 1986).

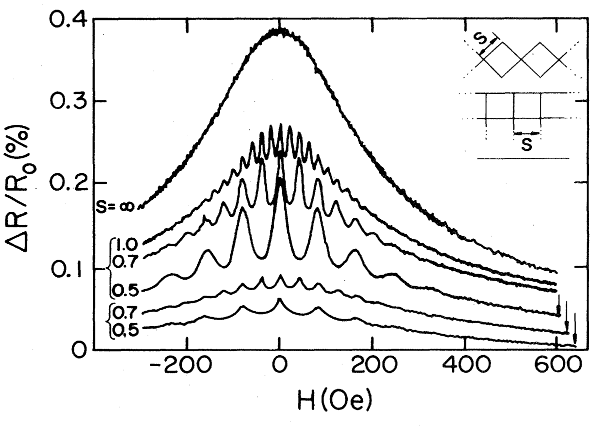

FIG. 12. Transverse magnetoresistance $R(H)$ for $T=0.13$ K for lithium strips 21 ± 1 nm thick and 55 ± 7 nm wide. Uppermost curve, wire control sample; next three curves, three necklace arrays; bottom two curves, two meshes. Some of the curves have been displaced vertically for clarity. The size S in μm of the unit cell side is indicated next to each curve. The upper right-hand sketches define the necklace, mesh, and control geometries, respectively. Measurements of $R(H,T)$ on the control sample yielded values for the diffusion L_φ of $(1.85\pm0.1)\times T^{-1}$ μm, $L_{so}=2.3\pm0.2$ μm, $L_s=3.1\pm0.2$ μm (Dolan *et al.*, 1985).

The dependence of the shape and amplitude of the $\Delta R(H)$ oscillations on the type of multiply connected structures made up of quasi-one-dimensional conductors was theoretically studied by Douçot and Rammal (1985) for the following cases.

(i) Ladders or arrays consisting of two or more loops. It was found that in the case of two loops of different area,

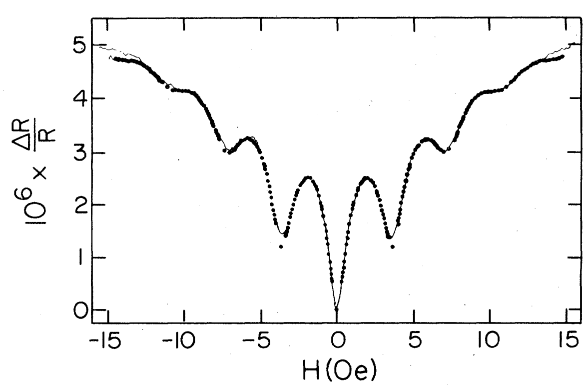

FIG. 14. Experimental vs theoretical magnetoresistance of a honeycomb network. Solid curve, transverse magnetoresistance $R(H)$ for $T=133$ mK for a planar copper honeycomb network of 2.7×10^6 hexahedra with a side of 1.5 μm, strip thickness 80 nm, $R_{300}/R_{4.2}=3$ (Pannetier, Chaussy, Rammal, and Gandit, 1984); ● theoretical calculations made by Douçot and Rammal (1985) for magnetoresistance of a network of the above size for $L_\varphi=5.36$ μm and $L_{so}=3.12$ μm.

regular oscillations appear when the ratio of magnetic fluxes through the loops is of the form n/m, with n and m both integer. In the case of interconnected identical loops with a perimeter $4S$ with $L_\varphi/S \to \infty$, the oscillation shape is distorted in a manner qualitatively similar to that for a multiple traversal of a ring by two conjugated waves, where, according to Eq. (2.12), the oscillation shape, which is sinusoidal for small L_φ/S, becomes Lorentzian. These results do not depend on whether one considers arrays of rings with common nodes or ladders in which loops have common sections (Fig. 15, curves a and b).

(ii) Infinite square networks. The localization correction to the resistivity $\Delta R(H)$ in this case is based on the solution to the problem of the energy spectrum of an electron propagating in a square lattice with a period S in a magnetic field H normal to the lattice (Hofstadter, 1976, and references therein). Despite the complexity of the spectrum, the curve $\Delta R(\phi/\phi_0)$, where $\phi = S^2 H$, turns out to be periodic and to have maxima at integer values of $2\phi/\phi_0$ (Fig. 15, curve c).

For $H=0$ and $S \ll L_\varphi$, the resistivity correction in case (ii) will be

$$\frac{\Delta R}{R} = \frac{e^2}{\pi^2 \hbar \sigma} \frac{L_\varphi}{q} \frac{S}{L_\varphi} \ln \frac{L_\varphi}{S} , \tag{2.21}$$

where σ is the conductivity and q is the wire cross-section area. One readily sees that Eq. (2.21) can be derived from Eq. (2.3) for the correction to the resistivity of a solid film with the mean free path l replaced by $S \gg l$.

The relative magnitude of effects for samples of different type (single-loop, ladder, network) and size prepared of the same wire is determined for $S \ll L_\varphi$ by the following relations: (A) Near the minima, i.e., at the noninteger values of $2\phi/\phi_0$, $\Delta R/R \propto S/L_\varphi$ for samples

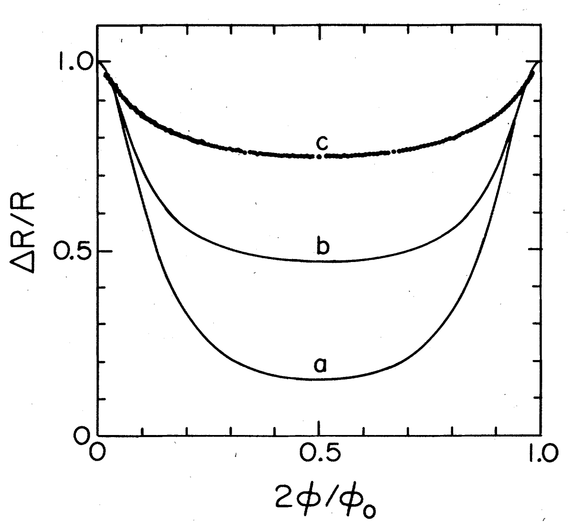

FIG. 15. Reduced magnetoresistance $\Delta R/\Delta R_{\max}$ vs reduced flux $2\phi/\phi_0$ for three types of samples made up of square loops with a side S, for $S/L_\varphi = 0.2$. (a) Single loop; (b) ladder; (c) network (Douçot and Rammal, 1985).

of any type. (B) At the maxima, i.e., at the integer values of $2\phi/\phi_0$,

$$\Delta R/R \propto \begin{cases} L_\varphi/S & \text{for a loop ,} \\ \text{const} & \text{for a ladder ,} \\ (S/L_\varphi)\ln L_\varphi/S & \text{for a network .} \end{cases}$$

Figure 15 presents $\Delta R(\phi)/R$ curves calculated for these cases, neglecting the flux through the conductors, for a particular case of $S/L_\varphi = 0.2$.

In a qualitative agreement with these results, the peaks in the experimental curves in Figs. 12 and 13 decrease in amplitude and become sharper for networks compared with arrays made up of loops of the same size.

(iii) Honeycomb networks. To carry out a qualitative comparison with the results of Pannetier *et al.* (1985), Douçot and Rammal (1985) took into account the dependence of L_φ on H defined by Eq. (2.15), where the ring wall thickness a is replaced by the width of the strips making up the network. The spin-orbit coupling reversing the sign of magnetoresistance was also included in the calculations. Figure 14 shows the close resemblance in the shapes of the theoretical and experimental curves reached for a copper honeycomb network.

3. Experiments on mesoscopic samples

A number of theoretical studies have analyzed the possibility of observing oscillations with a period ϕ_0 in disordered conductors (Browne *et al.*, 1984; Carini *et al.*, 1984; Gefen *et al.*, 1984). In particular, Gefen *et al.* (1984) investigated the resistance between two points of a one-dimensional ring with two scatterers additionally switched into the branches of the circuit [Fig. 3(a)].

If filaments are strictly one dimensional, the presence of scatterers does not affect the fact that two waves propagating along two branches of a circuit with a flux ϕ accumulate an additional phase difference of $2\pi\phi/\phi_0$. Therefore the coefficient of electron wave transmission through the ring and, hence, the ring conductance oscillate with a period of ϕ_0. In addition to this, oscillations with a period $\phi_0/2$ also arise from the interference of conjugated waves (see Sec. II.A), which, however, have a smaller amplitude in a one-dimensional circuit. Indeed, if the transmission intensity coefficient through one branch of the circuit is t_1, then the oscillations with the period ϕ_0 in the transmitted wave intensity are of order t_1, and those with the period $\phi_0/2$, of order t_1^2 (Gefen *et al.*, 1984).

It would seem that as the number of parallel channels N grows, the oscillations with the period ϕ_0 will add in a random way, resulting in the falling off of relative magnitude of the effect as $N^{-1/2}$, whereas the oscillations with period $\phi_0/2$ should add coherently, and thus in the case of a large number of channels the oscillations with the period ϕ_0 will vanish (Gefen *et al.*, 1984). As will be shown later, however, at $T=0$ the oscillations with the period ϕ_0 do not vanish for any number of channels; instead it is found that the total dephasing results here only from inelastic processes.

Experiments on quasi-one-dimensional rings aimed at revealing regular resistivity oscillations with periods of ϕ_0 and $\phi_0/2$ led to a discovery of a fluctuative dependence of the conductivity of these rings on magnetic field (Umbach *et al.*, 1984; Webb *et al.*, 1984), which did not vary with time. The experiments were carried out on rings made of gold and $Au_{60}Pd_{40}$ alloy. Figure 16 presents the magnetic field dependence of the resistivity of a gold ring 320 nm in diameter and 45 nm wide at different temperatures. This dependence is seen to have a complex fluctuative character, with an amplitude of the order of $10^{-3}R_0$ (R_0 is the sample resistivity). It was not possible to identify a dominant periodicity of the magnetoresistance in these experiments. As the temperature was lowered, this structure grew in amplitude while not shifting with magnetic field. Moreover, no decay of the random oscillations with magnetic field raised up to 8 T was detected. The magnetoresistance of wires made of the same materials was found to have a similar structure.

In their subsequent experiments on gold rings, Webb *et al.* (1985) discovered an oscillation spectrum with a clearly pronounced peak corresponding to the period ϕ_0 (Fig. 17). The observation of this phenomenon was made possible by increasing the ratio of ring diameter to width (784 nm/41 nm). The upper panel of Fig. 17 displays the magnetic field dependence of the resistivity, which is seen to contain, besides a fluctuating component, oscillations with a periodicity close to ϕ_0. Shown below is the Fourier power spectrum of the magnetoresistance:

$$\left| R\left[\frac{1}{\Delta H}\right] \right|^2 = \int_{-\infty}^{\infty} dH \int_{-\infty}^{+\infty} dh_0 e^{ih_0/\Delta H} \times R(H)R(H+h_0) , \tag{2.22}$$

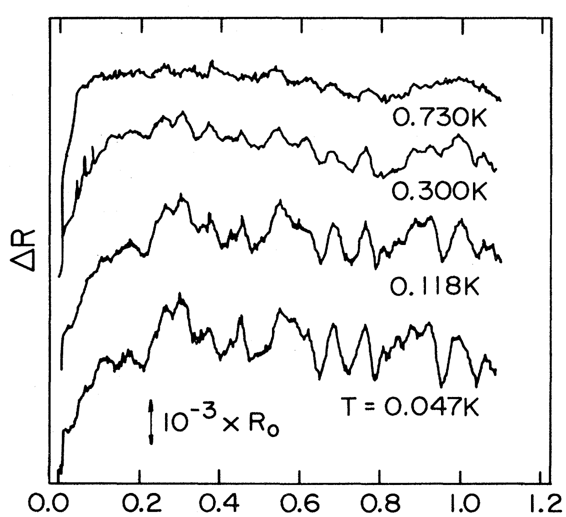

FIG. 16. Magnetoresistance of a gold ring vs magnetic field for different temperatures (Umbach *et al.*, 1984). Ring resistivity for $H=0$ is $R_0=7.7\ \Omega$.

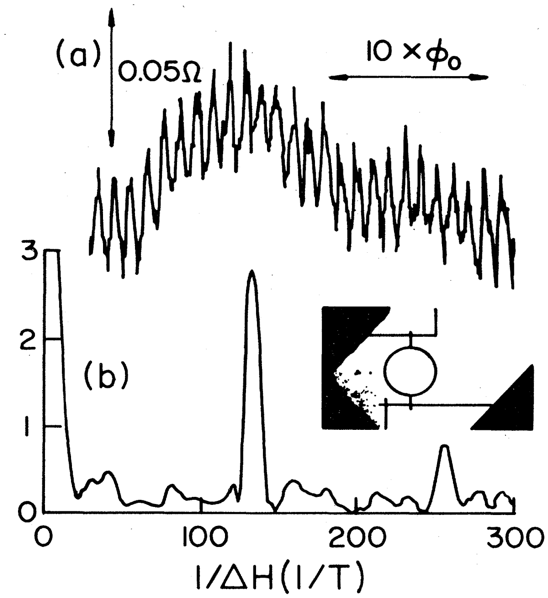

FIG. 17. (a) Gold ring magnetoresistance at 0.01 K (Webb *et al.*, 1985). (b) Fourier power spectrum in arbitrary units. Inset: picture of the ring.

where $R(1/\Delta H)$ is the Fourier power spectrum of the sample impedance.

The Fourier power spectrum of the magnetoresistance has peaks at $1/\Delta H=0$, 131, and 260 T^{-1}. The maximum at $1/\Delta H=131\ T^{-1}$ corresponds to the presence in the fluctuation spectrum of regular oscillations of a period of ϕ_0/S, and that at $1/\Delta H=260\ T^{-1}$ of a period $\phi_0/2S$ (S being the ring area).

Chandrasekhar *et al.* (1985) studied magnetic field effects on one-dimensional aluminum and silver rings. They succeeded in observing, in both cases, resistivity oscillations with a periodicity of $\phi_0/2$; the silver rings revealed at the same time the presence of oscillations of period ϕ_0. Figure 18 presents magnetic field dependences of the magnetoresistance for a silver ring. The observed behavior at low fields can be fitted by a sum of two oscillatory contributions, namely, one of period $\phi_0/2$ (the periodicity in the magnetic field being 24 Oe), described by Eq. (2.20b), and one of period ϕ_0. However, the most convincing evidence for the existence of oscillations of period ϕ_0 was obtained in the high-field domain, where the $\phi_0/2$ oscillations have already decayed. In this domain one sees clearly oscillations with a Fourier power spectrum peaking at $\Delta H \simeq 52$ Oe, which corresponds within 10% to the flux period ϕ_0.

Datta *et al.* (1985) investigated interference effects in parallel quantum wells in the GaAs-AlGaAs structure and observed resistance oscillations with the periodicity ϕ_0. The oscillation amplitude was approximately 10^{-3} of the total resistance. Washburn *et al.* (1985) studied the temperature dependence of the amplitude of oscillations with ϕ_0 and $\phi_0/2$ periods in polycrystalline Au films. They found that the Fourier spectrum amplitude of the

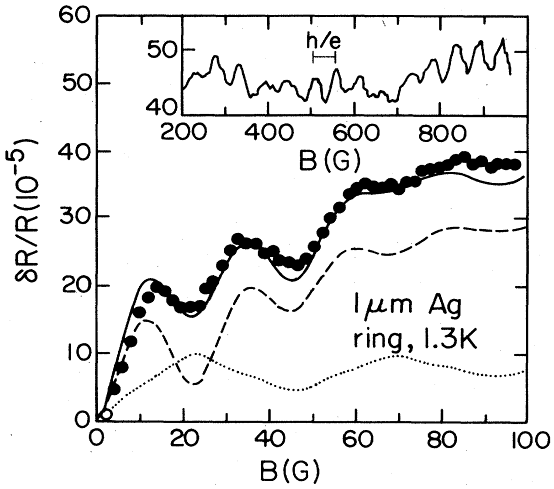

FIG. 18. Magnetoresistance of a 1-μm-diameter silver ring (Chandrasekhar *et al.*, 1985); ●, experiment; dashed curve, calculated oscillations according to Eq. (2.20a), with a periodicity of $hc/2e$; dotted curve, oscillations of period hc/e; solid curve, sum of the two contributions. Inset: oscillations at a high field with a period hc/e (periodicity in magnetic field $\Delta B \simeq 52$ G).

mesoscopic fluctuations in small rings is described by a temperature dependence of $T^{-1/2}$.

Umbach *et al.* (1984), Webb *et al.* (1984), Chandrasekhar *et al.* (1985), and Webb *et al.* (1985), showed experimentally that the complex magnetic field dependence of the magnetoresistance is closely related to the statistical properties of the sample proper. Umbach *et al.* (1986) studied the magnetoresistance oscillations of arrays of submicron-size silver rings connected with one another in series by bridges. The arrays consisted of 1, 3, 10, and 30 rings. It was found that the amplitude of oscillations of period ϕ_0 falls off with increasing number of series-connected rings N as $1/\sqrt{N}$, whereas that of oscillations with a periodicity of $\phi_0/2$ does not change. We should stress that if the fluctuations are large compared to the average conductance, the dominant periodicity in a single loop is ϕ_0, while in the opposite case it is $\phi_0/2$ (Büttiker *et al.*, 1985; Stone and Imry, 1986).

Similar statistical phenomena were found earlier in strongly localized systems by Fowler *et al.* (1982) and Kwasnick *et al.* (1984), who revealed a strongly fluctuating dependence of the conductivity of quasi-one-dimensional channels in metal-oxide-semiconductor field effect transistors (MOSFET's) on gate voltage. At first, Azbel' (1983) attempted to explain these results as due to the resonance tunneling of electrons in a one-dimensional system. Lifshiftz and Kirpichenkov (1979) had shown earlier that the phenomenon of electron resonance tunneling is essential in the statistical fluctuations of the transverse film resistance. In this model, the resistance is dominated by a random resonance pair of states. The probability of their appearance depends on the specific arrangement of impurities and fluctuates strongly. The conductance fluctuations should manifest themselves as large narrow spikes with variation of Fermi energy, their width being on the order of the width of the resonance levels. However, the experimental behavior of conductance in one-dimensional MOSFET's is at odds with this picture. Lee (1984) was able to explain these results qualitatively as due to fluctuations in one-dimensional Mott variable-range hopping conduction. In the one-dimensional case the conduction is controlled by the bottleneck effect. As the chemical potential varies, the impurity pair controlling the electron motion is replaced by another in a random way.

These and other studies (Altshuler, 1985; Altshuler and Khmelnitskii, 1985; Lee and Stone, 1985; Stone, 1985) have led to the understanding that the phenomena in question reflect the existence of a random potential in the sample, the properties of tiny samples being determined by the actual form of the random potential, which may fluctuate from sample to sample.

It was earlier believed that in disordered quantum systems the length scale on which any physical quantity becomes averaged out is the mean separation between impurities, $N_i^{-1/3}$, the relative fluctuations being proportional to $1/\sqrt{N_i}$. However, Altshuler (1985), Lee and Stone (1985), and Stone (1985) showed that this length scale exceeds by far $N_i^{-1/3}$, tending to infinity for $T \to 0$. Samples of size smaller than this length scale are now conventionally called mesoscopic (Stone, 1985). In such samples no total averaging of properties over the impurity distribution occurs, which results in destruction of crystal symmetry and the appearance in such samples of a number of phenomena typical of low-symmetry crystals (Altshuler and Khmelnitskii, 1985).

Stone (1985) performed numerical simulations of the magnetic field behavior of a mesoscopic sample, using the model of Lee and Fisher (1981), and revealed the appearance of an aperiodic structure in the dependence of the resistivity on magnetic field, similar to that observed in experiments.

In our discussion of fluctuation phenomena we are going to investigate the Fourier power spectrum of magnetoresistance (Altshuler and Khmelnitskii, 1985; Lee and Stone, 1985):

$$F_{\alpha\beta\gamma\delta}(h_0,H) = \langle \sigma_{\alpha\beta}(H)\sigma_{\gamma\delta}(H+h_0)\rangle - \langle \sigma_{\alpha\beta}(H)\rangle \langle \sigma_{\gamma\delta}(H+h_0)\rangle \ .$$

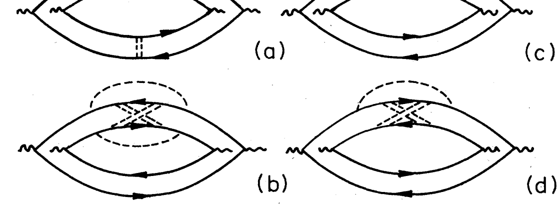

FIG. 19. Diagrams yielding main contribution to the correlation function of conductivities. Diagrams (c) and (d) cancel.

Here $\sigma_{\alpha\beta}(H)$ is the conductivity of the sample, and the angular brackets denote averaging over the various random potentials. The quantity $\sqrt{F(0,H)}$ characterizes the amplitude of a random variation of $\sigma(H)$, and the magnetic field scale h_0 on which $F(h_0,H)$ varies determines the field scale H typical of the variation of $\sigma(H)$. Using the diagrammatic technique for an electron in the field of impurities (Abrikosov *et al.*, 1963), one can represent the quantity $F(h_0,H)$ as a sum of diagrams (Fig. 19) corresponding to the expression (Altshuler and Khmelnitskii, 1985)

$$F_{\alpha\beta\gamma\delta}=4\left[\frac{\sigma}{\pi\nu V}\right]^2\int d\varepsilon\, d\varepsilon'\frac{\partial n}{\partial\varepsilon}\frac{\partial n}{\partial\varepsilon'}\int d\mathbf{r}\{(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\beta\gamma}\delta_{\alpha\delta})[\,|P^{(D)}_{\varepsilon-\varepsilon'}(0,\mathbf{r})|^2+|P^{(C)}_{\varepsilon-\varepsilon'}(0,\mathbf{r})|^2]$$
$$+\delta_{\alpha\beta}\delta_{\gamma\delta}\,\mathrm{Re}[P^{(D)}_{\varepsilon-\varepsilon'}(0,\mathbf{r})P^{(D)}_{\varepsilon-\varepsilon'}(\mathbf{r},0)+P^{(C)}_{\varepsilon-\varepsilon'}(0,\mathbf{r})P^{(C)}_{\varepsilon-\varepsilon'}(\mathbf{r},0)]\}\ . \tag{2.23}$$

Here V is the sample size, $n=[\exp(+\varepsilon/T)+1]^{-1}$. The calculation includes contributions from graphs that differ in the direction of the arrows. The graphs 19(c) and 19(d) cancel. We assume the average conductivity σ to be isotropic. The first term in the curly brackets is due to fluctuations in the diffusion coefficients [Fig. 19(a)], and the last one originates from the density-of-states fluctuations [Fig. 19(b)] (Altshuler and Shklovskii, 1986). The quantity $P_\omega^{(D),(C)}$ is the diffusion (cooperon), differing from the one introduced earlier in Eq. (2.9) and satisfying the equation

$$\left[-i\omega+D\left[-i\partial_{C,D}-\frac{e}{c\hbar}\mathbf{a}\right]^2+\frac{1}{\tau_{\rm in}}\right]P_\omega^{(C,D)}(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}')\ . \tag{2.24}$$

Here $\partial_D=\nabla$, $\partial_C=\nabla-(2ie/c\hbar)\mathbf{A}$, for which $\mathrm{curl}\,\mathbf{A}=\mathbf{H}$, $\mathrm{curl}\,\mathbf{a}=\mathbf{h}_0$. Note that the equations for $P^{(C)}$ and $P^{(D)}$ contain the same time $\tau_{\rm in}$, which, generally speaking, does not coincide with the energy relaxation time. The reason for this is that *any inelastic* collision destroys *any* correlation between the conductivities of two samples (Lee and Stone, 1985).

The magnetic vector potential $\mathbf{a}$ appears in the equation for diffusion because we are actually interested here in the correlation in the motion of a particle and hole propagating in different fields, $\mathbf{A}+\mathbf{a}$ and $\mathbf{A}$, respectively. Therefore the wave functions will have a phase difference

$$\Delta\varphi=\frac{e}{c\hbar}\int_{\mathbf{r}_1}^{\mathbf{r}_2}dl\,\mathbf{a}(\mathbf{r})\ .$$

The same considerations apply to the cooperon, which reflects the existence of correlation in the motion of two electrons propagating in different fields, $\mathbf{A}+\mathbf{a}$ and $\mathbf{A}$. The total phase difference accumulated by the two electrons in their motion will be

$$\Delta\varphi=\frac{e}{c\hbar}\int_{\mathbf{r}_1}^{\mathbf{r}_2}dl(2\mathbf{A}+\mathbf{a})\ .$$

Calculation of the correlation function [Eq. (2.23)] for $T=0$ predicts size-independent fluctuations in conductance of order $e^2/\hbar$. This implies that the length scale on which the fluctuations self-average is much larger than usually expected and tends to infinity as T goes to zero.

Lee and Stone (1985) calculated the Fourier power spectrum for the conductivity of a thin wire in magnetic fields and came to the following conclusions.

(1) The fact that Eq. (2.24) for the diffusion contains only the quantity h_0 implies that the fluctuations should remain practically unchanged up to the classically strong fields, which is exactly what is observed in experiments.

(2) For $T=0$ the magnetic field scale on which the Fourier power spectrum of the conductivity fluctuations undergoes variation is defined by the condition

$$\Delta\phi_c=\phi_0\sqrt{3}\ ,$$

where $\Delta\phi_c$ is the magnetic flux variation through the sample area perpendicular to the magnetic field. This field scale was obtained earlier from the numerical simulations of Stone (1985) and is likewise in accord with experiments.

The appearance of nondiagonal components of the conductivity tensor in the mesostructure, due to the breaking of space symmetry, results in a dependence of fluctuations on the sign of magnetic field. The Onsager relation is $\sigma_{ik}(H)=\sigma_{ki}(-H)$. When magnetoresistance is measured with the four-terminal method, symmetry under magnetic field reversal will be observed only if the voltage and current probes are interchanged. Büttiker (1986) discussed this asymmetry in terms of the Landauer formula (Landauer, 1970). He has shown that the same phenomenon takes place in Aharonov-Bohm experiments. If voltage is applied to probes 2 and 4 and current is measured between probes 1 and 3, then the oscillatory part of the loop resistance is

$$\delta R_{13,24}=\delta R\cos\left[\frac{2\pi\Delta\phi}{\phi_0}+\varphi_\alpha\right],$$

where φ_α is the phase, which is specific to the sample. If the voltage and current probes are interchanged, then

$$\delta R_{24,13}=\delta R\cos\left[\frac{2\pi\Delta\phi}{\phi_0}-\varphi_\alpha\right].$$

If the resistance $\delta R_{14,23}$ is measured, then

$$\delta R_{14,23}=\delta R \cos\left[\frac{2\pi\Delta\phi}{\phi_0}+\varphi_\beta\right]$$

and $\varphi_\alpha \neq \varphi_\beta$.

The first observations of asymmetry were made by Umbach *et al.* (1984) and Webb *et al.* (1985). A detailed comparison of the experimental and theoretical results was carried out by Benoit *et al.* (1986), which confirmed Büttiker's theory (1986).

We are going first to present the form of the Fourier power spectrum for a ring of radius R placed in a field of a vector potential $\mathbf{A}$ directed tangentially to the circumference, after which we shall discuss how the result should change when the magnetic field in the annulus itself is taken into account. The calculation of $F(\Delta\phi,\phi)$ is identical to that of $\Delta\sigma_1$ in Sec. II.B. For $L_T=\sqrt{D\hbar/2T} \ll L_{\rm in}$ the contribution due to the density-of-states fluctuations can be disregarded, yielding

$$\begin{aligned}\mathscr{F}(\Delta\phi,\phi)&=\langle \delta G(H)\delta G(H+h_0)\rangle\\&=\frac{16}{3}\left[\frac{e^2}{\pi\hbar}\frac{L_T}{R}\right]^2\sum_{l=-\infty}^{+\infty}\left[\frac{1}{\left[\frac{R}{L_{\rm in}}\right]^2+\left[l-\frac{\Delta\phi}{\phi_0}\right]^2}+\frac{1}{\left[\frac{R}{L_{\rm in}}\right]^2+\left[l-\frac{2\phi+\Delta\phi}{\phi_0}\right]^2}\right]\\&=\mathscr{F}_D(\Delta\phi)+\mathscr{F}_C(2\phi+\Delta\phi)\ ,\end{aligned} \tag{2.25}$$

where $G(H)$ is the conductance of a ring,[1] and for the flux we obtain

$$\Delta\phi=\oint \mathbf{a}\,dl=Sh_0$$

(S being the ring area).

As can be seen from Eq. (2.25), the conductance Fourier power spectrum is periodic in $\Delta\phi$ with a period ϕ_0, and in ϕ with a period $\phi_0/2$. Just as in Sec. II.B, in the presence of magnetic field, to $1/\tau_{\rm in}$ one should add the decay $1/\tau_H$ [Eq. (2.15)] due to the ring's being of finite thickness a, so that different magnetic fluxes are confined within different electron trajectories. Therefore the first and second terms in Eq. (2.25) will decay in different ways. For the first term the decay will be

$$\gamma_D=\frac{1}{\tau_{\rm in}}+\frac{1}{\tau_D}=\frac{1}{\tau_{\rm in}}+\frac{Da^2}{12}\left[\frac{h_0}{2\pi\phi_0}\right]^2, \tag{2.26a}$$

whereas for the second

$$\gamma_C=\frac{1}{\tau_{\rm in}}+\frac{1}{\tau_C}=\frac{1}{\tau_{\rm in}}+\frac{Da^2}{3}\left[\frac{H+h_0/2}{2\pi\phi_0}\right]^2. \tag{2.26b}$$

From Eqs. (2.25), (2.26a), and (2.26b) one sees that the cooperon part of the conductance fluctuations, unlike the diffusion part, falls off with increasing magnetic field, and for high fields H the conductance fluctuations depend on h_0 only and do not decrease as the absolute value of H is increased. The maximum of the function $\mathscr{F}(\Delta\phi,\phi)$ falls off with increasing temperature as $\tau_{\rm in}(T)/T \propto T^{-(1+p)/2}$.

If the condition $T \ll \hbar/\tau_{\rm in}$ is fulfilled, then

$$\mathscr{F}(\Delta\phi,\phi)=24\left[\frac{e^2}{\pi^2\hbar}\right]^2\sum_{l=-\infty}^{+\infty}\left[\frac{1}{\left[\left[l-\frac{\Delta\phi}{\phi_0}\right]^2+\left[\frac{R}{L_D}\right]^2\right]^2}+\frac{1}{\left[\left[l-\frac{2\phi+\Delta\phi}{\phi_0}\right]^2+\left[\frac{R}{L_C}\right]^2\right]^2}\right]. \tag{2.27}$$

As can be seen from Eq. (2.27), for $T\to 0$ and $L_{C,D}\to\infty$, the width of the maxima of the function $\mathscr{F}(\Delta\phi,\phi)$ tends to zero, thus implying the absence of phase relaxation effects when no inelastic processes are involved.

Equation (2.25) can be rewritten in a form convenient for comparison with Fourier spectra,

[1]Conductance fluctuations of a one-dimensional ring depend on more than just the material of the ring. Indeed, at $T=0$ they depend also on the experimental geometry. Therefore Eq. (2.25) is valid for $L_{\rm in}<2\pi R$. Isawa *et al.* (1986) have discussed the effect of probes on experimental results. Di Vincenzo and Stone (unpublished) have investigated the difference between the two- and four-terminal configurations. We are grateful to A. D. Stone, who drew our attention to the importance of the actual measurement methods used.

$$\mathscr{F}_D(\Delta\phi)=\frac{16\pi}{3}\left[\frac{e^2}{\pi\hbar}\right]^2\frac{L_T^2L_D}{R^3}\left[1+2\sum_{n=1}^{\infty}e^{-(2\pi R/L_D)n}\cos 2\pi n\frac{\Delta\phi}{\phi_0}\right],$$
$$\mathscr{F}_C(\Delta\phi+2\phi)=\frac{16\pi}{3}\left[\frac{e^2}{\pi\hbar}\right]^2\frac{L_T^2L_C}{R^3}\left[1+2\sum_{n=1}^{\infty}e^{-2\pi Rn/L_C}\cos 2\pi n\frac{\Delta\phi+2\phi}{\phi_0}\right]. \tag{2.28}$$

When comparing theory with experiment, one should bear in mind that in the calculation of an experimental spectrum one performs averaging only over magnetic field [see Eq. (2.22)]. Therefore the theoretical results obtained by averaging over the impurity distribution may be compared with experimental data only when these two kinds of averaging are equivalent or, in other words, *when the ergodic hypothesis is valid* (Lee and Stone, 1985). Rigorous proof of the validity of this ergodicity has been presented by Altshuler, Kravtsov, and Lerner (1986), who use a nonlinear σ model.

It should be noted in conclusion that Altshuler and Spivak (1985) and Feng *et al.* (1986) drew attention to the fact that the resistivity fluctuations are extremely sensitive to a small variation of the random potential. As follows from their estimate, a change in the position of only one impurity may change the conductance of a mesoscopic film at $T=0$ by a finite amount of the order of $e^2/\hbar$. This sensitivity would permit measurement of superslow processes in metals, for instance, the time scale of processes in spin glasses and metal glasses or the diffusion of impurities at low temperatures.

Feng *et al.* (1986) showed that mesoscopic fluctuations of resistance could be responsible for room-temperature $1/f$ noise in disordered metals and an anomalous low-temperature $1/f$ noise in metallic glasses due to motion of a single scattering center. At room temperature the resistance fluctuations are due to thermally activated motion of defects, while at low temperatures in metallic glasses they are due to the tunneling in two-level systems.

Skocpol *et al.* (1986) have observed random time switching between different "magnetofingerprints" on Si MOSFET's with low carrier mobility. Most likely they correspond to different configurations of scatterers.

III. OSCILLATORY EFFECTS IN SUPERCONDUCTORS

As pointed out in the Introduction, Little and Parks (1962) predicted and experimentally revealed (Parks and Little, 1964) the oscillatory dependence of the superconducting transition temperature T_c on the magnetic flux through a thin-walled cylinder. This effect is intimately connected with the phenomenon of magnetic flux quantization in superconductors. For a quantitative evaluation of this effect, consider the Ginzburg-Landau equation for a dirty superconductor (see, for example, the monograph of Saint-James *et al.*, 1969),

$$\left[\frac{\pi}{8}\frac{D}{\hbar T_c}\left[-i\hbar\nabla-\frac{2e\mathbf{A}}{c}\right]^2+\frac{T-T_c}{T_c}\right]\Delta(\mathbf{r})+\frac{7\xi(3)}{8\pi^2T_c^2}|\Delta(\mathbf{r})|^2\Delta(\mathbf{r})=0\,, \tag{3.1}$$

where T_c is the superconducting transition temperature in the absence of magnetic field and $\Delta(\mathbf{r})$ is the order parameter for the superconductor.

If the cylinder wall thickness a is much less than the magnetic field penetration depth $\lambda(T)$ and coherence length $\xi(T)$, then in a magnetic field parallel to the cylinder axis the absolute magnitude of the order parameter $|\Delta|$ and the superfluid current should be constant within the cylinder walls. One may therefore assume that

$$\Delta=|\Delta|\exp(i\alpha y)\,, \tag{3.2}$$

where y is measured along the cylinder circumference.

From the condition of the order parameter's remaining single-valued when traveling around a contour it follows that

$$\alpha=\frac{2\pi n}{L}=\frac{n}{R}\,,\quad n=0,1,2,\ldots, \tag{3.3}$$

where R is the cylinder radius. Substituting Eq. (3.2) in Eq. (3.1) and taking into account Eq. (3.3) yields for the difference between the free energies of a superconductor and of a normal metal

$$\delta F_S=\frac{\pi}{16}\frac{\hbar D}{T_cR^2}|\Delta|^2\left[n-\frac{2\phi}{\phi_0}\right]^2+\frac{T-T_c}{T_c}\frac{|\Delta|^2}{2}+\frac{7\zeta(3)}{32\pi^2T_c^2}|\Delta|^4\,, \tag{3.4}$$

which becomes a minimum each time the quantity $(n-2\phi/\phi_0)^2$ is minimal. Therefore (Tinkham, 1963)

$$\delta T_c=-\frac{\pi}{8}\frac{\hbar D}{R^2}\min\left[n-\frac{2\phi}{\phi_0}\right]^2=-T_c\left[\frac{\xi_0}{R}\right]^2\min\left[n-\frac{2\phi}{\phi_0}\right]^2, \tag{3.5}$$

where $\xi_0=(\pi D/8T_c)^{1/2}$.

In these states the superfluid current is likewise minimal,

$$j_s = \frac{\pi\sigma}{8eT_c}\left[i\left[\Delta\frac{\partial\Delta^*}{\partial y} - \Delta^*\frac{\partial\Delta}{\partial y}\right] - \frac{4e}{\hbar c}A\,|\Delta|^2\right]$$
$$= \frac{\pi}{4}\frac{|\Delta|^2}{T_c}\frac{\sigma}{eR}\min\left| n - \frac{2\phi}{\phi_0}\right| . \qquad (3.6)$$

As follows from Eqs. (3.5) and (3.6), both the shift of the superconducting transition temperature and the magnitude of the superfluid current are functions of magnetic flux, with a periodicity of $\phi_0/2$. Figure 22 below shows the phase diagram of a superconducting cylinder in an external magnetic field.

The Little-Parks effect was studied by observing the variation of the cylinder resistivity in a longitudinal magnetic field at a temperature close to T_c. If the transition point shifts by an amount greater than the temperature spread of the transition proper, then the oscillations should be asymmetric in shape, corresponding to the resistivity's switching alternately on and off.

The temperature dependence of the cylinder conductance in the vicinity of T_c is related to the fluctuation conductivity. In the region $T - T_c \ll \hbar/\tau_\varphi$ the major contribution to the fluctuation component in the conductivity comes from the Aslamazov-Larkin (1968) correction. For zero magnetic field and for a square film, this correction has the form

$$\delta G = \frac{e^2}{16\hbar}\frac{T_c}{T-T_c} .$$

The fluctuation correction of Aslamazov and Larkin to the conductance of a thin-walled cylinder in the field of a vector potential **A** was studied by Kulik and Mal'chuzhenko (1971). For $T > T_{c0}$ it can be written as

$$\delta G = \frac{\pi e^2}{16\hbar}\frac{\xi_0}{b}\sum_{n=-\infty}^{+\infty}\left[\frac{T-T_c}{T_c} + \frac{\xi_0^2}{R^2}\left[n - \frac{2\phi}{\phi_0}\right]^2\right]^{-3/2} . \qquad (3.7)$$

As is evident from Eq. (3.7), the temperature at which ΔG tends to infinity is a periodic function of magnetic flux, in

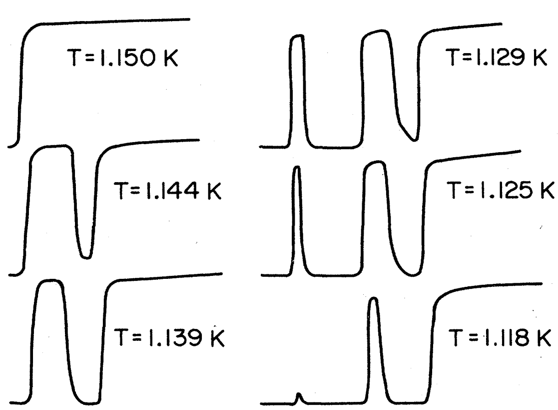

FIG. 20. The asymmetric shape of resistivity oscillations of a 1.33-μm-diameter aluminum cylinder (Groff and Parks, 1968).

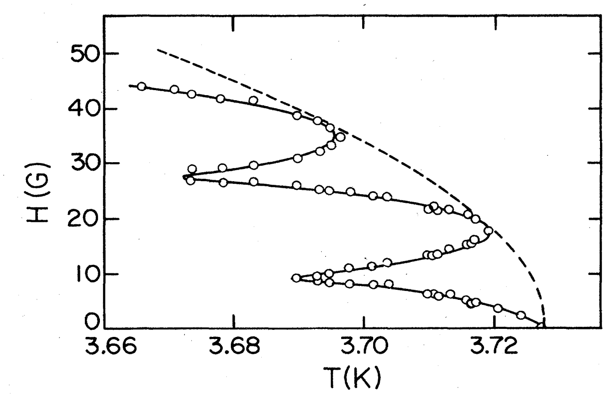

FIG. 21. Phase diagram $H(T)$ of a 1.2-μm-diameter tin cylinder: dashed curve, Eq. (3.8) for $\lambda(0) = 642$ Å; solid curve, Eq. (3.10) for $H_c = 303$ Oe, $T_c = 3.73$ K, $\lambda(0) = 620$ Å (Groff and Parks, 1968).

accordance with Eq. (3.5). Equation (3.7) can be transformed to

$$\delta G = \frac{e^2}{16\hbar}\frac{2\pi R}{b}\frac{T_c}{T-T_c}\left[1 + 2\sum_{p=1}^{\infty}\frac{2\pi Rp}{\xi}K_1\left[\frac{2\pi R}{\xi}p\right] \times \cos 2\pi p\frac{2\phi}{\phi_0}\right] , \qquad (3.7a)$$

where $\xi = [\pi D/8(T-T_c)]^{1/2}$ is the coherence length in a superconductor and $K_1(x)$ is the Macdonald function. As can be seen from Eq. (3.7a), the magnitude of the conductance oscillations falls off exponentially with increasing cylinder perimeter for $2\pi R \gg \xi$. However, for $2\pi R \simeq \xi$ the oscillatory term and the monotonic component become comparable.

Parks and Little (1964) and Groff and Parks (1968) carried out extensive observations of this effect on superconducting cylinders of tin and aluminum. Groff and Parks (1968) succeeded in observing an asymmetric shape in the oscillations (Fig. 20) and drawing an experimental phase diagram (Fig. 21). The phase diagram is seen to differ from the theory (Fig. 22). This may be accounted for by

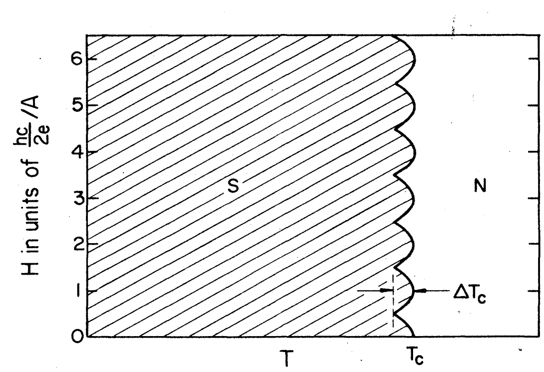

FIG. 22. Phase diagram of a superconducting cylinder with magnetic flux (Little and Parks, 1962). A is the cylinder cross-section area.

the fact that the magnetic field monotonically shifts the superconducting transition point to lower temperatures. By including this effect (Groff and Parks, 1968), one can fit the envelope (dashed line in Fig. 21) by the following expression for $a \ll 2R$ (R is the mean radius):

$$H_{\mathrm{envelope}}(T) \simeq H_{\|}(T)\left[1+\frac{7}{10}\left[\frac{a}{2R}\right]^2\right], \tag{3.8}$$

where

$$H_{\|}(T)=2\sqrt{6}\frac{\lambda(T)}{a}H_c(T) \tag{3.9}$$

is the parallel critical magnetic field of the film, $H_c(T)$ is the thermodynamic critical field of a superconductor, and $\lambda(T)$ is the penetration depth. For the $H(T)$ diagram we now have

$$\frac{T_c(0)-T_c(H)}{T_c(0)}=\frac{\phi_0^2}{32\pi^2\lambda^2(0)R^2H_c^2(0)}\left\{\left[\frac{2\phi}{\phi_0}-n\right]^2\left[1+\left[\frac{a}{2R}\right]^2\right]+\frac{n^2}{3}\frac{a^2}{R^2}+\frac{4}{3}n\left[\frac{2\phi}{\phi_0}-n\right]\left[\frac{a}{2R}\right]^2\right\}. \tag{3.10}$$

The solid line in Fig. 21 is the theoretical curve according to Eq. (3.10), which thus is seen to fit well to the experimental data.

Shablo *et al.* (1974) observed magnetoconductivity oscillations in a thin-walled cylinder of aluminum with an oxygen impurity up to 8 K. These experiments demonstrated that the Aslamazov-Larkin contribution [Eq. (3.7a)] could not account for the temperature dependence of the effect. The development of the theory of oscillations in multiply connected disordered conductors has made it clear that in these experiments what was observed for the first time was the Maki-Thompson contribution to the magnetoresistance oscillations [Eq. (2.20a)]. Gordon (1984), Gordon *et al.* (1984), and Gijs *et al.* (1984b) repeated the experiments of Shablo *et al.* (1974) on samples of pure aluminum and performed a quantitative comparison with theory. Figure 23 shows the experimental data of Gordon *et al.* (1984) for different tempertures. The solid lines present graphically the theoretical relations (2.20a). Figure 24 shows the temperature behavior of the first magnetoresistance oscillation. This dependence is seen to be fitted well by Eq. (2.20a), the Aslamazov-Larkin correction [Eq. (3.7a)] becoming predominant only in the immediate vicinity of the transition point.

Gijs *et al.* (1984b) analyzed the oscillation amplitude and the monotonic contribution in the weak magnetic

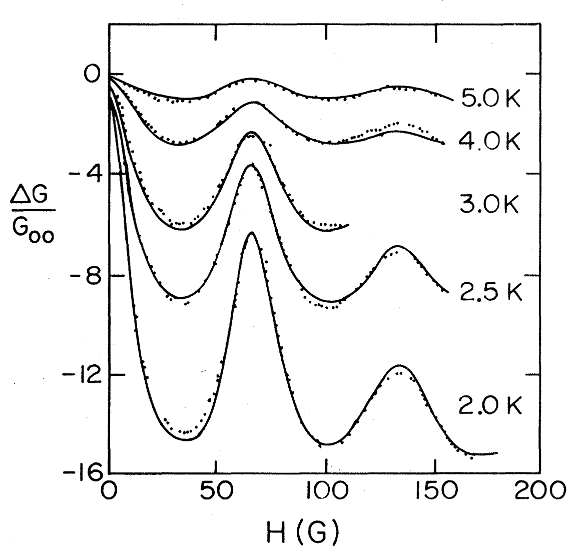

FIG. 23. Magnetoconductance oscillations of an aluminum cylinder (Gordon *et al.*, 1984). Solid curves: theoretical calculations according to Eq. (2.20a). $G_{00}=e^2/\pi^2\hbar$.

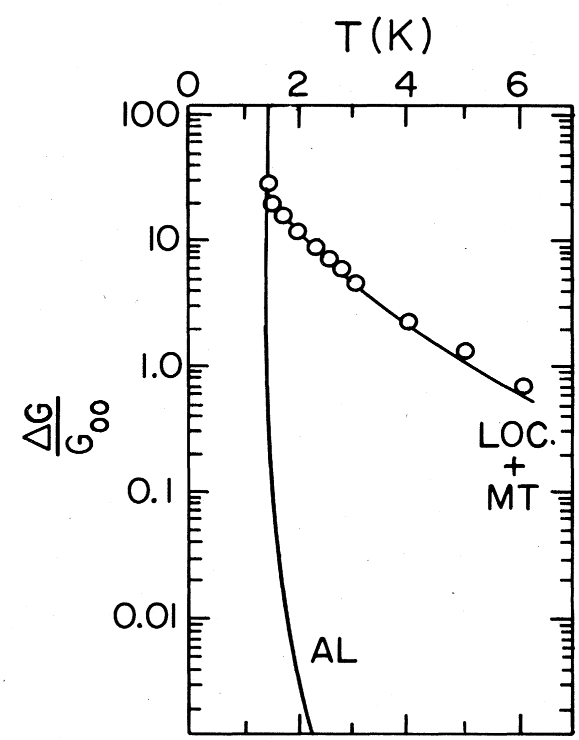

FIG. 24. Amplitude of the first magnetoresistance oscillation vs temperature for aluminum cylinders (Gordon *et al.*, 1984). The curve labeled Loc + MT represents Eq. (2.20a), and that labeled AL presents the Aslamazov-Larkin correction, Eq. (3.7a).

field domain (where β does not depend on magnetic field) and obtained the temperature dependence of the parameter β (Fig. 25). The solid line in Fig. 25 displays the theoretical behavior of $\beta(T)$ predicted by Larkin (1980). In the vicinity of T_c, $\beta(T)$ grows logarithmically. Shown in the same figure is the temperature dependence of τ_φ, which does not disagree with the predicted temperature course $\tau_\varphi \propto T^{-1}$ due to electron-electron scattering (Altshuler, Aronov, and Khmelnitskii, 1982b).

The numerous experiments on superconducting samples have shown that the Little-Parks effect and the magnetoresistance oscillations (Maki-Tompson contribution) in superconducting metals are intimately related and undergo transitions into one another as the temperature varies. The Aslamazov-Larkin contribution to this effect has to our knowledge not yet been revealed unambiguously.

We should stress that the resistance oscillations connected with the Little-Parks effect differ from those in a normal metal. The Aslamazov-Larkin correction describes the contribution of fluctuatively existing Cooper pairs near the transition point T_c to the conductivity.

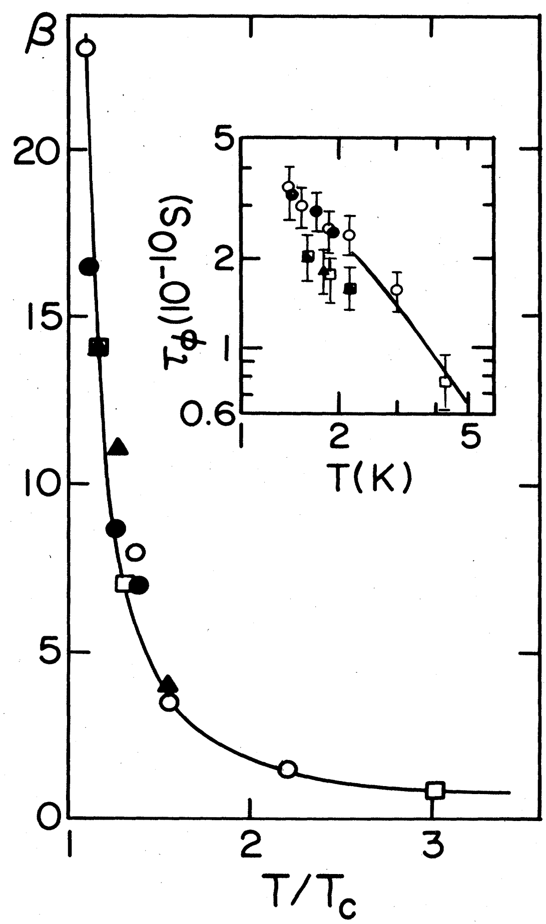

FIG. 25. $\beta(T)$ dependence derived by Gijs *et al.* (1984b) from resistance oscillations (●,▲) and monotonic magnetoresistance (○,□) measured in two Al cylinders. Solid curve, theory by Larkin (1980); Inset: τ_φ vs temperature plot.

The velocity of these pairs is quantized in multiconnected conductors. The oscillation effect in a normal metal is due to the interference of noninteracting electron waves on conjugated trajectories and does not depend on electron-electron interaction. The major corrections due to the electron-electron interaction are Maki-Tompson corrections. They describe the additional contribution to conductivity, originating from the interaction of normal excitations with fluctuative Cooper pairs (and, far from the transition point, with correlated electron pairs with opposite spins and momenta, which have a quantized angular momentum in the multiconnected conductor).

IV. MAGNETORESISTANCE OSCILLATIONS IN HOPPING CONDUCTION

In the preceding sections we have considered resistivity oscillations in multiconnected samples placed in a magnetic field with ϕ_0 and $\phi_0/2$ periods for the metallic conductivity. We now turn to the question of magnetic flux effects in subbarrier electron motion. If there is no subbarrier scattering, then the resistance of a dielectric interferometer has an oscillatory behavior in a magnetic field with a ϕ_0 period. Inclusion of the subbarrier scattering of electrons in Mott's variable-range hopping (VRH) conductivity is a more complex problem.

If the impurity configuration is random, the overlap integral between any two impurities is also a random quantity, falling off exponentially with increasing distance. If the subbarrier scattering is not taken into account, the electron wave function decreases with increasing distance on the scale of one center localization radius. If the subbarrier electron scattering is included (Lifshitz and Kirpichenkov, 1979), the radius of the wave function grows with increasing impurity concentration.

The hopping conductivity is determined by the square of the modulus of the overlap integral, or more exactly, by the log square of the modulus of the overlap integral, which is averaged over the impurity configurations. The question arises whether a periodicity of $\phi_0/2$ exists in the oscillations in the insulating state with Mott's VRH conduction. If it does exist, does it occur simultaneously with the oscillations of period ϕ_0, or does one regime replace the other? And how are the periods replaced? These questions were raised by Nguen *et al.* (1985a, 1985b), who showed that the transition from oscillations of period ϕ_0 to those of period $\phi_0/2$ may occur with increasing impurity concentration or degree of compensation as a phase transition of the second kind. This change in the oscillation period turns out to be intimately connected with a specific phase transition in the sign structure of the Green's function. It should be stressed once more that this new phase transition is not connected in any way with the metal-insulator transition and occurs deep in the insulating state.

Nguen *et al.* (1985a) studied this quantity in the Anderson model on a square lattice consisting of $(n+1)^2=N$ nodes, whose Hamiltonian is

$$\mathscr{H}=\sum_{i=1}^{N}\varepsilon_i a_i^\dagger a_i+\sum_{i\neq j}V_{ij}a_i^\dagger a_j \ , \tag{4.1}$$

where ε_i is the energy at the ith node and where $V_{ij}=V$ for nearest neighbors and zero otherwise. All the energies ε_i, except for ε_1 and ε_N, between which the transition occurs, take on randomly the values $-W$ and W with the probabilities x and $(1-x)$, respectively, where $W\gg|V|$. The energies $\varepsilon_1=\varepsilon_N=0$ and $a_i^\dagger, a_i$ are the creation and annihilation operators at the ith node. For the effective overlap integral between 1 and N one can write

$$I=V\sum_{\{r\}}\prod_{\{i_r\}}\Bigg|_{\varepsilon=0}=V\left[\frac{V}{W}\right]^{2n-1}J \ , \tag{4.2}$$

$$J=\sum_{\{r\}}\prod_{\{i_r\}}\alpha_i \ . \tag{4.3}$$

The summation in Eq. (4.3) is performed over the whole set of paths connecting nodes 1 and N. $\{i_r\}$ are the nodes of such a path other than 1 and N, and $\alpha_i=\pm1$ for $\varepsilon_i=\pm W$. In calculating J one can disregard the contribution coming from paths involving returns, since they have an additional smallness $V/W\ll1$ (Fig. 26).

Such a formulation of the problem simulates a situation typical of the hopping conduction when an electron making a hop between states with energies close to Fermi level μ becomes scattered on the way by many other impurities (see, for example, the monograph by Shklovskii and Efros, 1984). The probability of this hop and the corresponding inverse resistivity of the Miller-Abrahams network are proportional to $|I|^2$. The probability x corresponds to the fraction of impurities with $\varepsilon_l<\mu$, which is determined by the degree of compensation.

Numerical simulations by this model have revealed the existence of a critical value of x_c (in semiconductors it corresponds to a certain degree of compensation or impurity concentration) at which the distribution function of J changes its character.

If $x<x_c$ (in two dimensions $x_c=0.05$), then in most cases $J>0$, and for $x>x_c$ the probabilities for J to acquire positive (P_+) and negative (P_-) values become equal ($P_+=P_-$). Figure 27 shows the dependence of $\Delta P=P_+-P_-$ on x for a 100×100 lattice. As can be seen from the figure, at x_c a phase transition of the second kind occurs for the quantity ΔP, which is the probability to determine the sign of the Green's function.

While the above phase transition resembles that considered in percolation theory, it is at the same time characterized by anomalously large values of the indices. Taking ΔP for the order parameter of this transition, and assuming $\Delta P=(x_c-x)^\beta$, we obtain for the index $\beta\simeq1$. At the transition point the sign correlation function falls off with distance as $r^{-\eta}$, where $\eta\simeq0.7$. The scaling relation for the indices (Shklovskii and Efros, 1984) $2\beta=\nu(\eta+d-2)$ yields $\nu\simeq3$ for $d=2$ [where ν is the index of the correlation radius, $\xi=(x-x_c)^{-\nu}$]. For the two-dimensional case percolation theory yields $\beta_2\simeq0.14$; $\nu_2\simeq1.33$; and $\eta\simeq0.2$ (see monograph by Shklovskii and Efros, 1984).

In studies of the magnetic flux effect on ring resistivity, the hole in the ring was simulated by setting all $\alpha_i=0$ within a 7×7 square at the center of the lattice in question. The magnetic flux ϕ through the hole was modeled by multiplying the quantities α_i on all lattice nodes along the cut (dashed line in Fig. 26) by $e^{i\varphi}$, where $\varphi=2\pi\phi/\phi_0$. Figure 28 presents in graphical form the dependence of

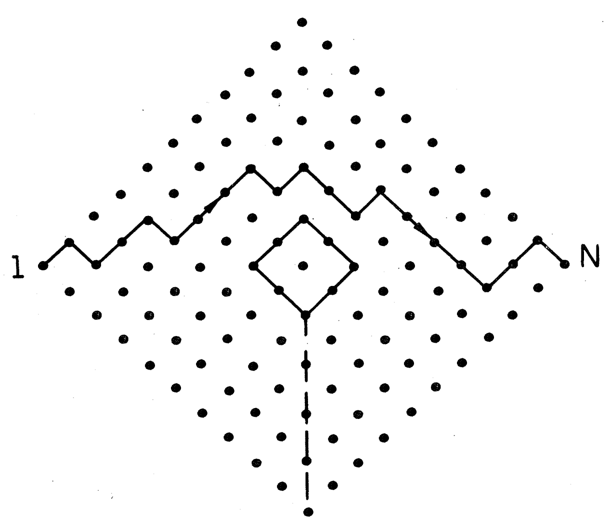

FIG. 26. The lattice in Anderson's model used by Nguen *et al.* (1985a). Solid curve, typical trajectory included in the calculation of J.

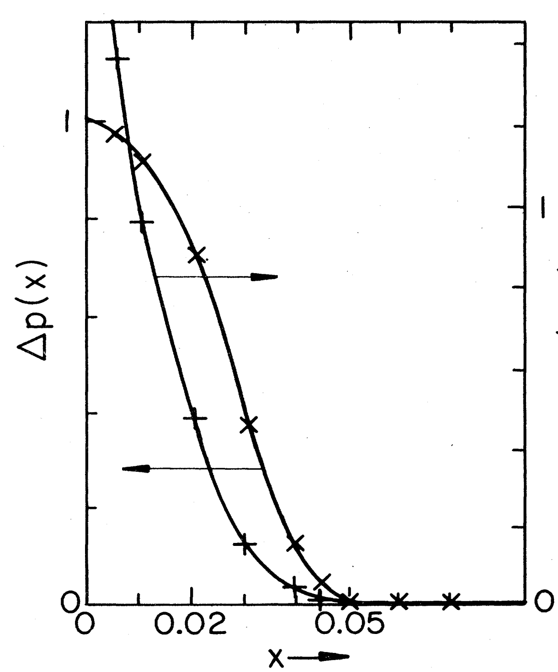

FIG. 27. ΔP and $L(\pi,x)$ vs concentration x (Nguen *et al.*, 1985a).

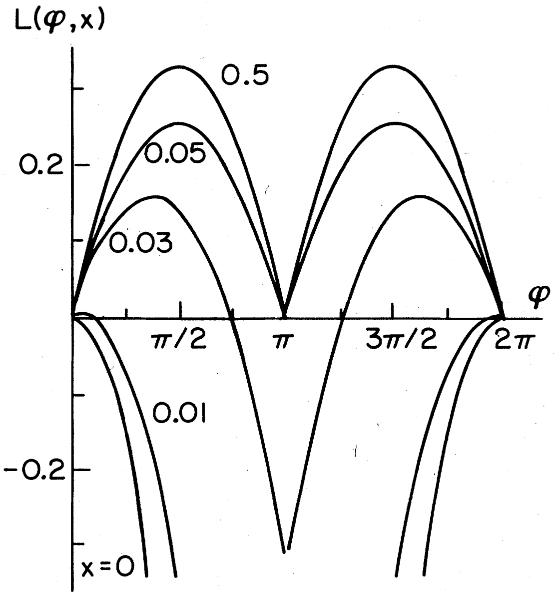

FIG. 28. Dependence of $L(\varphi,x)$ on reduced magnetic flux $\varphi=2\pi\phi/\phi_0$ (Nguen *et al.*, 1985a).

the quantity $L(\varphi,x)$ on φ for different x:

$$L(\varphi,x)=\overline{\ln|J(\varphi)/J(0)|^2} . \tag{4.4}$$

The overbar denotes averaging first over all possible situations, and then over all hopping ranges. The quantity $L(\varphi,x)$ is proportional to $\ln\sigma(\varphi)/\sigma(0)$ (Shklovskii and Efros, 1984). Examining Fig. 28 it is evident that for $x<x_c$ the magnetoresistance has a periodicity of 2π. As x increases, the magnitude of $|L(\pi,x)|$ decreases and vanishes at $x=x_c$ (Fig. 27). For still greater x, we have $L(\pi,x)=0$, the magnetoresistance being negative for all φ and exhibiting a periodicity of π. Thus the transition to the phase disordered in J at $x=x_c$ is accompanied by a change in the oscillation period from ϕ_0 to $\phi_0/2$.

If J_1 and J_2 are the sums over the trajectories running around the hole in the ring from above and from below ($J=J_1+J_2$), then, as pointed out by Nguen *et al.* (1985a), there is a relation between the change in oscillation period and the form of the distribution function for the quantities J_1 and J_2, $F(J_1,J_2)$, in terms of which the quantity

$$L(\varphi,x)=\int\int_{-\infty}^{+\infty} dJ_1 dJ_2 F(J_1,J_2)\ln\left|\frac{J_1+J_2e^{i\varphi}}{J_1+J_2}\right|^2 \tag{4.5}$$

is expressed. For $x=0$

$$F(J_1,J_2)=\delta\left[J_1-\frac{J}{2}\right]\delta\left[J_2-\frac{J}{2}\right] ,$$
$$L(\varphi,0)=\ln\left|\frac{1+\cos\varphi}{2}\right| , \tag{4.6}$$

and $L(\varphi,0)$ is periodic with a period 2π (Fig. 28). If starting with $x=x_c$, the function $F(J_1,J_2)$ becomes an even function of J_1 and hence of J_2, then Eq. (4.5) may be rewritten as

$$L(\varphi,x)=2\int\int_0^{\infty} dJ_1 dJ_2 F(J_1,J_2)\times\ln\left[1+\frac{2J_1^2J_2^2}{(J_1^2-J_2^2)^2}(1-\cos2\varphi)\right] \tag{4.7}$$

and $L(\varphi,x)$ will have a period of π and be positive for all φ. This implies that the magnetoresistance is negative. Note that for small φ, as can be seen from Fig. 28, negative magnetoresistance exists even for $x \ll x_c$.

The phase transition in the sign of the Green's function can be physically explained as follows. Let the concentration of centers with a negative scattering amplitude, $b<0$, in the system be n. After a scattering on such a center the wave function will acquire the form (Nguen *et al.*, 1985b)

$$\psi\propto\left[e^{-z/a}+\frac{b}{z}e^{-r/a}\right] \tag{4.8}$$

(a being the Bohr radius). It will be negative within a region of length $z=b$ and transverse size $\sqrt{|ba|}$. The condition of overlap of shadow regions from different scattering centers determines the phase transition point in concentration n_c, where memory of the original sign of the wave function is lost. In the three-dimensional case this condition is

$$n_c b^2 a \simeq 1 . \tag{4.9}$$

Since, for resonance scattering, $b\propto(\varepsilon_i-\varepsilon_j)^{-1}$, then if there is no gap in the density of states there will always be states with condition (4.9) satisfied, no matter the configuration. Therefore no phase transition will exist. Thus, for the transition to exist, the spectrum of impurity states should have a gap. In real conditions, the impurity band of semiconductors contains a Coulomb gap (Shklovskii and Efros, 1984), which makes this phase transition possible.

It should also be stressed that negative magnetoresistance appears only when averaging $\ln\sigma$ (the index of the overlap integral). It is this quantity that is self-averaging. When the variable-range hopping conductivity is calculated, the averaging of $\ln\sigma(\varphi)/\sigma(0)$ over all possible situations is reduced to a calculation of the resistance of the Miller-Abrahams random network (Shklovskii and Efros, 1984). The resistance of this network is greater than that of a parallel connection, but it is smaller than the resistance of a series connection. Therefore let us consider the limiting cases: the parallel and series connections of the rings.

The crossover of the ϕ_0 to $\phi_0/2$ periodicity is connected with the averaging over random realizations. In the single loop, the oscillation period is equal to ϕ_0 (mesoscopic effect), and its conductance is $\sigma_0+\sigma_1\cos2\pi\phi/\phi_0$. The total conductance of a parallel connection of the rings is

$$G=G_0+\sum_i\sigma_{1i}\cos2\pi\phi/\phi_0 .$$

If the interference part of the conductance of the rings σ_{1i} differs in sign only, and the sign of σ_{1i} is random, then $G=G_0$ and no oscillations are present. In the general case, if $P_\pm$ is the probability of finding σ_{1i} with given sign, then

$$G=G_0+N(P_+-P_-)\,|\,\sigma_1\,|\cos\frac{2\pi\phi}{\phi_0}\ ,\quad G_0=\sigma_0 N$$

and positive magnetoresistance will occur (N is the number of rings). For a series connection of rings the resistance R is

$$R=\sum_i\left[\sigma_0+\sigma_{1i}\cos\frac{2\pi\phi}{\phi_0}\right]^{-1}$$

$$=N\frac{\sigma_0-|\,\sigma_1\,|(P_+-P_-)\cos 2\pi\dfrac{\phi}{\phi_0}}{\sigma_0^2-\sigma_1^2\cos^2\dfrac{2\pi\phi}{\phi_0}}\ .$$

For $P_+=P_-$, the resistance of series-connected rings oscillates with $\phi_0/2$ periodicity, and negative magnetoresistance will be observed.

As shown in Sec. II, the quantum corrections to conductivity, the anomalous magnetoresistance, and the cylinder resistivity oscillations with period $\phi_0/2$ in the metallic state are associated with returning trajectories. On the other hand, the above-mentioned effect in the hopping domain is not connected in any way with returning trajectories (they are not included in this model at all). The question arises of how these effects are interrelated. Is it the same effect but on different sides of the metal-insulator transition, or are these phenomena of different physical origin? At present there is no unambiguous answer to this question.

Browne *et al.* (1984) and Carini *et al.* (1984) have performed a numerical study of Anderson's model (1958) for a one-dimensional ring and a cylinder. They calculated a quantity P called the participation ratio,

$$P=\frac{\left[\dfrac{1}{N}\sum_i\psi^2(r_i)\right]^2}{\dfrac{1}{N}\sum_i\psi^4(r_i)}\ ,$$

where r_i is the position of the ith node in an N-node lattice. This quantity is related to the quantum correction to the metal conductivity (Carini *et al.*, 1984)

$$\delta\sigma\propto -P^{-1}\ .$$

In the metallic phase this quantity undergoes oscillations with a periodicity of $\phi_0/2$. It should be noted that in the insulating domain the results of Browne *et al.* (1984) and Carini *et al.* (1984) are inapplicable, since the quantity P calculated in these publications does not characterize the conduction in the localization regime (it is insensitive to variations in the tails of the wave functions responsible for the hopping conduction). The participation ratio P characterizes only the wave function at a given node and thus does not exhibit anomalous behavior in the insulating state.

The sensitivity of the conductivity quantum corrections to magnetic field in the metallic phase results in a shift of the mobility edge (Khmelnitskii and Larkin, 1981), which manifests itself in the onset of anomalous magnetoresistance in the critical region in the insulating phase (Altshuler, Aronov, and Khmelnitskii, 1982a). For a cylinder in the field of a vector potential, the mobility edge should oscillate with magnetic flux variation, which, in its turn, should produce oscillatory magnetoresistance with a period of $\phi_0/2$ in the critical region, where the wave-function radius depends on the actual closeness to the transition point and the hopping range is comparable with the cylinder radius.

The first attempts to observe the magnetic flux effect in a multiconnected VRH conductor have recently been reported by Poyarkov *et al.* (1986) for a two-dimensional network, prepared by an electron-lithography process from an oxidized lead-telluride film 1600 Å thick. The resistance of the sample could be varied over a wide range by exposure to light at helium temperatures. As the temperature dependence of the sample resistance in the helium region (Fig. 29) could be fitted by the relation

$$R=R_0\exp(T_0/T)^\nu\ ,$$

where $\nu=0.3\pm0.1$ similar to Mott's law, one can suppose that the electron transport in the sample is connected with the VRH mechanism.

The high noise level made magnetoresistance measurements rather difficult. The magnetoresistance curve in Fig. 30 was obtained by averaging 24 consecutive runs (solid line). One of the single runs is shown by a dotted line. The observed period of oscillations of 7 ± 0.5 Oe is consistent with the $\phi_0/2$ value of the period in flux units,

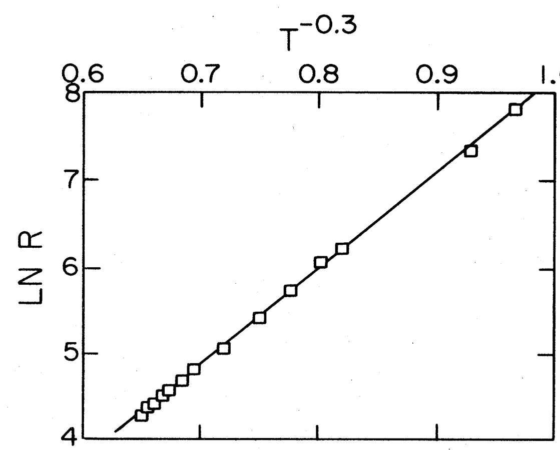

FIG. 29. Dependence of $R(G\Omega)$ on $T(K)$ for PbTe periodical network with 6×35 square loops. The period of the structure is 2 μm, the width of the network strips ≈0.5 μm. The measuring current is parallel to the short side of the network (Poyarkov *et al.*, 1986).

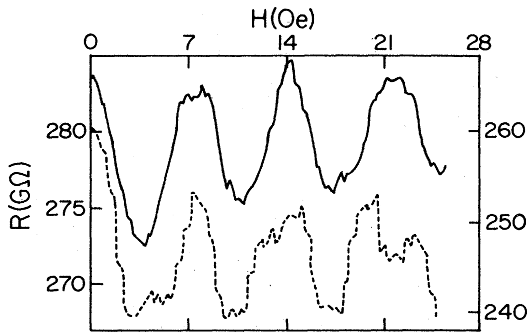

FIG. 30. $R(H)$ dependence at $T=1.12$ K for the same sample as in Fig. 29 (average of 24 runs). The $R(G\Omega)$ scale for the single-run curve (dotted line) is shown on the right (Poyarkov *et al.*, 1986).

and the ϕ_0 period is not seen because of the uncertainty due to the finite width of the network strips. The phase of the oscillations (maximum R at $H=0$) agrees with the prediction of Nguen *et al.* (1985a, 1985b) for the $\phi_0/2$ period.

The effective hopping range is evidently not much smaller than the perimeter of the loop and is on the order of 10^{-4} cm. Such a large range can hardly exist in a PbTe film with a homogeneous impurity distribution. One can suggest that it is the granular structure of the film with the tunnel barriers between the granules which is favorable for the appearance of large values of the hopping range.

V. CONCLUSIONS

A large number of experiments carried out in different geometries and on different metals have confirmed the major physical concepts associated with magnetic flux interference phenomena in solids. Among them are the following.

(1) The existence of oscillations in the transport properties of disordered metals with a period of $\phi_0/2=hc/2e$ related to the interference of conjugated waves.

(2) The existence of oscillatory phenomena in pure metals with a period of ϕ_0.

(3) Oscillations of the superconducting transition temperature and of the fluctuation conductivity due to the quantization of the macroscopic electron wave function in superconductors.

(4) The existence of oscillations with periods ϕ_0 and $\phi_0/2$ in a complex fluctuative dependence of the resistivity of multiconnected mesosystems on magnetic field.

(5) The existence of oscillations with period $\phi_0/2$ in multiconnected systems with VRH conductivity.

We have tried to show that present-day theory is capable of explaining satisfactorily the magnitude of the effects and their temperature and field behavior. It has become clear that the available experimental data are in fair agreement with theory. At the same time there are a number of unsolved problems that need to be answered from both theoretical and experimental viewpoints.

While the oscillatory effects of period ϕ_0 in normal metals and those of period $\phi_0/2$ in superconductors are known to be related to the gauge symmetry of Schrödinger's equations for a single electron and BCS theory, respectively, it is still unclear with what symmetry of the Schrödinger equation the oscillatory phenomena of periodicity $\phi_0/2$ in normal metals with a random potential should be associated. In the insulating phase, the change in symmetry resulting in a doubling of the period manifests itself in a phase transition as one oscillatory regime is replaced by another.

While one could present some qualitative considerations suggesting a relation between oscillations in the insulating phase and resistivity corrections originating from fluctuations in the metal, no quantitative analysis of this problem has been carried out. It is intimately connected with the metal-insulator transition in the presence of magnetic field. The available observations open the possibility of further investigations of flux effects in the VRH region. Of particular interest would be to study the change in oscillation period with increasing disorder. Such a study would also permit separation of the different mechanisms responsible for hopping conduction oscillations, namely, the magnetic-field-induced shift of the localization threshold and the sign transition of Nguen *et al.* (1985a, 1985b).

Networks made of superconducting metals are promising subjects for investigating magnetic flux effects. These effects are very sensitive to the topology of these structures, which can be both regular and disordered. Such structures may turn out to be good models of various systems, from an ordered lattice to "gauge glass," in which a change of magnetic field is equivalent to a transition from one spin glass replica to another (Pannetier, Chaussy, Rammal, and Villegier, 1984).

Mesosystems represent a new subject of physical study with highly individual properties. This individuality manifests itself in the shape of the magnetic spectrum of the magnetoresistance oscillations and in a number of remarkable properties predicted theoretically and yet unverified. It would be of particular importance to continue the investigation of these systems both theoretically and experimentally, including the use of magnetic flux effects.

As already mentioned, prediction of Altshuler and Spivak (1985) and Feng *et al.* (1986) and the observations of Skocpol *et al.* (1986) of the extreme sensitivity of the "magnetofingerprint" resistivity to small variations of the potential offers an important new means of studying the slow diffusion of heavy particles in conductors.

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