Massive neutrinos and neutrino oscillations

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The theory of neutrino mixing and neutrino oscillations, as well as the properties of massive neutrinos (Dirac and Majorana), are reviewed. More specifically, the following topics are discussed in detail: (i) the possible types of neutrino mass terms; (ii) oscillations of neutrinos (iii) the implications of *CP* invariance for the mixing and oscillations of neutrinos in vacuum; (iv) possible varieties of massive neutrinos (Dirac, Majorana, pseudo-Dirac); (v) the physical differences between massive Dirac and massive Majorana neutrinos and the possibilities of distinguishing experimentally between them; (vi) the electromagnetic properties of massive neutrinos. Some of the proposed mechanisms of neutrino mass generation in gauge theories of the electroweak interaction and in grand unified theories are also discussed. The lepton number nonconserving processes $\mu \to e\gamma$ and $\mu \to 3e$ in theories with massive neutrinos are considered. The basic elements of the theory of neutrinoless double-β decay are discussed as well. Finally, the existing data on neutrino masses, oscillations of neutrinos, and neutrinoless double-β decay are briefly reviewed. The main emphasis in the review is on the general model-independent results of the theory. Detailed derivations of these are presented.

CONTENTS

I. INTRODUCTION

Are neutrino masses different from zero? What kind of particles—Dirac or Majorana—are the neutrinos with definite masses? Does lepton mixing analogous to quark mixing take place? Do neutrinos that take no part in the standard weak interaction (the so-called "sterile" neutrinos) exist? The answers to these and many other fundamental questions of neutrino physics are not yet known to us. Numerous experiments are being performed and prepared at present in dozens of laboratories around the world with the aim of finding the answers to them.

Nonzero neutrino masses and neutrino mixing appear naturally in many different variants of unified theories representing generalizations of the theory of electroweak interaction of Glashow (1961), Weinberg (1967), and Salam (1968), viewed as standard today. Although remarkably successful phenomenologically, the Glashow-Weinberg-Salam theory seems incomplete from a theoretical point of view (see, for example, Harari, 1984). Consequently, the discovery of effects associated with nonzero neutrino masses would have a significant impact on the searches for possible ways to go beyond the standard theory.

One of the most interesting effects arising in the case when neutrino masses are different from zero and when neutrino mixing does take place are the oscillations of neutrinos. Oscillations of neutrinos were first considered by Pontecorvo (1957, 1958). At that time only one type of neutrino was known. The two-component neutrino theory was proposed (Landau, 1957; Lee and Yang, 1957; Salam, 1957) and was confirmed experimentally (Goldhaber, Grodzins, and Sunyar, 1958). The success of this theory was considered by many physicists as strong evidence that the mass of the neutrino is zero.

Pontecorvo assumed that there exists an analogy between lepton charge and strangeness and that not only the strangeness but also the lepton charge is not conserved by the weak interaction. In accordance with this hypothesis it was natural to assume (in analogy with the K^0-$\bar{K}^0$ system) that the neutrino state vector represents a superposition of the state vectors of two Majorana neutrinos with small (but different) masses (the analogs of K_1 and K_2). In this case the oscillations $\nu_L \rightleftarrows \bar{\nu}_L$ should take place in the neutrino beams, where $\bar{\nu}_L$ is a left-handed antineutrino, a particle that does not take part in the $(V\text{-}A)$ weak interaction (such particles were termed "sterile"; Pontecorvo, 1958). That the oscillations $\nu_L \rightleftarrows \bar{\nu}_L$ take place can be established by the "shortage" (at some distance from the source) of neutrinos of the initial type.

The idea of oscillations was applied to the case of two neutrino types (ν_e and ν_μ) by Pontecorvo in 1967. Both oscillations between "active" neutrinos $\nu_e \rightleftarrows \nu_\mu$ (taking part in the $V\text{-}A$ weak interaction) and oscillations between active and sterile neutrinos $\nu_e \rightleftarrows \bar{\nu}_{eL}, \nu_e \rightleftarrows \bar{\nu}_{\mu L}, \ldots,$ are possible in this case.

It should be indicated that the mixing of two massive neutrinos (as well as the mixing of baryons) was introduced and discussed in the works of Maki, Nakagawa, and Sakata (1962) and of Nakagawa *et al.* (1963). These authors noted that in the case of neutrino mixing "the weak neutrinos (ν_e and ν_μ) are not stable due to occurrence of virtual transmutations $\nu_e \rightleftarrows \nu_\mu$" and that the latter should be taken into account in the interpretation of the results of the famous Brookhaven neutrino experiment (Danby *et al.*, 1962) performed at that time.

Of significant importance for the development of the theory of neutrino oscillations was the work of Gribov and Pontecorvo published in 1969. In this work a consistent phenomenological theory of neutrino mixing and oscillations was formulated for the case of two neutrino types. The scheme of Gribov and Pontecorvo represents a minimal scheme of neutrino mixing: to the four states ν_e, ν_μ, $\bar{\nu}_e$, and $\bar{\nu}_\mu$ in this scheme there correspond the four states of two Majorana neutrinos with nonzero and different masses. Sterile neutrinos are not present.

Already in the first papers on neutrino oscillations it was pointed out that the possible effects of oscillations should be taken into account in the interpretation of experiments with solar neutrinos (Pontecorvo, 1958, 1967). This was done long before Davis *et al.* (1980) found the so-called solar neutrino problem.

Since the early seventies, work on neutrino oscillations and massive neutrinos has become more and more closely related to the gauge theories of electroweak interactions. In many studies the general case of n neutrino types is already considered. The case of mixing of n massive Majorana neutrinos was first discussed by Pontecorvo (1971).

The works of Bilenky and Pontecorvo (1976b), Eliezer and Swift (1976), and Fritzsch and Minkowski (1976) were based on the generalization of the quark-lepton analogy; it was assumed that lepton fields, like quark fields, enter into the weak charged current in a mixed form. The fields of flavor neutrinos are then orthogonal superpositions of the fields of Dirac neutrinos with definite masses. Finally, the mixing of an arbitrary number of neutrinos, in the case of active as well as of sterile neutrinos, was considered by Bilenky and Pontecorvo (1976a).

A detailed review of all these papers, including a discussion of possible methods by which to look for oscillations of neutrinos, was published in 1978 (Bilenky and Pontecorvo, 1978).

After the appearance of grand unified theories (Pati and Salam, 1973; Georgi and Glashow, 1974), the interest in neutrino oscillations and in the properties of massive neutrinos increased considerably. It was stimulated by the fact that nonzero masses and mixing of neutrinos arise naturally in these theories. In many extensions of the standard theory containing neutrinos with nonzero masses, and especially in the grand unified theories, the massive neutrinos are predicted to be Majorana particles. As a consequence, the properties of massive Majorana neutrinos and the physics they are associated with have been intensively studied since the beginning of the eighties. Significant progress has been made, in particular, in understanding the electromagnetic properties of Majorana neutrinos (Schechter and Valle, 1981b; Wolfenstein, 1981a; Kayser, 1982; Nieves, 1982; Shrock, 1982b; Kayser and Goldhaber, 1983; for earlier discussion see Majorana, 1937, and Case, 1957), which were shown to differ markedly from the electromagnetic properties of the Dirac neutrinos. Both the elementary-particle and the nuclear physics aspects of the theory of neutrinoless double-β decay $[(A,Z)\rightarrow(A,Z+2)+e^- +e^-]$, the process most sensitive to the existence of Majorana neutrinos coupled to the electron, were further developed (Wolfenstein, 1981a; Halprin *et al.*, 1983; Haxton *et al.*, 1982, 1984; Doi *et al.*, 1983a, 1984). As a result, a much better understanding of the relation between the existence of massive Majorana neutrinos and the existence of neutrinoless double-β decay and its rate emerged (Wolfenstein, 1981a; Schechter and Valle, 1982b; Nieves, 1984; Takasugi, 1984; Kayser *et al.*, 1986). The connection between the symmetry properties of the possible neutrino mass terms and the type of massive neutrinos they lead to was also thoroughly investigated. The studies of radiative decays of massive neutrinos within the modern gauge theories, begun in the late seventies (Lee and Shrock, 1977; Marciano and Sanda, 1977; Petcov, 1977b; see also Shrock, 1974), received new impetus and were actively pursued after it was realized (De Rújula and Glashow, 1980) that it might be feasible to observe the photon fluxes from such decays even if the neutrino radiative lifetimes exceed considerably the age of the universe. Finally, there has been a remarkable progress in the theory of neutrino oscillations in matter (Wolfenstein, 1978; Barger, Whisnant, *et al.*, 1980) in the last two years or so (Mikheyev and Smirnov, 1985; a summary is given, for example, in Mikheyev and Smirnov, 1986c).

In the present article we review the theory of neutrino mixing and neutrino oscillations as well as some of the basic properties of massive neutrinos (Dirac and Majorana). More specifically, the following topics are discussed in detail: (i) the phenomenological theory of neutrino mixing and oscillations; (ii) the implications of *CP* invariance for the mixing and oscillations in vacuum; (iii) the physical differences between massive Dirac and massive Majorana neutrinos and the possibility of distinguishing experimentally between them; (iv) the basic elements of the theory of neutrinoless double-β decay (with emphasis on its elementary-particle aspects); (v) the electromagnetic properties of massive neutrinos (including gauge theory predictions for neutrino radiative decays); and (vi) the status of lepton charges and mechanisms of neutrino mass generation in the gauge theories of the electroweak interaction. The predictions of the grand unified theories for neutrino masses and mixing angles as well as the lepton number nonconserving processes $\mu\rightarrow e\gamma$ and $\mu\rightarrow 3e$ in theories with massive neutrinos are also considered but in less detail. Finally, the results of the β-decay end-point and double-β-decay experiments searching for effects of finite neutrino masses and of the experiments searching for neutrino oscillations are briefly reviewed. We have also included two appendixes in which the properties of the free spin-$\frac{1}{2}$ Majorana field are summarized and an introduction to the theory of neutrinoless double-β decay is given.

In our discussion of the existing data on neutrino masses and mixing we have not included the results of experiments searching for relatively heavy neutrinos ($m_\nu \gtrsim$ few keV) that couple together with lighter neutrinos to e^- and/or μ^- in the weak lepton charged current. The physical implications of the existence of such neutrinos have been thoroughly studied by Shrock (1980, 1981a, 1981b, 1982a, 1982b; see also Gronau, Leung, and Rosner, 1984). A detailed description of the present limits on their masses and couplings as well as of the experiments in which they have been obtained can be found in Shrock (1983), Deutsch (1985), and Gall (1985).

The emphasis in this review is on general, model-independent results. We give a rather detailed exposition of the phenomological theory of neutrino mixing and oscillations. This theory is well established and has a direct relevance to the analysis and interpretation of the experimental data. All derivations in that part of the review can be easily followed by the reader (without reference to other literature). The electromagnetic properties of massive neutrinos and the basic elements of the theory of neutrinoless double-β decay are treated in a similar way.

As is well known, the existence of nonzero neutrino masses may have important cosmological and astrophysical implications. Their discussion lies outside the scope of the present review. Excellent reviews on the subject already exist (e.g., Steigman, 1979, 1984; Dolgov and Zeldovich, 1980).

Let us note finally that since the appearance in 1978 of the review by Bilenky and Pontecorvo, several other review articles treating the subject of massive neutrinos and neutrino oscillations have been published: Primakoff and Rosen (1981), Frampton and Vogel (1982), Boehm and Vogel (1984), Costa and Zwirner (1985), Wyler (1985), and Vergados (1986). There has been considerable progress in the understanding of the properties of massive neutrinos since the appearance of the articles by Frampton and Vogel and by Primakoff and Rosen, and we have tried to cover it. The reviews by Costa and Zwirner and by Vergados are devoted mainly to baryon and lepton number nonconservation in the gauge theories, while that of Boehm and Vogel considers the experimental constraints on neutrino masses and mixing obtained by 1984. These reviews, as well as the review lectures by Wyler (1985), overlap with our work only partially.

II. LEPTON CHARGES

All existing experimental data are compatible with the hypothesis that the lepton charges of particles are conserved. In the present introductory section we shall formulate the lepton charge (lepton number) conservation law and shall present the results of some of the latest experiments testing this law. We shall also give different possible formulations of the lepton charge conservation law.

The existence of the following three generations (families) of leptons and quarks has been established to date:

$$\begin{bmatrix} u & \nu_e \\ d & e \end{bmatrix}, \quad \begin{bmatrix} c & \nu_\mu \\ s & \mu \end{bmatrix}, \quad \begin{bmatrix} t & \nu_\tau \\ b & \tau \end{bmatrix}.$$

Let us assume that each generation of leptons has its own lepton charge, and let us define the lepton charges (elec-

TABLE I. Lepton charges of the particles.

Particles	L_e	L_μ	L_τ
e,ν_e	1	0	0
μ,ν_μ	0	1	0
τ,ν_τ	0	0	1
Hadrons, $W^\pm,Z^0,\gamma$	0	0	0

tron L_e, muon L_μ, and tau L_τ) according to Table I.[1] The lepton charges of the antiparticles are, by definition, opposite to the lepton charges of the corresponding particles. The lepton charge conservation law reads

$$\sum_i L_e^i = \text{const} ,\quad \sum_i L_\mu^i = \text{const} ,\quad \sum_i L_\tau^i = \text{const} . \tag{2.1}$$

Thus, in accordance with this law, the overall electron, muon, and tau lepton charges must be separately conserved in any process. Equation (2.1) is called the additive lepton charge conservation law. Below we shall discuss other possible formulations of the conservation law for lepton charges.

Let us now consider the results of experiments in which (2.1) has been tested. Some processes forbidden by the conservation law (2.1) are listed in Table II. The upper limits obtained for the ratio R of the probability (or cross section) of a given process and the total probability (cross section) of the corresponding allowed processes are shown in the second column of Table II.

At present a number of experiments are underway that are searching for the neutrinoless double-β decay $(A,Z)\to(A,Z+2)+e^-+e^-$ [$(\beta\beta)_{0\nu}$ decay; see Sec. IX], which is forbidden by (2.1). This process has not yet been observed. The best limits have been reached in experiments with ^{76}Ge. For the half lifetime of the decay

$$^{76}\text{Ge}\to{}^{76}\text{Se}+e^-+e^-$$

Caldwell *et al.* (1986) obtained the lower bound

$$T_{1/2}>2.5\times10^{23}\ \text{yr} .$$

It should be noted that only the most stringent limits on the probabilities of processes forbidden by the lepton charge conservation law are given in Table II. They have been obtained in experiments with muons (especially in the recent experiments performed at meson factories) and in experiments studying kaon decays.[2] The upper limits for the branching ratios of the τ decays, forbidden by (2.1), are much less stringent than the limits quoted in Table II. For example (Hayes *et al.*, 1982),

$$R(\tau^+\to\mu^++\gamma)<5.5\times10^{-4} ,$$

$$R(\tau^+\to\mu^++e^++e^-)<3.3\times10^{-4} .$$

We have discussed so far only the additive form of the lepton charge conservation law. A long time ago a multiplicative law of lepton charge conservation was introduced (Cabibbo, and Gatto, 1960; Feinberg and Weinberg, 1961). For the case of two lepton charges (electron and muon), the multiplicative law is formulated as follows:

$$\sum_i L_e^i+\sum_i L_\mu^i=\text{const} ,\quad (-1)^{\sum_i L_e^i}=\text{const} . \tag{2.2}$$

It is not difficult to see that the multiplicative law forbids $(\beta\beta)_{0\nu}$ decay and the processes listed in Table II as well. It is clear also that Eqs. (2.1) and (2.2) lead to the same consequences for processes involving only two leptons (for instance, both laws allow $\nu_\mu+N\to\mu^-+X$ and forbid $\nu_\mu+N\to e^-+X$, etc.). For processes with four leptons, the additive and the multiplicative laws of lepton charge conservation have different implications. According to (2.2), the decays

$$\mu^+\to e^++\nu_e+\bar\nu_\mu , \tag{2.3}$$

$$\mu^+\to e^++\bar\nu_e+\nu_\mu , \tag{2.4}$$

and the reactions

$$\nu_\mu+e^-\to\mu^-+\nu_e , \tag{2.5}$$

$$\bar\nu_\mu+e^-\to\mu^-+\bar\nu_e , \tag{2.6}$$

are allowed. The additive conservation law forbids processes (2.4) and (2.6).

The decay (2.4) has not been observed. The following upper limit has been obtained for the ratio $R(\mu^+\to e^+\bar\nu_e\nu_\mu)$ of the probability of the decay $\mu^+\to e^++\bar\nu_e+\nu_\mu$ and the probability of the standard decay $\mu^+\to e^++\nu_e+\bar\nu_\mu$ (Willis *et al.*, 1980):

$$R(\mu^+\to e^+\bar\nu_e\nu_\mu)<5\times10^{-2} .$$

[1]There are no direct proofs yet for the existence of neutrino of the third type ν_τ. The existing experimental data show, however, that ν_τ cannot coincide with ν_e, $\bar\nu_e$, ν_μ, and $\bar\nu_\mu$ (see, for example, Perl, 1980, and Winter, 1983).

[2]The limits on the $\mu^+\to e^+\gamma$, $\mu^+\to e^+\gamma\gamma$, $K^+\to\pi^+\mu^+e^-$ decay branching ratios and on the $\mu^-\to e^-$ conversion cross section are expected to be improved by 1—3 and 1 order of magnitude, respectively, in ongoing experiments at LAMPF, BNL, TRIUMF, and SIN; and in an experiment now underway at BNL a sensitivity $\sim10^{-11}$ with respect to the $K_L\to e^\pm\mu^\mp$ decay branching ratio is planned to be achieved (for a review of these experiments and corresponding references see Walter, 1985).

TABLE II. Upper limits for the branching ratios or relative cross sections of processes forbidden by the lepton charge conservation law.

Process	R	Reference
$\mu^+ \to e^+ + \gamma$	$< 4.9\times 10^{-11}$	Mischke *et al.*, 1986
$\mu^+ \to e^+ + e^- + e^+$	$< 2.4\times 10^{-12}$	Eichler *et al.*, 1984
$\mu^+ \to e^+ + \gamma + \gamma$	$< 7.2\times 10^{-11}$	Bolton *et al.*, 1984
$\mu^- + S \to e^- + S$	$< 7\times 10^{-11}$	Badertscher *et al.*, 1980
$\mu^- + T_i \to e^- + T_i$	$< 1.6\times 10^{-11}$	Bryman *et al.*, 1985
	$< 2\times 10^{-11}$	Eichler *et al.*, 1984
$K_L \to e^\pm + \mu^\mp$	$< 8\times 10^{-6}$	Fitch *et al.*, 1967 (see also Particle Data Group)
$K^- \pi^+ + e^+ + \mu^-$	$< 7\times 10^{-9}$	
$K^+ \to \pi^+ + e^- + \mu^+$	$< 5\times 10^{-9}$	Diamant-Berger *et al.*, 1976
$K^+ \to \pi^- + e^+ + \mu^+$	$< 7\times 10^{-9}$	
$\mu^- + I \to e^+ + Sb$	$< 3\times 10^{-10}$	Abela *et al.*, 1980

An upper limit for the ratio of the cross sections of processes (2.6) and (2.5) was obtained in experiments performed with high-energy neutrinos (Bergsma *et al.*, 1983):

$$\frac{\sigma(\bar{\nu}_\mu e^- \to \mu^- \bar{\nu}_e)}{\sigma(\nu_\mu e^- \to \mu^- \nu_e)} < 0.09 \; .$$

Thus the experimental data do not give evidence in support of the multiplicative law of lepton charge conservation.

Let us note finally that the existing data do not exclude the phenomenological possibility of conservation of the lepton charge L' equal to 1 for e^- and μ^+ (i.e., $L' = L_e - L_\mu$), which was introduced by Zeldovich (1952) and Konopinsky and Mahmoud (1953). If the charge L' is conserved, $(\beta\beta)_{0\nu}$ decay and the processes listed in Table II, with the exception of the last two, are forbidden. In a theory with conserved lepton charge L', the "phenomenological" neutrinos $\nu_e, \nu_\mu, \bar{\nu}_e, \bar{\nu}_\mu$ correspond, respectively, to a LH (left-handed) neutrino ν_L, LH antineutrino $\bar{\nu}_L$, RH (right-handed) antineutrino $\bar{\nu}_R$, and RH neutrino ν_R. We shall consider in detail the scheme with lepton charge L' and its generalization to the case of arbitrary even and odd numbers of charged leptons in Sec. VI.

As we have indicated earlier, no processes in which the electron, muon, and tau lepton charges are not separately conserved have been observed so far. In spite of that, the conservation of lepton charges is viewed today to be approximate only; correspondingly, lepton charges are considered as approximate phenomenological notions. This view of lepton charge conservation emerged, in part, from the idea of an analogy between leptons and quarks and from the results of extensive studies of the status of lepton charges in the modern gauge theories of electroweak interaction. The hypothesis of neutrino (lepton) mixing, which will be discussed in detail in the present review, is based on the assumption that the neutrino masses are different from zero and that the neutrino fields entering into the weak currents are linear combinations of the fields of neutrinos with definite masses. Neutrinoless double-β decay, the processes listed in Table II, and other lepton number nonconserving processes are allowed, in principle, in the theories with neutrino mixing. The fact that they are not observed experimentally is, possibly, a reflection of the smallness of the neutrino masses (see Sec. XI). In accordance with the neutrino mixing hypothesis, oscillations of neutrinos $\nu_l \rightleftarrows \nu_{l'}$, $l \neq l'$, should take place in the neutrino beams. The observation of neutrino oscillations would be a proof of the violation of the lepton charge conservation law. In the subsequent sections we shall consider in detail various aspects of the theory of massive neutrinos and neutrino oscillations, and in Sec. XII the relevant experimental data will be discussed.

III. ELEMENTS OF THE GLASHOW-WEINBERG-SALAM THEORY

It is well known that all existing data on the physics of weak and electromagnetic interactions are in wonderful agreement with the standard theory of electroweak interaction of Glashow (1961), Weinberg (1967), and Salam (1968). In the discussion of possible experiments in search for nonzero neutrino masses, neutrino oscillations, etc., we shall assume, consequently, that the interaction of neutrinos with quarks and leptons is described by the Lagrangian of the standard theory, that is, that the "phenomenological" neutrinos and antineutrinos $\nu_e, \nu_\mu, \nu_\tau, \bar{\nu}_e, \bar{\nu}_\mu, \bar{\nu}_\tau$ are particles which take part in the standard weak interaction. Accordingly, we think it is relevant to begin our review with a brief exposition of the Glashow-Weinberg-Salam theory. Besides, it is useful to recall the Higgs mechanism of mass generation on which the standard theory is based, before going into a discussion of the possible schemes of neutrino mixing.

Let us consider first the kinetic part of the Lagrangian of the neutrino, charged lepton, and quark fields and suppose that the neutrino fields and the LH components of the charged lepton and quark fields[3] form doublets of the group $SU(2)_L$,

$$\psi_{lL}=\begin{bmatrix}\nu'_{lL}\\ l'_L\end{bmatrix},\quad l=e,\mu,\tau,$$
$$\psi_{1L}=\begin{bmatrix}u'_L\\ d'_L\end{bmatrix},\quad \psi_{2L}=\begin{bmatrix}c'_L\\ s'_L\end{bmatrix},\quad \psi_{3L}=\begin{bmatrix}t'_L\\ b'_L\end{bmatrix}, \tag{3.1}$$

while the RH components of the charged lepton and quark fields are singlets with respect to this group. It is obvious that the Lagrangian of the fields of fundamental fermions is then invariant with respect to the global $SU(2)_L$ transformations. If we demand that the Lagrangian be invariant with respect to the local $SU(2)_L$ transformations as well, we shall arrive at a weak-interaction Lagrangian with "correct" (compatible with the observations) charged current. The unification of the weak and electromagnetic interactions requires, however, an extension of the symmetry group of the theory. The requisite minimal extension is the group $SU(2)_L\times U(1)$, where U(1) is the group of the weak hypercharge (Glashow, 1961).

The standard theory is based on the requirement of $SU(2)_L\times U(1)$ local gauge invariance. This invariance takes place if the following substitution is made in the free Lagrangian of the fields of fundamental fermions:

$$\partial_\alpha\psi_{lL}\to\left[\partial_\alpha-ig\frac{\tau}{2}\mathbf{A}_\alpha-ig'\tfrac{1}{2}Y_L^{\rm lep}B_\alpha\right]\psi_{lL},\quad l=e,\mu,\tau,$$
$$\partial_\alpha\psi_{aL}\to\left[\partial_\alpha-ig\frac{\tau}{2}\mathbf{A}_\alpha-ig'\tfrac{1}{2}Y_L^{\rm quark}B_\alpha\right]\psi_{aL},\quad a=1,2,3,$$
$$\partial_\alpha l'_R\to(\partial_\alpha-ig'\tfrac{1}{2}Y_R^{\rm lep}B_\alpha)l'_R,$$
$$\partial_\alpha q'_R\to(\partial_\alpha-ig'\tfrac{1}{2}Y_R^{\rm quark}B_\alpha)q'_R,\quad q=d,s,\ldots,t. \tag{3.2}$$

Here A_α^k and B_α are the gauge fields associated with the local symmetry groups $SU(2)_L$ and U(1), respectively, g and g' are dimensionless coupling constants, $Y_L^{\rm lep}$ is the hypercharge of the lepton doublets, etc.

It is assumed further that the interaction Lagrangian of the fundamental fermions and gauge vector bosons arises only as a result of the substitution (3.2) and does not contain other terms possible from the gauge invariance standpoint. Thus the fermion—gauge-boson interaction in the standard theory is the minimal interaction compatible with the requirement of gauge invariance.

To unify the weak and electromagnetic interactions in one electroweak interaction, we need to choose the weak hypercharges of the fermion doublets and singlets in such a way that the relation of Gell-Mann and Nishijima is satisfied,

$$Q=T_3^W+\frac{Y}{2}, \tag{3.3}$$

where Q is the charge (in units of the proton charge e) and T_3^W is the third component of the weak isospin. It follows from Eq. (3.3) that

$$Y_L^{\rm lep}=-1,\quad Y_L^{\rm quark}=\tfrac{1}{3},$$
$$Y_R^{\rm lep}=-2,\quad Y_R^{\rm quark}=2e_q, \tag{3.4}$$

e_q being the q-quark electric charge.

With the help of Eqs. (3.2) and (3.3) we get for the interaction Lagrangian

$$\mathscr{L}_I=ig\mathbf{j}_\alpha\mathbf{A}_\alpha+ig'\tfrac{1}{2}j_\alpha^YB_\alpha. \tag{3.5}$$

Here

$$j_\alpha^k=\sum_{a=1,2,3}\overline{\psi_{aL}}\gamma_\alpha\frac{\tau_k}{2}\psi_{aL}+\sum_{l=e,\mu,\tau}\overline{\psi_{lL}}\gamma_\alpha\frac{\tau_k}{2}\psi_{lL} \tag{3.6}$$

and

$$\tfrac{1}{2}j_\alpha^Y=j_\alpha^{\rm em}-j_\alpha^3, \tag{3.7}$$

where $j_\alpha^{\rm em}$ is the electromagnetic current of the quarks and leptons.

Separating the Lagrangian of the interaction of fermions with the charged vector bosons, we have from Eq. (3.5)

$$\mathscr{L}_I=\left[i\frac{g}{2\sqrt{2}}j_\alpha^{(+)}W_\alpha+\text{H.c.}\right]+\mathscr{L}_I^0. \tag{3.8}$$

Here

$$j_\alpha^{(+)}=2\sum_{a=1,2,3}\overline{\psi_{aL}}\gamma_\alpha\tau_+\psi_{aL}+2\sum_{l=e,\mu,\tau}\overline{\psi_{lL}}\gamma_\alpha\tau_+\psi_{lL} \tag{3.9}$$

is the weak charged current,

$$W_\alpha=\frac{1}{\sqrt{2}}(A_\alpha^1-iA_\alpha^2) \tag{3.10}$$

is the field of the charged vector bosons, and

$$\mathscr{L}_I^0=igj_\alpha^3A_\alpha^3+ig'(j_\alpha^{\rm em}-j_\alpha^3)B_\alpha \tag{3.11}$$

is the Lagrangian of the interaction of fermions and neutral vector bosons. Further, if instead of A_α^3 and B_α we introduce the fields

[3]We shall use the Pauli metric. In this metric $x=(\mathbf{x},ix_0)$; the Dirac equation has the form

$$(\gamma_\alpha\partial_\alpha+m)\psi(x)=0,$$

where γ_α are Hermitian matrices that satisfy the relation

$$\gamma_\alpha\gamma_\beta+\gamma_\beta\gamma_\alpha=2\delta_{\alpha\beta}.$$

The matrix γ_5 is defined as $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$. The left-handed and right-handed components of the field $\psi(x)$ are determined, respectively, by

$$\psi_L(x)=\tfrac{1}{2}(1+\gamma_5)\psi(x),$$
$$\psi_R(x)=\tfrac{1}{2}(1-\gamma_5)\psi(x).$$

$$Z_\alpha = A_\alpha^3 \cos\theta_W - B_\alpha \sin\theta_W ,$$
$$A_\alpha = A_\alpha^3 \sin\theta_W + B_\alpha \cos\theta_W , \quad (3.12)$$

where θ_W (the weak angle) is defined by the relation

$$\tan\theta_W = g'/g , \quad (3.13)$$

and require that the constants g and g' be connected with the charge e by the relation

$$g \sin\theta_W = e ,$$

then the electromagnetic interaction Lagrangian is extracted, obviously, from Eq. (3.11). One has

$$\mathscr{L}_I^0 = i \frac{g}{2\cos\theta_W} j_\alpha^0 Z_\alpha + i e j_\alpha^{\text{em}} A_\alpha . \quad (3.14)$$

Here

$$j_\alpha^0 = 2 j_\alpha^3 - 2\sin^2\theta_W j_\alpha^{\text{em}} \quad (3.15)$$

is the neutral current of the standard theory.

The $SU(2)_L \times U(1)$ gauge invariance can be exact only in the case when the masses of all particles are equal to zero. Consequently, the mass terms in the Lagrangian violate the gauge invariance of the theory. The standard theory of the electroweak interaction is based on the assumption that the $SU(2)_L \times U(1)$ gauge invariance is broken spontaneously [to the U(1) of the electromagnetism] via the Higgs mechanism (Englert and Brout, 1964; Guralnik, Hagen, and Kibble, 1964; Higgs, 1964, 1966). We shall consider this mechanism briefly here (for a more detailed discussion see Abers and Lee, 1973; Bernstein, 1974; Coleman, 1975).

In the simplest version of the theory, a doublet of scalar Higgs fields

$$\phi = \begin{pmatrix} \phi^{(+)} \\ \phi^0 \end{pmatrix} \quad (3.16)$$

whose hypercharge is equal to unity [in accordance with Eq. (3.3)] is introduced. The Higgs fields are supposed to interact both with the fields of vector particles and with the fermion fields. The corresponding couplings are introduced in such a way that the local gauge invariance is preserved.

A distinctive feature of the Higgs fields is the presence in the Lagrangian of the theory of the Higgs potential

$$V(\phi^\dagger\phi) = \kappa(\phi^\dagger\phi)^2 - \mu^2(\phi^+\phi) \quad (3.17)$$

(κ and μ^2 are positive constants), which leads to a degeneracy of the vacuum and to a nonzero vacuum expectation value $\langle\phi^0\rangle_0$ of the field ϕ^0. By choosing

$$\langle\phi^0\rangle_0 = \frac{\lambda}{\sqrt{2}} , \quad (3.18)$$

where $\lambda = (\mu^2/\kappa)^{1/2}$ (with this choice we fix the vacuum state), we can generate mass terms of the fields of intermediate vector bosons, of the fermions, and of the Higgs bosons (while photons remain massless). Since we are interested here in the neutrino masses, we shall consider the generation of fermion masses in the standard theory on the example of generation of quark masses.

Suppose that the fermion—Higgs-field interaction is described by Yukawa-type couplings. Two $SU(2)_L \times U(1)$ invariant Lagrangians of quark—Higgs-boson interaction of the indicated type can be constructed:

$$\mathscr{L}' = -\frac{\sqrt{2}}{\lambda} \sum_{\substack{a=1,2,3 \\ q=d,s,b}} \overline{\psi_{aL}} M_{aq}^{\text{down}} q_R' \phi + \text{H.c.} , \quad (3.19)$$

$$\mathscr{L}'' = -\frac{\sqrt{2}}{\lambda} \sum_{\substack{a=1,2,3 \\ q=u,c,t}} \overline{\psi_{aL}} M_{aq}^{\text{up}} q_R' \tilde{\phi} + \text{H.c.} \quad (3.20)$$

Here M^{down} and M^{up} are complex 3×3 matrices and

$$\tilde{\phi} = i\tau_2\phi^* = \begin{pmatrix} \phi^{0*} \\ -\phi^{(+)*} \end{pmatrix} \quad (3.21)$$

is a doublet of Higgs fields whose hypercharge is equal to (-1).

Taking into account Eq. (3.18) and using the gauge invariance of the Lagrangian, it is always possible to choose

$$\phi(x) = \begin{pmatrix} 0 \\ \dfrac{\lambda + H^0(x)}{\sqrt{2}} \end{pmatrix} , \quad \tilde{\phi}(x) = \begin{pmatrix} \dfrac{\lambda + H^0(x)}{\sqrt{2}} \\ 0 \end{pmatrix} \quad (3.22)$$

(unitary gauge), where $H^0(x)$ is the field of a neutral scalar physical Higgs particle. Inserting Eq. (3.22) in Eqs. (3.19) and (3.20), we obtain the following expressions for the quark mass terms:

$$\mathscr{L}^{\text{up}} = -\overline{p_L'} M^{\text{up}} p_R' + \text{H.c.} ,$$
$$\mathscr{L}^{\text{down}} = -\overline{n_L'} M^{\text{down}} n_R' + \text{H.c.} , \quad (3.23)$$

where

$$p_{L,R}' = \begin{pmatrix} u_{L,R}' \\ c_{L,R}' \\ t_{L,R}' \end{pmatrix} , \quad n_{L,R}' = \begin{pmatrix} d_{L,R}' \\ s_{L,R}' \\ b_{L,R}' \end{pmatrix} . \quad (3.24)$$

So, the elements of the quark mass matrices coincide (up to the factor $\sqrt{2}/\lambda$) with the constants of the quark—Higgs-boson Yukawa couplings.

Let us now proceed to the final stage of our consideration, that is, to the reduction of the mass terms (3.23) to a diagonal form. An arbitrary complex matrix can be reduced to a diagonal form with the help of the biunitary transformation (the proof is given in Sec. IV.B). One has

$$M^{\text{up}} = U_L m^{\text{up}} U_R^\dagger ,$$
$$M^{\text{down}} = V_L m^{\text{down}} V_R^\dagger , \quad (3.25)$$

where m^{up} and m^{down} are diagonal matrices with positive elements and $U_{L,R}$ and $V_{L,R}$ are unitary matrices. Substituting Eqs. (3.25) for M^{up} and M^{down} in (3.23), we obtain for the quark mass term

$$\mathcal{L}=\mathcal{L}^{\rm up}+\mathcal{L}^{\rm down}$$
$$=-\bar{p}m^{\rm up}p-\bar{n}m^{\rm down}n=\sum_{q=d,s,\ldots,t} m_q\bar{q}q \ . \quad (3.26)$$

Here

$$p=p_L+p_R=\begin{pmatrix} u \\ c \\ t \end{pmatrix}, \quad n=n_L+n_R=\begin{pmatrix} d \\ s \\ b \end{pmatrix}, \quad (3.27)$$

where

$$p_{L,R}=U_{L,R}^{\dagger}p'_{L,R} \ , \quad n_{L,R}=V_{L,R}^{\dagger}n'_{L,R} \ . \quad (3.28)$$

Hence $d(x),s(x),\ldots,t(x)$ are the fields of quarks possessing definite masses. The primed quark fields $[d'_{L,R}(x),s'_{L,R}(x),\ldots]$ forming the multiplets of the group $SU(2)_L$ are linear orthogonal combinations of the LH (RH) components of the fields of quarks with definite masses.

Finally, let us write the charged and the neutral weak currents of the standard theory in terms of the fields of quarks with definite masses. Exploiting the unitarity of the matrices $U_{L,R}$ and $V_{L,R}$ and using Eqs. (3.9), (3.15), and (3.28), we obtain the following expressions for the quark charged and neutral currents:

$$\begin{aligned} j_\alpha^{(+);\rm quark} &= 2\overline{p'_L}\gamma_\alpha n'_L = 2\overline{p_L}\gamma_\alpha U_{\rm KM} n_L \ , \\ j_\alpha^{0;\,\rm quark} &= \overline{p_L}\gamma_\alpha p_L - \overline{n_L}\gamma_\alpha n_L \\ &\quad -2\sin^2\theta_W[\tfrac{2}{3}\bar{p}\gamma_\alpha p+(-\tfrac{1}{3})\bar{n}\gamma_\alpha n] \ , \end{aligned} \quad (3.29)$$

where

$$U_{\rm KM}=U_L^{\dagger}V_L \quad (3.30)$$

is a unitary quark mixing matrix.

For the case of three generations of quarks, the mixing matrix $U_{\rm KM}$ was first considered by Kobayashi and Maskawa (KM) (1973). They showed that this matrix is characterized by three Euler angles and one phase and can be present in the form

$$U_{\rm KM}=\begin{pmatrix} c_1 & s_1c_3 & s_1s_3 \\ -s_1c_2 & c_1c_2c_3-s_2s_3e^{i\delta} & c_1c_2s_3+s_2c_3e^{i\delta} \\ s_1s_2 & -c_1s_2c_3-c_2s_3e^{i\delta} & -c_1s_2s_3+c_2c_3e^{i\delta} \end{pmatrix}, \quad (3.31)$$

where

$$s_i=\sin\theta_i \ , \quad c_i=\cos\theta_i \quad (i=1,2,3) \ .$$

We have considered the Higgs mechanism of generation of the quark masses. Analogously, the masses of the charged leptons result from couplings to the Higgs fields as a consequence of spontaneous symmetry breaking. We shall consider in detail various mechanisms of neutrino mass generation in Sec. VIII. Let us note here only that as a consequence of the chosen multiplet of Higgs fields and the absence of RH neutrino fields, it is impossible to generate a neutrino mass term (preserving the renormalizability), and the neutrinos are massless in the standard theory. For the lepton charged and neutral currents we have

$$\begin{aligned} j_\alpha^{(+);\,\rm lep} &= 2\sum_{l=e,\mu,\tau}\overline{\nu_{lL}}\gamma_\alpha l_L \ , \\ j_\alpha^{0;\,\rm lep} &= \sum_{l=e,\mu,\tau}\overline{\nu_{lL}}\gamma_\alpha\nu_{lL}-\sum_{l=e,\mu,\tau}\overline{l_L}\gamma_\alpha l_L \\ &\quad +2\sin^2\theta_W\sum_{l=e,\mu,\tau}\bar{l}\gamma_\alpha l \ . \end{aligned} \quad (3.32)$$

If the neutrino masses are different from zero, the fields $\nu_{lL}(x)$ that enter into Eq. (3.32) are given by the expression

$$\nu_{lL}=\sum_k U_{lk}\nu_{kL} \ , \quad (3.33)$$

where $\nu_k(x)$ is the field of a neutrino with mass m_k and U is the unitary matrix of lepton mixing. We shall show in later sections that neutrinos with definite masses can be Dirac as well as truly neutral Majorana particles. In theories representing extensions of the standard theory, the number of massive neutrinos may exceed the number of charged leptons.

The mixing described by Eq. (3.33) implies nonconservation of the lepton charges. We shall conclude the present section with a brief discussion of how, in the general case of the mixing (3.33), one should define the type of a "phenomenological" neutrino. The standard sources of neutrinos are the weak decays of the particles. Neutrinos are also analyzed by studying weak processes. Since all known weak processes are described by the Lagrangian of the standard theory, the type of a "phenomenological" neutrino or antineutrino is defined by Eq. (3.32) irrespective of whether lepton mixing takes place or not: we call a muon neutrino the LH "particle" (described in the case of mixing by a coherent superposition of states with different masses) which produces μ^- when interacting with the nucleons and which is born with μ^+ in the weak decays, etc.

IV. NEUTRINO MIXING SCHEMES

A. Introduction

In this section we shall consider possible neutrino mixing schemes. There exist several totally different schemes of neutrino mixing and only one possible scheme of quark mixing. This is connected with the fact that the neutrino electric charge is zero, in contrast to the quark charges. Correspondingly, neutrinos possessing definite masses can be Dirac as well as Majorana particles (quarks are Dirac particles). In addition (because of the zero electric charge of neutrinos), the number of massive Majorana neutrinos can exceed the number of charged leptons (i.e., the number of lepton flavors).[4]

It is common practice to classify neutrino mixing schemes according to the type of the mass terms, whose diagonalization leads to the corresponding mixings. It is not difficult to construct all possible (from a phenomenological point of view) mass terms. For this purpose let us introduce the columns

$$\nu_L = \begin{bmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \\ \vdots \end{bmatrix}, \quad \nu_R = \begin{bmatrix} \nu_{eR} \\ \nu_{\mu R} \\ \nu_{\tau R} \\ \vdots \end{bmatrix}. \tag{4.1}$$

The LH fields $\nu_{lL}(x)$ enter into the interaction Lagrangian of the standard electroweak theory. The index l takes n values $e,\mu,\tau,\ldots$ and characterizes the flavor of the corresponding neutrinos. The RH fields $\nu_{l'R}(x)$ do not enter into the interaction Lagrangian of the standard theory, but may be present in the mass terms. The index l' numbers the RH fields. We shall assume that l' also takes n values.[5]

In addition to ν_L and ν_R, let us consider[6]

$$(\nu_L)^c \equiv C\overline{\nu_L}^T, \quad (\nu_R)^c \equiv C\overline{\nu_R}^T. \tag{4.2}$$

Here C is the charge-conjugation matrix. The matrix C satisfies the conditions

$$C\gamma_\alpha^T C^{-1} = -\gamma_\alpha, \quad C^+C = \mathbb{1}, \quad C^T = -C. \tag{4.3}$$

It follows from Eqs. (4.2) and (4.3) that

$$\overline{(\nu_L)^c} = -\nu_L^T C^{-1}, \quad \overline{(\nu_R)^c} = -\nu_R^T C^{-1}. \tag{4.4}$$

The field $(\nu_{L(R)})^c$ transforms as $\nu_{L(R)}$ under the proper Lorentz transformations.

It is not difficult to show that $(\nu_L)^c$ is a RH field, while $(\nu_R)^c$ is a LH field. Indeed, we have

$$\tfrac{1}{2}(1-\gamma_5)(\nu_L)^c = C[\overline{\nu_L}\tfrac{1}{2}(1-\gamma_5)]^T. \tag{4.5}$$

Equation (4.5) was obtained by using the relation

$$C^{-1}\gamma_5 C = \gamma_5^T, \tag{4.6}$$

which is a consequence of Eq. (4.3). Further, taking into account that

$$\overline{\nu_L}\tfrac{1}{2}(1-\gamma_5) = \overline{\nu_L},$$

we get

$$\tfrac{1}{2}(1-\gamma_5)(\nu_L)^c = (\nu_L)^c.$$

Similarly, one finds

$$\tfrac{1}{2}(1+\gamma_5)(\nu_R)^c = (\nu_R)^c.$$

Let us proceed now to the construction of possible neutrino mass terms using the fields ν_L, $(\nu_L)^c$, ν_R, and $(\nu_R)^c$. If only the fields ν_L and ν_R enter into the mass term, one has

$$\mathcal{L}^D = -\overline{\nu_R} M^D \nu_L + \text{H.c.} \tag{4.7}$$

Using the flavor fields ν_L and $(\nu_L)^c$ present in the weak currents, one can build the mass term:

$$\mathcal{L}^M = -\tfrac{1}{2}\overline{(\nu_L)^c} M^M \nu_L + \text{H.c.} \tag{4.8}$$

Finally, the most general neutrino mass Lagrangian has the form[7]

$$\mathcal{L}^{D+M} = -\tfrac{1}{2}\overline{(\nu_L)^c} M_L^M \nu_L - \tfrac{1}{2}\overline{\nu_R} M_R^M (\nu_R)^c - \overline{\nu_R} M_1^D \nu_L + \text{H.c.} \tag{4.9}$$

In Eqs. (4.7)–(4.9), M^D, M^M, M_L^M, M_R^M, and M_1^D are $n\times n$ complex matrices.

The mass term $\mathcal{L}^D$ is invariant with respect to the global gauge transformations $\nu_L \to e^{i\Lambda}\nu_L$, $\nu_R \to e^{i\Lambda}\nu_R$. This invariance implies the conservation of the lepton charge. Neutrinos with definite masses are in this case Dirac particles. The mass term $\mathcal{L}^D$ is called a Dirac mass term.

Obviously, in the general case no global gauge transformations under which the Lagrangians $\mathcal{L}^M$ and $\mathcal{L}^{D+M}$ would be invariant exist. This is the reason why neutrinos with definite masses in the case of the mass terms $\mathcal{L}^M$ and $\mathcal{L}^{D+M}$ are Majorana particles. The Lagrangian $\mathcal{L}^M$ is called a Majorana mass term, while $\mathcal{L}^{D+M}$ is called a Dirac-Majorana mass term.

We proceed now to a detailed discussion of neutrino mixing resulting from the diagonalization of the three types of neutrino mass terms constructed above.

B. Dirac mass term

Let us assume first that the neutrino mass term has the form

[4]This possibility may be realized if, for example, neutral weak isosinglet particles that mix with the neutrinos exist. For further details see Sec. IV.D and the last paragraph of Sec. VII.B.

[5]Let us note that, in general, the number of RH neutrino fields $\nu_{l'R}$ may be greater or less than n.

[6]To avoid any misunderstanding, we would like to note the following. The charge-conjugation operator U_c, which changes the state vector of a particle with given momentum and helicity into a state vector of the corresponding antiparticle with the same momentum and helicity, transforms the operator ψ_L into $(\psi_R)^c \equiv C\overline{\psi_R}^T$:

$$U_c \psi_L(x) U_c^{-1} = \eta_c(\psi_R(x))^c,$$

where η_c is a phase factor. Thus a given Lagrangian will not be invariant with respect to the charge conjugation if it contains only the LH components of some fields [e.g., $\nu_{lL}(x)$].

[7]Obviously, the possible term $\overline{(\nu_L)^c} M_2^D (\nu_R)^c$ can be reduced to the third term in $\mathcal{L}^{D+M}$. Indeed, one has $\overline{(\nu_L)^c} M_2^D (\nu_R)^c = -\nu_L^T C^{-1} M_2^D C \overline{\nu_R}^T = \overline{\nu_R}(M_2^D)^T \nu_L$.

$$\mathscr{L}^D = -\overline{\nu_R} M^0 \nu_L + \text{H.c.}$$
$$= - \sum_{l,l'=e,\mu,\tau,\ldots} \overline{\nu_{l'R}} M^D_{l'l} \nu_{lL} + \text{H.c.} , \quad (4.10)$$

where M^D is a complex $n \times n$ matrix. It should be emphasized that in addition to the ordinary LH neutrino fields ν_{lL} $(l=e,\mu,\tau,\ldots)$, the RH fields $\nu_{l'R}$ $(l'=e,\mu,\tau,\ldots)$ not present in the interaction Lagrangian of the standard theory are needed to construct Lagrangian (4.10).

In order to reduce the mass term $\mathscr{L}^D$ to the standard form, it is necessary to diagonalize the matrix M^D. An arbitrary complex matrix can always be diagonalized by means of the biunitary transformation.[8] One has

$$M^D = V m U^\dagger . \quad (4.11)$$

Here V and U are unitary matrices and

$$m_{ik} = m_k \delta_{ik} , \quad m_k \geq 0 .$$

Inserting Eq. (4.11) into (4.10) we get

$$\mathscr{L}^D = -\overline{\nu'_R} m \nu'_L + \text{H.c.} = -\overline{\nu'} m \nu'$$
$$= - \sum_{k=1}^{n} m_k \overline{\nu_k} \nu_k . \quad (4.12)$$

Here

$$\nu'_L = U^\dagger \nu_L , \quad \nu'_R = V^\dagger \nu_R , \quad \nu' = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \end{pmatrix} . \quad (4.13)$$

From Eq. (4.12) one can conclude that ν_k is the field of a neutrino with mass m_k.[9]

Utilizing the unitarity of the matrix U, we obtain from Eq. (4.13)

$$\nu_L = U \nu'_L \quad (4.14)$$

or

$$\nu_{lL} = \sum_{k=1}^{n} U_{lk} \nu_{kL} , \quad l=e,\mu,\tau,\ldots . \quad (4.15)$$

Thus, if the neutrino mass term has the form (4.10), the flavor neutrino fields ν_{lL} present in the standard weak lepton currents are linear combinations of the LH components of the fields of neutrinos with definite masses. The unitary matrix U is called the lepton mixing matrix.

It is easy to see that the neutrinos ν_k are Dirac particles. Indeed, it is not difficult to convince oneself, using Eq. (4.15), that the Lagrangian of the theory containing the mass term (4.10) is invariant with respect to the global gauge transformations

$$\begin{aligned} &\nu_k(x) \to e^{i\Lambda} \nu_k(x) , \\ &l(x) \to e^{i\Lambda} l(x) , \quad l=e,\mu,\tau,\ldots , \end{aligned} \quad (4.16)$$

where Λ is a constant parameter.

Invariance with respect to the transformations (4.16) means that the lepton charge common to all charged leptons and all neutrinos ν_k is conserved.

Let us define the lepton charge

$$L = \sum_{l=e,\mu,\tau,\ldots} L_l . \quad (4.17)$$

Obviously, all charged leptons carry one unit of the charge L. It follows from Eq. (4.16) that the lepton charge L will be conserved if we assume that the charge L of all neutrinos with definite masses is equal to one. Thus neutrinos with definite masses are Dirac particles in the case of the mass term (4.10) (ν_k differs from $\overline{\nu_k}$ by the lepton charge; note that neutrinos ν_k differ from each other by their masses).

The fields $\nu_{lL}(x)$ are fields of the neutrinos taking part in the weak interaction. It is evident that the lepton charges L_l $(l=e,\mu,\tau,\ldots)$ will not be conserved in a theory with neutrino mass term (4.10) if the matrix M^D is not diagonal. It is not difficult to see, however, that such a theory will be invariant with respect to global gauge transformations,[10]

$$\begin{aligned} &\nu_{lL}(x) \to e^{i\Lambda} \nu_{lL}(x) , \quad \nu_{lR}(x) \to e^{i\Lambda} \nu_{lR}(x) , \\ &l(x) \to e^{i\Lambda} l(x) . \end{aligned} \quad (4.18)$$

[8]Here is a proof of this statement for the simple and physically interesting case of nondegenerate matrices M, $\det M \neq 0$. Consider the matrix $MM^\dagger$. Obviously, this matrix is Hermitian, and its eigenvalues are positive. We have then

$$MM^\dagger = V m^2 V^\dagger$$

where $V^\dagger V = V V^\dagger = \mathbb{1}$ and $(m^2)_{ik} = m_k^2 \delta_{ik}$. Further, we obtain

$$M = V m U^\dagger ,$$

where $U^\dagger = m^{-1} V^\dagger M$, while $m_{ik} = +(m_k^2)^{1/2} \delta_{ik}$. It is not difficult to convince oneself that U is a unitary matrix. Indeed, using the first equation above one gets

$$U^\dagger U = m^{-1} V^\dagger M M^\dagger V m^{-1} = \mathbb{1} .$$

[9]Using the unitarity of the matrices U and V we get for the kinetic terms in the neutrino Lagrangian

$$\mathscr{L}^0 = -\overline{\nu_L} \gamma_\alpha \partial_\alpha \nu_L - \overline{\nu_R} \gamma_\alpha \partial_\alpha \nu_R$$
$$= -\overline{\nu'} \gamma_\alpha \partial_\alpha \nu' = - \sum_{k=1}^{n} \overline{\nu_k} \gamma_\alpha \partial_\alpha \nu_k .$$

[10]It is implicitly assumed here that apart, possibly, from the neutrino mass term, the total Lagrangian of the theory possesses the indicated symmetry.

This implies that in the case of a Dirac mass term (4.10) the total lepton charge $L=\sum_{l=e,\mu,\tau,\ldots} L_l$ is conserved.

Thus processes like

$$\mu^+ \to e^+ + \gamma\ ,\ \mu \to e^+ + e^- + e^+\ ,$$
$$K^+ \to \pi^+ + \mu^\pm + e^\mp\ ,\ \mu^- + (A,Z) \to e^- + (A,Z)\ , \tag{4.19}$$

etc., are allowed in a theory with neutrino mixing given by Eq. (4.15). Oscillations $\overset{(-)}{\nu}_l \rightleftarrows \overset{(-)}{\nu}_{l'}$ $(l \neq l')$ should be observed in the neutrino beams. At the same time, neutrinoless double-β decay

$$(A,Z) \to (A,Z+2) + e^- + e^- \tag{4.20}$$

and processes like

$$\mu^- + (A,Z) \to e^+ + (A,Z-2)\ ,$$
$$K^+ \to \pi^- + e^+ + \mu^+\ ,\ \text{etc.}\ , \tag{4.21}$$

are forbidden as a consequence of the conservation of the total lepton charge L. In the following sections we shall consider schemes of neutrino mixing in which $(\beta\beta)_{0\nu}$ decay and other analogous processes are allowed in addition to oscillations and processes of the same type as Eq. (4.19).

Let us note in conclusion that the mixing of neutrinos generated by a Dirac mass term is analogous to Cabibbo-Kobayashi-Maskawa quark mixing. It was introduced by several authors (Maki *et al.*, 1962; Eliezer and Ross, 1974; Bilenky and Pontecorvo, 1976b; Fritzsch and Minkowski, 1976) with the purpose of constructing a theory in which there would be a complete analogy between the weak interaction of quarks and leptons.

C. Majorana mass term

We shall consider in this subsection neutrino mixing in the case of the Majorana mass term

$$\mathcal{L}^M = -\tfrac{1}{2}\overline{(\nu_L)^c} M \nu_L + \text{H.c.}$$
$$= -\tfrac{1}{2} \sum_{l,l'=e,\mu,\tau,\ldots} \overline{(\nu_{l'L})^c} M_{l'l} \nu_{lL} + \text{H.c.}\ , \tag{4.22}$$

where M is a complex $n \times n$ matrix.

In the simplest case of two neutrino types, the mass term (4.22) was first considered by Gribov and Pontecorvo (1969). The scheme of neutrino mixing introduced by these authors is the most "economical" one, as the neutrino mass term is formed only by the LH flavor neutrino fields. To the four neutrinos and antineutrinos taking part in the weak interaction there correspond four possible spin states of two massive Majorana neutrinos with different masses.

We shall consider the general case of n neutrino types. In order to cast the Lagrangian $\mathcal{L}^M$ in the standard form let us diagonalize the matrix M. In doing that one has to take into account that M is a symmetric matrix. Indeed, using Eqs. (4.3) and (4.4) (as well as the fact that a minus sign appears when interchanging two fermionic field operators), we have

$$\overline{(\nu_L)^c} M \nu_L = -(\nu_L^T C^{-1} M \nu_L)^T$$
$$= \nu_L^T (C^{-1})^T M^T \nu_L = \overline{(\nu_L)^c} M^T \nu_L\ .$$

This implies

$$M^T = M\ . \tag{4.23}$$

We shall be interested in the case of nondegenerate eigenvalues of the matrix M. A complex symmetric matrix can always be expressed in the form[11]

$$M = (U^\dagger)^T m U^\dagger\ , \tag{4.24}$$

where $U^\dagger U = U U^\dagger = \mathbb{1}$ and $m_{ik} = m_k \delta_{ik}$, $m_k \geq 0$. We now insert Eq. (4.24) into (4.22) and obtain

$$\mathcal{L}^M = -\tfrac{1}{2}\overline{(n_L)^c} m n_L - \tfrac{1}{2}\overline{n_L} m (n_L)^c\ , \tag{4.25}$$

where

$$n_L = U^\dagger \nu_L\ ,\ \ (n_L)^c = C\overline{n_L}^T\ . \tag{4.26}$$

Finally, we get for the neutrino mass term

$$\mathcal{L}^M = -\tfrac{1}{2}\bar{\chi} m \chi = -\tfrac{1}{2}\sum_{k=1}^{n} m_k \overline{\chi_k} \chi_k\ . \tag{4.27}$$

Here

$$\chi = n_L + (n_L)^c = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_n \end{pmatrix}\ . \tag{4.28}$$

---

[11]For an arbitrary matrix M we have

$$M = V m U^\dagger \tag{a}$$

where $VV^\dagger = \mathbb{1}$, $UU^\dagger = \mathbb{1}$, and $m_{ik} = m_k \delta_{ik}$, $m_k \geq 0$. We shall assume for simplicity that $m_i \neq m_k$ for $i \neq k$ and that $m_k > 0$. Obviously, $MM^\dagger = V m^2 V^\dagger$. On the other hand, $MM^\dagger = (U^\dagger)^T m^2 U^T$ since $M = M^T = (U^\dagger)^T m V^T$. Hence $(U^\dagger)^T m^2 U^T = V m^2 V^\dagger$. This implies

$$U^T V m^2 = m^2 U^T V\ .$$

Since m^2 is a diagonal matrix and $m_i \neq m_k$, it follows from the last relation that $U^T V$ is also a diagonal matrix. Further, $U^T V$ is a unitary matrix. Thus

$$U^T V = S$$

where $S_{ik} = e^{2i\alpha_k}\delta_{ik}$, α_k being real constants. Finally, inserting this expression into (a), we obtain

$$M = (U'^\dagger)^T m (U')^\dagger\ ,$$

where $U'^\dagger = S^{1/2} U^\dagger$, $(S^{1/2})_{ik} = e^{i\alpha_k}\delta_{ik}$.

We conclude on the basis of Eq. (4.27) that χ_k is the field of a neutrino with mass m_k.[12]

It is not difficult to see that the fields χ_k satisfy the condition

$$\chi_k(x)=C\overline{\chi_k}^T(x)\ ,\quad k=1,2,\ldots,n\ . \tag{4.29}$$

These conditions imply that $\chi_k(x)$ are the fields of Majorana neutrinos.[13] The properties of the Majorana fields will be discussed in detail in Appendix A. We should like to note here only that as a consequence of Eq. (4.29) the operator $\chi_k(x)$ has the form

$$\chi_k(x)=\int N_p[u^r(p)a_r(p)e^{ipx}+u^r(-p)a_r^\dagger(p)e^{-ipx}]d\mathbf{p}\ , \tag{4.30}$$

where

$$N_p=\frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2p_0}}$$

is a standard normalization factor, the spinor $u^r(p)$ describes a state with momentum p and helicity r, $u^r(-p)=C\overline{u^r}^T(p)$, and $a_r(p)$ and $a_r^\dagger(p)$ are the annihilation and creation operators of one and the same particle with momentum p and helicity r.

From Eqs. (4.26) and (4.28) one obtains

$$\nu_L=U\chi_L \tag{4.31}$$

and, consequently,

$$\nu_{lL}=\sum_{k=1}^{n}U_{lk}\chi_{kL}\ . \tag{4.32}$$

In this way, if the neutrino mass term has the form (4.22), the fields of ordinary LH flavor neutrinos are linear combinations of the LH components of the fields of Majorana neutrinos with definite masses.[14] The number of fields of Majorana neutrinos coincides with the number of flavor neutrinos (i.e., the number of charged leptons). Clearly, the $2n$ states with different helicity of the n massive Majorana neutrinos correspond to the $2n$ neutrinos and antineutrinos $(\nu_e,\nu_\mu,\nu_\tau,\bar\nu_e,\bar\nu_\mu,\bar\nu_\tau,\ldots)$ taking part in the weak interaction.

We should like to conclude this subsection with the following remarks.

(i) Evidently, there exist no global gauge transformations under which the mass term (4.22) in its most general form could be invariant. This implies that in the case under discussion no conserved lepton charges that could allow us to distinguish neutrino from antineutrino exist. As a consequence, the particles with definite mass in the case of the mass term (4.22) are Majorana neutrinos—truly neutral particles with spin $\frac{1}{2}$.

(ii) Using the language of the LH and RH field components, we can clarify the difference between the Dirac and Majorana cases as follows. From Eqs. (4.6) and (4.29) it follows that the LH and the RH components of the Majorana field $\chi_k(x)$ are connected by the relation

$$\chi_{kR}(x)=C\overline{\chi_{kL}}^T(x)\ . \tag{4.33}$$

The LH and the RH components of the Dirac field $\nu_k(x)$ are independent of each other.

Let us introduce the field $\nu'_{kL}=e^{i\Lambda}\nu_{kL}$ (Λ is a constant) instead of ν_{kL}. The charged currents and the charged lepton mass term would not change if simultaneously the charged lepton field were to undergo the transformation

$$l(x)\to l'(x)=e^{i\Lambda}l(x)\ .$$

Since the fields $\nu_{kL}(x)$ and $\nu_{kR}(x)$ are independent, the field $\nu_{kR}(x)$ can always be transformed $[\nu_{kR}(x)\to\nu'_{kR}(x)=e^{i\Lambda}\nu_{kR}(x)]$ so that the neutrino mass term be also invariant with respect to the indicated gauge transformations.

If the fields $\chi_k(x)$ and $l(x)$ in the Majorana case were transformed as

$$\chi_{kL}(x)\to\chi'_{kL}(x)=e^{i\Lambda}\chi_{kL}(x)\ ,$$
$$l(x)\to l'(x)=e^{i\Lambda}l(x)\ , \tag{4.34}$$

the weak charged currents and the charged lepton mass term, obviously, would not change. However, as a consequence of Eq. (4.33), $\chi'_{kR}(x)=e^{-i\Lambda}\chi_{kR}(x)$, and the neutrino mass term would not be invariant with respect to the

[12]For the kinetic term of the neutrino field Lagrangian one has

$$\mathcal{L}_0=-\overline{\nu_L}\gamma_\alpha\partial_\alpha\nu_L=-\overline{n_L}\gamma_\alpha\partial_\alpha n_L$$
$$=-\tfrac{1}{2}\overline{n_L}\gamma_\alpha\partial_\alpha n_L-\tfrac{1}{2}\overline{(n_L)^c}\gamma_\alpha\partial_\alpha(n_L)^c$$
$$=-\tfrac{1}{2}\bar\chi\gamma_\alpha\partial_\alpha\chi=-\tfrac{1}{2}\sum_{k=1}^{n}\overline{\chi_k}\gamma_\alpha\partial_\alpha\chi_k\ .$$

[13]Note that the mass eigenstate fields of $\mathcal{L}^M$ are determined by the procedure of diagonalization up to overall unphysical phase factors. We could have chosen them to be

$$\chi'_k(x)=e^{i\alpha_k}\chi_k(x)\ ,\quad k=1,2,\ldots,n \tag{a}$$

(where α_k is an arbitrary real constant). The choice of the phases is reflected in the Majorana condition

$$C\overline{\chi'}^T=\xi'_k\chi'_k(x)\ ,\quad \xi'_k=e^{-2i\alpha_k}\ . \tag{b}$$

[14]It should be emphasized that the diagonalization of the Majorana mass term $\mathcal{L}^M$ (4.22) may lead to massive Dirac neutrinos in the particular case when $\mathcal{L}^M$, as well as the theory it is part of, possesses some global symmetry (Bilenky and Pontecorvo, 1981; Wolfenstein, 1981b; Leung and Petcov, 1983). For further details see Secs. VI.C, VI.D, and VI.F.

transformation (4.34).

(iii) In a theory with Majorana mass term (4.22) none of the lepton charges L_l, $l=e,\mu,\tau,\ldots$ is conserved nor is the total lepton charge $L=\sum_l L_l$. Consequently, the theory allows not only processes like $\mu^+\to e^+ +\gamma$, $\mu^+\to e^+ + e^- + e^+$, etc., but also $(\beta\beta)_{0\nu}$ decay and other similar processes. The considered scheme of mixing with Majorana massive neutrinos leads to oscillations in the neutrino beams (like the scheme mixing of massive Dirac neutrinos discussed in the previous subsection).

We have considered so far neutrino mixing schemes in which the number of neutrinos (Dirac or Majorana) with definite masses coincides with the number of flavor neutrinos. We turn now to the discussion of a mixing scheme in which the number of massive neutrinos (of Majorana type) exceeds the number of flavor neutrinos.

D. Dirac-Majorana mass term

The fields $\nu_{lL}(x)$ enter into the interaction Lagrangian of the standard theory. Only these fields have been used in the construction of the Majorana mass term considered in Sec. IV.C. The Dirac mass term was built using the "active" flavor fields $\nu_{lL}(x)$ as well as the RH fields $\nu_{lR}(x)$ which do not enter into the interaction Lagrangian of the standard theory. This mass term is constructed so that global gauge invariance corresponding to conservation of the total lepton charge does take place. In the present section we shall consider the most general Dirac-Majorana mass term (4.9) (Bilenky and Pontecorvo, 1976a; Barger *et al.*, 1980; Bilenky, Hošek, and Petcov, 1980; Kobzarev *et al.*, 1980; Schechter and Valle, 1980). It is built by both the LH fields $\nu_{lL}(x)$ and the RH fields $\nu_{lR}(x)$ without imposing the condition of global gauge invariance.

The Dirac-Majorana mass term (4.9) can be written in the form

$$\mathcal{L}^{D+M}=-\tfrac{1}{2}\overline{(n_L)^c}Mn_L+\text{H.c.} \tag{4.35}$$

Here M is a complex $2n\times 2n$ matrix, and

$$n_L=\begin{bmatrix}\nu_L\\(\nu_R)^c\end{bmatrix},$$

where

$$\nu_L=\begin{bmatrix}\nu_{eL}\\\nu_{\mu L}\\\nu_{\tau L}\\\vdots\end{bmatrix},\quad (\nu_R)^c=\begin{bmatrix}(\nu_{eR})^c\\(\nu_{\mu R})^c\\(\nu_{\tau R})^c\\\vdots\end{bmatrix} \tag{4.36}$$

are column matrices formed by the LH fields.

Let us perform the standard procedure of diagonalization of the mass term. Taking into account the relation $\overline{(n_L)^c}=-(n_L)^TC^{-1}$, we get

$$\overline{(n_L)^c}Mn_L=-(n_L)^TC^{-1}Mn_L$$
$$=(n_L)^T(C^{-1})^TM^Tn_L=\overline{(n_L)^c}M^Tn_L\ .$$

So, as in the case of a Majorana mass term discussed in Sec. IV.C, the mass matrix M is symmetric:

$$M^T=M\ . \tag{4.37}$$

We shall assume that the eigenvalues of the matrix M are not degenerate. One has

$$M=(U^\dagger)^TmU^\dagger\ . \tag{4.38}$$

Here U is a unitary $2n\times 2n$ matrix, while

$$m_{ik}=m_k\delta_{ik}\ ,\quad m_k\geq 0\ ,\quad i,k=1,2,\ldots,2n\ .$$

Further, inserting Eq. (4.38) in (4.35), we get for the mass term under consideration

$$\mathcal{L}^{D+M}=-\tfrac{1}{2}\overline{(n_L')^c}mn_L'+\text{H.c.}\ , \tag{4.39}$$

where

$$n_L'=U^\dagger n_L\ . \tag{4.40}$$

From Eq. (4.39) we obtain

$$\mathcal{L}^{D+M}=-\tfrac{1}{2}\bar{\chi}m\chi=-\tfrac{1}{2}\sum_{k=1}^{2n}m_k\overline{\chi_k}\chi_k\ , \tag{4.41}$$

where

$$\chi=n_L'+(n_L')^c=\begin{bmatrix}\chi_1\\\chi_2\\\vdots\\\chi_{2n}\end{bmatrix}. \tag{4.42}$$

Obviously,

$$\chi_k(x)=C\overline{\chi_k}^T(x)\ . \tag{4.43}$$

So, if the neutrino mass term is given by Eq. (4.35), the particles with definite masses are Majorana neutrinos. From Eq. (4.41) we conclude also that $\chi_k(x)$ is the field of a neutrino with mass m_k.

It is not difficult to derive the relations connecting the flavor fields $\nu_{lL}(x)$ and the fields $(\nu_{lR}(x))^c$ not present in the standard weak-interaction Lagrangian with the LH components of the $2n$ Majorana fields $\chi_k(x)$. Indeed, using the unitarity of the matrix U, we find

$$n_L=Un_L'=U\chi_L\ . \tag{4.44}$$

This implies

$$\nu_{lL}=\sum_{k=1}^{2n}U_{lk}\chi_{kL}\ ,\quad (\nu_{lR})^c=\sum_{k=1}^{2n}U_{\bar{l}k}\chi_{kL}\ . \tag{4.45}$$

Note that the index l takes the values $e,\mu,\tau,\ldots$ (altogether n values); the index $\bar{l}$ numbers the n lower rows of the mixing matrix U.

Thus, if the neutrino mass term has the form (4.35), the n flavor fields $\nu_{lL}(x)$ are linear combinations of the LH components of $2n$ Majorana fields. It is essential that the "sterile" fields $(\nu_{lR}(x))^c$ are in this case linear combinations of the LH components of the same $2n$ Majorana fields. For the RH components we get from Eq. (4.44)

$$(n_L)^c = U^* \chi_R \ . \tag{4.46}$$

It follows from this relation that

$$(\nu_{lL})^c = \sum_{k=1}^{2n} U_{lk}^* \chi_{kR} \ , \quad \nu_{lR} = \sum_{k=1}^{2n} U_{\bar{l}k}^* \chi_{kR} \ . \tag{4.47}$$

It is evident that the Dirac-Majorana mass term is not invariant in the general case under any global gauge transformations of the neutrino fields. This implies that in the theory under discussion no lepton charges are conserved. Consequently, like the theories with a Majorana mass term considered in Sec. IV.C, a theory with a Dirac-Majorana mass term, in principle, allows processes of the type $\mu^+ \to e^+ + \gamma$, $\mu^+ \to e^+ + e^- + e^+$, etc., as well as total lepton charge nonconserving processes such as $(\beta\beta)_{0\nu}$ decay. An essential difference between the scheme with a Dirac-Majorana mass term and that with a Majorana (or Dirac) mass term shows up in neutrino oscillations. Namely, in addition to the oscillations between the "active" neutrinos (those taking part in the standard weak interaction), oscillations between the "active" and "sterile" neutrinos are possible in the case under consideration. The "sterile" neutrinos do not participate in the standard weak interaction. They are quanta of RH fields. The notion of sterile neutrinos was first introduced by Pontecorvo (1957). We shall discuss in detail in Sec. VII the phenomena of neutrino oscillations arising in the three mixing schemes.

E. A special case of mixing of neutrinos with Majorana masses

All neutrino mass terms considered so far have been constructed under the assumption that to each charged lepton there corresponds one LH flavor neutrino and that the RH fields are not present in the weak interaction Lagrangian. It is known, however, that the existing experimental data are not incompatible with the assumption that to two charged leptons (e.g., e and μ) there corresponds one four-component neutrino, whose LH and RH components enter into the weak lepton currents. We have in mind the scheme of Zeldovich, Konopinsky, and Mahmoud (ZKM; Zeldovich, 1952; Konopinsky and Mahmoud, 1953), according to which one lepton charge, the same for e^- and μ^+, is conserved. The charged-lepton current has in the ZKM scheme the form

$$j_\alpha^{(+)} = 2(\overline{\nu_L}\gamma_\alpha e_L + \overline{\nu_L^c}\gamma_\alpha \mu_L) \ , \tag{4.48}$$

where $\nu_L^c = \frac{1}{2}(1+\gamma_5)\nu^c$ and $\nu^c \equiv C\bar{\nu}^T$. The electron and muon neutrinos are, in the ZKM scheme, LH neutrino and LH antineutrino, respectively.

Let us assume now that the neutrino mass term does not conserve the ZKM lepton charge (Bilenky and Pontecorvo, 1980, 1981). In the case of two charged leptons the most general neutrino mass term has the form

$$\mathscr{L} = -\tfrac{1}{2}\overline{\nu_L^c} M \nu_L + \text{H.c.} \tag{4.49}$$

Here M is a symmetric 2×2 matrix,

$$\nu_L = \begin{pmatrix} \nu_L \\ \nu_L^c \end{pmatrix} .$$

Applying the standard diagonalization procedure we get

$$\nu_L = \sum_{k=1,2} U_{1k}\chi_{kL} \ , \quad \nu_L^c = \sum_{k=1,2} U_{2k}\chi_{kL} \ , \tag{4.50}$$

where $\chi_k(x)$ is the field of a Majorana neutrino with mass m_k.

It is apparent that this scheme is equivalent to the mixing scheme of Gribov and Pontecorvo (1969), corresponding to a Majorana mass term.

Let us discuss next the general case of $n>2$ charged leptons. We shall assume (generalizing the ZKM scheme) that one four-component neutrino field is associated with the fields of two charged leptons and that both the LH and the RH components of all neutrino fields enter into the lepton currents. Clearly such a scheme may be viable only if an even number (say, $2m$) of charged leptons exists. Generalizing Eq. (4.48) we obtain for the weak charged lepton current[15]

$$j_\alpha^{(+)} = 2(\overline{\nu_{1L}}\gamma_\alpha e_L + \overline{\nu_{2L}}\gamma_\alpha \tau_L + \overline{\nu_{1L}^c}\gamma_\alpha \mu_L + \overline{\nu_{2L}^c}\gamma_\alpha \xi_L + \cdots) \ . \tag{4.51}$$

The most general neutrino mass term that can be built using the fields ν_{kL} and ν_{kR} $(k=1,2,\ldots,m)$ has the form

$$\mathscr{L} = -\tfrac{1}{2}\overline{\nu_L^c} M \nu_L + \text{H.c.} \tag{4.52}$$

Here

$$\nu_L = \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \vdots \\ \nu_{mL} \\ \nu_{1L}^c \\ \vdots \\ \nu_{mL}^c \end{pmatrix} \tag{4.53}$$

[15] We have assumed that $\nu_e, \nu_\mu, \nu_\tau, \nu_\xi, \ldots$ are quanta of the fields $\nu_{1L}, \nu_{1L}^c, \nu_{2L}, \nu_{2L}^c, \ldots$ (the fourth lepton is denoted by ξ). It should be clear that the components of the four-component neutrino fields could be identified in a different way with the "phenomenological" neutrinos.

and M is a $2m \times 2m$ symmetric matrix. The method of diagonalization of the mass term (4.52) does not differ from those considered in detail in Secs. IV.C and IV.D. One has

$$\mathscr{L} = -\tfrac{1}{2} \sum_{k=1}^{2m} m_k \overline{\chi_k} \chi_k \tag{4.54}$$

and

$$\nu_{iL} = \sum_{k=1}^{2m} U_{ik} \chi_{kL} \ , \quad \nu_{iL}^c = \sum_{k=1}^{2m} U_{i+mk} \chi_{kL} \ . \tag{4.55}$$

Here U is a unitary mixing matrix and $\chi_k(x) = C\overline{\chi}_k^T(x)$ is the field of a Majorana neutrino with mass m_k.

The main difference between the scheme constructed here and the scheme based on a Majorana mass term is that the former requires an even number of charged leptons. (It should be obvious from Sec. IV.C that a scheme with a Majorana mass term can be constructed for any number of charged leptons.) The generalized ZKM scheme is appealing for its economy: all components of the neutrino fields enter into the lepton currents and neutrino mass term. If it turns out that there exists an even number of charged leptons, this may serve as an indication in favor of the validity of this theory.

Let us note in conclusion that the scheme considered here leads to oscillations only between neutrinos taking part in the standard weak interaction $\nu_l \rightleftarrows \nu_{l'}$ $(l \neq l')$. However, the ZKM-like lepton charge changes by two units in oscillations of the type $\nu_e \rightleftarrows \nu_\mu$ (see, however, footnote 15). It is natural to expect that oscillations of this type are suppressed as compared with the oscillations $\nu_e \rightleftarrows \nu_\tau$, $\nu_\mu \rightleftarrows \nu_\xi$.

V. *CP* INVARIANCE AND NEUTRINO MIXING

A. Mixing of neutrinos with Dirac masses

In all the mixing schemes considered above the mixing matrix is in the general case a complex unitary matrix. We shall discuss in this section the restrictions that follow from the assumption of *CP* invariance of the lepton—intermediate-boson interaction Lagrangian. We begin with the case of mixing of neutrinos with Dirac masses (Dirac mass term).

The operator of a Dirac field $\psi(x)$ transforms under *CP* as

$$U_{CP} \psi(x) U_{CP}^{-1} = \gamma_4 C \overline{\psi}^T(x') \ . \tag{5.1}$$

Here U_{CP} is the *CP*-conjugation operator, $x' = (-\mathbf{x}, ix_0)$, and C is the charge-conjugation matrix determined by Eqs. (4.3). From Eq. (5.1) we get for the LH and the RH components of the field $\psi(x)$

$$U_{CP} \psi_{L,R}(x) U_{CP}^{-1} = \gamma_4 C \overline{\psi_{L,R}}^T(x') \ . \tag{5.2}$$

From this relation we obtain

$$U_{CP} \overline{\psi_{L,R}}(x) U_{CP}^{-1} = -\psi_{L,R}^T(x') C^{-1} \gamma_4 \ . \tag{5.3}$$

Obviously, the free Lagrangian of the fields considered is invariant with respect to the *CP* transformation. It is not difficult to see that the interaction Lagrangian of the leptons and Z^0 boson is also *CP* invariant. Thus *CP* invariance in the lepton sector holds if

$$U_{CP} \mathscr{L}_I^{cc}(x) U_{CP}^{-1} = \mathscr{L}_I^{cc}(x') \ , \tag{5.4}$$

where

$$\mathscr{L}_I^{cc}(x) = \frac{ig}{2\sqrt{2}} \sum_{l,k} \overline{l_L}(x) \gamma_\alpha U_{lk} \nu_{kL}(x) \overline{W}_\alpha(x) + \frac{ig}{2\sqrt{2}} \sum_{l,k} \overline{\nu_{kL}}(x) \gamma_\alpha U_{lk}^* l_L(x) W_\alpha(x)$$

is the lepton—$W^\pm$-boson interaction Lagrangian. In this expression

$$\overline{W}_\alpha(x) = \eta_\alpha W_\alpha^\dagger(x) \ ,$$

where $\eta_\alpha = 1$, $\alpha = 1,2,3$, $\eta_4 = -1$. Using Eqs. (4.3), (5.2), and (5.3) we easily get

$$U_{CP} \overline{l_L}(x) \gamma_\alpha \nu_{kl}(x) U_{CP}^{-1} = \eta_\alpha \overline{\nu_{kL}}(x') \gamma_\alpha l_L(x') \ .$$

Further, taking into account that

$$U_{CP} W_\alpha(x) U_{CP}^{-1} = \eta_\alpha \overline{W}_\alpha(x') \ , \tag{5.5}$$

we find

$$U_{lk} = U_{lk}^* = O_{lk} \ , \quad O^T O = \mathbb{1} \ . \tag{5.6}$$

Thus, if *CP* invariance holds in the leptonic sector, the mixing matrix of neutrinos with Dirac masses is a real orthogonal matrix.

We conclude with the following remarks.

(i) Let us choose the arbitrary *CP* phases of the flavor neutrino fields equal to 1:

$$\begin{aligned} U_{CP} \nu_{lL}(x) U_{CP}^{-1} &= \gamma_4 C \overline{\nu_{lL}}^T(x') \ , \\ U_{CP} \nu_{lR}(x) U_{CP}^{-1} &= \gamma_4 C \overline{\nu_{lR}}^T(x') \ . \end{aligned} \tag{5.7}$$

Evidently, the standard interaction Lagrangian of the intermediate bosons and leptons in this case is invariant with respect to the *CP* transformations, and *CP* invariance would hold in the lepton sector if

$$U_{CP} \mathscr{L}^D(x) U_{CP}^{-1} = \mathscr{L}^D(x') \ , \tag{5.8}$$

where

$$\mathscr{L}^D(x) = -\sum_{l,l'} \overline{\nu_{l'R}}(x) M_{l'l} \nu_{lL}(x) + \text{H.c.} \tag{5.9}$$

From Eqs. (5.7) and (5.9) we obtain

$$M^* = M \ . \tag{5.10}$$

So, if *CP* invariance holds, the matrix M is real.

A real matrix can be reduced to a diagonal form via the biorthogonal transformation

$$M = O' m O^T \ , \tag{5.11}$$

where O' and O are orthogonal matrices and $m_{ik}=m_k\delta_{ik}$, $m_k>O$ (the proof is analogous to that given in footnote 8). From Eqs. (5.9) and (5.11) we find that

$$\nu_{lL}=\sum_{k=1}^{n} O_{lk}\nu_{kL} , \tag{5.12}$$

where $\nu_k(x)$ is the field of a Dirac neutrino with mass m_k. In this way, using a different method, we have arrived at the same conclusion: If CP invariance holds in the leptonic sector, the lepton mixing matrix in the case of Dirac massive neutrinos is a real orthogonal matrix.

(ii) In principle, there are phase factors associated with the CP transformations. Instead of Eq. (5.2) we have

$$U_{CP}\psi_{L,R}(x)U_{CP}^{-1}=\eta_{CP}\gamma_4 C\overline{\psi_{L,R}}^{T}(x') , \tag{5.13}$$

where $\eta_{CP}=e^{2i\alpha}$ and α is a real parameter. Taking into account these phase factors, we obtain from the condition of CP invariance of the Lagrangian, instead of Eq. (5.6),

$$\eta_{CP}^{*}(l)\eta_{CP}(\nu_k)\eta_{CP}(W)U_{lk}=U_{lk}^{*} . \tag{5.14}$$

This implies that the mixing matrix has the form

$$U_{lk}=e^{i\alpha_l}O_{lk}e^{-i\alpha_k}e^{i\alpha'} , \tag{5.15}$$

where O is an orthogonal matrix. We are going to show in Sec. VII that the phase factors present in Eq. (5.15) can be left out [owing to the fact that because of the nonobservability of the phases of the Dirac fields the factor η_{CP} in Eq. (5.13) can always be set equal to one].

We proceed now to a discussion of CP invariance in the case of mixing of neutrinos generated by a Majorana mass term.

B. Mixing of neutrinos with Majorana masses

The condition that $\chi_k(x)$ is a Majorana field can always be written in the form

$$C\overline{\chi_k}^{T}(x)=\xi_k\chi_k(x) , \tag{5.16}$$

where ξ_k is a phase factor. Since the weak-interaction Lagrangian is not invariant with respect to the charge conjugation, the phase factors ξ_k have no physical meaning and can be arbitrarily chosen. It will be assumed in what follows that $\xi_k=\pm1$. We shall see that the observable quantities do not depend on the factors ξ_k.

Under CP the Majorana field transforms as

$$U_{CP}\chi_k(x)U_{CP}^{-1}=\eta_{CP}(\chi_k)\gamma_4\chi_k(x') , \tag{5.17}$$

where $\eta_{CP}(\chi_k)$ is a phase factor and $x'=(-\mathbf{x},ix_0)$. From Eq. (5.17) we get

$$U_{CP}\,|p,r\rangle=\eta_{CP}^{*}\,|p',-r\rangle , \tag{5.18}$$

where $|p,r\rangle$ is the state vector of a Majorana neutrino with momentum p and helicity r and $p'=(-\mathbf{p},ip_0)$. It follows from Eq. (5.18) that η_{CP}^{*} is the CP parity of the Majorana neutrino. As is shown in Appendix A, the CP parities of Majorana particles can assume the values $\pm i$.

It is interesting to note that as early as in the first work on neutrino oscillations (Pontecorvo, 1958; one flavor neutrino was known at that time) it was emphasized that massive Majorana neutrinos can have different CP parities. We shall see further that the relative CP parities of Majorana neutrinos are, in principle, observable quantities.

Let us now consider the restrictions on the lepton mixing matrix which follow from the requirement of CP invariance of the Lagrangian of the system. The free Lagrangians of neutrinos with definite masses, of charged leptons, and of weak intermediate bosons, as well as the Lagrangian of the lepton—Z^0-boson interaction are, obviously, invariant with respect to the CP transformations. Consequently the total Lagrangian is CP invariant if

$$U_{CP}\mathcal{L}_I^{cc}(x)U_{CP}^{-1}=\mathcal{L}_I^{cc}(x') , \tag{5.19}$$

where

$$\mathcal{L}_I^{cc}(x)=\frac{ig}{2\sqrt{2}}\sum_{l,k}\overline{l_L}(x)\gamma_\alpha U_{lk}\chi_{k_L}(x)\overline{W}_\alpha(x)+\text{H.c.} \tag{5.20}$$

It has been indicated in Sec. V.A that the phase factors associated with the CP transformation of the Dirac fields have no physical meaning and can be chosen arbitrarily. In the case of mixing of Majorana neutrinos we consider it is convenient to choose the CP phase factors of the charged leptons to be equal to i:

$$U_{CP}l_L(x)U_{CP}^{-1}=i\gamma_4 C\overline{l_L}^{T}(x') . \tag{5.21}$$

From Eqs. (5.16) and (5.17) we get for the Majorana fields $\chi_{kL}(x)$

$$U_{CP}\chi_{kL}(x)U_{CP}^{-1}=\eta_{CP}(\chi_k)\xi_k\gamma_4 C\overline{\chi_{kL}}^{T}(x') . \tag{5.22}$$

Using Eqs. (5.5), (5.21), and (5.22) we obtain from (5.19) and (5.20) (Bilenky, Nedelcheva, and Petcov, 1984; Kayser, 1984)

$$U_{lk}\eta_k\xi_k=U_{lk}^{*} , \tag{5.23}$$

where

$$\eta_k=-i\eta_{CP}(\chi_k) . \tag{5.24}$$

The quantity η_k can take the values ±1.

Condition (5.23) differs substantially from the corresponding condition (5.6) in the case of Dirac massive neutrinos. As follows from Eq. (5.23), the phases of the elements of the mixing matrix are determined by the CP parities of the Majorana neutrinos as well as by (arbitrary) factors in the Majorana condition (5.16). The mixing matrix elements are real for $\eta_k\xi_k=1$ and purely imaginary for $\eta_k\xi_k=-1$ [the last statement is valid only if the CP phase factors of the fields $l(x)$ are chosen to be equal to i]. It is clear from Eq. (5.23) that the mixing matrix has the form

$$U_{lk}=O_{lk}e^{-i(\pi/4)(\eta_k\xi_k-1)} , \tag{5.25}$$

where O is a real orthogonal matrix.

From the derivation given above it should be obvious that these results are valid both for a Majorana mass term (the index k runs from 1 to n, where n is the number of charged leptons) and for a Dirac-Majorana mass term (the index k takes values from 1 to $2n$).

We shall consider next a different approach to the problem of CP invariance in the case of mixing of massive Majorana neutrinos. We have used in our treatment so far the transformation properties of the fields of neutrinos with definite masses (in addition to those of the charged-lepton and weak-intermediate-boson fields). Suppose that the flavor neutrino fields $\nu_{lL}(x)$ transform under CP as the LH components of Dirac fields:

$$U_{CP}\nu_{lL}(x)U_{CP}^{-1}=i\gamma_4 C\,\overline{\nu_{lL}}^T(x')\ . \tag{5.26}$$

Evidently, the Lagrangian of lepton—intermediate-boson interaction is CP invariant in this case. The total Lagrangian of the system will be CP invariant if the neutrino mass term is CP invariant. Consider first the Majorana mass term. One has

$$U_{CP}\mathscr{L}^M(x)U_{CP}^{-1}=\mathscr{L}^M(x')\ . \tag{5.27}$$

Here

$$\mathscr{L}^M=\tfrac{1}{2}\nu_L^T C^{-1}M\nu_L-\tfrac{1}{2}\overline{\nu_L}M^{\dagger}C\,\overline{\nu_L}^T\ , \tag{5.28}$$

where

$$\nu_L=\begin{pmatrix}\nu_{eL}\\ \nu_{\mu L}\\ \nu_{\tau L}\\ \vdots\end{pmatrix}$$

and M is an $n\times n$ symmetric matrix.

Using Eqs. (5.26)—(5.28) we obtain

$$M^{\dagger}=M\ . \tag{5.29}$$

Taking into account that $M^T=M$, we get

$$M^{*}=M\ . \tag{5.30}$$

Consequently, if CP invariance holds in the leptonic sector, the neutrino mass matrix M will be real [under the convention used for the CP phase factors in Eq. (5.26)].

Let us cast the matrix M in a diagonal form. For a real symmetric matrix one has

$$M=Om'O^T\ , \tag{5.31}$$

where O is a real orthogonal matrix and $m'_{ik}=m'_k\delta_{ik}$ $(i,k=1,2,\ldots,n)$, m'_k being the kth eigenvalue of the matrix M. The nonzero eigenvalues of a real symmetric matrix can be either positive or negative. We have

$$m'_k=m_k\rho_k\ , \tag{5.32}$$

where $m_k=|m'_k|$ and $\rho_k=\pm 1$. Further, if we use the identity $\rho_k=e^{i(\pi/2)(\rho_k-1)}$, the matrix M can, obviously, be expressed in the standard way:

$$M=(U^{\dagger})^T m U^{\dagger}\ , \tag{5.33}$$

where

$$U_{lk}=O_{lk}e^{-i(\pi/4)(\rho_k-1)}\ . \tag{5.34}$$

From Eq. (5.28), using (5.33), we obtain

$$\mathscr{L}^M=-\tfrac{1}{2}\sum_k m_k\overline{\chi_k}\chi_k\ , \tag{5.35}$$

where

$$\chi_k=(U^{\dagger}\nu_L)_k+(U^T C\,\overline{\nu_L}^T)_k \tag{5.36}$$

is the field of a Majorana neutrino with mass m_k. It follows from Eq. (5.36) that

$$\nu_{lL}=\sum_{k=1}^{n}U_{lk}\chi_{kL}\ . \tag{5.37}$$

Thus, if CP invariance holds in the leptonic sector, the lepton mixing matrix in the case of a Majorana mass term is given by Eq. (5.34). The phases of the elements of the mixing matrix take the values 0 ($\rho_k=1$) or $\pi/2$ ($\rho_k=-1$).

Let us see next how the massive Majorana neutrino fields behave under CP transformation. From Eq. (5.36) we get

$$\chi_{kL}=\sum_{l=e,\mu,\tau,\ldots}U_{lk}^{*}\nu_{lL}\ . \tag{5.38}$$

Using Eq. (5.26) one obtains

$$U_{CP}\chi_{kL}(x)U_{CP}^{-1}=i\sum_l U_{lk}^{*}\gamma_4 C\,\overline{\nu_{lL}}^T(x')\ . \tag{5.39}$$

Further, from (5.38) we find

$$\gamma_4 C\,\overline{\nu_{lL}}^T(x')=\sum_{k'}U_{lk'}^{*}\gamma_4 C\overline{\chi_{k'L}}^T(x')\ . \tag{5.40}$$

Finally, from (5.34) it follows that

$$U_{lk}^{*}=U_{lk}\rho_k\ . \tag{5.41}$$

Inserting now Eq. (5.40) into (5.39) and taking (5.41) into account, we get

$$U_{CP}\chi_{kL}(x)U_{CP}^{-1}=i\rho_k\gamma_4 C\overline{\chi_{kL}}^T(x')\ . \tag{5.42}$$

The Majorana fields $\chi_k(x)$ determined by Eq. (5.36) satisfy the condition $\chi_k=C\overline{\chi_k}^T$. Hence the factors ξ_k in the Majorana condition for these fields are equal to 1. By comparing Eqs. (5.42) and (5.22) we conclude that the CP parity of the Majorana field $\chi_k(x)$ is given by[16]

$$\eta_{CP}(\chi_k)=i\rho_k\ . \tag{5.43}$$

So, the CP parity of the field of a Majorana neutrino with mass m_k is determined by the sign of the corresponding

[16]Obviously, this result does not depend on the convention (5.26) used for the CP phases of the fields $\nu_{lL}(x)$.

eigenvalue of the neutrino mass matrix (Wolfenstein, 1981a; Bilenky, Nedelcheva, and Petcov, 1984). Note that Eq. (5.34) is compatible with the general expression (5.25) ($\xi_k = 1$, $\eta_k = \rho_k$).

The phases of the elements of the mixing matrix (5.34) are equal to zero for $\rho_k = 1$ and take the value $\pi/2$ if $\rho_k = -1$. At the same time the fields of the Majorana neutrinos satisfy the condition $C\overline{\chi_k}^T = \chi_k$. It is not difficult to see that the lepton mixing matrix can be made real by redefining the Majorana fields (Wolfenstein, 1981a). Indeed, inserting Eq. (5.31) into (5.28) we get the following expression for the Majorana mass term $\mathscr{L}^M$:

$$\mathscr{L}^M = -\tfrac{1}{2}\sum_{k=1}^{n} m_k \overline{\chi'_k}\chi'_k \ , \tag{5.44}$$

where

$$\chi'_k = (O^T \nu_L)_k + \rho_k (O^T C\,\overline{\nu_L}^T)_k \ . \tag{5.45}$$

Obviously, the field χ'_k of the Majorana neutrino with mass m_k satisfies the condition

$$C\overline{\chi'_k}^T(x) = \rho_k \chi'_k(x) \ . \tag{5.46}$$

From Eq. (5.45) we obtain[17]

$$\nu_{lL} = \sum_k O_{lk}\chi'_{kL} \ . \tag{5.47}$$

Thus, if the fields of neutrinos with definite mass satisfy the Majorana condition (5.46), the neutrino mixing matrix is a real orthogonal matrix.[18] By comparing Eqs. (5.36) and (5.45) we can conclude that the fields $\chi_k(x)$ and $\chi'_k(x)$ are connected by the relation

$$\chi'_k = e^{-i(\pi/4)(\rho_k - 1)}\chi_k \ . \tag{5.48}$$

This completes our discussion of the Majorana mass term, and we proceed to the analysis of the implications of CP invariance in the case of a Dirac-Majorana mass term [see Eq. (4.35)]. It is convenient to choose the arbitrary phase factors appearing in the CP transformation laws for both the LH flavor neutrino fields ν_{lL} and the RH fields ν_{lR} to be equal to i. In this case CP invariance implies that the symmetric $2n \times 2n$ mass matrix M in Eq. (4.35) is real. It should be apparent that the relations we have derived for the case of a Majorana mass term are also valid in this case.

To summarize, we have shown that in the case of CP invariance the CP parities of the massive Majorana neutrinos determine the phases of the elements of the lepton mixing matrix (or alternatively, enter into the Majorana condition). As was noted first by Wolfenstein (1981a), the relative values of the Majorana neutrino CP parities are, in principle, measurable quantities. It will be shown in Sec. IX that, in particular, the $(\beta\beta)_{0\nu}$-decay amplitude depends on them in a nontrivial way.

VI. MIXING OF TWO MASSIVE MAJORANA NEUTRINOS

A. The general case

In the present section we shall consider in detail the simplest possibility of neutrino mixing—the mixing of two neutrinos with Majorana masses. We shall treat first the general case. Then various limiting cases, which are interesting from a physical point of view, will be considered.

As has been shown in Secs. IV.C and IV.D, mixing of massive Majorana neutrinos arises both in the schemes with a Majorana mass term and in the schemes with a Dirac-Majorana mass term. Consider first the Majorana mass term. In the case of two neutrino types (say, ν_e and ν_μ) it has the form

$$\mathscr{L}^M = -\tfrac{1}{2}\overline{(\nu_L)^c} M \nu_L + \text{H.c.} \ , \tag{6.1}$$

where

$$\nu_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \end{pmatrix} ,$$

and M is a symmetric 2×2 matrix. We shall assume here that CP invariance holds in the leptonic sector. One has then $M^* = M$ (see Sec. V.B). Let us parametrize the matrix M as follows:

$$M = \begin{pmatrix} m_{ee} & m_{\mu e} \\ m_{\mu e} & m_{\mu\mu} \end{pmatrix} ,$$

where m_{ee}, $m_{\mu e}$, and $m_{\mu\mu}$ are real parameters. Equation (6.1) can then be written as

$$\mathscr{L}^M = -\tfrac{1}{2}\{ m_{ee}\overline{(\nu_{eL})^c}\nu_{eL} + m_{\mu\mu}\overline{(\nu_{\mu L})^c}\nu_{\mu L} + m_{\mu e}[\overline{(\nu_{eL})^c}\nu_{\mu L} + \overline{(\nu_{\mu L})^c}\nu_{eL}]\} + \text{H.c.} \tag{6.2}$$

The Lagrangian (6.2) was first constructed and the relevant scheme analyzed by Gribov and Pontecorvo (1969).

Next let us diagonalize the neutrino mass matrix. We have

$$M = O m' O^T \ , \tag{6.3}$$

[17]Let us note that the CP phase convention, as determined by Eqs. (5.21) and (5.26), was implicitly adopted and the fields $\chi'_k(x)$ were used by Halprin, Petcov, and Rosen (1983) and Petcov (1982a, 1982b, 1982c, 1983).

[18]It is useful to compare the two choices of Majorana fields considered. We can write the negative eigenvalues of the neutrino mass matrix as $m'_p = i m_p i$ ($m'_p < 0$). The factor i can be absorbed either by the element U_{lp} of the mixing matrix or by the Majorana field $\chi_p(x)$. All factors in the Majorana condition (5.16) are equal to 1 in the first case, and the phases of the mixing matrix elements can assume the values 0 or $\pi/2$. In the second case the mixing matrix is real. However, the factors ξ_k in the Majorana condition take the values ± 1. Obviously, the two choices are physically equivalent.

where O is an orthogonal matrix, $m'_{ik}=m'_k\delta_{ik}$, and m'_k is the kth eigenvalue of the matrix $M(i,k=1,2)$. For $m'_{1,2}$ we obtain

$$m'_{1,2}=\tfrac{1}{2}\{(m_{ee}+m_{\mu\mu})\pm[(m_{ee}-m_{\mu\mu})^2+4m_{\mu e}^2]^{1/2}\} . \tag{6.4}$$

An arbitrary orthogonal 2×2 matrix has the general form

$$O=\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix} , \tag{6.5}$$

where θ is a parameter. Inserting Eq. (6.5) into (6.3) one finds

$$m_{ee}=m'_1\sin^2\theta+m'_2\cos^2\theta , \quad m_{\mu\mu}=m'_1\cos^2\theta+m'_2\sin^2\theta , \tag{6.6}$$

$$2m_{\mu e}=(m'_1-m'_2)\sin^2\theta . \tag{6.7}$$

From Eq. (6.6) we get

$$m_{\mu\mu}-m_{ee}=(m'_1-m'_2)\cos2\theta . \tag{6.8}$$

Finally, it follows from Eqs. (6.7) and (6.8) that the angle θ is connected with the parameters m_{ee}, $m_{\mu\mu}$, and $m_{\mu e}$ by (Gribov and Pontecorvo, 1969)

$$\tan2\theta=\frac{2m_{\mu e}}{m_{\mu\mu}-m_{ee}} . \tag{6.9}$$

Obviously, this relation does not uniquely determine the angle θ. The latter can be determined unambiguously if in addition to $\tan2\theta$ the sign of $\sin2\theta$ is also known. From Eqs. (6.7) and (6.4) we find that

$$\sin2\theta=\frac{2m_{\mu e}}{[(m_{ee}-m_{\mu\mu})^2+4m_{\mu e}^2]^{1/2}} . \tag{6.10}$$

The nonzero eigenvalues of the matrix M can be both positive and negative. Assuming that $m'_{1,2}\neq0$, let us write

$$m'_k=m_k\rho_k, \quad k=1,2 \tag{6.11}$$

where $m_k=|m'_k|$ and $\rho_k=\pm1$. Inserting Eq. (6.3) into (6.1) we get the following standard expression for the neutrino mass term:

$$\mathscr{L}^m=-\tfrac{1}{2}\sum_{k=1,2} m_k\overline{\chi_k}\chi_k , \tag{6.12}$$

where

$$\chi_k=(O^T\nu_L)_k+\rho_k(O^TC\,\overline{\nu_L}^T)_k . \tag{6.13}$$

Obviously,

$$C\overline{\chi_k}^T=\rho_k\chi_k . \tag{6.14}$$

Hence $\chi_k(x)$ is the field of a Majorana neutrino with mass m_k. As has been shown in Sec. V, the CP parity of the Majorana neutrino χ_k is $i\rho_k$. Using Eqs. (6.5) and (6.13) we get

$$\begin{aligned}\chi_1&=(\sin\theta\,\nu_{eL}+\cos\theta\,\nu_{\mu L})\\&\quad+\rho_1[\sin\theta(\nu_{eL})^c+\cos\theta(\nu_{\mu L})^c] ,\\ \chi_2&=(-\cos\theta\,\nu_{eL}+\sin\theta\,\nu_{\mu L})\\&\quad+\rho_2[-\cos\theta(\nu_{eL})^c+\sin\theta(\nu_{\mu L})^c] .\end{aligned} \tag{6.15}$$

It is now easy to express the flavor neutrino fields $\nu_{eL}(x)$ and $\nu_{\mu L}(x)$ in terms of the LH components $\chi_{1L}(x)$ and $\chi_{2L}(x)$ of the two Majorana fields:

$$\begin{aligned}\nu_{eL}&=\sin\theta\chi_{1L}-\cos\theta\chi_{2L} ,\\ \nu_{\mu L}&=\cos\theta\chi_{1L}+\sin\theta\chi_{2L} .\end{aligned} \tag{6.16}$$

The simplest Dirac-Majorana mass term has the form

$$\mathscr{L}^{D+M}=-\tfrac{1}{2}\overline{(n_L)^c}Mn_L+\text{H.c.} , \tag{6.17}$$

where

$$n_L=\begin{bmatrix}\nu_L\\ (\nu_R)^c\end{bmatrix} \tag{6.18}$$

and

$$M=\begin{bmatrix}m_L & m_D\\ m_D & m_R\end{bmatrix} . \tag{6.19}$$

Here ν_L stands for any LH "active" neutrino field (ν_{eL} or $\nu_{\mu L},\ldots$), ν_R is a RH "sterile" field, and m_L, m_D, and m_R are parameters. In the case of CP invariance we shall consider m_L, m_D, and m_R to be real.

By comparing Eqs. (6.1) and (6.17) we see that all relations derived above would be valid for the case of a Dirac-Majorana mass term if we made the following substitutions in them: $m_{ee}\to m_L$, $m_{\mu\mu}\to m_R$, $m_{\mu e}\to m_D$, $\nu_{eL}\to\nu_L$, $\nu_{\mu L}\to(\nu_R)^c$. We have

$$\begin{aligned}\nu_L&=\sin\theta\chi_{1L}-\cos\theta\chi_{2L} ,\\ (\nu_R)^c&=\cos\theta\chi_{1L}+\sin\theta\chi_{2L} ,\end{aligned} \tag{6.20}$$

where the mixing angle θ is given by the expression

$$\tan2\theta=\frac{2m_D}{m_R-m_L} \tag{6.21}$$

and χ_1 and χ_2 are the fields of Majorana neutrinos with masses

$$m_{1,2}=\rho_{1,2}\tfrac{1}{2}\{m_R+m_L\pm[(m_R-m_L)^2+4m_D^2]^{1/2}\} \tag{6.22}$$

[the factor ρ_k is equal to 1 (-1) when the expression in parentheses is positive (negative)] and CP parities $\eta_{CP}(\chi_{1,2})=i\rho_{1/2}$. The relations derived so far in this subsection are exact. We shall discuss next some limiting cases that are interesting from a physical point of view.

B. Possible mechanism for generation of small Majorana neutrino mass

Consider the Dirac-Majorana mass term (6.17). Let us assume that

$$m_L=0,\ \ m_R \gg |m_D|\ . \tag{6.23}$$

For the eigenvalues of the matrix M we get from Eq. (6.4) up to terms linear in m_D/m_R

$$m'_1 \simeq m_R,\ \ m'_2 \simeq -m_D^2/m_R\ . \tag{6.24}$$

Thus, if the condition (6.23) apply, the particles possessing definite masses are two Majorana neutrinos, one of them being much heavier than the other:

$$m_1 \simeq m_R,\ \ m_2 \simeq \frac{m_D^2}{m_R},\ \ m_2 \ll m_1\ . \tag{6.25}$$

The "light" and the "heavy" Majorana neutrinos have opposite CP parities:

$$\eta_{CP}(\chi_1)=i,\ \ \eta_{CP}(\chi_2)=-i\ . \tag{6.26}$$

Further, from Eq. (6.21) as well as from (6.10) we get for the mixing angle

$$\theta \simeq \frac{m_D}{m_R}\ . \tag{6.27}$$

Finally, using Eqs. (6.20) and (6.27) one finds

$$\nu_L \simeq -\chi_{2L}+\frac{m_D}{m_R}\chi_{1L}\ ,\ \ (\nu_R)^c \simeq \chi_{1L}+\frac{m_D}{m_R}\chi_{2L}\ . \tag{6.28}$$

In this way we conclude that if the conditions (6.23) do hold, the LH flavor field ν_L represents a mixture of the LH component of a light Majorana neutrino [the relevant coefficient being (-1)] and the LH component of a heavy Majorana neutrino (with coefficient $m_D/m_R \ll 1$). This implies that effects of nonzero neutrino mass might be observed in experiments like those studying the electron spectrum in tritium β decay.

The mechanism for generating light neutrino mass described here is known as the mechanism of Gell-Mann, Ramond, Slansky (1979), and Yanagida (1979; see also Stech, 1980). It is used in a wide class of grand unified theories. If the appearance of nonzero neutrino masses is associated with this mechanism, the massive neutrinos will be Majorana particles. Consequently, total lepton number nonconserving processes like $(\beta\beta)_{0\nu}$ decay will take place.

C. Pseudo-Dirac neutrino

Suppose that in the case of a Majorana mass term

$$m_{ee}+m_{\mu\mu}=0\ . \tag{6.29}$$

From Eqs. (6.4) and (6.11) we obtain in this case

$$\begin{aligned} &m_1=m_2=(m_{\mu\mu}^2+m_{\mu e}^2)^{1/2}=m\ ,\\ &\rho_1=1,\ \rho_2=-1\ . \end{aligned} \tag{6.30}$$

The mass Lagrangian has the form

$$\mathcal{L}^M=-\tfrac{1}{2}m(\overline{\chi}_1\chi_1+\overline{\chi}_2\chi_2)\ . \tag{6.31}$$

Instead of χ_1 and χ_2 let us introduce the field

$$\psi=\frac{1}{\sqrt{2}}(\chi_1-\chi_2)\ . \tag{6.32}$$

Since $C\overline{\chi}_1^T=\chi_1$ and $C\overline{\chi}_2^T=-\chi_2$, one has

$$C\overline{\psi}^T=\frac{1}{\sqrt{2}}(\chi_1+\chi_2)\ . \tag{6.33}$$

Using Eqs. (6.31)–(6.33) we get

$$\mathcal{L}^M=-m\overline{\psi}\psi\ . \tag{6.34}$$

It follows from Eqs. (6.32)–(6.34) that $\psi(x)$ is a complex four-component $(C\overline{\psi}^T \neq \psi)$ field of a neutrino with mass m. Thus we have shown that the fields of two Majorana particles possessing equal masses and opposite CP parities are equivalent to the field of one Dirac particle having the same mass. With the help of Eqs. (6.13) and (6.32) it is not difficult to express the field $\psi(x)$ in terms of the flavor neutrino fields ν_{eL} and $\nu_{\mu L}$:

$$\psi=\cos\theta'\nu_{eL}+\sin\theta'\nu_{\mu L}-\sin\theta'(\nu_{eL})^c+\cos\theta'(\nu_{\mu L})^c\ , \tag{6.35}$$

where

$$\theta'=\frac{\pi}{4}-\theta\ .$$

Using Eqs. (6.9) and (6.10) we get

$$\tan2\theta'=\left[\frac{m_{\mu e}}{m_{\mu\mu}}\right]^{-1},\ \ \sin2\theta'=\frac{m_{\mu\mu}}{(m_{\mu\mu}^2+m_{\mu e}^2)^{1/2}}\ . \tag{6.36}$$

Finally, it follows from Eq. (6.35) that the LH flavor fields ν_{eL} and $\nu_{\mu L}$ are connected with ψ_L and ψ_L^c $[\psi_L^c=\frac{1}{2}(1+\gamma_5)\psi^c,\ \psi^c=C\overline{\psi}^T]$ as

$$\begin{aligned} &\nu_{eL}=\psi_L\cos\theta'-\psi_L^c\sin\theta'\ ,\\ &\nu_{\mu L}=\psi_L\sin\theta'+\psi_L^c\cos\theta'\ . \end{aligned} \tag{6.37}$$

The mass term (6.33) is invariant with respect to a global gauge transformation of the complex field $\psi(x)$. However, the lepton charge associated with this invariance (in the general case of $\theta' \neq 0$) is not conserved by the standard weak interaction. This is evident from the expression for the charged-lepton current:

$$\begin{aligned} j_\alpha^{(-)}=2[&\overline{e_L}\gamma_\alpha(\psi_L\cos\theta'-\psi_L^c\sin\theta')\\ &+\overline{\mu_L}\gamma_\alpha(\psi_L\sin\theta'+\psi_L^c\cos\theta')]\ . \end{aligned} \tag{6.38}$$

As a consequence, the weak interaction generates a Majorana mass for ψ that is much smaller than its Dirac mass. Effectively, it lifts the mass degeneracy between χ_1 and χ_2, splitting the Dirac neutrino ψ into two Majorana neutrinos with very close but different masses. In view of this the field $\psi(x)$ is called pseudo-Dirac (Wolfenstein, 1981b). It is a characteristic feature of the considered scheme of lepton number conservation that in the limit of zero Majorana mass corrections the flavor neutrino fields ν_{eL} and $\nu_{\mu L}$ are related through an orthogonal transformation with the components ψ_L and ψ_L^c of one Dirac field ψ.

Not only oscillations of neutrinos, but also processes like $\mu^+ \to e^+ + \gamma$ and $(\beta\beta)_{0\nu}$ decays are allowed in the case of pseudo-Dirac neutrinos. In fact, all processes typically associated with the existence of massive Majorana neutrinos are allowed, even in the limit of zero Majorana corrections to the mass of the pseudo-Dirac neutrino (Petcov, 1982a; Valle, 1983).

Let us add in conclusion that the conditions for the appearance of pseudo-Dirac neutrinos in the general case of n flavor neutrinos have been studied by Leung and Petcov (1983), while the Majorana corrections to the mass of the pseudo-Dirac neutrinos have been considered by Petcov (1982a), Doi *et al.* (1983b), Leung and Petcov (1983), and Petcov and Toshev (1984). Finally, it should be noted that a weak β-decay Lagrangian, which implies, in essence, a pseudo-Dirac neutrino, was discussed a long time ago by Pauli (1957).

D. The Zeldovich-Konopinsky-Mahmoud lepton charge

Let us consider the Majorana mass term (6.2) assuming that

$$m_{ee} = m_{\mu\mu} = 0, \quad m_{\mu e} > 0 \ . \tag{6.39}$$

As is clear from Eq. (6.4), the masses of the Majorana neutrinos coincide in this case:

$$m_{1,2} = m_{\mu e} = m \ , \tag{6.40}$$

while their CP parities are opposite,

$$\eta_{CP}(\chi_1) = i, \quad \eta_{CP}(\chi_2) = -i \ . \tag{6.41}$$

We have shown in Sec. VI.C that if the conditions (6.40) and (6.41) are fulfilled, the Majorana mass term is reduced to a Dirac mass term. It is useful to obtain this result in a different way.

In the case under consideration the neutrino mass term has the form

$$\mathscr{L}^M = -\tfrac{1}{2} m [\overline{(\nu_{\mu L})^c} \nu_{eL} + \overline{(\nu_{eL})^c} \nu_{\mu L}] + \text{H.c.} \tag{6.42}$$

The Lagrangian (6.42) is invariant with respect to the global gauge transformations

$$\begin{aligned} \nu_{eL}(x) &\to \nu'_{eL}(x) = e^{i\Lambda} \nu_{eL}(x) \ , \\ \nu_{\mu L}(x) &\to \nu'_{\mu L} = e^{-i\Lambda} \nu_{\mu L}(x) \ , \end{aligned} \tag{6.43}$$

where Λ is a constant parameter. This invariance implies that the mass term (6.42) is a Dirac mass term. Indeed, let us introduce the field $\nu(x)$ so that

$$\nu_L = \nu_{eL}, \quad \nu_R = (\nu_{\mu L})^c \ . \tag{6.44}$$

From Eqs. (6.42) and (6.44) we obtain

$$\mathscr{L}^M = -m \bar{\nu} \nu \ , \tag{6.45}$$

where $\nu(x)$ is a four-component Dirac field.

If simultaneously with (6.41) we perform the following transformation of the electron and muon fields,

$$\begin{aligned} e(x) &\to e'(x) = e^{i\Lambda} e(x) \ , \\ \mu(x) &\to \mu'(x) = e^{-i\Lambda} \mu(x) \ , \end{aligned} \tag{6.46}$$

the total Lagrangian of the system, obviously, will not change. This invariance implies that the lepton charge L', equal to $+1$ for e^- and ν_e, to -1 for μ^- and ν_μ, and to 0 for all other particles (i.e., $L' = L_e - L_\mu$) is conserved. Let us note that the charge L' was introduced a long time ago by Zeldovich (1952) and Konopinsky and Mahmoud (1953).

Thus we have shown in the present subsection that if the conditions (6.39) are realized we arrive at one massive Dirac neutrino and at the conservation of the ZKM lepton charge. The weak charged and neutral lepton currents then have the form

$$\begin{aligned} j_\alpha^{(+)} &= 2(\overline{\nu_L} \gamma_\alpha e_L + \overline{\nu_L^c} \gamma_\alpha \mu_L) \ , \\ j_\alpha^0 &= \overline{\nu_L} \gamma_\alpha \nu_L + \overline{\nu_L^c} \gamma_\alpha \nu_L^c \ . \end{aligned} \tag{6.47}$$

It is well known that such a theory is compatible with the existing data. The mixing of neutrinos in the ZKM scheme was considered by us in detail in Sec. IV.E.

It should be noted that the Dirac field arising as a result of the diagonalization of the mass term (6.42) differs substantially from the Dirac fields considered in Sec. IV.B. Whereas only the LH components of the Dirac fields discussed in IV.B are "active," both the LH and the RH components of the field $\nu(x)$ enter into the weak lepton currents. Clearly this difference is of a physical character. If, for example, a neutrino beam passes through a magnetic field, under certain conditions a LH neutrino transforms into a RH neutrino due to the neutrino magnetic moment. The RH neutrinos in the ZKM scheme can interact with nucleons and produce μ^+. In the standard case the RH neutrinos are sterile. The Dirac neutrinos arising in the two cases also possess different magnetic moments (Wolfenstein, 1981b; Petcov, 1982b). Let us note that some astrophysical effects of the rotation of the neutrino spin in a magnetic field have been studied by Fujikawa and Shrock (1980).

E. Schemes with maximal neutrino mixing

It is well known that (neglecting the small effects of CP violation), in the case of neutral kaons, (i) the mixing angle is equal to $\pi/4$, (ii) the particles with definite mass possess opposite CP parities, and (iii) the difference between the masses is much less than the masses of the mass eigenstate particles. The construction of schemes of neutrino mixing possessing these three characteristic features is of undoubted interest.

Consider first the Majorana mass term (6.1). Let us assume that (Bilenky and Pontecorvo, 1983).

$$m_{\mu e} \gg |m_{ee}|, |m_{\mu\mu}| \ . \tag{6.48}$$

It is apparent from Eqs. (6.9) and (6.10) that in this case $\theta \simeq \pi/4$. For the eigenvalues of the mass matrix we get from Eqs. (6.4) and (6.48)

$$m'_{1,2} \simeq \tfrac{1}{2}(m_{\mu\mu}+m_{ee}) \pm m_{\mu e} \ . \tag{6.49}$$

Consequently, if the inequalities (6.48) take place, the masses of the Majorana neutrinos are equal, respectively, to

$$m_{1,2} \simeq m_{\mu e} \pm \tfrac{1}{2}(m_{\mu\mu}+m_{ee}) \tag{6.50}$$

and their CP parities are opposite:

$$\eta_{CP}(\chi_1)=i, \quad \eta_{CP}(\chi_2)=-i \ .$$

Evidently,

$$|m_1-m_2| \ll m_1, m_2 \ . \tag{6.51}$$

Finally, it is not difficult to obtain from Eq. (6.16) the relation between the flavor neutrino fields and the fields of massive Majorana neutrinos:

$$\nu_{eL}=\frac{1}{\sqrt{2}}(\chi_{1L}-\chi_{2L}), \quad \nu_{\mu L}=\frac{1}{\sqrt{2}}(\chi_{1L}+\chi_{2L}) \ . \tag{6.52}$$

Let us turn now to the Dirac-Majorana mass term (6.16). Suppose that

$$m_D \gg |m_L|, |m_R| \ . \tag{6.53}$$

We get in this case

$$m_{1,2} \simeq m_D \pm \tfrac{1}{2}(m_L+m_R), \quad \theta \simeq \pi/4 \ ,$$
$$\eta_{CP}(\chi_1)=i, \quad \eta_{CP}(\chi_2)=-i \ . \tag{6.54}$$

Further, we have

$$\nu_L=\frac{1}{\sqrt{2}}(\chi_{1L}-\chi_{2L}), \quad (\nu_R)^c=\frac{1}{\sqrt{2}}(\chi_{1L}+\chi_{2L}) \ . \tag{6.55}$$

For the Majorana neutrino fields we obtain the expressions

$$\chi_1=\frac{1}{\sqrt{2}}(\nu+\nu^c), \quad \chi_2=\frac{1}{\sqrt{2}}(-\nu+\nu^c) \ . \tag{6.56}$$

The close analogy with the mixing of neutral kaons makes these two schemes of neutrino mixing rather attractive. Oscillations between two types of active neutrino ($\nu_e \rightleftarrows \nu_\mu, \nu_e \rightleftarrows \nu_\tau, \ldots$) are possible in schemes of the first type. In schemes of the second type, oscillations between active and sterile neutrinos ($\nu_{eL} \rightleftarrows \bar{\nu}_{eL}, \nu_{\mu L} \rightleftarrows \bar{\nu}_{\mu L}, \ldots$) are allowed. It should be noted, however, that the oscillation length [which is inversely proportional to the difference of the squares of neutrino masses (see Sec. VII)] in these schemes may turn out to be rather long. If the schemes of neutrino mixing considered here are realized in nature, we could face a situation in which the effects of relatively large masses m_1 and m_2 would be observed in experiments that directly measured the neutrino masses (the ^{3}H decay,..,), whereas neutrino oscillations, as well as the $(\beta\beta)_{0\nu}$ decay (see Sec. IX) would not be detected due to the small value of $|m_1-m_2|$.

Finally, it is not difficult to recognize in Eqs. (6.52) and (6.56) the two LH components ($\psi_{1,2L}$ and $\psi^c_{1,2L}$) of the fields ($\psi_{1,2}$) of two pseudo-Dirac neutrinos with masses $m_{\mu e}$ and m_D, respectively, split by small Majorana mass corrections [of order $(m_{\mu\mu}+m_{ee})$ and (m_L+m_R)] into pairs of Majorana neutrinos with different but very close masses.

F. Dirac neutrinos in the case of a Majorana mass term: some general results

We have seen in the preceding subsections that two mass-degenerate Majorana neutrinos may be equivalent to a Dirac neutrino. Consequently, under certain conditions, some of the neutrinos with definite masses in the case of a Majorana mass term may be Dirac particles. Our discussion of this possibility has been confined so far to examples of mass terms involving only two flavor neutrino fields. We shall formulate next some general results concerning the relation between the neutrino mass spectrum and the varieties of massive neutrinos generated by a Majorana mass term $\mathscr{L}^M$ [see Eq. (4.22)] and the symmetries it has in the case of n types of neutrinos ν_l. After that a brief qualitative discussion of the weak Majorana corrections to the masses of pseudo-Dirac neutrinos will be given.

The necessary and sufficient condition for the appearance of massive Dirac neutrinos in a theory with a neutrino mass term of Majorana type[19] $\mathscr{L}^M$ is the existence in the theory of a global gauge symmetry corresponding to the conservation of at least one lepton charge (Bilenky and Pontecorvo, 1981; Wolfenstein, 1981b; Leung and Petcov, 1983; Wyler and Wolfenstein, 1983). Let us consider for simplicity the case of one conserved lepton charge L'. The structure of the Majorana mass term [see Eq. (4.22)] implies that any conserved lepton charge should be, in general, a nonstandard linear combination of the ordinary lepton charges L_l:

$$L'=\sum_{l=e,\mu,\tau,\ldots}(-1)^{n_l}a_l L_l \ , \tag{6.57}$$

where

$$n_l=0 \text{ or } 1, \quad a_l=0 \text{ or } 1, \quad l=e,\mu,\tau,\ldots \tag{6.58}$$

($a_l \neq 0$ at least for one l). The number of massive Dirac neutrinos, as well as the number of massless neutrinos, is determined by the explicit form of this combination (Leung and Petcov, 1983; Wyler and Wolfenstein, 1983), namely by $\min[n_+(L'),n_-(L')]$ and $|n_+(L')-n_-(L')|$, respectively, where $n_+(L')$ $[n_-(L')]$ is the number of ordinary lepton charges L_l that enter into the

[19]The Majorana mass term $\mathscr{L}^M$ is assumed to contain all possible contributions to the neutrino mass matrix, except possibly the weak-interaction corrections.

expression for L' with a plus (minus) sign.[20] For $\min[n_+(L'),n_-(L')]\neq 0$ the lepton charge L' given by Eq. (6.58) is, obviously, a generalization of the ZKM lepton charge considered in Sec. VI.D. The properties of the Dirac neutrinos arising as a result of the conservation of the charge L' are thus analogous to the properties of the ZKM (i.e., nonstandard) Dirac neutrino.

Consider, for example, the case of three flavor neutrinos, ν_e, ν_μ, and ν_τ, possessing a Majorana mass term that conserves the charge $L'=L_e-L_\mu+L_\tau$, conserved also by the weak interactions (Petcov, 1982a):

$$\mathscr{L}'^M=-\tfrac{1}{2}[\overline{(\nu_{eL})^c(\nu_{\mu L})^c(\nu_{\tau L})^c}]\begin{bmatrix}0 & m_{e\mu} & 0\\ m_{e\mu} & 0 & m_{\mu\tau}\\ 0 & m_{\mu\tau} & 0\end{bmatrix}\begin{bmatrix}\nu_{eL}\\ \nu_{\mu L}\\ \nu_{\tau L}\end{bmatrix}+\text{H.c.}\,, \tag{6.59}$$

where $m_{e\mu}$ and $m_{\mu\tau}$ are mass parameters.

It is not difficult to show that this mass term conserves CP parity, and, under the CP phase convention (5.26), $m_{e\mu}$ and $m_{\mu\tau}$ should be real parameters. The diagonalization of $\mathscr{L}'^M$ leads to one massless Majorana neutrino χ_1 and two massive Majorana neutrinos χ_2 and χ_3, which are mass degenerate, possess opposite CP parities, and are equivalent to a massive Dirac neutrino ν:

$$\begin{bmatrix}\nu_{eL}+(\nu_{eL})^c\\ \nu_{\mu L}+(\nu_{\mu L})^c\\ \nu_{\tau L}+(\nu_{\tau L})^c\end{bmatrix}=\begin{bmatrix}\chi_1\cos\theta+\frac{1}{\sqrt{2}}(\chi_2+\gamma_5\chi_3)\sin\theta\\ \frac{1}{\sqrt{2}}(\chi_2-\gamma_5\chi_3)\\ -\chi_1\sin\theta+\frac{1}{\sqrt{2}}(\chi_2+\gamma_5\chi_3)\cos\theta\end{bmatrix}=\begin{bmatrix}\chi_1\cos\theta+(\nu_L+C\overline{\nu_L}^T)\sin\theta\\ (\nu_R+C\overline{\nu_R}^T)\\ -\chi_1\sin\theta+(\nu_L+C\overline{\nu_L}^T)\cos\theta\end{bmatrix}. \tag{6.60}$$

Here

$$\tan\theta=\frac{m_{e\mu}}{m_{\mu\tau}},\quad \nu(x)=\frac{1}{\sqrt{2}}[\chi_2(x)+\chi_3(x)]\,,$$

and $\eta_{CP}(\chi_1)=\eta_{CP}(\chi_2)=i$, $\eta_{CP}(\chi_3)=-i$. The mass of χ_2 and χ_3, and consequently of ν, is given by

$$m=(m_{e\mu}^2+m_{\mu\tau}^2)^{1/2}\,. \tag{6.61}$$

As a consequence of the conservation of the lepton charge $L'=L_e-L_\mu+L_\tau$, $\overset{(-)}{\nu}_e\rightleftarrows\overset{(-)}{\nu}_\tau$ oscillations are allowed, while $\overset{(-)}{\nu}_\mu$ cannot take part in the oscillations. Obviously $(\beta\beta)_{0\nu}$ decay is forbidden, but the processes $\mu^-+(A,Z)\to e^++(A,Z-2)$, $\tau^-\to e^-+\gamma$, $\tau^-\to e^-+e^++e^-$, $\tau^-\to e^-+\mu^++\mu^-$, etc., are not. However, the predicted cross section for the μ^--e^+ conversion and the rates for the indicated τ decays are unobservably small if the Dirac neutrino mass m does not exceed a few MeV (see, for example, Petcov, 1977b).

Further, as the specific example considered in Sec. VI.C suggests, pseudo-Dirac neutrinos can appear in a scheme with a Majorana mass term $\mathscr{L}^M$ if $\mathscr{L}^M$ possesses global symmetries that are not symmetries, say, of the weak interaction (Wolfenstein, 1981b). The weak interaction then induces Majorana mass corrections, splitting each pseudo-Dirac neutrino into a pair of Majorana neutrinos that are almost degenerate in mass. If CP invariance holds in the leptonic sector, these Majorana neutrinos will have opposite CP parities. As we have already noted in Sec. VI.C, all processes typical of massive Majorana neutrinos [like $(\beta\beta)_{0\nu}$ decay] are, in general, allowed in the case of pseudo-Dirac neutrinos, even in the limit of zero Majorana mass corrections (Petcov, 1982a; Valle, 1983). These processes are not necessarily suppressed relative to those involving Majorana neutrinos. Examples of schemes with pseudo-Dirac neutrinos, as well as the specific phenomenology they are associated with, were considered by Petcov (1982a), Singer and Valle (1983), and Doi *et al.* (1983b).

The indicated conditions for the existence of Dirac (or pseudo-Dirac) massive neutrinos in a scheme with a Majorana mass term imply, of course, restrictions on the couplings between the massive neutrinos and the charged leptons in the weak charged lepton current (Leung and Petcov, 1983; Petcov and Toshev, 1984). For example, the fields of Majorana and true Dirac neutrinos cannot be present simultaneously in the relation expressing a given flavor neutrino field $\nu_{lL}(x)$ in terms of fields of neutrinos with definite masses[21] unless all neutrinos of Majorana type have zero mass. Both fields corresponding to the Dirac neutrinos and their Dirac conjugates (multiplied by the charge-conjugation matrix) enter into the weak charged lepton current. However, they do not couple to the same charged leptons, nor to the charged leptons having couplings to massive Majorana neutrinos (otherwise they would correspond to pseudo-Dirac neutrinos).

These restrictions can be used to find the most general neutrino mass matrices leading to given numbers of mass-

[20]The diagonalization of the neutrino mass term $\mathscr{L}^M$ may reveal that some of the $\min[(n_+(L'),n_-(L')]$ Dirac neutrinos are massless. It can be shown, however, that they acquire radiatively induced Dirac mass terms at two-loop level [the diagrams generating these Dirac mass corrections are similar to the two-loop diagrams discussed by Petcov and Toshev (1984)]. Let us add that if several lepton charges are separately conserved, the total numbers of massive Dirac and massless neutrinos are given, respectively, by the sums of the numbers of massive Dirac and massless neutrinos associated with each of the conserved lepton charges via the formulas given above.

[21]Otherwise the Dirac neutrinos would be pseudo-Dirac.

less, massive Majorana, and massive Dirac neutrinos (Leung and Petcov, 1983; Wyler and Wolfenstein, 1983). For example, p massive Majorana and k massive Dirac neutrinos $(2k+p=n)$ are generated by the Majorana mass term (4.22) with

$$M=\begin{bmatrix} P & 0 & 0 \\ 0 & 0 & Q \\ 0 & Q^T & 0 \end{bmatrix}, \tag{6.62}$$

where P and Q are $p\times p$ and $k\times k$ (nonzero) matrices and $P^T=P$, while

$$M'=\begin{bmatrix} 0 & R \\ R^T & 0 \end{bmatrix}, \tag{6.63}$$

where R is a $k\times(k+p)$ nonzero matrix, is the most general mass matrix having p massless and k massive Dirac neutrino eigenstates. It is easy to check that $\min[n_+(L'), n_-(L')]$ and $|n_+(L')-n_-(L')|$ (where L' denotes the corresponding conserved lepton charge) do indeed give the numbers of massive Dirac and massless neutrinos in these cases.

We should like to conclude this section with a brief discussion of the weak Majorana corrections (Δm_M) to the mass of a pseudo-Dirac neutrino $(m_{\rm PD})$. They split the pseudo-Dirac neutrino into two Majorana neutrinos whose masses differ approximately by Δm_M. A complete calculation of these corrections can be performed in a renormalizable electroweak theory, in which the neutrino mass term arises in a self-consistent way (e.g., due to the Higgs mechanism). The precise value of Δm_M may depend on the Higgs sector of the theory. For models in which finite neutrino mass corrections arise at the one-loop level, such a calculation was discussed by Doi *et al.* (1984a), who also considered a particular example of a theory with pseudo-Dirac neutrinos. The one-loop weak corrections $\Delta m_M^{1\,\rm loop}$ are proportional (Wolfenstein, 1981b; Petcov, 1982a) to the initial mass of the pseudo-Dirac neutrino $m_{\rm PD}$: $\Delta m_M^{1\,\rm loop}\sim{\rm const}\times m_{\rm PD}$. The factor of proportionality depends, for example, on the ratio of the charged-lepton and $W^\pm$-boson masses squared and on products of the elements of the corresponding mixing matrix. It does not exceed approximately 10^{-6} in the case of three generations of leptons and may be as large as 10^{-3} if a fourth generation of leptons with a charged lepton of mass comparable to that of the $W^\pm$ boson exists. If the pseudo-Dirac neutrino possesses a mass in the region of 10 eV, neutrino mass differences of the order of $\Delta m_M^{1\,\rm loop}$ may be detected, in principle, in neutrino oscillation experiments. It should be added that under certain rather general conditions a pseudo-Dirac neutrino may acquire radiatively induced Majorana mass only at the two-loop level (Leung and Petcov, 1983). Some of the possible two-loop weak Majorana corrections are not proportional to the mass of the pseudo-Dirac neutrino (Petcov and Toshev, 1984). They depend linearly on the masses of the Majorana neutrinos coexisting with the pseudo-Dirac neutrino. In any case, the induced two-loop Majorana mass does not exceed roughly 10^{-4} eV if the neutrino masses are smaller than 100 eV.

VII. OSCILLATIONS OF NEUTRINOS

A. Dirac (Majorana) mass term

1. General expressions for the transition probabilities

We have considered in the preceding sections different possible schemes of neutrino mixing. The oscillations in neutrino beams are one of the most fundamental consequences of neutrino mixing.

The oscillations of neutrinos are analogous in their quantum-mechanical nature to $K^0\rightleftarrows\bar{K}^0$ oscillations. Suppose that the state vectors of the neutrinos taking part in the weak interactions $(\nu_e,\nu_\mu,\nu_\tau,\ldots)$ are superpositions of the state vectors of neutrinos (Dirac or Majorana) with different masses. What would be the behavior of a neutrino beam in this case? It is clear that at some distance from the source of neutrinos of a given type, the state vectors of neutrinos with different masses (because of the difference in the masses) would acquire different phase factors. The state vector of a neutrino would then be a superposition of the state vectors of neutrinos of different (all possible, in principle) types. It is obvious that the probability of finding a neutrino of a given type would be a periodic function of the distance between the source and the detector. This phenomenon was called neutrino oscillations (Pontecorvo, 1957, 1958).

In order for oscillations of neutral kaons, neutrinos, etc. to be possible, the following conditions have to be realized: (i) the particle interaction Lagrangian should contain terms that preserve some quantum numbers (strangeness in the case of kaons, lepton numbers in the case of leptons, etc.); (ii) the total Lagrangian (and the mass term) should not be diagonal with respect to these quantum numbers, and the relevant quantum-number-nonconserving couplings should be much weaker than those preserving the quantum numbers. The states with definite mass (and width) would then be superpositions of states possessing definite strangeness in the case of neutral kaons, definite lepton numbers in the case of neutrinos, etc.

We shall consider in this section the oscillations of neutrinos in vacuum in the case of a Dirac or a Majorana mass term. For the field operator of the flavor neutrino $\nu_{lL}(x)$ we have in these cases

$$\nu_{lL}(x)=\sum_{k=1}^{n}U_{lk}\nu_{kL}(x)\,, \tag{7.1}$$

where $\nu_k(x)$ is the field of a Dirac (or Majorana) neutrino with mass m_k, and U is a unitary $n\times n$ matrix (n is the number of charged leptons).

Consider a beam of neutrinos with momentum $\mathbf{p}$. We shall assume that

$$|\mathbf{p}| \gg m_k \ . \tag{7.2}$$

Neglecting the masses of the neutrinos ν_k in comparison with their momentum, we get from Eq. (7.1) for the state vector of the (flavor) neutrino ν_l, produced in the weak interaction,

$$|\nu_l\rangle = \sum_{k=1}^{n} U_{lk}^* \, |k,L\rangle \ . \tag{7.3}$$

Here $|k,L\rangle$ is the state vector of the neutrino (of Dirac or Majorana type) with momentum $\mathbf{p}$, mass m_k, and negative helicity. Equation (7.3) is based on the assumption that the differences in the masses of the neutrinos ν_k are so small that a coherent superposition of the state vectors of neutrinos with different masses is formed in the weak interaction. The conditions under which this takes place are discussed in the papers of Nussinov (1976) and Kayser (1981), as well as in reviews by Bilenky and Pontecorvo (1978) and Frampton and Vogel (1982).

Similarly, for the antineutrino state vector, we have

$$|\bar{\nu}_l\rangle = \sum_{k=1}^{n} U_{lk} \, |k,R\rangle \ , \tag{7.4}$$

where $|k,R\rangle$ is the state vector of a Dirac antineutrino (or Majorana neutrino) with mass m_k and positive helicity.

If in the initial moment $t=0$ the flavor neutrinos are described by the state vector $|\nu_l\rangle$, at the moment t they will be described by

$$|\nu_l\rangle_t = e^{-iH_0 t} |\nu_l\rangle \ , \tag{7.5}$$

where H_0 is the free Hamiltonian. One has

$$H_0 \, |k,L\rangle = E_k \, |k,L\rangle \ , \tag{7.6}$$

where $E_k = (\mathbf{p}^2 + m_k^2)^{1/2}$. From Eqs. (7.1), (7.3), and (7.6) we get [22]

$$|\nu_l\rangle_t = \sum_{k=1}^{n} e^{-iE_k t} U_{lk}^* \, |k,L\rangle \ . \tag{7.7}$$

As is well known, neutrinos are detected by observation of weak-interaction-induced reactions. To obtain the probability amplitude for finding a given type of neutrino in a beam of neutrinos described by the vector $|\nu_l\rangle_t$, we have to decompose $|\nu_l\rangle_t$ over the complete set of neutrino state vectors $|\nu_l\rangle$. Using the unitarity of the matrix U we get, with the help of Eqs. (7.3) and (7.6),

$$|\nu_l\rangle_t = \sum_{l'=e,\mu,\tau,\ldots} |\nu_{l'}\rangle \sum_{k=1}^{n} U_{l'k} e^{-iE_k t} U_{lk}^* \ . \tag{7.8}$$

Thus

$$a_{\nu_{l'};\nu_l}(t) = \sum_{k=1}^{n} U_{l'k} e^{-iE_k t} U_{lk}^* \tag{7.9}$$

is the probability amplitude for the transition $\nu_l \to \nu_{l'}$ for a time t in vacuum. The corresponding transition probability is equal to

$$P_{\nu_{l'};\nu_l}(t) = \left| \sum_{k=1}^{n} U_{l'k} e^{-iE_k t} U_{lk}^* \right|^2 \ . \tag{7.10}$$

Consider next the behavior of a beam of antineutrinos. From Eqs. (7.4)—(7.6) we obtain

$$|\bar{\nu}_l\rangle_t = \sum_{k=1}^{n} U_{lk} e^{-iE_k t} \, |k,R\rangle \ . \tag{7.11}$$

Further, we have

$$|\bar{\nu}_l\rangle_t = \sum_{l'=e,\mu,\tau,\ldots} |\bar{\nu}_{l'}\rangle a_{\bar{\nu}_{l'};\bar{\nu}_l}(t) \ , \tag{7.12}$$

where

$$a_{\bar{\nu}_{l'};\bar{\nu}_l}(t) = \sum_{k=1}^{n} U_{l'k}^* e^{-iE_k t} U_{lk} \tag{7.13}$$

is the probability amplitude of the transition of $\bar{\nu}_l$ into $\bar{\nu}_{l'}$ at a time t after the production of $\bar{\nu}_l$ at $t=0$. For the transition probability one has

$$P_{\bar{\nu}_{l'};\bar{\nu}_l}(t) = \left| \sum_{k=1}^{n} U_{l'k}^* e^{-iE_k t} U_{lk} \right|^2 \ . \tag{7.14}$$

Let us obtain some general relations which the vacuum oscillation probabilities should satisfy. Comparing Eqs. (7.10) and (7.14) we see that

$$P_{\nu_{l'};\nu_l}(t) = P_{\bar{\nu}_l;\bar{\nu}_{l'}}(t) \ . \tag{7.15}$$

This general relation is a consequence of *CPT* invariance (Cabibbo, 1978). Further, the sum of the probabilities of transitions of a given type of neutrino (antineutrino) into neutrinos (antineutrinos) of all possible types is, obviously, equal to unity:

$$\sum_{l'=e,\mu,\tau,\ldots} P_{\nu_{l'};\nu_l}(t) = 1 \ , \qquad \sum_{l'=e,\mu,\tau,\ldots} P_{\bar{\nu}_{l'};\bar{\nu}_l}(t) = 1 \ . \tag{7.16}$$

Clearly, these relations follow from the unitarity of the mixing matrix U.

Let us return to Eqs. (7.10) and (7.14) for the transition probabilities. We have assumed that $m_k \ll p(=\mathbf{p}_k)$, $k=1,2,\ldots,n$. Keeping only the terms linear in m_k^2/p^2, one has

$$E_k \simeq p + \frac{m_k^2}{2p} \ . \tag{7.17}$$

Using Eqs. (7.10), (7.14), and (7.17), we obtain for the probabilities of the transitions $\nu_l \to \nu_{l'}$ and $\bar{\nu}_l \to \bar{\nu}_{l'}$ the general expressions, respectively,

[22]The neutrinos with definite masses may be unstable. If, for example, $m_2 > m_1$, the decay $\nu_2 \to \nu_1 + \gamma$ is possible. The existing calculations show, however, that, as a rule, the radiative lifetime of a neutrino with mass smaller than 100 eV exceeds the age of the universe (see Sec. X.B). Therefore neutrino instability due to radiative decays can be neglected in the analysis of neutrino oscillations. Let us note, however, that the neutrino decay $\nu_2 \to \nu_1 + \gamma$ may have important astrophysical implications (De Rújula and Glashow, 1980). We note also that the possibility of "fast" invisible neutrino decays has been discussed by Gelmini and Valle (1984). We shall assume here that neutrinos are essentially stable.

$$P_{\nu_{l'};\nu_l}\left[\frac{R}{p}\right]=\sum_{k=1}^{n}|U_{lk}|^2|U_{l'k}|^2+2\sum_{k>j}|U_{l'k}U_{lk}^{*}U_{l'j}^{*}U_{lj}|\cos\left[\frac{m_k^2-m_j^2}{2p}R-\phi_{kj;l'l}\right], \quad (7.18)$$

$$P_{\bar{\nu}_{l'};\bar{\nu}_l}\left[\frac{R}{p}\right]=\sum_{k=1}^{n}|U_{lk}|^2|U_{l'k}|^2+2\sum_{k>j}|U_{l'k}U_{lk}^{*}U_{l'j}^{*}U_{lj}|\cos\left[\frac{m_k^2-m_j^2}{2p}R+\phi_{kj;l'l}\right]. \quad (7.19)$$

Here R is the source-detector distance and

$$\phi_{kj;l'l}=\arg(U_{l'k}U_{lk}^{*}U_{l'j}^{*}U_{lj})\ . \quad (7.20)$$

It should be noted that the transition probabilities are functions of the ratio R/p.

The probabilities $P_{\nu_{l'};\nu_l}$ and $P_{\bar{\nu}_{l'};\bar{\nu}_l}$ can also be written as

$$P_{\nu_{l'};\nu_l}(R/p)=\sum_{j,k}U_{l'j}U_{lj}^{*}U_{l'k}^{*}U_{lk}(e^{-i(E_j-E_k)t}-1+1)$$

$$=\delta_{l'l}+2\sum_{j>k}|U_{l'j}U_{lj}^{*}U_{l'k}^{*}U_{lk}|\left[\cos\left[\frac{m_j^2-m_k^2}{2p}R-\phi_{jk,l'l}\right]-\cos\phi_{jk,l'l}\right], \quad (7.21)$$

$$P_{\bar{\nu}_{l'};\bar{\nu}_l}(R/p)=\delta_{l'l}+2\sum_{j>k}|U_{l'j}U_{lk}^{*}U_{l'k}^{*}U_{lk}|\left[\cos\left[\frac{m_j^2-m_k^2}{2p}R+\phi_{jk;l'l}\right]-\cos\phi_{jk;l'l}\right]. \quad (7.22)$$

It is not difficult now to formulate the general conditions under which the probabilities of the transitions $\nu_l\to\nu_{l'}$ and $\bar{\nu}_l\to\bar{\nu}_{l'}$ ($l'\neq l$) are different from zero (i.e., under which neutrino oscillations take place). Suppose that all neutrino masses are equal. As follows from Eqs. (7.21) and (7.22), in this case

$$P_{\nu_{l'};\nu_l}=P_{\bar{\nu}_{l'};\bar{\nu}_l}=\delta_{l'l}\ . \quad (7.23)$$

Further, suppose that $U_{lj}=\delta_{lj}$ (there is no mixing). It is obvious from Eqs. (7.9) and (7.13) that the equalities (7.23) also hold in this case. Consequently active neutrinos of a given type may undergo transition in vacuum into active neutrinos of a different type (or into sterile neutrinos; see Sec. VII.B) only if (i) there exist as least two neutrinos that are nondegenerate in mass; (ii) neutrino mixing does take place (i.e., at least some nondiagonal elements of the lepton mixing matrix are different from zero).

Finally, let us assume that $p/R\gg|m_j^2-m_k^2|$, $j\neq k$, $j,k=1,2,\ldots,n$. It is clear from Eqs. (7.21) and (7.22) that we arrive at (7.23) under the above condition, as well. Thus the effects of neutrino oscillations may be observable if at least one difference of the squares of neutrino masses is of the order of or greater than p/R.

2. On the impossibility of distinguishing between Dirac and Majorana mass terms in oscillation experiments

We shall show in this subsection that if the neutrino mass term is of Dirac or Majorana type [Eqs. (4.10) and (4.22)], it will be impossible to determine what kind of particles—Dirac or Majorana—are the neutrinos with definite masses by studying the neutrino oscillations in vacuum (Bilenky, Hošek, and Petcov, 1980; Doi *et al.*, 1981b). We have seen that the probabilities of transitions between different flavor neutrinos are determined by the differences of the squares of neutrino masses and by the elements of the lepton mixing matrix. Let us find out first what is the difference between the mixing matrices in the Dirac and in the Majorana cases.

An $n\times n$ unitary mixing matrix U is characterized by n^2 real parameters: $n(n-1)/2$ Euler angles and $n(n+1)/2$ phases. Not all phases in the mixing matrix, however, enter into the expressions for the observable quantities.

Indeed, consider the mixing of neutrinos with Dirac masses. The expression for the charged lepton current has the form

$$j_\alpha^{(-)}=2\sum_{l,k}\bar{l}_L\gamma_\alpha U_{lk}\nu_{kl}\ .$$

It is well known that the multiplication of Dirac fields by constant phase factors has no effect on the physical results. Evidently, in the case of Dirac neutrinos ν_k, only those phases in the matrix U are physical which cannot be eliminated by the transformation

$$U\to U'=S(\alpha)US^{\dagger}(\beta)\ . \quad (7.24)$$

Here

$$S_{l'l}(\alpha)=e^{i\alpha_l}\delta_{l'l},\quad S_{ik}(\beta)=e^{i\beta_k}\delta_{ik}\ ,$$

where α_l and β_k are arbitrary real parameters.

Using Eq. (7.24) we can determine the number of physical phases present in the mixing matrix in the case of a Dirac mass term.

The matrices $S(\alpha)$ and $S(\beta)$ can always be cast in the form

$$S(\alpha)=e^{i\alpha_0}S(\alpha'),\quad S(\beta)=e^{i\beta_0}S(\beta')\ , \tag{7.25}$$

where $\det S(\alpha')=1$, $\det S(\beta')=1$. One has

$$\alpha_l=\alpha_0+\alpha_l',\quad \beta_k=\beta_0+\beta_k'\ ,$$

where

$$\sum_l \alpha_l'=0,\quad \sum_k \beta_k'=0\ . \tag{7.26}$$

It is obvious from Eqs. (7.24)–(7.26) that there are $(2n-1)$ arbitrary phases in $S(\alpha)$ and $S(\beta)$ $[2(n-1)$ phases α_l' and β_k' and one common phase $(\alpha_0-\beta_0)]$, which can be used to reduce the number of phases in the lepton mixing matrix. Thus, in the case of a Dirac mass term, the lepton mixing matrix contains

$$n_D=\frac{n(n+1)}{2}-(2n-1)=\frac{(n-1)(n-2)}{2} \tag{7.27}$$

physical phases.

We have shown in Sec. V that the mixing matrix of the neutrinos with Dirac masses should be real if CP invariance holds in the leptonic sector. Consequently the physical phases in the mixing matrix characterize the violation of CP invariance. It follows from Eq. (7.27) that, for $n=2$, CP invariance holds automatically. For $n=3$ there is one phase in the mixing matrix responsible for violation of CP invariance, etc. All these results are valid in the case of quarks as well, and are well known (Kobayashi and Maskawa, 1973).

Consider next the case of neutrino mixing induced by a Majorana mass term that does not have an analog in the quark sector. The charged lepton current in this case is given by

$$j_\alpha^{(-)}=2\sum_{l,k}\overline{l_L}\gamma_\alpha U_{lk}\chi_{kL}\ , \tag{7.28}$$

where the fields $\chi_k(x)$ satisfy the conditions

$$C\overline{\chi_k}^T(x)=\xi_k\chi_k(x),\quad k=1,2,\ldots,n \tag{7.29}$$

$(\xi_k=\pm1)$. As a consequence of Eq. (7.29), the fields $\chi_k(x)$ cannot absorb phase factors.[23] The phases of the charged lepton fields are not observable. Instead of $l(x)$ one can always introduce the field $l'(x)=e^{i\alpha_l}l(x)$. So, in the Majorana case, only the phases in the mixing matrix that cannot be eliminated by the transformation

$$U\to U'=S(\alpha)U\ , \tag{7.30}$$

where

$$S_{l'l}=e^{i\alpha_l}\delta_{l'l}$$

and α_l are arbitrary real parameters, are physical. Obviously, the number of physical phases in the case of a Majorana mass term is equal to

$$n_M=\frac{n(n+1)}{2}-n=\frac{n(n-1)}{2}\ . \tag{7.31}$$

Comparing Eqs. (7.27) and (7.31) we conclude that the number of physical CP-violating phases that may be present in the lepton mixing matrix in the Majorana case is greater than the number of physical phases that may enter into the mixing matrix in the Dirac case (Bilenky, Hošek, and Petcov, 1980; Kobzarev *et al.*, 1980; Schechter and Valle, 1980; Doi *et al.*, 1981b). In particular, for $n=2$ the Majorana mass term can be a source of CP violation, in contrast to the Dirac mass term. For the difference between the numbers of physical phases corresponding to the two cases we get from Eqs. (7.31) and (7.27)

$$n_M-n_D=n-1\ . \tag{7.32}$$

Is it possible to determine on the basis of this difference in number of phases in the mixing matrix the type of massive neutrinos (Dirac and Majorana) by studying neutrino oscillations? In order to find out, let us turn to Eqs. (7.10) and (7.14) for the probabilities of the transitions $\nu_l\to\nu_{l'}$ and $\bar\nu_l\to\bar\nu_{l'}$. It is obvious from these expressions that the probabilities $P_{\nu_{l'};\nu_l}(t)$ and $P_{\bar\nu_{l'};\bar\nu_l}(t)$ will not change if we replace the mixing matrix U by

$$U_{lk}'=e^{i\alpha_l}U_{lk}e^{-i\beta_k}\ , \tag{7.33}$$

where α_l and β_k are arbitrary real parameters. Thus, both in the Dirac and in the Majorana cases, the probabilities of the transitions $\nu_l\to\nu_{l'}$ and $\bar\nu_l\to\bar\nu_{l'}$ in vacuum may depend only on phases in the mixing matrix that cannot be absorbed by the transformation (7.33). The number of such phases is equal to $(n-1)(n-2)/2$. It is clear from Eq. (7.33) that the physical phases by which the mixing matrices in the cases of Dirac and Majorana mass terms differ do not enter into the transition probabilities $P_{\nu_{l'};\nu_l}(t)$ and $P_{\bar\nu_{l'};\bar\nu_l}(t)$.[24] Consequently, by studying neutrino oscillations in vacuum, it is impossible to answer the question what kind of particles—Dirac or Majorana—are the neutrinos with definite masses (Bilenky, Hošek, and Petcov, 1980; Doi *et al.*, 1981b). [The same conclusion can be shown to be valid in the case when neutrino oscillations take place in matter (Langacker *et al.*, 1987).] The observation, however, of the neutrinoless double-β

[23]If instead of $\chi_{kL}(x)$ we introduce $\chi_{kL}'(x)=e^{-i\beta_k}\chi_{kL}(x)$, the invariance of the Majorana condition (7.29) requires that instead of the component $\chi_{kR}(x)$ one should introduce $\chi_{kR}'(x)=e^{-i\beta_k}\chi_{kR}(x)$. Clearly, the neutrino mass term is not invariant with respect to such a transformation.

[24]The additional $(n-1)$ CP-violating phases, characteristic of the case of Majorana mass term, are always associated with effects whose magnitude is suppressed by the factor $(m_k/E_\nu)^2$, where E_ν is the neutrino energy in the relevant process and m_k is the mass of the Majorana neutrino taking part in the process (Schechter and Valle, 1981a).

decay would allow us to answer this question (see Sec. IX).

Finally, it follows from these considerations that for n neutrino flavors the oscillation probabilities $P_{\nu_{l'};\nu_l}(t)$ and $P_{\bar{\nu}_{l'};\bar{\nu}_l}(t)$ depend, in the case of a Dirac or a Majorana mass term, on $n(n-1)$ independent parameters: $(n-1)$ differences of the squares of neutrino masses, $n(n-1)/2$ mixing angles, and $(n-1)(n-2)/2$ CP-violating phases.

B. Oscillations of neutrinos in the case of a Dirac-Majorana mass term. Sterile neutrinos

We shall discuss now the oscillations of neutrinos in vacuum in the case of mixing arising as a result of the diagonalization of a Dirac-Majorana mass term. One has in this case (see Sec. IV.D)

$$\nu_{lL}=\sum_{k=1}^{2n} U_{lk}\chi_{kL}\ , \quad (\nu_{lR})^c=\sum_{k=1}^{2n} U_{\bar{l}k}\chi_{kL}\ , \tag{7.34}$$

where U is a unitary $2n\times 2n$ mixing matrix (n is the number of charged leptons), and $\chi_k(x)$ is the field of a Majorana neutrino with mass m_k.

Consider the behavior of a beam of neutrinos with momentum $\mathbf{p}$ under the condition that the mixing (7.34) takes place. We shall assume that $|\mathbf{p}|\gg m_k$. For the state vector of the flavor neutrino ν_l with momentum $\mathbf{p}$ one obtains from Eq. (7.34)

$$|\nu_l\rangle=\sum_{k=1}^{2n} U_{lk}^*\,|k,L\rangle\ . \tag{7.35}$$

Here $|k,L\rangle$ is the state vector of a neutrino with mass m_k, momentum $\mathbf{p}$, and negative helicity.

Suppose that as a result of some weak decays a beam of neutrinos ν_l is formed. In the initial moment ($t=0$) the neutrinos from the beam are described by the vector $|\nu_l\rangle$. The state of the neutrinos at time t is described by the vector

$$|\nu_l\rangle_t=\sum_{k=1}^{2n} e^{-iE_kt}\,|k,L\rangle U_{lk}^*\ , \tag{7.36}$$

where

$$E_k=(\mathbf{p}^2+m_k^2)^{1/2}\simeq p+\frac{m_k^2}{2p}\ .$$

As is well known, neutrinos are detected by observation of weak-interaction-induced reactions. In order to obtain the probability amplitudes to find neutrinos of different types in the beam of neutrinos described by the vector $|\nu_l\rangle_t$, we have to decompose $|\nu_l\rangle_t$ over a complete set of neutrino state vectors, which includes the vectors $|\nu_l\rangle$. It is essential that the vectors $|\nu_l\rangle$ do not form a complete set in the case of the mixing (7.34). It follows from Eq. (7.34) that in addition to $|\nu_l\rangle$ the complete set of state vectors also contains the vectors $|\bar{\nu}_{lL}\rangle$, describing LH antineutrinos which are quanta of the RH fields $\nu_{lR}(x)$. (It is clear that the complete set should be formed in the case under consideration by $2n$ state vectors.) The RH fields $\nu_{lR}(x)$ do not enter into the ordinary weak-interaction Lagrangian. This implies that the LH antineutrinos do not take part in the standard weak interaction. As has been indicated already in Sec. IV, such particles have been termed "sterile" (Pontecorvo, 1958).

Using the unitarity of the mixing matrix we get from Eq. (7.34)

$$|k,L\rangle=\sum_{l'=e,\mu,\tau,\ldots} U_{l'k}\,|\nu_{l'}\rangle+\sum_{l'=e,\mu,\tau} U_{\bar{l}'k}\,|\bar{\nu}_{l'L}\rangle\ . \tag{7.37}$$

Further, inserting Eq. (7.37) into (7.36) one has

$$|\nu_l\rangle_t=\sum_{l'=e,\mu,\tau,\ldots}|\nu_{l'}\rangle a_{\nu_{l'};\nu_l}(t)+\sum_{l'=e,\mu,\tau,\ldots}|\bar{\nu}_{l'L}\rangle a_{\bar{\nu}_{l'L};\nu_l}(t)\ , \tag{7.38}$$

where

$$a_{\nu_{l'};\nu_l}(t)=\sum_{k=1}^{2n} U_{l'k}e^{-iE_kt}U_{lk}^*\ ,\qquad a_{\bar{\nu}_{l'L};\nu_l}(t)=\sum_{k=1}^{2n} U_{\bar{l}'k}e^{-iE_kt}U_{lk}^*\ , \tag{7.39}$$

respectively, are the amplitudes of the transitions $\nu_l\to\nu_{l'}$ and $\nu_l\to\bar{\nu}_{l'L}$ for time t. For the probability of finding the active flavor neutrino $\nu_{l'}$ at a time t after the emission of the neutrino ν_l, we obtain

$$P_{\nu_{l'};\nu_l}(R/p)=|a_{\nu_{l'};\nu_l}(t)|^2=\sum_{k=1}^{2n}|U_{l'k}|^2|U_{lk}|^2+2\sum_{j>k}|U_{l'j}U_{lj}^*U_{l'k}^*U_{lk}|\times\cos\left[\frac{m_j^2-m_k^2}{2p}R-\phi_{jk;l'l}\right]\ . \tag{7.40}$$

Here $R=t$ is the distance between the source and the detector of the neutrinos, and

$$\phi_{jk;l'l}=\arg(U_{l'j}U_{lj}^*U_{l'k}^*U_{lk})\ . \tag{7.41}$$

The probability of finding a sterile neutrino $\bar{\nu}_{l'L}$ at a time t after the emission of ν_l is given by an analogous expression:

$$P_{\bar{\nu}_{l'L};\nu_l}(R/p)=|a_{\bar{\nu}_{l'L};\nu_l}(t)|^2=\sum_{k=1}^{2n}|U_{\bar{l}'k}|^2|U_{lk}|^2+2\sum_{j>k}|U_{\bar{l}'j}U_{lj}^*U_{\bar{l}'k}^*U_{lk}|\times\cos\left[\frac{m_j^2-m_k^2}{2p}R-\phi_{jk,\bar{l}'l}\right]\ . \tag{7.42}$$

So, in the case of a Dirac-Majorana mass term, neutrino mixing implies that both the probability of finding an active neutrino $\nu_{l'}$ and the probability of finding a sterile neutrino $\bar{\nu}_{l'L}$ (appearing as a result of the oscillations $\nu_l \rightleftarrows \nu_{l'}$ and $\nu_l \rightleftarrows \bar{\nu}_{l'L}$) in a beam of active neutrinos ν_l at some distance from the source may be different from zero. Let us recall that only oscillations between active neutrinos are possible in the case of a Dirac (Majorana) mass term.[25]

The sum of the probabilities of transitions of ν_l into all possible active and sterile neutrinos is equal to unity:

$$\sum_{l'=e,\mu,\tau,\ldots} P_{\nu_{l'};\nu_l}(R/p) + \sum_{l'=e,\mu,\tau,\ldots} P_{\bar{\nu}_{l'L};\nu_l}(R/p) = 1 \ . \tag{7.43}$$

This relation is a consequence of the unitarity of the $2n \times 2n$ mixing matrix U.

Consider next the behavior of a beam of antineutrinos. For the state vector of the active antineutrino $\bar{\nu}_l$ and the sterile RH neutrino ν_{lR} we obtain, with the help of Eq. (7.34),

$$|\bar{\nu}_l\rangle = \sum_{k=1}^{2n} |k,R\rangle U_{lk} \ , \quad |\nu_{lR}\rangle = \sum_{k=1}^{2n} |k,R\rangle U_{\bar{l}k} \ , \tag{7.44}$$

where $|k,R\rangle$ is the state vector of a RH Majorana neutrino with mass m_k and momentum $\mathbf{p}$ ($p \gg m_k$). If neutrinos $\bar{\nu}_l$ are born at some initial moment ($t=0$) in weak decays, the state of the neutrinos at a time t after their emission is described by the vector

$$|\bar{\nu}_l\rangle_t = \sum_{l'=e,\mu,\tau,\ldots} |\bar{\nu}_{l'}\rangle a_{\bar{\nu}_{l'};\bar{\nu}_l}(t) + \sum_{l'=e,\mu,\tau,\ldots} |\nu_{l'R}\rangle a_{\nu_{l'R};\bar{\nu}_l}(t) \ , \tag{7.45}$$

where the probability amplitudes $a_{\bar{\nu}_{l'};\bar{\nu}_l}(t)$ and $a_{\nu_{l'R};\bar{\nu}_l}(t)$ are given by

$$\begin{aligned} a_{\bar{\nu}_{l'};\bar{\nu}_l}(t) &= \sum_{k=1}^{2n} U_{l'k}^* e^{-iE_k t} U_{lk} \ , \\ a_{\nu_{l'k};\bar{\nu}_l}(t) &= \sum_{k=1}^{2n} U_{\bar{l}'k}^* e^{-iE_k t} U_{lk} \ . \end{aligned} \tag{7.46}$$

For the probabilities of the transitions $\bar{\nu}_l \to \bar{\nu}_{l'}$ and $\bar{\nu}_l \to \nu_{l'R}$ at time t we get

$$P_{\bar{\nu}_{l'};\bar{\nu}_l}(R/p) = \sum_{k=1}^{2n} |U_{l'k}|^2 |U_{lk}|^2 + 2\sum_{j>k} |U_{l'j}^* U_{lj} U_{l'k} U_{lk}^*| \cos\left[\frac{m_j^2 - m_k^2}{2p} R + \phi_{jk;l'l}\right] \ ,$$

$$P_{\overline{\nu_{l'R}};\bar{\nu}_l}(R/P) = \sum_{k=1}^{2n} |U_{\bar{l}'k}|^2 |U_{lk}|^2 + 2\sum_{j>k} |U_{\bar{l}'j}^* U_{lj} U_{\bar{l}'k} U_{lk}^*| \cos\left[\frac{m_j^2 - m_k^2}{2p} R + \phi_{jk;\bar{l}'l}\right] \ .$$

These probabilities satisfy the condition:

$$\sum_{l'=e,\mu,\tau,\ldots} P_{\bar{\nu}_{l'};\nu_l}(R/p) + \sum_{l'=e,\mu,\tau,\ldots} P_{\nu_{l'R};\bar{\nu}_l}(R/p) = 1 \ .$$

Comparing (7.39) and (7.46) we conclude that the transition probabilities satisfy also the following general relations (a consequence of the CPT invariance):

$$P_{\nu_{l'};\nu_l}(R/p) = P_{\bar{\nu}_l;\bar{\nu}_{l'}}(R/p) \ .$$

As we have seen, all three schemes of mixing considered (Dirac, Majorana, and Dirac and Majorana) lead to neutrino oscillations.[26] It is interesting to note that, in principle, one can distinguish the first two schemes from the third one by studying the oscillations (Barger *et al.*, 1980; Schechter and Valle, 1980). Indeed, suppose that the neutrinos are detected by observation of the processes of neutrino-nucleon neutral current scattering. Let the initial beam of neutrinos consist, e.g., of ν_μ. For the number of events $N^{NC}(R,p)$ at distance R from the source of ν_μ we have

$$N^{NC}(R,p) = \sum_{l=e,\mu,\tau,\ldots} P_{\nu_l;\nu_\mu}(R/p) N_0^{NC}(R,p) \ . \tag{7.47}$$

Here $N_0^{NC}(R,p)$ is the number of events expected in the absence of oscillations. Equation (7.47) was derived assuming (in accordance with the standard theory) that only the LH neutrino fields are present in the weak neutral current[27] and that e-μ-τ-$\cdots$ universality holds for the neutral-current couplings.

The sum of the probabilities on the right-hand side of Eq. (7.47) is equal to unity in the case of a Dirac (Majorana) mass term. Consequently, neutrino oscillation effects

[25]For a discussion of other schemes of neutrino mixing, possible in the theories with RH currents, and of the neutrino oscillations they imply, see Maalampi and Roos (1984).

[26]The oscillations between active neutrinos ($\nu_l \rightleftarrows \nu_{l'}, \bar{\nu}_l \rightleftarrows \bar{\nu}_{l'}$) and those between active and sterile neutrinos ($\nu_l \rightleftarrows \overline{\nu_{l'L}}, \bar{\nu}_l \rightleftarrows \nu_{l'R}$) are sometimes called oscillations of the first and of the second class, respectively (Barger *et al.*, 1980; Schechter and Valle, 1980).

[27]Other phenomenological possibilities, however, are not ruled out (Bilenky and Pontecorvo, 1984; Maalampi and Roos, 1984).

should not be observed in this case. The indicated sum of probabilities is given by

$$\sum_{l=e,\mu,\tau,\ldots} P_{\nu_l;\nu_\mu}(R/p)=1-\sum_{l=e,\mu,\tau,\ldots} P_{\bar{\nu}_{lL};\nu_\mu}(R/p) \tag{7.48}$$

if the neutrino mass term is of Dirac-Majorana type. In this case the quantity $\sum_{l=e,\mu,\tau,\ldots} P_{\nu_l;\nu_\mu}(R/p)$ may be smaller than one and may depend on R/p. Therefore the observation of neutrino oscillation effects in experiments in which neutrinos are detected through the processes of neutral-current elastic and/or deep-inelastic neutrino-nucleon scattering would be a proof of the existence of sterile neutrinos. It also would imply that the number of massive neutrinos exceeds the number of neutrino flavors. The explicit form of the periodic dependence of, say, $\sum_l P_{\nu_l;\nu_\mu}(R/p)$ on R/p is determined essentially by the number of independent neutrino mass square differences, which in turn is fixed by the total number of mixed active LH and sterile neutrinos. Hence a study of this periodic dependence may give, in principle, information about the number of sterile neutrinos taking part in neutrino oscillations of the second class.

Let us note finally that the scheme with a Dirac-Majorana mass term that we have considered is just one example of schemes containing sterile neutrinos that have sizable mixing (through mass terms) with active flavor neutrinos. Sterile neutrinos (fermions) that mix with active flavor neutrinos may appear, for example, in the extensions of the standard theory with mirror fermions (Maalampi and Roos, 1984), as well as in the supersymmetric theories with spontaneously broken R invariance (Ellis *et al.*, 1985). The neutrinos with definite masses in some of these schemes may well be Dirac particles (Maalampi and Roos, 1984). In all schemes with sterile neutrinos that mix with active neutrinos, transitions in vacuum of the active neutrinos into sterile states may take place.

C. *CP* invariance in the leptonic sector and neutrino oscillations in vacuum

We have shown in Sec. V that if CP invariance holds in the leptonic sector, the mixing matrix U satisfies the condition

$$U_{lk}=U_{lk}^{*} \tag{7.49}$$

in the case of a Dirac mass term, and the condition

$$U_{lk}\eta_k\xi_k=U_{lk}^{*} \tag{7.50}$$

in the case of a Majorana or Dirac-Majorana mass term. In Eq. (7.50) $\eta_k=-i\eta_{CP}(\chi_k)$, where $\eta_{CP}(\chi_k)$ is the CP parity of the Majorana neutrino with mass m_k, and ξ_k is an arbitrary sign factor in the Majorana condition. Using Eqs. (7.9), (7.13), (7.39), (7.46), (7.49), and (7.50) it is not difficult to convince oneself that in all three mixing schemes considered the probabilities of the transitions $\nu_l\to\nu_{l'}$ and $\bar{\nu}_l\to\bar{\nu}_{l'}$ should be equal if CP invariance holds in the leptonic sector (Cabibbo, 1978):

$$P_{\nu_{l'};\nu_l}(R/p)=P_{\bar{\nu}_{l'};\bar{\nu}_l}(R/p)\ . \tag{7.51}$$

In the case of a Dirac-Majorana mass term, CP invariance in addition implies

$$P_{\bar{\nu}_{l'L};\nu_l}(R/p)=P_{\nu_{l'R};\bar{\nu}_l}(R/p)\ . \tag{7.52}$$

Let us recall that as a consequence of CPT invariance one has

$$P_{\nu_{l'};\nu_l}(R/p)=P_{\bar{\nu}_l;\bar{\nu}_{l'}}(R/p)\ . \tag{7.53}$$

From Eqs. (7.51) and (7.53) we also have

$$\begin{aligned} &P_{\nu_{l'};\nu_l}(R/p)=P_{\nu_l;\nu_{l'}}(R/p)\ ,\\ &P_{\bar{\nu}_{l'};\bar{\nu}_l}(R/p)=P_{\bar{\nu}_l;\bar{\nu}_{l'}}(R/p)\ . \end{aligned} \tag{7.54}$$

A test of Eqs. (7.51) and (7.54) for $l\neq l'$ could enable us to find out whether CP invariance holds in the leptonic sector.[28] We shall discuss here one of the possible methods for a direct test of these relations (Bilenky and Niedermayer, 1981). Let us show first of all that as a consequence of CPT invariance and of the unitarity of the mixing matrix, Eq. (7.51) is valid in the case of oscillations involving two types of neutrinos (say, ν_e and ν_μ), even when CP invariance does not hold. Indeed, from Eq. (7.16) we get in this case

$$\begin{aligned} &P_{\nu_e;\nu_e}+P_{\nu_\mu;\nu_e}=P_{\bar{\nu}_e;\bar{\nu}_e}+P_{\bar{\nu}_\mu;\bar{\nu}_e}=1\ ,\\ &P_{\nu_e;\nu_\mu}+P_{\nu_\mu;\nu_\mu}=P_{\bar{\nu}_e;\bar{\nu}_\mu}+P_{\bar{\nu}_\mu;\bar{\nu}_\mu}=1\ . \end{aligned} \tag{7.55}$$

Using Eqs. (7.53) one finds

$$P_{\nu_\mu;\nu_e}=P_{\bar{\nu}_\mu;\bar{\nu}_e}=P_{\nu_e;\nu_\mu}=P_{\bar{\nu}_e;\bar{\nu}_\mu}\ . \tag{7.56}$$

Let us note that from Eqs. (7.55) and (7.56) one also has

$$P_{\nu_e;\nu_e}=P_{\nu_\mu;\nu_\mu}=P_{\bar{\nu}_e;\bar{\nu}_e}=P_{\bar{\nu}_\mu;\bar{\nu}_\mu}=1-P_{\nu_\mu;\nu_e}\ . \tag{7.57}$$

Thus it would be possible to perform a test of CP invariance in the leptonic sector by studying neutrino oscillations only if at least three types of neutrinos take part in the oscillations. Particularly suitable for the indicated test would be the beams of neutrinos originating purely from semileptonic decays of K_L mesons.[29] These beams would be mixtures of ν_e, $\bar{\nu}_e$, ν_μ, and $\bar{\nu}_\mu$. Up to small corrections (of order 10^{-3}) due to CP nonconservation in the K_L decays, the intensity of the ν_e (ν_μ) component of the beam in the region of neutrino production would be

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[28]Note that the equality $P_{\bar{\nu}_l;\bar{\nu}_l}(R/p)=P_{\nu_l;\nu_l}(R/p)$ follows from CPT invariance.

[29]Several high-energy physics laboratories plan to obtain such beams (see, for example, Loveless, 1980; Bryman, 1983; Hoffman, 1983).

equal to the intensity of the $\bar{\nu}_e$ ($\bar{\nu}_\mu$) component. In order to perform a test of *CP* invariance, it is necessary to compare the fluxes either of ν_e and $\bar{\nu}_e$, or of ν_μ and $\bar{\nu}_\mu$, or else of ν_τ and $\bar{\nu}_\tau$ at some distance R from the neutrino source (the K_L decay tunnel). Assuming the validity of the *CPT* theorem, we get for the intensities of the fluxes of ν_e, ν_μ, $\bar{\nu}_e$, and $\bar{\nu}_\mu$ at distance R from the source

$$\begin{aligned} I_{\nu_l}(R,p) &= P_{\nu_l;\nu_l}(R/p) I^0_{\nu_l}(p) + P_{\nu_l;\nu_{l'}}(R/p) I^0_{\nu_{l'}}(p) , \\ I_{\bar{\nu}_l}(R,p) &= P_{\bar{\nu}_l;\bar{\nu}_l}(R/p) I^0_{\nu_l}(p) + P_{\bar{\nu}_l;\bar{\nu}_{l'}}(R/p) I^0_{\nu_{l'}}(p) , \\ & l',l=e,\mu, \quad l'\neq l , \end{aligned} \tag{7.58}$$

where $I^0_{\nu_l}(p)$ is the initial intensity of the ν_l and $\bar{\nu}_l$ fluxes. The intensities of the ν_τ and $\bar{\nu}_\tau$ fluxes are given by

$$\begin{aligned} I_{\nu_\tau}(R,p) &= \sum_{l=e,\mu} P_{\nu_\tau;\nu_l}(R/p) I^0_{\nu_l}(p) , \\ I_{\bar{\nu}_\tau}(R,p) &= \sum_{l=e,\mu} P_{\bar{\nu}_\tau;\bar{\nu}_l}(R/p) I^0_{\nu_l}(p) . \end{aligned} \tag{7.59}$$

Obviously, the ratios

$$R_l = \frac{I_{\bar{\nu}_l}(R,p)}{I_{\nu_l}(R,p)} \tag{7.60}$$

would be equal to unity if Eq. (7.51) were valid. If any of the quantities R_l turns out to be different from unity, one can conclude that, first, oscillations of neutrinos do take place and, second, *CP* invariance does not hold in the leptonic sector.

Let us introduce the quantities

$$D_{l'l} = P_{\nu_{l'};\nu_l} - P_{\bar{\nu}_{l'};\bar{\nu}_l} . \tag{7.61}$$

CPT invariance implies

$$D_{l'l} = -D_{ll'} . \tag{7.62}$$

Using Eq. (7.16), we get

$$\sum_{l'} D_{l'l} = 0, \quad l=e,\mu,\tau,\ldots . \tag{7.63}$$

It follows from Eq. (7.62) that $D_{ll}=0$, $l=e,\mu,\tau,\ldots$. Evidently, $D_{l'l}$ can be different from zero for some l and l', $l\neq l'$, only if the leptonic weak interactions are not *CP* invariant and if at least three types of neutrinos take part in the oscillations. We shall show next that in the case of three types of oscillating neutrinos only one of the quantities $D_{l'l}$ ($l'\neq l$) is independent. Indeed, in this case we have

$$D_{\mu e} + D_{\tau e} = 0, \quad D_{e\mu} + D_{\tau\mu} = 0 , \quad D_{e\tau} + D_{\mu\tau} = 0 . \tag{7.64}$$

These relations imply (Barger, Whisnant, and Phillips, 1980; Bilenky and Niedermayer, 1981)

$$D_{\tau e} = D_{\mu\tau} = D_{e\mu} . \tag{7.65}$$

From Eqs. (7.58) and (7.59) we get

$$\begin{aligned} I_{\nu_e}(R/p) - I_{\bar{\nu}_e}(R/p) &= D_{e\mu} I^0_{\nu_\mu}(p) , \\ I_{\nu_\mu}(R/p) - I_{\bar{\nu}_\mu}(R/p) &= D_{\mu e} I^0_{\nu_e}(p) , \\ I_{\nu_\tau}(R/p) - I_{\bar{\nu}_\tau}(R/p) &= D_{\tau e} I^0_{\nu_e}(p) + D_{\tau\mu} I^0_{\nu_\mu}(p) . \end{aligned} \tag{7.66}$$

As a consequence of *CPT* invariance $D_{e\mu} = -D_{\mu e}$. If $I_{\nu_e}(R/p) \neq I_{\bar{\nu}_e}(R/p)$, then, as can be seen from Eq. (7.66), $I_{\nu_\mu}(R/p) \neq I_{\bar{\nu}_\mu}(R/p)$. Moreover, it follows from Eq. (7.66) that in the case when $I_{\nu_l}(R/p) \neq I_{\bar{\nu}_l}(R/p)$, $l=e,\mu$, the intensities of the neutrino fluxes satisfy the relation

$$\frac{I_{\nu_e}(R/p) - I_{\bar{\nu}_e}(R/p)}{I_{\nu_\mu}(R/p) - I_{\bar{\nu}_\mu}(R/p)} = -\frac{I^0_{\nu_\mu}(p)}{I^0_{\nu_e}(p)} . \tag{7.67}$$

This relation is based solely on *CPT* invariance. Therefore a test of the validity of Eq. (7.67) would constitute a test of the *CPT* theorem in such a subtle experiment as would be a neutrino oscillation experiment.

Using Eqs. (7.65) and (7.69), we also get

$$\begin{aligned} \frac{I_{\nu_\tau}(R,p) - I_{\bar{\nu}_\tau}(R,p)}{I_{\nu_e}(R,p) - I_{\bar{\nu}_e}(R,p)} &= \frac{I^0_{\nu_e}(p) - I^0_{\nu_\mu}(p)}{I^0_{\nu_\mu}(p)} , \\ \frac{I_{\nu_\tau}(R,p) - I_{\bar{\nu}_\tau}(R,p)}{I_{\nu_\mu}(R,p) - I_{\bar{\nu}_\mu}(R,p)} &= -\frac{I^0_{\nu_e}(p) - I^0_{\nu_\mu}(p)}{I^0_{\nu_e}(p)} . \end{aligned} \tag{7.68}$$

Equations (7.68) are based on the assumption that there exist only three types of oscillating neutrinos (ν_e, ν_μ, and ν_τ). Consequently a test of Eqs. (7.68) would provide information about the number of different types of neutrinos taking part in the oscillations.

If the probabilities $P_{\nu_{l'};\nu_l}(R/p)$ and $P_{\bar{\nu}_{l'};\bar{\nu}_l}(R/p)$ turn out not to be equal, $P_{\nu_{l'};\nu_l}(R/p) \neq P_{\bar{\nu}_{l'};\bar{\nu}_l}(R/p)$, that would be a proof of the *CP* noninvariance of the purely leptonic weak interaction. In this case the lepton mixing matrix should contain *CP*-violating phases. However, the effects of *CP* violation in neutrino oscillations may turn out to be practically unobservable under certain circumstances, even if there are large *CP*-violating phases in the lepton mixing matrix. We shall consider in conclusion two examples of conditions leading to such a situation.

(i) Under the conditions of a realistic neutrino oscillation experiment, the transition probabilities have to be averaged over the region of neutrino beam formation, the dimensions of the neutrino detector, the uncertainties in the spectrum of neutrinos, etc. Let us assume that for all $j\neq k$

$$\frac{R}{p} \gg (|m_j^2 - m_k^2|)^{-1}, \quad j\neq k . \tag{7.69}$$

Then, if R/p is sufficiently larger than all $(|m_j^2 - m_k^2|)^{-1}(k\neq j)$, the cosine terms in Eqs. (7.18) and (7.19) would acquire, as a result of the averaging, suppression factors rendering these terms negligible,[30] and for the averaged transition probabilities we would get

$$\bar{P}_{\bar{\nu}_{l'};\bar{\nu}_l}=\bar{P}_{\nu_{l'};\nu_l}=\sum_k |U_{l'k}|^2|U_{lk}|^2 . \tag{7.70}$$

Thus, if the conditions (7.69) are realized, Eq. (7.51) may take place even when *CP* invariance does not hold.

(ii) Let us number the neutrino masses in the following way:

$$m_1<m_2<\cdots<m_r$$

[$r=n$ for the Dirac (Majorana) mass term and $r=2n$ for the Dirac-Majorana mass term]. Suppose that

$$\frac{m_k^2-m_1^2}{2p}R \ll 1,\quad k=2,\ldots,r-1 . \tag{7.71}$$

For the amplitude of the transition $\nu_l\to\nu_{l'}$, we obtain in this case (neglecting terms of order $[(m_k^2-m_1^2)/2p]R$, $k\neq r$)

$$a_{\nu_{l'};\nu_l}=e^{-iE_1t}\left[\sum_k U_{l'k}(e^{-i(E_k-E_1)t}-1)U_{lk}^*+\delta_{l'l}\right]$$
$$\simeq e^{-iE_1t}[U_{l'r}U_{lr}^*(e^{-i(m_r^2-m_1^2)R/2p}-1)+\delta_{l'l}] . \tag{7.72}$$

Similarly, for the amplitude of the transition $\bar{\nu}_l\to\bar{\nu}_{l'}$, one has

$$a_{\bar{\nu}_{l'};\bar{\nu}_l}\simeq e^{-iE_1t}[U_{l'r}^*U_{lr}(e^{-i(m_r^2-m_1^2)R/2p}-1)+\delta_{l'l}] . \tag{7.73}$$

The corresponding probabilities for $l\neq l'$ are given by

$$P_{\nu_{l'};\nu_l}(R/p)=P_{\bar{\nu}_{l'};\bar{\nu}_l}(R/p)$$
$$=2|U_{l'r}|^2|U_{lr}|^2\left[1-\cos\frac{m_r^2-m_1^2}{2p}R\right] . \tag{7.74}$$

[30]Suppose, for example, that one has to average over the uncertainty Δp in the neutrino momentum p. As a result of the averaging, the cosine terms would acquire the factors

$$\left[\frac{\Delta p}{p}\frac{R}{p}|m_j^2-m_k^2|\right]^{-1},$$

$j\neq k$. These terms can be neglected provided

$$\frac{\Delta p}{p}\frac{R}{p}\gg(|m_j^2-m_k^2|)^{-1} .$$

Obviously, the validity of Eq. (7.69) is only a necessary condition for that. The present example illustrates the role of the conditions (7.69) in averaging over any other quantity, as a result of which the cosine terms in the expressions for the transition probabilities become negligible.

So, if the amplitudes of the transitions $\nu_l\to\nu_{l'}$ and $\bar{\nu}_l\to\bar{\nu}_{l'}$ depend effectively only on one difference of the squares of the neutrino masses [i.e., if the conditions (7.71) are realized], the *CP*-violating phases present in the mixing matrix do not enter into the expressions for the probabilities $P_{\nu_{l'};\nu_l}(R/p)$ and $P_{\bar{\nu}_{l'};\bar{\nu}_l}(R/p)$ and Eq. (7.51) is satisfied, in spite of the possible violation of *CP* invariance.

Finally, on the basis of Eq. (7.74) it is not difficult to derive the expression for the probability $P_{\nu_l;\nu_l}(R/p)$] [$P_{\bar{\nu}_l;\bar{\nu}_l}(R/p)$]. In the case of a Dirac (Majorana) mass term one has

$$P_{\nu_l;\nu_l}(R/p)=1-\sum_{l'\neq l}P_{\nu_{l'};\nu_l}(R/p) . \tag{7.75}$$

Taking into account that

$$\sum_{l'\neq l}|U_{l'n}|^2=1-|U_{ln}|^2 , \tag{7.76}$$

we obtain[31]

$$P_{\nu_l;\nu_l}(R/p)=P_{\bar{\nu}_l;\bar{\nu}_l}(R/p)$$
$$=1-2|U_{ln}|^2(1-U_{ln}|^2)$$
$$\times\left[1-\cos\frac{m_n^2-m_1^2}{2p}R\right] . \tag{7.77}$$

If we replace U_{ln} with U_{l2n} in this expression, it will correspond to the case of a Dirac-Majorana mass term.

D. The simplest cases of oscillations

1. Oscillations between two types of neutrinos

In the present subsection we shall consider in detail the cases of oscillations between two and three types of neutrinos in vacuum. Let us begin with the simplest possibility of oscillations involving two types of neutrinos:

$$\nu_l\rightleftarrows\nu_{l'},\quad l\neq l' .$$

The indices l and l' can be equal, respectively, to e and μ, or to e and τ, or to μ and τ, etc.

If the neutrinos with definite masses ν_1 and ν_2 are Dirac particles, the mixing matrix U is a real orthogonal 2×2 matrix. It has the following general form:

$$U=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} . \tag{7.78}$$

[31]Equations (7.74) and (7.77) appear, for example, in the schemes with a pseudo-Dirac neutrino (Petcov, 1982a).

It follows from Eq. (7.78) that the fields $\nu_{lL}(x)$ and $\nu_{l'L}(x)$ are connected with the LH components of the fields of neutrinos with definite masses $\nu_{1L}(x)$ and $\nu_{2L}(x)$ by the relations

$$\begin{aligned}\nu_{lL}&=\nu_{1L}\cos\theta+\nu_{2L}\sin\theta\ ,\\ \nu_{l'L}&=-\nu_{1L}\sin\theta+\nu_{2L}\cos\theta\ .\end{aligned}\tag{7.79}$$

The angle θ is the lepton mixing angle (leptonic analog of the Cabibbo angle).

From the general expressions (7.18) and (7.19) for the transition probabilities we obtain in the case under consideration

$$P_{\nu_{l'};\nu_l}(R/p)=P_{\bar\nu_{l'};\bar\nu_l}(R/p)=\tfrac{1}{2}\sin^2 2\theta\left[1-\cos\frac{\Delta m^2}{2p}R\right],\tag{7.80}$$

$$P_{\nu_l;\nu_l}(R/p)=P_{\nu_{l'};\nu_{l'}}(R/p)=1-\tfrac{1}{2}\sin^2 2\theta\left[1-\cos\frac{\Delta m^2}{2p}R\right].\tag{7.81}$$

Here $\Delta m^2=|m_1^2-m_2^2|$, where m_1 and m_2 are the neutrino masses.

If the massive neutrinos are Majorana particles (Majorana or Dirac-Majorana mass term), the mixing matrix will be characterized by one angle and one *CP*-violating phase. However, in the case under discussion, with two massive neutrinos, the probabilities of the neutrino transitions do not depend on the phase responsible for the violation of *CP* invariance. For the probabilities $P_{\nu_{l'};\nu_l}(R/p)$ and $P_{\nu_l;\nu_l}(R/p)$ in this case we get Eqs. (7.80) and (7.81).

If one expresses Δm^2 in units of eV^2, the source-detector distance R in meters, and the neutrino momentum p in MeV (note that we work in the system $\hbar=c=1$), Eq. (7.80), for example, takes the form

$$P_{\nu_{l'};\nu_l}(R/p)=\tfrac{1}{2}\sin^2 2\theta\left[1-\cos 2.54\frac{\Delta m^2}{2p}R\right].\tag{7.82}$$

Often the expression for $P_{\nu_{l'};\nu_l}(R/p)$ is written as

$$P_{\nu_{l'};\nu_l}(R/p)=\tfrac{1}{2}\sin^2 2\theta\left[1-\cos 2\pi\frac{R}{L}\right],\tag{7.83}$$

where

$$L=4\pi\frac{p}{\Delta m^2}\tag{7.84}$$

is the oscillation length in vacuum. For Δm^2 and p given in units of eV^2 and MeV, respectively, we have

$$L=2.5\frac{p}{\Delta m^2}m\ .\tag{7.85}$$

The introduction of the oscillation length permits us to formulate in a transparent way the conditions under which neutrino oscillations may be observed. It is clear from Eq. (7.83) that the necessary condition for the observation of neutrino oscillations (provided the value of $\sin^2 2\theta$ is sufficiently large) is

$$L\lesssim R\ .\tag{7.86}$$

If the oscillation length is much larger than the distance between the source and the detector, the oscillations will not have time to develop at distance R and, consequently, will not be observed.

Finally, if as a result of the averaging (over the region of neutrino beam formation, etc.) the term with cosine in Eq. (7.83) becomes negligible, the averaged probabilities are given by

$$\bar P_{\nu_{l'};\nu_l}\simeq\tfrac{1}{2}\sin^2 2\theta\ ,\quad \bar P_{\nu_l;\nu_l}\simeq 1-\tfrac{1}{2}\sin^2 2\theta\ .\tag{7.87}$$

Since the conditions leading to (7.87) are always realized for sufficiently large values of Δm^2, the probabilities (7.87) are often referred to in the literature as corresponding to the large-Δm^2 limit.

In practice, Eqs. (7.79) and (7.80) are always used in analyses of the data from neutrino oscillation experiments. A review of the most recent data will be presented in Sec. XII.

2. Oscillations involving three types of neutrinos

We shall consider next in detail the oscillations involving three types of neutrinos (ν_e,ν_μ,ν_τ). It follows from Eq. (7.33) that, for both Dirac and Majorana neutrino mass terms, the probabilities of the transitions depend in this case on the elements of a 3×3 unitary matrix which has the form of the Kobayashi-Maskawa matrix:

$$U=\begin{bmatrix} c_i & s_1c_3 & s_1s_3\\ -s_1c_2 & c_1c_2c_3-s_2s_3e^{i\delta} & c_1c_2s_3+s_2c_3e^{i\delta}\\ s_1s_2 & -c_1s_2c_3-c_2s_3e^{i\delta} & -c_1s_2s_3+c_2c_3e^{i\delta}\end{bmatrix}.\tag{7.88}$$

Here, $c_i=\cos\theta_i$, $s_i=\sin\theta_i$ $(i=1,2,3)$, and δ is the phase characterizing the violation of *CP* invariance.

The probabilities $P_{\nu_{l'};\nu_l}(R/p)$ and $P_{\bar\nu_{l'};\bar\nu_l}(R/p)$ in the general case can be written in the form

$$\begin{aligned}P_{\nu_{l'};\nu_l}(R/p)&=\sum_{k,j}U_{l'k}U_{lk}^*U_{l'j}^*U_{lj}(e^{-i(E_k-E_j)t}-1+1)\\ &=\delta_{l'l}-2\,\mathrm{Re}\sum_{k>j}U_{l'k}U_{lk}^*U_{l'j}^*U_{lj}(1-e^{-i\Delta_{kj}})\ ,\end{aligned}\tag{7.89}$$

$$P_{\bar\nu_{l'};\bar\nu_l}(R/p)=\delta_{l'l}-2\,\mathrm{Re}\sum_{k>j}U_{l'k}^*U_{lk}U_{l'j}U_{lj}^*(1-e^{-i\Delta_{kj}})\ ,\tag{7.90}$$

where

$$\Delta_{kj} = \frac{m_k^2 - m_j^2}{2p} R \ . \qquad (7.91)$$

Equations (7.89) and (7.90) are convenient to use for calculations of the transition probabilities in the case we are considering. The expressions one obtains for the probabilities are, however, rather cumbersome. We shall give here only expressions for those quantities which, in our opinion, are of greatest interest from an experimental point of view.

Using Eqs. (7.88)—(7.90), we get for the probabilities $P_{\nu_l;\nu_l}$ $(l=e,\mu)$,

$$P_{\nu_l;\nu_l} = P_{\bar\nu_l;\bar\nu_l} = 1 - 2A_{21}^l(1-\cos\Delta_{21}) - 2A_{31}^l(1-\cos\Delta_{31}) - 2A_{32}^l(1-\cos\Delta_{32}) \ , \qquad (7.92)$$

where

$$A_{21}^e = s_1^2 c_1^2 c_3^2, \quad A_{31}^e = s_1^2 s_3^2 c_1^2, \quad A_{32}^e = s_1^4 s_3^2 c_3^2 \ , \qquad (7.93)$$

and

$$\begin{aligned} A_{21}^\mu &= s_1^2 c_2^2 (c_1^2 c_2^2 c_3^2 + s_2^2 s_3^2 - 2c_1 c_2 c_3 s_2 s_3 \cos\delta) \ , \\ A_{31}^\mu &= s_1^2 c_2^2 (c_1^2 c_2^2 s_3^2 + s_2^2 c_3^2 + 2c_1 c_2 c_3 s_2 s_3 \cos\delta) \ , \\ A_{32}^\mu &= (c_1^2 c_2^2 c_3^2 + s_2^2 s_3^2 - 2c_1 c_2 c_3 s_2 s_3 \cos\delta) \\ &\quad \times (c_1^2 c_2^2 s_3^2 + s_2^2 c_3^2 + 2c_1 c_2 c_3 s_2 s_3 \cos\delta) \ . \end{aligned} \qquad (7.94)$$

Evidently, the quantities Δ_{kj} are related as

$$\Delta_{21} + \Delta_{32} + \Delta_{13} = 0 \ . \qquad (7.95)$$

Further, for the probabilities of the transitions $\nu_\mu \to \nu_e$ and $\bar\nu_\mu \to \bar\nu_e$ we obtain, respectively,

$$\begin{aligned} P_{\nu_e;\nu_\mu(\bar\nu_e;\bar\nu_\mu)}(R/p) &= 2s_1^2 c_1 c_2 c_3 \{c_1 c_2 c_3 (1-\cos\Delta_{21}) - s_2 s_3 [\cos\delta - \cos(\Delta_{21(\overset{+}{-})}\delta)]\} \\ &\quad + 2s_1^2 s_3 c_1 c_2 \{c_1 c_2 s_3 (1-\cos\Delta_{31}) + s_2 c_3 [\cos\delta - \cos(\Delta_{31(\overset{+}{-})}\delta)]\} \\ &\quad - 2s_1^2 s_3 c_3 \{(s_3 c_1^2 c_2^2 c_3 - s_2^2 s_3 c_3)(1-\cos\Delta_{32}) + s_2 c_1 c_2 c_3^2 [\cos\delta - \cos(\Delta_{32(\overset{+}{-})}\delta)] \\ &\quad - s_2 s_3^2 c_1 c_2 [\cos\delta - \cos(\Delta_{32(\overset{-}{+})}\delta)]\} \ . \end{aligned} \qquad (7.96)$$

If the conditions

$$\frac{R}{p} \gg (|m_k^2 - m_j^2|)^{-1}, \quad k \neq j \qquad (7.97)$$

are satisfied [see Eq. (7.69) and footnote 30], as a result of the averaging (over the neutrino spectrum, etc.) $\overline{e^{-i\Delta_{kj}}}$ $(k\neq j)$ may be neglected, and we get from Eqs. (7.89) and (7.90) in this case

$$\begin{aligned} \bar P_{\nu_{l'};\nu_l} &= \bar P_{\bar\nu_{l'};\bar\nu_l} \\ &= \delta_{l'l} - 2\,\mathrm{Re} \sum_{k>j} U_{l'k} U_{lk}^* U_{l'j}^* U_{lj} \\ &= \sum_k |U_{l'k}|^2 |U_{lk}|^2 \ . \end{aligned} \qquad (7.98)$$

In particular, using Eqs. (7.88) and (7.98), we find in the case of oscillations involving three types of neutrinos

$$\bar P_{\nu_e;\nu_e} = 1 - 2s_1^2 c_1^2 - 2s_1^4 s_3^2 c_3^2 \ . \qquad (7.99)$$

Let us note that for relatively large $|m_k^2 - m_j^2|$ ($|m_k^2 - m_j^2| \gg 10^{-8}$ eV^2) $k\neq j$, $\bar P_{\nu_e;\nu_e}$ represents the reduction coefficient of the flux of solar neutrinos, due to neutrino oscillations in vacuum.[32]

[32]For a more detailed discussion of the problem of solar neutrinos see, for example, Bilenky and Pontecorvo (1978) and Haxton (1984).

E. Neutrino oscillations in matter

Our discussion of neutrino oscillations has been confined so far to the case of oscillations in vacuum. We shall next consider briefly the oscillations of neutrinos propagating through matter (Wolfenstein, 1978).

The probabilities of neutrino transitions in matter may differ drastically from the probabilities of the corresponding transitions in vacuum (Wolfenstein, 1978; Barger, Whisnant, *et al.*, 1980; Mikheyev and Smirnov, 1985). In particular, under certain conditions the presence of matter can lead to a resonant amplification of the transitions between given types of neutrinos, even when the same transitions are strongly suppressed in vacuum due to a small mixing (Mikheyev and Smirnov, 1985). As was shown recently by Mikheyev and Smirnov (1985), for a wide range of values of neutrino oscillation parameters ($\Delta m^2 \simeq 10^{-8}$–$10^{-4}$ eV^2, $\sin^2 2\theta \geq 10^{-4}$) these conditions can take place in the sun. This implies that, even in the case of small vacuum mixing angles, the flux of electron neutrinos ν_e from the sun may be considerably smaller than the flux predicted by the standard solar model (see, for example, Bahcall *et al.*, 1982). The results of Mikheyev and Smirnov have been confirmed and somewhat extended in a number of subsequent studies (Barger, Phillips, and Whisnant, 1986; Bethe, 1986; Bouchez *et al.*, 1986; Haxton, 1986, 1987; Kolb, Turner, and Walker, 1986; Messiah, 1986; Parke, 1986; Rosen and Gelb, 1986, Langacker *et al.*, 1987). Matter effects may also be substantial for oscillations of neutrinos passing through the Earth (Barger, Whisnant, *et al.*, 1980; Cudwell and Gaisser, 1985; Carlson, 1986; Chechin, Ermilova, and Tsarev, 1986; Baltz and Weneser, 1987), as well as for the neutrinos emitted in collapsing stars (Mikheyev and Smirnov, 1986c).[33]

1. The evolution equation

We shall confine our discussion of neutrino oscillations in matter to the simplest possibility of oscillations involving two types of neutrinos $\overset{(-)}{\nu}_l$ and $\overset{(-)}{\nu}_{l'}$, $l \neq l'$: $l=e, l'=\mu$, or $l=e, l'=\tau$, or $l=\mu, l'=\tau$. We shall assume (for simplicity) that the neutrinos with definite mass in vacuum are Dirac particles, so that the mixing (7.79) takes place. However, the results we shall obtain can be proven to be independent of the type of the vacuum neutrino mass eigenstates (Langacker *et al.*, 1987). The $\overset{(-)}{\nu}_l \to \overset{(-)}{\nu}_{l'}$ and $\overset{(-)}{\nu}_l \to \overset{(-)}{\nu}_l$ transition probabilities in vacuum are given in the case of interest by Eqs. (7.80) and (7.81). The expressions for $P_{\overset{(-)}{\nu}_{l'};\overset{(-)}{\nu}_l}$ and $P_{\overset{(-)}{\nu}_l;\overset{(-)}{\nu}_l}$ can be derived by solving the equation of evolution in vacuum, e.g., for the neutrino state vector $|\overset{(-)}{\nu}_l(t)\rangle$. This method of calculating the neutrino transition probabilities is also applicable to the case of oscillations in matter, and we shall use it in our discussion of the latter.

The time evolution of the state vector $|\overset{(-)}{\nu}(t)\rangle$ of a flavor neutrino produced in some weak process is governed by the Schrödinger-type evolution equation

$$i\frac{d}{dt}|\overset{(-)}{\nu}(t)\rangle = H|\overset{(-)}{\nu}(t)\rangle , \tag{7.100}$$

where H is the Hamiltonian of the neutrino system. We shall choose the initial condition for Eq. (7.100) in the form

$$|\overset{(-)}{\nu}(0)\rangle = |\overset{(-)}{\nu}_l\rangle , \tag{7.101}$$

$|\overset{(-)}{\nu}(0)\rangle = |\overset{(-)}{\nu}_{l'}\rangle$ being a possible alternative. The vectors $|\overset{(-)}{\nu}_l\rangle$ and $|\overset{(-)}{\nu}_{l'}\rangle$ are the state vectors of the flavor neutrinos $\overset{(-)}{\nu}_l$ and $\overset{(-)}{\nu}_{l'}$ possessing definite momentum $\mathbf{p}$. In the two-neutrino oscillation case under consideration $|\overset{(-)}{\nu}(t)\rangle$ can always be written as

$$|\overset{(-)}{\nu}(t)\rangle = [\overset{(-)}{a}_l(t)|\overset{(-)}{\nu}_l\rangle + \overset{(-)}{a}_{l'}(t)|\overset{(-)}{\nu}_{l'}\rangle] \times \exp\left[-i\int_0^t \langle\overset{(-)}{\nu}_{l'}|H|\overset{(-)}{\nu}_{l'}\rangle dt'\right] . \tag{7.102}$$

Under the initial condition (7.101), $\overset{(-)}{a}_l(t)$ and $\overset{(-)}{a}_{l'}(t)$ represent the amplitudes of the probabilities for finding neutrinos $\overset{(-)}{\nu}_l$ and $\overset{(-)}{\nu}_{l'}$ at time t. The phase factor in Eq. (7.102) plainly has no physical significance. However, in view of our further considerations, it proves convenient to introduce it. Using Eq. (7.102) we obtain from Eq. (7.100) a system of coupled evolution equations for the amplitudes $\overset{(-)}{a}_l(t)$ and $\overset{(-)}{a}_{l'}(t)$ which is equivalent to Eq. (7.100):

$$i\frac{d}{dt}\begin{bmatrix}\overset{(-)}{a}_l(t)\\ \overset{(-)}{a}_{l'}(t)\end{bmatrix} = \overset{(-)}{M}{}^{(l';l)}\begin{bmatrix}\overset{(-)}{a}_l(t)\\ \overset{(-)}{a}_{l'}(t)\end{bmatrix} . \tag{7.103}$$

Here

$$\overset{(-)}{M}{}^{(l';l)} = \begin{bmatrix}\langle\overset{(-)}{\nu}_l|H|\overset{(-)}{\nu}_l\rangle - \langle\overset{(-)}{\nu}_{l'}|H|\overset{(-)}{\nu}_{l'}\rangle & \langle\overset{(-)}{\nu}_l|H|\overset{(-)}{\nu}_{l'}\rangle\\ \langle\overset{(-)}{\nu}_{l'}|H|\overset{(-)}{\nu}_l\rangle & 0\end{bmatrix} \tag{7.104}$$

is the evolution matrix of the system. Evidently, $\overset{(-)}{M}{}^{(l';l)}$ is a Hermitian matrix: $(\overset{(-)}{M}{}^{(l';l)})^\dagger = \overset{(-)}{M}{}^{(l';l)}$. The condition (7.101) corresponds to the following initial conditions for Eq. (7.103):

$$\overset{(-)}{a}_l(0) = 1, \quad \overset{(-)}{a}_{l'}(0) = 0 . \tag{7.105}$$

It obviously follows from Eqs. (7.103) and (7.104) that neutrino oscillations will take place only if

$$\langle\overset{(-)}{\nu}_l|H|\overset{(-)}{\nu}_{l'}\rangle \neq 0 . \tag{7.106}$$

The form of the evolution matrix $\overset{(-)}{M}{}^{(l';l)}$ implies that the neutrino oscillation probabilities are determined by the difference between the matrix elements of the neutrino Hamiltonian for the $\overset{(-)}{\nu}_l$ and the $\overset{(-)}{\nu}_{l'}$ states, and not by the values of each of the matrix elements.

In the case of neutrinos propagating in vacuum, one can express the elements of the corresponding evolution matrix $\overset{(-)}{M}{}_0^{(l';l)}$ in terms of the vacuum oscillation parameters by using Eqs. (7.78), (7.79), (7.2)–(7.5), (7.6) ($H=H_0$), and (7.17):

[33]We should like to note that the recent dramatic development in the theory of neutrino oscillations in matter (the number of publications that have appeared in the last six months exceeds 15) began after the present review had been essentially completed. The subject is still in a stage of development and it is impossible to cover it comprehensively in our article. Therefore we shall present here only some basic results.

$$\overset{(-)}{M}{}_0^{(l';l)}=\begin{bmatrix}(E_1-E_2)\cos 2\theta & \frac{1}{2}(E_2-E_1)\sin 2\theta \\ \frac{1}{2}(E_2-E_1)\sin 2\theta & 0\end{bmatrix}$$
$$=2\pi\begin{bmatrix}-\kappa\dfrac{\cos 2\theta}{L} & \kappa\dfrac{\sin 2\theta}{2L} \\ \kappa\dfrac{\sin 2\theta}{2L} & 0\end{bmatrix}, \qquad (7.107)$$

where

$$\kappa=\operatorname{sgn}(m_2^2-m_1^2) \qquad (7.108)$$

and $L=2\pi 2p/\Delta m^2$ is the vacuum oscillation length (see Sec. VII.D.1). The solutions of Eq. (7.103) with evolution matrix and initial conditions given by Eqs. (7.107) and (7.105) lead to the familiar expressions (7.80) and (7.81) for the vacuum oscillation probabilities $P_{\overset{(-)}{\nu}_{l'};\overset{(-)}{\nu}_l}$ and $P_{\overset{(-)}{\nu}_l;\overset{(-)}{\nu}_l}$.

Neutrinos propagating through matter can scatter off of the particles present in the matter (electrons and nucleons). The effects of incoherent (i.e., inelastic and nonforward elastic) scattering of neutrinos, which could cause attenuation of the neutrino flux, can be shown to be negligible in most of the cases of practical interest due to the small values of the corresponding neutrino scattering cross sections. However, forward $\overset{(-)}{\nu}_{l,l'}-e^-$ and $\overset{(-)}{\nu}_{l,l'}-N$ elastic scattering, which does not destroy the coherence of the neutrino states, may affect the oscillations, generating nontrivial indices of refraction of the $\overset{(-)}{\nu}_l$ and $\overset{(-)}{\nu}_{l'}$ waves propagating in matter.

The Hamiltonian of the neutrino system in matter has the form

$$H=H_0+H_{\rm int}^{\rm eff}, \qquad (7.109)$$

where $H_{\rm int}^{\rm eff}$ is the effective interaction Hamiltonian of the neutrinos with the electrons and nucleons of the matter. We shall assume that the interactions of neutrinos are described by the standard theory. In this case $H_{\rm int}^{\rm eff}$ is flavor diagonal, and one has[34]

$$\langle \overset{(-)}{\nu}_{l'}(\overset{(-)}{\nu}_l)\,|\,H_0+H_{\rm int}^{\rm eff}\,|\,\overset{(-)}{\nu}_l(\overset{(-)}{\nu}_{l'})\rangle$$
$$=\langle \overset{(-)}{\nu}_{l'}(\overset{(-)}{\nu}_l)\,|\,H_0\,|\,\overset{(-)}{\nu}_l(\overset{(-)}{\nu}_{l'})\rangle$$
$$=\kappa\frac{\sin 2\theta}{2L}\quad (l'\neq l). \qquad (7.110)$$

Furthermore, the $\overset{(-)}{\nu}_l-e^-$ and $\overset{(-)}{\nu}_l-N$ neutral-current weak interactions are universal for the different types of neutrinos and therefore cannot give (in the leading approximation) a nontrivial contribution in the upper diagonal element of the evolution matrix (7.104). Since the $\overset{(-)}{\nu}_{\mu,\tau}-e^-$ and $\overset{(-)}{\nu}_{\mu,\tau}-N$ elastic scattering amplitudes are determined by the neutral-current interaction, this implies that the $\overset{(-)}{\nu}_\mu \rightleftarrows \overset{(-)}{\nu}_\tau$ oscillations in matter in the case under consideration (oscillations involving only two neutrinos) will not differ from the oscillations in vacuum.[35] This is not valid, however, for oscillations involving the electron neutrino (antineutrino) $\overset{(-)}{\nu}_e$ ($\overset{(-)}{\nu}_e \rightleftarrows \overset{(-)}{\nu}_\nu$ or $\overset{(-)}{\nu}_e \rightleftarrows \overset{(-)}{\nu}_\tau$). As is well known, in addition to the contribution generated by the neutral-current weak interaction (Z^0-exchange diagram), the $\overset{(-)}{\nu}_e-e^-$ elastic scattering amplitude also receives a contribution generated by the charged-current weak-interaction ($W^\pm$-exchange diagram). Such a contribution is absent in the $\overset{(-)}{\nu}_{\mu,\tau}-e^-$ elastic scattering amplitude. As a consequence we have (Wolfenstein, 1978; Barger, Whisnant, *et al.*, 1980)

$$\langle \overset{(-)}{\nu}_e\,|\,H_0+H_{\rm int}^{\rm eff}\,|\,\overset{(-)}{\nu}_e\rangle-\langle \overset{(-)}{\nu}_{\mu,\tau}\,|\,H_0+H_{\rm int}^{\rm eff}\,|\,\overset{(-)}{\nu}_{\mu,\tau}\rangle$$
$$=-\kappa\frac{2\pi}{L}\cos 2\theta\,\overset{+}{\scriptstyle (-)}\,\frac{2\pi}{L_0}. \qquad (7.111)$$

Here

$$\overset{+}{\scriptstyle (-)}\frac{2\pi}{L_0}=\langle \overset{(-)}{\nu}_e\,|\,H_{\rm int}^{\rm eff}\,|\,\overset{(-)}{\nu}_e\rangle-\langle \overset{(-)}{\nu}_{\mu,\tau}\,|\,H_{\rm int}^{\rm eff}\,|\,\overset{(-)}{\nu}_{\mu,\tau}\rangle$$
$$=-\frac{2\pi}{p}N_e[F_{\overset{(-)}{\nu}_e}(0)-F_{\overset{(-)}{\nu}_{\mu,\tau}}(0)]$$
$$\simeq\overset{+}{\scriptstyle (-)}\sqrt{2}G_F N_e, \qquad (7.112)$$

where $F_{\overset{(-)}{\nu}_l}(0)$ is the $\overset{(-)}{\nu}_l-e^-$ forward scattering amplitude,[36] N_e is the density of the electrons in the medium, and $\overset{+}{\scriptstyle (-)}\sqrt{2}G_F N_e$ is the charged-current contribution to the real part of the $\overset{(-)}{\nu}_e-e^-$ forward scattering amplitude.[37] The quantity $L_0=2\pi(\sqrt{2}G_F N_e)^{-1}(L_0^{-1})$ charac-

[34]Note that, for example, $|\overset{(-)}{\nu}_l\rangle$ is now the state vector of the neutrino $\overset{(-)}{\nu}_l$ in matter: $|\overset{(-)}{\nu}_l\rangle\equiv|\overset{(-)}{\nu}_l(\mathbf{p})$; matter$\rangle$.

[35]This conclusion is not valid for neutrinos propagating in a very dense media (such as the interior of the neutron stars or of the collapsing stars) (Botella *et al.*, 1986).

[36]The amplitude $F_{\overset{(-)}{\nu}_l}(0)$ is normalized so that the total $\overset{(-)}{\nu}_l-e^-$ scattering cross section is given by $4\pi\,{\rm Im}F_{\overset{(-)}{\nu}_l}(0)/p$.

[37]A factor of $\sqrt{2}$ and a minus sign are missing in the expression for $[F_{\nu_e}(0)-F_{\nu_{\mu,\tau}}(0)]$ given in Wolfenstein (1978); the sign error propagated also in the works of Barger, Whisnant, *et al.* (1980) and of Lewis (1980), wherein the correct numerical factor in the formula for $[F_{\overset{(-)}{\nu}_e}(0)-F_{\overset{(-)}{\nu}_{\mu,\tau}}(0)]$ was found. The correct sign was obtained by Langacker, Leveillé, and Sheiman (1983).

terizes the medium and is called the eigenlength (frequency) of matter.

Equation (7.112) is valid for electrons at rest as well as for electrons with nonzero momentum if the forward scattering amplitude $F_{\overset{(-)}{\nu}_l}(0)$ is averaged over the direction of the electron momentum.

In ordinary matter, which is electrically neutral, one has

$$N_e = N_P = \frac{\rho_m}{m_N}\frac{N_P}{N_N} , \tag{7.113}$$

where $N_{P(N)}$ is the proton (nucleon) number density and ρ_m is the matter density. In most cases of interest (the Earth, the sun) $N_P/N_N \simeq$const.

It follows, for instance, from Eqs. (7.110)–(7.112) that the evolution matrix that determines the $\overset{(-)}{\nu}_e \to \overset{(-)}{\nu}_\mu$ transition amplitude has the form

$$\overset{(-)}{M}{}^{(\mu;e)} = 2\pi \begin{pmatrix} -\kappa\dfrac{\cos 2\theta}{L} \overset{+}{\underset{(-)}{}} \dfrac{1}{L_0} & \kappa\dfrac{\sin 2\theta}{2L} \\ \kappa\dfrac{\sin 2\theta}{2L} & 0 \end{pmatrix}$$

$$= (\overset{(-)}{M}{}^{(\mu;e)})^* . \tag{7.114}$$

A few additional comments are in order. First, only the vector couplings in $H_{\rm int}^{\rm eff}$ give a contribution to the neutrino forward scattering amplitudes (a coherent scattering of neutrinos can be caused by vector interactions). Second, in the language of optics, Eqs. (7.113) implies the existence of a difference between the $\overset{(-)}{\nu}_e$ and $\overset{(-)}{\nu}_{\mu,\tau}$ indices of refraction $n(\overset{(-)}{\nu}_e)$ and $n(\overset{(-)}{\nu}_{\mu,\tau})$ in the medium:

$$n(\overset{(-)}{\nu}_e) - n(\overset{(-)}{\nu}_{\mu,\tau}) = \frac{2\pi}{p^2}[F_{\overset{(-)}{\nu}_e}(0) - F_{\overset{(-)}{\nu}_{\mu,\tau}}(0)]N_e$$
$$= \overset{-}{\underset{(+)}{}} \frac{\sqrt{2}G_F N_e}{p} . \tag{7.115}$$

And third, the difference in the signs of $F_{\nu_l}(0)$ and $F_{\bar\nu_l}(0)$, which is reflected in Eqs. (7.111), (7.112), (7.114), and (7.115) and has important implications, is a consequence of the fact that neutrinos and antineutrinos carry opposite weak isospin charges.

It should be indicated that the effects of the neutral-current $\overset{(-)}{\nu}_l - e^-$ and $\overset{(-)}{\nu}_l - N$ forward scattering have to be taken into account in the case of oscillations involving an active ($\overset{(-)}{\nu}_l$) and a sterile ($\overset{(-)}{\nu}_x$) neutrino. The evolution equation for the probability amplitudes $\overset{(-)}{a}_l(t)$ and $\overset{(-)}{a}_x(t)$ of the transitions $\overset{(-)}{\nu}_l \to \overset{(-)}{\nu}_l$ and $\overset{(-)}{\nu}_l \to \overset{(-)}{\nu}_x$ can be obtained from Eq. (7.104) by making the formal change $l' \to x$. The matrix elements of H_0 which enter into the evolution matrix in this case are given again by Eq. (7.109), while for those of $H_{\rm int}^{\rm eff}$ we have ($\overset{(-)}{\nu}_x$ supposedly does not interact with matter)

$$\langle \overset{(-)}{\nu}_x | H_{\rm int}^{\rm eff} | \overset{(-)}{\nu}_l \rangle = 0 ,$$
$$\langle \overset{(-)}{\nu}_l | H_{\rm int}^{\rm eff} | \overset{(-)}{\nu}_l \rangle - \langle \overset{(-)}{\nu}_x | H_{\rm int}^{\rm eff} | \overset{(-)}{\nu}_x \rangle \tag{7.116}$$
$$\equiv \overset{+}{\underset{(-)}{}} \frac{2\pi}{L_0^l} = \langle \overset{(-)}{\nu}_l | H_{\rm int}^{\rm eff} | \overset{(-)}{\nu}_l \rangle$$
$$\simeq \overset{+}{\underset{(-)}{}} \sqrt{2}G_F \sum_a g_v^{la} N_a ,$$

where N_a is the number density of the particles a in matter and g_v^{la} is the vector coupling constant in the relevant effective $\nu_l - a$ interaction Hamiltonian. In the standard theory $g_v^{ee} = \frac{1}{2} + 2\sin^2\theta_W$, $g_v^{\mu e} = g_v^{\tau e} = -\frac{1}{2} + 2\sin^2\theta_W$, etc. For the oscillations $\overset{(-)}{\nu}_e \rightleftarrows \overset{(-)}{\nu}_x$ in an electrically neutral medium with neutron number density N_n one finds (see, for example, Mikheyev and Smirnov, 1986c)

$$\frac{2\pi}{L_0^e} = \sqrt{2}G_F(N_e - \tfrac{1}{2}N_n) .$$

2. The case of matter with constant density

For concreteness we shall assume in our further discussion that the neutrino produced at $t=0$ is $\overset{(-)}{\nu}_e$ and that $\overset{(-)}{\nu}_e$ may oscillate into $\overset{(-)}{\nu}_\mu$. Given the evolution matrix (7.114), it is not difficult to solve the evolution equation for $\overset{(-)}{a}_e(t)$ and $\overset{(-)}{a}_\mu(t)$ in the case of a medium with constant electron number density N_e. For this purpose it suffices to find the eigenvalues of $\overset{(-)}{M}{}^{(\mu;e)}$ and the orthogonal matrix that diagonalizes it. We shall give only the final result, namely the expressions for the $\overset{(-)}{\nu}_e \to \overset{(-)}{\nu}_e$ and $\overset{(-)}{\nu}_e \to \overset{(-)}{\nu}_\mu$ transition probabilities ($P^m_{\overset{(-)}{\nu}_e;\overset{(-)}{\nu}_e}$ and $P^m_{\overset{(-)}{\nu}_\mu;\overset{(-)}{\nu}_e}$). They have the same form as Eqs. (7.80) and (7.81) for the corresponding vacuum probabilities:

$$P^m_{\overset{(-)}{\nu}_e;\overset{(-)}{\nu}_e} = 1 - \tfrac{1}{2}\sin^2 2\overset{(-)}{\theta}_m \left[1 - \cos 2\pi \frac{R}{\overset{(-)}{L}_m}\right] , \tag{7.117}$$

$$P^m_{\overset{(-)}{\nu}_\mu;\overset{(-)}{\nu}_e} = \tfrac{1}{2}\sin^2 2\overset{(-)}{\theta}_m \left[1 - \cos 2\pi \frac{R}{\overset{(-)}{L}_m}\right] . \tag{7.118}$$

Here

$$\overset{(-)}{L}_m = \frac{L}{\left[1\,{}^{(-)}_{(+)}\,2\kappa \frac{L}{L_0}\cos 2\theta + \left[\frac{L}{L_0}\right]^2\right]^{1/2}} \tag{7.119}$$

is the $\overset{(-)}{\nu}_e \rightleftarrows \overset{(-)}{\nu}_\mu$ oscillation length in matter and

$$\sin 2\overset{(-)}{\theta}_m = \frac{\sin 2\theta}{\left[1\,{}^{(-)}_{(+)}\,2\kappa \frac{L}{L_0}\cos 2\theta + \left[\frac{L}{L_0}\right]^2\right]^{1/2}} , \tag{7.120}$$

$$\cos 2\overset{(-)}{\theta}_m = \frac{\cos 2\theta\,{}^{(-)}_{(+)}\,\kappa \frac{L}{L_0}}{\left[1\,{}^{(-)}_{(+)}\,2\kappa \frac{L}{L_0}\cos 2\theta + \left[\frac{L}{L_0}\right]^2\right]^{1/2}} , \tag{7.121}$$

where $\overset{(-)}{\theta}_m$ is the neutrino (antineutrino) mixing angle in matter:

$$\begin{aligned} |\overset{(-)}{\nu}_e\rangle &= |\overset{(-)}{\psi}{}_1^m\rangle \cos \overset{(-)}{\theta}_m + |\overset{(-)}{\psi}{}_2^m\rangle \sin \overset{(-)}{\theta}_m , \\ |\overset{(-)}{\nu}_\mu\rangle &= -|\overset{(-)}{\psi}{}_1^m\rangle \sin \overset{(-)}{\theta}_m + |\overset{(-)}{\psi}{}_2^m\rangle \cos \overset{(-)}{\theta}_m . \end{aligned} \tag{7.122}$$

In Eq. (7.122), $|\overset{(-)}{\psi}{}_{1,2}^m\rangle$ are the neutrino energy (and momentum) eigenstates in matter:

$$(H_0 + H_{\text{int}}^{\text{eff}})\,|\overset{(-)}{\psi}{}_{1,2}^m\rangle = \overset{(-)}{E}{}_{1,2}^m\,|\overset{(-)}{\psi}{}_{1,2}^m\rangle . \tag{7.123}$$

The oscillation length $\overset{(-)}{L}_m$ is determined by the energy difference

$$\begin{aligned} \overset{(-)}{E}{}_2^m - \overset{(-)}{E}{}_1^m &= (E_2 - E_1)\left[1\,{}^{(-)}_{(+)}\,2\kappa \frac{L}{L_0}\cos 2\theta + \left[\frac{L}{L_0}\right]^2\right]^{1/2} \\ &= 2\pi\kappa\,\overset{(-)}{L}{}_m^{-1} . \end{aligned} \tag{7.124}$$

Equation (7.124) can be derived by comparing the expression for $\overset{(-)}{M}{}^{(\mu;e)}$, obtained in terms of $(\overset{(-)}{E}{}_2^m - \overset{(-)}{E}{}_1^m)$ and $\overset{(-)}{\theta}_m$ from Eq. (7.104) by using (7.122) and (7.123), with the expression (7.114). Obviously, the mass eigenstate neutrinos in matter and in vacuum do not coincide: $\overset{(-)}{\psi}{}_{1,2}^m \neq \overset{(-)}{\nu}_{1,2L(R)}$. In matter the transitions $\overset{(-)}{\nu}_1 \rightleftarrows \overset{(-)}{\nu}_2$ become possible: as a consequence of $\langle \overset{(-)}{\nu}_{l'} | H_0 + H_{\text{int}}^{\text{eff}} | \overset{(-)}{\nu}_l \rangle \neq 0, ll' = e,\mu$, one has $\langle \overset{(-)}{\nu}_{2L(R)} | H_0 + H_{\text{int}}^{\text{eff}} | \overset{(-)}{\nu}_{1L(R)} \rangle \neq 0$. Hence $\overset{(-)}{\nu}_{1,2}$ are not eigenstates of the Hamiltonian of the neutrino system in matter.

The first interesting feature of the probabilities (7.117) and (7.118) to be noted (Langacker *et al.*, 1987) is that they are neither *CP* nor *CPT* invariant, in contrast to the probabilities of transitions in vacuum [see Sec. VII.C and Eqs. (7.80) and (7.81)]. Indeed, even for *CPT* and *CP*-conserving interactions of the neutrinos we have $\bar{\theta}_m \neq \theta_m$, $\bar{L}_m \neq L_m$, and consequently

$$P^m_{\nu_e;\nu_e} \neq P^m_{\bar{\nu}_e;\bar{\nu}_e} , \quad P^m_{\nu_\mu;\nu_e} \neq P^m_{\bar{\nu}_\mu;\bar{\nu}_e} \tag{7.125}$$

because ν_e and $\bar{\nu}_e$ scatter differently on electrons. This drastic difference between the properties of neutrino transition probabilities in vacuum and in matter is a consequence of the fact that, due to the absence of positrons and antinucleons, ordinary matter is neither *CPT* nor *CP*-symmetric, while the vacuum is both (at least to a good approximation). For the same reason, *CP* and *CPT* transformations cannot be defined for the mass eigenstate neutrinos $\overset{(-)}{\psi}{}_{1,2}^m$ in matter.

Further, it follows from Eqs. (7.117)–(7.120) that neutrino oscillations in matter are possible only if neutrino oscillations take place in vacuum (i.e., if $\theta \neq 0$ and $m_1 \neq m_2$). In a medium with "low" electron number density, when

$$\frac{L}{L_0} = \frac{2p}{|m_2^2 - m_1^2|}\sqrt{2}G_F N_e \ll 1 ,$$

neutrinos oscillate as in vacuum: $\overset{(-)}{L}_m \simeq L$ and $\overset{(-)}{\theta}_m \simeq \theta$. If the electron number density in the medium is relatively large, so that $L/L_0 \gg 1$, we have $\overset{(-)}{L} \simeq L_0$ and $|\sin 2\overset{(-)}{\theta}_m| \ll |\sin 2\theta|$. Thus the presence of matter suppresses neutrino oscillations ($\overset{(-)}{\nu}_e \rightleftarrows \overset{(-)}{\nu}_\mu$) in this case, even if they are not suppressed in vacuum.

However, the most striking feature of the dependence of $\sin 2\overset{(-)}{\theta}_m$ on L/L_0 is its resonance character (Barger, Whisnant, *et al.*, 1980; Mikheyev and Smirnov, 1985). Indeed, if

$$\kappa \cos 2\theta > 0 , \tag{7.126}$$

for any value of $\sin 2\theta$ there exists a value of L/L_0,

$$\left[\frac{L}{L_0}\right]_{\text{res}} = \kappa \cos 2\theta , \tag{7.127}$$

for which the neutrino mixing in matter is maximal:

$$|\sin 2\theta_m^{\text{res}}| = 1 . \tag{7.128}$$

This implies that the $\nu_e \to \nu_\mu$ transition in the case under consideration may be strongly enhanced in matter, even if the same transition is suppressed in vacuum due to a small value of the vacuum mixing angle. It follows from Eq. (7.120) that, depending on the sign of $\kappa \cos 2\theta$, resonant amplification is possible either for the transition $\nu_e \to \nu_\mu$ involving neutrinos or for the transition $\bar{\nu}_e \to \bar{\nu}_\mu$ involving antineutrinos, but not for both types of transitions. Analogous results can be obtained for the $\overset{(-)}{\nu}_e \rightleftarrows \overset{(-)}{\nu}_\tau$ and $\overset{(-)}{\nu}_l \rightleftarrows \overset{(-)}{\nu}_x$, $l = e,\mu,\tau$ oscillations.

We shall assume further for definiteness that Eq. (7.126) is satisfied, i.e., that neutrino oscillations $\nu_e \rightleftarrows \nu_\mu$ may be amplified in matter. From Eqs. (7.79) and (7.108) it can be concluded that (7.126) corresponds to a particular relation between the flavor and the mass eigenstate neutrinos: The dominant component in $|\nu_e\rangle$ in vacuum is the state of the lighter of the two neutrinos ν_1 and ν_2 (say, ν_1), while the dominant component in $|\nu_\mu\rangle$ is the heavier neutrino state. This is intuitively expected to take place and is predicted by some gauge models [like the SO(10) model], but it is by no means the only possibility.

Using Eq. (7.127) it is not difficult to find from Eq. (7.120) the width of the region in which $\sin^2 2\theta_m > \frac{1}{2}$ (resonance width):

$$\Delta \frac{L}{L_0} = 2\,|\sin 2\theta|\ . \tag{7.129}$$

The oscillation length and the energy difference $(E_2^m - E_1^m)$ at resonance can be obtained from Eqs. (7.119) and (7.124), taking (7.127) and (7.128) into account:

$$L_m^{\text{res}} = \frac{L}{|\sin 2\theta|}\ , \tag{7.130}$$

$$(E_2^m - E_1^m)_{\text{res}} = (E_2 - E_1)\,|\sin 2\theta|\ . \tag{7.131}$$

So, if the vacuum mixing angle is small ($\sin^2\theta \ll 1$), the resonance is narrow and the oscillation length at resonance exceeds considerably the vacuum oscillation length:

$$L_m^{\text{res}} \gg L\ . \tag{7.132}$$

Note that the diagonal element of the evolution matrix (7.114) vanishes, while for fixed p and Δm^2, $(E_2^m - E_1^m)$ and L_m take, respectively, their minimal and maximal values at resonance.

For neutrinos with a continuous energy spectrum, the resonance condition (7.127) determines the momentum at which the resonance takes place (resonance momentum):

$$p^{\text{res}} = \frac{\Delta m^2 \kappa \cos 2\theta}{2\sqrt{2} G_F N_e}\ .$$

The width Δp of the interval of momenta $(p^{\text{res}} - \Delta p/2, p^{\text{res}} + \Delta p/2)$ in which $\sin^2 2\theta_m > \frac{1}{2}$ can be found from (7.129) and is

$$\Delta p = 2p^{\text{res}}\,|\tan 2\theta|\ .$$

The case of small vacuum mixing angle, when

$$\sin\theta \ll \cos\theta \tag{7.133}$$

(one can choose $0 \le \theta < \pi/4$ without loss of generality), is of particular interest, since in this case the oscillations in vacuum are suppressed. In spite of Eq. (7.133), we will have $\sin^2 2\theta_m \sim 1$ if the resonance condition (7.127) is fulfilled. However, large mixing angles do not necessarily imply large oscillation probabilities. In matter with constant density, $P^m_{\nu_\mu;\nu_e}$ [Eq. (7.117)] may be large (i.e., close to unity) only if in addition to the resonance condition (7.127) the following inequality is satisfied:

$$2\pi R \gtrsim L_m^{\text{res}} = \frac{L}{|\sin 2\theta|}\ . \tag{7.134}$$

For small vacuum mixing angle we get from Eq. (7.127), using (7.126) and (7.133), that at resonance

$$L \simeq L_0\ . \tag{7.135}$$

Consequently, if (7.133) is valid, the condition (7.134) takes the form

$$2\pi R \gtrsim \frac{L_0}{\sin 2\theta}\ . \tag{7.136}$$

Thus $L_0/\sin 2\theta$ determines, in the case of a small vacuum mixing angle, the minimal distance from the neutrino source at which neutrino oscillation effects may be large. Let us estimate this distance for the neutrinos passing through the Earth. For this purpose it is a sufficiently good approximation to assume that the Earth consists of matter having a constant density ρ_m^E equal to its averaged density of 5.5 g/cm^3 and that it is isotopically symmetric: $N_e = N_p = \frac{1}{2} N_N$. In general, one has

$$L_0 = 2\pi(\sqrt{2} G_F N_e)^{-1}$$
$$\simeq 2\pi \times 2.64 \times 10^3\ \text{km}\,\frac{1}{\rho_m[\text{g/cm}^3]}\,\frac{1}{(N_e/N_N)}\ , \tag{7.137}$$

where $\rho_m[\text{g/cm}^3]$ is the matter density expressed in units of g/cm^3. For the case of the Earth we get

$$L_0^E \sim 2\pi \times 10^3\ \text{km}\ . \tag{7.138}$$

Thus it follows from Eqs. (7.136) and (7.138) that matter effects may be considerable for the neutrinos passing through the Earth ($R \le 1.3 \times 10^4$ km) only if $|\sin 2\theta| \ge 0.1$ (Bouchez *et al.*, 1986; Carlson, 1986; Chechin, Ermilova, and Tsarev, 1986; Baltz and Weneser, 1987). Using Eqs. (7.84), (7.85), (7.135), and (7.138), it is not difficult to find also that the resonance condition is satisfied in this case for

$$\frac{p}{\Delta m^2} \sim 2.5 \times 10^3\ \frac{\text{GeV}}{\text{eV}^2}\ . \tag{7.139}$$

Equation (7.139) implies that, depending on the value of Δm^2, the effects of matter may be important for solar neutrinos ($p < 14$ MeV) and/or for atmospheric neutrinos ($\langle p \rangle \sim 10$ GeV) passing through the Earth.

3. Propagation of neutrinos in matter with varying density

We shall next consider briefly the case of oscillations of neutrinos in matter with nonuniform spatial distribution of density (Mikheyev and Smirnov, 1985). Such a distribution, for example, supposedly takes place in the interior of the sun, where, according to the existing solar models (see, for example, Bahcall *et al.*, 1982), the density changes dramatically along the neutrino path from the core to the surface of the sun. Since only changes of density along the neutrino path are important, we shall be interested only in the variation of electron number density N_e (matter density ρ_m) with the distance $r = t$ traveled by the neutrinos: $N_e = N_e(r)$ $[\rho_m = \rho_m(r)]$. The point of neutrino production and the direction of the neutrino momentum, which, if given together with r, specify the position of the neutrino in space at time $t = r$, shall be assumed to be known.

As in the previous section we shall consider for definiteness the case of $\overset{(-)}{\nu}_e - \overset{(-)}{\nu}_\mu$ transitions under the initial condition $|\overset{(-)}{\nu}(0)\rangle = |\overset{(-)}{\nu}_e\rangle$. It will be assumed that inequalities (7.126) and (7.133) are satisfied, so that the $\nu_e \to \nu_\mu$ (and not the $\bar{\nu}_e \to \bar{\nu}_\mu$) transition can be expected to be amplified in matter.

Before discussing the case of interest let us also note that the transformation (7.122), with $\overset{(-)}{\theta}_m$ determined by Eqs. (7.120) and (7.121), actually diagonalizes the evolution matrix $\overset{(-)}{M}{}^{(\mu;e)}$ [Eq. (7.114)]:

$$\overset{(-)}{O}{}_m^T \overset{(-)}{M}{}^{(\mu;e)} \overset{(-)}{O}_m = \begin{bmatrix} \overset{(-)}{M}{}_1^d & 0 \\ 0 & \overset{(-)}{M}{}_2^d \end{bmatrix} . \tag{7.140}$$

Here

$$\overset{(-)}{O}_m = \begin{bmatrix} \overset{(-)}{c}_m & \overset{(-)}{s}_m \\ -\overset{(-)}{s}_m & \overset{(-)}{c}_m \end{bmatrix} \tag{7.141}$$

($\overset{(-)}{c}_m = \cos\overset{(-)}{\theta}_m$, $\overset{(-)}{s}_m = \sin\overset{(-)}{\theta}_m$) is the mixing matrix relating the flavor neutrino states and the states of neutrinos with definite mass in matter, and

$$\begin{aligned} \overset{(-)}{M}{}_1^d &= -(\overset{(-)}{E}{}_2^m - \overset{(-)}{E}{}_1^m)\, \overset{(-)}{c}{}_m^2 , \\ \overset{(-)}{M}{}_2^d &= (\overset{(-)}{E}{}_2^m - \overset{(-)}{E}{}_1^m)\, \overset{(-)}{s}{}_m^2 , \end{aligned} \tag{7.142}$$

where $(\overset{(-)}{E}{}_2^m - \overset{(-)}{E}{}_1^m)$ is determined by Eq. (7.124).

In the case of matter with (electron) density varying along the neutrino path, we have $\overset{(-)}{M}{}^{(\mu;e)} = \overset{(-)}{M}{}^{(\mu;e)}(t)$ and consequently $|\overset{(-)}{\psi}{}_{1,2}^m\rangle = |\overset{(-)}{\psi}{}_{1,2}^m(t)\rangle$, $\overset{(-)}{\theta}_m = \overset{(-)}{\theta}_m(t)$, $\overset{(-)}{M}{}_{1,2}^d = \overset{(-)}{M}{}_{1,2}^d(t)$, and $\overset{(-)}{O}_m = \overset{(-)}{O}_m(t)$. However, Eqs. (7.120)–(7.122) and (7.140)–(7.142) remain valid in this case for each value of t. Although the states $|\overset{(-)}{\psi}{}_{1,2}^m(t)\rangle$ are eigenstates of the evolution matrix $\overset{(-)}{M}{}^{(\mu;e)}(t)$ at each moment t,[38] they, in general, are not eigenstates of the Hamiltonian H, and Eq. (7.123) is not fulfilled. As the matter density varies, transitions between $\overset{(-)}{\psi}{}_1^m(t)$ and $\overset{(-)}{\psi}{}_2^m(t)$ are possible.

For given p and Δm^2 the resonance condition (7.127) determines a value of N_e (and therefore of ρ_m if $N_p/N_N \simeq$const) at which $\sin^2 2\theta_m = 1$ (resonance density):

$$N_e^{\mathrm{res}} = \frac{\Delta m^2 \cos 2\theta}{2\sqrt{2} p G_F} \tag{7.143}$$

$(0 \le \theta < \pi/4,\ \kappa > 0)$. It is not difficult to find from Eqs. (7.129), (7.112), and (7.143) the corresponding resonance width:

$$\Delta N_e = 2 N_e^{\mathrm{res}} \tan 2\theta . \tag{7.144}$$

If N_e and the derivative $dN_e(r)/dr$ are smooth functions of r, which they are in all physical examples of interest, the density width (7.144) corresponds to a spatial width given by

$$\Delta r = \frac{\Delta N_e}{\left| \left(\frac{dN_e}{dr} \right)_{\mathrm{res}} \right|} = \frac{\Delta N_e}{\left| \left(\frac{dN_e}{dt} \right)_{\mathrm{res}} \right|} , \tag{7.145}$$

where $(dN_e/dr)_{\mathrm{res}}$ is the value of the derivative at resonance.

It proves convenient to analyze the evolution of the neutrino system in the case under consideration using the evolution equation for the amplitudes $\overset{(-)}{a}{}_1^m(t)$ and $\overset{(-)}{a}{}_2^m(t)$ of the probabilities for finding the matter eigenstates $\overset{(-)}{\psi}{}_1^m$ and $\overset{(-)}{\psi}{}_2^m$ at time t:

$$|\overset{(-)}{\nu}(t)\rangle = [\overset{(-)}{a}{}_1^m(t)\, |\overset{(-)}{\psi}{}_1^m(t)\rangle + \overset{(-)}{a}{}_2^m(t)\, |\overset{(-)}{\psi}{}_2^m(t)\rangle] \times \exp\left[-i \int_0^t \langle \overset{(-)}{\nu}_\mu | H | \overset{(-)}{\nu}_\mu \rangle dt' \right] . \tag{7.146}$$

It follows from Eqs. (7.102) ($l = e, l' = \mu$), (7.122), and (7.146) that

$$\begin{bmatrix} \overset{(-)}{a}_e(t) \\ \overset{(-)}{a}_\mu(t) \end{bmatrix} = \overset{(-)}{O}_m(t) \begin{bmatrix} \overset{(-)}{a}{}_1^m(t) \\ \overset{(-)}{a}{}_2^m(t) \end{bmatrix} . \tag{7.147}$$

Taking $l = e$ and $l' = \mu$ and replacing

$$\begin{bmatrix} \overset{(-)}{a}_e(t) \\ \overset{(-)}{a}_\mu(t) \end{bmatrix}$$

with Eq. (7.147) in (7.103) we get, using Eqs. (7.114), (7.140), and (7.141),

$$i \frac{d}{dt} \begin{bmatrix} \overset{(-)}{a}{}_1^m(t) \\ \overset{(-)}{a}{}_2^m(t) \end{bmatrix} = \begin{bmatrix} \overset{(-)}{M}{}_1^d(t) & -i \dot{\overset{(-)}{\theta}}_m(t) \\ i \dot{\overset{(-)}{\theta}}_m(t) & \overset{(-)}{M}{}_2^d(t) \end{bmatrix} \begin{bmatrix} \overset{(-)}{a}{}_1^m(t) \\ \overset{(-)}{a}{}_2^m(t) \end{bmatrix} . \tag{7.148}$$

Here

$$\begin{aligned} \dot{\overset{(-)}{\theta}}_m(t) &= \frac{d}{dt} \overset{(-)}{\theta}_m(t) \\ &= \frac{1}{\Delta N_e} \frac{dN_e}{dt} \frac{\tan^2 2\theta}{\left[1 \underset{(+)}{-} \frac{N_e}{N_e^{\mathrm{res}}} \right]^2 + \tan^2 2\theta} , \end{aligned} \tag{7.149}$$

[38]The states $|\overset{(-)}{\psi}{}_{1,2}^m(t)\rangle$ are often called matter eigenstates in the literature (see, for example, Mikheyev and Smirnov, 1986b).

where the last equation has been obtained by using Eqs. (7.120), (7.121), (7.143), and (7.144). If neutrinos are born in a region with electron number density $N_e^0 = N_e(t=0)$, the initial condition for Eq. (7.148) corresponding to $|\overset{(-)}{\nu}(0)\rangle = |\overset{(-)}{\nu}_e\rangle$, as follows from (7.122), has the form

$$\begin{bmatrix} \overset{(-)}{a}{}_1^m(0) \\ \overset{(-)}{a}{}_2^m(0) \end{bmatrix} = \begin{bmatrix} \overset{(-)}{c}{}_m^0 \\ \overset{(-)}{s}{}_m^0 \end{bmatrix}, \tag{7.150}$$

where $\overset{(-)}{c}{}_m^0 = \cos\overset{(-)}{\theta}{}_m^0$, $\overset{(-)}{s}{}_m^0 = \sin\overset{(-)}{\theta}{}_m^0$, and $\overset{(-)}{\theta}{}_m^0 = \overset{(-)}{\theta}_m(t=0) = \overset{(-)}{\theta}_m(N_e^0)$.

For generality we have treated so far both the cases of neutrino and of antineutrino propagation. Equation (7.120) implies, however, that under the assumptions (7.126) and (7.133) antineutrino mixing, and therefore antineutrino oscillations ($\bar{\nu}_e \rightleftarrows \bar{\nu}_\mu$), can only be suppressed in matter. Therefore we shall consider further in this section the more interesting case of neutrino oscillations only.

If the density [and consequently $\dot{\theta}_m(t) \sim dN_e/dt$; see Eq. (7.149)] changes relatively slowly, so that the adiabatic condition

$$\left[\frac{2\dot{\theta}_m(t)}{M_2^d(t) - M_1^d(t)}\right]^2 \ll 1 \tag{7.151}$$

is fulfilled, one can neglect the nondiagonal terms $[\dot{\theta}_m(t)]$ in the neutrino evolution matrix in Eq. (7.148)[39] (Mikheyev and Smirnov, 1985, 1986c; Messiah, 1986). The solutions of the evolution equations can then readily be found:

$$\begin{aligned} a_1^m(t) &\simeq a_1^m(0)\exp\left[-i\int_0^t M_1^d(t')dt'\right], \\ a_2^m(t) &\simeq a_2^m(0)\exp\left[-i\int_0^t M_2^d(t')dt'\right]. \end{aligned} \tag{7.152}$$

It follows from Eq. (7.152) that in the adiabatic approximation (i.e., up to corrections of order $[2\dot{\theta}_m/(M_2^d - M_1^d)]^2$) transitions between the matter eigenstate neutrinos $\psi_1^m(t)$ and $\psi_2^m(t)$ do not take place, in spite of the variation of the density along the neutrino path. However, the flavor content of a neutrino beam can change substantially. Using Eqs. (7.152), (7.150), (7.141), and (7.142), we obtain from (7.147) for the amplitudes $a_e(t)$ and $a_\mu(t)$

$$\begin{aligned} a_e(t) &= \left[c_m^0 c_m(t) + s_m^0 s_m(t)\exp\left[-i\int_0^t [E_2^m(t') - E_1^m(t')]dt'\right]\right]\exp\left[-i\int_0^t M_1^d(t')dt'\right], \\ a_\mu(t) &= \left[-c_m^0 s_m(t) + s_m^0 c_m(t)\exp\left[-i\int_0^t [E_2^m(t') - E_1^m(t')]dt'\right]\right]\exp\left[-i\int_0^t M_1^d(t')dt'\right]. \end{aligned} \tag{7.153}$$

The corresponding probabilities have the form

$$\begin{aligned} P_{\nu_e;\nu_e}^m(t) &= |a_e(t)|^2 = [c_m^0 c_m(t) + s_m^0 s_m(t)]^2 - \tfrac{1}{2}\sin 2\theta_m^0 \sin 2\theta_m(t)\left[1 - \cos\left[\int_0^t (E_2^m - E_1^m)dt'\right]\right], \\ P_{\nu_\mu;\nu_e}^m(t) &= |a_\mu(t)|^2 = [-c_m^0 s_m(t) + s_m^0 c_m(t)]^2 + \tfrac{1}{2}\sin 2\theta_m^0 \sin 2\theta_m(t)\left[1 - \cos\left[\int_0^t (E_2^m - E_1^m)dt'\right]\right]. \end{aligned} \tag{7.154}$$

So, if the density changes adiabatically, for given p, Δm^2, and θ, the average probabilities $\bar{P}_{\nu_e;\nu_e}^m(t) = c_m^{02} c_m^2(t) + s_m^{02} s_m^2(t)$, $\bar{P}_{\nu_\mu;\nu_e}^m(t) = 1 - \bar{P}_{\nu_e;\nu_e}^m(t)$, and the amplitude of oscillations $A_p = \frac{1}{2}\sin 2\theta_m^0 \sin 2\theta_m(t)$ depend only on the values of the density at the beginning (N_e^0) and at the end point $r = t[N_e(t)]$ of the neutrino path [which via Eqs. (7.120) and (7.121) determines θ_m^0 and $\theta_m(t)$, respectively] and not on the density distribution along the path. Large-amplitude oscillations $\nu_e \rightleftarrows \nu_\mu$ take place in the regions with $N_e(t) \sim N_e^{\rm res}$ provided $N_e^0 \sim N_e^{\rm res}$ (or in the case $N_e^0 \ll N_e^{\rm res}$ if the vacuum mixing angle is relatively large). For $N_e^0 > N_e^{\rm res}$ ($N_e^0 < N_e^{\rm res}$) and $N_e(t) > N_e^{\rm res}$ [$N_e(t) < N_e^{\rm res}$] one has $s_m^{02} > 0.5$, $s_m^2(t) > 0.5$ [$c_m^{02} > 0.5, c_m^2(t) > 0.5$], and therefore $\bar{P}_{\nu_e;\nu_e}^m(t) > 0.5$. Further, if neutrinos are produced and/or end up in a region with density that exceeds considerably or is much smaller than the resonance density, and the vacuum mixing angle θ is small [Eq. (7.133)], then $|\sin^2\theta_m^0| \ll 1$ and/or $|\sin 2\theta_m(t)| \ll 1$ and the oscillatory term in Eq. (7.154) is negligible. Under these conditions practically oscillationless and almost total $\nu_e \to \nu_\mu$ conversion is possible (Mikheyev and Smirnov, 1986c). This can take place if, for example, (i) neutrinos are produced in a region with high electron number density,

$$\left.\frac{L}{L_0}\right|_{N_e = N_e^0} = \frac{N_e^0}{N_e^{\rm res}}\cos 2\theta \simeq \frac{N_e^0}{N_e^{\rm res}} \gg 1 \tag{7.155}$$

[we have used Eqs. (7.112), (7.143), and (7.133) in (7.155)]; (ii) the density decreases monotonically along the path of propagation of neutrinos to some minimal value $N_e^f = N_e(t_f) \ll N_e^0$; (iii) neutrinos pass through a resonance layer ($N_e^f < N_e^{\rm res}$) and

$$\left.\frac{L}{L_0}\right|_{N_e = N_e^f} \simeq \frac{N_e^f}{N_e^{\rm res}} \ll 1. \tag{7.156}$$

It follows from Eqs. (7.120), (7.121), and (7.133) that under the conditions (7.155) and (7.156)

[39]Indeed, it is not difficult to convince oneself that up to corrections of order $\{2\dot{\theta}_m(t)/[M_2^d(t) - M_1^d(t)]\}^2$, for example, the eigenvalues in the neutrino evolution matrix in (7.148) coincide in this case with $M_1^d(t)$ and $M_2^d(t)$ at any t.

$$\sin 2\theta_m^0 \simeq \frac{N_e^{\rm res}}{N_e^0}\tan 2\theta\left[1+\frac{N_e^{\rm res}}{N_e^0}\right], \quad \cos 2\theta_m^0 \simeq -1+\frac{1}{2}\left[\frac{N_e^{\rm res}}{N_e^0}\right]^2 \tan^2 2\theta\left[1+2\frac{N_e^{\rm res}}{N_e^0}\right],$$

$$\sin 2\theta_m(t_f) \simeq \tan 2\theta\left[1+\frac{N_e^f}{N_e^{\rm res}}\right], \quad \cos 2\theta_m(t_f) \simeq 1-\tfrac{1}{2}\tan^2 2\theta\left[1+2\frac{N_e^f}{N_e^{\rm res}}\right]. \tag{7.157}$$

Using Eq. (7.157) one obtains from (7.154) to leading order in $\sin^2\theta$, $N_e^{\rm res}/N_e^0$, and $N_e^f/N_e^{\rm res}$

$$P^m_{\nu_e;\nu_e}(t_f) \simeq \sin^2\theta\left[1+2\frac{N_e^f}{N_e^{\rm res}}+2\frac{N_e^{\rm res}}{N_e^0}\cos\left[\int_0^t (E_2^m - E_1^m)dt'\right]\right] \simeq \sin^2\theta ,$$

$$P^m_{\nu_\mu;\nu_e}(t_f) = 1 - P^m_{\nu_e;\nu_e}(t_f) \simeq \cos^2\theta . \tag{7.158}$$

Since $\sin^2\theta \ll \cos^2\theta$, we have indeed practically a total $\nu_e \to \nu_\mu$ conversion in this case. The neutrino state which at $t=0$ is described by $|\nu(0)\rangle = |\nu_e\rangle \simeq |\psi_2^m(0)\rangle$ transforms continuously into the orthogonal state $|\nu(t_f)\rangle \simeq |\nu_\mu\rangle \simeq |\psi_2^m(t_f)\rangle$ at t_f as the density decreases, passing through the resonance value. Since the density changes relatively slowly, the neutrino system, being at $t=0$ in a state which to a good approximation is an eigenstate of the Hamiltonian H, has time to adapt to the change and thus to follow it. The probability $P^m_{\nu_e;\nu_e}(t) \simeq \bar{P}^m_{\nu_e;\nu_e}(t)$ follows the behavior of $N_e(t)$, increasing and decreasing together with the electron density. The "rotation" of the neutrino flavor in the case under consideration is analogous to the well-known phenomenon of rotation of the spin of a spin-$\frac{1}{2}$ particle in a slowly varying magnetic field. Let us add that at resonance, as follows from Eqs. (7.128), (7.159), and (7.154), $P^m_{\nu_e;\nu_e}(t_{\rm res}) \simeq P^m_{\nu_\mu;\nu_e}(t_{\rm res}) \simeq \frac{1}{2}$ and that for $t > t_f$ practically no $\nu_\mu \rightleftarrows \nu_e$ oscillations will take place even if $N_e(t) \le N_e(t_f)$ since the neutrino system will be in a state with definite energy:

$$|\nu(t)\rangle \simeq \exp\left[-i\int_0^t E_2^m(t')dt'\,|\psi_2^m(t)\rangle\right] \simeq \exp\left[-i\int_0^{t_f} E_2^m(t')dt'\right] e^{-i(t-t_f)E_2}(|\nu_e\rangle\sin\theta + |\nu_\mu\rangle\cos\theta) .$$

In order for the adiabatic $\nu_e \to \nu_\mu$ conversion to be possible, the condition (7.151) should be satisfied at all moments $t \le t_f$. However, the validity of the adiabatic condition in the resonance layer is most important. At resonance $(M_2^d - M_1^d)^2 = (E_2^m - E_1^m)^2$ has a minimal value [see Eq. (7.124)], and the factor multiplying the derivative dN_e/dt in Eq. (7.149) for $\dot\theta_m(t)$ is maximal. Moreover, the most significant changes of the flavor content of the neutrino state take place precisely in the resonance layer. Using Eqs. (7.149), (7.145), and (7.124), we can rewrite the adiabatic condition at resonance in the form

$$\left[\frac{2L_m^{\rm res}}{2\pi\Delta r}\right]^2 \ll 1 . \tag{7.159}$$

Obviously, Eq. (7.159) will be satisfied if the resonance width exceeds the oscillation length at resonance density: $\Delta r \ge L_m^{\rm res}$. This condition resembles the necessary condition (7.134) for the existence of large oscillation effects in matter with constant density, although in the case under consideration oscillations practically do not occur.

Obviously, with minor modifications, the above results will be valid for the $\nu_e \to \nu_\tau$ and the $\nu_l \to \nu_x$, $l=e,\mu,\tau$, transitions. If $\kappa\cos 2\theta < 0$ and the vacuum mixing angle is small, oscillations involving neutrinos will be suppressed in matter with varying density, while transitions between antineutrinos $\bar\nu_e \to \bar\nu_{\mu(\tau)}$ and/or $\bar\nu_l \to \nu_x$, $l=e,\mu,\tau$, will be strongly enhanced.

Extensive numerical calculations (Mikheyev and Smirnov, 1985, 1986c; Barger, Phillips, and Whisnant, 1986; Bouchez *et al.*, 1986; Kolb, Turner, and Walker, 1986; Rosen and Gelb, 1986); performed in the framework of the standard solar model have shown that, for values of the vacuum oscillation parameters Δm^2 and $\sin^2 2\theta$ in the intervals $10^{-8} \lesssim \Delta m^2 \lesssim 10^{-4}$ eV^2, $\sin^2 2\theta \gtrsim 10^{-4}$, the adiabatic condition (7.151) as well as the conditions (i)—(iii) [Eqs. (7.155) and (7.156)] can take place in the sun, and thus a substantial depletion of the solar ν_e flux is possible.

With this remark we shall conclude our discussion of neutrino oscillations in matter. Among the questions not considered by us, of particular interest are those concerning the divergence of neutrino wave packets in matter and the transitions between the flavor neutrinos in the case of nonadiabatic density variation. They have been studied by Mikheyev and Smirnov (1986a, 1986c), Bouchez *et al.* (1986), Parke (1986), Petcov (1987), and Toshev (1987a), respectively. Three-neutrino oscillations in matter have also been intensively investigated recently (Baldini and Giudice, 1986; Kim *et al.*, 1986; Kuo and Pantaleone, 1986a, 1986b; Petcov and Toshev, 1986; Smirnov, 1986; Toshev, 1987b). A more detailed discussion of the subject in general can be found in Mikheyev and Smirnov (1986b).

VIII. MASSIVE NEUTRINOS IN GAUGE THEORIES

A. General remarks

After the foregoing rather extensive general analysis of the possible varieties of neutrino mass terms and massive neutrinos, we turn now to examples of neutrino mass generation in the framework of gauge theories with spontaneous symmetry breaking.[40] As is well known, no acceptable alternatives to the gauge theories as theories of elementary-particle interactions exist at present. Not only the electroweak but also the strong interactions of quarks are most successfully described by a theory with unbroken $SU(3)^C$ color gauge symmetry, or quantum chromodynamics (QCD). The symmetries playing a fundamental role in the construction of the gauge theories are dynamical. They fix unambiguously the character of the dynamics governing the basic particle interactions (electroweak, strong, etc.) and (apart from the sector responsible for the mass generation) ensure renormalizability once the symmetry group and the particle content of the theory are specified. It has been known for a long time (Lee and Yang, 1955) that the conservation of lepton L and baryon B charges could not possibly be associated with unbroken local symmetries.[41] This may reflect the existence of exact global symmetries which, however, are not inherent to, and have to be imposed as an additional constraint on, the gauge theories. In this sense global symmetries implying lepton and baryon charge conservation cannot be considered as fundamental in the context of the gauge theories. The latter admit violations of these symmetries whenever the requirement of local gauge invariance (and renormalizability) and the relevant multiplet content permit it. As a consequence, finite neutrino masses arise quite naturally in the gauge theories of electroweak interactions (see, for example, Cheng and Li, 1980) and especially in grand unified theories (GUT's) unifying the electroweak and strong interactions (see, for example, Ellis, 1981). In some GUT's, such as those based on the group SO(10), it is almost impossible to avoid finite neutrino masses. At the same time the simplest versions of these theories, namely, the standard $SU(2)_L \times U(1)$ electroweak theory, which contains no RH neutrino fields $\nu_{lR}(x)$, and its SU(5) grand unified generalization (Georgi and Glashow, 1974) predict massless neutrinos.

The neutrino mass matrix originates usually in gauge theories of electroweak interactions from Yukawa-type couplings of the lepton doublets and/or singlets with Higgs scalar fields, some components of which develop nonzero vacuum expectation values. In order not to spoil the renormalizability of the theory, these couplings have to be gauge invariant. The mass terms thus generated can be of the varieties we have considered, i.e., both massive Dirac and massive Majorana neutrinos are possible in the gauge theories.

[40]It is beyond the scope of the present review to discuss the structure of the gauge theories. There exist already several excellent reviews and books on the subject (Faddeev and Slavnov, 1980; Quigg, 1983; Cheng and Li, 1984).

[41]Otherwise the resulting gauge interactions would introduce a discrepancy in the Eötvös experiment, unless they were characterized by ultrasmall coupling constants.

Further, the properties of massive Dirac and massive Majorana neutrinos and the physics associated with them are very different. Massive Dirac neutrinos arise in theories in which some lepton charge is conserved. The simplest example of such a theory is the minimally extended standard electroweak theory that includes the RH neutrino fields $\nu_{lR}(x)$ as $SU(2)_L$ and U(1) singlets, wherein the total lepton charge L is assumed to be conserved. Besides the nonzero neutrino masses, the only predicted new phenomena that might lead to observable effects in this case are, in essence, the oscillations between neutrinos possessing different flavors (Petcov, 1977b). In contrast, massive Majorana neutrinos arise in theories with no conserved lepton charge, which represent considerable extensions of the standard theory (see, for example, Cheng and Li, 1980). As a rule, they predict the existence of a multitude of characteristic particles and processes. For instance, in the $SU(2)_L \times U(1)$ theories containing no $\nu_{lR}(x)$ fields, these could be relatively light neutral, charged, and doubly charged, as well as massless Higgs particles and a number of specific processes in which they might take part. For this reason, it is generally believed today that massive neutrinos could be the "visiting card" of some new physics beyond that predicted by the standard model.

As we shall see, the gauge theories offer essentially no clues about the precise values of the neutrino masses and leptonic mixing angles. As a rule, grand unified theories with massive neutrinos suggest a plausible explanation for the smallness of the neutrino masses, relating them to the ratio $M_{W,Z}^2/M_{\rm GUT}$ of the two mass scales typically present in GUT's: the scale of unification of electroweak and strong interactions $M_{\rm GUT} \sim 10^{15}$ GeV and the scale of unification of weak and electromagnetic interactions $M_{EW} \sim M_{W,Z^0} \sim 100$ GeV. The predicted values of the masses themselves are subject to uncertainties. Nevertheless, the predictions in general have a tendency to lie in the region between 10^{-5} eV and several tens of keV, which, in principle, can be explored in various experiments with neutrinos from accelerators, reactors, and the sun and with atmospheric neutrinos (see, for example, Bilenky and Pontecorvo, 1978). Specific values for the leptonic mixing angles as well as for the ratios of the neutrino masses can be obtained within a given theory if the theory, for example, is required to be invariant with respect to certain discrete transformations involving the neutrino fields. The angles and neutrino mass ratios are expressed then in terms of ratios of quark or charged-lepton masses (e.g., Harvey, Ramond, and Reiss, 1982). Usually, there is no *a priori* justification for the existence of such symmetries.

B. $SU(2)_L \times U(1)$ theories

1. Dirac neutrinos

Dirac-type mass terms [Eq. (4.10)] arise most naturally in the standard $SU(2)_L \times U(1)$ theory containing the RH neutrino fields $\nu_{lR}(x)$ as $SU(2)_L$ singlets (Petcov, 1977b). These terms are generated by the Yukawa-type interaction

$$\mathscr{L}_{l-\phi}=-\sum_{l,l'=e,\mu,\tau,\ldots} G_{l'l}\overline{\nu_{l'R}}\phi^{c\dagger}\begin{bmatrix}\nu_{lL}\\ l_L\end{bmatrix}+\text{H.c.}\ , \tag{8.1}$$

where $\phi^c=i\tau_2\phi^*$ is the charge conjugate of the standard Higgs doublet

$$\phi=\begin{bmatrix}\phi^{(+)}\\ \phi^0\end{bmatrix},$$

the neutral component of which has a nonzero vacuum expectation value $\langle\phi^0\rangle_0\neq 0$, and $G_{ll'}$ are, in general, complex constants. The coupling (8.1) gives rise to $\mathscr{L}^D$ [Eq. (4.10)] with $M_{l'l}=\langle\phi^0\rangle_0 G_{l'l}$. The neutrino mass spectrum as well as the lepton mixing matrix can be arbitrary. In this case neutrinos are treated on an equal footing with the other fermions of the theory, and there exists a complete analogy between leptons and quarks. In particular, the couplings (8.1) are similar to those inducing a mass term for the charged leptons:

$$\mathscr{L}'_{l-\phi}=-\sum_{l,l'=e,\mu,\tau,\ldots} G'_{l'l}\overline{l_R}\phi^{\dagger}\begin{bmatrix}\nu_{lL}\\ l_L\end{bmatrix}+\text{H.c.}\ , \tag{8.2}$$

where $G'_{l'l}$ are complex constants.

2. Majorana neutrinos

a. The model with a triplet of Higgs scalars

It is impossible to generate a Majorana mass term for neutrinos in the minimal $SU(2)_L\times U(1)$ theory [no $\nu_{lR}(x)$] via a renormalizable gauge-invariant coupling, as the product $(C^{-1}\nu_{lL})^T\nu_{l'L}$ changes the weak isospin by one unit, and the only Higgs field available is an isodoublet. However, if a triplet of neutral, charged, and doubly charged Higgs fields

$$H=\begin{bmatrix}-H^+/\sqrt{2} & H^{++}\\ H^0 & H^+/\sqrt{2}\end{bmatrix} \tag{8.3}$$

whose neutral component has a nonzero vacuum expectation value (i.e., $\langle H^0\rangle_0=v/\sqrt{2}\neq 0$) is introduced, the gauge-invariant coupling

$$\mathscr{L}_{l-H}=\frac{1}{\sqrt{2}}\sum_{l,l'=e,\mu,\tau,\ldots} h_{ll'}(\overline{\nu_{l'L}}\,\overline{l_L'})H^{\dagger}i\tau_2\begin{bmatrix}(\nu_{lL})^c\\ (l_L)^c\end{bmatrix}$$

$$+\text{H.c.}\ , \tag{8.4}$$

where $h_{ll'}$ are complex symmetric constants,[42] leads to $\mathscr{L}^M$ (Cheng and Li, 1980; Maiani, 1980), with

$$M_{ll'}=2\frac{\langle H^0\rangle_0}{\sqrt{2}}h^*_{ll'}=vh^*_{ll'}\ . \tag{8.5}$$

Stringent constraints on the value of $\langle H^0\rangle_0$ follow from the existing data on the ratio $M_W^2/(M_Z^2\cos^2\theta_W)$, which in the model is given by

$$\frac{M_W^2}{M_Z^2\cos^2\theta_W}=\frac{1+2\left[\dfrac{\langle H^0\rangle_0}{\langle\phi^0\rangle_0}\right]^2}{1+4\left[\dfrac{\langle H^0\rangle_0}{\langle\phi^0\rangle_0}\right]^2}=\frac{1+2\dfrac{v^2}{\lambda^2}}{1+4\dfrac{v^2}{\lambda^2}}\ .$$

From the experimental success of the standard model prediction $M_W^2/(M_Z^2\cos^2\theta_W)=1$ it follows that $v\ll\lambda$, where $\lambda\simeq 250$ GeV.

In addition to nonzero neutrino masses, the theory predicts the existence of three neutral, one charged, and one doubly charged physical Higgs scalar particles. Their masses and couplings with the leptons and quarks depend crucially on the mechanism used to break the global U(1) symmetry associated with the conservation of lepton charge L.[43] It is possible to assign two units of L to the Higgs triplet $H[L(H)=-2]$, so that the coupling (8.4) would conserve the lepton charge. The form of the U(1) symmetry breaking leading to $\langle H^0\rangle_0\neq 0$ and L nonconservation is determined by the assumed properties of the Higgs potential $V(\phi,H)$ of the theory. The breaking can be explicit when $V(\phi,H)$ is supposed to contain the trilinear L-nonconserving term $(\phi^{\dagger}H\phi^c)$. In this case the masses of the physical Higgs particles in the model can be arbitrarily large, e.g., they may considerably exceed, M_W. Consequently, except for the nonzero neutrino masses, there may be no specific observable effects at low energies due to the presence of the additional massive scalar particles in the theory. Moreover, no deviations from the standard big-bang cosmology are predicted.

The phenomenology of the theory changes drastically if one assumes that the lepton charge L is conserved by the Higgs potential $V(\phi,H)$, but the U(1) global symmetry associated with the conservation of L is broken spontaneously (Chikashige, Mohapatra, and Peccei, 1981) as H^0 develops a nonzero vacuum expectation value. The resulting model is due to Gelmini and Roncadelli (1981) and has been widely discussed (e.g., Georgi, Glashow, and Nussinov, 1981; Schechter and Valle, 1982a). Its most remarkable feature is the presence of a physical massless neutral scalar particle M^0 (the Goldstone boson of the broken global symmetry) called a Majoron, which has extremely weak pseudoscalar couplings with the charged leptons and quarks. The Majoron field $M^0(x)$ is a linear combination of the imaginary parts of the neutral com-

[42]The property $h_{ll'}=h_{l'l}$ is analogous to the symmetry property of the Majorana mass matrix of neutrinos.

[43]Note that this symmetry has nothing to do with the U(1) weak hypercharge symmetry of the standard theory.

ponents of the Higgs doublet field ϕ and the Higgs triplet field H. In the unitary gauge in which no unphysical fields are present in the Lagrangian of the theory, the couplings of M^0 can be obtained by the substitution

$$\phi^0(x)-\phi^{0*}(x)\to i\frac{2\sqrt{2}v}{(\lambda^2+4v^2)^{1/2}}M^0(x)\ , \tag{8.6}$$

$$H^0(x)-H^{0*}(x)\to -i\frac{\sqrt{2}\lambda}{(\lambda^2+4v^2)^{1/2}}M^0(x)\quad [M^{0*}(x)=M^0(x)]\ . \tag{8.7}$$

Let us note that in the same gauge we have effectively

$$\phi^{(+)}\to\frac{\sqrt{2}v}{(\lambda^2+2v^2)^{1/2}}H_T^+\ , \tag{8.8}$$

$$H^+\to\frac{\lambda}{(\lambda^2+2v^2)^{1/2}}H_T^+\ , \tag{8.9}$$

where H_T^+ is the physical massive charged scalar field in the theory, and the substitutions (8.8) and (8.9) should be made irrespective of the chosen mechanism of symmetry breaking.

The couplings of the Majoron to the neutrino mass eigenstates $\chi_K(x)$ and the charged leptons can be deduced from Eqs. (8.2), (8.4), (8.6), and (8.7):

$$\mathscr{L}_{\nu-M^0}=i\frac{\lambda}{(\lambda^2+4v^2)^{1/2}}M^0(x)\sum_{k=1}^{n}\frac{m_k}{2v}\overline{\chi_k}(x)\gamma_5\chi_k(x)\quad [C\overline{\chi_k}^T(x)=\chi_k(x)]\ , \tag{8.10}$$

$$\mathscr{L}_{l-M^0}=i\frac{\sqrt{2}v}{(\lambda^2+4v^2)^{1/2}}M^0(x)\times\sum_{l=e,\mu,\tau,\ldots}g\frac{m_l}{\sqrt{2}M_W}\bar{l}(x)\gamma_5 l(x)\ . \tag{8.11}$$

They are diagonal in the fields of leptons with definite masses, since the neutrino and charged lepton masses originate in the model from couplings to one Higgs field[44] (triplet and doublet, respectively). Further, the strength of the charged lepton—Majoron interaction is determined by the ratio $v/(\lambda^2+4v^2)^{1/2}$. Severe constraints on the possible value of v in the triplet Majoron model can be obtained from astrophysical considerations. It can be shown (Georgi, Glashow, and Nussinov, 1981) that the process $\gamma+e^-\to M^0+e^-$ inside red giant stars, which is possible due to $\mathscr{L}_{l-M^0}$, would lead to unacceptably large energy losses unless

$$v<100\text{ keV}\ . \tag{8.12}$$

Thus the charged lepton—Majoron coupling is exceedingly weak as, on account of Eq. (8.12), $v/\lambda<4\times10^{-7}$. The same conclusion applies to the quark-Majoron interaction under the condition (8.12). Hence, the Majoron may couple substantially only to neutrinos.

[44]For an analogous reason, the quark-Majoron couplings are also flavor diagonal.

Unlike the case of explicit L breaking, the Majoron model predicts physical scalar particles that are relatively light on the scale of electroweak symmetry breaking $\lambda\simeq250$ GeV. For instance, the mass of one of the neutral Higgs particles, whose couplings to the charged leptons and quarks are also suppressed by the factor v/λ, does not exceed v. Together with the Majoron the physical scalar particles should take part in or mediate a number of characteristic processes at low energies (Georgi, Glashow, and Nussinov, 1981). In particular, the charged and neutral ones may be produced in $W^\pm$-boson and Z^0-boson decays. The width of the Z^0 decay into a Majoron and its light neutral counterpart is twice that of the Z^0 decay into a neutrino-antineutrino pair of a given type. Being unobservable, this specific decay mode of the Z^0 would mimic decays into two new types of neutrino-antineutrino pairs. Consequently, a relatively accurate measurement of the width of the Z^0 invisible decays will provide a crucial test of the triplet Majoron model. The cosmological implications of the model are unconventional as well. For example, according to the model, neutrinos—in contrast to photons—could not survive during the evolution of the universe to form cosmic background. They are predicted to disappear, annihilating pairwise into pairs of Majorons (Georgi, Glashow, and Nussinov, 1981).

b. The model of Zee

An alternative mechanism for generating a Majorana mass term within the $SU(2)_L\times U(1)$ theories with a minimal fermionic content and an enlarged Higgs sector, including several Higgs doublets ϕ_i $(i=1,2,\ldots)$, was suggested by Zee (1980). As we shall see, the model of Zee is extremely rich in properties (Wolfenstein, 1980, 1981a; Petcov, 1982a, 1982b) that make it interesting from both a theoretical and an experimental point of view, and we are going to consider it in somewhat greater detail. The mechanism for neutrino mass generation in question relies on the fact that there exist more than one lepton family. For definiteness we shall assume three such families. The LH neutrinos ν_l acquire a radiatively induced Majorana mass term as a result of the introduction of an $SU(2)_L$ singlet charged Higgs field. H'. It couples to $SU(2)_L$ singlet combinations of two lepton doublets which are antisymmetric in the flavor indices:

$$\mathscr{L}_{l-H'}=\sum_{l,l'=e,\mu,\tau}f_{ll'}^0(\overline{\nu_{lL}l_L})H'^{\dagger}i\tau_2\begin{bmatrix}(\nu_{l'L})^c\\(l'_L)^c\end{bmatrix}+\text{H.c.}$$

$$=2\sum_{l,l'=e,\mu,\tau}f_{ll'}^0\,\overline{l'_L}H'^{\dagger}(\nu_{lL})^c+\text{H.c.}\ , \tag{8.13}$$

where $f_{ll'}^0=-f_{l'l}^0$ are, in general, complex constants. Utilizing the phase arbitrariness of the three pairs of lepton fields, one can make $\mathscr{L}_{l-H'}$ explicitly CP conserving, which implies [in the convention fixed by Eqs. (5.21) and (5.26)] that $f_{ll'}^0$ can be made real. According to Eq. (8.13)

H' can be assigned two units of the lepton charge L. The lepton-number-violation effects originate then from trilinear couplings of H' to ϕ_i, which have to be antisymmetric in the indices of the Higgs doublets in order to preserve the gauge symmetry

$$\mathscr{L}_{H'-\phi}=\sum_{j,k} c_{jk}\phi_j^{\dagger}\phi_k^{c}H'^{\dagger}+\text{H.c.}\ , \tag{8.14}$$

where $c_{jk}=-c_{kj}$ are, in general, complex constants. One can consider without loss of generality the simplest case of two Higgs doublets. In this case there is one constant in Eq. (8.14), c_{12}, and it can be made real. As in the standard model, the neutral components of the Higgs doublets $\phi_{1,2}^0$ develop nonzero vacuum expectation values ($\langle\phi_{1,2}^0\rangle_0\neq 0$), which break spontaneously the U(1) global symmetry associated with conservation of the weak hypercharge. Further, there are three charged scalar fields in the theory (H' and the upper components of the doublets $\phi_{1,2}$), but only two linear combinations formed by them ($H_{1,2}^+$) correspond to physical scalar particles. These particles possess nonzero masses.

Although absent from the Lagrangian of the theory, a neutrino mass term of Majorana type arises at the one-loop level due to the couplings (8.2), (8.13), and (8.14) (see Fig. 1). This term is finite and takes a particularly simply form (Wolfenstein, 1980) if only one of the Higgs doublets, say ϕ_1, couples to the leptons and the ϕ_1-lepton couplings are flavor diagonal:

$$M_{ll'}^{(H')}=f_{ll'}(m_l^2-m_{l'}^2),\ \ l,l'=e,\mu,\tau\ . \tag{8.15}$$

Here (Petcov, 1982b)

$$f_{ll'}=-f_{l'l}=f_{ll'}^0\frac{g}{\sqrt{2}M_W}\frac{\sin 2\beta}{32\pi^2}\cot\phi\,\ln\frac{M_2^2}{M_1^2}\ , \tag{8.16}$$

where $M_{1,2}$ are the masses of the charged Higgs particles $H_{1,2}^+$, and the mixing angles ϕ and β are defined through the relations

$$\tan\phi=\frac{\langle\phi_1^0\rangle_0}{\langle\phi_2^0\rangle_0}\ , \tag{8.17}$$

$$\tan 2\beta=\frac{2\dfrac{2\sqrt{2}}{g}c_{12}M_W}{\left[(M_1^2-M_2^2)^2-\left[2\dfrac{2\sqrt{2}}{g}c_{12}M_W\right]^2\right]^{1/2}}\ . \tag{8.18}$$

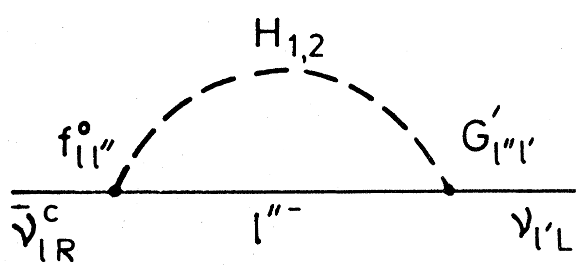

FIG. 1. Diagrams generating finite neutrino mass term of Majorana type in the model of Zee.

The existing data impose rather weak constraints on the parameters $M_{1,2}$, $f_{ll'}^0$, ϕ, and β, and consequently on $f_{ll'}$ (Petcov, 1982b). The search for charged scalar particles at PEP and PETRA has given negative results which imply $M_{1,2}>20$ GeV. Another constraint follows from the observed universality between the μ decay and β decay:

$$\overline{M}^{\,-2}[f_{e\tau}^{02}(f_{\mu\tau}^{02}+f_{e\mu}^{02})+f_{e\mu}^{02}(f_{e\mu}^{02}+f_{\mu\tau}^{02})]^{1/2}\lesssim 10^{-1}\frac{G_F}{\sqrt{2}}\ , \tag{8.19}$$

where

$$\overline{M}^{\,-2}=M_1^{-2}\cos^2\beta+M_2^{-2}\sin^2\beta\ . \tag{8.20}$$

It should be noted that $M_{ll}^{(H')}=0$, $l=e,\mu,\tau$, and therefore the relevant neutrino mass Lagrangian $\mathscr{L}_{\text{eff}}^M$ contains only flavor nondiagonal terms. This property of $\mathscr{L}_{\text{eff}}^M$ is a consequence of the chosen form of the lepton-ϕ_i couplings and does not hold in the general case. However, it is interesting from the theoretical point of view as it leads to very unusual consequences [e.g., for the $(\beta\beta)_{0\nu}$-decay rate].

Of special interest also is the possibility

$$|f_{e\mu}^0|\lesssim(f_{e\tau}^{02}+f_{\mu\tau}^{02})^{1/2}\ . \tag{8.21}$$

The character of the resulting neutrino mass spectrum, as well as the CP parities of the Majorana mass eigenstates in this case, is determined by the ratios of the charged-lepton masses squared. Of the three massive Majorana neutrinos, two, say $\chi_{2,3}$, are almost degenerate in mass, possess opposite CP parities, and are much heavier than the third χ_1:

$$\begin{aligned}
&m_1\simeq m_\mu^2\,|f_{e\mu}\sin 2\alpha|\ ,\\
&\eta_1^{CP}=-i\,\text{sgn}(f_{e\mu}\sin 2\alpha)\ ,\\
&m_{2,3}\simeq m_\tau^2(f_{e\tau}^2+f_{\mu\tau}^2)^{1/2}\pm\tfrac{1}{2}m_\mu^2 f_{e\mu}\sin 2\alpha\\
&\qquad\ =m\pm\tfrac{1}{2}m_1\text{sgn}(f_{e\mu}\sin 2\alpha)\ ,\\
&\eta_2^{CP}=-\eta_3^{CP}=i\ ,\\
&|m_2-m_3|\simeq m_1\lesssim\frac{m_\mu^2}{m_\tau^2}m_2\ ,
\end{aligned} \tag{8.22}$$

where

$$\tan\alpha=\frac{f_{\mu\tau}^0}{f_{e\tau}^0}\left[1-\frac{m_\mu^2}{m_\tau^2}\right]. \tag{8.23}$$

It follows from Eqs. (8.22) and (8.23) that, in the version of the model considered, the absolute values of the masses m_i cannot be predicted. However, $m_{2,3}$ may well lie in the range of 20 eV. The relations between $\nu_{lL}(x)$ and $\chi_i(x)$ $(i=1,2,3)$ are given by (Wolfenstein, 1980; Petcov, 1982a)

$$\begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} = \begin{pmatrix} \chi_{1L}\sin\alpha + \dfrac{\chi_{2L}-\chi_{3L}}{\sqrt{2}}\cos\alpha \\ -\chi_{1L}\cos\alpha + \dfrac{\chi_{2L}-\chi_{3L}}{\sqrt{2}}\sin\alpha \\ \dfrac{\chi_{2L}+\chi_{3L}}{\sqrt{2}} \end{pmatrix} . \tag{8.24}$$

Some of the predictions of the Zee model, as well as of the model with a triplet of Higgs scalars, will be considered by us later.

We shall restrict our discussion of possible mechanisms for generation of Majorana-type neutrino mass terms in the $SU(2)_L \times U(1)$ theories to the two examples considered above. A rather complete list of other possibilities can be found in Cheng and Li (1980).

Finally, it is amusing to note that a mass term of Dirac-Majorana type $\mathscr{L}^{D+M}$ with $M_L=0$ [see Eq. (4.9)] and, consequently, massive Majorana neutrinos may appear in the standard model containing the RH neutrino fields $\nu_{lR}(x)$. Indeed, besides the Dirac piece generated in the standard way [see Eq. (8.1)], the most general neutrino mass Lagrangian includes in this case a Majorana term for $\nu_{lR}(x)$. Since $\nu_{lR}(x)$ are $SU(2)_L$ and U(1) singlets, the latter is gauge invariant and does not spoil the renormalizability of the theory. The neutrino masses and lepton mixing angles thus arising can be arbitrary.

C. Grand unified theories

The grand unified theories (Pati and Salam, 1973; Georgi and Glashow, 1974) represent an attempt at a unified description of electroweak and strong interactions of leptons and quarks in the framework of gauge theories. According to the grand unified theories with unbroken color symmetry that we shall discuss, the electroweak and strong forces between the fundamental fermions, as described by the standard $SU(2)_L \times U(1)$ and $SU(3)^C$ gauge theories, should become universal in strength at ultrahigh energies characterized by a scale $M_{\rm GUT} \sim 10^{15}$ GeV (Georgi and Glashow, 1974). At these energies the difference between leptons and quarks is supposed to be inessential, reducing to a difference only between the values of the electroweak $[SU(2)_L \times U(1)]$ and strong $[SU(3)^C]$ charges they carry. Accordingly, the particle multiplets in GUT's are composed of both leptons and quarks. As a consequence, the baryon B and lepton L charges are not separately conserved, but the combination $(B\text{-}L)$ has a special status and may be preserved in GUT's. Further, nucleon decays like $p \to e^+\pi^0$, $p \to \mu^+K^0$, etc. are allowed and may proceed with rates close to the existing upper limits. Extensive searches for indications of the nucleon instability predicted by GUT's are being performed at present. However, the results obtained so far are either negative or not conclusive (Koshiba, 1984).

We shall focus our attention on neutrino mass generation in two classes of GUT's, namely the SU(5) (Georgi and Glashow, 1974) and SO(10) (Georgi, 1974; Fritzsch and Minkowski, 1975) theories. The first represents a minimal grand unified generalization of the $SU(2)_L \times U(1)$ electroweak and $SU(3)^C$ strong interaction theories. As in the $SU(2)_L \times U(1)$ theory, parity nonconservation is built into the SU(5) GUT's with the type of fermion multiplets chosen, and no attempt is made to relate it to some form of breaking of an initial symmetry of the theory. In contrast, one starts with parity-conserving fermion couplings in the SO(10) GUT's. The corresponding symmetry is assumed to be broken spontaneously or explicitly only in the Higgs sector. Through the interaction of fermions and gauge bosons with the Higgs fields, the symmetry breaking is conveyed to the gauge couplings of the fermions.

In the minimal SU(5) theory of Georgi and Glashow (1974), which is a generalization of the standard $SU(3)^C \times SU(2)_L \times U(1)$ theory, the fermions of each generation are assigned to the reducible $(\bar{5}+10)$ representation of the gauge group. There is no room in it for the RH neutrinos, which are SU(5) singlets. Thus neutrinos may acquire, in principle, only a mass term of Majorana type in the SU(5) theory.

For the first generation the $\bar{5}$-plets and 10_f-plets have the form

$$\bar{5} = \begin{pmatrix} d_L^{rc} \\ d_L^{yc} \\ d_L^{bc} \\ e_L \\ \nu_{eL} \end{pmatrix} , \tag{8.25}$$

$$10 = \begin{pmatrix} 0 & u_L^{bc} & -u_L^{yc} & u_L^r & d_L^r \\ -u_L^{bc} & 0 & u_L^{rc} & u_L^y & d_L^y \\ u_L^{yc} & -u_L^{rc} & 0 & u_L^b & d_L^b \\ -u_L^r & -u_L^y & -u_L^b & 0 & e_L^c \\ -d_L^r & -d_L^y & -d_L^b & -e_L^c & 0 \end{pmatrix} . \tag{8.26}$$

Here $u_L^{jc} = C\overline{u_R^j}^T$ and $d_L^{jc} = C\overline{d_R^j}^T$, where j is the color index of the quarks, $j=r,b,y$. The multiplets containing fermions of the second and third generations can be obtained from Eqs. (8.25) and (8.26) by the substitutions $\nu_{eL} \to \nu_{\mu L}$, $e \to \mu$, $u^j \to c^j$, $d^j \to s^j$, and $\nu_{eL} \to \nu_{\tau L}$, $e \to \tau$, $u^j \to t^j$, $d^j \to b^j$, respectively. The minimal set of Higgs fields necessary to realize the requisite spontaneous symmetry breaking in the theory $[SU(5) \to SU(3)^C \times U^{\rm em}(1)]$ consists of a 5-plet and a 24-plet. The indicated multiplet content, together with the requirement of gauge invariance of the Higgs field couplings, ensures, automatically the conservation of the $(B\text{-}L)$ charge. As a consequence, the neutrinos are predicted to be massless. The minimal SU(5) theory of Georgi and Glashow is essentially unique among the grand unified theories in allowing massless neutrinos to occur naturally.

The $SU(2)_L \times U(1)$ models with a triplet and singly charged Higgs field and a neutrino mass term of Majorana type, discussed in the previous subsection, can be accommodated within the SU(5) theory with an enlarged Higgs sector by adding a 15-plet and a 10-plet of Higgs fields, respectively (Cheng and Li, 1980). Among the SU(5) models with massive neutrinos in which neutrino mass terms arise as radiative corrections, the generalization of the model of Zee is unique in that it does not exclude sizable neutrino masses, mass differences, and mixing angles, and therefore, sizable effects of neutrino oscillations at the existing facilities (Nieves, 1981).

Massive neutrinos appear quite naturally in the SO(10) GUT's. Moreover, the suggested mechanisms of neutrino mass generation in these theories give rise to neutrino masses compatible with the existing experimental data, thus providing an explanation of the remarkable disparity between the neutrino and the charged-lepton and quark masses.

The generation of neutrino masses is intimately related to parity and $(B\text{-}L)$ nonconservation in the SO(10) theories (Ramond, 1979; Barbieri, 1980). Parity nonconservation is assumed to arise only as a result of spontaneous symmetry breaking or specific Higgs self-interactions (Georgi, 1974). Therefore both LH $[\nu_{lL}(x)]$ and RH $[\nu_{lR}(x)]$ neutrino fields are present in the relevant fermionic multiplets, and consequently neutrinos can acquire a mass term of Dirac type. Since, in addition, the $(B\text{-}L)$ charge is not necessarily conserved, no symmetry forbids the appearance of Majorana mass terms formed by $\nu_{lL}(x)$ and/or $\nu_{lR}(x)$. This implies that, in the SO(10) theories, the neutrino mass Lagrangian can be expected on rather general grounds to be of the Dirac-Majorana type considered by us in Sec. IV. D.

The fermions of each generation are assumed (Georgi, 1974; Fritzsch and Minkowski, 1975) to form a 16-dimensional irreducible spinor representation (16_f) of SO(10). Its SU(5) content is

$$16_f=\overline{5}_f+10_f+1_f\ , \tag{8.27}$$

where the $\overline{5}_f$- and 10_f-plets have been specified earlier [see Eqs. (8.25) and (8.26)]. The SU(5) singlet 1_f can be identified with the RH counterpart $[\nu_{lR}(x)]$ of the LH neutrino field $[\nu_{lL}(x)]$ in the multiplet. A neutrino mass term of Dirac type is inevitably generated by the fermion—Higgs-boson couplings that give rise to the quark and charged-lepton mass terms (see, for example, Langacker, 1981). However, due to the underlying symmetry of the theory and the irreducibility of the fermion representations used, it is not arbitrary in general. For example, in the cases of Higgs fields forming 10- and/or 126-dimensional representations[45] $(10_H,126_H)$ of SO(10), the resulting Dirac neutrino masses are of the same order as the charge-$\frac{2}{3}$ quark masses (Georgi, 1974; Georgi and Nanopoulos, 1979a, 1979b, 1979c). Obviously, such a possibility is ruled out by the existing data. The neutrino mass spectrum changes drastically, however, if, being SU(5) singlets, the RH neutrinos acquire a huge Majorana mass $M_{\nu_{lR}}$, say, of order of the unification mass $M_{\rm GUT}$ (Gell-Mann, Ramond, and Slansky, 1979; Yanagida, 1979; Stech, 1980). Neglecting for simplicity the possible inter-generation mixing, we then get the following mass term for the neutrinos of each generation:

$$\mathscr{L}^{D+M}_{\rm SO(10)}=-\tfrac{1}{2}[\overline{(\nu_L)^c}\,\overline{\nu_R}]\begin{bmatrix}0 & m_D\\ m_D & M_{\nu_R}\end{bmatrix}\begin{bmatrix}\nu_L\\ (\nu_R)^c\end{bmatrix}+\text{H.c.}\ , \tag{8.28}$$

where m_D is of order of the mass of the charge-$\frac{2}{3}$ quark from the same generation and flavor indices have been suppressed. As was shown earlier (see Sec. VI.B), the mass eigenstates of the Lagrangian $\mathscr{L}^{D+M}_{\rm SO(10)}$ are one light Majorana neutrino (χ) and one superheavy Majorana neutral lepton (N) with masses

$$m\simeq\frac{m_D^2}{M_{\nu_R}},\ \ M\simeq M_{\nu_R}\ , \tag{8.29}$$

respectively, and opposite CP parities. Further, up to corrections of the order of $m_D/M_{\nu_R}\ll 1$, we have $\nu_L(x)\simeq\chi_L(x)$, and $\nu_R(x)\simeq N_R(x)$, i.e., the admixture of superheavy neutral leptons in the LH charged lepton current is negligible. It can be shown that the effects of flavor mixing, as well as of nonzero Majorana masses smaller than m_D for the LH neutrinos,[46] do not qualitatively alter these results.

Large Majorana mass for the RH neutrinos can be generated via the Higgs mechanism by the coupling of two 16_f to 126_H, whose SU(5) singlet component has a nonzero vacuum expectation value (Gell-Mann, Ramond, and Slansky, 1979; Yanagida, 1979; Stech, 1980). This breaks the SO(10) symmetry to SU(5) and induces $(B\text{-}L)$ nonconservation. Large M_{ν_R} can also arise as a radiative correction at the two-loop level (Witten, 1980) if the Higgs fields chosen form only 10- and 16-dimensional representations $(10_H,16_H)$ of SO(10) (no 126_H present). The nonconservation of the $(B\text{-}L)$ charge is then associated with the trilinear Higgs boson coupling $(16_H16_H10_H)$. For M_{ν_R} one gets in these two cases, respectively,

$$\text{case A:}\ \ M_{\nu_R}\sim M_{\rm SO_{10}}\ ,$$

$$\text{case B:}\ \ M_{\nu_R}\sim\varepsilon\left[\frac{\alpha}{\pi}\right]^2\frac{m_D}{M_W}M_{\rm SO_{10}}\ ,$$

[45]One can use also the couplings of a 120-plet of Higgs fields to generate the requisite fermion mass spectrum. Although Dirac neutrino masses compatible with the observations could arise at tree level in this case, they may be changed dramatically by radiative corrections (for details see Barbieri *et al.*, 1980).

[46]Majorana masses smaller than m_D actually appear as radiative corrections in the case under consideration (Barbieri *et al.*, 1980; Magg and Watterich, 1980).

where $M_{SO_{10}}$ is the characteristic scale of breaking of the SO(10) symmetry to SU(5) and ε is a Higgs boson mixing parameter associated with the indicated trilinear coupling. The scale $M_{SO_{10}}$ is not fixed unambiguously. It can exceed the unification scale by several orders of magnitude, being possibly of order of the Planck mass $M_P \sim 10^{19}$ GeV: $M_{GUT} \lesssim M_{SO_{10}} \lesssim M_P$. As a consequence, there are large uncertainties in the predictions for neutrino masses. Using, for example, for the current masses of the u, c, and t quarks the values 5 MeV, 1.5 GeV, and 40 GeV, respectively, with $M_{SO_{10}} \sim 10^{15}$ GeV and $\varepsilon \sim 0.1$, we get the following light-neutrino mass spectra in cases (A) and (B):[47]

case A: $m_1 \simeq 10^{-12}$ eV, $m_2 \simeq 10^{-7}$ eV, $m_3 \simeq 0.7 \times 10^{-4}$ eV;

case B: $m_1 \simeq 2.3 \times 10^{-2}$ eV; $m_2 \simeq 7.0$ eV, $m_3 \simeq 181.8$ eV.

As we have indicated, variations by several orders of magnitude in the direction of smaller masses are possible due to the uncertainty in the value of $M_{SO_{10}}$. The effects of flavor mixing introduce additional uncertainties in the values of the light-neutrino masses.

It should be emphasized that it is possible to have zero mass and even light Dirac neutrinos in the SO(10) GUT's. This can be a consequence of the presence of SO(10) singlet fermions in these theories (Georgi, 1974; Georgi and Nanopoulos, 1979a, 1979b, 1979c; Roncadelli and Wyler, 1983; Wyler and Wolfenstein, 1983), which make it possible to maintain the conservation of the $(B\text{-}L)$ charge. Indeed, adding to each generation of leptons and quarks an SO(10) singlet fermion field $s_L(x)$ (the generation index is suppressed) and assuming that it forms an SO(10)-breaking but SU(5)- and $(B\text{-}L)$-preserving mass term with $\nu_R(x)$ $[-M_{SO_{10}}\overline{s_L}(x)\nu_R(x)+\text{H.c.}]$, one[48] obtains the following neutrino mass Lagrangian (Georgi and Nanopoulos, 1979a, 1979b, 1979c; see also Petcov, 1982a):

$$\mathscr{L}^D_{SO(10)} = -[\overline{\nu_L}\,\overline{(\nu_R)^c}\,\overline{s_L}] \begin{bmatrix} 0 & m_D & 0 \\ m_D & 0 & M_{SO_{10}} \\ 0 & M_{SO_{10}} & 0 \end{bmatrix} \times \begin{bmatrix} (\nu_L)^c \\ \nu_R \\ (s_L)^c \end{bmatrix} + \text{H.c.} \tag{8.30}$$

This conserves the corresponding lepton charge and therefore the $(B\text{-}L)$ charge, which is also conserved by the total Lagrangian of the theory. The mass eigenstates of the Lagrangian (8.30) are one massless neutrino, which up to corrections of order of $m_D/M_{SO_{10}}$ coincides with ν_L, and one Dirac neutral lepton with large mass of the order of $(M^2_{SO_{10}}+m^2_D)^{1/2} \simeq M_{SO_{10}}$.

The neutrino mass Lagrangian for a given generation would have one light and one heavy Dirac eigenstate if in addition to $\mathscr{L}^D_{SO(10)}$ it contained the lepton-number-conserving couplings to $\nu_L(x)$, $\nu_R(x)$, and $s_L(x)$ of a second SO(10) singlet fermion (Roncadelli and Wyler, 1983). The light Dirac neutrino couples in this case predominantly to the corresponding charged lepton in the LH charged current, the analogous coupling of the heavy neutral lepton being strongly suppressed.

Obviously, all these considerations can be extended to the case of intergeneration mixing.

Finally, let us note that the global symmetries associated with conservation of the lepton charges L_l and the total lepton charge L have the same status in theories with broken supersymmetry[49] as in ordinary gauge theories. They do not necessarily follow from the structure of the supersymmetric theories and are imposed as an additional constraint whenever L_l and/or L are assumed to be conserved. The possible mechanisms of lepton charge nonconservation have been extensively studied in the context of supersymmetric extensions of the standard theory (Hall and Suzuki, 1984; Lee, 1984; Dawson, 1985; Ellis *et al.*, 1985; Ross and Valle, 1985). It was found that neutrinos inevitably acquire nonzero masses in the theories with L_l and/or L nonconservation. However, the supersymmetric theories proposed so far offer no fundamentally new solutions of the neutrino mass problem.

IX. NEUTRINOLESS DOUBLE-β DECAY (ELEMENTARY-PARTICLE ASPECTS OF THE THEORY)

If neutrinos turn out to be massive, the question of what type of neutrinos they are (Dirac or Majorana) will inevitably arise. This will be a fundamental question to

[47]The u, c, and t quarks are assumed to have the masses quoted above at the scale of 1 GeV, while the neutrino masses are determined by the values of the quark masses evaluated at the unification scale (see, for example, Langacker, 1981). The latter are approximately a factor of 4.7 smaller than the former (Buras *et al.*, 1978).

[48]Such a mass term can be generated via the Higgs mechanism if the theory contains a 16-plet of Higgs fields whose SU(5) singlet component develops a nonzero vacuum expectation value (Georgi and Nanopoulos, 1979a). It can also be associated with the breaking of the left-right symmetry (Wyler and Wolfenstein, 1983). Then the mass parameter $M_{SO_{10}}$ is replaced by the characteristic mass of left-right symmetry, breaking, which can be as low as 1 TeV.

[49]For reviews of the phenomenological aspects of the supersymmetric theories see Fayet (1982), Ellis (1984), Ibañéz (1984), Nilles (1984); and Haber and Kane (1985).

answer, as it will concern the very nature of neutrinos and of the underlying symmetries of the electroweak interaction. We have seen in Sec. VII.A.2 that it is impossible to determine in practice the type of massive neutrinos in experiments studying neutrino oscillations. In fact, the existing large number of studies (e.g., Case, 1957; Ryan and Okubo, 1964; Schechter and Valle, 1981a; Kayser, 1982; Kayser and Shrock, 1982; Shrock, 1982b) allows us to conclude that the majority of effects typical only of massive Majorana neutrinos are very subtle, being suppressed by the factor $(m_k/E)^2$, where m_k is the Majorana neutrino mass and E is some characteristic energy scale for a given process involving real or virtual neutrinos. If neutrinos have nonzero masses smaller than 100 eV, the experiments most sensitive to the existence of the Majorana neutrinos (coupled to the electron) are at present those searching for neutrinoless double-β $[(\beta\beta)_{0\nu}]$ decay of certain nuclei (Racah, 1937),

$$(A,Z)\rightarrow(A,Z+2)+e^-+e^- . \tag{9.1}$$

Obviously, the electron lepton charge L_e is not conserved in this process.

Background considerations imply that the most favorable nuclei for experiments on $(\beta\beta)_{0\nu}$ decay are those for which the ordinary β decay is either forbidden (e.g., by energy conservation) or strongly inhibited (e.g., by large spin changes). Typical examples of nuclei that satisfy this requirement are ${}^{48}_{20}$Ca, ${}^{76}_{32}$Ge, ${}^{82}_{34}$Se, ${}^{100}_{42}$Mo, ${}^{128}_{52}$Te, ${}^{130}_{52}$Te, and ${}^{150}_{60}$Nd. The corresponding decays of interest have the form

$$\begin{aligned}
&{}^{48}_{20}\mathrm{Ca}\rightarrow{}^{48}_{22}\mathrm{Ti}+e^-+e^- ,\\
&{}^{76}_{32}\mathrm{Ge}\rightarrow{}^{76}_{34}\mathrm{Se}+e^-+e^- ,\\
&{}^{82}_{34}\mathrm{Se}\rightarrow{}^{82}_{36}\mathrm{Kr}+e^-+e^- ,\\
&{}^{100}_{42}\mathrm{Mo}\rightarrow{}^{100}_{44}\mathrm{Ru}+e^-+e^- ,\\
&{}^{128}_{52}\mathrm{Te}\rightarrow{}^{128}_{54}\mathrm{Xe}+e^-+e^- ,\\
&{}^{130}_{52}\mathrm{Te}\rightarrow{}^{130}_{54}\mathrm{Xe}+e^-+e^- ,\\
&{}^{150}_{60}\mathrm{Nd}\rightarrow{}^{150}_{62}\mathrm{Sm}+e^-+e^- .
\end{aligned} \tag{9.2}$$

A comparison of the phase-space factors entering into the probabilities of the $(\beta\beta)_{0\nu}$ decay (9.1) and of the competing two-neutrino L_e-conserving decay (Goeppert-Mayer, 1935)

$$(A,Z)\rightarrow(A,Z+2)+e^-+e^-+\bar{\nu}_e+\bar{\nu}_e \tag{9.3}$$

induced by the standard weak interaction (in second order in G_F) reveals that the former exceeds the latter typically by a factor of 10^5-10^6 (see, for example, Wu, 1980). The remarkable sensitivity of the process (9.1) to the existence of Majorana neutrinos and lepton-number-nonconserving interactions has its roots in the indicated strong phase-space rate enhancement.

Let us note that none of the decays (9.2) has been observed. The most stringent experimental limits have been obtained so far for the $(\beta\beta)_{0\nu}$-decay lifetime of ${}^{76}_{32}$Ge. [See Sec. XII, where the existing data on $(\beta\beta)_{0\nu}$ decay are briefly reviewed.]

There exist many extensive reviews of the theory and phenomenology of the double-β decays (9.1) and (9.3). Two published very recently are by Doi, Kotani, and Takasugi (1985) and by Vergados (1986). We shall consider in this section only some of the elementary-particle aspects of $(\beta\beta)_{0\nu}$-decay theory. These will be illustrated with examples of predictions of the gauge theories with massive Majorana neutrinos. An introductory exposition of the nuclear physics aspects of the theory is given in Appendix B. Our consideration will be based on some of the results derived in Appendix B.

We shall assume first that the effective weak β-decay Hamiltonian has the form

$$H_W^\beta=\frac{G_F}{\sqrt{2}}2(\overline{e_L}\gamma_\alpha\nu_{eL})j_{L\alpha}+\text{H.c.} \tag{9.4}$$

[$j_{L\alpha}(x)$ is the standard left-handed strangeness-conserving charged hadron current] and that neutrino mixing takes place,

$$\nu_{eL}(x)=\sum_{k=1}^{n}U_{ek}^L\chi_{kL}(x) . \tag{9.5}$$

Here $\chi_k(x)$ is the field of a Majorana neutrino with mass m_k, and U^L is a unitary mixing matrix. The field $\chi_k(x)$ satisfies the condition

$$C\overline{\chi_k}^T(x)=\xi_k\chi_k(x) , \tag{9.6}$$

where ξ_k are unphysical phase factors (see Secs. IV.C and V). We shall choose $\xi_k=\pm1$. It will be assumed also that the neutrino mixing (9.5) arises as a result of the diagonalization of the Majorana mass term [Eq. (4.22)]. Note that, as follows from Eqs. (4.24)–(4.30) and Eqs. (a) and (b) of footnote 13, the unitary matrix used to diagonalize the corresponding neutrino mass matrix differs from the mixing matrix U^L in (9.4) when $\xi_k\neq1$:

$$U_{lk}^L=U_{lk}e^{-i\alpha_k},\quad l=e,\mu,\tau,\ldots,\quad k=1,2,\ldots,n , \tag{9.7}$$

where

$$e^{-2i\alpha_k}=\xi_k , \tag{9.8}$$

and the matrix U does not depend on ξ_k.

Obviously, the $(\beta\beta)_{0\nu}$ decay (9.1) can be generated by the interaction (9.4): by exchanging a virtual Majorana neutrino, two neutrons in the initial nucleus can undergo transition into two protons and a pair of free electrons due to (9.4). In this way, a nontrivial contribution to the $(\beta\beta)_{0\nu}$-decay amplitude $A_{(\beta\beta)_{0\nu}}$ may arise in the second order of perturbation theory in the Fermi coupling constant G_F. A detailed derivation of the expression for the amplitude in this case is presented in Appendix B. We shall be interested in the dependence of the $(\beta\beta)_{0\nu}$-decay amplitude on the masses of Majorana neutrinos and on the elements of the lepton mixing matrix. Theses quantities enter into $A_{(\beta\beta)_{0\nu}}$ through an effective mass factor

$\langle m \rangle$, information about whose absolute value is obtained from the data on $(\beta\beta)_{0\nu}$ decay [see Appendix B, Eqs. (B17)–(B24)]:

$$A_{(\beta\beta)_{0\nu}} \sim \langle m \rangle = \sum_k^n (U_{ek}^L)^2 \xi_k m_k F(m_k, A) = \sum_k^n U_{ek}^2 m_k F(m_k, A) , \quad (9.9)$$

where

$$F(m_k, A) = \left\langle \frac{e^{-m_k r}}{r} \right\rangle \Big/ \left\langle \frac{1}{r} \right\rangle \quad (9.10)$$

and where we have made use of Eqs. (9.7) and (9.8) ($\xi_k^2 = 1$ due to the choice of ξ_k).[50]

The variable r is the distance between two neutrons within the nucleus undergoing $(\beta\beta)_{0\nu}$ decay: $r \le 2R$, $R = 1{,}2 A^{1/3}$ F being the radius of the initial nucleus. The average is with respect to the two-nucleon correlation function appropriate to a nucleus of atomic number A.

Let us note that the experiments on $(\beta\beta)_{0\nu}$ decay that are in progress or should be performed in the near future are expected to be sensitive to values of $|\langle m \rangle|$ down to ~ 1 eV (see Sec. XII and, for example, Caldwell, 1985). The existing lower bound on the $(\beta\beta)_{0\nu}$-decay lifetime of ^{76}Ge implies (Haxton, Stephenson, and Strottman, 1984)

$$|\langle m \rangle| < 6.3 \text{ eV} . \quad (9.11)$$

Since there are theoretical uncertainties in the calculation of the corresponding $(\beta\beta)_{0\nu}$-decay rate, the upper limit on $|\langle m \rangle|$ may actually be smaller approximately by a factor of 6 (Doi, Kotani, and Takasugi, 1985; Klapdor, 1986).

It follows from Eq. (9.9) that the $(\beta\beta)_{0\nu}$-decay amplitude vanishes in the limit of zero neutrino masses. As is shown in Appendix B, in the weak interaction theory with LH charged current and Majorana neutrinos, the lepton charge is conserved in this limit.

For $m_k \lesssim$ a few MeV, $k = 1, \ldots, n$, $F(m_k, A) \approx 1$ ($m_k r \ll 1$) and

$$\langle m \rangle \sim \sum_k (U_{ek}^L)^2 \xi_k m_k = \sum_k U_{ek}^2 m_k . \quad (9.12)$$

Let us consider this possibility in greater detail.

The first thing to be noted is that the value of $\langle m \rangle$ should be universal for all nuclei if neutrinos have masses smaller than a few MeV. Further, it is obvious from Eq. (9.12) [and (9.9)] that the effective neutrino mass $\langle m \rangle$ associated with the $(\beta\beta)_{0\nu}$ decay may differ substantially from the mass measured in experiments on direct neutrino mass determination [like the tritium β-decay experiment of Lubimov *et al.* (Boris *et al.*, 1984)]. For example, even if neutrinos with masses in the range of 15 eV exist and they are Majorana particles, we nevertheless may have $|\langle m \rangle| \ll 15$ eV, i.e., the $(\beta\beta)_{0\nu}$-decay rate may be suppressed to a level compatible with the observations. Such a suppression may be caused by a destructive interference between the contributions in the $(\beta\beta)_{0\nu}$-decay amplitude corresponding to the exchange of different Majorana neutrinos when the different terms in $\langle m \rangle$ tend to cancel each other. More specifically, this cancellation may occur due to the complexity of the mixing matrix elements in the case of *CP* nonconservation (Doi *et al.*, 1981b), as U_{ek}^2, $k = 1, \ldots, n$, are not necessarily positive then (see Sec. VII.A.2). If *CP* invariance holds, it may be a consequence of the existence of Majorana neutrinos with opposite *CP* parities (Wolfenstein, 1981a). Indeed, using the constraint (5.23) which is imposed on the mixing matrix U^L by the requirement of *CP* invariance, we find that the $(\beta\beta)_{0\nu}$-decay rate depends on the relative *CP* parities of the Majorana neutrinos:

$$\langle m \rangle = \frac{1}{i} \sum_k |U_{ek}|^2 \eta_{CP}(\chi_k) m_k , \quad (9.13)$$

where $\eta_{CP}^*(\chi_k)$ is the *CP* parity of the neutrino χ_k ($|U_{ek}^L| = |U_{ek}|$). This implies, in particular, that in the case of *CP* invariance the relative *CP* parities of Majorana neutrinos are in principle observable quantities. As is shown in Appendix A, $\eta_{CP}(\chi_k) = \pm i$ and, consequently, the terms in $\langle m \rangle$ corresponding to the exchange of Majorana neutrinos with opposite *CP* parities tend to mutually cancel. These cancellations may even be complete.

A remarkable example of the realization of the latter possibility is provided by the version of the Zee model considered in detail in Sec. VIII.B.2.b. In this version the model of Zee contains two almost mass-degenerate Majorana neutrinos, χ_2 and χ_3, which are much heavier than a third one χ_1: $|m_2 - m_3| = m_1 \lesssim (m_\mu^2/m_\tau^2) m_2 \ll m_{2,3}$. In particular, values of $m_{2,3}$ as large as 10–20 eV are possible. Further, *CP* is conserved and $\eta_{CP}(\chi_2) = i$, while $\eta_{CP}(\chi_3) = -i$. Using Eqs. (8.22), (8.24), and (9.13), it is not difficult to convince oneself that the specific form of the lepton mixing matrix, together with the indicated relation between the neutrino masses and the values of the *CP* parities, leads to exact mutual compensation of the three terms in $\langle m \rangle$ (Wolfenstein, 1981a):

$$\langle m \rangle = 0 ,$$

for the model of Zee. Tiny contributions from diagrams with exchange of virtual charged Higgs bosons make $A_{(\beta\beta)_{0\nu}}$ different from zero but very much suppressed in this theory (Petcov, 1982b). Similar cancellations were shown to take place in most of the SO(10) models as well (Chang and Pal, 1982), although they are not as effective as in the case considered above.

The suppression of the $(\beta\beta)_{0\nu}$-decay rate is simply related to the form of the Majorana mass term (4.22) of the neutrinos. Indeed, it follows from Eqs. (9.12) and (4.24) that (Wolfenstein, 1981a)

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[50]It is not difficult to prove that the effective mass $\langle m \rangle$ [and, consequently, the $(\beta\beta)_{0\nu}$-decay rate] does not depend on the unphysical phase factors in the Majorana conditions (9.6) for any choice of ξ_k.

$$\langle m \rangle = M_{ee} \ . \tag{9.14}$$

Consequently $(\beta\beta)_{0\nu}$ decay would be strongly suppressed or forbidden, respectively, if the term $\overline{(\nu_{eL})^c}\nu_{eL}$ were not present in the leading approximation or at all in the neutrino mass Lagrangian. The first possibility takes place in the model of Zee considered, wherein $M_{ee}=0$ in the one-loop approximation [see Eq. (8.15)] and nonzero contributions to M_{ee} arise only at the two-loop level.

The mass parameter $|M_{ee}|$ is not bounded to be very small and, consequently, the $(\beta\beta)_{0\nu}$-decay rate to be particularly suppressed in a theory with pseudo-Dirac neutrinos coupled to the electron (Petcov, 1982a). For example, in the scheme with a pseudo-Dirac neutrino discussed in Sec. VI.C, $M_{ee}(\equiv m_{ee})$ is a free parameter, and from Eqs. (6.16), (6.29)—(6.37), and (9.12) we get

$$\langle m \rangle_{\rm PD} = m_{ee} = m \sin 2\theta' \ ,$$

where m is the mass of the pseudo-Dirac neutrino and θ' is a mixing angle. Note that $A_{(\beta\beta)_{0\nu}}$ is proportional to the mass of the pseudo-Dirac neutrino and not to the Majorana correction to the mass. This is a general property of the contributions in $A_{(\beta\beta)_{0\nu}}$ arising due to exchange of pseudo-Dirac neutrinos for which the weak Majorana mass corrections arise at one-loop level.

It was shown in Secs. VI.C and VI.D that a massive Dirac neutrino is equivalent to two mass-degenerate Majorana neutrinos that possess opposite CP parities. For example, in the ZKM scheme,

$$\nu_{eL}=\nu_L=\frac{1}{\sqrt{2}}(\chi_{1L}-\chi_{2L}), \quad m_1=m_2=m \ ,$$

$$\eta_{CP}(\chi_1)=i, \quad \eta_{CP}(\chi_2)=-i \ ,$$

where $\nu(x)$ is the field of a Dirac neutrino with mass m. The absence of $(\beta\beta)_{0\nu}$ decay in the case when ν_e is a massive Dirac particle can then be viewed as a consequence of exact cancellation of the contributions in $\langle m \rangle$ of two Majorana neutrinos with equal masses having opposite CP parities:

$$\langle m \rangle_D = \tfrac{1}{2}(m_1-m_2)=0 \ .$$

Let us suppose next that some of the Majorana neutrinos possess masses considerably exceeding a few MeV. In this case one has

$$\langle m \rangle = \sum_{k_0} U_{ek_0}^2 m_{k_0} + \sum_{k'} U_{ek'}^2 m_{k'} F(m_{k'},A) \ , \tag{9.15}$$

and, if CP invariance holds,

$$\langle m \rangle = \frac{1}{i}\sum_{k_0} |U_{ek_0}|^2 \eta_{CP}(\chi_{k_0}) m_{k_0} + \frac{1}{i}\sum_{k'} |U_{ek'}|^2 \eta_{CP}(\chi_{k'}) m_{k'} F(m_{k'},A) \ , \tag{9.16}$$

where the index k_0 labels the "light" neutrinos ($m_{k_0} \lesssim$ a few MeV, $k_0=1,2,\ldots,n_0$) and the index k' numbers the "heavy" neutrinos[51] ($m_{k'} \gg$ a few MeV, $k'=n_0+1,\ldots,n$). We shall now present a brief qualitative discussion of the characteristic features of the terms in $\langle m \rangle$ arising due to exchange of heavy Majorana neutrinos (Halprin, Petcov, and Rosen, 1983).

For physically sensible correlation functions, $m_{k'}F(m_{k'},A)$ becomes a monotonically decreasing function of $m_{k'}$ for sufficiently large values of $m_{k'}$. If, say, we use the correlation function suggested by Halprin *et al.* (1976), then for $m_{k'} \gtrsim 4$—5 GeV, the contribution due to exchange of $\chi_{k'}$ in the $(\beta\beta)_{0\nu}$-decay amplitude is compatible with the experimental limit (9.11). However, because of the possibility of cancellations between different terms in the expression for $\langle m \rangle$,[52] the existence of heavy neutral Majorana leptons that couple with appreciable strength to the electron and have masses below 4 GeV cannot be ruled out by the experimental data on $(\beta\beta)_{0\nu}$ decay.

It is important to note also that the correlation function that determines the form of the function $F(m_{k'},A)$ varies with the nuclear species and typically depends strongly upon the nuclear radius R. Consequently, if the exchange of heavy Majorana leptons (particles) plays an important role in the $(\beta\beta)_{0\nu}$ decay, the effective mass $\langle m \rangle$ should vary from one nucleus to another, and this variation may be quite dramatic. This is in sharp contrast to the case of light neutrinos [see Eq. (9.12)], for which $\langle m \rangle$ is independent of the type of decaying nucleus.

In order to be more specific, we shall consider one example with a relatively simple correlation function that contains a hard-core repulsion between two nucleons in a nucleus (Halprin *et al.*, 1976):

$$\rho(r)=\{\tfrac{4}{3}\pi[(2R)^3-r_c^3)]\}^{-1}\theta(r-r_c)\theta(2R-r) \ , \tag{9.17}$$

where $r_c=0.5$ F is the hard-core radius. From Eqs. (9.10) and (9.17) one obtains

$$F(m_{k'},A)=\tfrac{1}{2}(m_{k'}R)^{-2}[(1+m_{k'}r_c)e^{-m_{k'}r_c} - (1+2m_{k'}R)e^{-2m_{k'}R}] \ . \tag{9.18}$$

Since for all nuclei of interest ($A \geq 48$) the hard-core radius is much smaller than the nuclear radius (i.e., $r_c \ll R = 1{,}2A^{1/3}$ F), $F(m_{k'},A)$ and, consequently, $\langle m \rangle$ vary with the nuclear radius, in this case as R^{-2}, and therefore with the atomic number as $A^{-2/3}$. The effective mass $\langle m \rangle$ exhibits a stronger dependence on the atomic number of the decaying nucleus if, for instance, $F(m_{k'},A)$ is evaluated with the two-nucleon correlation

[51]We shall also use the more appropriate term "heavy neutral leptons" for such neutrinos.

[52]The possibility of cancellation between the contributions due to exchange of light and of heavy Majorana neutrinos has been studied in detail by Halprin, Petcov, and Rosen (1983), Leung and Petcov (1984), and Langacker, Sathiapalan, and Steigman (1986).

function used by Doi *et al.* (1981a), $\rho(r)\sim\delta(r-R)$. In this case $F(m_{k'},A)=1.67e^{-m_{k'}R}$. These considerations suggest that studies of the $(\beta\beta)_{0\nu}$ decay of several nuclei should be simultaneously performed.

We should like to conclude with several remarks concerning the dependence of the $(\beta\beta)_{0\nu}$-decay amplitude on the Majorana neutrino masses in the case when RH currents are present in the weak β-decay Hamiltonian. The most general weak β-decay Hamiltonian with RH currents has the following form in the case of neutrino mixing:[53]

$$H_W^\beta=2\frac{G_F}{\sqrt{2}}[\overline{e_L}\gamma_\alpha\nu_{eL}(j_{L\alpha}+\kappa j_{R\alpha}) + \overline{e_R}\gamma_\alpha\nu_{eR}(\eta j_{L\alpha}+\lambda j_{R\alpha})]+\text{H.c.}\ , \tag{9.19}$$

$$\nu_{eL}=\sum_k U_{ek}^L\chi_{kL},\quad \nu_{eR}=\sum_k U_{ek}^R\chi_{kR}\ , \tag{9.20a}$$

$$C\overline{\chi_k}^T(x)=\xi_k\chi_k(x),\quad \xi_k=\pm 1\ . \tag{9.20b}$$

Here $(\overline{e_R}\gamma_\alpha\nu_{eR})$ and $j_{R\alpha}$ are RH charged lepton and hadron currents, respectively, κ, η, and λ are constant parameters, and U^L and U^R are mixing matrices. The existing experimental limits on the weak interaction with RH currents imply $|\kappa|,|\eta|,|\lambda|\ll 1$ (see, for example, Carr *et al.*, 1983; Doi, Kotani, and Takasugi, 1984).

The amplitude of the $(\beta\beta)_{0\nu}$ decay generated by the interaction (9.19) in second order of perturbation theory in G_F can be represented as

$$A_{(\beta\beta)_{0\nu}}=A^{LL}_{(\beta\beta)_{0\nu}}+A^{LR}_{(\beta\beta)_{0\nu}}+A^{RR}_{(\beta\beta)_{0\nu}}\ , \tag{9.21}$$

where $A^{LL(RR)}_{(\beta\beta)_{0\nu}}$ is the matrix element of the operator in the S matrix containing the product of two LH (RH) lepton currents

$$\overline{e_{L(R)}}(x_1)\gamma_\alpha\nu_{eL(R)}(x_1)\overline{e_{L(R)}}(x_2)\gamma_\beta\nu_{eL(R)}(x_2)\ ,$$

while $A^{LR}_{(\beta\beta)_{0\nu}}$ is the matrix element of the operator that includes the product of the LH and the RH lepton currents[54]

$$\overline{e_L}(x_1)\gamma_\alpha\nu_{eL}(x_1)\overline{e_R}(x_2)\gamma_\beta\nu_{eR}(x_2)\ .$$

The dependence of the terms $A^{LL}_{(\beta\beta)_{0\nu}}$, $A^{LR}_{(\beta\beta)_{0\nu}}$, and $A^{RR}_{(\beta\beta)_{0\nu}}$ on the neutrino masses m_k can be found in a way analogous to that used in Appendix B to derive the dependence on m_k of the $(\beta\beta)_{0\nu}$-decay amplitude in the case when the weak β-decay Hamiltonian is given by Eqs. (9.4) and (9.5) [see Eqs. (B5)—(B7)]. It is confined to the following factors in the matrix elements:

$$A^{LL(RR)}_{(\beta\beta)_{0\nu}}:\sum_k\frac{(U_{ek}^{L(R)})^2\xi_k m_k}{q^2+m_k^2}\ , \tag{9.22}$$

$$A^{LR}_{(\beta\beta)_{0\nu}}:\sum_k\hat{q}\frac{U_{ek}^L U_{ek}^R\xi_k}{q^2+m_k^2}\ , \tag{9.23}$$

where $(q^2+m_k^2)^{-1}$ is the propagator of the virtual neutrino χ_k and the form of the dependence is determined by the chiral structure of the corresponding lepton current operators.[55] For instance, the dependence on m_k in $A^{LR}_{(\beta\beta)_{0\nu}}$ appears due to the following neutrino factor in the corresponding S-matrix operator:

$$\nu_{eL}(x_1)\nu_{eR}^T(x_2)=-\tfrac{1}{2}(1+\gamma_5)\sum_k U_{ek}^L U_{ek}^R\xi_k S_k(x_1-x_2)\tfrac{1}{2}(1-\gamma_5)C\ ,$$

where $S_k(x_1-x_2)$ is the propagator of the Majorana neutrino χ_k [see Eq. (A8)] and we have made use of Eqs. (9.20a) and (A9). This equality leads to Eq. (9.23), since

$$\tfrac{1}{2}(1+\gamma_5)S_k(x_1-x_2)\tfrac{1}{2}(1-\gamma_5)=-\frac{1}{(2\pi)^4}\int\frac{\hat{q}}{q^2+m_k^2}e^{iq(x_1-x_2)}dq\ \tfrac{1}{2}(1-\gamma_5)\ .$$

Obviously, in the limit of zero neutrino mass $A^{LL(RR)}_{(\beta\beta)_{0\nu}}=0$. Let us consider the behavior of $A^{LR}_{(\beta\beta)_{0\nu}}$ in this limit.

If the neutrino mixing (9.20) is introduced phenomenologically (Enz, 1957; Pauli, 1957; see also Primakoff and Rosen, 1981), the mixing matrices U^L and U^R need not be related. [We may have, for example, $\nu_{eL}(x)=\chi_L(x)$ and $\nu_{eR}(x)=\chi_R(x)$, where $\chi(x)$ is the field of a Majorana

[53]We have assumed for simplicity that mirror leptons do not exist. It can be shown, however, that the results derived in the remaining part of the present section are valid also in the case when mirror leptons do exist. For a discussion of the form of the weak β-decay Hamiltonian in the theories with RH currents and mirror leptons see, Doi, Kotani, and Takasugi (1985).

[54]One can derive the general expression for the amplitude $A_{(\beta\beta)_{0\nu}}$ in the case under consideration using the methods exploited in Appendix B.

[55]Let us note that the dependence on m_k of the nuclear matrix elements appearing in $A^{LL}_{(\beta\beta)_{0\nu}}$ and $A^{RR}_{(\beta\beta)_{0\nu}}$ is given by the function $F(m_k,A)$ defined in Eq. (9.10),

$$A^{LL(RR)}_{(\beta\beta)_{0\nu}}\sim\langle m\rangle_{LL(RR)}=\sum_k(U_{ek}^{L(R)})^2\xi_k m_k F(m_k,A)$$

(see, Doi, Kotani, and Takasugi, 1984). Consequently, the results for the $(\beta\beta)_{0\nu}$-decay amplitude obtained in the present section for the case when the weak β-decay Hamiltonian is given by Eqs. (9.4) and (9.5) are valid for the amplitudes $A^{LL}_{(\beta\beta)_{0\nu}}$ and $A^{RR}_{(\beta\beta)_{0\nu}}$ as well.

neutrino.] It follows then from Eq. (9.23) that even if $m_k=0$, $A^{LR}_{(\beta\beta)_{0\nu}}$ may be different from zero. Thus one arrives at the conclusion that if RH currents exist, $(\beta\beta)_{0\nu}$ decay can take place, even in the case of massless Majorana neutrinos (Enz, 1957; Primakoff and Rosen, 1969). Although true in the framework of the phenomenological approach, this conclusion is not valid in the gauge theories of electroweak interactions, and we are going to show that next.

Let us assume that the effective weak β-decay Hamiltonian (9.19) and the neutrino mixing (9.20) arise in a gauge theory. As has been discussed in Sec. VIII, neutrino mixing results in the gauge theories from the diagonalization of the neutrino mass term. Since RH neutrino fields $[\nu_{eR}(x),\dots]$ are present in the theory, the mixing (9.20) can be generated only by a mass term of Dirac-Majorana type [see Sec. IV.A, Eq. (4.9)]. In this case the matrices U^L and U^R are not independent and one has

$$\sum_k U^L_{ek} U^R_{ek} \xi_k = 0 \; . \tag{9.24}$$

Indeed, it follows from Eqs. (4.35)—(4.45) and Eqs. (a) and (b) of footnote 13 that

$$\nu_{eL} = \sum_k U_{ek} e^{-i\alpha_k} \chi_{kL} \; , \tag{9.25}$$

$$(\nu_{eR})^c = C\,\overline{\nu_{eR}}^T = \sum_k U_{\bar{e}k} e^{-i\alpha_k} \chi_{kL} \tag{9.26}$$

$(e^{-2i\alpha_k}=\xi_k)$. Here U_{lk} and $U_{\bar{l}k}$ $(l=e,\mu,\tau,\dots)$ are elements of a unitary matrix[56] and, therefore, satisfy the unitarity conditions

$$\sum_k U_{lk} U^*_{l'k} = \delta_{ll'} \; , \tag{9.27a}$$

$$\sum_k U_{\bar{l}k} U^*_{\bar{l}'k} = \delta_{\bar{l}\bar{l}'} \; , \tag{9.27b}$$

$$\sum_k U_{lk} U^*_{\bar{l}'k} = 0 \; , \tag{9.27c}$$

$$\sum_l U^*_{lk} U_{lk'} + \sum_{\bar{l}} U^*_{\bar{l}k} U_{\bar{l}k'} = \delta_{kk'} \; . \tag{9.27d}$$

From Eq. (9.26) we obtain, using the equality $C^{-1}\chi_{kL}(x) = \xi^*_k \overline{\chi_{kR}}^T(x)$,

$$\nu_{eR} = \sum_k U^*_{\bar{e}k} e^{i\alpha_k} \xi_k \chi_{kR} \; . \tag{9.28}$$

By comparing Eqs. (9.20) with (9.25) and (9.28) one finds that

$$U^L_{ek} = U_{ek} e^{-i\alpha_k}, \quad U^R_{ek} = U^*_{\bar{e}k} e^{i\alpha_k} \xi_k \; . \tag{9.29}$$

With the help of Eq. (9.29) we get

$$\sum_k U^L_{ek} U^R_{ek} \xi_k = \sum_k U_{ek} U^*_{\bar{e}k} \; , \tag{9.30}$$

which, on account of (9.27c) leads to (9.24).[57] Note that since U_{lk} and $U_{\bar{l}k}$ are independent of ξ_k, Eq. (9.30) implies that $A^{LR}_{(\beta\beta)_{0\nu}}$ does not depend on ξ_k.

It follows from Eqs. (9.23) and (9.24) that, in the gauge theories with RH currents and neutrino mixing (9.20), (i) $A^{LR}_{(\beta\beta)_{0\nu}}=0$ if $m_k=0$, and (ii) for $m_k\neq 0$ the amplitude $A^{LR}_{(\beta\beta)_{0\nu}}$ is suppressed by a mechanism analogous to the mechanism of Glashow, Iliopoulos, and Maiani[58] (see, for example, Doi, Kotani, and Takasugi, 1984; Enqvist, Maalampi, and Mursula, 1984). Our considerations show that, if the electroweak interaction is described by a gauge theory and the $(\beta\beta)_{0\nu}$ decay can be generated only by the mechanism discussed, the observation of the $(\beta\beta)_{0\nu}$ decay would imply the existence of massive Majorana neutrinos, irrespective of whether RH currents exist or not (Kayser, Petcov, and Rosen, 1986).[59] Moreover, at least one Majorana neutrino should have a mass exceeding the value of the parameter $|\langle m\rangle|$ which would be inferred from the corresponding data (Kayser, 1986; Kayser, Petcov, and Rosen, 1986).

X. ELECTROMAGNETIC PROPERTIES OF MASSIVE NEUTRINOS

There are at least two circumstances that contribute to the considerable interest in the electromagnetic properties of massive neutrinos that we shall review next. First, the physical differences between the massive Dirac and the massive Majorana neutrinos (particles) are very clearly exhibited in their electromagnetic properties. And second, numerous studies have revealed that even extremely weak neutrino-photon effective couplings may lead, if they exist, to important and directly observable astrophysical and cosmological effects (see, for example, Cisneros, 1980; Fujikawa and Shrock, 1980; Turner, 1981). In particular,

[56] U_{lk} and $U_{\bar{l}k}$ are elements of two block matrices forming the unitary matrix, with the help of which the corresponding neutrino mass matrix is diagonalized (see Sec. IV.D). Obviously, U_{lk} and $U_{\bar{l}k}$ are independent of ξ_k.

[57] It can be shown that Eq. (9.24) also follows—in the gauge theories with RH currents and neutrino mixing given by (9.20)—from the general requirement of correct (i.e., compatible with the unitarity of the S matrix) high-energy behavior of the scattering amplitudes in these theories (Kayser, Petcov, and Rosen, 1986).

[58] Let us add that in the case of CP invariance $A^{LR}_{(\beta\beta)_{0\nu}}$ in contrast to $A^{LL}_{(\beta\beta)_{0\nu}}$ and $A^{RR}_{(\beta\beta)_{0\nu}}$ is independent of the CP parities of the Majorana neutrinos (Kayser, Petcov, and Rosen, 1986).

[59] For different analyses of the relation between the existence of the $(\beta\beta)_{0\nu}$ decay and the existence of massive Majorana neutrinos which lead to a similar conclusion see Schechter and Valle (1982b), Nieves (1984), and Takasugi (1984).

as was shown by De Rújula and Glashow (1980), it may be feasible to detect experimentally photon fluxes from the radiative decays of neutrinos in galactic halos and/or of relic neutrinos with masses greater than a few eV, even if the corresponding neutrino lifetimes exceed considerably the age of the universe ($\sim 10^{10}$ yr). Searches for photon fluxes from neutrino decays are being performed at present (Auriemma *et al.*, 1985) and are likely to continue with improving sensitivity in the future (Shipman and Cowsik, 1981).

We shall begin with a detailed analysis of the general properties of the electromagnetic current operator's matrix element $J_\alpha^{\text{em}}(x)$ between one-particle massive neutrino states, following from the Hermiticity, *CPT*-, and *CP*-transformation properties of $J_\alpha^{\text{em}}(x)$. The electromagnetic properties of massive Dirac and massive Majorana neutrinos will be compared and the basic differences between them will be outlined. Then examples of predictions of the gauge theories for the values of neutrino radiative lifetimes and the magnetic moments of Dirac neutrinos, as well as the existing limits on these quantities, will be briefly considered.

A. General analysis

The amplitude of the neutrino radiative decay

$$\nu_i \to \nu_j + \gamma \ , \tag{10.1}$$

where ν_i and ν_j are two Dirac neutrinos with definite masses m_i and m_j ($m_i > m_j$), can be written as

$$A(\nu_i \to \nu_j + \gamma) = ie\langle \nu_j(r_j,p_j) \,|\, J_\alpha^{\text{em}}(0) \,|\, \nu_i(r_i,p_i)\rangle \times \frac{1}{(2\pi)^{3/2}} \frac{\varepsilon_\alpha^\lambda(q)}{\sqrt{2q_0}} \times (2\pi)^4 \delta(p_i - p_j - q) \ . \tag{10.2}$$

Here $|\nu_j(r_j p_j)\rangle$ is the state vector of the neutrino ν_j, with four-momentum $p_j = (\mathbf{p}_j, ip_{j0})$ and projection r_j of the spin on the momentum $\mathbf{p}_j$; $J_\alpha^{\text{em}}(0)$ is the operator of the electromagnetic current in the Heisenberg representation at $x=0$; $\varepsilon_\alpha^\lambda(q)$ and q are the polarization vector and the four-momentum of the photon, etc. The matrix element of the electromagnetic current operator has the standard form

$$\langle \nu_j(r_j,p_j) \,|\, J_\alpha^{\text{em}}(0) \,|\, \nu_i(r_i,p_i)\rangle = iN_{ji} \overline{u^{r_j}}(p_j) \Gamma_\alpha(p_j,p_i) u^{r_i}(p_i) \ , \tag{10.3}$$

where $N_{ji} = [(2\pi)^3 \sqrt{4p_{i0}p_{j0}}]^{-1}$ is a normalization factor, $u^{r_i}(p_i)$ and $\overline{u^{r_j}}(p_j) = (u^{r_j}(p_j))^\dagger \gamma_4$ are Dirac spinors describing the free initial and final neutrinos, respectively, and $\Gamma_\alpha(p_j,p_i)$ is a vertex function whose explicit form depends on the dynamics governing the $\nu_i \to \nu_j$ radiative transition. Lorentz invariance and current conservation imply that for $q^2 \neq 0$ the function $\Gamma_\alpha(p_j,p_i)$ has the following general structure:

$$\Gamma_\alpha(p_j,p_i) = \sigma_{\alpha\beta} q_\beta [F_{ji}^V(q^2) + \gamma_5 F_{ji}^A(q^2)] + (\gamma_\alpha q^2 - \hat{q} q_\alpha)[G_{ji}^V(q^2) + \gamma_5 G_{ji}^A(q^2)] \ . \tag{10.4}$$

Here $\hat{q} = q_\alpha \gamma_\alpha$, and $F_{ji}^{V,A}(q^2)$ and $G_{ji}^{V,A}(q^2)$ are four independent form factors characterizing the process $\nu_i \to \nu_j +$ "γ", where "γ" is a virtual photon. The neutrino radiative decay amplitude $A(\nu_i \to \nu_j + \gamma)$ is determined by the transition moments $F_{ji}^{V,A} = F_{ji}^{V,A}(q^2)\,|_{q^2=0}$ only.

It follows from the Hermiticity of the electromagnetic current operator

$$(J_\alpha^{\text{em}}(x))^\dagger = \eta_\alpha J_\alpha^{\text{em}}(x), \quad \eta_\alpha = \begin{cases} 1, & \alpha = 1,2,3 \\ -1, & \alpha = 4 \ , \end{cases} \tag{10.5}$$

that

$$\begin{aligned} (F_{ji}^{V(A)}(q^2))^* &= \underset{(-)}{+} F_{ij}^{V(A)}(q^2) \ , \\ (G_{ji}^{V,A}(q^2))^* &= G_{ij}^{V,A}(q^2) \ . \end{aligned} \tag{10.6}$$

Indeed, Eq. (10.5) implies

$$(\langle \nu_j(r_j,p_j) \,|\, J_\alpha^{\text{em}}(0) \,|\, \nu_i(r_i,p_i)\rangle)^* = \eta_\alpha \langle \nu_i(r_i,p_i) \,|\, J_\alpha^{\text{em}}(0) \,|\, \nu_j(r_j,p_j)\rangle \ . \tag{10.7}$$

Using Eq. (10.3), we obtain from (10.7)

$$\gamma_4 \Gamma_\alpha^\dagger(p_j,p_i) \gamma_4 = -\eta_\alpha \Gamma_\alpha(p_i,p_j) \ . \tag{10.8}$$

Equations (10.6) can be easily derived now from (10.8), taking into account (10.4) and that

$$\begin{aligned} &\gamma_4 \sigma_{\alpha\beta} \gamma_4 = \eta_\alpha \eta_\beta \sigma_{\alpha\beta}, \quad \gamma_4 \gamma_\alpha \gamma_4 = -\eta_\alpha \gamma_\alpha \ , \\ &\gamma_4 \gamma_5 \gamma_4 = -\gamma_5 \ . \end{aligned} \tag{10.9}$$

Under the *CPT* transformation (in the case of a *CPT*-invariant S matrix), we have

$$J_\alpha^{\text{em}}(0) \overset{CPT}{\to} U_{CPT} J_\alpha^{\text{em}}(0) U_{CPT}^\dagger = -(J_\alpha^{\text{em}}(0))^\dagger \ , \tag{10.10}$$

where U_{CPT} is the antiunitary *CPT* conjugation operator. This implies

$$_{CPT}\langle \nu_j(r_j,p_j) \,|\, J_\alpha^{\text{em}}(0) \,|\, \nu_i(r_i,p_i)\rangle_{CPT} = -\langle \nu_i(r_i,p_i) \,|\, J_\alpha^{\text{em}}(0) \,|\, \nu_j(r_j,p_j)\rangle \ . \tag{10.11}$$

Here

$$|\nu_k(r_k,p_k)\rangle_{CPT} = U_{CPT} \,|\, \nu_k(r_k,p_k)\rangle = \eta_{CPT}(\nu_k) \,|\, \overline{\nu}_k(-r_k,p_k)\rangle, \quad k = i,j \ , \tag{10.12}$$

where $|\overline{\nu}_k(-r_k,p_k)\rangle$ is the state vector of the free antineutrino $\overline{\nu}_k$ with four-momentum p_k and projection $(-r_k)$ of the spin on the momentum $\mathbf{p}_k$ and $\eta_{CPT}(\nu_k)$ is an unphysical phase factor.[60] One has

$$_{CPT}\langle \nu_j(r_j,p_j) \mid J_\alpha^{\rm em}(0) \mid \nu_i(r_i,p_i) \rangle_{CPT} = iN_{ji}\overline{u_{CPT}^{r_i}}(-p_i)\overline{\Gamma}_\alpha(p_j,p_i)u_{CPT}^{r_j}(-p_j) . \quad (10.13)$$

Here

$$u_{CPT}^{r_k}(-p_k)=\gamma_4 V_T^{-1}(C\overline{u^{r_k}}^T(p_k))^*, \quad k=i,j , \quad (10.14)$$

is the CPT conjugate of the spinor $u^{r_k}(p_k)$, where V_T is the time-reversal matrix defined by

$$V_T^{-1}\gamma_\alpha^* V_T=\gamma_\alpha, \quad V_T^T=-V_T, \quad V_T^\dagger=V_T^{-1} , \quad (10.15)$$

and

$$\overline{\Gamma}_\alpha(p_j,p_i)=\sigma_{\alpha\beta}q_\beta[\overline{F}_{ji}^V(q^2)+\gamma_5\overline{F}_{ji}^A(q^2)] +(\gamma_\alpha q^2-\hat{q}q_\alpha)[\overline{G}_{ji}^V(q^2)+\gamma_5\overline{G}_{ji}^A(q^2)] . \quad (10.16)$$

The form factors $\overline{F}_{ji}^{V,A}(q^2)$ and $\overline{G}_{ji}^{V,A}(q^2)$ characterize the radiative transition $\overline{\nu}_i \to \overline{\nu}_j +$ "γ", where $\overline{\nu}_i$ and $\overline{\nu}_j$ denote antineutrinos. From Eq. (10.11) we get, using Eqs. (10.3), (10.13), and (10.14), as well as the properties of the matrices V_T and C,

$$CV_T\overline{\Gamma}_\alpha(p_j,p_i)V_T^{-1}C^{-1}=\Gamma_\alpha(p_i,p_j) . \quad (10.17)$$

Inserting Eqs. (10.4) and (10.16) in (10.17) and exploiting Eqs. (4.3) and (10.15), we find that as a consequence of CPT invariance

$$\overline{F}_{ji}^{V,A}(q^2)=-F_{ij}^{V,A}(q^2) , \quad \overline{G}_{ji}^{V,A}(q^2)=-G_{ij}^{V,A}(q^2) . \quad (10.18)$$

Further, if CP invariance holds, then under the CP transformation

$$J_\alpha^{\rm em}(0) \xrightarrow{CP} U_{CP}J_\alpha^{\rm em}(0)U_{CP}^\dagger=\eta_\alpha J_\alpha^{\rm em}(0) , \quad (10.19)$$

and, consequently,

$$_{CP}\langle \nu_j(r_j,p_j) \mid J_\alpha^{\rm em}(0) \mid \nu_i(r_i,p_i) \rangle_{CP} =\eta_\alpha\langle \nu_j(r_j,p_j) \mid J_\alpha^{\rm em}(0) \mid \nu_i(r_i,p_i) \rangle . \quad (10.20)$$

Here

$$\mid \nu_k(r_k,p_k) \rangle_{CP}=U_{CP} \mid \nu_k(r_k,p_k) \rangle = \mid \overline{\nu}_k(-r_k,p_k') \rangle , \quad p_k'=(-\mathbf{p}_k,ip_{k0}), \quad k=i,j \quad (10.21)$$

is a CP conjugate state describing a free antineutrino $\overline{\nu}_k$ with four-momentum p_k' and projection $(-r_k)$ of the spin on the momentum $(-\mathbf{p}_k)$. We have

$$_{CP}\langle \nu_j(r_j,p_j) \mid J_\alpha^{\rm em}(0) \mid \nu_i(r_i,p_i) \rangle_{CP} =iN_{ji}\overline{u_{CP}^{r_i}}(-p_i')\overline{\Gamma}_\alpha(p_j',p_i')u_{CP}^{r_j}(-p_j') , \quad (10.22)$$

where

$$u_{CP}^{r_k}(-p_k')=\gamma_4 C\overline{u^{r_k}}^T(p_k), \quad k=i,j . \quad (10.23)$$

With the help of Eqs. (10.22), (10.23), and (10.3) we get from (10.20)

$$\gamma_4 C\overline{\Gamma}_\alpha^T(p_j',p_i')C^{-1}\gamma_4=-\eta_\alpha\Gamma_\alpha(p_j,p_i) . \quad (10.24)$$

Finally, by using Eqs. (10.4), (10.16), and (4.3), it is not difficult to derive from (10.24) the relations between the neutrino and the antineutrino electromagnetic transition form factors which should hold in the case of CP invariance:

$$\overline{F}_{ji}^{V(A)}(q^2)=\underset{(+)}{-}F_{ji}^{V(A)}(q^2) , \quad \overline{G}_{ji}^{V,A}(q^2)=-G_{ji}^{V,A}(q^2) . \quad (10.25)$$

In the diagonal case, when in Eq. (10.1) $j=i$, eF_{ii}^V and ieF_{ii}^A $[F_{ii}^{V,A}=F_{ii}^{V,A}(0)]$ are just the induced magnetic and electric dipole moments of the Dirac neutrino:[61]

$$\mu_i=eF_{ii}^V, \quad d_i=ieF_{ii}^A . \quad (10.26)$$

It follows from Eqs. (10.18) and (10.25) that $F_{ii}^A(q^2)=0$. Therefore, if CPT invariant holds, the electric dipole moment of a Dirac neutrino, like the electric dipole moment of the neutron, can be different from zero only if CP invariance does not hold.

The amplitude $A(\chi_i \to \chi_j+\gamma)$ of the radiative decay of a Majorana neutrino χ_i into a lighter Majorana neutrino χ_j $(m_i > m_j)$ and a photon

$$\chi_i \to \chi_j+\gamma \quad (10.27)$$

has the same general structure as that of the decay (10.1) involving Dirac neutrinos [Eqs. (10.2)—(10.4)]. The matrix element of the electromagnetic current operator $J_\alpha^{\rm em}(0)$ entering into the expression for $A(\chi_i \to \chi_j+\gamma)$ is characterized in the general case $(q^2 \neq 0)$ by four independent form factors:

$$\langle \chi_j(r_j,p_j) \mid J_\alpha^{\rm em}(0) \mid \chi_i(r_i,p_i) \rangle =iN_{ji}\overline{u^{r_j}}(p_j)\widetilde{\Gamma}_\alpha(p_j,p_i)u^{r_i}(p_i) , \quad (10.28)$$

$$\widetilde{\Gamma}_\alpha(p_j,p_i)=\sigma_{\alpha\beta}q_\beta[\widetilde{F}_{ji}^V(q^2)+\gamma_5\widetilde{F}_{ji}^A(q^2)] +(\gamma_\alpha q^2-\hat{q}q_\alpha)[\widetilde{G}_{ji}^V(q^2)+\gamma_5\widetilde{G}_{ji}^A(q^2)] , \quad (10.29)$$

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[60]The unphysical phase factors associated with the CPT and CP transformations of the massive neutrino states are chosen further in this section to be equal to unity (for a discussion in which these phase factors are kept arbitrary (see, for example, Kayser and Goldhaber, 1983).

[61]In the nonrelativistic limit the amplitude (10.2) corresponds to the interaction Hamiltonian

$$-e(F_{ii}^V\boldsymbol{\sigma}\cdot\mathbf{H}+iF_{ii}^A\boldsymbol{\sigma}\cdot\mathbf{E}) ,$$

where $\boldsymbol{\sigma}$ are the Pauli matrices and $\mathbf{H}$ and $\mathbf{E}$ denote magnetic and electric fields, respectively.

where the notations are obvious. The Hermiticity of $J_\alpha^{\rm em}(x)$ [Eq. (10.5)] leads to constraints on the form factors $\widetilde{F}_{ji}^{V,A}(q^2)$ and $\widetilde{G}_{ji}^{V,A}(q^2)$ similar to Eq. (10.6):

$$(\widetilde{F}_{ji}^{V(A)}(q^2))^* = {}_{(-)}^{+}\widetilde{F}_{ij}^{V(A)}(q^2) \,, \quad (\widetilde{G}_{ji}^{V,A}(q^2))^* = \widetilde{G}_{ij}^{V,A}(q^2) \,. \tag{10.30}$$

However, since the Majorana particles possess no distinctive antiparticles, the implications of *CPT* invariance and of *CP* invariance for the electromagnetic form factors of Majorana neutrinos differ drastically from those for the form factors of Dirac neutrinos (Schechter and Valle, 1981b; Wolfenstein, 1981a; Kayser, 1982; Nieves, 1982; Pal and Wolfenstein, 1982; Shrock, 1982b; Kayser and Goldhaber, 1983; Kayser, 1984).

Consider first the constraints on the form factors $\widetilde{F}_{ji}^{V,A}(q^2)$ and $\widetilde{G}_{ji}^{V,A}(q^2)$ arising as a consequence of *CPT* invariance. It follows from Eq. (10.10) that in the case of *CPT* invariance

$$_{CPT}\langle \chi_i(r_i,p_i) \,|\, J_\alpha^{\rm em}(0) \,|\, \chi_j(r_j,p_j)\rangle_{CPT} = -\langle \chi_j(r_j,p_j) \,|\, J_\alpha^{\rm em}(0) \,|\, \chi_i(r_i,p_i)\rangle \,. \tag{10.31}$$

Under *CPT* the state vectors of Majorana neutrinos transform as follows (Kayser and Goldhaber, 1983):

$$|\chi_k(r_k,p_k)\rangle \xrightarrow{CPT} |\chi_k(r_k,p_k)\rangle_{CPT} = U_{CPT}\,|\chi_k(r_k,p_k)\rangle = (-1)^{r_k-1/2}\,|\chi_k(-r_k,p_k)\rangle, \quad k=i,j \,. \tag{10.32}$$

As a consequence of Eq. (10.32) the matrix element on the left-hand side of Eq. (10.31) can be written as

$$_{CPT}\langle \chi_i(r_i,p_i) \,|\, J_\alpha^{\rm em}(0) \,|\, \chi_j(r_j,p_j)\rangle_{CPT} = iN_{ij}\overline{u_{PT}^{r_i}}(p_i)\widetilde{\Gamma}_\alpha(p_i,p_j)u_{PT}^{r_j}(p_j) \,, \tag{10.33}$$

where

$$u_{PT}^{r_k}(p_k) = \gamma_4 V_T^{-1}(u^{r_k}(p_k))^* \,. \tag{10.34}$$

From Eq. (10.32), with the help of Eqs. (10.3), (10.28), (10.33), and (10.34), one obtains

$$V_T^{-1}\widetilde{\Gamma}_\alpha^T(p_i,p_j)V_T = -\widetilde{\Gamma}_\alpha(p_j,p_i) \,. \tag{10.35}$$

Using this equality as well as Eqs. (10.29) and (10.15) we find

$$\widetilde{F}_{ji}^{V,A}(q^2) = -\widetilde{F}_{ij}^{V,A}(q^2) \,, \quad \widetilde{G}_{ji}^{V(A)} = {}_{(+)}^{-}\widetilde{G}_{ij}^{V(A)}(q^2) \,. \tag{10.36}$$

It follows from Eq. (10.36) that in the case of *CPT* invariance $\widetilde{F}_{ii}^{V,A}(q^2)=0$ and $\widetilde{G}_{ii}^{V}(q^2)=0$. Consequently, in contrast to the massive Dirac neutrinos, the massive Majorana neutrinos can have neither magnetic nor electric dipole moments if *CPT* invariance holds; they can couple only to a virtual photon, and this coupling is characterized by one form factor $\widetilde{G}_{ii}^{A}(q^2)$ (Schechter and Valle, 1981b; Kayser, 1982; Nieves, 1982), the particle anapole form factor (Zeldovich, 1957).

Further, under *CP* transformation

$$|\chi_k(r_k,p_k)\rangle \xrightarrow{CP} U_{CP}\,|\chi_k(r_k,p_k)\rangle = \eta_{CP}^*(\chi_k)\,|\chi_k(-r_k,p_k')\rangle \,, \quad p_k' = (-\mathbf{p}_k, ip_{k0}) \,, \tag{10.37}$$

where $\eta_{CP}^*(\chi_k) = -i\eta_k = \pm i$ is the *CP* parity of the Majorana neutrino χ_k (see Sec. V.B and Appendix A). In the case of *CP* invariance we obtain from Eqs. (10.19) and (10.37)

$$\eta_{CP}(\chi_j)\eta_{CP}^*(\chi_i)\langle \chi_j(-r_j,p_j') \,|\, J_\alpha^{\rm em}(0) \,|\, \chi_i(-r_i,p_i')\rangle = \eta_\alpha\langle \chi_j(r_j,p_j) \,|\, J_\alpha^{\rm em}(0) \,|\, \chi_i(r_i,p_i)\rangle \,. \tag{10.38}$$

One has

$$\langle \chi_j(-r_j,p_j') \,|\, J_\alpha^{\rm em}(0) \,|\, \chi_i(-r_i,p_i')\rangle = iN_{ji}\overline{u_p^{r_j}}(p_j')\widetilde{\Gamma}_\alpha(p_j',p_i')u_p^{r_i}(p_i') \,, \tag{10.39}$$

where

$$u_p^{r_k}(p_k') = \gamma_4 u^{r_k}(p_k), \quad k=i,j \,. \tag{10.40}$$

As a consequence of Eq. (10.38), we get, using Eq. (10.28), (10.39), and (10.40),

$$\eta_{CP}(\chi_j)\eta_{CP}^*(\chi_i)\gamma_4\widetilde{\Gamma}_\alpha(p_j',p_i')\gamma_4 = \eta_\alpha\widetilde{\Gamma}_\alpha(p_j,p_i) \,. \tag{10.41}$$

Taking into account Eqs. (10.29) and (10.9), it is easy to obtain, from (10.41),

$$\widetilde{F}_{ji}^{V(A)}(q^2) = {}_{(+)}^{(-)}\eta_j\eta_i\widetilde{F}_{ji}^{V(A)}(q^2) \,, \quad \widetilde{G}_{ji}^{V(A)}(q^2) = {}_{(+)}^{-}\eta_j\eta_i\widetilde{G}_{ji}^{V(A)}(q^2) \,. \tag{10.42}$$

Thus, the amplitude of the radiative transition of a given Majorana neutrino into another Majorana neutrino and a photon depends in the case of *CP* invariance on the relative *CP* parity of the initial and final Majorana neutrinos (Wolfenstein, 1981a). As a consequence of Eq. (10.42), the electromagnetic current operator matrix element (10.28) is characterized by only two form factors (Nieves, 1982):

$$\widetilde{F}_{ji}^{V}(q^2) = \widetilde{G}_{ji}^{V}(q^2) = 0, \quad \eta_{CP}(\chi_j) = \eta_{CP}(\chi_i)$$

and

$$\widetilde{F}_{ji}^{A}(q^2) = \widetilde{G}_{ji}^{A}(q^2) = 0, \quad \eta_{CP}(\chi_j) = -\eta_{CP}(\chi_i) \,.$$

Correspondingly, the neutrino radiative lifetimes are determined in the case of Majorana neutrinos by one transition moment [the $\chi_i \to \chi_j + \gamma$ transition is of magnetic dipole type ($\widetilde{F}_{ji}^{A}=0$) if $\eta_{CP}(\chi_j) = -\eta_{CP}(\chi_i)$ and of electric dipole type ($\widetilde{F}_{ji}^{V}=0$) when $\eta_{CP}(\chi_j)=\eta_{CP}(\chi_i)$].

We should like to conclude this subsection with a remark that may be of practical use in calculations of the amplitude of the radiative transition between two Majorana neutrinos. The form factors $\widetilde{F}_{ji}^{V,A}(q^2)$ and $\widetilde{G}_{ji}^{V,A}(q^2)$

describing the transition can be formally obtained from the form factors $F_{ji}^{V,A}(q^2)$ and $G_{ji}^{V,A}(q^2)$ calculated as if the neutrinos taking part in the transition were Dirac particles. Indeed, it can be shown, assuming the validity of *CPT* invariance (for details see Nieves, 1982; Kayser, 1982; Shrock, 1982b), that

$$\tilde{F}_{ji}^{V,A}(q^2)=F_{ji}^{V,A}(q^2)-F_{ij}^{V,A}(q^2)\ ,$$
$$\tilde{G}_{ji}^{V,A}(q^2)=G_{ji}^{V,A}(q^2)-G_{ij}^{V,A}(q^2)\ . \tag{10.43}$$

If *CP* invariance also holds, then (Nieves, 1982; Pal and Wolfenstein, 1982; Petcov, 1982c)

$$\tilde{F}_{ji}^{V(A)}(q^2)=(1\underset{(+)}{-}\eta_j\eta_i)F_{ji}^{V(A)}(q^2)\ ,$$
$$\tilde{G}_{ji}^{V(A)}(q^2)=(1\underset{(+)}{-}\eta_j\eta_i)G_{ji}^{V(A)}(q^2)\ . \tag{10.44}$$

Note that both Eqs. (10.43) and (10.44) are compatible with the corresponding constraints (10.36) and (10.42), respectively.

Finally, it should be obvious that the results obtained in the present subsection for the radiative transitions of Dirac and of Majorana neutrinos are general and hold true for the radiative transitions between any pair of neutral Dirac and spin-$\frac{1}{2}$ Majorana particles, respectively.

B. Neutrino radiative decays and magnetic moments: experimental constraints and theoretical predictions

The general expressions for neutrino radiative lifetimes can be easily derived given the amplitudes $A(\nu_i\to\nu_j+\gamma)$ and $A(\chi_i\to\chi_j+\gamma)$ [Eqs. (10.2)—(10.4) and (10.28)] and have the form

$$\overset{(\sim)}{\tau}_{ji}=\tau(\nu_i(\chi_i)\to\nu_j(\chi_j)+\gamma)$$
$$=\left[\frac{\alpha}{2}m_i^3\left[1-\frac{m_j^2}{m_i^2}\right]^3(|\overset{(\sim)}{F}{}_{ji}^{V}|^2+|\overset{(\sim)}{F}{}_{ji}^{A}|^2)\right]^{-1}\ . \tag{10.45}$$

Bearing in mind the numerical estimates we shall make further, it is convenient to express $\overset{(\sim)}{\tau}_{ji}$ as

$$\overset{(\sim)}{\tau}_{ji}=2.4\times10^{16}\ \text{yr}\left[\frac{30\ \text{eV}}{m_i}\right]^5\left[\left[1-\frac{m_j^2}{m_i^2}\right]^3(100\ \text{Gev})^4\left[\frac{|\overset{(\sim)}{F}{}_{ji}^{V}|^2}{m_i^2}+\frac{|\overset{(\sim)}{F}{}_{ji}^{A}|^2}{m_i^2}\right]\right]^{-1}\ . \tag{10.46}$$

It should be mentioned that the relations and the expression obtained above for the case of Dirac initial and final neutrinos are valid also for the decay of a Dirac neutrino to a Majorana neutrino. However, if a Majorana neutrino (χ_i) can decay into a Dirac neutrino (ν_j), it will decay to its antiparticle ($\bar{\nu}_j$) as well, the second transition being characterized (on account of *CPT* invariance) by the moments $(-F_{ij}^{V,A})$. Recalling that the Hermiticity of $J_\alpha^{\text{em}}(x)$ implies $F_{ij}^{V(A)}=\underset{(-)}{+}(F_{ji}^{V(A)})^*$ and noting that the amplitudes of the two decays do not interfere, we get (Pal and Wolfenstein, 1982)

$$\tau'_{ji}=[\tau^{-1}(\chi_i\to\nu_j+\gamma)+\tau^{-1}(\chi_i\to\bar{\nu}_j+\gamma)]^{-1}$$
$$=\tfrac{1}{2}\tau(\chi_i\to\nu_j+\gamma)\ .$$

The magnetic moment μ_i of a massive Dirac neutrino ν_i is given, as was already shown, by $\mu_i=eF_{ii}^V$.

We shall assume further in this section that the neutrino undergoing radiative decay is much heavier than the neutrino appearing in the final state and that the neutrinos possess masses satisfying the cosmological bound (Gershtein and Zeldovich, 1966; Cowsik and McClelland, 1972; see also Steigman, 1984):

$$\sum_{i=1}^{n}g_im_i\lesssim100\ \text{eV}\ ,$$
$$g_i=\begin{cases}2 & \text{if } \nu_i \text{ is a Dirac particle,}\\ 1 & \text{if } \nu_i \text{ is a Majorana particle .}\end{cases} \tag{10.47}$$

The analyses of the existing astrophysical and cosmological data [upper limits to (i) astronomical photon backgrounds, (ii) photon fluxes from discrete sources, (iii) possible distortion of the microwave background radiation, etc.], performed under the assumption that neutrinos decay predominantly radiatively, exclude then lifetimes smaller than 10^{15}—10^{16} yr for neutrinos with masses exceeding approximately 1 eV (for a review and extensive list of references see, Turner, 1981, wherein constraints on the radiative lifetimes of neutrinos heavier than 100 eV are also discussed). It may well be possible (Shipman and Cowsik, 1981) to improve the sensitivity of detection of astrophysical fluxes of photons with energies in the far-ultraviolet region (i.e., exceeding 1 eV) by several orders of magnitude in the not too distant future.

There are stringent limits as well on the possible values of the neutrino magnetic moments:

$$|\mu_{\nu_e}|<1.5\times10^{-10}\mu_B\ , \tag{10.48a}$$

$$|\mu_{\nu_\mu}|<1.2\times10^{-9}\mu_B\ , \tag{10.48b}$$

from accelerator data (Kyuldjiev, 1984);

$$|\mu_{\nu_l}|<2\times10^{-11}\mu_B\ , \tag{10.48c}$$

based on cosmological arguments[62] (Morgan, 1981);

$$\left[\sum_{l=e,\mu,\tau,\ldots} |\mu_{\nu_l}|^2\right]^{1/2} < 8.5\times 10^{-11}\mu_B \ , \qquad (10.48d)$$

from astrophysical considerations (footnote 62) (Bég *et al.*, 1978) ($\mu_B = e/2m_e$ is the Bohr magneton). The bounds obtained on the basis of the existing accelerator data are expected to be improved approximately by an order of magnitude in the $\overset{(-)}{\nu}_\mu$-e^- and ν_e-e^- elastic scattering experiments that are being performed or are in preparation at present. Let us note also that magnetic moments of the order of or greater than 10^{-14}–10^{-13} $(e/2m_e)$ may have important astrophysical implications (Cisneros, 1980; Fujikawa and Shrock, 1980; Okun *et al.*, 1986).

We shall consider next examples of the electroweak gauge theory predictions for the neutrino radiative lifetimes[63] and magnetic moments. Let us discuss first the predictions of the minimally extended standard $SU(2)_L\times U(1)$ theory with massive Dirac neutrinos (see Secs. IV.B and VIII.B.1). The fields of the LH flavor neutrinos $\nu_{lL}(x)$ in this theory are linear combinations of the LH components of the fields of Dirac neutrinos $\nu_{kL}(x)$ with definite masses: $\nu_{lL}(x)=\sum_k U_{lk}\nu_{kL}(x)$, where U is a unitary lepton mixing matrix. At the one-loop level the amplitude of the neutrino radiative decay $\nu_i\to\nu_j+\gamma$ is generated by the diagrams with exchange of virtual $W^\pm$ bosons and charged leptons shown in Fig. 2(a). Here the suppression mechanism of Glashow, Iliopoulos, and Maiani (1970) is operating, and in the case of three generations of leptons we have (Lee and Shrock, 1977; Marciano and Sanda, 1977; Petcov, 1977b)

$$F_{ji}^{V(A)} \simeq_{(+)}^{-} \frac{G_F(m_i \overset{+}{(-)} m_j)}{8\pi^2\sqrt{2}} \sum_{l=e,\mu,\tau} \tfrac{3}{4} U_{lj}^* U_{li} \frac{m_l^2}{M_W^2} \ . \qquad (10.49)$$

Taking into account the experimental values of the τ-lepton and $W^\pm$-boson masses ($m_\tau = 1.78$ GeV, $M_W = 82$ GeV) and using Eq. (10.46), it is easy to derive a lower bound for the corresponding radiative lifetimes:

$$\tau_{ji} \gtrsim 10^{29}\ \text{yr}\left[\frac{30\ \text{eV}}{m_i}\right]^5 .$$

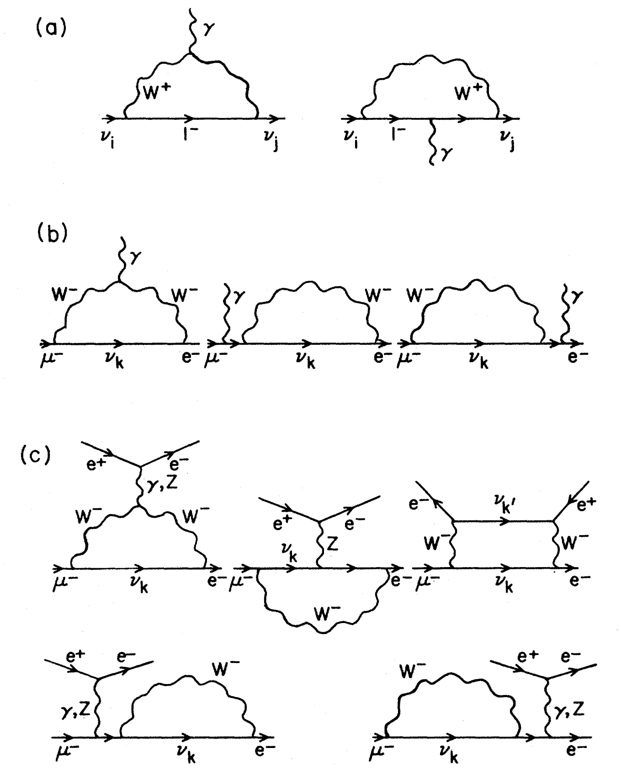

FIG. 2. Diagrams corresponding to (a) $\nu_i\to\nu_j+\gamma$; (b) $\mu\to e\gamma$; (c) $\mu\to 3e$ decays in the minimally extended standard electroweak theory with massive Dirac neutrinos.

Clearly, the radiative decays of neutrinos with masses satisfying the cosmological restriction (10.47) will be beyond observation if this bound is valid.

Much shorter lifetimes are possible if there exists a fourth generation of sequential leptons with a relatively heavy charged lepton σ, i.e., $m_\sigma \gtrsim M_W$ (De Rújula and Glashow, 1980; Pal and Wolfenstein, 1982). The Glashow-Iliopoulos-Maiani suppression mechanism turns out to be ineffective then. The decay rate is maximal for $m_\sigma \gg M_W$, when the moments $F_{ji}^{V,A}$ take the form of Eq. (10.49) with the factors

$$\sum_l \tfrac{3}{4} U_{lj}^* U_{li} \frac{m_l^2}{m_W^2}$$

replaced by $\frac{3}{4}U_{\sigma j}^* U_{\sigma i}$ and $\tau_{ji} \gtrsim 10^{23}$ yr $(30\ \text{eV}/m_i)^5$.

Unlike the radiative decay rates, the neutrino magnetic moments in the theory under consideration depend weakly on the lepton mixing matrix and charged-lepton masses and are extraordinarily small (Lee and Shrock, 1977; Marciano and Sanda, 1977; Fujikawa and Shrock, 1980):

$$\mu_i \sim \frac{3eG_F m_i}{8\pi^2\sqrt{2}} \sim 10^{-17}\left[\frac{m_i}{30\ \text{eV}}\right]\left[\frac{e}{2m_e}\right] . \qquad (10.50)$$

Note that μ_i vanishes in the limit of zero neutrino mass, when the difference between Dirac and Majorana neutri-

[62]Let us note that the cosmological bound on μ_{ν_l} was obtained in the framework of the big bang theory by requiring that the synthesis of ^{4}He in the early universe not be affected by the excitation of additional neutrino helicity states due to the electromagnetic interaction of the neutrinos. The astrophysical bound is based on considerations of the allowed energy losses due to neutrino pair emission by degenerate dwarf stars and is valid for neutrinos with masses smaller than 10 keV.

[63]For an early discussion of neutrino radiative decays in the framework of the intermediate vector-boson theory with lepton mixing, see Nakagawa *et al.* (1963).

nos in a theory with LH weak charged currents disappears (Case, 1957; Ryan and Okubo, 1964).

Massive Dirac neutrinos appear in some electroweak gauge theories with RH currents as well, but we shall not consider the predictions of these theories here. They have been discussed in detail by Lee and Shrock (1977), Smirnov and Zatzepin (1978), and Shrock (1982b). Let us note only that as a rule both neutrino radiative decay rates and neutrino magnetic moments in theories with RH currents are much larger than those discussed above.

We turn now to the predictions for radiative lifetimes of Majorana neutrinos. The results of the general analysis [Eq. (10.43)] show that the radiative lifetimes of the Majorana and Dirac neutrinos, determined by interactions common to both types of massive neutrinos (as is the standard weak interaction) do not differ substantially. However, massive Majorana neutrinos arise in the gauge theories due to specific couplings (most often between leptons and Higgs bosons and/or Higgs bosons themselves) changing the lepton charges by two units (see Sec. VIII and, for example, Cheng and Li, 1980). Such couplings might enhance considerably the neutrino radiative decay rates, even if the corresponding theory does not contain RH weak currents. The model with a triplet of Higgs scalars and the model of Zee, considered by us in Sec. VIII, provide examples of schemes in which this possibility may be realized.

As we have indicated in Sec. VIII.V.2.a, a combination of the charged scalar field (H^+) present in the triplet and of that in the standard Higgs doublet ($\phi^{(+)}$) corresponds in the model with the triplet of Higgs scalars (8.5) to a physical charged scalar particle H_T^+ [see Eqs. (8.8) and (8.9)]. It couples to leptons with a strength typical of Higgs bosons:[64]

$$\mathscr{L}_{l\text{-}H^+} = -\frac{g}{\sqrt{2}M_W} \times \sum_{l,k} \overline{\chi_k} U^\dagger_{kl}(m_l \tan\alpha_0 l_R + \eta_k m_k \cot\alpha_0 l_L) H_T^+ + \text{H.c.}\,, \tag{10.51}$$

where $\tan\alpha_0 = \sqrt{2}v/\lambda$ (if CP is not conserved, η_k should be taken to be equal to unity). If the mass of H_T^+ is in the range 20—80 GeV, the contribution to $A(\chi_i \to \chi_j + \gamma)$ determined by the lepton-H_T^+ interaction is dominant. In this case (Pal and Wolfenstein, 1982),

$$F_{ji}^{V(A)} \sim \mathop{-}_{(+)} \frac{G_F(m_i \mathop{+}_{(-)} m_j)}{8\pi^2\sqrt{2}} \times \sum_l U^*_{lj} U_{li} \frac{m_l^2}{M_H^2}\left[1-\frac{m_l^2}{M_H^2}\right]^{-2} \times \left[\ln\frac{M_H^2}{m_l^2} - 1 + \frac{m_l^2}{M_H^2}\right],$$

where M_H is the H_T^+ mass. The corresponding decay rate is maximal in the case of four generations of leptons with a heavy charged lepton σ, $m_\sigma \sim M_W$. Then we have

$$\tilde{\tau}_{ji}^{(T)} \gtrsim 10^{22}\ \text{yr}\left[\frac{30\ \text{eV}}{m_i}\right]^5 .$$

Much shorter lifetimes are possible in the model of Zee (Petcov, 1982b) discussed in detail in Sec. VIII.B.2.b. The expressions one obtains for the moments $\tilde{F}_{ji}^{V,A}$ are particularly simple in the version of the model with two Higgs doublets, in which the neutrino mass matrix is given by Eq. (8.15). In this case the $\chi_i \to \chi_1 + \gamma$, $i=2,3$, transition moments have the following unusual form (Petcov, 1982b):

$$\tilde{F}_{1i}^{V(A)} \sim \mathop{+}_{(-)} \sqrt{2} m_i \frac{m_\mu^2}{m_\tau^2} C_\mu \left[1 - \frac{C_\tau}{C_\mu}\cos 2\alpha\right](1 \mathop{-}_{(+)} \eta_1\eta_i)\,,$$

where

$$C_l = \frac{1}{\ln\dfrac{M_{H_2}^2}{M_{H_1}^2}}\left[\frac{\ln\dfrac{M_{H_2}^2}{m_l^2} - 1}{M_{H_2}^2} - (2 \to 1)\right], \quad l=\mu,\tau$$

and where $M_{H_{1,2}}$ are the masses of the two physical charged Higgs particles $H_{1,2}^+$ present in the theory. It should be noted that the diagrams giving the leading contribution to $A(\chi_i \to \chi_1 + \gamma)$ differ from those in Fig. 1 generating the neutrino mass matrix (8.15) only by the photon line and vertex. For this reason the coupling constants appearing in the expressions for $\tilde{F}_{1i}^{V,A}$ can be combined (further details can be found in Petcov, 1982b) to form the factor

$$\frac{m_i}{m_\tau^2}\frac{1}{\ln(M_{H_2}^2/M_{H_1}^2)} .$$

The corresponding neutrino radiative lifetimes may be as low as 10^{17} yr for $m_{2,3} \sim 30$ eV and relatively light charged Higgs bosons (i.e., $M_{H_2} \sim 20$ GeV and $M_{H_1} \sim 100$ GeV),

$$\tilde{\tau}_{1i}^{(\text{Zee})} \gtrsim 10^{17}\ \text{yr}\left[\frac{30\ \text{eV}}{m_i}\right]^5 .$$

As far as we know, this is the shortest neutrino radiative lifetime possible in the gauge theories with LH weak charged current.

One final remark. If, say, $f_{e\mu}^0 = 0$ in the model of Zee,

[64]This coupling can be deduced from Eqs. (8.2)—(8.5) by using Eqs. (8.8), (8.9), and the relation $M_W^2 = \frac{1}{4}g^2(\lambda^2 + 2v^2)$ valid in the model.

then $m_1=0$, the charge $L'=L_e+L_\mu-L_\tau$ is conserved, $m_2=m_3$, and the Majorana neutrinos χ_2 and χ_3 are equivalent to a Dirac neutrino

$$\psi=\frac{1}{\sqrt{2}}(\chi_2-\chi_3)$$

of the nonstandard (ZKM) variety having a mass $m=m_2=m_3$ (see Sec. VI). Such neutrinos possess as a rule nonstandard magnetic moments, which may differ substantially from the magnetic moment (10.50). Indeed, the magnetic moment of the neutrino ψ is given by (Petcov, 1982b)

$$\mu_{23}\sim -4emC_\tau \ .$$

For $m\sim 20$ eV it may be as large as $10^{-15}-10^{-14}(e/2m_e)$.

To summarize, the examples we have considered indicate that neutrinos with masses ~ 20 eV may have radiative lifetimes in the range $10^{17}-10^{22}$ yr, which can be explored experimentally in the near future, and that Dirac neutrinos may have magnetic moments of the order of $10^{-14}(e/2m_e)$, which may cause detectable astrophysical effects. However, neutrino radiative lifetimes and magnetic moments of the indicated magnitudes are possible only if there exist new particles and couplings beyond those present in the minimally extended standard theory with massive Dirac neutrinos.

XI. THE PROCESSES $\mu\to e\gamma$ AND $\mu\to 3e$ IN GAUGE THEORIES WITH MASSIVE NEUTRINOS

We have seen in the preceding sections that in the theories with massive neutrinos and neutrino mixing the lepton charges L_e, L_μ, L_τ are not conserved. As a consequence, lepton-number-nonconserving processes like $\mu^\pm\to e^\pm+\gamma$, $\mu^+\to e^++e^-+e^+$, $K^\pm\to\pi^\pm+e^++\mu^-$, etc. are allowed in these theories. As has been discussed in Sec. II, stringent experimental upper limits for the probabilities of such processes exist. The following questions naturally present themselves: are the predictions of the theories with massive neutrinos and neutrino mixing compatible with these limits, and if they are, what are the mechanisms that suppress the rates of the lepton-number-nonconserving reactions and decays to a level compatible with observations? With the aim of elucidating the status of lepton-number-nonconserving processes in gauge theories with massive neutrinos, we shall review briefly the predictions of these theories for the probabilities of $\mu\to e\gamma$ and $\mu\to 3e$ decays.

The sources of lepton number nonconservation in the minimally extended Glashow-Weinberg-Salam theory with massive Dirac neutrinos, considered in Sec. VIII.B.1, are the lepton—Higgs-boson couplings (8.1) which give rise to the neutrino mass term. The coupling constants in Eq. (8.1) multiplied by the vacuum expectation value $\langle\phi^0\rangle_0$ form the neutrino mass matrix. It is therefore natural to expect that in this theory the magnitude of the lepton-number-nonconservation effects will depend strongly on the neutrino masses and on the lepton mixing matrix elements.

The leading contribution to the $\mu\to e\gamma$ decay amplitude $A(\mu\to e\gamma)$ in the theory is shown diagrammatically in Fig. 2(b). The terms in $A(\mu\to e\gamma)$, corresponding to diagrams with exchange of different virtual neutrinos, tend to compensate each other, i.e., the suppression mechanism of Glashow, Iliopoulos, and Maiani (1970) is operating. The $\mu\to e\gamma$ decay rate and branching ratio are given by (Bilenky, Petcov, and Pontecorvo, 1977; Petcov, 1977b)

$$\Gamma(\mu\to e\gamma)=\frac{G_F^2 m_\mu^5}{192\pi^3}\frac{3}{32}\frac{\alpha}{\pi}\left|\sum_k U_{\mu k}^* U_{ek}\frac{m_k^2}{M_W^2}\right|^2 , \qquad (11.1)$$

$$B(\mu\to e\gamma)=\frac{\Gamma(\mu\to e\gamma)}{\Gamma(\mu\to e\bar\nu_e\nu_\mu)} =\frac{3}{32}\frac{\alpha}{\pi}\left|\sum_k U_{\mu k}^* U_{ek}\frac{m_k^2}{M_W^2}\right|^2 , \qquad (11.2)$$

where m_k is the mass of the Dirac neutrino ν_k, M_W is the $W^\pm$-boson mass, U_{lk}, $l=e,\mu$, are elements of the lepton mixing matrix, and Eqs. (11.1) and (11.2) were derived assuming that $m_k \ll M_W$.

The $\mu\to 3e$ decay amplitude is determined as depicted in Fig. 2(c). For neutrino masses much smaller than the $W^\pm$-boson mass, $m_k\ll M_W$, the contributions of the diagrams with exchange of a virtual Z boson and of those with two virtual $W^\pm$ bosons in $A(\mu\to 3e)$ are logarithmically enhanced and dominate over the contribution of the diagrams with exchange of a virtual photon (Petcov, 1977b). In the approximation in which the latter is neglected, the $\mu\to 3e$ decay branching ratio has the form (Lee and Shrock, 1977; Petcov, 1977b)

$$B(\mu\to 3e)=\frac{\Gamma(\mu\to 3e)}{\Gamma(\mu\to e\bar\nu_e\nu_\mu)} =\frac{3}{16}\frac{\alpha^2}{\pi^2}\left|\sum_k U_{\mu k}^* U_{ek}\frac{m_k^2}{M_W^2}\ln\frac{M_W^2}{m_k^2}\right|^2 . \qquad (11.3)$$

Note that the naive estimate $\Gamma(\mu\to 3e)/\Gamma(\mu\to e\gamma)\sim\alpha/\pi$ may not be valid in the theory under consideration. Due to the logarithmic enhancement of the $\mu\to 3e$ decay rate, we may have $\Gamma(\mu\to 3e)/\Gamma(\mu\to e\gamma)\gg\alpha/\pi$.

It obviously follows from Eqs. (11.2) and (11.3) that the $\mu\to e\gamma$ and $\mu\to 3e$ decay branching ratios would be extraordinarily small if the neutrino masses m_k satisfied the cosmological bound, being smaller than 100 eV. Taking into account that $M_W\simeq 82$ GeV and using the value 0.1 for the relevant product of the lepton mixing matrix elements, we obtain in this case

$$B(\mu\to e\gamma)\lesssim 0.5\times 10^{-41},\ \ B(\mu\to 3e)\lesssim 4\times 10^{-41} .$$

These bounds are some 30 orders of magnitude lower than the corresponding experimental upper bounds 4.9×10^{-11} and 2.4×10^{-12}, respectively (see Sec. II, Table II). Clearly, the $\mu\to e\gamma$ and $\mu\to 3e$ decays would be unobservable under the indicated conditions. As can be shown, the same result is valid for all possible lepton-number-nonconserving reactions and decays in the theory under consideration.

Thus we see that if the lepton charges L_l are not conserved, the $\mu\to e\gamma$, $\mu\to 3e$, $K^{\pm}\to\pi^{\pm}e^{-}\mu^{+}$, etc. decays may proceed with rates that are not observable in practice. If neutrinos possess nonzero masses and neutrino mixing does take place, this possibility may be realized provided the neutrino masses are much smaller than the $W^{\pm}$-boson mass. Under this condition, the only lepton-number-nonconserving process that might lead to observable effects would be the oscillations of neutrinos (Petcov, 1977b).

One arrives at an analogous conclusion in the case of three generations of leptons, even if relatively heavy neutrinos exist. In this case the mass of the heaviest neutrino, say ν_3, cannot exceed the experimental limit on the τ-neutrino mass: $m_3<70$ MeV (Albrecht *et al.*, 1985). Searches for heavy neutrinos that couple both to the electron and to the muon and have masses in the range 1—70 MeV imply that $|U^*_{\mu 3}U_{e3}|\lesssim 10^{-3}$ (for a summary, see Lubimov, 1984). This in turn implies

$$B(\mu\to e\gamma)\lesssim 10^{-22},\quad B(\mu\to 3e)\lesssim 10^{-22}\ ,$$

which is far below the sensitivity of the experiments on $\mu\to e\gamma$ and $\mu\to 3e$ decays performed at present.

The form of Eqs. (11.2) and (11.3) suggests that the $\mu\to e\gamma$ and $\mu\to 3e$ decay probabilities may be close to the corresponding experimental upper limits if, say, heavy leptons with masses in the range of several GeV or larger[65] exist (Bilenky, Petcov, and Pontecorvo, 1977; see also Bjorken, Lane, and Weinberg, 1977; Cheng and Li, 1977; Lee and Shrock, 1977; Marciano and Sanda, 1977; Wilczek and Zee, 1977). In particular, in the theory under consideration, this could be a heavy neutrino ν_4 (i.e., a heavy neutral lepton) belonging to a fourth sequential generation of leptons (Lee *et al.*, 1977). Assuming for illustration that $m_4\sim 10$ GeV and $|U^*_{\mu 4}U_{e4}|\sim 10^{-2}$, we obtain from Eqs. (11.2) and (11.3)

$$B(\mu\to e\gamma)\sim 4.8\times 10^{-12}\ ,$$
$$B(\mu\to 3e)\sim 4\times 10^{-13}\ , \tag{11.4}$$
$$\frac{\Gamma(\mu\to 3e)}{\Gamma(\mu\to e\gamma)}\sim 0.08\ .$$

In the model of Zee the lepton charges L_l, $l=e,\mu,\tau$ are not conserved by the lepton—charged-Higgs-boson interaction (8.13), which is not associated directly with neutrino mass generation. The leading contribution in the $\mu\to e\gamma$ decay amplitude may arise due to this interaction.[66] As a consequence of the nonstandard form of the couplings (8.13) ($\bar{\nu}_\tau$, for example, is coupled both to μ^- and e^-), the contribution in $A(\mu\to e\gamma)$ they generate is not suppressed by the Glashow-Iliopoulos-Maiani mechanism. The $\mu\to e\gamma$ decay probability is different from zero even in the limit of zero neutrino masses, and in the case of three generations of leptons we have (Petcov, 1982b; see also Leontaris, Tamvakis, and Vergados 1985, and Tamvakis and Vergados, 1985)

$$B(\mu\to e\gamma)\simeq\frac{\alpha}{48\pi}\left[\frac{f^0_{e\tau}f^0_{\mu\tau}}{\overline{M}^2G_F}\right]^2\ , \tag{11.5}$$

where $f^0_{e\tau}$, $f^0_{\mu\tau}$, and $\overline{M}$ are defined by Eqs. (8.13) and (8.20). For $\overline{M}\sim 50$ Gev and $|f^0_{e\tau}f^0_{\mu\tau}|\sim 10^{-5}$, $B(\mu\to e\gamma)\sim 10^{-12}$, which is close to the best experimental limit of 4.9×10^{-11} (see Table II in Sec. II). In fact, this limit imposes rather stringent constraints on the parameters of the theory. For instance, if $\overline{M}\lesssim 10^3$ Gev, then $|f^0_{e\tau}f^0_{\mu\tau}|\lesssim 1.6\times 10^{-2}$. The existing experimental limits on the $\mu\to 3e$ branching ratio do not imply stronger restrictions on $|f^0_{e\tau}f^0_{\mu\tau}|$, since, as can be shown, in the model of Zee (Petcov, 1982b; Leontaris, Tamvakis, and Vergados, 1985)

$$\frac{\Gamma(\mu\to 3e)}{\Gamma(\mu\to e\gamma)}\sim\frac{\alpha}{\pi}\ . \tag{11.6}$$

The $\mu\to e\gamma$ and $\mu\to 3e$ decay rates may also be close to their experimental upper limits in the $SU(2)_L\times U(1)$ model with the triplet of Higgs scalars, discussed in Sec. VIII.B.2.a. Of particular interest are the predictions for $\Gamma(\mu\to e\gamma)$ and $\Gamma(\mu\to 3e)$ of the version of the model with a majoron, and we shall focus our attention on them.

In the case of relatively light neutrinos, the diagrams giving the largest possible contributions in $A(\mu\to e\gamma)$ and $A(\mu\to 3e)$ in the model are induced by the lepton—Higgs-triplet L_l-nonconserving couplings (8.4). It is most convenient to calculate and analyze these contributions assuming that $\langle H^0\rangle_0=v/\sqrt{2}\simeq 0$. In this limit the flavor neutrinos ν_l do not acquire a mass term at tree level and the relevant lepton—Higgs-boson couplings have the form

$$\mathscr{L}'_{l\text{-}H}=\sum_{l,l'}h^*_{ll'}\left[-\overline{(\nu_{lL})^c}l'_LH^+_T+\frac{1}{\sqrt{2}}\overline{(l_L)^c}l'_LH^{++}\right]+\text{H.c.}\ , \tag{11.7}$$

where $H^+_T(x)$ and $H^{++}(x)$ are the fields of physical massive singly charged and doubly charged Higgs particles (see Sec. VIII.B.2.a). Let us note that in the model of interest, with the triplet majoron, the masses of H^+_T and

[65]Note that Eqs. (11.2) and (11.3) for $B(\mu\to e\gamma)$ and $B(\mu\to 3e)$ are not valid in the case of contributions of "neutrinos" with masses of the order of or exceeding M_W.

[66]The corresponding diagrams can be obtained formally from the diagrams shown in Fig. 2(b) by replacing the $W^{\pm}$-boson lines with lines of the virtual Higgs bosons $H^+_{1,2}$ and ν_k by χ_k.

H^{++} (M_H and M'_H, respectively) are not independent,

$$M_H \simeq \frac{1}{\sqrt{2}} M'_H \ , \qquad (11.8)$$

and that the mass of H^{++} cannot exceed roughly $G_F^{-1/2} \sim 300$ GeV, $M'_H \lesssim 300$ GeV (Georgi, Glashow, and Nussinov, 1981).

Obviously, the lepton-H_T^+ couplings in Eq. (11.7) are analogous to the couplings (8.13) generating the leading contribution in $A(\mu \rightarrow e\gamma)$ in the model of Zee. As a consequence, the term in $B(\mu \rightarrow e\gamma)$ that corresponds to the contribution of the one-loop diagrams with exchange of virtual Higgs boson H_T^+ and virtual neutrinos can be obtained from Eq. (11.5) by replacing $(f^0_{e\tau} f^0_{\mu\tau})^2$ with $\frac{1}{16} \mid \sum_l h^*_{l\mu} h_{le} \mid^2$ and $\overline{M}$ with the mass M_H of H_T^+. It is clear from Eqs. (11.7) and (11.8) that the contribution in $A(\mu \rightarrow e\gamma)$ arising due to the l-H^{++} couplings in (11.7) is of the same order as that generated by the lepton-H_T^+ couplings.[67] Taking both contributions into account, one obtains

$$B(\mu \rightarrow e\gamma) = \frac{\alpha}{48\pi} \frac{25}{16} \left[\frac{\left| \sum_l h^*_{l\mu} h_{le} \right|}{M_H^2 G_F} \right]^2 . \qquad (11.9)$$

In contrast to $A(\mu \rightarrow e\gamma)$, the $\mu \rightarrow 3e$ decay amplitude receives a nontrivial contribution already in the tree approximation in this model: due to the l-H^{++} interaction (11.7), a μ^+ can emit a real e^- and a virtual H^{++}, which can decay into a pair of real e^+. The corresponding $\mu \rightarrow 3e$ branching ratio is given by (Bernabeu, Pich, and Santamaria, 1985)

$$B(\mu \rightarrow 3e) = \frac{1}{4} \left[\frac{1}{2} \frac{\mid h^*_{e\mu} h_{ee} \mid}{M'^2_H G_F} \right]^2 . \qquad (11.10)$$

The existing data on the $(\beta\beta)_{0\nu}$ decay imply that $\mid h_{ee} \mid \lesssim 10^{-3}$ (Georgi, Glashow, and Nussinov, 1981). If, for instance, $\mid h_{e\mu} \mid \sim \mid h_{ee} \mid$ and $M'_H \sim 100$ GeV, then

$$B(\mu \rightarrow 3e) \sim 4 \times 10^{-13} \ .$$

Comparing Eqs. (11.9) and (11.10) we arrive at the interesting conclusion that in the model with the triplet Majoron one may naturally have

$$\frac{\Gamma(\mu \rightarrow 3e)}{\Gamma(\mu \rightarrow e\gamma)} \sim \frac{\pi}{\alpha} \gg 1 \ . \qquad (11.11)$$

Thus our considerations show that there exist natural mechanisms for suppressing the rates of the lepton-number-nonconserving reactions and decays in theories with massive neutrinos and neutrino mixing. The probabilities of the $\mu \rightarrow e\gamma$ and $\mu \rightarrow 3e$ decays predicted by these theories may be close to the existing experimental upper bounds if, for example, heavy leptons and/or charged Higgs particles exist. As follows both from our considerations and from earlier studies (Bjorken and Weinberg, 1977; Lee *et al.*, 1977; Petcov, 1977a; Wilczek and Zee, 1977), the predicted value of the ratio $\Gamma(\mu \rightarrow 3e)/\Gamma(\mu \rightarrow e\gamma)$ is very sensitive to the mechanism assumed for the $\mu \rightarrow e\gamma$ decay: it varies with the theory from α/π to π/α. Therefore, the measurement of both $\mu \rightarrow e\gamma$ and $\mu \rightarrow 3e$ decay rates might be used to distinguish between different mechanisms that might cause muon number nonconservation. These results imply that both decays should be investigated with better accuracy.

XII. EXPERIMENTAL TESTS OF THE NONZERO NEUTRINO MASS AND NEUTRINO OSCILLATION HYPOTHESES

A. Introduction

Neutrino masses and mixing constitute one of the outstanding problems of modern elementary-particle physics. In many laboratories around the world experiments designed to search for effects due to Majorana or Dirac masses and mixing of neutrinos are being performed. In spite of the considerable experimental progress achieved in recent years, the problem of neutrino masses still remains open. The data obtained most recently, however, have permitted us to narrow substantially the region of possible values of neutrino masses and mixing angles.

We shall first briefly discuss the results of experiments in search of neutrino oscillations. Due to the interference nature of the oscillations, searching for this phenomenon is one of the most sensitive methods of looking for effects of finite neutrino mass differences. We shall present only the data obtained in the last few years. Excellent reviews of the early experiments on neutrino oscillations performed before 1982 are given in Baltay (1982) and Wachsmuth (1982).

B. Neutrino oscillation experiments

In order to prove that neutrino oscillations do take place, it is necessary to make certain that the probability of a transition of a neutrino (antineutrino) of a given type ν_l ($\bar{\nu}_l$) into a neutrino (antineutrino) of a different type $\nu_{l'}$ ($\bar{\nu}_{l'}$), $l' \neq l$, $l = e,\mu$, $l' = e,\mu,\tau$ (or into a sterile neutrino) is different from zero and depends periodically on the quantity R/p (where R is the source-detector distance and p is the neutrino momentum). There exist two types of experiments searching for oscillations.

1. Appearance experiments. In experiments of this type one is looking for the appearance of neutrinos of a given kind $\nu_{l'}$ ($l' = e,\mu,\tau$) at some distance from the source of neutrinos of a different kind ν_l ($l = e,\mu$, $l \neq l'$). Finding such neutrinos would constitute evidence in favor of the oscillations $\nu_l \rightleftarrows \nu_{l'}$.

[67]For a calculation of the contribution in $A(\mu \rightarrow e\gamma)$ arising due to the l-H^{++} interaction see Pich, Santamaria, and Bernabeu (1984) and Leontaris, Tamvakis, and Vergados (1985).

TABLE III. Values of the parameter $(\Delta m^2)_0$, characterizing qualitatively the sensitivity of a given experiment searching for neutrino oscillations. p and R are the neutrino momentum and the source-detector distance typical of the experiment.

Neutrino source	P (MeV)	R (m)	$(\Delta m^2)_0$ (eV^2)
Reactor	1	10^2	10^{-2}
Meson factory	10	10^2	10^{-1}
High-energy accelerator	10^3	10^3	1
Atmospheric neutrinos	10^4	10^7	10^{-3}
Sun	1	10^{11}	10^{-11}

2. Disappearance experiments. In experiments of this type, neutrinos of the same kind are detected at some distance from the source of ν_l. If the measured flux of neutrinos should turn out to be less than the flux expected in the absence of oscillations, that would constitute evidence in favor of the oscillations $\nu_l \rightleftarrows \nu_x$ (where ν_x is an active or sterile neutrino).

Since oscillations of neutrinos have not yet been found, in most of the literature the experimental data are analyzed under the simplest assumption, that of oscillations between two states.[68] For the transition probabilities we have in this case [see Eqs. (7.79) and (7.80)]

$$\begin{aligned} P_{\nu_{l'};\nu_l}(R/p) &= P_{\bar{\nu}_{l'};\bar{\nu}_l}(R/p) \\ &= P_{\nu_l;\nu_{l'}}(R/p) \\ &= P_{\bar{\nu}_l;\bar{\nu}_{l'}}(R/p) \\ &= \tfrac{1}{2}\sin^2 2\theta \left[1-\cos 2.54\frac{\Delta m^2 R}{p}\right], \end{aligned} \tag{12.1}$$

$$\begin{aligned} P_{\nu_l;\nu_l}(R/p) &= P_{\nu_{l'};\nu_{l'}}(R/p) \\ &= P_{\bar{\nu}_l;\bar{\nu}_l}(R/p) \\ &= P_{\bar{\nu}_{l'};\bar{\nu}_{l'}}(R/p) \\ &= 1-\tfrac{1}{2}\sin^2 2\theta \left[1-\cos 2.54\frac{\Delta m^2 R}{p}\right]. \end{aligned} \tag{12.2}$$

Here $\Delta m^2 = |m_1^2 - m_2^2|$ is the difference of the squares of neutrino masses in eV^2, R is the distance between the source and the detector in meters, p is the neutrino momentum in MeV/c, and θ is the leptonic mixing angle. The indices l and l' can assume the values e and μ, or e and τ, or else μ and τ. Expressions analogous to (12.1) and (12.2) apply in the case of oscillations between active and sterile neutrinos.

It is obvious from Eqs. (12.1) and (12.2) that neutrino oscillations would not be observed in a given experiment if the difference of the squares of neutrino masses Δm^2 were so small that for all R and p characteristic of the experiment the cosine argument were much smaller than unity. Oscillations of neutrinos may be observed if the values of R and p typical of a given experiment satisfy the inequality

$$\Delta m^2 \gtrsim \frac{p}{R} . \tag{12.3}$$

Clearly, in order to observe oscillations it is also necessary that the oscillation amplitude $\sin^2 2\theta$ be large enough.

The inequality (12.3) implies that the parameter

$$(\Delta m^2)_0 = \frac{p}{R} \tag{12.4}$$

characterizes qualitatively the sensitivity of an experiment searching for neutrino oscillations (Bilenky and Pontecorvo, 1978): the smaller this parameter, the smaller the values of the differences of the squares of neutrino masses which can be "registered" in a given experiment. Typical values of the parameter $(\Delta m^2)_0$ for experiments with neutrinos from different sources are given in Table III.

As can be seen from Table III, the most sensitive (with respect to Δm^2) experiments with neutrinos from terrestrial sources are the reactor experiments. We shall begin with a discussion of the results obtained in the most recent reactor experiments.

During the last four years the group of Mössbauer has been performing neutrino oscillation experiments in Gösgen (Switzerland) at a reactor having a power of 2.8 GWth. Measurements were taken at three distances between the center of the active zone of the reactor and the detector: 37.9 m (Vuilleumier *et al.*, 1982), 45.9 m (Gabathuler *et al.*, 1984), and 64.7 m (Zacek *et al.*, 1985).

The reactor is an intensive source of low-energy (up to 8 MeV) electron antineutrinos, formed in the decays of the fission products. The antineutrinos are detected experimentally by observation of the process of inverse β decay of the neutron

$$\bar{\nu}_e + p \to e^+ + n . \tag{12.5}$$

The detector of the Mössbauer group comprises a sandwich of scintillator counters and proportional chambers filled with ^{3}He. Events were selected by coincidence in the appearance of positrons (detected by the scintillator counters) and neutrons (detected by the proportional chambers). Altogether there were registered

[68]The work of Blümer and Kleinknecht (1985), however, analyzes the experimental data under the more general assumption of oscillations involving three neutrino states. For earlier discussions of the three neutrino oscillations see De Rújula *et al.* (1980) and Barger *et al.* (1980).

$10\,930 \pm 220$ events at 37.9 m; $10\,590 \pm 190$ events at 45.9 m; 8787 ± 325 events at 64.7 m. At each of the three distances the spectrum of the positrons from the reaction (12.5) was measured.

The data of the measurements at all the three distances were analyzed by two methods (Zacek *et al.*, 1985). The first method consisted in comparison of the data obtained at different distances. Information about the spectrum of the initial antineutrinos was not entered in advance in this analysis. The antineutrino spectrum was represented in the form

$$S(E) = \exp\left[-\sum_{n=0}^{2} A_n E^n\right],$$

and the free parameters A_0, A_1, and A_2 were determined by fitting the experimental data. It was shown in this way that the positron spectra at the three distances were well described under the condition that neutrino oscillations were absent. If one assumes that neutrino oscillations take place and are characterized by the parameters Δm^2 and $\sin^2 2\theta$, then restrictions on the possible values of these parameters could be derived from the data. A comparison of the data obtained at the three distances is presented in Fig. 3. The region of values of the parameters Δm^2 and $\sin^2 2\theta$ located to the right of the solid curve is excluded (at 90% C.L.).

As is seen from Fig. 3, not only an upper limit on Δm^2 but a lower limit as well emerges if one compares the data at different distances. Let us note that this is connected with the fact that for sufficiently large Δm^2 the cosine term in Eq. (12.2) vanishes as a result of averaging, and the oscillation effect reduces to a multiplication of the initial flux intensity by a constant factor smaller than unity. Clearly, it is impossible to establish the presence of such a factor by merely comparing data taken at different distances.

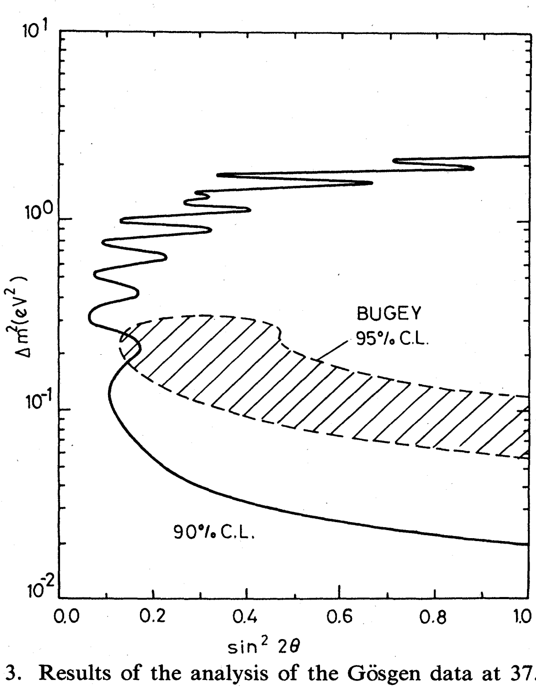

FIG. 3. Results of the analysis of the Gösgen data at 37.9 m, (Vuilleumier *et al.*, 1982), 45.9 m (Gabathuler *et al.*, 1984), 64.7 m and (Zacek *et al.*, 1985). The excluded region of values of the parameters Δm^2 and $\sin^2 2\theta$ is to the right of the solid curve. The shaded region corresponds to values of Δm^2 and $\sin^2 2\theta$ allowed by the Bugey data.

In the second method of analysis of the Gösgen data, information about the spectrum of the antineutrinos coming out of the reactor was used. The basic fission isotopes in the Gösgen reactor are ^{235}U, ^{239}Pu, ^{238}U, and ^{241}Pu. Their corresponding contributions to the power of the reactor are approximately 60%, 28%, 7%, and 5%. The spectra of the antineutrinos from fission products of the isotopes ^{235}U and ^{239}Pu giving the main contribution were determined from measurements of the β spectra of the fission products (Feilitzsch *et al.*, 1982). The calculations of Vogel *et al.* (1981) were used to derive the spectra of the antineutrinos from ^{238}U and ^{241}Pu. The results of the analysis, using the spectrum of the initial antineutrinos obtained in this way, are shown in Fig. 4. The excluded region of values of the parameters Δm^2 and $\sin^2 2\theta$ is to the right of the solid curve (90% C.L.). From Fig. 4 it follows that

$$\Delta m^2 \le 0.019 \text{ eV}^2 \quad (\sin^2 2\theta = 1),$$

$$\sin^2 2\theta \le 0.18 \quad (\Delta m^2 \ge 5 \text{ eV}^2).$$

The results of another experiment searching for oscillations in the beam of antineutrinos from a reactor have been published recently (Cavaignac *et al.*, 1984) by a group working in Bugey (France). The reactors in Bugey and Gösgen are very similar. The detectors of the two

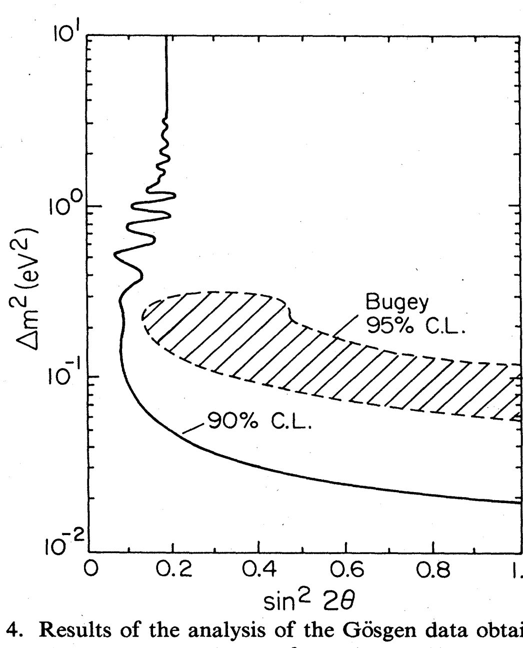

FIG. 4. Results of the analysis of the Gösgen data obtained at distances 37.9, 45.9, and 64.7 m from the reactor core. Information about the initial antineutrino spectrum was used in the analysis. The region to the right of the solid curve is excluded at 90% C. L. The shaded region of values of Δm^2 and $\sin^2 2\theta$ is allowed by the Bugey data.

groups are identical. The experiments in Bugey were performed at two distances, which, however, are smaller than those exploited in Gösgen: 13.6 and 18.3 m. This permitted the accumulation of considerably larger statistics in Bugey than in Gösgen, namely, 39 881±262 events obtained at 13.6 m and 23 345±310 events registered at 18.3 m.

The ratio of the positron yields measured in Bugey at the distances 13.6 and 18.3 m for different positron energies (in the interval from 1.5 to 6.5 MeV) and corrected for the difference in spatial angle and the burning of the nuclear fuel is presented in Fig. 5. For the ratio of the integral yields at these two distances the following value was found:

$$Y_1/Y_2=1.102\pm0.014\ (\mathrm{stat})\pm0.028\ (\mathrm{syst})\ .$$

The authors of the work performed in Bugey interpret the data they have obtained as evidence in favor of neutrino oscillations. For the allowed values of the parameters Δm^2 and $\sin^2 2\theta$ the shaded region in Figs. 3 and 4 is found. The best description of the Bugey data is achieved for

$$\Delta m^2=0.2\ \mathrm{eV}^2,\ \ \sin^2 2\theta=0.25\ .$$

In the work of Zacek *et al.* (1985) attention is paid to the contradiction between the Gösgen and Bugey data. Practically the whole region of possible values of Δm^2

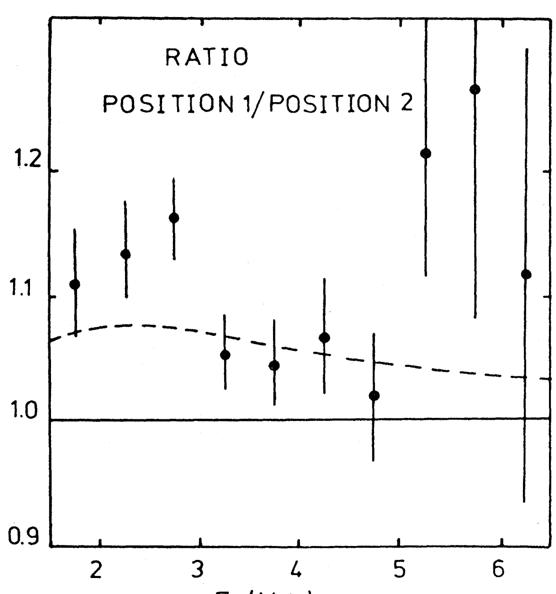

FIG. 5. The ratio of positron yields at distances 13.6 and 18.3 m, obtained in the Bugey experiments (Cavaignac *et al.*, 1984). Corrections for the difference in the spatial angles and the burning of the nuclear fuel are taken into account. The dashed curve corresponds to prediction of the neutrino oscillation theory for $\Delta m^2=0.2$ eV^2 and $\sin^2 2\theta=0.25$.

and $\sin^2 2\theta$ obtained from the Bugey data lies in an area excluded by the Gösgen data taken at the three distances (see Fig. 3). If information about the spectrum of the initial antineutrinos is used in the analysis of the Gösgen data, the whole region of values of Δm^2 and $\sin^2 2\theta$ determined from the Bugey data lies inside the region of values of these parameters excluded by the Gösgen data (see Fig. 4). Further, all positron spectra measured by the group of Mössbauer are well reproduced by using a spectrum extracted from the Bugey data obtained at 13.6 m under the assumption that neutrino oscillations do not take place (Zacek *et al.*, 1985). The positron spectrum measured at the distance of 18.3 m in Bugey cannot be derived in the same way. Finally, if one assumes that neutrino oscillations characterized by $\Delta m^2=0.2$ eV^2 and $\sin^2 2\theta=0.25$ do take place, and if one extracts the antineutrino spectrum from the 13.6-m Bugey data, it is impossible to reproduce by using this spectrum the Gösgen positron spectra as well as the 18.3-m Bugey spectrum. On the basis of this analysis Gabathuler (1985) expressed doubts about the correctness of the 18.3-m Bugey data.

Thus the experiments with reactor antineutrinos do not yet give an unambiguous answer to the question of whether neutrino oscillations take place. These experiments continue. Let us note that apart from those in Gösgen and Bugey, experiments designed to search for oscillations in beams of reactor antineutrinos are being performed also at the Rovno Power Station in the Soviet Union (Borovoi *et al.*, 1980) and at the Savannah River Reactor in the United States (Baumann *et al.*, 1984).

We proceed now to a discussion of the results of experiments looking for oscillations in neutrino beams from accelerators. Recently, the results of three new disappearance experiments searching for $\nu_\mu \rightleftarrows \nu_x$ oscillations have been published (Bergsma *et al.*, 1984; Dydak *et al.*, 1984; Stockdale *et al.*, 1984). The first two experiments were performed at CERN, the third in Batavia. In these experiments for the first time the neutrino events $\nu_\mu+N\to\mu^-+X$ were registered by two detectors. In the experiment of the CDHS Collaboration (Dydak *et al.*, 1984) the detectors were located at 130 and 885 m from the proton target; in the experiment of the CHARM Collaboration (Bergsma *et al.*, 1984) they were at 123 and 903 m. Finally, in the experiment of the CCFR Collaboration (Stockdale *et al.*, 1984) the detectors were placed at 715 and 1116 m from the center of the decay tunnel. For the experiments at CERN a special neutrino beam from the proton synchrotron (PS) with relatively low energy ($\sim$1 GeV) was formed. In Batavia a neutrino beam with energy in the interval from 40 to 230 GeV was used.

Evidence for the existence of neutrino oscillations was not found in all three experiments. The region of values of the parameters Δm^2 and $\sin^2 2\theta$ excluded by the data obtained in these experiments is depicted in Fig. 6 (the confidence level is 90%; the excluded regions are located to the right of the curves). As can be seen from Fig. 6, in the case of maximal mixing ($\sin^2 2\theta=1$), the data obtained in the experiments with two detectors permit one to exclude the values of the parameter Δm^2 in the intervals

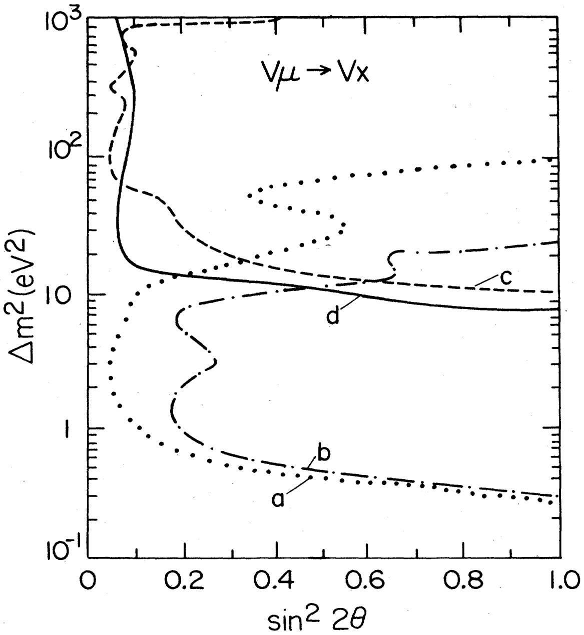

FIG. 6. Results of analyses of the data on $\nu_\mu \rightleftarrows \nu_x$ oscillations. The regions of values of Δm^2 and $\sin^2 2\theta$ to the right of the curves are excluded (at 90% C.L.) by (a) CDHS (Dydak *et al.*, 1984); (b) CHARM (Bergsma *et al.*, 1984); (c) CCFR (Stockdale *et al.*, 1984); and (d) Serpukhov (Belikov *et al.*, 1983).

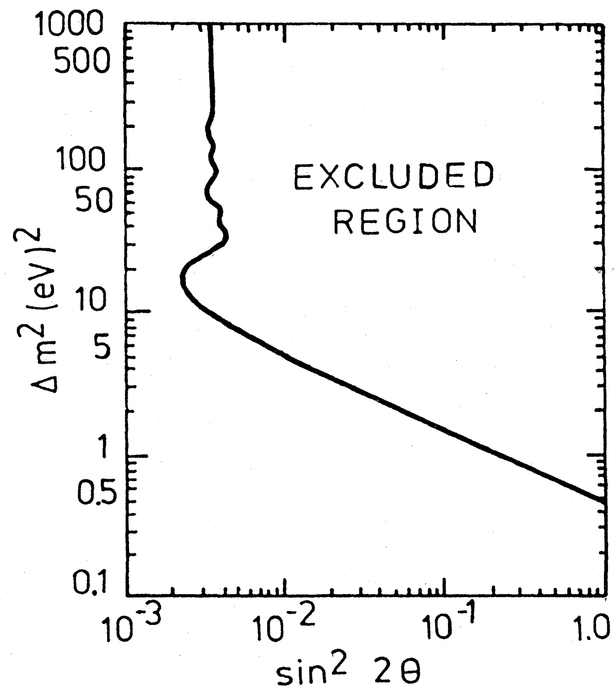

FIG. 7. The region in the plane $\Delta m^2, \sin^2 2\theta$ excluded by the data obtained at BNL in the experiment of Ahrens *et al.* (1985) searching for $\nu_\mu \rightleftarrows \nu_e$ oscillations.

$$0.26 < \Delta m^2 < 96 \text{ eV}^2 \quad \text{CDHS} ,$$

$$0.29 < \Delta m^2 < 22 \text{ eV}^2 \quad \text{CHARM} ,$$

$$15 < \Delta m^2 < 1600 \text{ eV}^2 \quad \text{CCFR} .$$

It was also found from the data that

$$\sin^2 2\theta < 0.053 \quad \text{for } \Delta m^2 \simeq 2.5 \text{ eV}^2 \quad \text{CDHS} ,$$

$$\sin^2 2\theta < 0.20 \quad \text{for } \Delta m^2 \simeq 2 \text{ eV}^2 \quad \text{CHARM} ,$$

$$\sin^2 2\theta < 0.02 \quad \text{for } \Delta m^2 \simeq 100 \text{ eV}^2 \quad \text{CCFR} .$$

The CCFR Collaboration performed a search for $\bar{\nu}_\mu \rightleftarrows \bar{\nu}_x$ oscillations (Stockdale *et al.*, 1985) with the same two detectors used in the $\nu_\mu \rightleftarrows \nu_x$ oscillation experiment. No evidence for $\bar{\nu}_\mu$ disappearance was found. For $\sin^2 2\theta = 1$ the CCFR data imply that $\Delta m^2 < 15$ eV^2 or $\Delta m^2 > 10^3$ eV^2 (90% C.L.); the most stringent limit on the relevant neutrino mixing angle obtained in this experiment is $\sin^2 2\theta < 0.02$ (for $\Delta m^2 \simeq 110$ eV^2).

We shall give next a brief exposition of the results of the most recent appearance experiments searching for the oscillations $\nu_\mu \rightleftarrows \nu_e$. In the experiment of Ahrens *et al.* (1985), performed at the Brookhaven (BNL) accelerator (with an average neutrino energy of 1.5 GeV), the quasielastic reactions $\nu_\mu + n \to \mu^- + p$ and $\nu_e + n \to e^- + p$ were detected. The distance between the proton target and the detector was 100 m. The ratio of the fluxes of ν_e and ν_μ was determined from the data at different neutrino energies. It turned out that the ratio of the ν_e and ν_μ fluxes measured in the experiment coincides within the errors with that expected in the absence of oscillations. The region of values of the parameters Δm^2 and $\sin^2 2\theta$ excluded by the BNL data is shown in Fig. 7. As is seen from Fig. 7, for large Δm^2,

$$\sin^2 2\theta < 3.4 \times 10^{-3} .$$

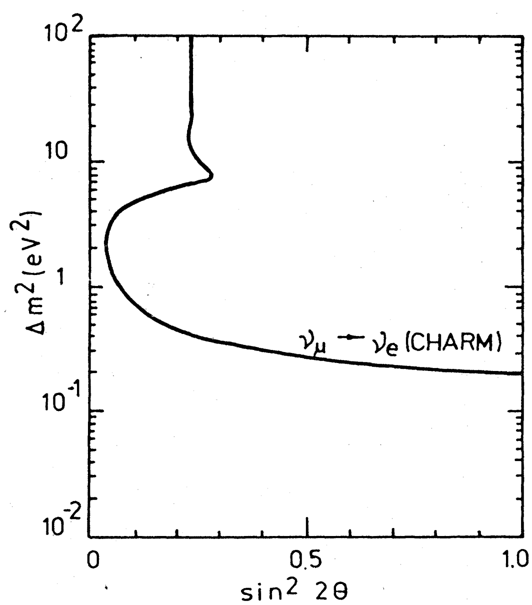

FIG. 8. Results of the analyses of the CHARM data obtained in an experiment looking for $\nu_\mu \rightleftarrows \nu_e$ oscillations with two detectors exposed to the neutrino beam from the PS.

If the parameter Δm^2 is so small that $\Delta m^2 R/p \ll 1$, then from Eq. (12.2) we have

$$P_{\nu_e;\nu_\mu} \sim \sin^2 2\theta \left[1.27 \frac{\Delta m^2 R}{p} \right]^2 .$$

From the data obtained at BNL it follows that, for small Δm^2,

$$\Delta m^2 |\sin 2\theta| < 0.43 \text{ eV}^2 .$$

In an experiment with two detectors the CHARM Collaboration looked for the oscillations $\nu_\mu \rightleftarrows \nu_e$ in addition to the oscillations $\nu_\mu \rightleftarrows \nu_x$ (Bergsma *et al.*, 1984). Quasielastic ν_e-induced events were detected (with an average neutrino energy of 1.5 GeV). An upper bound of 2.7×10^{-2} was obtained for the probability of the transition $\nu_\mu \to \nu_e$.

The results of the analysis of the CHARM data are presented in Fig. 8 (the region of values of Δm^2 and $\sin^2 2\theta$ located to the right of the curve is excluded at 90% C.L.). It follows from this analysis that

$$\Delta m^2 < 0.20 \text{ eV}^2 \ \ (\sin^2 2\theta = 1) ,$$

$$\sin^2 2\theta < 0.04 \ \ (\Delta m^2 \simeq 2 \text{ eV}^2) .$$

Experiments searching for oscillations of neutrinos have also been performed at the meson factory in Los Alamos (Nemethy *et al.*, 1981; Wotschack, 1984). Neutrinos in these experiments originated from the decays of stopped pions ($\pi^+ \to \mu^+ \nu_\mu$) and from the subsequent decay $\mu^+ \to e^+ \nu_e \bar{\nu}_\mu$. The admixture of $\bar{\nu}_e$ in the initial neutrino flux was $\sim 10^{-3}$, and the detector was located at 10 m from the target. The experiments looked for the oscillations $\bar{\nu}_\mu \rightleftarrows \bar{\nu}_e$ and performed a search for $\bar{\nu}_e + p \to e^+ + n$ events. Evidence for the existence of neutrino oscillations was not obtained in these experiments either. It was found that

$$\Delta m^2 < 0.49 \text{ eV}^2 \ \ (\sin^2 2\theta = 1) ,$$

$$\sin^2 2\theta < 0.028 \ \ (\text{for } \Delta m^2 = 2 \text{ eV}^2) .$$

The possibility of $\bar{\nu}_\mu \rightleftarrows \bar{\nu}_e$ and $\bar{\nu}_\mu \rightleftarrows \bar{\nu}_\tau$ oscillations was investigated in an appearance experiment carried out at Fermilab with the 15-foot bubble chamber, using a narrow-band $\bar{\nu}_\mu$ beam (Taylor *et al.*, 1983). Under better background conditions than those attained in the earlier accelerator $\bar{\nu}_\mu \rightleftarrows \bar{\nu}_e$ and $\bar{\nu}_\mu \rightleftarrows \bar{\nu}_\tau$ oscillation experiments, a search for excess over the background of events with e^+ in the final state, which could originate from the reactions $\bar{\nu}_e + N \to e^+ + X$ and $\bar{\nu}_\tau + N \to \tau^+ + X$, $\tau^+ \to e^+ \nu_e \bar{\nu}_\tau$, was performed. The average $\bar{\nu}_\mu$ energy in this experiment was 50 GeV, and the detector was located at a distance of 1400 m from the proton target (the length of the $\pi^\pm$- and $K^\pm$-decay tunnel was 352 m). The following limits on neutrino oscillation parameters were obtained at 90% C.L.

$\bar{\nu}_\mu \rightleftarrows \bar{\nu}_e$:

$$\Delta m^2 < 2.4 \text{ eV}^2 \ \ (\sin^2 2\theta = 1) ,$$

$$\sin^2 2\theta < 0.013 \ \ (\Delta m^2 \to \infty) .$$

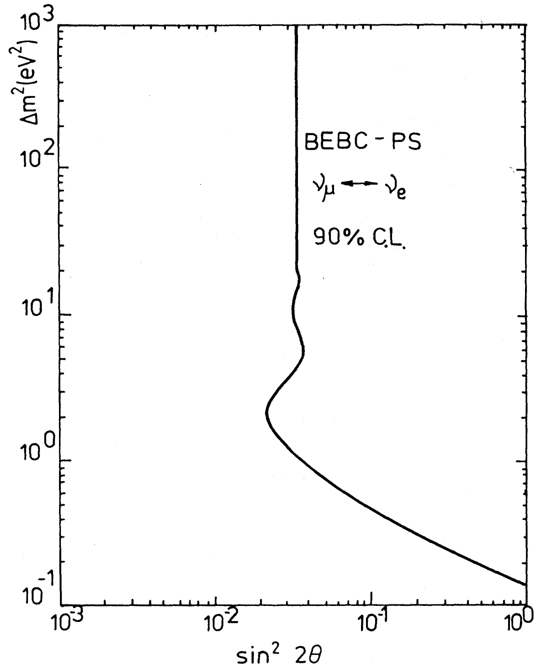

FIG. 9. Results of the analyses of the BEBC data obtained with the neutrino beam from the PS in search of the oscillations $\nu_\mu \rightleftarrows \nu_e$. The region of values of Δm^2 and $\sin^2 2\theta$ located to the right of the curve is excluded at 90% C.L.

$\bar{\nu}_\mu \rightleftarrows \bar{\nu}_\tau$:

$$\Delta m^2 < 7.4 \text{ eV}^2 \ \ (\sin^2 2\theta = 1) ,$$

$$\sin^2 2\theta < 0.088 \ \ (\Delta m^2 \to \infty) .$$

At the EPS international conference on high-energy physics in Bari the results of an appearance experiment searching for oscillations $\nu_\mu \rightleftarrows \nu_e$, performed at CERN with the bubble chamber BEBC, were reported (Baldo-Ceolin *et al.*, 1985). A flux of relatively low-energy muon neutrinos from the PS (with average neutrino energy $\simeq 1$ GeV) was used in this experiment. The distance from the target to the chamber was 820 m. Electrons that could appear as a consequence of the oscillations $\nu_\mu \rightleftarrows \nu_e$ were looked for in the chamber. Evidence that neutrino oscillations take place was not found. Altogether 434 $\nu_\mu + N \to \mu^- + X$ events and 4 $\nu_e + N \to e^- + X$ events were observed; the latter can be explained by the admixture of ν_e in the initial neutrino beam. The results of the standard analysis (the Δm^2, $\sin^2 2\theta$ plot) of the BEBC data are shown in Fig. 9. This analysis implies[69] that

$$\Delta m^2 < 0.14 \text{ eV}^2 \ \ (\sin^2 2\theta = 1) ,$$

$$\sin^2 2\theta < 0.02 \ \ (\text{for } \Delta m^2 \simeq 2.2 \text{ eV}^2) .$$

[69]For a review of the results obtained in the experiments with bubble chambers performed at CERN and Fermilab prior to 1982, see Baltay (1982).

TABLE IV. Expected neutrino fluxes from different processes taking place in the sun (Bahcall *et al.*, 1982). The expected rates of the reactions $\nu_e+{}^{37}Cl \to e^- + {}^{37}Ar$ and $\nu_e+{}^{71}Ga \to e^- + {}^{71}Ge$ (Bahcall, 1986) are given (in SNU) in the last two columns.

Process	Interval of neutrino energies (MeV)	Neutrino flux (10^{10} cm^{-2} sec^{-1})	Rate of the reaction $\nu_e+{}^{37}Cl \to e^- + {}^{37}Ar$	Rate of the reaction $\nu_e+{}^{71}Ga \to e^- + {}^{71}Ge$
$p+p\to D+e^+ +\nu_e$	0–0.42	6.0	0	69.3
$p+e^- +p\to D+\nu_e$	1.44	0.015	0.20	3.2
${}^7Be+e^-\to {}^7Li+\nu_e$	0.86 (90%) 0.38 (10%)	0.475	1.10	34.5
${}^8B\to {}^8Be+e^+ +\nu_e$	0–14	5.4×10^{-4}	5.75	16.2
${}^{13}N\to {}^{13}C+e^+ +\nu_e$	0–1.2	0.06	0.10	3.7
${}^{15}O\to {}^{15}N+e^+ +\nu_e$	0–1.7	0.05	0.35	5.9

The possibility of neutrino oscillations is also studied in experiments detecting atmospheric neutrinos, i.e., neutrinos originating from the decays of pions, kaons, and muons, produced as a result of the interaction of cosmic rays with the Earth's atmosphere. The detectors of these atmospheric neutrinos are located in underground laboratories.

In an experiment performed at the Baksan neutrino laboratory (Boliev *et al.*, 1981), muons produced by atmospheric neutrinos coming from the other side of the Earth and consequently traveling a distance of $\sim 10^7$ m were detected. The average neutrino energy was $\simeq 10$ GeV. The detected flux of muon neutrinos I_{exp} was compared to the expected flux in the absence of oscillations I_0. Thus this experiment was a disappearance experiment, looking for $\nu_\mu \rightleftarrows \nu_x$ oscillations.

Evidence in favor of the existence of oscillations was not observed in the Baksan experiment. The ratio $R=I_{exp}/I_0$ was found to be

$$R=0.98\pm0.20 .$$

It follows from these data that

$$\Delta m^2 < 6\times10^{-3}\ \mathrm{eV}^2 \quad (\sin^2 2\theta=1) ,$$

$$\sin^2 2\theta < 0.65 \quad (\text{for large } \Delta m^2) .$$

The results of another experiment on atmospheric neutrinos were published recently by LoSecco *et al.* (1985). The neutrino events were registered in this experiment by a large detector (fiducial mass of 3300 tons) situated underground at a depth of 1600 mwe (meters of water equivalent) and designed to detect proton decay. 135 neutrino events were detected. The average neutrino energy was $\simeq 920$ MeV. In order to test the hypothesis of neutrino oscillations, the authors compared the number of upward-going neutrinos in $\frac{1}{5}$ of the spatial angle with the number of neutrinos going downward in a spatial angle of the same magnitude. The upward- and downward-going neutrinos traveled, respectively, distances of $\sim 10^7$ and $\sim 10^4$ m. Hence the experiment compared numbers of neutrinos whose sources were located at different distances from the detector. No evidence for existence of neutrino oscillations was found in this experiment. For the case of maximal mixing ($\sin^2 2\theta=1$) the following region of values of the parameter Δm^2 was excluded by the data:

$$2.2\times10^{-5} < \Delta m^2 < 11.2\times10^{-5}\ \mathrm{eV}^2 .$$

A few words in conclusion about experiments aimed at detection of neutrinos from the sun. Solar neutrinos are being detected still only in the experiment of Davis *et al.* (see, for example, Haxton, Davis, and Deutsch, 1984). This experiment is based on the radiochemical method of Pontecorvo (1946) and detects ${}^{37}Ar$ formed in the reaction

$$\nu_e+{}^{37}Cl\to e^- + {}^{37}Ar . \tag{12.6}$$

The detector of Davis comprises a tank filled with 615 tons of liquid C_2Cl_4 ($\sim 2\times10^{30}$ atoms of ${}^{37}Cl$). The detector is placed deep underground in a shaft at a depth of 1400 m (4400 m of water equivalent). The argon produced in the reaction (12.6) "decays" via a K capture with a half lifetime of 35.1 days. ${}^{37}Ar$ is extracted from the tank by blowing helium through the liquid (in recent years ${}^{37}Ar$ has been extracted six times per year) and is placed in a proportional counter, in which the act of K capture is detected. The average number of atoms of ${}^{37}Ar$ formed per day is 0.48. As a result of many years of observations (from 1970 to 1984), Davis *et al.* have obtained[70]

$$\bar{I}_{exp}=2.1\pm0.3\ \mathrm{SNU} .$$

To test the hypothesis of oscillations, this quantity has to be compared with the flux of solar neutrinos expected in the absence of oscillations. The processes making the major contributions to this flux, according to the standard solar model (Bahcall *et al.*, 1982), are listed in Table IV. The neutrino fluxes from each of these processes and their contributions to the rate of the reaction $\nu_e+{}^{37}Cl$

[70]The quantity $\bar{I}$ characterizes the rate of production of ${}^{37}Ar$ and represents the product of the neutrino flux and the cross section of the process (12.6). The Solar Neutrino Unit (SNU) is equal to

$$10^{-36}\frac{\nu_e\ \text{captures}}{\sec\times(\text{target atom})} .$$

$\rightarrow e^- + {}^{37}\mathrm{Ar}$ are given[71] in the third and fourth columns of Table IV. Because of the relatively high threshold of the reaction (12.6) (0.814 MeV), the chlorine-argon method permits detection of only a small fraction ($\sim 10^{-4}$) of the total neutrino flux (mainly the neutrinos from ^{8}B decay).

So, if the standard model of the sun and the existing data about the relevant nuclear processes are used in the calculation of the expected flux $\bar{I}_0$, one obtains (Bahcall *et al.*, 1980; Bahcall, 1986):

$$\bar{I}_0 = 7.5 \pm 2.5 \ \ \mathrm{SNU} ,$$

where the error corresponds to a 3 standard deviation (s.d.) effective uncertainty. This was obtained by taking into account a 3 s.d. uncertainty in the experimental values of the quantities used as input in the calculation of $\bar{I}_0$ (cross sections, etc.).

Thus the flux of solar neutrinos observed by Davis *et al.* is considerably smaller than the expected flux. The existence of neutrino oscillations could be the most natural explanation of this "solar neutrino puzzle" (Pontecorvo, 1958, 1967; Bahcall and Frautschi, 1969; Mikheyev and Smirnov, 1985, 1986c; Bethe, 1986). Indeed, for $\Delta m^2 > 10^{-4}$ eV^2 and large vacuum mixing angles, neutrino oscillations would lead to a considerable reduction of the solar neutrino flux (the effects of solar matter are negligible). This reduction would be independent of the neutrino energy, since the dimensions of the neutrino source ($\sim 10^5$ km) are much larger than the neutrino oscillation length ($L \lesssim 3.5 \times 10^2$ km) [see Eqs. (7.69) and (7.70) and the related discussion]. In the case of oscillations involving n different types of neutrinos, the flux could be reduced at most by a factor of $1/n$ (Bilenky and Pontecorvo, 1978). If neutrino oscillation parameters Δm^2 and $\sin^2 2\theta$ have values $10^{-8} \lesssim \Delta m^2 \lesssim 10^{-4}$ eV^2 and $10^{-3.5} \lesssim \sin^2 2\theta < 0.5$ and $[\mathrm{sgn}(m_2^2 - m_1^2)]\cos 2\theta > 0$, a conversion of the electron neutrinos on their way out of the sun into different flavor neutrinos (ν_μ and/or ν_τ) and/or into sterile neutrinos (ν_x) would take place (Mikheyev and Smirnov, 1985; see also Barger, Phillips, and Whisnant, 1986; Bethe, 1986; Bouchez *et al.*, 1986; Kolb, Turner, and Walker, 1986; Rosen and Gelb, 1986). The corresponding $\nu_e \rightarrow \nu_{\mu(\tau;x)}$ transition probability is energy dependent, and the reduction of the solar flux would vary with the neutrino energy in this case. The dependence of the reduction factor $\bar{P}^{\,m}_{\nu_e;\nu_e}$ for the p-p, ^{7}Be, and ^{8}B neutrino fluxes on the ratio $E/\Delta m^2$, assuming $\nu_e \rightarrow \nu_{\mu(\tau)}$ transitions, is shown in Fig. 10 for three different values of $\sin^2 2\theta$ (0.16, 0.04, and 0.01; Mikheyev and Smirnov, 1986a). Note, in particular, that a reduction of the ^{8}B flux by a factor of ~ 3 could take place together with a strong suppression (e.g., by a factor of 10) of the low-energy part of the solar ν_e flux generated in the p-p reaction. Finally, a noticeable depletion of the flux of ν_e from the sun would also take place for $10^{-11} \lesssim \Delta m^2 \lesssim 10^{-8}$ eV^2 provided the vacuum mixing angles were relatively large. This would have a typical oscillatory dependence on the neutrino energy.

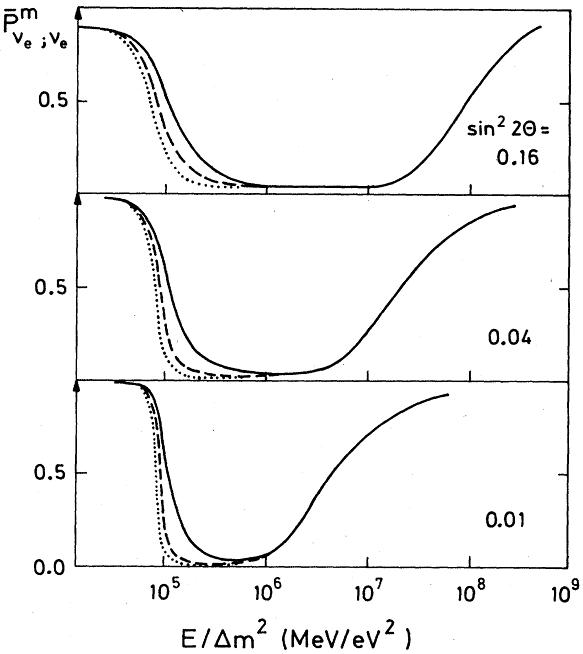

FIG. 10. The dependence of the matter suppression factor $\bar{P}^{\,m}_{\nu_e;\nu_e}$ for the p-p (solid curve), ^{7}Be (dashed curve), and ^{8}B (dotted curve) solar ν_e fluxes on the ratio $E/\Delta m^2$ for three values of $\sin^2 2\theta$ (0.16, 0.04, and 0.01) in the case of $\nu_e \rightarrow \nu_{\mu(\tau)}$ transitions.

At present, however, no definite conclusions concerning the oscillations can be made on the basis of the results of the Davis experiment. By making use of the chlorine-argon method of ν_e detection, this experiment is sensitive only to a small fraction of the total ν_e flux from the sun whose magnitude strongly depends on the temperature in the central region of the sun and other solar model parameters (Bahcall, 1978; Bahcall *et al.*, 1982; Filippone and Schramm, 1982; Schatzman, 1984).

At present, several new experiments aimed at detecting and studying the flux of solar neutrinos are in preparation. The gallium-germanium radiochemical method based on observation of the reaction (Kuzmin, 1965)

$$\nu_e + {}^{71}\mathrm{Ga} \rightarrow e^- + {}^{71}\mathrm{Ge}$$

has been developed. The threshold of this reaction is 0.235 MeV. As can be seen from Table IV, the gallium-germanium method will permit, consequently, the detection of neutrinos produced in the basic reaction $p + p \rightarrow D + e^+ + \nu_e$. The flux of such neutrinos can be

[71]Let us note that, according to the standard model of the sun, the p-p reaction generating the major part of the neutrino flux takes place in regions located at distances between approximately $0.03 R_\odot$ and $0.2 R_\odot$ from the center of the sun, while the ^{8}B neutrinos are produced in a spherical region with radius $0.1 R_\odot$, where $R_\odot \simeq 7 \times 10^5$ km is the solar radius.

reliably predicted on the basis of general thermodynamical considerations (Bahcall *et al.*, 1982; Bahcall, 1986; Hampel, 1986). The expected contributions of neutrino fluxes originating from different reactions and decays in the sun to the rate of the reaction $\nu_e + {}^{71}\mathrm{Ga} \rightarrow e^- + {}^{71}\mathrm{Ge}$ are given in the fifth column of Table IV. The total predicted rate in the standard solar model is 133 SNU. Let us note that a rate below 75 SNU cannot be obtained by a variation of the model of the sun (see, for example, Hampel, 1986). The observation of such a rate of ${}^{71}\mathrm{Ge}$ formation would be a strong indication of the existence of neutrino oscillations. Two ${}^{71}\mathrm{Ga}$-${}^{71}\mathrm{Ge}$ solar neutrino experiments that will use 60-ton (Zatsepin, 1983) and 30-ton (Kirsten, 1986) detectors are in preparation at present. Measurements of the solar neutrino flux with these detectors are planned to begin in 1988 and 1989, respectively.

Let us note also finally that a large argon detector is being designed, in which solar neutrinos will be detected by the reactions $\nu_e + {}^{40}\mathrm{Ar} \rightarrow e^- + {}^{40}\mathrm{K}^*$, ${}^{40}\mathrm{K}^* \rightarrow 2\gamma + {}^{40}\mathrm{K}$, and $\nu_e + e^- \rightarrow \nu_e + e^-$. This detector will allow the determination of the direction of the neutrino momentum (Bahcall *et al.*, 1985). [A detailed discussion of solar neutrino experiments capable of detecting neutrino-electron scattering is contained in the article by Bahcall which appears in *Reviews of Modern Physics* (Vol. 59, p. 505).]

C. Direct neutrino mass measurement experiments

1. Experiments studying the tritium β spectrum

The classical method of neutrino mass determination consists in a precision measurement of the spectrum of electrons from tritium decay in the end-point region. The electron spectrum in the allowed transition

$$ {}^{3}\mathrm{H} \rightarrow {}^{3}\mathrm{He} + e^- + \bar{\nu}_e \qquad (12.7) $$

is determined by the statistical weight and has the form

$$ \frac{dN}{dE} = n(E) = C^2 F(E) p E p_\nu E_\nu \ . \qquad (12.8) $$

Here p and E are the absolute value of the momentum and the total energy of the electron, $F(E)$ is the Fermi function (which takes into account the Coulomb interaction between ${}^{3}\mathrm{He}$ and e^-), and C is a constant, while

$$ E_\nu = E_0 - E, \ \ p_\nu = [(E_0 - E)^2 - m_\nu^2]^{1/2} \ , \qquad (12.9) $$

are the neutrino energy and momentum. In Eq. (12.9) E_0 is the total energy released in the decay (12.7). We have presented the electron spectrum in the simplest case of absence of neutrino mixing. In this case m_ν is the mass of the electron antineutrino. The possible modifications of the spectrum due to the existence of nontrivial neutrino mixing were discussed by Kobzarev *et al.* (1980) and by Shrock (1980). As a rule, the data of the ${}^{3}\mathrm{H}$ neutrino mass experiments are analyzed and the results are presented assuming that neutrino mixing does not take place.

Information about the neutrino mass is usually extracted from the Kurie function (Kurie plot)

$$ K(E) = \left[\frac{n(E)}{F(E) p E} \right]^{1/2} $$

$$ = C\{(E_0 - E)[(E_0 - E)^2 - m_\nu^2]^{1/2}\}^{1/2} $$

$$ = C\{(Q - T)[(Q - T)^2 - m_\nu^2]^{1/2}\}^{1/2} \ . \qquad (12.10) $$

Here $T = E - m_e$ is the kinetic electron energy and $Q = E_0 - m_e$. In the case of tritium decay, $Q \simeq 18.58$ keV. It is obvious from Eq. (12.10) that for $m_\nu = 0$ the Kurie function $K(T)$ is a straight line crossing the abscissa in the point $T_{\max} = Q$. If the neutrino mass is different from zero, the plot of the Kurie function deviates from a straight line in the region $Q - T \sim m_\nu$ and crosses the abscissa in the point $Q - T_{\max} = m_\nu$.

The electron spectrum measured in experiments represents a convolution of $n(E)$ with the response function of the apparatus:

$$ n_{\exp}(E) = \int n(E') R(E', E) dE' \ . $$

The instrumental response function $R(E', E)$ is determined in specific calibration experiments, in which conversion electrons with energies close to the end-point energy of the tritium β spectrum are used.

The highest sensitivity to date in the measurement of the hard part of the tritium β spectrum has been achieved in the work of the ITEP group (Boris *et al.*, 1984). In this work a toroidal magnetic spectrometer specially designed for neutrino mass measurements is used. A compound of tritium and valine is used as a source.

As is well known, the group from ITEP has claimed since 1980 that the neutrino mass is different from zero. At the high-energy physics conference in Leipzig this group presented data, from which it follows that (Boris *et al.*, 1984)

$$ m_\nu > 9 \ \mathrm{eV} \ \ (90\% \ \mathrm{C.L.}) \ . \qquad (12.11) $$

The most recent result of the group [obtained by using the spectrum of the final states of the T-valine molecule calculated by Kaplan *et al.* (1984)] reads (Lubimov, 1986)

$$ 17 < m_\nu < 40 \ \mathrm{eV} \ . \qquad (12.12) $$

The Kurie plots obtained by the ITEP group are presented in Fig. 11 (for three different sources). The solid curves were found from fits of the data for $m_\nu \neq 0$. The dashed curves were obtained by fitting the data in the case $m_\nu = 0$.

Let us note that the ITEP work has been criticized recently in several articles (Vilov, 1983; Simpson, 1984; Bergkvist, 1985). The authors of these articles hold that the ITEP group overestimates the accuracy with which the group has determined the instrumental response function, which, it is claimed (Simpson, 1984; Bergkvist, 1985;

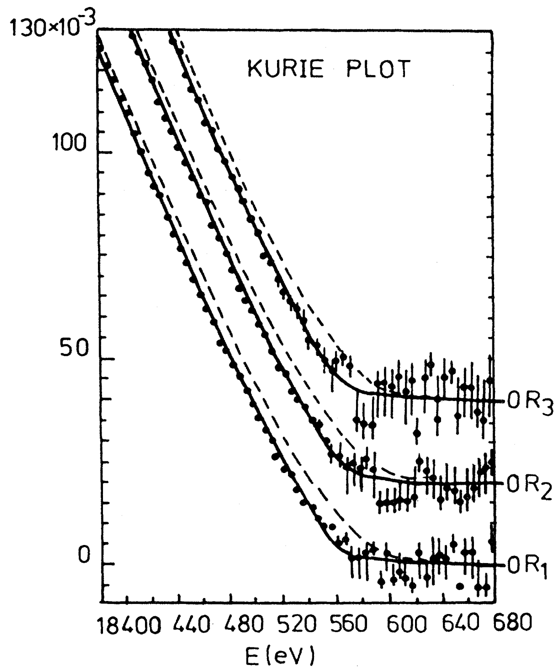

FIG. 11. The data of the ITEP group for the hard part of the spectrum of electrons from tritium decay (three different sources). The curves were obtained by fitting the data. In these fits the neutrino mass was either a free parameter (the solid curves) or set to zero (the dashed curves). The spectrum of the final states of the T-valine molecule was taken into account in the fits.

see also Robertson, 1985), may invalidate the conclusion that $m_\nu \neq 0$.

The ITEP work aroused enormous interest in setting up new experiments for a detailed study of the spectrum of electrons from tritium decay, with the aim of measuring the neutrino mass. Three experiments of this type are being performed (Fritschi *et al.*, 1986; Kawakami *et al.*, 1986; Robertson *et al.*, 1986) and at least twelve others are in preparation at present. These experiments make use of a variety of different sources (gaseous tritium, frozen tritium, compounds T-C, T-Ti, T-Al, etc.) and different types of spectrometers (see, for example, Robertson, 1985).

First results have been reported recently by the SIN group (Fritschi *et al.*, 1986). The β spectrum was studied in this experiment with a toroidal field, magnetic spectrometer of the type used by the ITEP group, modified with a radial, electrostatic retarding field around the source. The resolution of the spectrometer is 27 eV. The three sources used were prepared by implantation of ^3H ions into carbon, evaporated onto aluminum backing. Most important, the depth density distributions of the implanted ^3H in the sources were measured with a 50-Å resolution using the nuclear recoil technique (see, for example, Ross *et al.*, 1984). This permitted a reliable and rather accurate determination of the energy-loss spectrum of the electrons in the sources. The spectrometer resolution function used in the analysis of the data was calculated by Monte Carlo simulation. Conversion electron energy measurements are claimed to have confirmed the correctness of the calculated shape and width of the function (e.g., the calculated and the measured widths were found to differ by less than 10%).

The spectrum of the electrons in the end-point region, measured in four different runs in this experiment, is shown in Fig. 12 (Kurie plot). The curves are the best fits of the data points and correspond to $m_\nu = 0$. No indications for a nonzero neutrino mass were found in this experiment. A statistical upper limit for m_ν^2 of 106 eV^2 at 95% C.L. was obtained. Adding the estimated systematic error leads to the upper limit

$$m_\nu < 18 \text{ eV} , \qquad (12.13)$$

which constitutes the basic result of the experiment. Obviously, Eq. (12.13) is marginally compatible with the "model-dependent" result (12.12) of the ITEP group.

Many of the tritium experiments that are being performed or prepared at present aim to achieve an accuracy in neutrino mass measurements which would permit a check of the ITEP results. Undoubtedly, considerable progress in the study of the neutrino mass problem by the tritium method can be expected in the near future.

2. Limits on the masses of $\overset{(-)}{\nu}_\mu$ and $\overset{(-)}{\nu}_\tau$

The existing experimental upper limits on the masses of $\overset{(-)}{\nu}_\mu$ (m_{ν_μ}) and $\overset{(-)}{\nu}_\tau$ (m_{ν_τ}) are much less stringent than

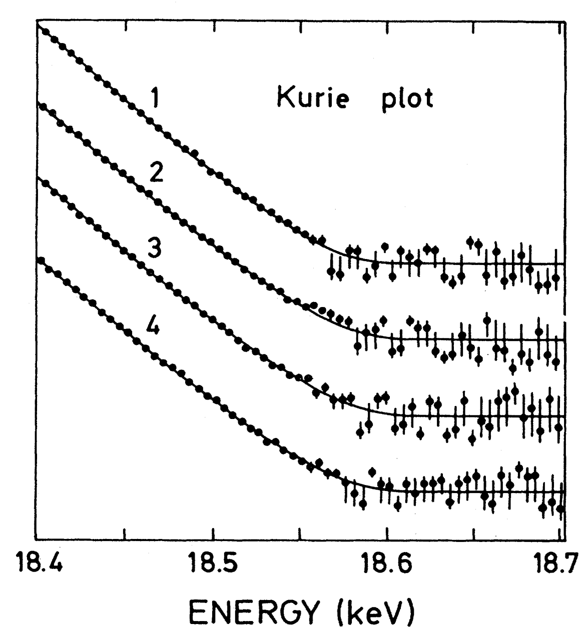

FIG. 12. The data of Fritschi *et al.* (1986) for the hard part of the electron spectrum from tritium decay obtained in four different runs. The solid lines are the best fits of the data and correspond to $m_\nu = 0$.

those on the mass of $\overset{(-)}{\nu}_e$:

$$m_{\nu_\mu} < 250 \text{ keV at } 90\% \text{ C.L.} \quad \text{(Abela } et\ al.\text{, 1984)}, \qquad (12.14)$$

$$m_{\nu_\tau} < 70 \text{ MeV at } 95\% \text{ C.L.} \quad \text{(Albrecht } et\ al.\text{, 1985)}. \qquad (12.15)$$

These limits were derived under the assumption that $\overset{(-)}{\nu}_\mu$ and $\overset{(-)}{\nu}_\tau$ are mass eigenstate neutrinos. The limit on m_{ν_μ} was obtained at SIN in a high-precision experiment in which the momentum of μ^+ originating from the decay $\pi^+ \to \mu^+ \nu_\mu$ at rest was measured. It represents an improvement by a factor of 2 over the previously existing limit on m_{ν_μ}.

The limit on m_{ν_τ} has been improved several times in the last four years by studying the hadron-invariant mass distributions in various exclusive hadronic decays of $\tau^\pm$:

$$\tau^{(\pm)} \to 3\pi^{(\pm)}\pi^0 \overset{(-)}{\nu}_\tau \quad \text{(Matteuzzi } et\ al.\text{, 1985)},$$

$$\tau^{(\pm)} \to K^\pm K^\mp \pi^{(\pm)} \overset{(-)}{\nu}_\tau \quad \text{(Mills } et\ al.\text{, 1985)},$$

$$\tau^{(\pm)} \to \pi^+\pi^-\pi^{(\pm)} \overset{(-)}{\nu}_\tau \quad \text{(Albrecht } et\ al.\text{, 1985)},$$

and

$$\tau^{(\pm)} \to 5\pi^{(\pm)} \overset{(-)}{\nu}_\tau \text{ and } \tau^{(\pm)} \to 5\pi^{(\pm)}\pi^0 \overset{(-)}{\nu}_\tau \quad \text{(Abachi } et\ al.\text{, 1986)}.$$

The result (12.15) is from ARGUS Collaboration at DESY and is the lowest upper limit on m_{ν_τ} obtained so far.

D. Results from neutrinoless double-β-decay experiments

Searching for the process

$$(A,Z) \to (A,Z+2) + e^- + e^-$$

is an extremely sensitive method of searching for a nonzero Majorana mass of neutrinos. So far the neutrinoless double-β decay has not been observed. However, in recent years the lower bounds on the $(\beta\beta)_{0\nu}$-decay lifetimes of different nuclei have been substantially increased.

The best limits on the lifetimes were obtained in experiments studying the decay

$$^{76}\text{Ge} \to {}^{76}\text{Se} + e^- + e^- .$$

In these experiments germanium detectors (normal germanium contains 7.76% of ^{76}Ge) that have good energy resolution and 4π geometry are used. One looks for peaks in the distribution with respect to the sum of the electron energies, corresponding to the $0^+ \to 0^+$ and $0^+ \to 2^+$ $(\beta\beta)_{0\nu}$ transitions $^{76}\text{Ge} \to {}^{76}\text{Se}$ (the energies released in the $0^+ \to 0^+$ and $0^+ \to 2^+$ transitions are equal to 2040.7 and 1481.6 keV, respectively). Record limits on the lifetime of the $0^+ \to 0^+$ $(\beta\beta)_{0\nu}$ decay of ^{76}Ge were obtained by three groups (68% C.L.):

$$T_{1/2}(0^+ \to 0^+) > 1.2 \times 10^{23} \text{ yr} \quad \text{(Bellotti } et\ al.\text{, 1984)}, \qquad (12.16a)$$

$$T_{1/2}(0^+ \to 0^+) > 2.5 \times 10^{23} \text{ yr} \quad \text{(Caldwell } et\ al.\text{, 1986)}, \qquad (12.16b)$$

$$T_{1/2}(0^+ \to 0^+) > 1.4 \times 10^{23} \text{ yr} \quad \text{(Avignone } et\ al.\text{, 1985)}. \qquad (12.16c)$$

The restrictions on the parameter

$$\langle m \rangle = \frac{1}{i} \sum_k m_k \eta_{cp}(\chi_k) \mid U_{ek} \mid^2$$

(see Sec. IX and Appendix B) that can be derived from these data depend on the theoretical model used in the calculation of the nuclear matrix element. If we assume that RH currents do not exist, then it follows from Eq. (12.16b) that (Doi, Kotani, and Takasugi, 1985; Klapdor, 1986)

$$\mid \langle m \rangle \mid < 1.0\text{–}6.3 \text{ eV}, \qquad (12.17)$$

where the uncertainty in the upper bound reflects the uncertainty existing at present in the theoretical calculations of the relevant nuclear matrix element.

The transition $0^+ \to 2^+$ is possible if RH charged currents enter into the weak interaction Hamiltonian. The best limit on the lifetime of the $0^+ \to 2^+$ $(\beta\beta)_{0\nu}$ transition $^{76}\text{Ge} \to {}^{76}\text{Se}$ were obtained in the recent work of Caldwell *et al.* (1986) and of Ejiri *et al.* (1986) (68% C.L.), respectively:

$$T_{1/2}(0^+ \to 2^+) > 5 \times 10^{22} \text{ yr},$$

$$T_{1/2}(0^+ \to 2^+) > 5.6 \times 10^{22} \text{ yr}.$$

Let us note that, to reduce the background in these experiments, electrons with a total energy $\simeq 1.48$ MeV were detected in coincidence with photons with an energy of 0.56 MeV (originating from the excited-to-ground-state transition of the ^{82}Se nucleus).

Experiments studying the $(\beta\beta)_{0\nu}$ decay of ^{76}Ge continue in many laboratories (see, for example, Caldwell, 1985). It is expected that the lower bound on $T_{1/2}$ will be increased in the next few years.

With the help of different methods the $(\beta\beta)_{0\nu}$ decay of nuclei other than ^{76}Ge is also being studied at present. Let us mention the work of Klimenko *et al.* (1984) who obtained for the $(\beta\beta)_{0\nu}$ decay of ^{150}Nd, at 90% confidence level, $T_{1/2} > 2.3 \times 10^{21}$ yr.

An upper bound on $\mid \langle m \rangle \mid$ in the range of a few eV has also been obtained by using geochemical data on the abundances of the xenon isotopes ^{128}Xe and ^{130}Xe in a natural one-billion-year-old tellurium ore (Kirsten, Richter, and Jessberger, 1983; see also Klapdor, 1986). The analysis used to derive the bound was based on the assumption that the measured concentrations of ^{128}Xe and ^{130}Xe in the ore were formed due to the $\beta\beta$ decays of ^{128}Te and ^{130}Te ($^{128,130}\text{Te} \to {}^{128,130}\text{Xe} + e^- + e^- +$ anything) in the period after the formation of the ore.[72] The present-day understanding of our planet formation and of the mineralization and cooling of ores indeed suggests that the initial concentrations of xenon isotopes in the tellurium ore studied were negligible. On the basis of earlier geochemical analyses, it was concluded (Kirsten *et al.*, 1968) that the 2ν decay (9.2) of ^{130}Te takes place and that the $\beta\beta$-decay half lifetime of ^{130}Te is (Kirsten, 1983) $T_{1/2}(^{130}\text{Te}) = (2.55 \pm 0.20) \times 10^{21}$ yr. [Similar analyses performed for the ^{82}Se-^{82}Kr system gave (see Kirsten, 1983) $T_{1/2}(^{82}\text{Se}) = (2.05 \pm 0.30) \times 10^{20}$ yr.] More recently a value for the ratio $R_T = T_{1/2}(^{130}\text{Te})/T_{1/2}(^{128}\text{Te})$ of the $\beta\beta$-decay half lifetimes of ^{130}Te and ^{128}Te compatible with zero and, correspondingly, a lower bound on $T_{1/2}(^{128}\text{Te})$ have been obtained (Kirsten, Richter, and Jessberger, 1983):

$$R_T = (1.01 \pm 1.13) \times 10^{-4} , \tag{12.18}$$

$$T_{1/2}(^{128}\text{Te}) > 8 \times 10^{24} \text{ yr (95\% C.L.)} . \tag{12.19}$$

It should be noted that the value of R_T is rather sensitive to the existence of the $(\beta\beta)_{0\nu}$ decay (Pontecorvo, 1968). In this case[73] $(\Gamma_{0\nu}^{128}/\Gamma_{2\nu}^{128}) \gg (\Gamma_{0\nu}^{130}/\Gamma_{2\nu}^{130})$, and therefore we should have $R_T = (\Gamma_{0\nu}^{128} + \Gamma_{2\nu}^{128})/(\Gamma_{0\nu}^{130} + \Gamma_{2\nu}^{130}) > (\Gamma_{2\nu}^{128}/\Gamma_{2\nu}^{130})$. Moreover, the determination of R_T from geochemical data is free from many of the errors associated with the determination of absolute values of $T_{1/2}(^{128}\text{Te})$ and $T_{1/2}(^{130}\text{Te})$ (e.g., errors in the age of the ore, absolute quantities of ^{128,130}Te and ^{128,130}Xe present in the ore, etc.). It was suggested (Pontecorvo, 1968) that the theoretical uncertainties in the calculation of R_T would be smaller than the uncertainties in the calculations of $T_{1/2}(^{128,130}\text{Te})$. However, this does not seem to be the case at present, as the most recent results in the evaluation of the $(\beta\beta)_{2\nu}$- and $(\beta\beta)_{0\nu}$-decay nuclear matrix elements for ^{128,130}Te indicate (see, for example, Klapdor, 1986).

Using the lower bound (12.19) and the values of the relevant $(\beta\beta)_{0\nu}$-decay nuclear matrix element derived by Klapdor and Grotz (1985), and assuming that RH currents do not exist, one finds (Klapdor, 1986)

$$|\langle m \rangle| < 0.35 - 1.6 \text{ eV} . \tag{12.20}$$

Experimental studies of neutrinoless double-β decay are continuously increasing. Several groups are developing new methods of searching for the $(\beta\beta)_{0\nu}$ decay of ^{136}Xe (e.g., Caldwell, 1985). Considerable progress in the study of this extremely important process (from the standpoint of solving the neutrino mass problem) will undoubtedly be achieved in the next few years.

ACKNOWLEDGMENTS

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APPENDIX A: MAJORANA NEUTRINO FIELDS

In this appendix we shall give information about the Majorana field, necessary for the understanding of this review (see Majorana, 1937; Mannheim, 1980). The Majorana particles are truly neutral particles (particles all of whose additive charges are equal to zero) with spin $\frac{1}{2}$. We shall denote the Majorana field by $\chi(x)$.

The operator of a free Majorana field satisfies the Dirac equation

$$(\gamma_\alpha \partial_\alpha + m)\chi(x) = 0 \tag{A1}$$

and the Majorana condition

$$C\bar{\chi}^T(x) = \xi\chi(x) , \tag{A2}$$

where ξ is a phase factor (we shall choose $\xi = \pm 1$), and the matrix C satisfies the conditions

$$C\gamma_\alpha^T C^{-1} = -\gamma_\alpha, \quad C^\dagger C = \mathbb{1}, \quad C^T = -C . \tag{A3}$$

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[72] Obviously, the ^{128,130}Te decay rates determined by geochemical methods represent sums of the rates of all possible ^{128,130}Te decays in which ^{128,130}Xe is formed:

$$\Gamma_{\beta\beta}^{128(130)} = \Gamma_{0\nu}^{128(130)} + \Gamma_{2\nu}^{128(130)} + \cdots ,$$

where

$$\Gamma_{\beta\beta}^{128(130)} \equiv \Gamma(^{128(130)}\text{Te} \to {}^{128(130)}\text{Xe} + e^- + e^- + \text{anything}) ,$$

$$\Gamma_{0\nu}^{128(130)} \equiv \Gamma(^{128(130)}\text{Te} \to {}^{128(130)}\text{Xe} + e^- + e^-) ,$$

and

$$\Gamma_{2\nu}^{128(130)} \equiv \Gamma(^{128(130)}\text{Te} \to {}^{128(130)}\text{Xe} + e^- + e^- + \bar{\nu}_e + \bar{\nu}_e) .$$

[73] The ratio $(\Gamma_{0\nu}^{128}/\Gamma_{2\nu}^{128})/(\Gamma_{0\nu}^{130}/\Gamma_{2\nu}^{130})$ is determined approximately by the ratios of the phase-space factors in $\Gamma_{0\nu,2\nu}^{128}$ and $\Gamma_{0\nu,2\nu}^{130}$ and is roughly given by $(\varepsilon_0'/\varepsilon_0)^6 \simeq 600$, where $\varepsilon_0 = 1.7$ and $\varepsilon_0' = 4.96$ are the kinetic energies (in units of m_e) released in the ^{128}Te and ^{130}Te $\beta\beta$ decays (see Doi, Kotani, and Takasugi, 1985).

The operator $\chi(x)$ has the form

$$\chi(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{1}{\sqrt{2p_0}}[u^r(p)a_r(p)e^{ipx}+\xi u^r(-p)a_r^\dagger(p)e^{-ipx}]d\mathbf{p}$$

$$=\chi^{(+)}(x)+\chi^{(-)}(x)\ , \tag{A4}$$

where

$$u^r(-p)=C[\overline{u^r}(p)]^T \tag{A5}$$

and the spinors $u^r(p)$ are normalized by the condition

$$\overline{u^r}(p)u^r(p)=2m\delta_{r'r}\ . \tag{A6}$$

The operators $a_r(p)$ and $a_r^\dagger(p)$ satisfy the canonical anticommutation relations

$$[a_r(p),a_{r'}(p')]_+=0\ ,$$
$$[a_r(p),a_{r'}^\dagger(p')]_+=\delta_{rr'}\delta(\mathbf{p}-\mathbf{p}') \tag{A7}$$

and, respectively, are the annihilation and creation operators of a Majorana particle with four-momentum p and helicity r.

For the propagator

$$\overline{\chi(x_1)\overline{\chi}(x_2)}$$

we obtain with the help of Eqs. (A4) and (A7)

$$\overline{\chi_\alpha(x_1)\overline{\chi}_\beta(x_2)}=\begin{cases}[\chi_\alpha^{(+)}(x_1),\overline{\chi}_\beta^{(-)}(x_2)]_+, & x_{10}>x_{20}\\ -[\overline{\chi}_\beta^{(+)}(x_2),\chi_\alpha^{(-)}(x_1)]_+, & x_{20}>x_{10}\end{cases}$$

$$=\frac{-1}{(2\pi)^4}\int\left[\frac{1}{\gamma p-im}\right]_{\alpha\beta}e^{ip(x_1-x_2)}dp=S_{\alpha\beta}(x_1-x_2)\ . \tag{A8}$$

Thus the Majorana particle propagator

$$\overline{\chi(x_1)\overline{\chi}(x_2)}$$

coincides with the standard Dirac propagator

$$\overline{\psi(x_1)\overline{\psi}(x_2)}\ .$$

The essential distinction between a Majorana field and a Dirac field consists in the fact that for a Majorana field the propagators

$$\overline{\chi(x_1)\chi^T(x_2)}$$

and

$$\overline{\overline{\chi}^T(x_1)\overline{\chi}(x_2)}$$

are different from zero as well; obviously,

$$\overline{\psi(x_1)\psi^T(x_2)}=0$$

and

$$\overline{\overline{\psi}^T(x_1)\psi(x_2)}=0\ .$$

Using Eqs. (A2) and (A3), we have

$$\overline{\chi(x_1)\chi^T(x_2)}=-\xi S(x_1-x_2)C\ ,$$
$$\overline{\overline{\chi}^T(x_1)\overline{\chi}(x_2)}=\xi C^{-1}S(x_1-x_2)\ . \tag{A9}$$

These relations are a consequence of $\chi(x)$'s being the field of an absolutely neutral particle. They are used in the calculation, for example, of the $(\beta\beta)_{0\nu}$-decay amplitude.

We shall show next that the CP parity of a Majorana particle can assume the values $\pm i$. Let us denote by U_{CP} the CP conjugation operator. One has

$$U_{CP}\chi(x)U_{CP}^\dagger=\eta_{CP}\gamma_4\chi(x')\ . \tag{A10}$$

Here η_{CP} is a phase factor [the CP parity of the Majorana field $\chi(x)$] and $x'=(-\mathbf{x},ix_0)$. From Eq. (A10) using (A3) we obtain

$$U_{CP}C\overline{\chi}^T(x)U_{CP}^\dagger=-\eta_{CP}^*\gamma_4C\overline{\chi}^T(x')\ . \tag{A11}$$

Taking into account Eq (A2) and comparing (A11) with (A10), we find[74]

$$\eta_{CP}^2=-1,\quad \eta_{CP}=\pm i\ . \tag{A12}$$

Let us note that the CP parity of the Majorana particle χ with momentum p is given by η_{CP}^*.

In conclusion, consider the bilinear form

$$\overline{\chi}_2O\chi_1\ ,$$

where $\chi_1(x)$ and $\chi_2(x)$ are Majorana fields and O is any of the Dirac matrices $(\mathbb{1},\gamma_\alpha,\sigma_{\alpha\beta},\gamma_\alpha\gamma_5,\gamma_5)$. Using the Majorana condition (A2), we get

$$\overline{\chi}_2O\chi_1=-\chi_1^TO^T\overline{\chi}_2^T=\xi_1\xi_2\overline{\chi}_1CO^TC^{-1}\chi_2\ . \tag{A13}$$

Further, with the help of Eq. (A3) one finds

$$C\sigma_{\alpha\beta}^TC^{-1}=-\sigma_{\alpha\beta},\quad C\gamma_5^TC^{-1}=\gamma_5,$$
$$C(\gamma_\alpha\gamma_5)^TC^{-1}=\gamma_\alpha\gamma_5\ .$$

Thus we have

[74]It is well known that the product of the intrinsic parities of a spin-$\frac{1}{2}$ particle and its antiparticle is equal to -1. One can easily show that the product of the CP parities of a spin-$\frac{1}{2}$ particle and its antiparticle is equal to -1 as well. In the Majorana case the particle coincides with its antiparticle. As a result, we arrive at Eq. (A12).

$$\begin{aligned}
&\overline{\chi}_2\chi_1=\xi_1\xi_2\overline{\chi}_1\chi_2 \,,\\
&\overline{\chi}_2\gamma_\alpha\chi_1=-\xi_1\xi_2\overline{\chi}_1\gamma_\alpha\chi_2 \,,\\
&\overline{\chi}_2\sigma_{\alpha\beta}\chi_1=-\xi_1\xi_2\overline{\chi}_1\sigma_{\alpha\beta}\chi_2 \,,\\
&\overline{\chi}_2\gamma_\alpha\gamma_5\chi_1=\xi_1\xi_2\overline{\chi}_1\gamma_\alpha\gamma_5\chi_2 \,,\\
&\overline{\chi}_2\gamma_5\chi_1=\xi_1\xi_2\overline{\chi}_1\gamma_5\chi_2 \,.
\end{aligned} \tag{A14}$$

In particular, it follows from Eq. (A14) that the vector current of the Majorana field is identically equal to zero:

$$\overline{\chi}(x)\gamma_\alpha\chi(x)=0 \,. \tag{A15}$$

APPENDIX B: NEUTRINOLESS DOUBLE-β DECAY (BASIC ELEMENTS OF THE THEORY)

In this appendix we shall give a brief exposition of the basic elements of the theory of neutrinoless double-β decay[75]

$$(A,Z)\to(A,Z+2)+e^-+e^- \,. \tag{B1}$$

(For more detailed treatments, see Primakoff and Rosen, 1969, 1981; Vergados, 1981; Haxton, Stephenson, and Strottman, 1982, 1984; Doi *et al.*, 1983a; Doi, Kotani, and Takasugi, 1984.) The lepton charge is not conserved in this process, which can take place only if neutrinos are Majorana particles.[76]

We shall consider the process (B1), assuming that the weak β-decay Hamiltonian has the standard form

$$\mathscr{H}_W^\beta=\frac{G_F}{\sqrt{2}}2(\overline{e_L}\gamma_\alpha\nu_{eL})j_\alpha+\text{H.c.} \,, \tag{B2}$$

where $j_\alpha(x)$ is the strangeness-conserving charged hadron current, and that neutrino mixing does take place,

$$\nu_{eL}=\sum_k U_{lk}^L\chi_{kL} \,. \tag{B3}$$

Here $\chi_k(x)$ is the field of a Majorana neutrino with mass m_k and U is a unitary mixing matrix. The fields $\chi_k(x)$ satisfy the Majorana condition, which we shall write in the general form

$$C\overline{\chi_k}^T(x)=\xi_k\chi_k(x) \,, \tag{B4}$$

where ξ_k is a phase factor. We shall choose $\xi_k=\pm 1$.

Clearly $(\beta\beta)_{0\nu}$ decay can occur in second order of perturbation theory in the weak interaction. The following term in the S matrix gives the contribution to the matrix element of the process (B1) in second order of perturbation theory in G_F:

$$S^{(2)}=-\frac{(-i)^2}{2}4\left[\frac{G_F}{\sqrt{2}}\right]^2\int N[\overline{e_L}(x_1)\gamma_\alpha\overline{\nu_{eL}(x_1)\nu_{eL}^T}(x_2)\gamma_\beta^T\,\overline{e_L}^T(x_2)]T\left[j_\alpha(x_1)j_\beta(x_2)\exp\left[-i\int\mathscr{H}_{\text{str}}(x)dx\right]\right]dx_1dx_2 \,. \tag{B5}$$

Here $\mathscr{H}_{\text{str}}(x)$ is the strong-interaction Hamiltonian. Note that in Eq. (B5) the strong interaction is taken into account exactly. With the help of Eqs. (A9), (B3), and (B4) we easily obtain

$$\overline{\nu_{eL}(x_1)\nu_{eL}^T}(x_2)=-\sum_k(U_{ek}^L)^2\xi_k\frac{1+\gamma_5}{2}S_k(x_1-x_2)\frac{1+\gamma_5}{2}C \,, \tag{B6}$$

where $S_k(x_1-x_2)$ is the propagator of the Majorana neutrino with mass m_k [see Eq. (A8)]. Further, one has

$$\frac{1+\gamma_5}{2}S_k(x_1-x_2)\frac{1+\gamma_5}{2}=\frac{-i}{(2\pi)^4}m_k\int\frac{e^{iq(x_1-x_2)}dq}{q^2+m_k^2}\frac{1+\gamma_5}{2} \,.$$

Using Eqs. (B5) and (B6), we obtain for the matrix element of the process (B1)

$$\begin{aligned}
\langle f\,|\,S^{(2)}\,|\,i\rangle=&-\left[\frac{G_F}{\sqrt{2}}\right]^2\frac{1}{\sqrt{4p_{10}p_{20}}}\frac{1}{(2\pi)^3}\sum_k(U_{ek}^L)^2m_k\xi_k\overline{u}(p_1)\gamma_\alpha(1+\gamma_5)\gamma_\beta C\overline{u}^{\,T}(p_2)\\
&\times\int e^{-ip_1x_1-ip_2x_2}\frac{(-i)}{(2\pi)^4}\int\frac{e^{iq(x_1-x_2)}dq}{q^2+m_k^2} \,,\\
&\langle p'\,|\,T[J_\alpha(x_1)J_\beta(x_2)]\,|\,p\rangle dx_1dx_2-(p_1\rightleftarrows p_2) \,.
\end{aligned} \tag{B7}$$

Here p_1 and p_2 are the four-momenta of the electrons, p and p' are the four-momenta of the initial and final nuclei, and $J_\alpha(x)$ is the weak charged current in the Heisenberg representation.

[75]The first calculation of the probability for such a decay to occur was carried out by Furry (1939).

[76]That $(\beta\beta)_{0\nu}$ decay can take place if the neutrino emitted together with the electron in β decays is a Majorana particle, was realized first by Racah (1937).

The second term in (B7) arises due to the identity of the final-state electrons. It is not difficult to show that it coincides with the first term. This can be done by using the relation

$$\bar{u}(p_1)\gamma_\alpha(1+\gamma_5)\gamma_\beta C\bar{u}^{\,T}(p_2)=\bar{u}(p_2)C^T\gamma_\beta^T(1+\gamma_5)^T\gamma_\alpha^T\bar{u}^{\,T}(p_1)=-\bar{u}(p_2)\gamma_\beta(1+\gamma_5)\gamma_\alpha C\bar{u}^{\,T}(p_1)$$

as well as the possibility of interchanging the current operators under the sign of the T product.

As can be seen from Eq. (B7), the amplitude of the $(\beta\beta)_{0\nu}$ decay vanishes if the masses of the Majorana neutrinos are equal to zero. This is connected with the fact that the lepton charges are conserved if only LH fields enter into the weak-interaction Lagrangian and the neutrinos are massless. Indeed, consider the current

$$j_\alpha=(\overline{e_L}\gamma_\alpha\chi_L)\ , \tag{B8}$$

where $\chi(x)$ is the field of a massless Majorana neutrino. It is not difficult to see that

$$\chi'(x)=e^{i\alpha\gamma_5}\chi(x) \tag{B9}$$

(α is an arbitrary real parameter) is also a Majorana field. One has

$$\chi'_L(x)=e^{i\alpha}\chi_L(x),\ \ \chi'_R(x)=e^{-i\alpha}\chi_R(x)\ . \tag{B10}$$

It follows from Eqs. (B8) and (B10) that the Lagrangian of the system under consideration is invariant with respect to the transformations

$$e(x)\to e'(x)=e^{i\alpha}e(x),\ \ \chi_L(x)\to\chi'_L(x)=e^{i\alpha}\chi_L(x) \tag{B11}$$

[in the case of zero neutrino mass $\chi_R(x)$ is not present in the Lagrangian]. This invariance implies conservation of the lepton charge, which is equal to 1 for e^- and the LH neutrino, to (-1) for e^+ and the RH neutrino, and to zero for all other particles. It should be noted that the invariance with respect to the transformations (B11) lies at the root of the theorem stating the equivalence (for $m_\nu=0$) of the theory with Dirac neutrino and the theory with Majorana neutrino (Case, 1957; Ryan and Okubo, 1964).

Bearing in mind the approximations we shall make further, let us perform the integration over the time variables in (B7). Taking into account that

$$\langle p'\,|J_\alpha(x_1)J_\beta(x_2)\,|\,p\rangle=\sum_n\langle p'\,|J_\alpha(\mathbf{x}_1)\,|\,n\rangle\langle n\,|J_\beta(\mathbf{x}_2)\,|\,p\rangle e^{-i(E'-E_n)x_{10}}e^{-i(E_n-E)x_{20}}$$

[where E and E' are the energies of the initial and final nuclei, E_n is the energy of the intermediate state, and $J_{\alpha(\beta)}(\mathbf{x},0)\equiv J_{\alpha(\beta)}(\mathbf{x})$], and defining the integrals correctly using the standard procedure of adiabatic switch-off of the interaction at $x_0\to\pm\infty$,

$$\int_{-\infty}^0 e^{ia\tau}dt\Longrightarrow\lim_{\varepsilon\to 0}\int_{-\infty}^0 e^{i(a-i\varepsilon)\tau}d\tau=\lim_{\varepsilon\to 0}\frac{-i}{a-i\varepsilon}\ ,\quad \int_0^\infty e^{-ia\tau}d\tau\Longrightarrow\lim_{\varepsilon\to 0}\int_0^\infty e^{-i(a-i\varepsilon)\tau}d\tau=\lim_{\varepsilon\to 0}\frac{-i}{a-i\varepsilon}\ ,$$

we get from Eq. (B7) for the matrix element of the $(\beta\beta)_{0\nu}$ decay

$$\begin{aligned}\langle f\,|S^{(2)}|\,i\rangle=i\left[\frac{G_F}{\sqrt{2}}\right]^2\frac{1}{\sqrt{4p_{10}p_{20}}}\frac{1}{(2\pi)^3}\sum_k(U^L_{ek})^2m_k\xi_k\bar{u}(p_1)\gamma_\alpha(1+\gamma_5)\gamma_\beta C\bar{u}^{\,T}(p_2)\\ \times\int d\mathbf{x}_1d\mathbf{x}_2e^{-i\mathbf{p}_1\mathbf{x}_1-i\mathbf{p}_2\mathbf{x}_2}\left[\frac{1}{(2\pi)^3}\int\frac{e^{i\mathbf{q}(\mathbf{x}_1-\mathbf{x}_2)}d\mathbf{q}}{q_{0k}}\right]\\ \times\sum_n\left[\frac{\langle p'\,|J_\alpha(\mathbf{x}_1)\,|\,n\rangle\langle n\,|J_\beta(\mathbf{x}_2)\,|\,p\rangle}{E_n+q_{0k}+p_{20}-E}\right.\\ \left.+\frac{\langle p'\,|J_\beta(\mathbf{x}_2)\,|\,n\rangle\langle n\,|J_\alpha(\mathbf{x}_1)\,|\,p\rangle}{E_n+q_{0k}+p_{10}-E}\right]2\pi\delta(E'+p_{10}+p_{20}-E)\ ,\end{aligned} \tag{B12}$$

where $q_{0k}=(\mathbf{q}^2+m_k^2)^{1/2}$.

All expressions obtained so far have been exact. Next we shall consider briefly the hadron part of the matrix element. We shall discuss only the $O^+\to O^+$ nuclear transitions.

The following three approximations are usually made in calculations of the $(\beta\beta)_{0\nu}$-decay amplitudes[77] (Primakoff and

[77]Let us note that many recent papers have been devoted to the calculation of corrections to the closure approximation and other approximations usually used (e.g., Doi, Kotani, and Takasugi, 1984; Haxton, Stephenson, and Strottman, 1984; Klapdor and Grotz, 1984).

Rosen, 1959, 1961).

(1) Closure approximation. The energies of the intermediate states $|n\rangle$ in (B12) are replaced by an averaged energy $E_n \to \langle E_n \rangle$. This allows us to perform the summation over the complete system of states $|n\rangle$ in (B12).

(2) Long wave approximation. Since $|\mathbf{p}_{1,2}|R \ll 1$, where R is the radius of the decaying nucleus, the substitution $e^{-i\mathbf{p}_1\mathbf{x}_1 - i\mathbf{p}_2\mathbf{x}_2} \to 1$ is made in Eq. (B12).

(3) Nonrelativistic momentum approximation and the two-nucleon mechanism. The following approximate expression is assumed for the current operator in (B12):

$$J_\alpha(\mathbf{x}) = \sum_n (\tau_+)_n [\delta_{\alpha 4} + i g_A \delta_{\alpha k} (\sigma_k)_n] \delta(\mathbf{x} - \mathbf{x}_n) , \tag{B13}$$

where the sum is over all nucleons in the initial nucleus. In Eq. (B13) g_A is the axial constant ($g_A \equiv 1.25$).

In the approximation (B13) one has

$$J_\alpha(\mathbf{x}_1) J_\beta(\mathbf{x}_2) = J_\beta(\mathbf{x}_2) J_\alpha(\mathbf{x}_1) . \tag{B14}$$

To convince oneself of the validity of (B14), it is necessary to take into account that $(\tau_+)_n(\tau_+)_n = 0$. If, further, we write Eq. (B12) in the form

$$\langle f | S^{(2)} | i \rangle = \bar{u}(p_1) \gamma_\alpha \gamma_\beta (1-\gamma_5) C \bar{u}^T(p_2) A_{\alpha\beta} ,$$

then from Eq. (B14) it follows that

$$A_{\alpha\beta} = A_{\beta\alpha} . \tag{B15}$$

Let us write

$$\gamma_\alpha \gamma_\beta = \delta_{\alpha\beta} + \tfrac{1}{2}(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha) . \tag{B16}$$

It is clear from Eq. (B15) that the second term on the right-hand side of (B16) does not contribute to the matrix element of the process under consideration. As a result, we obtain the following expression for the matrix element of the $(\beta\beta)_{0\nu}$ decay in the laboratory system:

$$\begin{aligned}\langle f | S^{(2)} | i \rangle = {} & \frac{i}{4(2\pi)^4} \left[\frac{G_F}{\sqrt{2}}\right]^2 \frac{1}{\sqrt{p_{10}p_{20}}} \left[\sum_k (U^L_{ek})^2 m_k \xi_k \right] \bar{u}(p_1)(1-\gamma_5) C \bar{u}^T(p_2) \\ & \times \left\langle \psi_f \left| \sum_{n,m} [H_1(|\mathbf{x}_n - \mathbf{x}_m|; m_k) + H_2(|\mathbf{x}_n - \mathbf{x}_m|; m_k)](\tau_+)_n(\tau_+)_m (1 - g_A^2 \boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_m) \right| \psi_i \right\rangle \\ & \times (2\pi)\delta(p_{10} + p_{20} + M' - M) .\end{aligned} \tag{B17}$$

Here ψ_i and ψ_f are the wave functions of the initial and final nuclei, M and M' are the masses of the initial and final nuclei (we have neglected the recoil of the final nucleus), and

$$H_j(|\mathbf{x}|; m_k) = \frac{1}{2\pi^2} \int \frac{e^{i\mathbf{q}\cdot\mathbf{x}} d\mathbf{q}}{q_{0k}(q_{0k} + a_j)}, \quad j = 1,2 , \tag{B18}$$

where $a_j = \langle E_n \rangle + p_{0j} - M$. Let us rewrite the considered matrix element as follows:

$$\begin{aligned}\langle f | S^{(2)} | i \rangle = {} & \frac{i}{2(2\pi)^3} \left[\frac{G_F}{\sqrt{2}}\right]^2 \frac{1}{\sqrt{p_{10}p_{20}}} \left[\sum_k (U^L_{ek})^2 m_k \xi_k \right] \frac{1}{R} \\ & \times \bar{u}(p_1)(1-\gamma_5) C \bar{u}^T(p_2)(M_F - g_A^2 M_{\mathrm{GT}}) \delta(p_{10} + p_{20} + M' - M) .\end{aligned} \tag{B19}$$

Here

$$M_F = \left\langle \psi_f \left| \sum_{n,m} h(|\mathbf{x}_n - \mathbf{x}_m|; m_k)(\tau_+)_n(\tau_+)_m \right| \psi_i \right\rangle ,$$
$$M_{\mathrm{GT}} = \left\langle \psi_f \left| \sum_{n,m} h(|\mathbf{x}_n - \mathbf{x}_m|; m_k)(\tau_+)_n(\tau_+)_m \boldsymbol{\sigma}_n \boldsymbol{\sigma}_m \right| \psi_i \right\rangle \tag{B20}$$

are the Fermi and the Gamow-Teller matrix elements and R is the radius of the intial nucleus, while

$$h(|\mathbf{x}|; m_k) = \tfrac{1}{2} R [H_1(|\mathbf{x}|; m_k) + H_2(|\mathbf{x}|; m_k)] . \tag{B21}$$

Neglecting a_j in comparison with q_{0k} in (B18) (a_j is of the order of several MeV, $|\mathbf{q}| \sim 35$ MeV), we get

$$h(|\mathbf{x}|;m_k) \simeq \frac{R}{|\mathbf{x}|} e^{-m_k|\mathbf{x}|} . \tag{B22}$$

When the masses of the Majorana neutrinos are sufficiently small ($m_k \lesssim$ a few MeV) one has

$$h(|\mathbf{x}|;m_k) \simeq \frac{R}{|\mathbf{x}|} . \tag{B23}$$

The neutrino masses enter into the matrix element of the $(\beta\beta)_{0\nu}$ decay in this case essentially through the factor

$$\langle m \rangle = \sum_k (U^L_{ek})^2 m_k \xi_k . \tag{B24}$$

Consequently, by using the existing experimental data, it is possible to obtain upper limits on the quantity $|\langle m \rangle|^2$ (see Sec. XII.B.5).

Let us now derive the expression for the probability of the considered process. For the differential probability of the decay we find from Eq. (B19)

$$d\Gamma_{0\nu} = \frac{1}{2}\frac{G_F^4 m_e^5}{(2\pi)^5} |\langle m \rangle|^2 \frac{1}{R^2} |M_F - g_A^2 M_{\rm GT}|^2 (1-\cos\theta)$$
$$\times (\varepsilon_0 - \varepsilon + 1)^2 (\varepsilon+1)^2 d\varepsilon \sin\theta \, d\theta \, F^2(Z) . \tag{B25}$$

Here ε is the kinetic energy of the electron in units of the electron mass m_e, θ is the angle between the final electrons,

$$\varepsilon_0 = \frac{1}{m_e}(M - M' - 2m_e)$$

is the kinetic energy of the final electrons, and

$$F(Z) = \frac{2\pi\alpha(Z+2)}{1-\exp[-2\pi\alpha(Z+2)]} \tag{B26}$$

is the Fermi factor of Coulomb corrections.[78] Note that we have neglected the mass of the electron in comparison with its energy in the calculation of the differential probability $d\Gamma_{0\nu}$.

As can be seen from Eq. (B25), the $(\beta\beta)_{0\nu}$-decay probability vanishes at $\theta=0$ in the approximation considered by us. This is connected with the fact that ultrarelativistic electrons possess negative helicity in the case when the interaction Hamiltonian is given by (B2) and as a consequence of angular momentum conservation cannot be emitted in one direction in the $O^+ \to O^+$ transition.

From Eq. (B25) we obtain for the total probability of the $(\beta\beta)_{0\nu}$ decay

$$\Gamma_{0\nu} = \frac{1}{2}\frac{G_F^4 m_e^5}{(2\pi)^5} |\langle m \rangle|^2 \frac{1}{R^2} |M_F - g_A^2 M_{\rm GT}|^2 F^2(Z)$$
$$\times \tfrac{1}{15}(\varepsilon_0^5 + 10\varepsilon_0^4 + 40\varepsilon_0^3 + 60\varepsilon_0^2 + 30\varepsilon_0) . \tag{B27}$$

The calculation of the matrix elements M_F and $M_{\rm GT}$ requires a knowledge of the wave functions of complicated nuclei and is a rather complicated problem (see, for example, Haxton, Stephenson, and Strottman, 1982).

[78]Let us note that the probability of $(\beta\beta)_{0\nu}$ decay increases considerably if one takes into account the relativistic Coulomb wave function of the electron (Haxton, Stephenson, and Strottman, 1982; Doi *et al.*, 1983a).

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