On waves in gases. Part II: Interaction of sound with magnetic and internal modes

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This work completes a two-part review on waves in gases, of which the first part [Rev. Mod. Phys. **58**, 117 (1986)] dealt with the modern aspects of acoustics of jets, turbulence, and ducts; this second part extends the range of topics from sound to magnetic, internal, and (to a lesser extent) inertial waves, thus considering all four restoring forces (pressure, gravity, and Lorentz and Coriolis forces). The motivations for the study of these waves were outlined in the introduction to Part I. Part II reviews the coupling of acoustic, magnetic, and internal waves, in four stages: in Sec. I dispersion relations are used to study the propagation and radiation of magneto-acoustic-gravity-inertial waves in media for which the wave speeds and scattering scales are constant; in Sec. II the case of linear waves in stratified media, with nonuniform propagation velocity, is then discussed by means of special functions, appearing as exact solutions of second-order problems; in Sec. III the study of linear waves with variable propagation speeds is extended to certain classes of higher-order problems including a discussion of cutoff frequencies, critical levels, partition of energy, mode coupling and conversion, etc; in Sec. IV the preceding studies are extended to damped and nonlinear waves, to include dissipation with variable damping scales and large disturbances in media under nonuniform external forces, such as magnetic flux tubes. The conclusion (Sec. V) sums up both parts of the review, in the sense that it deals with all types of waves in fluids; it mentions a few currently controversial topics, points out some directions for future research, and indicates methods available to address these issues.

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We extend the study of waves in fluids (Brekhovskikh, 1960; Chandrasekhar, 1961; Tolstoy, 1973; Whitham, 1974; Lighthill, 1978), from the classical case of acoustics (Rayleigh, 1877; Landau and Lifshitz, 1953; Beranek, 1954; Mason, 1964—1973; Morse and Ingard, 1968; Goldstein, 1976; Levine, 1978; Pierce, 1981; Brekhovskikh and Lysanov, 1982; Dowling and Ffowcs-Williams, 1983) to other types of waves, viz., magnetic (Alfvén, 1948; Landau and Lifshitz, 1956; Cowling, 1960; Alfvén and Falthammar, 1962; Ferraro and Plumpton, 1963; Shercliff, 1965; Cabannes, 1970), internal (Pedlosky, 1960; Yih, 1965; Philips, 1966; Delloue and Halley, 1972; Turner, 1973; Beer, 1974; Hines, 1974; Gossard and Hooke, 1975; Kraus, 1977), and inertial (Eckart, 1960; Tolstoy, 1963; Greenspan, 1968; Acheson and Hide, 1973; Moffatt, 1978; Gill, 1982). In the study of the interaction of these four types of waves, we should consider a fluid under all four restoring forces, viz., pressure, gravity, and Coriolis and Lorentz forces, e.g., a compressible, self-gravitating, rotating gas under magnetic fields. The main examples of such a fluid are to be found in astrophysics, e.g., in galaxies or stars, of which the closest, which has been the subject of more detailed observation, is the sun (Bray and Loughhead, 1974; Athay, 1976; Bruzek and Durrant, 1977; Gabriel and Elliot, 1980; Bonnet and Dupree, 1981; Stenflo, 1982; Gough, 1986). The main aim of the present work is to examine the physics of waves under all four restoring forces. In order to provide a brief illustration of their properties, while avoiding lengthy "morphological" descriptions, we choose as a standard example or "test laboratory" the phenomena in the outer, visible layers of the sun. Although much progress has been made in the study of the physics of the solar atmosphere (Parker, 1979; Gabriel and Mason, 1982; Priest, 1982a, 1982b), the interpretation of most observations is still a subject of controversy between many competing theories. Although we do provide, by means of references (Leibacher, 1985), an indication of alternative theories, our aim is not to say the last word on solar physics, but only to use the sun as a demonstration laboratory (provided by nature) to study the physics of interacting acoustic, magnetic, and internal waves.

The subject has many ramifications, some of which were mentioned in the introduction to Part I, so that we need add here only a few more references, adequate for the beginning of a literature search. The sun is the star closest to the Earth, and thus the most accessible probe of theories of stellar structure and evolution (Eddington, 1926; Chandrasekhar, 1957; Clayton, 1968; Reddish, 1978) and of dynamical and atmospheric processes (Mihalas, 1939; Chandrasekhar, 1942; Unsold, 1955; Athay, 1972; Swihart, 1981). The various types of stars (Rosseland, 1949; Schatzman, 1958; Shklovskii, 1968; Kopal, 1978), together with interstellar matter (Spitzer, 1968, 1978), form galaxies and galactic clusters (Mihalas and Binney, 1968; Fall and Lynden-Bell, 1981), which are the largest "structures" considered in astrophysics (Ambartsumy'an, 1958; Pacholczyk, 1970; Kourganoff, 1980) and cosmology (Hawking and Ellis, 1973; Segal, 1976). Astrodynamics and celestial mechanics (Danby, 1962; Herrick, 1971; Hagihara, 1972) are also relevant at the opposite end of the scale, that of the solar system (Kuiper, 1953; Alfvén, 1954; Lyttleton, 1968; Pottasch, 1984); our knowledge of our solar system's constituent bodies (Urey, 1952; Lovell, 1954; Krinov, 1960; Kopal, 1962) has improved significantly in recent years as a consequence of space exploration (Kaufmann, 1978) by satellites and probes (Helvey, 1960). Although the Earth (Jeffreys, 1924a; Chapman and Bartels, 1962; Stacey, 1969; Jacobs, 1975) is also an astronomical body, the study of its dynamics (Todhunter, 1873; Love, 1911; Scheidegger, 1963; Melchior, 1978; Lapwood and Usami, 1981) is considered to be a part of geophysics. The latter also includes the physics of the ionosphere (Stormer, 1955; Van Allen, 1956; Ratcliffe, 1960, 1972), which results from the trapping of the solar wind by the Earth's magnetic field, and relates to solar-terrestrial physics (Akasofu and Chapman, 1972). A scientific discipline of common interest to geophysics and astrophysics is the study of phenomena in plasmas (Dungey, 1958; Drummon, 1961; Shohet, 1971; Ecker, 1972; Sitenko, 1982; Nicholson, 1983). The references given are only a small sample, since progress in most of the topics mentioned is sufficiently rapid to justify the publication, on average, of a new book every few years [recently the pace has accelerated, e.g., in solar physics (Thomas and Athay, 1961; Bray and Loughhead, 1974; Athay, 1976; Bruzek and Durrant, 1977; White, 1977; Priest, 1982a, 1982b; Sturrock, Holzer, Mihalas, and Ulrich, 1986)], besides several conference proceedings and review volumes every year. Although physical processes in the sun (Parker, 1985) have analogs elsewhere, e.g., in other stars (Linsky, 1985; Noyes, 1985), for purposes of illustration of various phenomena, we shall concentrate on the solar case, in order to try to build up a global, though arguable, picture (Sec. IV.C.6—IV.C.8; see Table III below).

I. DISPERSION RELATIONS FOR ANISOTROPIC WAVES

Waves in fluids and other media can be classified as "large" or "small" in amplitude, depending on the magnitude of the perturbation relative to the mean state. Even if the equations of motion are nonlinear (as is generally the case with fluids), they can be linearized for waves of small amplitude, allowing the use of the principle of superposition. Thus waves of small amplitude can be described by linear partial differential equations, which can be subclassified into those with "variable" and "constant" coefficients. The variable coefficients correspond to media whose properties with regard to waves are nonuniform or unsteady, e.g., the propagation speeds, scattering scales, damping rates, etc., depend on position or time. Such linear wave equations can be solved exactly only for simple forms of the coefficients, requiring the use of special functions, which describe a rich variety of physical phenomena underlying their peculiar mathematical properties. The simplest to study are linear waves in media with constant parameters, i.e., uniform wave speeds, damping rates, scattering scales, etc. They are described by linear equations with constant coefficients, which can always be solved by Fourier analysis, that is, which permit (Sec. I.A) plane-wave solutions. It then becomes a matter of algebra to determine the dispersion relation (Sec. I.B), expressing frequency as a function of wave vector; from the latter the properties (Sec. I.C) of propagation and radiation can be derived for a wide variety of waves, isotropic and anisotropic, dispersive and nondispersive, dissipative or undamped. Thus the dispersion relation is the starting point for a study of wave properties, and we choose as reference waves in fluids under all four restoring forces.

A. Magneto-acoustic-gravity-inertial waves

Of the four types of waves in fluid, sound has been a subject of study (and speculation) since antiquity, inertial waves (Kelvin, 1880) and internal waves (Stokes, 1847) have been studied since the last century, and purely magnetic modes (Alfvén, 1942) are the last to have been discovered. Thus magnetic waves in ionized fluids were found long after electromagnetic waves in vacuo or dielectrics (Maxwell, 1873; Stratton, 1941), and also later than piezoelectric waves in crystals (Voigt, 1898; Cady, 1946). An incidental parallel is that electromagnetic waves were predicted theoretically (Maxwell, 1864) before being observed experimentally (Hertz, 1888), as were magnetic waves eighty years later: they were predicted theoretically to exist (Alfvén, 1942) and to play a major role in solar physics (Alfvén, 1945, 1947) and then were found in laboratory experiments (Lundquist, 1949; Lehnert, 1951), before being observed in the interplanetary medium (Belcher, Davis, and Smith, 1969; Belcher and Davis, 1971), and in the solar wind (Burlaga and Turner, 1976; Denskat and Burlaga, 1977) and atmosphere (Sawyer, 1974; Giovanelli and Beckers, 1982). The coupling of Alvén waves with sound, as magneto-acoustic waves (Aström, 1950; Herlofson, 1950; Banos, 1955; Lighthill, 1960; Campos, 1977), received more attention than the effects of rotation (Lehnert, 1954, 1955) and gravity (Howe, 1969) during the 1950s and 1960s, which were a period (Bullard, 1955;

Weinberg, 1962) of rapid expansion of MHD (magnetohydrodynamics). The study of three-wave couplings, viz., the consideration of MHD waves in stratified media, such as oscillations of compressible, ionized atmospheres, started with studies of reflections at interfaces (Ferraro, 1954; Stein, 1971) and rapidly evolved to the study of magneto-acoustic-gravity waves in gradually varying media (Yu, 1965; McLellan and Winterberg, 1968; Bel and Mein, 1971; Chen and Lykoudis, 1972; Michalitsanos, 1973; Stein and Leibacher, 1974; Yeh, 1974; Leroy and Bel, 1979; Campos, 1982; Thomas, 1982). In the context of magneto-atmospheric waves it is possible to find "pure" acoustic waves (Campos, 1984a) or shocks (Foukal and Smart, 1981) guided by the magnetic field, as well as more complex phenomena, e.g., coupling with rotation, either uniform (Suess, 1975; El Mekki, 1985) or differential (Fearn and Proctor, 1983).

1. Pressure, buoyancy, Lorentz, and Coriolis forces

In order to describe waves in gases, we start from the general equations of compressible fluids under gravity, magnetic, and inertial forces, in the presence of dissipation by viscosity, electrical resistance, and thermal conduction and radiation. The momentum equation states that the viscous stresses (right-hand side) balance all the forces present, viz., the inertial force (mass multiplied by total ≡ local + convective acceleration), the forces associated with rotation (Coriolis, nonuniform rotation term, and centrifugal), the gas pressure gradient, gravity, and magnetic force, i.e.,

$$\Gamma[\dot{\mathbf{V}}+(\mathbf{V}\cdot\nabla)\mathbf{V}+2\mathbf{\Omega}\Lambda\mathbf{V}+\dot{\mathbf{\Omega}}\Lambda\mathbf{X}+\mathbf{\Omega}\Lambda(\mathbf{\Omega}\Lambda\mathbf{X})] +\nabla P+\mathbf{G}\Gamma-(\mu/4\pi)\mathbf{H}\Lambda(\nabla\Lambda\mathbf{H}) =\nu_1\nabla^2\mathbf{V}+(\nu_2+\nu_1/3)\nabla(\nabla\cdot\mathbf{V}) , \quad (1)$$

where an overdot denotes partial derivative with regard to time. For example, if $\mathbf{X}$ is the fluid particle displacement, then $\dot{\mathbf{X}}\equiv\partial X/\partial t=\mathbf{V}$ is the flow velocity, and the remaining variables appearing in Eq. (1) are the mass density Γ, gas pressure P, acceleration of gravity $\mathbf{G}$, and magnetic field $\mathbf{H}$, and μ,ν_1,ν_2 denote, respectively, the magnetic permeability and incompressible and compressible kinematic viscosities. In the MHD approximation it is assumed that the ionized fluid has no charge, i.e., positive and negative charges balance, although, since ions and electrons usually have different velocities, a net conduction current can exist. The displacement current is neglected relative to the conduction current, allowing the exclusion of electromagnetic waves, and the elimination of Maxwell's equations for the magnetic field $\mathbf{H}$ yields the induction equation:

$$\dot{\mathbf{H}}+\nabla\Lambda(\mathbf{V}\Lambda\mathbf{H})=(c_*^2/4\pi\mu\sigma)\nabla^2\mathbf{H} +(c^2/4\pi\mu\eta)\nabla\Lambda[\mathbf{H}\Lambda(\nabla\Lambda\mathbf{H})] , \quad (2)$$

where c_* is the speed of light in vacuo. Equation (2) states that in a perfectly conducting medium, $\sigma=\infty=\eta$, the magnetic field is frozen into the fluid and "lags" in the presence of finite Ohmic σ or Hall η conductivities.

2. General equations of self-gravitating magnetohydrodynamics

The nonrelativistic gravity field is potential and created by mass:

$$\nabla\Lambda\mathbf{G}=0 , \quad (3a)$$

$$\nabla\cdot\mathbf{G}=4\pi\Gamma k_* , \quad (3b)$$

where k_* denotes the gravitational constant. The conservation of mass is stated by the equation of continuity:

$$\dot{\Gamma}+\nabla\cdot(\Gamma\mathbf{V})=0 \quad (4)$$

for a single chemical species, in the absence of mass diffusion. The equation of energy states that production of entropy s, in a convected frame, is due to dissipative processes associated with the incompressible and compressible viscosity, electrical resistance, and thermal conduction and radiation (the latter for a "grey" body of opacity υ):

$$\Gamma(\dot{s}+\mathbf{V}\cdot\nabla s)=\nu_1\left[\frac{\partial V_i}{\partial x_j}+\frac{\partial V_j}{\partial x_i}+\frac{2}{3}(\nabla\cdot\mathbf{V})\delta_{ij}\right]^2 +\nu_2(\nabla\cdot\mathbf{V})^2+(c_*^2/16\pi^2\sigma)(\nabla\Lambda\mathbf{H})^2 +\nabla\cdot(\kappa\nabla T)+(\bar{\sigma}/\upsilon)T^4 , \quad (5)$$

where T denotes the temperature, and $\kappa,\bar{\sigma}$ the thermal conductivity and Stefan-Boltzmann constant, respectively. Since we have 12 variables, consisting of three vectors (displacement $\mathbf{X}$ or velocity $\mathbf{V}$, magnetic field $\mathbf{H}$, and gravity $\mathbf{G}$) and three scalars (density Γ, entropy s, and pressure P, or temperature T), we need the same number of equations, viz., three vector—momentum (1), induction (2), and gravity [Eqs. (3a) and (3b) are equivalent to $\mathbf{G}=\nabla\psi$ where the gravitational potential ψ satisfies Poisson's equation $\nabla^2\psi=4\pi k_*\Gamma$]—plus three scalar equations—continuity (4), energy (5), and the equation of state, in the form $P(\Gamma,s)$, relating pressure to density and entropy. The preceding system of equations has been restricted only in neglecting plasma and relativistic effects and in making various simplifying assumptions concerning diffusion terms.

3. Undamped waves of small amplitude

We shall consider, in the first instance, undamped waves, for which all dissipation terms on the right-hand side (rhs) of Eqs. (1), (2), and (5) can be omitted; in this case, the equation of energy (5) states that the entropy s is conserved, $ds/dt=0$, in a convected frame $d/dt\equiv\partial/\partial t+\mathbf{V}\cdot\nabla$, and the equation of state $P(\Gamma,s)$ implies that pressure and density Γ are related by

$$(\dot{P}+\mathbf{V}\cdot\nabla P)=C^2(\dot{\Gamma}+\mathbf{V}\cdot\nabla\Gamma)\ , \tag{6a}$$

$$C^2\equiv(\partial P/\partial\Gamma)_s\ , \tag{6b}$$

where C denotes the adiabatic sound speed. The equations of self-gravitating mass [(3a) and (3b)] can be important for the study of global oscillations (Unno, Osaki, Ando, and Shibaiashi, 1979; Cox, 1980) of large fluid bodies such as stars (Eddington, 1926; Schwarzschild, 1958; Skilling, 1968; Tassoul, 1978). These equations are uncoupled from the wave motion in the Cowling (1941) approximation, which neglects (Pekeris, 1938; Ledoux and Walraven, 1957) the perturbation in the gravitational potential. For oscillations of physical extent much smaller than the planetary or stellar scale, the gravity $\mathbf{G}=\mathbf{g}$ can be treated as a constant, external force field, much as the angular velocity $\mathbf{\Omega}$ is supposed to be given; the nonrelativistic treatment limits the displacement $\mathbf{X}$ from the rotation axis to tangential velocities $|\mathbf{\Omega}\Lambda\mathbf{X}|^2 \ll c_*^2$ much smaller than the speed of light. For waves of small amplitude, we may decompose the total flow variables, viz., the displacement $\mathbf{X}$, velocity $\mathbf{V}$, pressure P, density Γ, and magnetic field $\mathbf{H}$, into a steady, nonuniform mean state of rest $(\mathbf{r},0,p,\rho,\mathbf{B})$ and a perturbation $(\boldsymbol{\xi},\mathbf{v},\bar{p},\bar{\rho},\mathbf{h})$:

$$\mathbf{X},\mathbf{V},P,\Gamma,\mathbf{H}(\mathbf{x},t)=\mathbf{r},0,p,\rho,\mathbf{B}(\mathbf{x})+\boldsymbol{\xi},\mathbf{v},\bar{p},\bar{\rho},\mathbf{h}(\mathbf{x},t)\ . \tag{7a}$$

We may now substitute Eq. (7a) in the general equations, subtract the mean-state equations, neglect nonlinear terms, and retain only the terms linear in the perturbations:

$$\ddot{\boldsymbol{\xi}}+2\mathbf{\Omega}\Lambda\dot{\boldsymbol{\xi}}+\dot{\mathbf{\Omega}}\Lambda\boldsymbol{\xi}+\mathbf{\Omega}\Lambda(\mathbf{\Omega}\Lambda\boldsymbol{\xi})+\rho^{-1}\nabla\bar{p}-(\bar{\rho}/\rho)\mathbf{g} -(\mu/4\pi\rho)[\mathbf{B}\Lambda(\nabla\Lambda\mathbf{h})+\mathbf{h}\Lambda(\nabla\Lambda\mathbf{B})]=0\ , \tag{7b}$$

for the linearized, inviscid momentum equation.

4. Wave operator for particle displacement

The linear, nondiffusive equations of induction (2), continuity (4), and adiabaticity (6a) can be used to express all wave variables, viz., the magnetic field $\mathbf{h}$, density perturbations $\bar{\rho}$, and pressure perturbations $\bar{p}$, in terms of the displacement

$$\mathbf{h}=\nabla\Lambda(\mathbf{B}\Lambda\boldsymbol{\xi})\ , \tag{8a}$$

$$\bar{\rho}=-\nabla\cdot(\rho\boldsymbol{\xi})\ , \tag{8b}$$

$$\bar{p}=-\rho c^2(\nabla\cdot\boldsymbol{\xi})-\rho\mathbf{g}\cdot\boldsymbol{\xi}-(\mu/4\pi)\boldsymbol{\xi}\cdot[\mathbf{B}\Lambda(\nabla\Lambda\mathbf{B})]\ , \tag{8c}$$

where all other quantities, i.e., mass density ρ, gravity $\mathbf{g}$, and magnetic field $\mathbf{B}$, refer to the mean state. In the first and second terms of Eq. (8c),

$$-c^2(\bar{\rho}+\boldsymbol{\xi}\cdot\nabla\rho)=-\rho c^2(\nabla\cdot\boldsymbol{\xi})$$

and $-\boldsymbol{\xi}\cdot\nabla p$, respectively, we have used the linearized adiabatic sound speed c and the equation of momentum for the mean state:

$$c^2\equiv(\partial p/\partial\rho)_s=\gamma p/\rho=\gamma RT\ , \tag{9a}$$

$$\nabla p-\rho\mathbf{g}-(\mu/4\pi)\mathbf{B}\Lambda(\nabla\Lambda\mathbf{B})=0\ . \tag{9b}$$

In Eq. (9a), γ denotes the ratio of specific heats, and we have used the equation of state for a perfect gas, $p=\rho RT$, with R the gas constant. Equation (9b) specifies the magneto-hydrostatic equilibrium of the atmosphere. Substitution of Eqs. (8a)—(8c) into Eq. (7b) yields the linear, nondissipative wave equation for the particle displacement:

$$\ddot{\boldsymbol{\xi}}+2\mathbf{\Omega}\Lambda\dot{\boldsymbol{\xi}}+\dot{\mathbf{\Omega}}\Lambda\boldsymbol{\xi}+\mathbf{\Omega}\Lambda(\mathbf{\Omega}\Lambda\boldsymbol{\xi})-\rho^{-1}\nabla\cdot[\rho c^2(\nabla\cdot\boldsymbol{\xi})]-\rho^{-1}\nabla[\rho(\mathbf{g}\cdot\boldsymbol{\xi})]-\mathbf{g}\nabla\cdot(\rho\boldsymbol{\xi})-(\mu/4\pi\rho)\nabla\{\boldsymbol{\xi}\cdot[\mathbf{B}\Lambda(\nabla\Lambda\mathbf{B})]\} -(\mu/4\pi\rho)\{\mathbf{B}\Lambda[\nabla\Lambda\nabla\Lambda(\mathbf{B}\Lambda\boldsymbol{\xi})]+[\nabla\Lambda(\mathbf{B}\Lambda\boldsymbol{\xi})]\Lambda(\nabla\Lambda\mathbf{B})\}=0\ . \tag{10}$$

The magneto-acoustic-gravity-inertial wave equation (10) balances the acceleration (first term) against the combined effects of rotation (second-to-fourth terms), compressibility (fifth), gravity (sixth and seventh), and magnetic field (eighth and ninth); it allows for the presence of three external force fields, namely, nonuniform and unsteady rotation $\Omega(\mathbf{x},t)$, nonuniform but steady magnetic field $\mathbf{B}(\mathbf{x})$, and arbitrary gravity $\mathbf{g}$. It is fairly general, since it applies to adiabatic propagation in any fluid (it may be a liquid or a gas, perfect or not) with any stable density or temperature stratification; the general wave operator (10) extends the result for magneto-atmospheric waves (McLellan and Winterberg, 1968; Bray and Loughhead, 1974; Campos, 1983b; Thomas, 1983) to include the effects of rotation together with arbitrary stratification of the fluid.

5. Conservation of vorticity and propagation of dilatation

The simplest instance of the wave equation (10), concerns "pure" sound in a compressible fluid, in the absence of external fields $\mathbf{\Omega}=\mathbf{B}=\mathbf{g}=0$, viz.,

$$0=\ddot{\boldsymbol{\xi}}-\rho^{-1}\nabla\cdot[\rho c^2(\nabla\cdot\boldsymbol{\xi})] =\ddot{\boldsymbol{\xi}}-c^2\nabla(\nabla\cdot\boldsymbol{\xi})-\gamma(\nabla\cdot\boldsymbol{\xi})\rho^{-1}\nabla p\ , \tag{11}$$

where the last term vanishes for a homogeneous medium. Taking the curl and div of the first two terms, we have

$$\partial^2(\nabla\Lambda\boldsymbol{\xi})/\partial t^2=0\ , \tag{12a}$$

$$(\partial^2/\partial t^2-c^2\nabla^2)(\nabla\cdot\boldsymbol{\xi})=0\ , \tag{12b}$$

which shows that for sound waves the vorticity is conserved (12a) and the dilatation propagates (12b) at sound speed. In the case of a homogeneous perfect gas at constant temperature, the sound speed c is a constant, and Eq. (12b) has plane-wave solutions:

$$\boldsymbol{\xi}(\mathbf{x},t)=\int_{-\infty}^{+\infty}\mathbf{X}(\mathbf{k},\omega)\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]d^3k\,d\omega\ , \tag{13}$$

with a spectrum $\mathbf{X}$ that depends on frequency ω and wave vector $\mathbf{k}$. The conservation of vorticity $\nabla\Lambda\boldsymbol{\xi}=0$ implies that the fluid particle displacement is aligned with the wave normal $\mathbf{k}\Lambda\mathbf{X}=0$, i.e., the wave is longitudinal, and consists of compressions and rarefactions without shear. The classical wave equation (12b) for the dilatation in a medium at rest (the generalization to flowing media was given in Sec. II.B of Part I) leads to the dispersion relation,

$$\omega=\pm ck\ , \tag{14a}$$

$$u\equiv\omega/k=\pm c\ , \tag{14b}$$

$$\mathbf{w}=\partial\omega/\partial\mathbf{k}=\pm c\mathbf{n}\ , \tag{14c}$$

and hence to the phase speed (14b) and group velocity (14c). These demonstrate the well-known fact that acoustic waves travel at sound speed c in the wave-normal $\mathbf{n}\equiv\mathbf{k}/k$ direction, i.e., their propagation is isotropic and nondispersive. The speed of propagation of sound is independent of direction and wavelength, because the corresponding restoring force, the gas pressure, is isotropic, and in a homogeneous medium there is no length scale to define an interaction parameter together with the wavelength.

6. Coupling of internal and acoustic modes

The isotropy and nondispersive property of sound are both broken in the presence of gravity, which introduces a preferred direction and causes a density stratification. If we choose gravity to be vertically downwards, $\mathbf{g}=(0,0,-g)$, the equation of hydrostatic equilibrium [Eq. (9b) with $\mathbf{B}=0$] implies

$$g=-\rho^{-1}\nabla p=-(c^2/\gamma)\nabla\ \ln\rho\equiv c^2/\gamma L\ , \tag{15a}$$

$$L\equiv-(\nabla\ln\rho)^{-1}=RT/g\ , \tag{15b}$$

in an isothermal atmosphere of a perfect gas, for which the sound speed c [Eq. (9a)] and density scale height L [Eq. (15b)] are both constants. The wave equation (10), without rotation and magnetic field $\mathbf{\Omega}=0=\mathbf{B}$, simplifies to

$$\ddot{\xi}-c^2\nabla(\nabla\cdot\xi)-(\gamma-1)\mathbf{g}(\nabla\cdot\xi)-\nabla(\mathbf{g}\cdot\xi)=0\ , \tag{16}$$

which is the acoustic-gravity wave operator. It has plane-wave solutions (13) in an isothermal atmosphere, implying that

$$\omega^2\mathbf{X}-c^2\mathbf{k}(\mathbf{k}\cdot\mathbf{X})+i(\gamma-1)\mathbf{g}(\mathbf{k}\cdot\mathbf{X})+i(\mathbf{g}\cdot\mathbf{X})\mathbf{k}=0\ . \tag{17}$$

We choose the x_3 axis opposite to gravity, and the x_1 axis so that the wave vector $\mathbf{k}$ lies in the (x_1,x_3) plane, i.e., $\mathbf{k}=(k_{\parallel},0,k_{\perp})$. It follows from Eq. (13) that the wave particle displacement $\mathbf{X}$ lies in the plane of gravity $\mathbf{g}$ and the wave vector $\mathbf{k}$:

$$\begin{bmatrix} \omega^2-c^2k_{\parallel}^2 & -c^2k_{\parallel}k_{\perp}-igk_{\parallel} \\ -c^2k_{\parallel}k_{\perp}-i(\gamma-1)gk_{\parallel} & \omega^2-c^2k_{\perp}^2-i\gamma gk_{\perp} \end{bmatrix} \begin{bmatrix} X_1 \\ X_3 \end{bmatrix}=0\ . \tag{18}$$

For waves to exist, the vector $\mathbf{X}=(X_1,0,X_3)$ should not vanish, so the determinant of the dispersion matrix in Eq. (18) must be zero, yielding the dispersion relation

$$\omega^4-(c^2k^2+i\gamma gk_{\perp})\omega^2+(\gamma-1)g^2k_{\parallel}^2=0\ , \tag{19}$$

which reduces to that for pure sound (14a), in the absence of gravity $g=0$.

7. Cutoff frequencies due to stratification and compressibility

In the presence of gravity $g\neq0$, it is convenient to solve the dispersion relation (19) for the vertical wave number $k_{\perp}$ (in the direction opposite to gravity):

$$k_{\perp}^2+(i/L)k_{\perp}-[\omega^2/c^2-k_{\parallel}^2+(\gamma-1)(gk_{\parallel}/\omega c)^2]=0 \tag{20}$$

as a function of frequency ω and horizontal wave number $k_{\parallel}$ (perpendicular to the direction of stratification), viz.,

$$k_{\perp}=-i/2L\pm[(\omega^2-\omega_2^2)/c^2-k_{\parallel}^2(1-\omega_1^2/\omega^2)]^{1/2}\ , \tag{21}$$

where ω_1 and ω_2 define, respectively, the internal (or Brünt-Vaisala) and acoustic cutoff frequencies:

$$\omega_1\equiv\sqrt{\gamma-1}(g/c)=\sqrt{\gamma-1}(c/\gamma L)\ , \tag{22a}$$

$$<\omega_2\equiv c/2L=\gamma g/2c\ . \tag{22b}$$

The reason for this designation is that, for frequencies between the cutoffs $\omega_1<\omega<\omega_2$, the vertical wave number is pure imaginary $k_{\perp}=i\alpha$, and only standing modes exist:

$$\exp(ik_{\perp}x_3)=\exp(-\alpha x_3)\ .$$

Above the upper $\omega>\omega_2$ and below the lower $\omega<\omega_1$ cutoffs, the vertical wave number has a real $\mathrm{Re}(k_{\perp})\neq0$ propagating part, and the imaginary part $\mathrm{Im}(k_{\perp})=-1/2L$ corresponds to amplitude growth with altitude,

$$|\exp(ik_{\perp}x_3)|=\exp[-\mathrm{Im}(k_{\perp})x_3]=\exp(x_3/2L)\ ,$$

on twice the scale height. Above the upper cutoff $\omega>\omega_2$, we have acoustic waves, unaffected by gravity for very high frequencies ($k_{\perp}\sim\omega/c$ for $\omega^2\gg\omega_2^2$) and modified by gravity as the cutoff ω_2 is approached; below the lower cutoff $\omega<\omega_1$, we have gravity modes, unaffected by compressibility for very low frequencies ($k_{\perp}\sim k_{\parallel}\omega_1/\omega$ for $\omega^2\ll\omega_1^2$) and modified by compressibility as the cutoff ω_1 (or Brünt-Vaisala frequency) is approached. The gravity mode cannot propagate vertically, and in this case, $k_{\parallel}=0$, only the one cutoff remains, namely, ω_2 for acoustic-gravity waves. Both cutoff frequencies are decreasing functions of temperature, $\omega_1,\omega_2\sim T^{-1/2}$, and the corresponding cutoff periods increase with the square root of temperature, $\tau_1,\tau_2\equiv2\pi/\omega_1,\ 2\pi/\omega_2\sim T^{1/2}$.

8. Mass supply to the corona and spicules

The property described above may be relevant to waves in the solar atmosphere, which consists (De Jager, 1971;

Athay, 1976; Bruzek and Durrant, 1977; Priest, 1982a) of three "layers": (i) The brightest, below which direct observation is impossible, is the photosphere, where the temperature decays with altitude, as required by radiative-convective processes. (ii) Above this is the chromosphere, which gives the sun its color, where the temperature gradient is reversed to positive, through a temperature minimum. (iii) The temperature then increases sharply, from less than 10^4 K in the chromosphere, across a "thin" transition region, to over 10^6 K in the corona, which is the outermost layer, readily visible during eclipses. Acoustic waves with a five-minute period, which have been observed for a long time (since Leighton, Noyes, and Simon, 1962), can propagate in the photosphere and corona, but become evanescent in the chromosphere, in the region near the temperature minimum, where the cutoff period $\tau_* = 200$ s is less than the wave period $\tau = 300$ s. Acoustic waves have a zero mean velocity v and density $\rho' = \rho v / c$ perturbations, but the mass flux $j \equiv \rho' v = \rho v^2 / c \equiv \rho v_*$ is nonzero, to second order, and implies a mass transport of a magnitude equal to that associated with a steady flow of velocity $v_* = v^2/c$. The actual existence of a steady flow, i.e., wave-induced streaming, should be checked by a nonlinear calculation, including induced pressure gradients. For the purpose of estimating the mass flux, we take $c = 10$ km/s for the sound speed and $v = 1$ km/s for the velocity perturbation in the photosphere; the equivalent "steady" flow velocity is $v_* = 10^4$ cm s^{-1}, and, for a mass density $\rho = 3 \times 10^{-7}$ g cm^{-3}, the mass flux is $j_0 = 3 \times 10^{-3}$ g cm^{-2} s^{-1}. As the waves propagate upward through the chromosphere, they traverse an evanescent region about 14 scale heights thick, and thus there is a reduction of $\varepsilon = e^{-7} = 8.3 \times 10^{-7}$ in the mass flux, to $j_1 = \varepsilon j_0 = 2.5 \times 10^{-9}$ g cm^{-2} s^{-1} at the base of the corona. This corresponds to the net upward mass flux in spicules, which are (Beckers, 1968, 1972) jetlike flows in the low corona carrying a mass flux of $j_2 = 0.2 j_1 = 5.0 \times 10^{-10}$ g cm^{-2} s^{-1}; we note here that the excess of upward over downward jets is 20%. This mass flux is sufficient to supply the corona with the mass flux $j_3 = 3.0 \times 10^{-11}$ g cm^{-2} s^{-1} it loses in the solar wind (Ulmschneider, 1971a), with an excess of mass $\Delta j = j_2 - j_3 = 4.7 \times 10^{-11}$ g cm^{-2} s^{-1}, which falls back through the transition region, causing (for a mass density $\rho_1 = 3.0 \times 10^{-17}$ g cm^{-3}) downflow velocities $v_1 = \Delta J / \rho_1 = 1.6 \times 10^6$ cm s^{-1} of about 16 km/s, which are in the range 10–25 km/s of observations (Pneuman and Kopp, 1977, 1978; Mein, Simon, Vial, and Shine, 1982; Engvold, Tandberg-Hanssen, and Reichmann, 1984). Thus acoustic waves could establish the mass balance in the solar atmosphere by supplying, from the photosphere to the corona, the mass the latter loses in the solar wind.

B. Wave properties and propagating fields

This outline of the mass transport in the solar atmosphere would suggest that the spicules, i.e., upward jets in the solar atmosphere, are acoustic waves, of large amplitude, guided along the magnetic field. It is possible to construct a model based on acoustic-gravity waves (Campos, 1984a) which complies with most of the observations of spicules (Mouradian, 1967; Pasachoff, Noyes, and Beckers, 1968; Michard, 1974; Mouradian and Simon, 1975; Bohlin *et al.*, 1977; Mouradian and Soru-Escaut, 1976; Moore *et al.*, 1977; Mosher and Pope, 1977; Labonte, 1979; Kulidzanishvili, 1980; Poletto, 1980; Rabin and Moore, 1980; Ajmanova, Ajmanov, and Gulyaev, 1982; Withbroe, 1983; Gaizauskas, 1984; Hasan and Keil, 1984; Landman, 1984b, 1986). For example, the spicule velocity corresponds to the speed of an acoustic-gravity wave (Campos, 1984a), and the sun's general magnetic field is sufficient to guide it, in the low corona, since there the magnetic pressure exceeds the gas pressure (Sec. I.B.8). The dissipation of the large-amplitude acoustic wave (by "eddy" viscosity), when balanced against thermal radiation, leads (Fig. 1) to a theoretical profile of temperature versus altitude [Fig. 1(b)], which is consistent with empirical data (Beckers, 1972); the observed profile of mass density [Fig. 1(a)] corresponds to an atmosphere in hydrostatic equilibrium, upon which is superimposed a compression front associated with the wave. There exist many other models of spicules, based on (i) magneto-

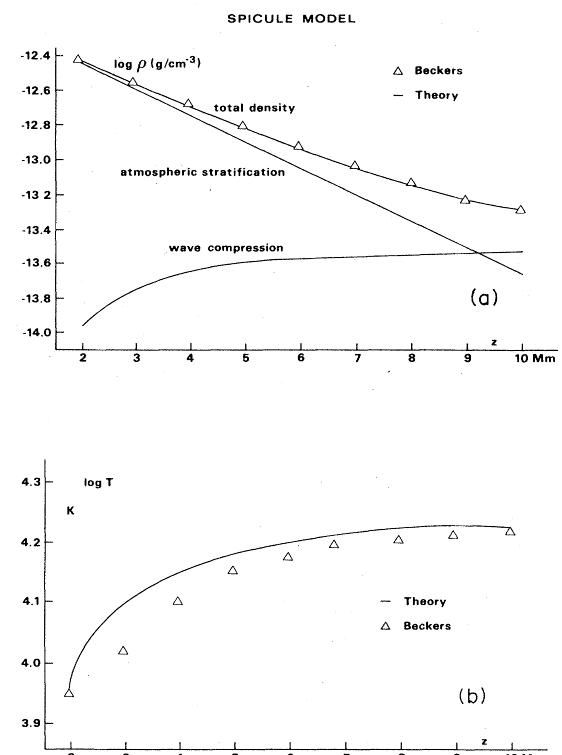

FIG. 1. Profiles of (a) mass density and (b) temperature vs altitude, observed in spicules (triangles; Beckers, 1968), compared with the predictions (solid curve; Campos, 1984a) of the theory of acoustic-gravity waves of large amplitude, with viscous and radiative damping.

hydrodynamic shocks (Thomas, 1948; Uchida, 1961) that displace the transition region upward (Osterbrock, 1961; Hollweg, 1982a); (ii) ejection of material by magnetic field reconnection (Petschek, 1964; Pikelner, 1971), possibly with other supporting forces (Uchida, 1969; Blake and Sturrock, 1985); (iii) instabilities (Kuperus and Athay, 1967; Defouw, 1970, Sparks and Van Hoven, 1985), which may (Roberts, 1979) or may not (Hollweg, 1979; Venkatakrishnan and Hasan, 1982) develop into upward motions; (iv) various types of jets (Hasan and Venkatakrishnan, 1981; Shibata, 1982; Suematsu *et al.*, 1982; Suematsu, 1985; Shibata and Uchida, 1986) and energy exchange mechanisms (Athay and Holzer, 1982; Athay, 1982, 1984; Kulidzanishvili and Zhugzhda, 1983; Hollweg, 1984a). It is beyond the scope of the present review to discuss in any detail these spicule models; they demonstrate that waves can carry mass and energy in atmospheres, and support a continuation of the study of wave properties.

1. General second-order wave equation

We have discussed two particular instances, namely, sound (Sec. I.A.5) and acoustic-gravity modes (Sec. I.A.6), of magneto-acoustic-gravity-inertial waves (Sec. I.A.4). Before proceeding to analyze (Sec. I.B.4—I.B.7) further instances of this, we note that we shall always be dealing with particular cases of the general, linear second-order vector wave equation:

$$A_{jlmn}\partial^2\xi_l/\partial x_m\partial x_n+B_{jlm}\partial^2\xi_l/\partial t\partial x_m+C_{jl}\partial^2\xi_l/\partial t^2 +D_{jlm}\partial\xi_l/\partial x_m+E_{jl}\partial\xi_l/\partial t+F_{jl}\xi_l=0 \ . \tag{23}$$

Equation (23) includes, as particular cases, the wave equation for the displacement, in nondissipative fluids, in isotropic and anisotropic elastic bodies (Love, 1927; Achenbach, 1973; Hudson, 1978), and piezoelectric waves in crystals (Cady, 1927). Assuming that all coefficients in Eq. (23) are constant, there exist plane-wave solutions [Eq. (13)], and the wave operator (23) leads to the dispersion matrix

$$\Pi_{jl}\equiv(A_{jlmn}k_mk_n-B_{jlm}k_m\omega+C_{jl}\omega^2-F_{jl}) +i(E_{jl}\omega-D_{jlm}k_m) \ . \tag{24}$$

The wave spectrum $\mathbf{X}$ satisfies

$$\Pi_{jl}(\mathbf{k},\omega)X_l(\mathbf{k},\omega)=0 \ , \tag{25a}$$

$$\mathrm{Det}[\Pi_{jl}(\mathbf{k},\omega)]=0 \ , \tag{25b}$$

and in order for waves to exist $\mathbf{X}$ cannot vanish identically, implying that the determinant of the dispersion matrix is zero. Equation (25b) is a polynomial equation, whose roots specify the dispersion relation

$$\omega=f(\mathbf{k}) \ , \tag{26a}$$

and hence the phase speed

$$u\equiv\omega/k=k^{-1}f(\mathbf{k}) \tag{26b}$$

and group velocity

$$\mathbf{w}=\partial\omega/\partial\mathbf{k}=\partial f/\partial\mathbf{k} \ . \tag{26c}$$

Since the determinant (25b) is at most of rank 3, there are no more than three pairs of roots, and it follows that general magneto-acoustic-gravity-inertial waves have, at most, three pairs of modes. The number may be less, if certain roots coincide, e.g., acoustic waves have one mode (Sec. I.A.5) and acoustic-gravity waves two modes (Sec. I.A.6); it is possible to have three distinct pairs of modes by coupling only two restoring forces, e.g., for magneto-acoustic waves (Sec. I.B.7).

2. Homogeneous operator and nondispersive waves

Suppose that in Eq. (23) all coefficients of terms of first and zero order vanish, $D_{jlk}=E_{jl}=F_{jl}=0$, so that the wave equation is homogeneous and of the second degree,

$$A_{jlmn}\partial^2\xi_l/\partial k_m\partial x_n+B_{jlm}\partial^2\xi_l/\partial t\partial x_m+C_{jl}\partial^2\xi_l/\partial t^2=0 \ , \tag{27}$$

the dispersion matrix is a quadratic function of wave vector and frequency,

$$\Pi_{jl}=A_{jlmn}k_mk_n-B_{jlm}k_m\omega+C_{jl}\omega^2 \ , \tag{28}$$

and the roots of its determinant (25b) are linear relations between frequency and wave vector,

$$\omega=A_lk_l \ , \tag{29a}$$

$$u=A_ln_l \ , \tag{29b}$$

$$w_l=A_l \ . \tag{29c}$$

The linear dispersion relation (29a) implies that the coefficient $\mathbf{A}$ coincides with the group velocity (29c), while the phase speed (29b) is the group velocity projected in the wave normal direction $\mathbf{n}\equiv\mathbf{k}/k$. Since neither the phase speed (29b) nor the group velocity (29c) depends on the wave number $k\equiv|\mathbf{k}|$ or on wavelength $\lambda\equiv2\pi/k$, the waves are nondispersive, i.e., a packet of waves of different lengths remains together, as all wave components move at the same speed; conversely, if the wave equation contains derivatives of different orders, the dispersion matrix is not homogeneous, its roots yield nonlinear relations between frequency and wave vector, the phase speed and group velocity depend on wavelength, and waves are dispersive. In other words, a wave packet spreads out as it moves, because components of different wavelengths move at distinct speeds, i.e., "lag" or "advance" relative to each other. On inspection of the equation of linear, nondissipative waves in fluids (10), it is clear that acoustic and magnetic terms have only second-order derivatives (in homogeneous media), whereas gravity and rotation introduce first-order derivatives. Thus the only nondispersive modes are acoustic (Sec. I.A.5), magnetic (Sec. I.B.6), and magneto-acoustic (Sec. I.B.7) waves in homogeneous media; all other waves in fluids are dispersive, viz., (i) internal waves (Sec. I.B.4) and inertial waves (Sec. I.B.5); (ii) all five two-wave couplings (except magneto-acoustic), such as acoustic-gravity waves (Sec. I.A.6); (iii) all four three-wave couplings; and (iv) the single four-wave coupling.

3. Laplacian operator and isotropic waves

Suppose that in the wave equation (23) all coefficients are isotropic tensors, i.e., are given by

$$A_{jlmn}=A_1\delta_{jm}\delta_{ln}+A_2\delta_{jl}\delta_{mn} \ ,$$
$$B_{jln}=Be_{jln} \ ,$$
$$C_{jl}=C\delta_{jl}, \quad D_{jln}=De_{jln} \ ,$$

etc., in terms of Kronecker δ_{jl} or permutation e_{jln} tensors. The wave equation can be written in the vector form

$$A_1\nabla(\nabla\cdot\xi)+A_2\nabla^2\xi+B(\nabla\Lambda\dot{\xi}) +C\ddot{\xi}+D(\nabla\Lambda\xi)+E\dot{\xi}+F\xi=0 \ . \quad (30)$$

Taking the divergence of Eq. (30), it follows that the dilatation $\nabla\cdot\xi$ satisfies a scalar wave equation

$$(A\nabla^2+C\partial^2/\partial t^2+E\partial/\partial t+F)(\nabla\cdot\xi)=0 \ , \quad (31)$$

with $A\equiv A_1+A_2$, where spatial derivatives appear only through the Laplacian operator $\nabla^2\equiv\partial^2/\partial x_l\partial x_l$. The dispersion polynomial depends on the wave vector $\mathbf{k}$ only through the wave number $k\equiv|\mathbf{k}|$, that is,

$$\Pi(k,\omega)=Ak^2+C\omega^2-F+i\omega E \ ; \quad (32)$$

its roots are generally nonlinear relations between frequency and wave number:

$$\omega=f(k) \ , \quad (33a)$$
$$u=k^{-1}f(k) \ , \quad (33b)$$
$$\mathbf{w}=(df/dk)\mathbf{n} \ , \quad (33c)$$

implying that the group velocity (33c) lies in the wave-normal direction $\mathbf{n}\equiv\mathbf{k}/k$, and the phase speed (33b) is isotropic. Conversely, if the wave equation involves spatial derivatives other than the Laplacian, the wave normal appears in the dispersion polynomial (32) and dispersion relation, and thus the group velocity is no longer parallel to $\mathbf{n}$, nor is the phase speed independent of $\mathbf{n}$. In the case of isotropic waves, the propagation speed is independent of direction, and wave fronts are spherical, whereas, with anisotropic waves, the propagation speed is direction dependent, and the wave fronts are not spherical. On inspection of the magneto-acoustic-gravity-inertial wave equation (10), it is clear that only the acoustic term is isotropic, and the presence of gravity, magnetic field, or rotation introduces anisotropy. In order to have waves that are both nondispersive and isotropic, we have to satisfy Eqs. (29a)—(29c) and (33a)—(33c) simultaneously; this leads to the dispersion relations (14a)—(14c) for sound, and shows that the only isotropic, nondispersive wave equation $A\nabla^2+B\partial^2/\partial t^2+C$ is essentially the classical wave equation (with $BA<0$, plus a constant, as in Helmholtz's equation).

4. Internal oscillations along line of steepest descent

Of the four types of waves in fluids under a single restoring force, we have considered so far only acoustic waves (Sec. II.A.5), which are the high-frequency limit ω/ω_1, ω/ω_2, $L\to\infty$ of acoustic-gravity waves (21),

$$k_\perp^2=\omega^2/c^2-k_\parallel^2 \ . \quad (34a)$$

The opposite, low-frequency limit ω_1/ω, $\omega_2/\omega\to\infty$ is

$$k_\perp=-i/2L+k_\parallel(\omega_1^2/\omega^2-1) \ . \quad (34b)$$

From Eq. (34a) we obtain (14a) for the total wave number $k^2\equiv k_\parallel^2+k_\perp^2$; it is real, showing that plane sound waves have constant amplitude, and the linear relation between frequency and wave number shows that acoustic waves are isotropic and nondispersive. For gravity waves, the vertical wave number has an imaginary part $\mathrm{Im}(k_\perp)=-1/2L$, implying that the wave amplitude grows exponentially with altitude $z\equiv x_3$ on twice the scale height

$$|\exp(ik_\perp z)|=\exp(z/2L) \ ;$$

the total wave number $k^2=[\mathrm{Re}(k_\perp)]^2+k_\parallel^2$ depends both on frequency ω and the angle θ of the wave normal $\mathbf{n}$ to the vertical (opposite to gravity):

$$\omega=\omega_1k_\parallel/k=\omega_1\sin\theta=\omega_1(\mathbf{k}\cdot\mathbf{m}/k) \ . \quad (35a)$$

From Eq. (35a) it is clear that internal waves cannot propagate vertically along or opposite to the gravity field ($\omega=0$ for $k_\parallel=0$ or $\theta=0,\pi$), and as the wave frequency ω increases, the wave vector $\mathbf{k}$ tilts closer to the horizontal, until, for horizontal propagation it equals the buoyancy frequency ($\omega=\omega_1$ for $k_\parallel=k$ or $\theta=\pi/2$), which is the cutoff frequency (21b) above which internal waves are evanescent. From Eq. (35a), where $\mathbf{m}$ is the unit horizontal vector in the plane of the wave normal $\mathbf{n}=\mathbf{k}/k$ and gravity $\mathbf{g}$, we can calculate the group velocity

$$\mathbf{w}=\partial\omega/\partial\mathbf{k}=(\omega_1/k^3)[(\mathbf{m}k^2-(\mathbf{k}\cdot\mathbf{m})\mathbf{k}] =(\omega_1/k)[\mathbf{n}\Lambda(\mathbf{m}\Lambda\mathbf{n})] \ , \quad (35b)$$

showing that it is larger for longer waves $|\mathbf{w}|\sim1/k\sim\lambda$. The fluid particle displacement and the wave energy flux are directed along the component of $\mathbf{m}$ transverse to $\mathbf{n}$, i.e., (i) they lie in the plane orthogonal to the wave normal direction, as appropriate to an incompressible, transverse wave; (ii) in this phase plane, the oscillation takes place along the line of steepest descent, for which buoyancy is most effective at balancing inertia.

5. Inertial flux along direction of maximum separation

The other restoring force, besides gravity, that leads to anisotropic and dispersive waves is the Coriolis force; wave dispersion is associated with the presence of a cutoff frequency for internal waves, and we shall show that this also applies, in a different way, to inertial waves. We consider transverse motions $\nabla\cdot\mathbf{v}=0$, in the absence of gravity and magnetic field, $\mathbf{g}=0=\mathbf{B}$, and in the presence of uniform rotation, $\mathbf{\Omega}=$constant. The linearized momentum equation (8) takes the form

$$\dot{\xi}+2\mathbf{\Omega}\Lambda\dot{\xi}+\rho^{-1}\nabla p+\nabla[(\mathbf{\Omega}\Lambda\xi)^2/2]=0\ , \qquad (36a)$$

where the centrifugal force $\mathbf{\Omega}\Lambda(\Omega\Lambda\xi)$ per unit mass has been written as the gradient of the centrifugal energy $(\Omega\Lambda\xi)^2/2$. For incompressible motions, it is usual to make the Boussinesq (1878) approximation, which neglects gradients of mean state density, e.g., in

$$\nabla\Lambda(\rho^{-1}\nabla p)=\nabla(\rho^{-1})\Lambda\nabla p=\rho^{-2}\nabla p\Lambda\nabla\rho\ .$$

Omitting the latter term when taking the curl of (36a), we obtain (36b):

$$\nabla\Lambda\dot{\xi}=-(\mathbf{\Omega}\cdot\nabla)\xi\ , \qquad (36b)$$

$$\nabla^2\ddot{\xi}-(\mathbf{\Omega}\cdot\nabla)^2\xi=0\ , \qquad (36c)$$

and thus, by applying $\partial/\partial t\nabla\Lambda$ once more, bearing in mind that, for incompressible motions $\nabla\cdot\xi=0$ we have $\nabla\Lambda(\nabla\Lambda\xi)=\nabla^2\xi$ follows (36c). The dispersion relation (36c) for inertial waves,

$$\omega=(\mathbf{\Omega}\cdot\mathbf{k})/k=\Omega k_\perp/k=\mathbf{\Omega}\cdot\mathbf{n}=\Omega\cos\theta\ , \qquad (37a)$$

has both similarities to and differences from Eq. (35a) for internal waves: (i) the cutoff frequency is the angular velocity Ω instead of the buoyancy frequency; (ii) the wave-vector component along the axis of rotation $k_\perp\equiv\mathbf{\Omega}\cdot\mathbf{k}/k\Omega$ replaces the component transverse to gravity $k_\parallel\equiv|\mathbf{k}\Lambda\mathbf{g}|/kg$. Thus inertial waves cannot propagate transversely to the axis of rotation ($\omega=0$ for $k_\perp=0$ or $\theta=\pi/2$), their frequency increases as the wave vector tilts closest to the angular velocity, and for propagation along the rotation axis it reaches the angular velocity ($\omega=\pm\Omega$ for $k_\perp=\pm k$ or $\theta=0,\pi$), which is the cutoff frequency beyond which only standing modes exist. The group velocity

$$\mathbf{w}=\partial\omega/\partial\mathbf{k}=[\mathbf{\Omega}/k-k^{-3}(\mathbf{\Omega}\cdot\mathbf{k})\mathbf{k}]=k^{-1}[\mathbf{n}\Lambda(\mathbf{\Omega}\Lambda\mathbf{n})] \qquad (37b)$$

is larger for longer waves $|\mathbf{w}|\sim 1/k\sim\lambda$, as for internal waves; in contrast with the latter, for which oscillation takes place along the line in the phase plane making the least angle with gravity, inertial waves maximize the angle with the rotation axis, so that rotational forces are more effective at balancing fluid acceleration. Thus, for inertial waves, the energy flux travels along the projection of the angular velocity $\mathbf{\Omega}$ in the direction transverse to the wave normal $\mathbf{n}$, in the direction of $\mathbf{n}\Lambda(\Omega\Lambda\mathbf{n})=\mathbf{\Omega}-(\mathbf{n}.\mathbf{\Omega})\mathbf{n}$, giving the largest possible separation from the axis of rotation.

6. Alfvén waves along magnetic field lines

Having considered acoustic (Sec. I.A.5), internal (Sec. I.B.4), and inertial (Sec. I.B.5) waves, we now turn to the remaining type of wave in gases under a single restoring force, namely, Alfvén waves in the presence of a magnetic field. Omitting gravity and rotation, $\mathbf{\Omega}=0=\mathbf{g}$, in Eq. (10), we obtain, for incompressible modes $\nabla\cdot\xi=0$, in the presence of a uniform magnetic field, $\mathbf{B}=\text{const}\neq 0$, the Alfvén wave equation

$$\ddot{\xi}-a^2(\mathbf{b}\cdot\nabla)^2\xi+a^2(\mathbf{b}\cdot\nabla)\nabla(\mathbf{b}\cdot\xi)=0\ , \qquad (38a)$$

$$a^2\equiv\mu B^2/4\pi\rho\ , \qquad (38b)$$

where $\mathbf{b}=\mathbf{B}/B$ is the unit vector along magnetic field lines and a the Alfvén speed. From Eq. (38a) it follows that the component of the fluid displacement along the magnetic field $\xi\cdot\mathbf{b}$ is conserved, $\partial^2(\xi\cdot\mathbf{b})/\partial t^2=0$, and the transverse component is propagated:

$$(\partial^2/\partial t^2-a^2\partial^2/\partial b^2)(\xi-(\xi\cdot\mathbf{b})\mathbf{b})=0\ , \qquad (39a)$$

along magnetic field lines,

$$\partial/\partial b\equiv\mathbf{b}\cdot\nabla=B^{-1}(\mathbf{B}\cdot\nabla)\ . \qquad (39b)$$

Thus Alfvén waves in three-dimensional space propagate one dimensionally along the magnetic field, as transverse motions along a stretched elastic string; the analogy is exact, since the propagation speed is $\sqrt{T/\rho}$ for oscillation of a string under elastic tension T, and for Alfvén waves [Eq. (38a)] we have $a=\sqrt{T/\rho}$, with $T\equiv\mu B^2/8\pi$ the magnetic tension. The dispersion relation corresponding to Eq. (39a),

$$\omega=\pm a(\mathbf{k}\cdot\mathbf{b})\ , \qquad (40a)$$

implies that the group velocity

$$\mathbf{w}=\pm a\mathbf{b} \qquad (40b)$$

is the Alfvén speed along magnetic field lines (of direction $\mathbf{b}\equiv\mathbf{B}/B$), and the phase speed

$$u=\pm a(\mathbf{n}\cdot\mathbf{b}) \qquad (40c)$$

is its projection on the wave-normal direction $u=\mathbf{w}\cdot\mathbf{n}$. We can classify the waves in fluids as (i) isotropic and nondispersive: acoustic (or sound); (ii) anisotropic and nondispersive: Alfvén (or magnetic); (iii) anisotropic and dispersive: internal (or gravity) and inertial (or gyroscopic). Besides these $\binom{4}{1}=4$ waves under a single restoring force, there are $\binom{4}{2}=6$ two-wave couplings under two restoring forces, $\binom{4}{3}=4$ three-wave couplings excluding one restoring force, and one four-wave coupling including all restoring forces. All the multiwave couplings are dispersive and anisotropic, since they involve either gravity or rotation or both, with a single exception: magnetoacoustic waves are anisotropic and nondispersive, and we take them as the last example of dispersion relations for waves in fluids.

7. Slow and fast modes and weak and strong fields

If we neglect gravity and rotation, $\mathbf{g}=0=\mathbf{\Omega}$, and consider a homogeneous medium under a constant magnetic field, we obtain from Eq. (10) the magneto-acoustic wave equation

$$\ddot{\xi}-c^2\nabla(\nabla\cdot\xi)-a^2[(\mathbf{b}\cdot\nabla)^2\xi-(\mathbf{b}\cdot\nabla)\nabla(\mathbf{b}\cdot\xi)] -a^2[\mathbf{b}(\mathbf{b}\cdot\nabla)(\nabla\cdot\xi)-\nabla(\nabla\cdot\xi)]=0\ , \quad (41)$$

whose terms may be classified as follows: (a) second-order time dependence, allowing waves propagating in opposite directions, and their superposition into standing modes; (b) acoustic term [Eq. (11)], involving the sound speed [Eq. (9a)] and dilatation $\nabla\cdot\xi$ [Eq. (12b)]; (c) Alfvén-wave term [Eq. (38a)], involving the Alfvén speed [Eq. (38b)] and propagation along magnetic field lines; (d) magneto-acoustic coupling, involving the Alfvén speed a and dilatation $\nabla\cdot\xi$. Considering a plane-wave solution [Eq. (13)] of Eq. (41), we obtain

$$\omega^2\mathbf{X}-(c^2+a^2)\mathbf{k}(\mathbf{k}\cdot\mathbf{X})-a^2[(\mathbf{k}\cdot\mathbf{b})^2\mathbf{X}-\mathbf{k}(\mathbf{k}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{X}) -\mathbf{b}(\mathbf{k}\cdot\mathbf{b})(\mathbf{k}\cdot\mathbf{X})]=0\ . \quad (42)$$

Choosing the x_1 axis along the wave vector $\mathbf{k}=(k,0,0)$ and the x_2 axis such that the magnetic field $\mathbf{b}=(b_1,b_2,0)$ lies in the (x_1,x_2) plane, we have

$$\begin{bmatrix} c^2+a^2b_2^2-u^2 & -a^2b_1b_2 & 0 \\ -a^2b_1b_2 & a^2b_1^2-u^2 & 0 \\ 0 & 0 & a^2b_1^2-u^2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}=0\ , \quad (43)$$

where $u\equiv\omega/k$ denotes the phase speed. The decoupling of the 3×3 dispersion matrix (43) into two parts shows that (i) the displacement X_3 transverse to the plane of wave vector $\mathbf{k}$ and magnetic field $\mathbf{B}$ corresponds $[u=ab_1=a(\mathbf{b}\cdot\mathbf{n})]$ to an Alfvén wave [Eq. (40b)], which extends without modification from incompressible (Sec. I.B.6) to compressible (Sec. I.B.7) fluids, since it involves no dilatation $\nabla\cdot\xi=0$; (ii) the displacement (X_1,X_2) in the plane of the wave vector and magnetic field $(\mathbf{k},\mathbf{B})$ corresponds to two coupled modes, satisfying the dispersion relation

$$u^4-(a^2+c^2)u^2+a^2c^2(\mathbf{b}\cdot\mathbf{n})^2=0\ , \quad (44)$$

which has two roots:

$$u_\pm=\tfrac{1}{2}(\,|c\mathbf{n}+a\mathbf{b}|\pm|c\mathbf{n}-a\mathbf{b}|\,)\ . \quad (45)$$

Thus acoustic waves, propagating at sound speed c in the wave-normal direction $\mathbf{n}$, combine in two ways with Alfvén waves, traveling at Alfvén, speed a along magnetic field lines $\mathbf{b}$, to form slow u_- and fast u_+ waves, so designated because the phase speed of the former does not exceed, and the phase of the latter does not fall below, the sound speed c or Alfvén speed a $[u_-\le\min(a,c)$ and $u_+\ge\max(a,c)]$.

8. Heating of chromosphere by Alfvén waves

The slow and fast modes decouple in the hydrodynamic limit of a weak magnetic field, or large plasma $\beta\equiv c^2/a^2$, for which the sound speed is much larger than the Alfvén speed, and the gas pressure predominates over the magnetic pressure. They also decouple in the magnetodynamic limit of a strong magnetic field or small plasma, for which the sound speed is much smaller than the Alfvén speed, and the gas pressure is dominated by the magnetic pressure:

$$c^2\gg a^2:\ u_-=a(\mathbf{b}\cdot\mathbf{n}),\ \ u_+=c\ ; \quad (46a)$$

$$a^2\gg c^2:\ u_-=c(\mathbf{b}\cdot\mathbf{n}),\ \ u_+=a,\ \ u_0=a(\mathbf{b}\cdot\mathbf{n})\ . \quad (46b)$$

Thus, in the weak-field limit [Eq. (46a)], the sound waves are the fast mode, and the Alfvén waves the slow mode, i.e., the three MHD wave modes coalesce into two; in the strong-field limit [Eq. (46b)], the Alfvén wave remains u_0, propagating at Alfvén speed a along magnetic field lines, the fast mode u_+ propagates at Alfvén a speed in all directions, and the slow mode u_- is an acoustic wave c constrained to move along magnetic field lines $\mathbf{b}$. This shows that, in the strong-field limit, gas pressure cannot overcome the magnetic stresses, and thus acoustic waves are guided along magnetic field lines as if the latter were rigid tubes; the guidance of acoustic waves along the magnetic field lines was mentioned (Sec. I.B) in connection with spicules and the mass balance in the solar atmosphere. Concerning the energy balance, it should be established by hydromagnetic rather than hydrodynamic waves, since it is well known that magnetic regions of the sun are hotter than nonmagnetic regions. Of the three magneto-acoustic wave modes, Alfvén waves propagate along a magnetic field of arbitrary direction, and their transverse magnetic field perturbations $\mathbf{h}$ are associated with electric current $\mathbf{j}=(c/4\pi)\nabla\Lambda\mathbf{h}$. The heating by Joule effect $q=j^2/\sigma$, where σ is the conductivity, may be balanced against thermal radiation losses, to yield a theoretical temperature profile [Fig. 2(b)] consistent with observational data; we have chosen for comparison the empirical models known as the BCA (Bilderberg continuum atmosphere; Gingerich and De Jager, 1968), HSRA (Harvard-Smithsonian reference atmosphere; Gingerich, Noyes, Kalkofen, and Cuny, 1971), and VAL (Vernazza, Avrett, and Loeser, 1973, 1976, 1981); the consistency of theory and observation extends from the profile of temperature versus altitude [Fig. 21(b)], to that of mass density [Fig. 2(a)], calculated on the basis of hydrostatic equilibrium.

C. Dynamic and magnetic generation and radiation

The theoretical and empirical temperature profiles [Fig. 2(b)] demonstrate the existence in the chromosphere of a temperature minimum; the latter is a feature of all the empirical chromosphere models developed over the last two decades (Athay, 1965, 1966a, 1985; Lambert, 1971; Ayres and Linsky, 1976; Morrison and Linsky, 1978; Basri, Linsky, Bartoe, Bruecker, and Van Hoosier, 1979; Cram and Damé, 1983; Skumanich, Lean, White, and Livingston, 1984). The temperature minimum gives evidence of mechanical heating, since in equilibrium the temperature would decrease monotonically with altitude, and there are no local energy sources sufficient to compensate for radiative losses. The mechanical heating was

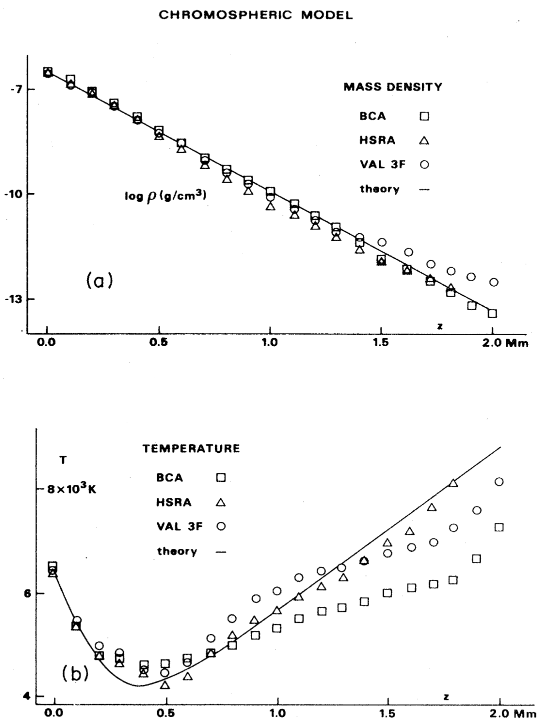

FIG. 2. Profiles of (a) mass density and (b) temperature vs altitude, as predicted (solid curve; Campos, 1984b) by resistive dissipation of Alfvén waves in the presence of thermal radiation, in comparison with three empirical models: □, the Bilderberg continuum atmosphere (BCA; Gingerich and De Jager, 1968); △, the Harvard-Smithsonian reference atmosphere (HSRA; Gingerich, Noyes, Kalkofen, and Cuny, 1972); ○, the VAL standard III model F (VAL 3F; Vernazza, Avrett, and Loeser, 1981).

originally attributed to waves, either magnetic (Alfvén, 1945, 1947) or acoustic (Biermann, 1946, 1948; Schwarzschild, 1948), whose presence in the chromosphere was subsequently substantiated by observation (Bonnet *et al.*, 1982; Damé, Gouttebroze, and Malherbe, 1984). The theories based on heating by acoustic waves/shocks were developed for the sun (Schatzman, 1949; Ulmschneider, 1971a, 1971b) and proposed for other stars (Ulmschneider, 1979, 1982) until observations definitely established that the acoustic energy flux decayed rapidly with altitude due to reflections, and is several orders of magnitude short (Mein, 1978; Provost and Mein, 1979; Mein and Mein, 1980; Schmieder and Mein, 1980; Bruner, 1981; Mein and Schmeider, 1981) of that needed to compensate for radiative losses in the middle and high chromosphere, transition region, and corona. Thus, only hydromagnetic waves remain as a viable mechanism for the heating of the solar atmosphere, and there is universal agreement that their energy flux, at photospheric levels, is sufficient, $\sim 10^7 - 10^8$ erg cm^{-2} s^{-1}. There is, however, a long-standing and still unresolved controversy over the mechanisms involved. Some argue that most of the Alfvén wave flux is not deposited in the chromosphere or low corona, i.e., it either propagates through or is reflected (Osterbrock, 1961; Hollweg, 1972, 1978, 1981a; Bel and Leroy, 1981; Leroy, 1981; Schwartz and Leroy, 1982), while others propose various physical mechanisms capable of dissipating enough wave energy to compensate for thermal radiative losses (Alfvén, 1948; Uchida and Kaburaki, 1975; Ionson, 1982, 1984; Heyvaerts and Priest, 1983; Campos, 1984b; Hollweg, 1984b, 1984c, 1984d; Hollweg and Sterling, 1984; Nocera, Leroy, and Priest 1984; Sakurai and Granik, 1984). The fundamental idea under consideration is whether waves could establish mass and energy balances in the solar, and possibly other stellar, atmospheres (Withbroe and Noyes, 1977; Linsky, 1980; Campos, 1984c); a detailed modeling of these physical processes depends on the mechanisms of generation (Lighthill, 1952, 1954; Kulsrud, 1955; Parker, 1964; Stein, 1967, 1981; Campos, 1977, 1978a; Ulmschneider and Stein, 1982) and radiation (Lighthill, 1960, 1964, 1967, 1978; Campos, 1982, 1983b; Adam, 1982a; Adam and Thomas, 1984) of waves in fluids, to which we now turn.

1. Waves forced by multipole sources

The linear, nondissipative magneto-acoustic-gravity-inertial wave equation (10) is a vector equation of second order and leads to a system of partial differential equations of second, fourth, and sixth orders in the cases having, respectively, one, two, and three modes. The order of the system is increased by the presence of (i) viscosity, which increases the order of the linearized momentum equation from two in Eq. (7b) to three, by the addition of the viscous stresses

$$\nu_1 \nabla^2 \dot{\xi} + (\nu_2 + \nu_1/3) \nabla(\nabla \cdot \dot{\xi})$$

from Eq. (1); (ii) electrical resistance, which increases the order of the linearized induction equation from one in Eq. (8a) to two, by the addition of the Ohmic $(c^2/4\pi\mu\sigma)\nabla^2 \mathbf{h}$ and Hall $(c^2/4\pi\mu\eta)$ $\nabla\Lambda[\mathbf{B}\Lambda(\nabla\Lambda\mathbf{h}) + \mathbf{h}\Lambda(\nabla\Lambda\mathbf{B})]$ terms from Eq. (2); (iii) thermal diffusion, e.g., conduction, convection, or radiation, which replaces the adiabatic equation (6a), which is of first order, by the energy equation (5) specifying entropy production, which is of second order, e.g., in the linear conduction term $\nabla \cdot (\kappa \nabla T)$; (iv) perturbations in the gravitational field $\mathbf{G} = \nabla \psi$ or potential ψ, which add to the system [(3a) and (3b)], a pair of equations equivalent to the Poisson equation $\nabla^2 \psi = 4\pi k_* \rho$, which is of second order. Thus the system of equations for waves in fluids can be increased in order up to 8 by the inclusion of perturbations in the gravitational potential, and up to 10, 11, or 12 by accounting for single, double, or triple diffusion. In all cases, as long as all the coefficients of the linear partial differential equations are constant, it is possible to eliminate for any wave variable, e.g., the displacement ξ, leading to the equation

$$[\Box_{jl}(\partial/\partial\mathbf{x}, \partial/\partial t)]\xi_l(\mathbf{x},t) = \partial^{n+m} S_{i_1 \cdots i_n} / \partial t^m \partial x_{i_1} \cdots \partial x_{i_n} \partial x_j , \quad (47)$$

where the wave operator $\Box_{jl}$ on the lhs is a matrix whose terms are polynomials of spatial $\partial/\partial\mathbf{x}$ and temporal $\partial/\partial t$ derivatives of arbitrary order. The forcing term on the rhs models the sources generating the waves; it may be determined from the nonlinear terms of the fundamental equations by the "wave analogy" Part I (Sec. II.A.1), of which we give a detailed example below (Sec. I.C.5). For the moment, it is sufficient to note that we allow the source term to be a monopole S, a dipole S_i, a quadrupole S_{ij}, or an arbitrary multipole $S_{i_1}\dots{}_{i_n}$, to which a time derivative $\partial^m/\partial t^m$ of any order m may be applied.

2. Exact integration over wave-number space

The source multipole has a Fourier spectrum:

$$\widetilde{S}_{i_1\dots i_n}(\mathbf{k},\omega)=(2\pi)^{-4}\int_{-\infty}^{+\infty}S_{i_1\dots i_n}(\mathbf{x},t)\times e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}d^3k\,d\omega\ , \quad (48)$$

where $\mathbf{k}$ denotes the wave number and ω the frequency. The wave operator $\Box_{jl}$ has constant coefficients, and thus plane-wave solutions [Eq. (13)] exist, whose spectrum X satisfies an algebraic relation (49a), which replaces the differential equation (47),

$$\Pi_{jl}\mathbf{X}_l=\overline{S}_j\ , \quad (49a)$$

$$\overline{S}_j\equiv(-)^m i^{n+m+1}\omega^m k_{i_1}\dots k_{i_n}k_j\widetilde{S}_{i_1\dots i_n}\ . \quad (49b)$$

Equation (49b) is the forcing spectrum, and Π_{jl} the dispersion matrix, which coincides with the wave operator $\Box_{jl}$ with the substitutions $\partial/\partial\mathbf{x}\to i\mathbf{k}$, $\partial/\partial t\to -i\omega$:

$$\Pi_{jl}(i\mathbf{k},-i\omega)\equiv\Box_{jl}(\partial/\partial\mathbf{x},\partial/\partial t)\ , \quad (50a)$$

$$\Pi\equiv\mathrm{Det}(\Pi_{jl})\ . \quad (50b)$$

The determinant (50b) specifies through $\Pi=0$ the dispersion relations for all wave modes. The algebraic relation (49a) may be inverted, for arbitrary $(\mathbf{k},\omega)$, as

$$X_j=\Lambda_{jl}\Pi^{-1}\overline{S}_l\ , \quad (51a)$$

$$\Lambda_{jl}\Pi_{lm}\equiv\Pi\delta_{jm}\ , \quad (51b)$$

where Λ_{jl} is the matrix of cofactors, and $\Pi^{-1}\Lambda_{jl}$ the inverse of Π_{jl}. Substituting Eq. (51a) into (13), we obtain the wave field

$$\xi_j(\mathbf{x},t)=\int_{-\infty}^{+\infty}[(\Lambda_{jl}\overline{S}_l)/\Pi]e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}d^3k\,d\omega\ , \quad (52)$$

in terms of the source spectrum [Eqs. (48) and (49b)]. The integral (52) has poles for the frequencies ω such that $\Pi(\mathbf{k},\omega)=0$, i.e., $\Pi(\mathbf{k},\omega)$ is evaluated by a sum of residues, one for each dispersion relation $\omega(\mathbf{k})$ of each mode. In the case of simple poles, the wave field is given by

$$\xi_j(\mathbf{x},t)=2\pi i\sum_m\int^{\omega=\omega_m(\mathbf{k})}\Lambda_{jl}\overline{S}_l(\partial\Pi/\partial\omega)^{-1}\times e^{i[\mathbf{k}\cdot\mathbf{x}-\omega_m(\mathbf{k})t]}d^3k\ , \quad (53)$$

where the sum $\sum_m$ extends over all simple roots $\omega_m(\mathbf{k})$ of $\Pi(\mathbf{k},\omega)=0$, i.e., over all wave modes, and the integration, for each mode $\omega_m(\mathbf{k})$, is performed over the wave-number space $\omega=\omega_m(\mathbf{k})$.

3. Asymptotics of nondissipative and damped waves

In order to simplify further the exact solution [Eq. (53)] of the wave equation (47), we seek an asymptotic approximation to the radiation field received by a distant observer, i.e., an approximation evaluated to lowest order in $|\mathbf{x}|^{-1}$. If the wave-number space $\omega_m(\mathbf{k})$ is flat, as it is for a nondissipative plane wave, all elements radiate to the far field, and no simplification of Eq. (53) is possible, even asymptotically. If it has a single curvature, then the observer in the far field will receive most of the radiation from those elements for which the normal $\partial\omega_m/\partial\mathbf{k}\equiv\mathbf{u}_m$ or group velocity points to him. If the wave-number space has a single curvature R_0, it can be approximated in the vicinity of the x_1 axis, which we take to be pointing to the observer, by the tangent cylinder:

$$\omega_m(\mathbf{k})=\omega_m(\mathbf{k}^0)+\tfrac{1}{2}R_0(k_1-k_1^0)^2+O((k_1-k_1^0)^3)\ . \quad (54)$$

Substitution of Eq. (54) into (53) allows us to evaluate to lowest order in $1/t$ the integral in dk_1, which is of Gaussian type:

$$\int_{-\infty}^{+\infty}\exp[-(i/2)R_0(k_1-k_1^0)^2t+O((k_1-k_1^0)^3)t]dk_1=\sqrt{2\pi/|R_0|t}\,\exp(i\phi_0)[1+O(t^{-1/2})]\ , \quad (55a)$$

$$\phi_0\equiv(\pi/4)\mathrm{sgn}(R_0)\ , \quad (55b)$$

where the sign function $\mathrm{sgn}(R_0)$ appearing in the phase shift (55b) is ±1, depending on whether the curvature is positive, $R_0>0$, or negative, $R_0<0$. The asymptotic wave field is given [Eqs. (53) and (55a)] by

$$\xi_j(\mathbf{x},t)\sim i(2\pi)^{3/2}\sum_m(|R_0|t)^{-1/2}\int^{k_1=k_1^0,\omega=\omega_m(\mathbf{k})}\Lambda_{jl}\overline{S}_l(\partial\Pi/\partial\omega)^{-1}\exp[i(k_1^0x_1+k_2x_2+k_3x_3-\omega t+\phi_0)]dk_2dk_3\ , \quad (56)$$

where the integration is performed for each mode ω_m along the subspace $k_1^0=k_1$, for a dissipative plane wave, and where we assume that the wave-number space has nonzero curvature [$R_0\neq0$ in Eq. (54)]. If $R_0=0$, i.e., at an inflexion, the tangent surface is of third degree and the Gaussian integral [(55a) and (55b)] is replaced by an Airy integral (Lighthill, 1978), which describes the transition between light and shadow near a caustic and leads to a decay like $t^{-1/3}$. Higher-order terms, up to n, could be treated similarly and would lead to a decay like $t^{-1/n}$.

4. Plane, cylindrical, and spheroidal waves

We thus have six cases to consider, by combining nondissipative or dissipative waves with wave fronts having none, one, or two curvatures. We have already given the exact wave field [Eq. (53)] for plane, *nondissipative waves* (no simplification in this case) and for plane, dissipative modes with curvature R_0 of the wave-vector space $\omega(\mathbf{k})$, the asymptotic form [Eq. (56)]. For nondissipative cylindrical waves, we have relations like Eq. (54) with R_1 the spatial curvature and ϕ_1 the corresponding phase shift, leading to a formula similar to Eq. (56) with $|\mathbf{x}|$ replacing t. For dissipative cylindrical waves, Eq. (54) is replaced by

$$\omega_m(\mathbf{k})=\omega_m(\mathbf{k}_1^0)+\tfrac{1}{2}R_0(k_1-k_1^0)^2+\tfrac{1}{2}R_1(k_2-k_2^0)^2 , \quad (57)$$

involving one space curvature R_1 and one time curvature R_0, so that we can evaluate asymptotically the dk_1dk_2 integrals in Eq. (53), obtaining for the wave field

$$\xi_j(\mathbf{x},t)\sim i\,4\pi^2\sum_m(|R_0R_1|t|\mathbf{x}|)^{-1/2}\int^{k_1=k_1^0,k_2=k_2^0,\omega_n=\omega_m(\mathbf{k})} \times\Lambda_{jl}\bar{S}_l(\partial\Pi/\partial\omega)^{-1}\exp[i(k_1^0x_1+k_2^0x_2-k_3x_3-\omega t+\phi_0+\phi_1)]dk_3 , \quad (58)$$

which involves only one integration. For nondissipative spheroidal waves, we have Eq. (57) with R_1,R_2 the principal curvatures of the wave front. Since t is replaced by $|\mathbf{x}|$, the wave field is given by Eq. (58) with the following changes: (i) the term $(R_0R_1t|\mathbf{x}|)^{-1/2}$ is replaced by $R^{-1/2}|\mathbf{x}|^{-1}$, where $R\equiv R_1R_2$ is the Gaussian curvature, so that the amplitude of the wave increases as radiation takes place through a smaller fraction $1/R$ of the solid angle $4\pi^2$, i.e., through a "pencil beam"; (ii) the phase term $\phi\equiv\phi_1+\phi_2\equiv(\pi/4)[\mathrm{sgn}(R_1)+\mathrm{sgn}(R_2)]$ vanishes for anticlastic beams, which have principal curvatures with opposite signs ($\phi=0$ for $R_1R_2<0$), e.g., a parabolic hyperboloid, while for synclastic beams ($R_1R_2>0$) we have $\phi_+=\pi/2$ for divergent ($R_1,R_2>0$) and $\phi_-=-\pi/2$ for convergent ($R_1,R_2<0$) beams, so that the phase jump across a focus is $\Delta\phi=\phi_+-\phi_-=\pi$ (Landau and Lifshitz, 1966). In the case of dissipative spherical waves, we have two space curvatures R_1,R_2 and one time curvature R_0:

$$\omega_m(\mathbf{k})=\omega_m(\mathbf{k}_1^0)+\tfrac{1}{2}\sum_{j=1}^{3}R_{j-1}(k_j-k_j^0)^2 , \quad (59)$$

allowing the evaluation of all integrals in Eq. (53), leading to

$$\xi_j(\mathbf{x},t)\sim i(2\pi)^{5/2}\sum_m(t|R_0R_1R_2|)^{-1/2}|\mathbf{x}|^{-1}\Lambda_{jl}\bar{S}_l \times(\partial\Pi/\partial\omega)^{-1}\exp\{i[\mathbf{k}_0\cdot\mathbf{x}-\omega_m(\mathbf{k}_0)t+\phi_0+\phi_1+\phi_2]\} \quad (60)$$

as an explicit formula for the asymptotic wave field.

5. Hydrodynamic and hydromagnetic source dipoles

In order to apply Eqs. (53), (56), (58), and (60) to the calculation of radiation fields, we need the dispersion matrix (50a) for the modes in question (51b), and the forcing spectrum (49b) for the source multipole (48). Since we have already discussed dispersion relations in some detail (Secs. I.A and I.B), we now concentrate on modeling the "sources" of waves by using the "wave analogy" (Part I, Sec. II.A.1), in a form that generalizes the acoustic case to include the effects of external magnetic fields. We start from the general equations of magneto-hydrodynamics [Eqs. (1), (2), (4)] without gravity or rotation, $\mathbf{G}=0=\mathbf{\Omega}$, and neglect dissipative terms for simplicity. We consider the flow to consist [Eq. (7a)] of a homogeneous, mean state of rest (constant $\rho,p,\mathbf{B}$) plus a nonuniform and unsteady perturbation. When substituting Eq. (7a) in the general equations, we retain, besides the linear terms in Eqs. (7b), (8a), and (8b), all the nonlinear terms, which we collect on the rhs, so that the equations remain exact:

$$\dot{\mathbf{h}}+\mathbf{B}(\nabla\cdot\mathbf{v})-(\mathbf{B}\cdot\nabla)\mathbf{v}=\nabla\Lambda(\mathbf{v}\Lambda\mathbf{h}) , \quad (61a)$$

$$\dot{\bar{\rho}}+\rho\nabla\cdot\mathbf{v}=-\nabla\cdot(\bar{\rho}\mathbf{v}) , \quad (61b)$$

$$\dot{\mathbf{v}}+(b^2/\rho)\nabla\bar{\rho}+(\mu/4\pi\rho)[\nabla(\mathbf{B}\cdot\mathbf{h})-(\mathbf{B}\cdot\nabla)\mathbf{h}] =-\partial(\bar{\rho}\mathbf{v}/\rho)/\partial t-\rho^{-1}\partial^2T_{ij}^0/\partial t\partial x_j , \quad (62a)$$

where T_{ij}^0 denotes the tensor

$$T_{ij}^0\equiv\rho v_iv_j+(p-c^2\bar{\rho})\delta_{ij}-(\mu/4\pi)(h_ih_j-\tfrac{1}{2}h^2\delta_{ij}) . \quad (62b)$$

Eliminating between the linear terms on the lhs of Eqs. (61a), (61b) and (62a), we obtain the magneto-acoustic wave equation (41), which we denote $\Box_{ij}v_j=0$, choosing the velocity perturbation $\mathbf{v}$ as the wave variable. If we retain the nonlinear terms on the rhs of Eqs. (61a), (61b), and (62a), then an exact equation follows, where the propagation operator is forced:

$$[\Box_{ij}(\partial/\partial\mathbf{x},\partial/\partial t)]v_{ij}(\mathbf{x},t)=-\rho^{-1}\partial^2T_{ij}/\partial t\partial x_j , \quad (63)$$

by the tensor

$$T_{ij}\equiv T_{ij}^0+T_{ij}^4 , \quad (64a)$$

$$\partial T_{ij}^4/\partial t\equiv-(\mu/4\pi)[\nabla\Lambda(\mathbf{v}\Lambda\mathbf{h})]_i*B_j . \quad (64b)$$

Here we have used the notation a_i*b_j for the symmetric tensor product of two vectors:

$$a_i*b_j\equiv a_ib_j+a_jb_i-\tfrac{1}{2}(\mathbf{a}\cdot\mathbf{b})\delta_{ij} , \quad (65)$$

which satisfies $a_i*b_i=\tfrac{1}{2}(\mathbf{a}\cdot\mathbf{b})$. The tensor T_{ij}, defined by Eqs. (64a), (64b), and (62b), consists of nonlinear terms and is significant only in regions of large disturbances, which act as hydrodynamic and hydromagnetic sources of waves.

6. Reynolds and Maxwell stress quadrupoles

If, in the source tensor (62b), we separate the pressure $p=\bar{p}+1/2\rho v^2$ into the dynamic pressure $\frac{1}{2}\rho v^2$ and a "compressible" part $\bar{p}$, it contains the following hydrodynamic and hydromagnetic tensors:

$$T_{ij}^1 \equiv \tfrac{1}{2}\rho v_i * v_j = \rho v_i v_j - \tfrac{1}{2}\rho v^2 \delta_{ij} , \tag{66a}$$

$$T_{ij}^2 \equiv -(\mu/8\pi) h_i * h_j = -(\mu/4\pi) h_i h_j + (\mu/8\pi) h^2 \delta_{ij} . \tag{66b}$$

Equation (66a) consist of the Reynolds stresses $\rho v_i v_j$, from which is subtracted the dynamic pressure, and Eq. (66b) is the Maxwell stress tensor; they can be derived, respectively, from the dynamic $\frac{1}{2}\rho v_i v_i$ and magnetic $-(\mu/8\pi)h_i h_i$ pressures, by replacing the inner product $a_i a_i \equiv \mathbf{a}\cdot\mathbf{a}$ by the symmetric tensor product $a_i * a_j$, defined by Eq. (65). They represent the generation of magneto-acoustic waves by hydromagnetic turbulence that acts as a quadrupole source. The remaining terms in the source tensor also divide into a dynamic and a magnetic contribution, respectively, $T_{ij}^3 \equiv (\bar{p}-c^2\bar{\rho})\delta_{ij}$ and T_{ij}^4 given by Eq. (64b). The dynamic contribution is nonvanishing if $\bar{p} \neq c^2\bar{\rho}$, i.e., for nonadiabatic propagation or nonuniform sound speed. The magnetic contribution (64b) vanishes for a perfectly conducting fluid, for which the velocity $\mathbf{v}$ and magnetic field $\mathbf{h}$ perturbations are aligned $\mathbf{v}\Lambda\mathbf{h}=0$, and is due to finite conductivity. The two terms T_{ij}^3, T_{ij}^4 represent the generation of magneto-acoustic waves by ionized, inhomogeneous regions, and are equivalent to force dipoles, e.g.,

$$\partial[(\bar{p}-c^2\bar{\rho})\delta_{ij}]/\partial x_j = \nabla(\bar{p}-c^2\bar{\rho}) .$$

The hydrodynamic contribution $T_{ij}^v \equiv T_{ij}^1 + T_{ij}^2 - \sigma_{ij}$, from which the viscous stresses σ_{ij} are subtracted, is the Lighthill (1952, 1954) tensor, which is the quadrupole source in aerodynamic acoustics [Eq. (67a)],

$$T_{ij}^v = \tfrac{1}{2}\rho v_i * v_j + (\bar{p}-c^2\bar{\rho})\delta_{ij} - \sigma_{ij} , \tag{67a}$$

$$\sigma_{ij} = \nu_1(\partial v_i/\partial x_j + \partial v_j/\partial x_i) + (\nu_2 - \tfrac{2}{3}\nu_1)(\nabla\cdot\mathbf{v})\delta_{ij} . \tag{67b}$$

In order to obtain the total source of magneto-acoustic waves $T_{ij} \equiv T_{ij}^v + T_{ij}^h$, we have to add to Lighthill's hydrodynamic tensor [Eqs. (67a) and (67b)] a hydromagnetic tensor $T_{ij}^h = T_{ij}^2 + T_{ij}^4 - \zeta_{ij}$ (Campos, 1977), consisting of analogous nondissipative terms [Eqs. (66b) and (64b)]:

$$\partial T_{ij}^h/\partial t = -(\mu/2\pi)\dot{h}_i * h_j - (\mu/4\pi)[\nabla\Lambda(\mathbf{v}\Lambda\mathbf{h})]_i * B_j - \partial\zeta_{ij}/\partial t , \tag{68a}$$

with a dissipative term

$$\zeta_{ij} \equiv (c^2/4\pi\mu\sigma)(\nabla^2\mathbf{h})_i * B_j , \tag{68b}$$

involving the Ohmic resistivity $1/\sigma$, which takes the place of viscous stresses [Eq. (67b)]. Recalling the decoupling of magneto-acoustic modes [Eq. (43)], we see that the wave generation equation (63) implies that the components of the source tensor T_{ij} may be classified as follows: (i) the compressions/tractions T_{11}, T_{22} and shear stresses $T_{12}=T_{21}$ in the plane (x_1,x_2) of the magnetic field $\mathbf{B}$ and wave vector $\mathbf{k}$ generate slow and fast modes only; (ii) the transverse compressions/tractions T_{33}, along the axis x_3 orthogonal to the $(\mathbf{B},\mathbf{k})$ plane, generate Alfvén waves only; (iii) the cross shears $T_{13}=T_{31}$ and $T_{23}=T_{32}$ generate all three (slow, fast, and Alfvén) modes.

7. Monopole sources and the intensity law

The quadrupole source T_{ij} scales like the dynamic and magnetic pressures, respectively, in the hydrodynamic [Eq. (66a)] and hydromagnetic [Eq. (66b)] terms, so that $\bar{T} \sim \frac{1}{2}\rho U^2 + \mu B^2/8\pi$, and the forcing spectrum [Eq. (49b)] $\bar{S}_j = \omega k_l T_{jl}/\rho$ scales as $\bar{S} \sim \omega k(U^2+\mu B^2/4\pi\rho) \sim \omega k(U^2+A^2)$, where A is the Alfvén speed $A \equiv \mu B^2/4\pi\rho$. For a nondissipative spherical wave, the form analogous to Eq. (60) involves as factors $R^{-1/2} \sim k$ and

$$\Lambda_{jl}\,|\,\partial\Pi/\partial\mathbf{k}\,|^{-1} \sim k\Pi^{-1}\Lambda_{jl} \sim k\Pi_{jl}^{-1} \sim k\omega^{-2} ,$$

implying an overall factor $\sim k^3\omega^{-1}(A^2+U^2)l^3$, where l is the scale and $\sim l^3$ the volume of the source region. The wave number $k=\omega/u$ is related, through the phase speed u, to the frequency $\omega \sim U/l$, which is determined by the flow velocity U in the source region and the length scale l. Thus, the asymptotic velocity perturbation scales as

$$v \sim (l/u)\,|\,\mathbf{x}\,|^{n}(U/u)^{m}(U^2+A^2) , \tag{69}$$

with the exponent $n=0$, $-\frac{1}{2}$ or -1 for spatial decay given, respectively, for plane, cylindrical, and spherical waves, and the power $m=2,1,0$ for quadrupole, dipole, and monopole sources, i.e., stresses, forces, and mass/energy supply, which involve progressively fewer factors in the forcing spectrum (49b). The intensity of radiation scales as $I \sim \rho u v^2$ per unit area for a plane wave, is multiplied by the perimeter $\sim |\mathbf{x}|$ of a cylinder (per unit length) for a cylindrical wave, and is multiplied by the area of a sphere $\sim |\mathbf{x}|^2$ for a spherical wave. Thus the spatial dependence drops out for nondissipative waves, and damped waves have an additional time decay that prevents them from reaching the far field. In the former (nondissipative) case, the intensity of radiation in the far field, $I \sim \rho u v^2 |\mathbf{x}|^{-2n}$, is given by the magneto-hydrodynamic scaling law

$$I \sim \rho(l^2/u)(U/u)^{2m}(U^2+A^2)^2 , \tag{70}$$

which applies to waves of all geometries, apart from a directivity factor omitted in Eq. (70) and contained in Eq. (60). Since the waves are assumed to be generated by a flow with velocity U small relative to the phase speed u, the factor $(U/u)^2 \ll 1$ shows that emission by monopoles $(m=0)$ is much more efficient than by dipoles $(m=1)$, and the latter emission in turn predominates over that of quadrupoles ($m=2$). Two particular cases of Eq. (70) are

$$I_v \sim \rho(l^2/c)(U/c)^{2m}U^4 , \tag{71a}$$

$$I_h \sim \rho(l^2/u)(U/u)^{2m}A^4 . \tag{71b}$$

These are, respectively, the hydrodynamic law of intensity of sound radiation (c speed) in aerodynamic acoustics (Part I, Sec. II.A), when the magnetic field is absent, and the magnetodynamic law of intensity of radiation of magneto-acoustic waves, when the magnetic pressure dominates the gas pressure. When they are comparable, the general law (70) should be used.

8. Generation of waves by the photospheric granulation

In the sun, waves can be observed in the deepest visible layer, the photosphere, which is the "base" of the atmosphere and overlies the convection zone. The existing models of the photosphere (e.g., Morrison and Linsky, 1978) overlap to some extent with those of the chromosphere above (see earlier references in Sec. I.C). The main distinguishing feature is the presence in the chromosphere of granulation, i.e., a network of convection cells extending deeper into the convection zone; the motions and energy processes in the photospheric granulation have been the subject of detailed observations (Wittman, 1981; Mattig, Mehltretter, and Nesis, 1981; Bassgen and Deubner, 1982; Muller, 1985; Berton, 1986; van Ballegooijen, 1986) and simulations (Nordlund, 1982, 1985; Wohl and Nordlund, 1985). The motions are predominantly upward at the centers and downward at the boundaries of granules and supergranules, where most of the magnetic field emerges in the form of intense kG-strength flux tubes. At the level of observational resolution presently achieved (about $0.5-2.0'' \sim 700-1500$ km) it is possible to conclude that the magnetic field is below detection threshold (less than about 5 G) at granule centers (it could be absent), and in contrast is strongly concentrated at granule boundaries. Magnetic field strengths of 1—2 kG have consistently been reported there (Weiss, 1978; Golub, Rosner, Vaiana, and Weiss, 1981; Howard and Labonte, 1981; Stenflo, 1982; Stenflo and Harvey, 1985), but the scale of the magnetic flux tubes (about $0.1'' \sim 150$ km) is still beyond the available resolution from the Earth, so that direct observation may be possible only from future space-based magnetic sensors. The granule boundaries are ionized, inhomogeneous regions and act as dipole sources of waves. Although they occupy a small fraction (less than 1%) of the solar disk, they radiate about half of the energy flux. Since the magnetic fields are weak or absent (a few Gauss at most) in cell centers, these regions of hy-

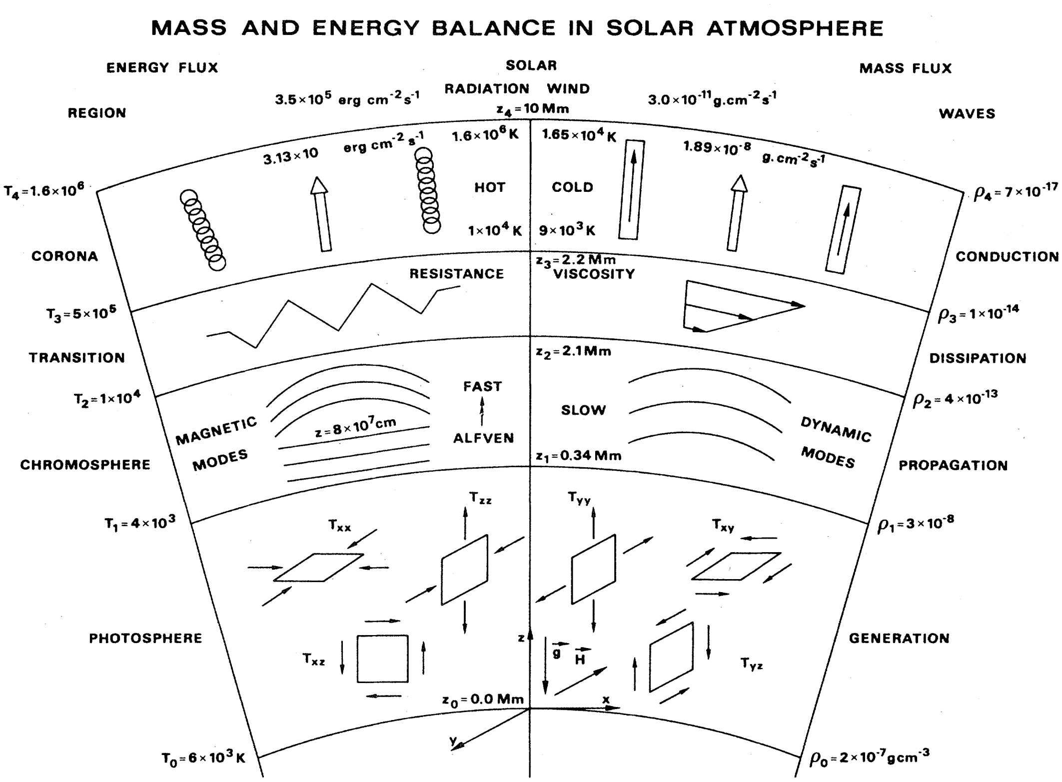

FIG. 3. Diagram (bottom) of stress components in the photosphere responsible for the generation of hydrodynamic and hydromagnetic waves, and outline (right/left) of the physical processes whereby these could establish the mass and energy balances, respectively, in the solar atmosphere.

dromagnetic turbulence act as quadrupole sources and are less efficient than dipoles; thus although they occupy a much larger area, they could radiate about as much as cell boundaries. The source mechanisms represented by the relevant stress-tensor components (Fig. 3) generate all three types of magneto-acoustic modes. The hydrodynamic modes, viz., sound waves, may be expected to dominate the mass balance (Sec. I.A.7), while the hydromagnetic modes, e.g., Alfvén waves, may dominate the energy balance (Sec. I.B.7). As evidence of this we have shown that acoustic waves can supply to the corona, in their compression front, a mass flux of 3×10^{-11} $\mathrm{g\,cm^{-2}\,s^{-1}}$; multiplying by the solar disk are a $D=\pi R^2=1.52\times10^{22}$ cm^2, where $R=6.95\times10^{10}$ cm is the solar radius, yields the mass loss 4.55×10^5 $\mathrm{g\,s^{-1}}$ in the solar wind. Concerning the energy balance, we use the intensity equation (70), with the proportionality factor 5 (Campos, 1986b) for Alfvén waves $u=A$, viz., $I_0=5\rho(D/A)(U^2+A^2)^2$. We have assumed monopole efficiency, since (Parker, 1964), in an atmosphere, the density effects of stratification enhance wave generation mechanisms, as for volume changes. Using $\rho=3\times10^{-7}$ $\mathrm{g\,cm^{-3}}$ for the mass density, $U=1$ km/s for the flow velocity, and $A=5$ km/s for the Alfvén speed, we obtain an estimate of intensity $I_0\simeq3\times10^{33}$ $\mathrm{erg\,s^{-1}}$, which is of the order of the solar radiation loss. Although these estimates are not precise and depend on parameters some of which have considerable observational scatter, they show that waves can carry mass and energy fluxes that are of the order of magnitude of solar losses.

II. LINEAR WAVES IN STRATIFIED MEDIA

We have seen that Fourier analysis provides a convenient and powerful method (Sec. I.A) for studying wave propagation and amplification/decay through the dispersion relation (Sec. I.B), and generation and radiation by means of asymptotic approximations (Sec. I.C). This method applies to linear partial differential equations with constant coefficients (with or without forcing), i.e., to waves of small amplitude in media for which the wave speeds (and other parameters, such as damping or scattering lengths) are constant, that is, homogeneous media. The method fails for linear waves in inhomogeneous media such that the wave speed changes on a scale comparable to (or shorter than) a wavelength, since in that case the waveforms cease to be sinusoidal, i.e., a complex exponential is *not* generally a solution of a linear differential equation with variable coefficients. For example, in the case of acoustic-gravity waves in an atmosphere, since the sound speed depends only on temperature for a perfect gas, the Fourier method applies strictly only in isothermal conditions. A different approach must be used in nonisothermal atmospheres (Sec. II.A). Concerning purely magnetic modes, i.e., Alfvén waves, the Alfvén speed is constant in a homogeneous medium (of uniform density) under a constant magnetic field (in strength and direction), in which case the waves are sinusoidal.

In an atmosphere the mass density $\rho(z)$ decays with altitude, and the Alfvén speed is constant only if the magnetic field decays in strength like $B(z)\sim\sqrt{\rho(z)}$; moreover, the Maxwell equation $\nabla\cdot\mathbf{B}=0$ implies that $B_z=0$, i.e., it must be horizontal. If the magnetic field does not decay in the required manner, or if it has a nonzero vertical component, as it does in the presence of an oblique uniform magnetic field, the Alfvén speed varies with altitude in an atmosphere and distorts the waveform into a nonsinusoidal shape. The restrictions stated before apply together in the case of magneto-acoustic waves in an atmosphere, i.e., waves are sinusoidal (Yu, 1965; Adam, 1977a; Thomas, 1982) only if (i) the atmosphere is isothermal, so that the sound speed is a constant, and the mass density $\rho(z)\sim e^{-z/L}$ decays exponentially with altitude z normalized to the scale height L; (ii) the magnetic field is horizontal and decays exponentially $B(z)\sim e^{-z/2L}$, but on twice the scale height. These restrictions guarantee uniform sound and Alfvén speeds, and thus a constant ratio of gas to magnetic pressure, which excludes the possibility of wave transformation; more generally, the sound and Alfvén speeds may be nonuniform and evolve differently with altitude, in which case not only is the waveform distorted into a nonsinusoidal shape, but also mode conversion becomes possible (Sec. II.C), e.g., an acoustic wave in a region of predominant gas pressure may become a magnetic mode as it propagates into a region of dominant magnetic pressure.

A. Acoustic-gravity waves in nonisothermal atmospheres

Internal and acoustic-gravity waves (Eckart, 1960; Pedlosky, 1960; Tolstoy, 1963; Beer, 1974; Gossard and Hooke, 1975) were studied first in isothermal conditions (Stokes, 1847; Lamb, 1879; Rayleigh, 1890; Biermann, 1948), and the extension to nonisothermal atmospheres has been made by considering "arbitrary" temperature profiles with gradients everywhere small (Moore and Spiegel, 1948; Brekhovskikh, 1960; Liu and Yeh, 1974; Lighthill, 1978), or specific temperature profiles allowing for arbitrarily large gradients (Lamb, 1910; Groen, 1948; Thorpe, 1968; Lindzen, 1970; Campos, 1983c). The common occurrence of these waves in the ocean (Philips, 1976; Kraus, 1977) and atmosphere (Delloue and Halley, 1972; Hines, 1974) has motivated the study of the effects of temperature (and salinity) gradients on acoustic-gravity waves. These waves have also been extensively studied in the solar atmosphere, in connection with mass and energy transport (see Secs. I.B and I.C for references) by waves and related oscillatory phenomena (Tavakol and Tworkowski, 1981; Leibacher, Gouttebroze, and Stein, 1982).

A curious natural phenomenon is the generation of acoustic-gravity waves in the Earth's atmosphere during solar eclipses, as the shadow of the moon moves supersonically, cooling the air mass and producing a "bow wave" (Seykora, Bhatnager, Jain, and Streete, 1985). Acoustic-gravity waves in an isothermal atmosphere (Part I, Sec.

V.A) have (i) a sinusoidal waveform, since the sound speed is constant; (ii) a velocity perturbation amplitude that grows exponentially $v \sim 1/\sqrt{\rho} \sim e^{z/2L}$ as a result of conservation of the energy flux $F = \rho v^2 c$. In the presence of small temperature gradients, in the sense that the length scale l over which the temperature changes is much larger than the wavelength λ, we may model the waves as "rays" and expect the properties (i) and (ii) to hold "locally." This corresponds to the WKBJ approximation, named after those who used it first in formal analysis (Jeffreys, 1924b) and quantum mechanics (Wentzel, 1926; Kramers, 1926; Brillouin, 1926), although it had been used before for water waves (Green, 1837) and in the acoustics of horns (Rayleigh, 1916). This approximation, when applied to the gravity and acoustic cutoff frequencies, would specify the latter as slowly-varying functions of altitude, determining the level at which a wave of a given frequency is reflected. An exact theory, allowing for significant temperature gradients on the scale of a wavelength, shows that the cutoff frequency, separating propagating waves from standing modes, is a fixed quantity, independent of altitude. Moreover, in the presence of strong temperature gradients, the properties (i) and (ii) do not hold, i.e., the waveforms are nonsinusoidal and amplitude growth is not exponential. When the temperature profile levels off to uniform, the sinusoidal waveform and exponential growth is regained, but there is a complex factor due to the history of the propagation of the wave through the temperature gradients. The modulus of this complex factor is the reflection coefficient, and the argument is the phase shift experienced by the wave when crossing the nonisothermal region. The WKBJ approximation neglects reflections (reflection coefficient equal to zero for propagating waves, and unity for a standing mode) and estimates phase shifts by integrating the local vertical wave number over altitude.

1. Convective stability and adiabatic temperature gradient

Before considering waves in a nonisothermal atmosphere, it is appropriate to discuss the conditions of equilibrium and stability of the mean state. In the absence of a magnetic field, the condition of hydrostatic equilibrium [Eq. (9b)] states that the pressure gradient balances the weight of fluid:

$$dp/dz = -\rho g = -p/L \ , \tag{72a}$$

where we have introduced the pressure scale height $L = -[d(\ln p)/dz]^{-1}$. When calculated for a perfect gas, L is

$$L \equiv p/\rho g = RT/g = c^2/\gamma g \ . \tag{72b}$$

A uniform magnetic field, $\mathbf{B} \sim$ const, exerts no force, i.e., the last term of Eq. (9b) vanishes, so that it does not affect the hydrostatic equilibrium [Eq. (72a)]. The latter is convectively stable if a fluid parcel, when disturbed from its equilibrium position, tends to return to it, and neither to remain in the new position (neutral equilibrium) nor to be displaced farther (instability). Thus a necessary and sufficient condition for stability is that the actual atmospheric density gradient, in modulus $|d\rho/dz|$, be larger than the adiabatic value $|(d\rho/dz)_{ad}|$, so that a fluid parcel displaced adiabatically upward will be denser than its surroundings and will tend to fall back to its original position, while a parcel displaced downward will be less dense than its surroundings and will tend to rise to its original position. Since the mass density gradient in an atmosphere is negative, we have

$$d\rho/dz < (d\rho/dz)_{ad} = c^{-2} dp/dz = -\rho g/c^2 = -\rho^2 g/\gamma p \ , \tag{73a}$$

where adiabatic pressure and density changes $dp = c^2 d\rho$ are related by the sound speed (9a) and we have used the condition (72a) of hydrostatic equilibrium. For a perfect gas, $p = \rho RT$, the stability condition (73a) reads

$$dT/dz = (\rho R)^{-1} dp/dz - (p/\rho^2 R) d\rho/dz > -g/R + g/\gamma R \ . \tag{73b}$$

Since the displaced fluid element is in pressure balance with its surroundings, condition (73a), in which the mass density gradient is smaller than the adiabatic value (Newcomb, 1961), implies that the temperature gradient (73b) is larger than the adiabatic value (Eddington, 1926; Landau and Lifshitz, 1953):

$$\begin{aligned} dT/dz > -g(1-1/\gamma)/R &= -g/C_p \\ &= -RT/C_p L = -c^2/\gamma C_p L \\ &\equiv (dT/dz)_{ad} \ , \end{aligned} \tag{74}$$

where we have introduced the specific heat at constant pressure C_p (Callen, 1970), the scale height L [Eq. (72b)], and the sound speed c [Eq. (9a)]. We conclude that a perfect gas is stable in an isothermal atmosphere or any atmosphere with temperature increasing with altitude, as well as for moderate negative temperature gradients, not exceeding in modulus the value g/C_p.

2. Buoyancy force and internal frequency

The displacement ζ of a fluid parcel in an atmosphere is determined by the balance of inertia and buoyancy forces,

$$d^2\zeta/dt^2 = F_b \zeta \ , \tag{75}$$

$$F_b = (g/\rho)[(d\rho/dz) - (d\rho/dz)_{ad}] \ , \tag{76}$$

where the buoyancy force per unit mass and height (76) is proportional to the acceleration of gravity and to the difference in mass density gradients between the actual atmosphere and a "virtual" adiabatic displacement. The equation of motion (75a) can be written in the form of a harmonic oscillator,

$$d^2\zeta/dt^2 + \omega_1^2 \zeta = 0 \ , \tag{77a}$$

$$\omega_1^2 \equiv -F_b = -g d(\ln\rho)/dz - g^2/c^2 \ , \tag{77b}$$

where ω_1 is the buoyancy (or Brünt-Vaisala) frequency and we have used Eq. (73a) in (77b). For a perfect gas [Eq. (9a)] in hydrostatic equilibrium [Eq. (72a)], we have

$$\begin{aligned} d(\ln\rho)/dz + d(\ln T)/dz &= d(\ln p)/dz \\ &= p^{-1}dp/dz \\ &= -\rho g/p = -\gamma g/c^2 , \end{aligned} \quad (78a)$$

so that the buoyancy frequency (77b) can be written

$$\begin{aligned} \omega_1^2 &= (g/T)dT/dz + (\gamma-1)g^2/c^2 \\ &= (\gamma-1)g^2/c^2[1-(dT/dz)/(dT/dz)_{ad}] . \end{aligned} \quad (78b)$$

Here we have introduced the adiabatic temperature gradient (74). From Eqs. (77a) and (78b) we conclude that (i) if the temperature gradient is subadiabatic, $dT/dz < (dT/dz)_{ad} < 0$, which implies $(dT/dz)/(dT/dz)_{ad} > 1$, we have an imaginary frequency $\omega_1^2 < 0$ in Eq. (78b), the displacement ζ is monotonic in Eq. (77a), and the atmospheric equilibrium is unstable; (ii) the condition of marginal stability, that the displaced fluid parcel stay in equilibrium with its surroundings, $d^2\zeta/dt^2 = 0$, is that $\omega_1 = 0$ in Eq. (78b) and that the temperature gradient $dT/dz = (dT/dz)_{ad}$ equal the adiabatic value [Eq. (74)]; (iii) in a stable stratification [Eq. (74)] with superadiabatic temperature gradient $dT/dz > (dT/dz)_{ad} < 0$, so that $(dT/dz)/(dT/dz)_{ad} < 1$, the buoyancy frequency ω_1 is real [Eq. (78b)], and a displaced fluid parcel oscillates [Eq. (77a)] at this frequency around its original equilibrium position.

3. Vertical velocity for oblique waves

The velocity perturbation of a three-dimensional acoustic-gravity wave satisfies $\mathbf{v} = \partial\boldsymbol{\xi}/\partial t$, the same equation (16) as the displacement $\boldsymbol{\xi}$, where, in a nonisothermal atmosphere of temperature profile $T(z)$ as a function of altitude z, the sound speed $C(z)$ is nonuniform [Eq. (9a)]. Thus the waveform will be sinusoidal in time t and on the horizontal coordinate x, but not in altitude z, so that we may introduce the Fourier representation

$$\mathbf{v}(z,x,t) = \int_{-\infty}^{+\infty}\int \mathbf{W}(z,k_{\parallel},\omega)\exp[i(k_{\parallel}x-\omega t)]dk_{\parallel}d\omega , \quad (79a)$$

where $\mathbf{W}$ is the velocity perturbation spectrum, at altitude z, for a two-dimensional wave of frequency ω and horizontal wave number $k_{\parallel}$ in the x direction (we neglect propagation in the y direction $k_y = 0$). The vertical and horizontal components of the acoustic-gravity wave equation (16) read

$$\omega^2 W_z + c^2(W_z'' + ik_{\parallel}W_x') - (\gamma-1)ik_{\parallel}gW_x - \gamma g W_z' = 0 , \quad (79b)$$

$$(\omega^2 - k_{\parallel}^2c^2)W_x + ik_{\parallel}(c^2W_z' - gW_z) = 0 , \quad (79c)$$

where primes denote derivative with regard to altitude, that is, $W_z' \equiv dW_z/dz$. We can eliminate the horizontal velocity spectrum W_x from the system [Eqs. (79b) and (79c)], thus obtaining a single scalar wave equation of second order for the vertical velocity perturbation spectrum W_z:

$$c^2W_z'' - \gamma g W_z' + \Lambda^2 W_z = 0 , \quad (80a)$$

$$\Lambda^2 \equiv \omega^2 - c^2k_{\parallel}^2 + (\gamma-1)(k_{\parallel}g/\omega)^2 . \quad (80b)$$

In Eq. (80a) we have omitted the term

$$(c^2)'[W_z' - g(k_{\parallel}/c)^2W_z]/[(\omega/k_{\parallel}c)^2 - 1] , \quad (80c)$$

which vanishes (i) for vertical waves, $k_{\parallel} = 0$, in nonisothermal atmospheres, $c' \neq 0$ and (ii) for oblique waves, $k_{\parallel} \neq 0$, in isothermal conditions, $c' = 0$. For (iii) oblique waves in nonisothermal atmospheres, $k_{\parallel} \neq 0 \neq c'$, the term (80c) is negligible provided that the scale height (72a) varies slowly $|dL/dz| \ll 1$, i.e., for moderate temperature gradients $|dT/dz| \ll g/R$, which are consistent with atmospheric stability [Eq. (74)].

4. Conditions for validity of the ray approximation

In an atmospheric layer of thickness small compared with the radius of a planet (e.g., the Earth) or a star (e.g., the sun), we may use the plane-parallel approximation [Eqs. (79a)–(79c)] with uniform acceleration of gravity; if the layer is isothermal, the sound speed c is also uniform for a perfect gas [Eq. (9a)], and the wave equation (80a) has constant coefficients, i.e., a complex exponential solution exists:

$$W_z(z) = W_0\exp(ik_{\perp}z) , \quad (81a)$$

$$c^2k_{\perp}^2 + ik_{\perp}\gamma g - \Lambda^2 = 0 , \quad (81b)$$

with constant amplitude W_0 and vertical wave number $k_{\perp}$, the latter satisfying the same polynomial equation (81b) of the second degree as before [Eq. (20)]. Its roots show that the vertical wave number is generally complex, $k_{\perp} = -i/2L \pm K$, implying that

$$W_z(z) = W_0 e^{z/2L}e^{\pm iKz} , \quad (81c)$$

$$K^2 \equiv \Lambda^2/c^2 - 1/4L^2 , \quad (81d)$$

i.e., the wave amplitude grows exponentially on twice the scale height [Eq. (81c)], and the waveform is sinusoidal with effective wave number given by Eq. (81d). In a nonisothermal atmosphere, $dT/dz \neq 0$, with length scale $l \equiv [d(\ln T)/dz]^{-1}$ for variation of temperature, the solution (81c) will hold "locally," i.e., with the scale height L (72a) and effective wave number K (81d) calculated as a function of altitude z, provided that the wavelength $\lambda \equiv 2\pi/K$ be short, in the sense $K^2l^2 \gg 1$ of the WKBJ or ray approximation, viz., $\lambda^2 \ll 4\pi^2l^2$ (also used in Sec. IV.C.2 of Part I). The effective wave number [Eqs. (81d) and (80b)] is real:

$$K^2 = [(\omega/\omega_2)^2 - 1]/4L^2 - k_{\parallel}^2[1-(\omega_1/\omega)^2] , \quad (82a)$$

for propagating gravity and acoustic modes, of frequencies, respectively, below ω_1 and above ω_2, where the cutoff frequencies are given by Eqs. (22a), (22b), and (78b) as a function of altitude:

$$\omega_{1,2}(z)=\{\sqrt{1-1/\gamma}g/\sqrt{R},(g/2)\sqrt{\gamma/R}\}[T(z)]^{-1/2}. \tag{82b}$$

Note that the gravity cutoff (82b) coincides with the "locally isothermal" form of the internal frequency $\omega_1^2=(\gamma-1)g^2/c^2=(1-1/\gamma)g^2/RT$ [Eq. (78b)], and the acoustic cutoff (82a) in a "locally isothermal" atmosphere coincides with $\omega_2^2=\gamma g^2/4RT=\gamma^2g^2/4c^2=c^2/4L^2$, the cutoff for sound in an exponential horn (Sec. IV.B.1, in Part I), for which the mass per unit length varies like the mass per unit altitude in the isothermal atmosphere. Since the cutoff frequencies [Eq. (82b)] increase with altitude in an atmosphere cooling with altitude, $dT/dz<0$, a gravity wave of frequency ω such that it propagates at altitude z_0, i.e., $\omega<\omega_1(z_0)$, will propagate at all higher altitudes $z>z_0$, since $\omega<\omega_1(z_0)<\omega_1(z)$; an acoustic mode propagating at an altitude z_0, for which $\omega>\omega_2(z_0)$, may become evanescent at a higher altitude z_r such that $\omega=\omega_2(z_r)$, implying that this is the level at which it is reflected. Corresponding results can be obtained for downward propagation or atmospheres heating with altitude.

5. Three-parameter family of atmospheric models

In order to study the propagation of linear acoustic-gravity waves in a nonisothermal atmosphere, without relying on the ray approximation, we must specifiy a temperature profile $T(z)$, allowing for non-negligible gradients. We choose the three-parameter family of atmospheric models, with temperature profiles

$$T(z)=T_\infty+(T_0-T_\infty)e^{-z/l}=T_\infty(1-\alpha e^{-z/l}), \tag{83a}$$

$$\alpha\equiv 1-T_0/T_\infty=1-(c_0/c_\infty)^2, \tag{83b}$$

which allows independent choice of (i) the initial $T_0=T(0)$ and asymptotic $T_\infty=T(\infty)$ temperatures, e.g., atmospheres heating ($T_\infty>T_0$ or $0<\alpha<1$) or cooling ($T_\infty<T_0$ or $\alpha<0$) with altitude, including the isothermal case ($T_\infty=T_0$ or $\alpha=0$) as separation; (ii) the temperature gradient

$$dT/dz=(dT/dz)_0e^{-z/l}, \tag{84a}$$

$$(dT/dz)_0=(T_\infty-T_0)/l, \tag{84b}$$

which is everywhere of the same sign (the temperature profile is monotonic), attains its maximum value (in modulus) at the base (level $z=0$) of the atmosphere, where the temperature gradient can be chosen by specifying the length scale l for temperature change. The condition of convective stability [Eq. (74)] is satisfied by Eq. (84a) at all altitudes, for any atmosphere heating with altitude $T_\infty>T_0$, in the isothermal case $T_\infty=T_0$, and also for atmospheres cooling with altitude $T_\infty<T_0$ to an asymptotic temperature $T_\infty>T_0-gl/C_p$, only in the latter case we must restrict the scale of temperature change to $l<(T_0-T_\infty)C_p/g$. For vertical waves $k_{\|}=0$, Eq. (80a) with $\Lambda=\omega$ in (80b) is valid in nonisothermal atmospheres, without further restriction. For oblique waves in nonisothermal atmospheres, $k_{\|}\neq 0$, the neglect of the term (80c) in (80a) requires that $|dT/dz|\ll g/R$, and thus imposes a more severe restriction [Eqs. (84a) and (84b)] $l\gg R|T_\infty-T_0|/g=|L_\infty-L_0|$ on the length scale l of temperature change, for both atmospheres that are heating and those that are cooling with altitude. The temperature profile [Eqs. (83a) and (83b)] coincides with that of the sound speed squared for a perfect gas [Eq. (9a)], and the wave equation [(80a) and (80b)] for the vertical velocity perturbation spectrum reads

$$(1-\alpha e^{-z/l})W_z''-L^{-1}W_z'+[\omega^2/c_\infty^2+(\gamma-1)(k_{\|}g/c_\infty\omega)^2-k_{\|}^2(1-\alpha e^{-z/l})]W_z=0, \tag{85}$$

where c_∞ denotes the asymptotic sound speed $c_\infty^2=\gamma RT_\infty=\gamma L_\infty g$, and c_0 [appearing in Eq. (83b)] the initial value. We perform in Eq. (85) the obvious change of independent variable

$$\zeta=\alpha e^{-z/l}, \tag{86a}$$

$$W_z(z;k_{\|},\omega)=\psi(\zeta), \tag{86b}$$

as well as the related change of dependent variable (86b), so that the wave equation (85) with exponential coefficients in altitude transforms into one with polynomial coefficients in the new variable ζ,

$$(1-\zeta)\zeta^2\psi''+(1+l/L-\zeta)\zeta\psi'+\{(\omega l/2\omega_2L)^2+k_{\|}^2l^2[(\omega_1/\omega)^2-1]+k_{\|}^2l^2\zeta\}\psi=0, \tag{87}$$

where the prime now denotes derivative with regard to the variable ζ, viz., $\psi'\equiv d\psi/d\zeta$. In the wave equation (87), the cutoff frequencies [Eq. (22b)] and scale height [Eq. (15b)] are calculated for the asymptotic temperature T_∞, suggesting that the latter alone determines the filtering conditions for acoustic-gravity waves in the family of nonisothermal atmospheric models [Eq. (83a)].

6. Cutoff frequency as a global property

We define generally a cutoff frequency as the frequency separating the wave spectrum into two ranges: one range with phases, corresponding to propagating waves, and the other range without phases, corresponding to standing modes. Thus the cutoff frequencies separate real from complex solutions of the wave equation (87). The latter has polynomial coefficients of degree three, suggesting that we perform the change of dependent variable

$$W_z(z;k_{\|},\omega)=\psi(\zeta)=\zeta^{vl/L}\Phi(\zeta), \tag{88a}$$

where the constant v may be chosen at will. After substitution of Eq. (88a) into (87), we choose v so as to cancel the coefficient of Φ, which is constant, i.e., not multiplied by ζ:

$$v^2-v+(\omega/2\omega_2)^2-k_{\parallel}^2L^2[1-(\omega_1/\omega)^2]=0\ . \qquad (88b)$$

This choice of v implies that all terms of the wave equation have a common factor ζ, which can be dropped out, lowering the degree of the polynomial coefficients from cubic in Eq. (87), to quadratic:

$$(1-\zeta)\zeta\Phi''+[(1+l/L+2vl/L)-(1+2vl/L)\zeta]\Phi' +[k_{\parallel}^2l^2-(vl/L)^2]\Phi=0\ . \qquad (89)$$

Although the two equations (88b) and (89) are the consequence of the same mathematical transformation (88a), we consider here the physical interpretation of the former (88b), and defer to the next section the consideration of Eq. (89). Equation (88b) is readily identified as the dispersion relation [Eq. (20) with (22a) and (22b), or equivalently (82a) with (81b)] for acoustic-gravity waves in an isothermal atmosphere at asymptotic temperature T_∞, where

$$v=-ik_{\perp}L=-\tfrac{1}{2}+iKL\ , \qquad (90a)$$

$$k_{\perp}=-i/2L-K\ , \qquad (90b)$$

and where K is the effective vertical wave number [Eq. (82a)]. It is clear that for frequencies between the cutoffs, $\omega_2<\omega<\omega_1$, the effective wave number K is pure imaginary [Eq. (82a)], hence v is real [Eq. (90a)], and so the solutions of Eq. (89) are real, i.e., have no phases and represent standing modes. For frequencies below the lower cutoff $\omega<\omega_1$ or above the upper cutoff $\omega>\omega_2$, the effective wave number K is real [Eq. (82a)], and hence v is complex [Eq. (90a)], and phase terms appear in the solutions of Eq. (89), which represent propagating waves, viz., gravity and acoustic modes, respectively. Note that the ray approximation would predict, for the model (83a), that the "local" cutoff frequencies would vary with altitude [Eq. (82b)] as $\omega_{1,2}(z)\sim(1-\alpha e^{-z/l})^{-1/2}\omega_{1,2}(\infty)$, where $\omega_{1,2}(\infty)$ are the cutoffs for the asymptotic temperature T_∞; the exact theory shows that the global cutoff frequencies are constants, $\omega_{1,2}(\infty)$, and in this case coincide with the cutoffs for an isothermal atmosphere at temperature T_∞. The result that the constant cutoffs are determined from the asymptotic temperature is model dependent, i.e., it holds for the family of atmospheric models (83a) and might not hold for other temperature profiles. The conclusion that the global cutoff frequencies are constants applies to acoustic-gravity waves in any nonisothermal atmosphere, since the separation between standing modes and propagating waves, without and with phases, respectively, always divides the spectrum into ranges, independent of altitude.

7. Reflection factor due to temperature gradients

The result in ray theory that the cutoff frequencies $\omega_{1,2}(z)$ vary with altitude serves mainly to determine the altitude z_r at which a short wave of high-frequency ω becomes evanescent, $\omega=\omega(z_r)$, and is "totally" reflected. Thus the ray approximation allows only two conditions: first, the wave is not reflected below z_r, i.e., the reflection coefficient is zero [$R(z)=0$ for $z<z_r$], and second, on reaching z_r, the wave is totally reflected, i.e., the reflection coefficient becomes unity [$R(z)=1$ for $z>z_r$]. The exact theory shows that long waves, of low frequency, do not suffer such abrupt "total" reflection, but are gradually reflected by temperature gradients, leading to a reflection coefficient $R(z)$ varying continuously with altitude. This reflection coefficient is generally complex for propagating waves, including a phase shift due to the variation of wave speed with altitude, which reduces to an integration of the "local" phase only in the case of high-frequency waves satisfying the ray approximation. We illustrate the calculation of the exact reflection coefficient in the case of acoustic-gravity waves in an atmosphere with temperature profile (83a), for which the second-order wave equation has been transformed to the type (89), with quadratic coefficients. A linear, second-order differential equation with quadratic coefficients is always reducible to the hypergeometric type (Kamke, 1944); in fact, Eq. (89) is already in the hypergeometric form, with parameters α,β,γ (Caratheodory, 1953) satisfying

$$\gamma_0=1+(l/L)(1+2v)\ , \qquad (91a)$$

$$\alpha_0+\beta_0=2vl/L\ , \qquad (91b)$$

$$\alpha_0\beta_0=(vl/L)^2-k_{\parallel}^2l^2\ , \qquad (91c)$$

i.e., using Eq. (90a), we obtain

$$\alpha_0,\beta_0=vl/L\pm k_{\parallel}l=-l/2L\pm k_{\parallel}l+iKl\ , \qquad (92a)$$

$$\gamma_0=1+2iKl\ , \qquad (92b)$$

where $k_{\parallel}$ is the horizontal wave number (79a) and K the effective vertical wave number (82a). The solution of the hypergeometric equation (89) is

$$\Phi(\zeta)=A_+F(\alpha,\beta;\gamma;\zeta) +A_-\zeta^{1-\gamma}F(1+\alpha-\gamma,1+\beta-\gamma;2-\gamma;\zeta)\ , \qquad (93)$$

where $A_\pm$ are arbitrary constants of integration. They are interpreted as the amplitudes of upward-propagating A_+ and downward-propagating A_- waves, since the two particular integrals in Eq. (93) scale, according to Eq. (88a), as

$$W_z^{\pm}(z;k_{\parallel},\omega)=\zeta^{vl/L}F(\alpha_0,\beta_0;\gamma_0;\zeta),\zeta^{1+vl/L-\gamma_0}F(1+\alpha_0-\gamma_0,1+\beta_0-\gamma_0;2-\gamma_0;\zeta)$$
$$\sim e^{z/2L}e^{\pm iKz}F(-l/2L+k_{\parallel}l\pm iKl,-l/L-k_{\parallel}l\pm iKl;1\pm2iKl;\alpha e^{-z/l})\ , \qquad (94)$$

where we have used Eqs. (88a) and (86a).

8. Evolution of amplitude and phase with altitude

Choosing an upward- or downward-propagating wave, i.e., respectively, the plus or minus sign in Eq. (94) and first or second term in Eq. (93), and determining the amplitude $A_\pm$ from the initial perturbation, we obtain

$$W_z^\pm(z;k_{||},\omega)=W_z^\pm(0;k_{||},\omega)\times e^{z/2L}e^{\pm iKz}\{1-R^\pm(z;\alpha,k_{||}l,Kl)\}\ , \qquad (95)$$

which corresponds to an acoustic-gravity wave in an isothermal atmosphere at asymptotic temperature T_∞ (first three factors), modified by the reflection factor

$$R^\pm(z;\alpha,k_{||}l,Kl)\equiv 1-F_0^\pm(\alpha e^{-z/l})/F_0^\pm(\alpha)\ , \qquad (96a)$$

$$F_0^\pm(\xi)\equiv F(-l/2+k_{||}l\pm iKl,-l/2-k_{||}l\pm iKl;1\pm 2iKl;\xi)\ , \qquad (96b)$$

which is specified by the hypergeometric functions (96b) for the temperature profile (83a). In the isothermal case ($T_\infty=T_0$ or $\alpha=0$) there is no reflection of acoustic-gravity waves at any altitude $R(z)=0$, and the last factor in (95) can be omitted. The latter factor is important in the presence of temperature gradients ($T_\infty\neq T_0$ or $\alpha\neq 0$), since then the reflection factor varies from zero initially, $R(0)=0$, to become a complex function of altitude, $R(z)$, consisting of an amplitude change $|1-R(z)|$ and a phase shift $\arg[1-R(z)]$ in Eq. (95). Stronger reflections and larger phase shifts occur in the region $\alpha e^{-z/l}\lesssim 1$ or larger temperature gradients. Asymptotically, at high altitude $\alpha e^{-z/l}\ll 1$ or $z\gg l\ln\alpha$, as the temperature stabilizes, $T(z)\rightarrow T_\infty$, there is no further reflection or phase shifting, and the reflection factor tends to the constant value $R_\infty^\pm=1-1/F_0^\pm(\alpha)$, where $F_0^\pm(\alpha)$ is [Eq. (96b)] the inverse of the transmission factor. Acoustic-gravity waves in a nonisothermal atmosphere [Eq. (83a)] are illustrated in the case of vertical propagation $k_{||}=0$, for which the present solution involves no approximations; vertical propagation is possible only for the acoustic mode, at frequencies ω above the acoustic cutoff $\omega>\omega_2\equiv c_\infty/2L$. The logarithms of wave amplitude and phase difference, normalized to the initial value, are plotted, respectively, on the lhs and rhs of Fig. 4 as a function of altitude z, made dimensionless by dividing by the scale height. We choose as a reference case A, a frequency twice the cutoff $\omega=2\omega_2$, an asymptotic temperature 1 order of magnitude larger than the initial value $T_\infty/T_0=10$, and an initial temperature gradient (84b) specified by $l=L$ as $T'(0)=(T_\infty-T_0)/L$. We change in turn B the asymptotic temperature to a case of cooling with altitude $T_\infty=T_0/2$; C, the frequency to a high, almost "ray" value $\omega=10\omega_2$; D, the temperature gradient to a steep rise $l=L/4$. It is clear that faster (slower) amplitude growth and larger (smaller) phase shifts occur for (i) larger (smaller) wave frequency ω relative to the cutoff ω_2; (ii) steeper (shallower) temperature rise, or larger (smaller) temperature gradients, between the same initial and asymptotic temperatures; (iii) atmospheres heating (cooling) with altitude.

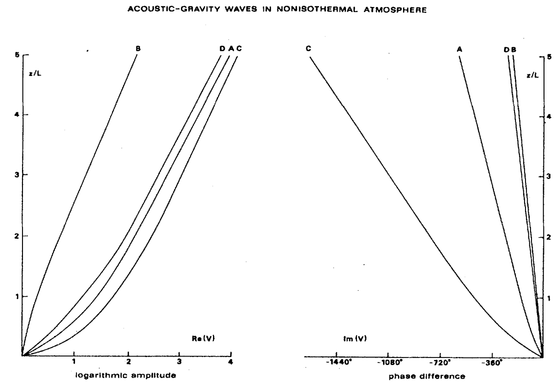

FIG. 4. Logarithm of amplitude (left) and phase (right) vs altitude z divided by scale height L, for acoustic-gravity waves propagating vertically in a nonisothermal atmosphere with temperature profile (83a), in the following cases: A, wave frequency ω twice the cutoff value ω_2 i.e., $\omega=2\omega_2$, asymptotic temperature T_∞ an order of magnitude larger than initial temperature T_0, i.e., $T_\infty=10T_0$, and temperature gradient parameter l equal to asymptotic density scale height $L=l$ (reference case); B, like the reference case, but with asymptotic temperature half the initial value, $T_\infty=T_0/2$; C, like the reference case, with wave frequency 10 times the cutoff, $\omega=10\omega_2$; D, like the reference case with temperature gradient parameter one-quarter the asymptotic density scale height, $l=L/4$.

B. Alfvén modes and waveform shearing

Acoustic-gravity waves are observed in the sun, not only as motions in the outer, atmospheric layers, but also as global oscillations or p modes (Brookes, Isaak, and Van Der Raay, 1976; Deubner, 1981; Fossat, Grec, and Pomerantz, 1981; Scherrer, Wilcox, Christensen-Daalsgard, and Gough, 1982). The inversion of solar oscillation frequencies has been used to construct models of the solar interior (Christensen-Daalsgaard and Gough, 1981; Scuflaire, Gabriel, and Noels, 1981; Shibaiashi and Osaki, 1981), and the subject has developed into the rapidly expanding field of helioseismology (Deubner and Gough, 1984; Leibacher, 1985; Deming *et al.*, 1986). The observation of rotational splitting of these global oscillations (Duvall and Harvey, 1984) has been used to deduce the rotation curve as a function of depth (Duvall *et al.*, 1984), demonstrating the existence of a core (Claverie, Isaak, McLeod, and Van Der Raay, 1981) rotating much faster than the surface (Howard, 1984). Asymptotic approximations have been used for the identification of the modes for higher orders or degrees (Tassoul, 1980), which reduce essentially to the WKBJ approximation (Gough, 1984). The high-quality data thus obtained on the solar interior has raised fresh issues, such as an apparent discrepancy between the solar neutrino flux (Bahcall *et al.*, 1982; Joseph, 1984; Bahcall, 1986) and the nuclear reaction rates in the solar core implied by the temperature profile, derived by inversion of p-mode frequencies. The rapid rotation of the solar core testified to by the rotational splitting raises the question of its magnetic decoupling from the outer layers of the solar interior, which rotate much more slowly. Other issues include the calculation of the quadrupole moment of the sun (Campbell and Moffat, 1983), the existence of torsional oscillations (Labonte and Howard, 1982; Labonte, 1984), the observation of g modes (Kuhn, Libbrecht, and Dicke, 1986; Wentzel, 1986), the properties of the core (Faulkner, Gough, and Vahia, 1986; Libbrecht, 1986) and convection zone (Schmitt, Rosner, and Bohn, 1982; Guenther and Gilman, 1985), and oscillations perpendicular to the galactic disk (Shuter and Klatt, 1986). The study of solar oscillations is relevant not only as an example of stellar pulsation theory (Cowling, 1941; Ledoux and Walraven, 1957; Tassoul, 1978; Cox, 1980; Gabriel, Noels, Scuflaire, and Mathys, 1985), but also as an important test case of many aspects of the theory of formation and evolution of stars (Acheson, 1978; Woolfson, 1979; Lebovitz, 1981; Hill and Logan, 1984). In the outer, atmospheric layers, the global oscillations correspond (Foing, Bonnet, and Bruner, 1986; Schmieder and Mein, 1986) to acoustic modes (Leibacher and Stein, 1971; Stein, 1982), and it has been reported, without subsequent confirmation from other sources, that gravity modes could also exist (Hill, Goode, and Stebbins, 1982). The acoustic modes are mostly trapped, standing waves (Lites, Chipman, and White, 1982), which are attenuated in sunspots (Thomas, Cram, and Nye, 1982), where the magnetic field strength (Deinzer, 1965) is of the order of 2 kG. The sunspots were first observed by Galileo in the XVIth century, as dark patches on the solar surface, and their number and evolution during the solar cycle (Moore and Rabin, 1985; Schatten and Mayr, 1985; Gilman and Howard, 1985, 1986) affects solar irradiance (Willson and Hudson, 1981; Willson, Gulkis, Janssen, Hudson, and Chapman, 1981) and is an indicator of solar rotation (Schröter, 1985) at the surface (Balthasar, Lustig, Stark, and Wohl, 1986). The sunspots in the photosphere, together with coronal arches and loops, are examples of regions of strong magnetic field, i.e., magnetic pressure comparable to or larger than gas pressure, in the solar atmosphere; the physical conditions in these solar magnetic regions support Alfvén waves, which have been the subject of an extensive literature (Ferraro, 1954; Hide, 1955; Ferraro and Plumpton, 1958; Zhugzhda, 1971; Hollweg, 1972, 1978, 1981a, 1981b, 1984a, 1984b, 1984c; Thomas, 1978; Leroy, 1980, 1981, 1983; Nye and Hollweg, 1980; Bel and Leroy, 1981; Zhugzhda and Locans, 1982; Campos, 1983a, 1983d; Parker, 1984a; Schwartz, Cally, and Bel, 1984; Mariska and Hollweg, 1985).

1. Alfvén speed and the magnetic "gas"

For acoustic-gravity waves in an atmosphere, the variation of sound speed with altitude is a crucial factor. For Alfvén modes we should start by considering the Alfvén speed (38b), which is given by

$$a^2=2P/\rho \; , \tag{97a}$$

$$P\equiv\mu B^2/8\pi \; , \tag{97b}$$

where P is the magnetic pressure and ρ the mass density. Thus both the sound [Eq. (9a)] and Alfvén [Eq. (97c)] speeds scale as the square root of pressure divided by density, with the gas p and magnetic P pressures appearing in the expressions for, the sound c and Alfvén a speeds, respectively. The coefficient in the adiabatic sound speed [Eq. (9a)] is the ratio $\gamma=C_p/C_v$ of specific heats at constant pressure C_p and volume C_v, which is given, for a perfect gas whose molecules have only translational and rotational degrees of freedom (no vibration or quantum effects), by (Landau and Lifshitz, 1967)

$$\gamma=1+2/N=2,\tfrac{5}{3},\tfrac{7}{5},\tfrac{4}{3} \; , \tag{98a}$$

$$N=2,3,5,6 \; , \tag{98b}$$

where N is the number of degrees of freedom of a molecule: (i) for a monatomic gas, molecules have only $N=3$ degrees of freedom (all translational), and the adiabatic exponent is $\gamma=\frac{5}{3}=1.67$, e.g., for fully ionized hydrogen; (ii) for a diatomic gas or polyatomic molecules whose atoms are aligned, there are two rotational degrees of freedom, for a total of $N=5$, corresponding to an adiabatic exponent $\gamma=\frac{7}{5}=1.40$, e.g., for oxygen at room temperature; (iii) for a polyatomic gas, with three-dimensional molecules, the number of degrees of freedom is that of a rigid body, $N=6$ (three each for translation and rotation), and the adiabatic exponent is $\gamma=\frac{4}{3}=1.33$; (iv) for the Alfvén speed, the adiabatic exponent is replaced by the value $\gamma=2$, corresponding to two degrees of freedom, $N=2$, i.e., the "magnetic gas" consists of molecules whose only permissible motion is two-dimensional translation, transverse to the magnetic field lines **B**. Thus acoustic waves are isotropic and longitudinal because the gas pressure is independent of direction and is a normal stress. Alfvén waves are anisotropic and transverse because the waves propagate along the magnetic field lines, which support tension through transverse displacements.

2. Equations for velocity and magnetic perturbations

The Alfvén speed [Eq. (97a)] is constant in a homogeneous medium under a uniform magnetic field, in which case the waves are sinusoidal. The waves cease to be sinusoidal, in a homogeneous medium, in the presence of a nonuniform magnetic field, e.g., one that is increasing for magnetic "focusing." In an atmosphere the mass density $\rho(z)$ decays with altitude, and preservation of a sinusoidal waveform requires a constant Alfvén speed [Eq. (38b)] and hence a magnetic field decaying in strength as $B(z)\sim\sqrt{\rho(z)}$. If we require a one-dimensional magneto-hydrostatic equilibrium, depending only on altitude z, the Maxwell equation stating the nonexistence of magnetic charges, $0=\nabla\cdot\mathbf{B}=dB_z/dz$, implies that the magnetic field can have no vertical component $B_z=0$, i.e., it must be purely horizontal. Thus the Alfvén speed will not be uniform and the Alfvén waves will be nonsinusoidal if the magnetic field has a vertical component. For example, in an isothermal atmosphere (see Sec. V.A.1 in Part I), the mass density decays exponentially with height $\rho(z)=\rho_0 e^{-z/L}$ on the scale height L [Eq. (15b)], and thus, in the presence of a uniform external magnetic field $\mathbf{B}$, the Alfvén speed increases exponentially on twice the scale height:

$$a(z)=a_0 e^{z/2L}, \tag{99a}$$

$$a_0^2 \equiv \mu B^2/4\pi\rho_0, \tag{99b}$$

from an initial value a_0 at the base $(z=0)$ of the atmosphere. For a pure Alfvén wave in an atmosphere, under a uniform magnetic field $\mathbf{B}$, in the absence of rotation, bearing in mind that the motion is incompressible $(\nabla\cdot\mathbf{v}=0)$ and transverse $(\mathbf{h}\cdot\mathbf{B}=0)$ and causes no density or pressure perturbation $(\bar{\rho}=0=\bar{p})$, the linearized induction and momentum equations [(8a) and (7b)] read

$$\dot{\mathbf{h}}=(\mathbf{B}\cdot\nabla)\mathbf{v}, \tag{100a}$$

$$\dot{\mathbf{v}}=(a^2/B)(\mathbf{B}\cdot\nabla)\mathbf{h}, \tag{100b}$$

in terms of the velocity $\mathbf{v}\equiv\partial\boldsymbol{\xi}/\partial t$ and magnetic field $\mathbf{h}$ perturbations. Eliminating between [(100a) and (100b)], and bearing in mind that the external magnetic field $\mathbf{B}$ is uniform but the Alfvén speed may not be, we obtain for the velocity [Eq. (75)] and magnetic field [Eq. (76)] perturbations, respectively, the following wave equations:

$$\ddot{\mathbf{v}}-a^2\partial^2\mathbf{v}/\partial b^2=0, \tag{101a}$$

$$\ddot{\mathbf{h}}-\partial(a^2\partial\mathbf{h}/\partial b)/\partial b=0, \tag{101b}$$

where $\partial/\partial b$ denotes the derivative along magnetic field lines [Eq. (39b)]. If the Alfvén speed is constant, e.g., if the medium is homogeneous under a uniform magnetic field, the velocity and magnetic field perturbations satisfy the same wave equation (101a), i.e., the classical wave equation in one dimension, as for the transverse oscillations on a string of uniform thickness. If the Alfvén speed is nonuniform, e.g., for an atmosphere under a magnetic field with a nonzero vertical component, the Alfvén wave equation for the velocity $\mathbf{v}$ is similar to that for a string (101a) of varying thickness, e.g., a "whip" of exponentially decreasing cross section, for isothermal conditions and uniform magnetic field [Eq. (99a)]. For the magnetic field perturbation $\mathbf{h}$, the wave equation (101b) has an extra term, $-2a(\partial a/\partial b)\partial\mathbf{h}/\partial b$, relative to that for the velocity $\mathbf{v}$ [Eq. (101a)], and this additional scattering effect implies that the two wave variables evolve differently.

3. Balance of energy density and flux

Associated with the velocity $\mathbf{v}$ and magnetic field $\mathbf{h}$ perturbations of an Alfvén wave are the kinetic and compression energy densities,

$$E_v=\tfrac{1}{2}\rho v^2, \tag{102a}$$

$$E_h=\mu h^2/8\pi. \tag{102b}$$

Their rate of change in time can be obtained by multiplying the momentum equation (100b) by $\rho\mathbf{v}$ and the induction equation (100a) by $\mu\mathbf{h}/4\pi$. Adding the resulting formulas, we obtain the equation of energy

$$\dot{E}+\nabla\cdot\mathbf{F}=0, \tag{103a}$$

$$E\equiv E_v+E_h, \tag{103b}$$

$$\mathbf{F}=(\mu/4\pi)\mathbf{B}(\mathbf{h}\cdot\mathbf{v}), \tag{103c}$$

stating the balance (103a) of the total (kinetic plus magnetic) energy density (103b) and the energy flux (103c). If the Alfvén speed is constant, sinusoidal waves exist:

$$\mathbf{v},\mathbf{h}(\mathbf{x},t)=(\mathbf{v}_0,\mathbf{h}_0)e^{i(\mathbf{k}.\mathbf{x}-\omega t)}, \tag{104a}$$

$$\mathbf{h}_0/B=\mathbf{v}_0/a, \tag{104b}$$

and the amplitudes of the velocity $\mathbf{v}_0$ and magnetic field $\mathbf{h}_0$ perturbations are related by Eq. (104b), which is obtained by substituting Eq. (104a) into (100a) or (100b) and using the dispersion relation $\omega=-a(\mathbf{k}\cdot\mathbf{b})$ for sinusoidal Alfvén waves [Eq. (40a)]. Equation (104b) implies the equipartition of kinetic and compression energies,

$$E_h=\mu h^2/8\pi=\mu h_0^2/8\pi=\mu B^2 v_0^2/8\pi a^2 = \tfrac{1}{2}\rho v_0^2=\tfrac{1}{2}\rho v^2=E_v,$$

so that the total energy density (103b) and flux (103c) can be given equivalently by either of two expressions:

$$E=2E_v=2E_h=\rho v_0^2=\mu h_0^2/4\pi, \tag{105a}$$

$$\mathbf{F}=\rho v_0^2 a\mathbf{b}=(\mu h_0^2/4\pi)a\mathbf{b}=Ea\mathbf{b}, \tag{105b}$$

i.e., the energy $\mathbf{F}$ is equal to the energy density E multiplied by the Alfvén speed a, in the direction $\mathbf{b}\equiv\mathbf{B}/B$ of magnetic field lines. If the Alfvén speed is nonuniform, the velocity perturbations $\mathbf{v}$ and magnetic field perturbations $\mathbf{h}$ evolve differently as functions of space $\mathbf{x}$, because they satisfy distinct wave equations. Thus Eq. (104b) does not hold for all $\mathbf{x}$, and the theorem of equipartition of energy (105a) and simple formula (105b) for the energy flux break down.

4. Properties in the WKBJ or ray approximation

Alfvén waves in an atmosphere differ from those in a homogeneous medium, for the same magnetic field; the reason is that, although gravity does not appear explicitly in the wave equations (101a) and (101b), it causes the mass density and hence the Alfvén speed to vary with altitude, which in turn changes significantly the properties of the waves: (i) the waveforms become nonsinusoidal and (ii) equipartition of kinetic and magnetic energies is violated, even for a uniform external magnetic field. We shall now discuss the substitute forms of properties (i) and

(ii) for Alfvén waves in general nonisothermal atmospheres under arbitrary magnetic fields consistent with one-dimensional magneto-hydrostatic equilibrium. Using the wave frequency ω, scale height L, and Alfvén speed a, it is possible to form only one dimensionless parameter, namely, the compactness:

$$\varepsilon(z)\equiv L(z)\omega/a(z)=\varepsilon_0 e^{-z/2L}\ , \tag{106a}$$

$$\varepsilon_0\equiv\omega L/a_0\ . \tag{106b}$$

The compactness decreases exponentially with altitude in the case of an isothermal atmosphere under a uniform magnetic field, for which the scale height L is a constant [Eq. (72b)] and the Alfvén speed is given by Eq. (99a). For a wave of high frequency, in the sense of compactness initially large $\varepsilon_0^2\gg 1$, the ray approximation applies, and the energy flux (105b) must be constant in the absence of reflections; thus we obtain the following scaling laws (Campos, 1983d) for the velocity v_0 and magnetic field h_0 perturbations:

$$v_0(z)\sim[\rho(z)a(z)]^{-1/2}\sim e^{z/4L}\ , \tag{107a}$$

$$h_0(z)\sim[a(z)]^{-1/2}\sim e^{-z/4L}\ . \tag{107b}$$

We find that the velocity increases while the magnetic field decays on four times the scale height, in the case (Wallen, 1944) of an isothermal atmosphere under a uniform magnetic field. In such a case both the kinetic [Eq. (102a)] and magnetic [Eq. (102b)] energies decay on twice the scale height $E_v\sim e^{-z/2L}\sim E_h$, so that (i) the initial equipartition of energies is preserved, as long as $[\varepsilon(z)]^2\gg 1$, i.e., for the altitude range $z\ll z_0\equiv 2L\ln(\varepsilon_0)$; (ii) the total energy density (77b) decays, $E\sim e^{-z/2L}$, in a manner that compensates for the increase [Eq. (99a)] in the Alfvén speed a, leading to a constant energy flux $Ea=F\sim$const. The preceding results will break down, even for a wave of high frequency, $\omega^2\gg(a_0/L)^2$, after the wave has propagated a sufficient distance, $z\gg z_0\equiv 2L\ln(\omega L/a_0)$; they will hold nowhere, not even initially, if the wave frequency is low, $\omega\lesssim a_0/L$.

5. Exclusion or "rigging" of the magnetic field

We shall now consider the high-altitude or asymptotic limit of small compactness, which holds for low-frequency waves everywhere and for high-frequency waves at a sufficient distance. For any nonisothermal atmosphere, the density $\rho(z)\to 0$ decays with altitude, and provided that the nonuniform magnetic field decays to a finite value, $B(z)>\delta>0$ as $z\to\infty$, or decays to zero, but more slowly than $\sqrt{\rho(z)}$, the Alfvén speed [Eq. (99a)] diverges $a(z)\to\infty$ as $z\to\infty$. It follows from the wave equations for the velocity (101a) and magnetic field (101b), that is, from the terms $a^2\partial^2 v/\partial z^2$ and $2a(da/dz)\partial h/\partial z$, respectively, that $\partial^2 v/\partial z^2$, $\partial h/\partial x\to 0$ as $z\to\infty$ for vertical propagation, implying that the velocity perturbation grows linearly:

$$v(z,t)\sim(d_1 z+d_2)e^{-i\omega t}\ , \tag{108a}$$

$$h(z,t)\sim i(B_z/\omega)d_1 e^{-i\omega t}\ , \tag{108b}$$

where d_1,d_2 are constants and where the induction equation (100a) with $(\mathbf{B}\cdot\nabla)=B_z\partial/\partial z$ was used to obtain the magnetic field perturbation [Eq. (108b)], which is bounded. In an isothermal atmosphere, the kinetic energy density [Eq. (102a)] decays as $E_v\sim z^2 e^{-z/L}$, and the magnetic energy density [Eq. (102b)] tends to an asymptotic value $E_h\sim(\mu/8\pi)(B_z d_1/\omega)^2$, which is a constant fraction $(d_1/\omega)^2$ of the background magnetic energy $\mu B_z^2/8\pi$. Thus, for a vertically propagating Alfvén-gravity wave, all energy is asymptotically magnetic, i.e., the opposite of equipartition. The interpretation is that the magnetic field perturbation of the wave is rigged into the background magnetic field, and the atmosphere oscillates up and down like a "lid," preserving the fraction of background magnetic energy corresponding to the wave. For a standing mode, perfectly reflected from infinity, the amplitude of the velocity perturbation is asymptotically finite [$d_1=0$ in (108a)], and the magnetic field perturbation decays to zero. In an isothermal atmosphere the decay proceeds in powers of $e^{-z/L}$, or submultiples of the scale height L, e.g., the kinetic energy density [Eq. (102a)] decays on the scale height $E_v\sim e^{-z/L}$, and since the magnetic field perturbation decays like the density on the same scale $h\sim\rho\sim e^{-z/L}$, the magnetic energy density (102b) decays twice as fast, $E_h\sim e^{-2z/L}$. Thus, for standing Alfvén waves, the total energy is biased in favor of the kinetic energy, and it decays $E\sim E_v\sim e^{-z/L}\sim\rho$ like the density. Thus reflections cause the energy and the field perturbation of the wave to be excluded from the upper layers of the atmosphere.

6. Propagating waves in an oblique field

We confirm Secs. II.B.4 and II.B.5 by calculating exactly the wave fields for an isothermal atmosphere under a uniform magnetic field; the low-altitude and high-altitude limits yield, respectively, Eqs. (107a) and (107b) and (108a) and (108b) and specify the constants appearing in these formulas. For an oblique magnetic field making an angle θ with the vertical, the wave equation (101a) for the velocity perturbation reads

$$[\partial^2/\partial t^2-a_0^2 e^{z/L}(\sin\theta\partial/\partial x+\cos\theta\partial/\partial z)^2]v_y(z,x,t)=0\ . \tag{109}$$

We seek a solution in the form of Eq. (79a), with

$$W_y(z;k_{\parallel},\omega)=\exp(-ik_{\parallel}z\tan\theta)\psi(z)\ , \tag{110a}$$

where $k_{\parallel}$ denotes the horizontal wave number and ω the frequency, and the altitude dependence is specified by the function $\psi(z)$, which satisfies

$$\psi''+(\omega/a_0)^2\sec^2\theta e^{-z/L}\psi=0\ . \tag{110b}$$

Using as an independent variable twice the compactness

[Eq. (106a)], with the Alfvén speed projected in the vertical direction,

$$\zeta \equiv (2\omega L/a_0\cos\theta)e^{-z/2L} , \tag{111a}$$

$$\Phi(\zeta)\equiv\psi(z) , \tag{111b}$$

transforms Eq. (101b) into a Bessel equation of order zero in the variable ζ, with Bessel J_0, Neumann Y_0, and Hankel $H_0^{(1,2)}$ functions of order zero (Watson, 1944),

$$\Phi(\zeta)=A_1J_0(\zeta)+A_2Y_0(\zeta)=A_-H_0^{(1)}(\zeta)+A_+H_0^{(2)}(\zeta) , \tag{112a}$$

where $A_{1,2}$ and $A_\pm$ are arbitrary constants of integration. For example, for an upward or downward propagating wave, traveling in the direction of increasing or decreasing z, or, by Eq. (111a), decreasing or increasing ζ, we choose the Hankel function $H_0^{(2)}/H_0^{(1)}$ and determine the amplitude A_+/A_- from the initial velocity perturbation:

$$W_y(z;k_{\|},\omega)=W_y(0;k_{\|},\omega)\exp(-ik_{\|}z\tan\theta)\{H_0^{(1,2)}[(2\omega L/a_0\cos\theta)e^{-z/2L}]/H_0^{(1,2)}(2\omega L/a_0\cos\theta)\} . \tag{112b}$$

In the case of vertical propagation $\theta=0$, we obtain for the velocity W_y and magnetic field H_y perturbation spectra, respectively,

$$W_y(z;0,\omega)=W_y(0;0,\omega)\{H_0^{(1,2)}[(2\omega L/a_0)e^{-z/2L}]/H_0(2\omega L/a_0)\} , \tag{113a}$$

$$H_y(z;0,\omega)=i(B/a_0)W_y(0;0,\omega)e^{-z/2L}\{H_1^{(1,2)}[(2\omega L/a_0)e^{-z/2L}]/H_0^{(1,2)}(2\omega L/a_0)\} . \tag{113b}$$

The Bessel function approximations (McLachlan, 1934) for large and small ζ [Eq. (111a)] yield, respectively, the low-altitude limit

$$W_y(z;0,\omega)=W_y(0;0,\omega)e^{z/4L} , \tag{114a}$$

$$H_y(z;0,\omega)=H_y(0;0,\omega)e^{-z/4L} , \tag{114b}$$

which confirms the results (107a) and (107b) of the ray approximation (Sec. II.B.4), and the high-altitude limit

$$W_y(z;0,\omega)\sim[W_y(0;0,\omega)/H_0^{(1,2)}(2\omega L/a_0)] \times[iz/\pi L+1-i2\gamma_*/\pi - i(2/\pi)\ln(2\omega L/a_0)] , \tag{115a}$$

$$H_y(z;0,\omega)\sim-(B/\pi\omega L)[W_y(0;0,\omega)/H_0^{(1,2)}(2\omega L/a_0)] , \tag{115b}$$

where γ_* is Euler's constant and specifies the constants d_1,d_2 appearing in the asymptotic laws [Eqs. (108a) and (108b)]. In the case of an oblique magnetic field $\theta\neq0$, the velocity perturbation (108a) leads to a magnetic field perturbation $H_y\sim d_1z+d_2$ which diverges with altitude and implies a diverging magnetic energy density, unless we set $A_2=0$ in Eq. (112a); in this case, the velocity and magnetic field perturbations, as well as the magnetic energy, are all bounded, and the wave field is specified by J_0 as a standing mode, i.e., Alfvén waves in an oblique magnetic field are reflected into standing modes, and cannot propagate through the atmosphere to infinity.

7. Eigenfrequencies and eigenfunctions of standing modes

When considering Alfvén waves in an atmosphere consisting of multiple isothermal layers, we can take the solution (112a), involving the superposition of upward- and downward-propagating waves. These waves have amplitudes of $A_\pm^{(n)}$ in the nth layer $n=1,\ldots,N-1$, together with upward propagation or reflection in $A_+^{(N)}$ in the uppermost, or Nth layer, depending on whether the wave is vertical or oblique. The $2N-1$ constants are determined by $2N-1$ boundary conditions, that is, the initial velocity at the base of the layer $n=0$, and the $2(N-1)$ conditions of continuity of velocity and total (gas plus magnetic pressure) at the $N-1$ interfaces between the N layers. The ratio $A_-^{(n)}/A_+^{(n)}$ indicates the strength of the downward-reflected wave in the nth layer when compared with the upward-incident wave. An extreme case is that of total reflection $A_+=A_-$, equivalent to $A_2=0$ in Eq. (112a), so that the velocity perturbation is specified by Eq. (112b) with Bessel functions J_0 replacing the Hankel functions $H_0^{(1,2)}$. In this case the velocity perturbation is bounded at high altitude, since $J_0(\zeta)\rightarrow1$ as $z\rightarrow\infty$ and $\zeta\rightarrow0$ in Eq. (111a). In the case of vertically standing modes, perfectly reflected from a layer several scale heights away, so that $\zeta\ll1$, Eqs. (113a) and (113b) with J_0 replacing $H_0^{(1,2)}$ show that resonances of the velocity perturbation occur when the denominator vanishes, i.e., for the eigenfrequencies

$$\omega_n=a_0j_n/2L , \tag{116a}$$

$$\omega_n/\omega_1=j_n/j_1 , \tag{116b}$$

where j_n denotes the roots of the Bessel function J_0. Note that while the absolute values of the eigenfrequencies depend on atmospheric properties [Eq. (116a)], their ratios do not, but are specified by fixed numbers [Eq. (116b)]. Substituting Eq. (113a) into the Fourier integral (79a), which for vertical propagation involves only an integration in frequency,

$$v_y(z,t)=\int_{-\infty}^{+\infty}W_y(0;\omega)\{J_0[(2\omega L/a_0)e^{-z/2L}]/J_0(2\omega L/a_0)\}e^{-i\omega t}d\omega , \tag{117}$$

we find that the eigenfrequencies correspond to roots of the denominator, i.e., simple poles. Thus the integral can be evaluated by the theorem of residues, as

$$v_y(z,t) = -\pi i \sum_n W_y(0;\omega_n)[J_1(j_n)]^{-1}\exp(-i\omega_n t)J_0(j_n e^{-z/2L}) , \tag{118a}$$

$$h_y(z,t) = -(\pi B/\omega L)e^{-z/2L} \sum_n W_y(0;\omega_n)[j_n/J_1(j_n)]\exp(-i\omega_n t)J_1(j_n e^{-z/2L}) . \tag{118b}$$

The velocity (118a) and magnetic field (118b) perturbations thus consist of a superposition of standing modes, of all eigenfrequencies ω_n present in the initial spectrum $W_y(0;\omega_n)\neq 0$. The low-altitude approximation to Eqs. (118a) and (118b) satisfies the ray law (Sec. II.B.4); the velocity perturbation (118a) is bounded asymptotically at high altitude, and the magnetic field perturbation (118b) decays like the density $h_y \sim e^{-z/L} \sim \rho$ (in agreement with the predictions of Sec. II.B.5).

8. Oscillations in sunspot umbras and prominences

The property of Alfvén waves, of being reflected in atmospheres for which the density decays and of rapidly increasing in speed with altitude, could lead to the appearance of standing modes. In fact, oscillations have been observed in magnetic regions of the sun, such as prominences (Landman, Edberg, and Laney, 1977; Bashkirtsev, Kobanov, and Mashnich, 1983; Jensen, 1983; Bashkirtsev and Mashnich, 1984; Wiehr, Stellmacher, and Balthasar, 1984; Landman, 1985; Balthasar, Knolker, Stellmacher, and Wiehr, 1986), filaments (Malherbe, Schmieder, and Mein, 1981), active region spicules (Kulidzanishvili and Zhugzhda, 1983; Berton and Rayrole, 1985), and most notably coronal loops and sunspot atmospheres. Several of these oscillations have been attributed to Alfvén waves (Jensen, 1983; Sterling and Hollweg, 1984), and we concentrate here on the sunspot modes. There are extensive observations of oscillations in the umbrae of sunspots (Staude, 1981; Nicolas, Kjeldseth-Moe, Bartoe, and Bruecker, 1982; Gurman and Athay, 1983; van Ballegooijen, 1984; Yun, Beebe, and Bagget, 1984; Lites, 1986a, 1986b), where the magnetic field is nearly vertical (Beckers and Schröter, 1968; Henze *et al.*, 1982; Adam, 1985; Berton, 1985; Ye and Jin, 1986), demonstrating the presence of five-minute and three-minute modes, as well as shorter periods (Rice and Gaizauskas, 1973; Schröter and Soltau, 1976; Soltau, Schröter, and Wöhl, 1976; Gurman *et al.*, 1982; Lites, White, and Packmann, 1982; Kneer and Uexküll, 1983; Uexküll, Kneer, and Mattig, 1983; Lites, 1984; Nye, Thomas, and Cram, 1984; Soltau and Wiehr, 1984; Thomas, Cram, and Nye, 1984; Kneer and Uexküll, 1985; Uexküll *et al.*, 1985). The umbral oscillations have been modeled in terms of Alfvén (Uchida and Sakurai, 1975; Campos, 1986b), acoustic (Leibacher, Gouttebroze, and Stein, 1982; Gurman and Leibacher, 1984), and flux tube modes (Hollweg and Roberts, 1981; Cally, 1983). Two competing theories based on fast (Scheuer and Thomas, 1981; Thomas and Scheuer, 1982; Thomas, 1984) and slow (Zhugzhda and Makarov, 1982; Zhugzhda, 1984; Zhugzhda, Staude, and Locans, 1984; Staude, Zhugzhda, and Locans, 1985; Zhugzhda, Locans, and Staude, 1985) magnetohydrodynamic modes have been pursued most persistently. We illustrate in Fig. 5 the waveforms for the magnetic field (b) and velocity (a) perturbations, for the first three modes of Alfvén waves, trapped between the anchoring level of magnetic field lines in the convection zone and the sunspot atmosphere, which acts as an upper reflector. The frequencies and amplitudes of the modes are consistent (Campos, 1986b) with available observations and indicate that the anchoring depth of magnetic field lines in the convection zone, about 10 mm below the photosphere, corresponds to the depth of origin (Parker, 1984b) of ephemeral magnetic regions on the solar surfaces. The considerable variability ($\pm 10\%$) observed in the wave periods can be explained by small displacements ($\lesssim 0.1$ Mm) of the anchoring depth of magnetic field lines, and the model may also be supported by some evidence (Mullan and Owens, 1984), in the solar wind, of the presence of Alfvén waves originating from sunspots.

C. Magnetosonic waves and critical levels

Having considered separately acoustic-gravity (Sec. II.A) and Alfvén (Sec. II.B) waves, which are due to the balance of inertia and the gas and magnetic pressures, respectively, we now consider their coupling into magnetosonic-gravity waves. The latter are described by a second-order wave equation for oblique waves in a horizontal magnetic field, and we defer to subsequent consideration (Sec. III) the more general case of an oblique magnetic field, which leads to a wave equation of the fourth order. Horizontal magnetic fields do occur in the solar atmosphere, at the top of loops, arches, and other closed magnetic structures in the corona, as will be discussed subsequently (Sec. III.C). Here we merely mention in passing that the magnetic field in a sunspot emerges nearly vertically in the umbra, where the oscillations are observed (Sec. II.B.7), and then tilts towards the horizontal in the penumbra, where running waves have been observed and interpreted as magnetosonic-gravity modes (Nye and Thomas, 1974, 1976b; Cally and Adam, 1983; Zhugzhda and Dzhalilov, 1984c). The study of magnetosonic-gravity waves (Nye and Thomas, 1976a; Summers, 1976; Adam, 1977b; Campos, 1983a, 1983d) shows that they resemble acoustic-gravity waves in regions of predominant gas pressure and become compressive Alfvén waves in regions of dominant magnetic pressure. The transition between the two regimes involves a hydromagnetic critical level, whose properties have been considered in some detail in the literature (Thomas, 1976;

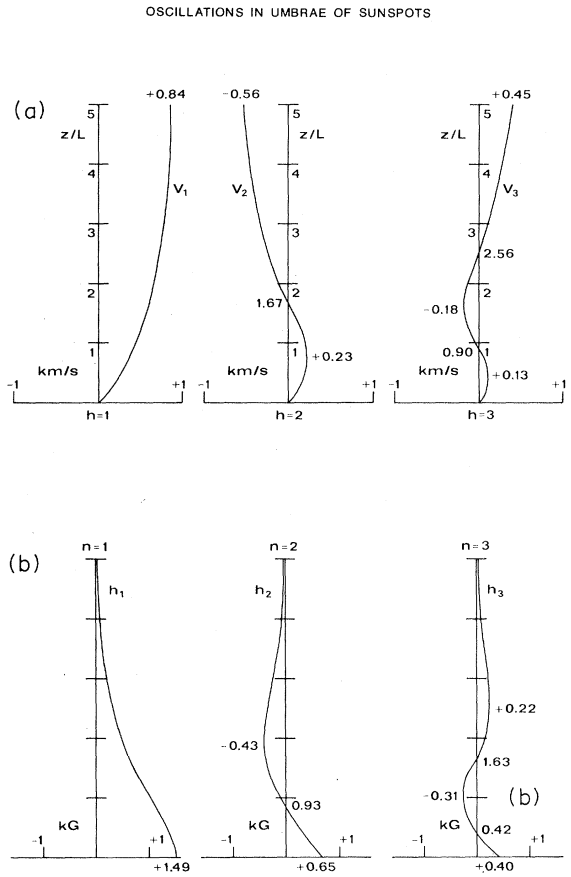

FIG. 5. Waveforms of (a) velocity perturbations and (b) magnetic field perturbations vs altitude z divided by scale height, for the first three standing modes of Alfvén waves, along magnetic field lines anchored at altitude $z=0$ and perfectly reflected at a distant layer $z/L \gg 1$, as a model of oscillations in sunspot umbrae.

El Mekki, Eltayeb, and McKenzie, 1978; Adam, 1984; Schwartz, Bel, and Cally, 1984; Campos and Leitão, 1985; Campos, 1987b). A critical level appears mathematically as a regular singularity (Ince, 1926) in the wave equation and corresponds physically to a layer where waves are reflected, or absorbed, or transformed into another mode, as a consequence of the interplay between two competing effects. Critical levels were first studied for internal waves propagating against a stream (Bretherton, 1966; Booker and Bretherton, 1967), they occur where the velocity of the shear flow balances the wave speed, so that waves can propagate no further. Similar phenomena occur in the tracing of sound rays in a wind (Lighthill, 1978), with either absorption of tangential rays or cusp reflections at the critical level. Critical levels can also occur in the presence of rotation (Acheson, 1972; El Mekki, 1983; Ritchie, 1985) and magnetic fields (McKenzie, 1973; Eltayeb, 1977; Rudraiah and Venkatachalappa, 1979; Rudraich, Venkatachalappa, and Sekar, 1982). The aspects of plasma physics related to the magnetosonic-gravity wave and its associated hydromagnetic critical level include the questions of stability (Sakai, 1983), resonances (Southwood, 1974; Rae and Roberts, 1982a), and the Alfvén continuum (Kieras and Tataronis, 1982; Connor, Tang, and Taylor, 1983; Mahajan and Ross, 1983; Mahajan, Ross, and Chen, 1983). If the wave amplitude becomes large at the critical level it may be necessary to consider nonlinear effects (see Sec. IV.C), and if the phase varies very rapidly, leading to steep gradients, dissipation may be significant (see Sec. IV.A).

1. Mode conversion from acoustic to Alfvén

In the absence of rotation, $\mathbf{\Omega}=0$, and in the presence of a uniform external magnetic field, Eq. (10) simplifies to the equation for magneto-acoustic-gravity waves, where we use as a wave variable the velocity perturbation $\mathbf{v}=\dot{\boldsymbol{\xi}}$,

$$\ddot{\mathbf{v}}-c^2\nabla(\nabla\cdot\mathbf{v})-(\gamma-1)\mathbf{g}(\nabla\cdot\mathbf{v})-\nabla(\mathbf{g}\cdot\mathbf{v}) =a^2[(\mathbf{b}\cdot\nabla)^2\mathbf{v}-(\mathbf{b}\cdot\nabla)\nabla(\mathbf{b}\cdot\mathbf{v})-\mathbf{b}(\mathbf{b}\cdot\nabla)(\nabla\cdot\mathbf{v})+\nabla(\nabla\cdot\mathbf{v})] \; . \tag{119}$$

The terms on the lhs, involving the sound speed c, adiabatic exponent γ, and acceleration of gravity $\mathbf{g}$, correspond to acoustic-gravity waves [Eq. (16)], while the terms on the rhs, involving the Alfvén speed a and directions $\mathbf{b}$ of the magnetic field, are the same as in the magneto-acoustic wave equation (41). Equation (16) has constant coefficients in an isothermal atmosphere (Sec. I.A.4), and Eq. (41) also has constant coefficients in a homogeneous medium under a uniform magnetic field (Sec. II.B.7); however, when gravity and magnetism are coupled, e.g., in an isothermal atmosphere under a nonuniform magnetic field, the Alfvén speed a varies with altitude [Eq. (99a)], and we can use a Fourier representation [Eq. (79a)] only in frequency ω and horizontal wave number $k_{\parallel}$ for two-dimensional waves. The dependence on altitude z is specified by substituting into the wave equation (119), where we assume a horizontal magnetic field $\mathbf{b}=(1,0,0)$. We obtain

$$\omega^2 W_z^2+c^2(W_z''+ik_{\parallel}W_x')-ik_{\parallel}g(\gamma-1)W_x -\gamma g W_z'+a^2(W_z''-k_{\parallel}^2W_z)=0 \; , \tag{120a}$$

$$(\omega^2-c^2k_{\parallel}^2)W_x+ik_{\parallel}(c^2W_z'-gW_z)=0 \; , \tag{120b}$$

where the only difference from the acoustic-gravity case (79b) and (79c) is an additional term involving the Alfvén speed in the first equation (120a). Eliminating W_x we obtain the wave equation for the vertical velocity perturbation spectrum:

$$[c^2+(1-k_{\parallel}^2c^2/\omega^2)a^2]W_z''-\gamma g W_z'+\Lambda_*^2W_z=0 \; , \tag{121a}$$

$$\Lambda_*=(\omega^2-c^2k_{\parallel}^2)(1-k_{\parallel}^2a^2/\omega^2)+(\gamma-1)(k_{\parallel}g/\omega)^2 \; , \tag{121b}$$

which reduces to Eqs. (80a) and (80b) in the absence of a magnetic field $a=0$. The ratio of sound [Eq. (9a)] and Alfvén [Eq. (97a)] speeds squared, $(c/a)^2=(\gamma/2)(p/P)$, scales, through half the adiabatic exponent γ, on the ratio of gas pressures p and magnetic pressures P. Thus, in a region where the gas/magnetic pressure predominates, the sound speed is much larger/smaller than the Alfvén speed, and the magnetosonic-gravity wave equation [(121a) and (121b)] reduces either to that of acoustic-gravity waves [Eqs. (80a) and (80b)] or to that of compressible Alfvén modes:

$$a^2W_z''-[\gamma g/(1-k_{\parallel}^2c^2/\omega^2)]W_z'+[(\omega^2-k_{\parallel}^2a^2)+(\gamma-1)(k_{\parallel}g/\omega)^2/(1-k_{\parallel}^2c^2/\omega^2)]W_z=0\ . \tag{122}$$

The property of magnetosonic-gravity waves include, as particular cases, acoustic and Alfvén-type behavior, raises a number of issues: (i) Bearing in mind that acoustic-gravity waves have cutoff frequencies (Sec. II.A), while Alfvén waves are not filtered in an atmosphere (Sec. II.B), do magnetosonic-gravity waves inherit the cutoff frequencies of the former, or are these modified by the magnetic field? (ii) Bearing in mind that, when a magnetosonic-gravity wave propagates from a region of dominant gas pressure to one of dominant magnetic pressure, the waves transform from acoustic to Alfvén-type, what are the properties in the region of comparable gas and magnetic pressures, where mode conversion occurs?

2. Methods of calculation of cutoff frequencies

We discuss issue (i), namely, the cutoff frequency, first for vertical waves, for which the calculations are simpler than for oblique modes, while the arguments involved are much the same in both cases. Vertical acoustic-gravity waves are specified by Eq. (79a) with $k_{\parallel}=0$:

$$c^2W_z''-\gamma gW_z'+\omega^2W_z=0\ . \tag{123a}$$

The presence of a horizontal magnetic field is accounted for by replacing the sound speed squared by its sum with the Alfvén speed squared,

$$(c^2+a^2)W_z''-\gamma gW_z'+\omega^2W_z=0\ , \tag{123b}$$

which is the equation for vertical magnetosonic-gravity waves [Eqs. (121a) and (121b) with $k_{\parallel}=0$]. Since vertical acoustic-gravity waves have [see Eq. (22b)] the cutoff frequency $\omega_2=c/2L$ in an isothermal atmosphere, it might be supposed that the same transformation $c^2\to c^2+a^2$ as from Eqs. (123a) to (123b) would specify the cutoff frequency for vertical magnetosonic-gravity waves, as

$$\omega=\sqrt{c^2+a^2}/2L=\omega_2\sqrt{1+1/\beta}\ , \tag{124a}$$

$$\beta\equiv c^2/a^2=\gamma p/2P\ , \tag{124b}$$

where the effects of the magnetic field appear through the plasma β. The result [Eq. (124a)] is often quoted in the literature (e.g., Bel and Mein, 1971; Michalitsanos, 1973; Bray and Loughhead, 1974; Stein and Leibacher, 1974; Athay, 1976; Priest, 1982a), as the cutoff frequency for vertical magnetosonic-gravity waves; however, the arguments that could be used to prove Eq. (124a) face insurmountable objections (Thomas, 1982; Campos, 1985a), as will be shown with three examples. First, it is known that for propagation perpendicular to the magnetic field [$\mathbf{b}\cdot\mathbf{n}=0$ in (45)] the phase speed of fast magnetosonic waves is $u_+=(c^2+a^2)^{1/2}$; although this result explains the transformation from the acoustic wave equation (123a) to the magnetosonic-gravity wave equation (123b), it cannot justify the cutoff frequency (124a), since the theory of magneto-acoustic waves applies to a homogeneous medium, and such media have no cutoffs. Second, the acoustic cutoff frequency $\omega_2=c/2L$ can be deduced from the dispersion relation for vertical acoustic-gravity waves, i.e., Eq. (20) with $k_{\parallel}=0$,

$$k_{\perp}^2+(i/L)k_{\perp}-\omega^2/c^2=0\ , \tag{125a}$$

$$k_{\perp}=-i/2L\pm(\omega/c)\sqrt{1-(\omega_2/\omega)^2}\ , \tag{125b}$$

since in an isothermal atmosphere all coefficients of Eq. (123a) are constant, and a sinusoidal solution $W_z\sim\exp(ik_{\perp}z)$ exists. A similar argument fails to apply to the magnetosonic-gravity wave equation (123b), which was deduced for a uniform external magnetic field, since the condition of constant sound speed c, i.e., an isothermal atmosphere, leads to an Alfvén speed a varying with altitude [Eq. (99a)], so that Eq. (123b) has no sinusoidal solutions. Third, Eq. (124a) could be tentatively justified on the basis of the WKBJ approximation, i.e., assuming an approximately sinusoidal waveform $W_z\sim\exp(ik_{\perp}z)$, leading to a "local" dispersion relation similar to Eqs. (125a) and (125b) with c^2+a^2 replacing c^2; bearing in mind that the Alfvén speed varies on twice the scale height [Eq. (99a)], we see that the WKBJ approximation applies in this case to wavelengths shorter than the scale height $\lambda^2\ll 4L^2$, so that the cutoff limit of infinite wavelength $\lambda\to\infty$ cannot be applied, and the reasoning leading to Eq. (124a) fails once more.

3. Compressive filtering and magnetic transparency

The preceding examples indicate that, since the cutoff frequency corresponds to the limit of infinitely spaced nodes $\lambda\to\infty$, it can only be derived from an exact solution of the wave equation. In the case of acoustic-gravity waves in an isothermal atmosphere, the wave equations (79b) and (79c) have constant coefficients; thus a sinusoidal waveform solution $W_z\sim\exp(ik_{\perp}z)$ together with the dispersion relation (20) is an exact solution, from which valid cutoff frequencies [Eqs. (22a) and (22b)] can be deduced. In the case of magnetosonic-gravity waves in an isothermal atmosphere, under an external uniform horizontal magnetic field, the wave equations [(121a) and (121b)] involve a constant sound speed [Eq. (7a)] and variable Alfvén speed [Eq. (99a)], and thus we derive an exact, nonsinusoidal solution, before reconsidering the matter of cutoff frequencies. If we perform the changes of independent ζ and dependent Φ variable,

$$\zeta = -[(c/a_0)^2/(1-k_{\parallel}^2c^2/\omega^2)]e^{-z/L} , \tag{126a}$$

$$W_z(z;k_{\parallel},\omega) = \exp(k_{\parallel}z)\Phi(\zeta) , \tag{126b}$$

the wave equation [(121a) and (121b)] transforms into a hypergeometric type:

$$\zeta(1-\zeta)\Phi'' + [(1+2k_{\parallel}L) - 2(1+k_{\parallel}L)\zeta]\Phi' - [k_{\parallel}L + (\omega/2\omega_2)^2 + (k_{\parallel}L\omega_1/\omega)^2]\Phi = 0 , \tag{127}$$

with parameters $\alpha_1,\beta_1,\gamma_1$ satisfying

$$\gamma_1 = 1+2k_{\parallel}L = \alpha_1+\beta_1 , \tag{128a}$$

$$\alpha_1\beta_1 = K^2L^2 + (k_{\parallel}L + \tfrac{1}{2})^2 , \tag{128b}$$

where we have introduced the effective vertical wave number K [Eq. (82a)], leading to the explicit forms for the parameters

$$\alpha_1,\beta_1 = 1/2 + k_{\parallel}L \pm iKL , \tag{129a}$$

$$\gamma_1 = 1+2k_{\parallel}L . \tag{129b}$$

It is now clear that for frequencies between the cutoffs $\omega_1 < \omega < \omega_2$ the effective wave number K is pure imaginary [Eq. (82a)], the coefficients (129a) and (129b) are all real, and thus the wave has no phase, i.e., is a standing mode. Wave propagation is only possible for real effective wave number K, i.e., below the lower cutoff $\omega < \omega_1$ [Eq. (22a)] for a magneto-gravity mode modified by compressibility, or above the upper cutoff $\omega > \omega_1$ [Eq. (22b)], for the magneto-acoustic mode modified by gravity. Thus we conclude that the cutoff frequencies are the same for acoustic-gravity waves in the absence of a magnetic field and for magnetosonic-gravity waves in the presence of a uniform horizontal magnetic field, that is, they are given by Eqs. (22a) and (22b) for oblique (Thomas, 1982) and Eq. (22b) for vertical (Campos, 1983d) propagation. Thus Eqs. (124a) and (124b) are incorrect, since a horizontal magnetic field does not affect wave filtering; this conclusion does not extend to oblique magnetic fields, which will be shown subsequently (Secs. III.C.2 and III.C.3) to affect cutoff frequencies. In conclusion, the filtering of magnetosonic-gravity waves is a purely compressive effect, as for acoustic-gravity waves; it is not changed by the presence of a horizontal magnetic field, which is "transparent" to the wave spectrum, much in the same way as Alfvén waves are not filtered in an atmosphere.

4. Power-type and logarithmic leading terms

We now turn to question (ii) concerning wave transformation, from the point of view of general critical level theory. A second-order wave equation can be written in the form

$$W'' + P_1(z)W' + P_0(z)W = 0 , \tag{130}$$

where W denotes a wave variable, a prime denotes a derivative with regard to altitude z, and the functions $P_0(z)$, $P_1(z)$ specify the atmospheric properties. If these functions are analytic in z, then an analytic solution of Eq. (130) exists at all altitudes, i.e., no singularities or critical levels occur. For example, in the case of Alfvén-gravity waves [Eqs. (110b)], we have $P_1(z)=0$ and $P_0(z) \sim e^{-z/L}$, and the exact solution, in terms of Hankel functions (112a), is a power series (possibly including a logarithmic term) that converges at all finite altitudes. In the case of magnetosonic-gravity waves [Eq. (121a)], we have

$$[c^2 + (1-k_{\parallel}^2c^2/\omega^2)a^2][P_1(z),P_0(z)] = -\gamma g,\Lambda_*^2 , \tag{131}$$

so that a singularity occurs when the term in square brackets vanishes, i.e.,

$$a^{-2} + c^{-2} = (k_{\parallel}/\omega)^2 , \tag{132a}$$

$$z_c = L \ln[(c/a_0)^2/(1-k_{\parallel}^2c^2/a_0^2)] . \tag{132b}$$

Therefore, in the case of an isothermal atmosphere [constant sound speed c given by Eq. (9a)], under a uniform magnetic field [Alfvén speed a specified by Eq. (99a)], the critical level is located at the altitude z_c [Eq. (132b)]. A singular point $z=z_c$ of the differential equation (130) is called regular (Poole, 1937) if the functions $P_0,P_1(z)$ have, at most, a single or a double pole, respectively, at z_c,

$$P_1(z) = (z-z_c)^{-1}p_1(z) , \tag{133a}$$

$$P_0(z) = (z-z_c)^{-2}p_0(z) , \tag{133b}$$

where $p_0,p_1(z)$ are analytic functions, i.e., have convergent power series

$$p_1(z) = \sum_{n=0}^{\infty} \varphi_n(z-z_c)^n , \tag{134a}$$

$$p_0(z) = \sum_{n=0}^{\infty} \psi_n(z-z_c)^n , \tag{134b}$$

with known coefficients φ_n,ψ_n. For example, in the case of magnetosonic-gravity waves, the critical level [Eqs. (132a) and (132b)] is a regular singularity, since P_1,P_0 in Eq. (131) both have a simple pole at $z=z_c$. In the neighborhood of a regular singularity [Eqs. (133a) and (133b)], the wave equation (130) has a power series solution:

$$W_\sigma(z) = (z-z_c)^\sigma \sum_{n=0}^{\infty} W_n(z-z_c)^n , \tag{135a}$$

starting with a power of exponent σ. Substituting Eqs. (135a) and (134a) and (134b) into (130), and equating coefficients of successive powers of $(z-z_c)$ to zero, yields to lowest order the indicial equation

$$\sigma^2 + (\varphi_0 - 1)\sigma + \psi_0 = 0 , \tag{135b}$$

specifying σ, and the higher orders specify the coefficients W_n of the solution (135a) in terms of φ_n,ψ_n in Eqs. (134a) and (134b). The indicial equation of (135b) is of the second degree and has two roots, σ_1,σ_2. If they are distinct and do not differ by an integer, the corresponding particular integrals W_{σ_1} and W_{σ_2} are linearly independent; otherwise, say, for a double root σ_0, it can be shown (Forsyth, 1885; see also Secs. III.B.4 and III.B.5) that a second particular integral, linearly independent from W_{σ_0} in Eq. (135a), is given by

$$\overline{W}_{\sigma_0}(z)=\lim_{\sigma\to\sigma_0}\partial[W_\sigma(z)]/\partial\sigma\sim(z-z_c)^{\sigma_0}\ln(z-z_c)\ , \quad (136)$$

which has a logarithmic singularity. We conclude that the nature of the singularity of the wave field at a critical level, viz., at a branch point for complex singularity in Eq. (135a), or logarithmic singularity in Eq. (136), can be predicted on inspection of the indicial equation (135b) if we assume that the series (135a) following the leading term converges.

5. Nature of singularities and convergence of solutions

In much of the literature on critical levels it is not proven that the series (135a) following the leading term converges, perhaps because such a demonstration usually requires an explicit formula for the nth coefficient W_n, which may be difficult or tedious to obtain. If the series does not converge, then it introduces another singularity, competing with that in the leading term, and so three possibilities arise: (a) if the singularity arising from the series is stronger than the leading term, the latter is irrelevant, since the wave properties at the critical level are determined by the former; (b) if the series gives rise to a singularity opposite to the leading term, they cancel, and the wave field is regular at the critical level; (c) if the series yields a singularity weaker than the leading term, the latter determines wave behavior at the critical level, as in the case of a convergent series. We shall give an example of case (b), i.e., cancellation of singularities, which occurs in the case of the critical level for magnetosonic-gravity waves. This critical level is located at an altitude z_c [Eq. (132b)], which transforms to the point $|\zeta|=1$ for the variable (126a). The hypergeometric equation (127) can be written, in the vicinity of the unit point $\zeta=1$, in the form of Eq. (130), with

$$P_1(\zeta)=(\zeta-1)^{-1}[2(1+k_{\parallel}L)-(1+2k_{\parallel}L)\zeta]\ , \quad (137a)$$

$$P_0(\zeta)=(\zeta-1)^2\{[(k_{\parallel}L+\tfrac{1}{2})^2+K^2L^2](1-1/\zeta)\} \quad (137b)$$

emphasizing the functions $p_0,p_1(\zeta)$ in Eqs. (133a) and (133b), which appear in brackets in Eq. (137a) and in curly brackets in (137b) and are analytic at $\zeta=1$, with leading terms $\varphi_0=1$, $\psi_0=0$ [Eqs. (134a) and (134b)]. Thus the indicial equation (135b) is $\sigma^2=0$, i.e., has a double root, and if the series that follows converges, the wave field at the critical level is a linear combination of Eqs. (135a) and (136) with $\sigma=0$, i.e., has a logarithmic singularity:

$$W_z(z;k_{\parallel},\omega)\sim[A_1+A_2\ln(z-z_c)][1+O(z-z_c)]\ . \quad (138a)$$

This conclusion appears to be confirmed if we seek a solution of the hypergeometric equation (127), with parameters [(129a)—(129c)] in terms of the variable $1-\zeta$, since such a solution appears (Erdelyi, 1953) as a linear combination of hypergeometric functions of the first F and second G kinds:

$$\Phi(\zeta)=B_1F(\alpha_1,\beta_1;1;1-\zeta)+B_2G(\alpha_1,\beta_1;1;1-\zeta)\ , \quad (138b)$$

and the former F is regular, whereas the latter G has a logarithmic singularity. Thus it has been repeatedly stated in the literature (McKenzie, 1973; Adam, 1977b; El Mekki, Eltayeb, and McKenzie, 1978; Thomas, 1983) that magnetosonic-gravity waves have a logarithmic singularity at the critical level. For nonevanescent waves, $\omega>k_{\parallel}c$, and the variable ζ is less than zero in Eq. (126a), so that the critical level is located at $\zeta=-1$ and not at $\zeta=1$. It follows that the hypergeometric series (138b) have variables greater than unity, $1-\zeta>1$, and thus do not converge, i.e., the existence of the logarithmic singularity is not proven by Eqs. (138a) or (138b). The logarithmic singularity does exist if (138b) converges, i.e., for $1-\zeta<1$ or $\zeta<0$, which corresponds [by Eq. (126a)] to evanescent waves $\omega<k_{\parallel}c$. The numerical calculations of compressible modes in atmospheres with magnetic fields tilted as close as 5° to the horizontal show no evidence of a singularity being approached (Schwartz and Bel, 1984), but they do not prove the nonexistence of the singularity, since the critical level only exists for a strictly horizontal magnetic field (Schwartz, Cally, and Bel, 1984; Campos, 1985a; see Sec. III.C). In conclusion, the existence of a logarithmic singularity for magnetosonic-gravity waves, in an isothermal atmosphere, under a uniform horizontal magnetic field, is proven only for evanescent waves $\omega<k_{\parallel}c$; it will be shown below that no such singularity occurs for propagating waves $\omega>k_{\parallel}c$ or $\zeta<0$.

6. Reflection of oblique and absorption of vertical waves

In order to clarify the properties of waves at a critical level, it is important to prove the validity and convergence of the solutions of the wave equation. The hypergeometric equation has solutions in terms of hypergeometric functions of any (Forsyth, 1885) of the six variables

$$\zeta,\ 1-\zeta,\ 1/\zeta,\ 1/(1-\zeta),\ \zeta/(\zeta-1)\ . \quad (139)$$

In the present problem, the variable $\zeta<0$ in Eq. (126a), the worst choices of variable among the six are $1-\zeta>1$ and $1-1/\zeta>1$, since they lead to hypergeometric series with variable greater than unity, that diverge, as, for example, in Eq. (138b). If we choose $1/\zeta$, we obtain a solution valid for large ζ, viz., $|1/\zeta|<1$ or $|\zeta|>1$, which by Eq. (126a) corresponds to low altitude z, i.e., z below the critical level z_c [Eq. (132b)]. Thus the wave field is given for $|\zeta|>1$ by (Gradshteyn and Rhyzik, 1980)

$$\Phi(\zeta)=A_+(-\zeta)^{-\alpha}F(\alpha,1+\alpha-\gamma;1+\alpha-\beta;1/\zeta)$$
$$+A_-(-\zeta)^{-\beta}F(\beta,1+\beta-\gamma;1+\beta-\alpha;1/\zeta)\ , \quad (140)$$

where $A_\pm$ are arbitrary constants of integration. Using Eqs. (126) and (129a)—(129c), it is clear that, below the critical level $0<z<z_c$, the vertical velocity perturbation spectrum is given by

$$W_z(z;k_{||},\omega)=A_+W_z^+(z;k_{||},\omega)+A_-W_z^-(z;k_{||},\omega)\,, \tag{141a}$$

as a linear combination of upward-propagating (W^+) and downward-propagating (W^-) waves, respectively,

$$W_z^{\pm}(z;k_{||},\omega)\equiv e^{z/2L}e^{\pm iKz}F(\tfrac{1}{2}+k_{||}L\pm iKL,\tfrac{1}{2}-k_{||}L\pm iKL;1\pm 2iKL;1/\zeta)\,. \tag{141b}$$

Here we have acoustic-gravity waves (first two factors) modified by the magnetic field (hypergeometric function). Above the critical level, $\infty>z>z_c$, we have $|\zeta|<1$ and use the solution (93) in the variable ζ, in other words, the wave field

$$W_z(z;k_{||},\omega)=A_1W_z^1(z;k_{||},\omega)+A_2W_z^2(z;k_{||},\omega) \tag{142a}$$

is a superposition of growing W^1 and evanescent W^2 waves, respectively,

$$W_z^{1,2}(z;k_{||},\omega)\equiv\exp(\pm k_{||}z)F(\tfrac{1}{2}\pm k_{||}L+iKL,\tfrac{1}{2}\pm k_{||}L-iKL;1\pm 2k_{||}L;\zeta)\,. \tag{142b}$$

The amplitudes below the critical level $A_\pm$ [Eqs. (141a) and (141b)] and above A_1,A_2 [Eqs. (142a) and (142b)] are related as the hypergeometric functions of variables ζ and $1/\zeta$ (Abramowitz and Stegun, 1964)

$$A_\pm=\{\Gamma(1\pm 2k_{||}L)\Gamma(-2iKL)/[\Gamma(1\pm k_{||}L-iKL)]^2\}A_{1,2}+\{\Gamma(1-2k_{||}L)\Gamma(+2iKL)/[\Gamma(1\pm k_{||}L+iKL)]^2\}A_{2,1}\,. \tag{143}$$

For oblique waves $k_{||}\neq 0$, in order that the wave field be bounded at high altitude [Eq. (142a)], we must set $A_1=0$, so that generally $A_+\neq 0\neq A_-$, i.e., we have upward (incident) and downward (reflected) waves below the critical level [Eq. (141b)], which thus acts as a reflector. For vertical waves $k_{||}=0$, the two particular integrals (142b) coincide, and the wave field is given, above the critical level, by a linear combination of hypergeometric functions of the first F and second G kinds:

$$\begin{aligned}W_z(z;0,\omega)&=A_1F_1[-(c/a_0)^2e^{-z/L}]\\&\quad+A_2G_1[-(c/a_0)^2e^{-z/L}]\,,\\F_1,G_1(\xi)&\equiv F,G(\tfrac{1}{2}+iKL,\tfrac{1}{2}-iKL;1;\xi)\,.\end{aligned} \tag{144}$$

Since, at high altitude, $z\to\infty$, we have $\zeta\equiv-(c/a_0)^2e^{-z/L}\to 0$, and the function of the first kind tends to unity, $F(\zeta)\to 1$, whereas that of the second kind has a logarithmic singularity $G(\zeta)\to\ln\zeta$, the velocity perturbation of a vertical magnetosonic-gravity wave grows linearly at high altitude:

$$W_z(z;0,\omega)\sim A_++A_-[i\pi-z/L+2\ln(c/a_0)]\,, \tag{145}$$

as in the case of Alfvén-gravity waves [Eqs. (108a) and (115a)]. The reduction of the rate of growth, from exponential far below the critical level, [Eq. (141b)] to linear far above it [Eq. (145)] demonstrates the partial absorption of vertical magnetosonic-gravity waves in its vicinity.

7. Phase limiting and amplitude selection

Although we now have all the elements needed to analyze reliably wave properties at the critical level, it is convenient to summarize briefly the method of approach that we have adopted. A critical level is a singularity of the wave equation at an intermediate altitude $z=z_c$; thus if a wave has one critical level, the corresponding wave equation may have up to three singularities, the other two specifying the initial $z=0$ and asymptotic $z=\infty$ wave fields. By a suitable change of independent variable, e.g., from altitude z to ζ, which is proportional to the mass of the atmosphere $\sim e^{-z/L}$, the critical level may be placed at the unit point $\zeta(z_c)=1$ and the initial and asymptotic fields at $\zeta=0,\infty$. A linear, second-order differential equation with regular singularities at $\zeta=0,1,\infty$ is necessarily of the hypergeometric type, and thus it is not surprising that the wave equation (122) was transformed into this form. The transformations between the six variables of Eq. (139), which allow solution in terms of hypergeometric functions, are such that they interchange among themselves the points $\zeta=0,1,\infty$, i.e., they preserve the location of the singularities of the equation. The $5!=120$ transformations between the six variables of Eq. (139) form a group (Klein, 1933), i.e., any sequence of transformations leads to another transformation of the same set. In fact, all transformations can be obtained from any two of them, which form a generating set. We may investigate the wave field at the critical level by using either the low-altitude [Eqs. (141a) and (141b)] or high-altitude [Eqs. (142a) and (142b)] solutions to approach it from below and above, respectively. The critical level (132b) is located at the point $\zeta=-1$, which lies on the radius of convergence $|\zeta|=1=|1/\zeta|$ of both solutions, but outside the real axis. The low-altitude solution (141b) has parameters $\alpha_2=1/2+k_{||}L\pm iKL$, $\beta_2=1/2-k_{||}L\pm iKL$, $\gamma_2=1\pm 2iKL$, satisfying $\alpha_2+\beta_2-\gamma_2=0$, while the high-altitude solution (142b) has parameters $\alpha_3=1/2\pm k_{||}L+iKL$, $\beta_3=1/2\pm k_{||}L-iKL$, $\gamma_3=1\pm 2k_{||}L$, which also satisfy $\alpha_3+\beta_3-\gamma_3=0$. Since the hypergeometric series (with parameters α,β,γ such that $\alpha+\beta=\gamma$) converges (Whittaker and Watson, 1902) at the point $\zeta=-1$ on its radius of convergence, we have two distinct proofs that the amplitude and phase of the wave are finite at the critical level; they are specified by the expansions (141a) and (141b) or (142a) and (142b), which converge there (Campos and Leitão, 1986; Campos, 1987b). Thus the logarithmic "singularity" in the leading terms of Eqs. (138a) and (138b) must be balanced by the divergent series that follow, since the whole expression is finite. The statement that the energy flux of magnetosonic-gravity waves has a discontinuous jump at the critical level (El Mekki, Eltayeb, and McKenzie, 1978) can only hold in the presence of a singularity. The inclusion of dissipation by thermal radiation (Cally, 1984) shows that the energy flux changes continuously across the critical level. The preceding analysis proves that, even for linear magnetosonic-gravity waves, in the absence of dissipation, the amplitude and phase are finite at the critical level, and thus the energy is reduced continuously, as oblique waves are reflected and vertical waves partially absorbed, in its vicinity.

We have shown that linear, nondissipative magnetosonic-gravity waves, in an isothermal atmosphere, under a uniform horizontal magnetic field, have a critical level, which may be of one of the following three types: type I, a singular layer, where the amplitude is logarithmically infinite [Eqs. (138a) and (138b)], for evanescent waves $\omega > k_{\|}c$; type II, a reflection layer for oblique $k_{\|}\neq 0$, propagating $\omega < k_{\|}c$ waves, which are reflected [Eqs. (141a) and (148b)] at the critical level [Eqs. (132a) and (132b)]; type III, a transition layer for vertical waves $k_{\|}=0$, which are transformed from exponential amplitude and linear phase below the critical level [Eqs. (141a) and (148b)] to linear amplitude and bounded phase above the critical level [Eqs. (144) and (145)] at $z_c=2L\ln(c/a_0)$. Note that, in the case of the singular layer or critical level of type I, the singularity is at $\zeta=1$ in the range $0\leq\zeta<\infty$ of the variable, whereas in the cases of reflection and transition layers (i.e., critical levels of types II and III, respectively) it lies on the unit circle outside the real axis $|\zeta|=1\neq\zeta$.

8. Characteristic curves for normalized wave variables

The process of conversion can be demonstrated in more detail by deducing an exact expression for the wave field, valid at all altitudes, and using it to calculate the amplitude and phase at the critical level and plot the waveforms in its vicinity. Of the six variables in Eq. (139), bearing in mind that $\zeta<0$ in Eq. (126a) for all altitudes, two are smaller than unity everywhere, viz., $1/(1-\zeta)<1$ and $\zeta/(\zeta-1)<1$, and thus lead to global solutions of the wave equation. The identity

$$F(\alpha,\beta,\gamma;1/\zeta)=(1-1/\zeta)^{-\alpha}F[\alpha,\gamma-\beta;\gamma;1/(1-\zeta)] \tag{146}$$

suggests that we use the former variable $1/(1-\zeta)$, and together with Eq. (141b) specifies the upward-propagating W_+ and downward-propagating W_- waves, respectively, by

$$W_z^{\pm}(z;k_{\|},\omega)=(1-1/\zeta)^{-k_{\|}L}(\zeta-1)^{-1/2\mp iKL}\times F_2^{\pm}[1/(1-\zeta)]\,, \tag{147a}$$

$$F_2^{\pm}(\xi)\equiv F[\tfrac{1}{2}+k_{\|}L\pm iKL,\tfrac{1}{2}+k_{\|}L\pm iKL;1\pm 2iKL;1/(1-\zeta)]\,. \tag{147b}$$

Equations (147a) and (147b) specify exactly the wave field at all altitudes, including at the critical level z_c [Eq. (132b)] or $\zeta(z_c)=-1$, where the amplitude and phase of the wave field are given by the modulus and argument of

$$W_z^{\pm}(z_c;k_{\|},\omega)=2^{-k_{\|}L}2^{KL-i/2}F_2^{\pm}(\tfrac{1}{2})\,. \tag{148}$$

The wave field is given at all altitudes by Eq. (141a) as a linear combination of upward-propagating W_z^+ and downward-propagating W_z^- waves [Eqs. (147a) and (147b)], with amplitudes A_+ and A_- specified by two suitable initial boundary or asymptotic conditions. We choose for illustration an upward-propagating wave, normalizing the vertical velocity perturbation spectrum (147a) to its value at the critical level [Eq. (148)] and choosing as independent variable the distance from the critical level divided by the scale height:

$$W_*(Y;\omega)\equiv W_z^+(z;0,\omega)/W_z^+(z_c;0,\omega)\,, \tag{149a}$$

$$Y\equiv(z-z_c)/L\,. \tag{149b}$$

We choose a vertical wave $k_{\|}=0$, which suffers absorption in the vicinity of the critical level, and for which the ratio of Eqs. (147a) and (148) simplifies to

$$W_*(Y,\omega)=[2/(1+e^{-Y})]^{1/2+iKL}\times\{F_*[1/(1+e^{-Y})]/F_*(1/2)\}\,, \tag{150a}$$

$$F_*(\xi)\equiv F(\tfrac{1}{2}+iKL,\tfrac{1}{2}+iKL+2iKL;\xi)\,. \tag{150b}$$

The wave field [(150a) and (150b)], normalized relative to the critical level, is independent of the magnetic field strength, which affects only the location [Eq. (132b)] of the critical level $z_c=2L\ln(c/a_0)$ and the absolute amplitude and phase there [Eq. (148)]. The relative values [(150a) and (150b)] depend only on the ratio of wave ω to cutoff frequency ω_2, $\Omega_*=\omega/\omega_2=2\omega L/c$, which appears in the effective wave number $K=\mathrm{Re}(k_\perp)$ for vertical waves [Eq. (125b)], $2KL=\sqrt{\Omega^2-1}$. The normalized amplitude and phase difference, that is, the modulus and argument of Eq. (150a), are plotted for several values of the dimensionless frequency Ω_*, in Fig. 6, as a function of distance Y from the critical level [Eq. (149b)], showing that (i) the amplitude growth is independent of frequency far below the critical level (acoustic-gravity waves), and the wave absorption in the vicinity of the critical level results in a smaller amplitude growth above, the reduction being larger for waves of lower frequency; (ii) the phase differences increase as the ratio of wave ω to cutoff frequency ω_* becomes larger, but phase growth is reduced across the critical level, from linear far below to bounded

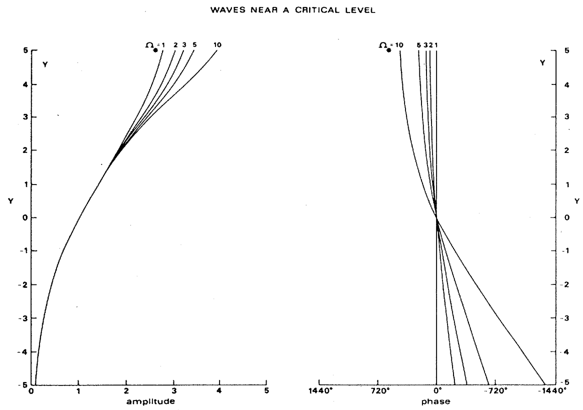

FIG. 6. Amplitude (left) and phase (right) of magnetosonic-gravity waves, normalized to their value at the critical level z_c, and plotted vs dimensionless distance $Y \equiv (z - z_c)/L$ (altitude difference divided by scale height L), for a wave propagating vertically upward, with a frequency ω multiplying the cutoff value ω_2 by a factor $\Omega_* \equiv \omega/\omega_2 = 1,2,3,5,10$.

far above. Thus the critical level limits the phase of magnetosonic-gravity waves and selects their growth according to frequency.

III. HIGH-ORDER WAVES WITH VARIABLE SPEED

We have so far considered the exact theory of linear waves in atmospheres, with variable propagation speed (Sec. II), in cases for which the wave equation is of second order and solutions can be obtained in terms of well known special functions, such as Bessel (Sec. II.B) and hypergeometric (Secs. II.A and II.C) types. The second-order wave equations arise for general acoustic-gravity waves (Sec. II.A), in the absence of a magnetic field, and for purely transversal Alfvén waves (Sec. II.B), propagating along a magnetic field of arbitrary direction. The coupling of compressibility and magnetism leads (a) to a second-order wave equation, for oblique propagation, only in the case of magnetosonic-gravity waves (Sec. II.C), in a purely horizontal magnetic field; and (b) to wave equations that are also second order for vertical propagation, in a vertical magnetic field, because the Alfvén and acoustic modes are decoupled. An acoustic wave propagating along the magnetic field is unaffected by it, since the gas motion does not displace magnetic field lines or generate magnetic tension. The occurrence of second-order magneto-atmospheric wave equations in both cases (a) and (b) above can be understood in terms of decoupling of slow u_- and fast u_+ modes [Eq. (45b)] of magnetohydrodynamic waves, which occurs only for propagation either (a) transverse to, or (b) along, the magnetic field. If $\mathbf{b} \cdot \mathbf{n} = 0$, we have $u_+^2 = c^2 + a^2$ and $u_- = 0$, corresponding in an atmosphere to magnetosonic-gravity waves (Sec. II.C); if $\mathbf{b} \cdot \mathbf{n} = 1$, we have $u_+ = c$, $u_- = a$, corresponding in an atmosphere to decoupled acoustic-gravity (Sec. II.A) and Alfvén (Sec. II.B) waves. In other cases, the slow and fast modes are generally coupled, implying a dispersion relation of the fourth degree in the magnetohydrodynamics of homogeneous media and a wave equation of the fourth order with variable coefficients for magneto-atmospheric waves. Thus the wave equation is of the fourth order, for example, for oblique waves in a vertical magnetic field (Ferraro and Plumpton, 1958; Weymann and Howard, 1958; Lust and Scholer, 1966; Scheuer and Thomas, 1981; Leroy and Schwartz, 1982; Schwartz and Leroy, 1982; Zhukov, 1985). It is likewise of fourth order for waves (vertical or oblique) in an oblique magnetic field (Zhugzhda and Dzhalilov, 1981, 1984a, 1984b, 1984c; Zhugzhda and Makarov, 1982; Schwartz and Bel, 1984; Zhugzhda, 1984; Campos, 1985a). More complex geometries, including rotation and the addition of other effects, such as dissipation by fluid viscosity, electrical resistance, and thermal conduction or radiation, can lead to wave-diffusion equations of higher order (Sec. I.C.1) as, for example, in the theory of magnetoconvection (Proctor and Weiss, 1982). Wave equations of higher than second order also occur for other types of waves in fluids, e.g., sound in vortical flows (Möhring, Müller, and Obermeier, 1984) and instability waves in viscous flows (Drazin and Reid, 1981), besides problems in thermoelasticity, magnetoelasticity, and plasma physics. With applications to the latter field in mind, a method has been developed to solve linear wave equations of arbitrary order, with linear coefficients (Gambier and Schmitt, 1983). For waves in isothermal atmospheres, the coefficients are usually either constant or exponentials of altitude, and a method of solution of linear wave equations, of arbitrary order, with exponential coefficients, is appropriate (Campos, 1985a).

A. Initial and asymptotic wave regimes

The study of high-order (≥ 3) waves in atmospheres is usually approached in one of three ways: (a) numerical solutions (Weymann and Howard, 1958; Lust and Scholer, 1966; Scheuer and Thomas, 1981), which are expedient but somewhat lacking in physical interpretation; (b) analytical methods, e.g., power series or matrix expansions (Ferraro and Plumpton, 1958; Lyons and Yanowitch, 1974; Leroy and Schwartz, 1982; Schwartz and Leroy, 1982; Schwartz and Bel, 1984), which become tedious to calculate beyond the first few terms; (c) exact solutions in terms of special functions (Zhugzhda and Dzhalilov, 1981, 1984a, 1984b, 1984c; Zhugzhda, 1984) of generalized hypergeometric type (Bailey, 1935) or Meijer G-type (Luke, 1975), which are complicated, requiring transformation diagrams for their interpretation. We shall present an analytic and explicit method of solving wave equations, in which the mathematical procedures have a ready (or "parallel") interpretation in terms of physical properties. The method applies to linear differential equations of any order, whose coefficients, e.g., propagation seeds, scattering scales, or damping rates, are either constant or exponential functions of altitude. The method relies on the transformation of the wave equation into a standard type, to which can be reduced equations with exponential coefficients, and also equations with certain types of double power coefficients. Once the wave equation is written in the standard form (Sec. III.A.1), it is possible to deduce, on inspection, with no need to solve it, a number of properties, such as (i) the asymptotic laws (Secs. III.A.6 and III.A.7) for the amplitude and phase of wave variables at high altitude, and thus the scaling of velocity, magnetic field, density, gas, and magnetic pressure, as well as kinetic, magnetic, compression, and potential energy densities; (ii) the cutoff frequencies (Secs. III.C.2 and III.C.3) separating standing modes from propagating waves, both in the cases of constant coefficients, when dispersion relations can be used, and for variable coefficients, when the WKBJ approximation becomes unreliable for this purpose; (Secs. II.A.6 and II.C.7); (iii) the existence and location of critical levels (Sec. III.C.4) and the properties of the wave field there (Sec. III.C.5), e.g., whether the amplitude and phase are finite, or whether logarithmic or power-type singularities occur. The method also leads readily, without any further manipulation, and in a standard way, to exact solutions (Sec. III.B) for the wave fields, converging for all frequencies at all altitudes, except possibly at critical levels, where the nature of the singularity, if it exists, is specified. These exact solutions involve special functions equivalent to the generalized hypergeometric type (Sec. III.C.6), of three kinds: (a) for nondegenerate cases (Secs. III.B.2 and III.B.3), functions of the first kind Φ, generalizing to higher order the Bessel J or hypergeometric F type, and having power singularities; (b) in degenerate cases, functions (Secs. III.B.4. and III.B.5) of the second kind Ψ, generalizing the Neumann Y or hypergeometric G type, which have logarithmic singularities; (c) for propagating waves, in the direction of increasing/decreasing ζ (Sec. III.C.6), functions of the third kind $\Theta^{(1,2)}(\zeta)$, generalizing the Hankel $H^{(1,2)}$ type, which are linear combinations $\Phi+b_{1,2}\Psi$ of the first two kinds, with the constants $b_{1,2}$ chosen so as to satisfy the appropriate radiation condition.

1. Standard form of wave equation

Waves of small amplitude, in a medium for which the wave speeds, scattering scales, and damping scales are constant or vary exponentially only in one direction, e.g., an isothermal atmosphere under constant external force fields, satisfy a linear partial differential equation of order N, and of the following type:

$$\left\{\sum_{j=0}^{N}[a_j(\partial/\partial t,\partial/\partial \mathbf{y}) + e^{-z/L}b_j(\partial/\partial t,\partial/\partial \mathbf{y})]L^j d^j/dz^j\right\} w(z,\mathbf{x},t)=0\ , \tag{151}$$

where the wave variable w generally depends on time t and space $(\mathbf{x},z)$ and where we have singled out derivatives d/dz in the direction of stratification or altitude z and left temporal $\partial/\partial t$ and horizontal spatial $\partial/\partial \mathbf{y}$ derivatives in the coefficients a_j,b_j, which are arbitrary polynomials of their arguments. Using Fourier analysis in the horizontal plane $\mathbf{y}$ and time t, we obtain

$$w(z,\mathbf{x},t)=\int\int_{-\infty}^{+\infty}\int W(z;\mathbf{k}_{||},\omega)\times\exp[i(\mathbf{k}_{||}\mathbf{y}-\omega t)]d^2k_{||}d\omega\ . \tag{152}$$

The perturbation spectrum W, for a wave of frequency ω and horizontal wave vector $\mathbf{k}_{||}$, satisfies the ordinary differential equation

$$\sum_{j=0}^{N}[a_j(-i\omega,i\mathbf{k}_{||})+e^{-z/L}b_j(-i\omega,i\mathbf{k}_{||})]L^j d^j W/dz^j=0\ . \tag{153}$$

Performing the change of independent variable,

$$\zeta=-\zeta_0 e^{-z/L}\ , \tag{154a}$$

$$W(z;\mathbf{k}_{||},\omega)=\Phi(\zeta)\ , \tag{154b}$$

where $\zeta_0>0$ is an arbitrary positive constant, we transform the wave equation (153) into the standard type

$$[R(\zeta d/d\zeta)-\zeta S(\zeta d/d\zeta)]\Phi(\zeta)=0\ , \tag{155}$$

where R,S are the polynomials

$$R(\xi)\equiv\sum_{j=0}^{r}(-)^j a_j\xi^j\ , \tag{156a}$$

$$S(\xi)\equiv\sum_{j=0}^{s}(-)^j b_j\xi^j\ , \tag{156b}$$

of degrees, respectively, $r,s\le N$ (the degrees will be $r,s=N$ if $a_N\neq 0\neq b_N$, and lower, if one of the leading coefficients vanish). Not only Eq. (153), but any wave equation of the type

$$\sum_{j=0}^{N}(\bar{a}_j\zeta^j+\bar{b}_j\zeta^{j+1})d^j\Phi/d\zeta^j=0 \tag{157}$$

can be rearranged into the form of Eq. (155), with the coefficients a_j,b_j of the polynomials R,S determined from $\bar{a}_j,\bar{b}_j$.

2. Atmosphere under oblique magnetic field

It is clear from Eq. (153) that the horizontal wave vector $\mathbf{k}_{||}$ and frequency ω appear in the standard form of the wave equation (155) only in the coefficients a_j,b_j of the polynomials (155a) and (158b); thus the method of analysis with regard to altitude z is basically the same for vertical $\mathbf{k}_{||}=0$ and oblique $\mathbf{k}_{||}\neq 0$ waves, although the latter case tends to have a more tedious algebra, i.e., more complicated coefficients a_j,b_j. For the purpose of illustrating the method we choose vertical waves in an atmosphere under a magnetic field of arbitrary direction. Taking the x axis in the plane of gravity $\mathbf{g}$ and the magnetic field $\mathbf{B}$, we have $\mathbf{g}=(0,0,-g)$ and $\mathbf{B}=B(m,0,n) \equiv B(\sin\theta,0,\cos\theta)$, where θ is the angle of the magnetic field with the vertical, and B the magnetic field strength (modulus of $\mathbf{B}$). In the absence of rotation, $\mathbf{\Omega}=0$, and presence of a uniform magnetic field, $\mathbf{B}=$const, the linearized momentum equation (7b) and induction equation (8a) read

$$\ddot{\mathbf{v}}-c^2\nabla(\nabla\cdot\mathbf{v})-(\gamma-1)\mathbf{g}(\nabla\cdot\mathbf{v})-\nabla(\mathbf{v}\cdot\mathbf{g}) -(\mu/4\pi\rho)[(\mathbf{B}\cdot\nabla)\mathbf{h}-\nabla(\mathbf{B}\cdot\mathbf{h})]=0\,, \tag{158a}$$

$$\dot{\mathbf{h}}-(\mathbf{B}\cdot\nabla)\mathbf{v}+\mathbf{B}(\nabla\cdot\mathbf{v})=0\,, \tag{158b}$$

in terms of the velocity $\mathbf{v}\equiv\dot{\xi}$ and magnetic field $\mathbf{h}$ perturbations, where the equations of continuity (8b) and adiabaticity (8c) were used to simplify Eq. (7b) into the form (158a). The Maxwell equation $\nabla\cdot\mathbf{h}=0$ implies that, for a vertical wave, $\partial h_z/\partial z=0$ and the vertical component of the magnetic field perturbation h_z is conserved. The horizontal components of the magnetic field perturbation (h_x,h_y) and the three components of the velocity perturbation (v_x,v_y,v_z) satisfy Eqs. (158a) and (158b):

$$\ddot{v}_z-c^2v_z''+\gamma g v_z'+(a^2/B)m\dot{h}_x'=0\,, \tag{159a}$$

$$\ddot{v}_x-(a^2/B)n\dot{h}_x'=0\,, \tag{159b}$$

$$\dot{h}_x+B(mv_z'-nv_x')=0\,, \tag{159c}$$

$$\ddot{v}_y-(a^2/B)n\dot{h}_y'=0\,, \tag{159d}$$

$$\dot{h}_y-nBv_y'=0\,, \tag{159e}$$

where we have introduced the Alfvén speed a [Eq. (38b)], and where the overdot(s) and prime(s) denote derivatives with respect to time t and altitude z, respectively, i.e., $\dot{f}\equiv\partial f/\partial t$ and $f'\equiv\partial f/\partial z$. The time derivatives of the mass density $\bar{\rho}$ [Eq. (8b)] and gas pressure $\bar{p}$ [Eq. (8c)] perturbations are specified, respectively, by

$$\dot{\bar{\rho}}=-(\rho v_z)'\,, \tag{159f}$$

$$\dot{\bar{p}}=-\rho c^2 v_z'+\rho g v_z\,, \tag{159g}$$

in terms of the velocity perturbation v_z and atmospheric density ρ.

3. Decoupling into second-order waves

It is clear that Eqs. (159d) and (159e) decouple from the remaining three equations of the system (159a)—(159e), and eliminating the wave variables v_y,h_y between (d) and (e), we obtain

$$\ddot{v}_y-n^2a^2v_y''=0\,, \tag{160a}$$

$$\ddot{h}_y-n^2(a^2h_x')'=0\,, \tag{160b}$$

which coincide with Eqs. (101a) and (101b) for Alfvén-gravity waves, where the derivative along the magnetic field lines is $\partial/\partial b=\cos\theta\partial/\partial z=n\partial/\partial z$, with θ denoting the angle of the magnetic field with the vertical. The velocity perturbations (v_x,v_z) in the plane of gravity and the external magnetic field $(\mathbf{g},\mathbf{B})$, and the horizontal magnetic field perturbation in the same plane h_x, are generally coupled [Eqs. (159a)—(159c)] and form a fourth-order system in the case of a strictly oblique external magnetic field, i.e., one that is neither vertical nor horizontal. In the case of a vertical magnetic field $(n=1,\ m=0)$, the fourth-order system (159a)—(159c) decouples into two second-order waves: (i) the horizontal velocity v_x and magnetic field h_x satisfy

$$\ddot{v}_x-(a^2/B)\dot{h}_x'=0\,, \tag{161a}$$

$$\dot{h}_x-Bv_x'=0\,, \tag{161b}$$

which lead to the Alfvén-wave equations

$$\ddot{v}_x-a^2v_x''=0\,, \tag{162a}$$

$$\ddot{h}_x-(a^2h_x')'=0\,, \tag{162b}$$

which are indistinguishable from those in the y direction [Eqs. (160a) and (160b)], since in a vertical magnetic field $(n=1)$ all horizontal directions are equivalent; (ii) the vertical velocity v_z lies along the external magnetic field and hence is not affected by it. Thus Eq. (159a),

$$\ddot{v}_z-c^2v_z''+\gamma g v_z'=0\,, \tag{163}$$

coincides with that for vertical acoustic-gravity waves (see Sec. V.A.3 of Part I), or Eq. (32). In the case of a horizontal magnetic field $(n=0,\ m=1)$, the velocity perturbation v_x aligned with the field is conserved, $x=0$, by Eq. (159b), and the system goes from fourth to second order, i.e., the vertical velocity perturbation v_z and horizontal magnetic field perturbation h_x, both in the plane of gravity and the external magnetic field, satisfy Eqs. (159a) and (159c):

$$\ddot{v}_z-c^2v_z''+\gamma g v_z'+(a^2/B)\dot{h}_x'=0\,, \tag{164a}$$

$$\dot{h}_x+Bv_z'=0\,, \tag{164b}$$

which lead to the wave equations

$$\ddot{v}_z-(c^2+a^2)v_z''+\gamma g v_z'=0\,, \tag{165a}$$

$$\ddot{h}_x-[(c^2+a^2)h_x']+\gamma g h_x'=0\,, \tag{165b}$$

corresponding to vertical magnetosonic-gravity waves [Eq. (165a) leads to (123b), for a wave of frequency ω, that is, $v_z=W_z e^{-i\omega t}$]. We have thus shown that vertical waves in an atmosphere, under a uniform, nonoblique magnetic field, are always of second order, viz., Alfvén [Eqs. (162a) and (162b)] and acoustic-gravity [Eq. (163)] waves for a vertical field, and magnetosonic-gravity waves [Eqs. (165a) and (165b)] for a horizontal, external magnetic field.

4. Fourth-order hydromagnetic-gravity waves

In the case of a strictly oblique magnetic field, i.e., one that is neither horizontal ($n \neq 0$) nor vertical ($m \neq 0$), the Alfvén wave perturbations [Eqs. (160a) and (160b)] remain in the direction perpendicular to the gravity and magnetic field, and in the latter plane $(\mathbf{g},\mathbf{B})$ we have a fourth-order hydromagnetic-gravity wave, specified by Eqs. (159a)—(159c), so designated because it includes, as particular cases, all the second-order waves considered before, viz., acoustic-gravity [Eq. (163)], Alfvén [Eqs. (162a) and (162b)], and magnetosonic-gravity [Eqs. (165a) and (165b)]. The system of equations (159a)—(159c) for hydromagnetic-gravity waves can be reduced for any of its three variables, leading to wave equations of the fourth order:

$$\ddddot{v}_z - (c^2+a^2)\ddot{v}_z'' + \gamma g \ddot{v}_z' - n^2 a^2 \gamma g v_z''' + n^2 a^2 (c^2 v_z'')'' = 0 , \tag{166a}$$

$$\ddddot{v}_x - a^2 \ddot{x}_x'' - a^2 (c^2 a^{-2} \ddot{v}_x)'' + \gamma g \ddot{v}_x' + n^2 a^2 (c^2 v_x'')'' - \gamma g n^2 a^2 v_x''' = 0 , \tag{166b}$$

$$\ddddot{h}_x - [(c^2+a^2)\ddot{h}_x']' + \gamma g \ddot{h}_x' - \gamma g m^2 (a^2 h_x')'' + n^2 [c^2 (a^2 h_x')'']' = 0 , \tag{166c}$$

for the vertical [Eq. (166a)] and horizontal [Eq. (166b)] velocities, and horizontal magnetic field [Eq. (166c)] perturbations, respectively. All the wave equations [(160a), (160b), (162a), (162b), (163a), (165a), (165b), (166a), (166b), (166c)] apply to nonisothermal atmospheres under a uniform magnetic field of arbitrary direction, for which both the sound speed c [Eq. (9a)] and Alfvén speed a [Eq. (38a)] are generally nonuniform; in the isothermal case, the sound speed is a constant [Eq. (9a)], but the Alfvén speed is not [Eqs. (99a) and (99b)]. Thus in all cases the wave equations are different for distinct wave variables. For vertical waves $\mathbf{k}_{\|}=0$, the Fourier decomposition (79a) is restricted to frequency:

$$v_z, v_x, h_x(z,t) = \int_{-\infty}^{+\infty} W_z, W_x, H_x(z;\omega) e^{-i\omega t} d\omega , \tag{167}$$

and we choose as wave variable for subsequent study the vertical velocity perturbation spectrum $W_z \equiv W$, for all waves except the Alfvén-gravity mode, for which we choose $W_x \equiv W$ (since in this case W_z does not propagate). The equations for Alfvén [Eq. (162a)], acoustic-gravity [Eq. (163)], magnetosonic-gravity [Eq. (165a)], and hydromagnetic gravity [Eq. (166a)] waves thus yield, respectively,

$$L^2 W'' - L W' + \alpha W = 0 , \tag{168a}$$

$$W'' - \alpha \beta e^{-z/L} W = 0 , \tag{168b}$$

$$(1+\beta e^{-z/L}) L^2 W'' - \beta e^{-z/L} (L W' - \alpha W) = 0 , \tag{168c}$$

$$n^2 (L^4 W'''' - L^3 W''') + \alpha (1+\beta e^{-z/L}) L^2 W'' - \alpha \beta e^{-z/L} (L W' - \alpha W) = 0 , \tag{168d}$$

where W is the appropriate velocity perturbation spectrum, and the parameters

$$\alpha_0 \equiv \omega^2 L^2 / c^2 \sim (2\pi\lambda / L)^2 , \tag{169a}$$

$$\beta_0 \equiv c^2 / a_0^2 = (\gamma/2)(p/P) , \tag{169b}$$

are (i) the square of the acoustic compactness, which compares the wavelength λ of sound to the scale height L and is large in the ray limit $\alpha_0 \gg 1$; (ii) the initial plasma β, or ratio of squares of sound c [Eq. (9a)] and Alfvén a_0 [Eq. (99a)] speeds, which is proportional, through half the adiabatic exponent γ, to the ratio of gas p and magnetic P pressures [Eq. (99b)]. We shall proceed in parallel with all four cases (168a)—(168d). The earlier results on acoustic-gravity waves (Sec. II.A), Alfvén waves (Sec. II.B), and magnetosonic-gravity waves (Sec. II.C) can be used to check the present method as well as to demonstrate that this approach is more straightforward. Moreover, the hydromagnetic-gravity waves [Eq. (167d)] include all of the earlier three modes as particular cases and demonstrate the application of the present method to a higher-order problem.

5. Column mass as altitude variable

All of the preceding wave equations [(168a)—(168d)] can be transformed to the standard form of Eq. (155) by means of a change of variable [Eq. (154a)], where the constant ζ_0 is chosen for convenience to be

$$\zeta = -e^{-z/L}, \quad -\alpha_0 \beta_0 e^{-z/L}, \quad \text{or } -\beta_0 e^{-z/L} , \tag{170a}$$

$$W(z;\omega) = \Phi(\zeta) . \tag{170b}$$

These variables apply to the following cases, respectively: (i) For the acoustic-gravity wave equation (168a), which has constant coefficients, the choice of ζ_0 is immaterial, e.g., we take $\zeta_0 = 1$. (ii) For the Alfvén-gravity wave equation (168b) we take $\zeta_0 = \alpha_0 \beta_0 = (\omega L / a_0)^2$, which is [Eqs. (169a) and (169b)] the square of the compactness for the initial Alfvén speed a_0. (iii) For the magnetosonic-gravity wave [Eq. (168c)] and hydromagnetic-gravity wave [Eq. (168d)], we take $\zeta_0 = \beta_0$, the plasma β at altitude $z=0$, so that $-\zeta = \beta_0 e^{-z/L} = (c/a)^2$ is the plasma β at arbitrary altitude z. The mass of the isothermal atmosphere, per unit area, above the altitude z, is given by

$$m(z) = \int_z^{\infty} \rho(\xi) d\xi = \rho_0 \int_z^{\infty} e^{-z/L} d\xi = \rho_0 L e^{-z/L} , \tag{170c}$$

and thus in all cases described by Eq. (170a) the choice of variable ζ amounts (to within a constant factor) to measuring altitude z on the scale of the column mass [Eq. (170c)]. Since the changes of variable (170a) transform all four wave equations [(168a)—(168d)] into the same standard type [Eq. (155)], each wave is characterized by the polynomials R,S:

$$R_1(\xi)=\xi^2+\xi+\alpha_0=(\xi-\tfrac{1}{2}-iKL)(\xi-\tfrac{1}{2}+iKL)\,, \quad (171a)$$

$$S_1(\xi)=0\,, \quad (171b)$$

$$R_2(\xi)=\xi^2\,, \quad (172a)$$

$$S_2(\xi)=\alpha_0\,, \quad (172b)$$

$$R_3(\xi)=\xi^2\,, \quad (173a)$$

$$S_3(\xi)=\xi^2+\xi+\alpha_0=(\xi-\tfrac{1}{2}-iKL)(\xi-\tfrac{1}{2}+iKL)\,, \quad (173b)$$

$$R_4(\xi)=n^2\xi^2(\xi^2+\xi+\alpha_0/n^2)$$
$$=n^2\xi^2(\xi-\tfrac{1}{2}-iK_0L)(\xi-\tfrac{1}{2}+iK_0L)\,, \quad (174a)$$

$$S_4(\xi)=\alpha_0(\xi^2+\xi+\alpha_0)$$
$$=\alpha_0(\xi-\tfrac{1}{2}-iKL)(\xi-\tfrac{1}{2}+iKL)\,, \quad (174b)$$

for acoustic-gravity [(171a) and (171b)], Alfvén [(172a) and (172b)], magnetosonic-gravity [(173a) and (173b)], and hydromagnetic-gravity [(174a) and (174b)] waves, respectively. $K\equiv\mathrm{Re}(k_\perp)$ denotes the effective vertical wave number, as for acoustic-gravity waves [Eq. (125a)], or magnetosonic-gravity waves in a horizontal magnetic field, and K_0 denotes its modification by an oblique field, $n\neq 1$,

$$KL=(\alpha^2-\tfrac{1}{4})^{1/2}\,, \quad (175a)$$

$$K_0L=(\alpha^2/n^2-\tfrac{1}{4})^{1/2}\,. \quad (175b)$$

The acoustic-gravity wave equation is the only one with constant coefficients, so that one polynomial (171b) vanishes, $S_1=0$; for all other wave equations, with variable coefficients, none of the polynomials vanish. The highest degree of the polynomials R,S is the order of the wave equation, i.e., second order for acoustic-gravity, Alfvén-gravity, and magnetosonic-gravity, and fourth order for hydromagnetic-gravity waves.

6. Scaling laws for amplitude and phase

As a first step in the use of the polynomials R,S [Eqs. (156a) and (156b)] to specify the properties of waves Φ described by Eq. (155), we consider the asymptotic wave fields, at high altitude, as $z\to\infty$ and $\xi\to 0$. The wave equation (151) then reduces to one with constant coefficients, viz.,

$$\sum_{j=0}^{r} a_r d^r w/dz^r=0\,, \quad (176a)$$

$$w(z)\sim e^{-\sigma z/L}=\exp(-\sigma_r z/L)\exp(-i\sigma_s z/L)\,, \quad (176b)$$

where σ is a single root $R(\sigma)=0$ of the polynomial (156a), generally complex $\sigma=\sigma_r+i\sigma_s$. Equations (176a) and (176b) hold asymptotically for wave equations with variable coefficients, e.g., Alfvén-gravity and hydromagnetic-gravity, and at all altitudes for wave equations with constant coefficients, e.g., acoustic-gravity. It shows that the single roots σ of the polynomial $R(\sigma)=0$ specify the asymptotic wave field as follows: (i) the real part $\sigma_r\equiv\mathrm{Re}(\sigma)$ specifies the amplitude law, viz., bounded for $\sigma_r=0$, and exponentially growing/decaying, respectively, for $\sigma_r<0$ or $\sigma_r>0$, with length scale $L/|\sigma_r|$; (ii) the imaginary part $\sigma_s=\mathrm{Im}(\sigma)$ specifies the phase law, viz., standing mode for $\sigma_s=0$, linear phase for $\sigma_s\neq 0$, with upward or downward propagation, respectively, for $\sigma_s<0$ or $\sigma_s>0$, with effective wave number $|\sigma_s|/L$. If σ is a root of multiplicity q of the polynomial $R(\sigma)=0$, the preceding law [Eqs. (176a) and (176b)] is modified by a complex polynomial of degree $q-1$,

$$W(z)\sim\exp(-\sigma_r z/L)\exp(-i\sigma_s z/L)\left[\sum_{j=0}^{q-1}c_j z^j\right]\,, \quad (176c)$$

which affects both amplitude and phase. We apply these results to acoustic-gravity, Alfvén-gravity, and hydromagnetic-gravity waves (deferring further consideration of magnetosonic-gravity modes until Sec. III.C). For acoustic-gravity waves [Eq. (171a)], the polynomial R_1 has two distinct, complex conjugate roots, $\sigma=-\frac{1}{2}\pm iK_0L$, and thus waves [Eq. (176b)] grow exponentially on twice the scale height and have effective wave number K_0 [Eq. (175b)], in agreement with Eq. (81a) for a vertical field $n=1$. For Alfvén-gravity waves [Eq. (172a)], the polynomial R_2 has a double $q=2$ root $\sigma=0$, while the wave field [Eq. (176c)] grows linearly with altitude, in agreement with Eq. (108a), and has a bounded phase, specified by the difference between $\mathrm{Im}(d_1)=\pi/2$ [since d_1 is positive imaginary by Eq. (115a)] and the initial phase. In the case of hydromagnetic-gravity waves [Eq. (174a)], the polynomial R_4 has four roots, viz., the complex-conjugate pair $\sigma=-\frac{1}{2}\pm iKL$ of acoustic-gravity waves, and the double root $\sigma=0$ of Alfvén-gravity waves. Asymptotically, the velocity perturbation of the former component is aligned with the external magnetic field, $v_z/v_x\sim n/m$, i.e., the motion consists of compressions along magnetic field lines, while the velocity perturbation of the latter component is aligned transverse to the external magnetic field, $v_z m+v_x n\sim 0$, i.e., the motion consists of transverse deflections of magnetic field lines. Thus, we conclude that the fourth-order hydromagnetic wave consists of two second-order components coupled together, namely, a dynamic and a magnetic component, resembling asymptotically an acoustic-gravity and an Alfvén-gravity wave, respectively.

7. Kinetic, compression, potential, and magnetic energies

The initial wave fields, at low altitude, in the sense of small z and $\xi\gg 1$, can be obtained (as in Sec. II.B.4) from the ray approximation, provided that the energy density and flux be known. Since the acoustic- and Alfvén-gravity waves are particular cases of the dynamic and magnetic components, respectively, of the hydromagnetic-gravity wave, we need only consider the latter. Its energy density consists of kinetic energy E_v [Eq. (102a)], magnetic energy E_h [Eq. (102b)], potential energy E_g, and compression energy E_p, with the last two specified by

$$E_g=\tfrac{1}{2}\rho(\omega_1/\omega)^2 v_z^2 \;, \tag{177a}$$

$$E_p=\tfrac{1}{2}(c^2/\rho)\bar{p}^2 \;, \tag{177b}$$

where ω_1 [Eq. (22a)] is the gravity (or Brünt-Vaisala) cutoff frequency and $\bar{\rho}$ the density perturbation. For the dynamic component, since the sound speed is a constant, the conservation of the energy flux implies that of the kinetic energy [Eq. (102a)], and so the velocity perturbation $v \sim \rho^{-1/2} \sim e^{z/2L}$ grows exponentially on twice the scale height; the mass density and gas pressure perturbations [Eqs. (159f) and (159g)] decay on the same scale, $\bar{\rho},\bar{p} \sim \rho v_z \sim e^{-z/2L}$. The magnetic field perturbation is a higher-order effect for the dynamic mode, i.e., it decays as the atmospheric density $h_x \sim \rho \sim e^{-z/L}$ exponentially on the scale height, while the magnetic pressure perturbation $P \equiv \mu h_x^2/8\pi \sim e^{-2z/L}$ decays twice as fast. These results apply in four cases: Case A is the dynamic component, whether standing or propagating, at high or low altitude. Case B is the magnetic component at low altitude, for which the argument of Sec. II.B.4 also applies, and shows that the velocity perturbation grows, $v \sim e^{z/4L}$, and magnetic field perturbation decays exponentially, $h \sim e^{-z/4L}$, on four times the scale height [as in Eqs. (107a) and (107b)]. The corresponding results for the mass density and gas pressure, $\bar{\rho},\bar{p} \sim \rho v \sim e^{-3z/4L}$, indicate an exponential decay on $\frac{4}{3}$ the scale height, and for the magnetic pressure $P \sim e^{-z/2L}$ a decay on half the scale height. Case C is the magnetic component at high altitude, with propagating waves, for which the velocity perturbation diverges linearly $v \sim z$ and the magnetic field perturbation tends to a constant $h \sim h_\infty$ [as in Eqs. (108a) and (108b)]; it follows that the mass density and gas pressure decay as $\bar{\rho},\bar{p} \sim \rho v \sim z e^{-z/L}$, and the magnetic pressure tends to a constant $P \sim \mu h_\infty^2/8\pi \equiv E_\infty$. The remaining instance, case D, is the standing magnetic component at high altitude, for which the velocity perturbation is bounded, $v \sim v_\infty$, and thus the mass density, gas pressure, and magnetic field perturbations $\bar{\rho},\bar{p},h \sim \rho \sim e^{-z/L}$ all decay exponentially on the scale height, while the magnetic pressure $P \sim e^{-2z/L}$ decays twice as fast. Note that the dynamic component satisfies the same laws (case A), at low or high altitude, for propagating waves and standing modes; for the magnetic component, we have to distinguish the low-altitude field (case B), from the high-altitude limit, which differs for propagating waves (case C) and standing modes (case D).

8. High-speed particle streams from coronal holes

We have indicated in Table I, for all cases (dynamic and magnetic component, standing or propagating, at low or high altitude) the scaling of wave variables (velocity, magnetic field, mass density, and gas and magnetic pressure perturbations), as well as of the energy densities (kinetic, compression, potential, magnetic, and total). For the dynamic component (case A), the dynamic energy densities (kinetic $E_v \sim E_0$, compression $E_p \sim E_0$, and potential $E_g \sim E_1$) are constant (with equipartition of the first two), and the magnetic energy decays $E_h \sim e^{-z/L}$, so the total energy is constant and equal to the dynamic contribution, $E \sim E_v + E_p + E_g \sim 2E_0 + E_1$. For the magnetic component at low altitude (case B), all energies decay on twice the scale height, $E_v,E_p,E_g,E_h \sim e^{-z/2L}$, i.e., inversely to the Alfvén speed a [Eq. (99a)], and the same applies to the total energy $E = E_v + E_p + E_g + E_h \sim e^{-z/2L}$, for which all four contributions are important. For the standing magnetic component at high altitude (case D), all energies decay, viz., the dynamic $E_v,E_p,E_g \sim e^{-z/L}$ on the scale height, and the magnetic $E_p \sim e^{-2z/L}$ twice as fast, so that the total energy $E \sim E_v + E_p + E_h \sim e^{-z/L}$ is dynamic and decaying. In the remaining instance (case C), for the propagating magnetic component at high altitude, the dynamic energies (kinetic, compression, and potential) decay as $E_v,E_p,E_g \sim z^2 e^{-z/L}$, and the magnetic energy is constant, $E_h \sim \mu_\infty^2/8\pi \equiv E_\infty$, so that the total energy $E \sim E_h \sim E_\infty$ is magnetic and constant. Strictly oblique magnetic fields, i.e., fields that are nowhere vertical or horizontal, occur in the solar atmosphere in open magnetic structures, such as coronal holes (Levine, Altschuler, Harvey, and Jackson, 1977; Munro and Jackson, 1977; Pineau des Forets, 1979; Pneuman, 1980; Raymond and Doyle, 1981; Summers, 1983; Osherovich, Gliner, Tzur, and Kuhn, 1985). The high-speed particle streams observed in the solar wind originate in coronal holes, and it is believed that waves accelerate the particles (Habbal and Leer, 1982; Hu, 1982; Fla, Habbal, Holzer, and Leer, 1984; Davila, 1985); indeed, they have been detected in spacecraft observations (Belcher, Davis, and Smith, 1969; Belcher and Davis, 1971; Denskat and Burlaga, 1977). The waves in high-speed streams in the solar wind have a magnetic energy that is a constant and substantial fraction of the background magnetic field, as for the magnetic mode of hydromagnetic-gravity waves; these waves also have a significant thermal energy contribution, indicating that the dynamic component is also present. The waves are nonsinusoidal, giving evidence of waveform shearing by the nonuniform Alfvén speed, typical of magneto-atmospheric waves. The oscillations are neither purely transversal (Alfvénic) nor purely longitudinal (acoustic) relative to the external magnetic field, showing that both the dynamic and magnetic components of the hydromagnetic-gravity waves are present. Thus we conclude that the observations of waves in high-speed particle streams in the solar wind, issuing from coronal holes, are consistent with the properties of fourth-order hydromagnetic-gravity waves, with coupled second-order dynamic and magnetic components.

B. Exact solutions at all altitudes and frequencies

We have argued that the two components of the solar wind (Armstrong and Woo, 1981; Wu, Steinholfson, and Tandberg-Hansen, 1981; Bruecker and Bartoe, 1983; Eyni and Steinitz, 1983; Cuperman, Tzur, and Dryer, 1984; Axford, 1985) have distinct origins: (i) the average solar wind (Parker, 1960, 1965; Hollweg, 1970) results from the

TABLE I. Evolution of wave variables and energies: Case A, dynamic component, standing or propagating, low or high altitude; Case B, magnetic component, standing or propagating, low altitude; Case C, magnetic component, propagating, high altitude; Case D, magnetic component, standing, high altitude.

Field component	Dynamic	Magnetic	
Wave type	Standing or propagating	Standing mode	Propagating wave
Low altitude:	Case A	Case B	Case B
Wave amplitude			
Fluid velocity	$e^{z/2L}$	$e^{z/4L}$	$e^{z/4L}$
Magnetic field	$e^{-z/2L}$	$e^{-z/4L}$	$e^{-z/4L}$
Mass density	$e^{-z/2L}$	$e^{-3z/4L}$	$e^{-3z/4L}$
Gas pressure	$e^{-z/2L}$	$e^{-3z/4L}$	$e^{-3z/4L}$
Magnetic pressure	$e^{-z/L}$	$e^{-z/2L}$	$e^{-z/2L}$
Energy density			
Kinetic	E_0	$e^{-z/2L}$	$e^{-z/2L}$
Compression	E_0	$e^{-z/2L}$	$e^{-z/2L}$
Potential	E_1	$e^{-z/2L}$	$e^{-z/2L}$
Magnetic	$e^{-z/L}$	$e^{-z/2L}$	$e^{-z/2L}$
Total	$2E_0+E_1$	$e^{-z/2L}$	$e^{-z/2L}$
High altitutde	Case A	Case D	Case C
Wave amplitude			
Fluid velocity	$e^{z/2L}$	v_∞	z
Magnetic field	$e^{-z/2L}$	$e^{-z/L}$	h_∞
Mass density	$e^{-z/2L}$	$e^{-z/L}$	$ze^{-z/L}$
Gas pressure	$e^{-z/2L}$	$e^{-z/L}$	$ze^{-z/L}$
Magnetic pressure	$e^{-z/L}$	$e^{-2z/L}$	$\mu h_\infty^2/8\pi \equiv E_\infty$
Energy density			
Kinetic	E_0	$e^{-z/L}$	$z^2e^{-z/L}$
Compression	E_0	$e^{-z/L}$	$z^2e^{-z/L}$
Potential	E_1	$e^{-z/L}$	$z^2e^{-z/L}$
Magnetic	$e^{-z/L}$	$e^{-2z/L}$	E_∞
Total	$2E_0+E_1$	$e^{-z/L}$	E_∞

thermal expansion of the corona (Newkirk, 1967; Suess, 1982; Osherovich, Gliner, and Tzur, 1985), whose mass loss is resupplied by acoustic-gravity waves in spicules (Sec. II.A.8); (ii) the high-speed particle streams originate from coronal holes, where the hydromagnetic-gravity waves accelerate matter as they propagate outwards with the solar wind (Sec. III.A.8). Stars other than the sun also have a mass loss, and in some cases the stellar winds have much larger mass fluxes (Lago, 1982; Penston and Lago, 1983; Lago, Penston, and Johnstone, 1985; Sá, Penston, and Lago, 1986), with evidence of acceleration by hydromagnetic waves of Alfvénic type (Underhill, 1963; Lago, 1984). In solar spicules, the magnetic field is nearly vertical, and thus (Sec. III.A.3) acoustic-gravity and Alfvén-gravity waves could coexist. The acoustic-gravity waves have a vertical velocity perturbation and are associated with a compression front, which carries the mass flux. Spicules are nearly isothermal, and thus the phase u and group w velocities may be calculated from the effective wave number $K=\mathrm{Re}(k_\perp)$ in Eqs. (125b) or (175a) as follows:

$$u_0=\omega/K=c\sqrt{1-(\omega_2/\omega)^2}\ , \tag{178a}$$

$$w_0=\partial\omega/\partial K=c/\sqrt{1-(\omega_2/\omega)^2}\ . \tag{178b}$$

The temperature $T=1.8\times10^4$ K in spicules corresponds [Eqs. (9a) and (15a)] to a sound speed $c=1.58\times10^6$ cm s^{-1} and scale height $L=5.47\times10^7$ cm, and thus [Eq. (22b)] to a cutoff period $\tau_2=4\pi L/c=420$ s. For a wave with a five-minute period, $\tau=300$ s, we have $\omega_2/u=\tau/\tau_2=0.71$. Note that the group velocity [Eq. (178b)] of the energy flux, $w_0=11$ km/s, is always lower than the sound speed, $c=16$ km/s, while the phase speed [Eq. (178a)] of the compression front, $u_0=22$ km/s, is always larger, $u_0>c>w_0$. The speed of the compression front, $u_0=22$ km/s, agrees with the observed velocity of mass motions, 20—30 km/s, which is remarkably constant over the considerable height (10^4 km) of spicules (Beckers, 1968, 1972). These velocities are vertical. Horizontal velocities, increasing linearly with altitude, have also been observed (Kulidzanishvili, 1980), $v_x=c_1z+c_2$ with $c_1=9.3\times10^{-2}$ s^{-1} and $c_2=-1.4\times10^5$ cm s^{-1}. These horizontal perturbations can be explained by an Alfvén-gravity wave, propagating vertically, for which the asymptotic velocity [Eqs. (108a) and (115a)] is also linear. The coefficients d_1,d_2 are given by

$$H_0^{(2)}(2\omega L/a_0)\{d_1,d_2\}=v_0/\pi L,(2v_0/\pi)[\gamma_*+\ln(2\omega L/a_0)]\ . \tag{179}$$

These coefficients, for an initial Alfvén speed $a_0 = 1.52\times 10^7$ cm s^{-1} and velocity perturbation $v_0 = 1.00\times 10^5$ cm s^{-1} in spicules, lead to the values $c_1 = 7.5\times 10^{-2}$ s^{-1} and $c_2 = -1.07\times 10^5$ cm s^{-1}, which are broadly consistent with observation.

1. Regular singularity and series expansion

The asymptotic laws for the wave fields can be deduced qualitatively [see Eq. (108a)] by simple arguments (as in Sec. II.B.5), but the explicit form of the coefficients, Eq. (179), is determined by taking the limiting form of an exact solution, as in Sec. II.B.6. The present method can be used not only to yield asymptotic approximations (Sec. III.A.6), but also to provide exact solutions, valid at all altitudes and for all frequencies, for waves of any order. To show this, let us take as applications second-order Alfvén and fourth-order hydromagnetic-gravity waves. The general wave equation (157) can be written in the form

$$\zeta^N d^N\Phi/d\zeta^N + \sum_{j=0}^{N-1} \zeta^j p_j(\zeta) d^j\Phi/d\zeta^j = 0 , \tag{180a}$$

$$p_j(\zeta) \equiv (A_j + B_j\zeta)/(A_N + B_N\zeta) = A_j/A_N[1 + O(\zeta)] , \tag{180b}$$

where the functions (180b) are analytic at the origin $\zeta = 0$, which is thus a regular singularity of the Nth-order differential equation (180a). This is a generalization of Eqs. (130), (133a), and (133b), which for a regular singular point $z = z_c$ of a second-order equation reads

$$(z - z_c)^2 w'' + (z - z_c)p_1(z)w' + p_0(z)w = 0 , \tag{181}$$

with p_0, p_1 analytic functions of z. As before in Eq. (135a), the general wave equation (155), now of arbitrary order, has a power series solution:

$$\Phi_\sigma(\zeta) = \zeta^\sigma \sum_{j=0}^{\infty} c_j \zeta^j , \tag{182}$$

where the exponent of the leading power, called the index σ, and the recurrence formula for the coefficients, expressing all of $c_1, c_2, \ldots$, in terms of c_0, are to be determined. Note that c_0 is undetermined, since Eq. (155) is linear, and thus we can set $c_0 \equiv 1$; the leading term ζ^σ corresponds, by Eq. (154a), to the asymptotic approximation (176b) considered in Sec. III.A.6. and we need all of the following terms for an exact solution.

2. Single indices and functions of the first kind

The operator $\zeta d/d\zeta$ has the property that, when applied to a power ζ^σ, it yields the same power multiplied by the exponent σ, that is, $\zeta(d/d\zeta)\zeta^\sigma = \sigma\zeta^\sigma$. This property extends to an arbitrary polynomial of the same operator, $R(\zeta d/d\zeta)\zeta^\sigma = R(\sigma)\zeta^\sigma$, and renders straightforward the substitution of the power series (182) into the wave equation (155) of arbitrary order:

$$\begin{aligned} 0 &= \sum_{j=0}^{\infty} c_j[R(\sigma+j)\zeta^{\sigma+j} - S(\sigma+j)\zeta^{\sigma+j+1}] \\ &= c_0 R(\sigma)\zeta^\sigma + \sum_{j=1}^{\infty} \zeta^{\sigma+j}[c_j R(\sigma+j) - c_{j-1}S(\sigma+j-1)] . \end{aligned} \tag{183}$$

Equating to zero the coefficients of $\zeta^\sigma, \zeta^{\sigma+1}, \ldots$, we obtain the recurrence formula for the coefficients:

$$\begin{aligned} c_j &= [S(\sigma+j-1)/R(\sigma+j)]c_{j-1} \\ &= \prod_{l=1}^{j} [S(\sigma+l-1)/R(\sigma+j)] , \end{aligned} \tag{184a}$$

using $c_0 = 1$. Note that if $c_0 = 0$ we would have all $c_j = 0$, and a trivial solution $\Phi = 0$ would result [Eq. (182)]. Equating to zero the coefficient of ζ^σ, we have $c_0 R(\sigma) = 0$, and since $c_0 \neq 0$, the index σ is a root σ_j of the polynomial (156a):

$$0 = R(\sigma) = \sum_{j=0}^{r} a_j(-\sigma)^j = (-)^r a_r \prod_{j=1}^{r} (\sigma - \sigma_j) . \tag{184b}$$

The condition $R(\sigma) = 0$ is precisely the same as that used in Sec. III.A.6 for the asymptotic expansion [Eq. (176a)] in $z \to \infty$, which is the leading term (154a) in $\zeta \to 0$. Substituting Eq. (184a) into (182) we conclude that the wave equation (155) has, for particular integrals, functions of the first kind:

$$\Phi_j(\zeta) \equiv \zeta^{\sigma_j} \left[1 + \sum_{j=1}^{\infty} \zeta^j \prod_{l=1}^{j} \frac{S(\sigma_j + l - 1)}{R(\sigma_j + l)} \right] , \tag{185a}$$

which have at most a power-type singularity whose exponent σ_j is a root of Eq. (184b). If $R(\sigma)$ has only single roots, i.e., if $\sigma_j = \sigma_l$ for $j \neq l$, the particular integrals Φ_j, Φ_l are linearly independent, and the general integral of Eq. (155) is

$$\Phi(\zeta) = \sum_{j=1}^{r} C_j \Phi_j(\zeta) , \tag{185b}$$

where the C_j are arbitrary constants of integration.

3. Convergence and self-transformation of the equation

We have yet to prove the convergence of the series (185a) for a function of the first kind, whose ratio of succeeding coefficients (184a) is $c_j/c_{j-1} \sim j^{s-r} \to 0$, as $j \to \infty$, provided that $r > s$. Thus, if the polynomial R is of greater degree than S, the series (185a) for a function of the first kind converges for all $\zeta < \infty$. Moreover, in this case Eq. (155) is of degree r, so that the general integral (185b) has r constants of integration $(C_1, \ldots, C_r)$, as it should have. If, instead, we had $s > r$, then $c_j/c_{j-1} \sim j^{s-r} \to \infty$ as $j \to \infty$, and the series (185a) would diverge for all $\zeta \neq 0$. The reason is that, in the case $s > r$, the variable ζ is not suitable, as is also indicated by the fact that Eq. (185b) has a number of constants of integration r less than the order s of Eq. (155). In the case $s>r$, we should use the variable

$$\xi=1/\zeta \ , \tag{186a}$$

$$\overline{\Phi}(\xi)=\Phi(\zeta) \ , \tag{186b}$$

which transforms Eq. (155) into another of the same type,

$$[S(-\xi d/d\xi)-\xi R(-\xi d/d\xi)]\overline{\Phi}(\xi)=0 \ , \tag{187a}$$

with the roles of the polynomials R,S interchanged. Thus the particular integrals of Eq. (187) are

$$\overline{\Phi}_j(1/\zeta)=\zeta^{-\nu_j}\left[1+\sum_{j=1}^{\infty}\zeta^{-j}\prod_{l=1}^{j}[R(1-l-\nu_j)/S(-l-\nu_j)]\right] , \tag{187b}$$

by analogy with Eq. (185a), where the indices ν_j are now the roots of the polynomial S,

$$0=S(-\nu)=\sum_{j=0}^{s}b_j\nu^j=b_s\prod_{j=1}^{s}(\nu-\nu_j) \ . \tag{188a}$$

The ratio of succeeding coefficients of Eq. (187a) is

$$R(1-l-\nu_j)/S(-l-\nu_j)\sim j^{r-s}\to 0$$

as $j\to\infty$, since $s>r$, providing that the series (187a) converges for all $\zeta\neq 0$. If all the roots of (188a) are distinct, the particular integrals (187a) are linearly independent, and the general integral is given by

$$\Phi(\zeta)=\sum_{j=1}^{s}\overline{C}_j\overline{\Phi}_j(1/\zeta) \ , \tag{188b}$$

where the number of constants of integration $\overline{C}_j$ equals the order $s\,(>r)$ of Eq. (155).

4. Multiple indices and functions of the second kind

We have obtained solutions of the wave equations [(155), (156a) and (156b)] in the cases $r>s$ when the polynomial R is of higher degree than s [Eqs. (184b), (185a), and (185b)], and also in the reverse case $s>r$ [Eqs. (187b), (188a), and (188b)] (we defer until Sec. III.C. the consideration of $s=r$). Both general integrals (185b) and (188b) involve only functions of the first kind, on the assumption that indicial equations, (184b) and (188a), respectively, have all roots distinct. If a root, say σ_0 of Eq. (184b), is of multiplicity $q\geq 2$, then the corresponding q constants of integration in Eq. (185b) coalesce into one, i.e., it cannot be the general integral of an equation of order r, since only $r-q+1$ constants of integration remain. In the case of a root of multiplicity q, we need to find $(q-1)$ new particular integrals, which will turn out to be functions of the second kind Ψ. They must also be linearly independent, both among themselves, and from functions of the first kind. Functions of the second kind are also needed if two indices differ by an integer, e.g., $\sigma_2-\sigma_1=m>0$, since in this case $\Phi_1(\zeta)$ involves a factor $R(\sigma_1+m)=R(\sigma_2)=0$ in the denominator, and the solution of the first kind [Eq. (185a)] breaks down, being replaced by one of the second kind. The solution of the first kind, which corresponds to single roots of Eq. (184b), is associated with an exponential asymptotic wave field, and the solution of the second kind, which corresponds to multiple roots of Eq. (184b), introduces the polynomial modification (176c) in the asymptotic law. In order to introduce the function of the second kind exactly, rather than just to leading order, we start from the wave equation (155), and seek a solution of the form (182), with coefficients c_j satisfying the recurrence relation (184a) and arbitrary index σ, so that the whole equation equals the first term of Eq. (183):

$$[R(\zeta d/d\zeta)-\zeta S(\zeta d/d\zeta)]\Phi_\sigma(\zeta)=c_0\zeta^\sigma R(\sigma) \ . \tag{189}$$

Taking σ to be any single root σ_j of Eq. (184b), we obtain the corresponding function of the first kind (185a). If σ_0 is a root of multiplicity $q\geq 2$, then

$$R(\sigma)=O((\sigma-\sigma_0)^q) \ , \tag{190a}$$

$$\lim_{\sigma\to\sigma_0}\partial^l[R(\sigma)]/\partial\sigma^l=0 \ , \tag{190b}$$

for all $l=0,1,\ldots,q-1$. Thus, differentiating Eq. (189) l times with regard to σ, and letting $\sigma\to\sigma_0$, we find that the rhs vanishes, and we obtain q solutions of Eq. (155):

$$[R(\zeta d/d\zeta)-\zeta S(\zeta d/d\zeta)]\Psi_\sigma^{(l)}(\zeta)=0 \ , \tag{191a}$$

$$l=0,1,\ldots,q-1, \quad \Psi_{\sigma_0}^{(l)}(\zeta)\equiv\lim_{\sigma\to\sigma_0}\partial^l[\Phi_0(\zeta)]/\partial\sigma^l \ , \tag{191b}$$

where the function $\Psi_\sigma^{(0)}(\zeta)=\Phi_\sigma(\zeta)$ is of the first kind, and $\Psi_\sigma^{(l)}(\zeta)$ for all other $l=1,\ldots,q-1$ are of the second kind.

5. Logarithmic singularities and complementary functions

From Eq. (191b) it follows that the leading term of the lth function of the second kind is

$$\Psi_{\sigma_0}^{(l)}(\zeta)\sim\lim_{\sigma\to\sigma_0}\partial^l(\zeta^\sigma)/\partial\sigma^l\sim(\ln\zeta)^l\zeta^{\sigma_0} \ , \tag{192}$$

in agreement with Eqs. (154a) and (176c). Equation (192) also shows that all the particular integrals are linearly independent, both from functions of the first kind (different σ_0) and from other functions of the second kind (different l). The lth function of the second kind,

$$\Psi_\sigma^{(l)}(\zeta)=(\ln\zeta)^l\Phi_\sigma(\zeta)+\sum_{m=0}^{l-1}(\ln\zeta)^m\chi_\sigma^{(l,m)}(\zeta) \ , \tag{193}$$

consists of the function of the first kind, $\Phi_\sigma(\zeta)\sim O(\zeta^\sigma)$, multiplied by a logarithmic singularity to the lth power, followed by "less singular" terms; the latter are a sum of powers of logarithms $(\ln\zeta)^m$, with $m=0,\ldots,l-1$, multiplied by succeeding complementary functions $\chi_\sigma^{(l,m)}(\zeta)\sim O(\zeta^{\sigma+m})$, each of which is a power series converging for all $\zeta<\infty$ as a function of the first kind, but of higher order, $\chi_\sigma^{(l,m)}(\zeta)\sim\zeta^m\Phi_\sigma(\zeta)$. The simplest (and most common) case is that of a double root, $q=2$, for which we have [Eqs. (182) and (184a)] a function of the first kind,

$$\Phi_\sigma(\zeta)\equiv\psi_\sigma^{(0)}(\zeta)=\zeta^\sigma\left[1+\sum_{j=1}^{\infty}\zeta^j\prod_{l=1}^{j}[S(\sigma+l-1)/R(\sigma+l)]\right], \tag{194}$$

and one function of the second kind,

$$\Psi_\sigma(\zeta)\equiv\Psi_\sigma^{(1)}(\zeta)=\partial[\Phi_\sigma(\zeta)]/\partial\sigma=(\ln/\zeta)\Phi_\sigma(\zeta)+\chi_\sigma(\zeta)\ . \tag{195a}$$

Here a simple logarithmic singularity multiplies the function of the first kind [Eq. (194)], to which is added the complementary function

$$\chi_\sigma(\zeta)\equiv\chi_\sigma^{(1,1)}(\zeta)=\sum_{j=1}^{\infty}\zeta^j\prod_{l=1}^{j}[S(\sigma+l-1)/R(\sigma+l)]\sum_{m=1}^{l}[S'(\sigma+l-1)/S(\sigma+l-1)-R'(\sigma+l)/R(\sigma+l)]\ , \tag{195b}$$

where a prime denotes derivative with regard to σ, or the argument of the polynomial, e.g., $R'(\xi)\equiv dR/d\xi$.

6. Radiation conditions and functions of the third kind

For high-frequency waves at low altitude, the ray approximation implies that functions of both the first and second kinds are a superposition of downward- and upward-propagating waves:

$$\Phi(\zeta),\psi(\zeta)\sim(C_{11},C_{12})e^{+ikz}+(C_{21},C_{22})e^{-ikz}\ . \tag{196a}$$

The functions of the third kind,

$$\Theta^{(1,2)}(\zeta)\equiv\Phi(\zeta)-(C_{11}/C_{12},C_{21}/C_{22})\Psi(\zeta)\ , \tag{196b}$$

are a linear combination of the first and second kinds, such that $\Theta^{(2)}/\Theta^{(1)}$ satisfy the radiation condition in the direction of decreasing or increasing ζ, respectively, i.e., upward or downward propagating:

$$\Theta^{(2,1)}(\zeta)\sim(\Delta/C_{22},-\Delta/C_{12})e^{\pm ikz}\ , \tag{197a}$$

$$\Delta\equiv C_{11}C_{22}-C_{12}C_{21}\ . \tag{197b}$$

Functions of the third kind have a logarithmic singularity,

$$\Theta^{(2,1)}(\zeta)\sim\zeta^\sigma[1-(C_{11}/C_{12},C_{21}/C_{22})\ln\zeta]\ , \tag{198}$$

like functions of the second kind. As an example of the use of the three kinds of functions, we consider Alfvén waves, which are described by Eq. (155), with the polynomials (172a) and (172b). The polynomial R_2 is of higher degree, $r=2$, than the polynomial S_2, which is a constant, $s=0$, and the appropriate indicial equation (184b), $\sigma^2=0$, has a double root. The function of the first kind [Eq. (185a)] is

$$\Phi_0(\zeta)=1+\sum_{j=1}^{\infty}\zeta^j\prod_{l=1}^{j}(\alpha/l^2)=\sum_{j=0}^{\infty}(\alpha\zeta)^j(j!)^{-2}=J_0(2\sqrt{-\alpha\zeta})=J_0[(2\omega L/a_0)e^{-z/2L}]\ , \tag{199}$$

which is a Bessel function of variable

$$2\sqrt{-\alpha\zeta}=2\sqrt{\alpha\beta}e^{-z/L}=(2\omega L/a_0)e^{-z/2L}\ ,$$

where we have used Eqs. (170a), (169a), and (169b). The complementary function (195b), together with the function of the first kind (199) specify the Neumann function:

$$\chi_0(\zeta)=1+\sum_{j=1}^{\infty}\zeta^j\prod_{l=1}^{j}(\alpha/l^2)\sum_{m=1}^{l}(2/m)=\sum_{j=0}^{\infty}(\alpha\zeta)^j(j!)^{-2}2\psi(j+1)=Z_0(2\sqrt{-\alpha\zeta})\ , \tag{200a}$$

$$\Psi_0(\zeta)=\ln(2\sqrt{-\alpha\zeta})J_0(2\sqrt{-\alpha\zeta})+Z_0(2\sqrt{-\alpha\zeta})=Y_0[(2\omega L/a_0)e^{-z/2L}]\ . \tag{200b}$$

The linear combination of functions of the first (199) and second (200b) kinds, for propagation in the direction of decreasing or increasing, respectively, i.e., upward or downward propagating, are Hankel functions:

$$H_0^{(2,1)}(2\sqrt{-\alpha\zeta})=J_0(2\sqrt{-\alpha\zeta})\pm iY_0(2\sqrt{-\alpha\zeta})=H_0^{(2,1)}[(2\omega L/a_0)e^{-z/2L}]\sim e^{z/4L}\exp(\pm i\omega z/a_0)\ , \tag{201}$$

which correspond to functions of the third kind (196b), with $iC_{11}=C_{12}$, $C_{21}=iC_{22}$. The Alfvén-gravity wave field is specified by a linear combination of functions of first (199) and second (200b) kinds, or of the third kind (201), in agreement with Eqs. (112a) and (112b) with the factor $\cos\theta$ in Eq. (112b) equal to unity for vertical propagation $\theta=0$.

7. Coupled dynamic and magnetic wave components

In the general case of hydromagnetic-gravity waves in an oblique magnetic field, the standard equation (155) has a polynomial R_4 of degree $r=4$ [Eq. (174a)], and another S_4 of degree $s=2$ [Eq. (174b)], so the appropriate indicial equation is Eq. (184b). It has four roots, viz., a complex-conjugate pair $\sigma_{1,2}=-\frac{1}{2}\pm iK_0L$ and a double root $\sigma_{3,4}=0$, which specify the dynamic and magnetic components, respectively, of the wave field. Concerning the latter, the function of the first kind (194) is

$$\Phi_0(\zeta)=1+\sum_{j=1}^{\infty}(\alpha_0\zeta/n^2)^j\prod_{l=1}^{j}[(l^2-l+\alpha_0)/l^2(l^2+l+\alpha_0/n^2)] , \tag{202a}$$

or, using Eqs. (169a) and (169b), (170a), and (170b), in terms of altitude,

$$F_0(z/L)\equiv\Phi_0(\zeta)=1+\sum_{j=1}^{\infty}(\omega L/a_0 n)^{2j}(j!)^{-1}e^{-jz/L}\prod_{l=1}^{j}[(l-\tfrac{1}{2}-iKL)(l-\tfrac{1}{2}+iKL)/(l+\tfrac{1}{2}+iK_0L)(l+\tfrac{1}{2}-iK_0L)] , \tag{202b}$$

showing that it is a standing mode (real, no phase), with bounded amplitude $F(z/L)\to 1$ as $z\to\infty$. The function of the second kind,

$$\Psi_0(\zeta)=\Phi_0(\zeta)(\ln\zeta)+X_0(\zeta)=X_0(-\beta e^{-z/L})+\ln(\beta-i\pi-z/L)F_0(z/L)\equiv H(z/L) , \tag{203a}$$

$$X_0(\zeta)\equiv\sum_{j=1}^{\infty}\zeta^j\prod_{l=1}^{j}[S(l-1)/R(l)]\sum_{m=1}^{l}[S'(m-1)/S(m-1)-R'(m)/R(m)] , \tag{203b}$$

involves a complementary function (203b) of higher order, $X_0(\zeta)\sim 0(\zeta)$, than the function of first kind, $\Phi_0(\zeta)\sim O(1)$. The asymptotic field is specified by the term $\ln\zeta=-z/L+i\pi+\ln\beta$, implying a linearly diverging amplitude and bounded phase. For the dynamic component the complex-conjugate roots $\sigma_{1,2}=-\frac{1}{2}\pm iK_0L$ are distinct, and only functions of the first kind [Eq. (194)] appear:

$$\Phi_\pm(\zeta)=\zeta^{-1/2\pm iK_0L}\left[1+\sum_{j=1}^{\infty}(\alpha_0\zeta/n^2)^j e^{-jz/L}\prod_{l=1}^{j}[S(l-\tfrac{3}{2}\pm iK_0L)/R(l-\tfrac{1}{2}\pm iK_0L)]\right] , \tag{204a}$$

i.e., they are complex conjugates $\Phi_+^*=\Phi_-$. Using Eqs. (169a) and (169b), (170a) and (170b), and (175a) and (175b), we may give these as functions of altitude, by

$$G_\pm(z/L)\equiv(-\beta)^{-1/2\pm iK_0L}\Phi_\pm(\zeta)=e^{z/2L}e^{\pm iK_0z}\times\left[1+\sum_{j=1}^{\infty}(\omega L/a_0n)^{2j}(j!)^{-2}e^{-jz/L}\times\prod_{l=1}^{j}\{[(l-1\pm iK_0L)^2+K^2L^2]/[(l-\tfrac{1}{2}\mp iK_0L)(l\mp 2iK_0L)]\}\right] . \tag{204b}$$

In other words, to leading order, $G_\pm$ are upward- or downward-propagating acoustic-gravity waves. Thus the dynamic [Eqs. (204a) and (204b)] and magnetic [Eqs. (202a) and (202b) and (203a) and (203b)] components of the fourth-order hydromagnetic-gravity wave appear to leading order as acoustic-gravity and Alfvén-gravity waves, respectively, with coupling in all terms of higher order.

8. Application of boundary and initial conditions

The vertical velocity perturbation spectrum of a hydromagnetic-gravity wave is given generally by a linear combination of the standing [Eq. (202b)] and propagating [Eq. (203a)] magnetic components and the upward G_+ and downward G_- dynamic components:

$$W_z(z;\omega)=A_1F(z/L)+A_2H(z/L)+A_3G_+(z/L)+A_4G_-(z/L) , \tag{205}$$

where the constants A_1 to A_4 are determined by boundary, initial, or radiation conditions. For example, for a standing mode we suppress $H(z/L)$ by setting $A_2=0$, and combine the complex-conjugate pair $G_+=G_-^*$ into the real expression $G_++G_-=2\,\mathrm{Re}(G_\pm)$ by setting $A_4=A_3$, leaving only two constants A_1,A_3 to be determined,

$$W_z(z;\omega)=A_1F(z/L)+A_3[G_+(z/L)+G_-(z/L)] . \tag{206a}$$

For an upward- or downward-propagating wave we select $G_\pm$ for the dynamic component and the appropriate function of the third kind [Eq. (196b)] for the magnetic component:

$$W_z(z;\omega)=A_1\Theta^{(2,1)}(-\beta e^{-jz/L})+A_3G_\pm(z/L) . \tag{206b}$$

In both the standing [Eq. (206a)] and propagating [Eq. (206b)] cases, the two remaining constants A_1,A_3 could be determined from initial conditions, e.g., specifying the initial spectrum of the vertical $W_z(0;\omega)$ and horizontal $W_x(0;\omega)$ velocity perturbation spectra. In general, in a multilayer model, the solution (205) would be used in each layer, along with four boundary conditions, applied at each interface between two layers: (i) and (ii) continuity of horizontal W_x and vertical W_z velocity perturbations (assuming there is no shear flow); (iii) continuity of total pressure, i.e., gas P_g plus magnetic $P_h=\mu H_x^2/8$; (iv) continuity of the horizontal magnetic field component H_x, if there are no surface electric currents. In the presence of surface electric currents J_0 the horizontal magnetic field would have a jump $[H_x]=(c_*/4\pi)J_0$.

C. Wave filtering, absorption, and transformation

The components of the hydromagnetic-gravity wave, both dynamic [Eq. (204b)] and magnetic [standing (202b) or propagating (203a)], are given by expressions that are exact and that converge for all altitudes and values of parameters. The convergence is rapid, both in altitude, since the series proceed in powers of Eq. (154a), $[(e^{-z/L})^j=e^{-jz/L}]$ and in order, since the jth coefficient scales like $c_j \sim (j!)^{-2}$, which is a convergence faster than the ordinary transcendental functions $c_j \sim (j!)^{-1}$. Thus the series can be calculated rapidly (Campos and Leitão, 1987), to high accuracy, with a few terms; for example, the plots in Fig. 7 took less than one second computing time in an IBM 8031 computer. The hydromagnetic-gravity wave fields depend on three parameters, namely, the compactness or dimensionless frequency [Eq. (169a)], the initial plasma β, defined for Eq. (169b) at altitude $z=0$, and the angle θ of inclination of the magnetic field to the vertical, $n \equiv \cos\theta$. We choose for illustration in Fig. 7 the standing magnetic component, whose amplitude is bounded at infinity and normalized to unity. The basic case is taken to be

$$\Omega_* = \omega L/c = 3 , \tag{207a}$$

$$\beta = (c/a_0)^2 = 1.0 , \tag{207b}$$

$$\theta = 45^\circ , \tag{207c}$$

i.e., a wave of dimensionless frequency $\Omega_* = 6(\omega/\omega_2)$ equal to 6 times the cutoff $\omega_2 = c/2L$, in an atmosphere for which the sound and Alfvén speeds are equal to altitude $z=0$ (207b), in the presence of a magnetic field of intermediate inclination $\theta=45°$. We change each parameter in turn, giving the values

$$\Omega_* = \omega/2\omega_2 = 1,2,3,4,5 , \tag{208a}$$

$$\beta = 0.1, 0.5, 1.0, 2.0, 10.0 , \tag{208b}$$

$$\theta = 15°, 30°, 45°, 60°, 75° . \tag{208c}$$

Thus we allow frequencies (208a) from twice the cutoff $\omega=2\omega_2$ to the ray limit $\omega=10\omega_2$, a plasma β (208b) ranging from initial dominance of magnetic $\beta \ll 1$ to initial dominance of gas $\beta \gg 1$ pressure, and five equally spaced inclinations (208c) to the magnetic field.

1. Effects of frequency, plasma β, and inclination

The effects of changing each of the three parameters in turn are illustrated in Fig. 7, as follows:

(a) As the frequency increases (left-hand side), the waveform oscillates more at low altitude $z \le 2L$, where $\beta e^{-z/L} \sim 1$ and the gas and magnetic pressure are comparable, but as the magnetic pressure predominates, $z > 3L$, the magnetic field resists transverse oscillations and the amplitude becomes almost uniform (it can be seen that there is a node at $z=0$ for $\Omega_*=3$, i.e., the first standing mode of the magnetic component of vertical hydromagnetic-gravity waves has the frequency $\omega = 6\omega_2 = 3c/L$, and the second mode a frequency

STANDING HYDROMAGNETIC-GRAVITY MODES

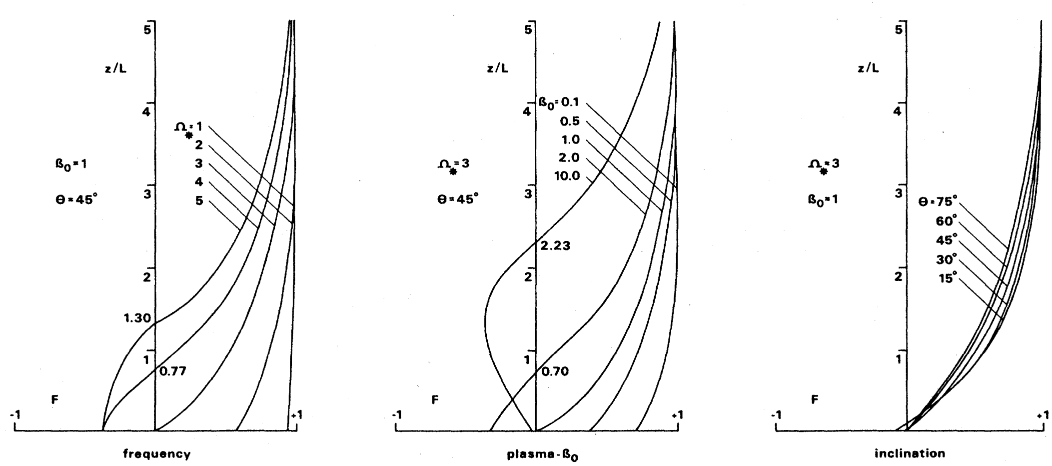

FIG. 7. Waveforms of the magnetic component of a fourth-order hydromagnetic-gravity wave, standing vertically in an atmosphere; the reference case is a wave of frequency ω, 3 times the cutoff value ω_1, i.e., $\omega=3\omega_1$, an initial plasma β (at altitude $z=0$) equal to unity, $\beta_0=1$, and a uniform external magnetic field at equal angle to horizontal and vertical, $\theta=45°$. The plots, vs altitude z divided by the scale height L, demonstrate the effects of changing (left) wave frequency to $\Omega_* \equiv \omega/\omega_1 = 1,2,3,4,5$, (center) initial plasma β to $\beta_0 = 0.1, 0.5, 1.0, 2.0, 10.0$, and (right) angle of inclination of magnetic field to the vertical to $\theta = 15°, 30°, 45°, 60°, 75°$.

$\omega > 10\omega_2 = 5c/L$, for initial plasma β unity and magnetic field tilted at equal angles to horizontal and vertical).

(b) As the initial plasma β decreases (center), the altitude at which sound c and Alfvén speed $a_0 e^{z/2L}$ become equal, $c = a(z_c)$, reduces, $z_c = 2L \ln(c/a_0) = L \ln\beta$, and the altitude of equal gas $p(z) = \gamma\rho_0 c^2 e^{-z/L}$ and magnetic $P = \rho_0 a^2/2$ pressures $p(z_p) = P$ also reduces, $z_p = L \ln(\gamma c^2/2a_0^2) = L \ln(\gamma\beta/2)$, so that, for small initial β ($\beta \ll 1$) the magnetic pressure dominates throughout, and the wave amplitude is almost uniform. For large initial β ($\beta \gg 1$) there is oscillation while the decaying gas pressure $p \sim e^{-z/L}$ is still larger than or comparable to the constant magnetic pressure P, but when the latter takes over, the $z \gg z_p$, the oscillation is checked.

(c) The inclination of the magnetic field (right-hand side) has little effect on the waveforms—the amplitude is slightly smaller at intermediate altitudes for a magnetic field less tilted to the vertical, which more effectively constrains the motion, showing that the main effect is that of magnetic field strength, not direction (for example, the frequency of the first standing mode $\Omega_* = 3$ or $\omega = 6\omega_2 = 3c/L$ applies to the initial plasma β of $\beta = 1$, i.e., a magnetic field strength $B = \sqrt{4\pi\rho_0}a = \sqrt{4\pi\rho_0}c$, and is nearly independent of magnetic field inclination in the range $15° \le \theta \le 75°$).

2. Criterion for identification of cutoff frequencies

We proceed with the general method for solving the wave equation in the standard form (155) by giving a criterion for the identification of cutoff frequencies. The latter can be calculated from the dispersion relation for wave equations with constant coefficients, e.g., acoustic-gravity waves in an isothermal atmosphere (Sec. I.A.7), but for wave equations with variable coefficients, the ray approximation is unreliable as a method of estimating cutoff frequencies, e.g., it gave incorrect values both for acoustic-gravity waves in nonisothermal atmospheres (Sec. II.A.4) and magnetosonic-gravity waves in an isothermal atmosphere under a horizontal magnetic field (Secs. II.C.2 and II.C.3). In the latter cases, the correct cutoff frequencies were found from the exact solutions of the wave equations, which have variable coefficients, requiring special transformations. The present method not only solves such equations in a standard way (Sec. III.B), but also reliably specifies the asymptotic fields (Sec. III.A.6) and cutoff frequencies (Sec. III.C.2) (the WKBJ approximation can do neither), on inspection of the wave equation, with no need to solve it. In order to obtain the criterion of identification of cutoff frequencies for the wave equation (155), we note that, if the polynomials R,S have complex roots, waves have phases and can propagate, whereas if all roots are real, there are no phases, and only standing modes exist. Thus the cutoff frequencies separating standing modes from propagating waves [Eq. (155)] correspond to double roots of the polynomials R,S [Eqs. (156a) and (156b)], separating real from complex roots. The first three cases of vertical waves in Sec. III.A.5 may serve as examples for comparison with earlier results: (i) for acoustic-gravity waves [Eqs. (171a) and (171b)] the polynomial of degree two R_1 has a double root $\alpha_0 = \frac{1}{4}$, corresponding by Eq. (169) to the cutoff frequency $\omega_2 = c/2L$ [Eq. (22b)] for vertical waves; (ii) for Alfvén-gravity waves [Eqs. (172a) and (172b)], the polynomial R_2 is independent of frequency, i.e., there is no cutoff frequency, and the waves are not filtered; (iii) for magnetosonic-gravity waves [Eqs. (173a) and (173b)], the polynomial R_3 is independent of frequency, like R_2 for Alfvén-gravity waves, and the polynomial S_3 coincides with R_1 for acoustic-gravity waves; thus the cutoff $\alpha_0 = \frac{1}{4}$, $\omega_2 = c/2L$ of acoustic-gravity waves is inherited, in the presence of a horizontal magnetic field, by magnetosonic-gravity waves (Sec. II.C.3).

3. Spectral separation and mode mixing

The most general case is that of vertical hydromagnetic waves in an oblique magnetic field [Eqs. (174a) and (174b)], for which both polynomials S_4, R_4 have double roots, respectively, for $\alpha_0 = \frac{1}{4}, n^2/4$, corresponding to the two cutoff frequencies

$$\omega_0 \equiv n\omega_2 = nc/2L \tag{209a}$$

$$\le \omega_2 = c/2L \ . \tag{209b}$$

The lower cutoff ω_0 depends, $n \equiv \cos\theta \le 1$, on the inclination of the magnetic field θ to the vertical, but not on its strength, whereas the upper cutoff ω_2 is totally independent of magnetic field (strength and direction). The upper and lower cutoffs ω_2 and ω_0 specify the unmodified and modified effective wave numbers K and K_0, respectively [see Eqs. (175a) and (175b)],

$$K = (\omega/c)\sqrt{1-(\omega_2/\omega)^2} \ , \tag{210a}$$

$$K_0 = (\omega/nc)\sqrt{1-(\omega_0/\omega)^2} \ . \tag{210b}$$

The lower cutoff ω_0 [Eq. (209a)] and modified effective wave number K_0 [Eq. (210b)] appear to leading order in the dynamic component of the wave field [Eq. (204b)]. The reason for the dependence on the inclination θ of the magnetic field to the vertical is that acoustic-gravity waves, at high altitude (where the magnetic dominates the gas pressure), must follow magnetic field lines, and thus a scale height L corresponds to a distance of propagation $L/\cos\theta \equiv L/n$, and the cutoff $c/2L$ for a vertical field becomes $c/2(L/n) = nc/2L$ for an oblique one. The upper cutoff ω_2 [Eq. (209b)] and unmodified effective wave number K [Eq. (210a)] appear in the magnetic component, e.g., Eq. (202b), so that each component of the hydromagnetic-gravity wave has its own cutoff and effective wave number. The two cutoffs [(209a) and (209b)] and the effective wave numbers [(210a) and (210b)] for vertical waves degenerate into one, in two cases: (i) for a vertical magnetic field, $n = 1$, the two cutoffs and wave numbers coincide, into those ($\omega_0 = \omega_2 = c/2L$, $K_0 = K$) for acoustic-gravity waves; (ii) for a horizontal magnetic field

$n=0$, the dynamic cutoff [Eq. (209a)] vanishes, $\omega_0=0$, and only the magnetic cutoff [Eq. (209b)] remains, the magnetosonic-gravity waves, which have the same cutoff $\omega_2=c/2L$ and effective wave number K [Eq. (210a)] as acoustic-gravity waves. In general, vertical hydromagnetic waves in strictly oblique (i.e., nonvertical and nonhorizontal) magnetic fields have two cutoff frequencies [(209a) and (209b)], the same number as oblique (i.e., nonvertical) acoustic-gravity waves in the absence of a magnetic field (Sec.II.A.4) and magnetosonic-gravity waves in a horizontal magnetic field (Sec. II.C.3). There is, however, an important difference: (i) the acoustic-gravity and magnetosonic-gravity waves are of second order, and so the cutoffs [Eqs. (22a) and (22b)] separate bands in the spectrum, viz., there is an evanescent band $\omega_1<\omega<\omega_2$, separating gravity and acoustic modes, respectively, below $\omega_1<\omega$ and above $\omega>\omega_2$; (ii) the hydromagnetic-gravity waves are of the fourth order, and thus the cutoff [(209a) and (209b)] apply independently to the dynamic ω_0 and magnetic ω_2 components, i.e., each is standing below and propagating above, its own cutoff. Thus the spectrum of hydromagnetic-gravity waves allows mode mixing as follows: (i) for frequencies below the lower cutoff, $\omega<\omega_0$, both the dynamic and magnetic components are standing; (ii) for frequencies above the upper cutoff ($\omega>\omega_2$), both components are propagating; (iii) for frequencies between the cutoffs ($\omega_0<\omega<\omega_2$), the dynamic component is propagating and the magnetic component standing. The situation inverse to (iii), i.e., dynamic component standing and magnetic component propagating, is not usually possible (since the conditions $\omega<\omega_0$ and $\omega>\omega_2$ are incompatible with $\omega_2>\omega_0$), and can only occur above both cutoffs $\omega>\omega_2>\omega_0$ if we choose suitable boundary conditions in Eq. (205), so as to reflect the dynamic ($A_3=A_4$) but not the magnetic ($A_2\neq 0$) component into a standing pattern.

4. Existence and location of critical levels

In order to conclude our presentation of the method of solving the wave equation (155), we still have to consider the case when the polynomials R,S [Eqs. (156a) and (156b)] are of the same degree, $r=s\equiv N$ (which was not treated in Sec. III.B). The case $r=s$ is important since, if $r\neq s$, the wave equation has solutions converging for all z, viz., Eqs. (184b), (185a), and (185b) for $r>s$ and (187b), (188a), and (188b) for $s>r$, so that there is no singularity of the wave equation at intermediate altitude, and critical levels do not occur. Thus a necessary condition for the wave equation (155) to have critical levels is that the polynomials R,S [Eqs. (156a) and (156b)] be of the same degree, $r=s\equiv N$, i.e., $a_N\neq 0\neq b_N$. This condition is not sufficient, i.e., if $r=s$, then one or more critical levels may or may not exist. In order to clarify this point, it is sufficient to consider the values of the coefficients a_N,b_N of the leading powers of R,S, as we now show. If the polynomials R,S [Eqs. (156a) and (156b)] are of the same degree, $r=s=N$, we can use two solutions of the wave equation (155): (i) the solutions (184a), (185a) and (185b) apply if

$$|\zeta| < \lim_{j\to\infty} |c_j/c_{j-1}| = \lim_{j\to\infty} |[S(\sigma+j-1)/R(\sigma+j)]| = |b_N/a_N| \equiv M , \tag{211a}$$

i.e., the series (285b) converges inside the radius of convergence M; it also applies (154a) in the high-altitude range $z>z_2\equiv L\ln(\zeta_0/M)$; (ii) the solutions found in (187b), (188a), and (188b) converge on the outside of the inverse of M,

$$|\zeta| > \lim_{j\to\infty} |R(1-j-\nu)/S(-j-\nu)| = |a_N/b_N| = 1/M , \tag{211b}$$

and applies [Eq. (154a)] in the low-altitude range $z<z_1\equiv L\ln(\zeta_0 M)$. Thus, three cases can arise, depending on the values of $M\equiv|b_N/a_N|$, which is the modulus of the ratio of the leading coefficients of S,M [Eqs. (156a) and (156b)]: (i) if $M>1$, the two regions [(211a) and (211b)] overlap, $z_2<z_1$, and since at least one analytic solution exists for all z (in fact, both solutions hold for $z_2<z<z_1$), there is no singularity or critical level; (ii) if $M=1$, then $z_2=z_1\equiv z_c$, and one solution applies below $z<z_c$ and the other above $z>z_c$; the altitude $z_c=L\ln\zeta_0$, which corresponds to a critical level; (iii) if $M>1$, then $z_2>z_1$, and neither solution applies in the annulus $1/M<|\zeta|<M$, corresponding to the altitude range $z_1<z<z_2$, so that both

$$z_1=L\ln(\rho_0/M) , \tag{212a}$$

$$z_2=L\ln(\rho_0 M) , \tag{212b}$$

are critical levels, and more could exist in between [this possibility could be investigated by continuation of the function $\Phi(\zeta)$ into the annulus]. Applying these criteria to the present case, we conclude that (i) for magneto-atmospheric waves [Eq. (155)], the polynomials R,S are of dissimilar degree [Eqs. (171)—(174)] in all cases except for a horizontal magnetic field [Eqs. (173a) and (173b)], i.e., no critical level exists for acoustic ($r=2>s=0$), Alfvén ($r=2>s=0$), and magnetosonic-gravity ($r=4>s=2$) waves, since (McKenzie, 1973), a nonzero vertical component of the magnetic field allows the wave to propagate through, instead of being absorbed at, the critical level; (ii) in the case of a horizontal magnetic field, the polynomials R_2,S_3 are of the same degree [Eqs. (173a) and (173b)], $r=2=s$, with identical leading coefficients $a_2=1=b_2$, so that $M=1$ in Eqs. (211a) and (211b) and magnetosonic-gravity waves have a critical level [Eqs. (212a) and (212b)] at the altitude $z_c\equiv z_1=z_2=L\ln\beta_0=2L\ln(c/a_0)$, determined by the initial plasma β [Eq. (169b)], in agreement with Eq. (132b) in the case of vertical waves $k_{||}=0$.

5. Amplitude and phase at the critical level

We have shown that, for the general wave equation (155), the possibility that a critical level exists depends on the polynomials R,S [Eqs. (156a) and (156b)] being equal in degree, $r=s\equiv N$, and that the location of this critical level is determined by the leading coefficients a_N,b_N. We now show that the coefficients a_{N-1},b_{N-1} of degree one unit lower decide whether or not the amplitude and phase of the wave are finite at the critical level. Mathematically, the question being posed is whether the series (185a) converges on the boundary $|\zeta|=M$ of the region (211a), i.e., at the critical level $z=z_1$ [Eq. (212a)], and similarly for Eqs. (187b), (211b), and (212b). In order to investigate the behavior of the series (185a) on its boundary of convergence, we need the ratio of two succeeding coefficients (184a) to $O(1/j)$,

$$c_j/c_{j-1}=(b_N/a_N)[1-\Lambda j/O(j-2)] , \tag{213a}$$

$$\Lambda\equiv a_{N-1}/b_N-b_{N-1}/b_N+N , \tag{213b}$$

where $\Lambda\equiv X+iY$ is generally complex. Note that, as $j\to\infty$, Eq. (213a) reduces to Eq. (211a). The real part $X\equiv\mathrm{Re}(\Lambda)$ of Eq. (213b) determines the behavior (case α) of the series (Bromwich, 1927) on the boundary of convergence $|\zeta|=M\equiv|b_N/a_N|$, except at the point $\zeta=b_N/a_N$, viz., (i) if $X<0$ the series diverges, (ii) if $x=0$ it oscillates, (iii) if $O<X\leq 1$ it converges conditionally, (iv) if $X>1$ it converges absolutely. On the point $\zeta=b_N/a_N$ the behavior (case β) of the series (Knopp, 1947) is given by X and $Y\equiv\mathrm{Im}(\Lambda)$, the imaginary part of Eq. (213b), viz., (i) if $X<1$ it diverges, (ii) if $X>1$ it converges, (iii) if $X=1$ it diverges for $Y=0$ and oscillates for $Y\neq 0$. The implications of the behavior of the series are as follows: (i) if it diverges (oscillates) the amplitude and/or phase of the wave are unbounded (indeterminate) at the critical level; (ii) if it converges the amplitude and phase of the wave are determined at the critical level by summing series, with rearrangement of terms allowable if it is absolutely convergent, but not allowable if it is conditionally convergent. For magneto-atmospheric waves, a critical level occurs only for a horizontal magnetic field, in which case [Eqs. (173a) and (173b)] the wave equation (155) has polynomials of degree $r=s=2\equiv N$, with two leading coefficients $a_2=b_2=b_1,a_1=0$, so that $\Lambda=1$ in Eq. (213b). The critical level $z_c=L\ln\beta_0$ corresponds [Eq. (170a)] to $\zeta=-\beta_0\exp(-z_c/L)=-1$, which lies on the circle of convergence $|\zeta|=|b_2/a_2|=1$, but does not coincide with the point $b_2/a_2=1$, i.e., we have case α above, conditional convergence, with $X=1$. Thus, for a magnetosonic-gravity wave, the amplitude and phase are finite at the critical level and may be obtained by summing the series solution [Eqs. (147a) and (147b)] without rearranging the terms (148).

6. Generalized and ordinary hypergeometric functions

The solution described above can also be obtained from the present method, as we now show. The general function of the first kind [Eq. (194)], with the polynomials R,S factorized in their roots, σ_j [Eq. (184b)] and ν_j [Eq. (188a)], respectively, are

$$\begin{aligned}\Phi_\sigma(\zeta)&=\zeta^\sigma\Bigg\{1+\sum_{j=1}^{\infty}[(-)^{s-r}(b_s/a_r)\zeta]^j\\&\qquad\times\prod_{l=1}^{j}[(\sigma+l-\nu_1-1)\cdots(\sigma+l-\nu_s-1)/(\sigma+l-\sigma_1)\cdots(\sigma+l-\sigma_r)]\Bigg\}\\&=\rho^\sigma{}_{s+1}F_r[\sigma-\nu_1,\ldots,\sigma-\nu_s,1;\sigma-\sigma_1+1,\ldots,\sigma-\sigma_r+1;(-)^{s-r}(b_s/a_r)\zeta] ,\end{aligned} \tag{214a}$$

where F is a generalized hypergeometric function,

$$\begin{aligned}{}_nF_m(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_m;\xi)&\equiv 1+\sum_{j=1}^{\infty}\xi^j\prod_{l=1}^{j}[(l+\alpha_1-1)\cdots(l+\alpha_n-1)/l(l+\beta_1-1)(l+\beta_m-1)]\\&=\frac{\Gamma(\beta_1)\cdots\Gamma(\beta_m)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)}\sum_{j=0}^{\infty}(\xi^j/j!)\prod_{l=0}^{j}\frac{\Gamma(l+\alpha_1)\cdots\Gamma(l+\alpha_n)}{\Gamma(l+\beta_1)\cdots\Gamma(l+\beta_m)} ,\end{aligned} \tag{214b}$$

with variable $\xi\sim\zeta\sim e^{-z/L}$ and parameters determined by ν_j,σ_j, which are the roots of R,S. Since σ is equal to one of $\sigma_1,\ldots,\sigma_r$, say $\sigma=\sigma_1$, the generalized hypergeometric function (214a) is always of the type ${}_sF_{r-1}$:

$$\Phi_1(\zeta)\equiv\lim_{\sigma\to\sigma_1}\Phi_\sigma(\zeta)=\zeta^{\sigma_1}{}_sF_{r-1}[\sigma_1+l-\nu_1-1,\ldots,\sigma_1+l+\nu_s-1;\sigma_1+l-\sigma_2,\ldots,\sigma_1+l-\sigma_r;(-)^{s-r}(b_s/a_r)\zeta] , \tag{215}$$

with variable $\xi\sim\zeta$, and s upper and $r-1$ lower parameters. It would be possible to introduce similarly the generalized hypergeometric functions of the second and third kinds, which correspond to Ψ_σ and $\Theta_\sigma^{(1,2)}$, respectively, and many of the properties indicated before could be justified in terms of these functions. The fact that Eqs. (155) and (157) are solvable in terms of generalized hypergeometric functions is known (Bailey, 1935; Erdelyi, 1953; Luke, 1975); and the direct approach to the problem adopted here yields, in a convenient form, the properties of waves described by the wave equations (151) and (153) with constant or exponential propagation speeds, scattering scales, or damping rates. In particular, in the case of second-order waves, $s=2=r$, the solution can always be expressed in terms of hypergeometric functions [Eq. (215)] of Gaussian or ordinary type ${}_2F_1$. For example, for magnetosonic-gravity waves in a horizontal magnetic field [Eqs. (173a) and (173b)] the solution can be expressed as follows: (i) below the critical level $|\zeta|>1$, the indices $\nu_{1,2}=-\frac{1}{2}\pm iKL$ are distinct [Eq. (188a)], and the solutions (187b)

$$\overline{\Phi}_{\pm}(1/\zeta)=\zeta^{-1/2\mp iKL}\left[1+\sum_{j=1}^{\infty}\zeta^{j}(j!)^{-1}\prod_{l=1}^{j}[(l-\tfrac{3}{2}\pm iKL)^2/(1\pm 2iKL)]\right]$$

$$=e^{z/2L}e^{\pm iKz}F(\tfrac{1}{2}\pm iKL,\tfrac{1}{2}\pm iKL;1\pm 2iKL;-\beta_0^{-1}e^{z/L})\,, \tag{216a}$$

appear in terms of functions of the first kind only; (ii) above the critical level $|\zeta|<1$, there is a double index $\sigma=0$, and the solution [Eq. (185a)] involves functions of the first kind:

$$\Phi_0(\zeta)=1+\sum_{j=0}^{\infty}\zeta^{j}(j!)^{-2}\prod_{l=1}^{j}[(l-3/2-iKL)(l-3/2+iKL)]=F(\tfrac{1}{2}+iKL,\tfrac{1}{2}-iKL;1;-\beta_0e^{-z/L})\,, \tag{216b}$$

as well as a function of the second kind G, with the same parameters and variable. The wave fields of magnetosonic-gravity waves, below the critical level (216a) and above it (216b), agree with Eqs. (142b) and (144b), respectively, in the case $k_{\parallel}=0$ of vertical propagation.

7. Approach to critical level conditions

We have illustrated (Fig. 6) the amplitude and phase of a magnetosonic-gravity wave, in the vicinity of the critical level, for a purely horizontal magnetic field (Sec. II.C.8). The extension to an oblique magnetic field leads to the case of hydromagnetic-gravity waves, which we have illustrated (Fig. 7) for the standing magnetic component (Sec. III.C.2), we now consider (Fig. 8) the upward-propagating dynamic component G_+ [Eq. (204b)], for the same three parameters as in Sec. III.C.1. Although in the case of an oblique magnetic field there is no critical level, we can illustrate the approach to it as the magnetic field tilts closer to the horizontal. In Fig. 8 we plot the logarithm of amplitude (lhs) and phase (rhs) of the dynamic component of an upward-propagating hydromagnetic-gravity wave versus altitude z made dimensionless by dividing by the scale height L. We consider Eqs. (207a)—(207c) as representing the basic case A, and give each of the parameters in turn another value, namely, the highest in Eqs. (208a)—(208c). From Fig. 8 we arrive at the following conclusions regarding the effect of each of the three parameters: case B, as the frequency increases from a moderate value $\Omega_*=3$ or $\omega=6\omega_2$ to the ray limit $\Omega_*=5$ or $\omega=10\omega_2$ (so that $\omega^2\gg\omega_2^2$), there is little change in the amplitude (lhs) and a marked increase in the phase (rhs), because for magnetic pressure larger than or equal to gas pressure the dynamic component resembles and acoustic-gravity wave, whose amplitude growth $\sim e^{z/2L}$ is independent of frequency and whose phase shift is proportional to frequency; case C, as the initial plasma β is increased from 1 to 10 at the base of the atmosphere, the gas pressure dominates over the magnetic pressure over the first two scale heights $z<2L$, allowing (lhs) a more marked amplitude growth, until it is checked at $z>3L$ as the magnetic pressure takes over, while phase (rhs) is little affected by magnetic field strength, since oscillations occur along magnetic field lines; case D, tilting the magnetic field from an intermediate angle $\theta=45°$ to near horizontal $\theta=75°$ has little effect on amplitude growth (lhs), which is determined mainly by the decay in atmospheric density, specified, for a uniform magnetic field, by hydrostatic equilibrium. The effect of magnetic field inclination is more marked on phase (rhs)—as the magnetic field tilts closer to the horizontal $\theta\to 90°$, the

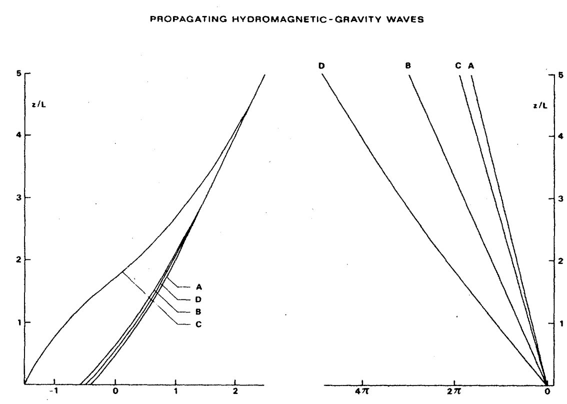

FIG. 8. Logarithm of amplitude (left) and phase (right) vs altitude z divided by scale height L, for dynamic component of fourth-order hydromagnetic-gravity wave, propagating vertically, in the following cases: A, reference case like that of Fig. 7, viz., wave frequency ω, 3 times the cutoff value ω_2, i.e., $\omega=3\omega_2$, initial plasma β unity, $\beta_0=1$, and magnetic field at $\theta=45°$ to the vertical; B, like the reference case, but with frequency 5 times the cutoff value, $\omega=5\omega_2$; C, like the reference case, with initial plasma β one order of magnitude larger, $\beta_0=10$; D, like the reference case, but with the magnetic field tilted $\theta=75°$ away from the vertical (i.e., 15° from the horizontal).

dynamic cutoff frequency [Eq. (209a)] reduces, $\omega_0 \rightarrow 0$, and thus the important ratio $\Omega_0 \equiv \omega/\omega_0 = \omega L/nc$ of wave ω to cutoff frequency increases, leading to larger phase shifts; thus, although the amplitude of the wave is bounded at the critical level, the phase varies rapidly in its vicinity, leading to large gradients of the waveform, i.e., to intense dissipation and heating, even if the diffusivities are small.

8. Closed and open magnetic regions

The absence or presence of critical levels can explain the basic difference [Fig. 9(a)] observed between closed and open magnetic regions on the solar corona. In the open magnetic regions, such as coronal holes, the magnetic field is nowhere horizontal, so critical levels do not exist, there is no intense heating, these regions remain dark, and as the waves are not absorbed, they are available to accelerate high-speed particle streams and propagate out with the solar wind, where they are observed (Sec. III.B.8). In the closed magnetic regions, such as coronal loops and arches, the magnetic field is horizontal at least at one point, the top, and very inclined in its vicinity, so that there is a critical level as well as marked phase changes, leading to efficient dissipation and explaining why these regions are bright and hot and dominate the coronal energy balance. The widely reported observation that coronal loops are bright and coronal holes dark agrees with the evidence for wave dissipation in the former (see references below) and wave propagation outwards from the latter (references in Sec. III.A.8). The distinction proposed here, between coronal holes without, and coronal loops with, critical levels, gives one possible physical explanation for those observations. Coronal loops have been observed in $H\alpha$, radio, ultraviolet, and x-ray wavelengths (Vaiana and Rosner, 1978; Pallavacini *et al.*, 1981; McConnel and Kundu, 1983; Loughhead and Bray, 1984; Bray and Loughhead, 1986). Two main areas of research have been static stability (Chiuderi, Einaudi, and Torricelli-Ciamponi, 1981; Landini and Mosignori-Fossi, 1981; Van Hoven, Ma, and Einaudi, 1981; Kuin and Martens, 1982; Wolfson, 1982; Batistoni, Einaudi, and Chiuderi 1985; Cramer and Donelly, 1985; McClymont and Craig, 1985a, 1985b) and dynamical heating processes (Habbal, Leer, and Holzer, 1979; Galeev, Rosner, Serio, and Vaiana, 1981; Torricelli-Ciamponi, Einaudi, and Chiuderi, 1982; Martens and Kuin, 1983; McNeice, 1985). Although the arches and loops are more readily visible as magnetic structures in the "average" corona (Pottasch, 1960, 1964; Athay, 1966b; Bessey and Liebenger, 1984; Fisher and Sime, 1984; Low, 1984; Osherovich, Tzur, and Gliner, 1984; Osherovich, Gliner, and Tzur, 1985; van Ballegooijen, 1985; Withbroe, Kohl, Weiser, and Munro, 1985; Wolfson, 1985), their "footprints" extend through the transition region to the chromosphere, where the temperature rises sharply from about the chromospheric value of about 10^4 K to the coronal value of over 10^6 K (Kopp and Kuperus, 1968; Burton, Jordan, Ridgeley, and Wilson, 1971; Moore and Fung, 1972; Jordan, 1976, 1980; Dere, Bartoe, and Bruecker, 1982; Mariska *et al.*, 1982; Antiochos, 1984). Significant mass motions are observed, in the form of, transients such as ejections in the corona (Low, 1982; Oran, Mariska, and Boris, 1982; Bruecker and Bartoe, 1983; Cargill and Pneuman, 1984; Fisher and Garcia, 1984; Karpen, Oran, and Boris, 1984; Low, 1984b; Schmieder, Mein, Matres, and Trandberg-Hanssen, 1984; Wagner, 1984; Jackson, 1985; Sawyer, 1985; Simnet and Harrison, 1985), and steady flows in the transition region (Dumont *et al.*, 1980b; Gebbie *et al.*, 1981, Dumont, Mouradian, and Pecker, 1982; Mein, Simon, Vial, and Shine, 1982; Mouradian *et al.*, 1982; Athay, Gurman, and Henze, 1983; Dumont *et al.*, 1983; Dufton, Kingston, and Keenan, 1984; Fang *et al.*, 1984; Mariska, 1984; Owocki and Canfield, 1986). The transition region from the chromosphere to the corona, where intense heating occurs, could be identified with the critical level, where vertical magnetosonic-gravity waves are absorbed. The critical level is located where the sound and Alfvén speeds are equal, i.e., the gas pressure $p=(2/\gamma)P=\mu B_{\parallel}^2/4\pi\gamma$, where $P \equiv \mu B_{\parallel}^2/8\pi$ is the magnetic pressure, for the horizontal magnetic field component. If we take $B_{\parallel}=2G$, the gas pressure at the critical level is estimated as $p \sim 1.7\times 10^{-1}$ dyn cm^{-2}, which is consistent with the gas pressure in the transition region, across which pressure varies slowly compared with temperature T and density ρ, i.e., $\rho \sim 1/T$. Above the critical level the velocity perturbation of the magnetosonic-gravity wave grows linearly with altitude, $v \sim z$, so that $e^{v} \sim e^{z/L} \sim 1/\rho \sim T$, and the velocity perturbation is proportional to the logarithm of temperature. This result is borne out by observations [Fig. 9(b)] of nonthermal velocities in the transition region (Vial, Lemaire, Artzner, and Gouttebroze, 1980), in the temperature range from $T \sim 10^4$ K (chromospheric) to $T \sim 10^6$ K (coronal).

IV. RESISTIVE DAMPING AND MAGNETIC STRUCTURES

The preceding analysis has shown that the presence of a magnetic field changes substantially the properties of waves in an atmosphere. Oblique Alfvén-gravity (Sec. II.B.6) and magnetosonic-gravity (Sec. II.C.6) waves are reflected; vertical Alfvén-gravity waves can propagate, with asymptotic properties (Sec. II.B.5) very different from those of a homogeneous medium; vertical magnetosonic-gravity waves are absorbed at a critical level (Secs. IV.C.7 and II.C.8); and oblique magnetic fields couple these modes into fourth-order waves with a new cutoff frequency (Secs. III.C.2 and III.C.3). Observations also suggest that the magnetic field plays a dominant role in the physics of the solar atmosphere (Gabriel, 1976; Svalgaard and Wilcox, 1978; Golub, Rosner, Vaiana, and Weiss, 1981; Howard and Labonte, 1981; Giovanelli and Jones, 1982; Anzer and Galloway, 1983; Snodgrass, 1983; Hoeksema, 1984; Athay, Jones, and Zirin, 1985; Stenflo, 1985; Stenflo and Vogel, 1986). This is the case for all visible layers: (i) in the photosphere, the magnetic field is concentrated in intense flux tubes, at the boundaries of granules; (ii) in the chromosphere, the "active" regions of strong magnetic field, e.g., above sunspots, are hotter than "quiet" regions, where the magnetic field takes smaller average values; (iii) in the transition region there are bright closed structures, such as loops and arches, and dark open structures, such as "holes," which dominate the energy and mass balances, respectively; (iv) in the corona, transients like ejecta, and quiescient structures like prominences, are associated with magnetic forces. The origin of the solar magnetic field is probably the dynamo effect, which manifests itself in field topology and solar activity (Moffatt, 1976,1978; Parker, 1977; Cowling, 1981; Durney, 1983; Eddy, 1983; Berger and Field, 1984; Yoshimura, Wu, and Wang, 1984; Pudovkin and Benevolenska, 1985; Ruzmaikin, 1985; Tong, Lu, Mao, and Han, 1985; Wood and Moffatt, 1985); the dynamo is probably located in the convection zone, and the magnetic field emerges in the photosphere through complex motions involving multiple (i.e., electrical, viscous, and thermal) diffusion, which expel the magnetic flux to the boundaries of convection cells, leading to the concentration of magnetic flux in intense tubes at the boundaries of granulation (Weiss, 1978; 1981a, 1981b; Knochbloch and Proctor, 1981; Arter, Proctor, and Galloway, 1984; Glatzmaier and Gilman, 1982; Proctor and Weiss, 1982; Proctor, 1983; Marcus, Press, and Teukolsky, 1983; Rabin, Moore, and Hagyard, 1984; Simon and Wilson, 1985; Wang, Zirin, and Shi, 1985; Zwaan, 1985). Magnetohydrostatic equilibrium (Nakagawa, 1974; Serio *et al.*, 1981; Aly, 1984; Low, 1984a; Melville, Hood, and Priest, 1984), can be used to model large-scale atmospheric magnetic fields (Gliner, 1984; Wolfson, 1985), as well as "local" fields in quiescent regions like coronal prominences (Tsubaki, 1975; Milne, Priest, and Roberts, 1979; Osherovich, 1982; Heasley and Milkey, 1983; Galindo-Trejo and Schindler, 1984; Landman, 1984; Leroy, Boomier, and Sahal-Bréchot 1984; Nikolsky, Kim, Koutchmy, and Stellmacher, 1984; Anzer and Priest, 1985; Hirayama, 1985; Osherovich, 1985). Most magnetic features in the solar atmosphere, e.g., spicules (Sec. I.A.8) and fibrils, are "dynamical" (Nakagawa and Levine, 1974; Nakagawa and Tanaka, 1974; Levine and Nakagawa, 1975; Parker, 1982a, 1982b, 1982c, 1982d; Low, 1984c; Bogdan and Lerche, 1985), and the most significant phenomena, such as oscillations and heating, are also unsteady; this is the reason why waves may be the key to understanding the global mass and energy balances in the solar atmosphere.

A. Ohmic dissipation of hydromagnetic waves

The heating of an atmosphere by waves requires the presence of dissipation mechanisms—in the case of the sun, viscosity (Yanowitch, 1967a, 1967b; Adam, 1975; Maeland, 1982), thermal conduction and radiation (Lyons and Yanowitch, 1974; Webb and Roberts, 1980; Press, 1981; Mihalas and Mihalas, 1983; Roberts, 1983a; Cally, 1984; Mihalas, 1984), and electrical resistance, which has been considered for homogeneous media (Alfvén, 1947; Lehnert, 1952), turbulent ionized fluids (Batchelor, 1950; Moffatt, 1978), loop resonances in the *LCR*-circuit analogy (Ionson, 1982, 1984, 1985), and magneto-atmospheric waves, using exact solutions (Campos, 1983e, 1983f) and the phase-mixing approximation (Heyvaerts and Priest, 1983; Nocera, Leroy, and Priest, 1984; Sakurai and Granik, 1984; Steinholfson, 1985). We have already considered atmospheric waves in the presence of viscous dissipation (Secs. V.A.2.—V.A.8, in Part I), and concentrate here on electrical resistance, deferring the case of thermal damping to the next section (IV.B.7 and IV.B.8). The *LCR*-circuit analogy for magneto-atmospheric waves neglects the variation of atmospheric properties with altitude, through the use of constant or "lumped" coefficients. This is equivalent to replacing oscillations in a tapering string by oscillations in a uniform string, or sound in a horn of varying cross section by sound in a uniform tube; any second-order wave, e.g., Alfvén-gravity or magnetosonic-gravity type, is reduced to the same *LCR* circuit, with the sole difference lying in the values of the "lumped" inductance, resistance, and capacitance, whose estimation is uncertain to the extent of variation of wave speeds and damping rates with altitude. The *LCR*-circuit analogy with forcing shows that, in resonance conditions, the energy extracted is independent of the electrical diffusivity, i.e., as long as the diffusivity is nonzero, the wave field evolves so as to yield a given rate of energy dissipation. This property is analogous to the absorption of vertical magnetosonic-gravity waves at the critical level, since energy deposition requires a nonzero diffusivity, but the amount of the energy lost by the waves is determined by the reduction of their rate of growth in the vicinity of the critical level, from exponential far below to linear far above. The same result, viz., a rate of dissipation independent of the diffusivity, is obtained by phase mixing of Alfvén waves in a region of rapidly varying wave speed, since adjoining wave components acquire different phases, and their mixing produces large waveform gradients that increase until the available diffusivity, however small but nonzero, produces the required dissipation rate. The process is relevant to magnetosonic-gravity waves near a critical level, since in its vicinity the propagation speed changes rapidly from an almost acoustic to almost Alfvénic form, causing large phase shifts, hence steep waveforms, and intense dissipation, even if the diffusivity is small. Since we have considered earlier the coupling of compressibility with viscous damping, i.e., viscous acoustic-gravity waves (Sec. V.A. of Part I), we concentrate now on the coupling of magnetism with electrical resistance, i.e., resistive Alfvén-gravity waves; this will lead us to the comparison of properties in Table II, which will be gradually justified in the exposition that follows.

1. Viscous and resistive Alfvén waves

We consider Alfvén waves, propagating in the z direction of the external magnetic field $B\mathbf{e}_z$, with velocity and

TABLE II. Hydrodynamic and hydromagnetic waves in atmospheres.

Wave	Hydrodynamic	Hydromagnetic	
Example	Viscous acoustic gravity	Resistive Alfvén gravity	
Perturbation	Velocity	Velocity	Magnetic field
Type	Longitudinal	Transversal	Transversal
Atmospheric properties:			
Propagation			
Speed	Sound	Alfvén	
Nonisothermal atmosphere	Bounded[a]	Unbounded[b]	
Isothermal case	Constant[c]	Exponential growth[d,e]	
Dissipation			
Mechanism	Kinematic viscosity	Electrical resistivity	
Nonisothermal atmosphere	Unbounded[e]	Bounded[a]	
Isothermal case	Exponential growth[g]	Constant[h]	
Cutoff frequency	Yes	No	
Energy densities	Kinetic, compression	Kinetic, magnetic	
Undamped waves			
Initial wave field			
Amplitude	Exponential growth[e]	Exponential growth[i]	Exponential decay[j]
Phase	Linear growth[k]	Linear growth[l]	Linear growth[l]
Equipartition	Yes	Yes	
Total energy	Kinetic + compression	Kinetic + magnetic	
Energy scaling	Constant	Exponential decay[m]	
Asymptotic, standing modes			
Amplitude	Exponential growth[e]	Constant	Exponential decay[n]
Equipartition	Yes	No	
Total energy	Kinetic + compression	Kinetic	
Energy scaling	Constant	Exponential decay[n]	
Asymptotic, propagating waves			
Amplitude	Exponential growth[e]	Linear growth	Constant
Phase	Linear growth[k]	Constant	Constant
Equipartition	Yes	No	
Total energy	Kinetic + compression	Magnetic	
Energy scaling	Constant	Constant	

[a] Atmosphere with bounded temperature.
[b] Nonzero magnetic field and decaying density.
[c] Perfect gas.
[d] Uniform magnetic field.
[e] On twice the scale height $\sim e^{z/2L}$.
[f] Nonzero kinematic viscosity and decaying density.

magnetic field perturbation in the x direction, $\mathbf{B}=(h,0,B)$ and $\mathbf{v}=(v,0,0)$, so that the viscous momentum [Eq. (1)] and resistive induction [Eq. (2)] are

$$\dot{v}-(a^2/B)\partial h/\partial z=\nu_1\nabla^2 v \ , \qquad (217a)$$

$$\dot{h}-B\partial v/\partial z=\chi\nabla^2 h \ . \qquad (217b)$$

Here $\nabla^2\equiv\partial^2/\partial z^2+\partial^2/\partial y^2$ is the Laplacian operator; we have neglected rotation $\mathbf{\Omega}=0$ and the Hall effect, respectively, and introduced the Alfvén speed a [Eq. (38a)] and the magnetic diffusivity $\chi=c_*^2/4\pi\mu\eta$, the latter with dimensions similar to those of the kinematic viscosity ν. Only the incompressible viscosity ν_1 [and not the compressible viscosity ν_2 in Eq. (217a)] appears, because Alfvén waves, being transversal, do not cause density or pressure perturbations. For this reason, we do not need the equation of energy (5), and Alfvén waves are not affected by thermal conduction and radiation at the linear level. The Alfvén wave equations (217a) and (217b) are exactly linear, even for waves of finite amplitude, since all nonlinear terms cancel. The convective acceleration $(\mathbf{v}\cdot\nabla)\mathbf{v}=0$ vanishes, since $\nabla\equiv\mathbf{e}_y\partial/\partial y+\mathbf{e}_z\partial/\partial z$ and $\mathbf{v}=v\mathbf{e}_x$; Alfvén waves propagate not only perturbations of velocity $\mathbf{v}$ and magnetic field $\mathbf{h}$, but also a nonzero electric current $\mathbf{j}=(c/4\pi)\nabla\Lambda\mathbf{h}$, producing a Lorentz force $c_*^{-1}\mathbf{j}\Lambda\mathbf{h}$, which couples nonlinearly to compressive fast and slow modes, and through them to thermal conduction and radiation. We defer the study of nonlinear effects to Sec. IV.C, and concentrate here on linear, dissipative waves, described by Eqs. (217a) and (217b) for the Alfvén type, where the Alfvén speed a, viscous diffusivity ν_1, and electrical diffusivity χ may all depend on the coordinate z, e.g., altitude. Elimination of $\mathbf{v}$ and $\mathbf{h}$ between (217a) and (218b) leads to fourth-order equations, for the veloci-

TABLE II. (*Continued*).

Wave	Hydrodynamic	Hydromagnetic	
Example	Viscous acoustic gravity	Resistive Alfvén gravity	
Perturbation	Velocity	Velocity	Magnetic field
Type	Longitudinal	Transversal	Transversal
Dissipative waves			
Critical level			
Type	Reflecting layer	Transition layer	
Low-altitude regime	Compressive	Resistive	
High-altitude regime	Viscous	Magnetic	
Cause	Viscous stress-strain	Joule effect	
Boundary condition	Dissipation	Damping	
Initial wave field			
Amplitude	Exponential growth[e]	Exponential or constant[o]	Exponential or constant[o]
Phase	Linear growth[k]	Linear growth	Linear growth[l]
Equipartition	Yes	No	
Total energy	Kinetic + compression	Kinetic + magnetic	
Energy scaling	Constant	Exponential or constant[p]	
Asymptotic field, standing mode			
Amplitude	Constant	Constant	Exponential decay[l]
Equipartition	No	No	
Total energy	Kinetic	Kinetic	
Energy scaling	Exponential decay[n]	Exponential decay[l]	
Asymptotic field, propagating wave			
Amplitude	Constant	Linear growth	Constant
Phase	Constant	Constant	Constant
Equipartition	No	No	
Total energy	Kinetic	Magnetic	
Energy scaling	Exponential decay[n]	Constant	

[g] On the scale height $\sim e^{z/L}$.
[h] Constant rate of ionization.
[i] On four times the scale height $\sim e^{z/4L}$.
[j] On four times the scale height $\sim e^{-z/4L}$.
[k] On effective wave number $\sim e^{iKz}$.
[l] On Alfvén wave number $\sim e^{i\omega z/a}$.
[m] On twice the scale height $\sim e^{-z/2L}$.
[n] On scale height $\sim e^{-z/L}$.
[o] As $\exp[z(1/L-\sqrt{\omega/2\chi})]$, i.e., exponential growth for $\omega<2\chi L^2$, decay for $\omega>2\chi L^2$, and constant for $\omega=2\chi L^2$.
[p] As the square of footnote n.

ty and magnetic field perturbations, respectively, of viscous and resistive Alfvén waves:

$$\ddot{v}-a^2v''-\nu_1\nabla^2\dot{v}-a^2\chi\nabla^2(a^{-2}\dot{v})+\nu_1\chi a^2\nabla^2(a^{-2}\nabla^2 v)=0\ , \tag{218a}$$

$$\ddot{h}-(a^2h')'-(\nu_1+\chi)\nabla^2\dot{h}+\nu_2\chi\nabla^4 h=0\ , \tag{218b}$$

where we have assumed constant electrical diffusivity χ and viscous diffusivity ν_1, and where the other parameters, viz., (a,ν_1) in Eq. (218a) and (a,χ) in Eq. (218b), may depend on z. Dot and prime denote derivatives with respect to time and altitude, $\dot{f}\equiv\partial f/\partial t$, $f'\equiv\partial f/\partial z$, and ∇^2 is the Laplacian $\nabla^2 f=f''+\partial^2 f/\partial y^2$. In the absence of dissipation, $\chi=0=\nu_1$, Eqs. (218a) and (218b) reduce to Eqs. (162a) and (162b). The wave equations for viscous and resistive Alfvén modes are generally distinct for the velocity perturbations [Eq. (218a)] and magnetic field perturbations [Eq. (218b)], and only coincide,

$$[\partial^2/\partial t^2-a^2\partial^2/\partial z^2-(\chi+\nu_1)\nabla^2\partial/\partial+\chi\nu_1\nabla^4]v_x,h_x(y,z,t)=0\ , \tag{219}$$

if the Alfvén speed a and viscous and electrical diffusivities, ν_1 and χ, respectively, are all constant.

2. Weak-damping and phase-mixing approximations

If the diffusivities ν_1,χ are small, we can neglect their product $\nu_1\chi$, and the equations of viscous and resistive Alfvén waves [Eqs. (218a) and (218b)] drop from fourth to second order in z; in the case of the velocity perturbation (218a) we obtain

$$\partial^2 v/\partial t^2-a^2\partial^2 v/\partial z^2=(\nu_1+\chi)(\partial^2/\partial z^2+\partial^2/\partial y^2)\partial v/\partial t\ , \tag{220}$$

where the external magnetic field $B(y)$ and Alfvén speed $a(y)$ can only depend on the coordinate y transverse to the plane (x,z) of the direction of propagation $\mathbf{B}=B\mathbf{e}_z$ and of the velocity and magnetic field perturbations $\mathbf{v}||\mathbf{h}||\mathbf{e}_x$. The waves may propagate along open or closed magnetic field lines, which correspond, respectively, to conservation of frequency ω or longitudinal wave number $k_{||}$ (parallel to B), related by $\omega=k_{||}a$. Since the medium may be inhomogeneous in the transverse or y direction, the wave number $k_{||}$ may depend on y, and we seek a solution of Eq. (195) in the form

$$v_x(y,z,t)=\int_{-\infty}^{+\infty}\int W_x(y,z)\exp[i(k_{||}(y)z-\omega t)]d\omega\, dk_{||} \ . \tag{221}$$

Substituting Eq. (221) into (220) we obtain

$$\omega^2W_x-[a^2-i\omega(\nu_1+\chi)](W_x''+2ik_{||}W_x'-k_{||}^2W_x)+i\omega(\chi+\nu_1)[\partial W_x/\partial y+2iz(dk_{||}/dy)\partial W_x/\partial y +iz(d^2k_{||}/dy^2)W_x-z^2(dk_{||}/dy)^2W_x]=0 \ . \tag{222}$$

Note that in the absence of dissipation, $\chi=0=\nu_1$, the waveform would be independent of z, and thus the weak-damping approximation may be stated $\partial/\partial z \ll k_{||}$; also, for altitudes or distances of propagation much larger than the length scale of variation of the wave number,

$$z \gg \{d[\ln k_{||}(y)]/dy\}^{-1}=l_1 \ ,$$

the phases of the waves may have evolved differently on adjoining magnetic field lines, leading to phase mixing. The combined weak-damping and phase-mixing approximations simplify Eq. (222) to the third and last terms,

$$W'=-z^2(\delta_*-i)^{-1}[(dk_{||}/dy)^2/2k_{||}]W \ , \tag{223a}$$

$$\delta_*\equiv a^2/\omega(\chi+\nu_1) \ , \tag{223b}$$

where δ_* is a dimensionless inverse damping parameter, i.e., large for small diffusivities, low-frequency waves, and large Alfvén speed. From Eq. (223a) it follows that the wave field decays at large distances as

$$W(y,z)=W(y,0)\exp[-(k_{||}z^3/l_1^2)/6(\delta_*-i)] \ , \tag{224a}$$

$$l_1\equiv k_{||}(dk_{||}/dy)^{-1} \ , \tag{224b}$$

with the increase of distance of propagation z measured on the wavelength $\lambda_{||}=2\pi/k_{||}$ and on the length scale (224b) of inhomogeneities, the effect being more marked for larger diffusivity $1/\delta_*$. The solution (224a) has a phase term, which was omitted in the original paper on phase mixing (Heyvaerts and Priest, 1983) under the assumption that $\delta_*\gg 1$ in Eq. (224a).

3. Competition of magnetism and electrical diffusivity

The equations of dissipative Alfvén-gravity waves [(218a) and (218b)] drop from fourth to second order in space, if one of the two diffusion processes, fluid viscosity or electrical resistance, is absent, regardless of the magnitude of the other. Assuming vertical waves, of frequency ω [Eq. (167)], we find that Eqs. (218a) and (218b) simplify to

$$W_x''+(\omega/a)^2W_x-(i/\delta)a^2(a^{-2}W_x)''=0 \ , \tag{225a}$$

$$a^{-2}(a^2H_x)'+(\omega/a)^2H_x-(i/\delta)H_x''=0 \ , \tag{225b}$$

where the inverse damping $\delta\equiv a^2/\omega\chi$ is calculated for the electrical diffusivity χ alone, and the external magnetic field $\mathbf{B}$ is assumed to be vertical and uniform. In a non-isothermal ionized atmosphere (Chapman and Cowling, 1949; Spitzer, 1956), with bounded temperature, the electrical conductivity η is finite and nonzero, and so the electrical diffusivity $\chi=c_*^2/4\pi\mu\eta$ is bounded; the Alfvén speed [Eq. (38b)] is unbounded for a nonzero external magnetic field, as the density decays to zero at high altitude. The combination of bounded diffusivity and unbounded propagation speed should lead to the appearance of a critical level, separating a low-altitude region of dominant diffusion from a high-altitude region where propagation predominates. This picture of resistive Alfvén waves in an atmosphere is precisely the opposite of that of viscous acoustic-gravity waves (Sec. V.A. of Part I), for which the sound speed is bounded and the viscous diffusivity is unbounded. In the latter case, of bounded propagation speed and unbounded diffusivity, a critical level also exists, but it separates a low-altitude region where propagation predominates from a high-altitude region where damping is dominant. We shall show, in due course, that this "inversion" in the location of the propagation and damping regions will change the character of the critical level (Sec. IV.A.8) and the type of boundary condition (Sec. IV.A.7) needed to render the wave field unique. In order to calculate the wave fields explicitly, we consider an isothermal atmosphere, for which the Alfvén speed squared [Eq. (99a)] increases exponentially on the scale height, and, if the rate of ionization is constant, the electrical diffusivity χ is a constant. This is the inverse of viscous acoustic-gravity waves in an isothermal atmosphere, for which the sound speed is constant (Sec. V.A.4 of Part I), but the viscous diffusivity increases exponentially on the scale height. We note, in passing, that the fourth-order wave equations [(218a) and (218b)] for viscous and resistive Alfvén waves, in an isothermal atmosphere with constant rate of ionization, have constant or exponential coefficients, i.e., are of the type (151), and can be solved by the general method of Sec. III.

4. Equations reducible to the hypergeometric type

Since we wish to make a direct comparison of viscous acoustic-gravity and resistive Alfvén waves, we shall take as a starting point the inviscid momentum equation [217(a), with $\nu_1=0$] and resistive induction equation (217b):

$$\dot{v}_x=(a^2/B)h_x' , \tag{226a}$$

$$\dot{h}_x-Bv_x'=\chi h_x'' . \tag{226b}$$

Bearing in mind that the Maxwell equation $\nabla\cdot\mathbf{B}=0$ implies that the external vertical magnetic field $\mathbf{B}$ is uniform, we see that Eqs. (226a) and (226b) can be eliminated for the velocity perturbation v and magnetic field perturbation h, respectively:

$$\ddot{v}_x-a^2v_x''-a^2[\chi(a^{-2}\dot{v}_x)']'=0 , \tag{227a}$$

$$\ddot{h}_x-(a^2h_x')'-\chi\dot{h}_x''=0 . \tag{227b}$$

The wave equations for the velocity and magnetic field perturbations of resistive Alfvén waves hold for Alfvén speed $a(z)$ and electrical diffusivity $\chi(z)$, varying in the direction of propagation z. In the case of vertical waves, in an isothermal atmosphere, under a uniform magnetic field, and with constant rate of ionization, we have constant B,χ and $a'/a=2L$, so that the vertical velocity perturbation spectrum satisfies

$$(a_0^2e^{z/L}-i\omega\chi)W_x''+2i(\omega\chi/L)W_x'+(\omega^2-i\omega\chi/L^2)W_x=0 . \tag{228}$$

The resistive Alfvén wave propagates not only a velocity perturbation v, but also a magnetic field perturbation h, which can be calculated from the former by Eqs. (226a) and (226b):

$$\dot{h}_x=Bv_x'+\chi[(B/a^2)\dot{v}_x]' , \tag{229a}$$

$$H_x(z;\omega)=i(B/\omega)[1-(i/\delta)e^{-z/L}]W_x' -(B/\omega L)\delta^{-1}e^{-z/L}W_x . \tag{229b}$$

Since the magnetic field perturbation h is transverse to the direction of propagation of the resistive Alfvén wave, the latter also propagates an electric current j, which is specified by the Maxwell equation $j_y=(c_*/4\pi)h_x'$ and momentum equation (226a) as

$$j_y=(c_*/4\pi)h_x'=(c_*B/4\pi a^2)\dot{v}_x , \tag{230a}$$

$$J_y(z;\omega)=-i(\omega c_*\rho/\mu B)W_x(z;\omega) =-(\omega c_*\rho_0/\mu B)e^{-z/L}W(z;\omega) . \tag{230b}$$

The wave equation (228) is of second order, with exponential coefficients, i.e., of type (153), with polynomials [Eqs. (156a) and (156b)] both of second degree, and thus its solution can be expressed (Sec. III.C.6) in terms of ordinary hypergeometric functions of type ${}_2F_1$. We have already encountered three other instances of reduction of linear second-order differential equations with exponential coefficients to the hypergeometric type: (i) vertical viscous acoustic-gravity waves in an isothermal atmosphere with uniform kinematic viscosity (Sec. V.A of Part I); (ii) oblique acoustic-gravity waves in a three-parameter family of nonisothermal atmospheres, with exponential-like temperature profile (Sec. II.A); and (iii) oblique magnetosonic-gravity waves in an isothermal atmosphere, under a uniform horizontal magnetic field (Sec. II.C). The present problem, viz., vertical resistive Alfvén-gravity waves, in an isothermal atmosphere, with uniform vertical magnetic field and constant rate of ionization, is the fourth such instance, as we now show.

5. Magnetic propagation in the high-altitude range

We perform the change of variable (154a), where ζ_0 is $-i/\delta$, with δ the inverse damping parameter:

$$\zeta=i(\omega\chi/a^2)e^{-z/L}\equiv(i/\delta)e^{-z/L} , \tag{231a}$$

$$W_x(z;\omega)\equiv\Phi(\zeta) . \tag{231b}$$

Equation (228) transforms to a hypergeometric equation:

$$(1-\zeta)\zeta\Phi''+(1-3\zeta)\Phi'-(1+i\varepsilon_0^2/\delta)\Phi=0 , \tag{232}$$

where ε_0 is the initial compactness [Eq. (106b)]. The parameters of the hypergeometric equation satisfy $\gamma_4=1$, and $\alpha_4+\beta_4=2$, $\alpha_4\beta_4=1+i\varepsilon_0^2/\delta=1+i\omega L^2/\chi$, so that they are given by

$$\gamma_4=1 , \tag{233a}$$

$$\alpha_4,\beta_4=1\pm\sqrt{-i\varepsilon_0^2/\delta}=1\pm(i-1)L\sqrt{\omega/2\chi} . \tag{233b}$$

Since the parameter γ_4 is unity, the general integral of the hypergeometric equation (23b) is a linear combination of the functions of the first and second kinds, F and G, respectively:

$$W_x(z;\omega)=A_1F_2[(i/\delta)e^{-z/L}]+A_2G_2[(i/\delta)e^{-z/L}] , \tag{234a}$$

$$F_2,G_2(\xi)\equiv F,G(1+(i-1)L\sqrt{\omega/2\chi},1-(i-1)L\sqrt{\omega/2\chi};1;\xi) . \tag{234b}$$

Asymptotically at high altitude, as $z\to\infty$ and $\zeta\to0$ in Eq. (231a), the function of the first kind tends to unity, $F_2\to1$, and that of the second kind has a logarithmic singularity,

$$G_2\sim\ln[-(i/\delta)e^{-z/L}]=-z/L-i\pi/2-\ln\delta .$$

Thus the velocity perturbation of resistive Alfvén-gravity waves grows linearly with altitude in the asymptotic regime:

$$W_x(z;\omega)\sim-A_2z/L+A_1-A_2[i\pi/2+\ln(a^2/\omega\chi)] . \tag{235a}$$

The perturbations of magnetic field [Eq. (229b)] and electric current [Eq. (230b)] associated with Eq. (235a) are

$$H(z;\omega)\sim-i(B/\omega L)A_1 , \tag{235b}$$

$$J(z;\omega)\sim-i(\rho\omega c_*/\mu BL)A_2ze^{-z/L} . \tag{235c}$$

We conclude that the asymptotic fields of resistive

Alfvén-gravity waves satisfy laws qualitatively similar [Eq. (108a)] to those of nondissipative Alfvén waves [compare Eqs. (235a) and (115a)], because in the high-altitude range propagation predominates over dissipation.

6. Dominant diffusion in the low-altitude range

The preceding solution [Eqs. (234a) and (234b)] is only valid for $|\zeta|<1$ in Eq. (231a), i.e., for the high-altitude range $z>z_3$, above the critical level, which is located, in the present case, at the altitude (236a) and for resistive Alfvén-gravity waves in atmospheres with general Alfvén speed $a(z)$ and electrical diffusivity $\chi(z)$ profiles, at z_3 satisfying (236b):

$$z_3=-L\ln\delta=L\ln(\omega\chi/a_0^2)\,, \tag{236a}$$

$$\chi(z_3)\omega=[a(z_3)]^2\,. \tag{236b}$$

Below the critical level $z<z_3$, we have $|\zeta|>1$ by Eqs. (236a) and (231a), and the solution (140), in terms of hypergeometric functions of variable $1/\zeta$, should be used. Thus the velocity perturbation of resistive Alfvén-gravity waves is specified, in the low-altitude range $z<z_3$, below the critical level, where diffusion predominates over propagation, by a linear combination

$$W_x(z;\omega)=A_+W_x^+(z;\omega)+A_-W_x^-(z;\omega) \tag{237a}$$

of the functions

$$W_x^\pm(z;\omega)=\exp[z/L\pm(i-1)z\sqrt{\omega/2\chi}\,]\times F(1\pm(i-1)L\sqrt{\omega/2\chi},1\pm(i-1)L\sqrt{\omega/2\chi};1\pm(i-1)\sqrt{2\omega/\chi};-i\delta e^{z/L})\,. \tag{237b}$$

The leading term of Eq. (237b), viz.,

$$W_x^\pm(z;\omega)\sim\exp(z/L\mp k_1z)\exp(\pm ik_1z)\,, \tag{238a}$$

$$k_1\equiv\sqrt{\omega/2\chi}\,, \tag{238b}$$

shows that the phase is determined by an "effective wave number" (238b), which is specified by the wave frequency ω and electrical diffusivity χ. This wave number is the real part, $k_1=\mathrm{Re}(k_*)$, of the wave number k_*, corresponding to the dispersion relations

$$i\omega=k_*^2\chi\,, \tag{239a}$$

$$\partial v/\partial t=\chi\partial^2 v/\partial z^2\,, \tag{239b}$$

for the classical diffusion equation (239b) for the velocity, and shows that indeed Ohmic dissipation dominates, to leading order, magnetic propagation, for resistive Alfvén waves in the low-altitude range. If we perform the substitution (146), we obtain from Eq. (237b) the wave fields $W_x^\pm$ in the form

$$W_x^\pm(z;\omega)=(1-i\delta e^{z/L})^{1\pm(1-i)L\sqrt{\omega/2\chi}}\exp[z/L\pm(i-1)k_1z]\times F(1\pm(i-1)L\sqrt{\omega/2\chi},\pm(i-1)L\sqrt{\omega/2\chi};1\pm(i-1)L\sqrt{2\omega/\chi};(1-i\delta e^{z/L})^{-1})\,. \tag{240a}$$

This form is valid over the entire altitude range, including at the critical level where the amplitude and phase are finite and specified by

$$W_x^\pm(z_3;\omega)=(1-i)^{1\pm(1-i)L\sqrt{\omega/2\chi}}\times F(1\pm(i-1)L\sqrt{\omega/2\chi},\pm(i-1)L\sqrt{\omega/2\chi};1\pm(i-1)L\sqrt{2\omega/\chi};(1+i)/2)\,. \tag{240b}$$

7. Damping condition for dissipative waves

Since the resistive Alfvén wave propagates an electric current, we must require that the total energy dissipated by Joule effect, in a column of gas extending over the entire height of the atmosphere (up to infinity), be finite:

$$\dot{E}_\chi=\frac{1}{\sigma}\int_0^\infty|J(z;\omega)|^2dz=\frac{c_*^2}{16\pi^2\sigma}\int_0^\infty|dH(z;\omega)/dz|^2dz\,. \tag{241}$$

Substitution of Eq. (235a) into (241) shows that this condition is satisfied for all A_2 and A_1 [in Eq. (234a)], i.e., since for resistive Alfvén waves in the high-altitude range propagation dominates dissipation, the dissipation condition (241) is always satisfied and does not serve to determine any of the constants of integration in Eqs. (234a) and (234b); this contrasts with the case of viscous acoustic-gravity waves, for which dissipation is dominant in the high-altitude range, and the dissipation condition (Sec. V.A.6 of Part I) eliminates one of the particular integrals. Thus, in the case of resistive Alfvén waves, we have to replace the dissipation condition by a different "damping" condition that will be effective in the low-altitude range. In a homogeneous medium, a damped wave has a decreasing amplitude, but we could not impose such a property on a dissipative atmospheric wave, because the density decrease with altitude (stratification effect) tends to increase wave amplitude—if stratification predominates, the wave still increases in amplitude in spite of dissipation, and conversely, if dissipation predominates, it decreases in amplitude, whereas if the two balance the amplitude is constant. Thus a damping condition (Campos, 1983d) should be a relative statement comparing dissipative and nondissipative waves in the same stratification conditions: the presence of dissipation should reduce the amplitude of the wave, when compared with a nondissipative wave in a medium with the same stratification. If we apply this damping condition in the low-altitude range [Eq. (238b)], it is clear that the presence of electrical resistance decreases the amplitude of the W_x^+ solution by $\exp(-k_1z)$, whereas it increases the W_x^- solution by $\exp(+k_1z)$, so that we must suppress the latter, and may retain the former, by setting $A_+\neq0=A_-$ in Eq. (237a). The remaining constant of integration A_+ is determined from the initial velocity perturbation. The choice of the solution W_x^+ in Eq. (240a) implies that the wave field in the low-altitude region [Eq. (238b)] has, to leading order, an amplitude evolution

$$|W_x^+(z;\omega)| \sim \exp(z/l_2) , \tag{242a}$$

$$1/l_2 = 1/L - \sqrt{\omega/2\chi} , \tag{242b}$$

which is exponential on a scale l_2; this scale coincides with the scale height L, reduced on account of electrical resistance. The effective length scale l_2 in Eq. (242b) demonstrates the competition between density stratification and electrical resistance, which tends to cause wave growth or decay with altitude. The net result is that there is wave decay for $\chi < \omega/2L^2$, i.e., high frequencies $\omega > 2\chi L^2$, and wave growth for $\chi > \omega/2L^2$, i.e., low frequencies $\omega < 2\chi L^2$. The amplitude is constant if the two effects balance, $\chi = \omega/2L^2$, i.e., for the frequency $\omega_* = 2\chi L^2$.

8. Critical level as a transition layer

The damping condition has succeeded in determining one of the constants of integration, in the case of resistive Alfvén waves in an atmosphere, for which purpose the dissipation condition was useless; conversely, the dissipation condition is capable of determining a constant of integration in the case of viscous acoustic-gravity waves (Sec. V.A.6 of Part I), for which the damping condition would be trivially met, since viscous acoustic-gravity waves always grow more slowly (at most linearly) than nondissipative acoustic waves (which grow exponentially). Thus the damping (Campos, 1983d) and dissipation (Yanowitch, 1967b) conditions may each be of use for waves in diffusive atmospheres, that is, for the cases in which dissipation predominates over propagation either in the low-altitude region, below critical level, or in the high-altitude region above critical level. We have thus found three types of critical levels for vertical waves in atmospheres: (type II) for viscous acoustic-gravity waves (Sec. V.A of Part I), the critical level acts as a reflecting layer, since viscous dissipation is dominant in the high-altitude range, and excludes wave propagation there; (type III) for resistive Alfvén waves (Sec. IV.A) the critical level acts as a transition layer, between a low-altitude region of dominant dissipation and a high-altitude region where waves may propagate unhindered by damping; (type I) for nondissipative magnetosonic-gravity waves (Sec. II.C), the critical level is of type III or II, i.e., a transition layer for vertical propagation, or a reflecting layer for oblique propagation, and in the case of evanescent waves is a singular layer, i.e., a critical level of type I. For viscous acoustic-gravity waves, letting viscosity vanish does not recover nondissipative acoustic-gravity waves, because the critical level recedes to infinity but still reflects the waves, producing downward-propagating components, which would not exist in the nondissipative case. For resistive Alfvén waves in the low-altitude range, where electrical diffusivity predominates, in the limit of perfect conductivity $\eta \to 0$ and $\chi \to \infty$, the waves grow [Eqs. (242a) and (242b)] exponentially on the scale height $|W_x^+| \sim \exp(z/L)$, which also differs from the amplitude law [Eq. (170a)] for nondissipative Alfvén waves at low altitude (Sec. II.B.4). Propagation predominates over dissipation in the high-altitude range, and we may expect the nondissipative solution to be recovered in the limit of vanishing diffusivity $\chi \to 0$, as we now confirm. When the diffusivity reduces, $\chi \to 0$, the parameters (233b) of the hypergeometric functions of the first and second kinds, F and G, respectively, become large, $\alpha_4, \beta_4 \sim \omega L^2/\chi \to \infty$, and the relation

$$\lim_{\alpha,\beta \to \infty} F,G(\alpha,\beta;\gamma,\xi) = J_0, Y_0(2\sqrt{-\alpha\beta\xi}) \tag{243}$$

shows that we obtain Bessel function's J_0 and Neumann functions Y_0

$$\lim_{\chi \to 0} F,G((i-1)L\sqrt{\omega/2\chi},(i-1)L\sqrt{\omega/2\chi};1;i(\omega\chi/a_0^2)e^{-z/L}) = J_0, Y_0[(2\omega L/a_0)e^{-z/2L}] , \tag{244}$$

which specify [Eqs. (111a) and (112a)] nondissipative Alfvén waves.

B. Magnetic slabs and flux tubes

Wave dissipation is one of the mechanisms proposed in connection with the controversial, and still unresolved, problem of heating the solar and stellar atmospheres. A variety of nonwave heating mechanisms have been proposed, based on magnetic, electrical, and thermodynamic effects (Lerche and Low, 1980; Lindsay, 1981; Parker, 1981a, 1981b; Sturrock and Uchida, 1981; Hinata, 1983; Mestel and Moss, 1983; Spicer, 1983; Heyvaerts and Priest, 1984; Qing, Zuang, and Youyi, 1984; Rabin and Moore, 1984; Schatten and Mayr, 1984; Kumar and Narain, 1985; Browning and Priest, 1986), as an alternative to heating theories based on dissipation of propagating or resonant magneto-atmospheric modes (Alfvén, 1947; Osterbrock, 1961; Hollweg, 1972, 1978, 1981a, 1981b, 1984a—1984d; Adam 1977a, 1977b, 1981, 1984; Leroy and Bel, 1979; Leroy, 1980, 1981, 1983, 1985; Bel and Leroy, 1981; Kuperus, Ionson and Spicer, 1981; Hollweg, Jackson, and Galloway, 1982; Ionson, 1982, 1984, 1985; Leroy and Schwartz, 1982; Schwartz and Leroy, 1982; Campos, 1983a, 1983b, 1984b, 1984c, 1985a; Hollweg and Sterling, 1984; Pasachoff and Landman, 1984; Schwartz and Bel, 1984; Schwartz, Cally, and Bel, 1984; Campos and Leitão, 1986). We have argued that the absorption of vertical magnetosonic-gravity waves at the critical level, which is located in the transition region, could explain the intense heating in closed coronal structures and the sharp temperature rise to over 10^6 K in the corona (Sec. III.C.8). This view was supported by a com-

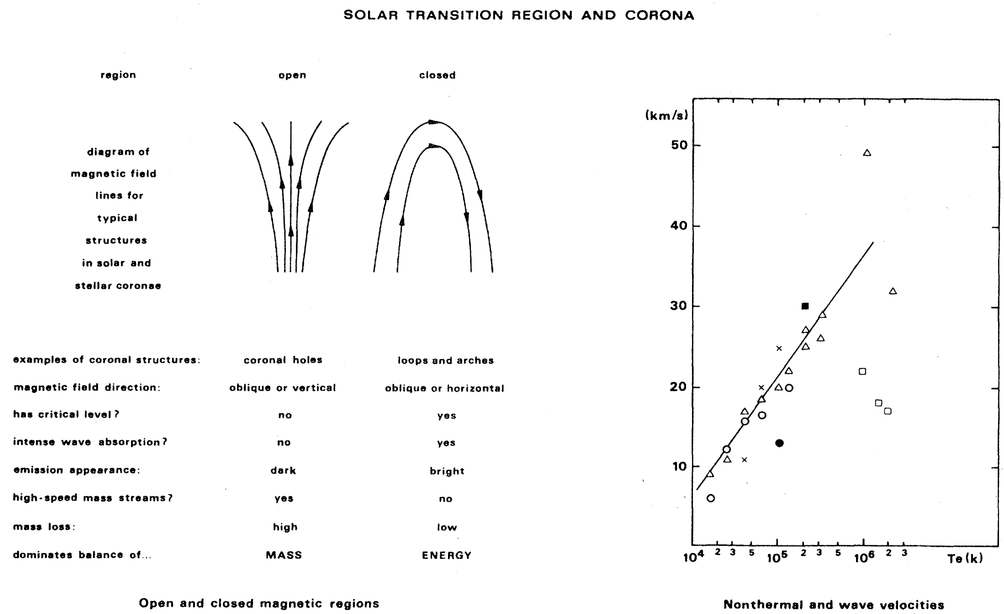

FIG. 9. Comparison (left) of closed and open magnetic regions in the solar corona, characterized, respectively, by the presence or absence of critical levels, and resulting features; comparison (right) of nonthermal velocities observed in the solar transition region (Vial, Lemaire, Artzner, and Gouttebroze, 1980), with the theoretical scaling (solid curve) of the velocity perturbation of vertical magnetosonic-gravity waves vs atmospheric temperature.

parison (Fig. 9) of the predicted wave velocity perturbation and observed nonthermal velocities, as a function of temperature. The dissipation of Alfvén waves is more gradual, and could explain (Sec. I.B.8) the more gradual temperature rise (and associated almost exponential density decay) in the chromosphere (Fig. 2). As further support for this view, we compare in Fig. 10 the horizontal nonthermal velocities observed in the chromosphere (Beckers and Canfield, 1975), with the (horizontal) velocity perturbation of an Alfvén wave, propagating vertically in [Fig. 10(a)] a uniform external magnetic field (Campos, 1984b) and [Fig. 10(b)] a magnetic flux tube (Hollweg, 1981a). Both plots are broadly consistent, showing that both magneto-atmospheric waves and magnetic flux tube modes are relevant to the physics of the solar atmosphere; the reason is that the magnetic field configuration in the solar atmosphere (Gabriel, 1976; Anzer and Galloway, 1983), sketched in Fig. 11, shows an evolution from intense magnetic flux tubes, concentrated at the boundaries of the granulation in the chromosphere, to an almost uniform magnetic field, in the high chromosphere, transition region, and corona, as the flux tubes fan out rapidly in the low chromosphere and merge in the mid chromosphere. The basic modeling of magnetic flux tubes concerns their equilibrium and dynamics (Parker, 1981a, 1981b; Spruit, 1981a, 1981b; Van Ballegooijen, 1982; Simon, Weiss, and Nye, 1983; Osherovich, 1984; Stenflo and Harvey, 1985; Pizzo, 1986), and the stability of the associated magneto-atmospheric structures (Zweibel, 1981; Adam, 1982a, 1982b, 1982c; An, 1984; Gaffet, 1984; Hasan, 1984, 1985; Roberts, 1984a; Vainshtein and Parker, 1986; Yeh, 1986). These magnetic structures can act as waveguides (Spruit and Roberts, 1983) for surface and body modes, viz., in flux tubes (Roberts and Webb, 1978, 1979; Hollweg, 1981a; Rae and Roberts, 1982b; Edwin and Roberts, 1983; Parker, 1983; Bogdan, 1984; Narayan and Somasundaram, 1985; Cally, 1986; Hassan, 1986; Lee and Roberts, 1986) and on interfaces or in slabs (Wentzel, 1979a; Roberts, 1980b; Rae and Roberts, 1981; Roberts, 1981a, 1981b; Edwin and Roberts, 1982; Somasundaram and Uberoi, 1982; Rae and Roberts, 1983a, 1983b; Roberts, Edwin, and Benz, 1983, 1984; Narayan and Somasundaram, 1985), both of which can support dissipation mechanisms (Wentzel, 1979b; Lee, 1980; Webb and Roberts, 1980; Spruit, 1982; Gordon and Hollweg, 1983; Roberts, 1983a).

1. Magneto-hydrostatic equilibrium and stability of a perfect gas

The magneto-atmospheric waves studied earlier (Secs. I, II, III, and IV.A) and the flux tube modes to be considered below (Secs. IV.B and IV.C) are the two possible cases of unsteady motions, in a gas under gravity and magnetic fields, for which stratification is one dimensional, i.e., mean-state quantities may depend on one coordinate, such as altitude z, but are uniform in the transverse (x,y) plane. The equation of magneto-hydrostatic equilibrium (9b), for gravity pointing downwards, $\mathbf{g}=(0,0,-g)$, and pressure p, density ρ, and magnetic field $\mathbf{B}$ depending only on altitude z, reads

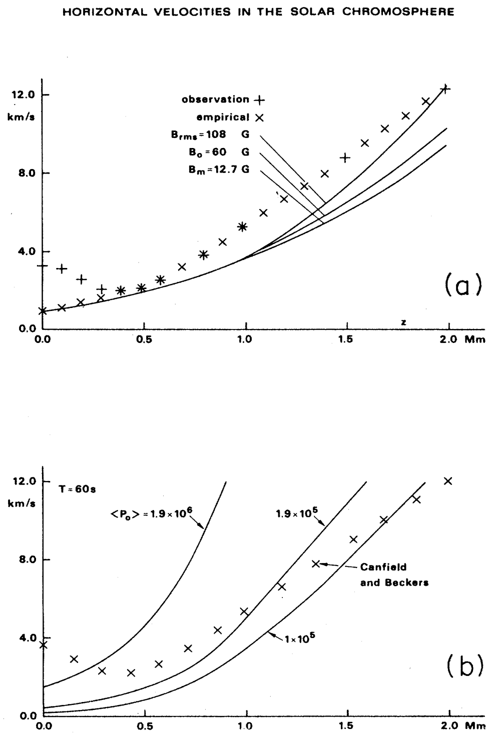

FIG. 10. Comparison of nonthermal horizontal velocities observed in the solar chromosphere ($\times$, Beckers and Canfield, 1975) with the velocity perturbation of Alfvén waves, calculated theoretically (solid curve) for vertical propagation in (a) an atmosphere under a uniform magnetic field of strength $B_m, B_0, B_{\rm rms} = 12.7$, 60, 108 G, corresponding to the mean B_m, root-mean-square $B_{\rm rms}$, and average $B_0 = (B_m + B_{\rm rms})/2$ magnetic field strength in the solar photosphere (Campos, 1984c); (b) a magnetic flux tube, in magneto-hydrostatic equilibrium with VAL-III model of the solar chromosphere (Hollweg, 1981a), for three values of the energy flux in erg cm^{-2} s^{-1}.

$$(p + \mu B_{\|}^2/8\pi)' + \rho g = 0 \; , \tag{245a}$$

$$B_{\perp} \mathbf{B}_{\|}' = 0 \; , \tag{245b}$$

where a prime denotes derivative with regard to altitude, the horizontal magnetic field

$$\mathbf{B}_{\|}(z) = B_x(z)\mathbf{e}_x + B_y(z)\mathbf{e}_y$$

is an arbitrary function of altitude, and the vertical $B_\perp$ magnetic field component $\mathbf{B} = \mathbf{B}_{\|} + B_\perp \mathbf{e}_z$ is constant, on account of Maxwell's equation $0 = \nabla \cdot \mathbf{B} = B_\perp'$. Equation (245b) shows that one-dimensional magneto-hydrostatic equilibrium is possible only in two cases: (i) $\mathbf{B}_{\|}' = 0$, in which case the external magnetic field is uniform, but of arbitrary direction relative to the vertical, i.e., the case of general hydromagnetic-gravity waves (Sec. III), including as particular cases acoustic-gravity, Alfvén-gravity and magnetosonic-gravity waves; (ii) the vertical magnetic field component is zero, $B_\perp = 0$, in which case the horizontal magnetic field $\mathbf{B}_{\|}(z)$ can have an arbitrary dependence on altitude, leading to waves in flux tubes, current sheets, and magnetic slabs. In the case (i) of a uniform magnetic field, Eq. (245a) reduces to the equation (72a) of magneto-hydrostatic equilibrium, that is, for a perfect gas $p = \rho R T$, the gas pressure p and mass density ρ stratification laws are

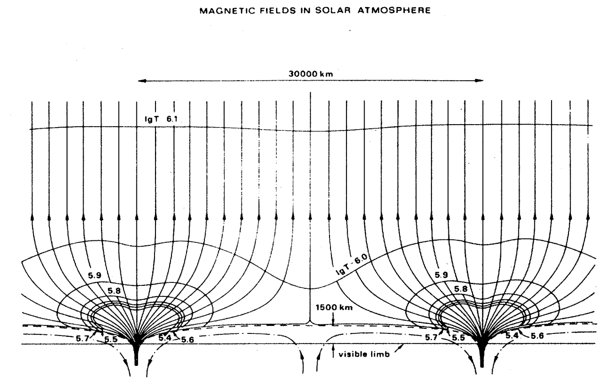

FIG. 11. Diagram of magnetic field pattern in the solar atmosphere (Gabriel, 1976), showing (bottom) the magnetic flux tubes emerging at the edges of granules in the photosphere, fanning out with altitude in the chromosphere, and eventually merging into a uniform magnetic field in the transition region to the corona.

$$p(z)/p(0) = \rho(z)T(z)/\rho(0)T(0) = \exp[-n(z)] \; , \tag{246a}$$

$$n(z) \equiv \int_0^z (g/R)[T(\xi)]^{-1} d\xi = \int_0^z [L(\xi)]^{-1} d\xi \; , \tag{246b}$$

for a given temperature $T(z)$ or scale height $L(z) = RT(z)/g$ profile. In this case the condition of convective stability for a perfect gas is Eq. (74). In the case (ii) of a horizontal nonuniform magnetic field, the magneto-hydrostatic equilibrium is affected, by adding to the gas pressure gradient p' the magnetic pressure P'

$$p' = -\rho g + P' = -p/L + P' \; , \tag{247a}$$

$$P(z) \equiv (\mu/8\pi)[B_{\|}(z)]^2 \; . \tag{247b}$$

The gas pressure profile (247a) is given by

$$p(z) = \left[p_0 + \int_0^z P'(\xi) \exp[n(\xi)] d\xi \right] \exp[-n(z)] \; , \tag{247c}$$

which reduces to Eqs. (246a) and (246b) in the case of magnetic field of constant strength, $P' = 0$. The condition of convective stability (73a) is not changed in the presence of a horizontal nonuniform magnetic field (Newcomb, 1961), on the condition that dp/dz is replaced by the gradient dP_*/dz of the total $P_* = p + P$ (gas plus magnetic) pressure. The evaluation of the temperature gradient in Eq. (73b) is modified, because (Thomas and Nye, 1975) the gas pressure gradient is specified by magneto-hydrostatic [Eq. (245a)] instead of hydrostatic [Eq. (72a)] equilibrium:

$$dT/dz > -g(1-1/\gamma)/R - (\rho R)^{-1} d(\mu B_{\|}^2/8\pi)/dz = (dT/dz)_{ad} - (\rho R)^{-1} dP/dz \; . \quad (248)$$

Thus a magnetic field increasing $dP/dz > 0$ with altitude stabilizes the atmosphere, since it opposes parcel displacements from the equilibrium position, while a field that decreases with altitude, $dP/dz < 0$, destabilizes the atmosphere, since it favors parcel displacements from the equilibrium position. An isothermal atmosphere, $dT/dz = 0$, may be destabilized, $dP/dz < \rho R (dT/dz)_{ad}$, by a magnetic field decaying rapidly with altitude. For example, in the case of constant Alfvén speed $B_{\|} \sim e^{-z/2L}$, $P \sim e^{-z/L}$, $dP/dz = -(P_0/L)e^{-z/L}$, convective stability, $\rho R P_0/L < g/C_p$, requires that the initial magnetic field strength B_0 in $P_0 = \mu B_0^2/8\pi$ satisfy $B_0^2 < 8\pi L g/C_p \mu$.

2. Wave speeds in a slender flux tube

The slender flux tube is a particular case of a horizontal nonuniform magnetic field $B_{\|}(z) = B_0(z)$, which is assumed to be uniform inside the tube and vanish outside, so that its nonuniformity reduces to a jump in magnetic field strength at the surface of the flux tube. If the tube is axisymmetric with radius $r = R(z)$, the jump of the magnetic field strength at the surface implies that the electric current density is a δ function, i.e., the "surface" of the tube is a current sheet. We denote by (p_i, ρ_i, T_i) and (p_e, ρ_e, T_e) the gas pressure, mass density, and temperature, respectively, inside and outside the tube. The conditions of thermal and mechanical equilibrium of the tube and surrounding medium require that the temperature be continuous across the interface and that the jump in gas pressure equal the magnetic pressure:

$$T_i(z) = T_e(z) \; , \quad (249a)$$

$$p_e(z) = p_i(z) + \mu[B_0(z)]^2/8\pi \; . \quad (249b)$$

Thus the gas pressure and mass density are always lower inside the tube than on the outside, and the sound speed is the same inside (c_i) and outside (c_e) the tube, $c_e = c_i \equiv c$, and has a constant ratio to the Alfvén speed:

$$\rho_e(z) = \rho_i(z) + \gamma a^2/2c^2 \; , \quad (250a)$$

$$c(z)/c(0) = a(z)/a(0) \; . \quad (250b)$$

Since the gas pressures decay exponentially in the same way [Eq. (246a) for constant or zero magnetic field strength] inside and outside the tube, the magnetic pressure decays in the same way, on the exponent n [Eq. (246b)], and the magnetic field strength on its half $n/2$:

$$B(z) = B(0)\exp[-\tfrac{1}{2}n(z)] \; , \quad (250c)$$

$$R(z) = R(0)\exp[\tfrac{1}{4}n(z)] \; , \quad (250d)$$

implying, by magnetic flux conservation $BR^2 \sim \text{const}$, that the radius of the flux tube increases with altitude on one-fourth of the same exponent [Eq. (250d)]. In the case of an isothermal atmosphere, $n(z) = z/L$, not only the sound speed c but also the Alfvén speed a is constant, since the decay of the mass density with altitude is compensated by the decrease of magnetic field strength, or fanning out of the magnetic field lines, according to

$$p(z)/p(0), \rho(z)/\rho(0), B(z)/B(0), R(z)/R(0) = e^{-z/L}, e^{-z/L}, e^{-z/2L}, e^{z/4L} \; . \quad (251)$$

The slender flux tube approximation requires that the radius of the tube be much smaller than the scale height, $R(z) \ll L$; the fanning out of the tube with altitude means that it eventually violates the slenderness condition; in the isothermal case, for example, it holds only for $z \ll L[\ln L/R(0)]$. The property that the sound and Alfvén speeds are both constant in isothermal conditions and vary with altitude in the same proportion for an arbitrary nonuniform temperature profile renders the study of waves in slender flux tubes analytically simpler than that of magneto-atmospheric waves in a uniform magnetic field. For unbounded waves in isothermal conditions the sound speed, but not the Alfvén speed [Eq. (99a)], is constant, and in nonisothermal conditions both speeds vary with altitude in dissimilar ways; on the other hand, the study of modes in an isolated flux tube is usually algebraically more tedious than that of waves in an unbounded atmosphere, since the flux tube requires the use of boundary conditions across the tube "wall" or current sheet. The matching of two or more wave fields across interfaces also occurs for multilayered atmospheres, and the boundary conditions to be applied are similar in both cases, e.g., continuity of displacement and total pressure.

3. Association in series and tube speed

The simplest, "two-dimensional" model of a flux tube, of infinite extent in the y direction, is the magnetic slab, consisting of two interfaces $z = \pm z_0$, with uniform but different magnetic fields inside $|z| < z_0$ and outside $|z| > z_0$. The magnetic slab is a double current sheet, and an even simpler case is a plane magnetic interface $z = 0$, separating uniform fields of different strengths. If we denote by $k_{\|} = \mathbf{k}\cdot\mathbf{B}/B$ the wave number in the direction of the magnetic field, and by $k_\perp = (k^2 - k_{\|}^2)^{1/2}$ the orthogonal component, the velocity perturbation in the latter direction is given by Eqs. (121a) and (121b):

$$W_z'' + k_\perp^2 W_z = 0 \; , \quad (252a)$$

$$k_\perp^2 \equiv (\omega^2 - k_{\|}^2 a^2)(\omega^2 - k_{\|}^2 c^2)/[\omega^2(a^2 + c^2) - k_{\|}^2 a^2 c^2] \; , \quad (252b)$$

where we have neglected gravity, $g = 0$, so that all coefficients are constant and the waves sinusoidal, $W_z \sim \exp(ik_\perp z)$. Since we are considering magnetohydrodynamic waves in a homogeneous medium under a uniform magnetic field, the transverse $k_\perp$ can be expressed in terms of the longitudinal $k_{\|}$ wave number, using the dispersion relation (44) for the phase speed

$u=\omega/k$, with $k^2=k_\perp^2+k_{||}^2$, which leads once more to Eq. (252b). This equation can be factorized

$$k_\perp^2=(\omega^2-k_{||}^2a^2)(\omega^2-k_{||}^2c^2)/(c^2+a^2)(\omega^2-k_{||}^2b^2)\ , \quad (253a)$$

$$b^{-2}\equiv a^{-2}+c^{-2}\ , \quad (253b)$$

where we have introduced the tube speed b, whose inverse square is given by the sum of the inverse squares of the sound speed c and Alfvén speeds a, as for the law of association of resistances in series, implying that the tube speed never exceeds that of the other two, $b\le \min(a,c)$. We may expect the tube speed to apply to the longitudinal wave propagation in flux tubes, which is affected both by gas pressures and by magnetic stresses, the latter acting as for sound in a collapsible tube, and this will be confirmed subsequently (in Sec. IV.B.5). In the present case, of magneto-hydrodynamic waves in a homogeneous medium, the propagating variables are, in addition to the transverse velocity perturbation, $v_\perp$, a longitudinal velocity perturbation $v_{||}$, and magnetic field perturbations $h_\perp$ (longitudinal) and $h_{||}$ (transverse);

$$v_{||}=v_\perp c^2k_{||}k_\perp/(\omega^2-k_{||}^2c^2)\ , \quad (254a)$$

$$h_{||},h_\perp=(B/\omega)v_\perp(k_\perp,-k_{||})\ , \quad (254b)$$

where we have used Eqs. (8a) and (79c). The gas pressure perturbation P_g and magnetic pressure perturbations P_h associated with Eqs. (254),

$$P_g=(\rho c^2/\omega)(v_{||}k_{||}+v_\perp k_\perp)$$
$$=(\rho/\omega)c^2k_\perp v_\perp/(1-k_{||}^2c^2/\omega^2)\ , \quad (255a)$$

$$P_h=\mu h_{||}B/4\pi=(\mu B^2/4\pi\omega)v_\perp k_\perp=(\rho/\omega)a^2k_\perp v_\perp\ , \quad (255b)$$

add to the total pressure:

$$P_*\equiv P_g+P_h$$
$$=(\rho/\omega)(c^2+a^2)k_\perp v_\perp[(\omega^2-k_{||}^2b^2)/(\omega^2-k_{||}^2c^2)]\ , \quad (256)$$

where we have introduced the tube speed.

4. Alfvén waves on a current sheet

We consider an interface $z=0$, separating two regions of uniform transverse and parallel magnetic fields (B_i for $z<0$ and B_e for $z>0$),

$$\mathbf{B}(z)=[B_i+(B_e-B_i)H(z)]\mathbf{e}_x\ , \quad (257a)$$

$$\mathbf{J}(z)=(c_*/4\pi\mu)(B_e-B_i)\delta(z)\mathbf{e}_y\ , \quad (257b)$$

where $H(z)$ denotes Heaviside's unit function (1 for $z>0$ and 0 for $z<0$), and $\delta(z)$ Dirac's δ function, showing that the interface is a current sheet, with surface current proportional to the difference of magnetic fields. [Equation (257b) follows from (257a) using Maxwell's equation, $\nabla\Lambda\mathbf{B}=(4\pi\mu/c_*)\mathbf{J}$.] Considering three-dimensional waves, the wave fields are given on the two sides of the interface by

$$v_z^{i,e}(x,y,z,t)=W_{i,e}(z)\exp[i(k_xx+k_yy-\omega t)]\ , \quad (258a)$$

$$W_{i,e}(z)=A_{i,e}\exp(ik_{i,e}z)\ , \quad (258b)$$

where we have separated the transverse wave numbers k_x,k_y which are continuous, from the wave number k_z normal to the interface, which is discontinuous and given by Eq. (252b):

$$k_{i,e}^2=\frac{(\omega^2-k_x^2a_{i,e}^2)(\omega^2-k_x^2c^2)}{[\omega^2(a_{i,e}^2+c^2)-k_x^2a_{i,e}^2c^2]}\ . \quad (259a)$$

Here $k_x\equiv k_{||}$ is the wave number along the magnetic field, and the sound speed c is continuous, but not the Alfvén speed $a_{i,e}$. Noting that the wave number transverse to the magnetic field is $k_\perp\equiv(k_y^2+k_{i,e}^2)^{1/2}$ we obtain for the total pressure [Eq. (256)]

$$P_{i,e}(z)=(\rho_{i,e}/\omega)(\omega^2-k_x^2a_{i,e}^2)$$
$$\times(k_y^2+k_{i,e}^2)^{-1/2}W_{i,e}(z)\ . \quad (259b)$$

The boundary conditions at the interface state the continuity of normal velocity W_z (or displacement $-iW_z/\omega$) and total pressure,

$$W_i(0)=W_e(0)\ , \quad (260a)$$

$$P_i(0)=P_e(0)\ . \quad (260b)$$

The first condition (260a) implies that the amplitudes of the waves [Eq. (258b)] are the same on both sides of the interface $A_i=A_e$, and the second condition (260b) yields, by Eq. (259b), the dispersion relation

$$\rho_i(\omega^2-k_x^2a_i^2)(k_y^2+k_e^2)^{1/2}+\rho_e(\omega^2-k_x^2a_e^2)(k_y^2+k_i^2)^{1/2}=0\ , \quad (261)$$

where we have taken positive roots of the radicals. The dispersion relation can be analyzed in three cases: (i) waves propagating on both sides of the interface $k_i^2,k_e^2>0$; (ii) waves propagating on one side and evanescent on the other, i.e., tunneling $k_i^2k_e^2<0$; (iii) waves evanescent on both sides, i.e., surface waves $k_i^2,k_e^2<0$. We give two examples of case (iii): (a) In the incompressible limit $c\to\infty$, for which $k_{i,e}^2\to -k_x^2$ by Eq. (259a), the dispersion relation (261) implies that

$$a_0^2\equiv\omega^2/k^2=(\rho_ia_i^2+\rho_ea_e^2)/(\rho_i+\rho_e)\ , \quad (262)$$

showing that Alfvén surface waves propagate at a phase speed a_0, which is the root mean square (rms) of the Alfvén speeds on the two sides of the interface, weighted by the mass densities. (b) This phase speed is also obtained from Eq. (261), in the limit of short waves transverse both to the interface and to the magnetic field $k_y^2\gg k_i^2,k_e^2$, which are compressible surface modes but also propagate at the rms Alfvén speed.

5. Cutoff frequencies for the tube modes

We now turn to vertical (instead of three-dimensional) motions, in a flux tube (instead of an interface), including gravity (which was omitted in Secs. IV.B.3 and IV.B.4). Thus we start from the general equation (10) without rotation, $\mathbf{\Omega}=0$, with uniform gravity pointing downwards, $\mathbf{g}=(0,0,-g)$, and horizontal magnetic field varying with altitude $B(z)$, leading to the following equation for the velocity $\mathbf{v}\equiv\dot{\xi}$ perturbation spectrum [Eq. (167)], with $W_z\equiv W$:

$$L^2W''-2LW'+[(\omega^2-\omega_0^2)L^2/b^2+(1-\gamma/2)(\omega_0L/c)^2]W=0\ , \qquad (263)$$

where the gravity cutoff is given, in nonisothermal conditions, by

$$\omega_0^2\equiv(g/L)(1-1/\gamma+L')\ , \qquad (264a)$$

$$W(z;\omega)=\exp[-n(z)/4]X(z;\omega)\ . \qquad (264b)$$

The transformation (264b) eliminates the middle term and shows that X satisfies the standard equation

$$X''+[(\omega^2-\omega_v^2)L^2/b^2]X=0\ , \qquad (265a)$$

$$\omega_v^2\equiv\omega_0^2+(b/L)^2(\tfrac{3}{4}-1/\gamma)(\tfrac{3}{4}-1/\gamma+L')\ , \qquad (265b)$$

with cutoff frequency ω_v for the velocity. The pressure perturbation is related to the velocity by Eq. (8c), so that

$$\bar{P}(z;\omega)=(\rho/\omega)[gW+(c^2-a^2/2L)W']\ , \qquad (266)$$

implying a generally different wave equation and cutoff frequency ω_p. In the case of an isothermal flux tube, all coefficients are constant, and the cutoff frequencies for the pressure ω_p and velocity [Eq. (265b)] coincide:

$$\omega_v^2=(g/L)(1-1/\gamma)^2+(b/L)^2(\tfrac{3}{4}-1/\gamma)^2=\omega_p^2\ . \qquad (267)$$

This result is analogous to the acoustics of horns (see Secs. IV.A.4 and IV.A.5 in Part I), for which the cutoff frequencies are generally different for the pressure ω_p and velocity ω_v, and only coincide, $\omega_p=\omega_v$, for an exponential duct of cross section $S(x)=S(0)\exp(\pm x/L)$, for which the mass of fluid per unit length varies exponentially $m(x)=m(0)\exp(\mp x/L)$. For longitudinal modes in a magnetic flux tube, the cutoffs coincide in the isothermal case, for which the mass density evolves exponentially, $\rho(x)=\rho(0)\exp(\mp x/L)$, respectively, for a wave propagating upward, $x=z$, or downward, $x=-z$, where x is the coordinate in the direction of propagation. Thus the analogs of acoustic waves in a rigid horn are tube modes in a flux tube, for which the sound speed c is replaced by the tube speed b [Eq. (253b)].

6. Duct, acoustic, and Alfvén analogies

In the isothermal case, the vertical velocity [Eq. (264b)] of the tube mode (265a) is given by

$$W(z;\omega)=W(0;\omega)e^{z/4L}\exp(\pm iK_*z)\ , \qquad (268a)$$

$$K_*=(\omega/b)[1-(\omega_v/\omega)^2]^{1/2}\ , \qquad (268b)$$

where the effective wave number K_* [Eq. (268a)] is given by the same expression (210a) as for acoustic-gravity waves, with the tube speed b [Eq. (253b)] replacing the sound speed c, and the cutoff frequency ω_v [Eq. (265b)] the acoustic cutoff ω_2 [Eq. (22b)]. The exponential amplitude growth, on four times the scale height, is different from that of acoustic-gravity waves, which grow exponentially on twice the scale height [Eq. (81c)]. The reason is that an acoustic-gravity wave samples a constant section of the atmosphere, so that the conservation of the energy flux, $F=\rho cW^2$, requires that the velocity perturbation

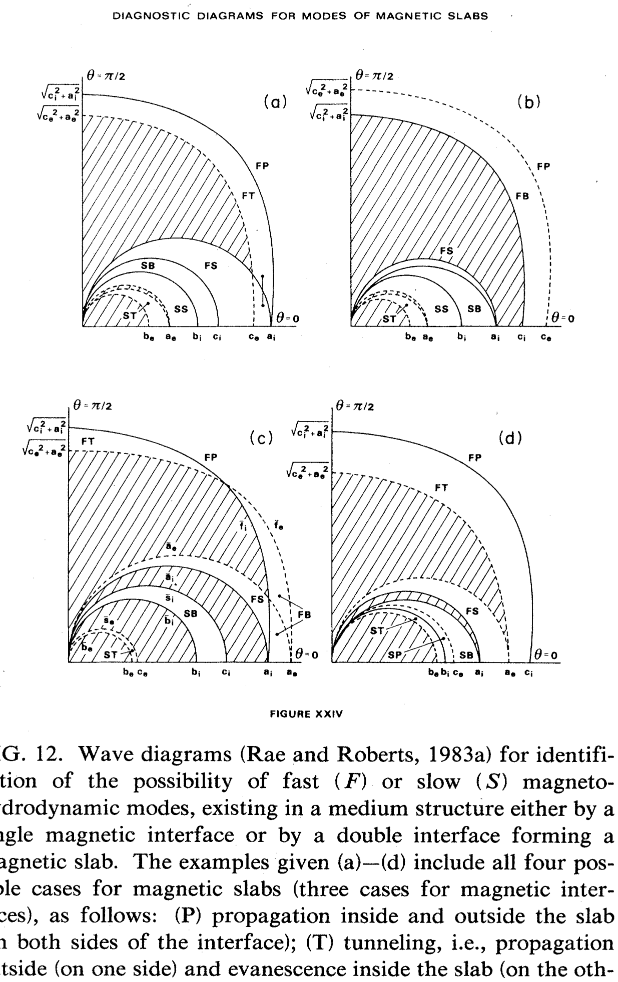

FIG. 12. Wave diagrams (Rae and Roberts, 1983a) for identification of the possibility of fast (F) or slow (S) magnetohydrodynamic modes, existing in a medium structure either by a single magnetic interface or by a double interface forming a magnetic slab. The examples given (a)—(d) include all four possible cases for magnetic slabs (three cases for magnetic interfaces), as follows: (P) propagation inside and outside the slab (on both sides of the interface); (T) tunneling, i.e., propagation outside (on one side) and evanescence inside the slab (on the other side of the interface); (S) surface waves, evanescent both outside and inside the slab (on both sides of the interface); (B) body waves, propagating inside the slab and evanescent outside. The polar coordinates are the phase speed for radius, and the angle θ of the wave vector with the normal to the interface(s); a,b,c denote the Alfvén, tube, and sound speeds, respectively; $\bar{a},\bar{b},\bar{s},\bar{f}=(a,b,s,f)\cos\theta$ are the projections of the Alfvén wave speed a, tube speed b, slow s, and fast f wave speeds along the normal to the interface(s); the suffixes i,e stand for interior and exterior of the slab, or the medium on each side of the interface.

grows on the inverse square root of the mass density $W\sim 1/\sqrt{\rho}$, i.e., exponentially on twice the scale height $W\sim e^{z/2L}$ in the isothermal case, viz., the velocity perturbation e-folds in two scale heights. A flux tube in an atmosphere fans out with altitude, so that the conservation of the total flux, $F=\rho b W^2 R^2$, introduces in the amplitude of the velocity perturbation a factor like the inverse square root of the tube's radius $W\sim 1/\sqrt{\rho R}$, which reduces the rate of growth (251) to half the former value, i.e., exponential of 4 times the scale height $W\sim e^{z/4L}$. This is the rate of growth (107a) for Alfvén waves of large compactness (Sec. II.B.4), which are dominated by the magnetic pressure; a similar scaling applies to compressive waves in a flux tube, because the boundary conditions imply that the gas pressure evolves in a similar way to the magnetic pressure. We have given examples of three types of magneto-hydrodynamic waves, in particular cases of magnetically structured atmospheres, viz., Alfvén and fast surface modes on a current sheet (Sec. IV.B.4), and slow tube modes in a slender flux tube (present section). The same three types appear in more complex structures, with a large number of coupling possibilities if there is more than one current sheet, as in the case of a magnetic slab, illustrated diagramatically in Fig. 12. Having considered nondissipative propagation in Secs. IV.B.3 to IV.B.6, we conclude the examples of linear waves in magnetically structured atmospheres by considering damped modes (Secs. IV.B.7 and IV.B.8). Of the main magneto-hydrodynamic (as distinct from plasma) dissipation mechanisms, we have already discussed fluid viscosity (Sec. V.A of Part I) and electrical resistance (IV.A), so we consider next thermal diffusion. The consideration of nonlinear acoustic waves has led to problems reducible to the heat conduction equation (Sec. V.C. of Part I), and thus we examine next thermal radiation, which is of particular importance in solar and stellar atmospheres (Kneer and Heasley, 1979; Kneer, 1980, 1983). In order to retain the comparison of magneto-atmospheric waves with flux tube modes we shall consider compressive motions in a flux tube with radiative relaxation.

7. Radiative damping in a magnetic cylinder

We shall consider radiative damping according to Newton's law of cooling, which replaces the energy equation (5) by

$$(\dot{P}+\mathbf{v}\cdot\nabla P)=c^2(\dot{\Gamma}+\mathbf{v}\cdot\nabla\Gamma)-P/\tau_R \ , \tag{269a}$$

where τ_R is the radiative damping time. In the limit of no damping, $\tau_R\to\infty$, Eq. (269a) reduces to the adiabatic condition (6a), where c is the adiabatic sound speed (6b); in the opposite limit, of radiative damping time much shorter than wave period τ, Eq. (269a) simplifies to $dP/dt=-P/\tau_R$ for $\tau_R\ll\tau$, showing that, in a convected frame, $d/dt\equiv\partial/\partial t+\mathbf{v}\cdot P$, the pressure $P(t)=P(0)\exp(-t/\tau_R)$ decays exponentially in the time scale τ_R. Linearizing Eq. (269a), and eliminating for v_z, together with the equations of momentum (7b), induction (8a) and continuity (8b), we obtain for a planar geometry, appropriate to free space or a magnetic interface or slab, an equation with plane-wave solutions

$$v_z''-(k_i/\omega)\ddot{v}_z=0 \ , \tag{269b}$$

$$v_z(z,x,t)=v_0\exp[i(k_0z+k_{\|}x-\omega t)] \ , \tag{269c}$$

where the wave number k_i in the presence of radiation is given by

$$k_i^2=\frac{(\omega^2-k_{\|}^2a^2)(\omega^2-k_{\|}^2c^2\psi)}{k_{\|}^2a^2c^2\psi-\omega^2(a^2+c^2)\psi} \ , \tag{270a}$$

$$\psi\equiv(i\omega+1/\tau_R\gamma)/(i\omega+1/\tau_R) \ , \tag{270b}$$

i.e., in the limit of no damping, $\tau_R\to\infty$, $\psi\to 1$ and $k_i\to k_\perp$, so that Eq. (270a) reduces to (252b). In the case of cylindrical geometry, the wave number is the same, but Eq. (269a) is replaced by a Bessel type for the radial direction

$$[r^{-1}(rv_r)']'-(k_i/\omega)^2v_r=0 \ , \tag{271a}$$

$$v_i(r,z,t)=A_i\bar{J}_1(k_ir)\exp[i(k_{\|}z-\omega t)] \ . \tag{271b}$$

We have chosen a modified Bessel function of the first kind, $\bar{J}_1$, which is finite on the axis of the cylinder (and suppressed the function of the second kind, which would have a logarithmic singularity). For waves in free space the dispersion relation [(270a) and (270b)] may be written, in terms of the horizontal $k_{\|}$ and total $k^2=k_i+k_{\|}^2$ wave number,

$$\omega^5+(i/\tau_R)\omega^4-k^2(a^2+c^2)\omega^3-(i/\tau_R)k^2(c^2/\gamma+a^2)\omega^2 +k^2k_{\|}^2c^2a^2\omega+(i/\tau_R)k^2k_{\|}^2a^2c^2/\gamma=0 \ , \tag{272a}$$

which is of fifth degree in the frequency ω and involves the adiabatic $c^2=\gamma RT$ and isothermal $c^2/\gamma=RT$ sound speeds. In the limit of no damping, $\tau_R\to\infty$, the dispersion relation becomes

$$\omega^4-k^2(a^2+c^2)\omega^2+k^2k_{\|}^2a^2c^2=0 \ , \tag{272b}$$

or that of ordinary magneto-hydrodynamic waves [Eq. (44) with $u=\omega/k$ and $\mathbf{k}\cdot\mathbf{b}=k_{\|}$], which is a biquadratic (degree one unit lower in ω) and features only the adiabatic sound speed. In order to obtain the dispersion relation for radiative modes in a magnetic cylinder, we need not only the interior solution (271b), but also the exterior solution:

$$v_e(r,z,t)=A_e\bar{Y}_1(k_er)\exp[i(k_{\|}z-\omega t)] \ , \tag{273a}$$

$$k_e^2\equiv k^2-\omega^2/c_e^2\psi_e \ , \tag{273b}$$

where we have chosen in (273a) the modified Bessel function of the second kind $\bar{Y}_1$, which is bounded at infinity, and the radial wave number (273b) calculated from the sound speed, assuming the magnetic field to be zero outside the cylinder. Matching the radial velocity v_r and total pressure P_* across the radius R yields

$$\rho_i(\omega^2-k_{\|}^2a^2)k_e\bar{I}_0(k_iR)\bar{Y}_1(k_eR)+\rho_e\omega^2k_i\bar{I}_1(k_iR)\bar{Y}_0(k_eR)=0\ , \quad (274)$$

which is a transcendental dispersion relation for a magnetic cylinder of arbitrary radius.

8. Isothermal and adiabatic sound speeds

We choose for comparison the following two cases: (a) acoustic-gravity waves in a radiating unbounded atmosphere, in the absence of a magnetic field, $a=0$, for which the dispersion relation (272a) drops from fifth to third order:

$$\omega^3+(i/\tau_R)\omega^2-k^2c^2\omega-(i/\tau_R)k^2c^2/\gamma=0\ ; \quad (275a)$$

(b) tube modes in a slender magnetic cylinder, $k_iR \ll 1$, for which the transcendental dispersion relation (274) reduces to a cubic:

$$(c^2+a^2)\omega^3+(i/\tau_R)(a^2+c^2/\gamma)\omega^2-k^2c^2a^2\omega-(i/\tau_R)k^2a^2c^2/\gamma=0\ . \quad (275b)$$

In the adiabatic limit $\tau_R\to\infty$, Eq. (275a) gives pure acoustic waves $\omega^2=k^2c^2$, and Eq. (275b) pure tube modes $\omega^2=c^2a^2k^2/(a^2+c^2)=k^2b^2$, where b is the tube speed [Eq. (253b)]. Dispersion relations with complex roots can be analyzed either in space or in time. Spatial analysis assumes a real frequency ω and yields complex roots $k=k_r+ik_s$ for the wave number, whose real part specifies the wavelength $\lambda=2\pi/k_r$, and whose imaginary part the amplitude law in space, e.g., growth $|\exp(ikz)|=\exp(-k_sz)$ for $k_s<0$, as used before for magneto-atmospheric waves. Temporal analysis assumes a real wave number k and yields complex roots $\omega=\omega_r+i\omega_s$ for the frequency, whose real part specifies the wave period $\tau=2\pi/\omega_r$, and whose imaginary part the amplitude law in time, e.g., decay $|\exp(-i\omega t)|=\exp(\omega_s t)$ for $\omega_s<0$, for damped modes, which we consider below. We solve both dispersion relations [(275a) and (275b)] for small and large values of the ratio τ_R/τ_S, where $\tau_S\equiv(kc)^{-1}$ is the time taken by an acoustic wave to propagate a distance $1/k=\lambda/2\pi$, which is $(2\pi)^{-1}$ times the wavelength λ. In the limit $\tau_R\gg\tau_S$, radiative damping is a weak modification of adiabatic propagation: (i) in free space the acoustic wave propagates at a phase speed slightly lower than the adiabatic sound speed c,

$$\omega_{1,2}^s=\pm kc[1-(\gamma-1)(\gamma+3)\cos^2\theta(\tau_S/\tau_R)^2/8\gamma^2]-(i/2)(1-1/\gamma)kc(\tau_S/\tau_R)\ , \quad (276a)$$

where θ is the angle of the wave number to the vertical, $k_{\|}=k\sin\theta$, and is damped; (ii) in a slender flux tube, the phase speed is slightly lower than the tube speed b,

$$\omega_{1,2}^t=\pm kb\{1-(\gamma-1)[3+(c^2+\gamma a^2)/(c^2+a^2)]\times(\tau_S/\tau_R)^2/8\gamma^2\}-(i/2)(1-1/\gamma)(b/c)^2/\tau_R\ , \quad (276b)$$

and there is also damping; (iii) in both cases there is a third, purely damped mode,

$$\omega_3^{s,t}=-ik(c/\gamma)(\tau_R/\tau_S)\ , \quad (277a)$$

$$\exp(-i\omega_3t)=\exp[-(c\tau_R/\gamma\tau_S)kt]\ . \quad (277b)$$

In the limit $\tau_R\ll\tau_S$, radiative damping is sufficiently strong to overcome adiabaticity, and imposes isothermal conditions: (i) in free space, the phase speed is slightly larger than the isothermal sound speed c_0,

$$\omega_{1,2}^s=\pm kc_0[1+(\gamma-1)(\gamma-5)/8\gamma\cos^2\theta(\tau_S/\tau_R)^2]-(i/2)(1-1/\gamma)kc/\cos\theta(\tau_S/\tau_R)\ , \quad (278a)$$

$$c_0^2=c^2/\gamma=RT\ ; \quad (278b)$$

(ii) in a slender flux tube, the phase speed is slightly larger than the isothermal tube speed b_0

$$\omega_{1,2}^t=\pm kb_0\{1+(\gamma-1)a^2[4c^2+(5-\gamma)a^2](\tau_R/\tau_S)^2/(a^2+c^2/\gamma)^2\}-(i/2)(1-1/\gamma)a^4(\tau_R/\tau_S)/c^2/\gamma+a^2\ , \quad (279a)$$

$$b_0^{-2}=\gamma/c^2+a^{-2}=c_0^{-2}+a^{-2}\ ; \quad (279b)$$

(iii) in the third, purely damped mode, the phase speed is different in free space,

$$\omega_3^s=-ikc(\tau_S/\tau_S)\ , \quad (280a)$$

and in the flux tube,

$$\omega_3^t=-i(kc/\gamma)(b/b)^2(\tau_S/\tau_R)\ . \quad (280b)$$

C. Nonlinear dispersive waves and solitons

The preceding account of linear wave modes in magnetically structured atmospheres has outlined some of their similarities with, and differences from magneto-atmospheric waves. Although the topic of flux tube and current sheet waves is barely a decade old, research on this subject has been pursued vigorously, and a reasonably exhaustive survey would require a full-size review, which unfortunately is not available, so that reference to the original papers is essential. Since the current limit of "high-resolution" observations of the sun (Bruecker, 1980) is about 0.5 arc seconds, equivalent to 700 km, and the transverse scale of flux tubes is about 0.1 arc seconds, a new generation of instruments, possibly space-based, with an order-of-magnitude greater resolution, will be needed before "sausage," "kink," or torsional tube modes can be observed; it has been claimed (Roberts, Edwin, and Benz, 1983, 1984) that certain observations of short-period coronal transients can be related to tube modes. An even more recent subject, developed in the last few years, is that of nonlinear, dispersive waves or "solitons" in magnetic flux tubes (Roberts and Mangeney, 1982; Roberts, 1983b, 1984b, 1986; Hollweg and Roberts, 1984; Ruderman and Merzlakov, 1984; Edwin and Roberts, 1986; Merzlakov and Ruderman, 1986). No direct observational evidence is yet claimed for the existence of "solitons" in the solar atmosphere, although such phenomena as coronal mass ejections or "plasmoids" and moving magnetic "features" in the chromosphere might qualify. Other types of nonlinear, transient disturbances in magnetic atmospheres have been considered, including waves (Chiu, 1971; Barnes and Hollweg, 1974; Adam, 1975; Lacombe and Mangeney, 1980; Shukla, 1983; Kalkofen, Ulmschneider, and Schmitz, 1984; Petrukhin and Fainshtein, 1984a, 1984b), shocks (Wentzel and Solinger, 1967; Hollweg, 1982b; Kumar, 1984) and nonlinear tube waves (Hollweg, Jackson, and Galloway, 1982; Herbold, Ulmschneider, Spruit, and Rosner, 1985; Maxwell, Pryer and McIntosh, 1985; Verma, Srivastava and Singh, 1985; Achterberg and Blandford, 1986; Srivastava, Leutloff, Vishwakarma, and Kumar 1986). An important nonlinear phenomenon in the solar atmosphere is magnetic reconnection (Priest, 1982b, 1985, 1986; Arion, 1984; Forbes and Priest, 1984; Heyvaerts and Priest, 1984; Sakai, Tajima, and Brunel, 1984; Forbes, 1986; Taylor, 1986), which is associated with the coalescence of null points, and thus rearrangement of the magnetic pattern. Such rearrangement certainly occurs during solar flares, a phenomenon that has been extensively studied. Observational studies include those of Svestka, 1972; Dennis, Frost, and Orwig, 1981; Orwig, Frost, and Dennis, 1981; Emslie, Brown, and Machado, 1981; van Beek *et al.*, 1981; Dere and Cook, 1983; Feldman, Doschek, and McKenzie, 1984; Fisher, Canfield, and McClymont, 1984; Fisher and Munro, 1984; Moore, Hurford, Jones, and Kane, 1984; Seely and Feldman, 1984; Tandberg-Hanssen *et al.*, 1984; Veck *et al.*, 1984; De Jager and Svestka, 1985; Pudovkin, Zaitseva, and Puchenkina, 1985; Raoult, Pick, Dennis, and Kane, 1985; Rust, Simnet, and Smith, 1985; Dulk, Bastian, and Kane, 1986; Tanaka and Zirin, 1986. Theoretical studies include those of Meyer, 1968; Levine and Nakagawa, 1974; Low, 1980; Ricchiazzi, and Canfield, 1983; MacNiece, McWhirter, Spicer, and Burgess, 1984; Ngai and Emslie, 1984; Fisher, Canfield, and McClymont, 1985a, 1985b, 1985c). The physics of solar flares is well documented in the literature, both in the form of symposia (Priest, 1982b) and of reviews (Priest, 1986); the applications of magnetic reconnection have been extended from the theory of solar flares to that of atmospheric heating (see Sec. V.A.2 below).

1. Semispectrum and Whitham's equation

We derive the general equation describing the propagation of weakly nonlinear, dispersive waves. We take as a starting point the dispersion relation for linear waves, or definition of phase speed $u(k)=\omega/k$,

$$\omega=u(k)k\ , \tag{281a}$$

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} W(k,\omega)e^{i(kz-\omega t)}dk\,d\omega + \frac{\partial}{\partial z}\int_{-\infty}^{+\infty} u(k)W(k,\omega)e^{i(kz-\omega t)}dk\,du=0\ , \tag{281b}$$

where (281b) follows from (281a) by assuming plane-wave propagation $\sim\exp[i(kz-\omega t)]$, for which $\omega\sim i\partial/\partial t$ and $k\sim -i\partial/\partial z$ when we multiply by the wave spectrum $W(k,\omega)$ and integrate over wave number k and frequency ω. We may introduce the semispectrum U, which depends on wave number and time:

$$U(k,t)=(2\pi)^{-1}\int_{-\infty}^{+\infty} W(k,\omega)e^{i\omega t}d\omega\ , \tag{282a}$$

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} U(k,t)e^{ikz}dk + \frac{\partial}{\partial z}\int_{-\infty}^{+\infty} u(k)U(k,t)e^{ikz}dk=0\ , \tag{282b}$$

where (282b) is obtained from (282a) by integrating in frequency; if we integrate further in wave number k, we obtain the velocity perturbation

$$v(z,t)=\int_{-\infty}^{+\infty}\int W(k,\omega)e^{i(kz-\omega t)}dk\,d\omega = \int_{-\infty}^{+\infty} U(k,t)e^{ikz}dk\ , \tag{283a}$$

$$\partial v/\partial t+\beta_* v\partial v/\partial z+(\partial/\partial z)\int_{-\infty}^{+\infty} u(k)U(k,t)e^{ikz}dk=0\ , \tag{283b}$$

where (283b) coincides with (282b) in the first and third terms, which are linear in v, and we have added arbitrarily the second, nonlinear term, in the form of a convective acceleration $v\partial v/\partial x$, multiplied by a parameter β_* that indicates the importance of nonlinearity relative to dispersion and is determined by the physics of the particular problem, as an input to the present general procedure. The third term of the wave equation (283b) involves the semispectrum $U(k,t)$, instead of the velocity perturbation $v(z,t)$. We express it in terms of the velocity perturbation by using the inverse of Eq. (283a):

$$\int_{-\infty}^{+\infty} U(k,t)u(k)e^{ikz}dk=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int u(k)v(y,t)e^{ik(z-y)}dydk \tag{284a}$$

$$=\int_{-\infty}^{+\infty} v(y,t)G(y-z)dy\equiv v_* G, G(z)\equiv(2\pi)^{-1}\int_{-\infty}^{+\infty} u(k)e^{-ikz}dk\ . \tag{284b}$$

$G(\xi)$ is the inverse Fourier transform of the phase speed [Eq. (284b)], and Eq. (284a) is its convolution with the velocity perturbation $v(z,t)$. Substituting Eq. (284a) into (283b), we obtain the Whitham integro-differential equation for weakly nonlinear, dispersive waves:

$$\partial v/\partial t+\beta_* v\partial v/\partial z+(\partial/\partial z)\int_{-\infty}^{+\infty}G(y-z)v(y,t)dy=0\ , \tag{285}$$

whose terms may be interpreted as follows, taking the wave variable $v(z,t)$ to be velocity perturbation: (i) the first term is the local acceleration, or local time derivative, for unidirectional waves; (ii) the second term is a convective acceleration $v\partial v/\partial z$, with a nonlinearity factor β_* determined from the equations of motion, e.g., by a multiple-scales approach; (iii) the last term is obtained from the phase speed $u(k)$ of a linear, dispersive wave, by taking its inverse Fourier transform (248b) to obtain the phase velocity $G(\xi)$, then convoluting with the wave perturbation, and finally differentiating with regard to z.

2. Long "sausage" modes in a nonisothermal slab

As a first example of the application of Whitham's equation (285), we consider waves in a magnetic slab, i.e., a region $z<|z_0|$ of uniform transverse magnetic field, $\mathbf{B}=B_0\mathbf{e}_x$, in a medium otherwise not subjected to magnetic fields:

$$\mathbf{B}=B_0[H(z+z_0)-H(z-z_0)]\mathbf{e}_x\ , \tag{286a}$$

$$\mathbf{J}=(c_*B_0/4\pi\mu)[\delta(z+z_0)-\delta(z-z_0)]\mathbf{e}_y\ , \tag{286b}$$

where H and δ denote Heaviside's unit function and Dirac's δ function, respectively, and Eq. (286b) shows that the interfaces $z=\pm z_0$ are current sheets. If we ignore gravity, the velocity perturbation of a two-dimensional wave in the plane (x,z) satisfies Eqs. (252a) and (252b) inside the slab:

$$W_i''+k_i^2W_i=0\ , \tag{287a}$$

$$k_i^2\equiv(\omega^2-k_{\|}^2a^2)(\omega^2-k_{\|}^2c_i^2)/(a^2+c_i^2)(\omega^2-k_{\|}^2b^2)\ , \tag{287b}$$

where b is the tube speed (253b) calculated for the Alfvén speed $a\equiv a_i$ and sound speed c_i inside the slab. Outside the slab there is no magnetic field, $a_0=0$, and if the temperature is different, the external sound speed c_e will be different from the internal value c_i, and Eqs. (287a) and (287b) will be replaced by the equations

$$W_e''+k_e^2W_e=0\ , \tag{288a}$$

$$k_e^2=k_{\|}^2-\omega^2/c_e^2\ . \tag{288b}$$

The solutions of Eqs. (287a) and (288a) are

$$W(z,\omega)=\begin{cases}A^+\exp[+k_e(z+z_0)],\quad z\le -z_0\ , & (289a)\\ A_+\cosh(k_iz)+A_-\sinh(k_iz)\ ,\quad -z_0\le z\le z_0\ , & (289b)\\ A^-\exp[-k_e(z-z_0)],\quad z\ge z_0\ , & (289c)\end{cases}$$

where outside the slab $|z|\ge z_0$ we have chosen exponential fields decaying at infinity [Eqs. (289a) and (289c)] with amplitudes $A^\pm$, and inside the slab [Eq. (289b)] there is a superposition of symmetric or "sausage" and unsymmetric or "kink" modes, respectively, the cosh and sinh terms with amplitudes A_+ and A_-. The four amplitudes, for two modes outside the slab ($A^\pm$) and two waves inside the slab ($A_\pm$), are related by four boundary conditions, stating the continuity of normal velocity (289) and total (gas plus magnetic) pressure (256) at the interface. This leads to the dispersion relation

$$\rho_i(\omega^2-k_{\|}^2a^2)k_e=\rho_e\omega^2k_i{}^{\tanh}_{\coth}(k_iz_0)\ , \tag{290a}$$

valid for $\omega^2\ge k_{\|}^2c_e^2$, where tanh and coth refer to "sausage" and "kink" modes, respectively. We consider the symmetrical "sausage" mode, in the limit of long wavelength, $k_iz_0\ll 1$, for which Eq. (290a) simplifies to

$$\rho_i(\omega^2-k_{\|}^2a^2)k_e=\rho_e\omega^2k_i^2z_0\ , \tag{290b}$$

$$\rho_ik_e(a^2+c_i^2)(\omega^2-k_{\|}^2b^2)=\rho_e\omega^2(\omega^2-k_{\|}^2c_i^2)z_0\ . \tag{290c}$$

The Alfvén mode $(\omega^2-k_{\|}^2a^2)$ in Eq. (290b) drops out when we use Eq. (287b) to deduce (290c). The latter shows that two long "sausage" modes are possible, $\omega^2\sim k_{\|}^2b^2, k_{\|}^2c_e^2$, in a nonisothermal magnetic slab:

$$\omega^2=k_{\|}^2b^2[1-kz_0\rho_ec_e(c_i^2-b^2)/\rho_i(c_i^2+a^2)(c_e^2-b^2)^{1/2}]\ , \tag{291a}$$

$$\omega^2=k_{\|}^2c_e^2\{1-[kz_0\rho_ec_e^2(c_e^2-c_i^2)/\rho_i(c_i^2+a^2)(c_e^2-b^2)]^2\}\ . \tag{291b}$$

These are (a) a "tube" mode $\omega\sim\pm k_{\|}b$, if [Eq. (291a)] the tube speed b is less than the external c_e sound speed, $b<c_e$; (b) an "acoustic" mode $\omega\simeq\pm k_{\|}c_e$, if [Eq. (291b)] the slab is either slightly cooler than the surrounding medium, $c_e>c_i$, or sufficiently hotter, $c_e<b<c_i$.

3. Korteweg-De Vries and Benjamin-Ono equations

In an isothermal slab, the only long "sausage" mode is the tube wave [Eq. (291a)], for which the phase speed $u(k)=\omega/k$ is

$$u_1(k)=b(1-\alpha_i|k|)\ , \tag{292a}$$

$$\alpha_i\equiv(z_0/2)(\rho_e/\rho_i)(1+a^2/c^2)^{-3/2}\ . \tag{292b}$$

In a nonisothermal slab sufficiently hotter than the environment, $b>c_e$, we have the acoustic mode [Eq. (291b)] with phase speed

$$u_2(k)=c_e-\alpha_ek^2\ , \tag{292c}$$

$$\alpha_e\equiv(c_ez_0^2/2)(\rho_e/\rho_i)[c_e^2(c_e^2-c_i^2)/(c_i^2+a^2)(c_e^2-b^2)]\ . \tag{292d}$$

If the slab is slightly cooler than the surrounding medium, $c_i<c_e$, we have both the "acoustic" [Eqs. (292c) and (292d)] and the "tube" [Eqs. (292a) and (292b)] modes, the latter with parameter α_i given by

$$\alpha_i=(z_0/2)(\rho_e/\rho_i)[(c_i^2-b^2)/(c_i^2+a^2)](1-b^2/c_e^2)^{-1/2} , \tag{292e}$$

which reduces to Eq. (292b) in the isothermal case $c_e=c_i\equiv c$. We may write the two phase speeds [(292a) and (292c)] together in the single expression:

$$u(k)=u_0(1-\alpha_i\,|\,k\,|\,)-\alpha_e k^2 , \tag{293a}$$

$$u_0\equiv u(0)=b,c_e , \tag{293b}$$

for the tube ($u_0=b,\alpha_e=0$) and acoustic ($u_0=c_e,\alpha_i=0$) modes, respectively. Taking the Fourier transform (284b) of Eq. (293a), and then (284c), we find that the convolution with the velocity perturbation yields

$$G(z)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}[u_0(1-\alpha_i\,|\,k\,|\,)-\alpha_e k^2]e^{-ikz}dk=u_0\delta(z)+\alpha_i/\pi z^2+\alpha_e\delta''(z) , \tag{293c}$$

$$V_*G=\int_{-\infty}^{+\infty}v(y,t)G(y-z)dy=u_0 v(z,t)+\frac{\alpha_i}{\pi}\frac{d}{dz}\int_{-\infty}^{+\infty}v(y,t)(y-z)^{-1}dy+\alpha_e(\partial^2/\partial z^2)v(z,t) , \tag{293d}$$

which, substituted in the Whitham equation (285), yields

$$\partial v/\partial t+u_0\partial v/\partial z+\beta_* v\partial v/\partial z+\alpha_e\partial^2 v/\partial z^3-(\alpha_i/\pi)(\partial^2/\partial z^2)\int_{-\infty}^{+\infty}(y-z)^{-1}v(y,t)dy=0 . \tag{293e}$$

This is the Kortweg-De Vries—Benjamin-Ono equation, whose terms are interpreted as a plane, unidirectional wave of speed u_0 (first two terms), with a convective effect with nonlinearity factor β_* (third term), and surface and internal dispersion with coefficients α_e,α_i (fourth and fifth terms, respectively), so that Eqs. (293e) is a nonlinear doubly dispersive wave equation. Setting $\alpha_e=0\neq\alpha_i$ and $u_0=b$ ($\alpha_e\neq 0=\alpha_i$ and $u_0=c_e$), we obtain the long longitudinal or "sausage" tube (acoustic) mode, in a magnetic slab, in an isothermal equilibrium slightly cooler than its surroundings, which satisfies the Benjamin-Ono (Korteweg-De Vries) equations, viz.,

$$\partial v/\partial t+b\partial v/\partial z+\beta_i v\partial v/\partial z+(\alpha_i/\pi)(\partial^2/\partial z^2)\int_{-\infty}^{+\infty}(y-z)^{-1}v(y,t)dy=0 , \tag{294a}$$

$$\partial v/\partial t+b\partial v/\partial z+\beta_e v\partial v/\partial z+\alpha_e\partial^3 v/\partial z^3=0 . \tag{294b}$$

Equation (294a) is the nonlinear internal wave equation, and (294b) the surface dispersion wave equation; the respective nonlinearity factors β_i and β_e are determined from the equations of motion, e.g., by a multipole scales approach. Both equations have "soliton" solutions, i.e., as "humps" of constant scale $1/l$, moving without deformation at constant speed s:

$$v(z,t)=v_0/[1+l_i^2(z-s_i t)^2] , \tag{295a}$$

$$l_i\equiv v_0\beta_i/4\alpha_i , \tag{295b}$$

$$s_i\equiv b+v_0\beta_i/4 , \tag{295c}$$

$$v(z,t)=v_0\mathrm{sech}^2[l_e(z-s_e t)] , \tag{296a}$$

$$l_e\equiv(v_0\beta_e/12\alpha_e)^{1/2} , \tag{296b}$$

$$s_e\equiv c_e+v_0\beta_3/3 . \tag{296c}$$

Thus we have an algebraic [Eqs. (295a)—(295c)] "hump" for internal nonlinear dispersive waves and a transcendental [Eq. (296a)—(296c)] "hump" for surface nonlinear dispersive waves.

4. Leibovich-Pritchard-Roberts equation in a magnetic cylinder

Since a magnetic slab is a degenerate two-dimensional flux tube, we may expect a three-dimensional, axisymmetric magnetic waveguide, which also supports tube modes, to lead to a modified Benjamin-Ono equation (Roberts, 1986), which may be called the Leibovich-Pritchard-Roberts equation, having been found, in the first instance, for weakly nonlinear waves in rotating fluids (Leibovich, 1970; Pritchard, 1970, studies the same problem but does not give the equation explicitly). A magnetic field $\mathbf{B}=B(z)\mathbf{e}_x$ varying transversally, i.e., oriented in the x direction but varying in the z direction, is compatible (Sec. IV.B.1) with one-dimensional magneto-hydrostatic equilibrium, with gravity $\mathbf{g}=-g\mathbf{e}_z$ in the z direction. If we consider a uniform field (or one varying by jumps across current sheets), in the absence of gravity, $\mathbf{g}=0$, we obtain Eqs. (287a) and (287b) for two-dimensional waves. If we allow the magnetic field $\mathbf{B}=B_0(z)\mathbf{e}_x$ and density $\rho(x)$ to vary continuously, then Eqs. (287a) and (287b) are replaced by

$$[\rho(\omega^2-k_{\|}^2a^2)W_z']'-\rho k_i^2(\omega^2-k_{\|}^2a^2)W_z=0 , \tag{297a}$$

$$(\omega^2-k_{\|}^2a^2)W_y=0 . \tag{297b}$$

In the case of a homogeneous medium and uniform magnetic field, ρ, $B_0\sim$const, the Alfvén mode $\omega=\pm k_{\|}a$ decouples in Eq. (297a) and we regain Eq. (287a). In the case of three-dimensional disturbances ($k_i\equiv k_2$, $k_{\|}\equiv k_x$, $k_y\neq 0$), the velocity perturbation in the y direction, transverse to the directions of stratification and of the magnetic field, is decoupled (as in Secs. II.A.2 and II.A.3), and the corresponding mode is an Alfvén wave [Eq. (297b)] leading, in the case of nonuniform Alfvén speed $a(z)$, to the appearance of the Alfvén continuum (Rae and Roberts, 1981). In the case of three-dimensional disturbances, we can also use cylindrical coordinates, with r,θ,x

the radial, azimuthal, and axial coordinates, respectively. For example, for an axisymmetric magnetic field $\mathbf{B}=B(r)\mathbf{e}_x$, i.e., parallel to the axis and varying radially, the radial velocity perturbation is given by

$$r^2W''+rW'-(n^2+k_\perp^2r^2)W=0 , \tag{298a}$$

$$v_r(r,\theta,x,t)=W(r)\exp[i(n\theta+k_\parallel x-\omega t)] , \tag{298b}$$

where the radial $k_\perp\equiv k_r$ and axial $k_\parallel\equiv k_x$ wave numbers are related in the same way as for a slab [Eqs. (252a), (253a), and (253b)]. We have assumed that the magnetic field is uniform, or varies by jumps across cylindrical surfaces, and for the fundamental ($n=0$) axisymmetric mode Eq. (298a) reduces to Eq. (271a) with $v_r\equiv W$. An example is the case of a magnetic cylinder of radius R, for which the dispersion relation (274) simplifies, in the long-wavelength limit $k_\perp r\ll 1$, to

$$2\rho_i k_e(\omega^2-k_\parallel^2a^2)\overline{Y}_1(k_eR)+\rho_e k_i^2R\omega^2\overline{Y}_0(k_iR)=0 , \tag{299a}$$

which corresponds to a tube wave, of phase speed $u_3=\omega/k_\parallel$ given by

$$u_3(k)=b-2\alpha_t k^2\overline{Y}_0[(b/a)\,|\,k\,|\,R] , \tag{299b}$$

$$\alpha_t\equiv(\rho_e/\rho_i)(b/a)^4bR^2/8 . \tag{299c}$$

Using the inverse Fourier transform of the modified Bessel function of the second kind $\overline{Y}_1$, we obtain

$$\begin{aligned}G(z)&=\int_{-\infty}^{+\infty}\{b-2\alpha_t k^2\overline{Y}_0[(b/a)\,|\,k\,|\,R]\}e^{-ikz}dz\\&=b\delta(z)+(\alpha_t/\pi)(d^2/dz^2)[(bR/a)^2+z^2]^{-1/2} .\end{aligned} \tag{299d}$$

Substitution in the Whitham equation (285) leads to the Leibovich-Pritchard-Roberts equation:

$$\partial v/\partial t+b\partial v/\partial z+\beta_t v\partial v/\partial z+(\alpha_t/\pi)(\partial^3/\partial z^3)\int_{-\infty}^{+\infty}[(bR/a)^2+(y-z)^2]^{-1/2}v(y,t)dy=0 , \tag{300}$$

which describes long longitudinal tube modes in a magnetic cylinder of radius R.

5. Benjamin-Ono-Burgers equation for thermal diffusion

As a final example of the Whitham equation, we reconsider long tube modes (Secs. IV.C.2 and II.C.3), in the presence of thermal conduction and radiation; the equation of adiabaticity (6a) is replaced, in the presence of radiative cooling according to Newton's law, by Eq. (269), and if we include thermal conduction ε_t, gives way to the equation of energy (5), in the form

$$\begin{aligned}&\Gamma C_p(\dot{T}+\mathbf{V}\cdot\nabla T)-(\dot{P}+\mathbf{V}\cdot\nabla P)\\&\qquad=\varepsilon_t\nabla^2T-\rho(\Gamma C_v/\tau_R)(T-T_0) ,\end{aligned} \tag{301a}$$

where C_v,C_p are the specific heats at constant volume and pressure, and T,T_0 the temperature of the gas and its value for isothermal equilibrium. A multiple-scales approach, applied to the equation of energy, together with continuity and momentum,

$$\begin{aligned}&\partial(\Gamma S)/\partial t+\partial(\Gamma SV)/\partial x\\&\qquad=0=\partial V/\partial t+V\partial V/\partial z+\Gamma^{-1}\partial P/\partial z ,\end{aligned} \tag{301b}$$

where S is the cross section of the magnetic flux tube, leads, for a perfect gas $P=R\Gamma T$, to (Edwin and Roberts, 1986)

$$\begin{aligned}&\partial v/\partial t+b\partial v/\partial z+\beta_i v\partial v/\partial z+(\alpha_i/\pi)(\partial^2/\partial z^2)\\&\qquad\times\int_{-\infty}^{+\infty}(y-z)^{-1}v(y,t)dy=\alpha_c\partial^2v/\partial z^2+\alpha_r v .\end{aligned} \tag{302a}$$

Here α_i is given by Eq. (292b) and the remaining coefficients $\beta_i,\alpha_c,\alpha_r$ by

$$\beta_i=a^2[(\gamma+1)a^2+3c^2]/2(c^2+a^2)^2 , \tag{302b}$$

$$\alpha_c=(\gamma-1)b^2\varepsilon_t/2\rho C_pc^2 , \tag{302c}$$

$$\alpha_r=(\gamma-1)b^2/2\gamma c^2\tau_R . \tag{302d}$$

The Benjamin-Ono-Burgers equation (302a), describes long nonlinear "sausage" modes in a thermally conducting and radiating magnetic flux tube, and consists of the following terms: (i),(ii) linear plane waves propagating at the tube speed [Eq. (253b)]; (iii) convective acceleration, with nonlinearity factor (302b); (iv) internal dispersion [Eq. (292b)], as in the Benjamin-Ono equation (294a); (v) diffusion by thermal conduction [Eq. (302b)], as in the Burgers equation [Eq. (283) of Part I]; (vi) diffusion by thermal radiation [Eq. (302d)], leading to a term $\sim v$, as in the Klein-Gordon equation. Comparing Eq. (302a) with the Whithan equation (285), it is clear that the nonlinearity factor is (302b), and the function $G(z)$ is given by

$$G(z)=b\delta(z)+\alpha_i/\pi z^2-\alpha_c\delta'(z)-\alpha_rH(z) , \tag{303a}$$

which reduces to Eq. (293c) in the absence of diffusion, $\alpha_c=0=\alpha_r$. The Fourier transform of Eq. (303a) yields the phase speed

$$c(k)=b-\alpha_ib\,|\,k\,|+i(\alpha_ck+\alpha_r/k) , \tag{303b}$$

for a tube mode in a thermally conducting and radiating gas. This could be checked by a long-wavelength approximation to the equation of sausage modes in a thermally conducting and radiating magnetic slab. The Benjamin-Ono-Burgers equation combines the two main types of waves in fluids, depending on the effect that balances nonlinearity. Equations balancing nonlinearity to second order with linear dissipation are generally of the Burgers type (Burgers, 1948, 1974; Lighthill, 1951; Adam, 1975; Crighton, 1979; Campos and Leitão, 1987); they describe "shock fronts" with smoothing and decay by diffusion and have been considered in Sec. V.C of Part I. Equations balancing nonlinearity with dispersion are of the Korteweg-De Vries, Benjamin-Ono, and generalized types (Korteweg and De Vries, 1895; Benjamin, 1967; Leibovich, 1970; Whitham, 1974; Ono, 1975; Miles, 1981; Roberts, 1986); they describe "solitons" of permanent or decaying shape and have been considered in Secs. IV.C.1—IV.C.5 of this paper.

6. Hydrodynamic waves and the mass balance

Since no comparisons with observation have been made in the literature on "solar" solitons, we may conclude by summarizing the evidence that arguably supports the existence of other types of waves in the solar atmosphere, starting with hydrodynamic waves and the mass balance. The solar corona loses mass in the "average" solar wind and is resupplied by the upward mass flow in spicules, which can be interpreted as acoustic-gravity waves: (a1) the spicules are more concentrated at the boundaries of the supergranulation, which are the regions of larger hydrodynamic and hydromagnetic stresses (Sec. I.C.6) responsible for wave generation in the photosphere; (a2) the mass flux in acoustic-gravity waves at the photosphere is reduced, on traversing an evanescent region around the chromospheric temperature minimum, to a value comparable to the mass flux in spicules at the base of the corona; (a3) the mass flux in spicules compensates for the coronal mass losses in the solar wind, with an excess of matter that falls back, explaining the downflow velocities ~ 10—20 km/s observed in the transition region; (a4) spicules trace the solar magnetic field, because in the corona the magnetic pressure exceeds the gas pressure, and acoustic waves travel along magnetic field lines (Sec. I.B.7); (a5) the velocity of matter in spicules, ~ 20—30 km/s, corresponds to the phase speed of the compression front of acoustic-gravity waves (see beginning of Sec. III.B); (a6) the temperature profile in spicules [Fig. 1(b)] corresponds to that of a thermally radiating atmosphere heated by viscous dissipation of acoustic-gravity waves; (a7) the viscous damping length and time correspond to the height $\sim 10^4$ km and lifetime ~ 10 min of spicules (Campos, 1984a); (a8) the density profile in spicules [Fig. 1(a)] corresponds to the superposition of a nonlinear wave compression upon an atmosphere in hydrostatic equilibrium; (a9) the acoustic-gravity wave becomes nonlinear as it propagates upward in an atmosphere of decaying density, and reaches shock strength at the top of the spicule, when matter breaks magnetic confinement and the spicule disappears (see Campos, 1984a, Fig. 3, bottom); (a10) macrospicules observed in coronal holes are larger than spicules in the ordinary corona because of the reduced density in the former regions, but they have comparable total energy, predominantly potential, and undergo similar physical processes; (a11) spicules are much cooler and denser than the surrounding corona because they are hydrodynamic modes subject to a weak viscous damping, whereas in the corona hydromagnetic modes can be dissipated by electrical resistance and other plasma diffusion mechanisms.

The high-speed particle streams originating from coronal holes can be explained by fourth-order hydromagnetic gravity waves: (b1) these have a dynamic component, similar to acoustic-gravity waves in spicules or macrospicules, and, in addition, a magnetic component, similar to an Alfvén wave (Sec. III.A.6); (b2) the coexistence of Alfvén and acoustic-gravity waves in spicules explains (see beginning of Sec. III.B) the observed linearly growing horizontal velocities by the former, and the vertical velocities and mass transport by the latter; (b3) coronal holes are regions where the magnetic field is never purely horizontal, i.e., critical levels do not exist (Sec. III.C.4), and the resulting lack of wave absorption explains why regions of open magnetic field in the solar corona are darker and cooler than their surroundings; (b4) since the hydromagnetic-gravity waves are not absorbed in open magnetic structures, they can propagate out with matter and accelerate it, forming the high-speed particle streams in the solar wind observed to originate from coronal holes (Sec. III.A.8); (b5) the waves in these high-speed solar winds have a magnetic energy that is a constant and significant fraction of the background magnetic energy, as predicted for the magnetic component of a fourth-order hydromagnetic-gravity wave (Fig. 7); (b6) the waves in the high-speed solar wind have, in addition to a magnetic energy, a thermal energy, showing that the dynamical component of the fourth-order hydromagnetic-gravity wave is also present (Sec. III.A.7); (b7) the waves in the high-speed solar wind are nonsinusoidal, due to the deformation of the waveform (Sec. II.B.1) caused by the rapid variation of Alfvén speed with altitude; (b8) the waves are neither longitudinal nor transverse, because the dynamic and magnetic components of the fourth-order hydromagnetic wave are coupled and thus propagate together.

7. Hydromagnetic waves and the energy balance

Having considered the mass balance, we proceed to the energy balance or heating of the solar atmosphere by hydromagnetic waves, which we consider separately for the chromosphere (c) and transition region and corona (d). The heating of the chromosphere could be explained by Alfvén waves: (c1) the solar chromosphere is hotter over granulation boundaries than over granulation centers, and the former are ionized inhomogeneous regions of enhanced hydromagnetic wave generation (Sec. I.C.7); (c2) Alfvén waves can propagate in the nearly vertical magnetic field over granule boundaries, and in the inclined magnetic field over cell centers (Sec. III.A.2), and thus can heat regions of the chromosphere above both granules and their boundaries; (c3) these two regions require comparable mechanical energy fluxes, and the energy flux in Alfvén waves, at photospheric level, is adequate, as well as comparable to the radiative losses of the sun (Sec. I.C.8); (c4) the temperature profile of the chromosphere

[Fig. 2(b)] can be modeled by balancing resistive dissipation of Alfvén waves against radiative losses in the grey approximation; (c5) the density profile associated with this temperature profile, on the basis of hydrostatic equilibrium, is also consistent with empirical data [Fig. 2(a)]; (c6) the empirical radiative losses of the solar chromosphere (Avrett, 1981) can be matched by dissipation of Alfvén waves (Campos, 1984c; Hollweg, 1984d); (c7) the nonthermal horizontal velocities observed in the chromosphere (Fig. 10) are consistent with the linear amplitude growth of Alfvén waves of small compactness in an atmosphere (Sec. II.B.5).

The heating of the transition region and corona could be explained by magnetosonic-gravity waves: (d1) these waves are similar to acoustic-gravity modes in the chromosphere, and to compressive Alfvén modes in the corona, and the mode conversion (Sec. II.C.1) occurs in the vicinity of the critical level in the transition region; (d2) the latter is a region of gas pressure varying slowly compared to mass density or temperature in agreement with the constant gas pressure near a critical level; (d3) the gas pressure in the transition region is comparable to the magnetic pressure due to the horizontal component of the magnetic field, as required (Sec. II.C.8) at a critical level; (d4) as the critical level is traversed, the growth of wave amplitude with altitude is reduced (Fig. 6), corresponding to absorption of wave energy by the medium; (d5) the critical level exists only for a purely horizontal magnetic field, i.e., at the top of coronal loops and arches, and the resulting heating explains why these closed magnetic regions are hotter and brighter than the surrounding corona (Fig. 9, left); (d6) as the magnetic field tilts to nearly horizontal at the top of closed magnetic structures, hydromagnetic-gravity waves, although they have finite amplitude, vary rapidly in phase (Fig. 8), and the resulting large gradients in the waveform allow effective dissipation by small diffusivities; (d7) the nonthermal velocities in the transition region (Fig. 8, right) grow linearly with the logarithm of temperature, as predicted (Sec. III.C.8) for magnetosonic-gravity waves; (d8) as the hydromagnetic-gravity waves propagate into the corona, after traversing the critical level, their amplitude grows slowly and phase is bounded (Fig. 6), so that no further heating occurs, and the corona is nearly isothermal, retaining the temperature of the top of the transition region; (d9) since magnetosonic-gravity waves are similar to acoustic-gravity waves when the gas pressure exceeds or is comparable to the magnetic pressure, they are not distinguishable in the chromosphere, where acoustic modes are often observed; (d10) the magnetosonic-gravity waves interact most strongly with the medium, i.e., are absorbed, at the critical level in the transition region, which is the region where empirical data indicate wave amplitude "saturation" occurs, and atmospheric heating is most intense.

8. The role of waves in the solar atmosphere

In the outline of the mass balances (Sec. V.C.6) and energy balances (Sec. IV.C.7) in the solar atmosphere, by hydrodynamic and hydromagnetic waves, respectively, we have considered only propagating waves, since standing modes have zero mass and energy fluxes in the absence of dissipation, although they can lose energy gradually by damping or substantially by resonance. Standing modes certainly exist on the sun: (e1) the large temperature "jump" from the chromosphere to the corona strongly reflects acoustic-gravity waves; (e2) oblique Alfvén (Sec. II.B.7) and magnetosonic-gravity (Sec. II.C.6) waves are substantially reflected even in an isothermal atmosphere by the large increase of the Alfvén speed with altitude, due to the decay in density over many scale heights. Standing oscillations have been observed in several regions of the sun, from the coronal prominences to the umbrae of sunspots. We choose as an example the latter, which can be explained as standing Alfvén-gravity waves; (e3) the frequencies of the first five modes, 300, 180, 135, 100, 70 s, agree in their ratios with the observed values, most notably the five- and three-minute oscillations; (e5) the absolute values of the frequencies indicate that the Alfvén waves, which propagate along magnetic field lines, are reflected at anchoring points at a depth into the chromosphere ($\sim 10^4$ km) consistent with the depth of origin of ephemeral magnetic regions on the sun (Parker, 1984); (e6) the upper reflection occurs either abruptly, at the temperature jump in the transition region, or gradually, in the mass density gradient in the chromosphere, and its exact location does not significantly affect the frequencies, since the Alfvén speed is large and the waves spend a small fraction of the time in the sunspot atmosphere; (e7) the observed $\pm 10\%$ variability of periods can be explained by small displacements, $\sim 10^2$ km, of the anchoring depth of magnetic field lines, i.e., the lower reflectors, near which the Alfvén speed is lowest and the wave spends most of the time; (e8) the magnetic field perturbation [Fig. 5(a)] oscillates mostly in the subphotospheric layers, above the velocity node at the anchoring points of the magnetic field lines, and then decays in the upper atmospheric layers; (e9) the velocity perturbation [Fig. 5(b)] also oscillates mostly in the deeper subphotospheric layers, but tends to a constant nonzero value in the atmospheric layers, consistent with the empirical data for the first two modes, for which data is available; (e10) the Alfvén wave, which has horizontal velocity perturbations in subphotospheric layers, couples nonlinearly with slow and fast compressive modes in the photosphere, leading to the observation of vertical velocity components in the chromosphere.

The hydromagnetic waves considered before are linear, since the velocity perturbation does not exceed the Alfvén speed, which increases rapidly with altitude; the acoustic-gravity waves in spicules become nonlinear because the sound speed increases more slowly than the velocity perturbation, and the latter eventually becomes comparable with the former. Nonlinear soliton-type waves could possibly explain coronal mass ejections, moving magnetic features, and other large disturbances of "permanent" shape. Phenomena giving rise to large-scale liberation of energy, such as the magnetic reconnection leading to

TABLE III. The role of waves in the physics of the solar atmosphere.

Process	Mass balance		Energy balance		Dynamics and activity	
Phenomenon	Average solar wind	High-speed streams in solar wind	Chromospheric temperature minimum	Temperature rise in the transition region	Oscillations in sunspot umbrae	Flares, surges, and particle events
Region	Mass supply to corona	Acceleration out of coronal holes	Heating of chromosphere	Heating of coronal loops	Modes arising from convection zone	Magnetic reconnection in photosphere
Wave	Propagating acoustic-gravity	Propagating hydromagnetic-gravity	Propagating Alfvén-gravity	Propagating magnetosonic-gravity	Standing Alfvén-gravity	Nonlinear and shocks
Restoring forces	Gas pressure Buoyancy	Gas pressure Magnetic force Buoyancy	Magnetic force	Gas pressure Magnetic force Buoyancy	Magnetic force	Gas pressure Magnetic force Buoyancy
Wave perturbations	Velocity Density Gas pressure	Velocity Magnetic field Density Gas pressure	Velocity Magnetic field Magnetic tension	Velocity Magnetic field Density Gas pressure Magnetic tension	Velocity Magnetic field	Velocity Magnetic field Density Gas pressure Magnetic tension
Dissipation mechanisms	Fluid viscosity Thermal condition Thermal radiation	Fluid viscosity Electric resistivity Thermal condition Thermal radiation	Fluid viscosity Electric resistivity	Fluid viscosity Electric resistivity Thermal condition Thermal radiation	Fluid viscosity Electric resistivity	Fluid viscosity Electric resistivity Thermal condition Thermal radiation
Wave modes	Two: gravity + acoustic	Two: dynamic + magnetic Coupled together	One	Two: gravity + acoustic With magnetic modification	One	Six types
Cutoff frequencies	Oblique: two Vertical: one	Vertical: two	None	Oblique: two Vertical: one	None	None
Vertical waves:						
Velocity perturbation	Longitudinal	Oblique	Transverse	Longitudinal	Transverse[a]	Oblique
Magnetic field	None	Transverse	Transverse	Transverse	Transverse	Oblique

[a] With longitudinal component in chromosphere due to coupling with compressibility.

flares, produce shock waves, which are regularly observed in the solar wind. The role of waves in the solar atmosphere is summarized in Table III, which should not be taken as definitive, as there is, at present, no consensus on the role of waves in the solar atmosphere. In order to indicate the extent of differences in current views, we give two examples: (a) for the same phenomenon several different explanations are proposed, e.g., models of umbral oscillations have been presented on the basis of Alfvén, slow and fast waves, and flux tube modes; (b) a given wave mode is used to explain widely different phenomena, e.g., Alfvén waves have been invoked to explain umbral oscillations, chromospheric heating, coronal loop resonances, and acceleration of the solar wind. Table III gives a coherent picture of the phenomena in the solar atmosphere backed by 55 [(a1)—(a11) + (b1)—(b7) + (c1)—(c7) + (d1)—(d10) + (e1)—(e10) = 55] pieces of "evidence," but it can be contested, on the basis of (i) accuracy or interpretation of empirical data, although we have relied mostly on "phenomena" reported by several observers; (ii) validity or magnitude of the effects reported, although most statements are supported by model calculations; (iii) availability of several other alternative explanations for each individual observation, although partial explanations tend not to add up to a coherent global picture. It is beyond the scope of the present review to go into much detail about the observation and modeling of solar phenomena, but the references given are adequate to start a literature search on most topics mentioned. In conclusion, Table III outlines a tentative global picture of the role of waves in the solar atmosphere and serves as a general indication of some of the physical phenomena that may be associated with unsteady motions in stratified and magnetized gases.

V. DISCUSSION

This review of waves in gases has been presented in two parts, with a common introduction (Sec. I of Part I) dealing with the motivations for the study of the subject and its applications in science and technology. The present concluding discussion is also common, as it is concerned with an outline of possible lines of future development of the subject, including (A) an indication of currently controversial issues, (B) suggestions for new research, and (C) comments on available methods. Part I (i.e., Campos, 1986a) was concerned with modern aspects of the classical subject of acoustics, while Part II dealt with the basic properties of magneto-acoustic-gravity waves, which have been the subject of substantial research in last few decades. It is perhaps a reflection of the somewhat perennial character of scientific methods that both subjects, old and new, could be organized similarly, i.e., the basic unity of the methods used to study waves in continuous media has allowed a parallel arrangement of the review of acoustic waves (Part I) and that of the generalization to magneto-acoustic-gravity-inertial waves (Part II), as can be seen by comparing the four main sections of each part. Section II of Part I deals with the extension of the classical Kirchhoff method of studying acoustic radiation by "known" sources placed in a medium at rest, to the generation of sound by "natural" sources in inhomogeneous, nonuniform flows; Sec. I of Part II, deals with the extension of the traditional method of Fourier analysis, used to study the propagation (isotropic or anisotropic, dispersive or not) of waves under one or more restoring forces (pressure, gravity, Lorentz and Coriolis forces), to the calculation of radiation fields by asymptotic methods. Section III of Part I is concerned with refraction of sound in inhomogeneous moving media, that is, the diffraction of acoustic rays in turbulence and the scattering of sound by irregular interfaces, which apply to short and long waves, respectively; Sec. II of Part II deals with the exact diffraction theory of linear waves, i.e., describes the propagation of waves of arbitrary wavelength in a compressible and ionized atmosphere. Section IV of Part I is concerned with the general properties of sound in ducts of nonuniform cross section, i.e., in horns and nozzles, respectively, both in the absence and in the presence of accelerated or decelerated mean flow; Sec. III of Part II presents a general method for studying higher-order waves in media with properties varying in one direction (wave speeds, scattering scales, and damping rates) according to exponential or bi-power laws, e.g., the coupling of second-order modes into fourth-order waves. The last section of Part I, is concerned with viscous dissipation of acoustic-gravity waves, nonlinear propagation in free space and ducts, and "burgulence," i.e., nonlinear, damped waves of "shock" type with dissipative limitation of gradients, and ultimate decay; the fourth section of Part II deals with resistive dissipation of hydromagnetic waves in atmospheres, propagation in magnetic structures such as interfaces, slabs, and flux tubes, and "solitons," i.e., nonlinear, dispersive waves of "permanent" form with dissipative decay. Table IV lists some of the topics considered and demonstrates the complementary nature of the two parts of the present review on waves in gases.

A. Subtle points and controversial issues

Part I of this review, on acoustics of inhomogeneous and moving media, is somewhat shorter than Part II, dealing with the coupling of compressibility, gravity, magnetism, and (to a lesser extent) rotation, partly because the presence of several restoring forces in the latter cases, compared to one in the former, can lead to more complex phenomena. Another reason for Part I to be shorter is that we have not reviewed the basic aspects of acoustics, which are well known and noncontroversial, excepting only the more involved applications, e.g., sound in nonuniform (accelerated, sheared, or turbulent) flows; concerning the magneto-acoustic-gravity waves in Part II, there are a number of contradictory results in the literature, on such basic issues as cutoff frequencies, amplitude and phase laws, and wave properties at the critical level, and a review on the subject must address these. When

TABLE IV. Classification of waves and associated phenomena.

Properties	
Amplitude	Linear (Secs. II-V.A of Part I; I-IV.B of Part II)
	Nonlinear (Secs. V.B and C of Part I; Sec. IV.C of Part II)
Directivity	Isotropic—acoustic (Secs. II and V of Part I)
	Anisotropic—all others (Secs. I-IV of Part II)
Dispersion	Nondispersive—acoustic, Alfvén, and magneto-acoustic (Secs. II-V of Part I; See I of Part II)
	Dispersive—all others (Secs. I-IV of Part II)
Type	Longitudinal—acoustic (Secs. II-V of Part I)
	Transverse—Alfvén, gravity, inertial, and couplings (Secs. I-IV of Part II)
	Mixed—all couplings with acoustic waves (Secs. I-IV of Part II)
Medium	
Composition	Homogeneous (Secs. II-IV of Part I; I of Part II)
	Stratified (Secs. II-IV of Part II)
Convection	At rest (Secs. II.A, IV.A and B, V.A of Part I; I-IV of Part II)
	In motion (Secs. II.A and B, III, IV.C, V.B and C of Part I)
Boundaries	Free space (Secs. II, III, V of Part I; I-IV.A of Part II)
	Waveguides (Sec. IV of Part I; IV.B and C of Part II)
Obstacles	Interfaces and walls (Secs. III.A, IV of Part I; IV.B and C of Part II)
	Turbulence and flows (Secs. III.B, IV.C of Part I)
	Continuous stratification (Sec. V.A of Part I; I-III, IV.A of Part II)
Process	
Generation	Monopole source (Secs. II.A and C of Part I)
	Force dipoles (Secs. II.A and C, IV.C of Part I)
	Quadrupole stresses (Sec. II.A of Part I; I.C of Part II)
Propagation	Dispersion relations (Secs. III.C, V.C of Part I; I, IV.B and C of Part II)
	Special functions (Secs. IV, V.A of Part I; II, III, IV.?.1 of Part II)
	Nonlinear transformations (Sec. V.C of Part I; IV.C of Part II)
Refraction	Scattering: low-frequency (Sec. III.A of Part I)
	Diffraction: high-frequency (Secs. III.C, IV.C of Part I; II.A and C of Part II)
	Refraction: all frequencies (Secs. IV, V.A of Part I; I-III of Part II)
Dissipation	Nondissipative (Secs. II-IV, V.B of Part I; I-III, IV.B of Part II)
	Dissipative (Secs. V.A and C of Part I; IV.A and C of Part II)
Consequences	
Diffusion	Viscosity (Secs. V.A and C of Part I)
	Electrical resistance (Sec. IV.A of Part II)
	Thermal (Sec. V.C of Part I; IV.C of Part II)
Methods	Exact (Secs. IV, V of Part I; I-III of Part II)
	Approximate (Secs. II, III of Part I; IV of Part II)
	Numerical (Secs. II, III, V of Part I; I-III of Part II)
Applications	Physics (Secs. I-V of Part I)
	Engineering (Secs. I-V of Part I)
	Astrophysics (Sec. I of Part I, Secs. I-IV of Part II)

presenting a topic, in either Part I or Part II, that has been the subject of incompatible statements in the literature, we have adopted the following procedures: (i) to outline each of the relevant arguments used in the literature, presenting them in a way that is adequate to show strong and weak points; (ii) to give a careful discussion of a method that might resolve the issue in an unambiguous manner; (iii) to mention other independent and alternative methods of proving or checking what is believed to be a "correct" result; (iv) to indicate the assumptions made, and the circumstances in which different results could be expected. Obviously, there is no assured method of resolving controversial issues, but it is felt that the procedure above gives a fair chance to all points of view, while trying to arrive at a conclusion that is as reliable as possible. A word of caution on controversial issues in the literature is perhaps always worthwhile, since (a) the newcomer to the field should be warned that the opinion on a controversial matter expressed in a few papers may not reflect a consensus, or be accepted without reservation; (b) the specialist in the subject should bear in mind that, whenever in the course of a work it is necessary to take sides on a controversial issue, that particular point, and deductions from it, may come under critical examination by other workers in the field. Thus, if it is not possible, in a given work, to use a method independent of, or invariant with regard to, controversial issues, it is at least advisable to check the suitability of the method and validity of the conclusions derived from it. We proceed to indicate some of the topics on waves in gases which have given rise to some disagreement in the literature, in order to point out the basic issues in question.

1. Critical levels of atmospheric waves

We begin our consideration of controversial topics with the subject of critical levels, which can occur for waves subject to two competing effects of varying relative magnitudes, such that one dominates in one region, and the other in another region, implying that "mode conversion" occurs between the two. For example, critical levels do not occur for viscous acoustic or resistive Alfvén waves in homogeneous media, with constant wave speeds and damping rates; but for both viscous acoustic-gravity waves (Sec. V.A of Part I) and resistive Alfvén waves (Sec. IV.A of Part II), the dissimilar laws of variation of wave speed and damping rate with altitude, in an atmosphere, lead to the appearance of a critical level. The critical level appears mathematically as a singularity in the wave equation, and may imply that the wave energy is not fully propagated across the critical level, as a consequence of absorption, reflection, or mode conversion, according to whether it is a critical level of type I, II, or III (i.e., singular, reflecting, or transition layer). Nondissipative acoustic-gravity waves, in an isothermal atmosphere, and Alfvén waves in an atmosphere under a uniform magnetic field, have no critical levels, since there are no varying competing effects in either case—the gas pressure and buoyancy decay at the same rate for the former, and the latter is affected by the constant magnetic pressure alone. Since the critical level occurs for these waves only in the presence of dissipation, it might be expected that the limit of zero diffusion would lead back to the nondissipative solution, but this may actually be true or untrue, depending on whether the critical level recedes to $+\infty$ or $-\infty$. In the case of viscous acoustic-gravity waves, the limit of zero viscous diffusivity corresponds to a critical level receding to $+\infty$, and still reflecting waves, so that nondissipative acoustic-gravity waves, which are not reflected in an isothermal atmosphere, are not regained (Sec. V.A.7 of Part I). For resistive Alfvén waves in an atmosphere, the limit of zero magnetic diffusivity implies a critical level at $-\infty$, so that it no longer affects waves, and the nondissipative case is indeed regained (Sec. IV.A.8). Thus a critical level can affect waves everywhere, although its effect is greater in its own vicinity, and the question arises of whether waves have a finite or continuous amplitude and phase at the critical level.

Wave properties near the critical level are usually studied (Bretherton, 1966; Booker and Bretherton, 1967) by means of a series expansion in its vicinity, assuming that the singularity is regular; the leading term of the series specifies the properties of the wave field at the critical level, where the remaining terms vanish by comparison, assuming that the series converges. The verification that the series expansion for the wave field near the critical level does converge is seldom made in the literature, perhaps because explicit formulas for the nth coefficient are needed and may be difficult or tedious to find in a particular application; however, if the series fails to converge, it introduces another singularity, which may dominate or cancel that in the leading term. An instance of the latter case is given by oblique propagating magnetosonic-gravity waves in an atmosphere under a horizontal magnetic field (Sec. II.C): (i) these are reported in the literature to have (Sec. II.C.4) a logarithmic singularity at the critical level, on the basis of the leading term of a divergent series; (ii) the latter exactly balances the singularity in the leading term, so that the wave has a finite amplitude and phase at the critical level, as shown by a convergent expansion (Sec. II.C.8) using a more suitable altitude variable.

2. Cutoff frequencies for nonuniform propagation

The existence of a critical level for waves in an atmosphere may require not only the presence of competing effects of varying relative magnitude, but also a suitable geometry of the restoring forces. For example, in the case of nondissipative magneto-atmospheric waves, the competition is between the decaying gas pressure and buoyancy on the one hand, and, on the other hand, the magnetic pressure, which does not decay if the magnetic field is uniform or in excess of a nonzero value. The critical level will exist only for a purely horizontal magnetic field, for a nonzero vertical component would allow waves to propagate through, instead of being absorbed. The case of magnetosonic-gravity waves, in an atmosphere under a horizontal magnetic field, is the subject of contradictory statements in the literature, not only concerning the wave amplitude at the critical level (Sec. V.A.1), but also with regard to the cutoff frequencies. Taking as reference the cutoff frequencies of acoustic-gravity waves, it has been stated that magnetosonic-gravity waves either (a) have the same filtering properties, or (b) have the cutoff frequencies modified by replacing the sound speed c by $(c^2+a^2)^{1/2}=c(1+1/\beta)^{1/2}$, where a is the Alfvén speed and $\beta\equiv c^2/a^2$ is the plasma β. In order to explain the origin of the discrepancy, we recall that the cutoff frequency separates propagating waves from standing modes and thus corresponds (Thomas, 1982) to the limit of infinitely spaced nodes, i.e., infinite wavelength $\lambda\rightarrow\infty$; thus the cutoff frequencies can only be obtained from exact solutions of the wave equation. For example, for acoustic-gravity waves in an isothermal atmosphere, the sound speed and scale height are constant, and the dispersion relation is an exact solution, from which filtering properties may be deduced reliably. In the case of magnetosonic-gravity waves in an isothermal atmosphere under a horizontal uniform magnetic field, (i) the Alfvén speed varies with altitude, and thus the "local" form of the dispersion relation, which is used to justify statement (b) above, is limited to short waves, so that the limit of infinite wavelength cannot be taken (Sec. II.C.2); (ii) an exact solution, valid for all wavelengths, can be obtained in terms of hypergeometric functions and proves the statement (a) that the cutoff frequencies are the same as for acoustic-gravity waves (Sec. II.C.3); (iii) the latter conclusion can be confirmed by the general criterion for the identification of cutoff frequencies, showing that a horizontal magnetic field has no effect on the cutoff frequency for vertical magnetosonic-gravity waves (Sec. III.C.2); (iv) the conclusion does not extend to oblique magnetic fields, since in that case vertical waves are coupled fourth-order modes, with two cutoff frequencies, one of which depends on magnetic field direction but not strength. This sequence of four statements (i)—(iv) is an example of the method, indicated at the beginning of Sec. V, of trying to clarify controversial issues in the literature.

3. Initial and asymptotic amplitude laws

The same method, in connection with another controversial aspect of propagating (i.e., nonevanescent) magnetosonic-gravity waves, in a uniform horizontal magnetic field, namely, their properties at the critical level, leads to the following conclusions: (i) the "logarithmic singularity" corresponds to the leading term of a divergent series expansion (Sec. II.C.4), and the singularity introduced by the series exactly cancels it; (ii) the latter statement follows from the exact solution of the problem in terms of a more suitable altitude variable, which specifies exactly the wave field at all altitudes, including the critical level (Sec. II.C.3); (iii) the result that the wave amplitude and phase are finite at the critical level are confirmed in two distinct ways, by proving the convergence of both the high- and low-altitude solutions at the critical level (Sec. II.C.7); (iv) the general method investigation of the properties of waves with exponential wave speeds, confirms the existence of the critical level and its location (Sec. III.C.4) and also confirms that the wave amplitude and phase are finite there (Sec. III.C.5).

In addition to the issues of the cutoff frequencies and critical levels, a third somewhat controversial issue occurs in the literature on magneto-atmospheric waves, namely, that of laws of variation of wave amplitude and phase with altitude. The situation is somewhat similar to the issue of the cutoff frequencies, in the sense that use of the dispersion relation yields reliable results, namely, exponential amplitude and linear phase evolution, for waves with constant speed of propagation, e.g., acoustic-gravity waves in an isothermal atmosphere (Secs. II.A.7 and II.A.4); in other circumstances "local" forms of the dispersion relation assume complex exponential wave fields and thus prove nothing concerning amplitude or phase laws, e.g., they are, respectively, not exponential and nonlinear for acoustic-gravity waves in the presence of temperature gradients (Sec. II.A.8). Concerning magnetic waves, since the Alfvén speed increases rapidly with altitude, the reference wavelength $\lambda = 2\pi a/\omega$ rapidly exceeds the scale height, so that the waves become nonsinusoidal; the ray approximation applies to high-frequency waves over a short distance $a/\omega \ll L/2\pi$, and leads, by the conservation of the energy flux, to velocity perturbations increasing exponentially at half the acoustic rate. For low-frequency waves at any altitude, and high-frequency waves at a sufficiently large altitude, the ray approximation breaks down, and the wave equations can be solved in the opposite limit of small compactness $(\omega L/a)^2 \ll 1$; this leads to linear amplitude growth for propagating waves and bounded amplitude for standing modes, with bounded phase in the former case. These amplitude and phase laws apply to the velocity perturbation in general atmospheres and can be checked from exact solutions for simple cases, e.g., isothermal atmospheres and uniform magnetic fields. The amplitude and phase laws for (a) acoustic-gravity and (b) Alfvén waves serve as reference cases of magneto-atmospheric waves, e.g., for magnetosonic-gravity waves in regions of predominant (a) gas pressure and (b) magnetic pressure, and also for the (a) dynamic and (b) magnetic component of fourth-order hydromagnetic-gravity waves.

4. Constraints for dissipative and stratified media

When considering waves in unbounded media, it is necessary to specify an appropriate boundary condition at "infinity," e.g., for second-order wave equations, the boundary value problem in one dimension involves two constants of integration, one determined by the initial wave field and the other by another condition; the latter condition is simple in the case of a bounded medium, e.g., zero displacement at a rigid reflector, or wave velocity perturbation equal to normal surface velocity at an "opaque" driver. In the case of an unbounded medium, the former reflection condition is replaced by an asymptotic condition; in the presence of one or more partially transmitting reflectors, the continuity of total pressure and normal displacement is used at each interface, but the asymptotic condition is still needed in the unbounded layer, beyond the last partial reflector. For waves in an unbounded, homogeneous medium, the radiation condition can be used to specify an outward-propagating wave, uncontaminated by waves propagating inward, which would bring energy from sources at "infinity." In the case of a stratified medium, the radiation condition may be inappropriate, e.g., waves propagating upward in an atmosphere may suffer reflections, which produce downward-propagating waves, in the absence of any source at infinity. For dissipative waves in homogeneous media, it is sometimes useful to apply a decay condition, requiring the amplitude to decay (in space or time), on account of dissipation; for dissipative waves in a stratified medium, the decay condition may not hold, e.g., waves tend to grow in amplitude in an atmosphere, as the density decays with altitude, and dissipation may be insufficient to reverse this. Thus the question arises of the choice of a second boundary condition for dissipative waves in an atmosphere; such a boundary condition must be valid, i.e., hold true, and also be relevant, i.e., serve to determine one constant of integration. Two valid boundary conditions for dissipative waves in stratified media are (i) the dissipation condition (Yanowitch, 1967a) stating that the total energy dissipated by the wave, over an infinite column of fluid, from zero to infinity, must be finite; (ii) the damping condition (Campos, 1983e), stating that the dissipative wave must have a smaller amplitude than a nondissipative wave in the same medium. While all dissipative waves in atmospheres satisfy both conditions, in some cases one is trivially satisfied and the other determines a constant of integration, i.e., the former is an irrelevant and the latter a relevant constraint. For example, for acoustic-gravity waves in a viscous atmosphere, the sound speed and static viscosity are bounded, and the kinematic viscosity increases as the mass density decays with altitude; it follows that the viscosity is dominant at high altitude, and the integral dissipation condition is (Sec. V.A.6 of Part I) the relevant constraint. In the case of Alfvén waves in an imperfectly conducting atmosphere, the magnetic diffusivity is bounded and the Alfvén speed unbounded; thus dissipation is negligible at high altitude, and the dissipation condition is irrelevant, but the damping condition (IV.A.7) serves as a constraint.

5. Boundary conditions at a moving interface

Another case in which the application of boundary conditions requires some care, and which has lead to a long-lasting controversy in the literature, is the scattering of waves by an interface separating two media in relative motion. For the purpose of discussion of the issue in question, it is sufficient to consider the simplest case of sound incident upon a plane interface separating a jet from a medium at rest; the laws of reflection and transmission of waves by a plane interface are traditionally obtained by applying suitable boundary conditions relating the former two to the incident wave field. For example, in the case (Landau and Lifshitz, 1967b) of elastic waves, the normal displacement and stress must be continuous, and for electromagnetic waves (Stratton, 1941) the jump in a normal electric and transverse magnetic field is determined by the surface electric charge and current densities, respectively. For wave reflection and transmission by a plane interface between two fluids at rest (or moving at the same velocity, i.e., in relative rest), the two boundary conditions are continuity of (i) the normal stress, e.g., gas pressure for sound, gas plus magnetic pressure for magneto-hydrodynamic waves, gas plus radiation pressure for acoustic waves in a radiating fluid, (ii) normal displacement or velocity, which differ in a factor involving the frequency, i.e., identical for media at rest, and involving the same Doppler shift for media moving at the same velocity. The former, dynamic boundary condition (i), expressing force balance, is universally adopted, whereas the latter, kinematic boundary condition (ii) is used variously in displacement or velocity form; this difference in (ii) is of no consequence for scattering by plane interfaces between media at relative rest, since the Doppler factor relating normal displacement to normal velocity is the same, and drops out of the equation, and continuity of either one implies that of the other. The situation is distinct for an interface separating media moving at different velocities, e.g., a jet in a still atmosphere, since then the Doppler-shifted frequency is different on both sides, and so is the relation between displacement and velocity; it follows that in the case of a plane interface between media in relative motion, the continuity of the normal components of displacement and velocity are incompatible, and different choices have been made in the literature (Sec. III.A.3 in Part I). The boundary conditions at an interface can be deduced by integrating the equations of motion across the discontinuity, i.e., applying the equations of motion across a thin layer matching the two media continuously, and letting the thickness tend to zero. Unfortunately there have been in the literature claims of mathematical proofs of the continuity of both velocity and displacement, and, inevitably, some dispute on the validity of such demonstrations, since they cannot both be correct. From a physical point of view, it is clear that the fluid must remain "attached" to the moving interface at all times, and this implies the continuity of the normal component of the displacement at the interface, in all cases: if the media on opposite sides are at relative rest, the normal velocity will also be continuous, whereas if they are in relative motion the normal velocity is discontinuous, on account of the different convection effects on the two sides of the interface. Thus we have adopted (Sec. II.A of Part I) the continuity of the normal component of displacement, which has led to results on sound transmission from jets that are consistent with observation, as concerns both directivity (Munt, 1977) and spectra (Campos, 1978a, 1978b). We must, however, bear in mind that relatively recent research work includes opposite claims as to whether continuity of velocity (Myers, 1980) or displacement (Poirée, 1982) should hold for scattering of sound by moving interfaces.

6. Correlation function for random phase shifts

The scattering of sound by plane interfaces causes only amplitude changes associated with the reflection or transmission of part of the wave energy. In the case of an irregular interface, the scattering occurs at different "heights" for distinct horizontal coordinates, resulting in phase shifts, which are random if the shape of the interface is aleatory; the transmission of sound through turbulence also leads to random phase shifts. Thus an irregular moving interface, or the layer of turbulence entrained with it, transforms a phase-coherent incident wave into a transmitted field for which wave variables, e.g., the acoustic velocity or pressure, have random phase shifts. Such incoherent wave fields can be characterized by the acoustic energy flux, which is quadratic in the wave variables and hence is determined by the statistics of the random phase shifts, including their correlation function. Taking a one-dimensional, spatial-only correlation for simplicity, it follows from the concept of correlation length L that phase shifts are almost uncorrelated over larger separation

$z > L$, and a "popular" simple form of the correlation function is $E_0(z) = \exp[-(z/L)^2]$. There are, however, some applications for which it is not suitable. For example, in the case of scattering by the moving irregular interface of a jet, the conservation of the volume occupied by the mean flow (assumed to be incompressible) requires the correlation function $E(z)$ to have zero integral from $-\infty$ to $+\infty$, which is not true of $E_0(z)$; the latter has positive integral, and implies an expansion of the jet. The concept of correlation scale L as the separation $z < L$ beyond which the correlation $E(z) \ll 1$ is small, is not affected if we multiply the preceding correlation function $E_0(z)$ by a polynomial $q(z/L)$. The correlation function $E(z) = q(z/l)E_0(z)$ is still small for $z \gg L$, when the negative exponential in $E_0(z) = \exp[-(z/L)^2]$ dominates; the polynomial $q(z/L)$ modifies the correlation function mainly at small separations, e.g., we can satisfy the condition of volume conservation or zero integral of $E(z)$ by choosing a symmetric polynomial of degree two, viz.,

$$q_1(z/L) = 1 - 2z^2/L^2 .$$

The type of correlation function $E(z/L) = q_1(z/L)E_0(z)$ has been observed for the random phase shifts of electromagnetic waves propagating in the atmosphere (Tatarski, 1971) and sound transmitted across turbulent jets (Ho and Kovasznay, 1976). The main difference between the two correlation functions is that $E_0(z) > 0$ for all z, so that the two wave components always have phases with the same sign, whereas $E_1(z)$ changes sign for $z = L/\sqrt{2}$, so that waves with larger separation have phases with opposite signs. The signs of the phases is important in connection with wave interference and affects the shape of the spectra for sound transmission across turbulent and irregular jets. The positive correlation function leads to hump-shaped spectra with a single maximum (observed by Candel, Jullienne, and Julliand, 1975), whereas the correlation function with sign reversals leads to spectra with sidebands (observed by Candel, Guédel, and Julienne, 1976); the latter are illustrated in Fig. 6 of Part I.

7. "Identification" and modeling of sources of sound

Scattering by interfaces and diffraction by turbulence mask the source of sound, thus changing the directivity pattern and energy spectrum of the wave source. Even when scattering and diffraction effects are absent, e.g., when we have a wave field in free space, without obstacles or reflectors, the "location" or "identification" of wave sources is intrinsically ambiguous. The literature on noise control contains what is perhaps an overabundance of claims to have "located" or "identified" the sources of sound in specific circumstances. It should be borne in mind that there are infinitely many different source distributions capable of producing the same wave field—if we choose an arbitrary point in space, there exists a superposition of multipoles concentrated at that point that produces the desired wave field, and a different set of multipoles at another point produces the same wave field. Thus, when considering the generation of waves, it is safer to present the discussion in terms of "model" sources, i.e., source distributions that produce the observed wave field; the actual "location" of the source region can be difficult task. For example, the noise of an aircraft in flight (Campos, 1984b, 1986c), arising from a turbine engine, may result not only from blade vibration, but also from shedding of vorticity, which emits sound as it is convected by the flow, leading to an "apparent" sound source somewhere downstream of the turbine, i.e., not on the turbine itself.

The identification of the physical mechanism of wave generation can be made by a method analogous to the "acoustic analogy" (Lighthill, 1952, 1978), which applies both to sound and to other types of waves (Campos, 1977, 1978a), justifying the designation of "wave analogy" in Sec. II.A.1 of Part I. A successful application of the "wave analogy" requires an unambiguous interpretation of all terms in the exact, nonlinear wave equation. In the original "acoustic analogy" (Lighthill, 1952), the linear, nondissipative terms coincide with the well-known classical wave equation, and all the remaining nonlinear and dissipative terms can be grouped in a source quadrupole, the "Lighthill tensor," which models the generation of sound by turbulence in a medium otherwise at rest. An attempt to model the generation of sound by shear flows has led to an equation (Lilley, 1973) containing a number of "interaction" terms, which have been variously interpreted as describing wave propagation, scattering by the shear flow, or generation mechanisms; the ambiguity in the interpretation of terms, has led to doubts on how this "analogy" should be applied. Even when, together with some *ad hoc* assumptions, it leads to results in agreement with observation (Mani, 1976a, 1976b), alternative methods of proof are sought (Dowling, Ffowcs-Williams, and Goldstein, 1978) as an additional justification. An example of unambiguous application of the "wave analogy" is the extension of the acoustic analogy to sound generation in nonuniform, steady flow: (i) the wave equation is deduced first (Sec. II.B of Part I), using a variational method (Campos, 1986a), valid for potential flow; (ii) the consideration of the exact equations of vortical, inhomogeneous flow shows that the preceding wave equation is forced (Sec. II.C of Part I) by dipole sources, corresponding to vorticity (Powell, 1964) and inhomogeneities (Howe, 1975). A similar two-stage procedure is adopted to extend the "acoustic analogy" to include electromagnetic forcing (Kulsrud, 1955; Campos, 1978a) or to model the generation of magneto-acoustic waves (Sec. I.C of Part II).

B. Open problems and suggestions for research

The purpose of a review is to outline the current state of knowledge in a given field, and thus it suggests, by implication, areas that require further research or that are still unexplored. Since it is not possible to describe in anything near an exhaustive manner the field of waves in gases, we have classified the subject into broad sections and discussed in some detail a few basic topics, leaving others as references; some applications were mentioned in a cursory manner, with the details of experiments, observations, modeling and calculations given in additional references. Thus the references listed in the support of the present review consist of (i) a moderate number of basic papers describing important advances in the field, which should be quoted regardless of age, (ii) a larger number of recent works, which demonstrates the main lines of enquiry pursued in the last few years and indicate the current state of knowledge in active research fields. We have limited the repetition of references in Parts I and II as much as possible to cases where the relevance of the works to the present subject makes such double quotation almost inevitable.

Whenever a result is stated under restrictive assumptions, the possibility exists, in principle, to generalize it, by removing the restriction; the pursuit of generality for its own sake may not always be productive, as some assumptions are of lesser practical importance than others. The assumptions made in a given problem may be classified into two broad categories: (i) assumptions that simplify the solution of a problem and are satisfied by most practical applications, so that the additional complexity of the generalization obtained by removing the assumption may be unwarranted; (ii) assumptions that are made in order to render a problem tractable by existing or modified methods, but that do restrict their domain of application, in the sense that there are situations in which the assumption is not met, and the importance of such cases warrants further study. The desirable generalizations may include applying methods developed in one field to another area, for which they are well suited, possibly with modifications; in some cases, undoubtedly rarer, a problem may suggest an entirely new or substantially new method, or a novel approach, e.g., in cases where substantial experimental or observational evidence exists, without supporting quantitative modeling.

1. Generation of internal and inertial waves

An example of a combination of a novel and a classical approach is the formulation of the "acoustic analogy" (Lighthill, 1952), which has played a pioneering role in the development of modern aerodynamic acoustics (Goldstein, 1976) by providing a way of modeling spontaneous sound generation in natural and engineering flows; the method extends into a "wave analogy," which is in principle applicable to any system of equations of motion (Sec. II.A.1 of Part I) and could possibly lead to similar developments concerning other types of waves in fluids, viz., magnetic, gravity, and inertial, as well as the coupling between them and with sound. The consideration of sound emission by ionized inhomogeneities points to some analogies between hydrodynamics and electromagnetism; e.g., the electric force exerted upon charges corresponds to the displacement force on blobs of density distinct from that of the surrounding fluid (Sec. II.C.3 of Part I), and the magnetic force exerted upon electric currents corresponds to Lamb's vector specifying the force on vortices (Sec. II.C.4 of Part I). The coupling of compressibility and magnetism, in connection with the generation of magneto-acoustic waves, also demonstrates a correspondence between dynamic and magnetic effects; e.g., the source quadrupole modeling the generation of such waves by hydromagnetic turbulence consists of modified Reynolds and Maxwell magnetic stresses (Sec. I.C.6 of Part II). Other analogies apply to dipole sources and dissipation tensors, and include, as a consequence, a correspondence between the gas and magnetic pressures, which is frequently used in transposing results from acoustic to magnetic waves, e.g., in the comparison of sound and Alfvén speeds (Sec. II.B.1). Thus the question is raised of whether the modeling of the generation of internal and inertial waves could give rise to analogies reminiscent of those relating acoustic to magnetic waves; such analogies, if found, would not only allow some of the substantial knowledge accumulated in acoustics to be transposed to other waves, but it could also contribute towards a more unified view of waves in fluids.

2. Multiple refraction by interfaces and turbulence

The process of distortion of a wave during refraction in a random medium, e.g., scattering by irregular interfaces or diffraction by turbulence (Campos, 1984d) is a common observation for acoustic, radar, radio, and other signals, but the modeling of such phenomena still leaves much scope for improvement. The simplest theory of scattering by an irregular interface in motion, replaces the interface by an assembly of flat, horizontal radiators, placed at correct heights relative to the mean position, to yield accurate phase shifts for plane waves. The interference pattern of the scattered waves would be modified if account were taken of the inclination of the facets, i.e., this would tend to increase the spectral broadening; moreover, if the Rayleigh-Born approximation were not made, i.e., if the local curvature of the interface were taken into account, the transmitted and reflected waves would cease to be plane, leading to further changes in their spectra. The diffraction of sound by turbulence is readily studied in the ray limit, for wavelengths much shorter than the scale of eddies, causing random phase shifts (Sec. III.C.2 of Part I), but no significant amplitude changes, in an incompressible mean flow. The case of sound of wavelength comparable to the scale of the eddies (Lighthill, 1953) demonstrates significant changes both in amplitude and phase, due to a combination of backscattering of some acoustic energy and redirection of the rest. The interaction of sound and flow of comparable scales can also trigger instabilities, as demonstrated by the acoustic excitation of jet noise, i.e., an acoustic tone can modify the structure of a turbulent jet, leading to a substantial increase in radiated noise (Bechert and Pfizenmaier, 1975). Both scattering by steeply irregular interfaces and diffraction by strong turbulence can lead to the phenomenon of multiple refraction; the tracking of many successive refractions in space-time may become a complicated task, in which case a description in wave-vector frequency space may be preferable (Howe, 1973). Multiple diffraction also occurs for sound transmission across double-sided jets, which can support multiple internal refractions (Sec. III.C.4 of Part I) and act as efficient noise suppressors in modern turbofan engines. Although most of these topics have been the subject of more or less research, our understanding of wave propagation with multiple refraction in random media still leaves much scope for improvement.

3. Acoustics of ducts with shear flow

The propagation of sound in a tube of constant cross section, without flow or with a uniform flow, is a classical waveguide problem. The propagation of sound in the ducts commonly found in engineering practice often involves two additional "complications," namely, (i) reflections from the walls if the cross section varies, e.g., as in a horn or nozzle (Campos, 1985b); (ii) refraction by a shear flow profile, e.g., with zero velocity at the walls and maximum velocity on the axis (Möhring, Müller, and Obermeier, 1983). Concerning the acoustics of nonuniform ducts, a number of general properties can be proven in the absence of a mean flow, when reflection from the walls is the only effect (Campos, 1984e). The addition of a mean flow implies, by volume conservation at low Mach number, that it must be accelerated as the cross section reduces or decelerated as the cross section increases, and the resulting nonuniform convection of sound complicates the calculation of the wave field (Campos, 1984f). The fact that the mean flow velocity must vanish at the walls implies that its profile in the duct must be sheared, and the transverse nonuniformity is represented by the presence of vorticity in the mean flow. Exact solutions of the acoustic equations in a shear flow in a uniform duct, are even more rare than for a "plug" flow in a nonuniform nozzle. The difficulties of accounting simultaneously for the effects on sound of (i) the nonuniform convection by accelerated or decelerated flow and (ii) the interaction with vorticity present in a sheared mean flow are considerable, since the mean flow velocity would vary in two directions. The tendency of the literature to split into either shear flow in uniform ducts or plug flow in nonuniform nozzles is thus understandable, as a way to gain insight into a single set of phenomena at one time. It is no less true, however, that in many practical applications the acoustic nozzles contain flows that are both sheared and accelerated (or decelerated); such problems have been tackled mostly by approximate methods, e.g., parametric expansions that apply with a few terms only (Nayfeh, Kaiser, and Telionis, 1975) if the reflections from the walls are weak. Thus the area of acoustics dealing with the interaction of reflection from tapered duct walls and convection by sheared mean flow still needs some fairly basic research.

4. Radiation patterns and scaling laws

When studying a "new" type of wave, in the sense of including additional restoring forces, scattering mechanisms, or dissipation processes, the method almost always used in the first instance is to consider the dispersion relation. This can be obtained by Fourier analysis, assuming the waves to be sinusoidal, and specifies their propagation properties, e.g., isotropic or anisotropic, dispersive or nondispersive (Secs. I.B.1—I.B.3). For anisotropic waves the question arises of how to calculate the directivity pattern, i.e., the amount of energy radiated in each direction; the radiated field can be calculated (Sec. I.C.1) by Fourier analysis applied to the "forced" wave equation, consisting of the propagation operator (on the lhs) and multipole sources (on the rhs). The radiation integrals are often tedious to evaluate, but they simplify asymptotically for an observer in the far field, who receives most of the radiation from the points on the wave-number surface $\omega(\mathbf{k})$ where the group velocity $\partial\omega/\partial\mathbf{k}$ (aligned with the normal) points directly to him, and the rule is readily applied to the many wave types for which wave-number surfaces are illustrated in the literature. The calculation of the radiation field depends (Secs. I.C.3 and I.C.4) on (i) whether the wave-number surface is flat, singly curved or doubly curved, i.e., whether the wave fronts are plane, cylindrical, or spheroidal; (ii) whether the waves are dissipative or nondissipative, since the frequency is conserved in the latter but not in the former case, requiring extension from the three-dimensional wave-vector $\mathbf{k}$ space, to the four-dimensional space $(\mathbf{k},\omega)$ obtained by including frequency ω as an additional coordinate; (iii) whether the wave-number surface is of second order at the radiating points, and the presence of inflexion edges, etc., leads to higher-order terms, corresponding to caustics and focusing phenomena. If the radiation laws are applied to the model sources (Sec. V.B.1) responsible for the generation of the waves in question, it is possible to obtain scaling laws for the intensity of radiation. Such scaling laws are useful in comparing the efficiency of radiation of a given wave by different physical mechanisms, i.e., in helping to decide which of several possible wave excitation processes is likely to predominate in a given set of circumstances. Moreover, if the constant factor in the scaling law is determined, say, from a particular case, the absolute intensity of radiation can be predicted for the same wave in similar circumstances, giving an indication of the importance of the wave phenomenon in energy terms. Since dispersion relations and wave-number surfaces have been presented in the literature for many types of waves, the extension to radiation laws and intensity scaling is a useful additional result, which is not difficult to obtain from them.

5. Mode coupling and high-order waves

The Fourier analysis or dispersion methods, which are a very convenient method for studying linear waves in homogeneous media, cease to apply in the presence of density stratification or nonuniform force fields, which cause the wave speeds, scattering scales, or damping rates to become nonuniform. In such cases, attempts to apply "local" dispersion relations through the ray approximation can lead to incorrect cutoff frequencies (Sec. V.A.6). The laws of variation of amplitude and phase with altitude cannot be determined either, since the WKBJ approximation assumes complex exponential wave fields, i.e., exponential or constant amplitude and linear or zero phase. Linear waves in stratified media have a wealth of important properties not shared by waves in homogeneous media, such as cutoff frequencies, critical levels, nonsinusoidal waveforms and nonlinear phases, associated with phenomena of spectral filtering, wave absorption, waveform deformation, and phase shifting, which can be described in detail by the exact solutions of linear wave equations with variable coefficients. Such exact solutions have been obtained mostly for simple forms of the coefficients, corresponding to atmospheres either isothermal or with simple temperature profiles, under magnetic fields that either are uniform or evolve in a simple way. Additional exact solutions of second-order wave equations, for acoustic-gravity, Alfvén, or magnetosonic-gravity waves in nonisothermal atmospheres and nonuniform magnetic fields, would be useful in their own right and as further checks on the general properties of magneto-atmospheric modes. The latter have several modes, and, except for particular geometries, couple second-order waves into motions described by wave equations of the fourth and higher orders. We have presented a method (Sec. III) for exact solution of linear wave equations of arbitrary order, with coefficients that are either a combination of constants and exponentials [Eq. (153)] or a combination of powers [Eq. (157)]. The method includes simple algebraic rules for the reliable determination of asymptotic amplitude and phase laws (Sec. III.A.6) and cutoff frequencies (Sec. III.C.2), as well as criteria for the existence and location of critical levels (Sec. III.C.4) and wave properties there (Sec. III.C.5). The method includes second-order waves as particular cases, and was illustrated by considering vertical hydromagnetic-gravity waves in an oblique magnetic field, which are described by a fourth-order wave equation (Sec. III.A.6). The method could also be applied to other fourth-order magneto-atmospheric waves discussed in less analytical detail in the literature, i.e., to oblique waves in a vertical magnetic field (Ferraro and Plumpton, 1958), or to oblique waves in oblique magnetic fields (Zhugzhda and Dzhalilov, 1984a), say, in cases where the horizontal wave number is either in the plane of gravity and the magnetic field, or orthogonal to it. The most general geometry for magneto-atmospheric waves in isothermal conditions, under uniform gravity and magnetic fields, that is, with gravity, the magnetic field, and the horizontal wave vector not coplanar, is amenable to analysis by the same method, with more complicated algebra.

6. Multiple diffusion and boundary layers

The method discussed in Sec. III is relevant to waves of any order with exponential (or certain types of power-law) wave speeds, scattering scales, or damping rates, and thus applies to a variety of problems concerning both atmospheric and other waves. For example, Alfvén waves in an atmosphere with electric resistance (Campos, 1983e) and magnetosonic-gravity waves in a thermally radiating atmosphere (Cally, 1984) are second-order cases; an instance of the fourth order is acoustic-gravity waves in a viscous atmosphere in the presence of thermal conduction (Lyons and Yanowitch, 1974). The method applies as well to other types of doubly or multiply diffusive atmospheric waves, e.g., Alfvén waves with viscous and resistive damping (Sec. IV.A.1), acoustic-gravity waves in the presence of thermal conduction and radiation, and magnetosonic-gravity waves under any combination of these four diffusion processes. The analogous problems of plasma waves in Epstein-type layers with exponential profiles can be dealt with by the same method, even for higher-order waves. Similar linear wave equations with variable coefficients occur in the acoustics of ducts, if dissipation by viscous or thermal processes is included, either in the absence of flow, e.g., wave damping in horns, or in its presence, e.g., acoustics of turbulent flow in nozzles, using an "eddy viscosity" to represent turbulent stresses. The equations describing the interaction of sound with vorticity (Möhring, Müller, and Obermeier, 1983) have been solved only for the linear velocity profile, which is associated with a particular decoupling, reducing the order of the wave equation from three to two; the case of an exponential velocity profile would preserve the coupling of vorticity and sound, resulting in a third-order wave equation, which is within the capabilities of the method. The exponential shear flow velocity occurs for the asymptotic suction profile, which is a case of a boundary layer having a uniform shape along a plate, obtained by applying uniform suction to withdraw slow "air" through the wall. The stability of the asymptotic suction profile is specified by a generalization of the Orr-Sommerfeld (Drazin and Reid, 1981) equation, which is of fourth order with exponential coefficients, i.e., solvable by the same method. The truncated Tollmien-Schlichting form of the stability equation for viscous flow, which remains of fourth order, and the inviscid case of the Rayleigh equation, which drops to second order, are again equations solvable by the method indicated in Sec. III. The stability of a boundary layer is modified, on a curved wall, by the presence of Görtler vortices, which are described by a set of simultaneous differential equations that have exponential coefficients, as in the case of the asymptotic suction profile. The method described in Sec. III could be used by expressing each of the velocity components in an exponential series, with the recurrence formula for the coefficients C_j in the case of one dependent variable Φ in Sec. III.B.1 replaced for N independent variables Φ_α with $\alpha=1,\ldots,N$, and with coefficients $C_{\alpha,j}$ related by a recurrence matrix specifying $C_{\alpha,j}$ in terms of $C_{\alpha,j-1}$.

7. Multidimensional stratification, flux tubes, and nonlinear dispersion

Returning to magneto-atmospheric waves, it has been shown that one-dimensional magneto-hydrostatic equilibrium (Sec. IV.B.1) is only possible for a uniform magnetic field or a horizontal magnetic field varying with altitude; the case of an oblique nonuniform magnetic field would lead, necessarily, to two-dimensional stratification, say, in the (x,z) plane, and if the latter were of exponential type, the method of Sec. III could be used in two independent variables $\zeta \sim e^{-z/L}$ and $\xi \sim e^{-y/l}$, leading to solutions in terms of Appell-type hypergeometric functions (Erdelyi, 1953). A simple way of having a magnetically structured atmosphere consistent with magneto-hydrostatic equilibrium is to consider magnetic flux tubes. The magneto-acoustic waves in flux tubes have some properties similar to those of sound in convergent ducts, with the reduction in cross section providing the decrease in mass density per unit length similar to an atmosphere's, and the sound speed modified by the magnetic stresses as for the elastic tension in a collapsible tube. This raises the question of the extent to which the general properties of sound in horns (Sec. IV.B of Part I) have analogs for flux tube modes, a positive example being given in Sec. IV.B.5 of Part II. A possible further analogy would relate the acoustics of nozzles (Sec. IV.C of Part I) to the modes in magnetic tubes with flow, e.g., as in straightened coronal loops or arcades. The study of waves in flux tubes relies on the assumption of slenderness (Sec. IV.B.2 of Part II), which becomes invalid as the tube flares out with height. The theory of waves in thick flux tubes is still in its infancy, with the first research into this topic appearing only recently (Pizzo, 1986). Other properties that are still subject to conjecture (Parker, 1979) include the association of flow with magnetic flux tubes and the interaction and merging of flux tubes. To give an example, tube modes (Sec. IV.B of Part II) probably occur (a) in the solar photosphere, where the magnetic field is concentrated in narrow tubes at granulation boundaries, and (b) in the high chromosphere, where the tubes have merged into a uniform magnetic field, so that magneto-atmospheric wave theory (Sec. II) applies. The transition between regimes (a) and (b) may involve wave propagation in merging flux tubes in the middle chromosphere, and has not been modeled in any detail. The two-dimensional analog of the flux tube is the magnetic slab, which consists of two surface currents enclosing a tangential internal magnetic field different from the exterior field; both the magnetic slab and the magnetic interface, which is a single current sheet, have been studied mostly in the absence of gravity, using dispersion relations resulting from boundary conditions. The dispersion relations for flux tube, magnetic slab, and current sheet modes can be put into Whitham's equation (Sec. IV.C) to describe weakly nonlinear waves, with or without dissipation; depending on the form of the terms in the dispersion relation, equations of the type named after Burgers, Korteweg and De Vries, Benjamin and Ono, Leibovich, Pritchard, and Roberts, can be obtained (Sec. IV.C.5). Wave modes with dispersion relations having different terms could lead to other "new" nonlinear wave equations.

C. Methods available, their advantages and scope

The presentation of a review usually gives preference to general methods, which are adequate for the solution of a class of problems as wide as possible, thus providing for economy of exposition as well as a powerful analytical tool. Since each method tends to be particularly suited to a certain class of problems, when dealing with several problems it may be appropriate to use different methods. The most suitable method for each problem is that which leads most simply to reliable results; by contrast, a less appropriate method may be intrinsically unreliable or involve intricate calculations that are prone to error. Among the range of methods that have been successfully applied to waves in gases are the following six groups: (i) direct elimination among the fundamental equations to arrive at a wave operator (Sec. V.B of Part I and I.A. of Part II) or use of a suitable variational principle (Sec. II.A of Part I); (ii) solution of linear wave equations with constant coefficients by Fourier analysis (Sec. I of Part II) for boundary-value problems, and by Laplace transforms for initial-value problems; (iii) exact solution of linear wave equations with simple forms of nonuniform coefficients by means of special functions (Sec. IV of Part I and Secs. II and III of Part II); (iv) approximate solution of linear wave equations with "arbitrary" variable coefficients satisfying certain constraints, allowing the use of ray, compactness, initial, or asymptotic methods (Sec. III of Part I); (v) exact solutions of nonlinear wave equations by special transformations in some cases, and more often by approximate methods, such as parametric expansions; (vi) numerical procedures whenever the analytical methods fail to give reliable or understandable results, e.g., at a "late" stage like summing a series solution or finding the roots of a dispersion relation, or at an "early" stage, like computing the solution of a wave equation by finite differences or elements. The two parts of this review have concentrated on the analytical methods that yield most readily information on the physics of waves, and it is therefore appropriate to conclude with a discussion of available methods that place more emphasis on the numerical aspects. An outline of the methods available for the study of waves in gases (Sec. V.C) is a sequel to the list of open problems requiring further research (Sec. V.B) and may also suggest alternative ways of examining the controversial issues presented in Sec. V.A.

1. Boundary-value and initial-value problems

The dispersion relation for linear wave equations with constant coefficients may be obtained either by substituting a complex exponential solution, or by Fourier or Laplace transforms. The Fourier transform is well suited to boundary-value problems, in which the constants of integration are determined by imposing conditions on the wave field at specific positions, for all time; it is also a convenient way of obtaining the radiation field of a source with a given spatial and temporal distribution, or equivalently, a given directivity and spectrum. The Laplace transform is suited to initial-value problems, for which the wave field is specified in all space at time $t=0$, and its subsequent evolution for all times $t>0$ is sought; the initial problem is useful when there is uncertainty as to how to meet causality, i.e., how to exclude waves coming from "infinity," or when there is some uncertainty as to which boundary condition applies at infinity, since the evolution of the initial wave field will indicate how it evolves ultimately as $t \rightarrow \infty$. The initial-value problem, as compared with the boundary-value problem, involves the inversion of a Mellin integral instead of a Fourier integral, complex analysis being often used in both cases. The discrete spectrum appears on the complex plane as a succession of isolated poles, whereas the continuous spectrum may correspond to branch cuts. The evaluation of residues at the poles can specify the eigenfunctions corresponding to each eigenfrequency, a technique that works even for linear waves in stratified media, for which wave speeds depend on space but not on time (Sec. II.B.7). Generally speaking, Fourier and Laplace transforms can be used in any spatial or temporal coordinate not appearing in the coefficients of the linear wave equation.

2. Ray, compactness, and asymptotic approximations

When the coefficients of the linear wave equation depend on one spatial coordinate, say z, with a length scale L, the wave field can be calculated approximately in the limits of short or long waves: (i) if the wavelength is much longer than the length scale $\lambda \gg L$ the inhomogeneity of the medium may be treated as an interface (Sec. III.A of Part I), reflecting and transmitting an incident wave, according to the appropriate boundary conditions (Sec. III.A.2); (ii) if the wavelength is much shorter than the length scale, $\lambda^2 \ll L^2$, then ray theory (Sec. III.C of Part I) can be applied, to specify significant phase shifts and moderate amplitudes, within its limitations. The two limits cannot be matched, since neither the compactness (i) nor the ray (ii) approximations hold in the case of wavelength comparable to the length scale $\lambda \sim L$. Another approximation that may hold for waves in homogeneous or stratified media is the consideration of initial and asymptotic wave fields, respectively, for small and large distances from the source. If the inner and outer regions do not overlap, then matching of the two solutions is again not possible. In the case of an atmospheric critical level, matching of the low- and high-altitude wave fields may be possible, but a solution valid in the neighborhood of the critical level may shed more light on the problem. Parameters other than distance or compactness may be used in expansions of the wave field, with overlapping inner and outer solutions, allowing their matching; the convergence of parametric expansions is often implicitly assumed without proof, and the matching procedure may require particular care (Crighton and Leppington, 1973). The description of the transition between the initial and asymptotic wave fields, and the matching of the ray and compactness limits across the intermediate frequency band, are both included in an exact solution of the wave equation, valid for all distances and frequencies.

3. Exact solutions and special functions

The exact solution of linear wave equations with variable coefficients can be obtained, for boundary-value problems, in terms of special functions, e.g., of Bessel or hypergeometric type. The forced wave equation can then be solved by a bilateral integral transform, using as kernel, instead of the exponential in the Fourier transform, the special function that is a solution of the unforced equation, e.g., a Hankel transform for a Bessel-type wave equation. The initial-value problem will require a unilateral transform in time t, with integration in the range $0-\infty$ [instead of a bilateral transform in space x, with integration in the range $(-\infty, +\infty)$ for the boundary-value problem]. The exact diffraction of waves in stratified media is associated with the properties of the special function specifying the wave field. Each special function has its own set of properties, although general approaches exist applying these to a variety of functions (Courant and Hilbert, 1953; Erdelyi, 1953; Morse and Feshbach, 1953; Campos, 1984g, 1985c, 1986f, 1986g). The criticisms sometimes made of exact solutions of linear wave equations with variable coefficients is that they (i) tend to be complicated, and (ii) are restricted to a particular stratification profile anyway. Concerning criticism (i), if by a simple solution is meant a finite combination of elementary functions, then a linear wave equation with variable coefficients only exceptionally (Sec. IV.A.6 of Part I) has such solutions, and the approximate methods seeking such simple solutions are necessarily of restricted validity, e.g., the case of the ray approximation. Moreover, even though special functions, identified with series expansions or integral representations, are required as exact solutions, certain wave properties, such as cutoff frequencies, critical levels, and asymptotic fields, can be determined by algebraic means, in cases for which approximate methods are unreliable (as shown in Sec. III). Concerning criticism (ii), it is sometimes possible to obtain general laws for waves in nonhomogeneous media, subject only to generic restrictions on the stratification profile; in these cases, an exact solution for a particular profile serves as a check on the general law and specifies in detail all coefficients or parameters appearing there.

4. Nonlinear waves and variational methods

The methods of solution of nonlinear wave equations are generally less effective than in the linear case, since the principle of superposition and other methods of gradual construction of a solution usually break down. The exact solution of nonlinear wave equations may be possible, in the absence of dissipation and dispersion, by the method of characteristics, and in the presence of dissipation or dispersion, by special transformations that render the equations linear in specific cases, e.g., the transformation of the Burgers equation into the heat equation, and the hodograph transformation linearizing plane homentropic flow equations by using the velocity as an independent variable (Von Mises, 1958). More often nonlinear wave equations can be solved, if at all, only by tedious procedures, such as the inverse scattering approximation (Whitham, 1974). Approximate methods, such as parametric expansions, appear to be generally possible (Nayfeh, 1973), but their convergence is seldom proven, and their application cumbersome beyond the lowest orders. It is generally possible to replace the equations of motion by a variational principle, the difficulties in eliminating for the wave equation giving way in this case to the manipulations required to find a Lagrangian. Once the Lagrangian is found, assuming that it is valid for nonlinear perturbations, the linear case is obtained by neglecting all terms beyond second order. Thus variational methods are usually not much more difficult for nonlinear than for linear waves. The Lagrangian is quadratic in the latter case, and has terms of higher order in the former. If the Lagrangian does not depend explicitly on time, the variational method leads readily to a conservation equation, in which the energy flux and density can be reliably identified. By contrast, trying to derive an energy equation from the equations of motion, for waves in a heterogeneous medium, may not be an easy task, bearing in mind the need to distinguish (Lighthill, 1978) between terms corresponding to the mean state, the wave perturbations, and their interaction. Such a distinction is readily made (Landau and Lifshitz, 1949, 1953) in the Lagrangian, since, on substitution in the Euler-Lagrange equation, (i) the zero-order terms disappear, i.e., are irrelevant in the Lagrangian, (ii) the first-order terms specify the mean state, i.e., do not affect the perturbations, (iii) the quadratic terms describe linear waves, (iv) the terms of higher than second order specify nonlinear effects.

5. Well-posed problem and fully justified algorithm

When trying to solve linear wave equations in stratified media with elaborate profiles, dealing with nonlinear wave equations for which no simplifying transformations are known, or solving problems with boundaries of complicated shape, there may be no alternative to the use of numerical methods. The point, sometimes raised, that direct numerical solution of a wave equation may appear to be "simpler" than an analytic method tends to ignore all that is needed to justify fully a numerical algorithm. In fact, it is necessary (i) to prove that the wave equation, together with boundary and initial conditions, has at least one solution, (ii) to establish the conditions under which the solution is unique, (iii) to deduce properties of the solution sufficient to allow the choice of an algorithm that converges to it, (iv) to choose parameters such that the convergence be assured, (v) to check that the possible accumulation of errors does not invalidate the numerical results. As examples of problems that fail to meet the criteria above we give the following: (i) if a wave problem is over-specified with incompatible boundary or initial conditions, the solution does not exist; (ii) if the radiation condition is not correctly applied in an unbounded medium, contamination by arbitrary waves coming from infinity may occur, leading to nonuniqueness; (iii) if a wave equation has a discontinuous solution, e.g., a shock, a method of solution applying to smooth functions may fail to approximate it; (iv) in a procedure using finite difference in space Δx and time Δt, the ratio $\Delta x : \Delta t$ should not exceed the wave speed, otherwise causality is violated, since signals cannot propagate at the required speed; (v) since a wave is a signal, the errors in the initial waveform propagate with it, e.g., in the case of multiple reflections the errors may accumulate and render meaningless the computation of higher-order harmonics. The preceding examples show that a valid application of numerical methods requires either a careful mathematical justification or a verification that the algorithm is consistent with the physical properties of waves, or preferably both.

6. Numerical instabilities and trial functions

There are situations in which a numerical procedure is applied *ad hoc* to a wave equation, without *a priori* mathematical proof of steps (i)—(v) in Sec. V.C.5, and in cases in which the wave properties are not sufficiently well known to give confidence in the algorithm. In these situations "numerical instabilities" may occur, for any of the following reasons, which lead to failure of the method: (i) the problem is ill posed, i.e., has no solution; (ii) the solution exists but is not unique, and the algorithm jumps among several possible solutions; (iii) the solution exists and is unique, but has properties such that the chosen algorithm cannot approximate it; (iv) the algorithm can potentially converge to a unique solution, provided that a more appropriate choice of parameters, e.g., step size, is made; (v) the algorithm has tended to the solution, but the accumulation of errors has rendered the result useless. The appearance of numerical instabilities in such cases is a warning that the algorithm has failed, and steps should be taken to modify it in a mathematically valid way, so that a reliable physical interpretation may be given to results. The worst case is perhaps that of an invalid algorithm, which has "converged" to something other than the solution of the problem, with no numerical instabilities to warn that something is amiss, thus lending credibility to an incorrect result, with the attendant risk of an erroneous interpretation. There exist algorithms, such as "trial solutions," that do not normally exhibit numerical instabilities, but their accuracy may be very difficult or impossible to estimate reliably, in the absence of solutions by other methods. A "trial solution" is a function, usually involving some parameters, which is offered as a guess at the solution of a problem. Substitution into the wave equation and all boundary and initial conditions should then specify uniquely the parameters. Usually some insight is required in guessing a true solution that satisfies all the conditions, and more often either the wave equation or some boundary or initial condition is met only to a certain level of approximation by the "trial solution." The method of trial solutions can be combined with a variational principle, i.e., the action integral can be calculated for the class of "trial functions" and the parameters chosen so as to yield an extremum, e.g., a relative minimum; the exact solution would yield a still lower absolute minimum. Since the exact solution is not known, the absolute minimum is undetermined, and thus we do not know either how close or how far the relative minimum in the class of trial functions is to the absolute minimum. This is an example of the difficulty of estimating the accuracy of a trial solution to a problem, short of knowing the exact solution, in which case the former is of less interest, except as a simpler approximation.

7. Combination of analytical and numerical methods

The preceding account of numerical methods (Secs. V.C.5 and V.C.6) is not meant to cast doubt on their usefulness, since there are problems for which they offer the only approach; it merely serves as a warning that numerical algorithms may be deceptively simple while concealing deeper mathematical problems, and that their application is no substitute for a clear understanding of the physics of the problem. It may be argued that the application of numerical methods should be deferred until as "late" as possible in a problem, on the following grounds: (a) the analysis of the problem may justify some of the steps (i) to (v) stated in Sec. V.C.5, allowing simpler proofs of the remaining points needed to validate the algorithm; (b) the use of numerical procedures at a later stage minimizes the accumulation of errors of computation; (c) a better knowledge of the solution can be used to construct a more efficient program, faster to run and with fewer requirements on memory; (d) analytic methods usually afford greater physical insight, which increases confidence in the interpretation of the numbers or graphs that appear as the output of a program.

The preceding remarks suggest the following gradual method of approach to wave problems, simple or complex: (i) to consider first linear disturbances in homogeneous media, using the dispersion relation to find whether the waves are isotropic or dispersive and to determine their phase and group velocities, the shape of wavenumber surfaces, and the wave variables that are propagated, e.g., longitudinal, transverse, or mixed; (ii) if the medium is stratified, to obtain approximations to the wave fields in various cases (initial and asymptotic, low and high frequency), for plausible forms of the stratification profiles; (iii) to check the results of (ii) by comparison, if possible, with exact solutions for waves in simple but relevant stratification profiles, determining cutoff frequencies, amplitude and phase laws, etc.; (iv) to examine, if appropriate, the energy density and flux associated with the waves, using variational methods or the equations of motion; (v) to consider nonlinear effects, if the disturbances are of large amplitude, by substituting the dispersion relation for linear waves in Whitham's equation (Sec. IV.C.1) and checking the latter against suitable approximations or expansions of the equations of motion or a variational principle accurate to higher than second order; (vi) to apply numerical methods to linear equations with coefficients of complicated functional form, or to nonlinear equations that are "untractable" by analytical methods, using all of the preceding steps as a background to validate the algorithm and interpret the result. It is felt that this combination of analytical and numerical methods is synergistic, that is, the two approaches reinforce each other; by employing analytic methods we gain physical insight into the phenomena, and then proceed to more complex problems by using numerical algorithms in a reliable manner.

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