Shell-model foundations of the interacting boson model

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The authors discuss the concepts on which the interacting boson model is based, in particular those of pairing and seniority, which lead to a connection with the shell model. They review some of the calculations performed so far using these concepts and briefly comment on their results.

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I. INTRODUCTION

Nuclei consist of many strongly interacting protons and neutrons. In spite of their complexity, many nuclei exhibit, in their low-lying spectra, a relatively simple structure. Many regularities of nuclear energy levels have been described by the shell model in terms of wave functions of single nucleons moving independently in a (spherical) static central potential well (Haxel, Jensen, and Suess, 1949; Mayer, 1949). In nuclei with few protons and neutrons outside (or missing from) closed shells (valence nucleons), level energies and other nuclear properties have been successfully calculated. Simple shell-model configurations have been used with Hamiltonians containing *effective* two-nucleon interactions. This is the situation in many light nuclei and in the neighborhood of magic numbers in medium-mass and up to heavy nuclei (deShalit and Talmi, 1963).

As the number of valence nucleons increases, the direct application of the shell model becomes prohibitively difficult. On the other hand, other regularities emerge from the experimental data. Energy levels of many nuclei can be grouped into bands characterized by electric quadrupole transitions within bands, which are very much enhanced over the single-nucleon values. In some regions of nuclei, levels can be grouped into rotational bands with energies proportional to $J(J+1)$. In such bands, enhanced electric quadrupole transitions are observed that can be derived from one intrinsic quadrupole moment. Such collective spectra found a simple interpretation in terms of a deformed shape of the nucleus rotating around an axis perpendicular to its axis of symmetry (Bohr, 1951, 1952; Bohr and Mottelson, 1953). Even-even nuclei, which are the nuclei considered here, were described in terms of a quadrupole deformation of the nuclear surface. Such a deformation is defined by five variables α_μ ($\mu=\pm2,\pm1,0$), forming the components of a quadrupole tensor. A possible scalar component was eliminated by requiring that the volume be conserved. An equivalent description was given by two variables β and γ defining the quadrupole deformation in the body-fixed frame of reference (that of its three principal axes). The orientation of this body-fixed frame in space was defined by the Euler angles adopted as the other three variables. A second-order differential operator in these variables was written down (Bohr Hamiltonian) whose eigenfunctions should give a good description of collective states in nuclei. Once a potential energy is defined as a function of β and γ in the Bohr Hamiltonian, $V(\beta,\gamma)$, it is possible to solve the differential equation. For certain potentials, approximate solutions were obtained that are particularly simple. These correspond to vibrational, rotational, or γ-unstable nuclei. Several attempts have been made to obtain solutions appropriate for any type of collective motion. The potential energy was either introduced phenomenologically (Gneuss and Greiner, 1971) or related to a microscopic calculation in a deformed basis (Kumar and Baranger, 1968; Kumar, 1983). In general, the solutions had to be obtained by rather involved numerical integrations.

An equivalent way to formulate the problem is to adopt as a complete set of orthogonal states the wave functions of the five-dimensional harmonic oscillator. Such states are solutions of the differential equation if the potential energy in the Bohr Hamiltonian is a quadratic function of the five variables α_μ. Such eigenstates can be conveniently expressed as states of bosons that are the components of a rank-2 spherical tensor. This is how d bosons (with angular momentum $J=2$) were introduced for the description of collective quadrupole states. In the case of

vibrations around a spherical equilibrium shape, such a description is rather simple. If, however, the equilibrium shape is deformed ($\beta_0 \neq 0$), the description of rotations and vibrations around that shape in terms of d bosons becomes very difficult. A step toward a simplified description of nuclear states in terms of d bosons was made by introducing states that form bases of irreducible representations of the group U(6) (Janssen, Jolos, and Dönau, 1974; Jolos, Dönau, and Janssen, 1975). The fully symmetric irreducible representations of U(6) are characterized by an integer N. This number determines the maximum number of d bosons that may appear in a state of a given representation, hence the name *truncated quadrupole model* (TQM) given to this model (Paar, 1979). The advantage of the U(6) group is that it contains as a subgroup the SU(3) group. The group SU(3) had been introduced earlier by Elliott (1958) in order to provide a shell-model description of rotational spectra. In simple analogy, rotational spectra could be obtained from states of d bosons. The actual construction of the relevant Hamiltonians and the irreducible representations is not a simple matter. The space associated with d bosons is five dimensional, and the generators of U(6) [and SU(3)] are complicated expressions in the creation and annihilation operators of d bosons.

Meanwhile, an alternative description was independently developed. In that approach, called the *interacting boson model* (IBM), s bosons (with $J=0$) are added to the d bosons, thereby constructing a six-dimensional space (Arima and Iachello, 1975). Thus representations and operators of U(6) can be conveniently constructed. The number N characterizing the fully symmetric irreducible representations of U(6) turned out to be simply the total number of s and d bosons. A simple Hamiltonian was constructed with single-boson terms and boson-boson interactions. The eigenvalues and eigenstates of this Hamiltonian could very well reproduce, for various values of the parameters adopted, various types of collective states like vibrational and rotational spectra.

The successful IBM (or IBM-1, as it is called now) can be viewed as a phenomenological model that can yield, in an elegant way, the solutions to problems posed by collective Hamiltonians. The d bosons could be viewed as quanta of surface vibrations, whereas the s bosons would be, in this view, an artificial device to simplify the mathematics. The integer N would just determine in some way the number of discrete (bound) states of the collective Hamiltonian. Furthermore, since it was shown that in the limit in which the number N is large the solutions of the boson Hamiltonian approach those of the collective model (Dieperink, Scholten, and Iachello, 1980), the boson model could be viewed as a simpler version of the collective model. Thus, from this point of view,

$$\text{IBM-1} = \text{TQM} \underset{N\to\infty}{\longrightarrow} \text{Collective Model} . \tag{1.1}$$

The question was raised whether the boson model had a deeper significance and could be related to the shell model. It was suggested that the simplest fermion states that can correspond to s and d bosons are pairs of identical valence nucleons coupled to $J=0$ and $J=2$ (Arima *et al.*, 1977; Otsuka *et al.*, 1978). Correlated $J=0$ (S) and $J=2$ (D) pairs of identical nucleons have a simple description in terms of generalized seniority (Talmi, 1971, 1973; Shlomo and Talmi, 1972). They can be used to construct shell-model eigenstates of valence protons or of valence neutrons, as in "semimagic nuclei." In nuclei with both valence protons and neutrons, the strong and attractive seniority-breaking interaction between protons and neutrons must play a dominant role. If a quadrupole-quadrupole interaction is adopted, eigenstates will include coherent admixtures of proton (π) and neutron (ν) states built of S_π, D_π and S_ν, D_ν pairs. This shell-model description led to a boson model with proton s_π and d_π bosons and neutron s_ν and d_ν bosons (Arima *et al.*, 1977; Otsuka *et al.*, 1978). The boson Hamiltonian should include a strong and attractive quadrupole interaction between proton and neutron bosons. This is a more detailed model in which effects of the numbers of valence protons and valence neutrons can be studied separately. It is called the interacting boson model 2 (IBM-2). It gives better agreement with experimental properties than IBM-1 and with parameters whose dependence on proton and neutron numbers is rather regular. Since it is derived from the shell model, it is sometimes called *interacting boson approximation* (IBA). The two abbreviations IBA and IBM have been used interchangeably in the literature. Although properly speaking the abbreviation IBA should be used for the approximation and IBM for the model, we shall henceforth follow common usage. Thus

$$\text{Shell Model} \rightarrow \text{IBA} \rightarrow \text{IBM-2} . \tag{1.2}$$

The interacting boson models 1 and 2 have attracted the attention of many authors. In particular, several schemes have been devised by means of which the parameters of the IBM-2 Hamiltonian and other operators can be related to the parameters of the shell model. In this article we review the *concepts* on which the majority of these procedures are based. We also briefly mention further extensions of the interacting boson models (IBM-3 and -4) and address some of the remaining problems.

II. THE INTERACTING BOSON MODEL 1

Before addressing the question of the relation between the interacting boson model and the shell model, we briefly review the original version of the model (interacting boson model 1). In this version (Arima and Iachello, 1975, 1976, 1978, 1979) it was assumed that low-lying collective states of even-even nuclei could be described as states of a given (fixed) number N of bosons. Each boson could occupy two levels, one with angular momentum $J=0$ (s boson) and another, usually with higher energy, with $J=2$ (d boson). In order to find the eigenstates of this assembly of bosons, an appropriate Hamiltonian was constructed and diagonalized. The simplest nontrivial Hamiltonian was assumed to contain only single-boson energies and

boson-boson interactions. Such Hamiltonians can be constructed by using boson operators which will now be defined.

It is convenient to use the formalism of second quantization. We thus introduce creation operators $d_\mu^\dagger$ and $s^\dagger$, which create a single d boson in a state with $J_z=\mu$ and a single s boson, respectively. These operators satisfy with the corresponding annihilation operators d_μ, s the usual Bose commutation relations. Using these operators, one can write down single-particle (boson) operators as follows:

$$d_\mu^\dagger d_{\mu'},\ d_\mu^\dagger s,\ s^\dagger d_\mu,\ s^\dagger s\ . \tag{2.1}$$

There are, altogether, 36 such independent operators. To construct a Hamiltonian that is invariant under rotations, it is more convenient to use certain linear combinations of the operators (2.1). These should be operators that transform in a simple way under rotations. The creation operators $d_\mu^\dagger$ transform under rotations like the states of a single d boson, $d_\mu^\dagger|0\rangle$. They transform as the components of an irreducible (spherical) tensor of rank 2. An irreducible tensor of rank 2 can be formed also from the annihilation operators d_μ using the definition

$$\tilde{d}_\mu=(-1)^{2-\mu}d_{-\mu}=(-1)^\mu d_{-\mu}\ . \tag{2.2}$$

One can now define a complete set of irreducible tensor operators by

$$(d^\dagger\times\tilde{d})_\kappa^{(k)}=\sum_{\mu\mu'}(2\mu 2\mu'\,|\,22k\kappa)d_\mu^\dagger\tilde{d}_{\mu'}\ ,\quad k=0,1,2,3,4, \tag{2.3}$$

as well as

$$d_\mu^\dagger s,\ s^\dagger\tilde{d}_\mu\ , \tag{2.4}$$

with rank 2 and the scalar (rank 0)

$$s^\dagger s\ . \tag{2.5}$$

The most general Hamiltonian with single-boson terms and boson-boson interactions, which is invariant under rotations (commutes with J), can now be constructed. It is a linear combination of all possible scalar products of the irreducible tensors in Eqs. (2.3)—(2.5). In addition, the two single-boson scalars can be added, i.e., Eq. (2.5) and the $k=0$ tensor in Eq. (2.3). Since all single-boson operators do not change the total number of s and d bosons, neither does the Hamiltonian constructed from them. It commutes with the number operator

$$\hat{N}=s^\dagger s+\sum_\mu d_\mu^\dagger d_\mu=s^\dagger s+(d^\dagger\cdot\tilde{d})\ , \tag{2.6}$$

whose eigenvalues N are appropriate quantum numbers for the eigenstates of the Hamiltonian.

The requirement that the boson Hamiltonian be Hermitian (and real) implies that the two quadrupole operators (2.4) should appear only in certain combinations. Still, the number of terms is high. There are altogether nine possible scalar products in addition to two single-boson terms. The scalar products, however, are not all linearly independent. Due to the symmetry of the boson states there are only three allowed states of two d bosons, with $L=0,2,4$. States with odd values of L are antisymmetric. Thus any two d-boson interactions can have at most three independent terms. Hence there are only three independent combinations of the five scalar products of the tensor operators [Eq. (2.3)]. In order to bring out this fact it is possible to transform all scalar products by changing the order of couplings. Due to the commutation relations, this will yield single-boson terms in addition to the two-boson interactions. The resulting Hamiltonian can then be expressed in the following form:

$$H=\varepsilon_s s^\dagger s+\varepsilon_d(d^\dagger\cdot\tilde{d})+\tfrac{1}{2}\sum_{L=0,2,4}c_L[(d^\dagger\times d^\dagger)^{(L)}\cdot(\tilde{d}\times\tilde{d})]^{(L)}+\tfrac{1}{2}\tilde{v}_0[(d^\dagger\times d^\dagger)_0^{(0)}s^2+(s^\dagger)^2(\tilde{d}\times\tilde{d})_0^{(0)}]$$
$$+\frac{1}{\sqrt{2}}\tilde{v}_2\{[(d^\dagger\times d^\dagger)^{(2)}\times\tilde{d}s]_0^{(0)}+[s^\dagger d^\dagger\times(\tilde{d}\times\tilde{d})^{(2)}]_0^{(0)}\}+\tfrac{1}{2}u_0(s^\dagger)^2s^2+\frac{u_2}{\sqrt{5}}s^\dagger s(d^\dagger\cdot\tilde{d})\ . \tag{2.7}$$

This Hamiltonian contains nine parameters, two coefficients of the single-boson energies and seven of the boson-boson interactions. The fact that the total number of bosons $\hat{N}=\hat{n}_s+\hat{n}_d$ commutes with the Hamiltonian can be used to simplify Eq. (2.7) to the form

$$H=\varepsilon_s N+\tfrac{1}{2}u_0N(N-1)+\varepsilon'(d^\dagger\cdot\tilde{d})+\tfrac{1}{2}\sum_{L=0,2,4}c_L'[(d^\dagger\times d^\dagger)^{(L)}\cdot(\tilde{d}\times\tilde{d})^{(L)}]$$
$$+\tfrac{1}{2}\tilde{v}_0[(d^\dagger\times d^\dagger)_0^{(0)}s^2+(s^\dagger)^2(\tilde{d}\times\tilde{d})_0^{(0)}]+\frac{1}{\sqrt{2}}\tilde{v}_2\{[(d^\dagger\times d^\dagger)^{(2)}\times\tilde{d}s]_0^{(0)}+[s^\dagger d^\dagger\times(\tilde{d}\times\tilde{d})^{(2)}]_0^{(0)}\}\ . \tag{2.8}$$

The single d-boson energy ε' and the coefficients c_L' in Eq. (2.8) are appropriate linear combinations of the parameters appearing in Eq. (2.7). The first two terms in Eq. (2.8) contribute equally to all states with given N. Thus only six parameters are available for the analysis of level spacings and eigenstates in the most general Hamiltonian [Eq. (2.8)].

To calculate rates of electromagnetic transitions, the eigenstates should first be found by diagonalization of the Hamiltonian in the space of N bosons. The matrix elements of the transition operator between these states should then be calculated. The most important transi-

tions, which reveal the nature of the low-lying collective states, are electric quadrupole (E2) transitions. It was assumed in the model described in this section that the E2 transition operator is a single-boson operator. The most general real and Hermitian single-boson quadrupole operator is given by

$$T^{(2)}_{\mu}=\alpha_2(d^{\dagger}_{\mu}s+s^{\dagger}\tilde{d}_{\mu})+\beta_2(d^{\dagger}\times\tilde{d})^{(2)}_{\mu} . \tag{2.9}$$

The E2 transition operator is thus determined by two numerical coefficients α_2 and β_2. Other operators can also be constructed in a similar way.

For a detailed comparison of the boson-model predictions with experiment, the Hamiltonian H is diagonalized in an appropriate basis, thus obtaining the level energies and eigenfunctions. Rates of electromagnetic transitions are then calculated by taking matrix elements of the appropriate operators between eigenstates of the system. The boson model has been used in this fashion to account for a variety of properties of nuclei (see, for example, Arima and Iachello, 1984, and the forthcoming article by Casten and Warner in this journal). When used in this form, the boson model is approximately equivalent to the collective model of Bohr and Mottelson, at least for the low-lying states, and several authors have considered in detail the relation between the two models. One particular aspect of the model, however, which has contributed to its success, is the fact that under certain conditions, i.e., with a certain choice of the parameters in Eq. (2.8), it is possible to calculate all properties of the system in a compact analytic form. In such limiting situations, the special structure of the resulting spectra is clearly displayed and can serve as a guide for the general case. Such limits also provide definite sets of eigenstates of the N boson system, which can be used as convenient bases for calculating the Hamiltonian matrix in any case. The analysis of such states relies on the use of the elegant and powerful methods of group theory. Since this aspect has been emphasized before several times, we shall present it here only briefly.

The single-boson operators (2.3), (2.4), and (2.5) are generators of the Lie algebra of the U(6) group. Acting on the set of single-boson states $d^{\dagger}_{\mu}|0\rangle, s^{\dagger}|0\rangle$, which span a six-dimensional space, they transform it into itself. The Hamiltonian (2.7), constructed from scalar products of these generators, has nonvanishing matrix elements only between states that transform irreducibly among themselves under these generators. Hence eigenstates of the Hamiltonian form bases of irreducible representations of U(6). Fully symmetric states of s and d bosons belong to the fully symmetric irreducible representations characterized by the integer N. The various limits are obtained by considering subgroups of U(6).

The simplest Lie subalgebra of U(6) contains the generators (2.3) and (2.5), which commute with the number operator $\hat{n}_d=(d^{\dagger}\cdot\tilde{d})$. This is the algebra of U(5)×U(1). The generator (2.5), $\hat{n}_s=s^{\dagger}s$, can be eliminated by using $\hat{N}=\hat{n}_d+\hat{n}_s$, as carried out in Eq. (2.8), and the subgroup considered is simply U(5). The most general U(5) Hamiltonian is obtained by putting $\tilde{v}_0=0$ and $\tilde{v}_2=0$ in Eq. (2.8). It can be expressed as the linear combination

$$\varepsilon\hat{n}_d+\tfrac{1}{2}\alpha\hat{n}_d(\hat{n}_d-1)+\beta(d^{\dagger}\cdot d^{\dagger})(\tilde{d}\cdot\tilde{d}) +\gamma[10(d^{\dagger}\times\tilde{d})^{(1)}\cdot(d^{\dagger}\times\tilde{d})^{(1)}-6(d^{\dagger}\cdot\tilde{d})] . \tag{2.10}$$

The eigenvalues of (2.10) can be written down in terms of those of the pairing term $n_d(n_d+3)-\tau(\tau+3)$ and of the square of the angular momentum $\hat{L}=\sqrt{10}(d^{\dagger}\times\tilde{d})^{(1)}$,

$$\varepsilon n_d+\tfrac{1}{2}\alpha n_d(n_d-1)+\beta[n_d(n_d+3)-\tau(\tau+3)] +\gamma[L(L+1)-6n_d] . \tag{2.11}$$

In (2.11), τ is the seniority of the d bosons and assumes the values n_d, $n_d-2,\ldots,$ 1, or 0. The spectrum given by (2.11) is that of an *anharmonic vibrator.*

Another subgroup of U(6) is O(6), the group of real orthogonal transformations in the six-dimensional space of s and d bosons. The generators of the O(6) Lie algebras are $(d^{\dagger}\times\tilde{d})^{(1)}$, $(d^{\dagger}\times\tilde{d})^{(3)}$, and $(d^{\dagger}s+s^{\dagger}\tilde{d})$. The most general O(6) Hamiltonian can be expressed as

$$A\tfrac{1}{4}[d^{\dagger}\cdot d^{\dagger}-(s^{\dagger})^2](\tilde{d}\cdot\tilde{d}-s^2) +B\tfrac{1}{3}\sum_{k=1,3}(d^{\dagger}\times\tilde{d})^{(k)}\cdot(d^{\dagger}\times\tilde{d})^{(k)} +C10(d^{\dagger}\times\tilde{d})^{(1)}\cdot(d^{\dagger}\times\tilde{d})^{(1)} . \tag{2.12}$$

The eigenvalues of the pairing operator in (2.12) are given in terms of the seniority σ as $N(N+4)-\sigma(\sigma+4)$, where $\sigma=N, N-2,\ldots,$ 1, or 0. The second term is diagonal in the seniority τ of d bosons only. Hence the eigenvalues of (2.12) are given by

$$\tfrac{1}{4}A[N(N+4)-\sigma(\sigma+4)]+\tfrac{1}{6}B\tau(\tau+3)+CL(L+1) . \tag{2.13}$$

The spectrum given by (2.13) corresponds to that of a *γ-unstable* nucleus.

There is still another subgroup of U(6) that contains as a subgroup the O(3) group induced by spatial rotations. This requirement is imposed by the rotational invariance of the Hamiltonian leading to eigenstates with definite angular momenta. The SU(3) group has generators

$$\hat{Q}=d^{\dagger}s+s^{\dagger}\tilde{d}-\frac{\sqrt{7}}{2}(d^{\dagger}\times\tilde{d})^{(2)} , \qquad \hat{L}=\sqrt{10}(d^{\dagger}\times\tilde{d})^{(1)} . \tag{2.14}$$

The most general SU(3) Hamiltonian can be written as

$$-2\kappa\hat{Q}\cdot\hat{Q}-\kappa'\hat{L}\cdot\hat{L} . \tag{2.15}$$

The eigenvalues of (2.15) can be expressed as

$$(\tfrac{3}{4}\kappa-\kappa')L(L+1)-\kappa(\lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu) . \tag{2.16}$$

The quantum numbers λ,μ characterize the irreducible representations of SU(3). For any given λ,μ, the spectrum given by (2.16) is a *rotational* spectrum.

The boson model discussed so far, IBM-1, can be considered and judged as a purely phenomenological model much in the same way as the collective model of Bohr and Mottelson. The model could be interpreted by giving physical meaning only to d bosons, which could be viewed as describing quadrupole surface vibrations around a spherical equilibrium shape. The s boson could be looked upon as a mathematical device that merely simplifies the construction of U(6) generators and representations. The integer N would then characterize the fully symmetric irreducible representations of U(6). It determines the total number of boson states in the given representation that would approximately correspond to the number of discrete (bound) states of the collective Hamiltonian (TQM).

There is an interesting illustration of such a physical situation. This is the boson model developed for bound states of diatomic molecules (Iachello, 1981; Iachello and Levine, 1982). The collective Hamiltonian contains in that case only three dynamical variables: the three components of the vector $\mathbf{r}$ connecting the nuclei of the two atoms. It contains the usual kinetic energy term and potential energy, which is a function of $r=|\mathbf{r}|$. This is in simple analogy with the nuclear collective Hamiltonian, in which the potential energy is a function of β and γ only. The three Euler angles appear only in the kinetic energy in the same way that the angles θ and φ do in the molecular problem (Fig. 1).

If the potential has a minimum at $r=0$, for small values of r the collective Hamiltonian is that of a three-dimensional harmonic oscillator. Instead of solving the Schrödinger equation, the simple algebraic approach could be used by introducing three creation and annihilation operators $p_\mu^\dagger$ and p_μ, which are the three components of a p boson. The appropriate Hamiltonian is $(p^\dagger\cdot\tilde{p})+\frac{3}{2}$ with eigenvalues $n_p+\frac{3}{2}$. A more difficult problem arises when the minimum of the potential is at $r_0\neq 0$ (dipole deformation). If the potential well is sufficiently deep, the low-lying levels will form a rotational band with energies proportional to $L(L+1)$. Higher rotational bands are based on higher vibrational levels, and energies are given approximately by $a+bv+cv^2+BL(L+1)$, where v is the vibrational quantum number.

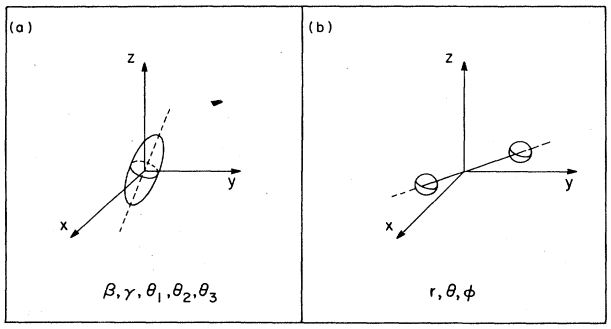

FIG. 1. Illustration of the collective degrees of freedom of nuclei (a) and molecules (b). For a deformed body with quadrupole deformation there are five degrees of freedom $(\beta,\gamma,\theta_1,\theta_2,\theta_3)$, while for a diatomic molecule there are only three (r,θ,φ).

The use of p bosons alone to obtain solutions corresponding to this more difficult case appears to be rather complex. However, both solutions, spherical and deformed, could be simply obtained by introducing an s boson with $J^P=0^+$ in addition to p (with $J^P=1^-$). The corresponding space is four dimensional with group structure U(4), the *vibron model.* The Hamiltonian for this model is constructed from the operators of U(4) much in the same way as Eq. (2.7),

$$H=\varepsilon_s s^\dagger s+\varepsilon_p(p^\dagger\cdot\tilde{p})+\tfrac{1}{2}\sum_{L=0,2}c_L[(p^\dagger\times p^\dagger)^{(L)}\cdot(\tilde{p}\times\tilde{p})^{(L)}]+\tfrac{1}{2}\tilde{v}_0[(p^\dagger\times p^\dagger)_0^{(0)}s^2+(s^\dagger)^2(\tilde{p}\times\tilde{p})_0^{(0)}]+\tfrac{1}{2}u_0(s^\dagger)^2s^2$$
$$+\frac{u_1}{\sqrt{3}}s^\dagger s(p^\dagger\cdot\tilde{p})\ . \tag{2.17}$$

In the case of diatomic molecules, p bosons as well as s bosons do not have any microscopic underlying picture. The number N simply determines the number of states in the irreducible representation that approximately correspond to the bound eigenstates of the Schrödinger equation. This boson model gives a very good description of low-lying levels of actual diatomic molecules. It is an ideal example of a successful boson model of a collective Hamiltonian. The collective degrees of freedom in the present case are the components of a dipole (given by the vector $\mathbf{r}$). The collective electromagnetic transitions are thus of electric dipole type E1, and the corresponding real and Hermitian operator is

$$T_\mu^{(1)}=\alpha_1(p_\mu^\dagger s+s^\dagger\tilde{p}_\mu)\ . \tag{2.18}$$

As in the case of the interacting boson model, for special values of the parameters in Eq. (2.17), the eigenvalues of the Hamiltonian can be written down in a closed formula. Such is the case if the Hamiltonian is constructed from the generators of O(4), which form a Lie subalgebra of the U(4) one. The generators of O(4) are given by the operator in Eq. (2.18) and the angular momentum vector $\sqrt{6}(p^\dagger\times\tilde{p})^{(1)}$. The most general O(4) Hamiltonian can be expressed by a linear combination of the scalar products of these two operators, or in analogy with Eq. (2.12) by

$$A\tfrac{1}{4}[p^\dagger\cdot p^\dagger-(s^\dagger)^2](\tilde{p}\cdot\tilde{p}-s^2)+B6[(p^\dagger\times\tilde{p})^{(1)}\cdot(p^\dagger\times\tilde{p})^{(1)}]\ . \tag{2.19}$$

Hence the eigenvalues of (2.19) are given by

$$\tfrac{1}{4}A[N(N+2)-\omega(\omega+2)]+BL(L+1)\ , \tag{2.20}$$

where the seniority ω is equal to $\omega=N,\ N-2,\ \ldots,\ 1$, or 0. For any given value of ω, there is according to (2.20) a rotational band with $L=\omega,\ \omega-1,\ \omega-2,\ \ldots,\ 0$. The quantum number ω can be related to the vibrational quantum number v by $v=(N-\omega)/2$. Another simple limit is obtained when one considers only operators that commute with $\hat{n}_p=(p^\dagger\cdot\tilde{p})$. The corresponding algebra is that of $U(3)\otimes U(1)$. Eliminating the operator $\hat{n}_s=s^\dagger s$ yields simply U(3). The U(3) Hamiltonian can be written in analogy with Eq. (2.10) as

$$\varepsilon\hat{n}_p+\tfrac{1}{2}\alpha\hat{n}_p(\hat{n}_p-1)+\gamma 6[(p^\dagger\times\tilde{p})^{(1)}\cdot(p^\dagger\times\tilde{p})^{(1)}]\ , \tag{2.21}$$

with eigenvalues

$$\varepsilon n_p+\tfrac{1}{2}\alpha n_p(n_p-1)+\gamma L(L+1)\ . \tag{2.22}$$

The spectrum given by (2.22) is that of a three-dimensional anharmonic oscillator.

III. THE INTERACTING BOSON MODEL 2

A. The model

Although the collective model and the interacting boson model 1 successfully describe many situations in nuclei, it seems desirable to go beyond them and attempt a microscopic calculation of collective properties of nuclei. There are, in principle, two ways to compute microscopically collective properties of nuclei. One is to start from the spherical shell model, which, in addition to being successful, provides a complete basis for any microscopic calculation of nuclear properties, and the other is to start from independent nucleons moving in a deformed potential well. For the collective model, the avenue starting from the deformed potential well has been investigated in detail by Kumar (1983). The avenue starting from the spherical shell model appeared to be the natural one for deriving the collective Hamiltonian within the framework of the interacting boson model. We shall henceforth call the spherical shell model simply "the shell model" and the model using a deformed potential well "the Nilsson model" (Nilsson, 1955). The avenue starting from the shell model seemed more appealing, since nuclei with a few valence nucleons outside of (or missing from) closed shells are well described by it, while a description of the same nuclei in terms of the Nilsson model encounters serious difficulties.

The shell-model description of collective states in nuclei has been a long-standing problem. For nuclei with many valence nucleons, the straightforward application of the shell model becomes prohibitively difficult. Shell-model wave functions constituting a complete set could have been combined to yield all kinds of collective states. This, however, would have little value as the complexity of the problem seemed enormous. No simple coupling scheme was found that could give an adequate description of the large variety of collective states found in nuclei. The best and most successful attempt has been the elegant SU(3) model of Elliott (1958). This model demonstrated for the first time how the motion of individual nucleons can lead to collective rotational spectra. It cannot, however, accommodate in a straightforward way the large spin-orbit interaction that leads to the shell model. Thus this model gives good results in some light nuclei, but it cannot be applied directly to the situation in many medium-mass and heavy nuclei.

The success of the boson model in describing in a unified manner the various kinds of collective spectra, vibrational, rotational, etc., offered a renewed challenge. A scheme within the shell model that would correspond to the set of s- and d-boson states would offer a successful shell-model description of collective states in nuclei. The basic elements of the boson model are s bosons and d bosons defined in a fixed frame of reference, and thus corresponding building blocks must be found in the shell model. In looking for these, as we shall presently see, the presence of s bosons and the conservation of N play a most important role.

The simplest building blocks with $J=0$ and $J=2$ are pairs of identical nucleons coupled to $J=0$ and $J=2$, respectively. Such pairs could be pairs of valence nucleons, but they could also be nucleon-hole pairs. The excitation of a nucleon from closed shells should lead to a state with high energy. Furthermore, to obtain a 0^+ pair the nucleon should be excited to the second higher shell. It was thus suggested that the basic building blocks of the interacting boson model are nucleon pairs (Arima *et al.*, 1977; Otsuka *et al.*, 1978). Another argument for considering nucleon pairs rather than nucleon-hole pairs is offered by the conservation of N. The total number of nucleon pairs is obviously conserved, whereas no simple conservation law would emerge from nucleon-hole pairs.

At this point we should recall that nuclei are built of protons and neutrons, a fact not explicitly recognized when IBM-1 or the collective model is used. We must specify which nucleon pairs we should use, proton-proton and neutron-neutron, or perhaps proton-neutron pairs. In most medium-mass and heavy nuclei where collective spectra appear, valence protons and valence neutrons are in different major shells. This prevents the coupling of a proton and neutron to $J=0$ and positive parity. We conclude that the natural approach is to consider pairs of valence protons and pairs of valence neutrons whose numbers are separately conserved as well as their sum $N=N_\pi+N_\nu$. The corresponding model is called the *interacting boson model 2,* or IBM-2, in order to emphasize that there are two types of collective degrees of freedom, proton bosons (π) and neutron bosons (ν) (Fig. 2).

Employing coupled pairs of protons and neutrons as building blocks for collective nuclear states does not imply that the proton-neutron interaction is ignored. As we shall see later, the strong and attractive interaction between protons and neutrons plays a dominant role in the present approach. It has been recognized in the past as giving rise to an attractive central potential as well as to nuclear deformation, and it will be introduced here as well

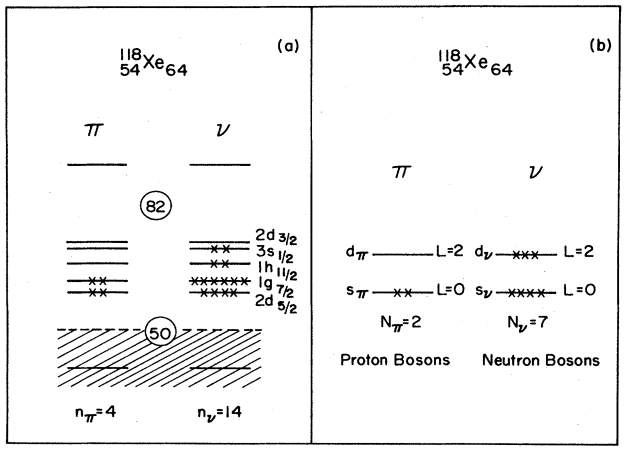

FIG. 2. (a) Schematic representation of the shell-model problem for $^{118}_{54}Xe_{64}$; (b) the boson problem, which replaces the shell-model problem for $^{118}_{54}Xe_{64}$. Both in part (a) and in part (b) the nucleons (a) or bosons (b) can be arranged in all possible ways consistent with the single-particle levels and Fermi (a) or Bose (b) statistics. Of all these possible ways only one is shown in the figure.

at the proper stage. When discussing the proton-neutron interaction, it should be stressed that any state with valence protons and neutrons can be constructed by coupling all possible proton states to all possible neutron states. If valence protons and neutrons are in different (major) shells, all such states will correspond to states with good isospin given by

$$T=(N_A-Z_A)/2\ , \tag{3.1}$$

where N_A and Z_A are the total number of neutrons and protons. In such situations, the proton-neutron interaction $(V^{T=1}+V^{T=0})/2$ may be quite different from proton-proton and neutron-neutron interactions, $V^{T=1}$.

Another point concerns the consideration of only valence nucleons. We aim at constructing from shell-model wave functions coherent combinations that will exhibit collective features. It is well known that even in simple shell-model configurations of a few valence nucleons polarization of the closed shells is evident. Still, if the polarizations by individual nucleons are small, the effect may be expressed by renormalization of the various single-nucleon operators. We take the same view for the boson model by introducing polarizations of the core through effective boson charges. More will be said later on this problem.

B. Nucleon pairs and seniority

Let us now turn our attention to the interacting boson model 2 and thus to pairs of identical nucleons. Pairs with $J=0$ have been used extensively in the past, beginning with the pioneering work of Racah (1943) for atomic electrons. This method was successfully applied to nuclei several years before the Bardeen, Cooper, and Schrieffer (BCS) theory of superconductivity. BCS theory has been extensively applied to nuclei, but there are three arguments against its use in the present context. First, it diagonalizes the rather artificial pairing interaction, which is a poor choice for the effective interaction between nucleons. Second, it does not conserve the particle numbers. This is no problem for electrons in metals, but it is a major problem for the case of a dozen or so valence nucleons. One way out of this difficulty is to use BCS theory projected onto the correct particle number. This theory, PBCS, is essentially equivalent to that discussed below. A third argument, particularly important here, is that the BCS theory treats $J=2$ pairs very differently from $J=0$ pairs. In trying to construct nucleon states similar to those of s and d bosons, a better approach is to treat them on the same footing. Such an approach is offered by the theory of generalized seniority (Talmi, 1971, 1973; Shlomo and Talmi, 1972). It is a generalization of Racah's seniority, which we shall now briefly outline.

The seniority scheme was introduced for identical fermions in a single j orbit (l orbit for electrons in atoms). It is based on considering pairs coupled to $J=0$ as created by the operator

$$S_j^\dagger=\tfrac{1}{2}\sum_m(-)^{j-m}a_{j,m}^\dagger a_{j,-m}^\dagger\ . \tag{3.2}$$

The operator that annihilates a pair is the Hermitian conjugate $S_j=(S_j^\dagger)^\dagger$. The operator $S_j^\dagger S_j$ measures in some way the amount of $J=0$ pairing. It has the eigenvalue $(2j+1)/2$ for the $(j^2J=0)$ state and 0 for all other states of the j^2 configuration. The seniority scheme in the j^n configuration is defined as the set of eigenstates of this operator.

An interesting property of the operators $S_j^\dagger, S_j$ and S_j^0,

$$S_j^0=\tfrac{1}{2}\sum_m a_{j,m}^\dagger a_{j,m}-\frac{2j+1}{4}\ , \tag{3.3}$$

is that they form a Lie algebra identical to that of the components of the angular momentum [SU(2) and O(3)], called *quasispin* (Kerman, 1961). In treating properties of the pairing interaction $S_j^\dagger S_j$, one can use either the quasispin labels $\mathscr{S},\mathscr{S}^0$ or the seniority v and particle number n. The particle number n is simply related to the expectation value $\mathscr{S}^0$ of S_j^0 by

$$\mathscr{S}^0=\frac{n}{2}-\frac{2j+1}{4}\ . \tag{3.4}$$

The seniority v is related to $\mathscr{S}$ by a similar expression,

$$\mathscr{S}=\frac{2j+1}{4}-\frac{v}{2}\ . \tag{3.5}$$

The seniority v of a state, is, loosely speaking, the number of unpaired nucleons. More precisely, a state in the j^v configuration with seniority v has no paired nucleons and satisfies

$$S_j\,|\,j^v vJM\rangle=0\ . \tag{3.6}$$

The eigenvalues of the pairing interaction $S_j^\dagger S_j$ are given by

$$\langle S_j^\dagger S_j \rangle = \frac{1}{2}\left[\frac{n-v}{2}\right](2j+3-n-v) . \tag{3.7}$$

The seniority scheme can be and has been used to derive many properties of shell-model states of the j^n configurations.

In most medium-mass and heavy nuclei, the valence nucleons are distributed over several j orbits in a major shell. In order to treat this situation, a generalization of the concept of seniority in a single j shell is called for. A simple generalization would be to consider the linear combinations $S^\dagger = \sum_j S_j^\dagger, S = \sum_j S_j$, and $S^0 = \sum_j S_j^0$. These operators would still form an SU(2) algebra, and one could use the elegant methods of group theory to analyze the situation (Kerman, 1961). This situation would be the case if, for example, the single-particle orbits j were all degenerate and the interaction were the pairing one. The actual situation in nuclei is rather different. One thus needs to consider a more complex form, called *generalized seniority* (Talmi, 1971), by using an operator that creates a correlated $J=0$ pair having the form

$$S^\dagger = \sum_j \alpha_j S_j^\dagger \tag{3.8}$$

with unequal amplitudes of α_j. This generalized operator has with its Hermitian conjugate rather complicated commutation relations, and no simple closed Lie algebra is obtained. Still, in analogy with $v=0$ states, states with $J=0$ of the type $(S^\dagger)^N|0\rangle$, where $N=n/2$ is the number of pairs, can be constructed and their properties investigated.

The concept of correlated pairs can be extended to include pairs with angular momentum J larger than 0. For example, we can define operators that create correlated pairs with $J=2$ by

$$D_\mu^\dagger = \sum_{jj'} \beta_{jj'} \frac{1}{\sqrt{1+\delta_{jj'}}} \sum_{mm'} (jmj'm'|jj'2\mu) a_{j,m}^\dagger a_{j',m'}^\dagger . \tag{3.9}$$

In analogy with $v=2$ states, states with one D pair and $(N-1)$ S pairs, $(S^\dagger)^{N-1}D^\dagger|0\rangle$ can be considered, and $G,I,\ldots,$ pairs can be constructed in a similar fashion.

The states with one D pair (or $G,I,\ldots$) can be assigned generalized seniority $v=2$. The situation becomes more complex if one attempts to create fermion states with more than one $D^\dagger$ operator, for example, $(D^\dagger \times D^\dagger)_M^{(J)}|0\rangle$. For $J=0$ this state will, in general, not be orthogonal to the state $(S^\dagger)^2|0\rangle$. In order to overcome this difficulty, one must orthogonalize the fermion states at each step when constructing states with many D pairs. In order to preserve as much as possible the advantages of the seniority scheme, one can project out of any state $(S^\dagger)^{N-v/2}(D^\dagger)_{JM}^{v/2}|0\rangle$ all components of the form $(S^\dagger)^{N-v/2+1}B_{JM}^\dagger|0\rangle$, where $B_{JM}^\dagger|0\rangle$ is a state with $v-2$ nucleons.

The generalized seniority scheme constructed in this way has two virtues. First, nuclei with only valence neutrons *or* protons (semimagic nuclei) appear to be well described by it. It was shown that within this scheme the states $(S^\dagger)^N|0\rangle$ and $(S^\dagger)^{N-1}D^\dagger|0\rangle$ have simple properties. Ground-state energies (binding energies) are linear and quadratic functions of N with no breaks corresponding to subshells. The spacings between ground state and the first excited $J=2$ states are independent of N. They are constant throughout the major shell considered. These features arise if the coefficients α_j and $\beta_{jj'}$ in Eqs. (3.8) and (3.9) are constant throughout the major shell. In some calculations (broken-pair approximations) wave functions like Eqs. (3.8) and (3.9) are used as *variational functions*, with coefficients α_j and $\beta_{jj'}$ determined for each N from a given Hamiltonian. If, for a given shell-model Hamiltonian, these coefficients are changing with N, the regular features of semimagic nuclei may not be reproduced. It is thus important to check that the results of the calculations produce coefficients α_j and $\beta_{jj'}$, which do not change appreciably with N. Finally, in the generalized seniority scheme, only one state, $(S^\dagger)^{N-1}D^\dagger|0\rangle$, has nonvanishing matrix elements of the quadrupole operator to the ground state $(S^\dagger)^N|0\rangle$. This last property holds for a single-nucleon quadrupole operator Q, which satisfies the condition (Shlomo and Talmi, 1972; Ginocchio and Talmi, 1980)

$$D_\mu^\dagger = \tfrac{1}{2}[Q_\mu, S^\dagger] . \tag{3.10}$$

An example of nuclei well described by the generalized seniority scheme are the tin isotopes. We show in Fig. 3 the location of the $J=2$ states, which can be ascribed to generalized seniority $v=2$, above the ground state. Despite the large number of valence neutrons (or neutron holes), up to 16, there is no indication of a real lowering of these $J=2$ states. The situation is similar for lead isotopes (below and above the magic neutron number of 126) and for the isotones with a neutron number of 82 (below and above the quasimagic proton number of 64). The small variations in 0—2 spacings in semimagic nuclei should be contrasted with the dramatic changes in 0—2 spacings when nucleons of the other kind are added, or with the changes of level spacing in odd nuclei. Thus one

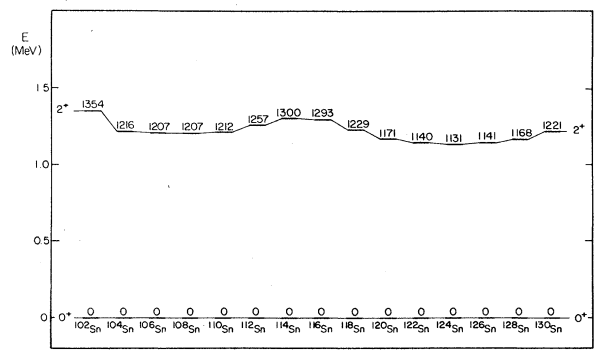

FIG. 3. Systematics of the $J^P=2^+$ levels in Sn isotopes.

must conclude that the basic effective interaction between identical nucleons is such that generalized seniority holds. One can use this empirical evidence in two ways, either by deriving the necessary and sufficient conditions that the fermion Hamiltonian H must meet in order that generalized seniority be satisfied (Talmi, 1971, 1973; Shlomo and Talmi, 1972), or by disregarding effective interactions that violate it. For example, the very popular pairing plus quadrupole interaction is unrealistic for *identical* particles.

The quadrupole portion of it violates generalized seniority, and it will lead to a lowering of the $J=0-J=2$ separation not observed experimentally in semimagic nuclei. Only small strengths of an explicit quadrupole interaction can be tolerated by experiment, as will be discussed below. It is also perhaps worth noting that in the case of the isotopes with a neutron number of 82, the (0—2) level spacings slightly increase with the number of valence protons. This would imply that, in addition to interactions that are diagonal in generalized seniority, there is in this case a small quadrupole term that is repulsive.

The second virtue of the generalized seniority scheme is that it leads naturally to a connection between fermion states built from $S^\dagger$ and $D^\dagger$ pair operators and states of s and d bosons. One may construct a one-to-one correspondence (mapping) in the following way:

Fermion space	Boson space
$(S^\dagger)^N\|0\rangle \quad J=0$	$(s^\dagger)^N\|0\rangle$
$(S^\dagger)^{N-1}D^\dagger\|0\rangle \quad J=2$	$(s^\dagger)^{N-1}d^\dagger\|0\rangle$
$\mathscr{P}(S^\dagger)^{N-2}(D^\dagger\times D^\dagger)^{(J)}\|0\rangle \quad J=4,2,0$	$(s^\dagger)^{N-2}(d^\dagger\times d^\dagger)^{(J)}\|0\rangle$
. . .	. . .

(3.11)

where the operator $\mathscr{P}$ denotes a projection into states from which components with $S^\dagger$ pairs have been projected out. The correspondence is such that the boson number N is half the particle number, while the number of d bosons is half the generalized seniority,

$$N=\frac{n}{2}, \quad n_d=\frac{v}{2} . \tag{3.12}$$

Equation (3.12) is the basic underlying microscopic structure of the interacting boson model.

In addition to constructing a correspondence (or mapping) of states, one can also construct boson Hamiltonians whose eigenstates are given by Eq. (3.11). As a simple example, one may consider a boson Hamiltonian that reproduces the features of generalized seniority. The boson Hamiltonian

$$V_0(s^\dagger s)+V_2(d^\dagger\cdot\tilde{d})+\tfrac{1}{2}W(s^\dagger)^2s^2+W(s^\dagger s)(d^\dagger\cdot\tilde{d}) \tag{3.13}$$

has eigenvalues $V_0N+\frac{1}{2}WN(N-1)$ in the states $(s^\dagger)^N|0\rangle$. It has the eigenvalues $V_0(N-1)+V_2+\frac{1}{2}WN(N-1)$ in the states $(s^\dagger)^{N-1}d^\dagger|0\rangle$, as can be verified by using the Bose commutation relations. Any state (3.11) with given n_s and n_d is an eigenstate of (3.13) with the eigenvalue

$$V_0n_s+V_2n_d+\tfrac{1}{2}Wn_s(n_s-1)+Wn_sn_d . \tag{3.14}$$

One should remark that in this example the boson Hamiltonian (3.13) is not the first term in a "boson expansion" of the shell-model Hamiltonian. The boson operators are not approximate creation operators of fermion pairs. Unlike $S^\dagger$ and $D^\dagger$, which have very complicated commutation relations with their Hermitian conjugates, $s^\dagger,d^\dagger_\mu,s,d_\mu$ satisfy exactly the boson commutation relations. The last two terms in (3.13) do not represent four fermion interactions that are not present in the shell-model Hamiltonian. The boson Hamiltonian (3.13) is just a model that reproduces the eigenvalues of the shell-model Hamiltonian for a limited and very special set of fermion states. It contains only single-boson terms and boson-boson interactions, and yet it reproduces *exactly* eigenvalues of nucleon states in which the Pauli principle is strictly obeyed.

The question can now be raised whether the mapping (3.11) includes shell-model states that correspond to every boson state. This is certainly not the case if $N>\Omega$, where $\Omega=\sum_j(j+\frac{1}{2})$ is the pair degeneracy of the shell. All fermion states of $N>\Omega$ vanish, whereas there are boson states for any N. Hence beyond $N=\Omega$ the correspondence between fermion pair states and boson states must simply be discontinued. As we shall see in the following, the correspondence should be modified even for $N>\Omega/2$. Even for smaller values of N there are cases in which the Pauli principle forbids certain couplings of the $D^\dagger$ operators. The answer to this question depends on the shell-model space considered. If the shell-model space is severely limited, the correspondence may not be possible. For realistic shell-model spaces such as those encountered in the shells 50—82, 82—126, . . . , the correspondence can be established, at least for the low-lying states. Ginocchio (1980) has shown that there are several semirealistic cases in which the correspondence can not only be established, but is exact. The model of Ginocchio makes use again of the elegant methods of group theory, and it is the generalization of the quasispin group SU(2) to cases that include D pairs. The operators defined in the space $k+\frac{3}{2}\geq j\geq k-\frac{3}{2}$ (integer k),

$$S^\dagger=\tfrac{1}{2}\sum_{jm}(-)^{j-m}a^\dagger_{j,m}a^\dagger_{j,-m},\quad (S^\dagger)^\dagger\ ,$$

$$D^\dagger_\mu=\sum_{jj'}(-)^{k+3/2+j}[(2j+1)(2j'+1)]^{1/2}\begin{Bmatrix} j & j' & 2\\ \frac{3}{2} & \frac{3}{2} & k\end{Bmatrix}(a^\dagger_j\times a^\dagger_{j'})^{(2)}_\mu,\quad (D^\dagger_\mu)^\dagger\ , \tag{3.15}$$

$$P^{(r)}_\mu=2\sum_{jj'}(-)^{r+k+3/2+j}[(2j+1)(2j'+1)]^{1/2}\begin{Bmatrix} j & j' & r\\ \frac{3}{2} & \frac{3}{2} & k\end{Bmatrix}(a^\dagger_j\times\tilde{a}_{j'})^{(r)}_\mu\ ,$$

form the closed Lie algebra of SO(8). In Ginocchio's model there is a one-to-one correspondence between boson and fermion states, and one can construct boson Hamiltonians that give *exactly* the same eigenvalues as the corresponding fermion Hamiltonians. It thus provides an example of a fermion system where the truncation to the S-D space is exact. The model has been generalized recently by Wu *et al.* (1986).

Another problem associated with the mapping (3.11) is that the fermion seniority v cannot exceed Ω. From this it follows that the set of states constructed by $N>\Omega/2$ s and d bosons cannot correspond to a set of states of N, S, and D pairs. The bosons are not limited by the Pauli principle, and their states also include states with $n_d>\Omega/2$ d bosons. There is, however, a natural way to establish the correspondence between boson states and fermion states beyond the middle of the shell. One can make use of the general property of Fermi systems that a configuration with $2N$ particles can be equally well described as a system of $2(\Omega-N)$ holes in the closed shell. Hence, *beyond the middle of the shell, we map onto boson states fermion-hole pair states rather than fermion pair states.*

The operator $S^\dagger$ that creates a correlated fermion pair with $J=0$ has been defined by Eq. (3.8). The operator that creates a hole pair (with $J=0$) in the closed shell S' should make $S'(S^\dagger)^\Omega\,|\,0\rangle$ proportional to $(S^\dagger)^{\Omega-1}\,|\,0\rangle$. It can be expressed by using the Hermitian conjugates of the $S^\dagger_j$ as $\sum_j\alpha'_jS_j$. The condition written above then implies that $\alpha'_j=1/\alpha_j$. We conclude that the pair annihilation operator, apart from normalization, is given by (Talmi, 1982)

$$S'=\sum_j\frac{1}{\alpha_j}S_j\ . \tag{3.16}$$

It is seen that for a single j orbit or for equal α_j, S' is simply the Hermitian conjugate of $S^\dagger$. For unequal α_j values, where j orbits with *larger* α_j are filled in the beginning of the shell, it follows from Eq. (3.16) that the j holes at the end of the shell are indeed those with *smaller* α_j (larger $1/\alpha_j$). A similar procedure can be used to construct the D-pair annihilation operator,

$$D'_\mu=\sum_{jj'}\beta'_{jj'}\frac{1}{\sqrt{1+\delta_{jj'}}}\sum_{mm'}(jmj'm'\,|\,jj'2\mu)a_{j',m'}a_{j,m}\ . \tag{3.17}$$

It can be constructed by (Talmi, 1982)

$$D'_\mu=\tfrac{1}{2}[Q_\mu,S']\ , \tag{3.18}$$

where Q_μ is the operator defined by Eq. (3.10). Using the relation between the coefficients α_j, $\beta_{jj'}$, and those of Q_μ, the $\beta_{jj'}$ can be written explicitly as (Johnson and Vincent, 1985)

$$\beta'_{jj'}=\beta_{jj'}/\alpha_j\alpha_{j'}\ . \tag{3.19}$$

C. The proton-neutron interaction and nuclear deformation

We have seen that the interaction between identical nucleons gives rise to the generalized seniority scheme as experimentally observed in semimagic nuclei. Among the main properties of the seniority scheme are constant spacings of the 0^+ and 2^+ states and a relatively small (a few single-particle units) B(E2) value between the 0^+ and 2^+ states. This situation is drastically changed when both valence protons and neutrons are present. In these nuclei the 0—2 spacing sharply decreases with increasing number of valence nucleons, and there is a corresponding increase in the B(E2) value for the 2—0 transition. This effect, related in the collective model to the occurrence of deformation, is thus a consequence of the proton-neutron interaction.

The interaction between a proton in the j orbit and a neutron in the j' orbit can be expressed as

$$\langle jj'JM\,|\,V_{\pi\nu}\,|\,jj'JM\rangle=\left\langle jj'JM\left|\sum_k F_k(U^{(k)}_\pi\cdot U^{(k)}_\nu)\right|jj'JM\right\rangle$$

$$=\sum_k F_k(-)^{j+j'+J}\begin{Bmatrix} j & j' & J\\ j' & j & k\end{Bmatrix}\ . \tag{3.20}$$

In Eq. (3.20) the irreducible tensor operators $U^{(k)}$ are defined as having reduced matrix elements equal to unity. The coefficients F_k of the tensor expansion can be obtained from experiments, if available, by inverting Eq. (3.20), and are given by

$$F_k=(2k+1)\sum_J(-)^{j+j'+J}(2J+1)\begin{Bmatrix} j & j' & J\\ j' & j & k\end{Bmatrix}V(jj'J)\ . \tag{3.21}$$

If the protons and neutrons are in the same orbit, an interaction [Eq. (3.20)] with $j=j'$ and coefficients F_k,

determined by Eq. (3.21) from actual energies, is the correct charge-independent interaction. It reproduces exactly the $T=1$ levels with even J values and the $T=0$ levels with odd J. If the j protons interact with j' neutrons and the neutron j' orbit is completely filled, then matrix elements are given by

$$V(j_\pi j'_\nu J)=\tfrac{1}{2}[V(jj'J,T=0)+V(jj'J,T=1)] . \tag{3.22}$$

In such cases, moreover, all states obtained by coupling j and j' are allowed, the tensor expansion is unique, and Eqs. (3.20) and (3.21) are valid. All states constructed with j protons and j' neutrons correspond to states with a well-defined isospin, which is equal to $(N_A-Z_A)/2$.

Information on the proton-neutron interaction is available from experimental data in some light and medium-mass nuclei, for example, from $1f_{7/2}$ configurations. This information indicates that among the various F_k values in Eq. (3.21), the coefficient of the quadrupole term F_2 is considerably larger (more attractive) than others (see, for example, Talmi, 1983). A drastic simplification of the proton-neutron interaction is to consider only a quadrupole-quadrupole interaction,

$$V_{\pi\nu}=F_2(U_\pi^{(2)}\cdot U_\nu^{(2)}) . \tag{3.23}$$

Analysis of the experimental data also indicates that there is a strong and attractive monopole ($k=0$) term. If one is interested only in the calculation of level spacings, this proton-neutron monopole term may not be important. However, it plays a major role in the evaluation of binding energies. After the quadrupole ($k=2$) and monopole ($k=0$) terms, the next important one is the hexadecapole ($k=4$) term, etc. The fact that the main proton-neutron interaction is of a quadrupole-quadrupole type is in contrast with the situation for the proton-proton and neutron-neutron interaction, which, as discussed in Sec. III.B, is of the seniority-conserving type. It may be surprising that there is apparently no quadrupole term in the interaction of identical nucleons. The reason is simply that the expansion [Eq. (3.20)] is not unique if only $T=1$ states are considered. A quadrupole term may well be present in the proton-proton and neutron-neutron interactions, but its seniority-breaking effect is canceled by the presence of other even-rank multipole terms. It is then possible to expand the $T=1$ interaction in terms only of odd tensors and a monopole term (deShalit and Talmi, 1963).

The approximation (3.23) suggests a possible truncation scheme for large-scale shell-model calculations that leads to the interacting boson model 2. In this scheme, one first constructs a complete set of states of the valence protons given by $|\alpha_\pi J_\pi M_\pi\rangle$, where α_π are additional quantum numbers (labels) of states that have the same values of J_π (and M_π). The analogous set of neutron states is given by $|\alpha_\nu J_\nu M_\nu\rangle$. A complete set of states of given numbers of valence protons and neutrons is obtained by coupling the proton and neutron states as

$$|\alpha_\pi J_\pi \alpha_\nu J_\nu JM\rangle=\sum_{M_\pi M_\nu}(J_\pi M_\pi J_\nu M_\nu|J_\pi J_\nu JM)\times|\alpha_\pi J_\pi M_\pi\rangle|\alpha_\nu J_\nu M_\nu\rangle . \tag{3.24}$$

In this set of states one can then evaluate matrix elements of the proton-neutron interaction. For example, for a quadrupole-quadrupole proton-neutron interaction [Eq. (3.23)], one has

$$\langle\alpha_\pi J_\pi\alpha_\nu J_\nu JM|(U_\pi^{(2)}\cdot U_\nu^{(2)})|\alpha'_\pi J'_\pi\alpha'_\nu J'_\nu JM\rangle=(-1)^{J_\nu+J'_\pi+J}\begin{Bmatrix}J_\pi & J_\nu & J\\ J'_\nu & J'_\pi & 2\end{Bmatrix}(\alpha_\pi J_\pi||U_\pi^{(2)}||\alpha'_\pi J'_\pi)(\alpha_\nu J_\nu||U_\nu^{(2)}||\alpha'_\nu J'_\nu) . \tag{3.25}$$

An effective truncation scheme should maximize the matrix elements (3.25). For that, a convenient set of states for the valence protons and valence neutrons are those discussed in Sec. III.B and provided by generalized seniority,

$$(S_\pi^\dagger)^{N_\pi}|0\rangle,\ (S_\nu^\dagger)^{N_\nu}|0\rangle,\ (S_\pi^\dagger)^{N_\pi-1}D_\pi^\dagger|0\rangle,\ (S_\nu^\dagger)^{N_\nu-1}D_\nu^\dagger|0\rangle,\ \ldots . \tag{3.26}$$

If the $U_\pi^{(2)}$ and $U_\nu^{(2)}$ in Eq. (3.23) are proportional to the corresponding quadrupole operators defined in Eq. (3.10), the states (3.26) will have large matrix elements of the proton-neutron interaction.

In the truncation scheme outlined above basis states are constructed with operators of the form

$$\begin{aligned}
S_\pi^\dagger&=\sum_{j_\pi}\alpha_{j_\pi}(-)^{j_\pi-m_\pi}a^\dagger_{j_\pi,m_\pi}a^\dagger_{j_\pi,-m_\pi},\quad S_\nu^\dagger=\sum_{j_\nu}\alpha_{j_\nu}(-)^{j_\nu-m_\nu}a^\dagger_{j_\nu,m_\nu}a^\dagger_{j_\nu,-m_\nu} ,\\
D^\dagger_{\pi,\mu}&=\sum_{j_\pi j'_\pi}\beta_{j_\pi j'_\pi}\frac{1}{\sqrt{1+\delta_{j_\pi j'_\pi}}}\sum_{m_\pi m'_\pi}(j_\pi m_\pi j'_\pi m'_\pi|j_\pi j'_\pi 2\mu)a^\dagger_{j_\pi,m_\pi}a^\dagger_{j'_\pi,m'_\pi} ,\\
D^\dagger_{\nu,\mu}&=\sum_{j_\nu j'_\nu}\beta_{j_\nu j'_\nu}\frac{1}{\sqrt{1+\delta_{j_\nu j'_\nu}}}\sum_{m_\nu m'_\nu}(j_\nu m_\nu j'_\nu m'_\nu|j_\nu j'_\nu 2\mu)a^\dagger_{j_\nu,m_\nu}a^\dagger_{j'_\nu,m'_\nu} .
\end{aligned} \tag{3.27}$$

They are given by the states

$$\{[(S_\pi^\dagger)^{n_{s_\pi}}(D_\pi^\dagger)^{n_{d_\pi}}_{\gamma_\pi J_\pi}]\times[(S_\nu^\dagger)^{n_{s_\nu}}(D_\nu^\dagger)^{n_{d_\nu}}_{\gamma_\nu J_\nu}]\}^{(J)}_M|0\rangle\ , \tag{3.28}$$

for which components of lower proton and neutron seniorities have been projected out. The numbers of proton and neutron pairs in the set (3.28), called the S-D space, obey the relations

$$n_{s_\pi}+n_{d_\pi}=N_\pi\equiv\tfrac{1}{2}n_\pi,\quad n_{s_\nu}+n_{d_\nu}=N_\nu\equiv\tfrac{1}{2}n_\nu\ . \tag{3.29}$$

This truncation is a tremendous simplification of the original shell-model problem. For example, in $^{154}_{62}Sm_{92}$, the 12 valence protons occupy the orbits in the 50-82 shell and the 10 valence neutrons the orbits in the 82-126 shell. The number of states with positive parity and $J=0$ is 41, 654, 193, 516, and 917; that of states with $J=2$ is 346, 132, 052, 934, and 889; and that of states $J=4$ is 530, 897, 397, 260, and 575. A full shell-model calculation for these nuclei is beyond the reach of today's computers. On the other hand, the number of $J=0$ states in the truncated space is about 30, producing a difficult but manageable problem.

Replacing the giant matrices of the proton-neutron problem by those constructed in the S-D space results in a tremendous reduction in size. Still, rather formidable complications face any actual shell-model calculation. The fermion states of the S-D space are still very complicated and difficult to work with due to the Pauli principle. One thus may attempt to replace the shell-model problem in the truncated space by a boson problem. A mapping of the fermi states of the S-D space onto boson states can be simply achieved using the results of Eq. (3.11). One introduces proton and neutron s-d bosons, $s_\pi d_\pi, s_\nu, d_\nu$ and constructs boson states as

$$\{[(s_\pi^\dagger)^{n_{s_\pi}}(d_\pi^\dagger)^{n_{d_\pi}}_{\gamma_\pi J_\pi}]\times[(s_\nu^\dagger)^{n_{s_\nu}}(d_\nu^\dagger)^{n_{d_\nu}}_{\gamma_\nu J_\nu}]\}^{(J)}_M|0\rangle\ . \tag{3.30}$$

The corresponding boson model is the interacting boson model 2 (IBM-2). The entire procedure (truncation and mapping) is schematically illustrated in Fig. 4.

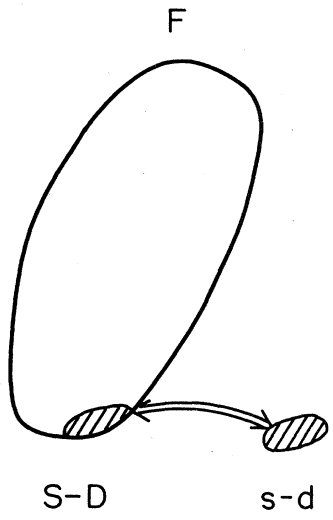

FIG. 4. A representation of the procedure suggested for constructing the boson Hamiltonian and transition operators. F is the full shell-model valence space. The dimensions of the S-D (and s-d) spaces are not in scale. The actual scaled space would be a tiny dot in F.

In addition to the truncation to the S-D space and mapping onto boson states, it is necessary to find a boson Hamiltonian that will have the same eigenvalues as the fermion Hamiltonian. We saw in Sec. III.B examples of boson Hamiltonians corresponding to Hamiltonians of identical nucleons. In the present case it is also necessary to find *proton-neutron boson* interactions that correspond to proton-neutron nucleon interactions. For an interaction of the type (3.23), it is sufficient to find boson quadrupole operators that will have between the states (3.30) the same matrix elements as the nucleon quadrupole operators between fermion states in the S-D space. For more complex interactions, one may need other boson operators.

IV. MICROSCOPIC CALCULATIONS

A. Calculations in a spherical basis

In previous sections we have outlined the connection between the shell model and the interacting boson model. There are several steps in going from actual shell-model calculations in a very large space to the boson system. The first step is the drastic truncation of the shell-model space into the S-D space. The second step is establishing a mapping between steps of the S-D space and states of $s_\pi, d_\pi, s_\nu, d_\nu$ boson space. Finally, a boson Hamiltonian should be constructed that will have the same eigenvalues as the shell-model Hamiltonian for corresponding fermion states. The same procedure must be repeated for all operators of interest, for example, for the quadrupole operator to be used for calculation of rates of E2 electromagnetic transitions. As mentioned above, there are simple cases in which this procedure is exact and can be carried out explicitly. In more realistic cases, approximate procedures must be devised. In order to check the usefulness of each approximation scheme, it is necessary to compare its results with those of exact calculations in cases where exact calculations can be done (i.e., for small numbers of valence nucleons). The first question that could be asked is how good is the truncation to the S-D space. Two remarks are in order here. First, the answer to this question depends on the effective interaction that is chosen. This poses a problem, since this interaction is not sufficiently well known. There is up to now no reliable way to obtain it from the interaction of free nucleons by using methods of many-body theory. In cases in

which good agreement was obtained between shell-model calculations and experimental data, the effective interactions were in fact determined from experiment. The self-consistent determination of the two-body matrix elements is possible only for simple configurations in which the number of matrix elements is small. In the cases considered here, hundreds of such diagonal and nondiagonal matrix elements are needed, and there is not much hope of obtaining all of them from experimental data. One may choose a particular interaction (for example, pairing plus quadrupole), but there is no guarantee that this interaction represents the actual situation. Another point that should be mentioned is that, even if the truncation to the S-D space leads to wave functions that do not have large overlaps with the full space, this truncation can still, in some cases, be made meaningful by renormalizing the operators. This is the situation that actually occurs when going from the free-nucleon interaction to the shell model. The interaction between free nucleons is strong and short ranged and leads to strong, short-range correlations between nucleons. The absence of these correlations in shell-model wave functions implies rather massive effects of renormalization. Nuclear many-body theory has been grappling with this problem for more than 30 years and is still far from the solution. No reliable way has been found so far to calculate matrix elements of the effective interaction. Moreover, no proof has been given for the validity of the shell model itself. No criteria have been formulated for nuclear interactions that give rise to the shell model as a good approximation. Yet the shell model seems to give excellent descriptions of nuclei, at least near closed shells. Despite these two drawbacks, the calculations carried out so far are of great importance to our understanding of the shell-model structure of the boson model. Their results provide insight and guidance in looking for interactions that yield better agreement with the model.

Microscopic calculations require, in practice, three steps:

(1) the determination of the values α_j and $\beta_{jj'}$ which appear in the $S^\dagger$ and $D^\dagger$ operators of protons and neutrons, Eq. (3.27);

(2) the construction of the boson Hamiltonian (and other operators) starting from the fermion space;

(3) the diagonalization of the boson Hamiltonian and evaluation of transition matrix elements.

The last step is obvious and we shall not discuss it here.

We consider first step one. Several approaches have been used to evaluate the coefficients α_j and $\beta_{jj'}$. The simplest is to solve the shell-model problem in the space of two particles (or two holes). More elaborate calculations have been performed, especially in order to test whether the coefficients α_j and $\beta_{jj'}$ depend on particle number. An example is shown in Fig. 5 (Bonsignori, Allaart, and van Egmond, 1983). It appears that most calculations produce coefficients that are, to a good approximation, constant within a major shell in accordance with the results of generalized seniority.

Step two is more difficult to perform. A simple

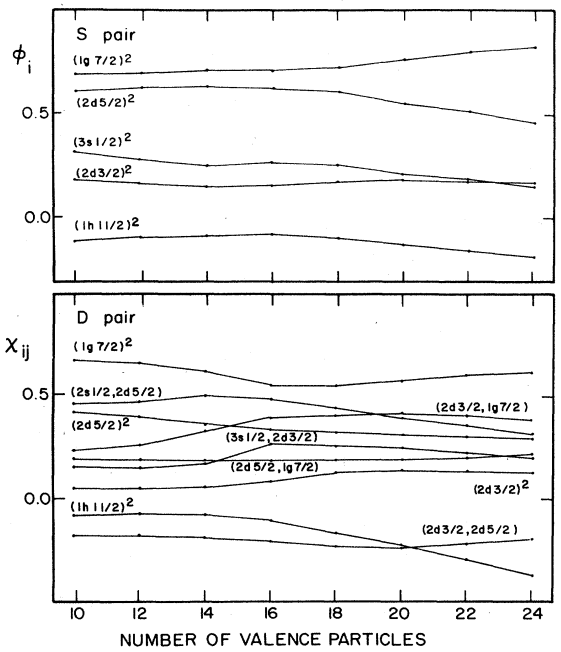

FIG. 5. Amplitudes of shell-model states in the S- and D-pair states as obtained by Bonsignori, Allaart, and van Egmond (1983). The amplitudes φ_i and χ_{ij} are such that their squares add up to one. The amplitudes φ_i and χ_{ij} are defined by

$$S^\dagger=\sum_i \varphi_i [a_j^\dagger \times a_j^\dagger]^{(0)}$$

and

$$D^\dagger=\sum_{ij} \chi_{ij} [a_i^\dagger \times a_j^\dagger]^{(2)} ,$$

i.e., in a slightly different way than Eqs. (3.8) and (3.9).

scheme is that of equating matrix elements of the appropriate operators in the fermion space with those in the boson space for some states (Otsuka, Arima, and Iachello, 1978). In particular, in order to construct a boson Hamiltonian, one needs to equate matrix elements for the two- and four-particle systems (one and two bosons). This procedure is called OAI mapping.

In order to illustrate this procedure, we consider the case of the quadrupole operator Q_μ. A single-boson operator can be written as

$$Q_\mu=\alpha_2(d_\mu^\dagger s+s^\dagger \tilde{d}_\mu)+\beta_2(d^\dagger \times \tilde{d})_\mu^{(2)} . \tag{4.1}$$

In the fermion space, the first term corresponds to the seniority-changing part of the quadrupole operator, while the second term corresponds to the seniority-conserving part. The coefficients α_2 and β_2 can thus be obtained by evaluating matrix elements of the fermion quadrupole operators $U^{(2)}$ of the type $\langle D||U^{(2)}||S\rangle$, $\langle D||U^{(2)}||D\rangle$. In the case of a single j shell this leads to the result

$$\alpha_2=\left[\frac{\Omega-N}{\Omega-1}\right]^{1/2}\left[\frac{1}{5}\right]^{1/2}\langle j^2(D)2||U^{(2)}||j^2(S)0\rangle\ ,$$
$$\beta_2=\left[\frac{\Omega-2N}{\Omega-2}\right]^{1/2}\left[\frac{1}{5}\right]^{1/2}\langle j^2(D)2||U^{(2)}||j^2(D)2\rangle\ . \tag{4.2}$$

An appropriate choice of the coefficients α_2 and β_2 given by Eqs. (4.2) will reproduce the fermion results for all values of N between states with $n_d=0$ and $n_d=1$. What can be said about matrix elements between states with higher values of n_d and n_d+1? This question has been investigated extensively. Again it is instructive to consider the case of a single j shell for which exact results can be derived. One obtains

$$\alpha_2=\left[\frac{\Omega+1-N-n_d}{\Omega+1-2n_d}\right]^{1/2}\left[\frac{1}{5}\right]^{1/2}\langle j^2(D)2||U^{(2)}||j^2(S)0\rangle,$$
$$\beta_2=\left[\frac{\Omega-2N}{\Omega-2n_d}\right]\left[\frac{1}{5}\right]^{1/2}\langle j^2(D)2||U^{(2)}||j^2(D)2\rangle\ . \tag{4.3}$$

We thus obtain for the boson operator (4.1) coefficients that depend on boson numbers N and n_d. As long as n_d is small compared to Ω, the Pauli principle is not very effective, and matrix elements between fermion pairs are proportional to those between corresponding boson states. The simple OAI mapping, however, becomes worse as n_d increases for fixed N. More elaborate mappings are needed to discuss situations in which n_d is comparable to Ω. An example of these elaborate mappings is provided by the method of Yang Li-Ming (1983).

1. The single j shell

The truncation-mapping procedure can be carried out in analytic form in the schematic case of a single j shell for protons and a single j shell for neutrons. In this case, the generalized seniority becomes exactly the fermion seniority. Otsuka (1978) has calculated the spectra of all even-even nuclei in the major shell 50-82, treated as a single j shell with $j=\frac{31}{2}$ (Fig. 6). He used a δ-function interaction between identical nucleons (proton-proton and neutron-neutron).

$$V(\mathbf{x}-\mathbf{x}')=-V_0\delta(\mathbf{x}-\mathbf{x}')\ , \tag{4.4}$$

with strength V_0 adjusted to give a separation of 1.4 MeV between the ground state and the first 2^+ state in semimagic nuclei. In addition, he used a quadrupole interaction between protons and neutrons,

$$V'=-k(U_\pi^{(2)}\cdot U_\nu^{(2)})\ . \tag{4.5}$$

Comparing Otsuka's calculation (Fig. 6) with the experimental situation (Fig. 7), one can see that the qualitative features of the experiment are reproduced by his schematic calculation. The calculation appears to yield not only the ground-state band, but also other low-lying excited bands. A more detailed effective interaction is expected to give better quantitative agreement with the data, but

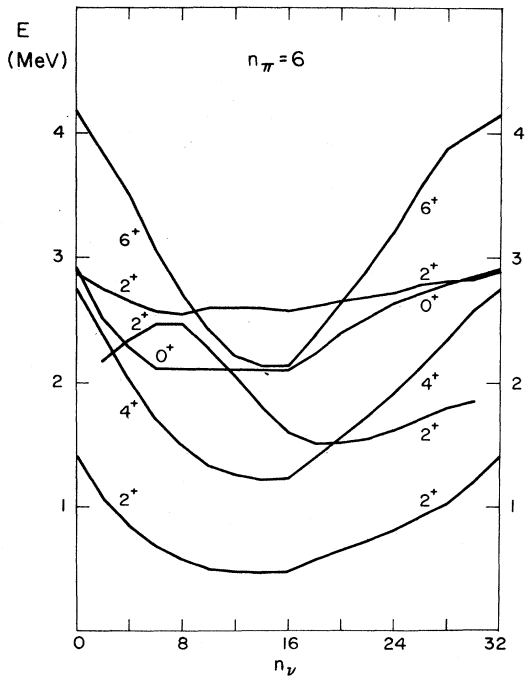

FIG. 6. Energy spectra of even-even nuclei for fixed proton number, $n_\pi=2N_\pi=6$, and varying neutron number, $0\le n_\nu\le 32$, in the single j-shell approximation.

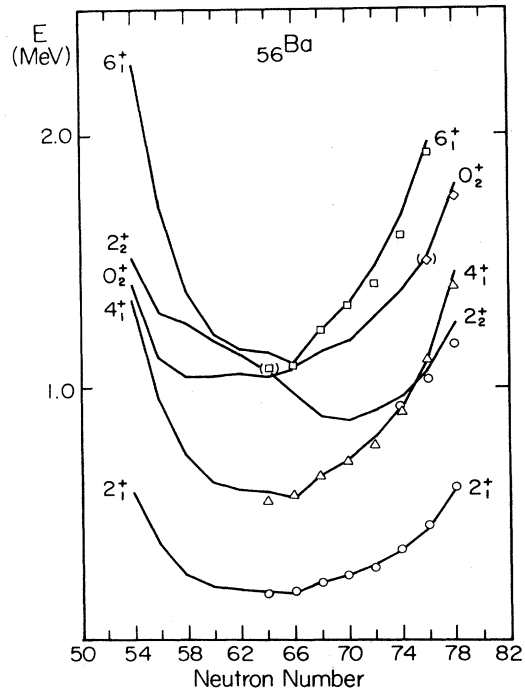

FIG. 7. Experimental energy spectra (circles, squares, and triangles) of Ba isotopes, $Z=56$ ($n_\pi=6$). The lines indicate phenomenological fits to the data (Puddu, Scholten, and Otsuka, 1980). The experimental spectra are shown here to point out the similarity with those calculated in the single j-shell approximation and shown in Fig. 6.

the gross features emerge even with the schematic interaction. These features seem to depend rather strongly on the existence of major closed shells and on the number of valence protons and neutrons, n_π and n_ν.

2. Several degenerate shells

Otsuka (1981a, 1981b) has extended his calculations to the case of degenerate j shells with the surface delta interaction (Arvieu and Moszkowski, 1966) between identical nucleons. In that case the coefficients α_j are equal [apart from a phase factor $(-)^l$] and the quasispin formalism can be applied directly. In order to obtain the coefficients $\beta_{jj'}$, Otsuka first constructed the $J=2$ pair operator according to Eq. (3.10),

$$F^\dagger_\mu=[S^\dagger,Q_{F,\mu}]\ , \tag{4.6}$$

where Q_F is the fermion quadrupole operator

$$Q_{F,\mu}=\sum_{jj'}\gamma_{jj'}(a^\dagger_j\times\tilde{a}_{j'})^{(2)} \tag{4.7}$$

with coefficients $\gamma_{jj'}$, taken from the surface δ interaction. The D-pair creation operator is then given by

$$D^\dagger_\mu=\mathscr{P}F^\dagger_\mu\ , \tag{4.8}$$

where $\mathscr{P}$ is a projection operator onto states of good seniority. The mapping was done as in the case of a single j shell. Otsuka considered the case of degenerate $1g_{7/2}$, $2d_{5/2}$, $2d_{3/2}$, and $3s_{1/2}$ shells. His intention was that of studying the "goodness" of the truncation-mapping procedure by comparing it with exact shell-model calculations. He found that the truncation-mapping procedure yields spectra that qualitatively agree with the exact spectra, Fig. 8, but that quantitatively differ.

Why are there quantitative differences? The answer may be simply that the boson model is not a good approximation in the schematic case of degenerate orbits and surface δ interaction. Still, there may well be other important effects responsible for this behavior. Whenever a space is truncated from large to small, one expects *renormalization effects* to occur. The interaction used in the large space must be renormalized when used in the small space. It thus becomes an "effective" interaction. Considerable effort in the last few years has gone into the construction of the effective boson interaction. A way to construct it is provided by Feshbach's projection method. In this method one divides the space into two parts (called P and Q). The effective interaction in the small space P is given by

$$H_{PP}=V_{PP}+V_{PQ}\frac{1}{E-H_{QQ}}H_{QP}\ . \tag{4.9}$$

In the case of the interacting boson model, it appears that the main renormalization effect is caused by the omission of pairs with angular momenta $J>2$. In particular, the largest contribution appears to come for G pairs ($J=4$). Although the percentage of G pairs contained in the low-lying states of nuclei might be small, *it has major effects on some parts of the effective interaction.* Otsuka treated these effects using Feshbach's projection method, and his results are shown in Fig. 8. One can see that although the structure of the spectrum remains unchanged, it is compressed, and it now agrees quite accurately with the exact shell-model result. The introduction of G pairs through renormalization effects may indeed be crucial for a quantitative comparison with experiment.

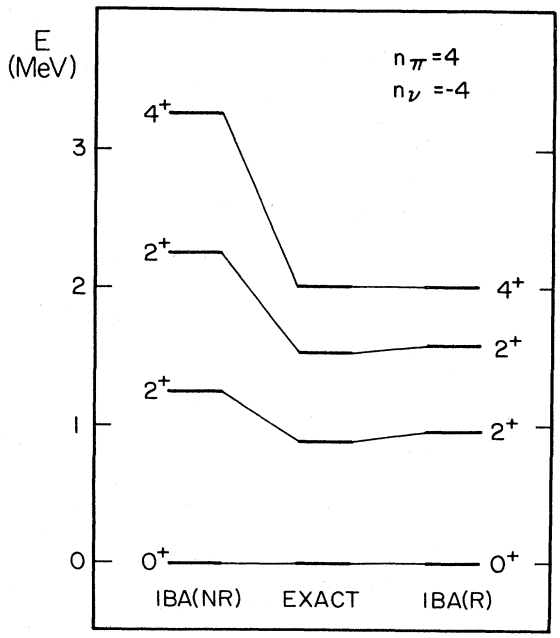

FIG. 8. Effects of the renormalization due to G pairs as calculated by Otsuka (1981a, 1981b). The exact shell-model calculations for four proton particles and four neutron holes (EXACT) is compared with the results of the truncation-mapping procedure without renormalization (IBA-NR) and with renormalization (IBA-R).

3. Several nondegenerate shells

The calculations described above of a single j shell and degenerate shells serve as guidelines for understanding the basic features of the results. For comparison with experiments, one needs to consider the realistic case of several nondegenerate shells. This problem has been attacked by several authors using a variety of methods. These are listed in Table I. We briefly comment here on those calculations in which the method is carried to its end, and spectra and electromagnetic decay properties of nuclei are explicitly calculated.

We begin by considering the results of Yang Li-Ming *et al.* (1984) for the Sm isotopes. These isotopes are particularly interesting since one observes here a transition from spherical to deformed. The input in this calculation are single-particle energies taken either from experiment or from previous calculations. The effective nucleon-nucleon interaction is taken to be pairing, quadrupole

TABLE I. Partial summary of current approaches to the microscopic structure of the interacting boson model in a spherical basis. Full references are given in the reference section at the end of this paper.

Authors	Method
T. Otsuka, 1978, 1981a, 1981b	Generalized seniority
A. Klein and M. Vallières, 1981	Kinematical method
H. B. Geyer and F. J. W. Hahne, 1981	Dyson representation
S. Pittel, P. D. Duval, and B. R. Barrett, 1982a, 1982b, 1983	Generalized seniority
Y. K. Gambhir, P. Ring, and P. Schuck, 1982a, 1982b	Broken-pair approximation
G. Bonsignori, K. Allaart, and A. van Egmond, 1983, 1984	Broken-pair approximation
D. Brink and M. Zirnbauer, 1982	Dyson representation
E. Maglione, F. Catara, A. Insolia, and A. Vitturi, 1982	Collective pair approximation
L. M. Yang, D. H. Lu, and Z. N. Zhou, 1984	Operatorized Bogoliubov transformation
Z. S. Yang, Y. Liu, and H. Qi, 1984	Modified Jancovici-Schiff substitution
G. O. Xu, 1984	Variable mean field
C. L. Wu and D. H. Feng, 1984	Composite particle representation
C. T. Li, 1984a, 1984b	Projected quasiparticle
A. Faessler and I. Morrison, 1984	Generalized seniority

pairing, and quadrupole-quadrupole for proton-proton, and neutron-neutron and quadrupole-quadrupole for proton-neutron. The corresponding strengths are shown in Table II. The calculated energy spectra are compared in Fig. 9 with experiment. The agreement is good. The calculations of Yang Li-Ming *et al.* (1984) include to some extent the effects of the Pauli principle through n_d-dependent coefficients in the boson Hamiltonian. They do not include G pairs. The effect of G pairs appears to be reflected in a renormalization of the fermion interaction. For example, the adoption of a quadrupole-quadrupole interaction between identical fermions is not in line with what has been said in Sec. III. It may be attributed to the neglect of G pairs and other configurations.

Similar results have been obtained by Yang Ze-Sen *et al.* (1984), who have computed the spectra of the Er isotopes. In this calculation the single-nucleon energies are assumed to be given by standard values. The interaction between nucleons is again given as before by pairing, quadrupole pairing, and quadrupole-quadrupole for identical particles, and by quadrupole-quadrupole for nonidentical particles. The values chosen (in MeV) are $G_0^\pi \sim 0.045$, $G_0^\nu \sim 0.045$, $G_2^\pi \sim 0.035$, $G_2^\nu \sim 0.020$, $\kappa^\pi \sim 0.09$, $\kappa^\nu \sim 0.06$, and $\kappa^{\pi\nu} \sim 0.04$. These values do not conform to the standard accepted values. In particular, the quadrupole-quadrupole interaction between like nucleons is even larger than in Table II. Although the computed spectra agree reasonably well with experiments, it seems that the method used in the calculations requires larger renormalization effects than that used in the calculation of Yang Li-Ming *et al.*

B. Calculations in a deformed basis

The scheme discussed above, namely, the truncation of the shell-model basis to S and D pairs and the subsequent mapping to a boson basis, has been carried out assuming as a starting point the *spherical* shell model. This approach emphasizes the seniority-type interaction through the use of a generalized seniority scheme. The proton-neutron quadrupole-quadrupole interaction is added subsequently. This interaction thoroughly mixes the basis states of the type

TABLE II. Parameters of the nucleon interaction (MeV) used in the calculation of Yang Li-Ming *et al.* (1984).

Isotope		86	88	90	92	94
Pairing	G_0^π	0.11	0.11	0.11	0.11	0.11
	G_0^ν	0.11	0.11	0.11	0.11	0.11
Quadrupole pairing	G_2^π	0.045	0.045	0.046	0.047	0.048
	G_2^ν	0.026	0.028	0.030	0.030	0.035
Quadrupole-quadrupole	κ^π	0.05	0.06	0.07	0.08	0.09
	κ^ν	0.06	0.06	0.07	0.08	0.09
Proton-neutron quadrupole-quadrupole	$\kappa^{\pi\nu}$	0.135	0.135	0.16	0.183	0.19

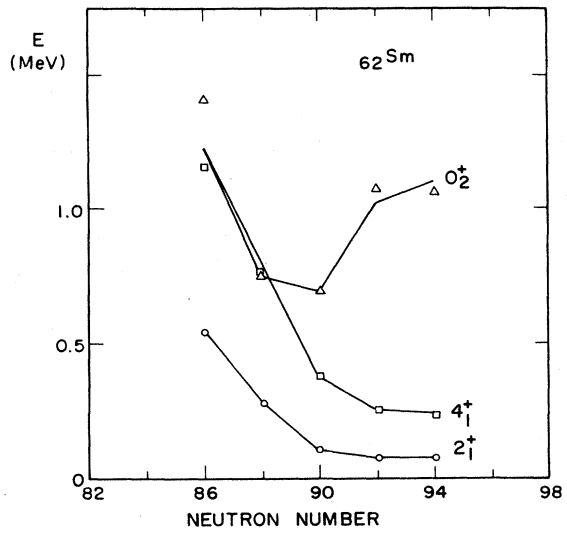

FIG. 9. Levels of $^{148-156}$Sm as calculated by Yang Li-Ming *et al.* (1984). The circles, squares, and triangles represent the experimental values.

$$\{[(S_\pi^\dagger)^{n_{s_\pi}}(D_\pi^\dagger)^{n_{s_\pi}}_{\gamma_\pi J_\pi}]\times[(S_\nu^\dagger)^{n_{s_\nu}}(D_\nu^\dagger)^{n_{d_\nu}}_{\gamma_\nu J_\nu}]\}_{JM}\,|\,0\rangle\ . \tag{4.10}$$

One can imagine a different scheme in which the starting point is the dominant role played by the quadrupole-quadrupole interaction (Bohr and Mottelson, 1980). This scheme can be implemented by constructing a truncation-mapping procedure based on nucleons moving in a *deformed* well (Nilsson model). A truncation procedure can then be performed as follows. The proton-neutron quadrupole interaction is first taken to create a deformed potential well for the independent motion of the nucleons. The proton-proton and neutron-neutron interactions are then introduced, and the result is that each pair of identical nucleons is distributed over *several* Nilsson orbitals. If this interaction is the simple-minded pairing interaction, the problem can be treated in the BCS approximation. A more accurate procedure is to carry out a Hartree-Fock-Bogoliubov calculation. In either way, the resulting wave functions (with definite nucleon number) can be expressed in terms of a *deformed* pair operator $\Lambda^\dagger$ creating a coherent mixture of Nilsson pair states,

$$\Lambda^\dagger=\sum_k \alpha_k \Lambda_k^\dagger=\sum_k \alpha_k \tfrac{1}{2} a_k^\dagger a_{-k}^\dagger\ , \tag{4.11}$$

where k now labels the quantum numbers of a particle in a deformed potential well. The ground state of a deformed nucleus with N_π proton pairs and N_ν neutron pairs can be written as

$$(\Lambda_\pi^\dagger)^{N_\pi}(\Lambda_\nu^\dagger)^{N_\nu}\,|\,0\rangle\ . \tag{4.12}$$

Since the operators $\Lambda^\dagger$ do not conserve angular momentum, the state (4.12) is an intrinsic state that contains many values of the angular momenta. This quantum number must be restored by projection is one wants to construct the actual states. The operator $\Lambda^\dagger$ can be rewritten in terms of spherical pairs

$$\Lambda^\dagger=\sum_J x_J A_0^{(J)^\dagger}\ , \tag{4.13}$$

where x_J denotes normalized amplitudes and $A_0^{(J)^\dagger}$ is obtained by projecting from $\Lambda^\dagger$ onto an angular momentum J and $M_J=0$ component. Using the notation of nucleon pairs we write

$$\Lambda^\dagger=x_0 S^\dagger+x_2 D_0^\dagger+x_4 G_0^\dagger+\cdots\ . \tag{4.14}$$

The states (4.12) are then mapped onto boson states

$$(\lambda_\pi^\dagger)^{N_\pi}(\lambda_\nu^\dagger)^{N_\nu}\,|\,0\rangle\ , \tag{4.15}$$

where the boson operator $\lambda^\dagger$ is given by

$$\lambda^\dagger=x_0 s^\dagger+x_2 d_0^\dagger+x_4 g_0^\dagger+\cdots\ . \tag{4.16}$$

The boson Hamiltonian and transition operators can then be obtained by equating matrix elements between corresponding fermion and boson states.

The truncation-mapping procedure in the deformed well has been used by several authors in order to study two problems:

(1) How good is the truncation to S and D pairs if the interacting boson model has to be consistent with the Nilsson model (or the results of Hartree-Fock-Bogoliubov calculations)?

(2) What is the explicit construction of the boson Hamiltonian and operators?

Although a full microscopic description of collective states that includes all nuclei from spherical to deformed can only be done within the framework of the spherical shell model (Sec. IV.A), one may view the mapping in a deformed basis as a convenient way to maximize the effects of the quadrupole-quadrupole interaction in deformed nuclei.

The two problems listed above have been attacked by several authors. Some of them are listed in Table III. The results of the calculations performed so far appear to indicate that the probability of having S-D pairs in the pair operator [Eq. (4.14)], $x_0^2+x_2^2$, is $\geq 85\%$ if one assumes the Nilsson model with a deformation parameter $\delta\sim 0.30$ and a pairing gap $\Delta\sim 1.0$ MeV. The same result is obtained by using an HFB approach. Although this result is very encouraging, several comments are in order. The first concerns the relation between the probability of having S-D pairs in the state (4.12) and that in the single-pair state (4.14). If the latter is smaller than 1, it is obvious that the larger the N, the smaller is the probability in the state (4.12). This probability, which may become very small for large N, is *not* a proper measure of the success of the truncation into the S-D space. A proper measure is provided by the matrix elements of the shell-model Hamiltonian in states $(\Lambda^\dagger)^N\,|\,0\rangle$. Since this Hamiltonian has only one- and two-body operators, the only important states are $\Lambda^\dagger\,|\,0\rangle$ and $(\Lambda^\dagger)^2\,|\,0\rangle$. The S-D probability for these states appears to be rather high.

TABLE III. Partial summary of current approaches to the microscopic structure of the interacting boson model in a deformed basis. Full references are given in the reference section at the end of this paper.

Authors	Method
A. Bohr and B. R. Mottelson, 1980	Nilsson + BCS
D. R. Bes, R. A. Broglia, E. Maglione, and A. Vitturi, 1982	Nilsson + BCS
T. Otsuka, A. Arima, and N. Yoshinaga, 1982	Nilsson + BCS
J. Dukelsky, G. G. Dussel, H. M. Sofia, and S. Pittel, 1982	Nilsson + BCS
J. Dukelsky and S. Pittel, 1983	HFB
E. Maglione, F. Catara, A. Insolia, and A. Vitturi, 1983, 1986a, 1986b	Nilsson + BCS
K. Sugawara-Tanabe and A. Arima, 1982	HFB
T. Otsuka, 1984a, 1984b	Nilsson + BCS
T. D. Cohen, 1985	Self-consistent mapping
W. Pannert, P. Ring, and Y. K. Gambhir, 1985	HFB
M. Sambataro, H. Schasser, and D. M. Brink, 1986	Generator Coordinate

A second point is the need for projecting good angular momentum from the state (4.12). In this state there are states with angular momenta considerably higher than $J=2N_\pi+2N_\nu$, which is an upper bound for states in the S-D space. All states with higher values of J do not include S and D pairs at all. Hence, in states with lower values of J, the weight of S-D states is higher than implied by the values of x_0 and x_2. Finally, whereas the expansion of a single-pair state (4.13) is unique, this is no longer true for states projected from $(\Lambda_\pi^\dagger)^{N_\pi}(\Lambda_\nu^\dagger)^{N_\nu}\,|\,0\rangle$. For example, when considering the case with $N_\pi=2$, $N_\nu=2$, the $J=0$ state due to two G pairs may well overlap with the $(D^\dagger\times D^\dagger)_0^{(0)}$ and $(S^\dagger)^2\,|\,0\rangle$ states.

A major conclusion of most calculations carried out so far in a deformed basis (as, for example, the recent calculations of Pannert, Ring, and Gambhir, 1985) is that the effects of G pairs cannot be neglected and must be included either explicitly or by renormalization. Otsuka (1984) has devised techniques to deal with these effects, as well as those of the Pauli principle. He has used his technique to evaluate the boson image of the quadrupole transition operator

$$\hat{T}^{(\mathrm{E2})}=e_\pi^B\hat{Q}_\pi^B+e_\nu\hat{Q}_\nu^B\,, \tag{4.17}$$

with

$$\hat{Q}_\rho^B=(d_\rho^\dagger s_\rho+s_\rho^\dagger\tilde{d}_\rho)+\chi_\rho(d_\rho^\dagger\times\tilde{d}_\rho)^{(2)}\,,\quad \rho=\pi,\nu\,. \tag{4.18}$$

Otsuka's results are shown in Table IV. He has also evaluated the strength of the proton-neutron boson interaction

$$V_{\pi\nu}=-\kappa\hat{Q}_\pi^B\cdot\hat{Q}_\nu^B\,, \tag{4.19}$$

also shown in Table IV. The calculations reported in the table are performed by using $\delta\sim0.25$, $\Delta\sim0.9$ MeV, $e_\pi^F=1.7\ e$, and $e_\nu^F=0.7\ e$. The values shown there were obtained by equating matrix elements of the appropriate operators between states in the ground-state band. Once this is done, one must test the consistency of the scheme by comparing boson and fermion calculations for other bands (β and γ). Some preliminary calculations have been reported by Otsuka (1984) and Vitturi and Maglione (1984).

V. OTHER TOPICS

A. *F* spin and Isospin

In the interaction boson model 2, collective states of nuclei are constructed from pairs of protons and neutrons. Instead of using explicitly proton and neutron labels, it has proven to be convenient to introduce a formalism similar to isospin, but applied to bosons. Proton and neutron bosons can be assigned a quantity, called F spin, of value $F=\frac{1}{2}$ (Arima *et al.*, 1977). Proton bosons then have $F_0=\frac{1}{2}$, while neutron bosons have $F_0=-\frac{1}{2}$. The basis states for the interacting boson model 2 are then

TABLE IV. Interacting boson model parameters as obtained by Otsuka (1984).

Parameter	With renormalization	Without renormalization
χ_π	−0.86	−0.80
χ_ν	−1.18	−1.03
$e_\pi^B(e\,\mathrm{fm}^2)$	12.0	10.6
$e_\nu^B(e\,\mathrm{fm}^2)$	10.0	6.7
κ (MeV)	0.094	0.120

constructed by multiplying proton basis states with those for neutrons, as in Eq. (3.30). It follows that one can obtain not only states that are symmetric in the proton and neutron degrees of freedom, but also states with partial symmetry. A convenient way to display the symmetry character of the resulting wave functions is through the use of Young tableaux. Since we have two types of particles, the resulting Young tableaux are two-rowed and can be obtained by multiplying the appropriate Young tableaux for protons and neutrons. For example, the wave functions corresponding to $N_\pi=1$ and $N_\nu=1$ can be obtained from the product

$$[1]\otimes[1]=[2]\oplus[11] . \tag{5.1}$$

The wave functions [2] are totally symmetric, while those with Young tableaux [11] are antisymmetric. In general, Young tableaux are of the form

N_1

□ □ ··· □
□ □ .
N_2 (5.2)

Instead of the quantities N_1 and N_2 one can use the quantities

$$N=N_1+N_2=N_\pi+N_\nu , \quad F=\frac{N_1-N_2}{2} , \tag{5.3}$$

to label the states, i.e., the total boson number and the value of the F spin. The occurrence of states that are not totally symmetric is a new aspect brought in by the coupling of protons and neutrons. Their prediction and resulting recent discovery (Bohle *et al.*, 1984) constitute one of the main achievements of the interacting boson model 2.

The introduction of F spin poses two questions:

(1) To what extent is F spin a good quantum number if one starts from a microscopic theory of the boson model?

(2) What is the relation between the interacting boson models 1 and 2?

We begin by considering the first question. It is clear here that F spin is an exact quantum number if the IBA-2 Hamiltonian is fully symmetric between protons and neutrons, i.e., if the Hamiltonian is a scalar in F spin or contains at most explicit dependence on $\hat{F}^2$ and F_0 (which are diagonal in F spin). In actual cases, the Hamiltonian will not have this property. In fact, the microscopic theory of the model leads to a Hamiltonian that can be schematically written as

$$H=\varepsilon(\hat{n}_{d_\pi}+\hat{n}_{d_\nu})+\kappa\hat{Q}_\pi\cdot\hat{Q}_\nu , \tag{5.4}$$

where the operators $\hat{Q}$ are given in Eq. (4.18). Since no $\hat{Q}_\pi\cdot\hat{Q}_\pi$ and $\hat{Q}_\nu\cdot\hat{Q}_\nu$ interaction is present in Eq. (5.4), this Hamiltonian is not fully symmetric. In many applications, another operator is added to Eq. (5.4), called the Majorana operator $\hat{M}_{\pi\nu}$,

$$H=\varepsilon(\hat{n}_{d_\pi}+\hat{n}_{d_\nu})+\kappa\hat{Q}_\pi\cdot\hat{Q}_\nu+\lambda'\hat{M}_{\pi\nu} . \tag{5.5}$$

This operator distinguishes between states with different symmetry character. It is a two-boson operator defined by

$$\begin{aligned}\hat{M}_{\pi\nu}=&\xi_1(d_\pi^\dagger\times d_\nu^\dagger)^{(1)}\cdot(\tilde{d}_\pi\times\tilde{d}_\nu)^{(1)}\\&+\xi_3(d_\pi^\dagger\times d_\nu^\dagger)^{(3)}\cdot(\tilde{d}_\pi\times\tilde{d}_\nu)^{(3)}\\&+\xi_2(d_\pi^\dagger s_\nu^\dagger-d_\nu^\dagger s_\pi^\dagger)\cdot(\tilde{d}_\pi s_\nu-\tilde{d}_\nu s_\pi) .\end{aligned} \tag{5.6}$$

The eigenvalues of Eq. (5.6) are 0 for symmetric states of two bosons, and, if $\xi_2=-2\xi_1=-2\xi_3=1$, the eigenvalues are $+1$ for antisymmetric states ($J=1,3$ and the antisymmetric $J=2$ state). The terms with ξ_1 and ξ_3 have a direct shell-model meaning. They express the interaction of a proton pair and a neutron pair in the $J=1$ and $J=3$ states. The term with ξ_2, however, can be shown to have no microscopic analog. Its occurrence may be attributed to renormalization effects (Otsuka, 1978; Scholten, 1983; van Egmond and Allaart, 1984). To see the effect of the Majorana operator in general, it is necessary to look at its eigenvalues. These can be obtained by using explicitly the F-spin operators of the bosons,

$$\sum_{i<k}\left[\frac{1-4(\mathbf{f}_i\cdot\mathbf{f}_k)}{4}\right]=\tfrac{1}{8}[N(N+2)-4F(F+1)] . \tag{5.7}$$

Hence, if λ' is positive, states of Eq. (5.5) with symmetry character $F<F_{\max}$ will be pushed upward. The question of whether or not F spin is a good quantum number depends on the interplay between the quantities λ' and κ in Eq. (5.5). Novoselski and Talmi (1985), using $\xi_2\sim 0$, find that for realistic values of the parameters used to describe the spectrum of ^{178}Hf, the 0^+ ground state of this nucleus contains 82.1% of $F=F_{\max}$ and 16.4% of $F=F_{\max}-1$. Other authors find smaller admixtures of $F=F_{\max}-1$. The amount of these admixtures in the low-lying states depends on the IBA-2 Hamiltonian used. A possible way to probe F-spin admixtures experimentally is by analyzing $M1$ transition rates between low-lying states.

A relation between the IBM-1 and IBM-2 can be obtained by projecting the operators of the interacting boson model 2 onto the set of states with maximum values of the F spin, $F-F_{\max}=N/2$ (Harter *et al.*, 1985; Novoselski and Talmi, 1985). The projected Hamiltonian can be identified with an IBM-1 Hamiltonian. It does not always give results that are as good as those obtained from the original IBA-2 Hamiltonian. This indicates that going from IBA-2 to IBA-1 involves a further renormalization of the parameters.

The introduction of F spin also raises the question of its relation to isospin. In medium-mass and heavy nuclei, protons and neutrons occupy different major shells, while neutrons occupy the 82-126 major shell. For these nuclei the introduction of F spin poses no major problem. However, if one wants to describe with the interacting boson model collective states in light nuclei, in which protons

and neutrons occupy the same single-particle orbits, one must take into account the charge independence property of the nucleon-nucleon interaction. This is one of the best established facts of nuclear physics.

As remarked above, isospin symmetry is not of much help in configurations in which valence protons are in a set of j orbits that are fully occupied by neutrons. All states of such configurations correspond to states with the same value of isospin, which is one-half the difference between the total number of neutrons and protons, $T=(N_A-Z_A)/2$. Thus for these nuclei all states constructed with proton and neutron pairs in different orbits correspond to states with the same value of isospin, though they may have different values of F spin. In light nuclei, this is no longer the case, and some care must be taken when applying the interacting boson model to such cases.

The simplest way to exploit the charge independence of the Hamiltonian when protons and neutrons occupy the same orbit is through the use of the isospin formalism. The use of this formalism is not mandatory. One could still use the proton-neutron formalism and obtain exact results, provided the interaction were charge independent. A boson Hamiltonian constructed under these conditions would be more complicated than Eq. (5.4). For example, it would contain other interactions between d_π and d_ν bosons, $(d_\pi^\dagger\times\tilde{d}_\pi)^{(k)}\cdot(d_\nu^\dagger\times\tilde{d}_\nu)^{(k)}$, with $k=1,3,4$, in addition to the $k=2$ term. We shall not discuss this possibility in detail. Instead we shall describe an elegant way introduced by Elliott and White (1980) to incorporate isospin into the boson model for cases in which protons and neutrons occupy the same set of orbits. These authors introduce, in addition to the $T=1$ proton-proton and neutron-neutron pairs (with $M_T=+1$ and $M_T=-1$), proton-neutron pairs with $T=1$, $M_T=0$. This model is called the interacting boson model 3, since it has three types of bosons. The proton-neutron $T=1$ pairs are called δ pairs. Instead of using explicitly the labels π, ν, and δ, one can use here the formalism of isospin T. The three types of bosons, π, ν, and δ, form an isotopic triplet with $T=1$.

Boson Hamiltonians with the same physics content as Eq. (5.4) can be written down in the isospin formalism. The single-boson terms $\varepsilon[(d_\pi^\dagger\cdot\tilde{d}_\pi)+(d_\nu^\dagger\cdot\tilde{d}_\nu)]$ can be replaced by a term proportional to $(d^\dagger\cdot\tilde{d})_{T=0}$, which also contains a $(d_\delta^\dagger\cdot\tilde{d}_\delta)$ term. The latter explicitly guarantees that the energy of a proton-neutron pair in $T=1$ states is the same as that of proton-proton and neutron-neutron pairs. Boson-boson interactions can be written down to yield given matrix elements in states of two bosons with $T=0$, $T=1$, and $T=2$. These should be different in order to describe the experimental situation. For example, the low-lying spectrum of ^{20}O ($T=2$) is qualitatively different from that of ^{20}Ne ($T=0$), despite the fact that both nuclei have four particles in the 8-20 shell. Their difference has been discussed in the previous sections and attributed to the fact that the basic interaction between identical nucleons is of the seniority-conserving type, while that between nonidentical nucleons is of the quadrupole type.

The interacting boson model 3 enlarges the scope of the boson model and provides a description of light nuclei. In the usual IBM-2, the boson description of ^{20}O ($T=2$) is in terms of two s_ν and d_ν bosons with an F spin value of $F=1$. In ^{20}Ne, however, it is the low-lying levels, with $T=0$, that have an approximate boson description of one s_π or d_π and one s_ν or d_ν bosons. These states also have maximum F spin ($F=1$), but different eigenvalues of the IBA-2 Hamiltonian. The shell-model structure of the $T=2$ levels in ^{20}Ne, which correspond to ^{20}O levels, does not allow a good IBM-2 boson description at all. In IBM-3, the $T=2$ levels of ^{20}Ne have instead a *good boson description* given by the $M_T=0$ projection of the $M_T=2$ boson states ^{20}O. Such $T=2$ levels are also well described by IBM-3 bosons in ^{20}F (with $M_T=1$). In addition, the introduction of δ bosons may be used to obtain a description of the $T=1$ levels of ^{20}F. The latter is an *odd-odd* nucleus, which is beyond the scope of IBM-2.

Finally, one may attempt a further extension of the boson model to include, in addition to $T=1$ pairs, proton-neutron pairs with $T=0$. This extension, called the interacting boson model 4, has been suggested by Elliott and Evans (1981). The microscopic structure of this model suggests an identification of the $T=1$ boson with nucleon pairs with $T=1$ (and spin $S=0$), while the $T=0$ bosons are identified with nucleon pairs with $T=0$ (and spin $S=1$). This microscopic structure thus relies on a spherical shell model with L-S coupling. It is appropriate only for light nuclei, since the presence of the spin-orbit interaction destroys L-S coupling in favor of j-j coupling. This IBM-4 model is the boson analog of the Wigner supermultiplet scheme.

The microscopic foundations of the interacting boson models 3 and 4 have been investigated recently by Evans, Elliott, and Szpikowski (1985) and by Halse (1985).

B. Valence nucleons versus all nucleons

The various forms of the interacting boson model discussed so far assume, in the majority of cases, as a microscopic starting point the spherical shell model with only active valence particles. An important question is to what extent this is a justifiable approximation. This approximation is somewhat in contrast with a picture of collective states in nuclei as rigid rotations of the *entire* nucleus. There are two physical quantities that are particularly sensitive to this question. The first quantity is related to magnetic properties of collective states in nuclei. The magnetic $M1$ operator can be written in IBM-2 as

$$\hat{T}^{(M1)}=g_\pi\hat{L}_\pi+g_\nu\hat{L}_\nu\ , \tag{5.8}$$

where g_π and g_ν are the proton and neutron boson g factors and $\hat{L}_\pi,\hat{L}_\nu$ the corresponding angular momentum operators. In heavy nuclei, some microscopic calculations of g_π and g_ν yield $g_\pi\sim1$, $g_\nu\sim0$, since the contribution of the spin part cancels out to some extent, and only the orbital contribution remains. Using Eq. (5.8), one can com-

pute the matrix elements of the $M1$ operator between the ground state and the first excited 1^+ state of deformed nuclei. This state corresponds to a rotation around the average axis of proton and neutron distributions. Due to zero-point oscillations of one distribution relative to the other, such a rotation is possible and is the lowest excited 1^+ mode. This mode is sometimes called a twisting oscillation. One then obtains (Dieperink, 1983)

$$B(M1;0_1^+\rightarrow 1_1^+)\approx \left[\frac{3}{4\pi}\right]\left[\frac{4N_\nu N_\pi}{N_\nu+N_\pi}\right](g_\pi-g_\nu)^2 . \tag{5.9}$$

This $B(M1)$ value is thus directly related to the number of proton and neutron particles taking part in the collective motion. Insertion of the appropriate values yields $B(M1;\uparrow)\sim 2\mu_N^2$ in ^{156}Gd. This value becomes instead $\sim 12\mu_N^2$ if all nucleons are supposed to take part in the collective motion. The experimental value $\sim 1.6\mu_N^2$ (Bohle *et al.*, 1984) appears to support the shell-model assumption that most of the contribution to the low-lying collective states comes from valence nucleons. The core is, to a large extent, inert, and the valence particles "slide" on it.

The second quantity that is particularly sensitive to this question is related to electric properties of collective states. The electric E2 operator can be written in IBM-2 as

$$\hat{T}^{(E2)}=e_\pi\hat{Q}_\pi+e_\nu\hat{Q}_\nu , \tag{5.10}$$

where e_π and e_ν are boson effective charges. In the microscopic theory of the interacting boson model, the effective boson charges e_π and e_ν are related to the fermion charges. Even with effects of the renormalization due to G pairs and other pair degrees of freedom, it is not possible to describe the experimental results starting from fermion effective charges $e_p=e$, $e_n=0$. One needs fermion effective charges of the order of $e_p\sim 1.7e$, $e_n\sim 0.7e$, as mentioned above in Sec. IV.B. This indicates that contributions from other shells are important. The need for effective quadrupole charges is well known in the shell model. Even a single nucleon outside closed shells may cause a polarization of the core. The polarization charge is attributed to the neglect of $2\hbar\omega$, $4\hbar\omega$, . . . , excitations.

Both in the shell model and in the boson model the effects of core polarization can be described by introducing effective charges. However, this description can be improved by explicitly including some of the omitted configuration. Park and Elliott (1986) have recently studied this problem by coupling to the low-lying states the giant monopole and quadrupole resonances (also described by bosons s' and d'). Similar results have also been obtained by Scholten (1984).

We remark here that it appears from phenomenological fits that as long as only B(E2) are considered, the use of effective charges is sufficient to describe the data. Once effective charges have been given to proton and neutron bosons, the boson model naturally produces the large

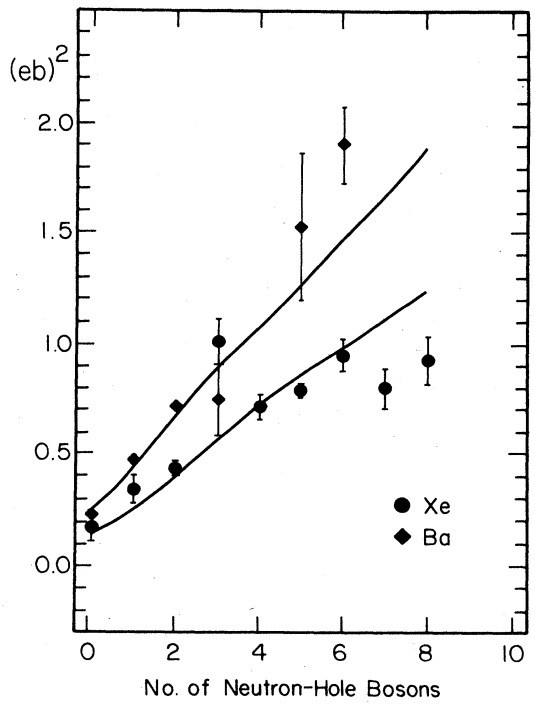

FIG. 10. B(E2) values of transitions from 0^+ ground states to first excited 2^+ levels plotted as a function of the number of neutron-hole bosons (Novoselski and Talmi, 1986).

enhancement of E2 transition strengths observed in the middle of major shells. This enhancement arises from coherent contributions of states of proton and neutron s and d bosons. No extra effect is needed to explain the data. An example is given in Fig. 10 (Noveselski and Talmi, 1986), in which one can observe that the same effective charges used in the middle of the shell describe the situation in semimagic nuclei (the points at neutron number 82). It should be stressed that such constant values of e_π and e_ν were determined from experiments only for nuclei with $O(6)$-like spectra. The situation may be more complicated in the case of strongly deformed nuclei. There it may be necessary to introduce further renormalization effects (see Table IV).

The E2 transition density may be measured (e.g., by electron scattering) not only at the photon point but also at other values of the momentum transfer. Such measurements give information about the radial dependence of the E2 operator due to core polarization. It appears that the effect of core polarization is not uniformly distributed over the nuclear volume, but is mostly located at the nuclear surface. Nonetheless, even for transition densities, once core polarization effects have been included through the use of boson effective form factors, the further enhancement observed in the middle of the major shells is automatically obtained in the boson model. In other words, the same transition densities can be used to

describe semimagic nuclei as well as nuclei in the middle of the shells (DeJager, 1984; Goutte, 1984).

VI. CONCLUSIONS

The main purpose of this paper has been to present the *conceptual* relationship between the shell model and the interacting boson model. We have also briefly surveyed detailed shell-model calculations that attempt to explore this relationship in a more quantitative way. Quantitative calculations are difficult for two reasons. The first difficulty is the lack of a detailed knowledge of the effective interactions in the shell model. The second is that approximations must be adopted to perform the calculations, and these give rise to renormalization effects that are difficult to evaluate. Several calculational schemes have been suggested to deal with these problems. These schemes have, in some cases, been implemented to their ends. The corresponding results are *very* encouraging. The overlap between the full shell-model wave functions and those in the S-D space seems to be appreciable. This indicates that it may be possible to take into account the contribution of states not included by renormalization of the parameters of the model. Estimates of such renormalization effects produce results that are in fair agreement with experiment. Use of more accurate effective interactions may improve the agreement even further.

The calculations starting from the spherical shell model are well defined, and their interpretation is straightforward. In cases where the effects of G pairs have been incorporated, they agree quantitatively with experimental data. Starting instead from a deformed potential well, there are additional complications, and the interpretation is not so clear cut. Most calculations here have been performed to test the validity of the S-D truncation rather than for comparison with experiment. Even here, the results seem to indicate that the S-D truncation is a reasonable approximation. There are physical observables that seem to be strongly affected by contributions from G pairs. However, inclusion of these pairs seems to reproduce the full results surprisingly well. Since the contributions of G pairs in the states considered are small, it is plausible that it will be sufficient to treat those admixtures in perturbation theory. Consequently, those contributions could most probably be included by renormalization of the parameters. However, further work remains to be done before we shall be able to compare the results of the calculations with experiments.

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