Numerical calculation of hadron masses in quantum chromodynamics*

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Recent numerical Monte Carlo simulations of the hadron spectrum are reviewed. After a general introduction, different ways of calculating the hadron masses in the "quenched approximation" (i.e., neglecting virtual quark loops) are described and the latest results are summarized. The pseudofermion method and the iterative hopping expansion method for the introduction of dynamical quarks is discussed, and the first results for the hadron spectrum including the effect of virtual quark loops are reviewed. A separate section is devoted to the discussion of questions related to scaling with dynamical quarks.

CONTENTS

I. INTRODUCTION

The numerical calculation of the hadronic mass spectrum is one of the great challenges in lattice quantum chromodynamics. As a result of many years of experimentation with strongly interacting particles, the masses of many hadrons are known to good precision. The confrontation of this important and extensive body of empirical knowledge with the quantum chromodynamics (QCD) theory has two important aspects: first, it can provide irrefutable evidence, which up to now has been missing, for QCD as the correct theory of strong interactions; second, in case of a successful reproduction of the known hadron masses we would have a marvelous demonstration of the capabilities of a new approach in theoretical particle physics, namely, large-scale computation. In fact, this would be the first time ever that masses of elementary particles would be theoretically calculated. The number of free parameters for the description of hundreds of hadron masses (and other static hadron properties) is remarkably small: besides the Λ parameter for the color gauge coupling there are only the quark masses for different flavors (six for the moment). It is quite certain that the experience gained in lattice QCD will be extremely useful beyond the theory of hadrons as well, in other relativistic quantum field theories. Large-scale numerical computation could help in the future to extend our theoretical understanding to areas where detailed phenomenological study is not (or not yet) feasible.

At present we are obviously only at the beginning of this, almost revolutionary, development. The first efforts to develop the numerical methods needed for hadron mass calculations were initiated only about three years ago. The first investigations necessarily had an exploratory character. Still, progress in the field seems to be rather rapid, and as is usual in such cases, a great number of papers (good and bad) has been produced. This makes a review somewhat difficult and certainly incomplete. Nevertheless, I have tried to give a coherent introduction and a detailed list of references to at least some of the interesting topics in the field.

II. CALCULATION OF HADRON MASSES

The quark fields in Euclidean lattice QCD are described by anticommuting (Grassmann) variables defined on lattice sites $x=(x_1,x_2,x_3,x_4)$; $1\le x_\mu \le N_\mu$ (the lattice size is $N_1N_2N_3N_4$). It is convenient to use dimensionless fields; therefore the connection between continuum fields and lattice fields is given by (a=lattice spacing)

$$\left[\frac{a^3}{2K}\right]^{1/2}\psi_{\rm cont}(ax)\to\psi_x,\quad \left[\frac{a^3}{2K}\right]^{1/2}\tilde{\psi}_{\rm cont}(ax)\to\tilde{\psi}_x\ . \tag{2.1}$$

Here K is, in general, an appropriately chosen normalization factor. For Wilson lattice fermions K is the "hopping parameter," which is related in the free-fermion case to the bare mass m by

$$K=(8r+2am)^{-1}\ , \tag{2.2}$$

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where r is the "Wilson parameter" satisfying $0<r\leq 1$. The quark part of the QCD lattice action with Wilson fermions (Wilson, 1974, 1977) is then

$$S_f=\sum_{xy}\tilde{\psi}_y Q_{yx}\psi_x\ ,$$
$$Q_{yx}=\delta_{yx}-K\sum_{\mu}(r+\gamma_\mu)U(x,\mu)\delta_{y,x+\hat{\mu}}\ . \tag{2.3}$$

The SU(3) link variables are denoted here by $U(x,\mu)$. They satisfy $U(x,\mu)^+=U(x+\hat{\mu},-\mu)$ ($\hat{\mu}$ is a unit vector in the direction μ). The Euclidean Dirac matrices are defined according to $\gamma_\mu=\gamma_\mu^+=-\gamma_{-\mu}$, and $\sum_\mu$ means a summation over both positive and negative directions: $\mu=\pm1,\pm2,\pm3,\pm4$.

The "quark matrix" Q in Eq. (2.3) is neither Hermitian nor anti-Hermitian, but obeys

$$Q_{yx}=\gamma_5 Q_{xy}^+\gamma_5\ , \tag{2.4}$$

with $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$, as usual. In the free case [$U(x,\mu)=1$] the quark propagator Q^{-1} can be easily obtained by Fourier transformation. On a finite lattice with periodic boundary conditions the μth component of the momentum $k_\mu=ap_\mu=2\pi\nu_\mu/N_\mu$ has values in the Brillouin zone (B_μ is an arbitrary integer):

$$B_\mu+1\leq\nu_\mu\leq B_\mu+N_\mu\ . \tag{2.5}$$

Introducing the notation

$$(k,x)\equiv 2\pi\left[\frac{\nu_1 x_1}{N_1}+\cdots+\frac{\nu_4 x_4}{N_4}\right], \tag{2.6}$$

we can write the free quark propagator with $N=N_1N_2N_3N_4$ as

$$G_{yx}\equiv Q^{-1}(U=1)_{yx}=N^{-1}\sum_k e^{-i(k,x-y)}\tilde{G}_k\ . \tag{2.7}$$

The momentum-space free propagator $\tilde{G}_k$ is

$$\tilde{G}_k=\left[1-2K\sum_{\mu>0}(r\cos k_\mu-i\gamma_\mu\sin k_\mu)\right]^{-1}$$
$$=\left[1-2K\sum_{\mu>0}(r\cos k_\mu+i\gamma_\mu\sin k_\mu)\right]$$
$$\times\left[\left[1-2K\sum_{\mu>0}r\cos k_\mu\right]^2+4K^2\sum_{\mu>0}\sin^2 k_\mu\right]^{-1}. \tag{2.8}$$

This form shows how the fermion doubling problem is solved for Wilson fermions. In the continuum limit $k_\mu\equiv ap_\mu\to 0$ the denominator is proportional to

$$\left[\frac{1-8rK}{2K}\right]^2+k_\mu k_\mu\to a^2(m^2+p_\mu p_\mu)\ . \tag{2.9}$$

At the other corners of the Brillouin zone, however, the mass in the denominator is $m_w=(m+2wra^{-1})$, if the number of momentum components with $k_\mu-\pi\equiv ap_\mu$ is $w=1,2,3,4$. Therefore the mass m_w tends to infinity for $a\to 0$, and the unwanted extra fermions decouple from the physical fermion with mass m.

The global symmetry properties of the Wilson lattice fermion action can be immediately seen in Eq. (2.3). For N_f flavors the quark matrix is block diagonal in flavor. In the individual blocks the hopping parameter has the value of K_f belonging to the bare quark mass m_f of the flavor in question ($f=u,d,s,c,b,t,\ldots$). For nondegenerate flavors the action has an exact $\mathrm{U}(1)^{\otimes N_f}$ symmetry corresponding to the conservation of the quark number in each flavor. For N_f flavors with degenerate mass, the global symmetry is $\mathrm{U}(N_f)=\mathrm{U}(1)\otimes\mathrm{SU}(N_f)$. The axial part of the global chiral $\mathrm{U}(1)\otimes\mathrm{SU}(N_f)\otimes\mathrm{SU}(N_f)$ symmetry is, however, explicitly broken by the Wilson term proportional to r, even in the case of zero bare mass $m_f=0$. The expected situation in QCD is that the axial-vector symmetry is spontaneously broken by the vacuum expectation value of $\bar{\psi}\psi$; therefore in the Wilson fermion formulation one has to assume that for vanishing lattice spacing the explicit breaking goes over into a spontaneous breaking.

The anticommuting Grassmann variables are not well suited for numerical calculations. Therefore it is convenient to perform the fermion integration by using the bilinearity of the action in fermion fields. In general, the expectation value of a quantity $F(U,\psi,\tilde{\psi})$ is defined as

$$\langle F\rangle=\frac{\int\prod_x\left[d\psi_x d\tilde{\psi}_x\prod_{\mu>0}dU(x,\mu)\right]e^{-S_g(u)-S_f(U,\psi,\tilde{\psi})}F(U,\psi,\tilde{\psi})}{\int\prod_x\left[d\psi_x d\tilde{\psi}_x\prod_{\mu>0}dU(x,\mu)\right]e^{-S_g(U)-S_f(U,\psi,\tilde{\psi})}}\ . \tag{2.10}$$

For a purely gluonic quantity, depending only on the gauge field U, this is equivalent to

$$\langle F\rangle=\frac{\int dU\exp[-S_{\mathrm{eff}}(U)]F(U)}{\int dU\exp[-S_{\mathrm{eff}}(U)]},\quad dU\equiv\prod_x\prod_{\mu>0}dU(x,\mu)\ . \tag{2.11}$$

The effective action $S_{\rm eff}$ in the gluonic sector is the sum of the pure gauge action S_g and the negative logarithm of the Matthews-Salam determinant encountered at the integration over the fermionic degrees of freedom:

$$S_{\rm eff}(U)=S_g(U)+S^q_{\rm eff}(U),\quad S^q_{\rm eff}(U)=-\ln\det Q(U)\ . \tag{2.12}$$

The quantities explicitly depending on the quark fields can also be evaluated from the effective action in the gluonic sector. For the product of purely fermionic variables we have, for instance,

$$\langle\psi_{r_1}\tilde{\psi}_{s_1}\psi_{r_2}\tilde{\psi}_{s_2}\cdots\psi_{r_n}\tilde{\psi}_{s_n}\rangle=\frac{\int dU\exp[-S_{\rm eff}(U)]\det_{(r_1\cdots r_n,s_1\cdots s_n)}[Q^{-1}(U)]}{\int dU\exp[-S_{\rm eff}(U)]}\ . \tag{2.13}$$

Here the determinant in the numerator is built from the matrix elements $Q^{-1}(U)_{rs}$ of the quark propagator in the background gauge field U. Note that the indices (r,s) are abbreviations for all sorts of indices of the quark field, namely, color, spin, and flavor indices.

The hadron masses can be calculated from the expectation values of correlation functions of composite operators carrying different quantum numbers. The choice of the composite operators is to a large extent arbitrary. In fact, for given values of the coupling constants one has to find the optimal operator, which has a strong enough coupling to the hadron in question and, at the same time, can be evaluated numerically without too great difficulty. In practice this means that in most cases the simplest local multiquark composite operators are taken. Let us now restrict ourselves to the ground-state mesons and baryons [in the sense of SU(6)] containing u, d, and s quarks. The spin dependence of the operators is dictated in this case by the relativistic generalization of SU(6) symmetry [for a review and references, see Pais (1966)]. The $J^{PC}=0^{-+}$ pseudoscalar mesons are described by bilinear composite operators like, for instance,

$$\varphi_x^{(\pi^+)}=\tilde{d}_{x\alpha a}\gamma_{5,\alpha\beta}u_x^{\beta a},\quad \varphi_x^{(K^+)}=\tilde{s}_{x\alpha a}\gamma_{5,\alpha\beta}u_x^{\beta a}\ . \tag{2.14}$$

Here $u_x^{\alpha a}$, $d_x^{\alpha a}$, and $s_x^{\alpha a}$ stand for the u-, d-, and s-flavor components of the quark field $\psi_x^{\alpha a}$, respectively. The indices $\alpha,\beta,\ldots$, denote Dirac spin indices, whereas $a,b,\ldots$, are the SU(3) color indices. The corresponding 1^{--} vector-meson fields are $(k=1,2,3)$

$$\varphi_{xk}^{(\rho^+)}=\tilde{d}_{x\alpha a}\gamma_{k,\alpha\beta}u_x^{\beta a},\quad \varphi_{xk}^{(K^{*+})}=\tilde{s}_{x\alpha a}\gamma_{k,\alpha\beta}u_x^{\beta a}\ . \tag{2.15}$$

For the baryons, the trilinear composite operators can be chosen in different ways (see, for example, Joffe, 1981). In the spin-$\frac{1}{2}$ octet one can use, for instance,

$$\begin{aligned}
&\varphi_{x\alpha}^{(p)}=\varepsilon_{abc}(C\gamma_5)_{\beta\gamma}u_x^{\alpha a}(u_x^{\beta b}d_x^{\gamma c}-d_x^{\beta b}u_x^{\gamma c}),\quad \varphi_{x\alpha}^{(\Sigma^+)}=\varepsilon_{abc}(C\gamma_5)_{\beta\gamma}u_x^{\alpha a}(u_x^{\beta b}s_x^{\gamma c}-s_x^{\beta b}u_x^{\gamma c})\ ,\\
&\varphi_{x\alpha}^{(\Lambda)}=\varepsilon_{abc}(C\gamma_5)_{\beta\gamma}[u_x^{\alpha a}(d_x^{\beta b}s_x^{\gamma c}-s_x^{\beta b}d_x^{\gamma c})+d_x^{\alpha a}(s_x^{\beta b}u_x^{\gamma c}-u_x^{\beta b}s_x^{\gamma c})-2s_x^{\alpha a}(u_x^{\beta b}d_x^{\gamma c}-d_x^{\beta b}u_x^{\gamma c})]\ ,\\
&\varphi_{x\alpha}^{(\Xi^0)}=\varepsilon_{abc}(C\gamma_5)_{\beta\gamma}s_x^{\alpha a}(s_x^{\beta b}u_x^{\gamma c}-u_x^{\beta b}s_x^{\gamma c})\ .
\end{aligned} \tag{2.16}$$

For the spin-$\frac{3}{2}$ decuplet one can consider

$$\begin{aligned}
&\varphi_{xk\alpha}^{(\Delta^{++})}=\varepsilon_{abc}(C\gamma_k)_{\beta\gamma}u_x^{\alpha a}u_x^{\beta b}u_x^{\gamma c},\quad \varphi_{xk\alpha}^{(\Sigma^{*+})}=\varepsilon_{abc}(C\gamma_k)_{\beta\gamma}(u_x^{\alpha a}u_x^{\beta b}s_x^{\gamma c}+u_x^{\alpha a}s_x^{\beta b}u_x^{\gamma c}+s_x^{\alpha a}u_x^{\beta b}u_x^{\gamma c})\ ,\\
&\varphi_{xk\alpha}^{(\Xi^{*0})}=\varepsilon_{abc}(C\gamma_k)_{\beta\gamma}(s_x^{\alpha a}s_x^{\beta b}u_x^{\gamma c}+s_x^{\alpha a}u_x^{\beta b}s_x^{\gamma c}+u_x^{\alpha a}s_x^{\beta b}s_x^{\gamma c}),\quad \varphi_{xk\alpha}^{(\Omega^-)}=\varepsilon_{abc}(C\gamma_k)_{\beta\gamma}s_x^{\alpha a}s_x^{\beta b}s_x^{\gamma c}\ ,
\end{aligned} \tag{2.17}$$

where ε_{abc} denotes, as usual, the totally antisymmetric SU(3) unit tensor, and C is the Dirac matrix for charge conjugation. In the numerical calculations it is customary to use the following representation of Euclidean Dirac matrices:

$$\begin{aligned}
&\gamma_1=\begin{vmatrix}0&0&0&i\\0&0&i&0\\0&-i&0&0\\-i&0&0&0\end{vmatrix},\quad \gamma_2=\begin{vmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{vmatrix},\quad \gamma_3=\begin{vmatrix}0&0&i&0\\0&0&0&-i\\-i&0&0&0\\0&i&0&0\end{vmatrix},\\
&\gamma_4=\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{vmatrix},\quad \gamma_5=\begin{vmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{vmatrix}=\gamma_1\gamma_2\gamma_3\gamma_4,\quad C=\begin{vmatrix}0&0&0&-1\\0&0&1&0\\0&-1&0&0\\1&0&0&0\end{vmatrix}=\gamma_1\gamma_3\gamma_5\ .
\end{aligned} \tag{2.18}$$

The expectation value of the product of two hadron operators can be expressed, using Eq. (2.13), by the products of the quark propagators in some background gauge field configuration U. For the quark flavors u, d, and s one has to take in the propagator the hopping parameter values K_u, K_d, and K_s, respectively. (The small mass difference between u and d

quarks is, however, usually neglected: $K_u = K_d$.) Writing out indices explicitly, let us introduce

$$U_{x\alpha a,y\beta b} \equiv Q^{-1}(U,K=K_u)_{x\alpha a,y\beta b},\quad D_{x\alpha a,y\beta b} \equiv Q^{-1}(U,K=K_d)_{x\alpha a,y\beta b},\quad S_{x\alpha a,y\beta b} \equiv Q^{-1}(U,K=K_s)_{x\alpha a,y\beta b}\ . \tag{2.19}$$

Then, for instance, for $\langle \varphi_x^{(\pi^+)} \varphi_y^{(\pi^-)} \rangle$ and $\langle \varphi_{xk}^{(\rho^+)} \varphi_{yl}^{(\rho^-)} \rangle$, one has to calculate, respectively,

$$\mathrm{Tr}_{sc}\{\gamma_5 U_{xy} \gamma_5 D_{yx}\},\quad \mathrm{Tr}_{sc}\{\gamma_k U_{xy} \gamma_l D_{yx}\}\ , \tag{2.20}$$

where Tr_{sc} stands for a trace over spin and color indices. Formulas such as Eq. (2.20) apply to all flavor-nondiagonal mesons. For mesons like $\eta,\eta',\omega,\varphi,\ldots$, some combination of flavor-diagonal operators such as

$$\varphi_x^{(\bar{u}u)} = \bar{u}_{x\alpha a} \Gamma_{\alpha\beta} u_x^{\beta a} \tag{2.21}$$

is needed (Γ is some Dirac matrix). For the expectation value $\langle \varphi_x^{(\bar{u}u)} \varphi_y^{(\bar{u}u)} \rangle$ the general expression (2.13) involves the combination

$$\mathrm{Tr}_{sc}\{\Gamma U_{xy} \Gamma U_{yx}\} - \mathrm{Tr}_{sc}\{\Gamma U_{xx}\}\mathrm{Tr}_{sc}\{\Gamma U_{yy}\}\ . \tag{2.22}$$

In order to obtain baryon masses, we must have the necessary combinations of quark propagators. For the proton (and similarly for Ξ^0 and Σ^+) these are

$$\varepsilon_{abc}\varepsilon_{def}(C\gamma_5)_{\beta\gamma}(C\gamma_5)_{\varepsilon\varphi}(U_{x\alpha a,y\delta d} U_{x\beta b,y\varepsilon e} D_{x\gamma c,y\varphi f} + U_{x\alpha a,y\varepsilon d} U_{x\beta b,y\delta e} D_{x\gamma c,y\varphi f})\ . \tag{2.23a}$$

For the Λ baryons they are

$$\begin{aligned}\varepsilon_{abc}\varepsilon_{def}(C\gamma_5)_{\beta\gamma}(C\gamma_5)_{\varepsilon\varphi}[\, & U_{x\alpha a,y\delta d} D_{x\beta b,y\varepsilon e} S_{x\gamma c,y\varphi f} + D_{x\alpha a,y\delta d} U_{x\beta b,y\varepsilon e} S_{x\gamma c,y\varphi f} + 4 S_{x\alpha a,y\delta d} U_{x\beta b,y\varepsilon e} D_{x\gamma c,y\varphi f} \\ & - U_{x\alpha a,y\varepsilon d} D_{x\beta b,y\delta e} S_{x\gamma c,y\varphi f} - D_{x\alpha a,y\varepsilon d} U_{x\beta b,y\delta e} S_{x\gamma c,y\varphi f} - 2 U_{x\alpha a,y\varepsilon d} D_{x\beta b,y\varphi e} S_{x\gamma c,y\delta f} \\ & - 2 D_{x\alpha a,y\varepsilon d} U_{x\beta b,y\varphi e} S_{x\gamma c,y\delta f} - 2 S_{x\alpha a,y\varepsilon d} D_{x\beta b,y\varphi e} U_{x\gamma c,y\delta f} - 2 S_{x\alpha a,y\varepsilon d} U_{x\beta b,y\varphi e} D_{x\gamma c,y\delta f}\,]\ .\end{aligned} \tag{2.23b}$$

For the Δ^{++} baryon (and similarly for Ω^-) the propagators are

$$\varepsilon_{abc}\varepsilon_{def}(C\gamma_k)_{\beta\gamma}(C\gamma_k)_{\varepsilon\varphi}(U_{x\alpha a,y\delta d} U_{x\beta b,y\varepsilon e} U_{x\gamma c,y\varphi f} + 2 U_{x\alpha a,y\varepsilon d} U_{x\beta b,y\delta e} U_{x\gamma c,y\varphi f})\ . \tag{2.23c}$$

For the Σ^{*+} baryon (and similarly for Ξ^{*0}) the propagators are

$$\varepsilon_{abc}\varepsilon_{def}(C\gamma_k)_{\beta\gamma}(C\gamma_k)_{\varepsilon\varphi}(U_{x\alpha a,y\delta d} U_{x\beta b,y\varepsilon e} S_{x\gamma c,y\varphi f} + 2 U_{x\alpha a,y\varepsilon d} U_{x\beta b,y\delta e} S_{x\gamma c,y\varphi f})\ . \tag{2.23d}$$

The numerical calculation of the hadron masses is based on the Källman-Lehmann representation of two-point functions. In the Euclidean region for a spinless field $\varphi(x)$ (for simplicity) we have

$$\langle 0 \mid T[\varphi(x)\varphi(y)] \mid 0 \rangle = \int_{m_0^2}^{\infty} dm^2 \rho(m^2) \Delta_E(x-y;m^2)\ , \tag{2.24}$$

with a positive spectral weight function $\rho(m^2)$ and the Euclidean propagator

$$\Delta_E(x;m^2) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik_\mu x_\mu}}{m^2 + k_\nu k_\nu}\ . \tag{2.25}$$

Projecting out the zero three-momentum intermediate states by an integration over three-space, one obtains for this "time slice"

$$\int d^3x \langle 0 \mid T\{\varphi(x)\varphi(y)\} \mid 0 \rangle = \int_{m_0}^{\infty} dm\, \rho(m^2) e^{-m\,|x_4 - y_4|}\ . \tag{2.26}$$

Stable single-particle states contribute by a δ-function term in ρ, whereas multiparticle intermediate states give a continuum contribution. For large Euclidean time separations the lowest mass m_0 dominates, and we have

$$m_0 = -\lim_{|x_4 - y_4| \to \infty} \frac{1}{|x_4 - y_4|} \times \ln \int dx^3 \langle 0 \mid T[\varphi(x)\varphi(y)] \mid 0 \rangle\ . \tag{2.27}$$

Another possibility is to perform a Fourier transformation,

$$\int dx_4 e^{-ip_4 x_4} \int d^3x \langle 0 \mid T[\varphi(x)\varphi(0)] \mid 0 \rangle = \int_{m_0^2}^{\infty} dm^2 \frac{\rho(m^2)}{m^2 + p_4^2}\ . \tag{2.28}$$

This shows the particle poles in the (real) energy variable $E = -ip_4$.

Both Eqs. (2.26) and (2.28) can, in principle, be used to extract the lowest masses from the expectation values of products of multiquark operators as in Eqs. (2.14)–(2.17). One has, however, to keep in mind that the formulas are exact only in the continuum limit. On a finite lattice there are $O(a)$ corrections due to the finite lattice spacing a, as well as finite size effects due to the finite physical extension of the lattice. [For some exact results about the spectrum of finite lattice pure gauge theory in the strong coupling region see, for instance, Schor (1983, 1984) and

O'Caroll (1984; O'Caroll and Barbosa, 1985).] An important modification to Eq. (2.26) is due to the periodic boundary conditions, which are introduced in most calculations in order to minimize finite-size effects. Due to the periodicity, a signal can propagate between two points in different ways. Neglecting propagations with more windings, we can replace the simple exponential behavior (2.26) in the case of mesons by

$$e^{-x_4 am}+e^{-(N_4-x_4)am} . \tag{2.29}$$

Here the time difference x_4 is given in lattice units, and N_4 is the lattice size in the Euclidean time direction. For baryons, the corresponding formula is more complicated because TCP invariance implies the propagation of the opposite parity (charge conjugate) state in the opposite time direction; therefore Eq. (2.29) is replaced by

$$(1+\gamma_4)(c_+e^{-x_4 am_+}+c_-e^{-(N_4-x_4)am_-}) +(1-\gamma_4)(c_+e^{-(N_4-x_4)am_+}+c_-e^{-x_4 am_-}) . \tag{2.30}$$

Here c_+ and c_- give the coupling strengths of the two opposite parity states with masses m_+ and m_-, respectively. In addition to showing the effects of finite time extension, the spectrum is, of course, also influenced by the finite spatial extension L of the lattice. A dimensionless measure of the finite size is $\xi\equiv Lm_0$ [with $m_0=m_0(L)$ the lowest mass in the given channel]. For large L, the mass on the finite lattice tends to the physical mass $M_0=\lim_{L\to\infty} m_0(L)$, and the deviation behaves as (Lüscher, 1984)

$$\delta_0\equiv\frac{M_0-m_0}{m_0}\simeq\frac{c_1}{\xi}e^{-c_2\xi} . \tag{2.31}$$

The constants $c_{1,2}$ depend on the quantum numbers. A more detailed formula (Lüscher, 1984) relates δ_0 to some elastic scattering amplitude, and hence c_1 turns out to be proportional to some coupling constant squared. The other constant c_2 is of order 1, so that finite-size effects should fast disappear once some critical size is reached.

III. MONTE CARLO CALCULATIONS IN THE QUENCHED APPROXIMATION

A. Generalities

The numerical evaluation of the fermion part of the effective action $S^q_{\rm eff}$ in Eq. (2.12) is rather time consuming, because the "quark determinant" $\det Q$ is essentially nonlocal. The nonlocality is due to the fact that $S^q_{\rm eff}$ describes the effect of closed virtual quark loops, and light virtual quarks can propagate to large distances. (For a discussion of the quark determinant see the next section.) In the "quenched" or "valence" approximation (Hamber and Parisi, 1981; Marinari *et al.*, 1981a; Weingarten, 1982) virtual quark loops are omitted by neglecting the dependence of $S^q_{\rm eff}(U)$ on the gauge configuration U. In this case in Eqs. (2.11) and (2.13) $S^q_{\rm eff}$ cancels out, and the effective gauge field action $S_{\rm eff}(U)$ can be replaced by the pure gauge action $S_g(U)$. The quenched approximation is expected to give a reasonable (say, within 10%) description of the hadron spectrum at least in the flavor nonsinglet channels. This expectation is based on the phenomenological Okubo-Zweig-Iizuka rule (Okubo, 1963; Zweig, 1964; Iizuka, 1966) and on some theoretical results obtained in the $1/N_c$ expansion (N_c = number of colors) ('t Hooft, 1974).

Since the gauge field configurations in the quenched approximation are distributed according to the pure gauge action, the scaling of the hadron masses m has to follow the renormalization-group equation (RGE) without quarks:

$$\left[-a\frac{\partial}{\partial a}+\bar\beta(g)\frac{\partial}{\partial g}\right]m=O(a) . \tag{3.1}$$

Here $\bar\beta(g)$ is the Callan-Symanzik β function of the pure gluon theory on the lattice, and the right-hand side is due to the scale-breaking lattice artifacts. (For $a\to 0$ it goes to zero by some power of a.) The solution of Eq. (3.1) is

$$m=a^{-1}\exp\left[-\int_{g_0}^{g}\frac{dx}{\bar\beta(x)}\right] . \tag{3.2}$$

If the integration constant g_0 is replaced by an overall factor c_m, one can write, in analogy with Eq. (5.4),

$$m=c_m a^{-1}(\beta_0 g^2)^{-\beta_1/2\beta_0^2}\exp\left[-\frac{1}{2\beta_0 g^2}-\int_0^g dx\left[\frac{1}{\bar\beta(x)}+\frac{1}{\beta_0 x^3}-\frac{\beta_1}{\beta_0^2 x}\right]\right]\equiv c_m\Lambda_{\rm latt} . \tag{3.3}$$

Here $\Lambda_{\rm latt}$ is the Λ parameter of pure lattice gauge theory, and β_0 and β_1 are the first two (universal) expansion coefficients of $\bar\beta(g)$ given by Eqs. (5.2) and (5.3) with $N_c=3$ and $N_f=0$. The integral piece in the exponent is not universal; it depends, for instance, on the particular form of lattice action chosen for $S_g(U)$. But, compared to the universal g^{-2} term, it becomes small in the continuum limit $g\to 0$.

In order to obtain the two-point functions of the hadronic multiquark operators in the quenched approximation, one has to calculate the expectation value of expressions like those in Eqs. (2.20)—(2.23). In the case of flavor nonsinglet mesons the required combination of quark propagators can be represented by Fig. 1 [see Eq. (2.20)]. For the flavor singlet mesons, like those in Eq. (2.22), one needs combinations as given in Fig. 2, whereas for the baryons in Eqs. (2.23a)—(2.23d) one has to calculate combinations like those shown in Fig. 3. In the cases of Figs.

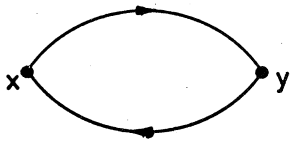

FIG. 1. The quark propagator configuration needed for flavor nonsinglet mesons.

1 and 3 it is enough to consider quark propagators originating from a single point, say x. This is because the orientation of the propagator lines can be reversed by using Eq. (2.4). This in turn means that for the flavor nonsinglet mesons one has to calculate the expectation value of expressions like

$$\mathrm{Tr}_{sc}(Q_{xy}^{-1}\Gamma\gamma_5 Q_{xy}^{-1+}\gamma_5\Gamma) . \tag{3.4}$$

Here Γ is some Dirac matrix describing the spin. In this connection let us note the special role of the flavor nonsinglet pseudoscalar mesons with $\Gamma\gamma_5=\gamma_5^2=1$. In this case there are no cancellations in the spin trace in Eq. (3.4), and the decrease of the hadron propagator for large distances is the slowest, corresponding to the smallest mass. This observation is the starting point for the derivation of several rigorous mass inequalities (see, for instance, Nussinov, 1983; Weingarten, 1983b; Witten, 1983).

The case of the flavor singlet mesons in Fig. 2 is much more difficult than the propagator configurations in Figs. 1 and 3. First of all, quark propagators starting from two different points are needed. This means that one has to evaluate the quark propagators from every starting point of at least several time slices. The second, potentially even more dangerous, difficulty is that the second term in Eq. (2.22) contains a nonconnected piece that has to be subtracted. This requires very high statistics, and even then the measurement of the correlation at large distances is rather questionable. (For a suggestion as to how to overcome these difficulties see Hamber *et al.*, 1983a.) The only attempt, up to now, to calculate flavor singlet meson masses and mixing with glueballs was made recently in the quenched approximation on a small $(4^3\times 8)$ lattice in SU(2) gauge theory (Fukugita *et al.*, 1984b).

B. Iterative methods for the calculation of quark propagators

The main task in quenched hadron mass calculations is the inversion of the quark matrix Q, in order to obtain

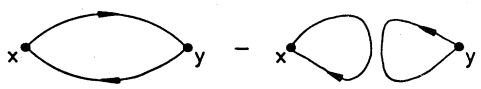

FIG. 2. The same as Fig. 1 for flavor singlet mesons.

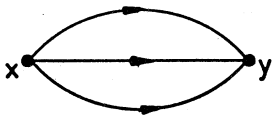

FIG. 3. The same as Fig. 1 for baryons.

the required matrix elements of the quark propagator Q^{-1}. Several standard numerical matrix inversion methods (Lanczos, 1950; Householder, 1964; Varga, 1965; Stoer and Bulirsch, 1980) have been tested and successfully applied. The most popular are the variants of the Gauss-Seidel method and the conjugate-gradient method.

Let us write the quark matrix Q in Eq. (2.3) as

$$\begin{aligned} &Q=1-KM , \\ &M_{x_2x_1}=\sum_{x,\mu}(r+\gamma_\mu)U(x,\mu)\delta_{x_2,x+\hat{\mu}}\delta_{x,x_1} . \end{aligned} \tag{3.5}$$

The simplest iteration for $p\equiv Q^{-1}i$ (i=some initial vector) is the "Jacobi iteration":

$$\begin{aligned} &p_{n+1}=i+KMp_n \quad (n=0,1,2\ldots) , \\ &p_0=i, \quad p=\lim_{n\to\infty}p_n . \end{aligned} \tag{3.6}$$

Iterating point by point, that is, taking on the right-hand side the already calculated elements of p_{n+1} instead of the old p_n, gives the "Gauss-Seidel iteration." This corresponds to the decomposition $M=M_l+M_u$, where M_l has nonzero elements only below the main diagonal (and there $M=M_l$; $M_u=0$). The iterative equation now becomes

$$p_{n+1}=i+K(M_lp_{n+1}+M_up_n) . \tag{3.7}$$

In order to improve convergence one can also introduce a relaxation parameter λ and set

$$p_{n+1}=(1-\lambda)p_n+\lambda[1+K(M_lp_{n+1}+M_up_n)] . \tag{3.8}$$

For small quark masses, still better convergence can be achieved by a "second-order" method. Returning to the simple expression in Eq. (3.6), even if point-by-point iteration is done, we can write the "first-order" iteration in Eq. (3.8) as

$$p_{n+1}=p_n+\lambda(1-Qp_n) . \tag{3.9}$$

Then, going a step further, with

$$p_{n+1}=\rho p_n+(1-\rho)p_{n+1}+\lambda\rho(1-Qp_{n+1}) , \tag{3.10}$$

one obtains the "second-order" iteration:

$$p_{n+2}=p_n+\lambda(1-\lambda\rho Q)(i-Qp_n) . \tag{3.11}$$

By an appropriate choice of the two parameters λ,ρ a good convergence can be achieved even for smaller values of the quark mass.

The other popular and effective method for the inversion of the quark matrix is the "conjugate-gradient" method. It begins with a guess at p_0 for $p=Q^{-1}i$. Then one has to calculate

$$r_0=Q^+(i-Qp_0)\ . \tag{3.12}$$

If the length $|r_0|=[(r_0^+,r_0)]^{1/2}$ is zero, than p_0 is the solution. Otherwise, for $n=0,1,2,\ldots$,

$$\begin{aligned} r_{n+1}&=r_n-\frac{|r_n|^2}{|Qh_n|^2}Q^+Qh_n\ , \\ p_{n+1}&=p_n+\frac{|r_n|^2}{|Qh_n|^2}h_n\ . \end{aligned} \tag{3.13}$$

If $|r_{n+1}|=0$, then p_{n+1} is the solution, because

$$\begin{aligned} Qp_{n+1}&=Qp_n+Q^{+-1}r_n \\ &=Qp_{n-1}+Q^{+-1}r_{n-1} \\ &=\cdots=Qp_0+Q^{+-1}r_0=i\ . \end{aligned} \tag{3.14}$$

For $|r_{n+1}|\neq 0$ one calculates

$$h_{n+1}=r_{n+1}+\frac{|r_{n+1}|^2}{|r_n|^2}h_n \tag{3.15}$$

and returns to Eq. (3.13) for the next n. It can be shown that the solution is always obtained in a finite number of steps.

On large lattices, computer memory limitations often pose a problem for the iterative methods because the iterated vectors have many components (and the gauge configuration itself takes a lot of storage space). The usual way of circumventing these difficulties is to partition the inversion of the quark matrix Q, which is possible because of the locality. A simple way to do this is to organize the iteration according to time slices. In this way it is enough to keep only a few (usually up to 3) time slices in the memory (Bowler, Kenway, *et al.*, 1984).

C. Hopping expansion method

Another way to obtain information about hadronic two-point (or many-point) amplitudes avoids the direct numerical inversion of the quark matrix by concentrating on the expansion coefficients in powers of the hopping parameter K. Knowing the hopping parameter expansion coefficients to sufficiently high orders, and assuming the analyticity of the amplitudes at $K=0$, one can investigate different features of the amplitudes at the physical values of K. The starting point (Wilson, 1977; Hasenfratz and Hasenfratz, 1981; Hasenfratz *et al.*, 1982a, 1982b; Lang *et al.*, 1982; Stamatescu, 1982) is a formal Taylor expansion like

$$(1-KM)^{-1}=\sum_{j=0}^{\infty}K^jM^j \tag{3.16}$$

or, for the fermion part of the effective action in Eq. (2.12),

$$\begin{aligned} S_{\rm eff}^q&=-\ln\det(1-KM)=-\mathrm{Tr}\ln(1-KM) \\ &=\sum_{j=1}^{\infty}\frac{K^j}{j}\mathrm{Tr}(M^j)\ . \end{aligned} \tag{3.17}$$

From the expansion coefficients in Eqs. (3.16) and (3.17) one can construct the expansion coefficients of the hadronic amplitude in question and then, either by direct application of the hopping parameter series (if the series converges) or by some analytical continuation method, one can calculate the amplitude at the desired value of K. Applying the explicit form of the "hopping matrix" M in Eq. (3.5), it is possible to represent the hopping parameter series as a sum over curves on the lattice. For instance, one can write $\mathrm{Tr}(M^n)$ as

$$\begin{aligned} \mathrm{Tr}(M^n)=\sum_{x_1\mu_1,\ldots,x_n\mu_n}&\delta_{x_1,x_n+\hat{\mu}_n}\delta_{x_n,x_{n-1}+\hat{\mu}_{n-1}}\cdots\delta_{x_2,x_1+\hat{\mu}_1}\mathrm{Tr}_c[U(x_n,\mu_n)\cdots U(x_2,\mu_2)U(x_1,\mu_1)] \\ &\times\mathrm{Tr}_s[(r+\gamma_{\mu_n})\cdots(r+\gamma_{\mu_2})(r+\gamma_{\mu_1})]\ . \end{aligned} \tag{3.18}$$

Due to the δ functions, the sum runs here over all closed loops. A similar representation of $(M^n)_{yx}$ is possible in terms of loops running from point x to point y. Although this representation is physically rather suggestive and appealing, it has the practical disadvantage that the number of curves at high orders is very large. For instance, at 12th order there are more than 4×10^6 closed curves going through a given link, and at 16th order already more than 6×10^9 (the number increases exponentially; Berg *et al.*, 1982). It is quite clear that the evaluation of the traces in Eq. (3.18) requires a prohibitively large number of multiplications already in these orders. This was the reason why the first numerical calculations of the hadron spectrum (Hasenfratz *et al.*, 1982a, 1982b) were restricted to low orders. A sufficiently high-order hopping expansion is possible with the numerical iterative method (Hasenfratz and Montvay, 1983, 1984). If one wants to obtain, for instance, the matrix element $\langle g|M^n|i\rangle$, then one uses

$$\langle g|M^n|i\rangle=\sum_h\langle g|M|h\rangle\langle h|M^{n-1}|i\rangle\ . \tag{3.19}$$

This shows how $\langle g|M^n|i\rangle$ is built up from the lower-order matrix elements $\langle h|M^{n-1}|i\rangle$. Due to the nearest-neighbor structure of the hopping matrix M [see Eq. (3.5)], the consecutive steps of the iteration for $\langle f|M^n|i\rangle$ can be visualized as represented in Fig. 4. From some starting point, in a given order, a finite number of points is reached. During the iteration it is possible to choose the boundary conditions for the quarks independently from the given boundary conditions (usually periodic) of the gauge configuration. In the case of the

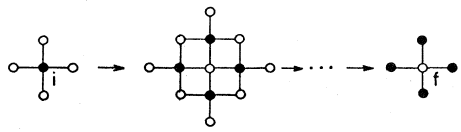

FIG. 4. Illustration of the iteration for the calculation of the hopping expansion coefficient $\langle f | M^n | i \rangle$. In every step the open points are calculated from the solid ones.

"periodic box" iteration, the quarks also obey periodic boundary conditions. In the "copied gauge field" iteration the quarks propagate without boundaries over the periodic gauge field background. A mixture of both approaches is also possible: "periodic box" in the space directions and "copied gauge field" in the time direction. The advantage of the copied gauge field iteration over the periodic box iteration is that the quark propagators are defined for continuous momenta [not just for the discrete values in Eq. (2.5)]. Therefore it is possible analytically to continue the hadron propagators to real energies by carrying out a Laplace transform with real $E=-ip_4$, instead of the Fourier transformation in Eq. (2.28). This allows us to look directly for the particle singularities (for fixed E in the hopping parameter variable) by a Padé-approximant technique. In such a way direct information is obtained on the nature of the singularity, and the localization of cuts or multiple poles [e.g., due to radial excitations (Hasenfratz and Montvay, 1984)] becomes possible. This is potentially a very useful technique in an unquenched spectrum calculation with light dynamical quarks, where the resonances $(\rho,\Delta,\ldots)$ are "hidden" behind multiparticle cuts. The price of a high-order calculation with copied gauge field iteration is the growing number of points reached in higher orders (and hence the growing length of the arrays in the computer program). For illustration, the number of points with given distance $l=0,1,\ldots,16$ on a four-dimensional hypercubic lattice is given in Table I.

The required order in the hopping expansion depends on the gauge coupling $\beta\equiv 2N_c g^{-2}$ and on the quark mass (it is higher for larger β and for smaller quark mass). At $\beta=5.7$ in SU(3) ($N_c=3$), for instance, good results can be achieved in 32nd order. In the test case of free Wilson fermions (with $r=1$), the position of branch-point singularities in the multiquark amplitudes are reproduced in 32nd order within 1—2 % in the case of mesons and 4—5 % in the case of baryons (Kunszt, 1983). For nonzero coupling ($g\neq 0$, $\beta<\infty$) the situation is most probably even better. The order of the quark propagator calculation is chosen in such a way that, for the required hadronic amplitude, some given order can be achieved. For instance, a 32nd-order calculation of the mesonic amplitude in Fig. 1 requires that the quark propagator iteration run up to the maximum distance $l=16$ from the given initial point (see Fig. 4). It can be easily seen that with this set of quark propagators the baryonic amplitude in Fig. 3 can be calculated up to 33rd order. It is also clear that mesonic amplitudes contain only even powers of K, whereas the baryonic ones contain both even and odd powers.

D. Results for Wilson fermions

Quenched hadron mass calculations with Wilson quarks have been performed by many authors; besides the references we have already mentioned (Hamber and Parisi, 1981; Hasenfratz, Hasenfratz, Kunszt, and Lang, 1982a, 1982b; Weingarten, 1982; Kunszt, 1983; Hasenfratz and Montvay 1984) see also Fucito, Martinelli, *et al.* (1982); Bernard, Draper, and Olynyk (1983); Bernard, Draper, Olynyk, and Rushton (1983); Bowler *et al.* (1983); Fucito *et al.* (1983); Fukugita *et al.* (1983, 1984a); Gupta and Patel (1983a, 1983b); Hamber and Parisi (1983); Lipps *et al.* (1983); Martinelli, Parisi, *et al.* (1983); Patel and Gupta (1983); Weingarten (1983a); Bowler, Chalmers, *et al.* (1984); König *et al.* (1984); Itoh *et al.* (1984); Langguth and Montvay (1984); Billoire *et al.* (1985). Qualitatively rather good results for the spectrum were reported even in the earliest pioneering papers, although the physical lattice was still very small, typically less than 1 fm. [The situation became even worse when more precise string-tension measurements (Gutbrod *et al.*, 1983; Parisi *et al.*, 1983; Barkai *et al.*, 1984; Hasenfratz *et al.*, 1984b; Otto and Stack, 1984) suggested an even smaller lattice spacing than was thought before.] The importance of some minimum lattice size was soon realized, however (Bernard, Draper, and Olynyk, 1983; Bowler, Pawley, *et al.*, 1983; Gupta and Patel, 1983a; Hasenfratz and Montvay, 1983; Martinelli, Parisi, *et al.*, 1983; Politzer, 1984). Some exploratory studies on larger lattices showed (Lipps *et al.*, 1983; Hasenfratz and Montvay,

TABLE I. The number of points N_l with given lattice distance (measured in links) in a four-dimensional hypercubic lattice.

l	0	1	2	3	4	5	6	7	8
N_l	1	8	32	88	192	360	608	952	1408
l	9	10	11	12	13	14	15	16	
N_l	1992	2720	3608	4672	5928	7392	9080	11 008	

TABLE II. The main results of some recent quenched hadron mass calculations with Wilson fermions. K_{cr} is the critical hopping parameter where the π mass vanishes (it is nearly equal to K_u for u and d quarks). The quark mass $m_q a$ is defined in Eq. (5.10).

	$\beta=5.7$			$\beta=6.0$		
	(Hasenfratz and Montvay, 1984; Kunszt and Montvay, 1984)	(Bowler, Chalmers, *et al.*, 1984)	(Langguth and Montvay, 1984)	(Lipps *et al.*, 1983)	(Billoire *et al.*, 1985)	(König *et al.*, 1984)
	16^4	$8^3\times16$ (copied)	$8^3\times16$ (copied)	$10^3\times20$	$10^3\times20$	$16^3\times28$ (blocked)
K_{cr}	0.1690±0.0005	0.1695±0.0007	0.1696±0.0016	0.1567±0.0001	~0.157	
$m_\rho a$	$0.58^{+0.12}_{-0.06}$	0.53±0.03	0.57±0.01	0.37±0.02	~0.35	0.37±0.03
$m_p a$	1.1±0.2	1.11±0.10	0.97±0.14	0.57±0.07	~0.45	0.69±0.04
$(m_\Delta - m_p)a$	0.13±0.06	0.02±0.01	0.25±0.08	0.10±0.10	~0.07	0.10±0.04
$(m_\pi a)^2/(m_q a)$	~3.2	2.85±0.15	2.5±0.3	~2.4	~2.4	

1984; König *et al.*, 1984; Billoire *et al.*, 1985) that the most drastic finite size effects go away if the spatial size of the lattice reaches 1.7—2.0 fm and the temporal size is roughly twice as much. The larger temporal size is needed for the separation of the lowest state from radial excitations. The elongated lattice for the calculation of quark propagators can, however, be prepared by copying a symmetric lattice periodically twice in the time direction. In the hopping expansion method this is done anyway, if the copied gauge field iteration is applied. In this case one has, however, to perform a high-enough-order calculation to exploit efficiently the distant time slices. In practice this means that, for an effective time elongation N_t, roughly an order $2N_t$ is required in the copied gauge field hopping expansion.

Assuming a string tension $\sqrt{\kappa}=420$ MeV, the recent SU(3) string-tension calculations give for the lattice spacing

$$\begin{aligned} &a(\beta=5.7)\simeq 0.21 \text{ fm} , \\ &a(\beta=5.8)\sim 0.16 \text{ fm} , \\ &a(\beta=6.0)\simeq 0.12 \text{ fm} . \end{aligned} \tag{3.20}$$

Therefore the minimum required lattice sizes are roughly $8^3\times16$ (at $\beta=5.7$), $12^3\times24$ (at $\beta=5.8$), and $16^3\times32$ (at $\beta=6.0$). Note that the lattice scale between $\beta=5.7$ and $\beta=6.0$ changes more rapidly than "asymptotic scaling" with the two-loop perturbative β function $\bar{\beta}(g) \to -\beta_0 g^3 - \beta_1 g^5$ in Eqs. (3.2) and (3.3) would require. Of course, hadron masses on the lattice should also scale according to Eq. (3.20), in order to be consistent with a continuum (scaling) behavior. The present situation is not in contradiction with such a behavior between $\beta=5.7$ and $\beta=6.0$ (see Table II). The errors are, however, still somewhat large, and the $\beta=6.0$ results presumably suffer from somewhat more finite-size effects. The results of König *et al.* (1984) may be better from this point of view, but it is not clear what is the influence of the (approximate) blocking procedure introduced by Mütter and Schilling (1984a, 1984b).

Note that the critical hopping parameter value K_{cr}, where the pion mass vanishes, is substantially larger at $\beta=5.7$ ($g^{-2}=0.95$) and $\beta=6.0$ ($g^{-2}=1.0$) than the one-loop perturbative value (Kawamoto, 1981; Stehr and Weisz, 1983):

$$K_{cr}^{\text{1-loop}} = 0.125 + 0.010\,178\,6\frac{N_c^2-1}{2N_c}g^2 . \tag{3.21}$$

This also shows that in this region important nonperturbative (or higher-order) effects are present.

Besides finite size effects, the other limiting feature of the existing calculations is the statistics. Most calculations use fewer than 20 propagators per K value. In Langguth and Montvay (1984) 80 meson propagators and 40 baryon propagators were collected, whereas König *et al.* (1984) had 72 propagators per K value. The experience with higher statistics shows that, for light quark

masses, the calculation of a few hundred propagators is probably not an exaggeration.

Comparing the numbers in Table II to experimental masses, it turns out that there is a rough agreement between the overall scale given by the masses and the scale (3.20) obtained from the string tension. The mass ratios, however, deviate from the right ones: the ratio m_p/m_ρ comes out around 1.8, and the Δ-p mass splitting has a tendency to be too small. [Note in this regard that Billoire *et al.* (1985) used nonrelativistic baryon operators instead of the standard relativistic ones of Eqs. (2.16) and (2.17). The standard operators would have given higher baryon masses.] The disagreement of mass ratios could come from the explicit breaking of chiral symmetry introduced by Wilson lattice fermions, but perhaps even more probably could be characteristic of the quenched approximation.

E. Kogut-Susskind fermions and variant actions

There exist also several quenched hadron spectrum calculations (Marinari *et al.*, 1981a; Hamber *et al.*, 1982; Hamber and Parisi, 1983; Billoire, Lacaze, *et al.*, 1984a, 1984b; Billoire, Marinari, *et al.*, 1984; Bowler, Chalmers, *et al.*, 1984; Gilchrist *et al.*, 1984a; Marinari *et al.*, 1984; Billoire *et al.*, 1985) with Kogut-Susskind lattice fermions (Banks *et al.*, 1977; Susskind, 1977; Kawamoto and Smit, 1981; Gliozzi, 1982; Kluberg-Stern *et al.*, 1983). In this formulation the problem of chiral symmetry and the associated proliferation of fermion degrees of freedom is treated differently. For zero bare quark mass there is an exact $U(1)_{\text{vector}} \otimes U(1)_{\text{axial}}$ symmetry of the lattice action, and the spontaneous breaking of the $U(1)_{\text{axial}}$ part implies the existence of a massless Goldstone boson in the strong coupling region (Blairon *et al.*, 1981; Kluberg-Stern *et al.*, 1981; Jolicoeur *et al.*, 1984). Flavor symmetry (like isospin, etc.) is, however, explicitly broken; therefore an important problem for numerical calculation is to study the masses of non-Goldstone pseudoscalar mesons. This was done for SU(2) gauge theory by Billoire, Lacaze, *et al.* (1984a, 1984b), and the result showed near $\beta=2.3$—2.4 strong evidence for various light flavored pseudoscalar mesons, in accordance with a Nambu-Goldstone realization of full chiral symmetry in the continuum.

The extraction of hadron masses from the hadronic two-point functions is a nontrivial task for Kogut-Susskind fermions, especially in the baryon sector. This is due to the mixing in flavor and spin-parity caused by the explicit symmetry-breaking terms. In the case of SU(3) color, up to now only the simplest (local) hadronic operators have been considered; therefore the interpretation of the numerical results in terms of the masses may have some systematic uncertainty. For a collection of some recent results see Table III. Comparing the values at $\beta=5.7$ (where finite size effects are presumably smaller) to those in Table II, we see reasonable agreement for the nucleon mass. In the case of the ρ mass, however, where the errors are small, there is a definite disagreement: the apparent lattice spacing seems to be about a factor 1.6 smaller for Wilson fermions. Correspondingly, there is no problem with the m_p/m_ρ ratio for Kogut-Susskind quarks. This is actually not a surprise, since this ratio is essentially correct already in strong coupling (Kluberg-Stern *et al.*, 1981; Jolicoeur *et al.*, 1984). In summary, the quenched hadron mass calculations with Kogut-Susskind fermions are promising, but the difficult problem of mixing (in flavor and in spin-parity) deserves further study, in particular in the case of nondegenerate flavor masses (Göckeler, 1984; Golterman and Smit, 1984a, 1984b).

In addition to changing the fermion part of the action, we can also change the gauge part, for instance, in order to improve the scaling properties in the intermediate coupling range. Some attempts in this direction have already been made (Bowler, Marinari, *et al.*, 1984; Itoh *et al.*, 1984; Marinari *et al.*, 1984), but within present precision there are no substantial deviations from the simple Wilson gauge action.

F. Other static hadron properties

Once the hadron mass calculation is under control, one can start to calculate a large number of different static hadronic matrix elements, which are of interest in strong and electroweak interactions (Bernard *et al.*, 1982; Fucito, Parisi, and Petrarca, 1982; Martinelli, Parisi, *et al.*, 1982; Ali and Montvay, 1983; Brower *et al.*, 1984; Cabibbo *et al.*, 1984; Gottlieb *et al.*, 1984; Velikson and Weingarten, 1985). Many of these matrix elements involve the electromagnetic or weak currents that are conserved in the continuum. Such conserved vector currents can be defined on the lattice, too (Karsten and Smit, 1981). For Wilson fermions the appropriate choice is (for $N_f=3$ degenerate quarks)

TABLE III. Some results of quenched hadron mass calculations with Kogut-Susskind fermions.

	$\beta=5.7$ (Gilchrist *et al.*, 1984a, 1984b) $10^3\times16$	(Bowler, Chalmers, *et al.*, 1984) $8^3\times16$ (copied)	$\beta=6.0$ (Billoire *et al.*, 1985) $10^3\times20$
$m_\rho a$	0.98±0.06	0.88±0.06	~0.37
$m_p a$	1.21±0.14	1.05±0.30	~0.5
$(m_\pi a)^2/(m_q a)$	7.6	7.0	~6.5

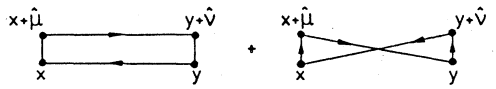

FIG. 5. The quark propagator configuration for a matrix element of flavor nonsinglet point-split current in Eq. (3.22).

$$V^s_{x,\mu}=Ka^{-3}\left[\tilde{\psi}_{x+\hat{\mu}}(r+\gamma_\mu)U(x,\mu)\frac{\lambda_s}{2}\psi_x - \tilde{\psi}_x(r-\gamma_\mu)U(x+\hat{\mu},-\mu)\frac{\lambda_s}{2}\psi_{x+\hat{\mu}}\right] . \tag{3.22}$$

The Gell-Mann matrices λ_s ($s=0,1,\ldots,8$) act here in flavor. It can be shown that, as a consequence of the equations of motion $\delta S_f/\delta\psi_x=\delta S_f/\delta\tilde{\psi}_x=0$, the current in Eq. (3.22) satisfies the "conservation equation"

$$\sum_{\mu>0}(V^s_{x,\mu}-V^s_{x-\hat{\mu},\mu})=0 . \tag{3.23}$$

The advantage of the conserved current in Eq. (3.22) over the "naive" local currents like const$\times\tilde{\psi}_x\gamma_\mu\psi_x$ is that, as a consequence of Eq. (3.23), the strength of $V^s_{x,\mu}$ is not renormalized. Therefore the matrix elements of the vector current in Eq. (3.22) can be directly compared to measurable quantities. In the case of other (for instance, axial-vector) currents, the only way to determine the multiplicative renormalization is, at present, one-loop perturbation theory (Martinelli and Zhang, 1983, 1983b; Meyer and Smith, 1983; Groot *et al.*, 1984; Martinelli, 1984), which is unreliable in the intermediate coupling constant range. [Examples of the failure of low-order perturbation theory are given by K_{cr}—see Eq. (3.21)—and by the "overshooting" of asymptotic scaling according to Eq. (3.20).]

The calculation of two-point current amplitudes from Eq. (3.22) requires the quark-propagator combinations depicted in Fig. 5. This is somewhat more difficult than the combination in Fig. 1 for the local currents, because the quark propagator has to be evaluated from two neighboring initial points. (For a first attempt see Ali and Montvay, 1983.)

Another way to calculate $q^2=0$ matrix elements (like nucleon magnetic moments; see Bernard *et al.*, 1982; Martinelli, Parisi, *et al.*, 1982) is to introduce an appropriate external classical field in the fermion part of the action and compare the results with and without such fields.

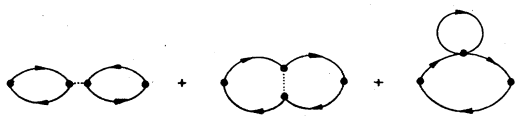

FIG. 6. Quark propagator configurations needed for nonleptonic decay matrix elements. The points connected by a dotted line are at the same site.

An important piece of information can be obtained by calculating matrix elements of the nonleptonic decay Hamiltonian (for K mesons, D mesons, F mesons, etc.). The required quark propagator diagrams (Brower *et al.*, 1984; Cabibbo *et al.*, 1984) are shown in Fig. 6. The last diagram in the figure is, unfortunately, as difficult as the second part of Fig. 2, but the evaluation of the first two quark propagator configurations already gives some useful information.

G. Outlook

In conclusion, the status of quenched lattice calculations can be considered as satisfactory. The quality of the present results corresponds reasonably to the invested effort. It is quite clear, however, that further improvement is both necessary and possible. Since the quenched calculation is, technically speaking, a part of the final task with dynamical quarks, high-standard quenched calculations are absolutely necessary. An example of a nice "two-star" quenched calculation of the hadron spectrum would be to take a $12^3\times24$ lattice at $\beta=5.7$, an $18^3\times36$ lattice at $\beta=5.8$, or a $24^3\times48$ lattice at $\beta=6.0$ with several thousand quark propagators per quark mass. The corresponding hopping expansion calculations would be 48th order on a 12^4 gauge field at $\beta=5.7$, etc. In this case the optimal iteration for the quark propagators is presumably on a periodic spatial box, with copied gauge field in the time direction. In such calculations finite size effects would probably be very small (in the range of a percent) and the statistics would be enough to have a good accuracy for light quark masses (perhaps 0.03—0.05 in lattice units).

IV. DYNAMICAL QUARKS

A. General formulas

As discussed at the beginning of the previous section, the quark part $S^q_{\rm eff}$ of the effective gauge field action in Eq. (2.12) describes the effect of closed virtual quark loops on the gauge field dynamics. The resulting interaction is inherently nonlocal, even if the original fermion action (before the integration over the anticommuting fermion variables) was local. This nonlocality is the reason why it is so difficult to include dynamical quarks in the numerical calculations.

In the updating procedure the change of the action is always needed for a given change of a link variable $U(x,\mu)$. From Eqs. (2.12) and (3.5) it follows that

$$\Delta S^q_{\rm eff}\equiv S^q_{\rm eff}(U')-S^q_{\rm eff}(U)=-\ln\det\frac{1-KM(U')}{1-KM(U)} . \tag{4.1}$$

Introducing the notations

$$D \equiv \frac{M(U')-M(U)}{1-KM(U)} = [1-KM(U)]^{-1}\Delta M \ ,$$
$$\Delta M_{x_2 x_1} = \sum_{x,\mu} (r+\gamma_\mu)\Delta U(x,\mu)\delta_{x_2,x+\hat{\mu}}\delta_{x,x_1} \ , \qquad (4.2)$$
$$\Delta U(x,\mu) \equiv U'(x,\mu) - U(x,\mu) \ ,$$

one obtains

$$\Delta S^q_{\rm eff} = -\ln\det(1-KD) = -\mathrm{Tr}\ln(1-KD)$$
$$= \sum_{j=1}^{\infty} \frac{K^j}{j}\mathrm{Tr}(D^j) \ . \qquad (4.3)$$

These formulas are actually valid for a single quark flavor with hopping parameter K. In the many-flavor case the Wilson fermion action is block diagonal in flavor, so $\Delta S^q_{\rm eff}$ is a sum over flavors with different hopping parameters $K \to K_q$ ($q = u,d,s,\ldots$).

In expectation values as Eqs. (2.11) and (2.13), the quark determinant $\det(1-KM)$ can, in principle, also be considered as a part of the gauge-field-dependent quantity to be evaluated with the pure gauge statistical distribution $dU \exp[-S_g(U)]$. Denoting such pure gauge field averages by $\langle\ \rangle_0$, we can write the expectation value $\langle F\rangle$ in Eq. (2.11), for instance, as

$$\langle F\rangle = \frac{\langle F e^{-S^q_{\rm eff}}\rangle_0}{\langle e^{-S^q_{\rm eff}}\rangle_0} \ . \qquad (4.4)$$

According to this formula one has to perform the Monte Carlo updating with the simple gauge field action $S_g(U)$, and then calculate $S^q_{\rm eff}$ on the given gauge configurations. We shall see below that such a procedure is impractical for small quark masses, but for theoretical understanding it could still be useful. In particular, if the quark part of the effective action $S^q_{\rm eff}$ can be considered small, one has the expansion (Joos and Montvay, 1983)

$$\langle F\rangle = \langle F\rangle_0 - [\langle S^q_{\rm eff}F\rangle_0 - \langle S^q_{\rm eff}\rangle_0\langle F\rangle_0] + \tfrac{1}{2}\langle {S^q_{\rm eff}}^2 F\rangle_0 - \langle {S^q_{\rm eff}}^2\rangle_0\langle F\rangle_0$$
$$-2[\langle S^q_{\rm eff}\rangle_0\langle S^q_{\rm eff}F\rangle_0 + 2\langle S^q_{\rm eff}\rangle_0^2\langle F\rangle_0] \pm \cdots \ . \qquad (4.5)$$

Here, on the right-hand side, only the fluctuations of $S^q_{\rm eff}$ matter; that is, with

$$\delta F \equiv F - \langle F\rangle_0, \quad \delta S^q_{\rm eff} \equiv S^q_{\rm eff} - \langle S^q_{\rm eff}\rangle_0 \ , \qquad (4.6)$$

one has

$$\langle F\rangle = \langle F\rangle_0 - \langle \delta S^q_{\rm eff}\delta F\rangle_0 + \tfrac{1}{2}\langle(\delta S^q_{\rm eff})^2\delta F\rangle_0 \pm \cdots \ . \qquad (4.7)$$

This shows how the nonlocality of $S^q_{\rm eff}$ reflects the physics; although $S^q_{\rm eff}$ extends over the whole lattice, in expectation values only its correlated fluctuations matter; therefore the nonlocality of $S^q_{\rm eff}$ is practically restricted to regions in which correlations are actually produced by virtual quark propagation. In Eq. (4.7) $\langle F\rangle_0$ corresponds to the quenched approximation, and the further terms on the right-hand side represent the corrections to it. The difficulty in the application of Eq. (4.4) or (4.7) to the calculation of unquenched averages lies in the fact that, in a gauge configuration ensemble produced by the pure gauge action, the fluctuations of $S^q_{\rm eff}$ are very large for light quarks. As an example, this is shown in Fig. 7 for the SU(2) gauge group on a 10^4 lattice with $N_f = 1$ flavors. Therefore, Eqs. (4.4) and (4.7) can be applied in practice only for heavy quarks. In the above example the meson masses could be determined only for dynamical quark masses $am_q \equiv (2K_q)^{-1} - (2K_{\rm cr})^{-1} \geq 0.2$ (roughly 200 MeV in physical units; Montvay, 1983).

Another way to represent the difficulty of numerical calculations with dynamical fermions is to recall the expectation that the bulk part of the quark determinant is needed just to produce the required renormalization of bare parameters. This means that a relatively small change in the physics (i.e., in mass ratios, etc.) is accompanied by a rather inconvenient shift in the scale. The shift in bare parameters can be seen already in the lowest-order approximations to $S^q_{\rm eff}$. According to Eq. (3.17) we have

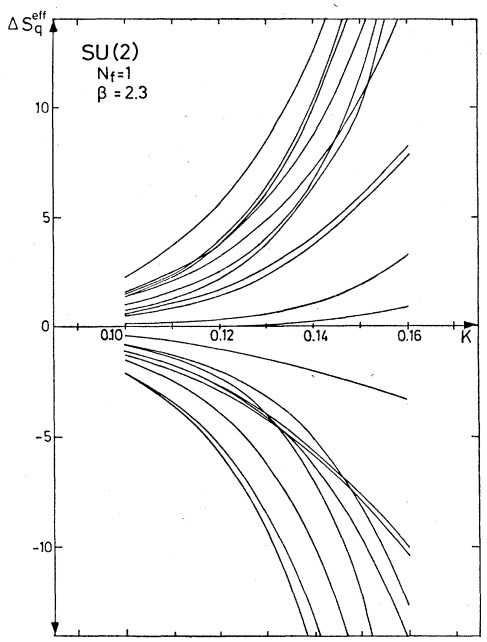

FIG. 7. The dependence of the fermion part of effective action $S^q_{\rm eff}$ (the average subtracted: $\delta S^q_{\rm eff} \equiv S^q_{\rm eff} - \langle S^q_{\rm eff}\rangle_0$) on the hopping parameter K, for 20 different gauge configurations (Joos and Montvay, 1983). The color group is SU(2) at $\beta = 2.3$ and $N_f = 1$ flavor is taken.

$$S^q_{\rm eff}=\sum_{j=1}^{\infty}\frac{K^j}{j}{\rm Tr}(M^j)\equiv\sum_{j=1}^{\infty}S^{q(j)}_{\rm eff}\ . \tag{4.8}$$

Let us now consider only the Wilson parameter value $r=1$. In this case Eq. (3.18) implies that the first nonvanishing term is $S^{q(4)}_{\rm eff}$ (and only even values of the index j contribute). It can be easily shown that the first two nonvanishing terms look like

$$\begin{aligned}
S^{q(4)}_{\rm eff}&=-16K^4N_f\sum_{\square}{\rm Re\,Tr}U=-4K^4N_f\sum_x\sum_{\mu>0}\sum_{\nu\neq\pm\mu}{\rm Re\,Tr}U_{(\mu\nu)}\ ,\\
S^{q(6)}_{\rm eff}&=\frac{K^6}{3}N_f\sum_x\sum_{\mu>0}\sum_{\nu_1\cdots\nu_5}\delta_{-\hat\mu,\hat\nu_1+\cdots+\hat\nu_5}{\rm Re\,Tr}U_{(\mu\nu_1\cdots\nu_5)}T_{(\mu\nu_1\cdots\nu_5)}\ ,\\
S^q_{\rm eff}&=S^{q(4)}_{\rm eff}+S^{q(6)}_{\rm eff}+\cdots\ .
\end{aligned} \tag{4.9}$$

Here N_f degenerate flavors are taken and $\square$ is a positively oriented plaquette, also denoted in the second form of $S^{q(4)}_{\rm eff}$ by $(\mu\nu)$. In the 6th-order term the factor $T_{(\mu\nu_1\cdots\nu_5)}$ is the Dirac trace given by

$$T_{(\mu\nu_1\cdots\nu_5)}=\begin{cases}-32 & \text{for } (\mu\nu_1\cdots\nu_5)=\\ -16 & \text{for } (\mu\nu_1\cdots\nu_5)=\\ -16 & \text{for } (\mu\nu_1\cdots\nu_5)=\end{cases} \tag{4.10}$$

As is shown by Eq. (4.9), the 4th-order term corresponds to a shift

$$\Delta\beta=16N_cN_fK^4 \tag{4.11}$$

in the coefficient $\beta\equiv 2N_cg^{-2}$ of the one-plaquette gauge action. Since $\Delta\beta$ is positive, the lattice spacing is decreased by $S^{q(4)}_{\rm eff}$ (and also by the whole $S^q_{\rm eff}$). The lowest-order terms in Eq. (4.9) show how, by the application of Eq. (3.18), the quark part of the effective action can be decomposed into a sum over closed Wilson loops multiplied by some Dirac trace and combinatoric factors. Since the mean values of more complicated Wilson loops are correlated to the single-plaquette expectation value, it is not very surprising that the mean value of $S^q_{\rm eff}$ can be approximated quite well by (Joos and Montvay, 1983)

$$S^q_{\rm eff}\simeq\frac{1}{W_{11}}S^{(0)}_{\rm eff}(K\sqrt{W_{11}})\ . \tag{4.12}$$

Here W_{11} is the single-plaquette expectation value and $S^{(0)}_{\rm eff}(K)$ is the free-quark effective action [see Eq. (2.8) for the derivation]

$$S^{(0)}_{\rm eff}(K)=-2N_cN_f\sum_k\ln\left[\left[1-2Kr\sum_{\mu>0}\cos k_\mu\right]^2+4K^2\sum_{\mu>0}\sin^2k_\mu\right]\ . \tag{4.13}$$

The approximation formula (4.12) works well only for the average; the fluctuations of $S^q_{\rm eff}$ are, unfortunately, not properly reproduced.

B. Pseudofermion method

For gauge field updating with dynamical fermions, according to Eqs. (4.1)—(4.3), matrix elements of the quark propagator $Q^{-1}\equiv(1-KM)^{-1}$ have to be calculated. In the "pseudofermion method" of Fucito, Marinari, Parisi, and Rebbi (1981) this is done by introducing a complex scalar "pseudofermion" field φ_x, having the same number of components as the anticommuting quark field ψ_x. The quark propagator matrix elements are obtained by running a separate Monte Carlo calculation for the pseudofermion field over a fixed gauge field configuration. The action of the pseudofermions is given by the matrix

$$\Omega\equiv Q^+Q=1-KM-KM^++K^2M^+M\ . \tag{4.14}$$

This is positive definite, as required for a Monte Carlo calculation, and, due to Eq. (2.4), its determinant is related to the quark determinant by

$$\det Q=\sqrt{\det\Omega}\ . \tag{4.15}$$

The matrix elements of the quark propagator can be obtained as

$$Q^{-1}_{ij}=\frac{\int d\varphi^+d\varphi\,\varphi^+_i(Q\varphi)_j\exp(-\varphi^+_k\Omega_{kl}\varphi_l)}{\int d\varphi^+d\varphi\exp(-\varphi^+_k\Omega_{kl}\varphi_l)}\ . \tag{4.16}$$

Since the quark matrix in Eq. (2.3) has only nearest-neighbor matrix elements, the pseudofermion action Ω extends up to next nearest neighbors. In the pseudofermion Monte Carlo it is convenient to introduce the auxiliary field (Hamber *et al.*, 1983a)

$$\chi_i=Q_{ij}\varphi_j\ . \tag{4.17}$$

With χ Eq. (4.16) can be written as

$$Q^{-1}_{ij}=\frac{\int d\varphi^+d\varphi\exp(-\chi^+_k\chi_k)\varphi^+_i\chi_j}{\int d\varphi^+d\varphi\exp(-\chi^+_k\chi_k)}\ . \tag{4.18}$$

In principle, the pseudofermion Monte Carlo has to be run after every change of a single link. This would cost, however, an enormous amount of time; therefore the matrix elements Q^{-1}_{ij} are usually kept for a full sweep over

the gauge variables and then evaluated again. This implies some violation of the detailed balance condition for the Markov process of Monte Carlo integration. Another approximation usually introduced in the pseudofermion method is, in the expansion (4.3), to take only the lowest-order term with $j=1$. Since D is proportional to the change ΔU of the link variable, this is a good approximation for small changes $\Delta U \to 0$.

The pseudofermion method was tested first in the two-dimensional Schwinger model (Marinari *et al.*, 1981b; Burkitt *et al.*, 1983; Otto and Randeira, 1983). First studies in QCD were carried out on small (2^4 and 4^4) lattices (Azcoiti and Nakamura, 1983; Bhanot *et al.*, 1983; Otto, 1984). Some results for the plaquette expectation value and $\langle\bar{\psi}\psi\rangle$ were obtained on an 8^4 lattice by Hamber *et al.* (1983b) with Kogut-Susskind fermions. We have seen in the previous section that, at least in the quenched approximation, good results for the hadron spectrum can be achieved on an 8^4 lattice copied at least twice in the time direction for the quark propagator calculation. Such a lattice size is not yet achieved in pseudofermion calculations of the hadron spectrum, but some studies on 4^4 (Azcoiti *et al.*, 1985) or four-times copied 4^4 (Hamber, 1985) lattices have been performed with $N_f=3$ light Wilson quarks.

C. Iterative hopping expansion method

The matrix elements of the quark propagator required for gauge field updating with dynamical fermions can also be directly evaluated in a hopping parameter expansion. The change in the quark part of the effective action is given in Eq. (4.3) by the matrix D, which has the detailed structure

$$
\begin{aligned}
D &= \begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix} , \\
D_{11} &= (1-KM)^{-1}_{x,x+\hat{\mu}}(r+\gamma_\mu)\Delta U(x,\mu) , \\
D_{12} &= (1-KM)^{-1}_{x,x}(r-\gamma_\mu)\Delta U(x,\mu)^+ , \\
D_{21} &= (1-KM)^{-1}_{x+\hat{\mu},x+\hat{\mu}}(r+\gamma_\mu)\Delta U(x,\mu) , \\
D_{22} &= (1-KM)^{-1}_{x+\hat{\mu},x}(r-\gamma_\mu)\Delta U(x,\mu)^+ .
\end{aligned}
\tag{4.19}
$$

In what follows only the case $r=1$ will be considered. In this case the nonzero contributions in the hopping expansion look like

$$
\begin{aligned}
D_{11} &= \sum_{l=3,5,\ldots} K^l M(U)^l_{x,x+\hat{\mu}}(1+\gamma_\mu)\Delta U(x,\mu) , \\
D_{12} &= \sum_{l=4,6,\ldots} K^l M(U)^l_{x,x}(1-\gamma_\mu)\Delta U(x,\mu)^+ , \\
D_{21} &= \sum_{l=4,6,\ldots} K^l M(U)^l_{x+\hat{\mu},x+\hat{\mu}}(1+\gamma_\mu)\Delta U(x,\mu) , \\
D_{22} &= \sum_{l=3,5,\ldots} K^l M(U)^l_{x+\hat{\mu},x}(1-\gamma_\mu)\Delta U(x,\mu)^+ .
\end{aligned}
\tag{4.20}
$$

The "periodic box iteration" of the hopping parameter series (Hasenfratz and Montvay, 1983, 1984) was adopted for unquenched updating by Montvay (1984). To speed up the code for the evaluation of the required matrix elements $M(U)^l$ in Eq. (4.20), a useful observation is that it is enough to compute only for half of the initial spin index values. In the Dirac matrix representation given by Eq. (2.18) this is trivial on links in the direction $\mu=4$, since $(1\pm\gamma_4)$ is nonzero for only half of the index values. For the other directions one can use, for instance, $k=1,2,3$,

$$
(1+\gamma_k)\frac{1-\gamma_4}{2} = (1+\gamma_k)\frac{1+\gamma_4}{2}\gamma_k\frac{1-\gamma_4}{2} . \tag{4.21}
$$

This shows how the two lower components of, say, $M(U)_{x,x+\hat{k}}(1+\gamma_k)$ can be expressed by the upper two.

The average relative weight of the different orders of the hopping parameter series in Eq. (4.20) is shown in Table IV for some representative cases. It can be seen that the hopping expansion converges, on the average, reasonably well within 16th order. At 16th-order calculation would, however, still take too much time. Calculations have actually been performed (Langguth and Montvay, 1984; Montvay, 1984)—on every link—to 8th or 12th order. This still takes a lot of time: one sweep on the 8^4 lattice (Langguth and Montvay, 1984) took $\sim$40 min in 8th order and $\sim$240 min in 12th order on the CYBER 205 at Karlsruhe University. It is very important to keep in mind that it is possible to correct, at least in the average, for the omitted higher orders. This is due to the fact that the higher-order coefficients are strongly correlated to the lower ones. Such a correlation is already suggested by the approximate validity of Eq. (4.12). Using the correlation allows one to estimate the result of the $l_{\max}=16$ order series from some lower-order (e.g., $l_{\max}=8$ or $l_{\max}=12$) calculation. For instance, in the 6^4 calculation of Montvay (1984), the $l_{\max}=16$ result could be obtained in the average by multiplying the $l_{\max}=8$ number by a factor $\lambda\cong 1.14$. The same factor needed from $l_{\max}=12$ to $l_{\max}=16$ was $\lambda\cong 1.03$. This achieved a substantial gain in computer time but, of course, increased the error for the quark determinant. Monitoring the difference from time to time on a few hundred links revealed that the estimate based on the extrapolation from $l_{\max}=8$ deviated on the average from the exact $l_{\max}=16$ value by $\sim$16%. The corresponding average deviation for $l_{\max}=12$ extrapolated to $l_{\max}=16$ was 5%. The error in the determinant ratio caused by this extrapolation was far from being normally distributed. In most cases the deviation was much less than the average, but sometimes (in a few percent of cases) errors on the order of 100% also occurred. It seems plausible that the effect of the few cases in which the error due to the extrapolation from the lower order to $l_{\max}=16$ is large averages out and does not influence the updating process in the long run. It is also possible to improve the extrapolation to higher orders by a more elaborate use of the covariance matrix between individual lower- and higher-order expansion coefficients.

TABLE IV. The average relative magnitude (in percent) of different orders of hopping expansion in $\Delta S^q_{\rm eff}$. The numbers in the table are obtained from the ratio of the absolute value of a given order divided by the sum of the absolute values up to 16th order. The first row was obtained on a 6^4 lattice (Montvay, 1984), the last two rows on an 8^4 lattice (Langguth and Montvay, 1984). In both cases 10 hits per link were done in the Metropolis updating and $N_f=3$ degenerate quark flavors were considered.

	Order	4	6	8	10	12	14	16
	$\beta=5.7$, $K=0.15$	44.0	27.9	15.1	7.4	3.3	1.5	0.8
A:	$\beta=5.4$, $K=0.163$	35.2	27.9	17.7	9.6	5.8	2.6	1.2
B:	$\beta=5.3$, $K=0.168$	24.2	23.3	17.0	13.7	10.1	7.7	4.0

First results on the hadron spectrum using the hopping expansion method in updating with light dynamical quarks were obtained by Langguth and Montvay (1984) on an 8^4 lattice. $N_f=3$ degenerate quark flavors were considered at two points in the (β,μ_q) plane $[\mu_q\equiv(2K_q)^{-1}]$:

$$\begin{aligned}&\text{point } A\text{: } \beta=5.4,\ \mu_q=3.0675\ldots\ (K_q=0.163)\ ,\\ &\text{point } B\text{: } \beta=5.3,\ \mu_q=2.9762\ldots\ (K_q=0.168)\ .\end{aligned}\tag{4.22}$$

For comparison, an 8^4 quenched calculation was performed, too, at $\beta=5.7$. Some planar Wilson-loop expectation values are given in Table V for these three cases. Planar and off-axis elongated Wilson-loop expectation values were also measured in order to determine the static energy E of an external SU(3) color charge pair by

$$aE(R)=-\lim_{T\to\infty}\frac{1}{T}\ln W(R,T)\ .\tag{4.23}$$

Here $W(R,T)$ stands for a Wilson loop with length T in the time direction and Euclidean distance R between the end points in fixed time slices. On the 8^4 lattice, T is, of course, restricted to $T\le 4$ by the periodic boundary conditions, and R has possible values $R=1$, $\sqrt{2}$, $\sqrt{3}$, 2, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{8}$, 3, and $\sqrt{10}$ (larger values of R were not considered because of statistics limitations). The obtained static energies are shown in Figs. 8(a)–8(c). The expected screening due to the virtual quark pairs (Joos and Montvay, 1983) cannot be seen. The static energies with dynamical quarks are, in fact, remarkably similar to the quark-antiquark potential in Fig. 8(c). Very probably, the distance between the external color charges is not large enough ($R=\sqrt{10}$ corresponds roughly to ~ 0.5 fm, as we shall see below).

The π, ρ, p, and Δ masses were determined on the 8^4 configurations by 32nd-order (for the baryons 33rd-order) "copied gauge field" iteration in the hopping parameter. (See Sec. III.) The results are shown in Figs. 9(a)–9(c) as a function of the quark mass $\mu\equiv(2K)^{-1}$ in the quark propagator. The quenched calculation [Fig. 9(c)] has already been discussed in Sec. III.D (see Table II). In the physical points, where the quark mass in the determinant is equal to the quark mass in the propagator ($\mu_q=\mu$), the unquenched result is

$$\begin{aligned}&\text{point } A\text{: } am_\pi=0.79\pm0.01,\ am_\rho=0.95\pm0.01\ ,\\ &\qquad am_p=1.62\pm0.02,\ am_\Delta=1.74\pm0.02\ ;\\ &\text{point } B\text{: } am_\pi=0.3^{+0.1}_{-0.2},\ am_\rho=0.62\pm0.05\ ,\\ &\qquad am_p=0.85\pm0.15,\ am_\Delta=1.09\pm0.15=am_p+(0.24\pm0.09)\ .\end{aligned}\tag{4.24}$$

TABLE V. Wilson-loop expectation values $W_{ij}=\frac{1}{3}\,\mathrm{Tr}C_{ij}$ in points A and B [see Eq. (4.22) for parameters]. The numbers in parentheses are the estimated errors in last numerals. In the last line the Wilson-loop expectation values on the configurations used for the quenched calculation at $\beta=5.7$ are given.

	W_{11}	W_{12}	W_{13}	W_{22}	W_{23}	W_{33}
A	0.5298(9)	0.2996(12)	0.1719(11)	0.1099(10)	0.0428(8)	0.0128(7)
B	0.5428(10)	0.3205(12)	0.1912(13)	0.1295(8)	0.0546(9)	0.0175(9)
$N_f=0$, $\beta=5.7$	0.5468(10)	0.3218(11)	0.1922(11)	0.1298(8)	0.0557(7)	0.0186(7)

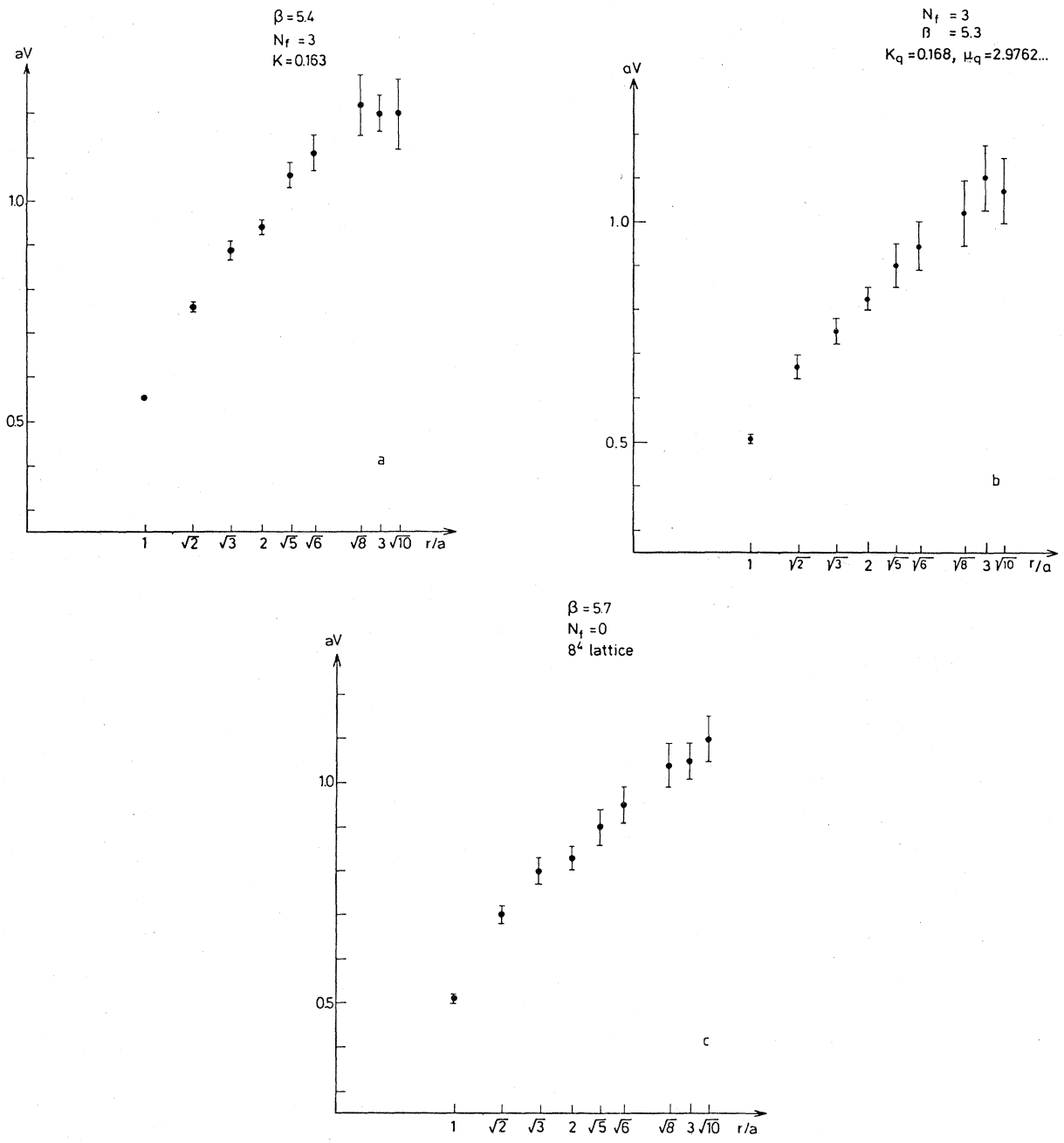

FIG. 8. The static energy of an external quark-antiquark pair as a function of the lattice distance: (a) $\beta=5.4$, $K_q=0.163$ with $N_f=3$ degenerate flavors (point A in Langguth and Montvay, 1984). (b) $\beta=5.3$, $K_q=0.168$ (point B in Langguth and Montvay, 1984). (c) $\beta=5.7$ in the pure gluon theory. The gauge configurations are those in the quenched calculation of Langguth and Montvay (1984).

It can be seen that point B is quite close to the critical line $\mu_{\rm cr}(\beta)$, where the pion mass (as well as the quark mass) vanishes (see Fig. 10). This is the reason for the deterioration of convergence for $\Delta S^q_{\rm eff}$, as shown by the last line of Table IV, and it implies an unknown systematic error in point B. Taking $(\mu_q-\mu_{\rm cr})\simeq 0.06$ in point A and $(\mu_q-\mu_{\rm cr})\simeq 0.01$ in point B as an estimate of quark mass in lattice units, and using Eqs. (5.15)—(5.17), one obtains for the renormalization-group invariant quark mass M_q and lattice spacing a

$$\begin{aligned} &\text{point } A\text{: } M_q\simeq 170 \text{ MeV}, \quad a\simeq 0.87 \text{ GeV}^{-1}\,, \\ &\text{point } B\text{: } M_q\simeq 30 \text{ MeV}, \quad a\simeq 0.76 \text{ GeV}^{-1}\,. \end{aligned} \tag{4.25}$$

[These numbers are different in Langguth and Montvay (1984), because there Eq. (5.16) with $c=1$ was taken. The

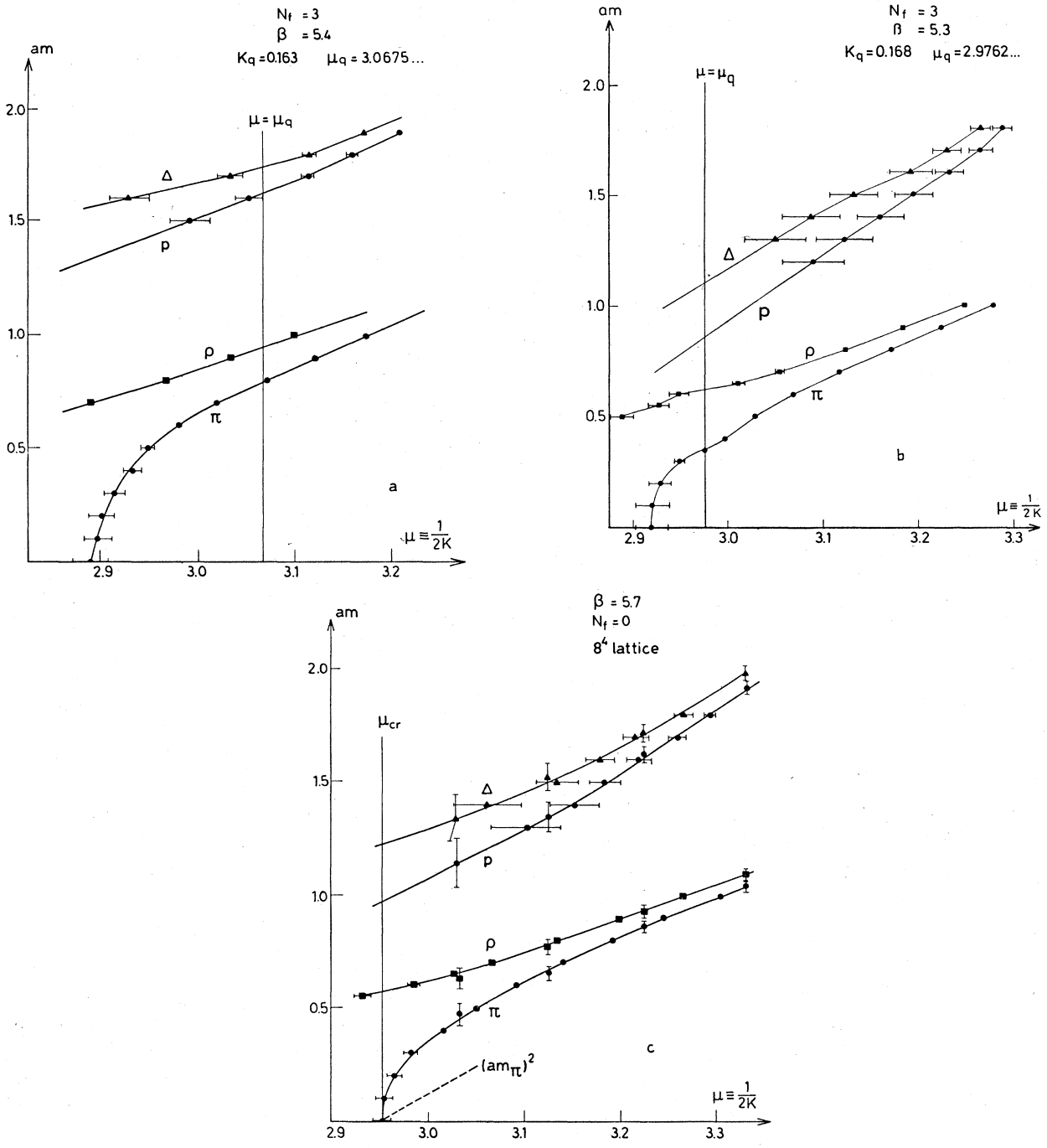

FIG. 9. Hadron masses as a function of quark-mass parameter in the quark propagator $\mu=(2K)^{-1}$: (a) $\beta=5.4$, $K_q=0.163$ with $N_f=3$ degenerate flavors (point A in Langguth and Montvay, 1984); (b) $\beta=5.3$, $K_q=0.168$ (point B in Langguth and Montvay, 1984); (c) quenched approximation for $\beta=5.7$ (Langguth and Montvay, 1984).

estimate for c in Eq. (5.17) is probably closer to reality.] Assuming the validity of asymptotic scaling (with zero quark mass) in point B, the obtained value of the Λ parameter is $\Lambda_{\rm lat}(N_f-3)\simeq 1.7$ MeV. This corresponds (Kawai *et al.*, 1981; Weisz, 1981) to $\Lambda_{\alpha=1}^{mom}\simeq 180$ MeV. Of course, the question of asymptotic scaling (or scaling in general) cannot be decided on the basis of only two points in the (β,μ_q) plane. Some evidence that both points A and B are within the scaling region comes, however, from the fact that rotation symmetry is well satisfied for the static energies shown by Figs. 8(a) and 8(b).

A direct comparison of the results in Eq. (4.24) with

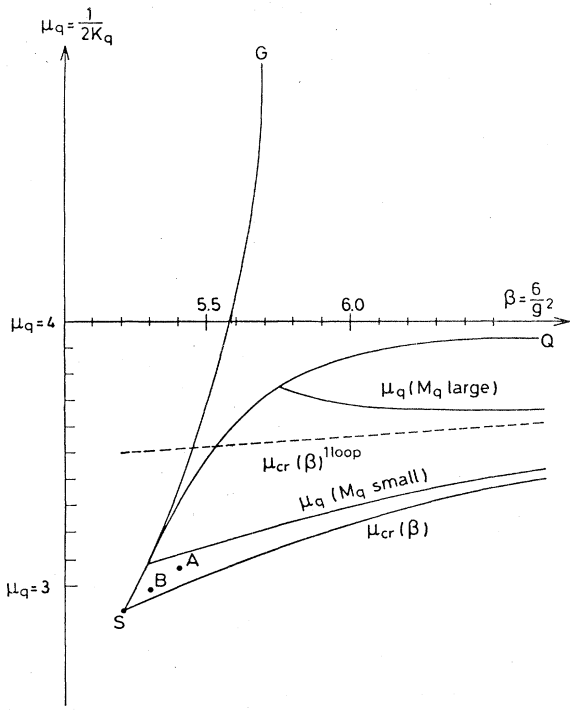

FIG. 10. Tentative shape of the scaling region in the (β,μ_q) plane with $N_f=3$ degenerate Wilson flavors. Scaling could be valid for purely gluonic quantities to the right of the line (SG), for all quantities to the right of the line (SQ). The critical line with zero quark mass is $\mu_{cr}(\beta)$, its perturbative one-loop approximation is the dashed line. The curves μ_q belong to constant RGI quark masses ("constant physics"). The position of the two points A and B measured in Langguth and Montvay (1984) is also shown.

the quenched masses in Table II is difficult because of the different quark mass dependences and because of the shift in scale. In spite of this, one can see already in point A, with quark mass $M_q\simeq 170$ MeV, that the p/ρ mass ratio is decreased if one compares similar values of $(\mu-\mu_{cr})$ in Figs. 9(a) and 9(c). The p/ρ ratio in point B with $M_q\simeq 30$ MeV is $m_p/m_\rho\simeq 1.35$, considerably lower than in the quenched case. The error is, however, still somewhat large to draw a definite conclusion.

V. SCALING WITH DYNAMICAL FERMIONS

Detailed study of scaling properties in pure gauge theory has shown [for SU(2) see Gutbrod and Montvay, 1984; Mackenzie, 1984; Patel *et al.*, 1984; for SU(3) see Gupta *et al.*, 1984; Hasenfratz *et al.*, 1984a; Bowler *et al.*, 1985] that asymptotic scaling (corresponding to the two-loop perturbative β function) is not yet reached in the intermediate coupling range, where most of the Monte Carlo calculations of physical quantities are performed. There is, however, an important region where dimensionless ratios of physical quantities are, to a good approximation, independent from the bare coupling. In this "scaling region," physics is dictated by the continuum theory, and the change of the lattice scale is given by some general nonperturbative β function. In order to perform an optimal numerical calculation, a precise knowledge of the scaling region and of the β function is very important.

In the case of a lattice gauge theory with dynamical fermions, a similar situation is to be expected. The additional complication in this case is, however, that the scaling properties depend on two parameters: not only on the gauge coupling, but also on the dynamical quark mass. Before discussing the quark mass dependence of the lattice renormalization scheme, let us first briefly summarize some facts about the renormalization-group equation (RGE) with quarks.

In a mass-independent renormalization scheme (for a review, see Peterman, 1979) the RGE for a physical quantity $P(\mu,g,m)$ depending on the renormalization-point mass parameter μ, the renormalized coupling g, and renormalized-quark mass m, is

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}-\gamma(g)m\frac{\partial}{\partial m}\right]P=0\ . \tag{5.1}$$

Here and in what follows it will be assumed, for simplicity, that the different quark flavors are degenerate in mass. The expansion of the renormalization-group functions is

$$\begin{aligned}\beta(g)&=-\beta_0 g^3-\beta_1 g^5-\beta_2 g^7-\cdots\ ,\\ \gamma(g)&=\gamma_0 g^2+\gamma_1 g^4+\cdots\ .\end{aligned} \tag{5.2}$$

In QCD with SU(N_c) color and N_f flavors we have

$$\begin{aligned}\beta_0&=\frac{1}{(4\pi)^2}\left[\frac{11N_c}{3}-\frac{2N_f}{3}\right],\\ \beta_1&=\frac{1}{(4\pi)^4}\left[\frac{34N_c^2}{3}-N_f\left[\frac{13N_c}{3}-\frac{1}{N_c}\right]\right],\\ \gamma_0&=\frac{1}{(4\pi)^2}\frac{3(N_c^2-1)}{N_c}\ .\end{aligned} \tag{5.3}$$

These are the universal expansion coefficients. All other coefficients depend on the renormalization scheme (lattice action, etc.).

The two standard solutions of the RGE (the so-called standard "renormalization-group invariants") are the Λ parameter and the renormalization-group invariant (RGI) quark mass M:

$$\begin{aligned}\Lambda&=\mu(\beta_0 g^2)^{-\beta_1/2\beta_0^2}e^{-1/2\beta_0 g^2}\exp\left[-\int_0^g dx\left[\frac{1}{\beta(x)}+\frac{1}{\beta_0 x^3}-\frac{\beta_1}{\beta_0^2 x}\right]\right],\\ M&=m(2\beta_0 g^2)^{-\gamma_0/2\beta_0}\exp\left[\int_0^g dx\left[\frac{\gamma(x)}{\beta(x)}+\frac{\gamma_0}{\beta_0 x}\right]\right].\end{aligned} \tag{5.4}$$

(The normalization of M here follows that of Gasser and Leutwyler, 1982.) Every physical quantity is a function of Λ and M only; therefore the "curves of constant physics" in the (μ,g,m) space are parametrized by $M,\Lambda=\text{const}$. Using the freedom of finite renormalizations, it is possible to introduce new renormalized parameters by

$$g'=gZ_1(g,\lambda),\quad \mu'\equiv\mu,\quad m'=mZ_2(g,\lambda)\ , \tag{5.5}$$

where $\lambda=M/\Lambda$ is a dimensionless measure of the RGI quark mass, and the functions Z_j $(j=1,2)$ are assumed to have the small-g expansion

$$Z_j(g,\lambda)=1+a_j(\lambda)g^2+\cdots\ . \tag{5.6}$$

Introducing the dimensionless variable $l=m'/\mu$ for the renormalized-quark mass, one obtains from Eq. (5.1) the new RGE

$$\left[\mu\frac{\partial}{\partial\mu}+\beta_g(g',l)\frac{\partial}{\partial g}+\beta_l(g',l)\frac{\partial}{\partial l}\right]P=0\ . \tag{5.7}$$

The new renormalization-group functions are related to the old ones by

$$\beta_g(g',l)=\left[\beta(g)\left[Z_1+g\frac{\partial Z_1}{\partial g}\right]\right]_{g=g(g',\lambda)}\ ,$$

$$\beta_l(g',l)=-l\left[1+\gamma(g)-\beta(g)\frac{\partial\ln Z_2}{\partial g}\right]_{g=g(g',\lambda)}\ , \tag{5.8}$$

$$\lambda\equiv\frac{M}{\lambda}\equiv\frac{m}{\mu}\varphi(g)=l\varphi(g)\ .$$

It can also be shown that the new and old RGI's are related by

$$\Lambda'=\Lambda\exp\left[\frac{a_1(\lambda)}{\beta_0}\right],\quad M'=M\ . \tag{5.9}$$

Using the lattice as a perturbative renormalization scheme, the simplest convention is to keep the mass-independent scheme corresponding to Eq. (5.1). This was, in fact, done in previous work on the RGI quark mass (Gonzalez-Arroyo *et al.*, 1982; Hamber and Wu, 1983; Göckeler, 1984; Golterman and Smit, 1984a, 1984b). In the nonperturbative region, however, where the numerical calculations are done, it is more convenient to define the renormalization scheme by the hadron masses. In the case of Wilson fermions one has to find first the critical line $\mu_{\rm cr}(\beta)$ in the (β,μ_q) plane [with $\beta\equiv 2N_c g^{-2}$ for the gauge coupling, as usual, and $\mu_q\equiv(2K_q)^{-1}$, where K_q is the hopping parameter of the dynamical quarks]. In perturbation theory this is equivalent to canceling the linear divergences in the quark self-energy. In general, $\mu_{\rm cr}(\beta)$ is the line where the lowest 0^- mass and the (bare, renormalized, and RGI) quark mass vanish. Having the value of the critical hopping parameter $K_{\rm cr}=(2\mu_{\rm cr})^{-1}$, one can define the bare quark mass parameter m_q (in lattice units) by

$$am_q\equiv\frac{1}{2K_q}-\frac{1}{2K_{\rm cr}}=\mu_q-\mu_{\rm cr}\ . \tag{5.10}$$

In the case of Kogut-Susskind fermions the critical (zero) quark mass is not renormalized, so this step is not necessary.

The next step, for any fermion formulation, is to find the lines where the hadron mass ratios are constant. These are the "lines of constant physics" where the RGI quark mass is constant. Of course, scaling for finite lattice spacing is always somewhat broken by lattice artifacts; therefore "constancy" always means within given errors and within a given class of hadron masses. To fix the renormalization scheme completely, it is reasonable to assume that the value of hadron masses (or equivalently, of a singled-out hadron mass) is constant along the "lines of constant physics." Furthermore, if one is free to define the value of, say the proton mass for quark mass values different from the physical case, one can arrange that the value of the Λ parameter be independent from the RGI quark mass. [See Eq. (5.9).] By this approach, the value of the lattice spacing a is fixed everywhere in the "scaling region" where the "lines of constant physics" can be defined at all. Since there seems to be no reason why this convention should define a quark-mass-independent scheme, the RGE on the lattice has a quark-mass-dependent form corresponding to Eq. (5.7):

$$\left[-a\frac{\partial}{\partial a}+\beta_g(g,\mu_q)\frac{\partial}{\partial g}+\beta_\mu(g,\mu_q)\frac{\partial}{\partial\mu_q}\right]P=O(a)\ . \tag{5.11}$$

Here the right-hand side represents the scale-breaking lattice artifacts, which in the continuum limit $a\to 0$ tend to zero at least as fast as (some power of) the lattice spacing.

The "lines of constant physics" $\mu_q=\mu_q(g)$ are determined by the differential equation

$$\frac{d\mu_q(g)}{dg}=\frac{\beta_\mu(g,\mu_q)}{\beta_g(g,\mu_q)}\ . \tag{5.12}$$

The different values of the RGI quark mass belong to solutions with different initial conditions. Defining the single variable β function for a given RGI quark mass by

$$\beta_q(g)\equiv\beta_g(g,\mu_q(g))\ , \tag{5.13}$$

we obtain the single variable RGE for this quark mass,

$$\left[-a\frac{\partial}{\partial a}+\beta_q(g)\frac{\partial}{\partial g}\right]P=O(a)\ . \tag{5.14}$$

The quark mass dependence of the β function $\beta_q(g)$ is assumed to appear only in the higher-order nonuniversal expansion coefficients $\beta_2,\gamma_1,\ldots$ [see Eq. (5.2)]. Hence for $g\to 0$ $(\beta\to\infty)$ the quark mass dependence disappears and the renormalization scheme becomes indistinguishable from the mass-independent scheme of lattice perturbation theory. This (perturbative) mass-independent regime is, however, presumably very difficult to reach by numerical hadron mass calculations.

Since the perturbative regime is presently out of range, in the intermediate coupling some pragmatic definition of the quark mass can be very useful. A possibility (Langguth and Montvay, 1984) is to introduce the RGI quark mass (M_q) by the lowest vector-meson mass m_{1^-}:

$$m_{1^-}=2M_q+E(M_q)\ . \tag{5.15}$$

For heavy quarks (like c, b, or t) $E(M_q)$ can be taken, to a good approximation, from the Schrödinger equation, assuming some quark-antiquark potential. For light quarks (u, d, and s) we can take, as an empirical value, $E(M_q)\simeq 0.75$ GeV, which agrees well with the ρ- and φ-meson mass. In order to fix the lattice scale, in addition to Eq. (5.15) still another relation is needed. Near the critical line $\mu_{\rm cr}$, where the RGI quark mass is small, one can assume

$$aM_q\simeq c(\beta)(\mu_q-\mu_{\rm cr})=c(\beta)am_q \tag{5.16}$$

with some function $c(\beta)$. Having nothing better, one can take for $c(\beta)$ the perturbative expression (Gonzalez-Arroyo *et al.*, 1982; Hamber and Wu 1983; Göckeler, 1984; Golterman and Smit, 1984a, 1984b) at some arbitrarily fixed coupling $\bar{\beta}(\bar{g})$

$$c(\bar{\beta})\simeq(2\beta_0\bar{g}^2)^{-\gamma_0/2\beta_0}\simeq 2.5\ . \tag{5.17}$$

The numerical value here corresponds to $N_f=3$ and $\bar{\beta}\simeq 5.4$ in SU(3).

An important question to ask at this point is what the scaling region might look like in the available part of the (β,μ_q) plane. For $\mu_q\to\infty$ the quark mass tends to infinity and the theory reduces to a pure gauge theory. In this case the gluonic quantities (such as string tension, glueball masses, gluonic energy density in thermodynamics, etc.) show approximate scaling for $\beta\gtrsim 5.7$. In the region of light quark masses the results of a recent calculation (Langguth and Montvay, 1984) indicate that a tentative shape of the scaling region for light dynamical quarks could look something like that shown in Fig. 10. Note the difference between the scaling regions of pure gluonic quantities and those of quantities containing heavy quarks explicitly (such as heavy-quark bound-state masses, quark energy density in thermodynamics, etc.).

VI. SHORT GUIDE TO THE MOST RECENT LITERATURE

These lecture notes were completed in December 1984. Since then several important steps have been taken on the way to our final goal, a reliable numerical calculation of the hadron masses in lattice QCD. Here I shall try to give a short overview of the most recent literature on a few selected topics. I hope this short guide will be useful even if the content of the papers is not discussed in detail.

Quenched hadron mass calculations. Considerable effort was recently invested in increasing the lattice size and statistics of the quenched calculations. The gauge coupling and the bare quark mass were pushed toward smaller values. In the computation of Barkai *et al.* (1985), the lattice size was $16^3\times 32$ and Kogut-Susskind quarks were considered at $\beta=6.0$ and $am_q\geq 0.01$. The Wuppertal group collected several hundreds of quark propagators on $16^3\times 28$ and $16^3\times 56$ lattices using the approximate block diagonalization method (König *et al.*, 1984; Mütter and Schilling, 1984a, 1984b; König, Mütter, and Schilling, 1985; König, Mütter, Schilling, and Smit, 1985). A quenched calculation was also performed by using a renormalization-group improved SU(3) gauge action with Wilson quarks on $16^3\times 32$ and $12^3\times 24$ lattices (Itoh *et al.*, 1986).

Flavor interpretation of Kogut-Susskind fermions. In the case of Kogut-Susskind ("staggered") fermions the projection to a single Dirac fermion species can be defined in momentum space (Sharatchandra *et al.*, 1981) or locally in coordinate space (Gliozzi, 1982; Kluberg-Stern *et al.*, 1983; see also Kitazoe *et al.*, 1978). In order to construct appropriate local hadron operators, one must first classify lattice operators according to the irreducible representations of the full lattice symmetry group of the staggered fermions (Golterman and Smit, 1984a, 1984b, 1985; Morel and Rodrigues, 1984; Parisi and Cheng, 1984; Golterman, 1986; Joos, 1986). Next one would like to know how these representations are imbedded in the representations of the continuum symmetry and whether the symmetries broken by the lattice regularization are restored at all in the continuum limit. In this respect remarkable progress was made recently by Jolicoeur *et al.* (1986), who were able to give the explicit connection between the momentum-space and coordinate-space flavor definitions and to support the correct flavor interpretation in the continuum by detailed calculations in the Gross-Neveu model.

Dynamical quark methods. The main research area in numerical QCD calculations recently has been the study and improvement of the numerical methods for dynamical fermions. A comparative study of different algorithms (Weingarten, 1985) showed that the asymptotic growth of the required CPU time for a large number of lattice points (N) and small quark masses (m_q) is expected to be proportional to N^{11}/m_q for the Metropolis algorithm proposed by Weingarten and Petcher (1981). For the pseudofermion algorithm, the same estimate gave N^8/m_q^2, although with a larger proportionality factor. The estimated total number of arithmetic operations necessary for a reliable computation of hadron masses is, according to this paper, almost prohibitively large ($\sim 10^{18}$). A very large number of arithmetic operations seems necessary also for the iterative hopping parameter algorithm if very light quarks (like the physical u and d quarks) are dynamically included (Montvay, 1985).

In addition to Monte Carlo updating, different algorithms based on differential equations have been tested and developed. In the framework of stochastic quantization (Parisi and Wu, 1981), the updating of the field configurations can be performed by the Langevin stochastic differential equations (Drummond *et al.*, 1983; Guha and

Lee, 1983; Halpern, 1983; Hamber and Heller, 1984). The Langevin method has been tested and further developed in the case of gauge theories with fermions in several papers (Batrouni *et al.*, 1985; Ukawa and Fukugita, 1985; Batrouni, 1986; Kronfeld, 1986; Martin *et al.*, 1986). The advantage of this method is its flexibility in allowing for specific improvements, for instance, in order to fight critical slowing down (Batrouni *et al.*, 1985). It is also possible to combine the Langevin algorithm with the microcanonical (or "molecular dynamics") equations (Duane, 1985; Duane and Kogut, 1985). Such a hybrid algorithm is able to combine the advantages of both methods: ergodicity (from Langevin) and relatively large step size (from molecular dynamics). Large-scale calculations with the hybrid algorithm showed that there is a sharp crossover between strong coupling behavior and approximate asymptotic scaling (Kogut, 1986a, 1986b). Similar conclusions were obtained with the pseudofermion method (Gavai and Karsch, 1986).

The pseudofermion algorithm has received a lot of attention. Its comparison with an "exact" algorithm on a 4^4 lattice showed a quantitatively good agreement (Gavai and Goksch, 1986). After such encouraging experiences several groups have begun larger hadron spectrum calculations with Kogut-Susskind fermions using the pseudofermion method. In the simpler case of SU(2) gauge fields, Laermann *et al.* (1986b) obtained reasonable meson masses on an $8^3\times 16$ lattice for β between 1.85 and 1.95 and $am_q=0.035$—0.20. This group also observed the expected screening of the interquark potential on the same size lattice for $\beta=1.85$—2.5 and $am_q=0.05$—0.20 (Laermann *et al.*, 1986a). In the physical case of an SU(3) gauge field, Fucito *et al.* (1986) calculated the hadron masses on $12^3\times 8$ lattices triplicated to $12^3\times 24$ for the propagator calculation. At $\beta=5.4$ and am_q between 0.1—0.02 their statistics is still not very large, but the first results look promising.

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