Some aspects of large-N theories

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This is a pedagogical review of some aspects of quantum field theories in the limit in which the number of internal degrees of freedom is large. The focus is on large-N QCD. The authors briefly discuss several well-known approaches to a solution of the $N=\infty$ limit: loop equations, classical actions, and master fields. Eguchi-Kawai models are discussed in detail, and some recently obtained numerical results are reviewed.

CONTENTS

I. INTRODUCTION

Most interesting quantum field theories and statistical systems contain internal-symmetry groups. In many cases the number of internal degrees of freedom may be regarded as a free parameter. In the limit in which N, which is some measure of the number of internal degrees of freedom, becomes large, the dynamics of such theories very often simplify. One could then develop a systematic approximation scheme by studying the $N=\infty$ limit and then considering finite-N corrections—leading to an expansion in powers of $1/N$. This "large-N approximation" has provided a valuable framework for studying several models. Frequently, the zero-order approximation (i.e., at $N=\infty$) is fairly close to the real finite-N theory, even when N is small.

In the context of particle physics the $1/N$ expansion was introduced by 't Hooft (1974), who proposed a generalization of the standard SU(3) gauge symmetry of QCD to SU(N) and an expansion in powers of $1/N$. In fact, $1/N$ is the only known free parameter in QCD (Witten, 1979a). We consider an SU(N) gauge theory coupled to N_f flavors of quarks in the fundamental representation, described by the Lagrangian

$$\mathscr{L}=\frac{1}{4g^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})+\bar{\psi}(x)\not{D}\psi(x)\ . \tag{1.1}$$

$F_{\mu\nu}$ is the standard non-Abelian gauge field, and $\psi(x)$ denotes the quark field. 't Hooft considered the limit

$$N\rightarrow\infty \quad \text{with } N_f, g^2N= \text{ fixed}\ . \tag{1.2}$$

The dominant Feynman graphs in this limit can be classified according to simple topological considerations ('t Hooft, 1974; Witten, 1979a; Coleman, 1982). This allows one to study meson phenomenology at $N=\infty$—which turns out to be remarkably similar to that in the real world.

In the real world $N=3$, and it may be argued that $\frac{1}{3}$ is not a terribly small number. However, the true expansion parameter in the large-N expansion is probably not simply $1/N$, but α/N, where α is some number. It is certainly possible that α is in fact very small—in that case the large-N approximation is reliable. A similar situation occurs in QED. Here the coupling constant e is about 0.3—certainly not too small. But the real expansion parameter in QED is $e^2/4\pi$, which is certainly small enough to ensure the reliability of the perturbation expansion (Witten, 1979a). In QCD we do not know yet how small α is, but the qualitative success of large-N meson phenomenology certainly indicates that α is small.

Veneziano (1976) has proposed a different large-N limit for QCD. This is defined by

$$N\rightarrow\infty N_f\rightarrow\infty\,,\ \ \frac{N_f}{N},g^2N,g^2N_f= \text{ fixed}\ .$$

The Veneziano limit provides a better explanation of certain aspects of low-energy phenomenology. The 't Hooft limit is, however, much simpler and has been studied in much more detail. In this paper we shall almost exclusively deal with the 't Hooft limit.

Over the past ten years there has been vigorous activity in the field of large-N expansions—both for four-dimensional QCD and for other two-dimensional models. Several classes of models can be solved exactly in the $N=\infty$ limit, leading to valuable physical insights (for a review, see Coleman, 1982). More recently, following the work of Eguchi and Kawai (1982), it has become clear that at $N=\infty$, field theories become equivalent to matrix models living at a single point. The advent of these "reduced models" (or Eguchi-Kawai models) has raised new hopes for a quantitative understanding of the $N=\infty$ limit of theories like QCD. In particular, large-N theories are now amenable to numerical simulations, which are providing interesting nonperturbative information.

In this paper we shall present an overview of some aspects of the large-N limit. This is not intended to be a comprehensive review of the subject; rather we shall concentrate on a few specific topics. We shall mostly talk about large-N QCD, but several other models will also be discussed mainly for illustrative purposes. Our main focus will be on Eguchi-Kawai models, and we shall pay more attention to those aspects of large-N formalism which are necessary for an understanding of these models.

In Sec. II we briefly discuss several phenomenological aspects of large-N QCD: mesons, baryons, and the η' problem. Most of the discussion consists of statements of results without proofs—detailed reviews on the subject already exist in the literature (Coleman, 1982).

In Sec. III we discuss more theoretical aspects of the large-N limit. Factorization and its consequences are explored. These include loop equations, saddle-point methods, and master fields. We derive the loop equations for the lattice gauge theory. The discussion of saddle-point methods and classical Hamiltonians is brief. Once again, these topics are covered in other review articles (Yaffe, 1982).

In Sec. IV we introduce Eguchi-Kawai models and quenched Eguchi-Kawai (QEK) models. The perturbation expansion of QEK models and their equivalences with field theories are discussed.

In Sec. V we discuss the twisted Eguchi-Kawai (TEK) models.

In Sec. VI we summarize some of the numerical results obtained with QEK and TEK models.

II. HADRON PHENOMENOLOGY

Perhaps the most immediate appeal of the large-N expansion lies in the fact that the phenomenology of QCD in the $N=\infty$ limit is remarkably similar to that of the real world. The dominant Feynman graphs at $N=\infty$ may be classified by simply counting the powers of N ('t Hooft, 1974; Veneziano, 1976; Witten, 1979a). For example, the graphs that contribute to the connected part of an n-point function of fermionic currents are all $O(N)$ and have the following properties:

(1) They are planar.

(2) There are no internal fermion loops.

(3) All current insertions are on a single fermion loop that forms the boundary of the graph.

Similarly the graphs contributing to connected Green's functions of gauge-invariant operators constructed out of gauge fields alone are $O(N^2)$ and (1) are planar and (2) contain no fermion loops. In general, each fermion loop costs a factor of $1/N$, while each nonplanar crossing is suppressed by $1/N^2$.

Assuming that the $N=\infty$ theory confines, so that propagating states are color singlets, it is now possible to study properties of hadrons. This is done by applying the above rules and analyzing the intermediate states that contribute to the various n-point functions. A detailed discussion may be found in the papers of Witten (1979a) and Coleman (1982). We shall simply quote the relevant results.

(a) Mesons. The properties of mesons at large N are qualitatively consistent with those in the real world.

(1) Mesons are stable: their decay amplitudes are $O(1/\sqrt{N})$.

(2) Mesons are noninteracting: scattering amplitudes are $O(1/N)$.

(3) Meson masses are finite, i.e., they are $O(1)$.

(4) The number of mesons are infinite.

(5) Exotics are absent.

(6) Zweig's rule holds.

In fact, the $1/N$ expansion is the only known framework within QCD that provides an explanation for Zweig's rule.

(b) Glueballs. A similar analysis of glueball states reveals the following.

(1) Glueballs are stable.

(2) Glueballs are noninteracting: a vertex involving l glueballs is suppressed by $O(1/N^{l-1})$.

(3) There are infinitely many glueballs.

(4) Glueballs do not mix with mesons: a vertex involving k mesons and l glueballs is of $O(1/N^{l+k/2-1})$.

(c) Baryons. Baryons pose a special problem at $N=\infty$. This is because a baryon in an SU(N) theory must be made out of N quarks, while a meson is always made out of a quark-antiquark pair, irrespective of N. This feature makes baryons behave in a fashion quite different from mesons (Witten, 1979a).

(1) Baryon masses are $O(N)$.

(2) The splitting of various excited baryonic states is $O(1)$.

(3) Baryons interact strongly amongst themselves: the typical baryon-baryon or baryon-antibaryon vertex is $O(N)$.

(4) Baryons interact with mesons with $O(1)$ couplings.

The above properties of baryons are remarkably similar to those of solitons in weakly coupled theories. Consider, for example, monopoles in a model with a weak coupling constant g^2. The monopole mass is $O(1/g^2)$, but the energies of excitations around the monopole background are $O(1)$. The monopole-antimonopole scattering amplitude is $O(1/g^2)$, while the monopole—electron scattering amplitude is $O(1)$. This led Witten to suggest that baryons are in some sense solitons of large-N QCD, with N playing the role of $1/g^2$ (Witten, 1979a).

The precise sense in which baryons are solitons was not clear until recently. Low-energy hadron phenomenology is well summarized by an effective $\mathrm{SU}(N_f)\times\mathrm{SU}(N_f)$ chiral model, where N_f denotes the number of flavors of quarks. The effective Lagrangian is given by

$$\mathscr{L}=f_\pi^2 \int d^4x \, \mathrm{Tr}(\partial_\mu^\dagger U)(\partial^\mu U) \, ,$$

with possible additions of Wess-Zumino terms to account for the anomalies (Wess and Zumino, 1971; Witten, 1983a). Now, f_π^2 is of order N; hence, at large N, f_π^2 can act as a semiclassical WKB parameter—and the theory can have solitonic sectors. In fact, it has been known for a long time (Skyrme, 1961) that the chiral model possesses topologically stable fermionic solitons—the "skyrmions"—that can be interpreted as baryons. This idea has been revived recently (Balachandran *et al.*, 1982; Witten, 1983b). The static properties of baryons computed in this framework seem reasonable (Adkins, Nappi, and Witten, 1983). At present this approach is being vigorously pursued. A different approach, which can, in principle, also deal with the chiral-symmetry-restored phase of QCD (at high temperatures), is based on a Nambu—Jona-Lasinio-type model (Dhar and Wadia, 1984).

(d) The η' problem. The large-N limit provides interesting insights concerning the U(1) problem. With three flavors of quarks, the standard Lagrangian of massless QCD has a $\mathrm{U}(3)\times\mathrm{U}(3)$ chiral symmetry at the classical level. However, the axial symmetries are spontaneously broken and the corresponding Nambu-Goldstone (NG) bosons appear as the light pseudoscalar mesons. In nature one observes eight light pseudoscalars—the π's, k's, and the η—instead of nine such mesons expected to arise from the breaking of axial U(3). The lightest SU(3) singlet pseudoscalar is the η', with a mass of about 1 GeV—much too heavy to be the expected ninth Nambu-Goldstone boson. The resolution of this problem lies in the fact that the U(1) axial current has an anomaly. The corresponding charge is actually not conserved, and hence there is no ninth NG boson. What then is the η'?

It might be argued that the η' would have been a NG boson had it not been for the anomaly: the anomaly splits the η' from π, k, and η. For this to make any sense there must exist a limit in which the anomaly turns off. The $N=\infty$ limit is precisely such a limit. This is because the anomaly equation reads

$$\partial_\mu J_\mu^5=\frac{g^2N_f}{16\pi^2}\mathrm{Tr}(\widetilde{F}_{\mu\nu}F^{\mu\nu}) \, .$$

In the limit $N_f=$ fixed, $N\to\infty$ with $g^2N=$ fixed, the right-hand side vanishes.

On the basis of results obtained in other models, Witten (1979b) argued that in the leading order of the $1/N$ expansion the vacuum energy of pure QCD depends on θ, the vacuum angle. Then the requirement that this θ dependence vanish in the zero-quark-mass limit leads, in the $1/N$ expansion, to the existence of a meson whose mass squared is of order $1/N$. This is precisely the η'. The η' is thus a genuine Nambu-Goldstone at $N=\infty$. For finite N, η' is a pseudo-Goldstone boson, with a $(\text{mass})^2$ proportional to the symmetry-breaking term—which is of order $1/N$.

III. FACTORIZATION, LOOP EQUATIONS, MASTER FIELDS, SADDLE POINTS, AND ALL THAT

A. Factorization of gauge-invariant quantities

The crucial feature of the large-N limit that gives rise to many of its intriguing theoretical properties is factorization. Stated in general terms this means that the connected Green's functions of invariant quantities are suppressed relative to the corresponding disconnected pieces by powers of $1/N$. Hence at $N=\infty$ expectation values of products of invariant quantities may be replaced by products of expectation values. Let us illustrate this in large-N QCD by using the perturbation rules stated in Sec. II. Let B_i denote fermionic current operators and G_i denote gauge-invariant operators made out of gluon fields alone. Then, according to the rules of Sec. II,

$$\begin{aligned}
&\langle B_1B_2\cdots B_n\rangle_c=O(N)\,,\\
&\langle B_1\cdots B_nG_1\cdots G_m\rangle_c=O(N)\,,\\
&\langle G_1\cdots G_m\rangle_c=O(N^2)\,.
\end{aligned}\tag{3.1}$$

From these equations it immediately follows that

$$\begin{aligned}
&\frac{\langle B_1\cdots B_n\rangle_c}{\langle B_1\rangle\langle B_2\rangle\cdots\langle B_n\rangle}=O\left[\frac{1}{N^{n-1}}\right],\\
&\frac{\langle B_1\cdots B_nG_1\cdots G_m\rangle_c}{\langle B_1\rangle\langle B_2\rangle\cdots\langle B_n\rangle\langle G_1\rangle\cdots\langle G_m\rangle}=O\left[\frac{1}{N^{n+2m-1}}\right],\\
&\frac{\langle G_1\cdots G_m\rangle_c}{\langle G_1\cdots\langle G_m\rangle}=O\left[\frac{1}{N^{2m-2}}\right].
\end{aligned}\tag{3.2}$$

Factorization may be proven also in the lattice strong-coupling expansion. As yet there has been no convincing general proof; it is, however, reasonable to assume that it is generally valid.

Do all gauge-invariant operators factorize? In general, no. Several examples have been cited in the literature (Haan, 1981; Green and Samuel, 1981). However, all "reasonable" operators do factorize. To determine which operators are "reasonable," one has to construct analogs of coherent states for the sequence of theories characterized by a given value of N. Let $|u\rangle$ and $|u'\rangle$ denote such coherent states. An operator A is called "classical" if its coherent-state matrix elements have a finite $N\to\infty$ limit, i.e.,

$$\lim_{N\to\infty}\frac{\langle u\,|\,\hat{A}\,|\,u'\rangle}{\langle u\,|\,u'\rangle}=\text{finite}\ . \qquad (3.3)$$

All such classical operators are reasonable and do factorize (Yaffe, 1982). Examples of such operators in QCD are Wilson loops, fermion bilinears (like B_i), and pure gauge operators like $\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}$ or $\mathrm{Tr}F_{\mu\nu}\tilde{F}^{\mu\nu}$. In fact, the important properties of the large-N limit discussed below are consequences of factorization of these classical operators (Yaffe, 1982).

B. Loop equations

One important consequence of factorization is that there exist closed Dyson-Schwinger equations relating invariant expectation values. For gauge theories the relevant quantities are Wilson loops. We shall refer to these as loop equations. The phenomenological success of string models suggests that the long-distance behavior of QCD is some kind of a string theory. It was suggested by Nambu (1979), Polyakov (1979), and Gervais and Neveu (1979) that the Wilson loop average may be regarded as the wave functional for a closed string. Equations for the Wilson loop were derived, and these resembled classical string equations. Later Makeenko and Migdal (1979) showed that at $N=\infty$ Dyson-Schwinger equations of Wilson loops form a closed system. (These equations are different from those obtained by the earlier authors.) We shall discuss these equations in the context of lattice gauge theories (Eguchi, 1979; Foerster, 1979; Weingarten, 1979).

Consider the pure U(N) gauge theory defined on a hypercubic lattice with the standard Wilson action:

$$S=\beta\sum_{x}\sum_{\mu>\nu}\mathrm{Tr}[U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x)+\text{H.c.}]\ , \qquad (3.4)$$

where $\beta=1/g^2$ and g^2 is the bare coupling. $U_\mu(x)$ is the standard link matrix belonging to U(N) in the direction μ and originating at the site x. We have, as usual,

$$U_{-\mu}(x)=U_\mu^\dagger(x-\mu)\ .$$

Let λ^a be the generators of U(N) normalized in the standard fashion. These obey

$$\sum_a(\lambda^a)_{ij}(\lambda^a)_{kl}=\delta_{jk}\delta_{il}\ . \qquad (3.5)$$

We now consider the quantity

$$X^a(C)=\int\prod_{x,\mu}dU_\mu(x)[\mathrm{Tr}\lambda^a U_\mu(x)U_\mu(x+\mu)\cdots]e^{-S}\ . \qquad (3.6)$$

The quantity within square brackets is the ordered product of links around the curve C shown in Fig. 1, with a λ^a in front of it. For the moment, we have chosen C to be simple, i.e., without any self-intersection. Note that $X^a(C)$ is identically zero. But that is irrelevant to our discussion.

Let us now make an infinitesimal change of variables on the link $U_\mu(x)$, keeping all the others fixed:

$$U_\mu(x)\to(1+i\varepsilon\lambda^a)U_\mu(x)\ ,$$

i.e.,

$$\delta_a U_\mu(x)=i\varepsilon\lambda^a U_\mu(x)\ ,$$
$$\delta_a U_\mu^\dagger(x)=-i\varepsilon U_\mu^\dagger(x)\lambda^a\ .$$

Evidently,

$$\sum_a\delta_a X^a(c)=0\ . \qquad (3.7)$$

The variation on the left-hand side of Eq. (3.7) consists of two types of terms.

(a) Source terms obtained by varying the operator. This is easily seen to be

$$i\varepsilon\int\prod_{x,\mu}dU_\mu(x)\left[\sum_a\mathrm{Tr}[\lambda^a\lambda^a U_\mu(x)\cdots]\right]e^S = i\varepsilon NZ\langle\mathrm{Tr}W(C)\rangle\ , \qquad (3.8)$$

where we have used the completeness relation, Eq. (3.5). $W(C)$ is the Wilson loop operator along the curve C:

$$W(C)=U_\mu(x)U_\mu(x+\mu)\cdots U_\mu(x-\mu)\ ,$$

and Z is the partition function:

$$Z=\int\prod_{x,\mu}dU_\mu(x)e^S\ .$$

(b) Equation-of-motion terms obtained by varying the action. This is given by

$$\delta_a S=\sum_{\nu\neq\mu}\mathrm{Tr}[\lambda^a U_p(x)-\lambda^a U_p^\dagger(x)]\ , \qquad (3.9)$$

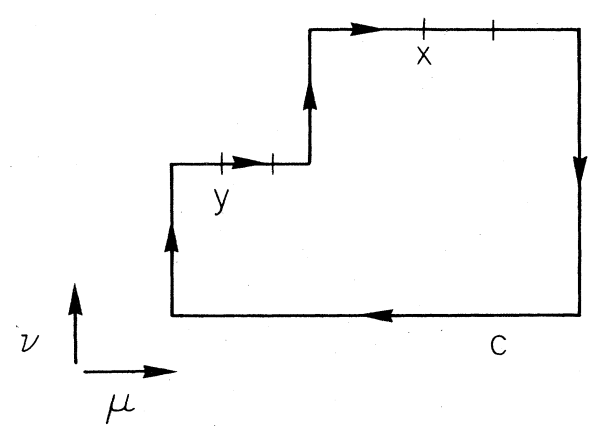

FIG. 1. The "Simple" Wilson loop.

where $U_p(x)$ denotes the plaquette in the $(\mu\nu)$ plane containing the link $U_\mu(x)$. The sum in Eq. (3.9) includes all such plaquettes,

$$U_p(x) \equiv U_\mu(x) U_\nu(x+\mu) U_\mu^\dagger(x+\nu) U_\nu^\dagger(x) , \quad (3.10)$$

and Eq. (3.9) contributes to $\sum \delta_a X^a(C)$ a term

$$\sum_{\nu \neq \mu} -i\varepsilon\beta Z \langle \mathrm{Tr}[W(C) U_p(x) - W(C) U_p^\dagger(x)] \rangle . \quad (3.11)$$

Combining Eqs. (3.8) and (3.11), we can now write Eq. (3.7) as

$$\frac{1}{N}\langle \mathrm{Tr} W(C) \rangle = \sum_{\nu\neq\mu} \frac{\beta}{N} \left[\frac{1}{N} \langle \mathrm{Tr} W(C) U_p(x) \rangle - \frac{1}{N} \langle \mathrm{Tr} W(C) U_p^\dagger(x) \rangle \right] , \quad (3.12)$$

which is the loop equation we wanted to derive. Equation (3.12) is shown diagrammatically in Fig. 2.

So far we have not used the factorization property. The reason we did not need factorization is that we started out with non-self-intersecting loops. However, as Fig. 2 immediately reveals, the loop equations relate simple loops to self-intersecting loops. Thus to obtain a closed set of equations for Wilson loops one must also consider the latter—and this is where factorization enters the game.

Self-intersecting loops on the lattice are loops in which a given link occurs more than once. For simplicity, we shall consider only those loops in which a given link can occur not more than twice. These can be of two types: one in which the links occur in the same direction [Fig. 3(a)] and one in which they occur in opposite directions. Consider a loop of the first kind. This may be written as

$$\mathrm{Tr}\, W(C) = \mathrm{Tr} W(C_1) W(C_2) ,$$

where $W(C_1)$ and $W(C_2)$ denote the Wilson loop operators along C_1 and C_2, respectively, with the link $U_\mu(x)$ appearing as the first link in both $W(C_1)$ and $W(C_2)$. To deduce loop equations one starts with the quantity

$$X^a(C_1 C_2) = \int \prod_{x,\mu} dU_\mu(x) e^S \mathrm{Tr}[\lambda^a W(C_1) W(C_2)] . \quad (3.13)$$

The equation-of-motion term in the variation of $X_a(C_1 C_2)$ is identical to that of a simple loop. The source term, however, contains two pieces. The first piece, coming from the variation of $U_\mu(x)$ in $W(C_1)$, is simply given by

$$i\varepsilon N Z \langle \mathrm{Tr} W(C_1) W(C_2) \rangle = i\varepsilon N Z \langle \mathrm{Tr} W(C) \rangle .$$

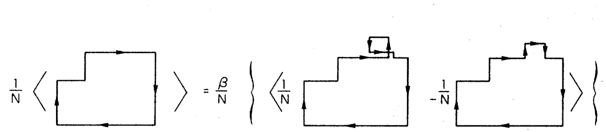

FIG. 2. Dyson-Schwinger equation for the simple loop.

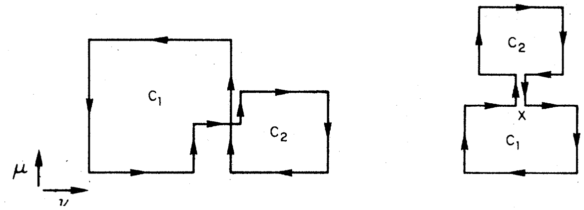

FIG. 3. Self-intersecting Wilson loops.

as in simple loops. The second piece occurs when the variation hits the $U_\mu(x)$ contained in $W(C_2)$. In the usual fashion [i.e., using Eq. (3.5)] this yields a term

$$i\varepsilon Z \langle \mathrm{Tr} W(C_1) \mathrm{Tr} W(C_2) \rangle . \quad (3.14)$$

For any finite N, the quantity (3.14) is not a Wilson loop operator, and one does not have a closed equation for loops. However, at $N=\infty$ Eq. (3.14) factorizes into

$$i\varepsilon Z \langle \mathrm{Tr} W(C_1) \rangle \langle \mathrm{Tr} W(C_2) \rangle , \quad (3.15)$$

so that one now has a product of Wilson loops. The full Dyson-Schwinger equations are now closed equations for Wilson loops alone. For self-intersecting loops of the second kind [Fig. 3(b)], the derivation is analogous. The extra source term, Eq. (3.15), now occurs with a negative sign. The final form of the loop equations reads (Wadia, 1981c)

$$\frac{1}{N}\langle \mathrm{Tr} W(C) \rangle \pm \frac{1}{N} \langle \mathrm{Tr} W(C_1) \rangle \langle \mathrm{Tr} W(C_2) \rangle = \frac{\beta}{N} \sum_{\nu=\mu} \left[\frac{1}{N} \langle \mathrm{Tr} W(C) U_p(x) \rangle - \frac{1}{N} \langle \mathrm{Tr} W(C) U_p^\dagger(x) \rangle \right] , \quad (3.16)$$

where the $+$ $(-)$ sign is for self-intersections of the first (second) type.

Similar loop equations may be derived in the presence of quark fields. This would, in general, involve relationships between Wilson loops and quark-string-antiquark operators. In the usual large-N limit (i.e., in which the number of flavors is held fixed), a string cannot split forming a quark-antiquark pair (since fermion loops are suppressed). However, this can happen in the Veneziano limit (Foerster, 1979; Das, 1983).

Continuum forms of the loop equations can also be derived (Makeenko and Migdal, 1979). These are essentially continuum versions of Eqs. (3.16), which now involve suitably defined derivatives of Wilson loops. These derivatives in loop space have to be regularized in an appropriate manner. Details of this formalism may be found in the review of Migdal (1983).

The existence of loop equations in the $N=\infty$ limit shows that QCD, in some sense, may be written as a string theory. However, the loop equations for the four-dimensional theory remain unsolved. Migdal and his collaborators have made some progress in this direction. They have shown that there exist self-consistent solutions in which the Wilson loops obey an area law. The theory has, in fact, been reduced to a fermionic string theory—which, however, remains unsolved. Recently there has been some progress in attempts to solve these equations numerically (Marchesini, 1984). One of the major difficulties in the program is the fact that the various Wilson loops are not all independent of each other.

Dyson-Schwinger equations may be derived for various other theories. Exact solutions are readily obtainable for vectorlike models—these can, however, be solved by various other methods. For most nontrivial models, like the matrix model and chiral models, there exists no exact solution as yet.

While it is true that a solution of the loop equations would provide all gauge-invariant Green's functions, it is certainly not true that they would provide all information about the theory. Examples of physical quantities that loop equations alone cannot determine are the spectrum and scattering amplitudes. These require, in the present framework, calculation of connected correlations of gauge-invariant operators—which vanish by factorization. Such quantities can, however, be obtained (in principle) in the classical Hamiltonian approach, which we shall briefly discuss below.

C. Master fields and saddle points

Consider two invariant classical operators A and B. Factorization implies

$$\langle AB\rangle=\langle A\rangle\langle B\rangle \quad \text{at } N=\infty \ .$$

When $A=B$ this becomes

$$\langle A^2\rangle=\langle A\rangle^2 \ , \tag{3.17}$$

which means that fluctuations vanish at $N=\infty$. This has led Witten (1980) to argue that at $N=\infty$ the functional integral is evaluated by a single field configuration called the master field. For gauge theories one has, of course, a master orbit—i.e., a trajectory in configuration space whose points are related to each other by a gauge transformation. While the master field certainly exists, it is not clear how to evaluate it except for trivially soluble models. Recently, recursive procedures have been developed to find the master field numerically (Yaffe, 1984), and equations obeyed by master fields have been obtained by several methods (Greensite and Halpern, 1983; Jevicki and Rodrigues, 1984).

The absence of fluctuations at $N=\infty$ also suggests that the large-N limit is some kind of a classical limit. To get a feeling for the nature of this limit we now discuss a soluble model in the framework of the quantum collective field method (Jevicki and Papanicolaou, 1980; Jevicki and Sakita, 1980, 1981; Sakita, 1980; Jevicki and Levine, 1981). Let us consider the linear U(N) sigma model involving a field $\varphi_i(x)$ in the fundamental representation of U(N). The action of the lattice is given by

$$S=\sum_x\left[\frac{1}{2}\sum_{\mu,i}|\varphi_i(x+\mu)-\varphi_i(x)|^2+\frac{1}{2}m^2\sum_i\varphi_i^*(x)\varphi_i(x)+\frac{\lambda}{N}\left[\sum_i\varphi_i^*(x)\varphi_i(x)\right]^2\right] , \tag{3.18}$$

and the partition function is

$$Z=\int\prod_{x,i}d\varphi_i^*(x)d\varphi_i(x)\exp(-S) \ . \tag{3.19}$$

We shall consider the limit

$$N\to\infty,\quad \lambda=\text{fixed} \ .$$

Now, each term in the action is of order N. By rescaling the variables we may bring N out in front of the entire action. One might think that for large N the integral is then dominated by the saddle point of the action. This is wrong. The reason is that the measure $\prod d\varphi_i^*(x)d\varphi_i(x)$ grows exponentially with N. In other words, there is a large entropy that must be taken into account in the minimization of the free energy. To extract the N dependence of the measure we go over to invariant collective variables defined by

$$\sigma'(x,y)=\sum_i\varphi_i^*(x)\varphi_i(y) \ , \tag{3.20}$$

and introduce

$$1=\int[d\sigma']\prod_{x,y}\delta\left[\sigma'(x,y)-\sum_i\varphi_i^*(x)\varphi_i(y)\right]$$

into the partition function, Eq. (3.19). Z now becomes

$$Z=\int[d\sigma']J[\sigma']e^{-S[\sigma']} \ , \tag{3.21}$$

where $S[\sigma']$ is the action written in terms of the σ''s,

$$S[\sigma']=\frac{1}{2}\sum_{x,y,\mu}K_\mu(x,y)\sigma'(x,y)+\frac{1}{2}m^2\sum_x\sigma'(x,x)-\frac{\lambda}{N}\sum_x[\sigma'(x,x)]^2 \tag{3.22}$$

and

$$K_\mu(x,y)=2\delta(x,y)-\delta(x,y+\hat{\mu})-\delta(x,y-\hat{\mu})$$

is simply the second derivative operator on the lattice. The Jacobian $J[\sigma']$ is given by

$$J[\sigma']=\int[d\varphi^*d\varphi]\prod_{x,y}\delta\left[\sigma'(x,y)-\sum_i\varphi_i^*(x)\varphi_i(y)\right] .$$

This may be evaluated by the saddle-point method at large N (Wadia, 1981b) by exponentiating the delta function:

$$J[\sigma']=\int [d\varphi^* d\varphi]\prod_{x,y} d\lambda(x,y)\exp\left[i\sum_{x,y}\lambda(y,x)\left[\sigma'(x,y)-\sum_i \varphi_i(x)\varphi_i^*(y)\right]\right] . \tag{3.23}$$

Performing the integration over φ and φ^*, one has

$$J[\sigma']=\int \prod_{x,y} d\lambda(x,y)\exp\left[\sum_{x,y}[i\lambda(y,x)\sigma'(x,y)-N\ln\lambda(x,y)\delta(y,x)]\right] . \tag{3.24}$$

Since each term in the exponent is of order N, $J[\sigma']$ is given by the saddle-point value

$$\lambda(x,y)=-iN\sigma'^{-1}(x,y) .$$

This yields

$$J[\sigma']=\exp\left[N\sum_x \ln\sigma'(x,x)\right] . \tag{3.25}$$

The whole partition function may now be written in terms of the order-one collective field $\sigma(x,y)$ defined by

$$\sigma(x,y)=\frac{1}{2N}\sigma'(x,y) ,$$

$$Z=\int [d\sigma]\exp(-NS_{\text{eff}}[\sigma]) , \tag{3.26}$$

where

$$S_{\text{eff}}[\sigma]=\sum_{x,y,\mu}K_\mu(x,y)\sigma(y,x) +m^2\sum_x\sigma(x,x)-\frac{4\lambda}{N}\sum_x[\sigma(x,x)]^2 -\sum_x \ln\sigma(x,x) . \tag{3.27}$$

In Eq. (3.26) both $S_{\text{eff}}[\sigma]$ and the measure $d\sigma$ are of order one. Hence for large N the integral may be evaluated by the saddle point of S_{eff}. The saddle-point equation is

$$\sum_\mu K_\mu(x,y)+m^2\delta(x,y)+\frac{4\lambda}{N}\sigma_0\delta(x,y)=\sigma^{-1}(x,y) , \tag{3.28}$$

where

$$\sigma_0=\sigma(x,x) .$$

In terms of the Fourier components defined by

$$\sigma(x,y)=\int_{-\pi}^{+\pi}\frac{d^dk}{(2\pi)^d}e^{ik\cdot(x-y)}\sigma_k , \tag{3.29}$$

one has

$$\sigma_k=\frac{1}{4\sum_\mu \sin^2 k_\mu/2+m^2+4(\lambda/N)\sigma_0} , \tag{3.30}$$

where σ_0 is determined by the self-consistent gap equation

$$\sigma_0=\sigma(x,x)=\int_{-\pi}^{+\pi}\frac{d^dk}{(2\pi)^d}\sigma_k ,$$
$$\sigma_0=\int\frac{d^dk}{(2\pi)^d}\frac{1}{4\sum_\mu \sin^2 k_\mu/2+m^2+4(\lambda/N)\sigma_0} . \tag{3.31}$$

In fact, Eq. (3.28) is simply the Dyson-Schwinger equation for the model. In this case Eqs. (3.30) and (3.31) provide all the correlation functions of the model—since all invariant n-point functions are products of two-point functions by virtue of factorization. However, the large-N effective action, Eq. (3.27), contains much more information than the loop equations. This is because one can now perform small fluctuations around the solution to the Dyson-Schwinger equations, thereby extracting the spectrum of the theory.

The collective field program has been carried out in the Euclidean (Jevicki and Sakita, 1981) as well as in the Hamiltonian framework (Jevicki and Sakita, 1980; Sakita, 1980). For the gauge theory the collective variables are the Wilson loop operators $W(C)$ along all possible loops C. In the Hamiltonian framework these loops are all spatial; in the Euclidean approach there are temporal loops as well. We shall not enter into the details of this formalism, but simply discuss the main issues.

The loop-space Hamiltonian may be written as (Sakita, 1980; Jevicki and Rodrigues, 1984)

$$H=\frac{g^2}{2a}\left[\sum_{c,c'}\pi(C)\Omega(C,C')\pi^\dagger(C) +\tfrac{1}{8}\sum_{cc'}\omega^\dagger(C)\Omega^{-1}(C,C')\omega(C') -\frac{2}{g^4}\sum_p[\varphi(P)+\varphi(\tilde{P})]\right] , \tag{3.32}$$

where

$$\Omega(C,C')=-2\sum_l\sum_\alpha[\hat{E}^\alpha(l)\varphi(C)][\hat{E}^\alpha(l)\varphi^\dagger(C')] ,$$
$$\omega(C,C')=2\sum_{l,\alpha}\hat{E}^\alpha(l)\hat{E}^\alpha(l)\varphi(C) , \tag{3.33}$$

and $\varphi(C)$ is the Wilson loop operator around the spatial loop C. $E^a(l)$ is the standard electric field operator along the link l. $\varphi(P)$ denotes the elementary plaquette Wilson loop and $\varphi(P)$ the conjugate loop. $\pi(C)$ denotes the momentum conjugate to $\varphi(C)$ in loop space. The procedure by which the above Hamiltonian is obtained is analogous to that used in obtaining the collective field action for the sigma model. One makes a change of variables from the links U_l to the Wilson loops $\varphi(C)$ (which form an overcomplete set of variables). Subsequently a canonical transformation is performed to go over to variables in terms of which H is explicitly Hermitian. Note that the $\varphi(C)$'s are not independent of each other. However, it has been argued that in the large-N limit, $\varphi(C)$'s and their conjugates $\pi(C)$'s may be regarded as independent variables.

By a rescaling of variables,

$$\varphi \to N\varphi,\ \ g^2 \to N^{-1}\lambda\ ,$$

N^2 factors out of the effective potential:

$$V_{\rm eff}(\varphi)=\frac{1}{8}\sum_{cc'}\omega^{\dagger}(C)\Omega^{-1}(C,C')\omega(C') - \frac{2}{\lambda^2}\sum_{P}[\varphi(P)+\varphi(\widetilde{P})]\ .$$

One might think that the expectation values of $\varphi(C)$ in the large-N limit are given by the saddle point of $V_{\rm eff}$,

$$\frac{\delta V_{\rm eff}(\varphi)}{\delta\varphi}=0\ .$$

This is, however, incorrect in the weak coupling region (Jevicki and Rodrigues, 1984) because of nontrivial inequalities coming from the fact that $\Omega(C,C')$ is positive definite. $V_{\rm eff}$ has to be minimized in the presence of these constraints. It has been shown, however, that a set of master variables can be introduced to transform the problem to that of an unconstrained minimization (Jevicki and Rodrigues, 1984). This approach has been pursued numerically for some models.

Another approach to the large-N limit is that of "constrained classical solutions" (Bardakci, 1981a; Halpern, 1981). We shall illustrate this method for a simple one-vector model consisting of a single N-component vector $x_i(t)$ evolving in time. The relevant matrix elements are vacuum expectation values of index-ordered products of operators, like

$$\langle 0\,|\,\hat{x}(t)\cdot\dot{\hat{x}}(t')\dot{\hat{x}}(t)\cdot\hat{x}(t'')\,|\,0\rangle\ .$$

Let us insert a complete set of quantum eigenstates after each field operator. Due to the restriction to index-ordered products, such intermediate states must transform either as $O(N)$ vectors or as $O(N)$ singlets. Factorization further implies that the only singlet state that can contribute to the leading large-N behavior is the ground state. One thus needs only the following matrix elements:

$$\langle n,i\,|\,\hat{x}_j\,|\,0\rangle=\delta_{ij}q_n/\sqrt{N}\ .$$

Here n labels the number of $O(N)$ vector eigenstates, i is an $O(N)$ index labeling the states within such a multiplet, and q_n is the "reduced" matrix element. Since all states are eigenstates of the Hamiltonian, the q_n's have a simple time dependence:

$$q_n(t)=e^{i\omega_n t}q_n(0)\ ,$$

where $w_n=E_n-E_0$ is the excitation energy of the nth eigenstate. Taking matrix elements of the quantum equation of motion,

$$\ddot{\hat{x}}_i+2V'(\hat{x}^2)\hat{x}_i=0$$

[where $V(\hat{x}^2)$ is an $O(N)$-invariant potential], and using factorization, one has the following equation for the reduced matrix elements:

$$\ddot{q}_n+2V'(q\cdot q^*)q_n=0\ .$$

Thus q_n's obey a classical equation of motion. These equations must, however, be supplemented by constraints obtained by taking vacuum expectation values of the commutation relations:

$$\sum(q_n^*\dot{q}_n-\dot{q}_n^*q_n)=1\ .$$

A similar set of constrained classical equations may be obtained and solved for the familiar vector models. The approach has also been extended to gauge theories (Bardakci, 1981b, 1982).

The precise nature of the "classical" limit at large N has been investigated in detail by Yaffe (1982). Essentially one constructs analogs of coherent states of quantum mechanics for the sequence of theories labeled by N. Under certain conditions (on the state space and operators), the expectation values of operators in these coherent states behave as classical dynamical variables in the $N=\infty$ limit. The quantum dynamics of the large-N theory reduces to classical dynamics governed by a classical Hamiltonian, which is just the coherent-state expectation value of the quantum Hamiltonian. This fact has been recently exploited to construct a numerical method to solve large-N theories (Brown and Yaffe, 1985). The relevant coherent states are generated by the action of a Lie group (the coherence group) on a fixed initial state (usually taken to be the strong coupling vacuum). For gauge theories, the coherence group is generated by the Lie algebra consisting of arbitrary linear combinations of the (untraced) loop operators and loop operators with one electric field insertion. The numerical method involves minimization of the classical Hamiltonian in the space of coherent states, suitably truncated, by a Newton minimization scheme. In this procedure, choice of Riemann normal coordinates in the classical phase space is extremely useful. Once the minimum is obtained, the large-N glueball or meson masses are computed by calculating the second derivative of the Hamitonian around the minimum. Similarly, particle decay widths are obtained from third derivatives, four-particle scattering amplitudes from fourth derivatives, and so on. There seem to exist consistent truncation schemes that render the method practicable. This method has been successfully tested for the exactly soluble one-plaquette model and applied to $(2+1)$-dimensional gauge theories. There also exists a Euclidean version of the method. Fermions can be included without much difficulty. A detailed presentation is contained in Brown and Yaffe (1986).

IV. EGUCHI-KAWAI MODELS AND QUENCHING

A. Basic ideas for reduction in large N

Recently, Eguchi and Kawai (1982) pointed out a remarkable consequence of factorization. They showed that under certain conditions one can completely forget about the space-time dependence of fields at $N=\infty$. We consider the standard U(N) lattice gauge theory. From this field theory one could obtain a matrix model by making the following replacement:

$$U_\mu(x) \to U_\mu \ . \tag{4.1}$$

The standard Wilson action becomes

$$S \to S_{\rm EK} = \beta \sum {\rm Tr}(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger + {\rm H.c.}) \ . \tag{4.2}$$

The quantity corresponding to a Wilson loop operator,

$$W(C) = {\rm Tr}[U_\mu(x) U_\nu(x+\mu) U_\mu(x+\mu+\nu) \cdots] \ , \tag{4.3}$$

is given by

$$W_R(C) = {\rm Tr}(U_\mu U_\nu U_\mu \cdots) \ , \tag{4.4}$$

which is just an ordered product of the reduced variables U_μ in the same order in which the corresponding links appeared in $W(C)$. The partition function of the reduced model is given by

$$Z = \int \prod_\mu dU_\mu \exp(-S) \ , \tag{4.5}$$

and reduced averages are obtained in the ensemble defined by Eq. (4.5):

$$\langle {\rm Tr} W_R(C) \rangle = \frac{1}{Z} \int \prod_\mu dU_\mu {\rm Tr} W_R(C) e^{-S_{\rm EK}} \ . \tag{4.6}$$

One could derive Dyson-Schwinger equations for $\langle W_R(C) \rangle$ in the same way as in the field theory. Consider the simple loop of Fig. 1 once again. The quantity $W_R(C)$ for this loop is given by

$$W_R(C) = U_\mu U_\mu U_\nu^\dagger \cdots U_\mu) \ .$$

To derive Dyson-Schwinger equations we start with the quantity

$$X_R^a(C) = \int \prod_\mu dU_\mu ({\rm Tr}\lambda^a U_\mu U_\mu \cdots) e^{-S_{\rm EK}} \ , \tag{4.7}$$

which is the direct analog of Eq. (3.6), and follow exactly the same steps as in Sec. III. The contribution from the variation of the action (the equation-of-motion term) is exactly the analog of Eq. (3.11), viz.,

$$\sum_{\nu \neq \mu} -i\varepsilon\beta Z \langle {\rm Tr}[W_R(C) U_{\mu\nu^+}] - {\rm Tr}[W_R(C) U_{\mu\nu^+}^\dagger] + {\rm Tr}[W_R(C) U_{\mu\nu^-}] - {\rm Tr}[W_R(C) U_{\mu\nu^-}] \rangle \ , \tag{4.8}$$

where

$$U_{\mu\nu^+} = U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger \ , \quad U_{\mu\nu^-} = U_\mu U_\nu^\dagger U_\mu^\dagger U_\nu \ . \tag{4.9}$$

We note that $U_{\mu\nu^+}$ ($U_{\mu\nu^-}$) would correspond [via Eq. (4.1)] to a plaquette in the (μ,ν) plane [$(\mu,-\nu)$ plane]. Thus Eq. (4.8) is the reduced version of the right-hand side of the equation in Fig. 2. The source terms come from variations of the U_μ's contained in $W_R(C)$. When the variation hits the first U_n in $W_R(C)$, one has, in analogy to Eq. (3.8),

$$i\varepsilon N Z \langle {\rm Tr} W_R(C) \rangle \ .$$

But now we have some extra source terms. These terms come from variations of all the other U_μ's contained in $W_R(C)$. Such terms are not present in the field theory case, since one could vary only the link $U_\mu(x)$—in fact such terms would occur only if the loop were self-intersecting. These extra source terms in the Eguchi-Kawai model are typically of the form

$$i\varepsilon \int \prod_\mu dU_\mu e^{-S_{\rm EK}} {\rm Tr}(\lambda^a U_\mu U_\mu \cdots \lambda^a U_\mu U_\mu U_\nu \cdots U_\mu) = i\varepsilon Z \langle {\rm Tr}(U_\mu U_\mu U_\nu^\dagger \cdots U_\mu) {\rm Tr}(U_\mu U_\mu U_\nu \cdots U_\mu) \rangle \ .$$

Here the variation has hit a U_μ that corresponds to the link starting at the point y in Fig. 1. Using factorization, the above quantity becomes

$$i\varepsilon Z \langle {\rm Tr}(U_\mu U_\mu U_\nu^\dagger \cdots U_\mu) \rangle \langle {\rm Tr}(U_\mu U_\mu U_\nu \cdots U_\mu) \rangle \ . \tag{4.10}$$

This is a product of U_μ's along open lines, i.e., the two open lines joining x and y. The Dyson-Schwinger equations for the Eguchi-Kawai model are identical to those of the field theory only if such open lines vanish.

The Eguchi-Kawai (EK) model, being a single-point theory, does not have any local gauge invariance. The action, Eq. (4.2) (as well as the measure), are, however, invariant under the following transformations:

$$U_\mu \to S U_\mu S^{-1} \ , \tag{4.11a}$$

$$U_\mu \to e^{i\theta_\mu} U_\mu \ . \tag{4.11b}$$

Equation (4.11a) is the remnant of local gauge invariance of the original field theory, while Eq. (4.11b) is a $[{\rm U}(1)]^d$ symmetry [Z_N^d for SU(N)]. The open-line traces in Eq. (4.10) are invariant under Eq. (4.11a), but not under Eq. (4.11b). Only Wilson loop operators along closed loops are invariant under both the symmetries. Eguchi and Kawai argued that the $[{\rm U}(1)]^d$ symmetry protects terms like Eq. (4.10) from acquiring a nonzero value—and hence the model (4.2) has the same Dyson-Schwinger equations as the parent lattice gauge theory. Assuming that the entire content of the $N=\infty$ limit is contained in the Dyson-Schwinger equations, it then follows that the reduced model described by Eq. (4.2) is completely equivalent to the standard Wilson theory at large N.

In the strong coupling region this is certainly true. The matrices U_μ are all fluctuating randomly: the eigenvalues of U_μ would be uniformly spread over the unit circle, thus maintaining the $[\mathrm{U}(1)]^d$ symmetry. In fact, this symmetry is unbroken for all coupling for dimensions less than or equal to two. It was, however, soon pointed out (Bhanot, Heller, and Neuberger, 1982a) that in weak coupling the symmetry (4.11b) is spontaneously broken for dimensions greater than two. The N eigenvalues of U_μ all tend to be equal to each other. The Eguchi-Kawai model as it stands is not equivalent to the standard lattice gauge theory in weak coupling—and hence certainly not in the continuum limit.

B. The quenched Eguchi-Kawai (QEK) model: φ^4 theory

Bhanot, Heller, and Neuberger (1982a) proposed a modification of the naive Eguchi-Kawai model, in which the above-mentioned $[\mathrm{U}(1)]^d$ symmetry does not break in weak coupling; their model is known as the quenched Eguchi-Kawai (QEK) model. We shall not describe the QEK model as originally formulated. Rather, we shall present it in the framework of more general considerations about the reduction mechanism in large-N theories.

A general formulation of reduced models has emerged in a series of papers beginning with the work of Parisi (1982). Consider a scalar field theory with the field $\varphi(x)$ in the adjoint representation of U(N). The lattice action is given by

$$S=\sum_x \left[\sum_\mu \tfrac{1}{2}\mathrm{Tr}\,|\varphi(x+\mu)-\varphi(x)|^2 + \tfrac{1}{2}m^2\mathrm{Tr}\varphi^2(x)+\frac{g}{N}\mathrm{Tr}\varphi^4(x)\right] \qquad (4.12)$$

[$\varphi(x)$ has been written as a $N\times N$ Hermitian matrix]. The large-N limit of this model is defined by

$$g=\text{ fixed},\quad N\to\infty .$$

The perturbation expansion of this model is very similar to that of the gauge theory—the leading-order diagrams are all planar.

A naive Eguchi-Kawai reduction prescription, i.e.,

$$\varphi(x)\to\varphi ,$$

does not lead to a model that is equivalent to Eq. (4.12). We consider, however, the reduction prescription

$$\varphi(x)\to D_k(x)\varphi D_k^\dagger(x) , \qquad (4.13)$$

where

$$[D_k(x)]_{ij}=\exp[i(k_i^\mu-k_j^\mu)x_\mu]\delta_{ij} \qquad (4.14)$$

is a matrix in the internal symmetry space. We shall refer to Eqs. (4.13) and (4.14) as the quenched momentum prescription. Applying this prescription to the action, Eq. (4.12), and factoring out the volume, one obtains the reduced action

$$S_{\mathrm{QEK}}^{(k)}=\tfrac{1}{2}\sum_{i,j}|\varphi_{ij}|^2\left[2d+m^2-2\sum_\mu\cos(k_i^\mu-k_j^\mu)\right] + \frac{g}{N}\mathrm{Tr}\varphi^4 \qquad (4.15)$$

which will be shown to be equivalent to the field theory, Eq. (4.12), at $N=\infty$. To spell out the precise sense in which these are equivalent, one must have a prescription that relates averages in the reduced theory to those in the field theory. Let us consider an invariant functional $f[\varphi(x)]$ of the field. The statement of equivalence then reads

$$\langle f[\varphi(x)]\rangle_{\text{field theory}}=\int\prod_{\mu,i}\left[\frac{dk_i^\mu}{2\pi}\right]\langle f[D_k(x)\varphi D_k^\dagger(x)]\rangle , \qquad (4.16)$$

where the average of a quantity $\widetilde{0}$ in the reduced model is defined by (for a fixed value of the k's)

$$\langle\widetilde{0}\rangle=\frac{1}{Z_k}\int\prod_{ij}d\varphi_{ij}e^{-S_{\mathrm{QEK}}^{(k)}}\widetilde{0} , \qquad (4.17)$$

$$Z_k=\int\prod_{ij}d\varphi_{ij}e^{-S_{\mathrm{QEK}}^{(k)}} . \qquad (4.18)$$

The origin of the epithet "quenched" is now clear. The action S_{QEK} defines an ensemble in which averages are to be taken for a fixed value of k. A quenched average over k is then performed. The k's are dynamical variables, but not on the same par as the φ_i's.

The form of the reduced action, Eq. (4.15), looks like the momentum-space action of the field theory, with $k_i^\mu-k_j^\mu$ behaving as the momentum. To make the connection precise, consider the zero-order propagator in the reduced model:

$$G_{ij}=\langle\varphi_{ij}^\dagger\varphi_{ji}\rangle=\frac{1}{2d-\sum_\mu\cos(k_\mu^i-k_\mu^j)+m^2} , \qquad (4.19)$$

which certainly looks like the usual momentum-space propagator. To show that this is really so, consider Eq. (4.16) with

$$f[\varphi(x)]=\varphi^\dagger(x)\varphi(0) .$$

The right-hand side becomes

$$\int\prod_{\mu i}\left[\frac{dk_i^\mu}{2\pi}\right]\sum_{ij}e^{i(k_i^\mu-k_j^\mu)x_\mu}\langle\varphi_{ij}^\dagger\varphi_{ji}\rangle=\int\prod_{\mu,i}\left[\frac{dk_i^\mu}{2\pi}\right]\sum_{ij}e^{i(k_i^\mu-k_j^\mu)x_\mu}\frac{1}{2d-\sum_\mu\cos(k_i^\mu-k_j^\mu)+m^2} . \qquad (4.20)$$

Note that Eq. (4.20) diverges badly for $i=j$. To avoid this we impose the constraint

$$\varphi_{ii}=0\ . \tag{4.21}$$

These are N constraints amongst N^2 variables. Hence they are irrelevant in the leading-order behavior at large N.

Equation (4.20) may be viewed in two equivalent ways.

(a) One could make a change of variables to

$$p^\mu=k_i^\mu-k_j^\mu,\ \ q^\mu=\tfrac{1}{2}(k_i^\mu+k_j^\mu)\ .$$

With this, Eq. (4.20) becomes

$$N(n-1)\int\frac{d^dp}{(2\pi)^d}e^{ip\cdot x}\frac{1}{2d-\sum_\mu \cos p_\mu+m^2}\ , \tag{4.22}$$

which is, up to $1/N$ corrections, equal to the usual propagator

$$\langle \mathrm{Tr}\varphi^\dagger(x)\varphi(0)\rangle \tag{4.23}$$

in the field theory. Note that the difference between Eqs. (4.22) and (4.23) is of order $1/N$ due to the presence of the constraint of Eq. (4.21).

(b) An alternative way of viewing this is to note that it is not necessary to perform the momentum integrations. This is because one can write

$$\sum_{i\neq j}f(k_i^\mu-k_j^\mu)=\sum_i\sum_{j\neq i}f(p_j^\mu)\ , \tag{4.24}$$

where

$$p_j^\mu=k_i^\mu-k_j^\mu\ ,$$

and f is any function. Now, p_j^μ lies in the Brillouin zone

$$-\pi<p_j^\mu<+\pi$$

(all momenta are in units of the inverse lattice spacing). Let us divide this hypercube in momentum space into N parts and choose the p_j^μ's densely and uniformly over the entire hypercube. In other words, each of the N parts in labeled by an index i which runs from 1 to N. The p_i^μ are chosen to be the particular momentum at the center of the cell labeled by i. Then, by the definition of a Riemann integral,

$$\int_{-\pi}^{+\pi}\frac{d^dp}{(2\pi)^d}f(p)=\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}f(p_i)\ . \tag{4.25}$$

Using this in Eq. (4.20) one gets the same result for $N=\infty$ as in Eq. (4.22).

The latter way of viewing the sum in Eq. (4.20) tells us how large N is. From Eq. (4.25) one sees that there is a total of N momenta to sum over. Now, if the original field theory is defined in a periodic box of side L, one has L^d momenta. Thus for the reduced model to be equivalent to the field theory one must have

$$N=L^d\ . \tag{4.26}$$

We have demonstrated Eq. (4.16) for the two-point function to $O(g^0)$. Of course, the equivalence holds order by order in the perturbation expansion. A perturbation expansion of the action $S_{\rm QEK}$ may be derived in the usual fashion. The lowest-order propagator suggests that we represent each propagator by a double line [Fig. 4(a)], with each line carrying a group index. This is the usual representation in the corresponding field theory ('t Hooft, 1974). However, here one assigns this double line a "momentum" (k_i-k_j). The propagator is, by definition, zero when $i=j$ (this follows from the constraint $\varphi_{ii}=0$). Vertices are similarly represented in Fig. 4(b). If k_i-k_j is to behave as a momentum, it must be conserved at each vertex. From Fig. 4(b) it is easily seen that this is true. In fact, the reason this is true is that each index line at the vertex flows in once and flows out once—as required by the internal symmetry of the theory. Thus the internal symmetry always guarantees momentum conservation. We shall present a more detailed explanation of this fact later.

Using the Feynman rules of Fig. 4 one can now compute any correlation function. Let us illustrate this for the $O(g^2)$ correction to the propagator. The relevant Feynman diagram is shown in Fig. 5. The contribution to $\langle\varphi_{ij}^\dagger\varphi_{ij}\rangle$ from this graph is given by

$$\frac{g^2}{N^2}\sum_{k\neq l}(G_{ij})^2G_{jl}G_{lk}G_{ki}\ . \tag{4.27}$$

The corresponding graph in the field theory is given by

$$g^2N^2\int\left[\frac{d\mathbf{q}}{2\pi}\right]\left[\frac{d\mathbf{r}}{2\pi}\right][G(p)]^2G(q)G(r)G(p-q-r)\ . \tag{4.28}$$

Renaming variables in Eq. (4.27),

$$\mathbf{k}_i-\mathbf{k}_j=\mathbf{p},\ \ \mathbf{k}_k-\mathbf{k}_l=\mathbf{r}\ ,$$

$$\mathbf{k}_l-\mathbf{k}_j=\mathbf{q},\ \ \tfrac{1}{4}(\mathbf{k}_i+\mathbf{k}_j+\mathbf{k}_k+\mathbf{k}_l)=\mathbf{Q}\ ,$$

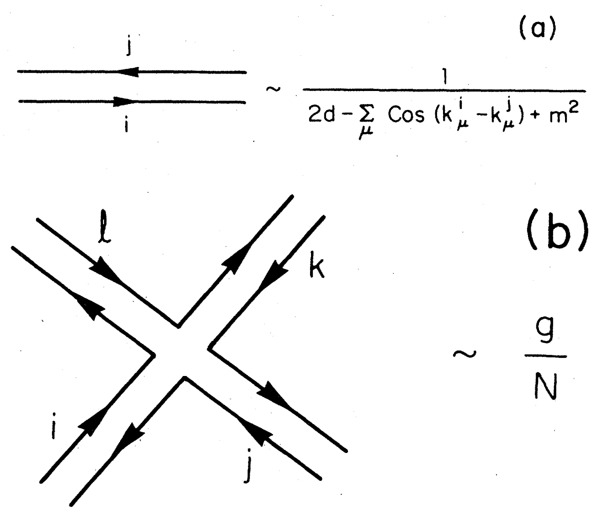

FIG. 4. Feynman rules for the φ^4 QEK model.

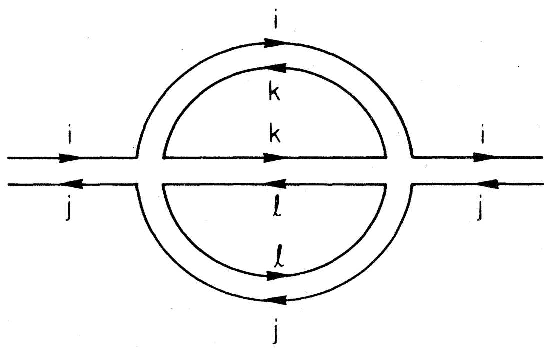

FIG. 5. Feynman graph for $O(g^2)$ contribution to the propagator.

one can now verify Eq. (4.16) explicitly in a way analogous to the zero-order case.

The equivalence stated in Eq. (4.16) holds only for planar graphs to all orders in perturbation theory because our way of assigning momenta to propagators in the reduced model does not work for nonplanar diagrams. In any nonplanar diagram of the field theory there is always at least one propagator that has its two indices equal to each other (e.g., Fig. 6). This would automatically be zero in the reduced model. Since the leading diagrams in the large-N limit are planar, the QEK model of Eq. (4.15) is equivalent to the field theory of Eq. (4.12) at $N=\infty$—at least to all orders of the perturbation expansion.

There is another way to understand this equivalence—within the framework of stochastic quantization. Any

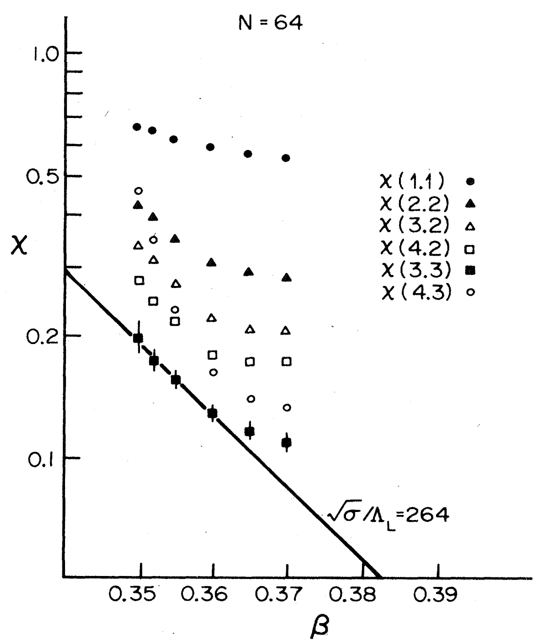

FIG. 6. χ ratios for the $N=64$ TEK model at zero temperature (reprinted from Fabricius and Haan, 1984).

quantum theory may be viewed as a dynamical statistical system evolving in a fifth "time" according to a Langevin equation with a Gaussian random noise (Parisi and Wu, 1981). The quantum averages are then equal to the long-time limit of stochastic averages of this equivalent Langevin system. In this framework it is easy to see how the space-time dependence of the fields factors out in the large-N limit, exactly according to the quenched momentum prescription (Alfaro and Sakita, 1982).

C. The QEK gauge theory

One might think that constructing a QEK model for the lattice gauge theory is straightforward: one simply needs to replace the original Eguchi-Kawai reduction prescription by a quenched momentum prescription for links. This is wrong. Consider the reduction ansatz

$$U_\mu(x)\to D_k(x)U_\mu D_k^\dagger(x)\ . \tag{4.29}$$

For a fixed value of $\{k\}$ the partition function becomes

$$Z_k=\int dU_\mu \exp\left[\beta\sum_{\mu>\nu}\mathrm{Tr}(U_\mu D_\mu^k U_\nu D_\mu^{k\dagger} D_\nu^k U_\mu^\dagger D_\nu^{k\dagger} U_\nu^\dagger + \mathrm{H.c.})\right],$$

where

$$(D_\mu^k)_{ij}=\exp(ik_i^\mu)\delta_{ij}\ . \tag{4.30}$$

Since the D_μ's commute, the QEK action may be rewritten as

$$S_{\rm QEK}=\beta\sum_{\mu>\nu}\mathrm{Tr}[(U_\mu D_\mu^k)(U_\nu D_\nu^k)(U_\mu D_\mu^k)^\dagger(U_\nu D_\nu^k)^\dagger] + \mathrm{H.c.} \tag{4.31}$$

One can, however, make a change of variables:

$$U_\mu\to U'_\mu=U_\mu D_\mu^k\ . \tag{4.32}$$

Since the Haar measure dU_μ is invariant, it is easy to see that in terms of U'_μ, Z_k is the partition function of the naive Eguchi-Kawai model. Replacing the naive EK reduction rule by the quenched momentum prescription did not change anything.

To get around this impasse, we need to alter either the integration measure (Das and Wadia, 1982; Gross and Kitazawa, 1982; Migdal, 1982) or the action (Chen, Tan, and Zheng, 1982). There is no unique way to change the measure. In the QEK model the remnant of gauge symmetry is

$$U_\mu D_\mu\to S(U_\mu D_\mu)S^{-1}\ . \tag{4.33}$$

One could first fix the gauge in a suitable fashion (say the Lorentz gauge) and introduce into the measure the constraint (Das and Wadia, 1982)

$$(\log U_\mu)_{ii}=0\ . \tag{4.34}$$

Another approach involving prior gauge fixing has been discussed by Parisi and Zhang (1983).

Gross and Kitazawa (1982) used a procedure that involved a gauge-invariant constraint and hence did not require prior gauge fixing. The measure they used may be written as

$$\int \prod_{\mu} dU_\mu C(U_\mu,D_\mu) , \tag{4.35}$$

where

$$C(U_\mu,D_\mu)=\prod_{\mu} \int dV_\mu \Delta(D_\mu)\delta(U_\mu D_\mu - V_\mu D_\mu V_\mu^\dagger) \tag{4.36}$$

and

$$\Delta(D_\mu)=\prod_{i<j} \sin^2\left[\frac{k_i^\mu - k_j^\mu}{2}\right] .$$

Here V_μ denotes a U(N) matrix. The delta function constrains the eigenvalues of $U_\mu D_\mu$ to be equal to those of D_μ. Since eigenvalues are invariant under the similarity transformation [Eq. (4.33)], this is an explicitly gauge-invariant constraint. A similar measure has also been proposed by Migdal (1982).

The effect of the constraints (4.34) and (4.36) is to destroy the invariance of the measure under the change of variables in Eq. (4.32). Recall that the naive EK model does not work because in weak coupling the eigenvalues of $(U_\mu D_\mu)$ tend to cluster around the same value. The constraint implied in Eq. (4.36) forces the eigenvalues to be equal to $e^{ik_i^\mu}$ and therefore to be randomly distributed over the unit circle, since the k's are totally random in the quenched model. This ensures that the correct vacuum is $U_\mu = I$. Equation (4.33) achieves the same end by constraining the diagonal elements of $\log U_\mu$. (Since the diagonal elements are not gauge invariant, one needs a prior gauge fixing.) Quenching thus prevents the $[\mathrm{U}(1)]^d$ from breaking, and hence forces all open lines to vanish.

To investigate the weak coupling perturbation expansion of the model, we expand U_μ around the vacuum,

$$U_\mu = \exp(igA_\mu) , \tag{4.37}$$

in powers of g and fix a gauge [which is already done if one uses the constraints specified by Eq. (4.33)]. There is a one-to-one correspondence between the Feynman graphs of the reduced model and those of the gauge theory, just as in the $\mathrm{Tr}\varphi^4$ model. In terms of the A_μ's the constraints (4.33) become

$$(A_\mu)_{ii}=0 , \tag{4.38}$$

which is the direct analog of the constraint $\varphi_{ii}=0$. The constraints in Eq. (4.36) also translate into equations relating $(A_\mu)_{ii}$ with the other $(A_\mu)_{ij}$, but those equations are different in different orders of perturbation theory. These, in general, generate new vertices apart from those contained in the action, leading to new tadpole graphs. Gross and Kitazawa (1982), however, showed that all such tadpole graphs vanish after the integration over the k's is performed.

While all the various types of constraints lead to reduced models that are equivalent to the gauge theory, for numerical purposes it is particularly convenient to use the measure (4.36), since it is explicitly gauge invariant. In fact, the QEK model with this measure is equivalent to the model proposed by Bhanot, Heller, and Neuberger (1982a). The full partition function is given by

$$Z_k = \int \prod_{\mu} dU_\mu dV_\mu \Delta(D_\mu)\delta(U_\mu D_\mu - V_\mu D_\mu V_\mu^\dagger) e^{-S_{\mathrm{QEK}}} , \tag{4.39}$$

with S_{QEK} given by Eq. (4.31). Now we integrate out the U_μ's. Due to the delta function this amounts to replacing U_μ by $V_\mu D_\mu V_\mu^\dagger D_\mu^\dagger$. Z_k now becomes

$$Z_k = \int \prod_{\mu} dV_\mu \Delta(D_\mu) \exp(-S'_{\mathrm{QEK}}) , \tag{4.40}$$

where

$$S'_{\mathrm{QEK}} = \beta \sum_{\mu>\nu} \mathrm{Tr}(V_\mu D_\mu V_\mu^\dagger V_\nu D_\nu V_\nu^\dagger V_\mu D_\mu^\dagger V_\mu^\dagger V_\nu D_\nu^\dagger V_\nu^\dagger + \mathrm{H.c.}) , \tag{4.41}$$

which is precisely the model of Bhanot, Heller, and Neuberger (1982a).

D. Quarks in QEK models

So far we have dealt with theories involving fields in the adjoint representation of the symmetry group. Fields in the fundamental representation may also be incorporated in a straightforward manner. In fact, a general quenched momentum prescription reads

$$\varphi(x) \to D(x)\cdot\varphi , \tag{4.42}$$

where the representation content of φ determines that of $D(x)$. Thus for a field in the fundamental representation,

$$\psi_i(x) \to D_{ij}^{(k)}(x)\psi_i ,$$

with the D's given by Eq. (4.14).

In gauge theories, internal quark lines are, of course, absent at $N=\infty$. However, one might study the meson spectrum by looking at, say, $\langle \bar\psi\psi(x)\bar\psi\psi(0)\rangle_c$. In the QEK model this connected correlation cannot be a function of x. This is because $\bar\psi\psi(x)$ is a local color singlet and hence translationally invariant in the reduced model. In index space this means that there can be no net index flow into a $\bar\psi\psi$ insertion, and hence no nonzero momentum. Gross and Kitazawa (1982), however, suggested that one can nevertheless force a net momentum to flow along the external quark lines. This would not jeopardize anything else, since there are no internal quark loops.

A more systematic approach is to consider a reduced model for the Veneziano limit of QCD. Such a model has been constructed and shown to be equivalent to the field theory (Levine and Neuberger, 1982a; see also Klinkhamer, 1984a).

E. Other models

The quenched momentum prescription may be applied to a variety of other models. For models involving fundamental representation fields only [e.g., the $(\varphi^2)^2$ theory] it readily yields an expression for the master field (Das and Wadia, 1982; Gross and Kitazawa, 1982). Consider the linear sigma model discussed in Sec. III. The two-point correlation function is given by

$$\sigma(x)=\frac{1}{N}\left\langle\sum_i \varphi_i^*(x)\varphi_i(0)\right\rangle$$
$$=\int_{-\pi}^{+\pi}\frac{d^dp}{(2\pi)^d}\frac{e^{ip\cdot x}}{4\sum_\mu \sin^2 p_\mu/2+m^2+4(\lambda/N)\sigma_0},$$

where the quantity σ_0 is determined by the self-consistent gap equation:

$$\sigma_0=\int_{-\pi}^{+\pi}\frac{d^dk}{(2\pi)^d}\frac{1}{4\sum_\mu \sin^2 k_\mu/2+m^2+4(\lambda/N)\sigma_0}.$$

Evidently, one would get the same equation in the QEK version of the model. The correlation function of the reduced fields is simply

$$\langle\varphi_i^*\varphi_i\rangle=\frac{1}{4\sum_\mu \sin^2 k_i^\mu/2+m^2+4(\lambda/N)\sigma_0},$$

and one obtains $\varphi(x)$ by the direct analog of Eq. (4.16). It is clear that Eq. (4.44) is obtainable from the reduced master field φ_i,

$$\tilde{\varphi}_i=\frac{1}{\left[4\sum_\mu \sin^2 k_i^\mu/2+m^2+4(\lambda/N)\sigma_0\right]^{1/2}},$$

which, when plugged back into the reduction prescription, leads to the master field of the full field theory:

$$\tilde{\varphi}_i(x)=\frac{e^{ik_i^\mu x_\mu}}{\left[4\sum_\mu \sin^2 k_i^\mu/2+m^2+4(\lambda/N)\sigma_0\right]^{1/2}}.$$

This correctly reproduces $\sigma(x)$ since, as argued earlier, a sum over the index i is equivalent to a momentum integration at $N=\infty$.

The master fields of other vectorlike models can be obtained in a similar manner. Gross and Kitazawa (1982) have also obtained the master field of two-dimensional pure QCD.

QEK models have been constructed for $SU(N)\times SU(N)$ chiral models (Heller and Neuberger, 1982a, 1982b; see also Bhanot, 1983; Green, 1983). In fact there has been some progress in attempts to solve the two-dimensional chiral model analytically (Bars, Gunaydin, and Yankelowicz, 1983).

F. Hamiltonian versions

Reduced models have been constructed for large-N Hamiltonian theories (Neuberger, 1982; Kitazawa and Wadia, 1983). This involves the reduction of the spatial dependence of fields, while retaining the temporal dependence. Thus typically the reduction prescription would read

$$\varphi(\mathbf{x},t)\to D_k(\mathbf{x})\varphi(t)D_k^\dagger(\mathbf{x}),$$

with $\mathbf{x}$ denoting the $(d-1)$-dimensional spatial position vector. The resulting model is simply a one-dimensional field theory, i.e., quantum mechanics. It has been argued that reduced Hamiltonians may be used to extract the glueball spectrum (Levine and Neuberger, 1982b). This cannot be done in Euclidean reduced models—one requires connected correlations of Wilson loops, which vanish due to factorization. Furthermore, Hamiltonian formulations can be used to obtain reduced models at finite temperature (Neuberger, 1983). This is done by simply restricting the total time extent of the box to a fixed value and imposing periodic boundary conditions in the usual manner.

G. QEK models in the continuum

All the above considerations may be applied to a field theory defined with a continuum regularization, e.g., a momentum cutoff. The quenched momentum prescription of Eq. (4.13) then readily yields the following expression for derivatives:

$$-i\partial_\mu\delta_{ik}\to(k_i^\mu-k_j)\delta_{ij}^\mu.$$

In fact, even in gauge theories a momentum cutoff provides a gauge-invariant regularization in the continuum (Gross and Kitazawa, 1982). This is because Ward identities are satisfied before integration over the momenta $\{k\}$.

H. The meaning of the quenched momentum prescription

We shall conclude this section by trying to investigate the meaning of quenched reduction. We consider the quenched momentum prescription once again,

$$\varphi(x)\to D_k(x)\varphi D_k^\dagger(x),$$
$$[D_k(x)]_{ij}=\delta_{ij}\exp[i(k_i^\mu-k_j^\mu)x_\mu].$$

Unlike the field in the naive EK model, the field $\varphi(x)$ is not translationally invariant at $N=\infty$. Rather, the translation group is represented within the internal symmetry group. At $N=\infty$ there are a large number of internal degrees of freedom. Some of these are used as "momenta." Since the translation group is Abelian, it is natural to represent it in the diagonal $[U(1)]^N$ subgroup of the internal $U(N)$ symmetry—and this is precisely what Eqs. (4.13) and (4.14) represent. In the next section we shall consider a different way of representing translations inside the internal symmetry group that works for an interesting class of models.

V. THE TWISTED EGUCHI-KAWAI (TEK) MODEL

A. General introduction

In the previous section we saw that quenched reduced models are obtained by representing translations within the diagonal subgroup of the internal symmetry group. In a sense this is a natural thing to do, since translations between two given points along different routes commute. However, if a theory contains fields that are in zero N-ality representations of SU(N) groups (like pure gauge theory), one has a much wider range of possibilities. One can now represent translations by matrices that fail to commute by an element of the center of the group, Z_N. Since zero N-ality fields are blind to the center, translations along different routes would still commute. Such a reduction scheme is the basis of twisted Eguchi-Kawai models (Eguchi and Nakayama, 1983; Gonzales-Arroyo and Okawa, 1983a). Consider a field theory defined on a lattice containing a field $\varphi(x)$ in the adjoint representation of SU(N). The twisted reduction prescription is

$$\varphi(x) \to D(x)\varphi D^{\dagger}(x) , \tag{5.1}$$

where

$$D(x) = \prod_{\mu} (\Gamma_\mu)^{x_\mu} \tag{5.2}$$

and Γ_μ are traceless SU(N) matrices obeying the 't Hooft algebra

$$\Gamma_\mu \Gamma_\nu = Z_{\nu\mu}\Gamma_\nu\Gamma_\mu . \tag{5.3}$$

$Z_{\mu\nu}$ is an element of the center of the group Z_N,

$$Z_{\mu\nu} = \exp\left[\frac{2\pi i}{N} n_{\mu\nu}\right] , \tag{5.4}$$

where $n_{\mu\nu}$ is an antisymmetric integer-valued $d\times d$ matrix (in d dimensions). Thus Γ_μ is the matrix that implements translations by one lattice spacing in the μ direction by means of adjoint action on φ. Since Γ_μ acts by adjoint action, the noncommutativity of the Γ_μ's does not lead to noncommutativity of translations. This would not be true if there were fields in the fundamental representation.

The reduced action is obtained by substituting Eq. (5.1) into the action of the field theory, i.e.,

$$S_{\rm TEK}(\varphi, n_{\mu\nu}) = \frac{1}{\rm vol} S[D(x)\varphi D^{\dagger}(x)] ,$$

and the partition function is given by

$$Z_{\rm TEK} = \int [d\varphi] \exp(-S_{\rm TEK}) , \tag{5.5}$$

for a *fixed* value of $Z_{\mu\nu}$. The expectation value of any functional of the reduced field φ is given by

$$\langle O(\varphi)\rangle_{\rm TEK} = \frac{1}{Z_{\rm TEK}} \int [d\varphi] O(\varphi) e^{-S_{\rm TEK}} . \tag{5.6}$$

The correspondence between correlation functions of the reduced model with those in the field theory is as follows. Let $f[\varphi(x)]$ be any invariant functional of the field $\varphi(x)$. Then

$$\langle f[\varphi(x)]\rangle_{\text{field theory}} = \langle f[D(x)\varphi D^{\dagger}(x)]\rangle_{\rm TEK} . \tag{5.7}$$

All these relations are for a fixed value of $Z_{\mu\nu}$. Note we are not summing over various translation matrices as in the QEK model. Of course, Eq. (5.7) would not hold for any $Z_{\mu\nu}$. *In fact, $Z_{\mu\nu}$ must be chosen so that Eq. (5.7) holds.* The choice of $Z_{\mu\nu}$, which respects this equivalence, depends on the specific model and on the dimensionality of space-time.

B. The TEK gauge theory

Let us now apply the twisted-reduction idea to lattice gauge theory and figure out what $Z_{\mu\nu}$ should be (Gonzales-Arroyo and Okawa, 1983a). The reduction rule for the link matrices is a direct generalization of Eq. (5.1):

$$U_\mu(x) \to D(x) U' D_\mu^{\dagger}(x) , \tag{5.8}$$

with $D(x)$ given by Eq. (5.2). The standard Wilson action now becomes (apart from the trivial volume factor)

$$S'_{\rm TEK} = \beta \sum_{\mu>\nu} {\rm Tr}(U'_\mu \Gamma_\mu U'_\nu \Gamma_\mu^{\dagger} \Gamma_\nu U'^{\dagger}_\mu \Gamma_\nu^{\dagger} U'^{\dagger}_\nu) + {\rm H.c.} \tag{5.9}$$

Using the algebra of Γ matrices in Eq. (5.3), this becomes

$$S'_{\rm TEK} = \beta \sum_{\mu>\nu} {\rm Tr}[Z_{\mu\nu}(U'_\mu\Gamma_\mu)(U'_\nu\Gamma_\nu)(U'_\mu\Gamma_\mu)^{\dagger}(U'_\nu\Gamma_\nu)^{\dagger}] + {\rm H.c.} \tag{5.10}$$

The partition function of the TEK gauge theory is given by

$$Z_{\rm TEK} = \int \prod_\mu dU'_\mu \exp(-S'_{\rm TEK}) , \tag{5.11}$$

where dU'_μ is the standard Haar measure. Making a change of variables,

$$U'_\mu \to U'_\mu \Gamma_\mu = U_\mu , \tag{5.12}$$

and using the invariance of the Haar measure, one gets

$$Z_{\rm TEK} = \int \prod_\mu dU_\mu \exp(-S_{\rm TEK}) , \tag{5.13}$$

$$S_{\rm TEK} = \beta \sum_{\mu>\nu} {\rm Tr}(Z_{\mu\nu} U_\mu U_\nu U_\mu^{\dagger} U_\nu^{\dagger}) + {\rm H.c.} \tag{5.14}$$

The reduced form for the Wilson loop operator is obtained by simply plugging in the reduction rule, Eq. (5.8). In terms of the U_μ variables, one has

$$W_R(C) = \left[\prod_{\mu\nu} (Z_{\mu\nu})^{N_{P_{\mu\nu}}}\right] {\rm Tr}(U_\mu U_\nu U_\mu \cdots) . \tag{5.15}$$

The quantity inside the trace is simply an ordered product of U_μ's in the same order in which they appeared in the field theory. $N_{P_{\mu\nu}}$ denotes the number of plaquettes in the $(\mu\nu)$ plane in the minimal surface spanning C.

Everything looks just like the naive Eguchi-Kawai model, apart from some Z_N factors. However, it is these Z_N factors which, when properly chosen, force the system to the correct vacuum at weak coupling.

The derivation of the Dyson-Schwinger equations for $W_R(C)$ in the TEK model is exactly like that in the Eguchi-Kawai model. Once again these equations are identical to the loop equations of the gauge theory, apart from products of traces of U_μ's along open lines. We consider such an open line extending from the origin to the point $\{k_\mu\}$. The remnant of gauge symmetry in the TEK model is the same as Eq. (4.11a). The $[\mathrm{U}(1)]^d$ symmetry is now a $(Z_N)^d$ symmetry [since we are dealing with an SU(N) theory],

$$U_\mu \to Z_\mu U_\mu (Z_\mu \varepsilon Z_N) \ . \tag{5.16}$$

Once again, in strong coupling this symmetry is unbroken—forcing all open lines to vanish. In weak coupling U_μ fluctuates around the vacuum value $U_\mu^{(0)}$, which minimizes the action. This is easily seen to be

$$U_\mu^{(0)} = \Gamma_\mu \ . \tag{5.17}$$

Thus in extreme weak coupling the trace of products of links along the open line from 0 to (k_μ) is easily seen to be

$$V(k) = Z \,\mathrm{Tr} \prod_\mu (\Gamma_\mu)^{k_\mu} \ .$$

where Z is a Z_N factor that depends on the particular route taken from 0 to $\{k_\mu\}$. To see whether this trace vanishes, let us first prove the following simple theorem.

Theorem: Let A and B be two SU(N) matrices and let

$$AB = e^{i\delta} BA \ , \tag{5.18}$$

such that $\delta \neq 2\pi k$ for any integer k. Then (i) $\delta = 2\pi n/N$ where n is an integer less then N and (ii) $\mathrm{Tr}AB = \mathrm{Tr}A = \mathrm{Tr}B = 0$. To prove (i), we take the determinant of both sides of Eq. (5.18):

$$(e^{i\delta N} - 1)\det(AB) = 0 \ .$$

Since $\det(AB) \neq 0$ and $\delta \neq 2\pi k$, one must have $\delta = 2\pi n/N$. To prove (ii), we take the trace of Eq. (5.18). This gives

$$(e^{i\delta} - 1)\mathrm{Tr}(AB) = 0 \ .$$

Since $e^{i\delta} \neq 1$, $\mathrm{Tr}(AB) = 0$. Similarly, from Eq. (5.18),

$$A = e^{i\delta} BAB^\dagger \ .$$

Now let us substitute

$$A = \Gamma_\mu \ \text{ and } \ B = V(k)$$

in the above theorem. By virtue of the algebra in Eq. (5.3), a relationship of the type (5.18) holds. Thus $\mathrm{Tr}V(k)$ can be nonzero only if

$$[V(k), \Gamma_\mu] = 0 \ \text{ for all } \mu \ . \tag{5.19}$$

Using the explicit form for $V(k)$ this leads to the condition

$$k_\mu n_{\mu\alpha} = q_\alpha N \ , \tag{5.20}$$

where q_α are integers (mod$N$).

(a) In two dimensions, $n_{\mu\alpha}$, being antisymmetric, must be of the form

$$n_{\mu\alpha} = n\,\varepsilon_{\mu\alpha} \quad (n = \text{integer}) \ .$$

Equation (5.20) may then be inverted to give

$$n k_\mu = \varepsilon_{\mu\nu} q_\nu N \ . \tag{5.21}$$

Now we choose $n = 1$. Then Eq. (5.21) means that for all open lines whose trace is nonzero, k_μ is proportional to N. We let the parent field theory be defined in a box of size N with periodic boundary conditions. Then the nonzero $V(k)$'s correspond to open lines in the field theory that run from one end of the box to the other and hence are closed by boundary conditions. However, these open lines are nonzero even in the field theory—and such terms are present in the loop equations of the field theory. All other open lines vanish. Hence the TEK model with $n = 1$ has identical loop equations with those of the field theory.

(b) In four dimensions we shall consider twists of the form

$$\tfrac{1}{4}\tilde{n}_{\mu\nu} n_{\mu\nu} = \sigma N \ , \tag{5.22}$$

where σ is an integer (mod$N$) and

$$\tilde{n}_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta} n_{\alpha\beta} \ . \tag{5.23}$$

Furthermore,

$$\tilde{n}_{\mu\nu} n_{\rho\nu} = \sigma N \delta_{\mu\rho} \ . \tag{5.24}$$

Equation (5.24) may be used to invert Eq. (5.20), leading to

$$\sigma k_\mu = \tilde{n}_{\mu\nu} q_\nu \ . \tag{5.25}$$

Let L be some integer, and let

$$N = L^2 \ . \tag{5.26}$$

Let us choose the symmetric twist:

$$n_{\mu\nu} = L \quad \text{for all } \nu > \mu \ . \tag{5.27}$$

Then $\sigma = 1$, and Eq. (5.25) means that k_μ must be proportional to L. Using the same argument as in the two-dimensional case, one sees that the TEK model with the twist given by Eqs. (5.26) and (5.27) is equivalent to the field theory defined on a periodic box of size L.

For odd numbers of dimensions, the matrix $n_{\mu\nu}$ is singular, and it is awkward to construct twists (see, however, Gocksch, Neri, and Rossi, 1983).

We have so far considered only simple twists. There can be in general a wide class of twists leading to interesting structures (Brihaye, Maiella, and Rossi, 1983; Fabricius, Haan, and Filk, 1984).

C. Twist-eating configurations

We now investigate the vacuum of the TEK theory. In d dimensions we need d traceless matrices Γ_μ satisfying the algebra

$$\Gamma_\mu\Gamma_\nu = Z_{\nu\mu}\Gamma_\nu\Gamma_\mu .$$

Since Γ_μ denotes the translation operator for a single lattice spacing along the μ direction, none of these matrices can be products of the others. Furthermore, these matrices are determined only up to unitary transformations.

van Baal (1983) has discussed a general procedure for constructing the twist-eaters, i.e., the Γ_μ's, given the twist matrix $n_{\mu\nu}$. We shall, however, restrict ourselves to the simple twists referred to above. For two dimensions, the algebra is given by

$$\Gamma_1\Gamma_2 = \exp\left[\frac{2\pi i}{N}\right]\Gamma_2\Gamma_1 .$$

These matrices have been constructed by 't Hooft (1981). They are given by, modulo unitary transformations:

$$\Gamma_1 = P = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & & & & \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$
$$\Gamma_2 = Q = \begin{bmatrix} 1 & & & \\ & e^{2\pi i/N} & & 0 \\ & & e^{4\pi i/N} & \\ & 0 & & \end{bmatrix}. \tag{5.28}$$

In four dimensions, with the twist given in Eq. (5.27), one must have four $L^2 \times L^2$ matrices satisfying

$$\Gamma_\mu\Gamma_\nu = e^{\pm 2\pi i/L}\Gamma_\nu\Gamma_\mu .$$

These may be constructed in a fashion entirely analogous to the construction of representations of Clifford algebras. A particularly convenient choice is given by the direct-product matrices:

$$\Gamma_0 = Q_L \otimes Q_L, \quad \Gamma_1 = Q_L P_L \otimes Q_L ,$$
$$\Gamma_2 = P_L \otimes Q_L, \quad \Gamma = 1 \otimes Q_L , \tag{5.29}$$

where P_L and Q_L are the $L \times L$ matrices given by Eq. (5.28), with N replaced by L.

From Eqs. (5.28) and (5.29) it is now clear why the $[Z_N]^d$ symmetry (which protects open lines from acquiring any nonzero value) is not broken even in weak coupling. The eigenvalues of each of the Γ_μ's are given by the set $(1, e^{2\pi i/L}, e^{4\pi i/L}, \ldots, e^{2\pi(L-1)i/L})$, which are thus uniformly distributed over the unit circle. This explicitly respects the $(Z_N)^d$ symmetry, since the action of the symmetry is simply to shuffle the eigenvalues.

Note that the Γ_μ's for the four-dimensional case obey

$$\Gamma_\mu^L = 1 .$$

This is simply a manifestation of the fact that Γ_μ is the translation operation in a periodic box of extent L.

D. Planar perturbation theory

In the quenched Eguchi-Kawai model, the reduced field φ_{ij} itself became the analog of the fields of the parent theory in the momentum representation. In the TEK model, "momenta" are generated from the Γ matrices themselves. The weak coupling perturbation expansion is performed by expanding U_μ about the vacuum Γ_μ:

$$U_\mu = e^{iga_\mu}\Gamma_\mu, \quad a_\mu = a_\mu^\dagger . \tag{5.30}$$

where $\beta = 1/g^2$. a_μ is the reduced gluon field. Usually one expands a_μ in a basis formed by the standard λ matrices. In our case it is useful to use the following basis in the Lie algebra of SU(N):

$$A(q) = \Gamma_0^{k_0}\Gamma_1^{k_1}\Gamma_2^{k_2}\Gamma_3^{k_3} , \tag{5.31}$$

where

$$k_\nu = \frac{1}{L}\tilde{n}_{\mu\nu} q_\mu , \tag{5.32}$$

and q_μ are integers in the range $1 \le q_\mu \le L$ [except $q_\mu = L$ for all μ to ensure tracelessness of $A(q)$]. The $A(q)$'s form a set of N^2-1 traceless, unitary, linearly independent matrices. Let us list some useful properties of $A(q)$:

$$A(L-q) = A(-q) , \tag{5.33}$$

$$A^\dagger(q) = A(-q)\exp\left[\frac{2\pi i}{N}\langle k \mid k\rangle\right] , \tag{5.34}$$

$$\mathrm{Tr}[A(q_1)A(q_2)\cdots A(q_n)] = N[\delta(\Sigma q_i)]\exp\frac{2\pi i}{N}\sum_{i<j}\langle k_i \mid k_j\rangle , \tag{5.35}$$

$$\mathrm{Tr}[A^\dagger(q_1)A(q_2)\cdots A(q_n)] = N\delta\left[-q_1 + \sum_{i=2}^{n} q_i\right]\exp\frac{2\pi i}{N}\sum_{i<j}\langle k_i \mid k_j\rangle\exp\left[\frac{2\pi i}{N}\langle k_1 \mid k_1\rangle\right] , \tag{5.36}$$

where

$$\langle k_i \mid k_j \rangle = \sum_{\mu>\nu} n_{\nu\mu}(k_i)_\mu (k_j)_\nu \ . \tag{5.37}$$

These relations may be easily derived from the basic commutation relations. The reduced field a_μ is expanded in the basis $\{A(q)\}$:

$$a_\mu = \frac{1}{L^\alpha} \sum_{\{q\}} a_\mu(q) A(q) \ . \tag{5.38}$$

The value of α will be derived below. To ensure the Hermiticity of a_μ, one must require

$$a_\mu^*(q) = a_\mu(-q) \exp\left[-\frac{2\pi i}{N} \langle k \mid k \rangle \right] \ . \tag{5.39}$$

The basic property of the $A(q)$'s, which allows one to interpret the q's as momenta, is

$$\Gamma_\mu A(q) \Gamma_\mu^\dagger = \exp\left[-\frac{2\pi i}{L} q_\mu \right] A(q) \ , \tag{5.40}$$

which can be easily shown from the commutation relations. Let us consider the field $a_\mu(x)$ in the parent field theory. The reduction prescription relates this to the reduced field a_μ by

$$a_\mu(x) = D(x) a_\mu D^\dagger(x) \ ,$$

where, as before,

$$D(x) = \prod_\mu (\Gamma_\mu)^{x_\mu} \ .$$

We consider a translation of a single unit in the μ direction in the field theory:

$$\begin{aligned} a_\mu(x+\mu) &= D(x+\mu) a_\mu D^\dagger(x+\mu) \\ &= D(x) \Gamma_\mu a_\mu \Gamma_\mu^\dagger D^\dagger(x) \ , \end{aligned} \tag{5.41}$$

which, by Eqs. (5.38) and (5.40), becomes

$$a_\mu(x+\mu) = \frac{1}{L^\alpha} \sum_q e^{-2\pi i/L q_\mu} a_\mu(q) D(x) A(q) D^\dagger(x) \ . \tag{5.42}$$

This clearly shows that

$$P_\mu = \frac{2\pi q_\mu}{L} \ , \tag{5.43}$$

where P_μ behave as the lattice momenta in a box of size L and $a_\mu(q)$ are the momentum-space components of the fields.

To perform the perturbation expansion one has of course, to, fix a gauge. The analog of the Lorentz gauge is, for example,

$$\sum_\mu (\Gamma_\mu a_\mu \Gamma_\mu^\dagger - a_\mu) = 0 \ . \tag{5.44}$$

The kinetic piece for a_μ now becomes

$$2\beta \sum_{\mu,\nu} \mathrm{Tr}(\Gamma_\mu a_\nu \Gamma_\mu^\dagger - a_\nu)^2 \ . \tag{5.45}$$

Using the expansion (5.38) and Eq. (5.40), we obtain

$$\Gamma_\mu a_\nu \Gamma_\mu^\dagger - a_\nu = \frac{1}{L^\alpha} \sum_q (e^{-2\pi i q_\mu/L} - 1) a_\nu(q) A(q) \ .$$

Plugging this into Eq. (5.45) and using

$$\mathrm{Tr} A^\dagger(q_1) A(q_2) = N \delta(q_1 - q_2)$$

[which follows from Eq. (5.36)], one has for the kinetic term

$$\frac{N}{L^\alpha} \sum_q \sum_\mu 2d - 2 \sum_\nu \cos\left[\frac{2\pi}{L} q_\nu \right] a_\mu^*(q) a_\mu(q) \ , \tag{5.46}$$

which shows readily that the zero-order propagator has the same form as that on an L^4 periodic lattice. Consider the zero-order propagator in coordinate space. Using the reduction rule and applying Eq. (5.40) repeatedly, one has

$$\begin{aligned} \mathrm{Tr}\langle a_\mu(x) a_\nu(0) \rangle &= \delta_{\mu\nu} \frac{N}{L^{2\alpha}} \sum_{\{q\}} e^{-2\pi i/L \mathbf{q}\cdot\mathbf{x}} \langle a_\mu^*(q) a_\mu(q) \rangle \\ &= \delta_{\mu\nu} \sum_{\{q\}} e^{-2\pi i/L \mathbf{q}\cdot\mathbf{x}} \frac{1}{2d - 2\sum_\nu (\cos 2\pi/L q_\nu)} \ . \end{aligned} \tag{5.47}$$

In the $L \to \infty$ limit, the sum over $\{q\}$ goes over to an integral over the Brillouin zone:

$$\sum_{\{q\}} \to L^d \int d^d q \ .$$

Thus, the claim of equivalence stated in Eq. (5.6) is true if

$$N^2 = L^d \ , \tag{5.48}$$

which is certainly true for the twists we considered for $d=2$ and 4. In fact, Eq. (5.48) is a general statement about the order of N in TEK models. This is to be contrasted with QEK models where one had $N = L^d$.

The various interaction terms in the reduced action may be written down in an entirely analogous fashion. The momenta $\{q\}$ are always conserved at each vertex, since a term involving a product of n gluon fields would have the trace

$$\mathrm{Tr}[A(q_1) \cdots A(q_n)] \ ,$$

which is proportional to $\delta(\Sigma q_i)$ by Eq. (5.35). The momentum dependence of the vertices is also identical to that in the field theory, apart from the phase factor

$$\exp\left[\frac{2\pi i}{N}\sum_{i<j=1}^{n}\langle k_i \mid k_j\rangle\right], \tag{5.49}$$

which comes from the above trace. The Feynman graphs for various Green's functions of the TEK model are thus in one-to-one correspondence with those of the field theory, with the following differences.

(a) There is an extra phase in each n-gluon vertex, as given by Eq. (5.49).

(b) If $a_\nu(q)$ is to be identified with the momentum-space gluon field, the propagator should be $\langle a_\mu(q)a_\nu(-q)\rangle$ rather than $\langle a_\mu^*(q)a_\nu(q)\rangle$. This gives an extra phase factor of $\exp[(-2\pi i/N)\langle k \mid k\rangle]$ for each propagator ($k_\nu = 1/L\tilde{n}_{\mu\nu}q_m$)—as evident from Eq. (5.39).

(c) In the graphs of the reduced theory there is no remaining trace over the internal symmetry group—the trace has already been performed when the action is written in terms of $a_\mu(q)$'s.

The presence of extra phase factors is a potential problem in arbitrary Feynman graphs unless they cancel. A typical phase factor has the form

$$\exp\left[\frac{2\pi i}{N}\langle k \mid k'\rangle\right] = \exp(iLA_{\mu\nu}P_\mu P_\nu),$$

where the p's are the lattice momenta $P_\mu = 2\pi q_\mu/L$, and $A_{\mu\nu}$'s are coefficients that can easily be determined. One thus has (in the limit $L\to\infty$) momentum integrals of the form

$$\int \frac{d^dP}{(2\pi)^d} e^{iLA_{\mu\nu}P_\mu P_\nu} f(\mathbf{P}).$$

For large L the phase factor rapidly oscillates, leading to a zero answer. In fact (provided the integral above is regularized in the ultraviolet and infrared), the Riemann-Lebesgue lemma states (Eguchi, 1983)

$$\lim_{N\to\infty}\int_0^1 e^{iNt}f(t)dt \sim O\left[\frac{1}{N}\right]. \tag{5.50}$$

Thus in d dimensions a diagram containing nonzero phase factors vanishes as $O(1/L^d)$.

It turns out, however, that in all planar diagrams the phase factors at vertices exactly cancel those coming from propagators. Furthermore, all nonplanar diagrams have nonzero phase factors. Hence they are suppressed by $O(1/L^d) = O(1/N^2)$ (Eguchi and Nakayama, 1983; Gonzales-Arroyo and Okawa, 1983a). We shall not repeat the demonstration of this cancellation. For the gauge theory this is discussed in detail in the original paper of Gonzales-Arroyo and Okawa, while a similar discussion for matrix models is contained in the work of Eguchi and Nakayama (1983).

In the field theory all planar diagrams have the same N dependence. This comes about by a combination of factors of N contained in the vertex (through N dependence of the coupling g^2, since g^2N is fixed) and those coming from the sum over color indices. As noted above, the diagrams of the TEK model do not contain any index sums. Thus all vertices in the TEK model must be $O(1)$ (Das, 1983). We consider the d-dimensional model. For generality let $N = L^m$. A term in the action involving n gluon fields has a sum over $(n-1)$ momenta—one of the momentum sums being killed by momentum conservation. In counting the powers of L in the n-gluon vertex, care must be taken to convert momentum sums into integrals, since these involve powers of L and hence powers of N. The L dependence of this vertex is then

(i) L^m from the trace over products of $A(q)$'s.

(ii) $L^{-\alpha n}$ from the normalization factor in Eq. (5.38).

(iii) $(L^d)^{n-1}$ from conversion of a sum over $(n-1)$ momenta into integrals.

(iv) $(N)^{-(n-2)/2} = L^{-m(n-2)/2}$ from the coupling. (The n gluon fields bring down a factor of g^n. Due to the overall $1/g^2$, one is left with g^{n-2}. Since g^2N is fixed, the above N dependence follows.)

Thus, the total L dependence is

$$L^{n(d-m/2-\alpha)+(2m-d)}.$$

For this to be $O(1)$ for *all* values of n, one must have

$$m = d/2, \quad \alpha = d - m/2, \tag{5.51}$$

which gives $\alpha = \frac{3}{2}$ for $d=2$, $\alpha=3$ for $d=4$, and our known results—$N=L$ for $d=2$ and $N=L^2$ for $d=4$. This ensures that all planar graphs in the reduced model have the same N dependence.

E. Quarks in TEK models

As mentioned earlier, it is not possible to construct TEK models for theories containing fields in the fundamental representation, since these fields carry a Z_N charge. Thus quarks cannot be incorporated in a straightforward fashion. However, if the number of flavors of quarks also goes to infinity, it is possible to undo the twist in color space by an opposite twist in the flavor space (Das, 1983). This yields a twisted reduced model for the Veneziano limit of QCD. Consider a quark field theory transforming as the $(N_c, \overline{N}_f)$ representation of the (color)$\times$(flavor) group $SU(N_c)\times SU(N_f)$, denoted by $\psi_{ia}(x)$. Here $i=1,\ldots,N_c$ is the color index, and $a=1,\ldots,N_f$ is the flavor index. The twisted reduction prescription is

$$\begin{aligned}\psi(x) &= \overline{D}(x)\psi P^\dagger(x),\\ \overline{\psi}(x) &= P(x)\overline{\psi}D^\dagger(x),\end{aligned} \tag{5.52}$$

where $D(x)$ is, as before,

$$D(x) = \prod_\mu (\Gamma_\mu)^{x_\mu}$$

and (5.53)

$$P(x) \equiv \prod_\mu (G_\mu)^{x_\mu}.$$

Translation invariance is maintained if the G_μ's obey the same algebra as Γ_μ:

$$G_\mu G_\nu = Z_{\nu\mu} G_\nu G_\mu \ .$$

Models for $N_f = N_c$ can now be readily constructed with the standard QCD Lagrangian and can be shown to be equivalent to the corresponding field theory in the Veneziano limit:

$$N_c, N_f \to \infty \ ,$$
$$N_f / N_c = 1, \ \ g^2 N_c = g^2 N_f = \text{fixed} \ .$$

It is problematic to incorporate Kogut-Susskind fermions in a fully reduced model, since by their very nature these fermions essentially live in a unit hypercube rather than a point. However, partially reduced models with Kogut-Susskind fermions can be constructed. This can, in fact, be done for a variable ratio N_f/N_c (Fabricius and Korthals-Altes, 1986).

F. Hot TEK models

With the quenched momentum prescription one could retain the temporal dependence of fields and reduce their spatial dependence. To get a finite-temperature theory, one could simply consider a finite temporal extent and impose periodic boundary conditions (Neuberger, 1982). In the TEK model there are difficulties in implementing this method in a straightforward fashion due to the singular nature of twist matrices in odd dimensions. Nevertheless, Gocksch *et al.* (Gocksch, Neri, and Rossi, 1984) have shown that with a spatially chosen spatial twist n_{ij} one can construct a partially reduced model (i.e., with no reduction along the temporal direction) that is equivalent to the finite-temperature theory up to one loop in perturbation theory. It is not clear, however, whether this equivalence persists to all orders or nonperturbatively.

There is, however, another way of constructing TEK models that is rigorously equivalent to a finite-temperature field theory, which we now discuss (Klinkhamer and van Baal, 1984).

The symmetric-twist TEK model [for an SU(N) gauge theory] is equivalent to the corresponding field theory defined in a periodic box of size L ($N = L^2$). This means that at $N = \infty$, the box size goes to infinity in all directions. If it were possible to construct twists such that at $N = \infty$ the spatial extent of the box went to infinity while the temporal extent remained finite, one would have a single-point model equivalent to a finite-temperature field theory (with the inverse temperature given by the temporal extent). Klinkhamer and van Baal (1984) have constructed several such twists. Let us write down the most useful one. The twist tensor is given by

$$\eta_{\mu\nu} = N_0 \begin{bmatrix} 0 & -2k^2(4k^2-1) & 2k(4k^2-1) & 2k^2(4k^2-1) \\ & 0 & 2k(2k+1) & 4k^2-1 \\ & & 0 & 2k(2k-1) \\ & & & 0 \end{bmatrix}, \quad (5.54)$$

where N_0 and k are integers. N is related to N_0 and k by

$$N = 2N_0^2 k(4k^2-1) \ . \quad (5.55)$$

The TEK model with the above twist is then equivalent, at $k = \infty$, to a gauge theory in a periodic box of sides $N_0 \times N_1 \times N_2 \times N_3$ where

$$N_1 = 2N_0 k(2k-1) \ ,$$
$$N_2 = N_0(4k^2-1) \ ,$$
$$N_3 = 2N_0 k(2k+1) \ .$$

This is obviously a finite-temperature theory. The lattice temperature T is

$$T = \frac{1}{N_0 a} \ ,$$

(where a is the lattice spacing) and becomes equal to the physical temperature in the limit $N_0 \to \infty$, $a \to 0$ with $(N_0 a) = \text{fixed}$.

At sufficiently high physical temperature the gauge theory is expected to deconfine. The order parameter for deconfinement is the Polyakov-Wilson line,

$$W = \text{Tr} \prod_{t=1}^{N_0-1} U_0(\mathbf{x}, t) \ , \quad (5.56)$$

where $U_0(\mathbf{x},t)$ is the timelike link originating at the site labeled by $(\mathbf{x},t)$ [$\mathbf{x}$ is the $(d-1)$-dimensional position vector]. W is thus the product of links along a straight timelike line running from one end of the box to the other and hence closed by virtue of periodic boundary conditions. In the confined phase, $W=0$, while $W \neq 0$ signals deconfinement.

In the above "hot"-twist TEK model the reduced Wilson line is simply given by

$$W_R = \text{Tr} U_0^{N_0} \ . \quad (5.57)$$

An extreme weak coupling, the functional integral is dominated by the following twist-eating configuration:

$$\begin{aligned} U_0^{(0)} &= Q_1^{-1} \otimes P_2^{2k(2k+1)(4k^2-1)} Q_2^{4k(1-4k^2)} \ , \\ U_1^{(0)} &= P_1^{k+1} \otimes P_2^{2k(2k+1)(k+1)} Q_2^{-(2k+1)^2} \ , \\ U_2^{(0)} &= P_1 \otimes P_2^{2k(2k+1)} Q_2^{-4k} \ , \\ U_3^{(0)} &= P_1^{1-k} \otimes P_2^{(1-2k^2)(2k-1)} Q_2^{(2k-1)^2} \ , \end{aligned} \quad (5.58)$$

where (P_1, Q_1) are $N_0 \times N_0$ matrices of the form given in Eq. (5.28), and (P_2, Q_2) are similar $M_2 \times M_2$ matrices where $M_2 = 2N_0 k(4k^2-1)$. Thus in weak coupling

$$W = \text{Tr} U_0^{N_0} \neq 0 \ ,$$

while in strong coupling $\text{Tr} U_0^{N_0} = 0$ due to standard reasons. Hence at some intermediate coupling there is a deconfining phase transition. Numerical results on this transition will be discussed in the next section.

The hot twist discussed above is one of several choices that generalize the TEK model to finite temperature. A general analysis of hot twists has been carried out by Fabricius and Korthals-Altes (1984).

Hot twists may also be used to write down Hamiltonians for TEK models (Klinkhamer, 1984b). This is done by considering the hot-twist model for $N_0=1$ and writing

$$Z_{\rm TEK}(N_0=1)={\rm Tr}\hat{T}_{\rm TEK}$$

for $a_0 \to 0$, and $\hat{T}_{\rm TEK}=\exp(-a_0\hat{H}_{\rm TEK})$, where $\hat{H}_{\rm TEK}$ is the desired Hamiltonian.

There is an alternative way to simulate finite-temperature effects in lattice gauge theories. This involves a symmetric box, i.e., the same number of lattice sites in all directions, but with asymmetric lattice spacings. Euclidean invariance in the continuum limit then necessitates use of asymmetric couplings, i.e., different couplings in front of spacelike and timelike plaquettes. Let a and a_τ be the spacelike and timelike lattice spacings. When

$$\xi=a/a_\tau$$

is large enough, the physical-temporal extent is much smaller than the spatial extent, and one has a finite-temperature situation. The action now reads

$$S=\sum_x \left[\beta_\sigma \sum_{i\neq j=1}^{3} P_{ij}+\beta_\tau \sum_{i=1}^{3} P_{0i}\right], \tag{5.59}$$

where P_{ij} and P_{0i} are the standard spacelike and timelike plaquettes, respectively. The two bare couplings $\beta_\sigma(a,\xi)$ and $\beta_\tau(a,\xi)$ are functions of ξ, but in the weak coupling region they are related to each other to respect Lorentz invariance (Karsch, 1982):

$$\begin{aligned}\beta_\sigma(a,\xi)&=\frac{1}{\xi g_E^2(a)}+\frac{1}{\xi}c_\sigma(\xi)+O(g_E^2)\,,\\ \beta_\tau(a,\xi)&=\frac{\xi}{g_E^2(a)}+\xi c_\tau(\xi)+O(g_E^2)\,.\end{aligned} \tag{5.60}$$

$g_E^2(a)$ is the Euclidean coupling on a symmetric lattice. The functions $c_\sigma(\xi)$ and $c_\tau(\xi)$ are known in perturbation theory.

A TEK version of the above model may easily be constructed (Das and Kogut, 1984c, 1984d). The reduced action now reads

$$\begin{aligned}S=&-\beta_\sigma \sum_{i\neq j=1}^{3} Z_{ij}{\rm Tr}(U_iU_jU_i^\dagger U_j^\dagger)\\ &-\beta_\tau \sum_{i=1}^{3} Z_{0i}{\rm Tr}(U_0U_iU_0^\dagger U_i^\dagger)+{\rm H.c.}\end{aligned} \tag{5.61}$$

The twists in Eq. (5.61) are the symmetric twists—the same as in the zero-temperature TEK model.

G. Other TEK models

TEK versions of other models containing zero N-ality fields may be constructed in a way essentially like that of the gauge theory. Several such models have been constructed and studied. Of particular interest are two-dimensional chiral models. These models share some features of the four-dimensional gauge theory: they are asymptotically free and they have the same Migdal-Kadanoff recursion relations. TEK chiral models have been constructed and studied using Monte Carlo methods (Eguchi and Nakayama, 1983; Aneva, Brihaye, and Rossi, 1984; Das and Kogut, 1984a; Gonzales-Arroyo and Okawa, 1984).

H. Continuum TEK models

TEK models for continuum theories may be constructed, at least formally (Gonzales-Arroyo and Korthals-Altes, 1983). We consider, for example, a two-dimensional model. The algebra of the twist matrices reads

$$\Gamma_0\Gamma_1=e^{\frac{2\pi i}{N}}\Gamma_1\Gamma_0\,. \tag{5.62}$$

Let us write $\Gamma_\mu=\exp(i\gamma_\mu)$. Then the γ_μ's obey the algebra

$$[\gamma_0,\gamma_1]=-\frac{2\pi i}{N}I\,, \tag{5.63}$$

where I is the identity matrix. One can now write

$$D(x)=\exp^{i\gamma_\mu x_\mu} \tag{5.64}$$

and proceed to reduce a field theory in the same way as we did for continuum QEK models. However, it is clear that the matrices γ_μ do not have any finite-dimensional representation. This limits the usefulness of this formulation.

I. QEK vs TEK

Let us conclude this section by a comparison of the two ways of reducing a large-N gauge theory. In the QEK reduction N is as large as the volume of the equivalent field theory, i.e.,

$$N=L^d\,.$$

In the TEK models, however,

$$N=L^{d/2}\,.$$

Thus, for a given N, finite-volume effects are less severe in TEK models. For numerical simulations of these one-point models the TEK model is much better since, for the same value of N, one is simulating a much larger system. The formulation of TEK models is, of course, much more elegant than their QEK counterparts. The integration measure is simple and does not involve constraints. Furthermore, even in the pure gauge theory the leading finite-N corrections in the QEK model are of order $1/N$ due to the presence of constraints. For the TEK model these corrections are of order $1/N^2$, just as in the full field theory. Moreover, since for TEK models $N^2=L^4$, finite-N corrections are simply finite-volume corrections.

One disturbing feature of all reduced models is that the large-N and thermodynamic limits have to be performed simultaneously. In a general field theory there is no *a priori* reason why these two limits should commute. It would be much nicer if one could obtain a reduced model for any finite volume. This would allow one to take the large-N limit for a finite volume and finally take the thermodynamic limit. Such models have not, however, been constructed so far.

VI. NUMERICAL RESULTS

With the advent of Eguchi-Kawai models it has become possible to simulate large-N field theories numerically. Monte Carlo and Langevin equation method studies have been carried out for several interesting models and have yielded important insight into the nonperturbative structure of these theories.

A. QEK models

Bhanot, Heller, and Neuberger (1982a) have performed Monte Carlo simulations on the naive Eguchi-Kawai model and have shown, by considering the order parameter $\langle (1/N)\mathrm{Tr}U_\mu \rangle$, that the $[\mathrm{U}(1)]^d$ symmetry protecting open lines from acquiring nonzero values is broken. They also showed that this symmetry is not broken in the QEK model. These calculations were performed with $N=5$.

The evidence for breaking of the $[\mathrm{U}(1)]^d$ symmetry for the EK model has been confirmed by more accurate studies by Okawa (1982a), where an efficient way of updating the links was used. This was done for various values of N up to $N=10$. Studies of the QEK model for higher values of N (up to $N=20$) (Bhanot, Heller, and Neuberger, 1982b; Okawa, 1982b) showed that this model has the same phase structure as that expected from the standard Wilson theory. In particular, the QEK model with the standard Wilson action has a first-order phase transition at about $\beta/N=0.3$. This is not a deconfining transition; rather, it has the same nature as the transition observed at $N=4$ and 5. It has also been checked that quantities like the internal energy behave in accordance with the results of a weak coupling perturbation expansion around the correct vacuum in the relevant region. Monte Carlo studies of the quenched chiral model in two dimensions have also been performed (Heller and Neuberger, 1982b; Bhanot, 1983). As opposed to earlier expectations, detailed studies show that there is no first-order phase transition in this model.

B. TEK models

As discussed earlier, TEK models are better suited for numerical work. Extensive numerical simulations of various TEK models have been carried out. In the following we summarize some of the important results.

1. Two-dimensional chiral models

Two-dimensional $\mathrm{SU}(N)\times\mathrm{SU}(N)$ chiral models possess several properties similar to those of the four-dimensional gauge theory. They are asymptotically free and possess a mass gap. Recently an exact solution to a chiral model for $N=2$ has been obtained (Polyakov and Wiegmann, 1983). The action on the lattice is given by

$$S=\beta\sum_x\sum_\mu \mathrm{Tr}[U^\dagger(x+\mu)U(x)+\mathrm{H.c.}]\ , \tag{6.1}$$

where $U(x)$ belongs to $\mathrm{SU}(N)$. The TEK version of this model is given by

$$S=\beta\sum_\mu \mathrm{Tr}(\Gamma_\mu U^\dagger \Gamma_\mu^\dagger U+\mathrm{H.c.})\ , \tag{6.2}$$

where the Γ_μ are the two-dimensional twist matrices. A particular representation of these twist matrices is simply provided by the matrices P and Q defined in Eq. (5.28). This model has been shown to be completely equivalent to the corresponding field theory (Aneva, Brihaye, and Rossi, 1984; Das and Kogut, 1984a), and has been studied by Monte Carlo methods for $N=12$, 24 (Das and Kogut, 1984a) and for $N=10$, 20, 30, and 50 (Gonzales-Arroyo and Okawa, 1984). Invariant quantities like the internal energy,

$$\langle E\rangle=\frac{1}{N}\mathrm{Re}\sum_\mu \langle \mathrm{Tr}U\Gamma_\mu U^\dagger\Gamma_\mu^\dagger\rangle\ , \tag{6.3}$$

agree very well with the corresponding object computed in the field theory in the strong and weak coupling limits. Both the studies also indicated that there is no first-order phase transition at intermediate couplings. The two-point correlation function,

$$G(x)=\frac{1}{N}\mathrm{Re}\langle \mathrm{Tr}UD(x)U^\dagger D^\dagger(x)\rangle\ , \tag{6.4}$$

was also computed to look for a mass gap (Das and Kogut, 1984a). While some evidence for an exponential falloff of $G(x)$ was found, the statistics were not good enough to compute a mass gap reliably in the continuum limit. The study of the correlation function, however, revealed a strange nonanalyticity in the weak coupling edge of the intermediate coupling region. In very long runs the system seemed to flip between a "normal" state and an "abnormal" state. In the normal state the behavior of various quantities was consistent with that at other values of β. In the abnormal state, however, the internal energy was slightly lower and, more dramatically, the correlation function was highly disordered, even becoming negative at large x. Of course, $G(x)$ cannot be negative in a field theory satisfying clustering properties—these effects would go away at large N, where the TEK model is equivalent to a field theory.

An explanation of this peculiar behavior has been offered in terms of instantonlike finite-action saddle points of the model (Klinkhamer, 1984c). Such nontrivial saddles in the TEK gauge theory have been found earlier (van Baal, 1983) and interpreted as analogs of torons. For the chiral model these are of the form

$$U=D(n)=\Gamma_1^n\Gamma_2^{-n} , \tag{6.5}$$

with a classical action equal to $8\pi^2 n^2$ for small n. The contribution of small fluctuations around such a saddle point to various quantities may be computed. The contribution to the internal energy E^n is given by

$$E^n=\cos\left[\frac{2\pi n}{N}\right](2+\langle E\rangle^0_{\text{Gaussian}}) , \tag{6.6}$$

while that to the correlation function G^n is given by

$$G^n(x)=\cos\left[\frac{2\pi n}{N}(x_1+x_2)\right] \times\left[1-\left\langle\sum_q\left[1-\cos\frac{2\pi}{N}q\cdot x\right]\right\rangle\right] . \tag{6.7}$$

The results for $n=1$ seem to be consistent with the behavior observed in the Monte Carlo runs. The abnormal behavior thus probably reflects the fact that the system falls into one of the nontrivial extrema. It is, however, not clear how this happens, in spite of the enormous suppression due to the Boltzmann factor. Equations (6.6) and (6.7) clearly show that the negativity of $G(x)$ for large x is a finite-N effect, for large N the cosine factor in front of the expression for G^n goes to one, and $G(x)$ becomes positive.

Gonzales-Arroyo and Okawa (1984) pointed out that in the TEK chiral model there are large finite-N corrections for noninvariant quantities. In particular they showed that $\langle \mathrm{Tr}U\rangle$ does not vanish in the weak coupling limit. However, the value of $\langle \mathrm{Tr}U\rangle$ in weak coupling decreases rapidly as N increases, so that at $N=\infty$, $\langle \mathrm{Tr}U\rangle=0$, as in the field theory.

2. Four-dimensional gauge theory at zero temperature

Detailed Monte Carlo studies of the four-dimensional pure gauge theory at zero temperature have been performed for $N=36$ (Gonzales-Arroyo and Okawa, 1983b) and for $N=64$ (Fabricius and Haan, 1984). In these studies both Wilson loops and internal energies were measured. The string tension is extracted from the χ ratio,

$$\chi(I,J)=-\ln\frac{W(I,J)W(I-1,J-1)}{W(I,J-1)W(I-1,J)} , \tag{6.8}$$

where $W(I,J)$ denotes a rectangular Wilson loop of size $I\times J$. These studies show that physical quantities do not depend significantly on N.

The standard TEK model with the Wilson action shows a first-order phase transition at $\beta/N=0.36\pm0.02$ (Gonzales-Arroyo and Okawa, 1983b). This is manifested by a jump in the internal energy by about 0.8 at this value of β/N. This transition is a bulk transition: it does not spoil confinement, but the string tension is discontinuous. The bulk transition is similar to the third-order phase transition found in the two-dimensional Wilson theory at $N=\infty$ (Gross and Witten, 1980; Wadia, 1979). The string tension measured on the weak coupling side of the transition shows some tendency towards asymptotic scaling. In particular, for $N=64$, while $\chi(3,3)$, $\chi(4,2)$, and $\chi(3,2)$ show some scaling, $\chi(4,3)$ definitely does not (Fabricius and Haan, 1984). These results are summarized in Fig. 7. It is fair to say that asymptotic scaling has not yet been established in TEK models on the basis of string tension studies. Nevertheless, let us quote the values of the string tension derived from the existing data:

$$\sqrt{\sigma}/\Lambda_L=280\pm20 \text{ (Gonzales-Arroyo and Okawa, 1983b)},$$
$$\sqrt{\sigma}/\Lambda_L<264 \text{ (Fabricius and Hann, 1984)} , \tag{6.9}$$

where Λ_L is the lattice Λ parameter. In terms of $\Lambda_{\min}$, the Λ parameter with minimal subtraction, these values are

$$\sqrt{\sigma}/\Lambda_{\min}=19\pm2 \text{ (Gonzales-Arroyo and Okawa, 1983b)} ,$$
$$\sqrt{\sigma}/\Lambda_{\min}<18 \text{ (Fabricius and Hann, 1984)} . \tag{6.10}$$

This may be compared with the corresponding values for SU(3) and SU(2):

$$\sqrt{\sigma}/\Lambda_{\min}=16\pm3 \text{ [SU(3)]}$$
(Bhanot and Rebbi, 1981; Pietarinen, 1981; Creutz and Moriarty, 1982) ,
$$\sqrt{\sigma}/\Lambda_{\min}=10\pm2 \text{ [SU(2)] (Creutz, 1980)} . \tag{6.11}$$

These values are not too different from those obtained at $N=\infty$, indicating that the $N=\infty$ theory has a behavior fairly similar to that of the realistic SU(3) theory.

Migdal *et al.* (1984) have used Langevin equation

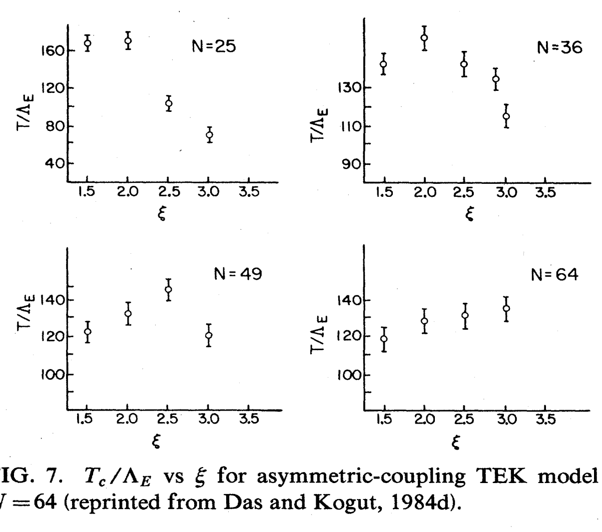

FIG. 7. T_c/Λ_E vs ξ for asymmetric-coupling TEK model at $N=64$ (reprinted from Das and Kogut, 1984d).

methods to study the TEK model for $N=9$, 16, 25, and 36. While they found plaquette energies to be independent of N for N greater than 16, larger Wilson loops showed detectable $1/N^2$ corrections. This is direct numerical evidence for the fact that finite-N corrections in the TEK model start at $O(1/N^2)$. Combining their data with those of Gonzales-Arroyo and Okawa (1983b), Migdal *et al.* obtained an improved value for the string tension:

$$\sqrt{\sigma}/\Lambda_L \simeq 345 \ . \tag{6.12}$$

Migdal *et al.* have also calculated the density of eigenvalues $\rho_{IJ}(\alpha)$ of the untraced Wilson loop matrix:

$$U_{IJ}=Z^{IJ}_{\mu\nu}U^I_\mu U^J_\nu U^{\dagger I}_\mu U^{\dagger J}_\nu \ . \tag{6.13}$$

In the strong coupling side of the phase transition the eigenvalues are distributed uniformly over the entire interval $(-\pi,\pi)$. However, for $\beta/N>0.36$, a clear gap is seen in the spectrum—the magnitude of the phases of the eigenvalues are all less than some number α_c, i.e.,

$$|\alpha| \le \alpha_c < \pi \ .$$

This behavior of $\rho(\alpha)$ is identical to that in the solvable two-dimensional theory (Gross and Witten, 1980; Wadia, 1979). In fact $\rho_{11}(\alpha)$ shows excellent agreement with the exact formula obtained in two dimensions. Such an agreement has also been observed in the SU(2) theory (Makeenko *et al.*, 1982; Belova *et al.*, 1984). Knowledge of the spectral density may be used to compute the various moments of the Wilson loop matrix:

$$\mu^{IJ}_n=\left\langle \frac{1}{N}\mathrm{tr}(U^n_{IJ})\right\rangle = \int_{-\pi}^{+\pi}\rho(\alpha)\cos n\alpha \, d\alpha \ . \tag{6.14}$$

Some of these moments turn out to be negative. It has been argued that this is evidence for a lack of correspondence between $N=\infty$ QCD and the naive Nambu-type string theory (Migdal *et al.*, 1984).

In the above-mentioned studies clear evidence for scaling has not been found. Clearly a much more careful investigation has to be carried out before drawing any firm conclusion about the physics.

3. Four-dimensional pure gauge theory at finite temperature

At a sufficiently high temperature gauge theories are expected to undergo a deconfinement phase transition. Such a phase transition may be observed in the laboratory in the near future. At the theoretical level the deconfining transition has indeed been observed and studied in SU(2) (Kuti *et al.*, 1981; McLerran and Svetitsky, 1981) and SU(3) (Celik *et al.*, 1983; Kogut *et al.*, 1983; Svetitsky and Fucito, 1983) pure gauge theories. For SU(2) the transition is second order, while SU(3) shows a strong first-order transition—in conformity with expectations based on general universality arguments (Svetitsky and Yaffe, 1982). For $N\ge 4$, universality arguments do not predict the order unequivocally. However, strong coupling mean-field studies show a first-order transition (Green and Karsch, 1984; Gross and Wheater, 1984; Ogilvie, 1984). Numerical studies for SU(4) seem to vindicate these predictions (Batrouni and Svetitsky, 1984; Wheater and Gross, 1984). It has been argued that the $N=\infty$ theory shows a first-order transition (Gocksch and Neri, 1983; Ogilvie, 1984; see, however, Pisarski, 1984).

The deconfinement transition in pure SU(∞) QCD has been studied quite extensively by Monte Carlo simulation of TEK models. This sheds important light on the confinement mechanism, and comparison of the results with those of the SU(3) theory provides a basis for examining the validity of the large-N approximation itself. Furthermore, deconfinement serves as an excellent laboratory for studying the continuum limit of lattice gauge theories. This is particularly so if the transition is first order. In that case it is fairly simple to pin down the critical temperature for deconfinement quite accurately. In terms of the critical coupling g_c^2, the deconfining temperature T_c is given by

$$T_c=\frac{1}{N_0 a(g_c^2)} \ , \tag{6.15}$$

where N_0 is the temporal extent of the box and $a(g_c^2)$ is the lattice spacing at coupling g_c. One could now measure g_c for various values of N_0 and test whether Eq. (6.15) is consistent with the asymptotic freedom prediction for $a(g_c^2)$. If so, one is simulating continuum physics and T_c is the physical deconfinement temperature. [The early SU(2) and SU(3) studies seemed to show such a scaling behavior. Recent work on SU(3) (Kennedy *et al.*, 1985), however, shows that asymptotic scaling does not set in before $N_0=10$.]

Gocksch *et al.* (1984) performed Monte Carlo simulation of their version of the hot TEK model for $N=11$ and $N_0=2$ and 3. They indeed found a sharp jump in the thermal Wilson line with evidence for coexisting phases and interpreted this to be a physical first-order deconfining transition. It is not clear, however, whether this was really so, as we shall see shortly. Furthermore, this model has been shown to be equivalent to the finite-temperature field theory only up to one loop in the perturbation expansion. An exact equivalence has yet to be shown. In addition, there is no evidence for scaling in the data.

There is a serious problem in studying deconfinement at $N=\infty$. The zero-temperature theory with the Wilson action has a first-order bulk phase transition. This transition is also present in the finite-temperature theory. Since the string tension drops discontinuously as one crosses this transition from the strong coupling side, the confinement length increases abruptly. For moderate values of N_0 this makes the confinement length larger than N_0, thus simulating a deconfining transition and forcing the Wilson line to jump discontinuously. The bulk transition, however, has nothing to do with physics—it is a lattice artifact. Thus the "deconfinement" it induces is not physical deconfinement. The interference between the bulk transition and the deconfinement transition has been observed in Monte Carlo simulations of the asymmetric-twist hot TEK model for $N_0=2,3$ (Das and Kogut, 1984b). Further simulations (Fabricius, Haan, and Klinkhamer, 1984) indicate that this interference persists up to $N_0=4$. To obtain any information about physical deconfinement, one must ensure that the two transitions are clearly separated.

In principle such a separation is possible. For sufficiently large N_0 the deconfinement transition is pushed into the weak coupling region, while the bulk transition remains where it is (around $\beta/N=0.350$). However, this is a rather impractical method. From Eq. (3.55) N grows as N_0^2. For the minimal value of K, i.e., $K=1$ (for which the above simulations have been performed), $N=96$ for $N_0=4$ and $N=150$ for $N_0=5$. This is extremely time consuming even on large supercomputers.

SU(N) lattice gauge theories with the Wilson action have bulk transitions for $N\geq 4$ which are artifacts of the particular action chosen. In fact, the interference between bulk and deconfinement transitions has been observed for $N=4$ (Batrouni and Svetitsky, 1984). For finite N, however, one can add a negative adjoint piece to the action and adjust the adjoint coupling to get rid of the bulk transition altogether. This allows one to study deconfinement freed of the effects of the bulk transition (Batrouni and Svetitsky, 1984). At $N=\infty$ this trick does not work, essentially because the "mixed" action theory is now equivalent to a Wilson theory with a redefined coupling (Makeenko and Polikarpov, 1982; Samuel, 1982; Das and Kogut, 1984c).

Nevertheless, it is indeed possible to decouple the transitions in the asymmetric coupling version of the hot TEK model (Das and Kogut, 1984c). This formulation has the advantage of having a continuously adjustable parameter—the asymmetry parameter ξ. Since the twists are the same as the symmetric twists of the zero-temperature TEK model, the possible values of N are much less restricted than those of the asymmetric-twist model. Monte Carlo simulations with $N=16$, 25, 36, 49, 64, and 81 (Das and Kogut, 1985a, 1985b) show that with a sufficiently large ξ the bulk transition disappears. The Wilson line, however, continues to jump in a discontinuous fashion, providing evidence for a first-order deconfining transition freed from the effects of any bulk transition. This is supported by the presence of two-state signals and hysteresis loops. In most cases this happens at a value of ξ for which the critical coupling is not in the weak coupling region. At $N=64$, $\xi=1.5$ and $N=81$, $\xi=1.5$ the bulk transition is still present, but is clearly on the strong coupling side of the deconfinement transition.

The $N=64$ data, in fact, show some tendency towards scaling. Let T_c denote the physical deconfining temperature. If $a(\beta_c/N)$ is the spatial lattice spacing at the critical coupling β_c and ξ is the asymmetry parameter, one has

$$T_c=\frac{\xi}{La(\beta_c/N)} . \tag{6.16}$$

If β_c is in the asymptotic scaling region, one would have

$$\frac{T_c}{\Lambda_E}=\frac{\xi}{L}\left[\frac{11}{48}\frac{N}{\beta_c}\right]^{51/121}\exp\left[\frac{24\pi^2}{11}\frac{\beta_c}{N}\right] , \tag{6.17}$$

where Λ_E is the "Euclidean" Λ parameter. Reversing the argument, one could calculate T_c/Λ_E using Eq. (6.17) and see whether this is independent of ξ and L. For N less than 64 one does indeed find a gross violation of scaling. For $N=64$, however, there is some tendency towards scaling. This is evident from Fig. 7, where T_c/Λ_E is plotted against ξ (a flat curve signifies perfect scaling).

To establish scaling properly much more work has to be done. Nevertheless we can get some idea of the value of T_c assuming that scaling has already set in. The $N=64$, $\xi=1.5$ data give

$$\frac{T_c}{\Lambda_E}=118\pm 6 .$$

Using the string tension data quoted earlier (Fabricius and Haan, 1984), one has

$$\frac{T_c}{\sqrt{\sigma}}=0.42\pm 0.05 ,$$

compared to

$$\frac{T_c}{\sqrt{\sigma}}=0.50\pm 0.05 \quad (N=3) .$$

The value of $T_c/\sqrt{\sigma}$ at $N=\infty$ is thus rather close to that at $N=3$. To get a really good number, however, one must evaluate σ on the asymmetric lattice. This involves computing the connected correlations of Wilson lines, which vanish in TEK models due to exact factorization.

Clearly more work is needed to establish scaling properly and to extract physically meaningful numbers. The numerical work done so far is certainly encouraging, though not definitive. The fact that the deconfinement temperature in physical units is close to the SU(3) value indicates that the confinement mechanisms at $N=\infty$ and 3 are similar. This means that the large-N approximation is probably a good approximation to the real world. It is certainly worthwhile to continue to investigate the large-N limit, particularly on the analytic front, where there is more chance of success than for the $N=3$ theory.

ACKNOWLEDGMENTS

I should like to thank John B. Kogut for extremely fruitful and enjoyable collaborations. I also thank Spenta Wadia for collaboration and many interesting discussions. This paper is based on a review talk given in 1984 at the Aspen Workshop on Lattice Gauge Theories. I should like to thank the organizers and the Aspen Center for Physics for what proved to be a stimulating workshop.

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