Spin glasses: Experimental facts, theoretical concepts, and open questions

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This review summarizes recent developments in the theory of spin glasses, as well as pertinent experimental data. The most characteristic properties of spin glass systems are described, and related phenomena in other glassy systems (dielectric and orientational glasses) are mentioned. The Edwards-Anderson model of spin glasses and its treatment within the replica method and mean-field theory are outlined, and concepts such as "frustration," "broken replica symmetry," "broken ergodicity," etc., are discussed. The dynamic approach to describing the spin glass transition is emphasized. Monte Carlo simulations of spin glasses and the insight gained by them are described. Other topics discussed include site-disorder models, phenomenological theories for the frozen phase and its excitations, phase diagrams in which spin glass order and ferromagnetism or antiferromagnetism compete, the Neél model of superparamagnetism and related approaches, and possible connections between spin glasses and other topics in the theory of disordered condensed-matter systems.

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*Present and permanent address.

I. INTRODUCTION AND OVERVIEW

Spin glasses are magnetic systems in which the interactions between the magnetic moments are "in conflict" with each other, due to some frozen-in structural disorder. Thus no conventional long-range order (of ferromagnetic or antiferromagnetic type) can be established. Nevertheless these systems exhibit a "freezing transition" to a state with a new kind of "order" in which the spins are aligned in random directions.

The nature of this new kind of order, and thus its appropriate theoretical description, is still heavily debated, as well as the character of the freezing transition: is it a new type of phase transition (Edwards and Anderson, 1975) or failure to establish complete thermal equilibrium during observation time? The problem of spin ordering in spin glasses is a problem belonging to the physics of structurally disordered materials, and does not arise in more conventional regular systems such as ideal crystals. Thus a number of challenging theoretical questions immediately emerge.

(i) Usually, the ordered phase is characterized by an order parameter. What is the order parameter for spin glasses?

(ii) The ordered phase generally has a lower symmetry than the corresponding disordered phase. Is there a "broken symmetry" here?

(iii) How does one deal with the statistical mechanics of a system with a large number of quenched variables which describe the frozen-in structural disorder?

(iv) As some relaxation times may exceed reasonable time scales of observation, nonergodicity becomes a serious practical problem. What is the proper way to generalize equilibrium statistical mechanics to treat such slowly relaxing metastable states? Etc.

Furthermore the characteristic phenomena observed in spin glasses, such as the rather sharp "cusp" in the frequency-dependent susceptibility in low fields, first found by Cannella and Mydosh (1972), are a fairly universal feature: the cusp occurs both in the dilute metallic alloy *Cu*Mn with 0.9% Mn (Fig. 1; Mulder *et al.*, 1981) and in the concentrated insulator $Eu_xSr_{1-x}S$ (Fig. 2; Maletta and Felsch, 1979). Rather than originating in random dilution of a crystal, as in these examples, the quenched-in disorder necessary to produce a spin glass may also be due to noncrystallinity, e.g., in $CoO{\cdot}Al_2O_3{\cdot}SiO_2$ (Wenger *et al.*, 1982). Analogous phenomena are also found in dielectric relaxation measurements on disordered ferroelectrics such as $KTaO_3$ diluted with Li (Höchli, 1982; Fig. 3), which is interpreted as an electric dipole glass, or in diluted molecular crystals such as $K(CN)_xBr_{1-x}$ (Loidl *et al.*, 1982), which is interpreted as a "quadrupolar" or "orientational" glass. Other phenomena exhibit a fairly universal character as well, and the number of materials now being categorized as spin glasses is truly abundant.

The physics of spin glasses raises many rather fundamental questions, and the phenomena observed are of a fairly general character—these two facts are probably the main reasons why this field has become one of the mainstreams of research in condensed-matter physics. Therefore the present review will not be able to give a complete survey of the field, but rather is intended as a tutorial introduction to it and a discussion of the most exciting questions that have come up. Of course, the judgment of what are the most exciting questions is highly subjective, and the present review is easily recognizable as a theorist's selection of topics: problems such as the infinite-range model (Sherrington and Kirkpatrick, 1975) and the lower critical dimension of short-range models will be discussed in detail; not much attention will be paid to effects of deviations from random mixing in dilute alloys (see, for example, Bouchiat *et al.*, 1981), change of properties with heat treatment (Beck, 1978), etc.

We shall now specify more precisely what are the main characteristic properties of a spin glass and what, it is generally believed, are the necessary ingredients of a theoretical model if it is to display such behavior (Sherrington, 1983). The moments in a spin glass are observed to be frozen over macroscopic times at temperatures below the freezing temperature T_f. However, the directions of the spins do not pick out any wave vector, so that (for Ising spins $S_i = \pm 1$)

$$\langle S_i \rangle_{t'} \neq 0 , \tag{1.1}$$

$$\frac{1}{N}\sum_i \langle S_i \rangle_{t'} \exp(i\mathbf{k}\cdot\mathbf{R}_i) = 0 \quad (N \to \infty) , \tag{1.2}$$

where $T < T_f$. The average $\langle\ \rangle_{t'}$ denotes a time average over an observation time $t_{\rm obs}$ much longer than any microscopic time.

As an accompaniment to spin freezing one observes a cusp at T_f in the magnetic susceptibility (see Cannella and Mydosh, 1972) and remanence (Tholence and Tour-

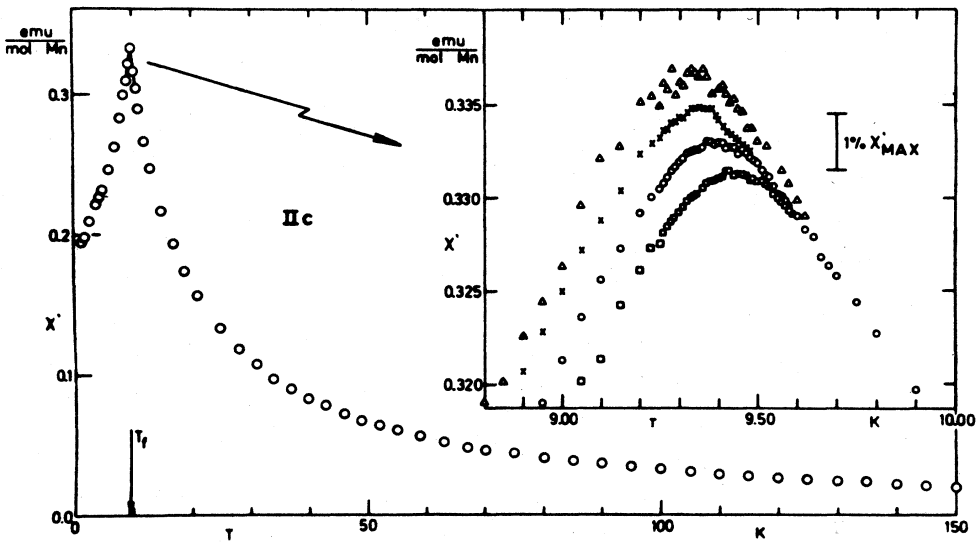

FIG. 1. Real part χ' of the complex susceptibility $\chi(\omega)$ as a function of temperature for sample IIc (*Cu*Mn with 0.94 at.% Mn, powder). Inset reveals frequency dependence and rounding of the cusp by use of strongly expanded coordinate scales. Measuring frequencies: □, 1.33 kHz; ○, 234 Hz; ×, 104 Hz; △, 2.6 Hz. From Mulder *et al.* (1981).

nier, 1974; Bouchiat and Monod, 1983) and hysteresis (Monod *et al.*, 1979, Prejean *et al.*, 1980) below the freezing temperature. This remanent magnetization is found, for instance, if one cools the spin glass in a field to $T<T_f$ and then switches the field off. Just above T_f relaxation times have a broad spectrum, and as $T\to T_f^+$ some of the weight in this spectrum lies at times very many orders of magnitude larger than any characteristic microscopic time. The range of correlations in space also increases as one approaches T_f from above. To see exactly what this means consider, again for Ising spins, the correlation function $\langle S_i S_j\rangle_{t'}$. In a spin glass the sign of this correlation will be random, so it is convenient to discuss the square $\langle S_i S_j\rangle_{t'}^2$. This decays to zero, generally exponentially to a first approximation, as the distance $|\mathbf{R}_i-\mathbf{R}_j|$ increases (to be precise one should average over many pairs of sites of a given relative separation). We call the range over which the decay takes place the spin glass correlation length ξ_{SG}; so, loosely speaking,

$$\langle S_i S_j\rangle_{t'}^2 \propto \exp(-|\mathbf{R}_i-\mathbf{R}_j|/\xi_{SG}) . \tag{1.3}$$

One observes that ξ_{SG} grows as $T\to T_f^+$, showing that

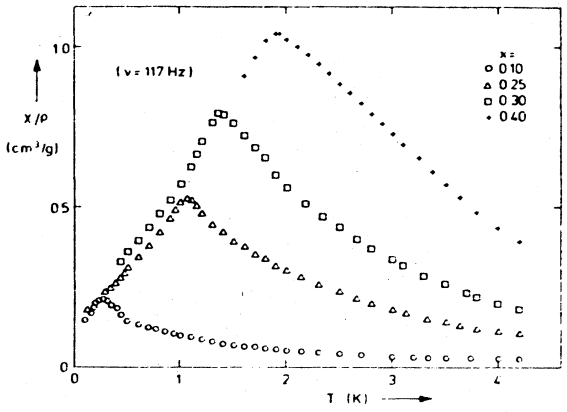

FIG. 2. Real part χ' of the complex susceptibility $\chi(\omega)$ as a function of temperature for $Eu_xSr_{1-x}S$, at $\omega=117$ Hz and various Eu concentrations as indicated in the figure. From Maletta and Felsch (1979).

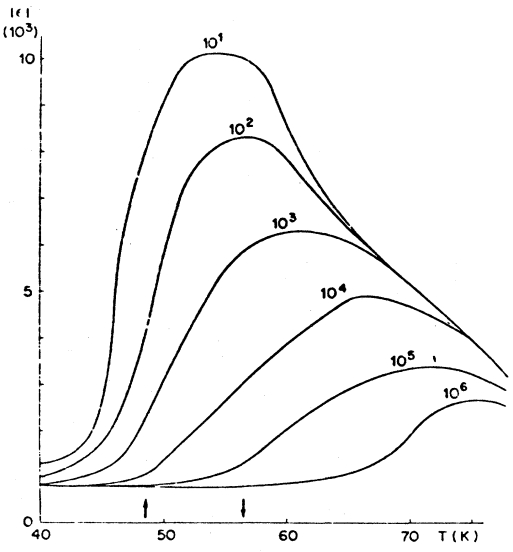

FIG. 3. Dielectric susceptibility of $K_{0.974}Li_{0.026}TaO_3$ as a function of temperature. Labels stand for the measuring frequencies, arrows for the maximum of the dielectric dispersion step (49 K) and the stability limit of remanent polarization (56 K). From Höchli (1982).

important collective effects are taking place, which presumably, in turn, give rise to the long relaxation times. The increase in ξ_{SG} seems to be less rapid than the growth of the characteristic times.

For the behavior at the freezing transition, as well as in the low-temperature phase, it matters whether we consider Ising spins, $S_i = \pm 1$, as done above, or isotropic spin glasses (XY spins confined to a plane, or three-component Heisenberg spins). It may also become necessary to consider crossover effects between these different classes of spin glasses: many real systems are nearly Heisenberg-like, but a weak anisotropy is present.

It is still a matter of debate whether or not the relaxation times and spin glass correlation length diverge at T_f, signifying a phase transition, or just become large and finite. If there is a phase transition, one can then replace the time averages in Eq. (1.1) by statistical mechanics averages and define (Edwards and Anderson, 1975) a spin glass order parameter

$$q = [\langle S_i \rangle_T^2]_{\rm av} , \tag{1.4}$$

which is nonzero below T_f. Here $\langle\ \rangle_T$ denotes a statistical mechanics average and $[\ ]_{\rm av}$ is a configurational average over the random interactions. In the simplest approximation, namely, mean-field theory, the spin glass order parameter is actually more complicated than this because spin glass ordering is nonunique: there exist a large number of degenerate thermodynamic states with the same macroscopic properties but with different microscopic configurations.

These are separated by infinitely high free-energy barriers in phase space. It is, however, unclear whether real systems have degenerate thermodynamic states separated by *infinite* barriers or whether this is an artifact of mean-field theory. In any case, the hypersurface describing the coarse-grained free energy in the space of appropriate phase-space coordinates exhibits many "valleys" (for a qualitative description see, for example, Fig. 4 and Binder, 1980a). In mean-field theory the complete spin glass order parameter is a function (actually a probability distribution) that describes not only ordering within a single valley but also how correlated the valleys are with each other in phase space.

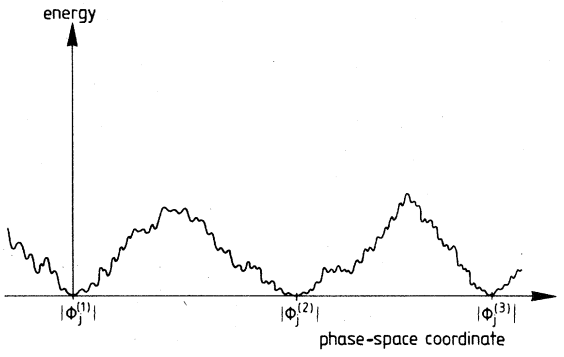

FIG. 4. Schematic plot of the coarse-grained free energy of a spin glass plotted vs a phase-space coordinate which measures the projection of the considered state on a particular ordered state. From Binder (1980a).

One difficulty of the theory of spin glasses is, of course, that it is difficult to give a precise meaning to vague but intuitive pictures like Fig. 4. The total free energy of a system, considered as a function of some other extensive variables, must be a simple convex function of these variables, as general thermodynamic principles tell us, and hence does not look like Fig. 4. One believes that one finds such a picture for the coarse-grained free energy $F_{\rm cg}(\{\psi_j\})$, defined as a restricted partition function $Z_{\rm cg} \equiv \exp[-F_{\rm cg}(\{\psi_j\})/k_B T] = {\rm Tr}\exp(-\mathscr{H}/k_B T)$, where the trace is calculated under the constraints of keeping fixed the set of phase-space variables $\{\psi_j\}$ that one considers to be the important variables for the considered problem. In a ferromagnet near the critical point, the important variables are the long-wavelength Fourier components of the magnetization. Thus coarse graining essentially amounts to integrating out the short-wavelength magnetization fluctuations. In a spin glass, however, identification of the important variables $\{\psi_j\}$ which should appear in the coarse-grained free energy is nontrivial; hence we shall see that various types of coarse-grained free energies for spin glasses have in fact been proposed, and will be discussed.

While we shall find, in our discussion of the mean-field theory in Sec. IV, that the thermodynamic state in the ordered phase is very degenerate, i.e., there are many valleys in Fig. 4 with exactly the same depth, it is not clear that this feature also applies to spin glasses with short-range interactions. In fact, it is conceivable (see, for example, Moore and Bray, 1985; Fisher and Huse, 1986) that the ordered phase of an Ising spin glass is only twofold degenerate (in zero field, there is an up-down symmetry of the Hamiltonian). But a picture of the type of Fig. 4 is still expected for the coarse-grained free energy, the only distinction being that two valleys (corresponding to fully ordered "monodomain" configurations) are somewhat deeper than all the other valleys (corresponding to metastable states, which might be viewed as configurations with large clusters of spins overturned relative to the monodomain configuration, for instance). In a ferromagnet or antiferromagnet with no frozen-in disorder, such configurations are also possible, but they are not separated by high free-energy barriers from each other and from the stable states and thus do not play a significant role in the dynamic behavior of the system.

A nontrivial spin glass model, then, must clearly have nonperiodic ground states and a complicated configuration space with many "valleys." To obtain these features it appears that a model must have a substantial amount of two ingredients: *randomness* and *frustration*. The latter refers to competition between different terms in the Hamiltonian so that no spin configuration simultaneously minimizes each term. There have been attempts to model spin glasses by having one of these ingredients without the other, but it appears that one must have both together

in order to describe spin glass behavior. The simplest model with these features is probably that proposed by Edwards and Anderson (1975), which we shall discuss extensively in this review. Note that there are certain other models (e.g., that of van Hemmen, 1982, 1983) with substantial randomness and frustration but without the complicated configuration space and slow dynamics of spin glasses (Choy and Sherrington, 1984; Morgenstern and van Hemmen, 1985). It is also possible for randomness and frustration to give states with long-range magnetic order. This occurs if the amount of randomness and frustration is rather weak. An example is the Ising ferromagnet in a small random magnetic field, discussed in Sec. VI.F.1, for dimension greater than two. The system is ferromagnetic at low temperatures and not therefore generally considered to be a spin glass, although it does have a multivalley structure of configuration space and a very broad spectrum of relaxation times. Hence randomness and frustration are necessary but may not be sufficient to obtain nontrivial spin glass behavior. A further complication is that, for certain ranges of parameters, the mean-field theory of spin glasses predicts long-range magnetic order plus degenerate thermodynamic states separated by infinite barriers. Such systems are commonly called spin glasses even though they have long-range order. The question of what is a spin glass is, therefore, not always completely clear cut.

The plan of this review is as follows: Section II discusses the experimental situation. We shall restrict ourselves to a few rather well-investigated systems (the reader interested in a more complete account of experimental work may consult Fischer, 1982, 1985).

Section III then contains theoretical considerations of a more general character and introduces the various theoretical descriptions. Section IV summarizes our current understanding of the mean-field theory of the Edwards-Anderson model (Edwards and Anderson, 1975) of spin glasses. This area has seen enormous activity in the last decade, which is understandable since for all cooperative phenomena a mean-field description is always a useful point of departure. In contrast to ordinary ordering phenomena, where the mean-field description often is rather trivial, the mean-field theory of spin glasses is extremely difficult, and it has taken several years to acquire a reasonable understanding of it. Several new concepts have emerged, which will be discussed rather thoroughly.

Section V describes the properties of short-range Edwards-Anderson models, which are still much less well understood. Much of our knowledge about the behavior of these models stems from computer simulations, which will hence be emphasized in this context. It was soon evident (Binder and Schröder, 1976a, 1976b) that many features of the short-range model are surprisingly similar to experimental findings. Thus there is nearly general agreement that the short-range Edwards-Anderson model is a reasonable first approximation, as far as the theoretical description of real spin glasses is concerned. But the question of in which spatial dimensions this model has a phase transition at nonzero temperature is still very controversial. We shall discuss this interesting question in detail.

Section VI is then devoted to a brief description of other approaches to spin glass theory and of problems that have some bearing on the subject of spin glasses but that are somewhat outside the main scope of this review.

As will have become evident by now, there are many questions in the field that are controversial (and there is not even a complete consensus as to what are the important questions). While it is a general strategy of this review to give some room for all major schools of thought in the field, the final section (Sec. VII) containing the conclusions necessarily will be more biased by the authors' point of view. And, last but not least, the authors must apologize to many colleagues that they were unable to mention their work in this review: complete coverage of the subject would be a very hard task and was not intended. Additional references are given in earlier reviews (Binder, 1980a; Rammal and Souletie, 1982; Fischer, 1983c, 1985; Chowdhury and Mookerjee, 1984) and the articles in van Hemmen and Morgenstern (1983).

II. EXPERIMENTAL RESULTS

In order to perform experiments on spin glasses, it is first of all necessary to make sure that a given material does indeed fall into this category and is not just a kind of (disordered) ferromagnet, antiferromagnet, or ordinary paramagnet at all temperatures. For several years there was a running controversy among some experimental groups whether one should make distinctions between spin glasses and so-called "mictomagnets" (Mydosh, 1977, 1978; Beck, 1978). The former category was reserved for dilute magnetic metallic alloys, and more concentrated or nonmetallic systems were put into the latter. It is now widely realized, however, that there is no fundamental distinction between these cases, and hence we shall consider all these systems as spin glasses. In any case, this discussion has clarified the question that an experimentalist must ask (Mydosh, 1978, 1983; Sherrington, 1983): what properties must a system have in order to be a spin glass? These defining properties are (i) frozen-in magnetic moments below some freezing temperature T_f and hence a peak in the frequency-dependent susceptibility (Figs. 1 and 2); (ii) lack of periodic long-range magnetic order; (iii) Remanence and magnetic relaxation on macroscopic time scales below T_f when there are changes of the magnetic field.

Of course, the question of whether moments are frozen in depends on the time scale of the observation; moreover, condition (ii) must be somewhat relaxed if one wishes to consider states with mixed spin glass and ferromagnetic ordering (see Secs. II.C.3 and II.D). Thus some imprecision in such a working definition necessarily remains.

In Sec. II.A a brief characterization of "subclasses" of the category spin glass will be given, because naturally the "universality" of spin glass properties is not strictly complete. There are differences depending on whether one

deals with Ising-type spins or vector spins, and the nature of the magnetic anisotropy is important.

A. Spin glass systems

1. Exchange interactions

The "classical" spin glass materials are noble metals (Au,Ag,Cu,Pt) weakly diluted with transition metal ions, such as Fe or Mn. The picture one has in mind is then that the scattering of the conduction electrons at the spins leads to an indirect exchange interaction (see Fig. 5). This RKKY interaction (Ruderman and Kittel, 1954; Kasuya, 1956; Yosida, 1957) oscillates strongly with distance $\mathbf{R}$,

$$J(\mathbf{R})=J_0\frac{\cos(2k_FR+\varphi_0)}{(k_FR)^3}\ ,\quad R\to\infty\ . \tag{2.1}$$

Here J_0 and φ_0 are constants, and k_F is the Fermi wave number of the host metal. Since the distances between the spins are random, some of the interactions of a considered spin with other ones will be positive, favoring parallel alignment, some negative, favoring antiparallel alignment; thus no spin alignment can be found that is satisfactory to all exchange bonds. This "frustration" of some of the bonds (Toulouse, 1977) will appear as the second basic ingredient, together with the frozen-in disorder, of spin glass behavior; this point will be discussed in more detail in Sec. III.E below.

In the limit where the alloy is very dilute, positive and negative signs of the exchange in Eq. (2.1) will be equally likely, and hence one might approximate Eq. (2.1) by $J(R)=\pm J_0'/(k_FR)^3$, where J_0' is another constant and the signs are chosen at random. For such a symmetric distribution $P(J)=P(-J)$ of bond strengths J, the magnetic susceptibility in the entire paramagnetic phase would be a simple Curie law $\chi\propto T^{-1}$; in addition, due to the decay of the interaction strength as the minus third power of distance, the so-called "concentration scaling laws" (Souletie and Tournier, 1969, 1971) hold.

It is now well established, however, that this picture is far too idealized. An attempt has recently been made by Morgownik and Mydosh (1983a, 1983b) to infer the actual exchange interaction $J(\mathbf{R})$ for several spin glass systems from a careful analysis of high-temperature susceptibility data. At least for values of $\mathbf{R}$ up to the fifth-neighbor distance, $J(\mathbf{R})$ is not in agreement with Eq. (1.2), since the conduction electron polarization around single Mn impurities [inferred from ^{63}Cu NMR measurements by Cohen and Slichter (1978)] has a rather different spatial dependence (see Fig. 6).

The situation is even more complicated as the same analysis reveals (Morgownik and Mydosh, 1983a, 1983b) a complicated interplay with atomic short-range order. It appears that deviations from ideal random mixing occur in such a manner as to enhance the probability of neighboring distances where $J(\mathbf{R})$ in Fig. 6 is ferromagnetic in all four systems investigated (*Cu*Mn, *Au*Mn, *Au*Fe, *Pt*Mn). For *Cu*Mn, this conclusion is also corroborated by earlier neutron scattering work (Cable *et al.*, 1982). These deviations from random mixing are very clearly seen in the complicated concentration dependence in the high-temperature susceptibility. In fact, in leading orders in reciprocal temperature, χ can be expressed as (see, for example, Binder, 1982a)

$$\chi=\frac{Nx(g\mu_B)^2S(S+1)}{3k_BT}\left[1+\frac{\theta(x)}{T}+\cdots\right], \tag{2.2}$$

with the Curie-Weiss temperature $\theta(x)$

$$\theta(x)=\frac{S(S+1)}{3k_B}\sum_{\mathbf{R}}J(\mathbf{R})P_x(\mathbf{R})\ . \tag{2.3}$$

Here N is the total number of atoms per unit volume, and x is the concentration of magnetic atoms. It is assumed that the latter have magnetic moment $g\mu_BS$ and spin quantum number S independent of concentration and temperature (while this is certainly true for magnetic mo-

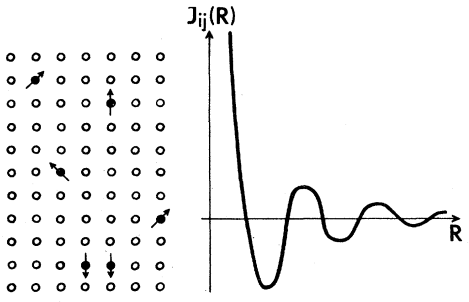

FIG. 5. Schematic sketch of magnetic moments randomly diluted in a metallic matrix, and the resulting RKKY exchange integral plotted as a function of distance. From Binder (1977a).

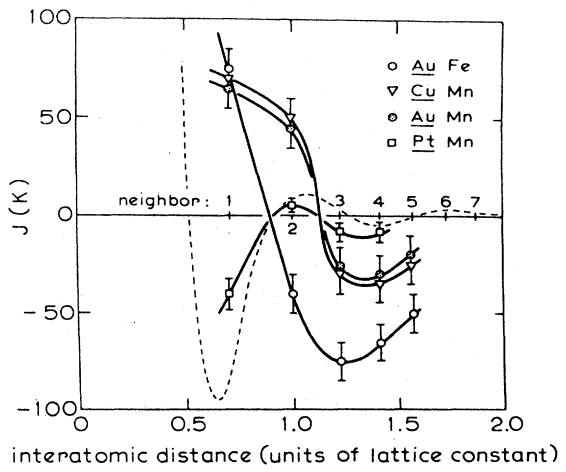

FIG. 6. Estimated exchange parameters J as a function of distance for four spin glass systems. The dashed line represents the RKKY conduction electron polarization around a Mn ion in Cu, according to Cohen and Slichter (1978). From Morgownik and Mydosh (1983a).

ments due to localized electrons, for itinerant electrons this clearly involves an approximation). The (conditional) probability $P_x(\mathbf{R})$ that a site a distance $\mathbf{R}$ apart from a magnetic ion is also taken by a magnetic ion is simply $P_x(\mathbf{R})=x$, independent of $\mathbf{R}$, in the case of random mixing, and then $\theta(x)$ is simply linear in x. Even rather small deviations from random mixing produce drastic effects in the susceptibility, as model calculations (Binder, 1982a) and experiment (Morgownik and Mydosh, 1983a) reveal. The striking conclusion of Morgownik and Mydosh (1983a) is that "ferromagnetic clusters are the 'building blocks' out of which the spin glass state is established, and they should be considered theoretically."

The conclusion we want to draw at this point is that, even for systems considered as "archetypal" spin glasses (Morgownik and Mydosh, 1983b), the effective magnetic Hamiltonian is not known precisely. But it is clear that there will be competition effects between ferromagnetic bonds and antiferromagnetic ones, as Fig. 6 shows.

In this respect, the situation is qualitatively similar to that for the nonmetallic spin glass $Eu_xSr_{1-x}S$ (Maletta and Felsch, 1979; Maletta, 1982). The magnetic Eu ions have $S=\frac{7}{2}$, $g=2$, and the exchange interactions are ferromagnetic between nearest neighbors, antiferromagnetic between next-nearest neighbors, with $J_{\mathrm{NNN}}/J_{\mathrm{NN}}\approx-\frac{1}{2}$; superexchange with more distant neighbors is negligibly small (Bohn *et al.*, 1980). As a consequence of this short range of the exchange forces, they can support a spin glass state only for concentrations exceeding the next-nearest-neighbor percolation threshold [$x_p^{\mathrm{NNN}}\simeq 0.136$, on the face-centered cubic lattice (Dalton *et al.*, 1964)]. The superparamagnetic state for concentrations below this threshold is also of interest (Eiselt *et al.*, 1979), since the view exists in the literature that spin glass behavior is not essentially different from "rock magnetism" (Wohlfarth, 1977a), i.e., superparamagnetism (Tholence and Tournier, 1974). We shall come back to this problem in Sec. III.G below.

Due to its well-defined magnetic moments and rather precisely known exchange parameters, which depend only weakly on concentration or temperature via lattice expansion (Köbler and Binder, 1980), $Eu_xSr_{1-x}S$ probably is the spin glass material for which interactions are best understood to date. For this reason we emphasize somewhat experimental results for this material as a prototype for nonmetallic spin glasses, although this material also has its problems (effects of ferromagnetic tendency, demagnetizing effects, and strong dipolar anisotropy are incompletely understood).

Related materials with different values of the ratio $J_{\mathrm{NNN}}/J_{\mathrm{NN}}$ are $Eu_xSr_{1-x}Se$ and $Eu_xSr_{1-x}Te$ or the mixed system $Eu_xSr_{1-x}S_{x'}Se_{1-x'}$ (Westerholt and Bach, 1981b). While in all the systems mentioned so far disorder is produced by random (or at least nearly so) occupation of *crystal* sites, there now exist a large number of materials in which the disorder is due to the *amorphous structure* of the material: metallic $Al_{0.63}Gd_{0.37}$ (Mizoguchi *et al.*, 1977; Malozemoff and Imry, 1981) as well as the insulating magnetic glasses $MnO\cdot Al_2O_3\cdot SiO_2$ (Kline *et al.*, 1976) or $CoO\cdot Al_2O_3\cdot SiO_2$ (Krusin-Elbaum *et al.*, 1979; Morgownik *et al.*, 1982). In addition, metallic glasses such as $(Fe_xNi_{1-x})_{75}P_{16}B_6Al_3$ (Rao *et al.*, 1983), $(Fe_xMn_{1-x})_{95}P_{16}B_6Al_3$ (Salamon *et al.*, 1981), and $(Fe_yMn_{1-x})_{75}P_{16}B_6Al_3$ (Geohegan and Bhagat, 1981; Manheimer, Bhagat, Kistler, and Rao, 1982a) are spin glasses. In these and similar materials, however, detailed magnetic interactions are not yet known. The fact that, nevertheless, many physical properties of the spin glass state are similar supports the point of view that the details of magnetic interactions are not so important—to get a spin glass one just needs competition between ferromagnetic and antiferromagnetic bonds due to some disorder (see, for example, Binder and Schröder 1976b). On the other hand, it is not so clear whether disorder together with purely antiferromagnetic bonds suffices to yield spin glass behavior: doped semiconductors such as Si:P, CdS:In (Kummer *et al.*, 1978, 1979; Andres *et al.*, 1981) are thought to be modeled by a spin $S=\frac{1}{2}$ Heisenberg Hamiltonian, but with $P(J)$ a distribution of purely antiferromagnetic bonds (Bhatt and Lee, 1981, 1982). In such systems the susceptibility is monotonically increasing with decreasing temperature, and no peak, characteristic of a spin glass, is found. Recent examples of spin glasses obtained by dilution of antiferromagnets will be discussed in Sec. II.D.

2. Anisotropy

While in the Introduction we considered Ising spins ($S=\pm 1$), the systems mentioned so far are all described by vector spins, i.e., a Heisenberg rather than an Ising model. It turns out, however, that anisotropies in addition to the exchange may play a crucial role (Walstedt and Walker, 1981; Morris *et al.*, 1986). One anisotropy that is always present is due to dipolar interactions,

$$\mathscr{H}_{\mathrm{dip}}=\sum_{i\neq j}[\boldsymbol{\mu}_i\cdot\boldsymbol{\mu}_j-3(\boldsymbol{\mu}_i\cdot\mathbf{R}_{ij})(\boldsymbol{\mu}_j\cdot\mathbf{R}_{ij})/R_{ij}^2]/R_{ij}^3 \ . \tag{2.4}$$

While the resulting anisotropies do account well for the behavior observed in $Eu_xSr_{1-x}S$, both in the very dilute (superparamagnetic) regime $x<x_p^{\mathrm{NNN}}$ (Eiselt *et al.*, 1979) and in the spin glass regime at higher concentrations (Binder and Kinzel, 1983a), they are probably less important in other systems: there the temperatures of interest are typically much larger, and at the same time the magnetic moments $g\mu_B S$ are much smaller than in $Eu_xSr_{1-x}S$. Using a pseudodipolar anisotropy for *Cu*Mn or *Au*Fe would require a prefactor making $\mathscr{H}_{\mathrm{dip}}$ orders of magnitude larger to yield about the observed freezing temperature (Walstedt and Walker, 1981). Pseudodipolar interactions may arise in *Au*Fe due to the orbital character of the moments, and additional single-ion anisotropy terms may arise from spin-orbit coupling of Au electrons (Levy and Fert, 1981). Single-ion anisotropies are of the form

$$\mathcal{H}_{\rm SI}=-D\sum_i (S_i^z)^2$$

(uniaxial crystals, z direction being the easy axis), (2.5a)

$$\mathcal{H}_{\rm SI}^{\rm random}=-D'\sum_i (\mathbf{p}_i\cdot\mathbf{S}_i)^2 ,$$

$$[p_i^\alpha]_{\rm av}=0 , \quad [p_i^\alpha p_j^\beta]_{\rm av}=\tfrac{1}{3}\delta_{\alpha\beta}\delta_{ij} , \tag{2.5b}$$

$$\mathcal{H}_{\rm SI}=-D''\sum_i [(S_i^x)^4+(S_i^y)^4+(S_i^z)^4]$$

(cubic crystals), (2.5c)

where $\mathbf{p}_i$ is a random unit vector and D,D',D'' are constants. The most interesting anisotropy arising in the metallic spin glasses is due to the interaction introduced by Dzyaloshinskii (1958) and Moriya (1960), which is due to spin-orbit scattering of a third atom. Fert and Levy (1980, 1981) discuss this mechanism in detail and show that it is particularly important if other impurities with strong spin-orbit coupling are present in the material (e.g., *Cu*Mn containing Pd, Fe, Co, or Pt impurities). The resulting effective interaction between a pair of spins at sites $\mathbf{R}_1$, $\mathbf{R}_2$ is (the third atom being located at the origin)

$$\mathcal{H}_{\rm DM}(\mathbf{R}_1,\mathbf{R}_2)\simeq -J_{\rm DM}\frac{\sin[k_F(|\mathbf{R}_1|+|\mathbf{R}_2|+|\mathbf{R}_2-\mathbf{R}_1|)+\varphi_{\rm DM}]}{[1+ck_F(|\mathbf{R}_1|+|\mathbf{R}_2|+|\mathbf{R}_2-\mathbf{R}_1|)]}\frac{(\mathbf{R}_1\cdot\mathbf{R}_2)(\mathbf{R}_1\times\mathbf{R}_2)\cdot(\mathbf{S}_1\times\mathbf{S}_2)}{|\mathbf{R}_1|^3|\mathbf{R}_2|^3|\mathbf{R}_1-\mathbf{R}_2|k_F^3} . \tag{2.6}$$

Fert and Levy (1980, 1981) also estimate the constants $J_{\rm DM},\varphi_{\rm DM},c$ and find that typically $J_{\rm DM}$ is of the order of $0.1J_0$, where J_0 is the amplitude of the RKKY interaction, Eq. (2.1). Evidence for the relevance of Eq. (2.6) in *Cu*Mn spin glasses has come from observation that the hysteresis loop is widened if Au or Pt impurities are added (Prejean *et al.*, 1980) and that the anisotropy field seen in ESR experiments increases (Okuda and Date, 1969; Monod and Berthier, 1980).

Due to coupling of the term $\mathbf{S}_1\times\mathbf{S}_2$ to the direction of $\mathbf{R}_1\times\mathbf{R}_2$, Eq. (2.6) leads to a macroscopic anisotropy. There is a remarkable distinction between the macroscopic anisotropies resulting from Eqs. (2.5a) and (2.5b), which are of uniaxial character, and that resulting from Eq. (2.6), which is of unidirectional character. In a metallic spin glass, the total macroscopic anisotropy energy then has the form (Henley *et al.*, 1982; Levy *et al.*, 1982)

$$E_{\rm anis}=-K_1\cos\vartheta-\tfrac{1}{2}K_2\cos^2\vartheta , \tag{2.7}$$

where K_1,K_2 are phenomenological anisotropy constants and ϑ is the angle by which the spins are rotated from their frozen metastable directions. When the spins are rotated by $\vartheta=\pi$, the uniaxial anisotropy energy $\frac{1}{2}K_2\cos^2\vartheta$ is the same as for $\vartheta=0$, while the unidirectional term $K_1\cos\vartheta$ has changed sign. Evidence for Eq. (2.7) will be discussed in Sec. II.C.2; the theoretical justification of Eq. (2.7) will be discussed in Sec. VI.B.

Anisotropies of the type of Eqs. (2.5b) and (2.6) will not lead to any macroscopic global anisotropy of the system in the disordered state: on the average no direction is preferred. The situation is different, of course, if one deals with uniaxial spin glasses for which Eq. (2.5a) applies, such as $(Ti_{1-x}V_x)_2O_3$ (Dumas *et al.*, 1979): there static susceptibilities $\chi_{\parallel},\chi_\perp$ have to be distinguished according to whether the field is applied parallel or perpendicular to the easy axis. In the limit where $\chi_\perp \ll \chi_{\parallel}$ the system may be approximated by Ising spins. While $(Ti_{1-x}V_x)_2O_3$ is quasimetallic, there exists also an example of an insulating uniaxial spin glass, namely Fe_2TiO_3 (Atzmony *et al.*, 1979). This material is a curiosity because it is crystalline and stoichiometric: the spin glass character results from the fact that magnetic Fe^{3+} ions and nonmagnetic Ti^{4+} ions are randomly distributed over the same types of crystallographic sites in the lattice.

Other examples of anisotropic spin glasses are *Zn*Mn (Albrecht *et al.*, 1982), where the easy axis is the c axis and the system is Ising-type, and *Cd*Mn (Albrecht *et al.*, 1982), an example of an *XY* spin glass: due to a negative sign of the constant D in Eq. (2.5a), the Mn spins prefer to lie in the basal plane, but no direction in this plane is then preferred. *Mg*Mn, on the other hand, is another example of a Heisenberg spin glass (Albrecht *et al.*, 1982).

In *Zn*Mn and *Cd*Mn the anisotropy is comparable to the exchange. Albrecht *et al.* (1982) attribute their single-site anisotropies to spin-orbit scattering of the type considered in the Dzyaloshinskii-Moriya mechanism. Clearly, just as a microscopic explanation of the exchange constants on a quantitative level is as yet hardly possible [Eq. (2.1) together with a free-electron picture for the conduction electrons hardly is appropriate; see Fig. 6], one should not expect in general that the anisotropy can be accounted for reliably by microscopic theories, particularly since an expression such as Eq. (2.6) involves three parameters ($J_{\rm DM}$, $\varphi_{\rm DM}$, and c, in addition to k_F).

B. Properties in small magnetic fields

While the random interactions in a spin glass want to freeze in the spins in random directions, a magnetic field wants to align them parallel to the field. Thus there is competition between spin glass order and the Zeeman energy, and in fact strong enough magnetic fields can destroy the spin glass state entirely. In the following, we shall consider fields to be "small" if they are believed to affect a spin glass state not very much (or, in the disordered regime, if their contribution to the nonlinear magnetization is small). Thus smallness of the field is here not understood in an absolute sense. We start with a discussion of the frequency-dependent susceptibility and then turn to specific heat, as well as to various measurements that have yielded insight into the dynamics of spin glasses. Questions associated with hysteresis and

remanence, the temperature-field "phase diagram," and the nonlinear susceptibility near the freezing temperature will be considered thereafter.

1. Time-dependent susceptibilities

Typical data for the zero-field frequency-dependent susceptibility have already been shown in Figs. 1 and 2. The susceptibility at very high temperatures follows a Curie-Weiss law, $\chi=\tilde{C}/(T-\theta)$ [cf. Eq. (2.2)], $\tilde{C}$ being the Curie constant. Even at temperatures T as high as about $5T_f$ the first deviations from Curie-Weiss behavior occur, both in metallic (Morgownik and Mydosh, 1983; Rao *et al.*, 1983) and insulating spin glasses (Köbler and Binder, 1980); thus on a local scale strong magnetic correlations develop far above the freezing temperature. If one were to disregard the frequency dependence of the susceptibility cusp in Figs. 1 and 2 (as well as its rounding), one could extract a spin glass order parameter q from it, applying the prediction resulting from the infinite-range model of spin glasses (Sherrington and Kirkpatrick, 1975; see Sec. IV.A),

$$\chi=\frac{\tilde{C}(1-q)}{T-\theta(1-q)}\ , \tag{2.8}$$

but treating $\tilde{C}$ and θ as effective temperature-dependent parameters rather than true constants. Such a procedure was tried for both the field-cooled (fc) and the zero-field-cooled (zfc) "static" susceptibilities in the case of *Cu*Mn by Nagata *et al.* (1979; Fig. 7), with the result that q vanished roughly linearly with temperature at T_f. A similar procedure was applied earlier by Mizoguchi *et al.* (1977) to $Al_{0.63}Gd_{0.37}$. The resulting temperature variation of the estimated spin glass order parameter is shown in Fig. 8, together with Monte Carlo data for a simple cubic Ising spin glass with symmetric Gaussian distribution of bond strength between nearest neighbors on the lattice

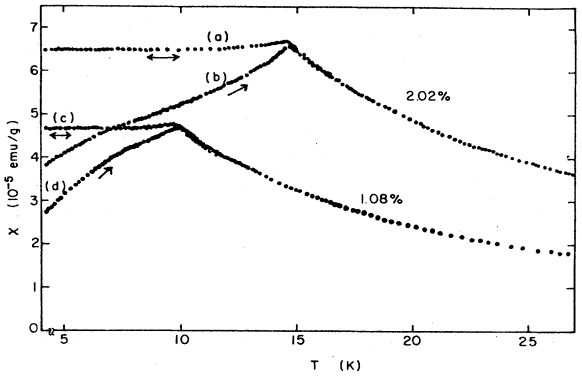

FIG. 7. Static susceptibilities of *Cu*Mn vs temperature for 1.08 and 2.02 at. % Mn. After zero-field cooling ($H<0.05$ Oe), initial susceptibilities (b) and (d) were taken for increasing temperature in a field of $H=5.9$ Oe. The susceptibilities (a) and (c) were obtained in the field $H=5.9$ Oe, which was applied above T_f before cooling the samples. From Nagata *et al.* (1979).

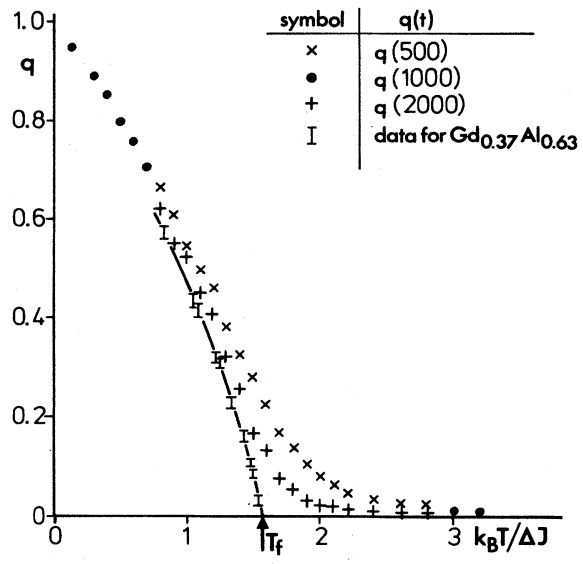

FIG. 8. Spin glass order parameter vs temperature for $Gd_{0.37}Al_{0.63}$ (from Mizoguchi *et al.*, 1977) compared to simulations of a $16\times16\times16$ Ising spin glass at various observation times t, measured in Monte Carlo steps per spin (from Binder, 1977a). ΔJ is the width of the Gaussian distribution of the simulation, and the experimental $k_BT_f/\Delta J$ is chosen arbitrarily.

[Binder (1977a); in this case $\theta=0$ and $\tilde{C}=K_B^{-1}$ if Eq. (2.8) is understood as susceptibility per spin, with $g\mu_B=1$]. The results of all these investigations are quite similar. However, one has to be very careful in interpreting these results: the frequency dependence of $\chi'(\omega)$ in Fig. 1, as well as the distinction between fc and zfc susceptibilities below T_f in Fig. 8, indicate that the establishment of full thermal equilibrium is a delicate matter. One interpretation that is frequently used (e.g., Elderfield and Sherrington, 1983a) is the zfc susceptibility measures the response of only one "valley" in phase space (see Fig. 4), and thus the order parameter measured would be that of Eq. (1.1), assuming that during the observation time one stayed in one valley. On the other hand, it is believed (Elderfield and Sherrington, 1983a) that "field cooling corresponds to averaging over all available states, the full Gibbs average," and the order resulting from the fc susceptibility would be that of Eq. (1.4). These identifications, also known as the "folklore mapping between mean-field theory and experiment," have to be viewed with considerable caution, however: Wenger and Mydosh (1984a) have observed that the fc susceptibility of the cobalt aluminosilicate spin glass $[(CoO)_{0.4}(Al_2O_3)_{0.1}(SiO_2)_{0.5}]$ depends on the cooling rate—which implies that it cannot be the thermal equilibrium susceptibility. While it is usually said that the magnetization in the field-cooled state is fully reversible, slight irreversibilities have recently been found in *Cu*Mn spin glasses by Lundgren Svedlindh, and Beckman (1982, 1983); Lundgren, Nordblad, Svedlindh, and Beckman

(1983, 1985), and since these authors find that the field-cooled magnetization slowly changes with time, their conclusion (Lundgren, Svedlindh, Nordblad, and Beckman, 1983) is that "the true equilibrium value of the magnetization is difficult to determine accurately." Moreover it is found (Lundgren, Svedlindh, Nordblad, and Beckman, 1983) that the zfc susceptibility depends strongly on the time the sample is kept at constant temperature after cooling prior to field application. These experiments show that in zero-field cooling thermodynamic equilibrium is not obtained until the time the sample has been kept at constant temperature exceeds the maximum value $\tau_{\max}$ of the relaxation time spectrum in equilibrium. Lundgren *et al.* (Lundgren, Svedlindh, Nordblad, and Beckman, 1983) interpret their results in terms of an effective waiting-time-dependent relaxation time spectrum, which is cut off at the waiting time itself (as long as it is smaller than $\tau_{\max}$). Since Lundgren *et al.* (Lundgren, Svedlindh, Nordblad, and Beckman, 1983) estimate $\tau_{\max}$ at $T=23$ K, the temperature at which the experiments were performed, as $\tau_{\max}\approx 10^{20}$ sec (while $T_f\simeq 26$ K for *Cu* with 4 at. % Mn), they conclude that "the spin glass state is a nonequilibrium one, in the thermodynamical sense."

In the picture of the spin glass free-energy hypersurface anticipated in Fig. 4 we can easily understand this behavior *qualitatively* by remarking that the coarse-grained free energy is strongly temperature dependent near T_f: above T_f all barriers between valleys are very small and become gradually larger as the temperature is lowered. In other words, all valleys are very shallow above T_f, but some of them become very deep as the temperature is lowered. Thus, on cooling, a system may well find itself at first in a side-minimum halfway between the bottom of the valley and a saddle point. Then a lot of rearrangements of clusters of spins may be necessary until the system can relax from the metastable side-minimum to the stable valley.

It is not absolutely certain that this behavior proves the nonequilibrium character of the spin glass state. If states with spin glass order exist as thermal equilibrium states, they are certainly highly degenerate (i.e., there are many order-parameter components). On cooling the system through its transition temperature one would expect to find that locally the system would start to form ordered regions of its various possible order-parameter components. As these regions grew, there would be misfit at their walls (like that at "antiphase domain boundaries" in ordered alloys or antiferromagnets), and hence the growth of the domains might be extremely slow. This mechanism to explain the slow relaxation encountered in spin glasses below T_f was first discussed by Stauffer and Binder (1978), who showed, simulating a Mattis (1976) spin glass, that for such a system it would not lead to slow enough relaxation. On the other hand, it is now known that in systems with larger order-parameter degeneracy, kinetics of domain growth is slower (Grest *et al.*, 1983; Sadiq and Binder, 1983) than in a system with only two ordered states; an additional slowing down of domain growth might be due to domain-wall pinning effects at strong bonds.

In view of this alternative possible interpretation of some of the experimental findings, it is important to establish whether there exists a unique static freezing temperature T_f. Mulder *et al.* (1981) define a frequency-dependent freezing temperature $T_f(\omega)$ from the position of the susceptibility maximum (see Fig. 1) and conclude

$$\left|\frac{d\ln T_f(\omega)}{d\ln\omega}\right| = 0.0022 \tag{2.9}$$

is independent of concentration for *Cu*Mn in the concentration range $0.94 < x < 6.4$ at. % Mn, while T_f itself is roughly proportional to the concentration. If Eq. (2.9) held down to $\omega\to 0$, the static freezing temperature would be zero; however, the frequency dependence of $T_f(\omega)$ over the observed frequency range is so weak that it is equally likely that $T_f(\omega)$ actually settles down to a static limit. In fact, "static" measurements of the susceptibility of *Cu*Mn $Al_{0.63}Gd_{0.37}$ spin glasses (Malozemoff and Imry, 1981), where time constants from minutes to 24 hours were used, did not show any shift of T_f with the time constant at all. Similarly, in amorphous $PrP_{0.85}$ (Guyot *et al.*, 1980) and in $Eu_{0.4}SR_{0.6}S$ (Ferré *et al.*, 1981) one finds that $T_f(\omega)$ seems to settle down towards a constant value when it is measured over a wide enough frequency range (Fig. 9). We shall come back to this behavior in Sec. V.E below. Here we only note that a comparative discussion of the frequency dependence of T_f in many spin glasses is given by von Löhneysen (1981), with the conclusion that "a common law for $\Delta T_f/(T_f\Delta\log\omega)$ is not visible" [for more recent additional data, see Lecomte, von Löhneysen, and Wassermann (1983)].

Particularly rewarding insight into the dynamics of spin glasses has recently been gained by experiments

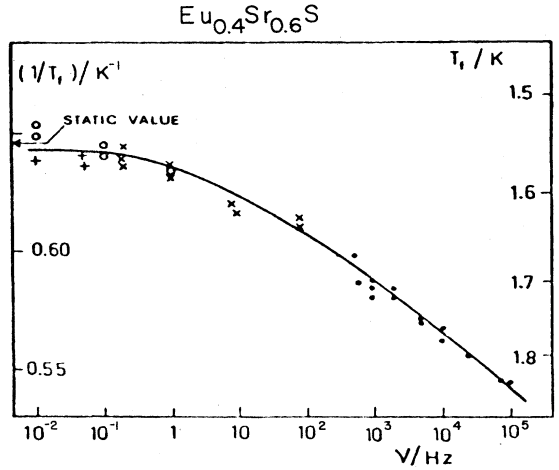

FIG. 9. Inverse freezing temperature $T_f^{-1}(\omega)$ of $Eu_{0.4}Sr_{0.6}S$ plotted vs logarithm of measurement frequency. Different symbols indicate different measurement techniques. From Ferré *et al.* (1981).

where both the real and the imaginary parts (χ',χ'') of the complex susceptibility $\chi(\omega)$ have been measured accurately. Such measurements have been possible for $Eu_{0.4}Sr_{0.6}S$ (Rajchenbach and Bontemps, 1983; Paulsen *et al.*, 1984), $Eu_{0.2}Sr_{0.8}S$ (Hüser *et al.*, 1983; see Fig. 10), $(Ho_2O_3)_{0.08}(B_2O_3)_{0.92}$, and $(CoO)_{0.4}(Al_2O_3)_{0.1}(SiO_2)_{0.5}$ spin glasses (Wenger, 1983). If the magnetization relaxed with a single relaxation time τ, χ' and χ'' could be expressed in terms of τ and the static isothermal and adiabatic susceptibilities χ_T,χ_S as follows (Casimir and du Pré, 1938):

$$\chi'(\omega)=\chi_S+\frac{\chi_T-\chi_S}{1+\omega^2\tau^2} , \tag{2.10a}$$

$$\chi''(\omega)=\omega\tau\frac{\chi_T-\chi_S}{1+\omega^2\tau^2} . \tag{2.10b}$$

An implication of Eqs. (2.10) is that a plot of χ'' vs χ' (with ω as a parameter) should yield a semicircle [this is known as the "Cole-Cole plot" in dielectric relaxation (Daniel, 1967) and the "Argand diagram" in magnetic relaxation]. If the relaxation of the magnetization is governed by a distribution of relaxation times rather than a single one, however, these plots are approximately described by arcs of semicircles. Then Eq. (2.10) is generalized as (Lundgren *et al.*, 1981; Hüser *et al.*, 1983; Wenger, 1983).

$$\chi'(\omega)=\chi_S+\int_{\tau_{\min}}^{\tau_{\max}}[\chi_T(\tau)-\chi_S(\tau)]\frac{g(\tau)d\ln\tau}{1+\omega^2\tau^2} , \tag{2.11}$$

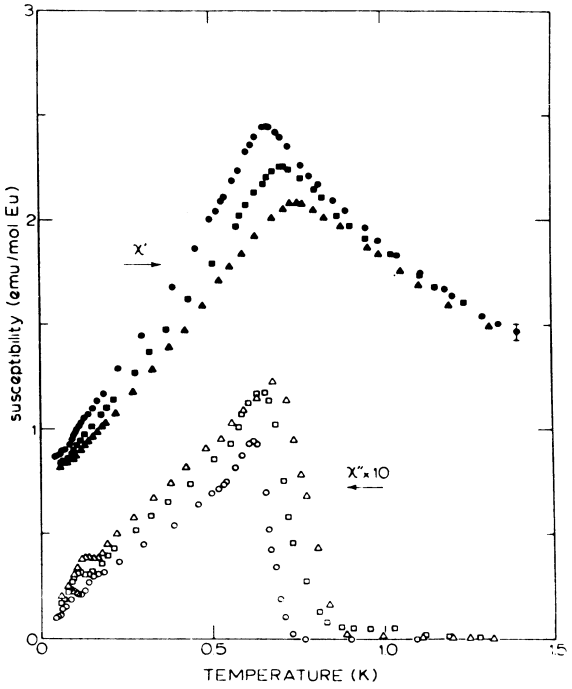

FIG. 10. Temperature dependence of the dispersion χ' (solid symbols) and absorption χ'' (open symbols) for $Eu_{0.2}Sr_{0.8}S$ at an applied field $H\approx 0.1$ Oe. Frequencies: ●,○,$\omega=10.9$ Hz; ■,□,261 Hz; ▲,△, 1969 Hz. From Hüser *et al.* (1983).

$$\chi''(\omega)=\int_{\tau_{\min}}^{\tau_{\max}}[\chi_T(\tau)-\chi_S(\tau)]\omega\tau\frac{g(\tau)d\ln\tau}{1+\omega^2\tau^2} , \tag{2.12}$$

where $\chi_T(\tau)$ and $\chi_S(\tau)$ are time-dependent generalizations of the isothermal and adiabatic susceptibilities, and $g(\tau)$ is a suitable distribution function of relaxation times. Of course, $\chi'(\omega)$ and $\chi''(\omega)$ obviously can only determine the product $[\chi_T(\tau)-\chi_S(\tau)]g(\tau)$, and not the factors of this product separately. Thus Eq. (2.11) is usually applied together with the assumption that in the time domain of interest (between minimum and maximum relaxation times) the τ dependence of $\chi_T(\tau)-\chi_S(\tau)$ can be neglected in comparison with that of $g(\tau)$ [see, for example, Wenger (1983)]. An example of the distribution function $g(\tau)$ resulting in this way for the cobalt aluminosilicate spin glass is shown in Fig. 11(a), and the temperature dependence of $\tau_{\min}$, $\tau_{\max}$, and $\tau_{\rm av}$ [defined from the frequency where $\chi''(\omega)$ is maximal, via $\tau_{\rm av}=1/\omega$] in Fig. 11(b) (Wenger, 1983). These data allow several important conclusions.

(i) The relaxation time spectrum is very broad, even far above T_f. As the temperature is lowered, it dramatically broadens and ultimately extends to the regime of macroscopic times at and below T_f.

(ii) The shortest relaxation time $\tau_{\min}$ is a microscopic time, of the order of 10^{-11}–10^{-10} sec in the cobalt aluminosilicate spin glass.

(iii) Both $\tau_{\min}$ and $\tau_{\rm av}$ are consistent with Arrhenius-type laws,

$$\tau=\tau_0\exp(E_{\rm act}/k_BT) , \tag{2.13}$$

where the prefactor τ_0 and activation energy $E_{\rm act}$ have physically reasonable values ($\tau_0\approx 10^{-11}$ sec, $E_{\rm act}$ being in the range between 14 and 30 K).

Such a behavior would be consistent with thermally activated processes, involving constant (i.e., temperature-independent) (free) energy barriers. However, the maximum relaxation time $\tau_{\max}$ is *not* consistent with this simple picture of thermally activated hopping over barriers (which we shall nevertheless investigate in more detail in Sec. III.G): if one were to apply Eq. (2.13), a fit to $\tau_{\max}$ would work only over rather restricted temperature ranges, and near T_f unphysically high values of τ_0 and $E_{\rm act}$ would result (e.g., $10^{30}\lesssim\tau_0\lesssim 10^{40}$ sec). In the context of glassy systems, a strong temperature variation such as that of $\tau_{\max}$, is often described by the law attributed to Vogel (1921) and Fulcher (1925),

$$\tau=\tau_0\exp[E_{\rm act}/k_B(T-T_0)] , \tag{2.14}$$

where T_0 is a characteristic temperature introduced in an *ad hoc* fashion. In fact, Tholence (1979, 1980) has shown that the frequency dependence of $T_f(\omega)$ for a number of spin glasses is better described by

$$[T_f(\omega)-T_0]^{-1}\propto\ln(\omega\tau_0) , \tag{2.15a}$$

which follows from Eq. (2.14), than by

$$T_f^{-1}(\omega)\propto\ln(\omega\tau_0) , \tag{2.15b}$$

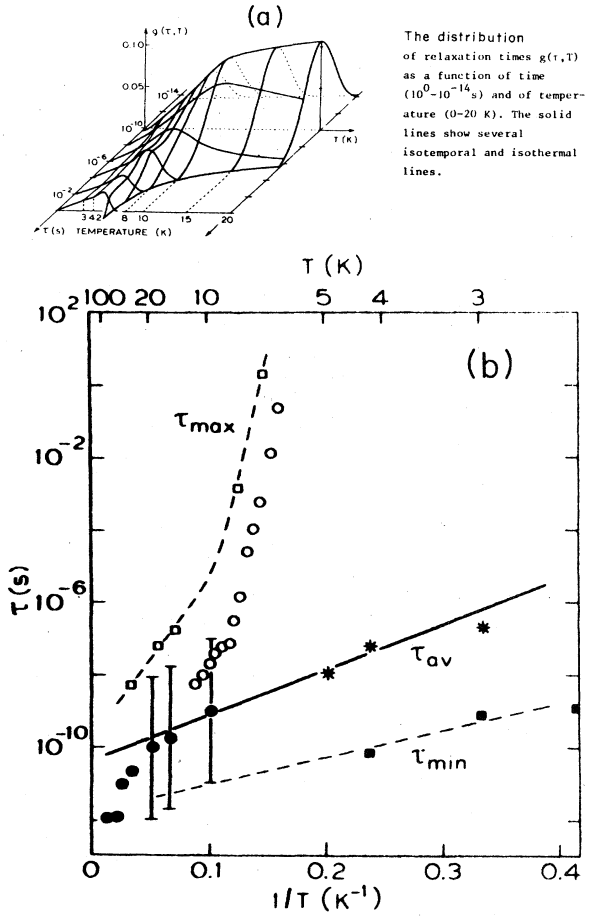

FIG. 11. (a) The distribution of relaxation times $g(\tau)$ as a function of time (10^0–10^{-14} sec) and of temperature (0–20 K). The solid curves shows several isotemporal and isothermal lines. All data were obtained from an analysis of χ',χ'' for $(CoO)_{0.4}(Al_2O_3)_{0.1}(SiO_2)_{0.5}$ in the frequency range from 0.64 Hz to 30 MHz. (b) Relaxation times as obtained from χ'' measurements, plotted vs inverse temperature: □, maximum relaxation time; *, average relaxation time; ■, minimum relaxation time. Open circles are deduced from the frequency dependence of $T_f(\omega)$ [where $\chi'(\omega)$ has its peak] and solid circles are the μSR measurements of Uemura, Huang, *et al.* (1981). From Wenger (1983).

which would follow from Eq. (2.13) (see Sec. III.G). On the other hand, the temperature T_0 does not coincide with the phase transition temperature T_f that is used when data such as those shown in Figs. 7 and 8 are interpreted in terms of a static order parameter (or other data involved in the analysis of the static equation of state of spin glasses; see Sec. II.C.4).

This behavior of the relaxation times even above T_f (if a static freezing temperature exists) is clearly rather hard to understand in terms of a phase transitions picture. There one would rather expect a standard critical slowing down (Hohenberg and Halperin, 1977),

$$\tau_{\max}=\tau_0(1-T_f/T)^{-z\nu}\ ,\quad T\to T_f\ , \tag{2.16}$$

where ν is the critical exponent of the correlation length and z the dynamic exponent. Very large values of z would be needed to account for the observed behavior. At ordinary phase transitions, $\tau_{\max}$ usually is very small, even rather close to the transition, and hardly ever reaches the regime of macroscopic times for the observable temperature differences (apart from cases where a transition temperature tends to zero, as in a ferromagnet near its percolation threshold).

Thus measurements of time-dependent susceptibilities in principle might yield detailed insight into the dynamics of freezing. At present, however, the lack of corresponding reliable analytic theories hampers the interpretation of these experiments, and the rather phenomenological analysis of Eq. (2.11) (see Fig. 11) affords only a somewhat qualitative description. The same difficulty also applies to other measurements of dynamic properties, which are discussed next.

2. Probes of behavior on short time scales: neutron scattering, Mössbauer effect, nuclear magnetic resonance, muon relaxation, electron-spin resonance

Two types of inelastic scattering have been applied to the study of the dynamics of spin glasses. The standard type of inelastic scattering experiment measures the cross section for the scattering of neutrons with wave vector $\mathbf{k}_0$ for a momentum transfer $\mathbf{K}=\mathbf{k}-\mathbf{k}_0$, where $\mathbf{k}$ is the wave vector of the outgoing neutrons, and the corresponding energy transfer $\hbar\omega=(k^2-k_0^2)\hbar^2/2m_N$ [where $\hbar$ is Planck's constant and m_N the neutron mass]

$$\frac{d^2\sigma}{d\Omega\, d\omega}\propto\frac{k}{k_0}\left[S(\mathbf{K})\delta(\omega)+\frac{2}{\pi}\frac{F^2(\mathbf{K})}{g^2\mu_B^2}\frac{1}{1-\exp(-h\omega/k_BT)}\times\mathrm{Im}\chi(\mathbf{K},\omega)\right]. \tag{2.17}$$

In this expression [see Marshall and Lovesey (1971) for a derivation], Ω denotes the angle under which the scattering is observed, $S(\mathbf{K})$ is the static scattering function, $F(\mathbf{K})$ is the magnetic form factor, and $\chi(\mathbf{K},\omega)$ is the wave-vector-dependent dynamic susceptibility. For $\mathbf{K}=0$, $\chi(\mathbf{K},\omega)$ reduces to the dynamic susceptibility discussed in the previous subsection.

In principle, Eq. (2.17) contains several kinds of information on the system: from the elastic $S(\mathbf{K})$ one infers the wave-vector-dependent static susceptibility $\chi(\mathbf{K})$,

$$(g\mu_B)^2S(\mathbf{K})/F^2(\mathbf{K})=\chi(\mathbf{K})=\sum_{i,j}\exp[i\mathbf{K}\cdot(\mathbf{r}_i-\mathbf{r}_j]\langle\mathbf{S}_i\cdot\mathbf{S}_j\rangle_T\ . \tag{2.18}$$

Clearly, the elastic part contains information both on

magnetic short-range order [if there were periodic magnetic long-range order, $\chi(\mathbf{K})$ would also exhibit the associated Bragg δ-function singularities] and on a frozen-in component, if $[\langle \mathbf{S}_i \rangle_T \langle \mathbf{S}_i \rangle_T]_{\rm av}$ is nonzero. The inelastic part reflects the dynamic relaxation of the spins. If the relaxation of magnetization fluctuations with wave vector $\mathbf{K}$ were described by a single relaxation time $\tau_{\mathbf{K}}$, we would have a Lorentzian line shape

$$\mathrm{Im}\chi(\mathbf{K},\omega)=\hbar\omega\chi(\mathbf{K})\frac{\hbar/\tau_{\mathbf{K}}}{\omega^2+(\hbar/\tau_{\mathbf{K}})^2} . \tag{2.19}$$

In early work on inelastic neutron scattering in spin glasses (Scheuer *et al.*, 1977, 1979) it was in fact attempted to analyze the data with the help of Eq. (2.19). While this assumption of a single relaxation time $\tau_{\mathbf{K}}$ is appropriate at temperatures far above freezing, the data show that it becomes inadequate near the freezing temperature. There the analysis of the inelastic scattering becomes very difficult: for small ω, one is limited by the finite energy resolution of the experiment, and relaxation times τ exceeding about 10^{-9} sec can no longer be resolved. In addition, the limited scattering intensity (and associated statistical errors of the data) make a detailed line-shape analysis difficult. A related systematic difficulty of this technique is the ambiguity of the separation between elastic and inelastic parts of the scattering. This problem has been studied in detail for *Cu* with 8 at. % Mn by Murani (1978a, 1978b, 1978c), Murani and Tholence (1977), and Murani and Heidemann (1978). These authors showed that by changing the energy resolution one changes the apparent elastic cross section, and hence one indirectly shows that there is a wide spectrum of relaxation times present. For that reason it is clear from the outset that any attempt to extract the Edwards-Anderson order parameter from neutron scattering is a very delicate matter (see also the discussion given by Soukoulis *et al.*, 1978). We emphasize, however, that neutron scattering is an invaluable tool for asserting that ordinary magnetic long-range order is absent (by the lack of corresponding Bragg peaks) and for analyzing the ferromagnetic or antiferromagnetic short-range order in the material. This application is particularly important when one studies phase diagrams in which spin glass and ferromagnetic orderings compete, such as those for $Eu_xSi_{1-x}S$ (Maletta and Convert, 1979; Maletta, 1982). We shall return to this problem in Sec. II.D. Another application of this type of inelastic neutron scattering is the search for spin waves in the frozen phase of spin glasses. While relaxing or diffusing modes yield a peak of $\mathrm{Im}\chi(\mathbf{K},\omega)$ at $\omega=0$, and such behavior is also observed in metallic spin glasses for temperatures below freezing (Scheuer *et al.*, 1979), spin waves should show up via peaks at $\pm\omega_{\mathbf{K}}\neq 0$. So far, no evidence for spin-wave excitations exists in the typical metallic spin glasses. In $Eu_xSr_{1-x}S$ for $x=0.52$, where one is just near the borderline concentration at which the system can no longer support ferromagnetic long-range order, the data of Maletta *et al.*, (1981) are consistent with some broad spin-wave peaks at larger values of $\mathbf{K}$. The dispersion of these spin waves is still nearly quadratic, resembling those occurring in the ferromagnetic phase at higher concentrations. Thus these spin waves probably merely reflect the existence of rather pronounced ferromagnetic short-range order in the material. These experiments, as well as related ones on Fe_xCr_{1-x} alloys (Fincher *et al.*, 1980; Shapiro *et al.*, 1981a, 1981b), are reviewed in more detail by Maletta (1982).

The second type of scattering is the neutron spin-echo method (Mezei, 1980). This method is somewhat better suited to the study of spin glass freezing because it yields directly the so-called intermediate scattering function,

$$S(\mathbf{K},t)\propto\sum_{ij}\langle \mathbf{S}_i(t)\cdot\mathbf{S}_j(0)\rangle_T\exp[i\mathbf{K}\cdot(\mathbf{r}_i-\mathbf{r}_j)] , \tag{2.20}$$

over a time domain of roughly $10^{-12}<t<10^{-8}$ sec. This technique has been applied to *Cu* with 5 at. % Mn (Mezei and Murani, 1979; Mezei, 1981; Murani *et al.*, 1981; see Fig. 12) and to $La_{0.7}Er_{0.3}Al_2$ (Mezei *et al.*, 1983; Mezei, 1983). It is clearly recognized that the relaxation is distinctly slower than exponential and changes rather gradually with temperature. Near and below the "static" freezing temperature ($T_f\simeq 27.5$ K in this system; see Tholence, 1980) the relaxation is nearly consistent with a logarithmic law, $S(\mathbf{K},t)\propto \mathrm{const}-\ln t$, over some intermediate range of times. The authors of these experiments interpret their results as clear evidence that the freezing of spins in spin glasses is a purely dynamical gradual process, without any static phase transition. However, Heffner *et al.* (1984) argue that the data in Fig. 12 can as well be fitted to a form $S(\mathbf{K},t)\propto \mathrm{const}+t^{-\zeta}$, where the exponent $\zeta\approx 0.24$ near T_f and $\zeta\approx 0.5$ for $T<T_f$. The constant in this form is nonzero only for $T<T_f$ and thus would imply a truly static ordering. Clearly, also, the data of Fig. 12 are too limited to establish the precise analytic form by which $S(\mathbf{K},t)$ decays.

The other methods to which we turn now, the Mössbauer effect, nuclear magnetic resonance (NMR), electron-spin resonance (ESR), and muon-spin resonance (μSR), all measure the local spin dynamics, rather than the global (or long-wavelength) dynamic relaxation accessible by the methods so far discussed. These "local probes" hence are also sensitive to the varying local envi-

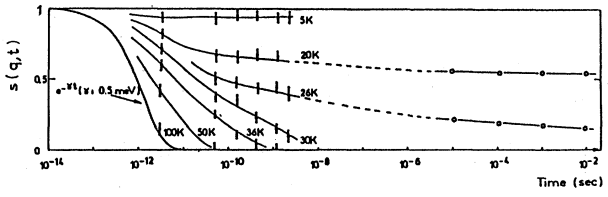

FIG. 12. Spin relaxation in *Cu*Mn with 5 at. % Mn, at various temperatures. Data points at times shorter than 10^{-8} sec were directly measured by neutron spin-echo techniques at $K=0.093$ Å^{-1} (Mezei and Murani, 1979); those beyond 10^{-6} sec were calculated from ac susceptibility results of Tholence (1980). The lines are guides to the eye only. From Mezei (1981).

ronments that the Mössbauer atom, the atom used for NMR, ESR, or the μ^+ particle, respectively, may encounter. All these techniques involve intrinsic time constants, and relaxation slower than these time constants is indistinguishable from truly static behavior. In μSR, for instance, the lifetime of the μ^+ particle, $\tau_\mu = 2.2.10^{-6}$ sec (Uemura and Yamazaki, 1982), limits the time range over which relaxation can be studied to $t < 10^{-5}$ sec; in the Mössbauer effect using ^{57}Fe (which is useful for *Au*Fe spin glasses; see Gonser *et al.*, 1965; Violet and Borg, 1965, 1967; Lauer *et al.*, 1981), relaxation can only be studied if it decays for times $t \lesssim 10^{-7}$ sec. So far, however, little analysis of Mössbauer spectra in terms of the relaxation dynamics near the freezing transition has been attempted [Meyer *et al.* (1985)]; rather, most experiments have been directed towards estimating the freezing temperature from the onset of a "peak splitting" in the Mössbauer spectrum, and towards an understanding of the spectrum in the "frozen" state. The latter question is nontrivial, since the resulting distribution of hyperfine fields seen in this spectrum depends in detail on the local atomic arrangement around the Fe atoms, and pronounced chemical short-range-order effects occur in these alloys (Beck, 1978, 1980, 1981, 1983; Dartyge *et al.*, 1982).

The Mössbauer effect has been used to trace out phase diagrams as functions of temperature and concentration, e.g., for amorphous $Fe_xNi_{78-x}Si_9B_{13}$ (Manns *et al.*, 1983).

NMR measurements have been carried out for *Cu*Mn spin glasses by MacLaughlin and Alloul (1976, 1977), Levitt and Walstedt (1977), and Alloul (1979a, 1979b). Again, the Larmor frequency corresponding to the applied fields is of the order of 10^7/sec. In the "fast motion region" (Abragam, 1961) the linewidth of the NMR signal corresponds to the spin-autocorrelation time

$$\tau \propto (1/N)\sum_i \int_0^\infty dt \langle \dot{\mathbf{S}}_i(t)\cdot\dot{\mathbf{S}}_i(0)\rangle \ . \tag{2.21}$$

The experiments indicate that τ rapidly increases as one approaches the freezing temperature, and there τ becomes comparable to the inverse cutoff frequency, which enters the expression for the linewidth (Abragam, 1961), and hence Eq. (2.21) no longer applies. These experiments also show that τ depends strongly on the applied fields, which are of the order kOe (as in the Mössbauer effect). Thus neither of these techniques is suited for studying the dynamics in small fields near the freezing temperature. The same limitation applies to ESR measurements, of course; from ESR linewidth measurements in *Cu* with 4.25 at.% Mn (Salamon and Herman, 1978; Salamon, 1979) and dilute *Ag*Mn (Dahlberg *et al.*, 1979) one can conclude that the spin-autocorrelation time increases strongly above the freezing temperature and that freezing occurs roughly in the same temperature ranges as measured by the bulk frequency measurements. However, the analysis of Salamon (1979) seems to be at variance with recent ESR work by Hoekstra *et al.* (1982).

While the information on the dynamics of freezing due to NMR and ESR linewidth experiments is rather qualitative, these techniques are very valuable at temperatures far below freezing, where all the spins in the sample are more or less rigidly locked together; observation of resonance modes themselves (rather than their linewidths) as a function of applied fields yields information on the macroscopic anisotropy of spin glasses (see Sec. II.C.2).

Recent observations of zero-field NMR spectra in *Cu*Mn, *Ag*Mn, and *Au*Mn spin glasses (Alloul and Mendels, 1985) for low temperatures ($T \lesssim 0.2T_f$) yield evidence that the decrease in the local magnetization $|\langle S_i \rangle_T|$ is proportional to T^2 with increasing temperature T. This reduction of the local magnetization is found to agree quantitatively with that observed for the remanent magnetization (Alloul *et al.*, 1986). These authors suggest that the dominant low-energy modes have diffusive character.

The theory of the ESR linewidth at $T > T_f$ has recently been considered by Barnes (1984) and Levy *et al.* (1983). Recent ESR measurements include (LaGd)Al_2 spin glasses (Zomack *et al.*, 1983) and AgMn spin glasses (Mozurkewich *et al.*, 1984). It appears that the quantitative analysis of such ESR measurements is a complicated problem.

More detailed information on the dynamics of spin freezing has been obtained by muon-spin depolarization measurements. Three different techniques have been applied.

(i) In transverse-field μSR, one measures the muon-spin precession in a transverse field. From the damping of the precession envelope one infers the depolarization rate. Murnick *et al.* (1976) observed that in *Cu*Mn and *Au*Fe spin glasses this rate became drastically slower near the freezing temperature. Similar results were obtained by Emmerich and Schwink (1981) in *Cu*Mn and by Brown *et al.* (1981) in *Ag*Mn spin glasses.

(ii) In zero-field μSR, one measures the decay positions by forward and backward counters, and extracts the muon-spin relaxation function from the time dependence of the observed asymmetry. This technique was introduced by Uemura *et al.* (1980) for *Au*Fe and *Cu*Mn spin glasses, and it was found that the relaxation time increases rapidly from about 10^{-10} sec at $T \sim 1.2T_f$ to 10^{-5} sec at $T \sim 0.5T_f$. Since by μSR one cannot measure any relaxation slower than 10^{-5} sec, this means that at $T \lesssim T_f$ one starts to observe a "static" component whose weight increases as the temperature is lowered further. In order to interpret these experiments quantitatively, however, it is necessary to make model assumptions about the shape of the muon-spin relaxation function. This problem is discussed in more detail by Uemura (1980; Uemura, Huang, *et al.*, 1981; Uemura, Nishiyama, *et al.*, 1981). In any case, the temperature dependence of the "average" relaxation time fitted to the μSR data is in reasonable agreement with the relaxation time fitted to the neutron spin-echo data (Fig. 12; see Fig. 13). This point is discussed in detail by Heffner and MacLaughlin (1984), who suggest that the data imply a power-law decay of the spin-autocorrelation function with time.

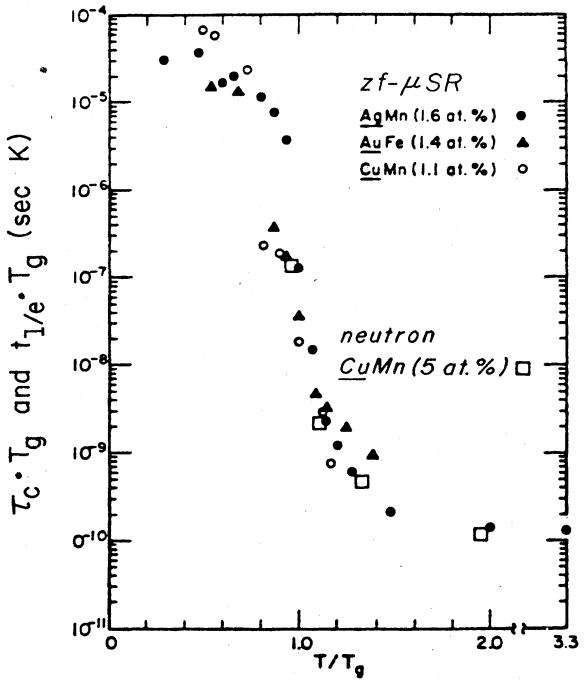

FIG. 13. Comparison of correlation times of *Cu*Mn, *Au*Fe, and *Ag*Mn spin glasses, as determined by zero-field μSR, to the decay time at which the spin-correlation function measured by neutron spin echo (Fig. 12) decays to $1/e$ in *Cu*Mn with 5 at. % Mn. The neutron experiment was performed by Mezei and Murani (1979), μSR data for *Ag*Mn are due to Heffner *et al.* (1982). Correlation times and temperatures for different specimens are scaled by the freezing temperatures in this figure. From Uemura (1981b).

(iii) In longitudinal-field μSR one proceeds as in zero-field μSR, but measures the relaxation as a function of the applied longitudinal field. The idea behind this technique (Uemura, Huang, *et al.*, 1981; Uemura, Nishiyama *et al.*, 1981; Uemura and Yamazaki, 1982) is to study the competition between the applied field and the "static" component of the internal (random) field. The field dependence clearly reveals that relaxation functions applying a single effective relaxation time are inadequate. In this spirit, Emmerich *et al.* (1983) tried to fit a rectangular spectrum of relaxation times to the zero-field μSR data on *Cu*Mn spin glasses. MacLaughlin *et al.* (1983) observe that in *Ag*Mn spin glasses far below freezing the relaxation rate observed as a function of field H can be fitted to a power law, $H^{-0.46\pm0.04}$, and suggest that this behavior indicates a nonexponential decay of spin correlations with time $t^{-\zeta}$, $\zeta \approx 0.54 \pm 0.05$. As a further support of this interpretation, they relate their results to recent zero-field NMR data of Alloul *et al.* (1983) and Alloul (1983) for *Cu*Mn spin glasses. MacLaughlin *et al.* (1983) claim that "the present data therefore agree quantitatively with the prediction of mean-field dynamic theories." However, a glance at their Fig. 1 immediately reveals that they observe a relaxation rate variation typically over about one-half order of magnitude only, and their data points have large error bars. Furthermore, near T_f their exponent ζ is about $\frac{1}{4}$, and the applied field might affect the relaxational behavior directly. In our opinion, the data are again evidence for a broad spectrum of relaxation times, and hence a nonexponential decay of the relaxation function, but one should be cautious about drawing any more definite conclusions.

Emmerich *et al.* (1985), analyzing zero-field μSR data on *Cu*Mn spin glasses, suggest that the data can only be explained by the onset of a spatially inhomogeneous local "static" order parameter ("static" again is meant on the time scale of the experiment). They also give evidence for spatial inhomogeneity of relaxation times and suggest that the formation of small regions of ordered spin starts at $T \approx 1.6 T_f$. Emmerich *et al.* (1985) do not detect any critical fluctuations near T_f.

3. Specific heat

The specific heat of spin glasses exhibits a rather broad peak at temperatures exceeding the freezing temperature by about 20%; at $T < T_f$ it varies approximately linearly with T; in a magnetic field, the specific-heat peak is progressively rounded. A few typical examples of this behavior are shown in Fig. 14 for *Cu* with 2.79 at. % Mn (Brodale *et al.*, 1983) and in Fig. 15 for $Eu_xSr_{1-x}S$ (Meschede *et al.*, 1980). A similar behavior has been seen in *Cu*Mn spin glasses at other concentrations (Zimmermann and Hoare, 1960; Wenger and Keesom, 1976; Martin, 1978, 1979, 1980a; Fogle *et al.*, 1978), in *Au*Fe (Martin, 1980b), in *Pt*Mn spin glasses (Nieuwenhuys *et al.*, 1973; Sacli *et al.*, 1974; Kimishima *et al.*, 1977; Wassermann, 1982), in amorphous $Zr_{30}Cu_{60}Fe_{10}$ (Lecomte *et al.*, 1981) and $GdAl_2$ (Coey *et al.*, 1977), in *Th*Gd spin glasses (Sereni *et al.*, 1979), etc.

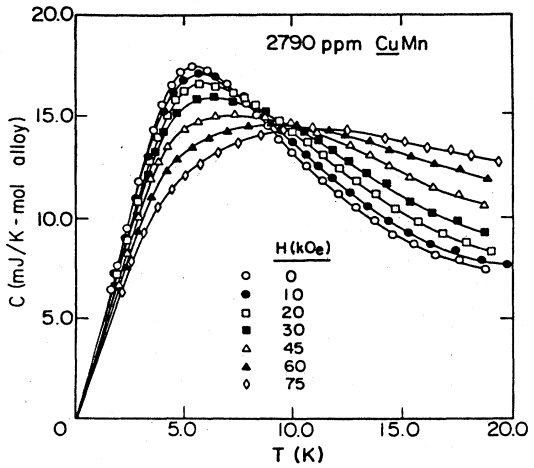

FIG. 14. Magnetic contribution of the specific heat of *Cu*Mn spin glasses with 2.79 at. % Mn plotted vs temperature in various magnetic fields. From Brodale *et al.* (1983).

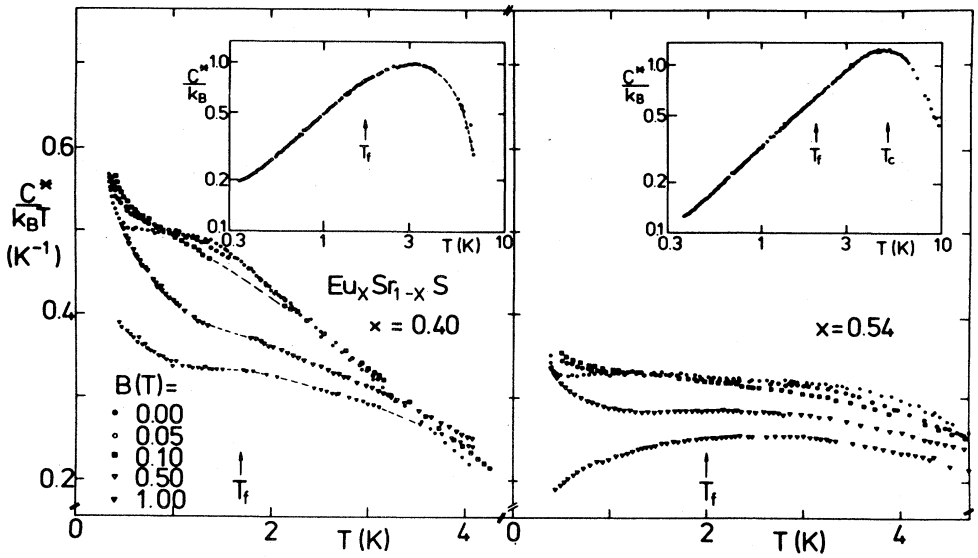

FIG. 15. Specific heat per Eu atom divided by $k_B T$ vs temperature T in various applied fields (units in tesla) for $Eu_xSr_{1-x}S$ with $x=0.40$ and $x=0.54$. Insets show the zero-field behavior in a log-log plot. The transition temperatures T_f (spin glass transition) and T_c (transition to a ferromagnetic phase, occurring for $x=0.54$) are indicated by arrows. From Meschede *et al.* (1980). Note that the upturn of the curves for C/T, seen at very low temperatures, is interpreted as being due to a Schottky term due to magnetically decoupled small clusters of Eu spins.

Closer inspection shows that the linear variation of the specific heat does not hold exactly, but rather $C_M \approx c_1 T + c_2 T^2$, where c_1, c_2 are constants (Martin, 1979) or $C_M \propto T^{3/2}$ (Coey *et al.*, 1977; Thomson and Thompson, 1981). The latter law is familiar from Heisenberg ferromagnets, and over some temperature ranges it might also apply to systems containing large ferromagnetic clusters, where spin waves with wavelength not exceeding the cluster size may occur at low temperatures. Thus this law $C_M \propto T^{3/2}$ would be plausible for $GdAl_2$ (Coey *et al.*, 1977), since for this system pronounced ferromagnetic short-range order is expected; it is less plausible for systems such as *Cu*Mn and *Ag*Mn spin glasses (Thomson and Thompson, 1981), however.

Typical data for *Cu*Mn spin glasses at low temperatures are shown in Fig. 16 (Martin, 1979). The data are cast into a scaling representation suggested by the "concentration scaling laws" (Souletie and Tournier, 1969), which predict

$$C_M(T,x)/x = \tilde{C}_M(T/x) \; . \tag{2.22}$$

It is evident that Eq. (2.22) is a rather crude approximation, even for alloys in the concentration range from $x=10^{-2}$ to 10^{-3} [for larger x, it becomes even worse, and for very small x, Eq. (2.22) breaks down due to the Kondo effect (for a review see Fischer, 1978)]. Thus we shall disregard "concentration scaling" in the following. But it is evident from Fig. 16 that C_M is linear in T over a rather wide range of temperatures; in any case the physical origin of the curvature seen at the smallest temperatures is unclear. Even the linear regime raises interesting questions: should one attribute it to "two-level systems," as one does in ordinary glasses (Anderson *et al.*, 1972)? We shall come back to this question at several places later in this paper.

Another remarkable fact is the lack of any singularity in the zero-field specific heat at the freezing temperature.

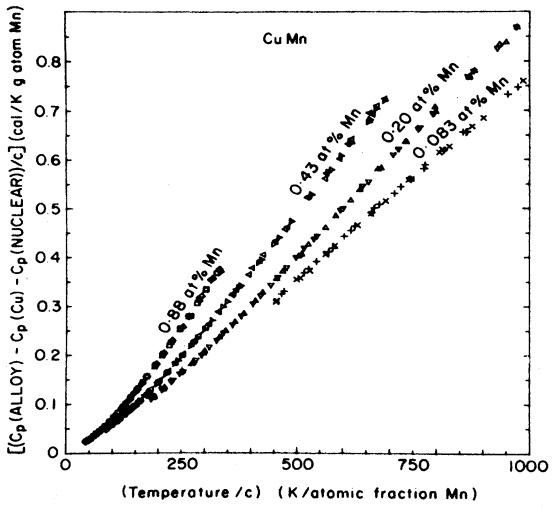

FIG. 16. Plot of scaled spin glass specific heat against scaled temperature (the concentration c of Mn atoms is expressed as an atomic fraction). In order to extract the magnetic contribution, the specific heat of pure Cu and the nuclear contribution of the Mn atoms are subtracted. From Martin (1979).

While some features of the experimental results are in surprisingly close agreement with mean-field predictions, this finding certainly is not: as will be discussed later, mean-field theory predicts a cusp in the specific heat at T_f, which is as pronounced as that of the susceptibility. Of course, the absence of a visible singularity at T_f does not prove the nonexistence of a phase transition either.

(i) The critical behavior might be subtle (e.g., a large negative value of the specific-heat exponent, such that the singular term is not the leading term near T_f).

(ii) The "critical region" for C_M might be so narrow that the asymptotic critical behavior is not seen. Examples of such a possibility occur even in diluted ferromagnets, like $Co_xZn_{1-x}(C_5H_5NO_6)(ClO_4)_2$ above the percolation threshold ($x_p \simeq 31$ at.%) up to $x = 50$ at.% (Algra *et al.*, 1979): no anomaly of C_M is found at the Curie temperature where the long-range ferromagnetic order sets in.

Since observation-time dependence has been a crucial parameter for understanding the behavior of the susceptibility, one may ask whether time-dependent effects are also seen in specific-heat measurements. So far clear evidence for energy relaxation has only been found to be induced by a relaxing remanent magnetization (Berton *et al.*, 1981). Sereni *et al.* (1979) use their measured specific heats of *Th*Gd spin glasses to estimate the magnetic entropy difference between the spin glass ground state and high temperatures, and find that this entropy difference is only about 70% of its expected value. If this result were correct, it would indicate a large "frozen-in entropy," as in a real glass (for a discussion of the latter, see Jäckle, 1981), and then the specific heat in the frozen regime must differ from its equilibrium behavior. We feel, however, that the analysis of Sereni *et al.* (1979) is hampered by uncertainties due to the use of (inaccurate) concentration scaling laws and the subtraction of lattice contributions at high temperatures; the (smaller) frozen-in entropy found in *Cu*Mn and *Au*Fe spin glasses (Martin, 1980a, 1980b) may suffer from similar problems, and further work on this question seems desirable.

A further check on the question of thermal equilibrium is the validity of the thermodynamic relations between cross-derivatives of the free energy ("Maxwell relations"). Fogle *et al.* (1981, 1982) studied the Maxwell relation

$$\left[\frac{\partial^2 M}{\partial T^2}\right]_H = \frac{1}{T}\left[\frac{\partial C_M}{\partial H}\right]_T ; \tag{2.23}$$

while at first it was suggested that Eq. (2.23) indeed failed below the freezing temperature (Fogle *et al.*, 1981), a more careful analysis revealed that this suggestion was due to an underestimation of nonlinear effects on the magnetization near T_f, and that no measurable deviation from Eq. (2.23) existed (Fogle *et al.*, 1982; Wenger and Mydosh, 1982; Gullikson and Schultz, 1982). We shall return to this question from a theoretical point of view in Sec. III.B.

4. Various other techniques

Due to magnetoelastic couplings, magnetic transitions are known to lead to anomalies in the sound velocity and damping (Lüthi *et al.*, 1970). Near the transition, the magnetic contribution to the change of the sound velocity is proportional to the magnetic specific heat. This technique was applied by Hawkins *et al.* (1976) to *Au* with 8 at.% Fe, by Hawkins *et al.* (1977) to various *Au*Cr spin glasses, and by Hawkins *et al.* (1979) to *Cu* with 5 at.% Mn. In no case was a clear anomaly at the freezing temperature observed. While the specific heat typically has a broad maximum at a temperature of about 30% above T_f, the change in sound velocity in *Cu* with 5 at.% Mn has a minimum at a temperature about 30% below T_f. Hawkins *et al.* (1979) nevertheless *assume* that the sound velocity change at T_f is singular and obtain from a fit a specific-heat exponent $\alpha = -1.9 \pm 0.2$. Obviously, this result would also be consistent with a regular variation ($\alpha = -2$). Moreover, the order of magnitude of the observed sound velocity change is in agreement with theoretical expectations (Beton and Moore, 1983). The more interesting effects expected from theoretical treatments (Fischer, 1981a, 1983c; Hertz *et al.*, 1981; Khurana, 1982; Beton and Moore, 1983) to be seen in sound attenuation seem not to have received experimental attention so far.

Herlach *et al.* (1983) have studied the phonon thermal conductivity in amorphous spin glasses $(PdCuSi)_{90}TM_{10}$, where TM stands for Mn,Fe,Co. They found that below the spin glass transition the conductivity in a field is slightly larger than without a field. Herlach *et al.* (1983) suggest that the phonons are coupled via spin-orbit interaction to magnetic two-level systems, which are thermally activated by inelastic scattering of the phonons. A similar interpretation has been given by Ayadi and Ferré (1983) to their observation that irradiation of cobalt aluminosilicate spin glasses with near-infrared light enhances the rate at which the thermo-remanent magnetization relaxes. The thermal conductivity of some insulating spin glasses has also been studied by Lecomte, von Löhneysen, and Zinn (1983) and Arzoumanian *et al.* (1983).

Particular attention has been paid to the electrical resistivity of metallic spin glasses, especially for various *Cu*Mn alloys (e.g., Ford and Mydosh, 1976; Schilling *et al.*, 1977) and *Au*Fe alloys (e.g., Ford *et al.*, 1970; Schilling *et al.*, 1974). One typically observes that the resistivity change (relative to the host metal) varies roughly linearly with temperature near T_f and has a broad maximum at a temperature T_m much higher than T_f. At low temperatures the resistivity change varies proportionally to $T^{3/2}$ (Ford and Mydosh, 1976) or proportionally to T^2 (Laborde and Radhakrishna, 1973). While the low-temperature variation of the resistivity has been linked theoretically to the temperature variation of the spin glass order parameter (Seiden, 1976) and to elementary excitations in spin glasses (Rivier, 1974; Adkins and Rivier, 1976, 1975; Fischer, 1979), the resistivity maximum is

linked to an interplay of spin glass properties and the Kondo effect (Larsen, 1976, 1978; Larsen *et al.*, 1977; Fischer, 1981a). Unfortunately, a detailed theory of spin glass dynamics seems to be required for an interpretation of the experimental resistivity data, which hence can yield only rather indirect information on the nature of the spin glasses.

At standard magnetic transitions the electrical resistivity has an energylike singularity (Fisher and Langer, 1968; Binder and Stauffer, 1976a). The lack of a (detectable) singularity in the specific heat at T_f hence is consistent with the fact that no anomaly of the electrical resistivity occurs there either.

Among related transport properties we mention the Hall resistivity, which has a peak similar to that of the susceptibility (McAlister and Hurd, 1976, 1978), and the magnetoresistance (Nigam and Majumdar, 1983). The latter is found to be nearly temperature independent below T_f, while it decreases above T_f and is hardly detectable above T_m. Again the analysis of the data is difficult and requires detailed theoretical work (Mookerjee, 1980). Finally we mention the thermopower of metallic spin glasses, which is found to have a pronounced maximum at low temperatures (MacDonald *et al.*, 1962; Cooper *et al.*, 1980). This behavior is explained, at least qualitatively, in terms of an interplay between spin glass properties and the Kondo effect (Fischer, 1981a).

While clearly all these phenomena are very interesting in their own right, their interpretation rests heavily on assumptions about the theory of spin glasses. Since for real three-dimensional systems the theory of spin glasses is still in a very rudimentary stage, as will be outlined later, we feel that it is not yet warranted to consider these phenomena further in the present article.

C. Properties in stronger magnetic fields

While the magnetic field in some of the work discussed above (NMR, EPR, magnetoresistance, etc.) was an important parameter, we have not yet studied the change of spin glass properties with the strength of the field. This subject will be discussed now. As we shall see below, the study of the magnetic field dependence of spin glass properties has many aspects, some of which are crucial to a proper understanding of both the nature of the "frozen state" and the "freezing transition." Hence it is necessary to describe the experimental findings in greater detail.

1. Hysteresis and remanence

A striking phenomenon in spin glasses is the observation of irreversible behavior in the temperature region of the freezing transition and at lower temperatures. Again the region in the temperature-field (T-H) plane, where the onset of irreversibility occurs, depends somewhat on the time scale of the measurement. This onset of irreversibility will be discussed in terms of various proposed H-T phase diagrams in Sec. II.C.3; at this point we shall discuss only the behavior well below this region.

After switching off the field, one finds a remanent magnetization that decays so slowly with time that a nonzero remanence is observed over macroscopic time scales. This remanent magnetization also depends in a detailed way on the "magnetic history" of the sample. The two "magnetic histories" most commonly used in the experiments are the following (Tholence and Tournier, 1974): to obtain the isothermal remanent magnetization σ_{IRM}, one cools the sample in zero field to the desired temperature to be studied; then a field of a chosen strength is applied for a macroscopic period of time and switched off again. A few typical examples of the field dependence of σ_{IRM} are shown in Fig. 17, for a metallic spin glass (*Au* with 0.5 at. % Fe), a nonmetallic spin glass ($Eu_{0.3}Sr_{0.7}S$; Maletta and Felsch, 1979), and a Monte Carlo simulation of a two-dimensional Ising square lattice with a Gaussian distribution of random nearest-neighbor bonds (Kinzel, 1979). To obtain the thermo-remanent magnetization σ_{TRM}, on the other hand, one applies the field at some initial temperature *above* T_f and then cools down the system slowly in constant field to the desired temperature, at which the field is then switched off. While σ_{TRM} does not depend on the precise value of the initial temperature, it does depend on the time t one waits before measuring σ_{TRM} after the field has been switched off (see, for example, Bouchiat and Monod, 1983); similarly σ_{IRM} depends on both the time the field is applied and the time one waits before measuring σ_{IRM} after the field has been switched off. Again different results would be obtained if other "magnetic histories" (paths of constant H/T, etc.) were to be considered. Thus it is somewhat hard to draw general conclusions about the behavior of the remanent magnetizations in spin glasses. One important observation, however, is that σ_{TRM} starts out linear with the field at small fields, while the field dependence of σ_{IRM} seems rather to be quadratic. At high fields both σ_{TRM} and σ_{IRM} tend to the same saturation value $\sigma_{\mathrm{sat}}(T)$; σ_{IRM} approaches it in a monotonic way, while one always finds a characteristic "overshooting" of σ_{TRM} (Fig. 17). Since for fields exceeding the field $H_c(T,t)$, where σ_{IRM} and σ_{TRM} merge, the history dependence and hence irreversible effects are negligible, measurements of the type shown in Fig. 17 are a method of defining a critical field $H_c(T,t)$ for the onset of irreversibility on a time scale t.

Bouchiat and Monod (1983) performed a rather systematic study of σ_{IRM} and σ_{TRM} for *Ag*Mn spin glasses in the concentration range from 1% to 24%, while varying both T and t. They found that their data—plotted in a scaled form, where $\sigma_{\mathrm{IRM}}, \sigma_{\mathrm{TRM}}$ are normalized with σ_{sat} and H is normalized with $H_c(T,t)$—are fairly independent of concentration and time, but do depend on the reduced temperature $T/T_f(t)$. Results for other spin glass systems (*Cu*Mn, *Au*Fe, *Pt*Mn, $LaAl_2$Gd, $Eu_{0.4}Sr_{0.6}S$, and the $Al_2O_3SiO_2MnO$ glass) are found to be consistent with this description.

We turn now to the magnetization in a field. After zero-field cooling the magnetization $M_{\mathrm{zfc}}(H)$ has a characteristic S-shaped form (i.e., the initial susceptibility

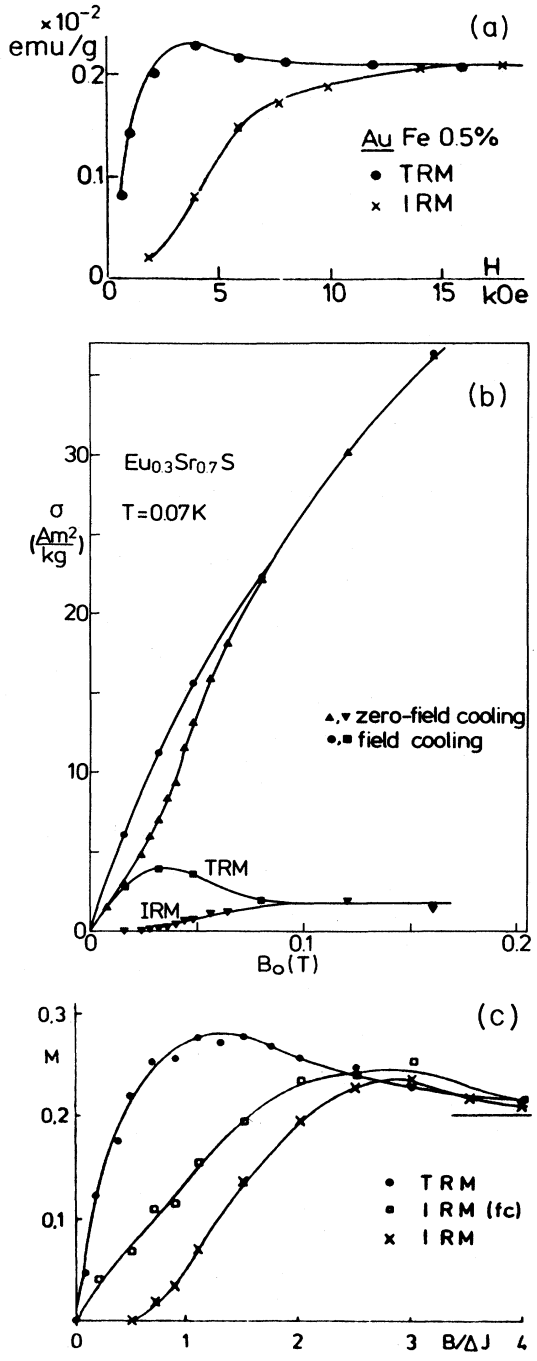

FIG. 17. (a) Field dependence of the isothermal remanent magnetization σ_{IRM} and of the thermo-remanent magnetization σ_{TRM} obtained after cooling from an initial temperature larger than T_f to $T=1.2$ K in a field H. From Tholence and Tournier (1974). (b) Magnetizations (upper two curves) and remanent magnetizations $\sigma_{IRM}, \sigma_{TRM}$ (lower two curves) of $Eu_{0.3}Sr_{0.7}S$ at $T=0.07$ K (Maletta and Felsch, 1979). (c) Monte Carlo results for $\sigma_{IRM}, \sigma_{TRM}$ for an Ising square lattice of size 50×50 with periodic boundary conditions, at a temperature $T=\Delta J/4k_B$ where ΔJ is the Gaussian distribution of exchange constants between neighboring spins. IRM (fc) is obtained by some mixed cooling procedure; see Kinzel (1979) for more details.

$\chi_{zfc}=[\partial M_{zfc}(H)/\partial H]_{H=0}$ is smaller than the susceptibility reached at an inflection point for nonzero H; see also Fig. 17(b)). This S-shaped behavior occurs for temperatures below T_f only, and is observed both in metallic spin glasses (for measurements of *Cu*Mn, see Schwink and Schulze, 1978; Emmerich and Schwink 1979; Knitter and Kouvel, 1980) and in nonmetallic ones such as $Eu_xSr_{s-x}S$ [Maletta and Felsch, 1979; Fig. 17(b)].

It is now well established, however, that $M_{zfc}(H)$ and thus χ_{zfc} are not observations that yield information on the thermal equilibrium behavior of spin glasses: rather one observes a slow increase of both quantities with observation time t [typically this increase is proportional to $T\ln t$ (Tholence and Tournier, 1974; Guy, 1975, 1977; Felten *et al.*, 1978; Knitter and Kouvel, 1980; Chamberlin *et al.*, 1982)]. These time effects are similar to the decay of the remanent magnetization in time, which will be discussed in more detail below. The magnetization $M_{fc}(H)$ found in field cooling shows only weak time effects (Beauvillain, Dupas, Renard, and Veillet, 1984; Lundgren *et al.* 1982; Lundgren, Svedlindh, and Beckman, 1983; Lundgren, Svedlindh, Nordblad, and Beckman, 1983; Wenger and Mydosh, 1984a), and thus some authors have suggested that this field-cooled magnetization is the true equilibrium magnetization of a spin glass (Monod and Bouchiat, 1982; Chamberlin *et al.*, 1982; Malozemoff and Imry, 1981). Hence we shall discuss $M_{fc}(H)$ more closely in the context of the equation of state of spin glasses near T_f (see Sec. II.C.3).

While changing the temperature at fixed field leads only to rather small irreversibility effects, changing the field at fixed temperature below T_f gives rise to pronounced irreversibility. In particular, if one cycles the field from positive to negative values and back, one observes hysteresis phenomena as in ferromagnets. However, there exists a wide variation in the shape of the hysteresis loops, which also depend on the magnetic history of the sample. In some cases, such as dilute *Au*Fe, the loops are rather narrow and flat and antisymmetric around the origin [Fig. 18(a), from Prejean *et al.*, 1980]. While in this example the initial state is a field-cooled one, in *Cu*Mn spin glasses one observes loops of this character if the initial state is a zero-field-cooled one: with a field-cooled initial state, one often observes displaced hysteresis loops [Beck, 1978; Monod *et al.*, 1979; Prejean *et al.*, 1980; for an example see Fig. 18(b)]. While in *Cu*Mn spin glasses at high concentrations the loops are quite smooth, at lower Mn concentrations loops have been found consisting of sharp steps [Schwink and Schulze, 1978; Monod *et al.*, 1979; Prejean *et al.*, 1980; for an example see Fig. 18(c)]. In the latter case almost the entire remanence is reversed in a very short time and at a very sharp value of the field, indicating a sharp, macroscopically coherent reversal of the magnetization. These phenomena have been taken by Monod *et al.* (1979) as proof of cooperative behavior among a large number of frozen spins. Recent theoretical work, in which approximate numerical mean-field calculations of various spin glass models are performed (Soukoulis *et al.*, 1983a, 1983b),

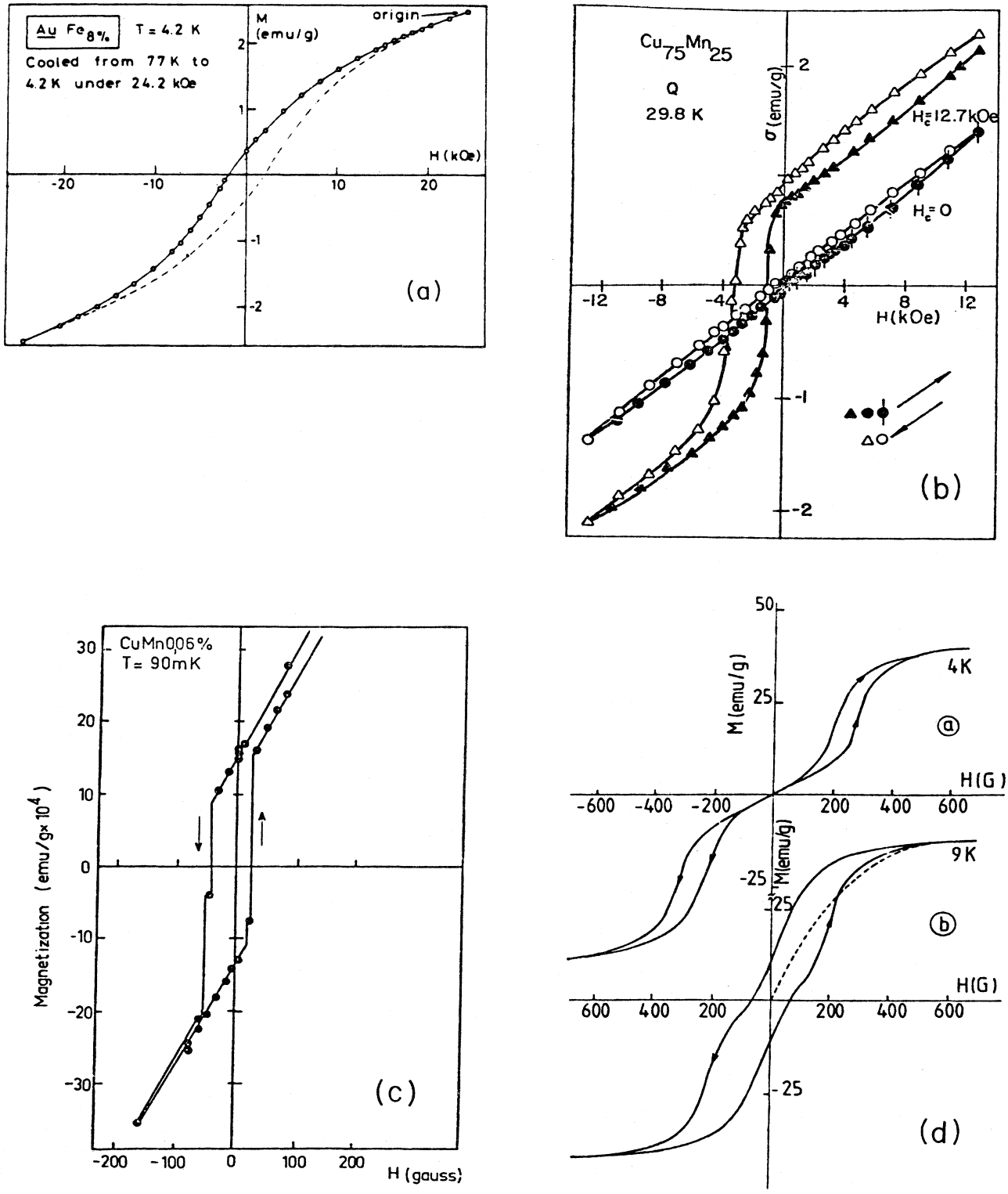

FIG. 18. (a) Hysteresis behavior of *Au*Fe with 8 at. % Fe cooled from 77 to 4.2 K in a field of 24.2 kOe. From Prejean *et al.* (1980). (b) Magnetization of *Cu*Mn with 25 at. % Mn vs field after zero-field cooling ($H_c=0$) and after cooling in a field of 12.7 kOe. From Beck (1978). (c) Hysteresis of *Cu*Mn with 0.06 at. % Mn at 0.09 K with saturated remanent magnetization. From Monod *et al.* (1979). (d) A set of hysteresis loops of *Ni*Mn with 21 at. % Mn cooled in zero field down to the temperatures indicated. From Senoussi (1984).

suggests that ferromagnetic short-range order (due to a nonzero mean value of the Gaussian distribution of interactions in the model) is essential for having such sharp magnetization reversals; in any case this behavior is limited to a small number of spin glass systems only. Particularly complicated behavior may occur in systems where spin glass and ferromagnetic orderings compete and in which states with mixed orderings can be produced. An example of hysteresis loops with more structure, found in *Ni* with 21 at.% Mn (Senoussi, 1984) is shown in Fig. 18(d).

A particularly interesting feature of this irreversible behavior in spin glasses is the slow decay of the various remanent magnetizations with time. Holtzberg *et al.* (1977) have suggested [Fig. 19(a)] that the decay law of any remanent magnetization $\sigma_{\rm RM}$ is a logarithm,

$$\sigma_R(t)=\sigma_0-S_{\rm RM}\ln t \ , \tag{2.24}$$

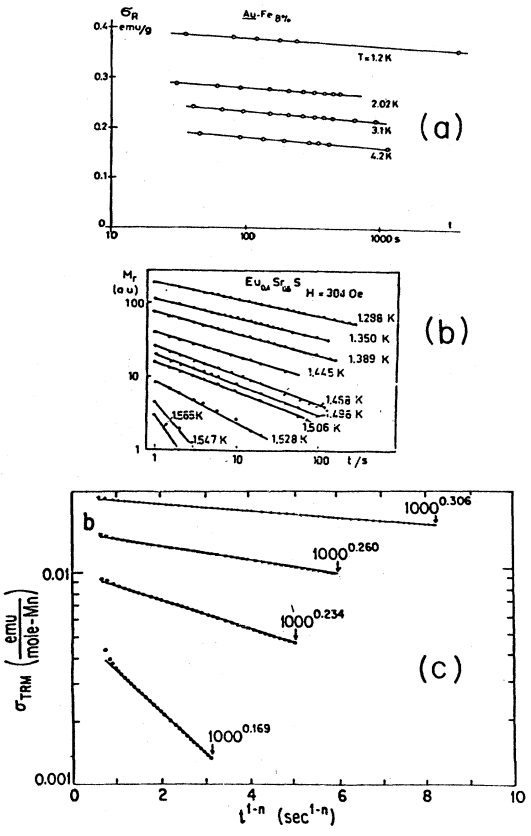

FIG. 19. (a) Isothermal remanent magnetization of *Au*Fe with 8 at.% Fe, plotted vs the logarithm of time. From Holtzberg *et al.* (1977). (b) Log-log plot of the saturated thermo-remanent magnetization $\sigma_{\rm TRM}$ vs time for $Eu_{0.4}Sr_{0.6}S$ at various temperatures close to $T_f=1.55$ K. From Ferré *et al.* (1981). (c) Logarithm of the saturation value of the thermo-remanent magnetization of *Ag*Mn with 2.6 at.% Mn plotted vs t^{1-n} at four temperatures, for a time range of about $1-10^3$ sec (Chamberlin *et al.*, 1984). Fitted exponents $1-n$ are indicated in the figure.

where σ_0 is a constant and the coefficient $S_{\rm RM}$ is called "magnetic viscosity" (e.g., Guy, 1978). Although recent work (Prejean and Souletie, 1980; Berton *et al.*, 1979; Omari *et al.*, 1984) confirms that the relaxation of the saturation value of the remanent magnetization is a function only of $[T\ln(t/\tau_0)]$, data of the type shown in Fig. 19(a) are not a proof for Eq. (2.24): they extend only over about one decade in time, and the remanent magnetization has decreased only very little during this time interval. In fact, measurements of the decay of the remanent magnetization of $Eu_{0.4}Sr_{0.6}S$ over two decades of time (Ferré *et al.*, 1981) show that Eq. (2.24) is not a valid description over the full time range studied; rather the data are more consistent with a power law [Fig. 19(b)],

$$\sigma_R(t)\propto t^{-a(T,H)} \ , \tag{2.25}$$

where the exponent $a(T,H)$ depends on both temperature and field. For temperatures not too close to the freezing temperature the remanent magnetization is proportional to the temperature and much less than unity, and hence $t^{-a(T,H)}=\exp[-a(T,H)\ln t]\approx 1-a(T,H)\ln t$, consistent with the behavior reported above. While Eq. (2.25) was first suggested from Monte Carlo simulations (Binder and Schröder, 1976a, 1976b), more careful experimental work (Ferré *et al.*, 1981; Chamberlin *et al.*, 1984; Hoogerbeets *et al.*, 1985) has revealed that Eq. (2.25) is only approximately valid. Chamberlin *et al.* (1984) pointed out that a better representation of their data for *Ag* with 2.6 at.% Mn is given by a fractional exponential decay,

$$\sigma_R(t)\propto \exp[-\text{const}\times t^{\,1-n_R}/(1-n_R)] \ , \tag{2.26}$$

the exponent $1-n_R$ being about $\frac{1}{3}$ for T not too close to T_f [Fig. 19(c)] and even smaller for T closer to T_f. It would be interesting to check whether Eq. (2.26) is valid for other spin glasses. However, Nordblad *et al.* (1986) have presented evidence that Eq. (2.26) reflects only an influence of the aging process of the spin glass.

Clearly there exists a wealth of experimental data on the irreversible behavior of spin glasses, which represents a challenge for theoretical interpretation; hence we shall come back to this problem at various points in this article (Secs. III.G, V.B.1, and V.E.4).

2. Torque, transverse susceptibility, and electron-spin-resonance measurements

In this subsection we review some experiments on low-temperature properties of spin glasses which elucidate their anisotropy properties. Already the data on displaced hysteresis loops in *Cu*Mn spin glasses [Fig. 18(b)] show that the spin system keeps some memory of the cooling field direction. While originally this unidirectional anisotropy was linked to an inhomogeneous description of the spin system (Kouvel, 1961), there is now experimental evidence for a macroscopic anisotropy with a triadic character (Fert and Hippert, 1982; Hippert and Alloul, 1982; Alloul, 1983; Alloul and Hippert, 1983). This experimental

evidence stems from torque measurements, measurements of the transverse susceptibility χ_t, and resonance experiments [Fig. 20(a)]. These experiments show that at temperatures far below T_f the spin system undergoes an essentially rigid rotation, when one exposes it to a small field $\mathbf{H}$ oriented in a direction different from the direction of the remanent magnetization σ_R. This behavior can be described in terms of a simple magnetostatic model (Saslow, 1980, 1982), in which the free energy of a spin glass is written as

$$F=(\mathbf{M}-\sigma_R)^2/2\chi_{\rm iso}-\mathbf{M}\cdot\mathbf{H}+E_{\rm anis}(\theta)\ . \tag{2.27}$$

Here $\mathbf{M}=\sigma_R+\chi_{\rm iso}\mathbf{H}$ is the total magnetization, while

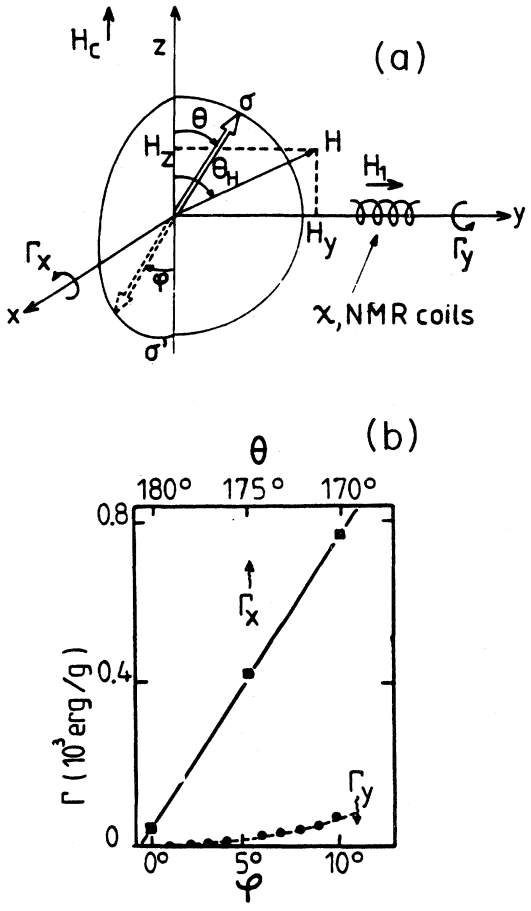

FIG. 20. (a) Geometry of various torque measurements: H_1 is the ac field in transverse susceptibility experiments and the radio-frequency field in NMR or ESR experiments. The remanent magnetization σ is rotated away from its direction in the z axis (obtained by orienting the cooling field H_c in this direction) by an angle θ in the yz plane, when the field is rotated an angle θ_H, with $\theta=\pi$ for $\theta_H=\pi$. Then a rotation of the field in the xz plane yields a rotation of σ by an angle φ. The respective torques acting on the sample are Γ_x,Γ_y. From Alloul (1983). (b) Torque measurements in a *Cu*Mn spin glass with 20 at. % Mn at 1.5 K. Both torques Γ_x and Γ_y measured after a rotation σ by an angle π in the yz plane are shown. From Fert and Hippert (1982).

$\chi_{\rm iso}\mathbf{H}$ is the isotropic part of the magnetization and $\chi_{\rm iso}$ the associated susceptibility [we consider an isotropic spin glass in the sense that there exists *a priori* no preferred direction—the form of Eq. (2.27) does not depend on the direction of the cooling field $\mathbf{H}_c$, which we denote as the z direction]. For small θ, Eq. (2.7) can be expanded as

$$E_{\rm anis}(\theta)\approx-\left[K_1+\frac{K_2}{2}\right]+\tfrac{1}{2}(K_1+K_2)\theta^2$$

$$=\text{const}+\tfrac{1}{2}K\theta^2\ , \tag{2.28}$$

i.e., only one effective anisotropy constant K enters. From Eqs. (2.27) and (2.28) one finds the transverse susceptibility χ_t taken along y,

$$\chi_t=\chi_{\rm iso}+\sigma_R(H_z+K/\sigma_R)^{-1}\ , \tag{2.29}$$

which is borne out by measurements (Hippert and Alloul, 1982). Similarly, in the limit $\sigma_R\gg\chi_{\rm iso}H_z$, one predicts an ESR mode (γ_0 is the gyromagnetic ratio)

$$\omega=\gamma_0(H_z+K/\sigma_R)\ , \tag{2.30}$$

consistent with observation (Monod and Berthier, 1980). NMR experiments also fit into this picture (Alloul, 1979a, 1979b). Furthermore, the initial torque $\Gamma_x\equiv dE_{\rm anis}/d\theta=K\theta$, which acts on the system if its magnetization is rotated by a field applied in the yz plane, yields corresponding results (Fert and Hippert, 1982; Hippert *et al.*, 1982). Now the key point of these observations is that this macroscopic anisotropy energy $E_{\rm anis}(\theta)$, which shows up in all these experiments, is independent of the magnitude of the remanence σ. Indeed it remains meaningful even in a zero-field cooled state, where no remanence exists, but then instead of Eq. (2.30) (note that now $\sigma_R\ll\chi_{\rm iso}H$) an ESR mode with frequency

$$\omega=\gamma_0(K/\chi_{\rm iso})^{1/2} \tag{2.31}$$

is predicted and found (Schultz *et al.*, 1981).

Thus K and hence $E_{\rm anis}(\theta)$ is an intrinsic property of the spin glass state. Since in this state the spin configuration minimizes the total free energy, taking into account both exchange and anisotropic couplings, any rotation of the spins with respect to the lattice costs some anisotropy energy. But this energy $E_{\rm anis}(\theta)$ will be the same, *whatever the axis of the rotation is,* since the spin glass is globally isotropic. Therefore the state of the spin glass cannot be characterized by an anisotropy axis, but rather an anisotropy triad (Saslow, 1982; theoretical background on this problem will be presented in Sec. VI.B).

If a rotation of the spin system by an angle $\theta=\pi$ around the x axis is induced by rotating the field from z to $-z$ in the yz plane [Fig. 20(a)], the triadic character of the anisotropy shows up when we consider further rotations from that state in the yz and xz planes. While for vector anisotropy the responses for these two cases should be the same, for triadic anisotropy $\Gamma_y\equiv0$ for a rotation of y in the xz plane, as the total rotation from the original state is still π. Indeed the measurements of the reversible

torque (Fert and Hippert, 1982) show $\Gamma_y \approx 0$ (and $\Gamma_y \ll \Gamma_x$) for small φ [see Fig. 20(b)]. In order to understand what happens at larger φ, one needs to study the irreversible torque as well, which is found to depend strongly on the magnitude of the remanent magnetization (Gyorgy *et al.*, 1983).

As will be discussed in Sec. VI.B, the triadic character of the anisotropy shows up most clearly in the ESR spectrum (Henley *et al.*, 1982), where (for $\sigma_R \neq 0$) three modes are predicted, two transverse modes and a longitudinal one. While there is some evidence for the transverse modes [Monod and Berthier, 1980; Schultz *et al.*, 1980, 1981; but see also Hoekstra *et al.*, 1982 and the early ESR work reviewed by Beck (1978)], the longitudinal mode has been detected only very recently (Gullikson *et al.*, 1983).

It is worth mentioning that the triadic character of the anisotropy shows up in a unidirectional angular dependence [the term $-K_1 \cos\theta$ in Eq. (2.7)] rather than in a uniaxial dependence (the term $-K_2 \cos^2\theta$), which would result from a vector anisotropy.

Microscopically, the unidirectional anisotropy is linked to the existence of Dzyaloshinskii-Moriya interaction (see Sec. II.A.2). Hence the measurements described here constitute evidence for these interactions. This evidence has been strengthened by measurements on ternary *Cu*Mn spin glasses, where the third component is a nonmagnetic impurity, and the anisotropy constant K is found to be proportional to the concentration of this third component (Hippert and Alloul, 1982; Prejean *et al.*, 1980). On the other hand, measurements of χ_t in an amorphous insulating spin glass (Velu *et al.*, 1981) have also yielded an anisotropy constant K independent of the magnitude of σ_R. A possible triadic character of the anisotropy in such a material would require a quite different microscopic origin.

3. The temperature-field phase diagram

In the mean-field theory of spin glasses (see Sec. IV) a discovery of critical lines in the plane spanned by the variables temperature and magnetic field has prompted a careful experimental search for analogous phenomena in real systems. The first of these lines was the so-called "AT line" (de Almeida and Thouless, 1978), which occurs in Ising spin glasses with infinite-range random interactions and behaves near the freezing temperature as

$$H_{\rm AT}(T)/\Delta J \propto (1 - T/T_f)^{3/2} . \tag{2.32}$$

While the equilibrium state of the spin glass for $H > H_{\rm AT}(T)$ is thought to be unique (i.e., the free-energy hypersurface contains a single "valley"), for $H < H_{\rm AT}(J)$ one expects that the spin glass can exist in many different orderings (many "valleys" in phase space; see Fig. 4). Since in a macroscopic system the transition from one valley to the next should take a macroscopically large time, and since one expects that the "topology" of the valleys changes in a complicated way when the field H is varied, it is plausible to associate the onset of irreversibility on macroscopic time scales with the AT line. In addition, according to the "projection hypothesis" of Parisi and Toulouse (1980), one would expect that $M(H,T)$ for $H < H_{\rm AT}(T)$ is a function of H alone; it has become clear that this hypothesis cannot be strictly correct (Toulouse *et al.*, 1982), but it is still thought to be a very good approximation.

For isotropic spin glasses (Heisenberg spins), the situation is more complicated, since in nonzero field longitudinal and transverse components are no longer equivalent. Gabay and Toulouse (1981) predicted that the freezing-in of the transverse components should occur at the "GT-line,"

$$H_{\rm GT}(T)/\Delta J \propto (1 - T/T_f)^{1/2} , \tag{2.33}$$

while the freezing-in of longitudinal components should occur at the AT line [Eq. (2.32)]. However, more recent work has shown that the longitudinal components should also feel the GT line (Cragg *et al.*, 1982; Moore and Bray, 1982). Thus the significance of the AT line for isotropic spin glasses—if it has any significance at all—is that of a smooth crossover region from "weak irreversibility" to "strong irreversibility" (Sherrington *et al.*, 1983).

Motivated by these predictions, Monod and Bouchiat (1982) performed a pioneering study of the temperature dependence of the field-cooled magnetization in *Ag* with 10.6 at. % Mn, assuming that in slow field-cooling experiments one obtains the equilibrium magnetization M. It is seen (Fig. 21) that the "susceptibility" M/H at high temperatures roughly follows a Curie law, while at low temperatures it approaches a temperature-independent "plateau" value, which weakly increases with decreasing field. While the plateau values are well defined, the field $H_c(T)$ where the plateau begins is not, because of the asymptotic approach to the plateau. Therefore $H_c(T)$ can only be found with considerable error bars (Fig. 21). In the spirit of the "projection hypothesis," Monod and Bouchiat (1982) identified $H_c(T)$ with the boundary of the spin glass phase, the "AT line" $H_{\rm AT}$ [Eq. (2.32)].

Berton *et al.* (1982) have studied *Cu*Mn spin glasses with 0.25 at. % Mn by the magnetocaloric effect (i.e., the variation of the temperature of an adiabatically isolated substance with external magnetic field), which via a Maxwell relation yields the temperature derivative of the magnetization

$$(\partial T/\partial H)_{S_M} = -\frac{T}{C_H}\left[\frac{\partial M}{\partial T}\right]_H \tag{2.34}$$

C_H is the specific heat at constant magnetic field, S_M the entropy. Here, of course, again the tacit assumption is made that it is the equilibrium behavior that is measured. Berton *et al.* (1982) find that $(\partial M/\partial T)_H$ tends to zero at low temperatures and has a pronounced minimum above T_f, the position of which strongly depends on the magnetic field (Fig. 22). Rather arbitrarily, Berton *et al.* (1982) define a critical field $H_c(T)$ from the inflection point (at the low-temperature side of the minimum) of the

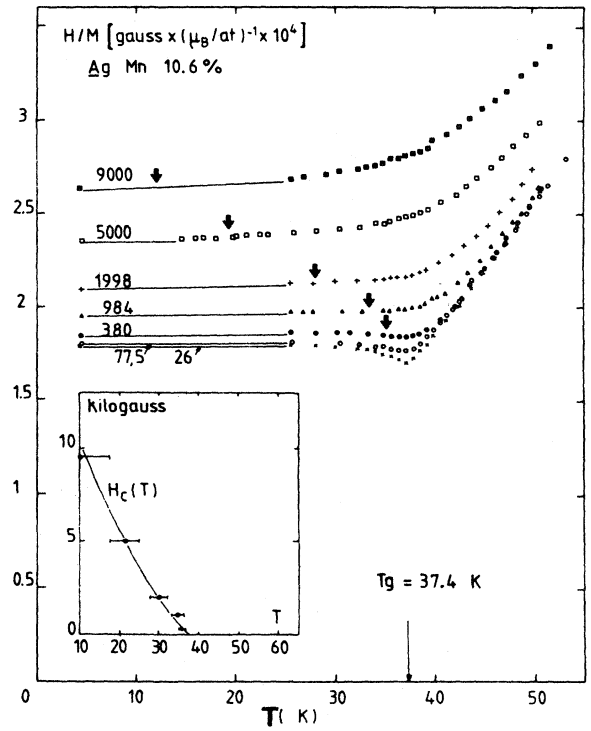

FIG. 21. Inverse of the "susceptibility" (H/M) for *Ag*Mn with 10.6 at. % Mn as a function of temperature for various magnetic fields (indicated on each curve in gauss). Data were obtained by slow cooling in constant applied field. The onset of the "plateau" (marked by arrows) is defined arbitrarily by the point of the $M(T)$ curve departing by 3% from its low-temperature value. The resulting boundary of the spin glass phase $H_c(T)$ is shown as an inset. From Monod and Bouchiat (1982).

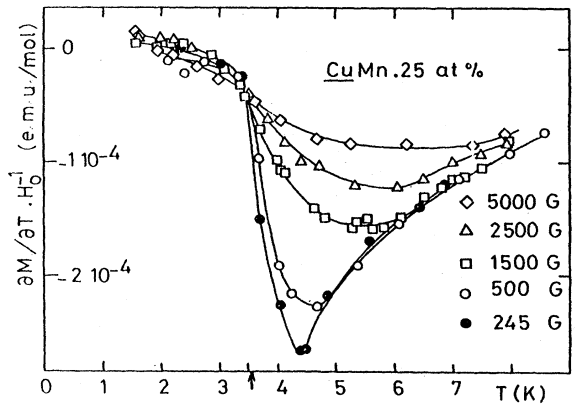

FIG. 22. Derivative of the magnetization, $H^{-1}(\partial M/\partial T)_H$ for a *Cu*Mn spin glass with 0.25 at. % Mn plotted vs temperature at various fields as indicated. Data were deduced from the magnetocaloric effect. Arrow marks the freezing temperature, where the ac susceptibility has a cusp. From Berton *et al.* (1982).

curves shown in Fig. 22, and associate this with the "AT line."

Finally Chamberlin *et al.* (1982) study several *Ag*Mn spin glasses and use the derivative of the zero-field-cooled magnetization, $dM_{zfc}(H,T)/dT$, which can be approximated near T_f by a succession of several straight-line portions with different slopes: the loci of intersection points of these straight lines yield several critical-field curves $H_c(T)$, which are discussed as "candidates" for the AT line by Chamberlin *et al.* (1982).

While some of these experimental curves $H_c(T)$ indeed bear at least a superficial resemblance to the AT line of the mean-field treatment, Eq. (2.32), it is interesting to note that other characteristic fields can be identified that behave quite differently. An interesting example is the position $T_p(H)$ of the maximum of the field-cooled "susceptibility" M/H. While in rather strong fields $T_p(H)$ is a curve of similar shape to the AT line (and to the data discussed above), for small fields the curve $T_p(H)$ bends backwards to lower temperatures. At least, such behavior has been established both for CuMn spin glasses and *Al*Gd with 37 at. % Gd (Barbara *et al.*, 1981a). A typical example is shown in Fig. 23 (Barbara and Malozemoff, 1983). This behavior of $T_p(H)$ is not predicted by mean-field theory, and hence it is not clear that it makes sense to compare other characteristic fields to mean-field predictions either. In fact, it has even been speculated (Binder and Kinzel, 1983b) that the curve $T_p(H)$ might bend back to zero temperature, $T_p(H) \propto H^{\Delta}$ as $H \to 0$, Δ being an associated exponent. However, fitting such a law to the data shown in Fig. 23 would require the exponent Δ to be very large, of the order of 100 or so. Hence it is certainly fair to say that the explanation of these data is an open question (Binder and Kinzel, 1983b).

We now turn to characteristic fields $H_c(T,t)$ defined in such a way that the time dependence of the observations is invoked explicitly (Fig. 24). There are various ways to do this: for instance, one can require that the imaginary part of the susceptibility $\chi''(\omega)$ become vanishingly small; of course, the resulting curve $H_c(T,\omega)$ must depend distinctly on the measurement frequency ω. Similarly, one

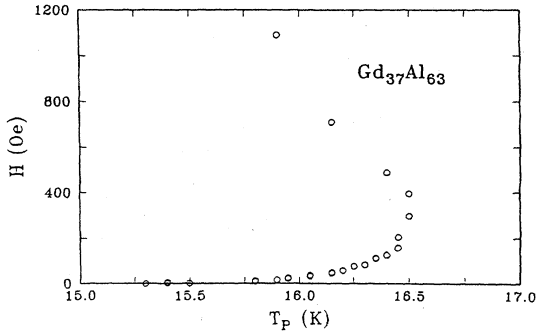

FIG. 23. Temperature $T_p(H)$ of *Al*Gd with 37 at. % Gd where the field-cooled magnetization has its maximum. From Barbara and Malozemoff (1983).

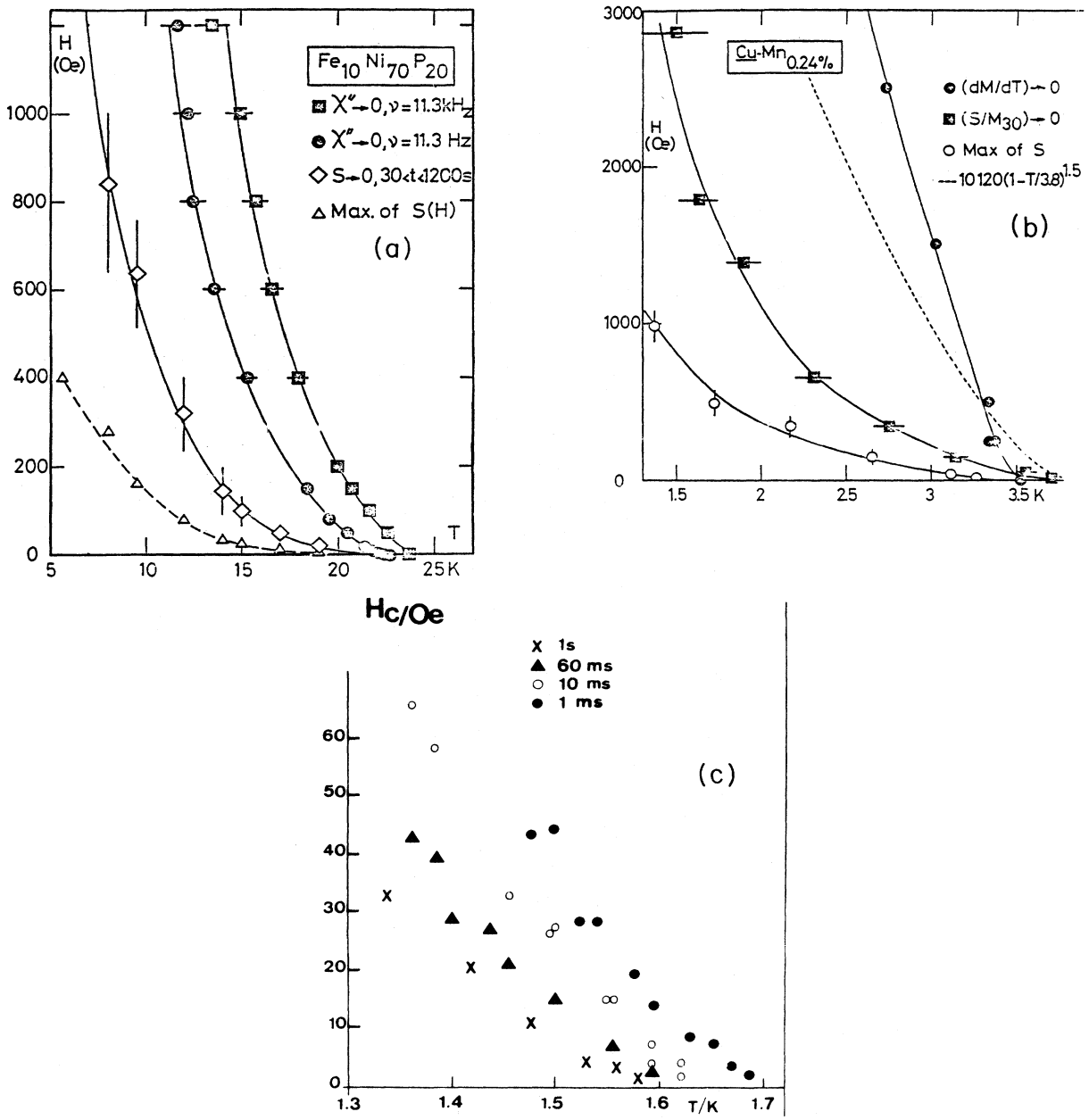

FIG. 24. (a) Critical field $H_c(T,\omega)$ [or $H_c(T,t)$, respectively] for the amorphous spin glass $Fe_{10}Ni_{70}P_{20}$ defined in various ways (see text). From Salamon and Tholence (1983). (b) Critical magnetic field for *Cu*Mn with 0.24 at. % Mn. From Tholence and Salamon (1983). (c) Critical magnetic field of $Eu_{0.4}Sr_{0.6}S$ plotted vs temperature for four different time scales. From Bontemps *et al.* (1983).

may define another critical field from the curve where the magnetic viscosity S_{RM} either vanishes or has its maximum. Of course, each of these arbitrary definitions defines a separate curve [Fig. 24(a)], but again the behavior is rather universal for different types of spin glasses [Figs. 24(b) and 24(c); see also Salamon and Tholence, 1982; Yeshurun *et al.*, 1982, 1984]. It is interesting to note that the time-dependent critical fields defined by these methods in general will intersect the static field $H_c(T)$ (as defined from $dM/dT=0$), as is shown in Fig. 24(b), where the data of Berton *et al.* (1982), derived from Fig. 22, are included as the solid circles.

Most illuminating have been studies of $Eu_xSr_{1-x}S$ in which the decay of the magnetization after a small field change (superimposed on a constant larger field) is measured by Faraday rotation (Bontemps *et al.*, 1983) or where the complex frequency-dependent susceptibility is measured (Rajchenbach and Bontemps, 1983; Paulsen *et al.*, 1984). Critical fields are defined from curves in the H-T plane where the irreversibility *on a considered*

time scale practically vanishes; as the time constant gets larger the whole curve is displaced towards lower temperatures [Fig. 24(c)]. While on this scale the curves are still similar to the AT line, a more careful study reveals a different structure at small fields (Fig. 25; Rajchenbach and Bontemps, 1983; Paulsen *et al.*, 1984): the curves $H_c(T,\omega)$ at $H=0$ merge with a perpendicular tangent, while Eq. (2.40) yields a vanishing tangent. It is obvious that it makes little sense to compare finite-time observations, which depend on the time in a really crucial manner, to the AT line, which even in the mean-field context is meaningful only in the infinite-time limit. Thus fully dynamical theories have been proposed to account for these observations (Fischer, 1983b; Binder and Young, 1984); we shall return to these theories in Secs. IV and V.

At this point, we stress only that the experimental discussion has concentrated on the AT line: although the experimental systems are more Heisenberg-type than Ising-type, and hence a GT line rather than an AT line should be expected, very little experimental evidence for a GT line has been found. Schulz and Wassermann (1983), in their study of the H-T diagram of *Pt*Mn spin glasses, find a $T_p(H)$ similar to that of Fig. 23 and suggest that this line be identified with the GT line; Kett *et al.* (1982), in a study of $Cd_{1-x}Mn_xTe$ spin glasses, claim that they can identify both the AT line and the GT line; in a view of the actually rather smooth data, however, the identification of any critical line seems rather unconvincing. Perhaps the most convincing evidence for a line similar to the GT line is offered by Mössbauer experiments in *Au*Fe spin glasses (Lauer and Keune, 1982; Campbell *et al.*, 1983) and torque measurements in *Cu*Mn and *Ag*Mn spin glasses (Campbell, De Courtenay, and Fert, 1984).

It seems to us that even if in real three-dimensional systems the spin glass freezing is a static phase transition, one should not expect to be able to describe it quantitatively by mean-field theory, as is sometimes attempted (Yeshurun and Sompolinsky, 1982). Rather, scaling concepts beyond mean-field theory are required (e.g., Malozemoff *et al.*, 1983 Barnes *et al.*, 1984; Suzuki, 1985). This point will be elaborated in the following subsection.

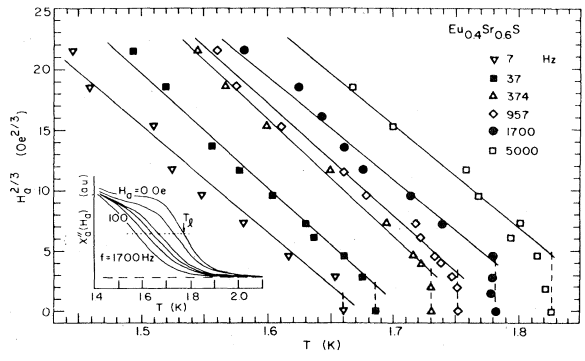

FIG. 25. Critical magnetic field (raised to the $\frac{2}{3}$ power to check for the AT behavior) plotted vs temperature for $Eu_{0.4}Sr_{0.6}S$. Measurement frequencies are indicated. From Paulsen *et al.* (1984).

4. Nonlinear susceptibility and the scaled magnetic equation of state near the freezing transition: evidence for a phase transition?

In spin glasses higher-order terms in the expansion of the equation of state in powers of the magnetic field are particularly interesting: these quantities are more sensitive to spin glass order than the zero-field susceptibility $\chi_0(T)$. The next-order term is the nonlinear susceptibility $\chi_{nl}(T)$,

$$M/H=\chi_0(T)-H^2\chi_{nl}(T)+O(H^4)\ . \tag{2.35}$$

First measurements of $\chi_{nl}(T)$ with ac techniques in the $(Ti_{1-x}V_x)_2O_3$ spin glass (Chikazawa *et al.*, 1980, 1981), due to their frequency dependence, are not easily interpreted. Hence more attention has been paid to the pioneering work of Monod and Bouchiat (1982) on the field-cooled magnetization of *Ag*Mn with 10.6% Mn, which these authors believe to yield the true equilibrium magnetization. Figure 26(a) show that their data for M/H are indeed consistent with a quadratic variation in H, for temperatures above the freezing temperature $T_f=37.4$ K. As one approaches T_f, the region of this quadratic variation becomes much smaller. Fitting these data to Eq. (2.35), Monod and Bouchiat (1982) concluded that the resulting nonlinear susceptibility was consistent with a critical divergence at T_f,

$$\chi_{nl}(T)\propto[(T-T_f)/T_f]^{-\gamma}\ , \tag{2.36}$$

where γ is a critical exponent characteristic of a static phase transition to a spin glass state. While they estimated [see Fig. 26(a)] that $1\lesssim\gamma\lesssim2$, most other work results in estimates of $3\leq\gamma\leq4$ (Barbara *et al.*, 1981b, 1982; Berton *et al.*, 1982; Omari *et al.*, 1983) notable exceptions are Mn aluminosilicate, for which $\gamma\approx1$ is obtained (Beauvillain, Dupas, Renard, and Veillet, 1984), *Au*Fe, for which Chikazawa *et al.* (1983) suggest a logarithmic divergence of χ_{nl} only, and $Eu_{0.48}Sr_{0.52}S$, for which $\chi_{nl}(T)$ is believed to diverge only at zero temperature [Fig. 26(b)]; Köbler *et al.*, 1984).

Unfortunately, none of these analyses is really convincing. The trouble is best illustrated by Fig. 26(c), taken from Chikazawa *et al.* (1983), where a log-log plot of $\chi_{nl}(T)$ vs $|1-T/T_f|$ is given: while for $|1-T/T_f|\gtrsim0.1$ the data are nicely consistent with an exponent $\gamma\gtrsim3$, and consistent with Omari *et al.* (1983), who use only that temperature range, one sees that χ_{nl} flattens off when one approaches T_f. Taking any value of the exponent γ from data of this kind seems to us not to be a very meaningful procedure. It may be that $\chi_{nl}(T)$ tends to a logarithm law, as suggested by Chikazawa *et al.* (1983), or to a finite constant at T_f, as suggested by Köbler *et al.* (1984). On the other hand, it may be that the rounding of this peak is not of an intrinsic nature, but

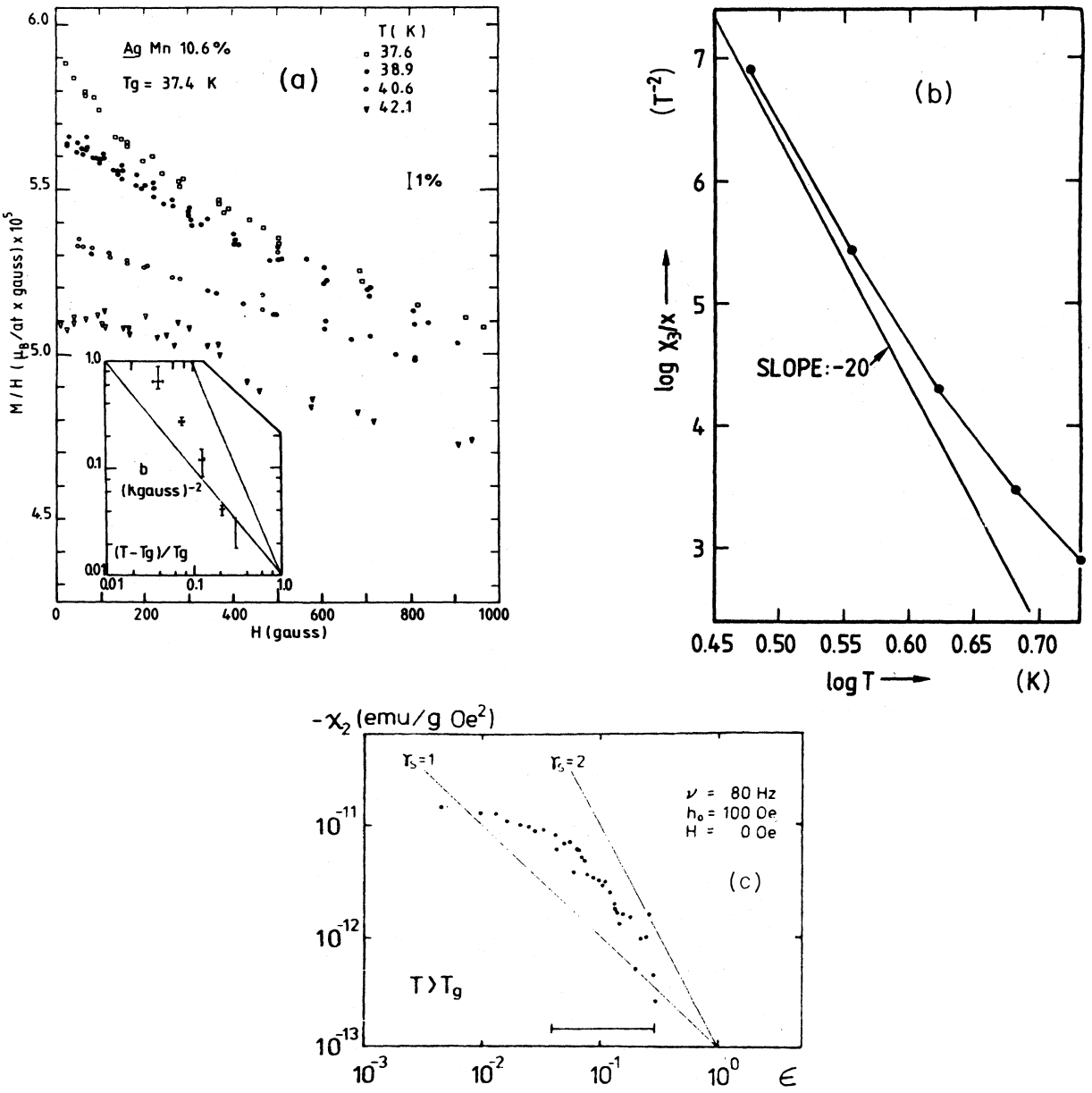

FIG. 26. (a) M/H plotted vs magnetic field in AgMn with 10.6 at. % Mn at several temperatures above the freezing temperature as indicated. Inset shows a log-log plot of the nonlinear susceptibility χ_{nl} vs the reduced temperature $(T-T_f)/T_f$ (straight lines indicate variations $\chi_{nl}=[(T-T_f)/T_f]^{-1}$ and $\chi_{nl}=[(T-T_f)/T_f]^{-2}$, respectively). From Monod and Bouchiat (1982). (b) Log-log plot of $\chi_{nl}(T)$ vs T for $Eu_{0.48}Sr_{0.52}S$. The straight line suggests an asymptotic law $\chi_{nl}(T)\propto T^{-20}$ at low temperatures. From Köbler *et al.* (1984). (c) Log-log plot of $\chi_{nl}(T)$ vs $|T-T_f|/T_f$ for AuFe with 0.97 at. % Fe (Chikazawa *et al.*, 1983). The horizontal bar indicates the temperature range used in (a).

due to sample inhomogeneities on large scales, in which case the "ideal behavior" may be closer to what is seen for $|1-T/T_f|\gtrsim 0.1$ (Omari *et al.*, 1983).

There is, moreover, a difficulty with part of the data (Barbara *et al.*, 1981b, 1982), namely, that Eq. (2.35) seems to fail: one sees no regime of quadratic variation with H^2 but rather a behavior as

$$M/H=\chi_0(T)-H^{[a(T)-1]}\chi_1(T)+\cdots , \tag{2.37}$$

where the exponent $a(T)$ decreases continuously from its value deep in the paramagnetic phase $[a(T)=3]$ to a value of about $a(T_f)\approx 1.7$ at the freezing temperature. Even if Eq. (2.37) represents the data nicely, it is not a permissible behavior in the paramagnetic phase for $T>T_f$ at all, since the free energy F there must be analytic in H. Due to the symmetry $H\Leftrightarrow -H$, only even powers of H can occur in the Taylor expansion of F, and hence $a(T)\equiv 3$ for all $T>T_f$. Again, the apparent

discrepancy between this conclusion and experiment is attributed to sample inhomogeneities (Barbara *et al.*, 1982). A further complication is that some of these systems have a lot of ferromagnetic short-range order, which may enhance $\chi_{nl}(T)$ considerably and lead to strong but noncritical background temperature dependence. In particular, close to reentrant ferromagnetic phase boundaries, rather sharp gigantic peaks of $\chi_{nl}(T)$ must occur, even if the system remains paramagnetic (Binder, 1982b).

Finally we mention a very recent interesting analysis of the critical behavior of the *Ag*Mn spin glass due to Bouchiat (1986); the conclusion of this work is that in the region $10^{-2} \lesssim 1-T_f/T \lesssim 10^{-1}$ one can identify an exponent $\gamma \approx 2.2 \pm 0.2$, while for temperatures farther away from T_f there is an apparently larger "effective exponent" due to corrections to scaling.

Other measurements concern the magnetic field dependence of the susceptibility right at T_f. One of the first measurements of this type was performed for $(Eu_xSr_{1-x})S$, for $x=0.15$ to $x=0.40$ (Maletta and Felsch, 1979; Maletta, 1980); in this experiment the decrease of the susceptibility maximum was studied. If there is Edwards-Anderson order, one can relate the susceptibility at T_f to the field dependence of the Edwards-Anderson order parameter $q(T,H)$ (see Sec. IV),

$$q(T_f,H) \propto 1-\chi^{(\max)}(H)/\chi^{(\max)}(0) \; . \tag{2.38}$$

As with the standard notation in critical phenomena, one may associate a critical exponent δ with this variation (Binder, 1977a; Chalupa, 1977b),

$$q(T_f,H) \propto (H^2)^{1/\delta} \; . \tag{2.39}$$

In this definition, we have made use of the fact that it is H^2 that is the field conjugate to q (Harris *et al.*, 1976; see also Sec. III.F.1), in contrast to the case of a ferromagnet, where simply H is the field conjugate to M.

The experimental data [Fig. 27(a)] are indeed compatible with Eq. (2.39) and would imply an exponent $\delta \approx 4.1$. This estimate, moreover, is in good agreement with numerical simulations [Stauffer and Binder, 1978; Fig. 27(b)]. However, again the situation is not completely convincing: the data are probably affected by the finite measurement frequency (or finite "observation time" of the simulation, respectively). Most measurements of the field-cooled magnetization at T_f, which is believed to be the thermal equilibrium magnetization (Malozemoff and Imry, 1981; Monod and Bouchiat, 1982)—at least the effects due to time dependence are smaller—have yielded exponent estimates in the similar range $4 \lesssim \delta \lesssim 6$ (Barbara *et al.* 1981b, 1982; Omari *et al.* 1983; Beauvillain, Dupas, Renard, and Veillet, 1984). In contrast, the result $\delta \approx 1.5$ obtained by Mulder *et al.* (1981) for *Cu*Mn with 2 at. % Mn from ac susceptibility measurements seems to be significantly different, as does the recent work of Bouchiat (1986) on *Ag*Mn spin glasses, which implies a somewhat smaller value, $\delta = 3.1 \pm 0.2$.

All these data are indicative of the existence of a static phase transition at nonzero temperature; they do not constitute an ultimate proof, however, since either the analyzed data may still be affected by nonequilibrium effects, or the observed nonanalytic variation with magnetic field, Eq. (2.39), is an artifact of probing the response in a regime of fields where it is highly nonlinear, and for much smaller fields Eq. (2.35) might still be valid.

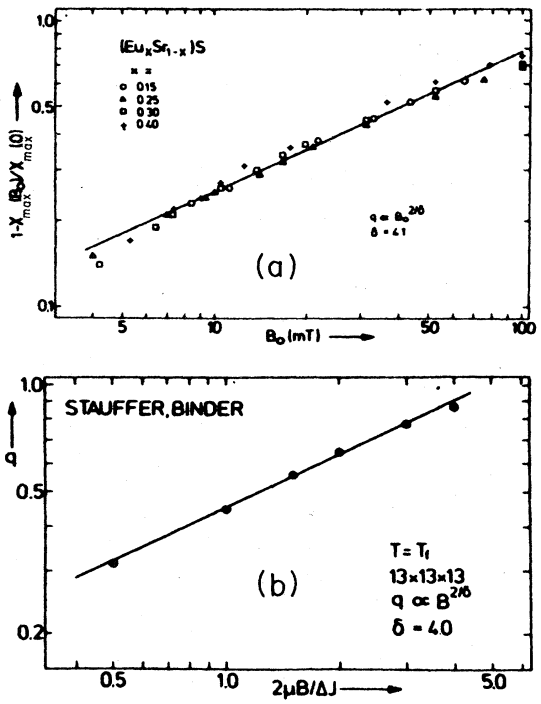

FIG. 27. (a) Depression of $\chi^{(\max)}$, from ac measurements at $\omega = 117$ Hz as a function of applied field (denoted as B_0 in the figure) in $Eu_xSr_{1-x}S$, for various concentrations x as shown in the figure. Straight line on this log-log plot indicates an exponent $\delta = 4.1$. From Maletta (1980). (b) Field dependence of the Edwards-Anderson order parameter at T_f, calculated from Monte Carlo simulations of a simple cubic Ising model with nearest-neighbor exchange distributed according to a Gaussian of width ΔJ. Data points are based on observations of 2000 Monte Carlo steps/spin. From Stauffer and Binder (1978).

None of these possible criticisms applies to measurements of the magnetic equation of state at temperatures sufficiently above T_f, like those of Omari *et al.* (1983). In this region, time effects are still unimportant, as shown by all analyses of frequency-dependent susceptibilities (see Sec. II.B.1). Furthermore, the equation of state is analyzed in terms of an expansion analogous to Eq. (2.35), namely,

$$\frac{M}{H} = \chi_0^{(\mathrm{ni})} \left[a_1 - \frac{1}{15} a_3 \left[\frac{\mu H}{T} \right]^2 + \frac{2}{305} a_5 \left[\frac{\mu H}{T} \right]^4 + \cdots \right] , \tag{2.40}$$

where μ is the magnetic moment per spin, $\chi_0^{(\mathrm{ni})}$ is the

zero-field susceptibility of a noninteracting paramagnet, and the expansion coefficients a_1, a_3, a_5 are obtained from the fit. (The normalization chosen is such that for a noninteracting paramagnet $a_1 = a_3 = a_5 = \cdots = 1$.) The advantage of this analysis is that one keeps track of higher-order nonlinear terms unlike Eqs. (2.35) and (2.37), and hence one can convince oneself that the range of field values used in the analysis is appropriate. A very interesting result emerging from this analysis is the prediction

$$a_5 \propto a_3^{2.25} , \tag{2.41}$$

which is particularly strong evidence for a transition, as one does not need to fit any value for T_f. If T_f is assumed to coincide with the freezing temperature taken from low-field, low-frequency ac susceptibility measurements, one finds that all the data on the nonlinear part of the magnetic equation of state, $M(T,H)/H - \chi_0(T)$, can be collapsed on a single scaling function [Fig. 28(a)]. As we shall discuss in later sections in more detail, static scaling at a spin glass transition can be cast in the form (Chalupa, 1977b; Suzuki, 1977; see also Binder, 1977a)

$$1 - \frac{M}{\chi_0 H} = \left[1 - \frac{T}{T_f}\right]^{\beta} \tilde{M} \left[\left[\frac{H}{T}\right]^2 \left[1 - \frac{T}{T_f}\right]^{-\gamma-\beta} \right] , \tag{2.42}$$

where $\tilde{M}$ is some scaling function. Figure 28(a) thus presents experimental estimates for this function.

What is surprising, of course, is that such strong evidence for scaling is obtained taking data rather far from T_f ($1.1 T_f \lesssim T \lesssim 4 T_f$), while scaling plots taking data closer to T_f yield more scatter [Barbara *et al.*, 1981b, 1982; Beauvillain, Dupas, Renard, and Veillet, 1984; see Fig. 28(b)]. In our view, the data for the aluminosilicate spin glass [Fig. 28(b)] could as well be taken as evidence that in this system scaling does not hold in the temperature region investigated. The nice scaling of *Cu*Mn spin glasses is also supported by the data taken by Berton *et al.* (1982) with the magnetocaloric effect; the possibility emerges that metallic and nonmetallic spin glasses in this respect might be different.

A rather convincing demonstration of scaling is also obtained in the recent work by Bouchiat (1986) on *Ag*Mn spin glasses, where data only in the range $10^{-2} \lesssim 1 - T_f/T \lesssim 10^{-1}$ are used, and fields are chosen such that the nonlinear part of the magnetization is still relatively small. The exponents obtained are $\beta = 1.0 \pm 0.1$, $\gamma = 2.2 \pm 0.2$, and $\delta = 3.1 \pm 0.2$.

From a theorist's point of view, data much closer to T_f and taken at smaller field, like those of Omari *et al.* (1983), clearly are desirable. Omari *et al.* (1983) actually use a scaling variable slightly different from that used in Eq. (2.36), namely $1 - T_f/T$ instead of $T/T_f - 1$, suggesting that the "critical region" is apparently wider. However, experience with standard critical phenomena shows that usually the critical region is limited by singular correction terms, in which case the change of variables suggested by Omari *et al.* (1983) does not help.

Recent interesting work on this question has been performed by Taniguchi *et al.* (1985), who study the coefficients a_3 and a_5 of the nonlinear susceptibility of the $Fe_{10}Ni_{70}Pd_{20}$ spin glass using higher harmonics of an ac drive field. This allows the use of much smaller fields, like those used by Omari *et al.* (1983). Taniguchi *et al.* (1985) obtain $\gamma = 2.3 \pm 0.2$ and $\delta = 5.2 \pm 0.5$. Note that Taniguchi *et al.* (1985) also locate AT and GT lines, but we are sceptical about the validity of this "folklore mapping" between mean-field theory and real systems, as noted above.

Data such as those obtained by Barbara *et al.* (1981b, 1982), Bouchiat (1986), and Omari *et al.* (1983) are now widely taken as evidence that real spin glasses do have a static phase transition at T_f. A comparison with corresponding simulations and pertinent theoretical predictions will reveal, however, that this issue is not as yet settled beyond doubt (Sec. V.E.5).

D. Temperature-concentration phase diagrams

As we have seen in Sec. II.A, many spin glass systems are produced by randomly diluting a system that, in the absence of such dilution, exhibits some more conventional magnetic long-range order (ferromagnetism or antiferromagnetism). Typically this ordinary order will exist over some range of concentration of the magnetic atoms, until one reaches some critical concentration, at which the long-range order breaks down and the spin glass state takes over. In metallic spin glasses, due to the long range of the RKKY interactions (see Sec. II.A), the spin glass exists down to arbitrarily small concentrations of magnetic atoms; e.g., in Pd diluted with Fe one still finds a transition down to Fe concentrations of 2.2×10^{-4} at.%, though T_f then is only $T_f \approx 0.19$ mK (Peters *et al.*, 1984). However, in short-range systems like $Eu_xSr_{1-x}S$ (Maletta and Convert, 1979; Maletta and Felsch, 1979, 1980) there exists another critical concentration x_c' at which spin glass order is no longer possible, and for $x < x_c'$ the system is a "superparamagnet" (see Sec. III.F). Of course, this is only true at not-too-low temperatures, where the short-range exchange interactions dominate, but the long-range dipolar interactions can be neglected: at low enough temperatures, these dipolar interactions are responsible for spin glass behavior at arbitrarily small concentrations, as demonstrated for $Eu_xSr_{1-x}S$ (Eiselt *et al.*, 1979).

The existence of such a critical concentration x_c' can easily be understood in the model where one assumes nearest-neighbor ferromagnetic exchange and next-nearest-neighbor antiferromagnetic exchange (Binder *et al.*, 1979), which is a reasonable approximation for $Eu_xSr_{1-x}S$ (Bohn *et al.*, 1980). Then the system of magnetic ions for $x < x_p^{\mathrm{NNN}}$, the next-nearest-neighbor percolation threshold, must break up into an assembly of finite clusters, which are magnetically decoupled from each other. As percolation of magnetic interactions is neces-

sary to carry any type of long-range order, we conclude $x_c' \geq x_p^{\mathrm{NNN}}$.

It turns out that the ferromagnetic phase in $Eu_xSr_{1-x}S$ exists at low temperatures for $x > x_c''$, while at higher temperatures the ferromagnetic phase is stable at even lower concentrations: the spin glass transition line $T_F(x)$ and the ferromagnetic phase boundary $T_c(x)$ meet at a point (T_m, x_m) in the T-x plane, with $x_m > x_c''$ (Fig. 29). Thus the ferromagnetic phase boundary is reentrant, and cooling the system at fixed concentration one observes a double transition (Fig. 30). Rather than the susceptibility cusp at a spin glass transition or the saturation of the susceptibilities at a plateau value (determined by sample geometry and demagnetization factor) at an ordinary fer-

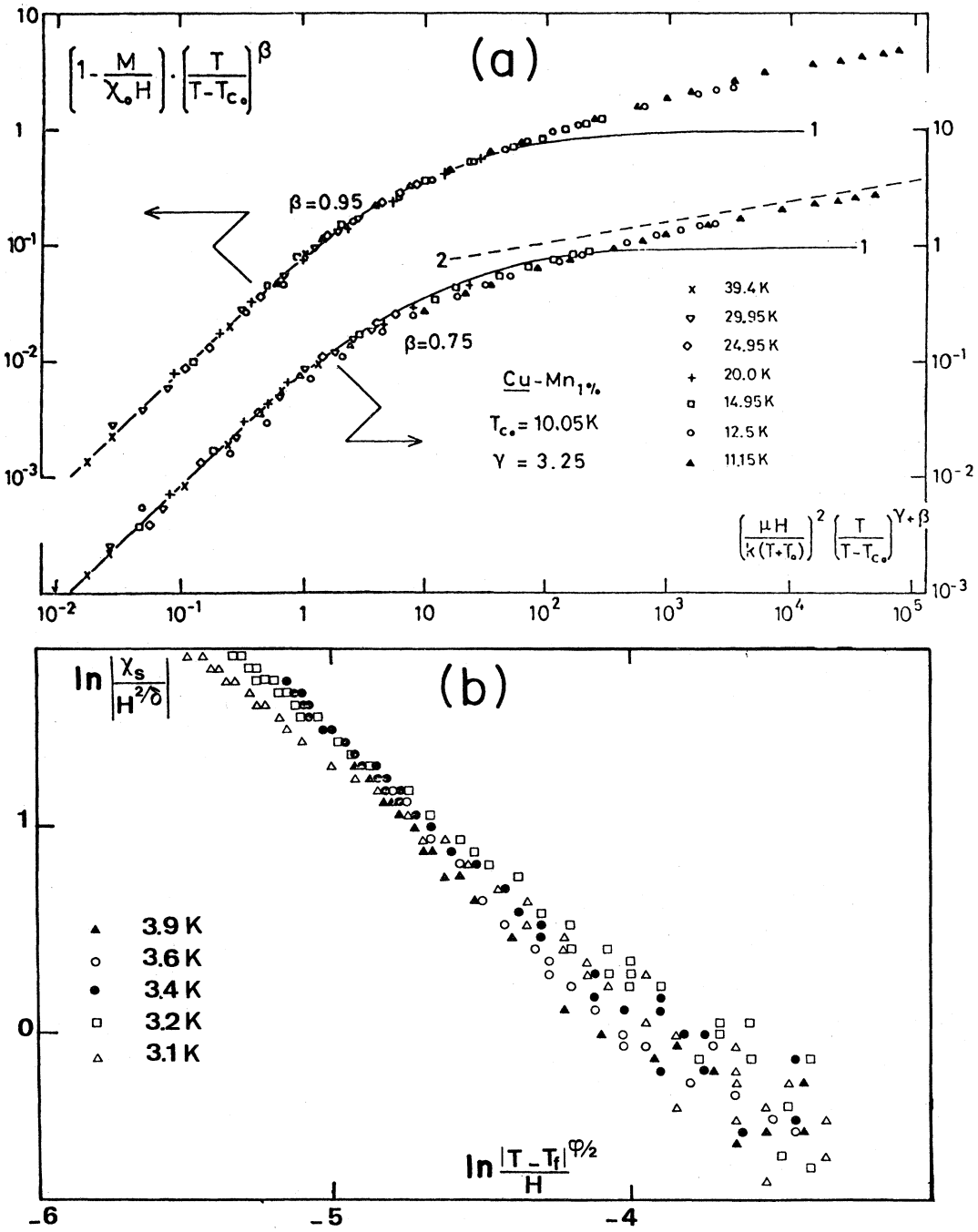

FIG. 28. (a) Scaling plot of the magnetization data for *Cu*Mn with 1% Mn for two choices of the exponent β (as indicated) and for $\gamma = 3.25$. From Omari *et al.* (1983). (b) Scaling plot of the magnetization data for amorphous manganese aluminosilicate. From Beauvillain, Dupas, Renard, and Veillet (1984).

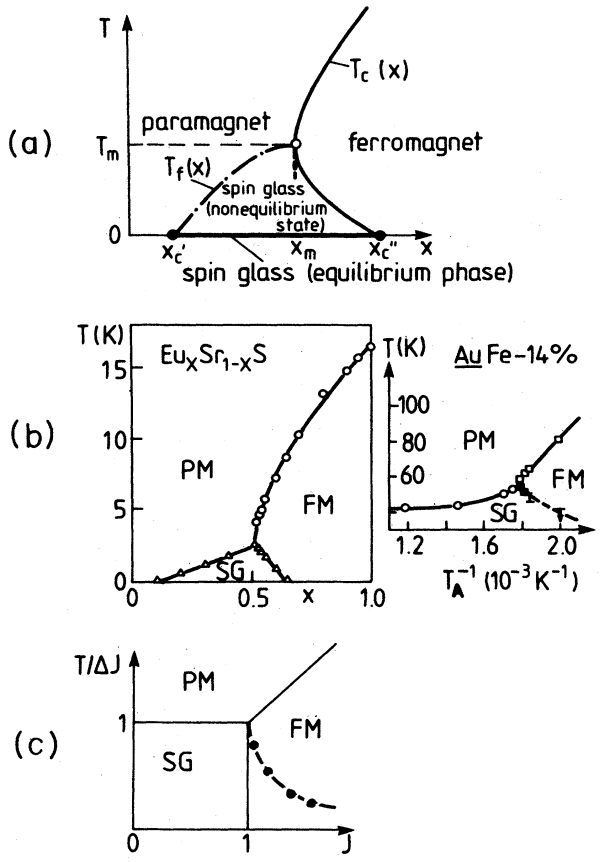

FIG. 29. Phase diagram of systems in which spin glass order and ferromagnetic order compete. (a) Theoretical phase diagram for a short-range system[4] (Kinzel and Binder, 1981). In the concentration range $x_c' < x < x_c''$ the ground state of the system is a spin glass. The ferromagnetic phase boundary exhibits a reentrance point (T_m, x_m). The spin glass state exists in the regime bounded by these lines and the line $T_f(x)$. For system dimensionalities exceeding the lower critical dimensions, a static phase transition occurs at $T_f(x)$ and hence the point (T_m, x_m) is a nontrivial multicritical point. For systems below their lower critical dimension, the spin glass exists as a nonequilibrium state only; $T_f(x)$ is—at least weakly—time dependent and hence hits the reentrance point only approximately. (b) Experimental phase diagrams proposed for $Eu_xSr_{1-x}S$ (Maletta and Convert, 1979) and for AuFe with 14% Fe (Crane and Claus, 1981). In the latter case T_A is the annealing temperature (by varying the annealing temperature, one varies the chemical short-range order parameters). PM=paramagnetic phase, FM=ferromagnetic phase, SG=spin glass phase. (c) Phase diagram of the Ising spin glass with infinite-range interaction [the SK model (Sherrington and Kirkpatrick, 1975)]. Here J and ΔJ are mean value and width of the Gaussian interaction (suitably normalized to ensure a sensible thermodynamic limit; see Sec. IV). Note that the original solution of Sherrington and Kirkpatrick (1975), yielding a phase diagram with reentrant ferromagnetic phase boundary, is unstable. The dash-dotted curve denotes the line where the replica-symmetric solution becomes unstable [it is just an extension of the AT line, Eq. (2.40), which in this larger parameter space $H/\Delta J$, $T/\Delta J$, $J/\Delta J$ becomes a surface]. From Parisi and Toulouse (1980).

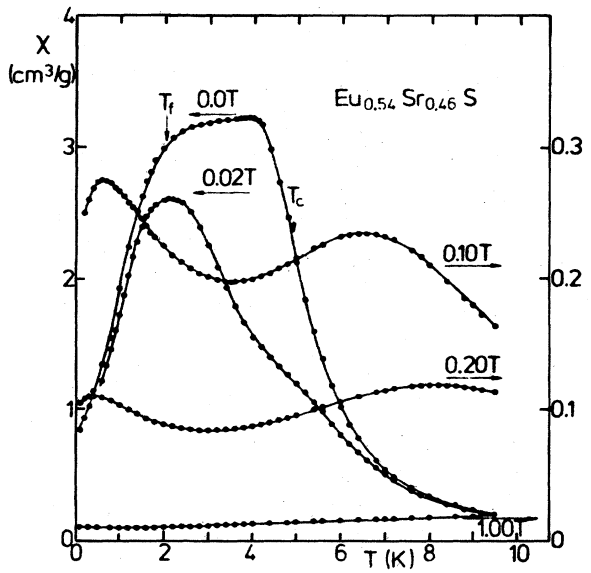

FIG. 30. Temperature dependence of the ac susceptibility at 117 Hz and 10^{-5} tesla for $Eu_{0.54}Sr_{0.46}S$ in various static magnetic fields, indicated at the curves (in tesla). Note the change of scale between $H=0.02$ tesla and $H=0.10$ tesla. From Maletta and Felsch (1980).

romagnetic transition, one now sees that the susceptibility saturates and then slightly decreases in the ferromagnetic region, before it has its strong falloff. Similar behavior has been observed in crystalline metals such as AlFe (Shull *et al.*, 1976), $(Pd_{1-y}Fe_y)_{1-x}Mn_x$ (Verbeek *et al.*, 1978; Nieuwenhuys *et al.*, 1979; Shapiro *et al.*, 1980), Fe_xCr_{1-x} (Sarkissian, 1981; Shapiro *et al.*, 1981a, 1981b; Burke *et al.*, 1983a , 1983b, 1983c), $(Cr_{1-y}V_y)_{0.84}Te$ (Ohta, Kurosawa, and Anzai, 1982), or *Au*Fe (Coles *et al.*, 1978; Crane and Claus, 1981), in amorphous metals such as $[Fe_x(TM)_{1-x}]_{75}P_{16}B_6Al_3$ (Yeshurun *et al.*, 1981; Salamon *et al.*, 1981; Geohegan and Bhagat, 1981; Lynn *et al.*, 1981; Manheimer, Bhagat, Kistler, and Rao, 1982; Manheimer, Bhagat, and Chen, 1982b), where (TM) stands for transition metals such as Ni, Mn, or Cr, and in insulators such as $KMn_xZn_{1-x}F_4$ (Cowley *et al.*, 1980).

The interpretation of susceptibility measurements such as those shown in Fig. 30 is, however, not at all obvious. In particular, in the case of *Au*Fe, where chemical clustering at the considered concentrations is very important, there is no complete agreement about the phase diagram in the literature (Carnegie and Claus, 1979; Carnegie *et al.*, 1979; Mydosh *et al.*, 1979; Crane and Claus, 1980, 1981). Griffith *et al.* (1985) observe that the susceptibility of the ferromagnet Pd with 0.4 at. % Fe closely resembles that of a spin glass if the sample is in a cold worked state; this finding again illustrates that the distinction between spin glasses with ferromagnetic short-range order and ferromagnets with disorder may be difficult. Magnetic neutron Bragg scattering would be a more convincing test for the presence of ferromagnetic long-range order than such susceptibility or magnetization measure-

ments. At the reentrant ferromagnetic phase boundary one would expect the Bragg intensities again to vanish. In *Au* with 17 at.% Fe, however, Murani (1980) found a monotonic increase of integrated Bragg intensities with decreasing temperatures; this result is hardly consistent with a reentrant ferromagnetic phase boundary. In $Eu_xSr_{1-x}S$ for $x_m < x < x_c''$, on the other hand, one finds no Bragg intensity in the reentrant spin glass region (Maletta *et al.*, 1982). This is confirmed by a study of spin dynamics (Shapiro *et al.*, 1985).

A closer analysis of the neutron data in this material, as well as in the amorphous metal $(Fe_{0.68}Mn_{0.32})_{75}P_{16}B_6Al_3$, shows (Aeppli *et al.*, 1984) that even in the regime between the upper and lower critical temperatures $T_c(x)$, for $x_m < x < x_c''$, one does not find any Bragg intensity, although at the critical temperatures the inverse correlation length κ seems to vanish [Figs. 31(a) and 31(b)]. Rather than finding κ to be finite again in the ferromagnetic region (where spontaneous magnetization should occur), one finds that κ remains zero throughout, and that there is no spontaneous magnetization, at least within resolution limits. The situation is rather subtle, as is shown by model calculations due to Binder (1982b) analyzing the "normal" behavior near a reentrant phase boundary: since the size of κ is related to the normal distance from the phase boundary, which in between the two critical temperatures $T_c^{(l)}, T_c^{(u)}$ cannot become large, κ must stay rather small over a broad temperature range [Fig. 31(c)]. While Binder (1982b) speculated that all that might happen in the experiments would be for the behavior of Figs. 31(c) to be somewhat smeared out due to finite resolution and sample heterogeneities, Maletta *et al.* (1983), Maletta (1983), and Aeppli *et al.* (1983, 1984) maintain that the experiments show the existence of a new type of phase, characterized by $\kappa = 0$ over a finite range of temperature. We also note that optical microscope observations reveal ferromagnetic domains in $Eu_xSr_{1-x}S$ for $x = 0.54$ but not for $x = 0.52$ (Dillon *et al.*, 1984). From elastic light scattering, Geschwind *et al.* (1984) infer ferromagnetic correlation lengths of the order of 10^3 Å in this region.

A recent review of neutron scattering studies from two reentrant spin glass systems, $Eu_xSr_{1-x}S$ and $Fe_{1-x}Al_x$, is given by Shapiro *et al.* (1986), with further references.

These experiments may also be relevant to the case of a mixed-state ferromagnet—spin glass, in which ferromagnetic long-range order in the z direction would coexist with frozen-in transverse spin components. Such a state is in fact found in the mean-field treatment of vector spin glasses (Gabay and Toulouse, 1981). Aeppli *et al.* (1983, 1984) and Maletta *et al.* (1982, 1983) argue, however, that in short-range systems such a state cannot exist, as the frozen-in transverse spin components would act as a random field on the ferromagnetic order and hence create an instability against domain formation (Imry and Ma, 1975). In fact, none of these experiments could be interpreted in terms of such a mixed state.

If the mixed spin-glass—ferromagnetic phase did exist, somewhere in the phase diagram [Figs. 29(a) and 29(b)] an additional phase boundary would occur. Moreover, the

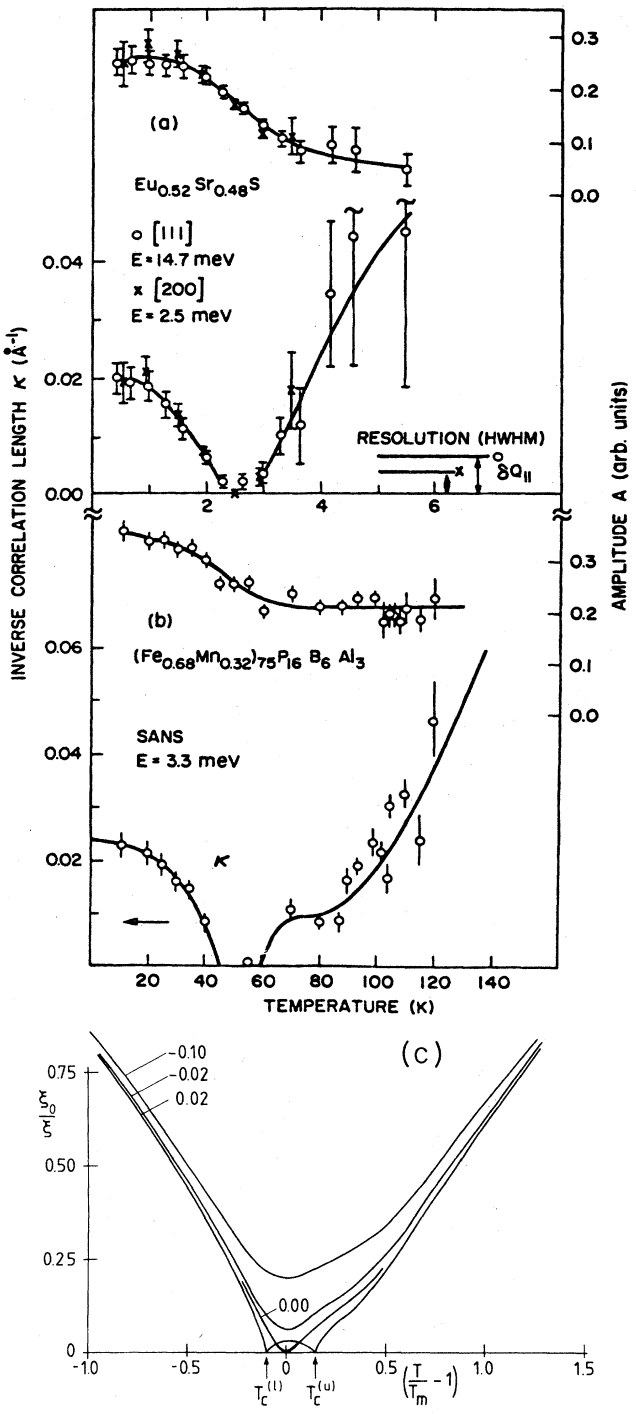

FIG. 31. Temperature dependence of inverse correlation length κ (left-hand scale and lower curve in each frame) and Lorentzian amplitude (right-hand scale and upper curves in each frame): (a) for $Eu_{0.52}Sr_{0.48}S$; (b) for $(Fe_{0.68}Mn_{0.32})_{75}P_{16}B_6Al_3$. From Aeppli *et al.* (1984). (c) Inverse ferromagnetic correlation length plotted vs temperature, for the phase boundary shown in (a), for four choices of the parameter $(x/x_m - 1)$. Arbitrarily the amplitude ξ_0 of the correlation length in the ferromagnetic region was chosen to be twice as large as in the paramagnetic region. For $x > x_m$ one cuts the ferromagnetic phase boundary both at lower and upper critical points $T_c^{(l)}, T_c^{(u)}$. From Binder (1982b).

proposed phase with infinite correlation length but no spontaneous magnetization ("frustrated ferromagnet"; Maletta *et al.*, 1983) needs to be distinguished from the ordinary ferromagnet (with spontaneous magnetization) by a phase boundary. Thus the phase diagrams shown in Fig. 29(b) conceivably could have a much richer structure, but this question plainly is not yet settled. In any case, what is established is the existence of a transition out of some kind of ferromagnetic state to a spin glass, which gives rise to a number of effects: the drop in the demagnetization-limited ac susceptibility (Verbeek *et al.*, 1978; Maletta and Felsch, 1980; see Fig. 30); vanishing of spontaneous magnetization in low-field magnetization measurements (Crane and Claus, 1981); reduction of spin-wave energies measured by inelastic neutron scattering (Lynn *et al.*, 1981; Shapiro *et al.*, 1981b; Aeppli *et al.*, 1982); changes in the hysteretic behavior (Geohegan and Bhagat, 1981); increase in the linewidth for ferromagnetic resonance (Bhagat *et al.*, 1981); onset of magnetic viscosity and history-dependent effects (Yeshurun and Sompolinsky, 1982; Manheimer, Bhagat, and Chen, 1982a); etc.

So far we have considered systems where a ferromagnet is mixed with a nonmagnetic component. Of course, it is of interest to consider other types of random systems as well, such as diluted antiferromagnets or mixed ferro-antiferromagnetic systems. All these random systems also share the possibility of exhibiting a spin glass phase somewhere in their phase diagrams.

An example of the mixture of a ferromagnet with an antiferromagnet is the EuS_ySe_{1-y} system (Westerholt and Bach, 1982; Fig. 32) while EuS is a ferromagnet, EuSe orders antiferromagnetically. It turns out that the two critical lines of ordering meet at a multicritical point; the ferromagnetic phase again is reentrant, and the phase in between the antiferromagnetic phase and the reentrant ferromagnetic one is interpreted as "myctomagnetic" by

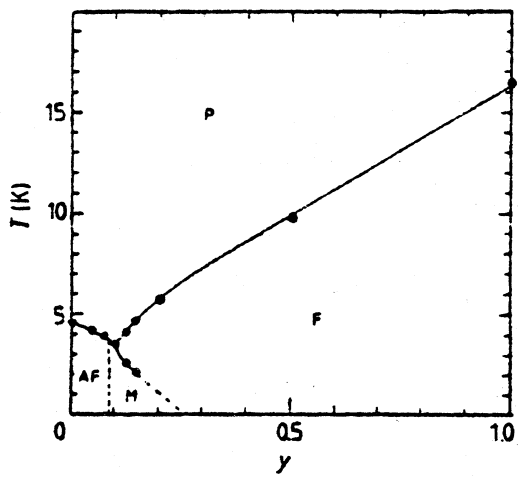

FIG. 32. Phase diagram of EuS_ySe_{1-y}: *P*, paramagnetic phase; AF, antiferromagnetic phase; *F*, ferromagnetic phase; *M*, mictomagnetic phase. From Westerholt and Bach (1982).

Westerholt and Bach (1982). This term has been used to describe spin glasses with a large degree of chemical short-range order (Beck, 1978, 1980). Theoretical work often predicts in such a phase diagram a region with mixed ferromagnetic and antiferromagnetic order (Aharony, 1977); it is not clear whether the experimental evidence can rule this possibility out. Another system in which ferro- and antiferromagnetic transition lines nearly meet is $GdAg_{1-x}Zn_x$ (Köbler *et al.*, 1985).

It is now possible to produce mixed compounds of the type $Eu_xSr_{1-x}S_ySe_{1-y}$ (Westerholt and Bach, 1981a, 1981b). For suitable choices of the sulfur concentration y ferromagnetic order breaks down with a very small Sr concentration $1-x$ [see Figs. 33(a) and 33(b)]; then one has a very wide range of concentration in which the spin glass state is stable. Since both EuS and EuSe are basically described as Heisenberg magnets, with exchange interactions J_1 between nearest neighbors and J_2 between next-nearest neighbors, where $J_2/J_1 \approx -0.5$ for EuS (Bohn *et al.*, 1980) and $J_2/J_1 \approx -1.1$ for EuSe (Zinn, 1976), one could say that in varying y one is effectively varying J_2/J_1. Within this approximation the wide range of stability of the spin glass phase can be explained by theoretical considerations confirmed by simulations (Binder *et al.*, 1979), which predict that $x_c'' \to 1$ at the multicritical point, where ferro- and antiferromagnetic order compete in the ground state [$J_2/J_1 = -1$; see Fig. 33(c)]. This theoretical work also predicts that the spin glass state exists for the case where the undiluted system orders antiferromagnetically. Experiments on the $Eu_xSr_{1-x}Se$ system (Westerholt and Bach, 1985) have indeed found spin glass behavior for $x < 0.7$, while the first-order antiferromagnetic transition gets smeared for $x \lesssim 0.85$.

Recently it was found (Fiorani *et al.*, 1983, 1984) that the frustrated antiferromagnet $ZnCr_2O_4$ (which has essentially only nearest-neighbor antiferromagnetic interactions between the Cr ions that occupy the octahedral sites of the spinel lattice) upon dilution with Ga turns into a spin glass.

Other examples of spin glasses obtained by dilution of an antiferromagnetic are $Fe_{1-x}Mg_xCl_2$ (Bertrand *et al.*, 1982; Wong *et al.*, 1985), though in this case it is probably again due to competition between antiferromagnetic and ferromagnetic bonds, and $Cd_{1-x}Mn_xTe$ (Galazka *et al.*, 1980).

Finally, we mention systems such as $Cu_{0.6}Zr_{0.4-x}Fe_x$, in which for $x \lesssim 0.05$ the Fe ions have no magnetic moment and the alloy has a phase transition to a superconducting state, while for $0.05 \lesssim x \lesssim 0.1$ superconductivity is suppressed and a spin glass state is found (von Löhneysen *et al.*, 1983).

E. Orientational glasses and ferroelectrics with glassy polarization behavior

As we have seen above, spin glass behavior results if disorder is introduced into magnets with competing in-

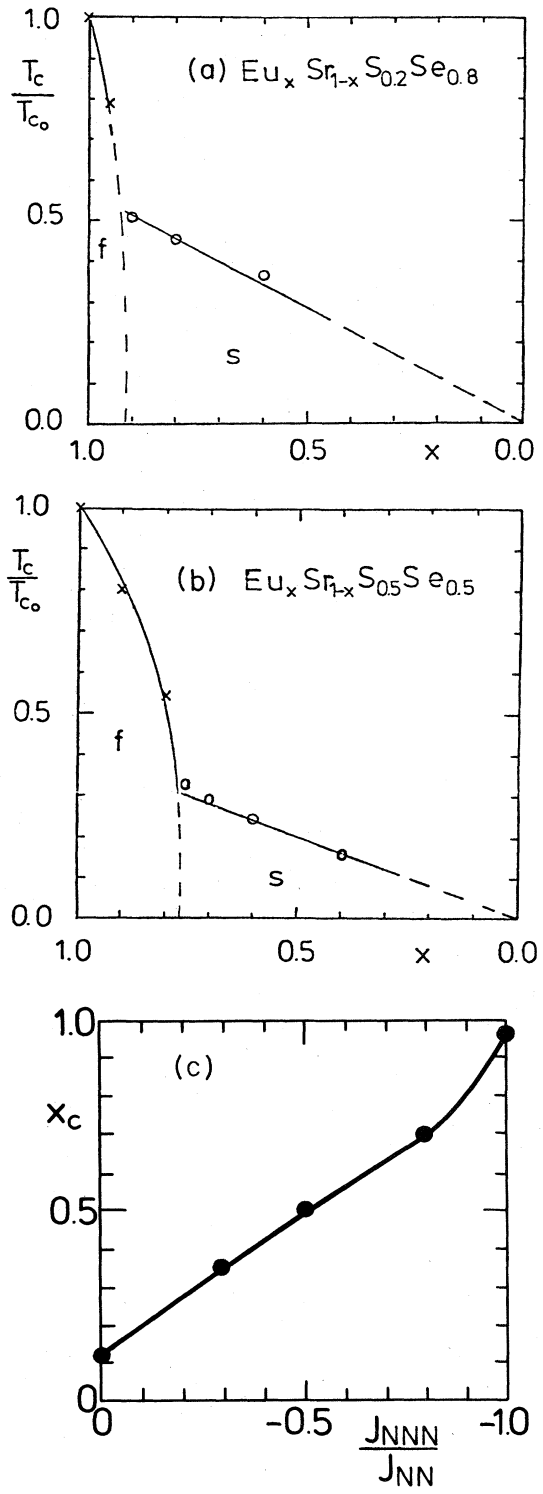

FIG. 33. (a) Phase diagram of $Eu_xSr_{1-x}S_{0.2}Se_{0.8}$. (b) Phase diagram of $Eu_xSr_{1-x}S_{0.5}Se_{0.5}$. From Westerholt and Bach (1981b). (c) Variation of critical concentration x_c'' between spin glass and ferromagnet at $T=0$, as obtained from Monte Carlo simulation of a classical Heisenberg ferromagnet on the face-centered cubic lattice, plotted as a function of the ratio of the next-nearest-neighbor and nearest-neighbor exchange constants. From Binder *et al.* (1979).

teractions. Thus it is immediately plausible to expect that similar behavior may occur for other ordering phenomena, if there is competition between various interactions and disorder is created, e.g., by random dilution. Candidates for such behavior are orderings associated with structural transitions as they occur in ferroelectrics or antiferroelectrics, or in molecular crystals. In mixed ferro-antiferroelectric systems (or diluted ferroelectrics, respectively) the local ordering is due to electric dipoles, and hence one is talking about "dipolar glasses" (Fischer and Klein, 1976; Bhattacharya *et al.*, 1982; Courtens, 1982; Höchli, 1982; see also Känzig *et al.*, 1964). From a theoretical point of view, the spin representing a magnetic moment now turns into a "pseudospin" representing the electric moment; dielectric rather than magnetic susceptibilities need to be considered, while otherwise the situation is similar to that for magnetic spin glasses. A more fundamental difference arises when one considers glassy behavior in mixed molecular crystals, where the orientational ordering has tensor rather than vector character. The local order parameter can then be considered as a quadrupole moment rather than a dipole moment, and hence one is talking of "quadrupolar glasses" (Sullivan *et al.*, 1978; Haase and Saleh, 1981; Sullivan and Esteve, 1981; Haase and Klenin, 1983; Sullivan, 1983).

Possible examples of electric dipole glasses are $K_{0.974}Li_{0.026}TaO_3$ (Höchli, 1982; see Fig. 3 for results of frequency-dependent dielectric susceptibility measurements) and the $Rb_{1-x}(NH_4)_xH_2PO_4$ system (Courtens, 1982, 1983, 1984a, 1984b; Slak *et al.*, 1984). The latter system has been studied by dielectric susceptibility (Courtens, 1982, 1983, 1984a, 1984b), birefringence (Courtens, 1982), and NMR measurements (Slak *et al.*, 1984). While pure RbH_2PO_4 orders ferroelectrically, $NH_4H_2PO_4$ orders antiferroelectrically, and mixed crystals exist over the full range of x (Courtens, 1982). This system, however, seems to be rather complicated, as can be seen from the conjectured phase diagram [Courtens, 1982; see Fig. 34(a)], where a mixed phase of unknown character occurs. The situation may be analogous to that of the EuS_ySe_{1-y} system discussed in the previous section (Fig. 32). While at the upper transition temperature at $x=0.34$ in Fig. 34(a) the birefringence vanishes and the temperature derivative of the dielectric constant ε_{11} has a (rounded) kink (1 is the tetragonal axis of ferroelectric ordering), the glass transition at about $T_f=27$ K is characterized by a kink in the real part of ε_{11} (and at the same time an imaginary part sets in). Recent work on this material has concentrated on the low-temperature dynamical properties of $Rb_{0.65}(NH_4)_{0.35}H_2PO_4$ (Courtens, 1984a, 1984b; Slak *et al.*, 1984). One finds that the freezing transition is characterized by a strong broadening of the distribution of relaxation times. It turns out that the dielectric susceptibility at low temperatures is quantitatively consistent with the Vogel-Fulcher law (Vogel, 1921; Fulcher, 1925), Eq. (2.14), if one represents the imaginary part $\underline{\varepsilon}''(\omega,T)$ of the dielectric susceptibility tensor in the form

$$\underline{\varepsilon}''(\omega,T)=\underline{h}(T)[1+\tanh(c_1u)][1-c_2\ln(\omega\tau_0)]\ . \qquad (2.43)$$

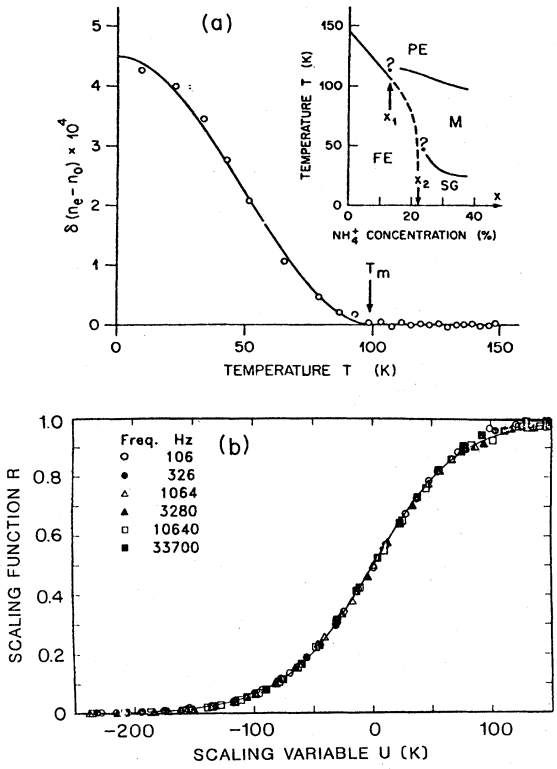

FIG. 34. (a) The remainder of the tetragonal birefringence $\delta(n_l - n_0)$ after subtraction of the lattice expansion contribution for $Rb_{0.66}(NH_4)_{0.34}H_2PO_4$. Inset: A tentative phase diagram exhibiting ferroelectric (FE), paraelectric (PE), dipolar glass (SG), and mixed (M) phases. T_m is the transition temperature from the mixed to the paraelectric phase. The precise ordering of the mixed phase, which has considerable antiferromagnetic short-range order, is unclear. Solid curves in the phase diagram are second-order transitions, the dashed line represents a first-order transition. The nature of the special points (indicated by question marks) remains to be elucidated. From Courtens (1982). (b) Scaling function $R(u) \equiv [1+\tanh(c_1 u)]/2$ plotted vs u, for $Rb_{0.65}(NH_4)_{0.35}H_2PO_4$. Frequencies ω indicated in the figure. From Courtens (1984b).

Here c_1, c_2 are constants, the tensor $\underline{h}(T)$ depends only on temperature, and the scaling variable u is related to ω, T via

$$u = c_3 - E_{\text{act}} , \tag{2.44}$$

where c_3 is another constant, and E_{act} is the activation energy in the Vogel-Fulcher law, Eq. (2.14), in which one puts $\tau \equiv \omega^{-1}$. Figure 34(b) shows that one obtains an excellent representation of all the data (extending over the temperature range from 4 to 35 K and the frequency range indicated); however, the number of adjustable parameters is rather large.

Measurements of the dipolar glass $K_{1-x}Li_xTaO_3$ in an electric field E (Höchli *et al.*, 1985) reveal a characteristic line in the T-E plane where irreversible behavior sets in, similar to the lines in the H-T plane for spin glasses identified as AT lines (Sec. II.C). In $K_{1-x}Na_xTaO_3$ a critical divergence of the nonlinear electric susceptibility is found (Maglione *et al.*, 1986).

Finally we draw attention to dielectric susceptibility measurements on the diluted ferroelectric $(KDP)_{1-x}(ADP)_x$ due to Choo and Kim (1983). The susceptibility shows a double transition, in striking similarity to the $Eu_xSr_{1-x}S$ data shown in Fig. 30. Choo and Kim (1983) interpret the upper transition as that between paraelectric and ferroelectric, and the lower one as a freezing temperature where the domain-wall mobility is frozen out. A study of crystalline ferroelectrics with glassy polarization behavior is also present by Burns and Dacol (1983, 1984).

We turn now to the freezing of orientation order in molecular crystals. Such behavior may already occur, even in undiluted molecular crystals, such as hexagonal ice or solid CO (Suga and Seki, 1974). Here we concentrate attention on diluted molecular crystals. In this area, much work has been directed towards understanding mixtures of orthohydrogen and parahydrogen (Ishimoto *et al.*, 1973, 1976; Sullivan *et al.*, 1975, 1978, 1979; Sullivan, 1976, 1983; Washburn *et al.*, 1980, 1981; Cochran *et al.*, 1980 and the $K(CN)_{1-x}Br_x$ system (Satija and Wang, 1978; Rowe *et al.*, 1979, 1983; Loidl *et al.*, 1980, 1981, 1982, 1983, 1984; Michel and Rowe, 1980; Kwiecien *et al.*, 1981; Bhattacharya *et al.*, 1982; Garland *et al.*, 1982; De Yoreo *et al.*, 1983). Related phenomena have also been found in $(N_2)_xAr_{1-x}$ mixtures (Press *et al.*, 1982). Figure 35 shows the phase diagram of the latter system and of the orthohydrogen-parahydrogen mixture (Sullivan *et al.*, 1978, Press *et al.*, 1982). Figure 36 (Sullivan, 1983) gives a qualitative impression of the orderings in the orientationally ordered and glasslike states.

In the case of the $(N_2)_{1-x}Ar_x$ system, neutron diffraction has been used (Press *et al.*, 1982) to show that for $x = 0.28$ there is no kind of orientational long-range order, while at the same time the inelastic scattering data indicate a gradual freezing in of the local orientations. For the o-H_2/p-H_2 mixtures, the evidence for the phase diagram (Fig. 36) mainly rests on NMR experiments. These experiments are somewhat difficult to interpret, and hence there has been a controversy (Sullivan *et al.*, 1978; Cochran *et al.*, 1980; Washburn *et al.*, 1981) as to whether a sharp transition to a glasslike state occurs, and where it is located. But it is clear that at very low temperatures the local orientations of the o-H_2 molecules are frozen in.

Perhaps the best studied system is the $(KCN)_x(KBr)_{1-x}$ mixed crystal system. Pure KCN undergoes a transition from the disordered cubic phase to an orientationally ordered orthorhombic phase (space group $Immm$) at 168 K (Matsuo *et al.*, 1968), which is weakly first order. At 83 K there is an order-disorder transition, in which the $(CN)^-$ ions order completely in a kind of antiferroelectric structure (space group $Pmmm$).

Interestingly, for a weak dilution ($x = 0.95$ or $x = 0.90$,

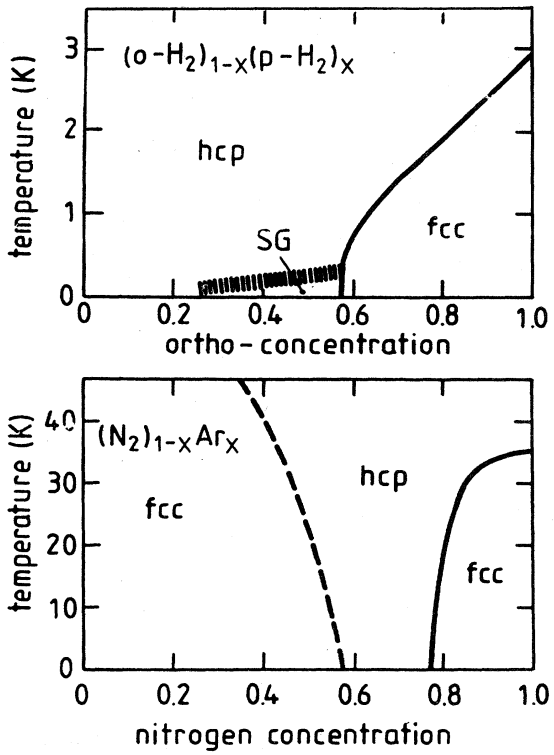

FIG. 35. Phase diagrams for solid solutions of o-H_2 and p-H_2 (upper part) and N_2 diluted with Ar (lower part). For large concentrations of the orientationally anisotropic component (o-H_2 or N_2, respectively), the phase diagrams are similar: the orientationally ordered phase has face-centered cubic structure, the disordered one is hexagonal close packed. The orientational glass phase (denoted by SG) is indicated in the upper part only. From Press *et al.* (1982).

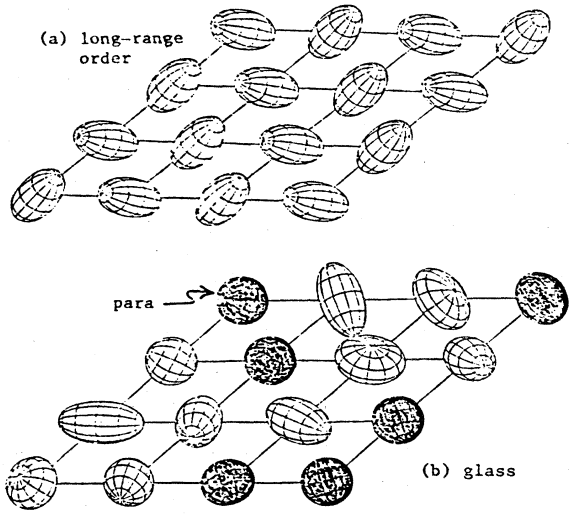

FIG. 36. Comparison of the orientational order for (a) a plane section of the Pa_3 structure of o-H_2 and (b) the quadrupolar glass phase in o-H_2/p-H_2 mixtures. From Sullivan (1983).

for instance), one finds a qualitatively different behavior (Rowe *et al.*, 1983): only part of the sample transforms to the same orthorhombic structure as pure KCN does, while the other part transforms into a monoclinic structure (space group Cc). For a larger dilution ($x=0.80$), only this monoclinic phase is present. These findings are an example of a long-range order stabilized by the disorder in the system, while the same order in the pure system would be metastable only.

Thus the phase diagram of the mixed (KCN)-(KBr) system is more complicated than the phase diagrams of the magnetic systems discussed previously, and probably it is not yet known in full. There is ample evidence, however, that a phase with frozen-in random orientational order exists at large dilutions $x \geq 0.050$ (Loidl *et al.*, 1981, 1982, 1983, 1984; Bhattacharya *et al.*, 1982; Rowe *et al.*, 1979; Satija and Wang, 1978). While in pure KCN the transition is seen by a strong softening of the elastic constant C_{44} in the cubic high-temperature phase (Haussühl, 1973; the softening is only almost complete, since the transition is weakly of first order), this softening is suppressed in the strongly diluted systems, as revealed by Brillouin scattering (Satija and Wang, 1978) and ultrasonic measurements (Loidl *et al.*, 1980, 1981). The neutron inelastic scattering data (Rowe *et al.*, 1979; Loidl *et al.*, 1983) show that the orientational motions do indeed become frozen in, at least on the time scales accessible by those measurements, and can be interpreted in terms of local orientational order very much like the Edwards-Anderson order of spin glasses (Michel and Rowe, 1980).

Frequency-dependent measurements of elastic constants (Loidl *et al.*, 1982) and of the dielectrics susceptibility (Bhattacharya *et al.*, 1982) show that an Arrhenius-type behavior of relaxation times produces a slowing down of the motions even above freezing; again the qualitative similarity with spin glasses or dielectric glasses (Höchli, 1982) is remarkable, since the electric dipole moment attached to the $(CN)^-$ dumbells freezes in at the same time as the molecule orientation is frozen, and the system can at the same time be considered as a "dipole glass" (Bhattacharya *et al.*, 1982; this fact is also responsible for the dielectric susceptibility's being a suitable tool for the study of this freezing transition). But since static electric fields have hardly any effect on the results (Bhattacharya *et al.*, 1982), it is clear that no dielectric interactions among the dipoles are responsible for the freezing, but rather interactions among the quadrupoles (such as those mediated by lattice strains via long-wavelength phonons; see, for example, Michel *et al.*, 1978; De Raedt *et al.*, 1981). Finally we emphasize that measurements of thermal conductivity and specific heat (De Yoreo *et al.*, 1983) show a behavior reminiscent of ordinary glasses.

F. Summary of the experimental results

Extensive experimental work over more than a decade has shown that the phenomena commonly attributed to spin glasses are fairly universal: a rather sharp peak in

the low-frequency-dependent susceptibility, which becomes progressively rounded with increasing frequency, and with only rather weak frequency dependence of the peak temperature; a spectrum of relaxation times that broadens far above T_f and extends to macroscopic time scales at and below T_f; at the same time, equilibrium spin glass correlations develop above freezing and lead to a dramatic increase of the static nonlinear susceptibility, and the magnetization can be brought into a scaled equation of state very similar to the behavior at ordinary magnetic phase transitions. Thus the spin glass combines some features characteristic of equilibrium phase transitions with some features characteristic of nonequilibrium systems such as ordinary glasses; despite a great deal of effort it is not yet completely clear whether all spin glasses behave qualitatively the same, or whether one needs different "universality classes" for metallic spin glasses and nonmetallic ones, Heisenberg versus Ising spins, etc. While the (unidirectional) anisotropy found in some metallic spin glasses clearly controls the low-temperature properties of the system, which then is nearly rigidly frozen, it is not clear to what extent anisotropy controls the behavior near the freezing transition. In the regime of temperatures intermediate between very low temperatures and freezing, experiment has produced a wealth of data on hysteresis, remanence, magnetic viscosity, etc., which must await theoretical explanation.

Much more work is still needed to clarify our understanding of phase diagrams in which spin glass and conventional orderings compete, and of related phenomena in dielectrics and molecular crystals, where analogous phenomena may occur. In the latter systems, the microscopic interactions that compete with each other often are less well understood than in their magnetic counterparts. In addition, one does not always have a parameter analogous to the magnetic field in the spin glass at one's disposal, and hence an experimental study of these systems may be even more difficult. But there are some indications that these systems may have properties intermediate between spin glasses and ordinary structural glasses, and hence their study may be rewarding.

One general conclusion is that most experimental techniques allow rather indirect conclusions about the nature of correlations in the glassy state. Hence it is crucial to combine many experimental techniques for the same sample, in order to gain a fairly complete picture. Thus far such an effort has been directed towards relatively few spin glass systems. While these "model systems" such as *Cu*Mn and *Eu*Sr spin glasses are rather nicely understood, the study of other systems may well lead to surprises.

Since most experimental information on magnetic correlations and order in spin glasses is rather indirect, theoretical work is crucial for the analysis of experiments. Both phenomenological concepts, such as the "Néel model" of superparamagnetic particles, and the mean-field theory of the infinite-range model have been used as a basis for such analysis, and although these concepts are quite diverse, both of them have had some success. Clearly there is a need for a more detailed theory as well as for more realistic models; if this goal is attained, a reanalysis of many experimental data may prove very rewarding.

III. GENERAL THEORETICAL CONCEPTS AND MODELS

In this section we wish to address first some of the general questions encountered in systems with frozen-in disorder: Supposing one wishes to perform an average over this disorder, how is this done? And will the results actually correspond to experimental observations, which usually refer to a simple sample only (Sec. III.A)? Since in a random system the mean value of a quantity does not always coincide with its most probable value, it is not obvious whether one has in a large system a sort of "self-averaging." In addition, there are technical difficulties in carrying out the desired averaging in practice, and the recipe for avoiding those difficulties, which goes under the heading "replica method," introduces other problems (Sec. III.C).

In addition, there is another fundamental problem related to the nature of averaging when the system may exist in several equivalent "ordered states" in the thermodynamic limit, because the standard canonical "Gibbs average" averages over all of them (Sec. III.B). What may really be needed is to break this ergodicity and restrict the average to one "valley" in phase space, for instance. At usual phase transitions, of course, this problem is handled by restricting the average to one choice of the possible values that the order parameter can take; in a spin glass the identification of order parameters is a problem (Sec. III.F).

Finally we shall be concerned with a survey of models on which explicit calculations are based. While the Edwards-Anderson model (Sec. III.D) is still most popular, there has always been the desire to introduce other models that are either simpler or more realistic (Sec. III.E). While it now is commonly accepted that "frustration" of the interactions is a necessary ingredient of any model Hamiltonian (Sec. III.F), concepts based on exploiting the related "gauge invariance" of the Hamiltonian have not led to the desired breakthrough so far.

A complementary approach to statistical mechanics based on a Hamiltonian is a theoretical description on a more macroscopic-phenomenological level. Concepts such as "superparamagnetic clusters" and "two-level systems" have often been advocated in the context of spin glasses, and hence we shall briefly review these ideas in Sec. III.G.

A. Averaging in random systems

Let us suppose we have a system described by some statistical variables. We denote them by S_i $(i=1,\ldots,N)$ to indicate that we are really interested in spins, though the discussion will be more general than this. Furthermore the system is assumed to be random, so in addition we

need to specify the variables $\{x\}$, say, describing the randomness. For instance, x_i may describe whether or not there is a spin on site i.

In general the random variables may themselves fluctuate with time; as an example, in a crystal mixed from magnetic and nonmagnetic species the atoms may diffuse through the lattice. It is therefore necessary to compare typical fluctuation time for the random variable $\tau_{\rm dis}$ with the observation time $\tau_{\rm exp}$. If $\tau_{\rm exp} \gg \tau_{\rm dis}$ the random variables come to thermal equilibrium and so are averaged over in a manner similar to the statistical averages; for instance, the free energy of the system becomes

$$F = -k_B T \ln[Z\{x\}]_{\rm av} , \qquad Z\{x\} = \operatorname{Tr}_{\{S_i\}} \exp[-\mathscr{H}\{x,S_i\}/k_B T] . \tag{3.1}$$

This is called an *annealed average.* In practice atomic diffusion times are huge at the low temperatures where spin glass behavior occurs in real systems, so this limit is not applicable. Furthermore simple annealed models, such as the Ising square lattice with bonds of random sign in its annealed version (Thorpe and Beeman, 1976), do not have interesting spin glass properties. In this particular model, one finds that a spontaneous magnetization appears at an upper critical temperature and vanishes again at a lower critical temperature. In the "disordered" phase at low temperature long-range correlations develop among the signs of the bonds.

We shall exclusively consider the opposite extreme $\tau_{\rm dis} \gg \tau_{\rm exp}$ from now on. This is called a *quenched average,* in which each of the random variables takes a unique value as the statistical variables fluctuate (Brout, 1959; Mazo, 1963). One therefore evaluates the partition function for a particular configuration of the random variable $Z\{x\}$, but we shall show that, unlike Eq. (3.1), the averaging of $Z\{x\}$ itself makes no sense. On the other hand, the unaveraged partition function is very cumbersome, since one has to specify the values of typically 10^{23} random variables for a macroscopic system. So we need to perform some sort of average over the probability distribution $p(\{x\})$ for the $\{x\}$. The question therefore arises: what quantities can be averaged?

Loosely speaking, the answer is that one can generally average extensive variables. To see this, we give a simple intuitive argument due to Brout (1959). Consider a single very large system and divide it up into a large number of subsystems such that each subsystem is macroscopic and clearly contains a different set of random variables. If we assume the coupling between subsystems is negligible (normally down by a surface-to-volume ratio, for systems with short-range interactions), then the value of any extensive variable (normalized per unit volume or per degree of freedom) for the whole system is equal to the average of the values of this quantity over the subsystems. If the original system is very big, we can average over a large number of subsystems and expect that the result of doing so will differ by only a small amount from the result of performing a complete average over all choices of $\{x\}$. In other words, choosing the magnetization per spin M as an example,

$$M\{x\} - [M]_{\rm av} \to 0 \quad (\text{for } N \to \infty) , \tag{3.2}$$

for any set $\{x\}$ that occurs with reasonable probability. The same result is expected for any other density of an extensive quantity. Systems that have this property are called "*self-averaging.*" One large system gives the same results for (densities of) extensive quantities as a configurational average. This property, which can be proved for certain models (van Hemmen and Palmer, 1982; van Enter and van Hemmen, 1983, 1984), is very important because the theory, which for convenience performs the average, can be compared with experiment, which is of course on a single large sample. Clearly self-averaging is in accord with experiment because variations between samples where the random variables have the same statistical properties have never, to our knowledge, been observed.

For finite N the Brout argument leads to a probability distribution for densities of extensive variables which is Gaussian and of width $N^{-1/2}$, i.e., for the free-energy density f

$$P(f) \propto \exp\{-N(f-[f]_{\rm av})^2/2(\Delta f)^2\} . \tag{3.3}$$

Since $Z = \exp(-Nf/k_B T)$ it is clearly not correct to average the partition function. In fact, if one does this, one obtains from Eq. (3.3)

$$f_{\rm ann} = -\frac{k_B T}{N} \ln[Z]_{\rm av} = [f]_{\rm av} + (\Delta f)^2/k_B T . \tag{3.4}$$

So although $\ln[Z]_{\rm av}$ is well defined, it does not give $[f]_{\rm av}$ nor the free energy of any sample that occurs with reasonable probability. Equation (3.4) shows that $f_{\rm ann} \geq [f]_{\rm av}$; similar inequalities also hold for derivatives of f, such as susceptibilities, and hence yield upper bounds for the corresponding quantities of the quenched system (Falk, 1975).

The fact that f (or equivalently $\ln Z$) must be averaged, rather than Z, is at the heart of the difficulty in doing statistical mechanics of random systems. A rather successful way of getting around this, known as "the replica trick," is discussed in Sec. III.C. Another way to avoid averaging the logarithm is mentioned in Sec. IV.D. Finally we mention that one can formally write any quenched average as an equivalent annealed average, at the price of implicitly defining a complicated temperature-dependent interaction among the variables (Morita, 1964; Falk, 1976). This formulation can also serve as a starting point for interesting approximations to some problems in the statistical mechanics of random systems (Schwartz, 1985a)

Consider, for a moment, the average magnetization

$$[M]_{\rm av} = (1/N) \sum_i [\langle S_i \rangle_T]_{\rm av} . \tag{3.5}$$

After configurational averaging, translational invariance is restored, so each term in the sum in Eq. (3.3) must give the same result. Consequently one has

$$[M]_{\rm av}=[\langle S_i\rangle_T]_{\rm av} \tag{3.6}$$

for any site i. From Eq. (3.2) we find that, for $N\to\infty$ and assuming self-averaging,

$$[\langle S_i\rangle]_{\rm av}=(1/N)\sum_i\langle S_i\rangle_T\ , \tag{3.7}$$

so the summation over sites for a single large sample is equivalent to a configurational average. Although self-averaging seems almost obvious, it turns out *not* to be true for some of the densities of extensive variables that appear in the theory of spin glasses, Sec. IV.F. These quantities are extremely sensitive to boundary conditions, and so the Brout assumption of neglecting the coupling between subsystems is not valid. However, these quantities, though essential for a complete understanding of the theory, are not directly measurable, so self-averaging should still work in experiments, as observed. Note that site averaging and configurational averaging, Eq. (3.7), are no longer equivalent when self-averaging fails. It is then necessary to perform the sum over sites for a given sample and then calculate the *probability distribution* for this result when many samples are taken.

Even if self-averaging holds for densities of extensive quantities, there exist other quantities which have a much more complicated behavior. An example of such quantities are spin-pair correlation function (McCoy and Wu, 1968; Derrida and Hilhorst, 1981; Kinzel and Binder, 1981; van Hemmen and Morgenstern, 1982; Derrida, 1984). The problem can be most easily understood by considering the special case of a one-dimensional Ising chain with random nearest-neighbor bonds J_{ij} between spins at sites $i,j=i+1$. The correlation function between the spin at the origin and a spin a distance R apart is found from the transfer matrix method as

$$\langle S_0S_{R_T}\rangle=\prod_{i=0}^{R-1}\tanh\left[\frac{J_{i,i+1}}{k_BT}\right]\ . \tag{3.8}$$

It is clear in this case that the correlation functions are products of random numbers. Thus for large R they have log-normal distributions like the partition function. This means that the probability distribution is concentrated for large R around the most probable value $\langle S_0S_R\rangle_{\rm mp}$ given by

$$\langle S_0S_R\rangle_{\rm mp}=\exp([\ln\langle S_0S_{R_T}\rangle]_{\rm av})\neq[\langle S_0S_{R_T}\rangle]_{\rm av}\ . \tag{3.9}$$

In this example, both $\langle S_0S_R\rangle_{\rm mp}$ and $[\langle S_0S_r\rangle]_{\rm av}$ have an exponential decay with distance, but the corresponding correlation lengths are different. More generally, one may introduce correlation lengths $\xi_p(T)$ for each moment of $\langle S_0S_R\rangle$ separately,

$$[\langle S_0S_R\rangle^p]_{\rm av}\underset{R\to\infty}{\to}\exp\{R\ln[(\tanh J_{ii+1}/k_BT)^p]_{\rm av}\}$$
$$=\exp[-R/\xi_p(T)]\ . \tag{3.10}$$

The existence of many different correlation lengths is expected for random systems in general; moreover the possibility has been raised (Derrida and Hilhorst, 1981) that these lengths might diverge at different temperatures and hence give rise to several phase transitions. Of course, for the Ising spin chain this does not happen; all lengths $\xi_p(T)$ diverge at $T=0$ only. If, on the other hand, we construct the (normalized) extensive quantity

$$C(R)=\frac{1}{N}\sum_{i=1}^{N}\langle S_iS_{i+R}\rangle_T$$

then for N sufficiently large, $C(R)$ *is* self-averaging. However, the necessary value of N for relative variations in $C(R)$ between samples to be small increases exponentially with R for the Ising chain. Consequently a numerical evaluation of $[\langle S_0S_R\rangle_T]_{\rm av}$ on a finite system is difficult because there will be huge sample-to-sample variations for large R. Even in higher dimensions one expects significant relative variations between samples for distances where the correlation function is small, though the dependence on R and N is not precisely understood. For typical cases studied by Monte Carlo and transfer matrix calculations (Morgenstern and Binder, 1980a, 1980b; Young, 1983a, 1984; Bhatt and Young, 1985a; Ogielski, 1985) averages over up to 10^2 samples seem to yield reasonable accuracy (for a discussion of this problem see van Hemmen and Morgenstern, 1982).

A study of distribution functions of spin pair correlation functions (Kinzel and Binder, 1981) is also relevant for the interpretation of "local measurements" such as the Mössbauer effect or NMR (often one talks about the distribution of the static local "effective field" probed by such an experiment). While some numerical results are available for specific models (Kinzel and Binder, 1981), not much is known about the distribution of correlations in general. Figure 37 shows an example of the probability distributions of correlations between spins at nearest- and next-nearest-neighbor distances in a model for $Eu_xSr_{1-x}S$ at $T=0$.

B. Ergodicity

An experimentalist performs his measurements over a certain (finite) amount of time. The rate at which fluctuations occur depends upon the spectrum of relaxation times $\{\tau\}$ in the problem. If the observation time $\tau_{\rm exp}$ is much greater than $\tau_{\rm max}$, the largest relaxation time, then the system explores all regions of phase space with the equilibrium probability: so the *time average* performed by the experiment will be equivalent to an average from equilibrium statistical mechanics (which we shall call a *Gibbs average*) where all microscopic states are included, weighted with the canonical distribution $\exp(-\mathcal{H}/k_BT)/Z$. We shall say that such a system is in *complete equilibrium.*

There are many instances, however, where these two averages are not the same. "Simple" examples are glasses like window glass or amorphous SiO_2, which should, according to the Gibbs average, exist as crystalline SiO_2. The observed amorphous structure is a *metastable state,* whose free energy is higher than the crystalline free ener-

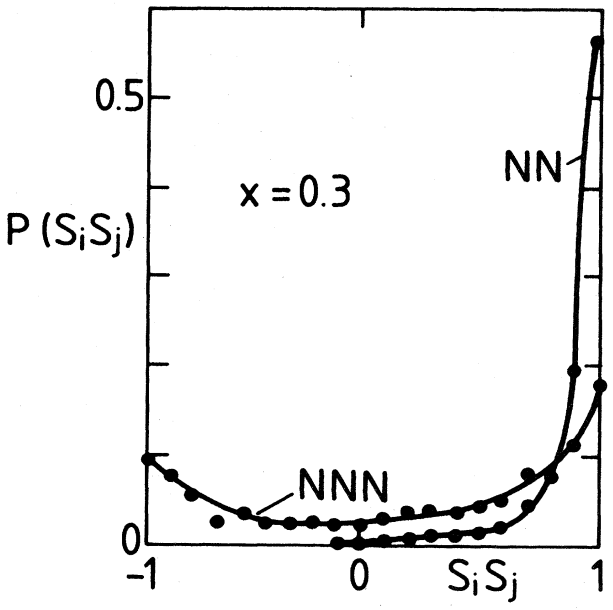

FIG. 37. Probability distribution of the spin pair correlation function at zero temperature for nearest-neighbor (NN) and next-nearest-neighbor distance on the fcc lattice, with 30% of the sites taken by magnetic atoms, ferromagnetic exchange J_1 between nearest neighbors, and antiferromagnetic exchange $J_2=J_1/2$ between next-nearest neighbors. From Kinzel and Binder (1981).

gy; this state will in a *finite* (but huge) time decay into the crystalline state by nucleating sufficiently large domains of the crystalline phase, which will then grow and cover the whole of the material (for theoretical descriptions of nucleations in general, see Zettlemoyer, 1969; Binder and Stauffer, 1976b). Clearly, though, the time required to do this is far larger than the time of any conceivable experiment.

There are also more extreme instances where the relaxation time, instead of being large but finite, actually diverges in the thermodynamic limit. This can happen when a phase transition occurs. For simplicity consider an Ising magnet, in which the spins can only point up or down and are coupled ferromagnetically. Below the Curie temperature a spontaneous magnetization appears and, in zero field, the system either goes into the state with all spins pointing up or the state with all spins down, with equal probability. For short-range systems the free-energy barrier ΔF, which has to be overcome to make a system in the down state flip over the up state, is of the order of the area of the domain wall that crosses the sample, i.e., proportional to L^{d-1} where L is the linear dimension of the d-dimensional sample, if the coupling is of short range. For infinite-range ferromagnetic exchange, ΔF would be proportional to the total volume L^d. The time needed to observe such a flip is of the order $\exp(\Delta F/k_BT)$, which is simply the ratio of Boltzmann factors between the initial state and the state with the wall. Hence the time to cross the barrier between the two phases diverges exponentially as N (or L^d) $\rightarrow\infty$.

We shall say that systems like this, where $\tau_{\max}\rightarrow\infty$ as $N\rightarrow\infty$, are *nonergodic*. Systems like window glass, in which the relaxation times are huge but finite, are *effectively nonergodic* on experimental time scales. It may not be possible to distinguish between these cases experimentally, but it is crucial to understand the difference for the theory of spin glasses. In the first case we are dealing with several thermodynamic states (the words pure states, phases, and ergodic components are also used), which have the same minimum free energy per spin, $f_{\min}$. The system cannot be converted from one state to another by nucleating domains of finite size because their radius tends to infinity when the free-energy difference per spin between the states tends to zero. Hence the domain size diverges as $N\rightarrow\infty$, so the barrier height and relaxation time also diverge in this limit. Consequently the different states are *stable* (for $N\rightarrow\infty$). In the second case we have a *metastable* state whose free energy per spin is greater than $f_{\min}$ and will decay in a finite time to a stable phase of minimum free energy.

This question of "*broken ergodicity*" has been discussed by Palmer (1982). Its importance for spin glasses was emphasized earlier by Anderson (1977, 1979).

Usually broken ergodicity arising from phase transitions does not pose severe problems when applying statistical mechanics. Referring to our example of the Ising ferromagnet, we know that this system will choose one of the two states that are related to each other by symmetry and will be in *partial equilibrium* with respect to the microscopic states sampled in this phase. Clearly, to reproduce the observed behavior by statistical mechanics, we need to restrict the microscopic states in the average to one thermodynamic state or the other. This can conveniently be done by applying a *symmetry-breaking field* H and letting $H\rightarrow 0$ only after taking the thermodynamic limit. Since the field weights the up (down) state by $\exp[+(-)NHm/k_BT]$, where m is the spontaneous magnetization per spin, such a field will have the desired effect.

In spin glasses the problem of broken ergodicity is more serious. According to mean-field theory (see Sec. IV), it appears that many thermodynamic states are available to the system (in the spin glass phase below the transition temperature T_f). Whether this also happens for realistic models with short-range interactions that have a transition is not clear at present. The details of these states will depend on the precise values of the interactions and so vary from sample to sample. However, they do, as expected, have the same free energy (van Enter and van Hemmen, 1984). More precisely, the differences in free energy per spin are of order N^{-1} and so vanish as $N\rightarrow\infty$. But unlike the Ising ferromagnet, these states are not related to each other by a symmetry of the Hamiltonian. Rather they appear because of *accidental degeneracy*, which in turn occurs because of randomness and frustration (see Sec. III.F) in the system. Of course, there may, in addition, be global symmetries. For instance, an Ising

spin glass, like the Ising ferromagnet, has the symmetry of $S_i \to -S_i$ provided no field is applied. Hence for every state there will be a "time-reversed" state with the sign of the spin expectation values reversed. Applying a uniform field (or a completely random staggered field) will destroy this symmetry and reduce by a factor of 2 the number of states. There will still be many left, however, because of accidental degeneracy. Since the site magnetizations are determined in a very complicated way by the set of interactions in the sample, it is impossible in practice to project out one particular state, either by a carefully chosen random staggered field or otherwise. Therefore a statistical mechanics calculation will inevitably be averaging over many states, and we must look into the consequences of this.

We should emphasize that degenerate thermodynamic states do occur in the infinite-range model discussed in detail in Sec. IV. It is, however, not clear whether this is also true for real three-dimensional systems or whether instead they have a unique thermodynamic state.

Phase space can be decomposed uniquely into microscopic states associated with different thermodynamic states [see van Hemmen (1983) and van Enter and van Hemmen (1984) and references therein]. Let us denote a particular state by l. It will have a free energy per spin f_l given by

$$\exp(-Nf_l/k_BT)=\sum_{\lambda\in l}\exp(-E_\lambda/k_BT)(=Z_l)\ , \tag{3.11}$$

where the sum is over all microscopic states associated with state l. Since the partition function is a sum over all microscopic states it follows that

$$Z=\sum_l Z_l=\sum_l \exp(-Nf_l/k_BT)\ , \tag{3.12}$$

and the statistical weight associated with a given thermodynamic state is

$$P_l=\frac{1}{Z}\exp(-Nf_l/k_BT)\ . \tag{3.13}$$

Just because all the f_l tend to the same value as $N\to\infty$ it does not necessarily mean that all the P_l are equal, because differences in the f_l of order $1/N$, for instance, can change P_l by a finite amount. Furthermore the mean field theory (Sec. IV) deals with an infinite-range model in which standard nucleation theories (Zettlemoyer, 1969; Binder and Stauffer, 1976b) do not apply and stable states (in the sense of infinite lifetime) can occur for $f>f_{\min}$ even when $N\to\infty$ (for a discussion of this fact for Ising infinite-range ferromagnets and related systems see Binder, 1973, 1984a, and Penrose and Lebowitz, 1971). In such a model it is essential to include the weight factor P_l. Now let us evaluate the Gibbs average of a partial quantity A, say,

$$\langle A\rangle_T=\frac{1}{Z}\sum_\lambda A_\lambda\exp(-E_\lambda/k_BT)\ , \tag{3.14}$$

where A_λ is the value of A in microscopic state λ. Splitting the sum into a restricted sum for a given phase l, followed by a sum over l, gives

$$\langle A\rangle_T=\frac{1}{Z}\sum_l\left[\frac{1}{Z_l}\sum_{\lambda\in l}A_\lambda\exp(-E_\lambda/k_BT)\right]Z_l\ . \tag{3.15}$$

From Eqs. (3.11) and (3.13) this becomes (Palmer, 1982; De Dominicis and Young, 1983a)

$$\langle A\rangle_T=\sum_l P_l\langle A\rangle_T^{(l)}\ , \tag{3.16}$$

where $\langle A\rangle_T^{(l)}$ is the partial equilibrium average of A in state l and is defined by

$$\langle A\rangle_T^{(l)}=\frac{1}{Z_l}\sum_{\lambda\in l}A_\lambda\exp(-E_\lambda/k_BT)\ . \tag{3.17}$$

Equation (3.16) is the main result of this discussion.

A further complication in applying statistical mechanics to spin glasses, which does not generally occur for ferromagnets, is that within each state below the transition temperature (if there is one), and even for the single state that exists above any possible transition, there are long (but finite) relaxation times due to metastable states. These times can be longer than experimental time scales, so it is hard to disentangle metastability from true ergodicity breaking. In a ferromagnet one would also expect very close to the critical point a regime in which relaxation times exceed observation times, due to critical slowing down, but in practice this regime is confined to an unobservably narrow neighborhood of the critical point.

In these circumstances one might question whether one can learn from statistical mechanics anything at all about spin glasses that is relevant to experiment. It is clearly extremely difficult to reproduce the observed irreversibility and hysteresis from a microscopic theory. However, it appears possible experimentally (see Sec. II), by techniques such as field cooling, to obtain reproducible results independent of the measuring time, as long as it is not too short. It is certainly plausible that these systems are close to equilibrium, and if they are below a transition they are probably in *one* of the thermodynamic states of near minimum-free energy. One should therefore calculate from statistical mechanics the properties of *one* state and compare this with experiment. We shall argue that all macroscopic properties are the same for each state (see van Enter and van Hemmen, 1984), not only the free energy, so it does not matter which is chosen. Thus if $\langle A\rangle_T^{(l)}$ actually does not depend on l, Eq. (3.16) even implies $\langle A\rangle_T=\langle A\rangle_T^{(l)}$. Furthermore the results are independent of the sample (see Sec. III.A). Surprisingly the mean field theory not only gives this information but also makes predictions for "overlap functions" (see Sec. IV) involving two (or more) *different* states. These are the *non-self-averaging* quantities discussed in Sec. III.A. In fact, one cannot, with the present mean-field theory (Sec. IV), calculate properties of one state without a knowledge of these overlap functions. Since the free-energy barriers between the different "valleys" in configuration space, corresponding to the different states, are infinitely high in

the thermodynamic limit, a time average of a system starting out in such a pure state samples only one particular "valley" and does not probe properties relating to other valleys, if the limit of observation time $t \to \infty$ is taken after the thermodynamic limit $N \to \infty$. However, if one takes these two limits, not one after the other, but suitably combined, it is possible to define "order parameters" that refer to a sampling of many "valleys" rather than just a single one (Sompolinsky, 1981a); this point will be discussed in more detail in the dynamic approach to the mean-field theory (Sec. IV.D.3).

While Eq. (3.16) shows that the Gibbs average of an extensive observable is simply an average of this observable over the various thermodynamic states or "components," this is no longer true for the root-mean-square width $\Delta(A)$ characterizing the probability of fluctuations (Palmer, 1982). Using Eq. (3.16) for A^2 we obtain

$$\langle A^2\rangle_T - \langle A\rangle_T^2 = \sum_l P_l \langle A^2\rangle_T^{(l)} - \left[\sum_l P_l \langle A\rangle_T^{(l)}\right]^2 \equiv \sum_l P_l [\langle A^2\rangle_T^{(l)} - (\langle A\rangle_T^{(l)})^2] + [\Delta^{(l)}(A)]^2 . \tag{3.18}$$

Thus the canonical mean-square fluctuation is the sum of the average *intracomponent* mean-square fluctuation and the *intercomponent* variance $[\Delta^{(l)}(A)]^2$ defined by Eq. (3.18). As an example, we mention the specific heat C_H,

$$C_H = \left[\frac{\partial E}{\partial T}\right]_H = \sum_l P_l \partial \langle \mathscr{H}\rangle_T^{(l)}/\partial T + \Delta C , \qquad \Delta C = k_B [\Delta^{(l)}(E)/T]^2 . \tag{3.19}$$

Applying Eq. (3.16) to the energy, we see that ΔC vanishes if P_l is temperature independent.

While Eqs. (3.18) and (3.19) constitute formal definitions of intercomponent fluctuation contributions (Palmer, 1982), it is an open problem whether these quantities are extensive. It is, however, possible that $\Delta C \propto N$ as well as C_H, even if $E^{(l)} = \langle \mathscr{H}\rangle_T^{(l)}$ is "sharp" in the thermodynamic limit: the relative spread $\Delta^{(l)}(E)/E$ of $E^{(l)}$ must then be of order $N^{-1/2}$.

We turn to a discussion of the free energy itself, emphasizing that its canonical value is not simply the component average $\bar{F}$. Since $Nf_l = F - k_B T \ln P_l$,

$$\bar{F} = \sum_l P_l N f_l = F + TI , \tag{3.20}$$

where the "complexity" I (Palmer, 1982) of the component ensemble is

$$I = -k_B \sum_l P_l \ln P_l . \tag{3.21}$$

This "complexity" can be interpreted as the average additional information needed to specify a particular state, given the *a priori* probabilities P_l. One can define an "effective number" K^* of states as

$$K^* = \exp(I/k_B) , \tag{3.22}$$

which lies between unity ($K^* = 1$ if there is no broken ergodicity) and the actual number of states $K \equiv \sum_l 1$ ($K^* = K$ if $P_l = 1/K$). However, clearly a number of the order $K = \exp(\text{const} \times N)$ of states is needed in order for the complexity I to be extensive. This occurs for the ground state of degenerate systems such as the $\pm J$ model of a spin glass; there is not yet any example of a system for which one could show that I is extensive at nonzero temperature as well. Even in mean-field theory K is of order unity (see Sec. IV.F), so one can neglect the difference between F and $\bar{F}$.

At this point we are also in a position to comment on possible violations of "Maxwell relations," which relate various derivatives of the free energy to each other. Such relations hold both for quantities that refer to a single component l and for quantities referring to the full canonical Gibbs ensemble. A violation of these relations is possible only if one compares on one side quantities referring to a single state, with quantities referring to the full Gibbs ensemble on the other side. Then intercomponent fluctuation contributions such as those considered in Eqs. (3.18) and (3.19) might become directly measurable in terms of the deviation of the corresponding Maxwell relation. We shall not elaborate on this possibility further, however, since any evidence that this case is experimentally relevant so far is lacking.

One may generalize the above thermodynamic description by allowing weights P_l for the various states differing from the canonical prescription, Eq. (3.13). In principle, such a different choice might express some "history dependence" describing the preparation of the system (Palmer, 1982). In practice, however, there is no clear-cut theoretical prescription of how to do this. For the freezing transition into a glassy state observed in fluids that are slowly cooled to low temperature, Jäckle (1981) suggested defining a temperature T^* such that the system is in full thermodynamic equilibrium for $T \geq T^*$ but that ergodicity is effectively broken for $T < T^*$. Thus at T^* suddenly many distinct states $\{l\}$ appear, while above T^* all these states are mutually easily accessible from each other and hence are not distinct: there is just one (metastable) fluid phase. Then P_l would have to be evaluated from Eq. (3.13) at $T = T^*$, while for $T < T^*$, Eq. (3.13) would not apply for the description of a glassy state. This assumption is useful, however, only if the full ergodicity breaking appears at once at T^* in a single "bifurcation" of the valley structure of phase space; for spin glasses one rather expects a picture of a "bifurcation cascade" going on as the temperature is lowered (Krey 1977, 1982; Palmer, 1982; Jäckle and Kinzel, 1983; Mézard *et al.*, 1984a, 1984b). We shall return to this problem in the light of more explicit theoretical calculations later.

C. The replica method

We have seen in Sec. III.A that a proper treatment of systems with quenched disorder involves averaging the free energy,

$$f=[f\{x\}]_{\rm av}=-\frac{k_BT}{N}[\ln Z\{x\}]_{\rm av}\ . \tag{3.23}$$

Obviously, this task is prohibitively difficult, since as a first step one needs to calculate the free energy $f\{x\}=\ln Z\{x\}$, which depends on the large set $\{x\}$ describing the disorder: one would have to do statistical mechanics with a Hamiltonian that contained infinitely many parameters in the thermodynamic limit and had no translational invariance. The final step, averaging over the set of disorder variables, does not really present any simplification if the procedure of Eq. (3.23) is taken literally, since the free energy is "self-averaging" in the thermodynamic limit, as discussed in Sec. III.A.

Rather than abandoning the idea of averaging over the disorder, we see a way to make better use of the average over the disorder variables, such that this average really simplifies the problem. The average in Eq. (3.23) cannot be performed directly because the random variables occur inside a logarithm. If the disorder is weak one could split $\mathcal{H}\{x\}$ into a nonrandom part $\mathcal{H}_0$ and a random perturbation $\delta\mathcal{H}\{x\}$, expand in powers of $\delta\mathcal{H}\{x\}$, and average term by term (e.g., Rudnick, 1980). Spin glasses, however, are highly random systems, and in many of the standard models $\mathcal{H}_0$, the nonrandom part, is zero. This technique is therefore not generally applicable, though even for spin glasses it is valid at high temperatures and can be used to generate high-temperature series expansions (Fisch and Harris, 1977; Ditzian and Kadanoff, 1979; Reed, 1979a; Rajan and Riseborough, 1983; Palmer and Bantilan, 1985). At low temperatures and certainly below a spin glass transition temperature one needs a nonperturbative way of averaging over disorder.

This is the idea behind the replica approach (Kac, 1968; Edwards, 1970, 1971; Edwards and Anderson, 1975; Lin, 1970; Emery, 1975; Grinstein and Luther, 1976; Jasnow and Fisher, 1976), which rests on the use of the exact relation

$$[\ln Z\{x\}]_{\rm av}=\lim_{n\to 0}\frac{1}{n}([Z^n\{x\}]_{\rm av}-1)=\lim_{n\to 0}\frac{\partial}{\partial n}[Z^n\{x\}]_{\rm av}\ , \tag{3.24}$$

[remember $Z^n=\exp(n\ln Z)\approx 1+n\ln Z$ as $n\to 0$]. For positive integer n, one can express $Z^n\{x\}$ in terms of n identical replicas of the system,

$$Z^n\{x\}=\prod_{\alpha=1}^{n}Z_\alpha\{x\}=\prod_{\alpha=1}^{n}\exp[-\mathcal{H}\{x,S_i^\alpha\}/k_BT]$$
$$=\exp\left[-\sum_{\alpha=1}^{n}\mathcal{H}\{x,S_i^\alpha\}/k_BT\right]\ , \tag{3.25}$$

where Z_α is the partition function of the αth replica. For positive integer n, it is indeed simple to carry out the average $[\ ]_{\rm av}$, and one can express the result formally in terms of an effective Hamiltonian $\mathcal{H}_{\rm eff}(n)$ that no longer contains any disorder and is *translationally invariant,*

$$Z_n\equiv[Z^n\{x\}]_{\rm av}\equiv\operatorname*{Tr}_{\{S_i^\alpha\}}\exp[-\mathcal{H}_{\rm eff}(n)/k_BT]\ ; \tag{3.26}$$

$\mathcal{H}_{\rm eff}(n)$ is defined in the space of variables $\{S_i^\alpha\}$ of all n replicas of the system. While before the averaging in Eq. (3.25) we have just a sum over the Hamiltonians of the various replicas, i.e., the replicas do not interact averaging over the disorder in Eq. (3.26) leads to coupling between different replicas. To be specific, we consider a random-bond problem, the sum over i,j running over all sites,

$$\mathcal{H}=-\tfrac{1}{2}\sum_{ij}J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j-H\sum_i S_i^z\ , \tag{3.27}$$

where the exchange constants J_{ij} are randomly chosen according to a fixed distribution $P(J_{ij})$. Then Eq. (3.26) becomes, for magnetic field $H=0$, and considering Ising rather than Heisenberg spins for simplicity,

$$Z_n=\operatorname*{Tr}_{\{S_i^\alpha\}}\prod_{\langle i,j\rangle}dJ_{ij}P(J_{ij})\exp\left[\frac{J_{ij}}{k_BT}\sum_{\alpha=1}^{n}S_i^\alpha S_j^\alpha\right]\ . \tag{3.28}$$

From a Taylor expansion of Eq. (3.28) one then finds (see also Lage, 1977; Sarbach, 1980)

$$\mathcal{H}_{\rm eff}(n)/k_BT=-\tfrac{1}{2}\sum_{ij}\sum_{k=1}^{\infty}\frac{1}{k!}\frac{J_{ij}^{\rm cum}(k)}{(k_BT)^k}\left[\sum_{\alpha=1}^{n}S_i^\alpha S_j^\alpha\right]^k\ , \tag{3.29}$$

where $J_{ij}^{\rm cum}(k)$ is the kth cumulant of J_{ij}:

$$J_{ij}^{\rm cum}(1)=[J_{ij}]_{\rm av}=\bar{J}\ , \tag{3.30a}$$

$$J_{ij}^{\rm cum}(2)=[J_{ij}^2]_{\rm av}-[J_{ij}]_{\rm av}^2\equiv(\Delta J_{ij})^2\ , \tag{3.30b}$$

etc. Obviously, in $\mathcal{H}_{\rm eff}(n)$ the terms with $k>1$ provide couplings between different replicas of the disordered system. Since $\mathcal{H}_{\rm eff}(n)$ is an effective Hamiltonian of a (fictitious) translationally invariant problem without disorder, all methods of statistical mechanics for ideal pure systems can now be applied. One expects that the simplest step will be a mean-field approximation: there one replaces—roughly speaking—the terms $S_i^\alpha S_j^\alpha$ by $S_i^\alpha\langle S_j^\alpha\rangle$, $S_i^\alpha S_i^\beta S_j^\alpha S_j^\beta$ by $S_i^\alpha S_i^\beta\langle S_j^\alpha S_j^\beta\rangle$, $S_i^\alpha S_i^\beta S_i^\gamma S_j^\alpha S_j^\beta S_j^\gamma$ by $S_i^\alpha S_i^\beta S_i^\gamma\langle S_j^\alpha S_j^\beta S_j^\gamma\rangle$, and so on. The "order parameters"

$$m_\alpha\equiv\langle S_j^\alpha\rangle\ ,\quad q_{\alpha\beta}\equiv\langle S_i^\alpha S_j^\beta\rangle_{\alpha\neq\beta}\ ,$$
$$q_{\alpha\beta\gamma}\equiv\langle S_j^\alpha S_j^\beta S_j^\gamma\rangle_{\alpha\neq\beta\neq\gamma}\ ,\quad \cdots \tag{3.31}$$

have then to be determined self-consistently (we defer a more thorough discussion of mean-field theory to Sec. IV).

All these results, Eqs. (3.26)—(3.29), refer to the case of a positive integer n. The crucial step needed to apply Eq. (3.24), however, is to continue $\mathcal{H}_{\rm eff}(n)$ analytically to arbitrary positive real n. Unfortunately, this "replica trick," Eq. (3.24), thus creates two problems. (i) In Eq. (3.24) the limits $n\to 0$ and the thermodynamic limit $N\to\infty$ are treated in the reverse order. Since it has been suspected for some years that this interchange of limits is responsible for some pathological results [such as the negative entropy of the Sherrington-Kirkpatrick (1975) model],

this problem has received much attention (see van Hemmen and Palmer, 1979, 1982, for a discussion). However, it is now believed that this step does not really create trouble (see the discussion at the end of Sec. IV.A). (ii) The analytic continuation of $\mathscr{H}_{\rm eff}(n)$ to real positive n near $n=0$ is nonunique. While it is obvious that $\mathscr{H}_{\rm eff}(n)$ is invariant under permutations of the indices of the replicas, as long as n is a positive integer, it is not obvious that this symmetry is preserved by the solution when n takes noninteger values and one takes the limit $n\to 0$. This fact leads to the idea of "*replica symmetry breaking*" (Blandin, 1978; de Almeida and Thouless, 1978; Bray and Moore, 1978, 1979a; Parisi, 1979, 1980a, 1980b, 1980c, 1980d, 1980e; Blandin *et al.*, 1980). This approach will be discussed further below.

In addition to calculating the free energy, one also needs to know how to evaluate correlation functions with the replica method. Let us begin with a simple example, the magnetization per spin

$$M=[\langle S_i\rangle_T]_{\rm av} =[(1/Z\{x\})\mathrm{Tr}S_i\exp(-\mathscr{H}\{x\}/k_BT)]_{\rm av} . \quad (3.32a)$$

Performing the configurational average is difficult because the random variables appear in both the numerator and the denominator. To get around this, multiply numerator and denominator by $(Z\{x\})^{n-1}$, so

$$M=\left[\frac{Z^{n-1}\mathrm{Tr}S_i\exp(-\mathscr{H}\{x\}/k_BT}{Z^n}\right]_{\rm av} . \quad (3.32b)$$

If we let $n\to 0$, the denominator is unity, so the random variables appear only in the numerator and can be averaged over. The numerator involves a trace over n replicas of the spins, one of these replicas, α say, being the spins that appear explicitly in Eq. (3.32b). Hence

$$M=\langle S_i^\alpha\rangle , \quad (3.32c)$$

where $\langle\ \rangle$ denotes an average over $\mathscr{H}_{\rm eff}$, i.e.,

$$\langle(\ )\rangle=\lim_{n\to 0}\mathrm{Tr}(\ )\exp[-\mathscr{H}_{\rm eff}(n)/k_BT] . \quad (3.32d)$$

For $n\to 0$ there is no need for a normalizing denominator in Eq. (3.32d) because $\mathrm{Tr}\exp[-\mathscr{H}_{\rm eff}(n)/k_BT]\to 1$ in this limit. If, however, one wishes to discuss the theory for nonzero n it is necessary to divide by $\mathrm{Tr}\exp[-\mathscr{H}_{\rm eff}(n)/k_BT]$. Note that α in Eq. (3.32c) can be *any* of the replicas, and the answer must be independent of this arbitrary choice, as indeed it is, because $\mathscr{H}_{\rm eff}(n)$ is invariant under permutation of the replicas. This was noted above.

Next we relate

$$q=[\langle S_i\rangle_T^2]_{\rm av}=\left[\frac{[\mathrm{Tr}S_i\exp(-\mathscr{H}\{x\}/k_BT)][\mathrm{Tr}S_i\exp(-\mathscr{H}\{x\}/k_BT)]}{Z^2}\right]_{\rm av} \quad (3.33a)$$

to averages; with respect to $\mathscr{H}_{\rm eff}(n)$. q is very important in spin glass theories and is an obvious choice for an order parameter in the low-temperature phase. Multiplying numerator and denominator by Z^{n-2}, letting $n\to 0$, and performing the configurational average, we find that the numerator again involves n replicas, two of which, α and β say (obviously $\alpha\neq\beta$), are for the two sets of spins that appear explicitly in Eq. (3.33a). One therefore finds

$$q=\langle S_i^\alpha S_i^\beta\rangle \quad (\alpha\neq\beta) . \quad (3.33b)$$

Again the answer must be independent of the arbitrary choice of replicas.

The generalization to arbitrary products of thermal averages is now obvious. For example,

$$[\langle S_i\rangle_T^k]_{\rm av}=\langle S_i^{\alpha_1}S_i^{\alpha_2}\cdots S_i^{\alpha_k}\rangle , \quad (3.34a)$$

where all the replicas α_i $(i=1,2,\ldots,k)$ are distinct. This result can be generalized to thermal averages of spins on different sites. The important point is that *for every thermal average on the left-hand side there is a distinct replica, and the result is independent of the arbitrary choice of these replicas.*

At this point it is convenient to anticipate some of the results of mean-field theory presented in Sec. IV. In this theory the $q_{\alpha\beta}=\langle S_i^\alpha S_i^\beta\rangle$ $(\alpha\neq\beta)$ emerge naturally as order parameters. However, the simple solution of Sherrington and Kirkpatrick (1975) with all $q^{\alpha\beta}$ equal is actually unstable below T_c (de Almeida and Thouless, 1978), and one must "break replica symmetry," as mentioned above, in order to find a stable solution. This means that $q_{\alpha\beta}$ depends on the choice of replicas, which seems to contradict the remark above that $\langle S_i^\alpha S_j^\beta\rangle$ does not depend on α and β. The resolution of this apparent paradox is that any solution which breaks replica symmetry is not unique (De Dominicis and Young, 1983a) because other equivalent solutions can be generated by permutations of the replicas. There is no reason to single out any one of these solutions, and they should all be included in evaluating spin averages. Thus $\langle S_i^\alpha S_j^\beta\rangle$ is obtained by averaging $q_{\alpha\beta}$ for fixed α and β, over all distinct solutions. With some reflection one concludes that this is equivalent to taking a single solution and averaging over all replicas, i.e.,

$$q=\lim_{n\to 0}\frac{1}{n(n-1)}\sum_{\alpha\neq\beta}q_{\alpha\beta} , \quad (3.33c)$$

where the $q_{\alpha\beta}$ are for a particular solution. The analog of Eq. (3.34a) is

$$[\langle S_i\rangle_T^k]_{\rm av}=\lim_{n\to 0}[n(n-1)\cdots(n-k+1)]^{-1} \times\sum_{(\alpha_j\cdots\alpha_k)}\langle S_i^{\alpha_1}S_i^{\alpha_2}\cdots S_i^{\alpha_k}\rangle \quad (3.34b)$$

when one has replica symmetry breaking. The notation $(\alpha_1 \cdots \alpha_k)$ means that one sums only over distinct replicas, and from now on the average $\langle\ \rangle$ refers to a single solution.

A detailed physical explanation of replica symmetry breaking will be deferred until Sec. IV.E, but here we should like to note that it is connected with broken ergodicity and the existence of many thermodynamic states, discussed in Sec. III.B. We do not know of any instances in which the stable solution for positive integer n has replica symmetry breaking. It appears only in the continuation $n \to 0$. Various other nonperturbative methods have been suggested for doing the quenched average (De Dominicis, 1980; Rudnick, 1980; Sompolinsky and Zippelius, 1981, 1982a; Parisi and Sourlas, 1979; De Dominicis *et al.*, 1980; Bray and Moore, 1980b). Whenever the replica-symmetric solution is correct these alternative approaches are also fairly straightforward to apply and give the same results as the replica method. On the other hand, where the replica solution involves replica symmetry breaking, complications have arisen in these other approaches, the resolution of which has only been accomplished by resorting back to the replicas (De Dominicis and Young, 1983a; Houghton *et al.*, 1983a, 1983b). An exception to this is Sompolinsky's (1981a) theory of dynamics, discussed in Sec. IV.D.

D. The Edwards-Anderson model

Clearly, it is a difficult task even in the replica framework formulated in the last section, to start from a realistic microscopic description of a spin glass by considering actual interactions, anisotropies, etc. (Secs. II.A.1 and II.A.2) and to perform the average over a realistic description of the disorder (site dilution, for instance). However, the general experience with all kinds of cooperative phenomena is that it is most helpful to start out with models providing a sort of coarse-grained description, which captures the essential points of the physics in question, but leaves out many of the microscopic details characteristic of a particular material.

It is this spirit which led Edwards and Anderson (1975) in their pioneering paper to suggest that one may describe a spin glass by a Hamiltonian of the type of Eq. (3.27), where spins are put onto the sites of a regular lattice, and disorder is introduced by a suitable distribution $P(J_{ij})$ of exchange bonds. The standard choice is the Gaussian Edwards-Anderson model (Edwards and Anderson, 1975; see also Sherrington and Southern, 1975),

$$P(J_{ij}) = [2\pi(\Delta J_{ij})^2]^{-1/2} \times \exp[-(J_{ij}-\bar{J}_{ij})^2/2(\Delta J_{ij})^2] \ . \tag{3.35}$$

In this case, irrespective of the range of this interaction, all cumulants $J_{ij}^{\rm cum}(k \geq 3) \equiv 0$, and hence $\mathscr{H}_{\rm eff}(n)/k_BT$ takes the particularly simple form [cf. Eq. (3.29); again we treat the Ising case only]

$$\mathscr{H}_{\rm eff}(n)/k_BT = -\frac{1}{2}\sum_{ij}\frac{\bar{J}_{ij}}{k_BT}\sum_{\alpha=1}^{n} S_i^\alpha S_j^\alpha - \frac{1}{4}\sum_{ij}\left[\frac{\Delta J_{ij}}{k_BT}\right]^2 \sum_{\alpha\beta} S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta \ . \tag{3.36}$$

The Sherrington-Kirkpatrick (SK) model (Sherrington and Kirkpatrick, 1975) is the infinite-range version of the Edwards-Anderson model: here one assumes that the same distribution $P(J_{ij})$ holds for any pair of spins irrespective of their distance. For this model Eq. (3.35) also holds, with $\bar{J}_{ij}$ and ΔJ_{ij} being independent of distance and scaling with N as

$$\bar{J}_{ij} \propto 1/N \ , \quad \Delta J_{ij} \propto 1/\sqrt{N} \tag{3.37}$$

to ensure a sensible thermodynamic limit. Regardless of the form of the distribution, higher cumulants $J_{ij}^{\rm cum}(k \geq 3)$ are irrelevant in this infinite-range case. For such an infinite-range model, mean-field theory ought to be exact: while the formulation of mean-field theory for an infinite-range ferromagnet is rather trivial, the formulation of mean-field theory for spin glasses has been—and still is—a major challenge of theoretical physics (see Sec. IV). A simple soluble case only arises in the spherical limit (number of spin components $m \to \infty$); see Kosterlitz *et al.*, 1976.

In the short-range case, it does make sense to also consider distributions $P(J_{ij})$ different from Eq. (3.35). An important example is the so-called "$\pm J$ model" or "frustration model" (Toulouse, 1977),

$$P(J_{ij}) = p_1\delta(J_{ij}-J) + (1-p_1)\delta(J_{ij}+J) \ , \tag{3.38a}$$

or its diluted version (e.g., Aharony and Binder, 1980),

$$P(J_{ij}) = p_1\delta(J_{ij}-J) + p_2\delta(J_{ij}+J) + (1-p_1-p_2)\delta(J_{ij}) \ , \tag{3.38b}$$

where the interaction is generally between nearest neighbors only. Here p_1 denotes the concentration of bonds $+J$, and p_2 the concentration of bonds $-J$. Due to residual ground-state entropy in the case of Eq. (3.38), one expects rather different physical properties at low temperatures.

We would also like to mention that Ising models with randomly mixed ferro- and antiferromagnetic bonds were studied prior to Edwards and Anderson (1975) by Katsura and Matsubara (1974) and later by other groups (Pekalski and Oguchi, 1975; Ueno and Oguchi, 1975); but these groups proposed the occurrence of "glasslike phases" (Matsubara and Sakata, 1976) or "random-ordered phases" (Ueno and Oguchi, 1976) only after Edwards and Anderson (1975). Therefore we shall continue to call models of this type "Edwards-Anderson" models.

When one wishes to compare predictions of any such model to experiment, clearly such comparison will at best be rather qualitative. For instance, in the model every lattice site is occupied with a spin, while in the real system, if it is produced by dilution (which it usually is), this is not the case. As a consequence, it is not clear that one

should associate one spin in the model with a single magnetic moment in the real material: rather, one may argue (e.g., Binder 1977a, 1977b) that a spin of the model might correspond to a strongly correlated cluster of magnetic moments in the real material. Such local clusters and their effects have been considered to be essential by many experimentalists, and thus the Edwards-Anderson model has not always been accepted as a reasonable starting point of the theory. We maintain, however, that it is a prototype model for the description of cooperative phenomena in strongly disordered magnets with competing interactions. Of course, it is still a valid question to ask "what happens if we change the model?" This question will be followed up in the next subsection. A comparative study of various models will in fact help to better identify the essential ingredients any reasonable model of a spin glass must have.

E. Site-disorder models

There are three main reasons why one might like to consider site-disorder models rather than the (bond-disordered) Edwards-Anderson model introduced in the previous subsection.

(i) Most real spin glasses (Sec. II) are simple dilution systems in which a fraction of magnetic atoms is randomly replaced by nonmagnetic ones. In such a case, a truly realistic model must be a site-disorder model.

(ii) Since it turns out that even the mean-field theory for the Edwards-Anderson model is so difficult (Sec. IV), it is legitimate to ask whether one may find a still simpler model which nevertheless contains the essential physics of a spin glass. It was with this hope in mind that the Mattis spin glass (Mattis, 1976) and variants thereof were invented. Although some of these models are indeed more easily solved, we shall see below that important features of spin glass behavior are lost.

(iii) Experience with phase transition problems (e.g., Fisher 1974a) leads us to ask whether there are different "universality classes" of spin glasses, and if so, which parameters classify them. For instance, in the percolation problem, where one introduces random quenched disorder in a ferromagnet with noncompeting interactions, one may do this in two ways: either one randomly removes the magnetic sites (site percolation) or one removes the bonds between them (bond percolation) until at the percolation threshold long-range magnetic order even in the ground state breaks down (Stauffer, 1979; Essam, 1980; Deutscher *et al.*, 1983). In this case with respect to critical exponents at the percolation threshold it does not matter whether one has site or bond disorder: both cases belong to the same universality class. "Directed percolation" (Deutscher *et al.*, 1983), on the other hand, belongs to a different universality class.

We now consider these various aspects of spin glass models.

1. Mattis-Luttinger models and their generalizations

The spin glass model of Mattis (1976) is of the form (3.27), but for the random exchange J_{ij}, instead of Eq. (3.35) or (3.38), one chooses the form

$$J_{ij}=J(\mathbf{R}_i-\mathbf{R}_j)\varepsilon_i\varepsilon_j \quad (\varepsilon_i=\pm 1 \text{ random}) , \tag{3.39}$$

where $J(\mathbf{R}_1-\mathbf{R}_j)$ is a nonrandom exchange depending on relative distance between the sites i,j only. Defining a new set of pseudospins τ_i by

$$\tau_i=\varepsilon_i\mathbf{S}_i , \tag{3.40a}$$

we transform Eq. (3.27) to

$$\mathscr{H}=-\tfrac{1}{2}\sum_{ij}J(\mathbf{R}_i-\mathbf{R}_j)\tau_i\cdot\tau_j-H\sum_i\varepsilon_i\tau_i^z . \tag{3.40b}$$

For zero magnetic field, this is the Hamiltonian of the m-vector model without any disorder. Thus information about the critical behavior of that model (e.g., Fisher, 1974a) immediately can be inferred: for short-range interactions, the model has a phase transition for all dimensions $d>d_l$ at a finite critical temperature $T_c>0$, while $T_c=0$ for $d<d_l$. This lower critical dimensionality is $d_l=1$ for $m=1$ (Ising spins), while $d_l=2$ for $m\geq 2$. Right at d_l, $T_c=0$, as well, except for the case of $m=2$, where a phase transition occurs at nonzero T_c to a state without long-range order but with an algebraic decay of correlation functions (Kosterlitz and Thouless, 1973). For $d>d_l$ we have standard critical behavior described by exponents $\alpha_m,\beta_m,\gamma_m,\nu_m$, etc., which are defined in terms of the specific heat C, order parameter M_τ, susceptibility χ_τ, and correlation length ξ_τ (Aharony and Imry, 1976, 1977a; Bidaux *et al.*, 1976),

$$\begin{aligned} &C\propto|1-T/T_c|^{-\alpha_m} , \\ &M_\tau\equiv\sum_i\tau_i^z/N=\sum_i\tau_i\varepsilon_iS_i^z/N\propto(1-T/T_c)^{\beta_m} , \end{aligned} \tag{3.41a}$$

$$\chi_\tau\propto|1-T/T_c|^{-\gamma_m} , \quad \xi_\tau\propto|1-T/T_c|^{-\nu_m} . \tag{3.41b}$$

It must be emphasized that χ_τ is the response of M_τ with respect to a staggered field $H_\tau^i=H\varepsilon_i$, rather than the physical susceptibility. Similarly, ξ_τ measures the range of the pseudospin correlation $\langle\tau_i\cdot\tau_j\rangle_T$ rather than that of the spin correlation, which trivially is (for classical spins of unit length)

$$\begin{aligned} [\langle\mathbf{S}_i\cdot\mathbf{S}_j\rangle_T]_{\text{av}}&=\langle\tau_i\cdot\tau_i\rangle_T[\varepsilon_i\cdot\varepsilon_j]_{\text{av}} \\ &=\langle\tau_i\cdot\tau_j\rangle\delta_{ij}=\delta_{ij} . \end{aligned} \tag{3.42a}$$

Clearly, the standard singularity of the specific heat of the m-vector model [Eq. (3.41)] is not a desirable property of any spin glass model; this model is a kind of spin glass simply because its phase transition does not lead to the onset of magnetization but rather to an order of the kind proposed by Edwards and Anderson (1975),

$$M=[\langle S_i^z\rangle_T]_{\text{av}}=\langle\tau_i^z\rangle_T[\varepsilon_i]_{\text{av}}=0 , \tag{3.42b}$$

$$q=[\langle S_i^z\rangle_T^2]_{\rm av}=\langle\tau_i^z\rangle_T^2[\varepsilon_i^2]_{\rm av}$$
$$=\langle\tau_i^z\rangle_T^2=M_\tau^2\propto(1-T/T_c)^{2\beta_m}\ . \qquad (3.42c)$$

Due to the onset of this order parameter q, the standard susceptibility also has a cusp,

$$\chi(H=0)=\frac{1}{T}\sum_i[\langle S_i^zS_j^z\rangle-\langle S_i^z\rangle_T\langle S_j^z\rangle_T]_{\rm av}$$
$$=\tfrac{1}{3}(1-q)/T\ , \qquad (3.42d)$$

in agreement with a relation between $\chi(H=0)$ and q first discussed by Fischer (1976). The singular behavior of this cusp (perpendicular slope for $T<T_c$, since $\beta_m<\frac{1}{2}$ for $d<d_u=4$) clearly is not in agreement with experiment; moreover Monte Carlo simulations of dynamic versions of this model (Bray *et al.*, 1978; Stauffer and Binder, 1978) show that this model does not exhibit the slow relaxation behavior characteristic of spin glasses, in contrast to simulations of corresponding Edwards-Anderson models (Binder and Schröder, 1976a, 1976b). It is now generally agreed that the Mattis model is not a reasonable model of a spin glass—due to the transformation to Eq. (3.40a) it really is a ferromagnet in disguise, and q in Eq. (3.42c) simply is a "secondary" order parameter to the "primary" order parameter M_τ in Eq. (3.41a).

One may consider similar models, however, where the disorder cannot be "gauged away" completely by the transformation, Eq. (3.40a). Consider a model for a quenched magnetic alloy of the form A_xB_{1-x}, both kinds of atoms being magnetic, with exchange integrals J_{AA},J_{BB},J_{AB} different from each other (Aharony, 1975; Aharony and Imry, 1976). The Hamiltonian of this model is again of the form of Eq. (3.27), but now the exchange J_{ij} is given by

$$J_{ij}=J_{AA}x_ix_j+J_{AB}[x_i(1-x_j)+x_j(1-x_i)]$$
$$+J_{BB}(1-x_i)(1-x_j)\ , \qquad (3.43a)$$

where $x_i=1$ if the site i is taken by an A atom, $x_i=0$ otherwise. This equation with $\varepsilon_i=2x_i-1$, is transformed into

$$J_{ij}=\tfrac{1}{4}[J_{AA}+2J_{AB}+J_{BB}+(J_{AA}-J_{BB})(\varepsilon_i+\varepsilon_j)$$
$$+(J_{AA}-2J_{AB}+J_{BB})\varepsilon_i\varepsilon_j]\ . \qquad (3.43b)$$

For $J_{AA}=J_{BB}=-J_{AB}=J$ and $x=\frac{1}{2}$, Eq. (3.43) thus reduces to the Mattis model, while in the general case the disorder in the interactions cannot be removed. But exact information on this model is not available, and apart from Monte Carlo simulations (Tatsumi, 1977, 1978a, 1978b) it has received little attention.

Another site-disorder model that can be solved exactly was proposed by Luttinger (1976). He assumes Eq. (3.27) for $n=1$ (Ising spins), with J_{ij} given by

$$J_{ij}=\frac{1}{N}[J_1+J_2\varepsilon_i\varepsilon_j+J_3(\varepsilon_i+\varepsilon_j)]\ ,$$
$$\varepsilon_i=\pm1 \text{ random}\ , \qquad (3.44)$$

where J_1,J_2,J_3 are constants, and this interaction is chosen for any pair of spins irrespective of distance, and therefore scaled inversely with the number of spins. For $J_1=J_3=0$ this is the mean-field version of the Mattis model, while otherwise it is the mean-field version of the above magnetic alloy model, Eq. (3.43), for $x=\frac{1}{2}$. A spin glass transition with a cusp of the susceptibility is found, and although disorder cannot be eliminated, one can calculate the quenched free energy exactly, so there is no need for the use of the replica trick.

Extension and variations of such models have been studied by various authors (Sherrington, 1976; Sherrington and Fernandez, 1977; Fernandez and Sherrington, 1978; van Hemmen, 1982, 1983; Provost and Vallee, 1983; van Hemmen *et al.*, 1983; Choy and Sherrington, 1984). The model of van Hemmen (1982, 1983) is defined by

$$J_{ij}=[J_1+J_2(\xi_i\eta_j+\xi_j\eta_i)]/N\ , \qquad (3.45)$$

where J_1 and J_2 are constants and the $\{\xi_i,\eta_i\}$ independent, identically distributed random numbers with an even distribution around zero and variance unity. Detailed studies (van Hemmen *et al.*, 1983) reveal that this model exhibits both a spin glass phase and a mixed spin-glass–ferromagnetic phase. The variation of the specific heat with temperature is found to be linear at low temperatures. The zero-field susceptibility has a flat plateau below the freezing temperature. Thus van Hemmen (1982, 1983) and van Hemmen *et al.* (1983) proposed that this mean-field model is a satisfactory alternative to the SK model. Choy and Sherrington (1984) demonstrate, however, that this model lacks a great multiplicity of metastable states, and that its "Glauber dynamics" have essentially conventional exponential decay to equilibrium away from the transition temperature, where critical slowing down occurs normally. Monte Carlo results of Morgenstern and van Hemmen (1985) confirm these findings and show that the ground state is only twofold degenerate. Despite having randomness and frustration, the system actually picks out a Mattis state and therefore lacks some features (such as a nontrivial ground state) that are generally considered integral to a true spin glass. Therefore we have omitted any more quantitative description of the results, as well as of the elegant mathematics by which they are derived, and rather refer the reader to the original literature for details. We feel, however, that the *finite-range* van Hemmen model, which does not form a Mattis state, would be a good spin glass model, and it would be interesting to check whether it has a similar transition to the one in the Edwards-Anderson model. We should also note that the van Hemmen model is very closely related to a model of neutral networks (Hopfield, 1982) that has aroused considerable interest.

2. Realistic models

If one wishes to study a nearly realistic model of a metallic spin glass such as *Au*Fe or *Cu*Mn, one may choose a fcc lattice and put classical spins randomly onto this

lattice with a prescribed concentration x. One introduces interactions between the spins by choosing the RKKY interaction, Eq. (2.1), with plausible assumptions for the constants involved. Systems in which isolated magnetic moments are suppressed by the Kondo effect but moments are stabilized by interactions with increasing concentration, so-called "Stoner glasses" (Hertz, 1979), will not be discussed in this review. Sometimes one may also wish to study the effects of anisotropy, usually assumed to be either pseudodipolar [Eq. (2.4)] or of the Dzyaloshinskii-Moriya type [Eq. (2.6)].

Clearly, a detailed investigation of such models is of great physical interest; in fact, the earliest mean-field approaches of Marshall (1960), Klein and Brout (1963), and Klein (1964) attempted to study this model, though the possibility of a spin glass transition was only realized (Riess and Klein, 1977) after Edwards and Anderson. Unfortunately, one can hardly obtain any reliable result on such a model by analytical methods: systematic expansion in powers of inverse temperature (Larkin and Khmelnitskii, 1970) or concentration (Larkin and Khmelnitskii, 1970; Matho, 1978, 1979; Owen, 1983) are limited to a few leading orders. Thus most information on the physical properties of such systems comes from numerical simulations (see Sec. VI.C). At this point, we only mention that the concentration expansion (also called "virial expansion") of any thermodynamic quantity $A(T,x)$ can be cast in the form (e.g., Owen, 1983)

$$A(T,x)=\sum_{n=1}^{\infty}\alpha(T)(x/T^*)^n\,, \tag{3.46a}$$

where $\alpha_n(T)$ are expansion coefficients and T^* is a reduced temperature defined by [cf. Eq. (2.1) where $\varphi_0=0$]

$$T^*=k_BT(k_Fa)^3/J_0s(s+1)\,, \tag{3.46b}$$

a being the lattice spacing. If $\alpha_n(T)$ were independent of T, Eq. (3.46) would imply the concentration scaling proposed by Souletie and Tournier (1969, 1971); actual calculation of the first two nontrivial terms in the virial expansion shows a temperature dependence of $\alpha(T)$ for the susceptibility in reasonable agreement with experiment.

Concentration scaling would be obtained if the $\cos(2k_FR)$ term in Eq. (2.1) led to a random distribution of bond strengths centered on zero. From this consideration it is clear that it is of interest to consider Edwards-Anderson models [Eq. (3.27)] in which the distribution of bond strengths is (Ueno and Okamoto, 1981; Kotliar *et al.*, 1983)

$$P(J_{ij})\propto\exp\{-J_{ij}^2/2[\Delta J(R_i-R_j)]^2\}\,,$$
$$\Delta J(R_i-R_j)\propto|\mathbf{R}_i-\mathbf{R}_j|^{-\sigma}\,, \tag{3.47}$$

where σ is an arbitrarily introduced exponent: for $\sigma=0$, one recovers the SK model, while for $\sigma=d$ Eq. (3.47) is an approximation to RKKY interactions. With this model one can also show for $\sigma>d/2$ that the free energy is self-averaging (van Enter and van Hemmen, 1983).

As a simple site-disorder model with possible application to insulating spin glasses such as $Eu_xSr_{1-x}S$, a model with competition between a nearest-neighbor ferromagnetic interaction J_1 and an antiferromagnetic next-nearest-neighbor interaction J_2 has been investigated (Binder *et al.*, 1979; Kinzel and Binder, 1981). Particularly simple is the Ising case on the square lattice, where one can analyze the ground state in terms of expansions in x [obtained through order x^9 (Binder *et al.*, 1979)] or $1-x$ [obtained through order $(1-x^5$]. In this model one can study the competition between ferromagnetic order and spin glass behavior and thus address the question of temperature-concentration phase diagrams.

In this model it is also easy to see how the combined effects of disorder and competition between interactions can lead to many local energy minima. For this purpose, it suffices to consider the ground state of small clusters of spins (Fig. 38). Depending on the geometrical configurations of the spins, some clusters are rigidly locked together [Figs. 38(a) and 38(b)], i.e., they have just a twofold ground-state degeneracy as in a case of uniform ferromagnetic interaction, while others [Figs. 38(c)—38(e)] have a higher ground-state degeneracy. While the cluster of Fig. 38(c) is degenerate between two states ($C1$,$C2$) only for the particular value $J_2/J_1=-\frac{1}{2}$ [in the pure system ($x=1$) the ground state changes from a ferromagnetic to

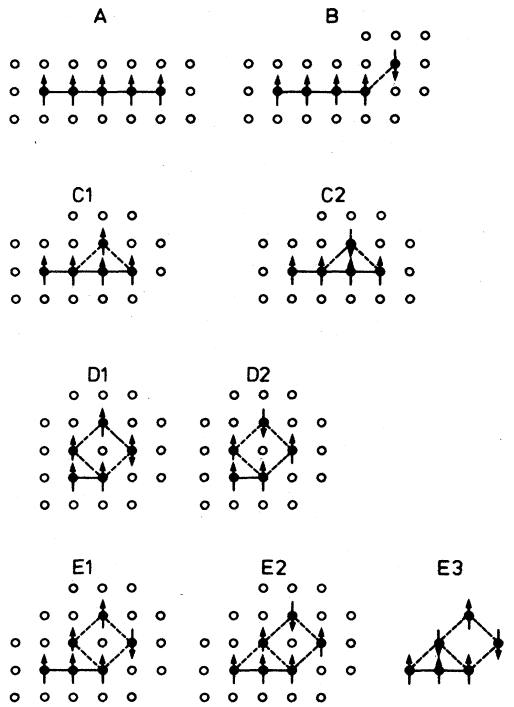

FIG. 38. Ground-state configurations of various clusters of magnetic atoms (solid dots), for a model with ferromagnetic nearest-neighbor bonds (solid lines) and antiferromagnetic next-nearest-neighbor bonds (dashed lines). Nonmagnetic atoms are shown as open circles. For further explanation, see text. From Binder *et al.* (1979).

a layered antiferromagnetic structure], the cluster of Fig. 38(d) has a sixfold degenerate ground state (configurations $D1$, $D2$, $D3$, and overall reversal of the spins) in the whole range $-\frac{1}{2} < J_2/J_1 < 0$: the ferromagnetic bonds will align the three spins in the left lower corner of this cluster parallel to each other, irrespective of the configuration of the remaining spins, which are coupled to this cluster only by antiferromagnetic bonds. Since it is impossible to satisfy the three antiferromagnetic bonds at the same time, there are three degenerate spin configurations, $D1, D2, D3$. This is an example of the "frustration" effect (Toulouse, 1977), which will be discussed in more detail in the next subsection.

While for clusters containing up to five spins there is only the special value $J_2/J_1 = -\frac{1}{2}$ where an enhanced degeneracy occurs, larger clusters exhibit enhanced degeneracy for other rational values of the ratio J_2/J_1. For instance, the cluster of six spins shown in Fig. 38(e) has a degenerate groundstate ($E1$, $E2$, and another configuration not shown) for $J_2/J_1 > -\frac{1}{3}$, while for $J_2/J_1 < \frac{1}{3}$ there is only one nondegenerate ground state configuration ($E3$), and for $J_2/J_1 = -\frac{1}{3}$ all these states are degenerate.

Similar considerations apply in the ferromagnetic state near $x = 1$: near dilution sites some spins may be aligned opposite to the direction of the spontaneous magnetization [Figs. 39(e)], and degenerate configurations occur as well (Binder *et al.*, 1979). Thus the spins bound to the percolating ferromagnetic cluster produce a state with a finite ground-state entropy. At the same time, one can understand qualitatively that ferromagnetic order may break down before one reaches the percolation threshold (see Sec. V.C).

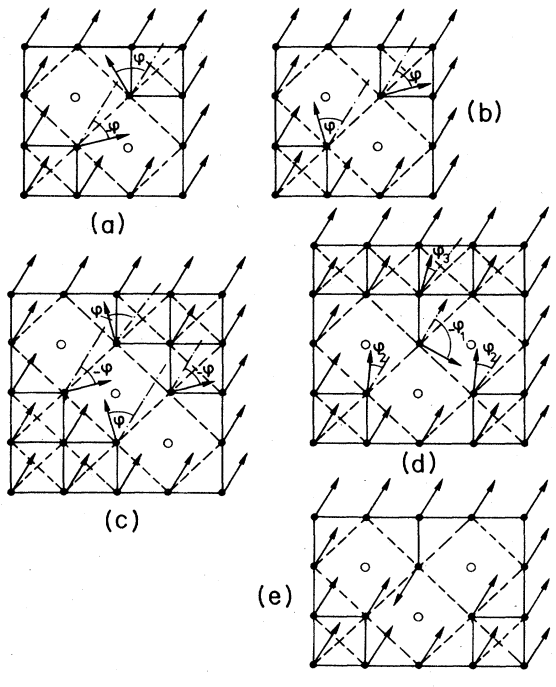

FIG. 39. Ground-state spin configurations near clusters of nonmagnetic atoms (indicated as open circles) in the XY case (a)—(d) and Ising case (e), for a model with nearest-neighbor ferromagnetic exchange (full bonds) and antiferromagnetic next-nearest-neighbor exchange (dashed bonds). From Kinzel and Binder (1981).

It must be emphasized that a somewhat different picture emerges if one considers continuous spins (XY or Heisenberg model) rather than the Ising case: then the spin configurations of clusters containing competing ferro- and antiferromagnetic bonds become noncollinear (Villain, 1979a; Kinzel and Binder, 1981). The cluster of Fig. 38(d) in the XY case has two degenerate states (in addition to the symmetry of a homogeneous rotation of all the spins). Near the ferromagnetic state $x = 1$, the situation is much more complicated: If one considers clusters of dilution sites [Figs. 39(a)—39(c)], again the ferromagnetic state is distorted because spins close to the dilution site are no longer aligned parallel to the magnetization direction. If one restricted this distortion to the sites closest to the defect, such as in Figs. 39(a) and 39(b), two spins would be misaligned by an angle $\pm\varphi$, with $\cos\varphi = \frac{3}{2} + J_1/J_2$, for $J_2/J_1 < -\frac{2}{5}$. This state is twofold degenerate [Figs. 39(a) and 39(b)]. In reality there will be a distortion of the long-range magnetization (Dunlop and Sherrington, 1985), where the deviations φ from the ferromagnetic axis decay with some inverse power of the distance from the defect. Due to the interaction of these spin deviation fields, even the description of the weakly diluted ferromagnet in the case of competing interactions is a difficult problem.

In any case it is clear that an isolated defect like that shown in Figs. 39(a) and 39(b) must lead to two equivalent states, i.e., it creates a "two-level system." Excitations in which the system changes from one state of a two-level system to the other are believed to be very important for structural glasses and possibly for spin glasses as well (Anderson *et al.*, 1972; W. A. Phillips, 1972). We shall explore the physical consequences of two-level systems in more detail in Sec. III.G.

For a description of the statistical mechanics of disordered isotropic magnets at low temperatures, one thus needs a variable [which Villain (1977b) called "chirality"] distinguishing states like those shown in Figs. 39(a) and 39(b), in addition to the degrees of freedom describing weak thermal spin misalignments (spin waves). We shall give a more precise definition of this additional degree of freedom in the next section.

It is clear, however, that a description of real systems such as $Eu_xSr_{1-x}Sr$ requires the use of three-dimensional lattices (fcc) and Heisenberg spins. Very little analytic information exists for this case; systematic high-temperature series expansions exist through fourth order only in inverse temperature (Binder *et al.*, 1979) and thus clearly are unsatisfactory. As in the case of RKKY spin glasses, most information on this model relies on Monte Carlo simulations (Sec. VI.C). The same remark holds for a similar model on the bcc model proposed to describe $Fe_{1-x}Al_x$ alloys (Grest, 1980).

3. Other models: random-energy model, disordered antiferromagnets, etc.

In this subsection we briefly mention a variety of models that have been discussed in the context of spin glasses.

Gross and Mézard (1984) studied a generalization of the SK model to include p-spin interactions, introduced by Derrida (1980),

$$\mathcal{H}=-\sum_{1\le i_1<i_2<\cdots<i_p\le N} J_{i_1 i_2\cdots i_p} S_{i_1} S_{i_2}\cdots S_{i_p} , \tag{3.48a}$$

where the exchange constants $J_{i_1 i_2\cdots i_p}$ are drawn from the distribution

$$P(J_{i_1 i_2\cdots i_p})=\left[\frac{N^{p-1}}{\pi p!}\right]^{1/2}\exp\left[-\frac{(J_{i_1 i_2\cdots i_p})^2}{(\Delta J)^2 p!}N^{p-1}\right] . \tag{3.48b}$$

For $p\to\infty$ this model becomes equivalent to the random-energy model of Derrida (1980, 1981), whose energy levels E also satisfy a Gaussian distribution,

$$P(E)=[2\pi N(\Delta J)^2]^{-1/2}\exp[-E^2/2N(\Delta J)^2] , \tag{3.48c}$$

and are statistically independent of each other. The statistical mechanics of this model is easily treated by a direct calculation (Derrida, 1980, 1981); at the same time it is a convenient testing ground for concepts such as replica symmetry breaking (Gross and Mézard, 1984). More details about the properties of this model will be given in Sec. IV.

Experiments on doped semiconductors, such as In in CdS (Kummer *et al.*, 1978; Walstedt *et al.*, 1979) and Si doped with P (Andres *et al.*, 1981), have led to the study of models such as Eq. (3.27) for both the Ising and the Heisenberg case in which the interactions are random in strength but always antiferromagnetic (Bhatt and Rice, 1980; Khanna and Sherrington, 1980; Rosso, 1980; Bhatt and Lee, 1981, 1982; McLenaghan and Sherrington, 1984). While such models may still exhibit antiferromagnetic order when the disorder is weak (McLenaghan and Sherrington, 1984), for stronger disorder antiferromagnetism is suppressed. In contrast to early expectations (Kummer *et al.*, 1978), there is no evidence of spin glass behavior for these models. On the other hand, diluted "frustrated antiferromagnets" (such as the nearest-neighbor triangular or fcc Ising antiferromagnets) are expected to have spin glass states, although they too have no ferromagnetic bonds (De Seze, 1977; Aharony, 1978; Grest and Gabl, 1979; Villain, 1979a; Aharony and Binder, 1980; Anderico *et al.*, 1982; Fernandez *et al.*, 1982; Shnidman and Mukamel, 1984). In the case of a diluted classical Heisenberg fcc antiferromagnetic, no clear evidence of spin glass behavior was obtained; see, however, Ching and Huber (1981). A discussion of possible experimental realizations of related models can be found in Villain (1979a). A possible nonmagnetic experimental realization of diluted triangular antiferromagnets would be argon-diluted nitrogen films adsorbed on graphite (Migone *et al.*, 1983; Shnidman and Mukamel, 1984).

A rather different model studied recently is a two-dimensional XY ferromagnet with random Dzyaloshinskii-Moriya interaction (Rubinstein *et al.*, 1983). For large amounts of randomness, the low-temperature phase of the XY model is destroyed entirely. For small amounts of randomness, the behavior with decreasing temperatures is first paramagnetic, then ferromagnetic, and finally paramagnetic again via a second, reentrant phase transition. These phase transitions are driven by an unbinding of vortices just as in pure XY models (Kosterlitz and Thouless, 1973). There is no evidence for a spin glass phase at finite temperature.

4. Phenomenological models of the Ginzburg-Landau type

For problems in the theory of critical phenomena it is standard wisdom to start from a Ginzburg-Landau-Wilson effective Hamiltonian (Stanley, 1971; Fisher, 1974a). It is very natural to attempt such an approach for spin glasses as well.

The first attack along these lines is due to Harris *et al.* (1976) and Chen and Lubensky (1977), with the motivation of providing renormalization-group expansions (see Sec. V.A.1). One starts within the framework of the replica method, i.e., the Hamiltonian Eq. (3.36), which has to be treated in the limit $n\to 0$. Rather than solving this problem in the case of infinite-range interactions, which will be the task considered in Sec. IV, one wants to develop a continuum theory, in which the partition function Z is expressed in terms of order-parameter densities of the magnetization M, staggered magnetization $\tilde{M}$, and spin glass order parameter q:

$$Z=\int d\{M\}d\{\tilde{M}\}d\{q\}\exp(-\mathcal{F}[M,\tilde{M},q]/k_B T) , \tag{3.49}$$

where the symbols $d\{\ \}$ indicate functional integration, and the Ginzburg-Landau-Wilson Hamiltonian, $\mathcal{F}$, is constructed from the replica method, assuming that short-wavelength fluctuations of the fields $M,\tilde{M},q$ can be averaged over. One defines a local analog of $q_{\alpha\beta}$ [Eq. (3.31)] by

$$q^{kl}_{\alpha\beta,i}=S^{\alpha}_{i,k}S^{\beta}_{i,l}(1-\delta_{\alpha\beta}) , \tag{3.50}$$

where $S^{\alpha}_{i,k}$ is the component of spin $\mathbf{S}^{\alpha}_i$ in the k direction, and $\delta_{\alpha\beta}$ is the Kronecker symbol. It is then argued from symmetry considerations that the phenomenological Hamiltonian of a spin glass should take the following form (Chen and Lubensky, 1977):

$$\mathcal{F}=\mathcal{F}_M+\mathcal{F}_{\tilde{M}}+\mathcal{F}_{Mq}+\mathcal{F}_{\tilde{M}q}+\mathcal{F}_q+\cdots , \tag{3.51}$$

where

$$\mathcal{F}_M/k_BT=\tfrac{1}{2}\int d^dr\left[r_M\sum_\alpha \mathbf{M}^\alpha(\mathbf{r})\cdot\mathbf{M}^\alpha(\mathbf{r})+\sum_\alpha\sum_{kl}\nabla_k M_l^\alpha(\mathbf{r})\cdot\nabla_k M_l^\alpha(\mathbf{r})\right.$$
$$\left.+2u_1\left[\sum_\alpha \mathbf{M}^\alpha(\mathbf{r})\cdot\mathbf{M}^\alpha(\mathbf{r})\right]^2+2v_1\sum_\alpha[\mathbf{M}^\alpha(\mathbf{r})\cdot\mathbf{M}^\alpha(\mathbf{r})]^2\right], \tag{3.52a}$$

$$\mathcal{F}_{\tilde{M}}/k_BT=\tfrac{1}{2}\int d^dr\left[r_{\tilde{M}}\sum_\alpha \tilde{\mathbf{M}}^\alpha(\mathbf{r})\cdot\tilde{\mathbf{M}}^\alpha(\mathbf{r})+\sum_\alpha\sum_{kl}\nabla_k\tilde{M}_l^\alpha(\mathbf{r})\nabla_k\tilde{M}_l^\alpha(\mathbf{r})\right.$$
$$\left.+2u_2\left[\sum_\alpha \tilde{\mathbf{M}}^\alpha(\mathbf{r})\cdot\tilde{\mathbf{M}}^\alpha(\mathbf{r})\right]^2+2v_2\sum_\alpha[\tilde{\mathbf{M}}^\alpha(\mathbf{r})\cdot\tilde{\mathbf{M}}^\alpha(\mathbf{r})]^2\right], \tag{3.52b}$$

$$\mathcal{F}_q/k_BT=\tfrac{1}{4}\int d^dr\left[r_q\sum_{\alpha\beta}\sum_{kl}q_{\alpha\beta}^{kl}(\mathbf{r})q_{\alpha\beta}^{kl}(\mathbf{r})+\sum_{klm}\nabla_k q_{\alpha\beta}^{lm}\nabla_k q_{\alpha\beta}^{lm}+4w_q\sum_{\alpha\beta\gamma}\sum_{klm}q_{\alpha\beta}^{kl}(\mathbf{r})q_{\beta\gamma}^{lm}(\mathbf{r})q_{\gamma\alpha}^{mk}(\mathbf{r})\right], \tag{3.53a}$$

$$\mathcal{F}_{Mq}/k_BT=w_1\int d^dr\sum_{\alpha\beta}\sum_{kl}q_{\alpha\beta}^{kl}(\mathbf{r})M_k^\alpha(\mathbf{r})M_l^\beta(\mathbf{r}), \tag{3.53b}$$

$$\mathcal{F}_{\tilde{M}q}/k_BT=w_2\int d^dr\sum_{\alpha\beta}\sum_{kl}q_{\alpha\beta}^{kl}(\mathbf{r})\tilde{M}_k^\alpha(\mathbf{r})\tilde{M}_l^\beta(\mathbf{r}). \tag{3.53c}$$

Here $r_M, r_{\tilde{M}}, r_q, u_1, u_2, v_1, v_2, w, w_1, w_2$ are phenomenological coefficients. The first three of these may change sign as a function of temperature,

$$r_M=r_M'(T-T_c),\quad r_{\tilde{M}}=r_{\tilde{M}}'(T+T_c),\quad r_q=r_q'(T-T_f); \tag{3.54}$$

one expects from the microscopic replica Hamiltonian [such as Eq. (3.36)] that $T_c\propto\bar{J}$ (which may be positive or negative), while $T_f\propto\Delta J$. The other coefficients $u_1,\ldots,$ are assumed to be positive [otherwise one would need to carry this expansion of Eqs. (3.52) and (3.53) to higher order]. If we consider a lattice, which may be divided into two sublattices (e.g., simple cubic), then the three-energy function is invariant under the transformation $[\bar{J}\to-\bar{J}(T_c\to-T_c), M\to\tilde{M}]$. This symmetry says that the phase diagram in the plane of variables $T/\Delta J, \bar{J}/\Delta J$ is symmetric around the line $\bar{J}=0$, with a ferromagnetic phase at sufficiently large positive $\bar{J}$ and a corresponding antiferromagnetic phase at sufficiently large negative $\bar{J}$. In between these conventionally ordered phases, we expect a spin glass phase, at least at high enough dimensionalities d. Because of the above symmetry, it suffices to study the ordering for $\bar{J}>0$ only, considering only the terms $\mathcal{F}_M+\mathcal{F}_q+\mathcal{F}_{Mq}$ in the above expansion.

In studying transitions with both ferromagnetic and spin glass order parameters, one has to allow for the fact that a nonzero magnetization leads to a nonzero $q_{\alpha\beta}^{kl}$. In an ordered ferromagnetic state, the magnetization in each replica should be identical. Therefore Chen and Lubensky (1977) write $M_k^\alpha=Me_k$ where $\{e_k\}$ is an m-component unit vector, and decompose $q_{\alpha\beta}^{kl}$ into parallel and perpendicular components $q_{||}, q_\perp$ as follows:

$$q_{\alpha\beta}^{kl}=[q_\perp(\delta_{kl}-e_ke_l)+q_{||}e_ke_l](1-\delta_{\alpha\beta}). \tag{3.55}$$

In this step, Chen and Lubensky (1977) have ignored the possibility of replica symmetry breaking. In the mean-field limit, their treatment hence simply reduces to an extension of the replica-symmetric theory of Sherrington and Kirkpatrick (1975; see Sec. IV). For completeness, we just mention that mean-field theory can be derived from Eqs. (3.52) and (3.53) simply by considering a spatially uniform state, from which the gradient terms can be omitted, and the spatial integration simply yields the volume V,

$$\frac{\mathcal{F}}{nk_BTV}=\tfrac{1}{2}r_MM^2+(nu_1+v_1)M^4-HM-w_1q_{||}M^2$$
$$+(n-1)\{\tfrac{1}{4}r_q[(m-1)q_\perp^2+q_{||}^2]$$
$$-w(n-2)[(m-1)q_\perp^3+q_{||}^3]\}, \tag{3.56}$$

where all terms of order q^4 have been omitted and an external magnetic field added. Equation (3.56) was also proposed by Suzuki (1977) for the special case of the Ising spin glass in which $m=1$ so $q_\perp$ does not exist and $q\equiv q_{||}$. We defer further discussion of Eq. (3.56) to Sec. IV.

Obviously, the approach outlined here relies heavily on the replica method and thus shares all its difficulties. Therefore it is tempting to try to bypass these problems by constructing a coarse-grained Ginzburg-Landau-Wilson Hamiltonian of the disordered magnetic system directly.

The first approach along such lines was the random-temperature Ginzburg-Landau model of Ma and Rudnick (1978); instead of Eq. (3.51) they considered the much simpler functional

$$\mathcal{F}/k_BT=\frac{1}{2}\int d^dr\left[r_M(\mathbf{r})\mathbf{M}(\mathbf{r})\cdot\mathbf{M}(\mathbf{r})\right.$$
$$+\sum_{kl}\nabla_kM_l(\mathbf{r})\nabla_kM_l(\mathbf{r})$$
$$\left.+\tfrac{1}{2}u_1[\mathbf{M}(\mathbf{r})\cdot\mathbf{M}(\mathbf{r})]^2\right], \tag{3.57}$$

where now the coefficient $r_M(\mathbf{r})$ is a random variable. While Ma and Rudnick (1978) suggested that the model Eq. (3.57) leads to spin glass order, it now is accepted (Sherrington, 1980) that Eq. (3.57) can describe only an impure ferromagnet, and not a spin glass. A generalization that may describe spin glass order, however, is

$$\mathcal{F}/k_BT=\frac{1}{2}\int d^dr\left[r_M(\mathbf{r})\mathbf{M}(\mathbf{r})\cdot\mathbf{M}(\mathbf{r})+\tfrac{1}{2}u_1(\mathbf{r})[\mathbf{M}(\mathbf{r})\cdot\mathbf{M}(\mathbf{r})]^2+\int d^dr'\mathbf{M}(\mathbf{r})\cdot\chi(\mathbf{r}-\mathbf{r}')\mathbf{M}(\mathbf{r}')\right], \tag{3.58}$$

where $\chi(\mathbf{r}-\mathbf{r}')$ is a fixed function of the separation but *not* restricted to be ferromagnetic (Sherrington, 1980, 1981a, 1981b). If $\chi(\mathbf{r}-\mathbf{r}')$ were ferromagnetic, a gradient expansion of $\mathbf{M}(\mathbf{r}')$ at $\mathbf{r}'=\mathbf{r}$ would again yield the Ma-Rudnick model, Eq. (3.57). However, various macroscopic spin glass models are expected to reduce to Eq. (3.58) where $\chi(\mathbf{r}-\mathbf{r}')$ for some distances is antiferromagnetic.

A related model was proposed by Hertz (1978), for Heisenberg spins,

$$\mathcal{F}/k_BT=\tfrac{1}{2}\int d^dr(r_M\mathbf{M}(\mathbf{r})\cdot\mathbf{M}(\mathbf{r})+\tfrac{1}{2}u_1[\mathbf{M}(\mathbf{r})\cdot\mathbf{M}(\mathbf{r})]^2+\{[\nabla_k-i\mathbf{Q}_k(\mathbf{r})\cdot\mathbf{L}]\mathbf{M}(\mathbf{r})\}^2), \tag{3.59}$$

where the components L_i of $\mathbf{L}$ are the 3×3 matrix representations of angular momenta which generate rotations about the three spin axes. The random vectors $\mathbf{Q}_k(\mathbf{r})$ describe the situation in which the kind of spin alignment (locally ferromagnetic or antiferromagnetic) randomly changes from place to place in the system. For XY spins that can be described by a complex scalar magnetization $M(\mathbf{r})$, a simpler expression is

$$\mathcal{F}/k_BT=\tfrac{1}{2}\int d^dr\{r_M|M(\mathbf{r})|^2+\tfrac{1}{2}u_1|M(\mathbf{r})|^4+|[\nabla-i\mathbf{Q}(\mathbf{r})]M(\mathbf{r})|^2\}, \tag{3.60}$$

$\mathbf{Q}(\mathbf{r})$ being the wave vector of the lowest-energy spin-density wave into which the spins condense below the ordering temperature. Evidently, the spin glass order in these models is perceived as a generalization of an antiferromagnetic structure, in which $\mathbf{Q}$ would be nonrandom and describe the position in reciprocal space where the magnetic superstructure Bragg peaks occur. The spin glass order in this model hence is described as a randomly distorted antiferromagnet. Of course, in the ideal case no particular value of $\mathbf{Q}$ is preferred.

It is not clear, however, whether models of this type really contain all the essential features of spin glasses, such as the existence of many equivalent orderings that are accidentally degenerate, or at least nearly so (Fig. 4). But, as will be discussed below, they provide a starting point for gauge theories of spin glasses (Sec. V.A).

F. Spin glass order parameters and correlations, frustration, and gauge invariance

1. Order parameters

In the section on the Mattis spin glass we have seen that a susceptibility cusp can arise [Eq. (3.42)] at a phase transition where the order parameter $q=[\langle S_i\rangle_T^2]_{\rm av}$ becomes nonzero. We now wish to discuss this, and related order parameters, in greater detail. Except for the end of this section we shall discuss Ising spins.

First we note that the canonical Gibbs average must be evaluated with a field $\tilde{H}$, which breaks the spin-reversal symmetry ($S_i\Longrightarrow -S_i$ for all i) of the Hamiltonian. This could be a completely random staggered field or a uniform field, and is allowed to tend to zero *after* the thermodynamic limit has been taken (see, for example, Young and Kirkpatrick, 1982; Young and Jain, 1983). As discussed in Sec. III.B, the symmetry-breaking field does not necessarily single out one particular state (valley), so the Gibbs average still includes a weighted sum over valleys (De Dominicis and Young, 1983a). Hence we define the statistical mechanics order parameter by

$$q=\lim_{\tilde{H}\to0}\lim_{N\to\infty}[\langle S_i\rangle_T^2]_{\rm av} \tag{3.61a}$$

$$=\lim_{\tilde{H}\to0}\lim_{N\to\infty}\left[\left[\sum_l P_l\langle S_i\rangle_T^{(l)}\right]^2\right]_{\rm av}. \tag{3.61b}$$

Because q is the square of a Gibbs average there are "interference terms" of the form $P_lP_{l'}\langle S_i\rangle_T^{(l)}\langle S_i\rangle_T^{(l')}$ involving *different* valleys, $l\neq l'$. However, an experiment or a computer simulation over a short time measures the properties of a *single* valley. We therefore need an order parameter appropriate for a single valley, for example,

$$q^{II}=\frac{1}{N}\sum_i(\langle S_i\rangle_T^{(l)})^2. \tag{3.62}$$

It is more convenient to calculate a weighted average of q^{II} over all valleys, so we define the Edwards-Anderson (1975) order parameter by

$$q_{\rm EA}=\left[\sum_l P_l(\langle S_i\rangle_T^{(l)})^2\right]_{\rm av}. \tag{3.63}$$

It seems intuitively plausible that q^{II} is the same for all valleys of minimum free energy, so $q_{\rm EA}(=q^{II})$ is the single-valley order parameter. Stronger arguments that q^{II} is the same for all valleys can be given for the Sherrington-Kirkpatrick model studied in detail in Sec. IV. For short-range interactions this conclusion seems to be implied by the work of van Enter and Griffiths (1983). It is not necessary to introduce a symmetry-breaking field into the definition, Eq. (3.63), since one expects that $q_{\rm EA}$ does not depend on the order of limits, $\tilde{H}\to0$, $N\to\infty$.

One would like to be able to calculate $q_{\rm EA}$ using statistical mechanics. It is not obvious how to do this, because we do not, in practice, know how to calculate a random field that will project out a single state. However Blandin

(1978) made the ingenious observation that, although *we* do not know how to do this, the system itself knows, and so we can calculate q_{EA} by introducing a second copy (or real replica) of the system in the following way. The two systems have the *same* interactions and are coupled by a term in the Hamiltonian

$$\delta\mathcal{H} = -\tilde{H}\sum_i S_i^1 S_i^2 , \tag{3.64}$$

where $\tilde{H}$ is positive, so that the two systems prefer to be in the same state. In other words, whichever valley system 1 is in, it causes a very special "random field" to act on system 2, so that the latter prefers to be in the same valley as 1. If we take the limit $N\to\infty$ before $\tilde{H}\to 0$, we expect that the *only* contribution to the Gibbs average will be where the two systems are in the same valley. Consequently q_{EA} can also be evaluated from

$$q_{EA} = \lim_{\tilde{H}\to 0}\lim_{N\to\infty}[\langle S_i^1 S_i^2\rangle_T]_{av} . \tag{3.65}$$

One may also define q and q_{EA} in terms of dynamics. As discussed in Sec. III.B, the barriers between valleys diverge in the thermodynamic limit, so if we take $N\to\infty$ before the time t tends to infinity, the system will stay in a single valley and one will measure q_{EA}. However, if we study the infinite-time limit of a finite system, then the system will move between different valleys in phase space and the time average will be equivalent to the Gibbs average, i.e.,

$$q_{EA} = \lim_{t\to\infty}\lim_{N\to\infty} q(t) , \tag{3.66a}$$

$$q = \lim_{\tilde{H}\to 0}\lim_{N\to\infty}\lim_{t\to\infty} q(t) , \tag{3.66b}$$

where

$$q(t) \equiv [\langle S_i(0)S_i(t)\rangle_{t'}]_{av} \tag{3.67}$$

and the time average $\langle\ \rangle_{t'}$, performed over an observation time t_{obs}, is defined by

$$\langle S_i(0)S_j(t_1)S_k(t_2)\cdots\rangle_{t'} = \frac{1}{t_{obs}}\int_0^{t_{obs}} dt'[S_i(t')S_j(t'+t_1)S_k(t'+t_2)\cdots] . \tag{3.68}$$

Of course it depends on the type of spin glass model whether it is necessary to make a distinction between q and q_{EA}. In the Mattis (1976) model (See Sec. III.D.1), there are just two ordered states related to each other by spin-reversal symmetry. When one applies a field, only a single state remains in the sum in Eq. (3.61b), so $q = q_{EA}$. However, this is not true for the Sherrington-Kirkpatrick model (see Sec. IV). It remains to be seen whether short-range Edwards-Anderson models have more than two ordered states (see Sec. V.E.5).

It is sometimes useful to define an order parameter Δ (Sommers, 1978) that is just the difference between q_{EA} and q, i.e.,

$$\Delta = q_{EA} - q . \tag{3.69}$$

A nonzero Δ signifies a multivalley structure in phase space and breakdown of ergodicity. Such a quantity appears naturally in the dynamical approach to spin glasses (Sompolinsky, 1981a; Sompolinsky and Zippelius, 1981, 1982a; Horner 1984a, 1984b), where one actually has two functions $q(x)$ and $\Delta(x)$, while Δ, defined in Eq. (3.69), is given by $\Delta \equiv \Delta(x=0)$.

For a symmetric bond distribution, and in the limit of small fields, the uniform susceptibility $\chi = \partial[\langle S_i\rangle_T]_{av}/\partial H$ is equal to the local susceptibility χ_{ii}, where

$$\chi_{ii} = \partial[\langle S_i\rangle_T]_{av}/\partial H_i .$$

Since

$$k_B T\chi = N^{-1}\sum_{i,j}[\langle S_i S_j\rangle_T - \langle S_i\rangle_T\langle S_j\rangle_T]_{av} \tag{3.70a}$$

$$= N^{-1}\sum_i (1-[\langle S_i\rangle_T^2]_{av}) \equiv k_B T\chi_{ii} ; \tag{3.70b}$$

where we have used the fact that

$$[\langle S_i S_j\rangle_T - \langle S_i\rangle_T\langle S_j\rangle_T]_{av} = 0$$

unless $i=j$ (Fischer, 1976). Since χ_{ii} is *always* related to q by $k_B T\chi_{ii} = 1-q$, we have

$$k_B T\chi = 1-q \quad (H\to 0, \bar{J}=0) , \tag{3.71}$$

so we can relate the magnetic susceptibility to the order parameter. It is interesting to generalize Eq. (3.71) to time-dependent quantities. Standard linear response theory (e.g., Forster, 1975; Ma and Mazenko, 1975), taken in the classical limit, gives

$$\frac{dq(t)}{dt} = -k_B T\chi(t) \quad (H\to 0, \bar{J}=0) , \tag{3.72a}$$

where $\chi(t)$ is the usual linear response function and we have used the result that $[\langle S_i(t)S_j(t')\rangle_T]_{av}$ is zero for $i\neq j$ with a symmetric bond distribution. Note that $\chi_{ii}(t)$ is *always* related to $q(t)$ by

$$\frac{dq(t)}{dt} = -k_B T\chi_{ii}(t) . \tag{3.72b}$$

Integration gives

$$k_B T\int_0^t \chi(t')dt' = 1-q(t) \quad (H\to 0, \bar{J}=0) . \tag{3.73}$$

It is useful to discuss Eq. (3.73) in the limits $t\to\infty$, $N\to\infty$. First of all we note that

$$\lim_{\tilde{H}\to 0}\lim_{N\to\infty}\lim_{t\to\infty}\int_0^t \chi(t')dt' = \chi , \tag{3.74a}$$

$$\lim_{t\to\infty}\lim_{N\to\infty}\int_0^t \chi(t')dt' = \lim_{\omega\to 0}\chi(\omega) , \tag{3.74b}$$

where χ is the equilibrium susceptibility evaluated from a

Gibbs ensemble average and $\chi(\omega)$ is the dynamical susceptibility. Hence when we take $t\to\infty$ before $N\to\infty$, Eq. (3.73) gives back Eq. (3.71), while in the opposite order of limits one has

$$k_B T \lim_{\omega\to 0}\chi(\omega)=1-q_{\rm EA} \quad (H\to 0,\bar{J}=0) . \tag{3.75}$$

From Eqs. (3.69), (3.71), and (3.75) one also has

$$\Delta=k_B T[\chi-\lim_{\omega\to 0}\chi(\omega)] \quad (H\to 0,\bar{J}=0) . \tag{3.76}$$

Hence there is a difference between the equilibrium (isothermal) susceptibility χ and the zero-frequency limit of the dynamical susceptibility $\chi(\omega)$. This is because $\chi(\omega)$ only probes the response in a single valley, whereas χ includes an extra contribution, $\Delta/k_B T$, from changes in the statistical weight of the valleys due to the applied perturbation (De Dominicis and Young, 1983a). Equations (3.75) and (3.76) also indicate how an experiment could measure the order parameters $q_{\rm EA}$ and Δ, at least in principle. Note that Eqs. (3.71), (3.72a), (3.73), (3.75), and (3.76) are *always* true if one replaces the uniform susceptibility by the local susceptibility.

One sometimes sees in the literature an "anomaly" Δ, defined by $k_B T\chi=1-q+\Delta$, which would violate the fluctuation-dissipation theorem in the form Eq. (3.70b), if $\Delta\neq 0$. However, we shall see in Sec. IV.F that there is no violation of the fluctuation-dissipation relation, and if one is referring to bond average quantities (which we are here) both (3.70a) and (3.70b) are true, so Δ defined in this way is zero. If one is referring to a particular sample, then one has to be more careful because the $i\neq j$ terms in Eq. (3.70a) do not vanish for each sample (see Sec. IV.F). Hence Eq. (3.70b) is not true without the configurational average, though (3.70a) would still be valid.

It is also illuminating to consider an order parameter defined as a time average over a finite but large observation time $t_{\rm obs}$ (Binder and Schröder, 1976a, 1976b; Binder, 1977a, 1977b),

$$\bar{q}(t_{\rm obs})=N^{-1}\left[\sum\langle S_i\rangle_{t'}^2\right]_{\rm av} =\frac{2}{t_{\rm obs}}\int_0^{t_{\rm obs}}\left[1-\frac{t}{t_{\rm obs}}\right]q(t)dt , \tag{3.77}$$

where the time average is defined in Eq. (3.68) and $q(t)$ in Eq. (3.67). In the last equality we have used the time-translational invariance of time-displaced correlation functions. If we now define a susceptibility from a time average of magnetization fluctuations we obtain (Stauffer and Binder, 1978)

$$\bar{\chi}(t_{\rm obs})=\frac{1}{Nk_B T}\left[\left\langle\left[\sum_i S_i\right]^2\right\rangle_{t'}-\left\langle\sum_i S_i\right\rangle_{t'}^2\right]_{\rm av}=\frac{1}{k_B T}[1-\bar{q}(t_{\rm obs})] , \tag{3.78}$$

making use of the fact that $[\langle S_j(t_1)S_k(t_2)\rangle_{t'}]_{\rm av}=0$ for $j\neq k$ with a symmetric bond distribution, even for $t_1\neq t_2$. Note that the order parameters q and $q_{\rm EA}$ and the susceptibilities χ and $\lim_{\omega\to 0}\chi(\omega)$ can also be obtained from $\bar{q}(t_{\rm obs})$ and $\bar{\chi}(t_{\rm obs})$, just as they can from $q(t)$ and $\chi(t)$, i.e.,

$$q=\lim_{\tilde{H}\to 0}\lim_{N\to\infty}\lim_{t_{\rm obs}\to\infty}\bar{q}(t_{\rm obs}) , \tag{3.79a}$$

$$q_{\rm EA}=\lim_{t_{\rm obs}\to\infty}\lim_{N\to\infty}\bar{q}(t_{\rm obs}) , \tag{3.79b}$$

$$\chi=\lim_{\tilde{H}\to 0}\lim_{N\to\infty}\lim_{t_{\rm obs}\to\infty}\bar{\chi}(t_{\rm obs}) , \tag{3.79c}$$

$$\lim_{\omega\to 0}\chi(\omega)=\lim_{t_{\rm obs}\to\infty}\lim_{N\to\infty}\chi(t_{\rm obs}) . \tag{3.79d}$$

These equations are analogous to Eqs. (3.66a), (3.66b), (3.74a), and (3.74b). Hence Eq. (3.78) also leads to Eqs. (3.71) and (3.75) when we consider different orders of the limits $t_{\rm obs}\to\infty, N\to\infty$.

Since many of the calculations use the replica trick, it is necessary to identify q and $q_{\rm EA}$ within the replica framework. This question has been a matter of some discussion (Young, 1981; Sommers, 1982, 1983a, 1983b; Hertz, 1983a, 1983b; Orland, 1983; Fischer and Hertz, 1983; De Dominicis and Young, 1983a). When there is only one thermodynamic state, then $q=q_{\rm EA}$ and one has a single-order-parameter description. This corresponds to the replica-symmetric solution, where all the order parameters $q_{\alpha\beta}$ are the same, i.e.,

$$q=q_{\rm EA}=q_{\alpha\beta} \quad \text{(replica-symmetric solution)} \tag{3.80}$$

for all $\alpha\neq\beta$. Where there are many states, replica permutation symmetry must be broken, and this will be studied in detail in Sec. IV, in the context of a particular scheme (Parisi, 1980b, 1980c, 1980d, 1980e) for the Sherrington-Kirkpatrick model. We have already noted in Sec. III.C that in these circumstances the order parameter q is obtained by averaging all $q_{\alpha\beta}$, i.e., (De Dominicis and Young, 1983a),

$$q=\lim_{n\to 0}\frac{1}{n(n-1)}\sum_{\alpha\neq\beta}q_{\alpha\beta} . \tag{3.81}$$

We have not yet discussed $q_{\rm EA}$ in replica language, but will show in Sec. IV that it is given by the components of the matrix $q_{\alpha\beta}$ which are as close as possible to the diagonal, i.e., we write symbolically

$$q_{\rm EA}=\lim_{n\to 0}\lim_{\alpha\to\beta}q_{\alpha\beta} . \tag{3.82}$$

Note that $q_{\alpha\alpha}$ itself does not enter the theory because $S_i^2=1$, a trivial constant (see Sec. III.C).

In a situation with many states, one can clearly define a large number of other order parameters in addition to q

and q_{EA}. For example, one might be interested in the "overlap functions" $q^{ll'}$, where

$$q^{ll'}=\frac{1}{N}\sum_i \langle S_i\rangle_T^{(l)}\langle S_i\rangle_T^{(l')} . \tag{3.83}$$

These contain information about correlations between the site magnetizations in different valleys. It is clearly too cumbersome to study the set of all overlap functions, and a statistical description is obtained from the probability distribution (Houghton *et al.*, 1983a; Parisi, 1983a);

$$P(q)=\left[\sum_{l,l'} P_l P_{l'}\delta(q-q^{ll'})\right]_{\rm av} . \tag{3.84a}$$

This will play a central role in the mean-field theory, described in Sec. IV. In terms of replicas, $P(q)$ becomes (see Sec. IV.E)

$$P(q)=\lim_{n\to 0}\frac{1}{n(n-1)}\sum_{\alpha\neq\beta}\delta(q-q_{\alpha\beta}) , \tag{3.84b}$$

which takes a rather simpler form in the case of the Parisi (1980a, 1980b, 1980c, 1980d, 1980e) replica symmetry-breaking scheme (Sec. IV.E). Clearly the statistical mechanics order parameter, Eq. (3.81), is just the first moment of $P(q)$.

It is also possible to define the overlap distribution microscopically (Young, 1983b, 1985), without direct reference to thermodynamic states. If one considers two sets of spins S_i^1, S_i^2, $i=1,\ldots,N$, each with the same interactions but no coupling between them, and define

$$P_N(q)=[\langle\delta(q-q^{12})\rangle_T]_{\rm av} \tag{3.84c}$$

where

$$q^{12}=\frac{1}{N}\sum_{i=1}^{N} S_i^1 S_i^2 \tag{3.84d}$$

as the probability that the two systems have overlap q^{12}, then it is straightforward to show (Young, 1985) that

$$\lim_{N\to\infty} P_N(q)=P(q) . \tag{3.85}$$

Equation (3.84c) has the advantage that it can be used even for finite systems in which it is not possible to define the various states since all barriers are finite. In fact, even for $N\to\infty$ it is not clear whether the states can be defined in a rigorous manner for the models with infinite-range interactions discussed in Sec. IV, so it is useful to be able to define the order-parameter distribution without having to do this.

One of the surprising recent developments in the mean-field theory has been the realization (see Sec. IV.F) that certain quantities in the theory are not self-averaging (see Sec. III.A; Mezard *et al.*, 1984a, 1984b; Young *et al.*, 1984). In particular, the overlap distribution of a single sample

$$P_J(q)=\sum_{ll'} P_l P_{l'}\delta(q-q^{ll'}) \tag{3.86}$$

varies from sample to sample even in the thermodynamic limit, so the bond averaging in Eq. (3.84) is essential and not just a theorist's convenience. Where we wish to emphasize that a quantity depends on the particular bond distribution we shall write in a subscript J, as in Eq. (3.86).

Plainly the order parameter in spin glasses is much more complicated than q_{EA}, at least in mean-field theory. The existence of many valleys leads to a probability distribution $P_J(q)$, and then lack of self-averaging implies that the "order parameter" is really the distribution (among different bond configurations) of the distribution $P_J(q)$. It is not completely clear whether are more layers of complexity remaining to be unraveled.

Various other order parameters have also been introduced and will now be discussed. An obvious generalization of Eq. (3.61a) is to other local order parameters q_k, defined by

$$q_k=\lim_{\tilde H\to 0}\lim_{N\to\infty}[\langle S_i\rangle_T^k]_{\rm av} , \quad k=3,4,\ldots , \tag{3.87}$$

which play a role in the mean-field theory of disordered spin systems with short-range interactions (Sarbach, 1980). [The expression for the q_k in replica language is given in Eq. (3.31).] In addition,

$$q_1=\lim_{\tilde H\to\infty}\lim_{N\to\infty}[\,|\langle S_i\rangle_T|\,]_{\rm av}$$

has been discussed (Aharony and Imry, 1976, 1977a; Young and Kirkpatrick, 1982).

A more interesting quantity, because it does not require a symmetry-breaking field, is the correlation function (Morgenstern and Binder, 1980a, 1980b)

$$g_{SG}(\mathbf{R}_i-\mathbf{R}_j)=[\langle S_i S_j\rangle_T^2]_{\rm av} , \tag{3.88}$$

which for short-range interactions depends on the relative distance R_i-R_j. It is sensible to consider the correlation length ξ_{SG} describing the exponential decay of $g_{SG}(\mathbf{R}_i-\mathbf{R}_j)$ in the paramagnetic phase (Morgenstern and Binder, 1980a, 1980b),

$$g_{SG}(\mathbf{R}_i-\mathbf{R}_j)\propto\exp(-|\mathbf{R}_i-\mathbf{R}_j|/\xi_{SG}) , \quad |\mathbf{R}_i-\mathbf{R}_j|\to\infty . \tag{3.89}$$

At T_c ξ_{SG} diverges, and if q is nonzero below T_c there is correlation between the spins at infinite distance (Binder, 1980a, 1980b), so one may define an order parameter $q^{(2)}$ by

$$q^{(2)}=\lim_{|\mathbf{R}_i-\mathbf{R}_j|\to\infty} g_{SG}(\mathbf{R}_i-\mathbf{R}_j) . \tag{3.90}$$

It is interesting to evaluate g_{SG} in terms of averages over valleys in phase space, discussed in Sec. III.B. One finds

$$g_{SG}(\mathbf{R}_i-\mathbf{R}_j)=\left[\left[\sum_l P_l\langle S_i S_j\rangle_T^{(l)}\right]^2\right]_{\rm av} . \tag{3.91}$$

The pure states should satisfy a "clustering property" (see, for example, Griffiths, 1972), which means that all connected correlation functions, evaluated within a single

phase, must vanish when relative distances tend to infinity; in particular,

$$\lim_{|\mathbf{R}_i-\mathbf{R}_j|\to\infty}\langle S_iS_j\rangle_T^{(l)}=\langle S_i\rangle_T^{(l)}\langle S_j\rangle_T^{(l)}\ , \tag{3.92}$$

so

$$q^{(2)}=\lim_{|\mathbf{R}_i-\mathbf{R}_j|\to\infty}\left[\left[\sum_l P_l\langle S_i\rangle_T^{(l)}\langle S_j\rangle_T^{(l)}\right]^2\right]_{\rm av}\ . \tag{3.93a}$$

For an infinite system one can replace Eq. (3.93a) by

$$q^{(2)}=\frac{1}{N^2}\sum_{i,j}\left[\left[\sum_l P_l\langle S_i\rangle_T^{(l)}\langle S_j\rangle_T^{(l)}\right]^2\right]_{\rm av} \tag{3.93b}$$

because the double sum is dominated by pairs of sites that are far apart. Hence it follows that

$$q^{(2)}\geq[q_J^2]_{\rm av}\geq q^2\ , \tag{3.94}$$

where

$$q_J=\lim_{\tilde{H}\to 0}\lim_{N\to\infty}\frac{1}{N}\sum_i\langle S_i\rangle_T^2 \tag{3.95}$$

is the statistical mechanics order parameter for a given bond configuration. The first inequality in Eq. (3.94) holds as an equality only if there is just a single state. In other words, when there are many states the clustering property

$$\lim_{|\mathbf{R}_i-\mathbf{R}_j|\to\infty}\langle S_iS_j\rangle_T=\langle S_i\rangle_T\langle S_j\rangle_T$$

holds for averages within a given state, but *not* for the full Gibbs average because of interference terms involving *different* states when one calculates $\langle S_i\rangle_T\langle S_j\rangle_T$. Furthermore we shall see in Sec. IV.F that lack of self-averaging always occurs when replica perturbation symmetry is broken, i.e., when there are many states. Hence the second inequality in Eq. (3.94) also only becomes an equality when there is a single state.

In terms of replicas, one can evaluate $g_{\rm SG}$ by a straightforward generalization of the argument that led to Eq. (3.34) and find

$$g_{\rm SG}(R_i-R_j)=\lim_{n\to 0}\frac{1}{n(n-1)}\sum_{\alpha\neq\beta}\langle S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta\rangle\ . \tag{3.96}$$

In the limit $|\mathbf{R}_i-\mathbf{R}_j|\to\infty$ the clustering property must hold for the replica Hamiltonian because it is translationally invariant, so

$$q^{(2)}=\lim_{n\to 0}\frac{1}{n(n-1)}\sum_{\alpha\neq\beta}q_{\alpha\beta}^2\ , \tag{3.97}$$

which is just the second moment of the distribution $P(q)$.

One may also consider generalizations of the order parameter $q^{(2)}$ to finite times, just as $q(t)$ in Eq. (3.66) is related to q, namely (Fernandez and Streit, 1982; Mackenzie and Young, 1982, 1983),

$$q^{(2)}=\lim_{N\to\infty}\lim_{t\to\infty}q^{(2)}(t)\ , \tag{3.98}$$

where

$$q^{(2)}(t)=\lim_{|\mathbf{R}_i-\mathbf{R}_j|\to\infty}[\langle S_i(0)S_j(0)S_i(t)S_j(t)\rangle_{t'}]_{\rm av}\ . \tag{3.99}$$

It is interesting to ask what happens if one reverses the limits in Eq. (3.98). With $N\to\infty$ the system stays in a single valley, so, because of the clustering property of single-valley averages, one obtains

$$\lim_{t\to\infty}\lim_{N\to\infty}q^{(2)}(t)=q_{\rm EA}^2\ . \tag{3.100}$$

For the infinite-range model all distinct pairs of spins are equivalent, so the limit of infinite distance in Eqs. (3.90) and (3.99) may be removed and any distinct pair ($i\neq j$) used to define $q^{(2)}$ and $q^{(2)}(t)$ (Kirkpatrick and Young, 1981). In this case $q^{(2)}$ is related to the energy E by (Bray and Moore, 1980a)

$$E\equiv[\langle\mathscr{H}\rangle_T]_{\rm av}/N=-(1-q^{(2)})(\Delta J)^2/(2k_BT)\ . \tag{3.101}$$

It is interesting to introduce the spin glass susceptibility $\chi_{\rm SG}$ (Chen and Lubensky, 1977; Fisch and Harris, 1977) by

$$\chi_{\rm SG}=N^{-1}\sum_{i,j}[(\langle S_iS_j\rangle_T-\langle S_i\rangle_T\langle S_j\rangle_T)^2]_{\rm av}\ , \tag{3.102}$$

which must diverge at the freezing temperature T_f, since the correlation length $\xi_{\rm SG}$ diverges. We shall frequently be interested in $\chi_{\rm SG}$ in the paramagnetic phase where, of course, $\langle S_i\rangle_T=0$. $\chi_{\rm SG}$ gives the change in q when a small random staggered field $H_r(i)$ is applied. To see this, note that from linear-response theory we have

$$\delta\langle S_i\rangle_T=\sum_j(\langle S_iS_j\rangle_T-\langle S_i\rangle_T\langle S_j\rangle_T)\delta H_r(j)/k_BT\ , \tag{3.103}$$

so, on squaring and averaging, one obtains

$$(k_BT)^2\frac{\partial q}{\partial[H_r^2]_{\rm av}}=\chi_{\rm SG}\ . \tag{3.104}$$

Furthermore it is straightforward to show (Jayaprakash *et al.*, 1977) that q itself is related to the derivative of the free energy with respect to $[H_r^2]_{\rm av}$ by

$$k_BT\frac{\partial f}{\partial[H_r^2]_{\rm av}}=(1-q)/2\ . \tag{3.105}$$

Because of Eqs. (3.104) and (3.105) it is natural to say that $[H_r^2]_{\rm av}$ is the "field" conjugate to the spin glass order parameter q and that $\chi_{\rm SG}$ is the corresponding response function. In addition, for a symmetric bond distribution *and to lowest order in the fields only* one can replace the random field by a uniform field through the change $[H_r^2]_{\rm av}\to H^2$. In this symmetric case $\chi_{\rm SG}$ is related to the expansion of the free energy in powers of the uniform magnetic field (Chalupa 1977a; Suzuki, 1977) in the paramagnetic phase. Writing the expansion of the magnetization as (Omari *et al.*, 1983)

$$m=-\frac{\partial f}{\partial H}$$
$$=\frac{H}{k_BT}\left[a_1-\frac{a_3}{3}\left[\frac{H}{k_BT}\right]^2+\frac{2a_5}{15}\left[\frac{H}{k_BT}\right]^4-\cdots\right], \tag{3.106}$$

where the coefficients $a_1,a_3,a_5,\ldots$, are defined to be unity for noninteracting Ising spins, we have

$$a_1=1\ , \tag{3.107}$$

which follows from Eq. (3.71) with $q=0$, while

$$a_3=3\chi_{\rm SG}-2 \tag{3.108}$$

(Kinzel and Binder, 1984). Comparing Eq. (3.106) with (2.35) one sees that a_3 is simply related to the nonlinear susceptibility $\chi_{nl}(T)$ by

$$\chi_{\rm nl}(T)=\frac{a_3}{3}\left[\frac{1}{k_BT}\right]^3=\left[\chi_{\rm SG}-\frac{2}{3}\right]\left[\frac{1}{k_BT}\right]^3. \tag{3.109}$$

Thus the nonlinear susceptibility must diverge at T_f if $\chi_{\rm SG}$ does. In the general case of a nonsymmetric distribution, $\langle S_iS_j\rangle_T$ no longer vanishes for $i\neq j$, and Eqs. (3.107) and (3.109) are replaced by

$$a_1=1+\frac{1}{N}\sum_{i\neq j}[\langle S_iS_j\rangle_T]_{\rm av}\ , \tag{3.110}$$

which is sensitive to ferromagnetic correlations, if there are any, and (Chalupa, 1977a)

$$\chi_{\rm nl}(T)=-\frac{1}{6N(k_BT)^3}\left[\left\langle\left[\sum_i S_i\right]^4\right\rangle_{T,c}\right], \tag{3.111}$$

where $\langle\ \rangle_{T,c}$ denotes a cumulant average. Thus $\chi_{\rm nl}(T)$ also feels ferromagnetic correlations. However, assuming that such correlations remain short range at the spin glass transition, the singular behavior would still be given by $\chi_{\rm SG}$, defined in Eq. (3.102), where of course one may set $\langle S_i\rangle_T=0$ since the present discussion refers only to the paramagnetic phase.

As we have noted in Sec. III.A, the correlation function between a given pair of spins is not self-averaging, and hence one many consider $[\langle S_iS_j\rangle_T^p]$ with $p\neq 2$ [cf. Eq. (3.10)] and corresponding susceptibilities defined, in the paramagnetic phase, by

$$\chi_{\rm SG}^{(p)}=N^{-1}\sum_{i,j}[\,|\langle S_iS_j\rangle_T|^p]_{\rm av}\ ,\quad p=1,2,3,\ldots\ . \tag{3.112}$$

So far only $\chi_{\rm SG}^{(2)}\equiv\chi_{\rm SG}$ and $\chi_{\rm SG}^{(1)}$ (Aharony and Binder, 1980) have received practical attention.

We have already discussed the "overlap functions" $q^{ll'}$, defined in Eq. (3.83). An alternative "projection order parameter" is $\psi^{(l)}$, in which the spin projection on a ground state ($\phi_j^{(l)}\equiv\langle S_j\rangle_{T=0}^{(l)}$) is studied (Binder, 1977a). This may be considered as the thermal average of a variable,

$$\psi^{(l)}=N^{-1}\sum_j\langle\psi_j^{(l)}\rangle_i\ ,\quad \psi_j^{(l)}=S_j\phi_j^{(l)}\ . \tag{3.113}$$

Again one may define an associated correlation function $g_\psi^{(l)}(\mathbf{R}_i-\mathbf{R}_j)$, order parameter $\psi^{(l)}$, and susceptibility χ_{ll} via the relations

$$g_\psi^{(l)}(\mathbf{R}_i-\mathbf{R}_j)=\langle\psi_i\psi_j\rangle_T^{(l)}$$
$$=\phi_i^{(l)}\phi_j^{(l)}\langle S_iS_j\rangle\ , \tag{3.114a}$$
$$k_BT\chi_{ll}=\sum_{ij}\langle\psi_i\psi_j\rangle_T^{(l)}/N$$
$$=\sum_{ij}\phi_i^{(l)}\phi_j^{(l)}\langle S_iS_j\rangle_T/N\ , \tag{3.114b}$$
$$|\psi|^2=\lim_{|\mathbf{R}_i-\mathbf{R}_j|\to\infty}g_\psi^{(l)}(\mathbf{R}_i-\mathbf{R}_j)$$
$$=\lim_{N\to\infty}k_BT\chi_{ll}/N\ . \tag{3.114c}$$

Alternatively, we may define χ_{ll} by an appropriately staggered field $\mathscr{H}'=\mathscr{H}-H_\psi^{(l)}\sum_i\psi_i$, taking the response function

$$\chi_{ll}=\partial^2 f/\partial|H_\psi^{(l)}|^2\ . \tag{3.115}$$

In a Mattis spin glass, of course, this order parameter $\psi^{(l)}$ is nothing but the "hidden" order parameter M_τ, $\chi_{ll}=\chi_\tau$, etc.; thus Eqs. (3.113)—(3.115) look for a kind of underlying "hidden" long-range order in the Edwards-Anderson spin glass, the only difference being that one must allow for many "order-parameter components" $\psi^{(l)}$ representing the various valleys $\{\langle S_i\rangle_T^{(l)}\}$ in phase space. The restrictive assumption of Eqs. (3.113)—(3.115), however, is that at finite temperature the valley structure is still determined by the valley structure at zero temperature. Indeed, for $T=0$ we have the "sum rules" (Aharony and Binder, 1980)

$$\sum_l q_\psi^{(l)}=\langle S_iS_j\rangle_{T=0}^2\ ,$$
$$k_BT\sum_l\chi_{ll}=\sum_{ij}\langle S_iS_j\rangle_{T=0}^2/N\ , \tag{3.116}$$

i.e., there is no difference between these correlations and the correlations considered in Eq. (3.88). At least in the infinite-range model, however, there is evidence that the valley structure in phase space at finite temperature is unrelated to the valleys at zero temperature (see Sec. VI.E). An additional disadvantage is that the ground states usually are not known analytically; thus the practical use of Eqs. (3.113)—(3.115) has been limited to numerical investigations.

Here, let us discuss briefly how one can define order parameters for vector spins. The standard order parameters q and $q_{\rm EA}$ become second-rank tensors $q^{kk'}$ and $q_{\rm EA}^{kk'}$ in spin space, where k and k' denote spin components and, for example,

$$q^{kk'}=[\langle S_i^k\rangle_T\langle S_i^{k'}\rangle_T]_{\rm av}\ . \tag{3.117}$$

In the replica method the order parameter is a second-rank tensor in spin space as well as in replica space, i.e.,

$$q^{kk'}=\lim_{n\to 0}\frac{1}{n(n-1)}\sum_{\alpha\neq\beta}q_{\alpha\beta}^{kk'}\ , \tag{3.118}$$

where $q_{\alpha\beta}^{kk'}=\langle S_i^{\alpha,k}S_i^{\beta,k'}\rangle$, $\alpha\neq\beta$. The probability distribution for the overlap functions now involves all $q^{kk'}$, so we write

$$P\{q^{kk'}\}=\lim_{n\to 0}\frac{1}{n(n-1)}\sum_{\alpha\neq\beta}\prod_{\langle k,k'\rangle}\delta(q^{kk'}-q_{\alpha\beta}^{kk'}) . \quad (3.119)$$

One frequently studies an isotropic spin glass in which all directions in spin space are equivalent. In this case second-rank tensors in spin space reduce to a Kronecker delta function $\delta_{kk'}$, so that, for example,

$$q_{\alpha\beta}^{kk'}=\delta_{kk'}q_{\alpha\beta} . \quad (3.120)$$

In general one defines the spin glass correlation function $g_{\rm SG}(\mathbf{R}_i-\mathbf{R}_j)$ and spin glass susceptibility $\chi_{\rm SG}$ in terms of scalar products, i.e.,

$$g_{\rm SG}(\mathbf{R}_i-\mathbf{R}_j)=[\langle\mathbf{S}_i\cdot\mathbf{S}_j\rangle_T^2]_{\rm av} , \quad (3.121a)$$

and, in the paramagnetic phase,

$$\chi_{\rm SG}=N^{-1}\sum_{i,j}[\langle\mathbf{S}_i\cdot\mathbf{S}_j\rangle_T^2]_{\rm av} , \quad (3.121b)$$

though, again, one could separate out different Cartesian components.

To conclude this section, we have seen that the order parameter in spin glasses is rather complicated, at least in mean-field theory. These complications, e.g., the fact that $q\neq q_{\rm EA}$, arise from there being many thermodynamic states with significant statistical weight. Ordering in a single state, described by $q_{\rm EA}$, is measurable experimentally. It does not seem feasible experimentally to determine off-diagonal overlap functions. However, for numerical simulations, which are inevitably on small samples in which thermodynamic states are not well defined, it is only the statistical mechanics order parameters, e.g., q and $q^{(2)}$, that are precisely defined. It is not clear whether this many-valley structure also occurs for short-range systems in three dimensions. If not, then there is just a single thermodynamic state, and the order parameter is a number describing ordering in this state.

2. Frustration and gauge invariance

The order parameters introduced above are not so useful for elucidating the spin structure in the ground state itself. For a discussion of ground-state spin configurations the concept of "frustration" (Toulouse, 1977) is particularly useful. We again consider the random-bond model, Eq. (3.27), for Ising spins in the absence of magnetic fields, with a distribution of bond strengths given by Eq. (3.38a). One now notes that the Hamiltonian is invariant under the local transformation

$$S_i\to -S_i,\quad J_{ij}\ (j\text{ adjacent to }i)\to -J_{ij} . \quad (3.122)$$

Obviously, if we perform this transformation in the case of a Mattis model for all negative bonds, the disorder is completely eliminated. Thus in a more general case of a disordered material, the need arises to distinguish between such a trivial disorder, which can be eliminated by simple local transformations that leave $\mathscr{H}$ and hence the free energy invariant, and the more serious disorder, which cannot be "gauged away." In fact, Eq. (3.122) may be called a "gauge transformation" by analogy with the local transformation that leaves the Hamiltonian of electrodynamics invariant: the electromagnetic fields $\mathbf{E},\mathbf{H}$ are (putting the velocity of light equal to unity)

$$E=-\frac{\partial\mathbf{A}}{\partial t}+\nabla\varphi,\quad H=\nabla\times\mathbf{A} , \quad (3.123)$$

where φ is the scalar potential and $\mathbf{A}$ the vector potential, which may be combined into the four-dimensional vector $\{A_\mu\}=\{\mathbf{A},i\varphi\}$. These fields are invariant against the gauge transformation (note $\nabla_\mu=\{\nabla,i\partial/\partial t\}$)

$$\{A_\mu\}\to\{A_\mu\}+\nabla_\mu\alpha , \quad (3.124a)$$

α being any function of space and time. In quantum mechanical problems. Eq. (3.124a) must be complemented with a transformation of the wave function ψ, e being the unit of charge

$$\psi\to e^{ie\alpha}\psi ; \quad (3.124b)$$

for simplicity here we have chosen units in which the particle mass and Planck's quantum are unity. Equations (3.122) and (3.124) are somewhat similar, if we compare the spin with the particle field, the factors ±1 with the phase factor $e^{ie\alpha}$, and the bonds J_{ij} with the vector potential. Of course, while the gauge group defined by Eq. (3.124) is the continuous group U(1), Eq. (3.122) involves the simpler discrete group Z_2 only.

In any case, this consideration leads to the concept of the relevant disorder's being gauge invariant. The problem arises how this "serious" disorder (which is not present in the Mattis model) can be measured.

One quickly realizes that "serious" (i.e., gauge-invariant disorder is due to the "frustration effect," i.e., competition between interactions that cannot all be satisfied by any spin configuration. Consider the elementary triangle of Fig. 40(a), with all bonds negative: there is no way of choosing the orientations of the spins around the triangle without "frustrating" at least one bond. This frustration effect can be expressed by the function ϕ_f

$$\phi_f=J_1J_2J_3/|J|^3 , \quad (3.125a)$$

or more generally for any closed contour on the lattice ($J_{ij}=\pm J$)

$$\phi_f=\prod_c J_{ij}/|J| , \quad (3.125b)$$

where the product is along all the bonds of the contour. Disorder is not serious when $\phi_f=+1$, while $\phi_f=-1$ when frustration is present. It is then appropriate to consider ϕ_f for the elementary units of the lattice (elementary triangles on the triangular lattice, elementary squares on the square lattice, the so-called plaquettes).

Since each bond on the square lattice is shared by two plaquettes, frustrated plaquettes appear pairwise. If there

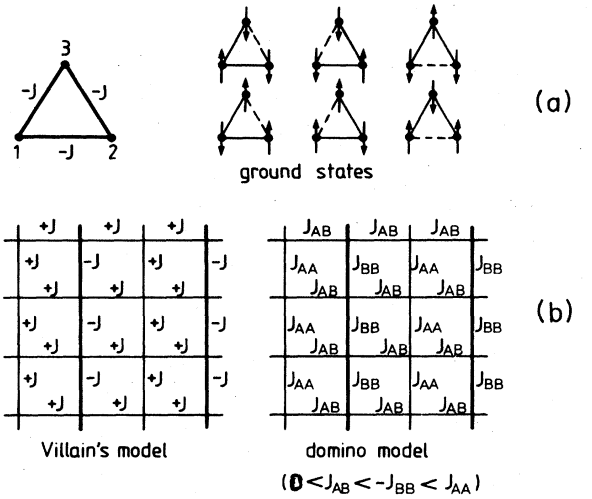

FIG. 40. (a) Frustrated triangle and its ground state: "unsatisfied" bond is denoted as a dashed line. (b) Two fully frustrated two-dimensional models, suggested by Villain (1977a) and Villain *et al.* (1980). From Binder (1980a).

is a lot of frustration in the system, the configuration of frustrated plaquettes clearly is a complicated network. If one considers a pair of frustrated plaquettes in an otherwise unfrustrated background, the extra energy due to frustration simply is proportional to the length of the minimal distance between the two plaquettes (the "string"). The ground state of the square lattice hence is defined by finding the set of minimal-length strings connecting the frustrated plaquettes. Of course, the solution to this problem is not at all unique: different solutions correspond to different ground states. Hence this consideration yields a precise construction for the various ordered states, at least for the nearest-neighbor $\pm J$ model in the square lattice. [For a discussion of frustration in higher-dimensional lattices, see for instance Fradkin *et al.* (1978) and Toulouse (1979).] Unfortunately, there is no analytic solution to this problem, and hence one can use this formulation for studying explicit ground-state properties only by means of numerical algorithms which solve this string-length optimization problem on a computer (Rammal *et al.*, 1979; Bieche *et al.*, 1980; Barahona *et al.*, 1982).

Clearly, there is some similarity of this problem to global optimization problems in general, such as the "traveling salesman" problem: in fact, efforts at numerically finding ground states, or at least low-lying states, in spin glasses have led to attractive ideas of how to handle such optimization problems (Kirkpatrick *et al.*, 1983): this may be the most important practical application that has emerged from spin glass research so far.

Since even the ground-state problem of the $\pm J$ model in two (as well as in higher) dimensionalities is a very difficult problem, intractable analytically, it is no surprise that the statistical mechanics of Ising systems with random quenched frustration is not very far developed. Though one can set up the program of calculating the gauge-invariant correlation functions formally (Fradkin *et al.*, 1978), explicit results either are rather qualitative or refer to simpler limiting cases. Thus we shall not discuss this work here.

The recognition of the central importance of the frustration effect, allowing a high ground-state degeneracy and thus creating the many "valleys" in phase space that are such an essential feature of spin glasses, has promoted interest in the somewhat simpler problem of Ising systems with periodically distributed frustration. Two examples of such lattices with frustration but without disorder are shown in Fig. 40(b); such systems exhibit very interesting and rich behavior, and therefore we shall discuss them briefly in Sec. VI.E. There is, however, more or less general consensus that such systems are rather different from spin glasses, contrary to the original suggestion that they may contain spin glass phases as well (Toulouse, 1977).

Rather different behavior occurs when we consider continuous rather than Ising spins. The situation in this case has been analyzed by Villain (1977b, 1978) and Fradkin *et al.* (1978). Consider an isolated plaquette on the square lattice for classical XY spins (Fig. 41), i.e., with spin components $(\cos\theta_i, \sin\theta_i)$ where θ_i is the angle with some quantization direction. For nonfrustrated plaquettes, the spin arrangement is collinear; however, for frustrated plaquettes the spin arrangement is canted, and $\theta_i - \theta_{i-1} = \pm\pi/4$. There are two different senses of rotation, and hence Villain (1977b) introduces a variable called "chirality" τ to distinguish them. If one puts a single frustrated plaquette in a two-dimensional, nonfrustrated "sea" at zero temperature, one finds that the spin direction rotates quasicontinuously along a large loop around the frustrated plaquette. The total rotation angle is again $\pm\pi$, and hence the chirality again can be introduced. The picture of the spin configuration is reminiscent of the vortices introduced by Kosterlitz and Thouless (1973) to describe excitations of two-dimensional XY magnets. Around a vortex, however, the spin rotation is $\pm 2\pi$. The spin configuration around a frustrated plaquette thus is a half vortex. This picture leads to a simplified treatment motivated by experience with the pure XY model: one allows for two types of independent vari-

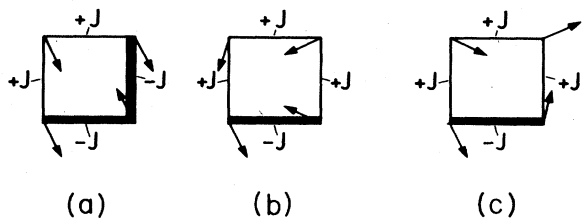

FIG. 41. Classical ground state of a set of four spins in the XY model with interactions $\pm J$ (thick bonds are antiferromagnetic, thin bonds are ferromagnetic). (a) Nonfrustrated plaquette; (b) frustrated plaquette, chirality $\tau = +1$; (c) frustrated plaquette, chirality $\tau = -1$.

ables. Since one has continuous spins, there must be small-amplitude oscillations (like magnons). Then there are discrete variables related to the chirality describing the large spin deviations near frustrated plaquettes. Although the positions of the bonds are frozen, the chirality is a dynamical variable because the spin configuration may jump from one state to the degenerate other one, whereby the sign of τ changes [see the simple example in Figs. 40(b) and 40(c)]. The discrete variables τ hence represent two-level systems of a sort, which are believed to contribute in an important way to the low-temperature excitations of spin glasses and window glasses (Anderson *et al.*, 1972; Phillips, 1972).

While all these considerations refer to the ideal nearest-neighbor $\pm J$ bond-disorder model, which hardly is a faithful description of any real material, we immediately recognize that the same considerations do in fact carry over to the spin configurations in more realistic site-disorder models (e.g., Figs. 38 and 39). These considerations also point towards the importance of the parameter "spin dimensionality m," which was not so evident from the discussion of the replica method and the Edwards-Anderson order parameter and related quantities.

The discussion of the low-energy spin configurations of the m-vector model for spin glasses profitably makes use of the concept of "topological defects" (Toulouse, 1979; Julia and Toulouse, 1979; Dzyaloshinskii, 1980). In the two-dimensional Ising case, the strings emerging from the frustrated plaquettes can be considered as "half-line defects." While for the two-dimensional Ising case the plaquettes can be associated with the sites of the dual lattice, in three dimensions the plaquettes are associated with dual links, giving the frustration in $d=3$ a vector character. The configuration of the frustration variable is then a network of closed loops of links on the dual lattice (Toulouse, 1977; Fradkin *et al.*, 1978). In the original lattice, this corresponds to a closed tube of frustrated plaquettes. Since the frustration network acts as a system of sources or sinks for defects in the spin configuration, the defects associated with these closed tubes are walls, the area of which is minimal in the ground state. While for $m=2, d=3$ we encounter plaquettes with half-vortices, as mentioned above, for $m=2, d=3$ where the plaquettes form closed loops we have associated "half-vortex-rings." For the Heisenberg model ($m=3$), in addition to line and wall defects, one must consider "textures"; the spin configurations, possibly stabilized in the ground state of Heisenberg spin glasses, are "half-textures" (Toulouse, 1979). A thorough recent discussion of these defects in Heisenberg spin glasses can be found in Henley (1984a); this work is also a useful guide for a corresponding numerical search for such defects in computer simulations (Henley, 1984b).

A general conclusion due to the concept of gauge invariance is the following: since the "serious" disorder is invariant against the gauge transformation Eq. (3.122), and it is this disorder that is relevant for true spin glass behavior, we should also investigate the properties of the spin system in terms of gauge-invariant properties. Of course, this point applies only to the disordered phase, where the symmetry is not yet broken. In fact, both the correlation functions $g_{SG}(\mathbf{R}_i-\mathbf{R}_j)$ [Eq. (3.98)] and the correlation $g_\psi(\mathbf{R}_i-\mathbf{R}_j)$ related to $g^{(l)}(\mathbf{R}_i-\mathbf{R}_j)$ of Eq. (3.114a) as an average over the L ground states

$$g_\psi(\mathbf{R}_i-\mathbf{R}_j)=\frac{1}{L}\sum_l g_\psi^{(l)}(\mathbf{R}_i-\mathbf{R}_j)$$
$$=\langle S_iS_j\rangle_{T=0}\langle S_iS_j\rangle_T \tag{3.126}$$

are gauge invariant, and so is the order parameter q. In contrast, the gauge symmetry is broken if we talk about one specific ordered state l. The idea that there is an approximate local gauge invariance, even if the system is frozen in such a low-temperature configuration in one valley in phase space is the basic starting point of some attempts to construct a gauge theory of spin glasses (Dzyaloshinskii and Volovik, 1978; see Sec. 6.A).

G. Superparamagnetism and two-level systems

In this section we are concerned with some simple concepts for the description of dynamic phenomena and irreversibility in spin glasses. The first of these concepts is the idea of ascribing certain effects in spin glasses, like the frequency-dependent peak of the ac susceptibility, to "magnetic clusters" or "magnetic clouds" (Tholence and Tournier, 1974), which are treated like superparamagnetic particles (Preisach, 1935; Néel, 1949, 1955; Bean and Livingstone, 1959; Brown, 1963; for a more recent review see Souletie, 1983). Of course, if in a dilute magnetic alloy the magnetic atoms were phase-separated from the nonmagnetic ones, there would not be any fundamental difference between such a material and a rock containing a random distribution of fine magnetic particles. Since real spin glass materials often exhibit chemical clustering, it is also possible that superparamagnetic behavior may obscure the effects characteristic of true spin glasses. Disentangling simple superparamagnetism from more subtle spin glass behavior hence is one motivation for studying the theory of superparamagnetism in more detail (Kumar and Dattagupta, 1983; Wenger and Mydosh, 1984b); in addition, some points such as the idea that the dynamics on large time scales is dominated by thermally activated processes, are common to most recent theories on spin glasses. It is clearly an oversimplification to claim that spin glasses exhibit nothing but rock magnetism (Wohlfarth, 1977a, 1979, 1980; Shtrikman and Wohlfarth, 1981). Although for a number of years the concepts of superparamagnetism have been very popular for analyzing experimental data (e.g., Verhelst *et al.*, 1975; Bieman *et al.*, 1978; Renard *et al.*, 1978; Schwink *et al.*, 1978; Meert and Wenger, 1981), the numbers obtained from such fits often have little physical significance. This fact is not so surprising, since a spin glass should only be modeled as a system of *interacting* super-

paramagnetic clusters, which again may be viewed as a renormalized Edwards-Anderson model (Binder, 1977a, 1977b, 1980a) and not as a system of independent clusters. Only in the latter case is a simple treatment possible.

This treatment starts by describing the system in terms of a distribution function $P(\mu)$ of the clusters with magnetic moment μ. In thermal equilibrium the mean magnetization of the N_c clusters in a magnetic field H is expressed in terms of the Langevin function $\mathcal{L}(y)$ as

$$M = N_c \int_0^\infty d\mu\, \mu P(\mu) \mathcal{L}(\mu H / k_B T) . \tag{3.127}$$

For small field this would lead to a Curie law for the susceptibility, since $\mathcal{L}(y) \to y/3$ for $y \to 0$, and hence

$$\chi = N_c \overline{\mu^2} / (3 k_B T), \quad \overline{\mu^2} = \int_0^\infty d\mu\, \mu^2 P(\mu) . \tag{3.128}$$

Now a peak of χ is obtained only by assuming that there may be incomplete thermal equilibrium: due to some anisotropy energy $E_\mu = K\mu$, where K is some constant, the reorientation of a cluster by the magnetic field may be "blocked" over the considered time scale. One assumes that the typical time τ_μ for reorienting a cluster is given by an Arrhenius formula,

$$\tau_\mu = \nu_A^{-1} \exp(K\mu / k_B T) , \tag{3.129}$$

ν_A being an "attempt frequency." Now, crudely speaking, for the time-dependent susceptibility $\chi(t)$ only such clusters should be included in the response function for which $\tau_\mu \leq t$. Consequently, Eq. (3.128) is replaced by

$$\chi(t) \approx \frac{N_c}{3 k_B T} \int_0^{(k_B T/K)\ln(\nu_A t)} d\mu\, \mu^2 P(\mu) . \tag{3.130}$$

Hence $\chi(t)$ will exhibit a peak at a temperature where $(k_B T/K)\ln(\nu_A t)$ roughly equals the peak position of the distribution $\mu^2 P(\mu)$, which is of the same order as $\overline{\mu^2}$. If this temperature of the peak is identified as the time-dependent freezing temperature $T_f(t)$, then a logarithmic variation with observation time (or inverse frequency, respectively) is obtained,

$$\begin{aligned} &T_f(t) \propto (K/k_B)(\overline{\mu^2})^{1/2} / \ln(\nu_A t) , \\ &T_f^{-1}(\omega) \propto \text{const} - \ln(\omega/\nu_A) . \end{aligned} \tag{3.131}$$

Of course, the above plausibility arguments can be put on a more rigorous mathematical basis by describing the dynamic response of the assembly of superparamagnetic particles in terms of a Fokker-Planck-equation description (Brown, 1963; Kumar and Dattagupta, 1983), but the essential conclusion remains the same. While it often is possible to adjust the unknown function $P(\mu)$ such that Eq. (3.130) reproduces the experimental susceptibility, one runs into problems interpreting the parameters K, ν_A: usually the anisotropy is much too large to be explained in terms of the physical anisotropies discussed in Sec. II.A.2, and ν_A often it is completely unphysical. Of course, one may also work with a distribution of barrier heights not simply related to the distribution $P(\mu)$ via Eq. (3.129) [see Souletie (1983)]. In addition, Eq. (3.131) is not really supported experimentally if one considers $T_f(\omega)$ over a wide enough frequency scale, as discussed in Sec. II.B.1. There are various approximate attempts to include effects due to interactions among the superparamagnetic clusters, which lead from the Arrhenius law [Eqs. (3.129), (3.131), and (2.15b)] to the Vogel-Fulcher law [Eqs. (2.14) and (2.15a)], e.g., Shtrikman and Wohlfarth (1981) and Cyrot (1981); however, our view is that a satisfactory derivation of this phenomenological relation has yet to be given.

It clearly is a major weakness of this approach that many rather arbitrary assumptions [such as $P(\mu), K, \nu_A$] need to be made. For a very dilute short-range system such as $Eu_xSr_{1-x}S$ in the range $x \lesssim 0.1$, however, the situation is different: since magnetic interactions extend to next-nearest neighbors only, the concept of clusters well separated from each other is well defined up to the percolation threshold [$x_p^{\text{NNN}} = 0.136$ on the fcc lattice (Shante and Kirkpatrick, 1971)]. Assuming random mixing, one may count the number $n_K(x)$ of clusters of configuration k with S_k spins in each of these clusters as

$$n_k(x) = g_k x^{S_k} (1-x)^{t_k} , \tag{3.132}$$

t_k being the perimeter of configuration k, i.e., the number of nonmagnetic atoms that are nearest or next-nearest neighbors of the magnetic atoms in the cluster, and g_k the multiplicity (i.e., g_k counts the number of different orientations that a cluster of given size and shape can have due to rotations or mirror images in the lattice). In insulating systems such as $Eu_xSr_{1-x}S$, it is the dipolar interaction between spins within the same cluster that is responsible for the anisotropy. Denoting the angle between the resulting easy axis and the magnetic field direction by φ_k, in thermal equilibrium the average magnetic moment μ_k in the field direction is

$$\mu_k = m_k \cos\varphi_k \tanh(m_k H \cos\varphi_k / k_B T) , \tag{3.133}$$

where m_k is the magnetic moment of this cluster. Summing up over all possible cluster configurations and averaging over the angle φ_k, one obtains the static equilibrium susceptibility (Eiselt *et al.*, 1979)

$$\chi = (1/k_B T) \sum_k g_k m_k^2 x^{s_k} (1-x)^{t_k} , \tag{3.134}$$

as an explicit version of Eq. (3.128), which now has no adjustable parameters (assuming rigid alignment of the spins within a given cluster). Calculating from the dipolar Hamiltonian, Eq. (2.4), the energy difference E_k between the two spin configurations where the spins are aligned parallel with the easy axis, and the saddle-point spin configuration in between them, one may estimate the rate R_k of cluster reorientation due to spin-lattice coupling as (Eiselt *et al.*, 1979)

$$R_k = S_k^{1/2} T 10^{11} \exp(-E_k / k_B T) \ (\text{sec}^{-1}\text{K}^{-1}) , \tag{3.135}$$

where the classical formula for the attempt frequency $\nu_A \approx k_B T/h$ at $T = 1$ K was chosen. With these assumptions, one may obtain the time-dependent susceptibility

$\chi(t)$ without any adjustable parameter whatsoever (Eiselt *et al.*, 1979; see also Kötzler and Eiselt, 1979),

$$\chi(t)=(1/k_BT)\sum g_k m_k^2 x^{s_k}(1-x)^{t_k}[1-\exp(-R_k t)] \; . \tag{3.136}$$

Figure 42 shows that this approach does yield a peak of $\chi(t)$ or $\chi(\omega)$, in quantitative agreement with experiment (Eiselt *et al.*, 1979). but it is obvious that such a peak due to superparamagnetism of small clusters leads to a much more dramatic frequency shift (which, moreover, is nicely consistent with Arrhenius behavior) and would be characteristic of any spin glass (see Figs. 1 and 2 and the discussion in Sec. II). In fact, the superparamagnetic state in very dilute $Eu_xSr_{1-x}S$ is distinct from the spin glass phase in the same material [see Fig. 29(b): $T_f(\omega)$ for the spin glass phase extrapolates to zero at about x_p^{NNN}].

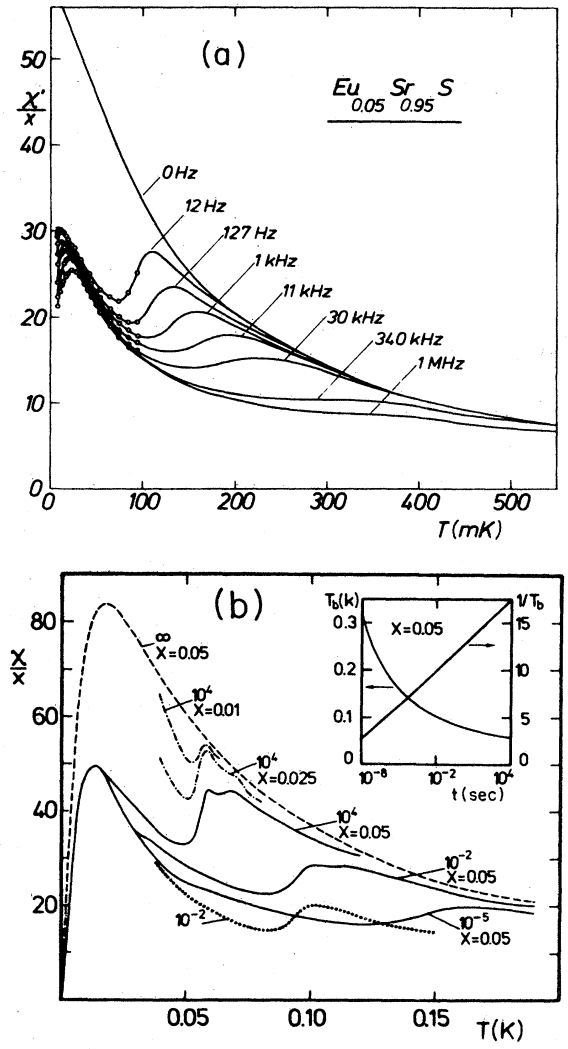

FIG. 42. Dynamic susceptibilities of $Eu_{0.05}Sr_{0.95}S$ plotted vs temperature as measured (a) and calculated (b). Parameters of the curves are measurement frequency (a) or time (b) up to which the field is applied. The dash-dotted lines in case (b) give results for $x=0.025$ and 0.01; the dotted line gives results for $x=0.05$ when all clusters larger than pairs are ignored. The inset shows the Arrhenius law for the position of the time-dependent maximum. From Eiselt *et al.* (1979).

Even in this nearly trivial description of superparamagnetism in very dilute $Eu_xSr_{1-x}S$, an inherent limitation of this approach becomes evident when one considers the region of very low temperatures, where $\chi(\omega)$ exhibits a second peak [Fig. 42(b)]. This peak can be attributed to dipolar interactions among different clusters (Eiselt *et al.*, 1979); in the theoretical description [Fig. 42(b)] it was included only qualitatively via a mean-field treatment on the level of the theory of Klein and Brout (1963). While in very dilute $Eu_xSr_{1-x}S$ "single-cluster superparamagnetism" [peak of $\chi(t)$ at higher temperature] and the collective effects (dipolar spin glass behavior showing up via the peak at lower temperature) are separate phenomena, because of the short range of the exchange interaction, superparamagnetism of strongly correlated magnetic clusters and spin glass behavior due to interactions among clusters in other cases cannot be separated, but occur at the same time in the system on different length scales.

We now turn to a different way of describing dynamic phenomena in spin glasses in terms of clusters: we consider the frozen state well below the freezing temperature on short and intermediate time scales. Then fluctuations are possible where the spin configuration is rearranged by overturning a cluster of spins relative to its frozen environment (Binder, 1977a, 1977b; Stauffer and Binder, 1978; Dasgupta *et al.*, 1979; Ma, 1980). We may view this process as a thermally activated hopping over one of the small barriers in the rugged landscape of the coarse-grained free energy depicted schematically in Fig. 4. Figure 43 shows such a small barrier for an Ising glass in more detail. Both the distance $\varepsilon_2-\varepsilon_1=\varepsilon$ between the energy levels and the barrier heights v in between them are random variables, if one studies these excitations in a random-bond model. Dasgupta *et al.* (1979) express the relaxation of energy, magnetization, and susceptibility in terms of the probability distribution $N(\varepsilon,v)$ that energy differences $|\varepsilon|$ between the levels and energy barriers v

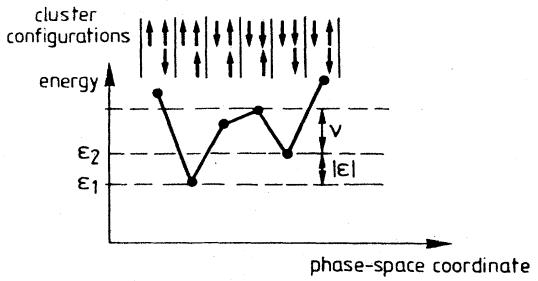

FIG. 43. Schematic plot of energy vs a phase-space coordinate representing various states of a "cluster." From Binder (1980a), following Dasgupta *et al.* (1979).

will occur: For instance, one considers an initial distribution with a finite total magnetization. Suppose that single-spin flips have brought the system to an energy minimum. Dasgupta *et al.* (1979) then concentrate on the further decay due to the flipping of clusters,

$$\sigma_R(t)=\bar{\sigma}_R+\sigma'(0)t^{-\alpha_M T}/\alpha_M \ , \tag{3.137a}$$

where

$$\bar{\sigma}_R=\int_0^\infty dv \int_{-\infty}^{\infty} d\varepsilon\, \bar{m}\ \mathrm{sgn}\varepsilon N(\varepsilon,v) \ , \tag{3.137b}$$

$$\sigma'(v)=\int_{\infty}^{0} d\varepsilon\, 2\bar{m}N(\varepsilon,v) \ , \tag{3.137c}$$

and

$$\alpha_M=-\{[d\sigma'(v)/dv]/\sigma'(v)\}_{v=0} \ . \tag{3.137d}$$

In these equations, $\bar{m}$ denotes the spin per cluster averaged over the $N(\varepsilon,v)d\varepsilon\, dv$ clusters. Obviously, for a suitable distribution of barrier heights, one obtains coefficients $\alpha_M>0$, and hence this mechanism may account for the apparent power-law decay of the remanent magnetization, seen over some range of times sometimes in experiments [Fig. 19(b)] and simulations (see Sec. V.B). This treatment also predicts that the exponent $a(T,H)$ in the phenomenological law Eq. (2.33) is linearly proportional to temperature.

Of course, it is a difficult task to calculate $N(\varepsilon,v)$ analytically. For a square nearest-neighbor Gaussian Ising spin glass, Dasgupta *et al.* (1979) estimate the leading contribution of $N(\varepsilon,v)$ due to clusters containing two spins only, and find reasonable agreement with corresponding Monte Carlo simulations. We shall return to this work and related studies (Takayama and Takase, 1981; Kinzel, 1982a, 1982b; Nemoto *et al.*, 1982; Takayama *et al.*, 1982, 1983a, 1983b; Morgenstern, 1983a; Nemoto and Takayama, 1983; Palmer, 1983) in Sec. V.B.

In this context, it is also of interest to consider the relaxation due to single spin flips which also becomes very slow at low temperatures. One starts (Binder and Schröder, 1976a, 1976b) from the master equation describing the dynamics of an Ising spin glass,

$$\frac{d}{dt}P(\{S_j\},t)=-\sum_i W(S_i\to -S_i)P(\{S_j\},t) + \sum_i W(-S_i\to S_i)P(\{S_{j\neq i},-S_i\},t) \ , \tag{3.138}$$

where $P(\{S_j\},t)$ is the probability that the spin configuration $\{S_j\}$ occurs at time t and $W(S_i\to -S_i)$ is a single spin-flip transition probability per unit time, which—following Glauber (1963)—one usually assumes to be

$$W(S_i\to -S_i)=\frac{1}{\tau_1}\frac{\exp(-\delta\mathscr{H}/k_BT)}{1+\exp(-\delta\mathscr{H}/k_BT)} \ . \tag{3.139}$$

Here the time τ_1 is the average time between spin flips in the absence of exchange interactions and fields, and $\delta\mathscr{H}$ is the energy change involved in a single spin flip, which is then calculated from Eq. (3.27). As is well known, the choice of Eq. (3.139) is rather arbitrary, but at least it satisfies detailed balance with the canonical equilibrium distribution $P_{\mathrm{eq}}(\{S_j\})=(1/Z)\exp(-\mathscr{H}/k_BT)$ [for a general discussion of kinetic Ising models see Kawasaki (1972)]. From Eqs. (3.138) and (3.139) one may derive the following exact equation of motion for the spin autocorrelation function (Bray and Moore, 1979c):

$$\frac{d}{dt}\langle S_i(t)S_i(0)\rangle_T=-\langle S_i(t)S_i(0)\rangle_T + \langle S_i(t)S_i(0) \times\tanh[H_i^{\mathrm{eff}}(0)S_i(0)/k_BT]\rangle_T \ , \tag{3.140a}$$

where the effective field $H_i^{\mathrm{eff}}(0)$ is

$$H_i^{\mathrm{eff}}(0)=\sum_{j(\neq i)} J_{ij}S_j(0) \ . \tag{3.140b}$$

For low temperatures most spins are aligned along their effective field, and hence one may put (Bray and Moore, 1979c) $H_i^{\mathrm{eff}}(0)S_i(0)\approx H_i^{\mathrm{eff}}(0)$. In addition, Eq. (3.140a) is factorized

$$\frac{d}{dt}\langle S_i(t)S_i(0)\rangle_T=-\langle S_i(t)S_i(0)\rangle_T\left[1-\tanh\frac{H_i^{\mathrm{eff}}}{k_BT}\right] \ , \tag{3.141}$$

$$\langle S_i(t)S_i(0)\rangle_T=\exp\left[-t\left[1-\tanh\frac{H_i^{\mathrm{eff}}}{k_BT}\right]\right] \ .$$

Averaging this equation with the distribution of effective fields $P(H_i^{\mathrm{eff}},T)$ yields

$$\frac{d}{dt}[\langle S_i(t)S_i(0)\rangle_T]_{\mathrm{av}}=\frac{d}{dt}\int_{-\infty}^{+\infty} dH_i^{\mathrm{eff}}P(H_i^{\mathrm{eff}},T)\exp\left[-t\left[1-\tanh\frac{H_i^{\mathrm{eff}}}{k_BT}\right]\right] \tag{3.142a}$$

which for $T\to 0$ reduces to

$$\frac{d}{dt}[\langle S_i(t)S_i(0)\rangle_T]_{\mathrm{av}}=-k_BTP(0,0)\int_0^\infty dy\,(1-\tanh y)e^{-t(1-\tanh y)}\underset{t\to\infty}{\to}-\frac{k_BTP(0,0)}{2t} \ . \tag{3.142b}$$

This result implies that

$$[\langle S_i(t)S_i(0)\rangle_T]_{\rm av}=\text{const}-\tfrac{1}{2}k_BTP(0,0)\ln t\ ,$$

i.e., the spin autocorrelation follows a logarithmic decay law. This result is interesting because the coefficient obtained in front of the logarithm is simply related to the coefficient of the specific heat, which may be written as (Anderson *et al.*, 1972; Dasgupta *et al.*, 1979)

$$C\approx\frac{\partial}{\partial T}\int_0^\infty dP(H_i^{\rm eff}=\varepsilon,T)\varepsilon/[1+\exp(\varepsilon/k_BT)]$$

$$\underset{T\to 0}{\to}\frac{\pi^2}{6}P(0,0)k_BT\ . \tag{3.143}$$

The physical picture for $T\to 0$ is a dilute gas of independently excited spins, similar to a free Fermi gas at low temperatures, with $P(0,0)$ being the density of states. Thus the magnetic susceptibility assumes the form of a Pauli susceptibility (Dasgupta *et al.*, 1979),

$$\chi=\int_0^\infty d\varepsilon\,P(\varepsilon T)2\frac{\partial}{\partial\varepsilon}\{1-2/[1+\exp(\varepsilon/k_BT)]\}$$

$$\to 2P(0,0)\ . \tag{3.144}$$

Of course, these considerations about single spins or clusters having two low-lying states separated by a distance ε have much in common with the concept of "two-level systems" introduced for glassy systems in general (Anderson *et al.*, 1972; Phillips, 1972). But an important distinction is that here we consider barrier hopping due to thermal activation only; in contrast to "window glasses" there is no evidence that tunneling between the two levels is important. Thus the dynamic description embodied in Eqs. (3.137)—(3.142) is reasonable for an Ising spin glass, but not for a true glass.

So far we have considered motion over single barriers only. In order to obtain a larger change of state, one must move over many barriers in the "rugged free-energy landscape" schematically depicted in Fig. 4. These many sequential correlated activation steps have recently been considered by Palmer *et al.* (1984). These authors construct a model containing a hierarchy of degrees of freedom, from fast to slow. The faster degrees of freedom successively constrain the slower ones. With some arbitrary but plausible assumptions, Palmer *et al.* construct a model which instead of Eq. (3.137) yields the "Kohlrausch law," Eq. (2.26) (Kohlrausch, 1847). This function is known to be a fairly accurate description of viscoelastic relaxation in polymers, dielectric relaxation in disordered materials [in these contexts it is called the "Williams-Watts function" (Williams and Watts, 1970)], etc. Thus although the model of Palmer *et al.* (1984) makes no explicit connection to a more microscopic model of a spin glass, the apparent universality of this relaxation in fact calls for a rather general mechanism as well. Other theoretical models that also lead to Eq. (2.26) (Ngai, 1979, 1980; Schlesinger and Montroll, 1984) seem even less related to spin glasses. An intriguing remark of Palmer *et al.* (1984) is that there may be a connection between the Kohlrausch law, Eq. (2.26), and Vogel-Fulcher type behavior, Eq. (2.14), of the maximum relaxation time in the system (on this time scale the relaxation is simply exponential).

While the treatments of Dasgupta *et al.* (1979) and Bray and Moore (1979b) sketched in Eqs. (3.137)—(3.142) refer explicitly to Ising spin glasses, most real systems are closer to Heisenberg spin glasses. It is not *a priori* obvious, however, that for continuous degrees of freedom such barrier-hopping mechanisms are similarly important. Ma (1980) considers clusters of spins in a frozen background of a quantum Heisenberg model and shows that, due to the coupling of the spins in the cluster to the environment, an effective anisotropy for the cluster arises, although the Hamiltonian is strictly isotropic. Ma (1980) finds qualitatively similar relaxation behavior to that in the Ising case [Eq. (3.137)].

While this treatment implies the existence of two-level systems in spin glasses with continuous spins, and corresponding structures have indeed been identified in numerical studies (Reed, 1979a; Walker and Walstedt, 1980; Grzonka and Moore, 1983, 1984; Henley, 1984b; see Sec. V.B), we anticipate an important distinction between Heisenberg spin glasses and "window glasses": for the latter, it is believed that two-level systems are the exclusive reason for the linear variation of the specific heat with temperature at low temperatures (Phillips, 1972; Anderson *et al.*, 1972). In Heisenberg spin glasses, numerical studies (Walker and Walstedt, 1977, 1980) have suggested that the linear specific heat can be explained quantitatively by a high density of states of delocalized modes (i.e., "spin waves" of a sort). Moreover, there should be other modes of relaxation, which might lead to relaxation similar to the laws of Eqs. (3.137) and (3.142), but have nothing to do with two-level systems. Bray and Moore (1982a) have suggested that local minima occur with a high density of directions for which the (free) energy surface is essentially flat. Bray and Moore (1982a) suggest that the system may stay in the vicinity of such a particular local minimum (which they call a "hole") for macroscopic times; but they also raise the question of whether their "hole story" really is the whole story, as far as slow relaxation in Heisenberg spin glasses is concerned.

IV. MEAN-FIELD THEORY

It was the observation by Cannella and Mydosh (1972) of a cusp in the ac susceptibility which showed the importance of cooperative effects in spin glasses and led theorists to the idea that there might be a sharp phase transition. The first theoretical paper to predict a phase transition was that of Adkins and Rivier (1974), though their idea of a "short-range order parameter" appears to predict a transition only because the RKKY interaction is long range, and has not been taken up subsequently. Most recent theoretical developments stem from the classic paper of Edwards and Anderson (1975; see Sec. III.D), who proposed a simpler model than the RKKY interac-

tion and investigated its properties within what they called a mean-field theory. Although some aspects of their work have to be refined to make a consistent mean-field theory, as we shall discuss in detail, their basic ideas have been amply vindicated. Shortly afterwards Sherrington and Kirkpatrick (1975) proposed a variant of the Edwards-Anderson (EA) model in which mean-field theory should be exact, and most of the subsequent work in this area has been in finding the exact solution of the SK model. In this section we shall summarize present understanding of this model, concentrating on the simplest case of Ising spins. Other models, such as those with Heisenberg spins, will be discussed relatively briefly in Sec. IV.H.

A. Sherrington-Kirkpatrick model and replica-symmetric solution

The mean-field theory (MFT) of ferromagnetism (e.g., Stanley, 1971) has the two desirable virtues of a successful approximation: (a) it is simple; (b) its predictions usually agree fairly well with experiment. Concerning (a), the theory is characterized by a single order parameter, the magnetization per spin M, which is determined from the solution of a single transcendental equation. As for (b), one usually finds that for three-dimensional systems the global features of the phase diagram are reproduced, though critical behavior close to the transition temperature is not so well described (Stanley, 1971). Furthermore MFT is the starting point for better theories (see, for example, Ma, 1976) which include fluctuation effects neglected in MFT. These theories predict that fluctuations destroy the transition completely if the space dimension d is less than a special value d_l, called the lower critical dimension. In the case of ferromagnetism $d_l=1$ for Ising spins and $d_l=2$ for vector spin models with continuous rotational symmetry of the spins (Stanley, 1971). Clearly MFT is in serious error if $d<d_l$.

For an Ising ferromagnet with no disorder the Hamiltonian is

$$\mathcal{H}=-\sum_{\langle i,j\rangle} J_{ij}S_iS_j\ , \tag{4.1}$$

where $S_i=\pm 1$ and the sum is over each pair (i,j) once. The MFT states that the magnetization M is given by the stable solution of

$$M=\tanh(\beta JM)\ , \tag{4.2}$$

where $J=\sum_{j\neq i}J_{ij}$. This is well known (Stanley, 1971) to have only a solution with $M=0$ for $T>T_c=J/k_B$, whereas two more solutions with M of equal magnitude and opposite sign occur for $T<T_c$. One can show that these solutions with nonzero M are stable for $T<T_c$ and that the solution with $M=0$ is a local maximum of the free energy. One can allow for more general ordered states such as antiferromagnets by permitting the local spin expectation value to depend on site and by calculating m_i $(=\langle S_i\rangle_T)$ from

$$m_i=\tanh\left[\beta\sum_j J_{ij}m_j\right]\ . \tag{4.3}$$

It would obviously be very useful to have a MFT for spin glasses, and one might ask whether Eq. (4.3) is a satisfactory MFT for the EA Ising spin glass model (see Sec. III.D). We shall see in Sec. IV.C that it is not, and that another term must be included in the argument of the tanh. However, even if Eq. (4.3) were adequate for spin glasses, we would still not have a complete theory. Equation (4.3) actually represents N equations which have to be solved for a particular set of J_{ij} and then averaged over the J_{ij} distribution. This is an extremely difficult task, as we shall see for the corrected equations in Sec. IV.C. We really need a theory with a small number of equations that involve the distribution of J_{ij} rather than a particular realization. The work of the Japanese school (Matsubara and Sakata, 1976; Ueno and Oguchi, 1976), which came slightly after Edwards and Anderson, starts with equations similar to (4.3), though at a higher level of approximation, such as Bethe-Peierls, and then proceeds to average over the disorder, which needs further approximations. We shall not discuss the details of this approach and its subsequent developments (e.g., Katsura, 1976; Oguchi and Ueno, 1977; Tamaribuchi and Takano, 1978; Katsura and Fujiki, 1979; Takano, 1980) because, interesting though it is, we feel that it has contributed rather less to an understanding of spin glasses than the Edwards-Anderson—Sherrington-Kirkpatrick approach. We have preferred, in this section, to give a fairly thorough and hopefully pedagogical account of what we feel has been the "mainstream" area, but this necessarily means that other interesting approaches are mentioned only briefly or not at all.

Instead of averaging the local mean-field equations, (4.3), analytically one can solve them numerically for a given set of J_{ij} by an iterative method (Soukoulis *et al.*, 1983a). This local mean-field approach is not quantitatively reliable, because it misses the "reaction field" term (see Sec. IV.C). However, it is straightforward to implement and probably gives a reasonably good physical picture of phenomena taking place on finite time scales where the system gets stuck in states that are not necessarily the lowest-energy state.

Since it is not *a priori* obvious what mean-field theory should be, Sherrington and Kirkpatrick (1975) made the following suggestion. The MFT of ferromagnetism, which is normally an approximation, becomes exact in the limit of infinite-range interactions (Stanley, 1971) where every spin couples equally with every other spin. Sherrington and Kirkpatrick therefore proposed that the MFT of spin glasses should be the exact solution of an infinite-range EA model in which the probability distribution $P(J_{ij})$ is the same for all pairs of spins no matter how far they are apart. This is clearly an unphysical assumption, but the infinite-range model of ferromagnetism leads to a theory that agrees fairly well with experiment, so it seems sensible to try an analogous approach for spin glasses. The SK model will be studied in detail in this section.

We shall see that its solution does not have the desired feature of simplicity, ingredient (a) above, but unfortunately other models either throw away important physics or are even more complicated. There is also controversy, discussed in Sec. V, as to whether d_l is greater than three, or not, for spin glasses, and hence it is disputed whether or not the model is relevant for describing experiments, ingredient (b). In this section we shall not discuss these aspects but confine our attention to the properties of the model itself. We shall see that they are quite novel.

The SK Hamiltonian is the same form as the EA model, namely

$$\mathscr{H}=-\sum_{\langle i,j\rangle} J_{ij}S_iS_j-\sum_i H_iS_i\ , \tag{4.4}$$

where a local field H_i has been included, the notation $\langle i,j\rangle$ means include each distinct pair once, and $P(J_{ij})$, a Gaussian distribution,

$$P(J_{ij})=\frac{1}{J}\left[\frac{N}{2\pi}\right]^{1/2}\exp[-N(J_{ij}-J_0/N)^2/2J^2]\ , \tag{4.5}$$

is the same for *all* pairs of spins, so

$$\begin{aligned}&[J_{ij}]_{\rm av}=J_0/N\ ,\\ &[J_{ij}^2]_{\rm av}-[J_{ij}]_{\rm av}^2=J^2/N\ .\end{aligned} \tag{4.6}$$

In this section the symbol J refers to the standard deviation of the distribution ($\times N^{1/2}$), whereas elsewhere we use ΔJ. The factors of N^{-1} in Eq. (4.6) will be necessary to get a sensible but nontrivial thermodynamic limit. Calculations indicate that a Gaussian distribution is not essential, but that for $N\to\infty$ any distribution whose first two moments are given by Eq. (4.6) and whose higher moments are bounded will give the same results.

Following Edwards and Anderson, Sherrington and Kirkpatrick studied their model by means of the replica trick (Sec. III.C). One starts with Eq. (3.25) for $[Z^n]_{\rm av}$, which here takes the form

$$[Z^n]_{\rm av}=\sum_{\{S_i^\alpha=\pm1\}}\int_{-\infty}^{\infty}\left[\prod_{\langle ij\rangle}P(J_{ij})dJ_{ij}\right]\exp\left[\beta\sum_{\langle i,j\rangle}J_{ij}\sum_{\alpha=1}^{n}S_i^\alpha S_j^\alpha+\beta\sum_i H_i\sum_{\alpha=1}^{n}S_i^\alpha\right], \tag{4.7}$$

where $\alpha=1,\ldots,n$, denotes a replica. The integral over the J_{ij} is easily computed for the Gaussian distribution, Eq. (4.5), by completing the square, and gives

$$[Z^n]_{\rm av}=\sum_{\{S_i^\alpha=\pm1\}}\exp\left[N^{-1}\sum_{\langle i,j\rangle}\left[\tfrac{1}{2}(\beta J)^2\sum_{\alpha,\beta}S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta+\beta J_0\sum_\alpha S_i^\alpha S_j^\alpha\right]+\beta\sum_i H_i\sum_\alpha S_i^\alpha\right]. \tag{4.8}$$

Noting that $(S_i^\alpha)^2=1$ and dropping some $1/N$ corrections in the exponent, one rewrites Eq. (4.8) as

$$[Z^n]_{\rm av}=\exp[\tfrac{1}{4}(\beta J)^2nN]\sum_{\{S_i^\alpha=\pm1\}}\exp\left[\frac{(\beta J)^2}{2N}\sum_{\alpha<\beta}\left[\sum_i S_i^\alpha S_i^\beta\right]^2+\frac{\beta J_0}{2N}\sum_\alpha\left[\sum_i S_i^\alpha\right]^2+\beta\sum_i H_i\sum_\alpha S_i^\alpha\right]. \tag{4.9}$$

The squared terms can be simplified using the Hubbard-Stratonovitch identity

$$\exp(\lambda a^2/2)=\left[\frac{\lambda}{2\pi}\right]^{1/2}\int_{-\infty}^{\infty}dx\exp\left[-\frac{\lambda x^2}{2}+a\lambda x\right], \tag{4.10}$$

and one obtains

$$\begin{aligned}[Z^n]_{\rm av}=\exp[\tfrac{1}{4}(\beta J)^2nN]\int_{-\infty}^{\infty}&\left[\prod_{\alpha<\beta}\left[\frac{N}{2\pi}\right]^{1/2}\beta J\,dq_{\alpha\beta}\right]\left[\prod_\alpha\left[\frac{N\beta J_0}{2\pi}\right]^{1/2}dm_\alpha\right]\\ &\times\exp\left[-\frac{N(\beta J)^2}{2}\sum_{\alpha<\beta}q_{\alpha\beta}^2-\frac{N\beta J_0}{2}\sum_\alpha m_\alpha^2+N\log{\rm Tr}\exp L[q_{\alpha\beta},m_\alpha]\right],\end{aligned} \tag{4.11}$$

where

$$L[q_{\alpha\beta},m_\alpha]=(\beta J)^2\sum_{\alpha<\beta}q_{\alpha\beta}S^\alpha S^\beta+\beta\sum_\alpha(J_0m_\alpha+H)S^\alpha\ , \tag{4.12}$$

the trace is now over the n spins S^α, and we have taken the field to be uniform for simplicity. Note that there are $n(n-1)/2$ independent variables $q_{\alpha\beta}$ with $\alpha<\beta$. The diagonal component $q_{\alpha\alpha}$ does not occur [because $(S^\alpha)^2=1$, a constant], and we shall define $q_{\alpha\beta}$ with $\alpha>\beta$ by making q a symmetric matrix, i.e., $q_{\alpha\beta}=q_{\beta\alpha}$.

If we assume that the limit $N\to\infty$ can be taken before $n\to0$ (see Sec. III.C), then the integrals in Eq. (4.11) can be evaluated by steepest descents because the argument of

the exponent is proportional to N. From Eqs. (4.11), (4.12), and (3.24) one then finds that the free energy per spin f is given by

$$-\beta f=\lim_{n\to 0}\left[\frac{(\beta J)^2}{4}\left[1-\frac{1}{n}\sum_{(\alpha,\beta)}q_{\alpha\beta}^2\right]+\frac{\beta J_0}{2}\frac{1}{n}\sum_{\alpha}m_\alpha^2+\frac{1}{n}\log \mathrm{Tr}\exp(L)\right], \tag{4.13}$$

where the $q_{\alpha\beta}$ and m_α are given self-consistently by the saddle-point conditions

$$\frac{\partial f}{\partial q_{\alpha\beta}}=\frac{\partial f}{\partial m_\alpha}=0 \tag{4.14}$$

or, in other words,

$$q_{\alpha\beta}=\langle S^\alpha S^\beta\rangle=\lim_{n\to 0}\frac{\mathrm{Tr}S^\alpha S^\beta \exp L\,[q_{\alpha\beta},m_\alpha]}{\mathrm{Tr}\exp L\,[q_{\alpha\beta},m_\alpha]}, \tag{4.15a}$$

$$m_\alpha=\langle S^\alpha\rangle=\lim_{n\to 0}\frac{\mathrm{Tr}S^\alpha \exp L\,[q_{\alpha\beta},m_\alpha]}{\mathrm{Tr}\exp L\,[q_{\alpha\beta},m_\alpha]}. \tag{4.15b}$$

The notation $\sum_{(\alpha,\beta)}$ means sum over all distinct replicas. From Eq. (4.13) we see that $\mathrm{Tr}\exp L$ must tend to unity as $n\to 0$, to cancel the factor of n^{-1}, so the denominators in Eqs. (4.15) can be neglected. The central problem of this section will be to find the correct (in a sense discussed later) solution of Eqs. (4.15).

For future reference we note that for $J_0=0$ the energy per spin $U\,[=-\partial(\beta f)/\partial\beta]$ is given by

$$-U=\lim_{n\to 0}\frac{J^2}{2T}\left[1+\frac{1}{n}\sum_{(\alpha,\beta)}q_{\alpha\beta}^2\right]+\frac{H}{n}\sum_\alpha m_\alpha . \tag{4.16}$$

Furthermore for $J_0=0$, $H\to 0$, the susceptibility χ is

$$\chi=\lim_{n\to 0}\frac{1}{T}\left[1+\frac{1}{n}\sum_{(\alpha,\beta)}q_{\alpha\beta}\right]. \tag{4.17}$$

For the rest of this section we set Boltzmann's constant to unity.

Edwards and Anderson and Sherrington and Kirkpatrick took the replica-symmetric solution with $q_{\alpha\beta}=q$, $m_\alpha=M$. One can then evaluate the trace by writing

$$q\sum_{(\alpha,\beta)}S^\alpha S^\beta=q\left[\left[\sum_\alpha S^\alpha\right]^2-n\right] \tag{4.18}$$

and using identity (4.10) one more. One finds

$$-\beta f=\frac{(\beta J)^2}{4}(1-q)^2-\frac{\beta J_0}{2}M^2+\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}dz\,e^{-z^2/2}\log[2\cosh\beta\tilde{H}(z)]dz \tag{4.19}$$

with q and M self-consistently given by

$$q=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}e^{-z^2/2}\tanh^2[\beta\tilde{H}(z)]dz , \tag{4.20}$$

$$M=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}e^{-z^2/2}\tanh[\beta\tilde{H}(z)]dz , \tag{4.21}$$

and $\tilde{H}(z)$ related to q and m by

$$\tilde{H}(z)=Jq^{1/2}z+J_0M+H . \tag{4.22}$$

From Sec. III.C we note that $M=[\langle S_i\rangle_T]_{\mathrm{av}}$ is the magnetization and $q=[\langle S_i\rangle_T^2]_{\mathrm{av}}$ is nonzero whenever $\langle S_i\rangle_T$ is nonzero, even if the $\langle S_i\rangle_T$ are random and have no preferred sign. Hence spin glass ordering is characterized by $q\neq 0$, $M=0$. From Eqs. (4.20) and (4.21) we see that $\tilde{H}(z)$ can be interpreted as the local molecular field acting on a site. Different sites have different fields because of disorder, and the distribution is Gaussian with mean J_0M+H and variance J^2q.

These properties of the local molecular field have also been obtained without replicas (Southern, 1976; Blandin, 1978; Chalupa, 1978) from the local mean-field equations, Eq. (4.3), according to which $\tilde{H}_i$, the molecular field on a site i, is given by

$$\tilde{H}_i=H+\sum_j J_{ij}\langle S_j\rangle_T . \tag{4.23}$$

Each factor of $J_{ij}\langle S_j\rangle_T$ is a random variable and, neglecting correlations between different terms, the central limit theorem tells us that the H_i have a Gaussian distribution. Also neglecting correlations between the J_{ij} and the $\langle S_j\rangle_T$ one finds

$$[\tilde{H}_i]_{\mathrm{av}}=H+J_0M , \tag{4.24}$$

and assuming further that there are no correlations between different $\langle S_j\rangle_T$, so that

$$[\langle S_i\rangle_T\langle S_j\rangle_T]_{\mathrm{av}}=M^2+(q-M^2)\delta_{ij} , \tag{4.25a}$$

one obtains

$$[\tilde{H}_i^2]_{\mathrm{av}}-[\tilde{H}_i]_{\mathrm{av}}^2=J^2q . \tag{4.25b}$$

Noting that configurational averaging and site averaging are normally equivalent (see Sec. III.A), this gives the statistical properties of the local molecular fields $\tilde{H}_i$ derived above from the replica method. While the present derivation is intuitively reasonable and has the advantage of avoiding replicas, it is unfortunately wrong. The local mean-field equations, Eq. (4.3), are incorrect (see Sec. IV.C), and would give twice the correct T_f. In order to obtain the right answer for the distribution of $\tilde{H}_i$ one must also make the wrong assumption of statistical independence of the J_{ij} and the S_i and the error in making it must exactly compensate for the error in assuming the local mean-field equations. This tells us that we have to be very careful about neglecting correlation effects in spin glasses.

It is straightforward to solve Eqs. (4.20) and (4.21) numerically for M and q and also obtain analytic results in limiting cases. For $H=0$ the phase diagram is shown in Fig. 44. The phases are paramagnetic with $M=q=0$, ferromagnetic with $M\neq 0, q\neq 0$, and spin glass with $M=0, q\neq 0$. Clearly if $M\neq 0$ then q must also be

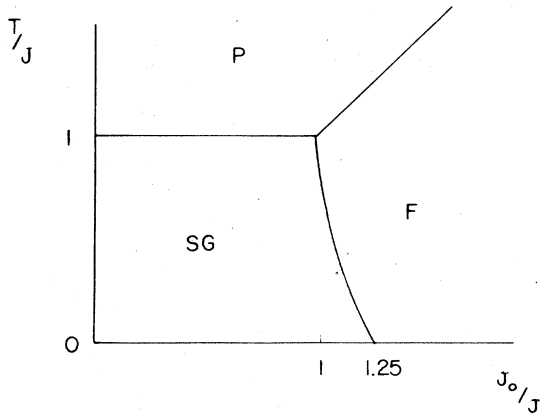

FIG. 44. Phase diagram for the SK solution of the infinite-range SK model. J and J_0 are related to the width and mean of the exchange distribution by Eq. (4.6). The phases are paramagnetic (P), ferromagnetic (F), and spin glass (SG). Note that for $1<J_0/J<1.25$ the system goes from paramagnet to ferromagnet to a "reentrant" spin glass phase as the temperature is lowered. This reentrant behavior does not occur in the exact solution of the SK model (see Fig. 49).

nonzero. Note that for $1\leq J_0/J\leq 1.25$ the system passes from a paramagnetic to a ferromagnetic to a "reentrant" spin glass phase as the temperature is reduced. It seems curious that at intermediate temperatures, where entropy plays a role, the system is ferromagnetic, but as $T\to 0$ one finds that the more disordered spin glass state actually has lower energy and so is preferred. In fact the SK calculation does not get the entropy correct at low temperatures (it becomes negative, which is impossible for any system with a countable number of states), and the exact solution does not, apparently, have a reentrant spin glass phase (see Sec. IV.B).

We discuss first the solution of Eqs. (4.19)—(4.22) for $J_0=H=0$ where, of course, $M=0$. Expanding the integrand in Eq. (4.19) one finds for small q that

$$\beta f=\beta f_0+\frac{1}{4}\left[1-\frac{T^2}{J^2}\right]Q^2-\frac{Q^3}{3}+\tfrac{17}{24}Q^4+\cdots\ , \tag{4.26}$$

where $Q=(\beta J)^2 q$ and $\beta f_0=-\log 2-\beta^2 J^2/4$. This is a special case of Eq. (3.56) with $M=0$ and for Ising spins, so $q_\perp$ does not exist and $q\equiv q_{||}$. The transition occurs when the coefficient of the quadratic term vanishes, so

$$T_f=J\ . \tag{4.27}$$

Finding the extremal values with respect to q of Eq. (4.26) and defining

$$\theta=\frac{T-T_f}{T_f}\ , \tag{4.28}$$

one obtains

$$q=|\theta|+\tfrac{1}{3}|\theta|^2+\cdots\ . \tag{4.29a}$$

It is also straightforward to show that

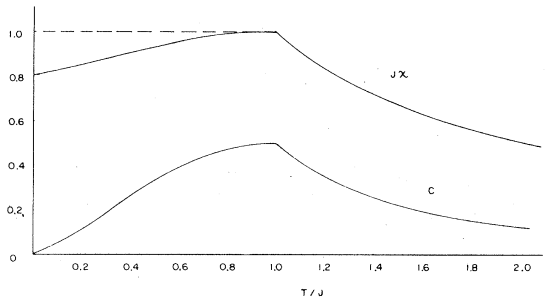

FIG. 45. Plot of the susceptibility χ and specific heat C for the SK solution of the SK model with $J_0=H=0$. Note that $T_f=J$. The dashed line is the prediction of the Parisi theory for χ.

$$\chi=\frac{1}{T}(1-q)\ , \tag{4.29b}$$

$$-U=\frac{J^2}{2T}(1-q^2)\ , \tag{4.29c}$$

$$f=-T\log 2-\frac{1}{4}\frac{J^2}{T}\quad (T>T_f)\ , \tag{4.29d}$$

$$C=\frac{J^2}{2T^2}\quad (T>T_f)\ , \tag{4.29e}$$

$$q=1-\left[\frac{2}{\pi}\right]^{1/2}\frac{T}{J}+\cdots\quad (T\to 0)\ , \tag{4.29f}$$

where C is the specific heat. The susceptibility and specific heat are shown in Fig. 45. A cusp is predicted at T_f both for C and χ, but experimentally only a cusp in χ is seen. Figure 46 shows the temperature dependence of q (labeled $q_{\rm SK}$ in the figure).

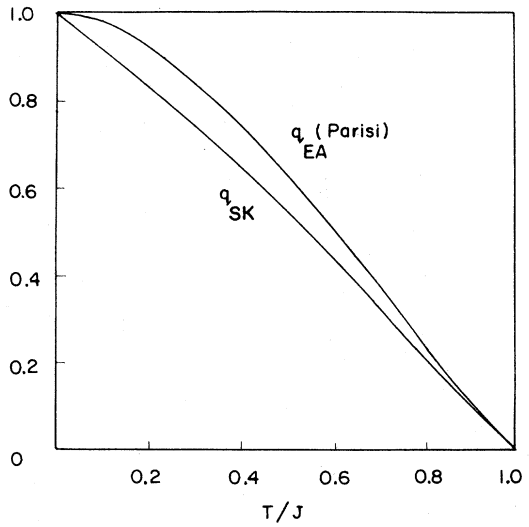

FIG. 46. $q_{\rm SK}$ is the spin glass order parameter for the SK solution of the SK model with $J_0=H=0$. $q_{\rm EA}$ is the approximate result (4.88) for $q(1)$ in the Parisi theory.

For $J_0=0$ but $H\neq 0$ one finds that M is a nonanalytic function of H at T_f, and for small H

$$M=\frac{H}{T}\left[1-\frac{H^2}{3T^2}\frac{T^2+2T_f^2}{T^2-T_f^2}+\cdots\right] \quad (T>T_f)\,, \tag{4.30}$$

$$M=\frac{H}{T_f}\left[1-\frac{|H|}{\sqrt{2}T_f}+\cdots\right] \quad (T=T_f)\,. \tag{4.31}$$

Hence $\chi_{\rm nl}\equiv\partial^3 M/\partial H^3$ diverges at T_f, and for $T\to T_f^+$

$$\chi_{\rm nl}\propto\theta^{-1}\,. \tag{4.32}$$

A divergence of $\chi_{\rm nl}$ is a general feature of spin glass transitions (see Sec. III.F; Katsura, 1976; Suzuki, 1977). A finite magnetic field rounds out the cusps in χ and C, though the effect is rather weaker than the rounding of χ observed experimentally.

Since q is the spin glass order parameter and $\chi_{\rm nl}$ is essentially (Sec. III.F) the corresponding susceptibility, one can define exponents β and γ by

$$\chi_{\rm nl}\propto(T-T_f)^{-\gamma} \quad (T\to T_f^+)\,, \tag{4.33a}$$

$$q\propto(T_f-T)^{\beta} \quad (T\to T_f^-)\,. \tag{4.33b}$$

The mean-field exponent values are therefore

$$\gamma=1\,, \tag{4.34a}$$

$$\beta=1\,, \tag{4.34b}$$

and the cusp in the specific heat shows that the corresponding exponent, α, has the value

$$\alpha=-1\,. \tag{4.34c}$$

This replica-symmetric solution is fairly simple and intuitively appealing in that a transition to a spin glass phase, characterized by a single order parameter q, is predicted at low temperatures. However, there are some curious features, which we now discuss. Consider $J_0=H=0$, for which $M=0$, so f is only a function of q. f is a maximum with respect to q for the "obvious" solutions, $q=0$, $T>T_f$, $q\neq 0$, $T<T_f$; furthermore, below T_f the $q=0$ solution has a lower free energy than the spin glass state (see Fig. 47). These results are just the opposite of what is expected.

The resolution of this paradox lies in the fact that the number of variables $q_{\alpha\beta}$ is $n(n-1)/2$, which becomes *negative* as $n\to 0$. The saddle-point calculation above makes sense only if the chosen solution is a local minimum (i.e., locally stable). That is to say, we form the matrix of second derivatives $\partial^2 f/\partial q_{\alpha\beta}\partial q_{\gamma\delta}$, evaluated at the saddle point, and the solution is locally stable if there are *no negative eigenvalues*. This is because the N^{-1} corrections to the saddle-point result involve Gaussian integrals about the saddle point which must converge. However, changing all the $q_{\alpha\beta}$'s simultaneously will *decrease* the free energy because there are a negative number of them.

Above T_f the $q=0$ solution is stable, but below T_f it is unstable (see Sec. IV.B) and so must be rejected, even though it has a lower free energy than the spin glass solution. In fact, below T_f the SK solution with $q\neq 0$ is also unstable (Sec. IV.B) and so must be rejected too. This is why the entropy becomes negative at low temperatures. The presumed correct solution (Parisi, 1979, 1980a, 1980b, 1980c, 1980d, 1980e) has an even higher free energy. Because of this, one frequently sees the statement that the free energy in spin glasses should be maximized. This is, however, rather misleading because one could have a situation with a first-order transition when there are two (or more) stable solutions. The saddle-point method says very clearly that in this case the solution of minimum free energy must be taken. Hence the correct statement is that we are looking for the *stable* solution of Eqs. (4.15) of *lowest* free energy. This has recently been demonstrated by Mottishaw and Sherrington (1985) for an exactly solvable model which has a first-order spin glass transition. It turns out, however, that local stability may not be enough for the solution to be truly stable, and there is at least one problem, discussed in Sec. IV.H.3, in which there are two locally stable solutions but thermodynamics forces you to choose the one of higher energy. The one of lower free energy must be metastable. Thus the problem of which solution to choose is sometimes not trivial. The statement that the free energy should be maximized often works, but should be treated with caution.

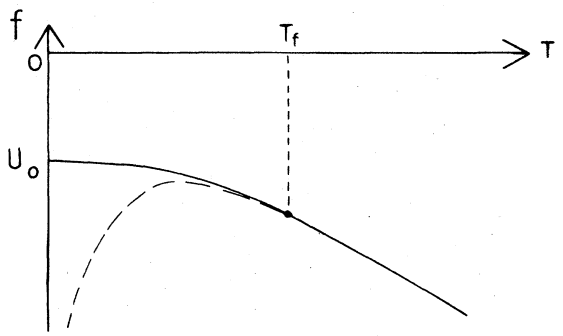

FIG. 47. Sketch of the free energy of the SK solution of the SK model with $J_0=H=0$. U_0 is the ground-state energy per spin ($=\sqrt{2/\pi}J=0.80J$) of the SK solution. The dashed line shows the analytic continuation of the paramagnetic solution below T_f, where it lies below the SK solution of the spin glass phase. The paramagnetic solution must, however, be rejected below T_f because it is unstable. The SK solution also turns out to be unstable below T_f, and the Parisi solution, which is believed to be exact, has a somewhat higher free energy than the SK value.

While this point of view seems to work, it is not completely satisfactory. In the saddle-point approach one really looks for the largest value of the exponent in Eq. (4.11). Usually this is at an extremum, but here the dominant contribution appears to be at the end points $q_{\alpha\beta}\to\pm\infty$. The standard approach is to ignore these regions. The fact that they seem to dominate is probably related to the interchange of limits ($N\to\infty$ followed by $n\to 0$ instead of the other way round). For the percola-

tion problem this has indeed been shown recently by Rudnick and Gaspari (1986). If the limits are performed in the correct order, the $q \to \pm\infty$ regions do not contribute for the percolation problem (Rudnick and Gaspari, 1986) and we expect the same to be true in spin glasses.

B. Replica symmetry breaking and Parisi's solution

We now discuss in more detail the instability of the replica-symmetric solution mentioned above and then discuss a stable solution that appears to be correct.

First of all it is useful to note that saddle-point fluctuations give us values for correlation functions such as the spin glass susceptibility χ_{SG}, defined by

$$\chi_{SG}=\frac{1}{N}\sum_{i,j}[(\langle S_iS_j\rangle_T-\langle S_i\rangle_T\langle S_j\rangle_T)^2]_{av} . \tag{4.35}$$

From the rules in Sec. III.C we express the individual terms as replica averages in the following way:

$$[\langle S_iS_j\rangle_T^2]_{av}=\lim_{n\to 0}\frac{1}{n(n-1)}\sum_{(\alpha,\beta)}\langle S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta\rangle ,$$

$$[\langle S_iS_j\rangle_T\langle S_i\rangle_T\langle S_j\rangle_T]_{av}=\lim_{n\to 0}\frac{1}{n(n-1)(n-2)}\times\sum_{(\alpha,\beta,\gamma)}\langle S_i^\alpha S_j^\alpha S_i^\beta S_j^\gamma\rangle , \tag{4.36}$$

$$[\langle S_i\rangle_T^2\langle S_j\rangle_T^2]_{av}=\lim_{n\to 0}\frac{1}{n(n-1)(n-2)(n-3)}\times\sum_{(\alpha,\beta,\gamma,\delta)}\langle S_i^\alpha S_j^\beta S_i^\gamma S_j^\delta\rangle ,$$

the average being over all sets of distinct replicas. In the thermodynamic limit it follows from the saddle-point calculation in Sec. IV.A that averages on different sites decouple, so, for the replica-symmetric solution, each expression in (4.36) is equal to q^2, and the terms in Eq. (4.35) cancel. It is therefore necessary to include fluctuations to calculate χ_{SG}.

Let us, for simplicity, evaluate χ_{SG} with $J_0=H=0$, in which case the m_α variables can be integrated out. It is also useful to add to the effective Hamiltonian in the exponent of Eq. (4.8) a set of fictitious fields $\Delta_{\alpha\beta}$ which couple to $\sum_i S_i^\alpha S_i^\beta$, i.e.,

$$[Z^n]_{av}=\sum_{\{S_i^\alpha=\pm 1\}}\exp\left[\frac{(\beta J)^2}{2N}\sum_{\langle i,j\rangle}\sum_{\alpha,\beta}S_i^\alpha S_j^\alpha S_i^\beta S_j^\beta+\sum_{\alpha<\beta}\Delta_{\alpha\beta}\sum_i S_i^\alpha S_i^\beta\right] . \tag{4.37}$$

Taking derivatives with respect to $\Delta_{\alpha\beta}$ and remembering that for $n\to 0$ there is no normalizing denominator in the definition of correlation functions [see Eq. (3.32d)], one has

$$\sum_i\langle S_i^\alpha S_i^\beta\rangle=\lim_{n\to 0}\frac{\partial}{\partial\Delta_{\alpha\beta}}[Z^n]_{av} , \tag{4.38}$$

$$\sum_{i,j}\langle S_i^\alpha S_i^\beta S_j^\gamma S_j^\delta\rangle=\lim_{n\to 0}\frac{\partial^2}{\partial\Delta_{\alpha\beta}\partial\Delta_{\gamma\delta}}[Z^n]_{av} . \tag{4.39}$$

In Eq. (4.39), α, β, γ, and δ can take any values, except that $\alpha<\beta$, $\gamma<\delta$. If we include $\Delta_{\alpha\beta}$, Eq. (4.11) becomes

$$[Z^n]_{av}=\int_{-\infty}^{\infty}\left[\prod_{\alpha\beta}\left(\frac{N}{2\pi}\right)^{1/2}\beta J\,dq_{\alpha\beta}\right]\exp(-Nn\beta f\{q\}) , \tag{4.40}$$

where

$$-\beta f\{q\}=\frac{(\beta J)^2}{4}\left[1-\frac{1}{n}\sum_{(\alpha,\beta)}q_{\alpha\beta}^2\right]+\frac{1}{n}\log\mathrm{Tr}\exp(L\{q\}) \tag{4.41}$$

with

$$L\{q\}=\sum_{\alpha<\beta}[(\beta J)^2q_{\alpha\beta}+\Delta_{\alpha\beta}]S^\alpha S^\beta . \tag{4.42}$$

Using new integration variables $u_{\alpha\beta}$ defined by

$$u_{\alpha\beta}=q_{\alpha\beta}+(\beta J)^{-2}\Delta_{\alpha\beta} , \tag{4.43}$$

we find that the $\Delta_{\alpha\beta}$ no longer appear in L but occur instead in the quadratic term in Eq. (4.41). The derivatives in Eqs. (4.38) and (4.39) can now be evaluated and give, for $\alpha<\beta$, $\gamma<\delta$,

$$\frac{1}{N}\sum_i\langle S_i^\alpha S_i^\beta\rangle=\lim_{n\to 0}\langle q_{\alpha\beta}\rangle , \tag{4.44}$$

$$\frac{1}{N}\sum_{i,j}\langle S_i^\alpha S_i^\beta S_j^\gamma S_j^\delta\rangle=\lim_{n\to 0}[N\langle q_{\alpha\beta}q_{\gamma\delta}\rangle-(\beta J)^{-2}\delta_{\alpha\beta,\gamma\delta}] , \tag{4.45}$$

where $\delta_{\alpha\beta,\gamma\delta}$ is one if the pair (α,β) is the same as the pair (γ,δ) and zero otherwise, and the averages are for the weight $\exp(-Nn\beta f\{q\})$ with $\Delta_{\alpha\beta}=0$. Evaluating Eq. (4.44) by steepest descents gives Eq. (4.15a), while Eq. (4.45) requires a study of the Gaussian fluctuations about the saddle point. We define

$$q_{\alpha\beta}=q_{\alpha\beta}^c+\delta q_{\alpha\beta} , \tag{4.46}$$

where $q_{\alpha\beta}^c$ is the saddle-point value ($=q$ in the SK solution), so from Eqs. (4.35), (4.36), (4.45), and (4.46) we obtain for the SK solution

$$\chi_{SG}=N\lim_{n\to 0}[\langle\delta q_{\alpha\beta}^2\rangle-2\langle\delta q_{\alpha\beta}\delta q_{\alpha\gamma}\rangle+\langle\delta q_{\alpha\beta}\delta q_{\gamma\delta}\rangle] , \tag{4.47}$$

where all replicas are different.

Having shown that saddle-point fluctuations determine χ_{SG}, we now evaluate these fluctuations. We expand $f\{q\}$ up to second order in the $\delta q_{\alpha\beta}$, obtaining, for any saddle point,

$$\beta f\{q\}=\beta f\{q^c\}+\lim_{n\to 0}\frac{1}{2n}\sum_{\substack{\alpha<\beta\\ \gamma<\delta}} R^{\alpha\beta,\gamma\delta}\delta q_{\alpha\beta}\delta q_{\gamma\delta}+\cdots\ , \tag{4.48}$$

where $f\{q^c\}$ is the saddle-point free energy, and

$$(\beta J)^{-2}R^{\alpha\beta,\gamma\delta}=\delta_{\alpha\beta,\gamma\delta}-(\beta J)^2[\langle S^\alpha S^\beta S^\gamma S^\delta\rangle - \langle S^\alpha S^\beta\rangle\langle S^\gamma S^\delta\rangle]\ , \tag{4.49}$$

the averages being over weight $\exp(L\{q\})$, Eq. (4.42), with $\Delta_{\alpha\beta}=0$ and $q_{\alpha\beta}=q^c_{\alpha\beta}$. To perform the Gaussian integrations over the $\delta q_{\alpha\beta}$, which come from Eqs. (4.40) and (4.48), we need to find the $n(n-1)/2$ eigenvalues of the quadratic form $R^{\alpha\beta,\gamma\delta}$. We see immediately that the saddle point is sensible only if none of the eigenvalues are negative. Otherwise the Gaussian integrals (which, as we have seen, give rise to physically meaningful quantities such as $\chi_{\rm SG}$ and also, it turns out, the $1/N$ corrections to f) diverge.

It is trivial to calculate the eigenvalues above T_c where $L\{q\}=0$, so $R^{\alpha\beta,\gamma\delta}$ is diagonal, i.e.,

$$R^{\alpha\beta,\gamma\delta}=\delta_{\alpha\beta,\gamma\delta}(\beta J)^2[1-(\beta J)^2]\ . \tag{4.50}$$

Hence all eigenvalues are equal and given by

$$\lambda=(\beta J)^2[1-(\beta J)^2]\ . \tag{4.51}$$

From Eqs. (4.48) and (4.50) it follows that

$$\langle \delta q^2_{\alpha\beta}\rangle=(N\lambda)^{-1}\ ,$$

$$\langle \delta q_{\alpha\beta}\delta q_{\alpha\gamma}\rangle=\langle \delta q_{\alpha\beta}\delta q_{\gamma\delta}\rangle=0\ ,$$

so, from Eq. (4.47),

$$\chi_{\rm SG}=\frac{1}{\lambda}-(\beta J)^{-2} \tag{4.52}$$

$$=\frac{T^2}{T^2-T_f^2}\quad (T>T_f)\ , \tag{4.53}$$

where the last line follows from $T_f=J$. Since $\chi_{\rm SG}$ is related to $\chi_{\rm nl}$ by (3.109), Eq. (4.53) agrees with the expression for $\chi_{\rm nl}$ in Eq. (4.30), obtained directly from thermodynamics. We see that $\chi_{\rm SG}$ diverges at T_f because $\lambda=0$, signaling the onset of an instability. As $T\to T_f^+$, λ is given by

$$\lambda=2\theta+\cdots\ , \tag{4.54}$$

where $\theta=(T-T_f)/T_f$.

Below T_f, $R^{\alpha\beta,\gamma\delta}$ is no longer diagonal. The diagonalization has been performed for the replica-symmetric solution by de Almeida and Thouless (1978), Pytte and Rudnick (1979), and Bray and Moore (1979a). They find that there are three distinct eigenvalues λ_i, $i=1,2,3$ for general n, but λ_1 and λ_2 become degenerate for $n\to 0$. The combination of fluctuations in Eq. (4.47) corresponds (Pytte and Rudnick, 1979) to the eigenvector for λ_3, called the "replicon" mode (Bray and Moore, 1979a). Hence $\chi_{\rm SG}$ is given by

$$\chi_{\rm SG}=\frac{1}{\lambda_3}-(\beta J)^{-2}\ , \tag{4.55}$$

where λ_3 is negative and for $T\to T_f^-$

$$\lambda_3=-\tfrac{4}{3}\,|\,\theta\,|^{\,2}\ . \tag{4.56}$$

By contrast $\lambda_{1,2}$, related to spin fluctuations by (Pytte and Rudnick, 1979)

$$\lambda_{1,2}^{-1}-(\beta J)^2=\chi_A=\frac{1}{N}\sum_{i,j}[\langle S_iS_j\rangle^2-4\langle S_iS_j\rangle\langle S_i\rangle\langle S_j\rangle + 3\langle S_i\rangle^2\langle S_j\rangle^2]_{\rm av} \tag{4.57}$$

is positive and given by

$$\lambda_{1,2}=2\,|\,\theta\,| \tag{4.58}$$

as $T\to T_f^-$.

Equation (4.56) spells disaster for the SK solution below T_f because $\chi_{\rm SG}$ is then predicted to be negative, which is clearly impossible from the definition (4.35).

The eigenvalues can be expressed as functions of n, and one finds that for positive integer n they are all positive. The instability only shows up when we continue to real n and let $n\to 0$. There is a critical value $n_c(\theta)$ below which λ_3 becomes negative, and for $|\,\theta\,|\to 0$ one can show that (Kondor, 1983)

$$n_c(\vartheta)=\tfrac{4}{3}\,|\,\theta\,|\ . \tag{4.59}$$

de Almeida and Thouless (1978) have found that when one includes a field or a nonzero mean in the distribution, λ_3 is positive provided

$$1>(\beta J)^2(1-2q+r)\ , \tag{4.60}$$

where

$$r=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}e^{-z^2/2}\tanh^4\tilde{H}(z)dz \tag{4.61}$$

and where $\tilde{H}(z)$ is given by Eq. (4.22). The borderline situation in which there is an equality in Eq. (4.60) is known as the Almeida-Thouless (AT) line. With $J_0=0$, $H\neq 0$, the AT line is shown in Fig. 48 in the H-T plane. In certain limits it can be calculated analytically with the results

$$\frac{\delta T_f}{J}=\left[\frac{3}{4}\right]^{1/3}\left[\frac{H}{J}\right]^{2/3},\quad \delta T_f\to 0\ , \tag{4.62}$$

where $\delta T_f=T_f-T_{\rm AT}(H)$, T_f is the zero-field transition temperature, and

$$\frac{T_{\rm AT}(H)}{J}=\frac{4}{3}\frac{1}{(2\pi)^{1/2}}\exp(-H^2/2J^2),\quad T\to 0\ . \tag{4.63}$$

For $H=0$, $J_0\neq 0$, the AT line is shown in Fig. 49 as the boundary between the ferromagnet phase F and a modified ferromagnet phase F' (sometimes called a mixed phase). The properties of this phase, F', and the boundary between it and the spin glass phase cannot be calculated from the SK solution. One needs the Parisi

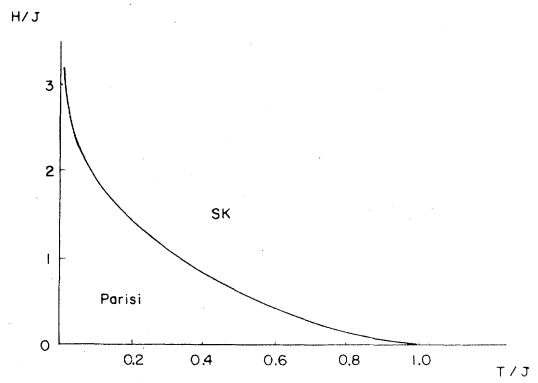

FIG. 48. Plot of the Almeida-Thouless (AT) line for the SK model with $J_0=0$. To the right of the line the SK solution with a single order parameter is correct, while to the left of the line the Parisi solution is believed exact. The Parisi solution represents the many-valley structure of phase space and nonergodic behavior. The AT line, therefore, signals the onset of irreversibility.

solution, to be discussed shortly. Above the AT line the SK solution is stable and does appear to be the exact solution.

How can we rectify the instability of the SK solution below the AT line? The eigenvalues of the paramagnetic solution, Eq. (4.51), go negative for $T<T_f$, signifying an instability. As a result the system breaks the symmetry in order to make the eigenvalues non-negative. This works for $\lambda_{1,2}$ [Eq. (4.58)], but an instability remains in λ_3 to second order in $|\theta|$ (the first-order instability of the paramagnetic solution having been cured). To stabilize λ_3 we must clearly break the symmetry in a different way.

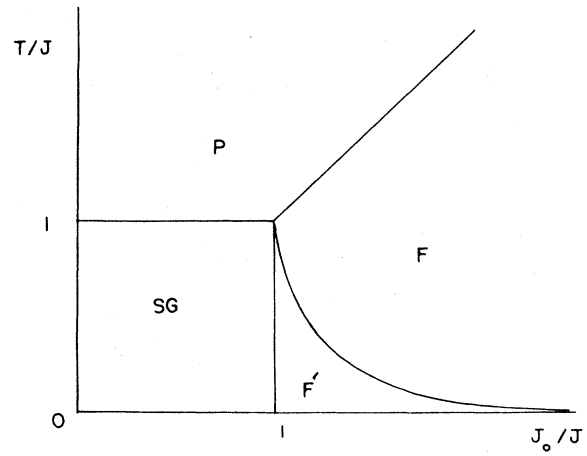

FIG. 49. A plot of the presumed exact phase diagram of the SK model with $H=0$. The P-SG and P-F boundaries are the same as in Fig. 44. F' is a ferromagnetic phase with replica symmetry breaking, i.e., irreversibility (this is often called a "mixed" phase) and is separated from F (where the SK solution is exact) by an AT line. The F'-SG phase boundary is vertical in the Parisi theory, as discussed in the text.

This must involve more than one order parameter, so we have to break the permutation symmetry of the replicas.

We have very little guidance as to how one should break this symmetry, and there are infinitely many possible schemes once we let $n\to 0$ (Parisi, 1980a). Some help is obtained, however, if we note that Eqs. (4.13), (4.16), and (4.17) have factors of n^{-1}, which must be compensated if a meaningful limit $n\to 0$ is to exist. This is obviously achieved if we make the replicas "equivalent" to each other in the sense that all quantities involving a single replica are replica independent. For example,

$$m_\beta = m_\gamma \tag{4.64a}$$

and

$$\sum_{\alpha=1}^{n} (q_{\alpha\beta})^k = \sum_{\alpha=1}^{n} (q_{\alpha\gamma})^k \tag{4.64b}$$

for any power k and for any β,γ. Although the replicas are equivalent for a solution that satisfies Eq. (4.64), the permutation symmetry of the replicas is broken if the $q_{\alpha\beta}$ are not all equal. Because the replicas are actually equivalent, the standard terminology of "replica symmetry breaking" is rather misleading; "breaking of replica permutation symmetry" would be more appropriate. However we shall use the conventional term from now on.

Initially several schemes were proposed (Blandin, 1978; Bray and Moore, 1978; Blandin *et al.*, 1980). However, the results of Bray and Moore do not satisfy Eq. (4.64b) and give an infinite free energy, while the solution of Blandin (1978) and Blandin *et al.* (1980) is unstable, like the SK solution. Shortly afterwards Parisi (1979, 1980a, 1980b, 1980c, 1980d, 1980e) proposed a bold generalization of Blandin's ideas. His theory predicts many novel results, satisfies Eq. (4.64), is stable, and agrees well with numerical results, so it is probably the exact solution of the SK model.

Parisi's ansatz for the order-parameter matrix $q_{\alpha\beta}$ can be represented by a tree, shown schematically in Fig. 50. The circles represent the n replicas, the vertical distance represents q, and $q_{\alpha\beta}$ is the value of q where the branches from α and β meet. There are an infinite number of bifurcations of the tree, so all values of q occur between a minimum value q and a maximum value $q_{\max}$, where $0\le q_{\min}\le q_{\max}\le 1$. The matrix itself can be drawn as follows. Take the $n\times n$ matrix (for notation we write $n\equiv m_0$) and divide it up into $m_1\times m_1$ blocks along the diagonal, as shown in Fig. 51. Give a value $q(m_0)$ to all elements except those in a diagonal block. Now take one of the $m_1\times m_1$ diagonal blocks and divide it, along the diagonal, into $m_2\times m_2$ blocks. Assign a value $q(m_1)$ to the elements not in a diagonal block. This procedure is repeated K times, say, so we have

$$n\equiv m_0\ge m_1\ge m_2\ge\cdots\ge m_K\ge 1 \tag{4.65}$$

and corresponding order parameters $q(m_0)$, $q(m_1),\ldots,q(m_K)$. This procedure makes sense for n a positive integer and K finite. We now assume it can be analytically continued for $n\to 0$ and $K\to\infty$. Hence

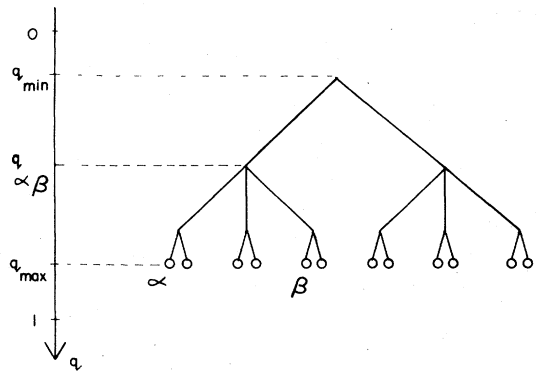

FIG. 50. Sketch of the replica symmetry-breaking scheme of Parisi, which can be represented by a tree. The circles at the bottom represent the n replicas and the vertical axis measures q increasing downwards. To find $q_{\alpha\beta}$ one traces back along the branches of the tree from α and from β until they join. $q_{\alpha\beta}$ is the value of q at this point. Branchings occur at all values of q between a maximum value q_{max} and a minimum q_{min}.

$0 \le m_i \le 1$ and, as $K \to \infty$, we can replace m_i by a continuous variable x, obtaining Parisi's order-parameter function $q(x)$, $0 \le x \le 1$. In other words, we now have an infinite number of order parameters. Note that the SK solution corresponds to the $K=0$ stage of Parisi's scheme, or equivalently to assuming that $q(x)$ is independent of x. Unless otherwise stated the rest of the discussion in this section will be for $J_0=0$.

It is straightforward to show that

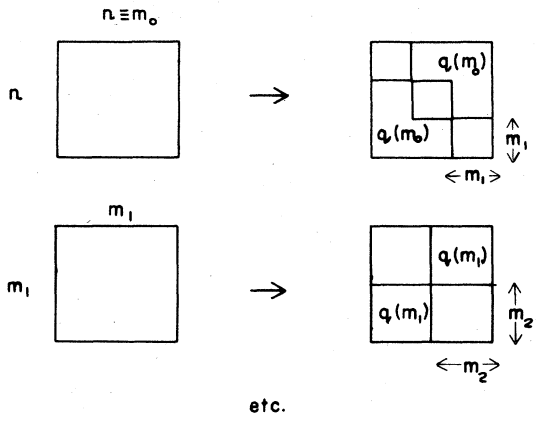

FIG. 51. Construction of the matrix $q_{\alpha\beta}$ for the Parisi replica symmetry-breaking scheme. One starts with the $n \times n$ matrix and for convenience writes $n \equiv m_0$. This is divided along the diagonal into $m_1 \times m_1$ blocks as shown. Outside the diagonal blocks, the order parameter is $q(m_0)$. Each diagonal block is then subdivided in a similar way into $m_2 \times m_2$ blocks. Replicas which are in the same block of m_1 replicas, but different m_2 blocks have the order parameter $q(m_1)$. This procedure is performed K times, and eventually one lets $K \to \infty$. The diagonal elements $q_{\alpha\alpha}$ are zero.

$$\lim_{n\to 0} \frac{1}{n(n-1)} \sum_{(\alpha,\beta)} q_{\alpha\beta}^k = \int_0^1 q^k(x)dx \tag{4.66}$$

for any power k, so that

$$\chi = \frac{1}{T}\left[1 - \int_0^1 q(x)dx\right] \tag{4.67a}$$

and

$$-U = \frac{J^2}{2T}\left[1 - \int_0^1 q^2(x)dx\right] + HM \tag{4.67b}$$

where M is the magnetization per spin

$$M = [\langle S_i \rangle_T]_{av} = \langle S_\alpha \rangle \ . \tag{4.68a}$$

The statistical mechanics order parameter q, defined by Eq. (3.61), can be obtained from Eqs. (3.81) and (4.66) and is given by

$$q = \int_0^1 q(x)dx \ . \tag{4.68b}$$

In Sec. IV.C we shall show that the Edwards-Anderson order parameter q_{EA}, defined by Eq. (3.63), is given by

$$q_{EA} = q(1) \ . \tag{4.68c}$$

The free energy is complicated and given for $J_0=0$ by (Parisi, 1980c; Duplantier, 1981)

$$-\beta f = \frac{(\beta J)^2}{4}\left[1 - 2q(1) + \int_0^1 q^2(x)dx\right] + \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-z^2/2} G(0, H + q(0)^{1/2} z)dz \ , \tag{4.69}$$

where, for a given $q(x)$, $G(x,y)$ is the solution of the partial differential equation

$$\frac{\partial G}{\partial x} = -\frac{J^2}{2}\frac{\partial q}{\partial x}\left[\frac{\partial^2 G}{\partial y^2} + x\left[\frac{\partial G}{\partial y}\right]^2\right] \tag{4.70}$$

with boundary condition

$$G(1,y) = \log[2\cosh(\beta y)] \ . \tag{4.71}$$

The extremal value, in the sense discussed at the end of Sec. IV.A with respect to variation of the function $q(x)$ has now to be found. Note that when we change $q(x)$ it is necessary to solve Eq. (4.70) again. The MFT of spin glasses is hence not very simple, as noted in Sec. IV.A. To obtain the SK solution one takes $q(x)$ to be independent of x, so $G(x,y) = \log[2\cosh(\beta y)]$ for all x, and hence Eq. (4.69) reduces to Eq. (4.19). Although the choice of the $q_{\alpha\beta}$ matrix seems arbitrary, and the analytic continuation to $n=0$ is not mathematically rigorous, the end result, Eqs. (4.69)—(4.71), looks perfectly respectable and is our first hint that sensible and interesting physics will emerge.

Equations (4.69)—(4.71) do not give the free energy or a self-consistent equation for $q(x)$ in closed form. This can be achieved, but the results will be deferred until Sec. IV.D, where a different formulation (Sompolinsky, 1981a)

that leads to essentially the same results will be presented.

The general solution for $q(x)$ is unknown, but it is possible to obtain $q(x)$ exactly close to T_f. Expanding the exponential in Eq. (4.13) and performing the trace one obtains the Landau expansion for arbitrary $q_{\alpha\beta}$ as (Bray and Moore, 1978; Pytte and Rudnick, 1979)

$$\beta f=\beta f_0+\lim_{n\to 0}\frac{1}{n}\left[\frac{1}{4}\left(\frac{T^2}{T_f^2}-1\right)\mathrm{Tr}Q^2-\tfrac{1}{6}\mathrm{Tr}Q^3-\tfrac{1}{8}\mathrm{Tr}Q^4 +\tfrac{1}{4}\sum_{(\alpha,\beta,\gamma)}Q_{\alpha\beta}^2Q_{\alpha\gamma}^2 -\tfrac{1}{12}\sum_{(\alpha,\beta)}Q_{\alpha\beta}^4+O(Q^5)\right], \tag{4.72}$$

where we have set $H=J_0=0$ and $Q_{\alpha\beta}=(\beta J)^2q_{\alpha\beta}$. Note that if $Q_{\alpha\beta}=Q$ for all (α,β), this reduces to the SK solution (4.26). Studying the stability of the SK solution (Bray and Moore, 1978; Pytte and Rudnick, 1979), one finds that it is the $\sum Q_{\alpha\beta}^4$ term in Eq. (4.72) that makes λ_3 negative. In fact,

$$\lambda_3=-16y\,|\theta|^2 \tag{4.73}$$

[cf. Eq. (4.56)], where $-y$ is the coefficient of the $\sum Q_{\alpha\beta}^4$ term. To obtain $q(x)$ to correct order in $|\theta|$ one can therefore neglect (Parisi, 1980a) the $\mathrm{Tr}Q^4$ and $\sum Q_{\alpha\beta}^2Q_{\alpha\gamma}^2$ terms in Eq. (4.72). This is known as the Parisi approximation. Actually Parisi (1980a) took $y=\frac{1}{8}$ instead of the value $y=\frac{1}{12}$ for the SK model, but this only changes some numerical factors in the answer. Taking the Parisi approximation, working to lowest order in $|\theta|$, and replacing sums over replicas by integrals over x, analogous to Eq. (4.66), one obtains (Parisi, 1980a; Thouless *et al.*, 1980)

$$\beta f=\beta f_0+\tfrac{1}{2}\int_0^1 dx\left[|\theta|q^2(x)+\frac{q^4(x)}{6}-\tfrac{1}{3}xq^3(x) -q(x)\int_0^x q^2(y)dy\right]. \tag{4.74}$$

Hence one can obtain f as a functional of $q(x)$ within a Landau expansion. A more general functional, applicable at all temperatures, will be discussed in Sec. IV.D. Variation of Eq. (4.74) with respect to $q(x)$ gives (Thouless *et al.*, 1980).

$$2|\theta|q(x)-xq^2(x)-\int_0^x q^2(y)dy -2q(x)\int_x^1 q(y)dy+\tfrac{2}{3}q^3(x)=0\,. \tag{4.75}$$

As discussed in Sec. IV.A, we seek a solution of Eq. (4.75) that is locally stable. However, the stability analysis is complicated and will only be discussed in Sec. IV.G. Differentiating Eq. (4.75) with respect to x one finds

$$2|\theta|-2xq(x)-2\int_x^1 q(y)dy+2q^2(x)=0 \tag{4.76}$$

or

$$q'(x)=0\,,$$

and a second derivative gives

$$q(x)=x/2 \quad \text{or} \quad q'(x)=0\,. \tag{4.77}$$

The SK solution corresponds to choosing the second solution in Eq. (4.77) for all x and gives $q(x)\equiv q_{\mathrm{SK}}=|\theta|$ [Eq. (4.29a)]. We now look for another solution. If we assume that $q(x)$ is continuous (which will turn out not to be true for Potts models—(see Sec. IV.H.3), the solution must be of the form

$$\begin{aligned} &q(x)=q(1),\quad 2q(1)=x_1\le x\le 1\,,\\ &q(x)=x/2,\quad x_0\le x\le x_1\,,\\ &q(x)=q(0),\quad 0\le x\le x_0=2q(0)\,. \end{aligned} \tag{4.78}$$

Substituting Eq. (4.78) into (4.75) gives

$$q(0)[2|\theta|-2q(1)+2q(1)^2]-\tfrac{4}{3}q(0)^3=0 \tag{4.79a}$$

and

$$q(1)[2|\theta|-2q(1)+2q(1)^2]-\tfrac{4}{3}q(0)^3=0 \tag{4.79b}$$

at $x=0$ and 1, respectively. From Eq. (4.79) we see that either $q(0)=q(1)$, which is just the SK solution again, or

$$q(0)=0 \quad (H=0) \tag{4.80}$$

and $2|\theta|=2q(1)-2q(1)^2$, so

$$q(1)=|\theta|+O(|\theta|^2)\,. \tag{4.81}$$

The solution given by Eqs. (4.78) and (4.80), and (4.81) is sketched in Fig. 52. It is also straightforward to include a small magnetic field H (Parisi, 1980b; Thouless *et al.*, 1980). The solution is still of the form (4.78) but now

$$q(0)=\frac{3}{4}\left[\frac{H^2}{J^2}\right]^{2/3}, \tag{4.82}$$

so a second "plateau" region appears (see Fig. 52). For

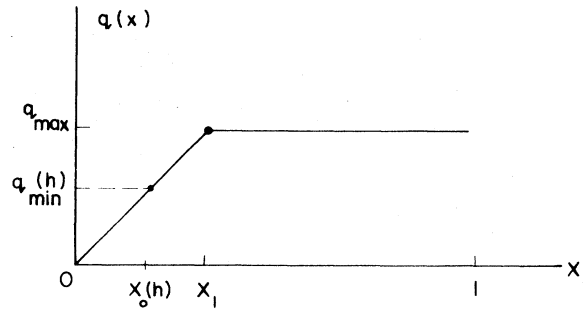

FIG. 52. The Parisi solution for $q(x)$ close to $T=T_f$. The solid line is for $H=0$ and the dashed line is for small H. The solution is independent of H, in this region of the H-T plane, except for the appearance of the second plateau. For $H=0$ one has $q_{\min}=x_0=0$. The general features of this figure persist at arbitrary temperatures and fields. There is a "plateau" region $q=q_{\max}$ for $x\ge x_1$, a region $x_0<x<x_1$, where $q(x)$ increases monotonically, and, for $H\neq 0$, a second plateau where $q=q_{\min}$ for $x\le x_0$.

$x \geq x_0 = 2q(0)$ the solution is unchanged to the order calculated. When $H = H_{AT}$, the value on the AT line [see Eq. (4.62)], $q(x)$ is independent of x, and for $H \geq H_{AT}$ one recovers the SK solution.

To higher order in $|\vartheta|$ one finds (Thouless *et al.*, 1980)

$$q(1) = |\theta| + |\theta|^2 - |\theta|^3 + O(|\theta|^4) , \tag{4.83}$$

while Eq. (4.80) seems to be true for all $T < T_f$ (Parisi, 1980b; Sompolinsky, 1981a). As a consequence of Eq. (4.80) one can show (Sompolinsky, 1981a) that

$$\int_0^1 q(x)dx = 1 - \frac{T}{T_f} \quad (H = 0) \tag{4.84}$$

exactly, so, from Eq. (4.67a),

$$\chi = \frac{1}{J} \quad (H = 0) , \tag{4.85}$$

a constant, below T_f, see Fig. 45. A constant susceptibility below T_f is similar to many field-cooled magnetization measurements (see, for example, Fig. 7), and is one of the striking features of the Parisi solution.

It is interesting to investigate changes in thermodynamic quantities close to the AT line. The magnetization has only a weak singularity, both M and $\partial M/\partial T$ being continuous, though $\partial^2 M/\partial T^2$ is discontinuous (Toulouse *et al.*, 1982; Sommers, 1984), so the AT transition is third order in the Ehrenfest sense. The AT line is characterized by $\lambda_3 = 0$, so χ_{SG} diverges [Eq. (4.55)]. However, in finite H, χ_{SG}, is not directly measurable, the relation (3.109) between χ_{SG} and χ_{nl} only holding for $H \to 0$. We shall see in the rest of this section that the AT line is most easily seen in dynamics because it marks the onset of irreversibility.

Although Eqs. (4.84) and (4.85) are exact, the function $q(x)$ is not known exactly except close to T_f. One approach to getting approximate results at arbitrary temperatures is to take only a finite number K of subdivisions of the order-parameter matrix. One finds for example that the ground-state entropy $S(T=0)$, which is -0.16 in the SK ($K=0$) solution, rapidly approaches zero for the $K=1$ and 2 approximations (Parisi, 1980c). It therefore seems that the Parisi theory cures the negative-entropy problem of the SK solution.

A different approximation has been proposed by Parisi and Toulouse (1980; see also Vannimenus *et al.*, 1981). They pointed out that one can obtain the solution for $q(x)$ and M fairly easily if one makes the following assumptions, known collectively as the PaT hypothesis (or projection hypothesis), everywhere below the AT line.

(i) M is independent of T (so S is independent of H).

(ii) $q(1)$ is independent of H.

(iii) $q(0)$ is independent of T.

(iv) outside the plateau regions $q(x)$ can be written

$$q(x) = f(xJ/T) , \tag{4.86}$$

where f is a universal scaling function. Equation (4.85) shows that (i) is true for $H \to 0$, while we have seen above that (ii) and (iii) are true close to T_f. There is less evidence to support assertion (iv). From the PaT hypothesis one sees that $q(0)$ at a given (H,T) is just q_{SK} at $(H, T_{AT}(H))$, and similarly $q(1)$ is q_{SK} at $(H_{AT}(T), T)$, so these can easily be calculated. For $T \to 0$ one finds

$$q(1) = 1 - \frac{3}{2}\left[\frac{T}{J}\right]^2 + \cdots . \tag{4.87}$$

As discussed in Parisi (1980c), Thouless *et al.* (1977), Bray and Moore (1979b), and Sec. IV.C, the result $1 - q(1) \propto T^2$ at low T is correct, and the coefficient, though not known exactly, cannot differ much from that in Eq. (4.87). In fact, the expansion away from T_f in Eq. (4.83) can be rewritten, neglecting $|\vartheta|^4$ and higher terms, as

$$q(1) \simeq 1 - 2\left[\frac{T}{J}\right]^2 + \left[\frac{T}{J}\right]^3 , \tag{4.88}$$

which also predicts $1 - q(1) \propto T^2$ as $T \to 0$, again with about the right coefficient. Thus Eq. (4.88) should be a good approximation at all temperatures and is simpler than the PaT result for $q(1)$, which has to be determined numerically at intermediate temperatures. Equation (4.88) is plotted in Fig. 46, where it is labeled q_{EA}. In the next section we shall show that $q(1)$ is indeed the Edwards-Anderson order parameter defined in Sec. III.F. M and S can equally be obtained from values on the AT line. Note that hypothesis (i) predicts a weak discontinuity in $\partial M/\partial T$ because the AT lines does not occur exactly where $\partial M/\partial T = 0$ in the SK solution (Vannimenus *et al.*, 1981), whereas $\partial M/\partial T$ is actually continuous (Sommers, 1984), as noted above. The PaT hypothesis is therefore not exact, but seems to be a good approximation. From Eq. (4.86) and knowledge about the AT line one can also determine (Vannimenus *et al.*, 1981) the scaling function f in Eq. (4.86) and the breakpoints x_0 and x_1. In particular, one finds

$$\lim_{T \to 0} x_1(T) = \tfrac{1}{2} . \tag{4.89}$$

Finally in this section we discuss the effect of a nonzero mean J_0 of the distribution within the Parisi theory. Toulouse (1980) has noted that thermodynamic properties can be deduced from the solution with $J_0 = 0$ but nonzero field. His argument is that if $A(T,M)$ is the thermodynamic potential with $J_0 = 0$, then with a nonzero mean it becomes simply

$$A(T,M,J_0) = A(T,M) - J_0 \frac{M^2}{2} . \tag{4.90}$$

This can, for example, be used to determine the phase boundary between the "mixed" phase F', where $M \neq 0$ but replica symmetry breaking occurs, and the spin glass phase SG (see Fig. 49). For small M

$$A(T,M) = A_0(T) + \tfrac{1}{2}\chi^{-1}M^2 + \cdots , \tag{4.91}$$

where χ^{-1} is the zero-field susceptibility with $J_0 = 0$.

Ferromagnetism disappears when the coefficient of M^2 in $A(T,M,J_0)$ vanishes, i.e.,

$$J_0=\chi^{-1} . \tag{4.92}$$

But $\chi^{-1}=J$ within the Parisi theory for all $T\leq T_f$ [see Eq. (4.85)], so the F'-SG phase boundary in Fig. 49 is vertical, and reentrant spin glass behavior, predicted by the SK theory, Fig. 44, does not occur.

We shall see later on in this section that the Parisi theory does appear to be the exact solution of the SK model and that one can give (Sec. IV.E) a simple, appealing interpretation of replica symmetry breaking and the function $q(x)$. Before that, however, we shall discuss two other approaches, in Secs. IV.C and IV.D, which avoid the replica trick and will give us a good deal of insight.

C. TAP equations

In this section and the next we discuss two approaches to a solution of the SK model that have attempted to avoid the replica trick.

Replicas appear when one performs the average over the disorder. Thouless, Anderson, and Palmer (1977) suggested that one defer this average to the end and first of all write down local mean-field equations for the site magnetizations for a given set of bonds. The naive guess is that Eq. (4.3) would be the appropriate set of equations. However, Thouless, Anderson, and Palmer realized that an extra term is necessary for spin glasses, even with infinite-range interactions. This is known as the Onsager reaction field term (discussed earlier by Brout and Thomas, 1967) and arises physically as follows. The magnetization m_i on site i comes from the magnetizations of the neighbors m_j. However, one should not take exactly m_j because the contribution to m_j from site i itself should be removed. In other words, the correct equations are

$$m_i=\tanh\left[\beta\left[\sum_j J_{ij}m_j'+H_i\right]\right] , \tag{4.93}$$

where

$$m_j'=m_j-\chi_{jj}J_{ij}m_i \tag{4.94}$$

and

$$\chi_{jj}=\frac{1}{T}(1-m_j^2) \tag{4.95}$$

is the local susceptibility of site j. We have added a local magnetic field H_i and taken the long-range limit, where each interaction J_{ij} is very small, in obtaining Eq. (4.94). Combining leads to

$$m_i=\tanh\left[\beta\left[\sum_j J_{ij}m_j\right]-\beta\sum_j J_{ij}^2(1-m_j^2)m_i+H_i\right] . \tag{4.96}$$

The set of Eqs. (4.96) are called the TAP equations. Since $J_{ij}\sim N^{-1/2}$ one might expect the correction term to be negligible compared with the expected term $\sum J_{ij}m_j$. However, this is not so because the signs in the expected term are random, so both it and the reaction field term are of order unity. Note, however, that for a ferromagnet, where $J_{ij}\propto z^{-1}$ with z the coordination number, the reaction field term is smaller by z^{-1} and is therefore unimportant in the long-range limit. The importance of the reaction field for spin glasses has been further emphasized by Cyrot (1979). It is reasonable to ask whether any additional terms are needed. Since other approaches such as the Bethe-Peierls approximation (Thouless, Anderson, and Palmer, 1977) and diagrammatic techniques (Thouless, Anderson, and Palmer, 1977; Sommers, 1978; De Dominicis, 1980) yield the TAP equations when applied to the SK model, it appears that there are no further corrections. Thouless *et al.* also pointed out that one can obtain a corresponding free-energy functional

$$\begin{aligned}F_{\rm TAP}=&-\tfrac{1}{2}\sum_{i,j}J_{ij}m_im_j-\sum_i H_im_i\\&+\frac{T}{2}\sum_i\{(1+m_i)\log[\tfrac{1}{2}(1+m_i)]\\&\qquad+(1-m_i)\log[\tfrac{1}{2}(1-m_i)]\}\\&-\frac{1}{4T}\sum_{i,j}(1-m_i^2)(1-m_j^2)J_{ij}^2 .\end{aligned} \tag{4.97}$$

Taking the variational derivative of Eq. (4.97) with respect to the m_i yields the TAP equations.

It is instructive to look at the Hessian matrix formed from second derivatives of $F_{\rm TAP}$. Defining

$$A_{ij}=\partial^2F_{\rm TAP}/\partial m_i\partial m_j , \tag{4.98}$$

one can evaluate the eigenvalues of A_{ij}. If Eq. (4.97) is to describe a sensible physical solution, one cannot have any negative eigenvalues, otherwise, for instance, the $1/N$ corrections to the free energy diverge (Thouless, Anderson, and Palmer, 1977). This leads to the stability condition (Bray and Moore, 1979b; Owen, 1982)

$$1-\beta^2J^2(1-2q_{\rm EA}+r)\geq 0 , \tag{4.99}$$

where

$$q_{\rm EA}=\frac{1}{N}\sum_i m_i^2 \tag{4.100}$$

and

$$r=\frac{1}{N}\sum_i m_i^4 , \tag{4.101}$$

which must be satisfied by any solution. Note that Eq. (4.99) is of the same form as the AT condition (4.60) for the stability of the SK solution. We return to this stability criterion below.

The TAP solutions can be investigated analytically close to the transition. Expanding the tanh in Eq. (4.96) (for $H_i=0$) and working to lowest order in the m_i gives

$$m_i-\beta\sum_j J_{ij}m_j+\beta^2m_i\sum_j J_{ij}^2=O(m^2) . \tag{4.102}$$

Now $\sum_j J_{ij}^2 = J^2$, independent of i, and the eigenvalues of J_{ij} are known (Mehta, 1967; Thouless, Anderson, and Palmer, 1977; Edwards and Jones, 1978) to have a semicircular distribution with largest eigenvalue $2J$. Writing Eq. (4.102) in terms of the eigenvalues J_λ and eigenvectors $\langle i \mid \lambda \rangle$, one has, with $m_\lambda = \sum_i m_i \langle i \mid \lambda \rangle$,

$$m_\lambda [1 - \beta J_\lambda + (\beta J)^2] = O(m^2) . \tag{4.103}$$

The transition occurs when the coefficient of m_λ vanishes for $\lambda = \lambda_{\max}$, the mode with the largest eigenvalue, $2J$. Hence $T_f = J$ in agreement with replica calculations (Secs. IV.A and IV.B). Note that T_f would have been overestimated by a factor of 2 without the reaction field term, as discussed in Sec. IV.A.

Results can also be obtained as $T \to 0$ by a combination of numerical and analytic techniques. At $T = 0$ each spin must lie parallel to the field acting on it, so for $H_i = 0$

$$m_i = \mathrm{sgn}(\tilde{H}_i) , \tag{4.104}$$

where

$$\tilde{H}_i = \sum_j J_{ij} m_j . \tag{4.105}$$

Numerically (Thouless, Anderson, and Palmer, 1977; Palmer and Pond, 1979) one finds that the distribution of the $\tilde{H}_i$, written $p(\tilde{H})$, is linear for small fields. Defining

$$p(\tilde{H}) = \frac{\tilde{H}}{h_0^2} \tag{4.106}$$

one finds from the TAP equations that as $T \to 0$

$$1 - q_{\mathrm{EA}} = \alpha \left[\frac{T}{J} \right]^2 , \quad (T \to 0) \tag{4.107}$$

where α is related to h_0 by

$$\left[\frac{h_0}{J} \right]^2 = \frac{\alpha}{4} + \frac{(2 \ln 2 + 1)}{3} + \frac{\ln 2}{\alpha} . \tag{4.108}$$

From Eq. (4.108) one sees that h_0 has a minimum as α is varied at $\alpha = 2(\ln 2)^{1/2} = 1.665$, and Thouless, Anderson, and Palmer have suggested that this is the correct value. On the other hand, Bray and Moore (1979b) have proposed that the entire spin glass phase below the AT line is marginally stable (see also below), so that Eq. (4.99) is satisfied as an equality. This gives $\alpha = 1.810$. We also noted in Eq. (4.87) that $\alpha = \frac{3}{2}$ within the PaT hypothesis. It has not been possible to determine numerically which, if any, of these three estimates for α is correct. The susceptibility of a single solution is related to q_{EA} by Eq. (3.75), so

$$\chi_{\mathrm{TAP}} \equiv \lim_{\omega \to 0} \chi(\omega) \propto T \quad (T \to 0) . \tag{4.109}$$

This differs from the equilibrium susceptibility $\chi = J^{-1}$ [see Eq. (4.85)] as discussed in Sec. III.F.1 and by Bantilan and Palmer (1981). Wohlfarth (1977b) has pointed out that Eq. (4.109) violates the third law of thermodynamics because of the Maxwell relation $\partial S / \partial h = \partial M / \partial T$ and the fact that $M = \chi_{\mathrm{TAP}} H$ as $H \to 0$, where M is the magnetization of a given solution. Since the Maxwell relations should hold for a single solution as well as for the full Gibbs average, this must mean that $\partial \chi / \partial T$ is nonzero at $T = 0$ only for H strictly zero. A model in which this occurs has been given by Sherrington and Fernandez (1977), but we are not aware of any calculations of χ_{TAP} in a field to check that this happens for the SK model. One finds from Eq. (4.106) that the entropy S varies at T^2 for $T \to 0$, which has been confirmed numerically (Ma and Payne, 1981; Young and Kirkpatrick, 1982).

We now return to the question of stability of the solutions mentioned above, which has been discussed in some detail by Bray and Moore (1979b) and Nemoto and Takayama (1985). Bray and Moore found numerically that a stable solution exists only for about 10% of bond configurations with $40 \leq N \leq 250$, but for those samples with a stable solution the smallest eigenvalue of the Hessian matrix seemed to extrapolate to zero as $N \to \infty$. Labeling a mode that diagonalizes the Hessian by λ and defining $\delta m_\lambda = \sum_i \langle \lambda \mid i \rangle \delta m_i$ to be a change in the magnetization in this direction, Nemoto and Takayama (1985) find that, even where there are no stable TAP solutions, most bond configurations have a line in the space of the m_i where $\partial F / \partial m_\lambda = 0$ for all λ except the mode λ_0 with smallest eigenvalue, and they determine the point on this line where the eigenvalue of λ_0 goes through zero. A sketch of the free energy against δm_{λ_0} for the cases in which there is (a) a stable TAP solution, and (b) no stable solution, is given in Fig. 53. Where there is no solution Nemoto and Takayama find that $\sum_i (\partial F / \partial m_i)^2$ at the point of zero eigenvalue seems to vanish as $N \to \infty$. Hence it appears that for $N \to \infty$ the physical TAP solutions (those with no negative eigenvalues) are really saddle points in the N-dimensional space of the m_i. Earlier, Thouless, Anderson, and Palmer had also argued that the solution would be a saddle point when the free energy is considered as a function of a single variable q_{EA}. Note that, as in the replica calculations [Secs. IV.A and IV.B], the paramagnetic solution has the lowest free energy but must be rejected because it is unstable.

We now discuss how many solutions of the TAP equations there are. Bray and Moore (1980b), De Dominicis *et al.* (1980), and Tanaka and Edwards (1980) have showed that there are an exponentially large number,

$$[N_S(H,T)]_{\mathrm{av}} \propto \exp[N \alpha(H,T)] , \tag{4.110}$$

where $\alpha(H,T) = 0$ at and above the AT line and $\alpha(0,0) = 0.20$. Above the AT line, averaging the unique TAP solution over the bonds gives the SK result. However, the SK solution cannot be correct below the AT line because it violates the stability condition (4.99). Bray and Moore (1980b) have also calculated the number of solutions of a given energy at $H = T = 0$ (see Fig. 54). Their solution is unstable for $E < -0.672J$ (shown by the dashed curve), but we shall see in Sec. IV.F that a linear variation behavior close to the minimum energy is still

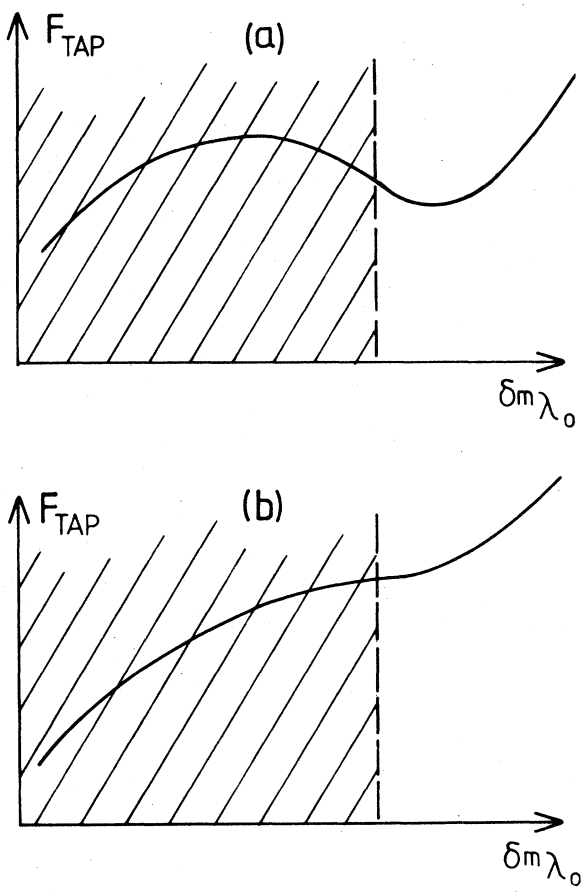

FIG. 53. Here m_{λ_0} is the component of magnetization of the spins along the eigenvector corresponding to the smallest eigenvalue λ_0 of the stability matrix $\partial^2 F_{TAP}/\partial m_i \partial m_j$. The lines show the Thouless, Anderson, and Palmer (TAP) free energy as a function of m_{λ_0} when it has been minimized with respect to magnetizations along all other eigenvector directions. The eigenvalue λ_0, which is just $\partial^2 F_{TAP}/\partial m_{\lambda_0}^2$, is negative to the left of the stability limit, indicated by a dashed line, and positive to the right. (a) The case of a stable TAP solution, which is at the minimum of F_{TAP}. (b) The situation when there is no TAP solution. As $N \to \infty$, the minimum in (a) approaches the stability limit, and for (b) the value of $(\partial F/\partial m_{\lambda_0})^2$ at the stability limit tends to zero. Hence one infers that TAP solutions are marginally stable in the thermodynamic limit. Note that the eigenvalues are calculated from the matrix of second derivatives, even where one of the first derivatives, $\partial F_{TAP}/\partial m_{\lambda_0}$, is nonzero. The figure is based on Nemoto and Takayama (1985).

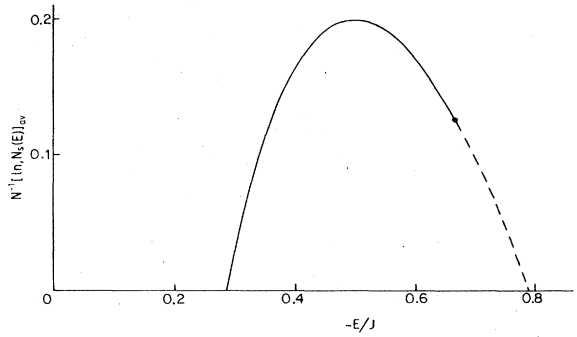

FIG. 54. Average of the logarithm of the number of TAP solutions for $H = T = 0$ as a function of energy, obtained by Bray and Moore (1980b). The total number of solutions is dominated by the maximum value of the curve and is given by Eq. (4.110) with $\alpha(0,0) \simeq 0.20$. Bray and Moore's solution is unstable in the region shown by the dashed line.

correct. Roberts (1981) has shown that the qualitative features of Fig. 54 persist in a nonzero field.

It is not clear that all these solutions are physical (i.e., that they have no negative eigenvalues of Hessian), though one can show that they are all stable at $T = 0$ (Bray and Moore, 1980b). Nonetheless it seems clear that there are many physical solutions below the AT line, and we naturally interpret each of them as a thermodynamic state (see Sec. III.B), since different solutions should have infinite free-energy barriers between them, otherwise thermal fluctuations would mix them together at finite temperature (Bray and Moore, 1981a). We do not therefore use the term metastable states to describe these solutions, but reserve this term for states that decay in a finite amount of time. Young (1981) pointed out that averages within a single solution, or state, could give different results from a full statistical mechanics average and that this could resolve some of the apparent paradoxes being discussed at that time, such as the difference between χ and χ_{TAP} noted above. Subsequently De Dominicis and Young (1983a) showed that by giving a Boltzmann weight to the solutions and averaging over J_{ij} using the replica trick one could recover the Parisi solution, discussed in Sec. IV.B, from the TAP approach. They were also able to show that the ordering in a single solution, q_{EA} [see Eq. (4.100)], is equal to $q(x = 1)$, as noted in Eq. (4.68c). Other weightings of the solutions are also of interest (Bray and Moore, 1981a; Bray *et al.*, 1984).

To conclude this section we emphasize that the TAP approach has been very valuable, and indeed has very recently provided the basis of a complete solution without the replica trick (Mézard *et al.*, 1986a, 1986b).

D. Dynamics

1. Introduction

Considerable progress has been made in understanding the SK model through its dynamics. Since a Hamiltonian with Ising spins has no intrinsic dynamics, it is necessary to introduce this separately through equations of motion that mimic the effect of coupling the spins to other degrees of freedom acting as a heat bath. For discrete Ising spins, $S_i = \pm 1$, one most commonly studies the kinetic Ising model of Glauber (1963), in which the probability that a spin is flipped per unit time is given by Eq. (5.8). This

type of stochastic dynamics is very conveniently simulated by Monte Carlo technique (see Sec. V.B). It is, however, often convenient to relax the length constraints on the spins and go to a "soft-spin" version (De Dominicis, 1978; Hertz and Klemm, 1979; Sompolinsky and Zippelius, 1981, 1982a; Hertz, 1983a, 1983b), in which each spin can take any value between $-\infty$ and $+\infty$. A suitable Hamiltonian is

$$-\beta\mathscr{H}=\frac{r}{2}\sum_i S_i^2+\frac{u}{4}\sum_i S_i^4+\beta\sum_{\langle i,j\rangle}J_{ij}S_iS_j+\beta H\sum_i S_i\ , \tag{4.111}$$

where r and u are parameters that give the single spin probability distribution, J_{ij} is the usual interaction, and H is a magnetic field. The simplest equation of motion is then (Ma, 1976; Hohenberg and Halperin, 1977)

$$\frac{\partial S_i(t)}{\partial t}=-\frac{1}{\tau_0}\frac{\partial(\beta\mathscr{H})}{\partial S_i}+\eta_i(t)\ , \tag{4.112}$$

where τ_0 sets the time scale and $\eta_i(t)$ is a Gaussian random noise with variance given by

$$\langle\eta_i(t)\eta_j(t')\rangle=2\tau_0^{-1}\delta_{ij}\delta(t-t')\ .$$

This model is the more convenient when one wishes to use many-body perturbation-theory techniques. Discrete spins are recovered in the limit $r\to-\infty, u\to\infty$ with $|r/u|=1$. Both discrete and soft-spin models give a Boltzmann distribution at long times for a finite system independent of initial spin state.

Apart from the intrinsic interest of dynamics, there is additional motivation in studying time-dependent phenomena of random systems because the formation apparently allows one to circumvent the replica trick (De Dominicis, 1978). The equations of motion (4.112) can be formally integrated from an initial time t_i to a final time t_f with a specified initial state $S_i(t_i)$. This can be converted into a probability distribution for the spins to follow a particular trajectory in phase space $S_i(t)$, given the state at $t=t_i$. For $t>t_i+\tau_{\rm eq}$, where $\tau_{\rm eq}$ is an equilibration time, the system will have lost the memory of its (arbitrary) initial state and will then sample states with the Boltzmann distribution. If one lets $t_i\to-\infty$ (and $t_f\to\infty$) the system will therefore be in equilibrium for all times *provided the equilibration time is finite* (Houghton *et al.*, 1983a, 1983b). Furthermore the probability distribution for a trajectory in phase space is normalized to unity (without need of a normalizing denominator involving the interactions) and so can be averaged over the random variables without replicas. Note that one never needs to introduce factors of $\exp(-\beta\mathscr{H})$ to weight the states (from which comes the need for replicas in statistical mechanics calculations—see Sec. III.C); the equation of motion automatically generates states with this probability. Further discussion of the formalism for dynamics is given in Bausch *et al.* (1976).

Near a second-order transition, time-dependent correlation functions take the form predicted by dynamical (Hohenberg and Halperin, 1977). We shall be particularly interested in the autocorrelation function $q(t)$, defined by

$$q(t)=[\langle S_i(0)S_i(t)\rangle_T]_{\rm av}\ , \tag{4.113}$$

where we note that the thermal average can be replaced by an average over a sufficiently long observation time $t_{\rm obs}$ [see Eq. (3.67)]. Dynamical scaling says that the decay of $q(t)$ is governed by a characteristic time τ, which diverges at T_f, i.e., we write

$$q(t)\propto t^{-\lambda}\tilde{q}_{\pm}(t/\tau)\ , \tag{4.114}$$

valid in the scaling region $t\to\infty$, $T\to T_f$. Here $\tilde{q}_+$ and $\tilde{q}_-$ are universal scaling functions for $T>T_f$ and $T<T_f$, respectively. It is conventional to write the time τ as

$$\tau\propto\xi_{\rm SG}^z\ , \tag{4.115a}$$

so

$$\tau\propto|\vartheta|^{-z\nu}\ , \tag{4.115b}$$

where

$$\vartheta=(T-T_f)/T_f\ ,$$

$\xi_{\rm SG}$ is the correlation length, which diverges with an exponent ν, and z is the dynamical exponent. Unfortunately one cannot define a correlation length for an infinite-range model. However, if one takes a finite-range model and uses the solution of the SK model as the mean-field approximation to it, one can define a correlation length exponent (see Sec. IV.G), with the result that $\nu=\frac{1}{2}$ in MFT. We therefore express the divergence of the relaxation time by Eq. (4.115) with the understanding that $\nu=\frac{1}{2}$ for the SK model.

From the expected behavior below T_f, one can obtain λ, since

$$\lim_{t\to\infty}q(t)=q_{\rm EA}\propto|\vartheta|^{\beta}\ , \tag{4.116}$$

so one requires $\lim_{x\to\infty}\tilde{q}_-(x)\propto x^{\lambda}$ to cancel the t dependence, and hence

$$\lambda=\beta/(z\nu)\ . \tag{4.117}$$

As well as the characteristic time τ it is useful to define an "average relaxation time" (Binder, 1977b; Kirkpatrick, 1980; Young, 1983a; Ogielski and Morgenstern, 1985) by

$$\tau_{\rm av}=\int_0^{\infty}q(t)dt\ , \tag{4.118}$$

if $q_{\rm EA}=0$. The simple modification for $q_{\rm EA}\neq0$ is given by Young (1983a). The divergence of $\tau_{\rm av}$ comes from the long time limit, so we can use Eqs. (4.114) and (4.117) and find (Ogielski, 1985)

$$\tau_{\rm av}\propto\vartheta^{-z_{\rm av}\nu}\ , \tag{4.119a}$$

where

$$z_{\rm av}=z-\beta/\nu\ . \tag{4.119b}$$

The difference between z and $z_{\rm av}$ is the same as that be-

tween the linear and nonlinear relaxation times of the order parameter at a ferromagnetic transition (Racz, 1976).

2. Dynamics above the AT line

In this section we discuss the dynamics of the SK model above the AT line (see Sec. IV.B). The much more difficult task of investigating the dynamics below the AT line will be deferred until the next section.

We start by tasking $H=0$, so we are approaching transition temperature at $T=T_f$ from above. The formulation for computing correlation functions in the soft-spin version has been discussed in detail by Sompolinsky and Zippelius (1982a) and Sommers and Fischer (1985). If $C_{\rm sing}(\omega)$ is the Fourier transform of the scaling expression for $q(t)$, one finds from Sommers and Fischer (1985) and Young (1982)

$$C_{\rm sing}(\omega) \propto \frac{1}{\vartheta}\tilde{C}(\omega/\vartheta^2) , \tag{4.120}$$

where ϑ is given by Eq. (4.116), $\tilde{C}$ is given by

$$\tilde{C}(x)=2^{1/2}[(1+x^2)^{1/2}+1]^{-1/2} , \tag{4.121}$$

and a nonuniversal constant representing the microscopic time scale has been absorbed into ω. Fourier-transforming Eq. (4.121), one gets (Sommers and Fischer, 1985; Young, 1982)

$$q(t) \propto t^{-1/2}\tilde{q}_+(t\vartheta^2) , \tag{4.122}$$

where

$$\tilde{q}_+(x)=e^{-x}-(\pi x)^{1/2}\mathrm{erfc}(x^{1/2}) \tag{4.123}$$

with

$$\mathrm{erfc}(y)=\frac{2}{\sqrt{\pi}}\int_y^\infty e^{-t^2}dt . \tag{4.124}$$

Equations (4.122) and (4.115) show that

$$\tau \propto \vartheta^{-2} \tag{4.125}$$

so $z\nu=2$ and hence

$$z=4 . \tag{4.126}$$

Furthermore the factor of $t^{-1/2}$ in Eq. (4.122) is consistent with Eq. (4.117) because $\beta=1$, $\nu=\frac{1}{2}$. Note that the average relaxation time (4.119) diverges as $\tau_{\rm av} \propto \vartheta^{-1}$, so

$$z_{\rm av}=2 . \tag{4.127}$$

From Eqs. (4.122) and (4.123) it follows that

$$q(t) \propto t^{-1/2} \quad (\vartheta=0) , \tag{4.128a}$$

$$q(t) \propto \frac{e^{-t\vartheta^2}}{2\vartheta^2 t^{3/2}} \quad (t\to\infty, \vartheta\neq 0) . \tag{4.128b}$$

The Glauber dynamics of fixed-length Ising spins has been discussed by Kinzel and Fischer (1977) and Kirkpatrick and Sherrington (1978). From their results one can also derive Eqs. (4.122) and (4.123), which shows the expected universality of the scaling function $\tilde{q}_+(x)$. The $t^{-1/2}$ variation at $T=T_f$ has also been found by Ma and Rudnick (1978), Hertz and Klemm (1979), and De Dominicis (1978). Numerical evidence for it has been given by Kirkpatrick and Sherrington (1978).

It is much more difficult to determine the dynamical correlations in a magnetic field as one approaches the AT line. One still expects the dynamical scaling form

$$q(t) \propto t^{-\beta/(z\nu)}\tilde{q}_+(t\vartheta^{z\nu}) \tag{4.129}$$

to hold (with $\beta=1$, $\nu=\frac{1}{2}$), where now ϑ measures the distance from the AT line, $\vartheta=[T-T_{\rm AT}(H)]/T_{\rm AT}(H)$. Surprisingly both the exponent z (Sompolinsky and Zippelius, 1981, 1982a) and the scaling function (Sommers and Fischer, 1985) are nonuniversal, and vary along the AT line. In particular z varies from 4 at $H=0$, $T=T_f$ to 5.06 as $H\to\infty$, $T\to 0$. [Note that ν in the notation of Sompolinsky and Zippelius is $\beta/z\nu$ $(=2/z)$ in our notation.] Continuously varying exponents are generally associated with a marginal variable in the renormalization-group sense (Ma, 1976). It would be interesting to know what is the marginal operator in this case.

One can also determine (Fischer, 1983d) the lines in the H-T plane where $\tau_{\rm av}$ has a fixed value, say t_0. The AT line (4.62) corresponds to $t_0=\infty$, whereas for t_0 finite $T(0,t_0)-T(H,t_0) \propto H^2$ as $H\to 0$ with crossover to AT-like behavior for larger H. This is similar to results of computer simulations in two dimensions (Kinzel and Binder, 1983, 1984; Young, 1983a) and experiment (Bontemps *et al.*, 1983; Paulsen *et al.*, 1984), but probably by coincidence (see Sec. V.B). Fischer and Kinzel (1984) have looked at the dynamical susceptibility near T_f and determined the form of the crossover from the expected form $\mathrm{Im}\chi(\omega) \propto \omega$ at high temperatures to $\mathrm{Im}\chi(\omega) \propto \omega^{1/2}$ [consistent with Eq. (4.128a)] at T_f. Fischer (1983d) and Togashi and Suzuki (1985) have made a mean-field decoupling of the Glauber equations of motion, incorporating fluctuation effects with an Onsager reaction term as in TAP (see Sec. IV.C). However, this approximation does not seem to be quite right, and the true Glauber equations of motion are more complicated (Shastry and Young, 1981). In particular, Togashi and Suzuki find that relaxation times are finite below T_f, whereas they actually remain infinite (Sompolinsky, 1981a; Sompolinsky and Zippelius, 1981; 1982a), as discussed in Sec. IV.D.3.

Of course, a calculation of dynamics also gives static properties. Furthermore one avoids the replica trick, as noted in Sec. IV.D.1 and can therefore obtain an independent derivation of the solution. Sompolinsky and Zippelius (1982a) have carried this out in detail and shown that in the limit of fixed-length spins one recovers the SK solution above the AT line. The TAP approach (Sec. IV.C) also gives the SK result in the region, so we have two independent replica-free derivations that show that the SK solution is correct in the region where it is locally stable. This supports the usual arguments (see Sec. IV.A)

that the *stable* solution coming from the replica method gives the correct physics.

3. Dynamics below the AT line

We have seen that replica, TAP, and dynamics approaches all give the SK solution above the AT line. Furthermore the replica and TAP calculations show this solution to be unstable below the line. As one would expect, an instability also occurs in dynamics (Sompolinsky and Zippelius, 1981, 1982a). The kinetic coefficient becomes negative, which implies that correlations diverge exponentially with time, clearly a nonsensical result.

The replica route has been salvaged by the Parisi solution (Sec. IV.B), which is stable (Sec. IV.G). The TAP approach gives the same result (De Dominicis and Young, 1983a), but only by using replicas, which one was trying to avoid. How can one obtain a stable theory for dynamics? One possibility is to combine dynamics and replicas (Houghton *et al.*, 1983a, 1983b), which again reproduces the Parisi theory for statics. Once more, this is a retrograde step because one was trying to avoid replicas by the dynamical approach (see Sec. IV.D.1). However, Sompolinsky (1981a) proposed a very ingenious way of finding a stable solution entirely within the framework of dynamics, and we shall now discuss this.

Slightly earlier Sompolinsky and Zippelius (1981, 1982a) had argued that infinite relaxation times occur (in an infinite system) and that as a result the equations are ill defined. Sompolinsky (1981a) then proceeded to apply the equations to a large but *finite* system in which all relaxation times must be finite. He found a spectrum of relaxation times τ_x that all diverge in the thermodynamic limit. For a finite system the autocorrelation function $q(t)$, Eqs. (4.113), varies on these time scales, and the $q(\tau_x)$ will become order parameters in the theory. One can fix the label x to lie in the interval from 0 to 1, and Sompolinsky assumes that x is continuous with $\tau_{x_1} \gg \tau_{x_2}$ if $x_2 > x_1$, i.e., that there is a broad continuous spectrum of time scales. From now on we shall change notation and write $q(\tau_x)$ as $q(x)$, i.e.,

$$q(x)=[\langle S_i(0)S_i(\tau_x)\rangle_T]_{\rm av} \quad (0\le x\le 1) . \tag{4.130}$$

Note that the Edwards-Anderson order parameter (3.66a) is given by

$$q_{\rm EA}=q(1) , \tag{4.131a}$$

and the statistical mechanics order parameter q, Eq. (3.66b), is

$$q=q(0) . \tag{4.131b}$$

Sompolinsky also looks at the local dynamical susceptibility $\chi_{ii}(t)$ (see Sec. III.F.1) and defines a second function $\Delta(x)$ by

$$T\int_0^{\tau_x}\chi_{ii}(t)dt = 1-q(1)+\Delta(x) . \tag{4.132}$$

Note that linear response theory (fluctuation-dissipation theorem), Eq. (3.72b), gives $\Delta(x)=q(1)-q(x)$. However, Sompolinsky's solution differs from this, as we shall discuss later. Nonetheless the fluctuation dissipation is assumed to be valid at finite time scales, $x=1$, so

$$\Delta(1)=0 . \tag{4.133}$$

Self-consistent equations for $q(x)$ and $\Delta(x)$ can then be established. Let us define (for $J_0=0$, which we assume in the rest of this section)

$$\tilde{H}=H+Jz_0[q(0)]^{1/2}+J\int_0^1 dx\, z(x)[q'(x)]^{1/2} -\beta J^2\int_0^1 dx\,\Delta'(x)\langle M\rangle_x , \tag{4.134}$$

$$M=\tanh\beta\tilde{H} , \tag{4.135}$$

and the $z(x)$ to be random variables with a symmetric Gaussian distribution and variance given by

$$\langle z(x_1)z(x_2)\rangle=\delta(x_1-x_2), \quad \langle z_0^2\rangle=1 . \tag{4.136}$$

Here and in the rest of this section, the angular brackets $\langle\ \rangle$ denote an average over the z's, and $\langle\ \rangle_x$ indicates an average over the $z(y)$ with $y>x$. The self-consistent equations are then (Sompolinsky, 1981a; Sommers, 1983a)

$$q(x)=\langle\langle M\rangle_x^2\rangle , \tag{4.137}$$

$$\Delta(x)=q(1)-1+(\beta J)^{-1}[q'(x)]^{-1/2}\langle\partial M/\partial z(x)\rangle , \tag{4.138}$$

$$\Delta(0)=q(1)-1+(\beta J)^{-1}[q(0)]^{-1/2}\langle\partial M/\partial z_0\rangle . \tag{4.139}$$

$\tilde{H}$ is clearly a local field, made up of contributions $z(x)$, which are frozen on a time scale τ_x. Evaluating Eq. (4.138) as $x\to 1$ where $\Delta(x)\to 0$, one finds

$$\beta^2J^2[1-q(1)+\langle M^4\rangle]=1 , \tag{4.140}$$

provided $\Delta(x)$ is not identically zero. If $\Delta(x)=0$ one obtains the SK solution, as we shall see. Equation (4.140) corresponds to the TAP stability condition (4.99) satisfied as an equality. Hence Eq. (4.140) is evidence for the marginal stability of the spin glass phase below the AT line discussed already in Sec. IV.C. Sompolinsky (1981a) finds that, as $H\to 0$, the only dynamically stable solution has

$$q(0)=0 . \tag{4.141}$$

Expanding Eq. (4.137) for small x, one finds $\chi J=1$, just as in the Parisi solution (4.85). Here χ, the equilibrium susceptibility, is given by $T\chi=1-q(1)+\Delta(0)$, from Eq. (4.132), and the relation $\chi_{ii}=\chi$ for $H\to 0$.

Equations (4.137)—(4.139) can be obtained from the functional

$$-\beta f=\frac{(\beta J)^2}{4}\left[[1-q(1)]^2+2\int_0^1 dx\,\Delta'(x)q(x)\right]+\left\langle\log(2\cosh\beta\tilde{H})+\frac{(\beta J)^2}{2}\int_0^1 dx\,\Delta'(x)\langle M_x\rangle^2\right\rangle \tag{4.142}$$

by differentiating with respect to $\Delta'(x)$, $q'(x)$, and $q(0)$. Later we shall see that Eq. (4.142), evaluated at the extremal values, is actually the free energy, though this is not obvious from the dynamical derivation. It is, however, straightforward to see that finding the extreme values of Eq. (4.142) does not determine $q(x)$ and $\Delta(x)$ uniquely. For suppose we replace x by a monotonic function $u(x)$ such that $u(0)=0$, $u(1)=1$, and let $z(x) \rightarrow \tilde{z}(u)(du/dx)^{1/2}$, then all the equations are the same but with x replaced by u. In other words, if $(\Delta(x),q(x))$ is a solution, one can change x into any monotonic function of x varying between 0 and 1 as x goes from 0 to 1, and one has another solution. There is therefore an infinite family of solutions, which differ by reparametrizations of the interval (0,1) and which give the same value of f. One says that the functional f is "gauge invariant." Although $\Delta(x)$ and $q(x)$ depend on choice of "gauge," Δ is uniquely determined as a function of q, and the end points $q(0)$, $q(1)$, $\Delta(0)$, and $\Delta(1)$ $(=0)$ are also gauge invariant.

In fact, we shall now see that the Sompolinsky theory is mathematically equivalent to the Parisi theory in the sense that the Parisi solution corresponds to choosing a particular gauge. To show this, we follow Elderfield (1983) and Goltsev (1984a) and define a function $G(x,y)$ by

$$G(x,y)=\left\langle \log[2\cosh\beta\tilde{H}(x,y)] + \frac{(\beta J)^2}{2}\int_x^1 dr\,\Delta'(r)\langle M\rangle_r^2 \right\rangle_x , \tag{4.143}$$

where

$$\tilde{H}(x,y)=y+J\int_x^1 dr\{z(r)[q'(r)]^{1/2} - \beta J^2\Delta'(r)\langle M(x,y)\rangle_r\} \tag{4.144}$$

and

$$M(x,y)=\tanh\beta\tilde{H}(x,y) , \tag{4.145}$$

so that Eq. (4.142) can be written

$$-\beta f=\frac{(\beta J)^2}{2}\left[[1-q(1)]^2+2\int_0^1 dx\,\Delta'(x)q(x)dx\right]+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-z_0^2/2}G(0,H+q(0)^{1/2}z_0)dz_0 . \tag{4.146}$$

By differentiation one can verify that (De Dominicis *et al.*, 1982; Elderfield, 1983; Goltsev, 1984a)

$$\frac{\partial G}{\partial x}=-\frac{J^2}{2}\left[q'\frac{\partial^2 G}{\partial y^2}-\Delta'\left[\frac{\partial G}{\partial y}\right]^2\right] , \tag{4.147}$$

where the prime denotes differentiation with respect to x, and clearly from Eq. (4.143) the boundary condition on (4.147) is

$$G(1,y)=\log(2\cosh\beta y) . \tag{4.148}$$

Equations (4.146)—(4.148) reduce to the Parisi equations (4.69)—(4.71) if one sets

$$\Delta'(x)=-xq'(x) . \tag{4.149}$$

In other words, if there is a gauge in the Sompolinsky theory where the solution satisfies Eq. (4.149), then the Parisi and Sompolinsky theories are mathematically equivalent. Now Eq. (4.149) can be written

$$\frac{d\Delta(q)}{dq}=-x(q) , \tag{4.150}$$

provided $q(x)$ is monotonic, so the inverse function can be defined, and we note that $\Delta(q)$ is uniquely determined. Since x can be replaced by any monotonic function $u(x)$ between 0 and 1 with $u(0)=0$ and $u(1)=1$, the equivalence between the Parisi and Sompolinsky solutions follows if one can show

(i) $q'(x)\geq 0$,

(ii) $d\Delta/dq$ is a monotonically decreasing function of q that lies between -1 and 0.

Part (i) can be shown (Sommers, 1983a; Elderfield, 1984b), and part (ii) is certainly true within perturbative calculations around T_f. A rigorous proof of (ii) for arbitrary temperatures seems to be lacking, though Elderfield (1984b) has shown that if (ii) is violated the Sompolinsky theory would not give a sensible order-parameter distribution (see Sec. IV.E). There seems, in fact, to be no doubt that the theories are equivalent.

From this discussion we see that Eq. (4.142) at the extremal solution is indeed the free energy of the Parisi-Sompolinsky solution, and furthermore it is written in closed form. It provides a rather convenient starting point for calculations in the vicinity of T_f, but unfortunately it does not seem to help one to develop expansions away from $T=0$.

One can also obtain the Sompolinsky functional (4.142) by a replica symmetry-breaking scheme (De Dominicis *et al.*, 1981), which effectively combines the Sommers (1978, 1979) and Parisi schemes, without recourse to dynamics. Dasgupta and Sompolinsky (1983) and Sommers *et al.* (1983) have given heuristic derivations of the Sompolinsky equations starting from the TAP equations.

Note that q_{EA} [Eq. (4.131a)] agrees with replica calculations in the manner of Parisi but starting from the TAP equations (De Dominicis and Young, 1983a), whereas q [Eq. (4.131b)] differs from the standard interpretation of the Parisi theory [Eq. (4.68b)].

The central feature of Sompolinsky's theory is the spectrum of time scales which diverge in the thermodynamic limit. It is clearly of interest to check this by numerical calculations, which can also determine how the relaxation times vary with N, a piece of information that is not

given by theory. Mackenzie and Young (1982, 1983) have carried out Monte Carlo studies of this question. They calculated $q^{(2)}(t)$, defined for the infinite-range model by

$$q^{(2)}(t)=\frac{1}{N(N-1)}\sum_{i\neq j}[\langle S_i(t_0)S_j(t_0)S_i(t_0+t) \times S_j(t_0+t)\rangle_{t'}]_{\rm av} \quad (4.151a)$$

[cf. Eq. (3.99) for the definition applicable to short-range systems] and found that data for $\Delta q^{(2)}(t)=q^{(2)}(t)-q^{(2)}(\infty)$ at a fixed temperature below T_f, but for different sizes, all lay on the same curve when plotted as a function of $\ln t/\ln\tau$ (see Fig. 55), where τ is size dependent and, in fact,

$$\ln\tau \propto N^x \quad (4.151b)$$

with $x\simeq\frac{1}{4}$ (Fig. 56). Although one cannot be absolutely certain from the numerical results that this increase of $\ln\tau$ with $N^{1/4}$ will continue as $N\to\infty$, the data do at least support the idea of nonergodic behavior in the SK model below the AT line. Figure 56 also shows that relaxation times remain finite above the AT line as N increases, as expected from analytic theories. Mackenzie and Young also showed that results for static quantities such as $q^{(2)}[\equiv q^{(2)}(\infty)]$, when extrapolated to $N=\infty$, agreed well with the predictions of the Parisi theory where the function $q(x)$ is interpreted in the "replica way" (see Sec. IV.E), slightly differently from Sompolinsky's dynamical interpretation. These data are shown in Fig. 57.

So far in this section we have concentrated on dynamics on the time scales that diverge as $N\to\infty$. Sompolinsky and Zippelius (1981, 1982a) have discussed fluctuations on finite time scales and found that $q(t)-q_{\rm EA}$ vanishes with a (nonuniversal) power of t, rather than the usual exponential decay, because of the marginality condition (4.140).

We have already noted that the Sompolinsky theory violates the fluctuation-dissipation theorem on time scales

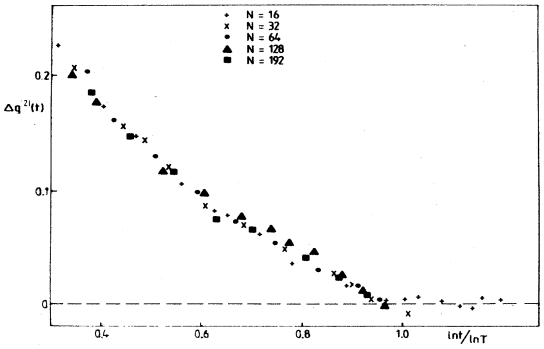

FIG. 55. $\Delta q^{(2)}(t)=q^{(2)}(t)-q^{(2)}(\infty)$ plotted against $\ln t/\ln\tau$ for a range of sizes for the SK model, with $T=0.4J, H=0$. The time τ is when $\Delta q^{(2)}(t)$ first reaches zero and varies with N as shown in Fig. 56. From Mackenzie and Young (1982, 1983).

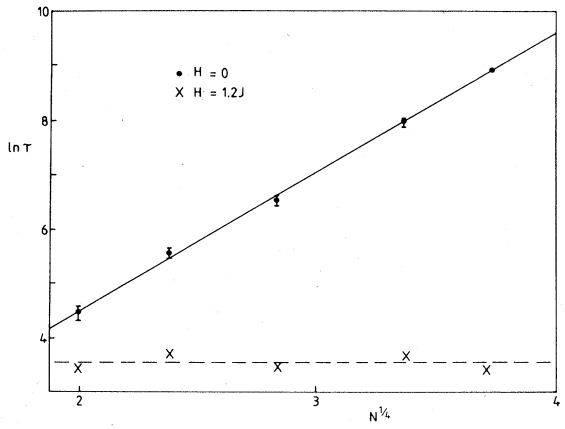

FIG. 56. $\ln\tau$ plotted against $N^{1/4}$ for the SK model, with several sizes between 16 and 192 at $T=0.4J$, $H=0$: The $\times$'s are for $T=0.4J$, $H=1.2J$, which is above the AT line. From Mackenzie and Young (1982, 1983).

that diverge as $N\to\infty$. This has aroused considerable discussion (Young and Kirkpatrick, 1982; De Dominicis and Young, 1983a, 1983b; Hertz, 1983a, 1983b; Houghton *et al.*, 1983a, 1983b; Sommers, 1983b; Horner, 1984a, 1984b). We argue (Secs. III.F and IV.E) that the fluctuation-dissipation theorem should actually hold on all time scales. If this is so, one would like to understand (a) what processes are being described by the Sompolinsky

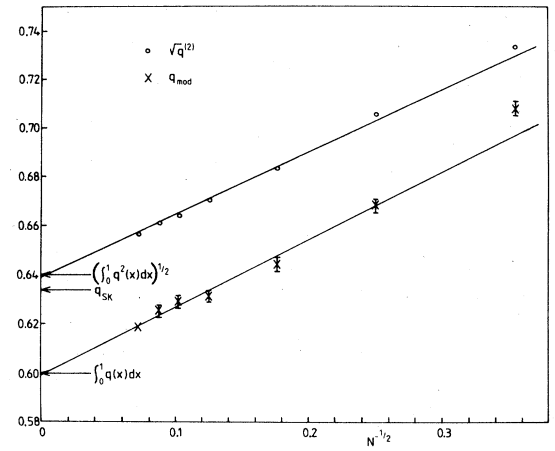

FIG. 57. Data for $(q^{(2)})^{1/2}$ and $q_{\rm mod}$ plotted against $N^{-1/2}$ for the SK model, with $T=0.4J$, $H=0$. $q_{\rm mod}$ is the long-time limit of $q_{\rm mod}(t)$ where

$$q_{\rm mod}(t)=N^{-1}\left[\left\langle\left|\sum_i S_i(t_0)S_i(t_0+t)\right|\right\rangle_{t'}\right]_{\rm av}.$$

The extrapolation to $N=\infty$ agrees well with the predictions of the Parisi theory, shown by the arrows. Also shown is $q_{\rm SK}$, the order parameter in the SK solution. From Mackenzie and Young (1983).

$q(t)$ on the long time scales, and (b) how $q(t)$ varies in equilibrium on these long time scales for a large but finite system. For point (a), Hertz (1983a, 1983b), Sommers (1983b), and De Dominicis and Young (1983b) have suggested that the theory may describe a system in which the initial state is not in equilibrium, but this seems rather speculative. Suggestions for the correct $q(t)$ in equilibrium have been made by Parisi (1983b) and De Dominicis (1983), of which only the latter satisfies the fluctuation-dissipation theorem on all time scales. Horner (1984a, 1984b) has proposed that the long time behavior can be rectified by using an annealed model in which the bonds are allowed to vary on a frequency Ω where $\Omega \to 0$ at the end.

Houghton *et al.* (1983a, 1983b) have suggested that one ensure that the system is in equilibrium by weighting the spin state at the initial time t_i with a Boltzmann factor and summing over all possible initial states. One is then forced to introduce replicas, which one was really trying to avoid by the dynamical approach. Since the thermodynamic limit is taken at the start, fluctuations on the divergent time scales are not seen in this theory.

To conclude this section, we have seen that the SK model appears to have a broad spectrum of relaxation times, which diverge as $N \to \infty$ below the AT line. Fluctuations on these long time scales are observable in a finite system, though there is some uncertainty about the correct theoretical description.

E. Physical interpretation of replica symmetry breaking

From the results of the last two sections we can put together a simple intuitive picture of the SK model, which will lead us to an interpretation of replica symmetry breaking in Parisi's theory.

In Sec. IV.C we learned that there are many solutions of the TAP equations, even at finite temperature, with a broad distribution of free energies. Not all the solutions may be minima (in the sense discussed in Sec. IV.C, i.e., there are no negative eigenvalues of the stability matrix); some are quite possibly saddle points (i.e., there are negative eigenvalues). Nonetheless it seems that we do have many minima, since all the solutions are indeed minima at $T=0$ (Bray and Moore, 1980b). Furthermore these minima should be separated by barriers that diverge in the thermodynamic limit (Bray and Moore, 1981a), otherwise thermal fluctuations would mix them together. Hence each minimum is a different thermodynamic state (see Sec. III.B) (the words pure state, ergodic component, and valley are also used to mean the same thing), in which an infinite system will stay forever if it is initially prepared in that state. For a finite system, these barriers will be finite, so it is presumably rare fluctuations over these barriers that give rise to the spectrum of long relaxation times in Sompolinsky's theory. As $N \to \infty$, the barrier heights appear to diverge and so do the relaxation times.

For an infinite system it is then necessary to distinguish (Young, 1981; De Dominicis and Young, 1983a; Sec. III.B) between averages within a single thermodynamic state and a full statistical mechanics (Gibbs) average, which is a weighted sum over all states. Above the AT line, however, these complications vanish; there is only one state, and its properties are given by the SK solution.

Below the AT line, fluctuations within a single state decay algebraically rather than exponentially because each state is only marginally stable (Bray and Moore, 1979b; Sompolinsky, 1981a; Nemoto and Takayama, 1985; see also Thouless *et al.*, 1977). For static properties, marginal stability means that the spin glass susceptibility of a single phase l, say, defined by

$$\chi_{\rm SG}^{(l)} = \frac{1}{N} \sum_{i,j} (\langle S_i S_j \rangle_T^{(l)} - \langle S_i \rangle_T^{(l)} \langle S_j \rangle_T^{(l)})^2 , \tag{4.152}$$

diverges as $N \to \infty$ (Bray and Moore, 1979b), though less rapidly than N because of the expected "clustering" property of the individual phases (see van Enter and van Hemmen, 1984, and references therein). This states that for a model with finite-range interactions

$$\lim_{|\mathbf{R}_i - \mathbf{R}_j| \to \infty} \langle S_i S_j \rangle_T^{(l)} - \langle S_i \rangle_T^{(l)} \langle S_j \rangle_T^{(l)} = 0 , \tag{4.153}$$

while for an infinite-range system the equivalent statement is

$$\lim_{N \to \infty} \langle S_i S_j \rangle_T^{(l)} - \langle S_i \rangle_T^{(l)} \langle S_j \rangle_T^{(l)} = 0 \quad (i \neq j) . \tag{4.154}$$

We shall now see that the Parisi order-parameter function is related to "overlap functions" between different states, Eq. (3.83). Let us calculate

$$q^{(k)} = [\langle S_1, S_2, \ldots, S_k \rangle_T^2]_{\rm av} , \tag{4.155}$$

where all sites are different. We shall do this two different ways and compare results. First of all, the replica method gives (see Sec. III.C and De Dominicis and Young, 1983a)

$$q^{(k)} = \lim_{n \to 0} \frac{1}{n(n-1)} \sum_{\alpha \neq \beta} \langle S_1^\alpha S_1^\beta S_2^\alpha S_2^\beta \cdots S_k^\alpha S_k^\beta \rangle \quad (\alpha \neq \beta) . \tag{4.156}$$

In replica mean-field theory, averages on different sites decouple (see Secs. IV.A and IV.B), so

$$q^{(k)} = \lim_{n \to 0} \frac{1}{n(n-1)} \sum_{\alpha \neq \beta} q_{\alpha\beta}^k , \tag{4.157}$$

which, according to Eq. (4.66), becomes

$$q^{(k)} = \int_0^1 q^k(x) dx \tag{4.158}$$

in the Parisi theory. A change of variables then gives

$$q^{(k)} = \int q^k \frac{dx}{dq} dq , \tag{4.159}$$

since $q(x)$ is monotonic (see the discussion in Sec. IV.D.3). This is our first result for $q^{(k)}$. We also evaluated $q^{(k)}$ as a weighted average over states, i.e.,

$$q^{(k)}=\left[\sum_{l,l'} P_l P_{l'}\langle S_1 S_2\cdots S_k\rangle_T^{(l)}\langle S_1 S_2\cdots S_k\rangle_T^{(l')}\right]_{\rm av} . \tag{4.160}$$

Because of the clustering property (4.154) and the fact that all sets of distinct sites are equivalent after configurational averaging, one can reexpress Eq. (4.160), for $N\to\infty$, by

$$q^{(k)}=\left[\sum_{l,l'} P_l P_{l'}(q^{ll'})^k\right]_{\rm av} , \tag{4.161}$$

where

$$q^{ll'}=\frac{1}{N}\sum_i \langle S_i\rangle_T^{(l)}\langle S_i\rangle_T^{(l')} \tag{4.162}$$

is the overlap between site magnetization in phases l and l'. Equation (4.161) trivially reduces to

$$q^{(k)}=\int q^k P(q)dq , \tag{4.163}$$

where

$$P(q)=\left[\sum_{l,l'} P_l P_{l'}\delta(q-q^{ll'})\right]_{\rm av} \tag{4.164}$$

is the probability that there are states (with Boltzmann weight) having overlap equal to q. Comparison of Eq. (4.163) with Eq. (4.159) shows that all moments of the distribution $P(q)$ are equal to the corresponding moments of dx/dq. Hence we have

$$\frac{dx}{dq}=P(q) . \tag{4.165}$$

This result was first proved by Parisi (1983a), though Houghton *et al.* (1983a) also noted that the equality of Eqs. (4.159) and (4.161) with $k=1$ suggested Eq. (4.165). Some aspects of this physical picture of replica symmetry breaking had been anticipated by Orland (1983). Hence it is rather natural in the Parisi theory to focus on the inverse function $x(q)$ [possible since $q(x)$ is monotonic]; then the derivative of this is a probability distribution.

Equation (4.165) shows that all of the Parisi function has physical significance, so it is of interest to evaluate it by numerical simulations. As it stands the distribution (4.164) is unsuitable for numerical computation because the distinct states cannot be defined for a finite system. However, there is a microscopic definition, $P_N(q)$, (3.84c) and (3.84d), which tends to Eq. (4.164) as $N\to\infty$ (Young, 1985) and can be directly computed. Note that, where there is only a single state, $P(q)$ is a single delta function, and one anticipates that $P_N(q)$ will be Gaussian with a width varying as $N^{-1/2}$. This behavior is found (Fig. 58) for the SK model above the AT line, confirming that there is only one state in this region. Below the AT line (Fig. 59; Young, 1983b), the distribution is much broader, having a tail, roughly size independent, with a finite weight down to $q=0$ and a peak at large q, which grows slowly with increasing size. The dashed line shows a calculation of dx/dq using an approximate form for $q(x)$. This consists of a delta function at the largest q plus a smooth part down to $q=0$. The upturn as $q\to 0$ is probably an artifact of the approximations used in determining $q(x)$. The smooth part at small q is well reproduced by the numerical data, though the peak in the data at large q grows so slowly with size that one cannot be sure it will become a delta function as $N\to\infty$. Nonetheless these data do support the picture of many valleys because the distribution is so broad, and the slow growth of the peak may be connected with the marginality of the individual states noted above.

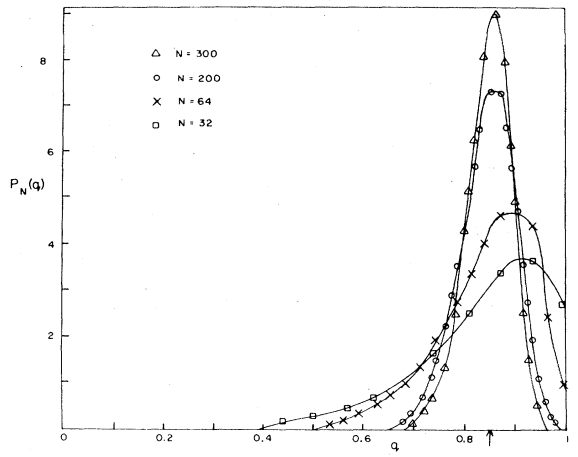

FIG. 58. Data for $P_N(q)$ for the SK model for several sizes at $T=0.4J$, $H=1.2J$, which is above the AT line. The results are consistent with a Gaussian distribution centered on the SK order parameter $q_{\rm SK}$, and whose width varies as $N^{-1/2}$.

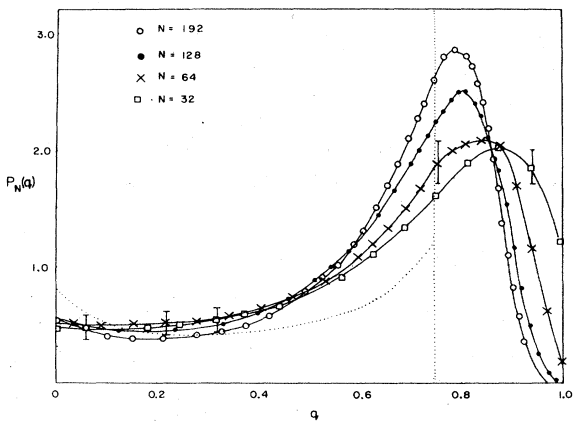

FIG. 59. Data for $P_N(q)$ for the SK model for several sizes at $T=0.4J$, $H=0$. The distribution has been symmetrized, so only $q\geq 0$ is shown. The dotted line is the prediction of an approximate solution of Parisi's equations and consists of a delta function of weight $\frac{4}{7}$ at $q=q_{\rm max}=0.744$ and a continuous part with a finite weight down to $q=0$. From Young (1983b).

We feel, therefore, that the interpretation of replica symmetry presented here is fairly well vindicated (see also Sec. IV.F). Horner (1984b, 1986) has investigated $P(q)$ from the dynamical approach.

To conclude this section we discuss in more detail the determination of thermodynamic averages as averages over all equivalent replica solutions (De Dominicis and Young, 1983a; Sec. III.C). First of all we evaluate χ_{SG} using full statistical mechanics averages [Eq. (4.35)], as opposed to Eq. (4.152), which is a restricted average in a single state. The different pieces in Eq. (4.35) are related to replica averages by Eq. (4.36).

It is straightforward to carry this out for the Parisi theory (Mézard *et al.*, 1984a, 1984b; Young *et al.*, 1984), with the result

$$\chi_{SG}=\frac{N}{3}\left[\int_0^1 q^2(x)dx-\left[\int_0^1 q(x)dx\right]^2\right]. \quad (4.166)$$

Hence $\chi_{SG}\propto N$, so the clustering property (4.153) and (4.154) does not hold for full thermal averages, only for averages within a single state.

Next we consider more carefully how many equivalent replica solutions there are. We have already noted (Sec. III.C) that the replica Hamiltonian is invariant under permutation of replicas. If $H=0$, it is *also* invariant under the time-reversal operations $S_i^\alpha\rightarrow -S_i^\alpha$ for all i on any subset of the replicas (De Dominicis and Young, 1983a), or, in other words,

$$q_{\alpha\beta}\rightarrow\varepsilon_\alpha\varepsilon_\beta q_{\alpha\beta} \quad (4.167)$$

where $\varepsilon_\alpha=\pm 1$. Hence Eq. (4.157) should be replaced by

$$q^{(k)}=\lim_{n\rightarrow\infty}\frac{1}{n(n-1)}\frac{1}{2^n}\sum_{\{\varepsilon_\alpha=\pm 1\}}\sum_{\alpha\neq\beta}(\varepsilon_\alpha\varepsilon_\beta q_{\alpha\beta})^k \quad (4.168)$$

for $H=0$, which leads to Eq. (4.158) for k even, but $q^{(k)}=0$ for k odd. This results in a *symmetric* distribution $P(q)$, with $P(|q|)=dx/dq$ ($\times\frac{1}{2}$ for normalization). The distribution in Fig. 59 has, for convenience, been symmetrized.

One can then ask, what happens if a small field is applied. Which of the 2^n choices of the ε_α (degenerate for $H=0$) gains most energy from this symmetry-breaking perturbation? The one that gains most will completely dominate the others for $N\rightarrow\infty$. Implicit in Parisi's work is the assumption that the choice with no negative $q_{\alpha\beta}$'s is the one that wins. In order to check this we expand the free energy (4.13) (with $J_0=0$) in powers of the field, obtaining

$$-\beta F(H)=-\beta F(0)+\lim_{n\rightarrow 0}\frac{1}{n}\left[\beta H\sum_\alpha\langle S^\alpha\rangle_0+\frac{(\beta H)^2}{2}\sum_{\alpha,\beta}\langle S^\alpha S^\beta\rangle_0+\cdots\right], \quad (4.169)$$

where $\langle\ \rangle_0$ is an average for $H=0$, with a particular choice of the ε_α. Since $\langle S^\alpha\rangle_0=0$, we must study the last term in Eq. (4.169), which is

$$\frac{(\beta H)^2}{2}\lim_{n\rightarrow 0}\left[1+\frac{1}{n}\sum_{(\alpha,\beta)}\varepsilon_\alpha\varepsilon_\beta q_{\alpha\beta}\right], \quad (4.170)$$

where we define our reference solution $q_{\alpha\beta}$ to have no negative components. We wish to maximize Eq. (4.170); comparing with Eq. (4.17), one sees this is equivalent to finding the solution that maximizes the susceptibility. It appears that the "obvious" solution is $\varepsilon_\alpha=1$ for all α as taken by Parisi. However, we must be more cautious because there are a negative number of $q_{\alpha\beta}$ as $n\rightarrow 0$, so

$$\lim_{n\rightarrow 0} n^{-1}\sum_{(\alpha,\beta)} q_{\alpha\beta}<0$$

even though $q_{\alpha\beta}\geq 0$ for all (α,β) [see Eq. (4.66)]. Hence, for the Parisi choice, $\varepsilon_\alpha=1$ for all α, (4.170) becomes

$$\frac{(\beta H)^2}{2}\left[1-\int_0^1 q(x)dx\right]. \quad (4.171)$$

We are not aware of any proof that Eq. (4.171) is the largest value that (4.170) can have, but we are unable to find a better choice. One example that we have tried is to take each of the smallest blocks of replicas (see Sec. IV.B) and make $\varepsilon_\alpha=1$ for half the replicas in each block and $\varepsilon_\alpha=-1$ for the other half. There is no contribution to the sum in Eq. (4.170) from terms where α,β lie in different blocks because of cancellations. However, the cancellation does not occur when α,β are in the same block because $q_{\alpha\alpha}=0\neq\lim_{\alpha\rightarrow\beta}q_{\alpha\beta}$. Noting that $q_{\alpha\beta}=q(1)$ for α,β in the same smallest block, we obtain with this choice

$$\frac{(\beta H)^2}{2}[1-q(1)], \quad (4.172)$$

which is smaller than Eq. (4.171) because $q(1)$ is the largest value of $q(x)$. Hence we believe that the Parisi choice, all $\varepsilon_\alpha=1$, is correct, but we are unable to prove it. Note that since the weight of a given solution is proportional to $\exp(-N\beta F)$ one needs $N(\beta H)^2\gg 1$ for one solution to dominate. Our discussion is also for $\beta H\ll 1$, and unfortunately one cannot simultaneously satisfy the conditions $1\gg\beta H\gg N^{-1/2}$ with sizes that are amenable to computer simulation. Hence numerical work has so far been unable to resolve this point.

It is also interesting to consider the overlap between the states of a system at two different fields (Parisi, 1983a) or temperatures (Sompolinsky, 1985). Let us do this by calculating

$$q_{12}=\left[\left\langle\frac{1}{N}\sum_i S_i^{(1)}S_i^{(2)}\right\rangle_T\right]_{\rm av}, \quad (4.173)$$

where $S_i^{(1)}$ and $S_i^{(2)}$ are two sets of spins, each with the same interactions and with no coupling between them.

We apply slightly different magnetic fields $H^{(1)},H^{(2)}$, where $H^{(1)}-H^{(2)}=\Delta H$, but the same temperature. A similar argument applies for the case of different temperatures. Replicating and carrying through standard manipulations (see Sec. IV.A), one obtains an expression similar to (4.11) (with $J_0=0$), except that there are $2n$ replicas and L [Eq. (4.12)] has an extra term,

$$\delta L=\frac{\beta\Delta H}{2}\left[\sum_{\alpha=1}^{n}S^{\alpha}-\sum_{\alpha=n+1}^{2n}S^{\alpha}\right], \tag{4.174}$$

which breaks the permutation symmetry of the $2n$ replicas. With $\Delta H=0$ one can clearly write a Parisi solution for $2n$ replicas and apply all possible permutations of them (De Dominicis and Young, 1983b). Hence, if $\Delta H=0$,

$$q_{12}=\int_0^1 q(x)dx \tag{4.175}$$

which is the usual order parameter. One now asks which subset of these permutations is preferred by the symmetry-breaking term (4.174), and for this subset what is the value of q_{12} where

$$q_{12}=\frac{1}{n^2}\sum_{\alpha=1}^{n}\sum_{\gamma=n+1}^{2n}q_{\alpha\gamma}\ , \tag{4.176}$$

i.e., q_{12} is the average of the elements in the off-diagonal $n\times n$ blocks in Fig. 60.

Following the arguments that lead to Eq. (4.170), one takes the average of $\exp(\delta L)$ up to second order and hence maximizes the expression

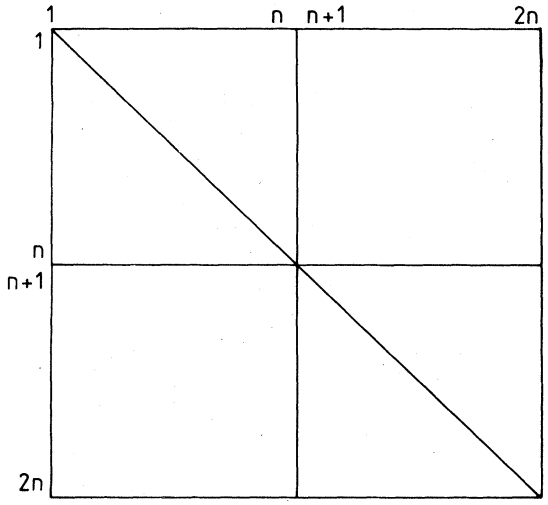

FIG. 60. The $2n\times 2n$ order-parameter matrix $q_{\alpha\beta}$ obtained when one considers two systems at slightly different temperatures or fields. Replicas $1,\ldots,n$ refer to system 1, and replicas $n+1$ to $2n$ refer to system 2. The replicas in each of these two groups can be separately permuted to obtain an equivalent solution, but except for zero temperature or field difference one cannot permute replicas from the first group with those in the second.

$$\left[\frac{\beta\Delta H}{2}\right]^2\lim_{n\to 0}\left[2+\frac{1}{n}\sum\left(q_{\alpha\beta}+q_{\gamma\delta}-q_{\alpha\gamma}-q_{\gamma\alpha}\right)\right], \tag{4.177}$$

where α,β run from 1 to n, γ,δ from $n+1$ to $2n$, and diagonal elements, $q_{\alpha\alpha},q_{\gamma\gamma}$ are zero. The optimal choice appears to be putting replicas $1,\ldots,n$ and replicas $n+1,\ldots,2n$ into different blocks at the first level of subdivision (see Sec. IV.B), which is possible if $2n/m_1$ is even. This means that all $q_{\alpha\gamma}$ are equal to $q(0)$, so Eq. (4.177) becomes

$$\tfrac{1}{2}(\beta\Delta H)^2\left[1-\int_0^1 q(x)dx\right], \tag{4.178}$$

and q_{12} is given by

$$q_{12}=q(0)\ . \tag{4.179}$$

Another choice (De Dominicis and Young, 1983b) is to make each of the $n\times n$ blocks $q_{\alpha\beta}$, $q_{\gamma\delta}$, $q_{\alpha\gamma}$, and $q_{\gamma\alpha}$ a Parisi solution of replicas [with the diagonal elements of the "off-diagonal blocks" in Fig. 60 equal to $q(1)$ rather than zero]. With this choice, Eq. (4.177) is

$$\tfrac{1}{2}(\beta\Delta H)^2[1-q(1)]\ , \tag{4.180}$$

which is smaller than (4.178). We believe that Eq. (4.178) is the best choice, and so q_{12} is given by Eq. (4.179), although we are unable to prove it. Equation (4.179) was first stated by Parisi (1983a). The above arguments are based on Sompolinsky (1985), who also shows that Eq. (4.179) holds when the two systems are at slightly different temperatures ΔT. One needs $\beta\Delta H$ (or $\beta\Delta T$) $\gg N^{-1/2}$ to observe Eq. (4.179); in the other limit the usual result (4.175) is obtained. Since the difference between Eqs. (4.178) and (4.180) comes from replica symmetry breaking and is always small, one may need a very large value of $\beta\Delta TN^{1/2}$ (or $\beta\Delta HN^{1/2}$) to see any deviation from Eq. (4.175).

F. Lack of self-averaging and ultrametricity

We discussed in Sec. III.A the usual argument (Brout, 1959) for self-averaging in random systems. One divides the large system into subsystems and, provided coupling between subsystems can be neglected, an average for the whole system effectively averages over independent subsystems with different bond configurations. This partially performs the configurational average, and the larger the system the more subsystems it can be divided into, so the more completely one is carrying out the configurational average. Clearly the argument depends on insensitivity to the boundary conditions at the surface of the subsystems. However, when a phase transition occurs, the state of the system can depend crucially on boundary conditions. Consider an Ising ferromagnet in zero field below T_f (see, for example, Sec. III.B); the boundary conditions can pick out either the state with M positive or

the one with M negative *because the weights of these phases are sensitive to boundary conditions.* The weight is proportional to $\exp(-N\beta f)$, so even a change in f of order N^{-1} produces a significant change in weight, and boundary conditions alter f by a surface-to-volume ratio which is bigger than this.

In Secs. III.B and IV.E we discussed the importance of the weights of the thermodynamic states in spin glass mean-field theory. It seems reasonable, then, that quantities depending on these weights will not be self-averaging, i.e., different bond configurations will give different results even for $N\to\infty$. We shall see in this section that this turns out to be true and that neither the weights P_l nor the overlaps between *different* states $q^{ll'}$ are self-averaging (Mézard *et al.*, 1984a, 1984b; Young *et al.*, 1984). Actually the Brout argument does not apply to an infinite-range model because this cannot be divided into subsystems. However, it turns out that quantities expected to be self-averaging in short-range systems, such as properties of a single state, are also self-averaging in the SK model and vice versa. Hence we do not believe that the infinite-range interactions are responsible for lack of self-averaging in the SK model; rather it is the existence of many degenerate states which contribute to the Gibbs average. It seems to us that this could, in principle, also occur in short-range systems.

As a first example of lack of self-averaging let us calculate

$$(\Delta q)^2=\left[\left[\frac{1}{N}\sum_i \langle S_i\rangle_T^2\right]^2\right]_{\rm av}-\left[\frac{1}{N}\sum_i \langle S_i\rangle_T^2\right]_{\rm av}^2 \tag{4.181}$$

$$=[q_J^2]_{\rm av}-q^2\ , \tag{4.182}$$

where

$$q_J=N^{-1}\sum_i \langle S_i\rangle_T^2 \tag{4.183}$$

is the statistical mechanics order parameter for a single bond realization. We shall use the subscript J to indicate quantities that are not configurationally averaged. Using the rules in Sec. III.C for replica averages, one has

$$[q_J^2]_{\rm av}=\lim_{n\to 0}\frac{1}{n(n-1)(n-2)(n-3)}\sum_{(\alpha,\beta,\gamma,\delta)} q_{\alpha\beta}q_{\gamma\delta}\ , \tag{4.184}$$

where all replicas are different. It is not hard to evaluate this in the Parisi theory, with the result (Young *et al.*, 1984)

$$[q_J^2]_{\rm ex}=\frac{2}{3}\left[\int_0^1 q(x)dx\right]^2+\frac{1}{3}\int_0^1 q^2(x)dx \tag{4.185a}$$

so

$$(\Delta q)^2=\frac{1}{3}\left[\int_0^1 q^2(x)dx-\left[\int_0^1 q(x)dx\right]^2\right]\ , \tag{4.185b}$$

which is nonzero where there is replica symmetry breaking, i.e., where there are many states.

Δq is nonzero because the distribution $P_J(q)$ is not self-averaging. A measure of this is the difference $[P_J(q_1)P_J(q_2)]_{\rm av}-P(q_1)P(q_2)$, which has been calculated by Mézard *et al.* (1984a, 1984b) and Elderfield (1984a). They find

$$[P_J(q_1)P_J(q_2)]_{\rm av}-P(q_1)P(q_2)=\tfrac{1}{3}[P(q_1)\delta(q_1-q_2)-P(q_1)P(q_2)]\ . \tag{4.186}$$

Equation (4.185) is recovered as a special case by taking the first moment of q_1 and q_2. From the second moment we find that $(\Delta q^{(2)})$, defined by

$$(\Delta q^{(2)})^2=[(q_J^{(2)})^2]_{\rm av}-(q^{(2)})^2\ ,$$

with

$$q_J^{(2)}=[N(N-1)]^{-1}\sum_{i\neq j}\langle S_iS_j\rangle_T^2\ ,$$

is given by

$$(\Delta q^{(2)})^2=\frac{1}{3}\left[\int_0^1 q^4(x)dx-\left[\int_0^1 q^2(x)dx\right]^2\right]\ . \tag{4.187}$$

Young *et al.* (1984) have evaluated $(\Delta q^{(2)})$ for very small samples and checked that the extrapolation to large N agrees well with the right-hand side of Eq. (4.187) evaluated in the Parisi theory (see Fig. 61).

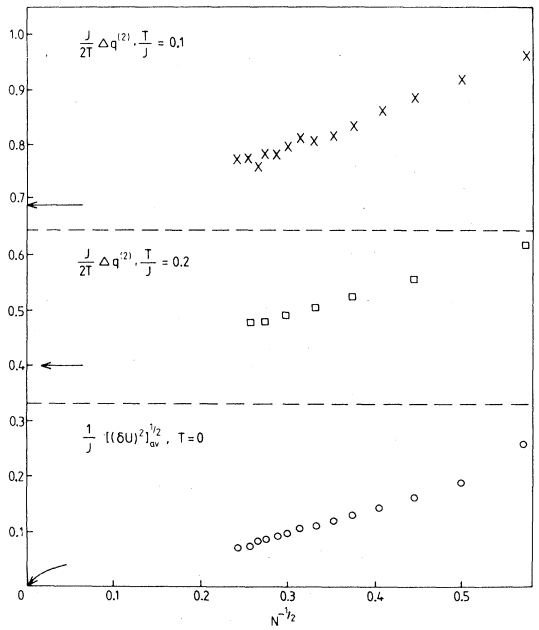

FIG. 61. Data for $\Delta q^{(2)}$, the standard deviation between samples of $q_J^{(2)}$, multiplied by T/J at $T=0.1J$ and $0.2J$, for the SK model with sizes between $N=3$ and 17 spins obtained by exact enumeration of all the states plotted against $N^{-1/2}$. The arrows show predictions from the Parisi theory. Also shown is the standard deviation of the energy, which seems to be extrapolating to zero for $N\to\infty$. From Young *et al.* (1984).

On the other hand, the energy (Fig. 61), magnetization (Elderfield, 1984a; Young *et al.*, 1984), q_{EA} (Mézard *et al.*, 1984a, 1984b), and mean-square staggered spin (Mézard and Parisi, 1984) are self-averaging. It thus appears that all quantities depending on the properties of a single state are self-averaging (and furthermore independent of state for a given set of bonds). Consequently it is difficult to think of an experiment that would detect lack of averaging, though this is very important theoretically.

Although M is self-averaging, the susceptibility χ_J is not (Young *et al.*, 1984) because the weights change rapidly as H is varied, probably on a scale $\Delta H \sim N^{-1/2}$, and the weights are sample dependent. Hence the operations of performing the field derivative and taking the thermodynamic limit do not commute (Sompolinsky and Zippelius, 1982a). For each sample, M has fluctuations, as H is varied, about an envelope curve whose slope $\Delta M/\Delta H$ is equal to the average susceptibility χ (see Fig. 62). Defining the differential susceptibility χ_J by

$$\chi_J = \frac{1}{NT}\sum_{i,j}(\langle S_i S_j\rangle_T - \langle S_i\rangle_T\langle S_j\rangle_T)\ , \tag{4.188a}$$

we have

$$\chi_J \neq \chi \quad (\equiv [\chi_J]_{av})\ . \tag{4.188b}$$

Calculating the fluctuations in χ_J, one finds that the $i\neq j$ terms in Eq. (4.188a) contribute even for $H\to 0$ because $\chi_{SG}\propto N$ [see Eq. (4.166)]. Hence one has

$$T\chi_J \neq 1-q_J \quad (H\to 0)\ , \tag{4.188c}$$

though an equality results if both sides are averaged [see Eq. (3.70b)]. Note that fluctuations in M about the envelope curve average to zero either by averaging over samples for a fixed H or averaging a single sample over a range of H much longer than $N^{-1/2}$. In other words,

$$\frac{\Delta M_J}{\Delta H} = \frac{1}{\Delta H}\int_H^{H+\Delta H}\chi_J(H)dH = \chi \tag{4.188d}$$

where $1 \gg \beta\Delta H \gg N^{-1/2}$. Young *et al.* (1984) referred to Eqs. (4.188b) and (4.188c) as violations of the fluctuation-dissipation theorem for a single sample. On reflection it seems preferable to define the fluctuation-dissipation theorem by Eq. (4.188a), *which is always valid,* and not by (4.188b) or (4.188c), but this is purely a question of semantics.

Let us now discuss more carefully the weights of the states. One can define the average number of states of weight between P and $P+dP$ by $f(P)dP$, where

$$f(P) = \left[\sum_l \delta(P-P_l)\right]_{av}\ . \tag{4.189}$$

Interestingly $f(P)$ can be calculated in closed form (Mézard *et al.* 1984a, 1984b),

$$f(P) = \frac{P^{y_1-2}(1-P)^{-y_1}}{\Gamma(y_1)\Gamma(1-y_1)}\ , \tag{4.190}$$

where

$$y_1 = 1-x_1 \tag{4.191}$$

and x_1 is the breakpoint in the function $q(x)$ (see Fig. 52), so y_1 is the width of the plateau. Note that

$$\int_0^1 Pf(P)dP = 1 \tag{4.192}$$

because $\sum_l P_l = 1$, and

$$\int_0^1 P^2 f(P)dP = \left[\sum_l P_l^2\right]_{av} = y_1\ . \tag{4.193}$$

The plateau in $q(x)$ gives a delta function of weight y_1 at $q=q(1)$ in $P(q)$. From Eqs. (4.164) and (4.193) we see that this is just the weight in the self-overlaps $q^{II}(=q_{EA})$. Hence we infer that (Mézard *et al.*, 1984a, 1984b)

$$q_{EA} = q(1)\ , \tag{4.194}$$

as already noted in Eq. (4.68c)

Now $\int_\varepsilon^1 f(P)dP$ diverges as $\varepsilon\to 0$, so there are an infinite number of states with infinitesimally small weight. However, there are also a small number with significant statistical weight because of Eq. (4.193) and the observation that y_1 is finite. Hence a small number of states *dominates* the statistical weight. In fact, as $y_1\to 1$, which occurs for $T\to T_f^-$, *one* state dominates.

It is interesting that Eq. (4.190) is universal, not depending explicitly on T,H but only indirectly on these quantities through y_1. This fact, as well as the dominance of a few states, has now been simply explained by Derrida and Toulouse (1985) and Mézard *et al.* (1985a). These authors noted that Eq. (4.190) follows directly, without needing replicas, if one assumes that the free en-

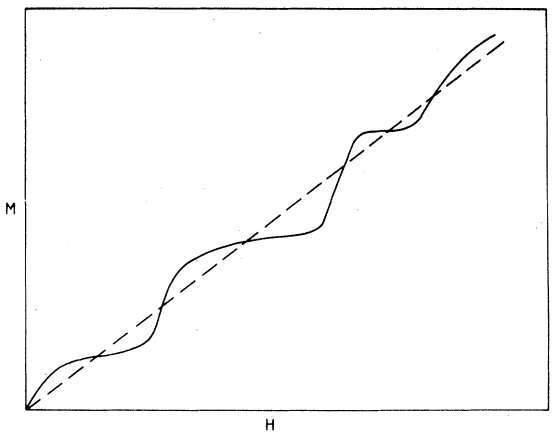

FIG. 62. The solid curve is a sketch of the magnetization M against field H for a single sample of the SK model below T_f. As $N\to\infty$ the curve tends closer and closer to the envelope curve, shown by the dashed line, though its slope does *not* tend to that of the dashed line. The fluctuations in the solid line away from the envelope curve, which probably take place in a range of M or H of order $N^{-1/2}$, are due to rapid changing of the weights of the thermodynamic states and average to zero when a configurational average is performed. From Young *et al.* (1984).

ergies of the states are independent random variables with an exponential distribution in the vicinity of the minimum free energy, i.e., the number with free energy per spin between f and $f+df$ is $\rho(f)df$, where

$$\rho(f) \propto \exp[x_1 N\beta(f-f_c)] \tag{4.195}$$

and f_c is the minimum free energy. It does not appear that any other distribution would give Eq. (4.190), so one argues that $\rho(f)$ in the SK model *must* have the form in Eq. (4.195) for f near f_c. This is also the distribution of energies in the random-energy model (Derrida, 1980, 1981). Clearly it is most improbable that there will be a state with f much less than f_c. There are an enormous number with $f \gg f_c$ (which is why the total number diverges as noted above), but these have negligible statistical weight when multiplied by the Boltzmann factor $\exp(-N\beta f)$, since $x_1 < 1$. Hence the states of interest have free energy very close to f_c. Dominance occurs because of gaps between the free energies of the states in a given sample.

We are now in a position to determine the "complexity" I [Eq. (3.21)] of the SK model, which is physically the extra entropy due to there being many phases. Equation (3.21) gives

$$\frac{I}{k_B} = -\int_0^1 f(P)P\log P\, dP \tag{4.196}$$

$$= \psi(1) - \psi(y_1) \tag{4.197}$$

from Eq. (4.190), where ψ is the digamma function. As $T \to T_f^-$, $I \to 0$, while as $T \to 0$, the PaT hypothesis predicts $x_1 \to \frac{1}{2}$ and hence $I/k_B = 2\log 2$. Note that I is the complexity of the whole system (not per spin), so I gives a negligible N^{-1} correction to the free energy. Hence the SK model is not very "complex" because of the dominance of a few states. The PaT prediction that the coefficient of f in the exponent in Eq. (4.195) should diverge as $T \to 0$ seems very surprising to us. On physical grounds we expect it to remain finite, in which case $x_1(T) \propto T$ as $T \to 0$. This would mean $I \to 0$ as $T \to 0$ (and as $T \to T_f^-$), which is physically reasonable and says that the lowest state completely dominates at low temperatures.

Having understood the distribution of free energies and weights fairly well, we turn our attention to fluctuations in the overlaps. Now $P_J(q)$ will consist of a set of delta functions, a few taking most of the weight, because of the dominance noted above, but with a large number of very small weight. Hence fluctuations in $P_J(q)$ will be enormous. A better behaved quantity turns out to be the cumulative probability distribution

$$Y_J(q) = \int_q^1 dq' P_J(q') , \tag{4.198}$$

where

$$[Y_J(q)]_{\rm av} = y(q) \equiv 1 - x(q) . \tag{4.199}$$

Mézard *et al.* (1984a, 1984b) have been able to determine accurately the distribution of $Y_J(q)$ by studying its moments. They find that the distribution depends on q, T, H only through $y(q)$, and so write it as $\Pi_y(Y)$. Hence $\Pi_y(Y)$ and $f(P)$ [Eq. (4.190)] are universal in that they make no reference to an order-parameter function $q(x)$ and are consequently the same for the random-energy model (Derrida, 1980, 1981) and the SK model (Derrida and Toulouse, 1985; Mézard *et al.*, 1985a). Figure 63 shows $\Pi_{0.7}(Y)$ obtained by Mézard *et al.* (1984b). As $Y \to 1$, $\Pi_y(Y)$ diverges like $(1-Y)^{-y}$ because the dominant configurations in this region are those in which one state dominates and the distribution of weights [Eq. (4.190)] diverges as $(1-P)^{-y}$ for $P \to 1$. Numerical calculations of Parga *et al.* (1984) are in good agreement with this.

Mézard *et al.* (1984a, 1984b) also investigated overlaps between three states labeled 1, 2, and 3 say. One might have thought that there would be no particular restriction on the three possible overlaps q^{12}, q^{23}, and q^{31}. However, it turns out that they cannot all be different; either all three are equal or the two smaller ones are the same. Such restrictions are characteristic of what is called an "ultrametric space." A good review of ultrametricity may be found in Rammal *et al.* (1986). The ultrametric structure follows fairly directly from the fact that there is a strong analogy between replicas and states [compare, for instance, Eq. (4.157) with Eq. (4.161)]. The Parisi ansatz has a treelike structure (see Fig. 50), which, after a moment's reflection, one can see, gives precisely these restrictions on the $q_{\alpha\beta}$. Hence we infer a treelike structure of the overlaps between thermodynamic states as well, just like Fig. 50, except that the ends of the tree on the bottom line denote states rather than replicas. This implies a hierarchical structure of "valleys within valleys within ...," which in turn suggests that barriers between states are larger when the overlap is smaller. It is also tempting to regard the vertical axis in Fig. 50 as temperature; the states then undergo successive bifurcations (or division

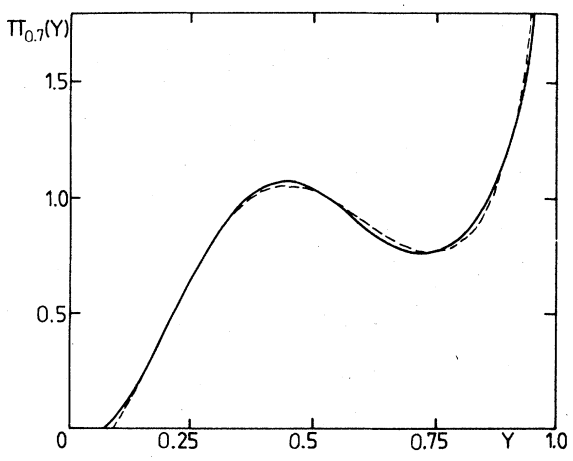

FIG. 63. The distribution $\Pi_{0.7}(Y)$ obtained by Mézard *et al.* (1984b). The two different curves correspond to different ways of extracting the distribution from its moments.

into more parts) as the temperature is lowered. This picture seems to have been proposed first by Krey (1977), though in a qualitative way. Note that ultrametricity must occur for each sample, even though the calculations generally discuss average behavior, because having a net probability of zero for three different overlaps means that such a situation can never occur in any sample.

Probably the most amazing aspect of ulrametricity in the SK model is the division of weight between the occurrences of cases of three equal q's and cases in which only two of the q's are equal. Let us consider all triples of phases where the largest overlap is fixed, and equal to $q_{\max}$, say. From the results of Mézard *et al.* (1984a, 1984b) one deduces that the probabilities are

$$\begin{aligned} &q^{12}=q^{23}=q^{31}=q_{\max} \quad (\text{prob. } \tfrac{1}{4}) , \\ &q^{12}=q^{23}<q^{31}=q_{\max} \quad (\text{prob. } \tfrac{1}{4}) , \\ &q^{23}=q^{31}<q^{12}=q_{\max} \quad (\text{prob. } \tfrac{1}{4}) , \\ &q^{31}=q^{12}<q^{23}=q_{\max} \quad (\text{prob. } \tfrac{1}{4}) . \end{aligned} \tag{4.200}$$

The equal partitioning among the four possibilities appears very trivially in the replica trick, basically from ratios of binomial coefficients ${}^nC_r$ in the limit $n\to 0$. It would be very interesting to understand physically how it arises.

Three factors make ultrametricity difficult to observe in numerical simulations. First, the distributions are significantly rounded out at sizes that can be studied; see, for example, Fig. 59 for $P(q)$. Second, there are triangular inequalities, for instance,

$$q_{\text{mid}}-q_{\min}\le 1-q_{\max} , \tag{4.201}$$

where q_{mid} and $q_{\min}$ are the intermediate and smallest of the q's. If $q_{\max}$ is close to unity, the two smaller q's are forced to be close together, and this can easily be confused with ultrametricity. Third, the distribution $P(q)$ has a pronounced peak (see Fig. 59), which means that there tends to be a lot of weight where the q's are almost equal. In view of these difficulties we feel it is very helpful to look at the *size dependence* of the moments of $q_{\text{mid}}-q_{\min}$ for a fixed $q_{\max}$, not close to unity. Parga *et al.* (1984) have studied sizes $N=32,64$ by working with $T=0$ solutions and a non-Boltzmann weight and found some evidence for ultrametricity. Recently Bhatt and Young (1986) have looked at sizes up to $N=512$ at $T=0.6T_c$ using Boltzmann weights. A preliminary finite size scaling analysis for $q_{\max}=\frac{1}{2}$ gives the first moment of the distribution of $q_{\text{mid}}-q_{\min}$ varying as $N^{-1/3}$. Hence there is some numerical evidence for ultrametricity, but it is rather hard to see convincingly.

One consequence of the ultrametric structure is that for any value of q one can partition the states into disjoint clusters such that the overlap of states in a cluster is greater than or equal to q, and the overlap between any pair of states in different clusters is less than q (Mézard *et al.*, 1984a, 1984b). If we denote the weight of cluster I by W_I, when the division is at a scale q, then

$$W_I=\sum_{l\in I} P_l . \tag{4.202}$$

It follows (Mézard *et al.*, 1984a, 1984b) that $Y_J(q)$, defined by Eq. (4.198), is given by

$$Y_J(q)=\sum_I W_I^2 , \tag{4.203}$$

and furthermore the distribution of the cluster weights W_I has the same universal form as Eq. (4.190) but with y_1 replaced by $y(q)$, i.e.,

$$f(W)=\frac{W^{y-2}(1-W)^{-y}}{\Gamma(y)\Gamma(1-y)} . \tag{4.204}$$

Once again no reference is made to the order-parameter function $q(x)$.

Recently, Mézard *et al.* (1986a, 1986b) have made an important advance by obtaining a complete solution, *including the function q(x),* without using the replica trick. We refer the reader to these references for further details.

G. Fluctuations and stability of the Parisi solution

At the end of Sec. IV.A we noted that a physically sensible solution must be stable, i.e., have no negative eigenvalues, and we showed in Sec. IV.B that the SK solution does not satisfy this condition below the AT line. Parisi's solution was then introduced and shown to give physically sensible and interesting results. Nonetheless the Parisi solution must still pass the stability test if it is to be taken as the exact solution of the SK model.

The problem, then, is to diagonalize $R^{\alpha\beta,\gamma\delta}$, a matrix of dimension $n(n-1)/2$ given by Eq. (4.49), in which the averages are with a weight $\exp(L\{q\})$, and $L\{q\}$ is given by Eq. (4.42) with $\Delta_{\alpha\beta}=0$ and $q_{\alpha\beta}$ equal to the Parisi solution. Since the matrix $q_{\alpha\beta}$ has a complicated structure, this is a highly nontrivial task. Some simplification occurs if one is close to $T=T_f$, $H=0$, so the free energy can be expanded as a functional of the $q_{\alpha\beta}$ [Eq. (4.72)]. Furthermore one can make the Parisi approximation, retaining out of the quartic terms in (4.72) only the $Q_{\alpha\beta}^4$ term, because this is the one responsible for replica symmetry breaking. Expanding Eq. (4.72) away from the saddle point up to second order and comparing with Eq. (4.48), one straightforwardly obtains, just below T_f (for $H=J_0=0$, which will be assumed in this section),

$$\begin{aligned} R^{\alpha\beta,\gamma\delta}=&(2\theta-2q_{\alpha\beta}^c)^2\delta_{\alpha\beta,\gamma\delta}-q_{\alpha\gamma}\delta_{\beta\delta}-q_{\alpha\delta}\delta_{\beta\gamma} \\ &-q_{\beta\gamma}\delta_{\alpha\delta}-q_{\beta}\delta_{\alpha\gamma} , \end{aligned} \tag{4.205}$$

where $\alpha<\beta,\gamma<\delta$, the $q_{\alpha\beta}^c$ refer to saddle-point values, and θ, given by Eq. (4.28), is negative. Eigenvalues in a subspace were found by Thouless *et al.* (1980). Subsequently, by a tour de force, De Dominicis and Kondor (1983) managed to completely diagonalize Eq. (4.205). Independently Goltsev (1983) has also found the eigenvalues of the R, though some of his results have been criticized by Kondor and De Dominicis (1983b). There are an in-

finite number of eigenvalues divided into two families, "replicon" and "longitudinal-anomalous," and the spectrum can be characterized as follows. (i) There are bands of eigenvalues that are all finite and positive. (ii) There are bands of eigenvalues that tend arbitrarily close to zero (an accumulation point). (iii) There are isolated zero eigenvalues.

A detailed description of the spectrum is given in De Dominicis and Kondor (1983, 1985a). Note that there are no negative eigenvalues, so the *Almeida-Thouless disease has been cured.*

One would still like to understand whether any problems arise from zero modes and those with arbitrarily small eigenvalues, (ii) and (iii) above.

We believe that there are three possible physical origins for these zero modes.

(a) Each thermodynamic state has directions in which it is very soft, corresponding to the marginal stability noted in Secs. IV.C and IV.D. We suspect that these will be of type (ii) above.

(b) There are degenerate thermodynamic states (see Sec. IV.E), and hence there are fluctuations that take the systems from one state to another.

(c) The Parisi free energy is invariant under a reparametrization of the plateau region $x_1 \le x \le 1$ [i.e., $x \to u(x)$ for $x_1 \le x \le 1$ with $u(x)$ monotonic and $u(x_1)=x_1$, $u(1)=1$]. Hence there will be zero modes corresponding to such a change of "gauge." These will probably be of type (iii) above. Note that they have no physical significance and will drop out of any calculation of observables.

It is obviously desirable to understand the physical significance of the various zero modes. We believe that this can be achieved by a statistical mechanics calculation of the spin averages in Eq. (4.36) as well as a calculation of analogous quantities, but with full statistical mechanics averages replaced by averages in a single state. For example, the first term in Eq. (4.36) would be just

$$N^{-1}\sum_{i,j}(\langle S_i S_j\rangle_T^{(l)})^2 .$$

It seems that these single state averages are obtained from replica averages similar to those in Eq. (4.36), but instead of averaging overall distinct sets of replicas one constrains the replicas to be in the same smallest block of the Parisi matrix (Sompolinsky, 1985). Though we have not seen a proof of this statement, it seems almost obvious from the ultrametric structure of the matrix (see Sec. IV.F and Fig. 50). As we shall see below, a calculation of these observable quantities requires the eigenvectors and degeneracies of the modes as well as the eigenvalues, but these have also been evaluated (De Dominicis and Kondor, 1984).

Both the full and single-valley averages in Eq. (4.36) are expected to be infinite because of the zero modes. However, it is of interest to know which modes contribute. We expect that those contributing to neither full nor single-state averages are associated with gauge changes, case (c), while those contributing to full but not single-state results are related to fluctuations between states, case (b). Modes contributing to single-state averages then describe case (a). De Dominicis and Kondor (1984) have evaluated the first term in Eq. (4.36) and have shown that many of the zero modes do disappear from the expressions for full and single-valley averages. Our interpretation is that these correspond to gauge transformations. As we discuss further below, De Dominicis and Kondor show that the full average is more strongly divergent than the single-valley average, demonstrating that there are zero modes contributing to the full average but not to single-state fluctuations. We interpret these as fluctuations *between* thermodynamic states.

We saw in Sec. IV.D that the Parisi solution corresponds to one choice of gauge in the Sompolinsky formulation. This refers to the saddle-point solution. However, the correspondence is even stronger because Kondor and De Dominicis (1983a) have shown that the fluctuation spectrum is also gauge invariant and hence the same in the two theories.

One motivation for studying the SK model is to use it as a starting point for calculating fluctuations in systems with short-range interactions. The first step is to determine the Gaussian fluctuations about the mean-field solution, which are related, as we shall see below, to the eigenvalues and eigenvectors discussed above. Hence it is appropriate to discuss this aspect of short-range systems in a section otherwise devoted to infinite-range models. The ultimate objective of the approach is to couple fluctuations together and develop a field theory describing the spin glass state of finite-range systems. From this one should be able to determine the critical dimension below which Parisi-type order disappears (see Sec. V.E). Unfortunately this has not yet been carried out.

Let us therefore discuss the Gaussian fluctuation about the mean-field theory of a finite-range, Edwards-Anderson model, as described in Sec. III.D, with Gaussian distribution of bonds. The replica Hamiltonian is given by Eq. (3.36) with $\bar{J}_{ij}=0$. Using standard techniques (e.g., Mühlschlegel and Zittartz, 1963), $[Z^n]_{\rm av}$ can be written, apart from a normalization constant, as

$$[Z^n]_{\rm av}=\exp[\tfrac{1}{4}(BJ)^2 nN]\int_{-\infty}^{\infty}\prod_{\alpha<\beta}dq^i_{\alpha\beta}\exp\left[-\tfrac{1}{2}\sum_{\alpha<\beta}\sum_{i,j}q^i_{\alpha\beta}(K^{-1})_{ij}q^j_{\alpha\beta}+\sum_i \log \mathrm{Tr}\exp(L[q_i])\right], \tag{4.206}$$

where

$$L[q_i]=(\beta J)^2\sum_{\alpha<\beta}q^i_{\alpha\beta}S_i^\alpha S_i^\beta , \tag{4.207}$$

$$A(\mathbf{k})=\sum_j(\beta\Delta J_{ij})^2 e^{i\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r}_j)} , \tag{4.208}$$

$$A(\mathbf{k}=0)\equiv(\beta J)^2 , \tag{4.209}$$

and

$$K_{ij}=(\beta\Delta J_{ij})^2/(\beta J)^4 , \tag{4.210}$$

so, Fourier transforming, we find

$$K^{-1}(\mathbf{k})=(\beta J)^2[1+\alpha k^2+\cdots](k\to 0) , \tag{4.211}$$

where α is a constant that depends on the form of ΔJ_{ij}. Equation (4.206) is the generalization to short-range interactions of Eq. (4.11). Introducing fields $\Delta^i_{\alpha\beta}$, which couple to $S_i^\alpha S_i^\beta$ as in Eq. (4.37), and following the steps which led to Eq. (4.45), we find

$$\langle s_i^\alpha s_i^\beta\rangle=\lim_{n\to 0}\langle q_{\alpha\beta}\rangle , \tag{4.212}$$

$$\langle q_{\alpha\beta}\rangle\equiv q^c_{\alpha\beta} , \tag{4.213}$$

where $q^c_{\alpha\beta}$ denotes the saddle-point value and the q averages are with weight $\exp(-Nn\beta f\{q\})$. Here

$$\beta f\{q\}=\beta f\{q^c\}+\lim_{n\to 0}\frac{1}{2n}\sum_{\substack{\alpha<\beta\\ \gamma<\delta}}\sum_{\mathbf{k}} R^{\alpha\beta,\gamma\delta}(\mathbf{k})\delta q_{\alpha\beta}(\mathbf{k})\delta q_{\gamma\delta}(-\mathbf{k}) \tag{4.214}$$

and $R^{\alpha\beta,\gamma\delta}(\mathbf{k})$, is given by

$$(\beta J)^{-2}R^{\alpha\beta,\gamma\delta}(\mathbf{k})=(1+\alpha k^2)\delta_{\alpha\beta,\gamma\delta}-(\beta J)^2[\langle S^\alpha S^\beta S^\gamma S^\delta\rangle-\langle S^\alpha S^\beta\rangle\langle S^\gamma S^\delta\rangle] \tag{4.215}$$

as $k\to 0$; the spin averages in Eq. (4.215) are evaluated exactly as in Eq. (4.49). Hence $R^{\alpha\beta,\gamma\delta}(\mathbf{k})$ is the same as the $R^{\alpha\beta,\gamma\delta}$ for the SK model, apart from the extra factor of αk^2 along the diagonal. In addition we find that

$$G^{\alpha\beta,\gamma\delta}(\mathbf{k})=\frac{1}{N}\sum_{i,j}\langle S_i^\alpha S_i^\beta S_j^\gamma S_j^\delta\rangle e^{i\mathbf{k}\cdot(\mathbf{R}_i-\mathbf{R}_j)} \tag{4.216}$$

with $\alpha<\beta,\gamma<\delta$ is given by

$$G^{\alpha\beta,\gamma\delta}(\mathbf{k})=\lim_{n\to 0}\left[Nq^c_{\alpha\beta}q^c_{\gamma\delta}\delta_{\mathbf{k},0}+\frac{A(0)}{A(\mathbf{k})}\langle\delta q_{\alpha\beta}\delta q_{\gamma\delta}\rangle-\frac{1}{A(\mathbf{k})}\delta_{\alpha\beta,\gamma\delta}\right] , \tag{4.217}$$

where again the δq averages are with weight $\exp(-Nn\beta f\{q\})$ and $f\{q\}$ given by Eq. (4.214). Decomposing $\delta q_{\alpha\beta}$ into eigenvectors, $|\mu\rangle$ of R, considering the long-wavelength limit, extracting the most singular part, and choosing units of length where α in Eq. (4.211) is unity, one obtains

$$G^{\alpha\beta,\gamma\delta}(\mathbf{k})=\lim_{n\to 0}\left[Nq^c_{\alpha\beta}q^c_{\gamma\delta}\delta_{\mathbf{k},0}+\sum_\mu\frac{\langle\alpha\beta|\mu\rangle\langle\mu|\gamma\delta\rangle}{\lambda_\mu+k^2}\right] , \tag{4.218}$$

where the λ_μ are eigenvalues of the mean-field stability matrix and $\langle\alpha\beta|\mu\rangle$ is the projection of the eigenvector $|\mu\rangle$ onto $\delta q_{\alpha\beta}$. Equation (4.218) is defined for $\alpha<\beta,\gamma<\delta$, and it is convenient to symmetrize it by defining $G^{\beta\alpha,\gamma\delta}(\mathbf{k})=G^{\alpha\beta,\gamma\delta}(\mathbf{k})$, $G^{\alpha\beta,\delta\gamma}(\mathbf{k})=G^{\alpha\beta,\gamma\delta}(\mathbf{k})$. The second term in Eq. (4.218) represents Gaussian fluctuations that are finite for a short-range system but vanish as $N\to\infty$ for the SK model because α in Eq. (4.215) is infinite, so $k\neq 0$ fluctuations are suppressed.

Note that there are zero modes, $\lambda_\mu=0$, so $G^{\alpha\beta,\gamma\delta}(\mathbf{k})$ will diverge as $k\to 0$. The wave-vector-dependent spin glass susceptibility $\chi_{\rm SG}(\mathbf{k})$, the Fourier transform of $(\langle S_iS_j\rangle_T-\langle S_i\rangle_T\langle S_i\rangle_T)^2$, is given by

$$\chi_{\rm SG}(\mathbf{k})=\lim_{n\to 0}\left[\frac{1}{n(n-1)}\sum_{(\alpha,\beta)}G^{\alpha\beta,\alpha\beta}(\mathbf{k})-\frac{2}{n(n-1)(n-2)}\sum_{(\alpha,\beta,\gamma)}G^{\alpha\beta,\alpha\gamma}(\mathbf{k})+\frac{1}{n(n-1)(n-2)(n-3)}\sum_{(\alpha,\beta\gamma,\delta)}G^{\alpha\beta,\gamma\delta}(\mathbf{k})\right] , \tag{4.219}$$

the summations being over all distinct sets of replicas. Note that $\chi_{\rm SG}(\mathbf{k}=0)\equiv\chi_{\rm SG}$ [Eq. (4.35)]. Equation (4.219) is the generalization to short-range interactions of (4.35) and (4.36). The corresponding quantity for a single valley $\chi^{(I)}_{\rm SG}(\mathbf{k})$, discussed above and in Eq. (4.152), is given by

$$\chi^{(I)}_{\rm SG}(\mathbf{k})=[\lim_{n\to 0}G^{\alpha_1\alpha_2,\alpha_1\alpha_2}(\mathbf{k})-2G^{\alpha_1\alpha_2,\alpha_1\alpha_3}(\mathbf{k})+G^{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{k})] , \tag{4.220}$$

where the α_i are distinct replicas in the same smallest block of the Parisi matrix.

De Dominicis and Kondor (1984) have evaluated $G^{\alpha\beta,\alpha\beta}(\mathbf{k})$ just below T. Writing this as $G_x(\mathbf{k})$ where x is defined by $q_{\alpha\beta}=q(x)$, they find, for small k,

$$G_x(\mathbf{k})\propto\begin{cases}k^{-4} & (x\ll k) , & (4.221a)\\ k^{-3} & (x_1\gg x\gg k) , & (4.221b)\\ k^{-2} & (x\to 1) & (4.221c)\end{cases}$$

and

$$\int_0^1 G_x(\mathbf{k})dx\propto k^{-3}\log k , \tag{4.222}$$

where x_1 is the breakpoint in the Parisi function (Fig. 52). In a magnetic field, but still below the AT line, De Dominicis and Kondor (1984) find

$$G_x(\mathbf{k})\propto\begin{cases}k^{-3} & (\text{small } x) , & (4.223a)\\ k^{-2} & (x\to 1) & (4.223b)\end{cases}$$

and

$$\int_0^1 G_x(\mathbf{k})\propto k^{-3} \tag{4.223c}$$

the ranges $x\ll k$ and $x_1\gg x\gg k$ both now giving the same k^{-3} divergence (for $k\ll h^{2/3}$).

Recently De Dominicis and Kondor (1985b) have evaluated $\chi^{(I)}_{\rm SG}(\mathbf{k})$, obtaining the remarkably simple result

$$\chi_{SG}^{(l)}(\mathbf{k})=1/k^2 . \tag{4.224}$$

Unfortunately, the full statistical mechanics average, $\chi_{SG}(\mathbf{k})$, is very complicated and has not yet been evaluated. We may give the following interpretation to results (4.221)—(4.224). Equations (4.221c) and (4.223b) show that fluctuations in a single state diverge as k^{-2}. This divergence reflects the marginal stability of each state, as noted in Secs. IV.C and IV.D. The full statistical mechanics average given by Eqs. (4.222) and (4.223c) diverges more strongly because of large fluctuations between thermodynamic states. If one were naive one would be tempted to estimate the lower critical dimension (see Sec. V.E) by integrating over all k and requiring that the result be finite (because it is a local quantity). Since the full average is more divergent than the single-state average, one might use Eq. (4.222), which would imply that $d_l=3$. However, Eq. (4.221a) diverges as k^{-4}, so one could also argue that $d_l=4$ (Sompolinsky and Zippelius, 1983). In any case, these arguments are too simple, and a complete field theory including interactions between the modes is necessary to determine d_l, the lower critical dimension for Parisi order.

The above results were obtained just below T_f; De Dominicis and Kondor (1985a) have noted that they should be essentially unchanged anywhere below T_f because they depend mainly on the ultrametric structure (Sec. IV.F), of the Parisi matrix. In fact, correlation functions at arbitrary temperatures have been found by Sompolinsky and Zippelius (1983) using the dynamical approach. Kondor and De Dominicis (1983a) have argued that these only give the $G^{\alpha\beta,\gamma\delta}(\mathbf{k})$ for certain limiting choices of the replicas. As an example they argue that $G_x(\mathbf{k})$ is obtained by Sompolinsky and Zippelius only for $x=0$, where indeed the two calculations agree. According to the dynamical interpretation of x, the statistical mechanics average is obtained with $x=0$, so the stronger divergence in Eq. (4.221a) is very significant. However, according to the more widely held interpretation, discussed in Sec. IV.E, statistical mechanics averages are given by integrals over x. Goltsev (1984b) has shown how Sompolinsky and Zippelius's results can be obtained from the replica method.

Equation (4.218) is very simple above T_f where, as noted in Sec. IV.B, all the λ_μ are equal to 2θ, so

$$G^{\alpha\beta,\gamma\delta}(\mathbf{k})=\frac{1}{2\theta+k^2}\delta_{\alpha\beta,\gamma\delta} . \tag{4.225}$$

Fourier transforming Eq. (4.225), one obtains

$$[\langle S_i S_j\rangle_T^2]_{\rm av}\propto\exp(-|\mathbf{R}_i-\mathbf{R}_j|/\xi_{SG}) \tag{4.226}$$

as $|\mathbf{R}_i-\mathbf{R}_j|\to\infty$ where $\xi_{SG}^{-2}=2\theta$. In general one defines an exponent ν by $\xi_{SG}\propto\theta^{-\nu}$ and hence the mean-field value is

$$\nu=\tfrac{1}{2} , \tag{4.227}$$

as noted in Sec. IV.D.1. Furthermore at $T=T_f$, $G^{\alpha\beta,\alpha\beta}(\mathbf{k})\propto k^{-2}$. This is conventionally written as $k^{-(2-\eta)}$ so, in mean-field theory, we have

$$\eta=0 . \tag{4.228}$$

Note from Eqs. (4.221)—(4.223) that fluctuations diverge more strongly below T_f, where ordering has occurred, than at T_f itself.

To conclude, we believe that the most important results presented in this section are the stability of the Parisi solution and the results in Eqs. (4.221)—(4.224). It would be very interesting to check whether the same divergences occur in the other averages that go into the expressions for $\chi_{SG}(\mathbf{k})$ [Eq. (4.219)] and whether any cancellations occur in forming this linear combination. A full field theory of interacting fluctuations below T_f is awaited with great interest.

H. Non-Ising models

In this section we review, relatively briefly, work on versions of the SK model that do not have Ising spins. The obvious generalization is to m-component vector spins; the properties of an isotropic vector spin glass are discussed in Sec. IV.H.1, while the effects of anisotropy, including a magnetic field, are treated in Sec. IV.H.2. Finally in Sec. IV.H.3 we look at results on some other mean-field models.

1. Isotropic vector spin glasses in zero field

Let us take the individual statistical variables to be classical m-component vectors $\mathbf{S}_i$, normalized so that

$$\mathbf{S}_i\cdot\mathbf{S}_i=m , \tag{4.229}$$

which ensures that T_f is independent of m, with an SK Hamiltonian,

$$\mathscr{H}=-\sum_{\langle i,j\rangle}J_{ij}\mathbf{S}_i\cdot\mathbf{S}_j-\sum_i\mathbf{H}\cdot\mathbf{S}_i , \tag{4.230}$$

where the interactions J_{ij} are infinite-range, statistically independent random variables with a Gaussian distribution, whose mean and variance are given by Eq. (4.6). (In this section we shall mainly set $H=J_0=0$.) The spin glass order parameter [Eq. (3.117)] is now a tensor in spin space,

$$q^{kk'}=[\langle S_i^k\rangle_T\langle S_i^{k'}\rangle_T]_{\rm av} , \tag{4.231}$$

where k,k' refer to spin components. In terms of replicas we have

$$q^{kk'}=\lim_{n\to0}\frac{1}{n(n-1)}\sum_{\alpha\neq\beta}q_{\alpha\beta}^{kk'} , \tag{4.232a}$$

where

$$q_{\alpha\beta}^{kk'}=\langle S_i^{\alpha k}S_i^{\beta k'}\rangle . \tag{4.232b}$$

With $H=J_0=0$ one expects that, on the average, the system will be isotropic in spin space, so that

$$q^{kk'}=q\delta_{kk'} , \tag{4.233}$$

i.e., all directions in spin space are equivalent. In replica theory the assumption of an isotropic state is usually taken to mean that one searches for a solution of the form

$$q_{\alpha\beta}^{kk'} = q_{\alpha\beta}\delta_{kk'} , \tag{4.234}$$

though we see from Eq. (4.232a) that only the sum over replicas has to be isotropic, not necessarily each replica pair separately. We shall discuss this again below.

As usual it is fairly straightforward to obtain the replica-symmetric solution. The order parameter q is given by the solution of de Almeida *et al.* (1978),

$$q = \frac{2^{1-m/2}}{\Gamma(m/2)} \int_0^\infty dz\, z^{m-1} e^{-z^2/2} \left[\frac{I_{m/2}[\beta\tilde{H}(z)]}{I_{(m-2)/2}[\beta\tilde{H}(z)]} \right]^2 , \tag{4.235}$$

where $I_m(x)$ is a modified Bessel function, $\Gamma(x)$ is the gamma function, and

$$\tilde{H}(z) = J(qm)^{1/2} z . \tag{4.236}$$

For $m=1$, Eq. (4.235) reduces to Eq. (4.20), and the result for $m=2$, with a different spin normalization, was given by Kirkpatrick and Sherrington (1978). In the case of $m=3$,

$$I_{3/2}(x)/I_{1/2}(x) = \coth(x) - 1/x ,$$

the familiar Langevin function. In the limit $m \to \infty$, Eq. (4.235) can be solved analytically, with the result (de Almeida *et al.*, 1978)

$$q = 1 - \frac{T}{T_f} . \tag{4.237}$$

For all m one has

$$T_f = J , \tag{4.238}$$

$q=0$ for $T > T_f$ and $q \to 1$ as $T \to 0$.

de Almeida *et al.* (1978) also studied the stability of the replica-symmetric solution along the lines discussed in Sec. IV.B. For vector spins there are nine eigenvalues below T_f (as opposed to two for the Ising model). Some eigenvalues of the stability matrix turn out to be negative, and these have the form

$$\lambda = -a(m)\,|\theta|^2 \quad (\theta \to 0^-) ,$$

where $\theta = (T - T_f)/T_f$ and $a(m) \to 0$ as $m \to \infty$. Hence for any finite m the replica-symmetry solution is unstable, and we shall search for a Parisi-type solution. Clearly $m = \infty$ is special, since the replica-symmetric solution is stable in the spin glass phase. It appears to be the correct solution in this limit, and following the standard interpretation of replica symmetry (see Sec. IV.E), we infer that there is only a single thermodynamic state for $m \to \infty$.

Support for this idea comes from a study of the analogs of the TAP equations (Bray and Moore, 1981b). The number of solutions is of the form $N_S = \exp[N\alpha(T,m)]$ [cf. Eq. (4.110)], where $\alpha \to 0$ as $m \to \infty$ (Bray and Moore, 1981b), consistent with there being only one TAP solution and hence one phase in this limit.

Turning to the Parisi solution, we find that this is very similar (Elderfield and Sherrington, 1982b; Gabay *et al.*, 1982) to the Ising case (Fig. 52). Just below T_f the solution is

$$\begin{aligned} q(x) &= \left[\frac{m+2}{6} \right] x, \quad 0 \le x \le x_1 = \frac{6|\theta|}{m+2} , \\ q(x) &= |\theta| , \quad x_1 \le x \le 1 \end{aligned} \tag{4.239}$$

[cf. Eqs. (4.78), (4.80), and (4.81) for the Ising case]. Note that for $m \to \infty$, $x_1 \to 0$, so replica symmetry breaking disappears, as expected from the above discussion. Elderfield (1982b) has shown that $\int q(x)dx = 1 - T/T_f$ and hence $\chi = J^{-1}$, a constant, for all $T \le T_f$, just as for the Ising model [see Eq. (4.85)].

Next we discuss the probability distribution for the overlaps between states. For a vector spin model this is a joint distribution $P\{q^{kk'}\}$ for all m^2 possible overlaps $q^{kk'}$, and is defined by

$$P\{q^{kk'}\} = \left[\sum_{l,l'} P_l P_{l'} \prod_{k,k'} \delta(q^{kk'} - q_{ll'}^{kk'}) \right]_{\rm av} , \tag{4.240a}$$

where

$$q_{ll'}^{kk'} = \frac{1}{N} \sum_i \langle S_i^k \rangle_T^{(l)} \langle S_i^{k'} \rangle_T^{(l')} \tag{4.240b}$$

is the overlap between the kth component of spins in state l with the k'th component in state l'. It is straightforward to show (Elderfield and Sherrington, 1984), for instance, along the lines of the derivation of Eq. (4.165), that $P\{q^{kk'}\}$ is given by Eq. (3.119), which, for the Parisi ansatz $q_{\alpha\beta}^{kk'} = q^{kk'}(x)$, becomes

$$P\{q^{kk'}\} = \int_0^1 dx \prod_{k,k'} \delta(q^{kk'} - q^{kk'}(x)) . \tag{4.241}$$

With the isotropic solution $q^{kk'}(x) = q(x)\delta_{kk'}$ a comparison with Eq. (4.240) shows that all off-diagonal elements ($k \neq k'$) of $q_{ll'}^{kk'}$ vanish and all diagonal elements are equal *for all pairs of states l,l'*. While it seems clear that each state separately should be isotropic and that the full average over all states should also be isotropic, this last remark is a stronger statement and arises because $q_{\alpha\beta}^{kk'}$ is isotropic for every pair of replicas α,β [see Eq. (4.234)], whereas isotropy only required the sum over all replica pairs to be proportional to $\delta_{kk'}$. (Notice once again the close analogy between replicas and states.) We are not aware of any attempts to look for solutions that are isotropic when averaged over all states but not for each pair separately.

Note that with $H=0$ one can rotate all the spins without any cost in energy. Hence the replica Hamiltonian is invariant under these rotations as well as replica permutations, i.e., for any solution $q_{\alpha\beta}^{kk'}$ we can find equivalent solutions (see Sompolinsky *et al*, 1984),

$$q_{\alpha\beta}^{kk'} \to \sum_{k'',k'''} R_\alpha^{kk''} R_\beta^{k'k'''} q_{\alpha\beta}^{k''k'''} , \tag{4.242}$$

where $R_\alpha^{kk''}$ is an $m\times m$ rotation matrix for replica α. Equation (4.242) is the generalization to vector spins of Eq. (4.167). The fully isotropic solution implies that there is always a rotation that will make $q_{\alpha\beta}^{kk'}$ proportional to $\delta_{kk'}$. A similar statement holds for the overlaps between states. Equation (4.241), together with this discussion, shows that dx/dq is the distribution of any of the following quantities:

$$Q=\left[\frac{1}{m}\sum_{k,k'}(q_{ll'}^{kk'})^2\right]^{1/2},$$

$$\max(q_{ll'}^{kk}) \text{ (not summed on } k),\tag{4.243}$$

$$\frac{1}{m}\max\left[\sum_k q_{ll'}^{kk}\right],$$

where max( ) denotes the maximum value with respect to rotation of spins in one of the thermodynamic states l or l'. Note that Q has the advantage of being rotational invariant and so does not have to be maximized in this way. One can equally well give a microscopic description in which $q_{ll'}^{kk'}$ is replaced by the overlap between spins in two different systems, analogous to Eq. (3.84c). Hence it would be useful to calculate the distribution of the quantities in Eq. (4.243) in a computer simulation to check the Parisi theory for vector spin glasses.

The theory implicitly assumes that a very small field would project out the rotations where all overlaps are diagonal, and one could presumably construct an argument for this along the lines of the discussion between Eqs. (4.167) and (4.172), which is for Ising spins.

Since the Parisi matrix is of the same form as for the Ising model, features such as lack of self-averaging and ultrametricity (Sec. IV.F) will also be present for vector spin glasses.

The relaxational dynamics of infinite-range isotropic spin glasses have been investigated by Sompolinsky and Zippelius (1982b). As for Ising spins, fluctuations within a single state on finite time scales decay with a power of the time t. The power is nonuniversal in that it depends on m and T except for $m=\infty$, where it is equal to $\frac{1}{2}$ for all T. A power-law, as opposed to an exponential, decay is characteristic of marginal stability for each thermodynamic state.

Heisenberg spins have intrinsic dynamics from precession of the spins about their local fields. It is therefore of interest to see whether this type of dynamics has the same critical behavior as the relaxational dynamics introduced in Sec. IV.D. Götze and Sjögren (1984) have looked at real Heisenberg dynamics within a mode-coupling approximation and find the same dynamical exponent, Eq. (4.126), and scaling function, Eq. (4.123), as obtained earlier for relaxation dynamics in an Ising model. This is striking because, for ferromagnetic systems, the dynamical behavior of the two models would be very different. However Götze and Sjögren (1984) also find exponential decay of $q(t)$ to $q_{\rm EA}$ as $t\to\infty$ below T_f, instead of the expected marginal stability (see Sec. IV.D.3). It appears that a mode-coupling theory is inadequate to describe the spin glass state. Götze and Sjögren (1984) claim that "the spin glass transition is of purely dynamical origin." We do not accept this view because, as we have seen, the nonlinear susceptibility, a four-spin correlation function, diverges, and it seems to us that the divergence in this *static* quantity (which goes with a divergent spin glass correlation length in a finite-range model; see Sec. IV.G) is what causes singularities in the dynamics. It is interesting to speculate on whether there is also a hidden divergent length in recent mode-coupling theories (Leutheusser, 1984; Das *et al.*, 1985) which state that the conventional glass transition is "purely dynamical."

To our knowledge the fluctuation spectrum about the Parisi solution has not been evaluated for vector spins, so we do not have any information on Gaussian fluctuations analogous to Eq. (4.221). However, we anticipate that the vector spin results will not be very different.

Sompolinsky *et al.* (1984) have evaluated the exchange stiffness of a Heisenberg spin glass. They consider a model with short-range interactions, but evaluate its properties in mean-field theory [cf. the discussion of Eq. (4.218)]. The exchange stiffness is related to the energy in a uniform twist of the spins and is a parameter in the hydrodynamic theory discussed in Sec. VI.B. If the system is constrained to stay in a single thermodynamic state, the exchange stiffness varies as $(T_f-T)^\mu$ where $\mu=3$. One expects that μ satisfies the Josephson relation $\mu=(d-2)\nu$ with $d=8$, the dimension below which some exponents deviate from their mean-field value (Fisher and Sompolinsky, 1985). Sompolinsky *et al.* (1984) have also shown that the exchange stiffness is *zero* if one allows the system to relax to a different thermodynamic state during the twist (i.e., if one performs a full statistical mechanics average).

We have noted several times that the $m=\infty$ model is particularly simple because there is no replica symmetry breaking. It is also special in having the upper critical dimension d_u (Sec. VI.A) equal to 8 (Green *et al.*, 1982), as opposed to $d_u=6$ for other spin glass models. At present it is not clear whether this model is completely pathological or whether, because of its relative simplicity, it is worth studying in more detail.

To conclude this section we see that, except for $m=\infty$, isotropic vector spin glasses are rather similar to the Ising case.

2. Isotropic vector spin glasses in nonzero field and anisotropic vector spin glasses

There are many ways one can induce anisotropy in an isotropic vector spin glass state described by Eqs. (4.233) and (4.234). One of the most interesting, and the one we discuss first, is the use of a magnetic field.

The Hamiltonian is given by Eq. (4.230), and unless explicitly stated we shall have zero mean, J_0, in the distribution. Let us define S_i^1 to be the spin component in the direction of the field, so that S_i^k, $k=2,\ldots,m$, comprise

the perpendicular components $\mathbf{S}_i^{\perp}$. In the presence of a field we always have

$$M \equiv [\langle S_i^1 \rangle_T]_{\rm av} \neq 0 \,, \qquad (4.244)$$
$$q_{||} \equiv [\langle S_i^1 \rangle_T \langle S_i^1 \rangle_T]_{\rm av} \neq 0 \,,$$

whereas the order parameter $q_{\perp}$, defined by

$$\delta_{kk'} q_{\perp} = [\langle S_i^k \rangle_T \langle S_i^{k'} \rangle_T]_{\rm av} \quad (k,k'=2,\ldots,m) \,, \qquad (4.245)$$

which represents transverse freezing, may or may not vanish. The line in the H-T plane where $q_{\perp}$ becomes nonzero was first discussed by Gabay and Toulouse (1981). For $H \to 0$ the Gabay-Toulouse (GT) line varies as

$$\frac{\delta T_f}{J} = \frac{(m+4)}{2(m+2)} \left[\frac{H}{J} \right]^2 , \qquad (4.246)$$

where $\delta T_f = T_f - T_{\rm GT}(H)$ and T_f is the zero-field freezing temperature. The GT line is sketched in Fig. 64. Notice that the shape of the line given by Eq. (4.246) is very different from the AT line near T_f [see Eq. (4.62)]. At temperatures above the GT line, $q_{\perp}=0$ and the transverse spin glass susceptibility $\chi_{\rm SG}^{\perp}$, defined by

$$\chi_{\rm SG}^{\perp} = \frac{1}{N} \sum_{i,j} [\langle \mathbf{S}_i^{\perp} \cdot \mathbf{S}_j^{\perp} \rangle_T^2]_{\rm av} \,, \qquad (4.247)$$

diverges as $T \to T_{\rm GT}(H)^+$ like

$$\chi_{\rm SG}^{\perp} \propto \frac{1}{T - T_{\rm GT}(H)} \,. \qquad (4.248)$$

There is an analogous transition when $H=0$ but $J_0 \neq 0$ (see Fig. 65), from a collinear ferromagnet to a "canted ferromagnetic state" in which the spins are not all parallel because $q_{\perp} \neq 0$ and $M \neq 0$. Several experiments have been interpreted as evidence for the GT line (Lauer and Keune, 1982; Campbell *et al.*, 1983, 1984; Fogle *et al.*, 1983) where transverse freezing occurs.

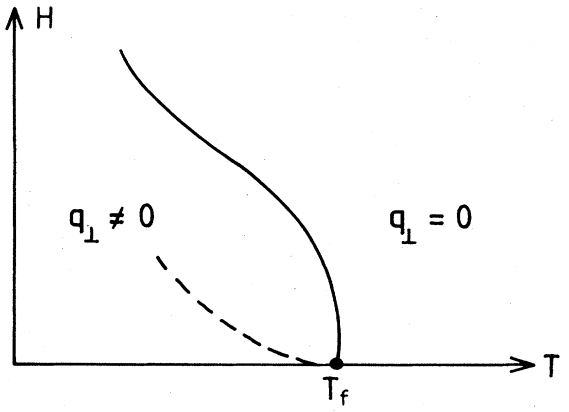

FIG. 64. Sketch of the Gabay-Toulouse (GT) line for an infinite-range vector spin glass. The low-temperature phase has nonzero transverse spin glass ordering. The Almeida-Thouless (AT) line, strictly speaking, no longer occurs, but there is a well-defined crossover region, indicated by the dashed line, which follows a similar curve.

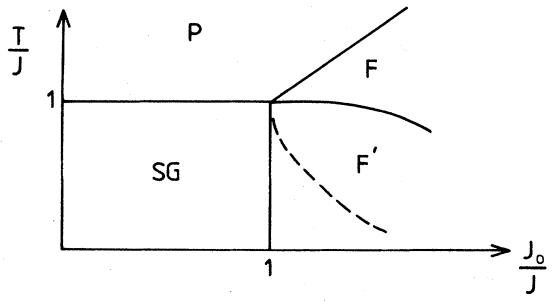

FIG. 65. Sketch of the phase diagram of an infinite-range vector spin glass with nonzero mean J_0 in the distribution. The phases are paramagnetic (P), spin glass (SG), ferromagnetic (F), and a "mixed" phase (F') in which ferromagnetism coexists with transverse spin glass order. Phases F and F' are separated by a GT line. The dashed line indicates a crossover region as in Fig. 64. The F'-SG phase boundary is vertical, just as for the Ising case shown in Fig. 49.

Gabay and Toulouse originally claimed that there was *no* replica symmetry breaking just below the GT line, but this is now known to be incorrect (Cragg *et al.*, 1982; Moore and Bray, 1982). Since the field does not couple to transverse components, the transition of the GT line is precisely *the same as an $m-1$ component isotropic vector spin glass* (Moore and Bray, 1982), so replica symmetry breaking occurs. Because of this, Fischer (1984) finds that the dynamical exponent z, which is nonuniversal for Ising spins and varies along the AT line (see Sec. IV.D.2), keeps its $H=0$ value of 4 everywhere along the GT line.

Elderfield and Sherrington (1982c) and Gabay *et al.* (1982) have investigated the region below the GT line by a Parisi solution (see Sec. VI.B). They find that just below the line replica symmetry breaking has only a weak effect on the longitudinal spin component, in the sense that $q_{||}(x)$ is only weakly dependent on x. It is only when $|\theta| \sim H^{2/3}$, just as for the AT line, that replica symmetry breaking of $q_{||}$ is large. We therefore see that a vestige of the AT line remains for vector spins, but instead of being a sharp line it represents a crossover region below which "strong irreversibility" occurs in the longitudinal component.

We must emphasize that both the AT line, $\delta T_f \propto H^{\psi_{\rm AT}}$, where $\psi_{\rm AT} = \frac{2}{3}$, and the GT line, $\delta T_f \propto H^{\psi_{\rm GT}}$, where $\psi_{\rm GT}=2$, are expected to have the *same* exponents for short-range models at dimension $d=6$ (Fisher and Sompolinsky, 1985). For $d<6$, $\psi_{\rm AT}=\psi=2/(\beta+\gamma)$, which gives $\psi=1$ in $d=6$, and $\psi \sim 0.55$–0.7 from recent Monte Carlo simulations (Bhatt and Young, 1985a; Ogielski and Morgenstern, 1985), as discussed in Sec. V.E.4. This important result of Fisher and Sompolinsky must now be taken into account when interpreting experimental data. In this spirit, doubts may be raised concerning experimental identifications of lines as "AT lines" or "GT lines,"

where usually ψ is compared with the respective mean-field predictions.

For Ising spins, the zero-field susceptibility is constant below T_f [see Eq. (4.85)]. Elderfield and Sherrington (1982c) have shown that the same is true for vector spin glasses. Hence the PaT hypothesis (see Sec. IV.B) should be a reasonable approximation, but only below the AT crossover region, not everywhere below the GT line.

We next consider the case of uniaxial anisotropy, where the Hamiltonian is

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j - D \sum_i (S_i^1)^2 \,. \tag{4.249}$$

If $D>0$, ordering is preferred in the "one" or longitudinal direction, whereas transverse ordering is favored if $D<0$. With $J_0=0$ the phase diagram, found by Cragg and Sherrington (1982) and Roberts and Bray (1982) within replica-symmetric theory, is shown in Fig. 66. As well as a paramagnetic phase P at high temperatures, there is a transverse phase T, with $q_{||}=0, q_\perp \neq 0$, a longitudinal phase L, where $q_{||} \neq 0, q_\perp = 0$, and a phase in which all components order, LT. The most interesting feature is the possibility of two transitions as temperature is lowered, either $P \to L \to LT$ if $D>0$, or $P \to T \to LT$ if $D<0$. A precise calculation of the L-LT and T-LT phase boundaries requires a Parisi-type solution, which has been carried out for small D by Elderfield and Sherrington (1982a). Experiments on anisotropic spin glass systems by Albrecht *et al.* (1982) and Fert *et al.* (1982) are consistent with the predictions of Cragg and Sherrington (1982) and Roberts and Bray (1982), but unfortunately the values of D do not seem to lie within the range where the second transition would be seen. Hence it is interesting that Bray and Viana (1983) and Viana and Bray (1983) have found related anisotropic models that have a second transition no matter how large the anisotropy is.

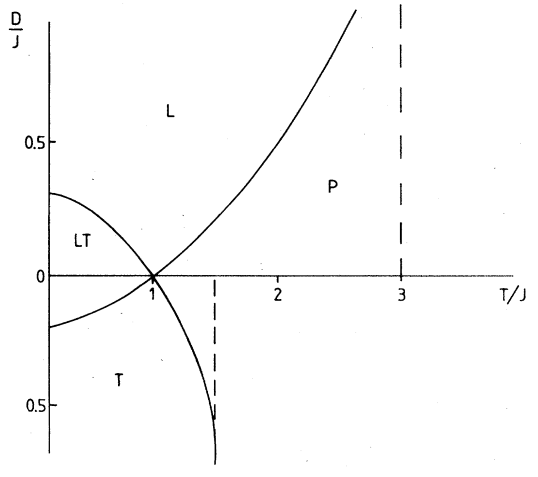

FIG. 66. Sketch of the phase diagram of an infinite-range vector spin glass model with uniaxial anisotropy D (following Roberts and Bray, 1982, and Cragg and Sherrington, 1982).

The rich behavior expected when there is a magnetic field as well as uniaxial anisotropy has been discussed by Elderfield and Sherrington (1983b).

Anisotropies that couple longitudinal and transverse spin components, such as Dzyaloshinksii-Moriya interactions (Fert and Levy, 1980), have been studied in the mean-field limit by Kotliar and Sompolinsky (1984), Sompolinsky *et al.* (1984), and Goldbart (1985). In particular, Kotliar and Sompolinsky (1984) showed that the transition line in a field $T_f(H)$ crosses over from the Ising, AT variation [Eq. (4.62)] for small H to the Heisenberg, GT form in Eq. (4.246) at larger fields. This crossover has apparently been seen by de Courtenay *et al.* (1984), but again we wish to emphasize that the AT and GT lines behave differently from the mean-field predictions (Fisher and Sompolinsky, 1985).

Vector spin glasses including magnetic field, anisotropy, and nonzero mean ferromagnetic interaction have also been studied in a semiquantitative manner by Soukoulis *et al.* (1983b) by a numerical mean-field approach, solving iteratively the coupled mean-field equations (without the reaction field correction) at high temperatures, where the solution is unique, and gradually lowering the temperature. This approach, which can be worked out both for the long-range model and for short-range models, is not believed to be quantitatively completely reliable, but may yield a useful qualitative picture. We are not giving details about this approach, and not even mentioning many other papers relating to the mean field of spin glasses, because it is our intention to describe in this section only what we think has been the "mainstream" of research in this area.

3. Other models

This section will discuss work on other infinite-range models with spin-glass-like features.

Kosterlitz *et al.* (1976) have studied a "spherical" spin glass model in which the length constraint $S_i^2=1$ on each Ising spin is relaxed to a single overall constraint $\sum_i S_i^2 = N$. The infinite-range limit can be solved without the replica trick and gives identical results to the $m=\infty$ isotropic vector spin glass (de Almeida *et al.*, 1978); see Sec. IV.H.1), at least in zero field. The equivalence apparently does not hold in a field because the vector spin glass has a GT line (see Sec. IV.H.2) where freezing of the transverse components occurs, and the spherical model presumably does not, because it has no transverse components.

We saw in Fig. 52 that the phase diagram of the SK model does not have a range of values of J_0/J where the system goes from paramagnet to ferromagnet and finally to a *reentrant* spin glass as the temperature is lowered. Motivated by the large number of systems that apparently do have a reentrant spin glass phase (e.g., Nieuwenhuys *et al.*, 1979; Aeppli *et al.*, 1982), Viana and Bray (1985) have proposed and solved a *dilute* infinite-range spin glass that can have a reentrant spin glass transition.

Virtually all spin glass theory has considered classical models. However, the quantum Heisenberg model with infinite-range interactions was investigated by Fischer (1975), Sherrington and Southern (1975), and in more detail by Bray and Moore (1980c). Although they were unable to determine the transition temperature exactly, they could show that a spin glass transition does occur, even for $S=\frac{1}{2}$, in contrast to a speculation of Klemm (1979).

We noted in Sec. IV.F that the free energies of the valleys in infinite-range spin glasses have an exponential distribution in the vicinity of the minimum free energy. Motivated by this result, De Dominicis *et al.* (1985) have looked at the dynamics of random levels with such a distribution. Relaxation to equilibrium follows a *stretched exponential* decay, which is widely observed in glasses (Jonscher, 1977; Ngai, 1979, 1980) and has also been seen in spin glasses (Chamberlin *et al.*, 1984; Hoogerbeets *et al.*, 1985).

The SK Hamiltonian in Eq. (4.4) has two-spin interactions. One of the most illuminating advances has been to generalize this to p-spin couplings (Derrida, 1980, 1981). The Hamiltonian is then

$$\mathcal{H}=-\sum_{i_1\cdots i_p} J_{i_1\cdots i_p} S_{i_1}\cdots S_{i_p} , \tag{4.250}$$

where the S_i are Ising spins, and the sum is over all groups of p spins in the system. The interactions have a Gaussian distribution suitably scaled with N and p to obtain a sensible limit as N and p tend to infinity. A simplification occurs for $p\to\infty$ because the energy levels become independent random variables (Derrida, 1980, 1981), so this is often called the random-energy model. As a result the free energy can be obtained straightforwardly. There is a transition at

$$\frac{T_f}{J}=(4\log 2)^{-1/2} , \tag{4.251}$$

and for $T>T_f$ the free energy is given by the same expression, Eq. (4.29d), as the SK model in the paramagnetic phase. For $T\le T_f$ the free energy is temperature independent, showing that the entropy vanishes and the system is frozen in a ground state everywhere below the transition, a rather striking result.

Further insight into the low-temperature phase of the $p=\infty$ model has come from the work of Gross and Mézard (1984). They showed that the order-parameter function $q(x)$ is given by just the first stage of the Parisi replica symmetry-breaking scheme. So $q(x)$ is discontinuous, as indicated in Fig. 67. For x larger than the breakpoint x_1, $q(x)=1$, which shows that the "self-overlap" of each state is 1. In other words, all the site magnetizations are ± 1 and the state is completely frozen, as noted above. For $x<x_1$ one has $q(x)=0$, showing that the overlaps between different states are zero. Gross and Mézard also obtained a replica-free derivation of this last result by showing that the magnetization of different TAP solutions (see Sec. IV.C) are uncorrelated. Furthermore the number of states with weight P is given by Eq. (4.190) (we noted the universality of this result in Sec. IV.F). Hence, as for the SK model, the total number of states is infinite, but a few dominate the statistical sum. The breakpoint is at $x_1=T/T_f$, so

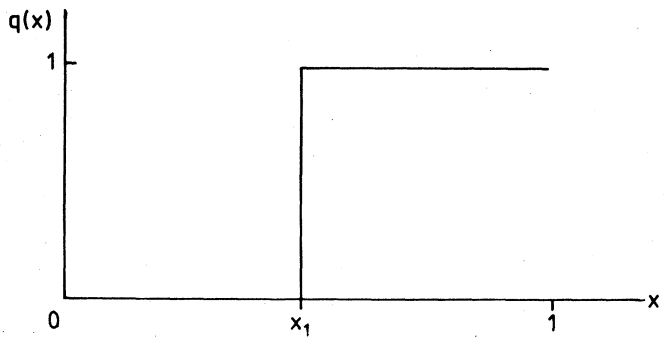

FIG. 67. The order-parameter function $q(x)$ for a p-spin infinite-range spin glass model when $p\to\infty$, following Gross and Mézard (1984). This is just the random-energy model of Derrida (1980, 1981).

$$\int_0^1 q(x)dx=1-T/T_f , \tag{4.252}$$

the same as in the SK model [Eq. (4.84)] and for isotropic vector spin glasses. Could this be a universal result for infinite-range spin glass models?

One might argue that, in a sense, the $p=\infty$ spin glass has a first-order transition because the self-overlap jumps discontinuously from 0 to 1 at T_f. However, statistical mechanics averages involve integrals over x, like Eq. (4.252), and these are continuous, so thermodynamically the transition is second order.

The $p=\infty$ infinite-range spin glass is important because it is the only model, as far as we are aware, that has a phase space with many valleys and whose solution can be expressed very simply at arbitrary temperatures. Derrida and Toulouse (1985) have suggested that it may be a suitable starting point for a replica-free solution of the SK ($p=2$) model.

So far we have discussed models in which the Hamiltonian is invariant under inversion of the spins, $S_i\to -S_i$ for all i, if there is no magnetic field. However, there are systems, such as dilute ortho-hydrogen (Sullivan *et al.*, 1984), that do not have this symmetry and are better described by randomly interacting Potts variables (see Wu, 1982, for a review of Potts models) a "Pott glass," or by quadrupole variables. These models turn out to have rather surprising behavior (Elderfield and Sherrington, 1983a; Erzan and Lage, 1983; Gross *et al.*, 1985). We shall discuss briefly the Potts glass state. A Potts model is characterized by the number of states p available to each statistical variable. For $p=2$ it becomes an Ising model, but for $p\neq 2$ one loses spin inversion symmetry. The Potts glass is found to have two transitions (Gross *et al.*, 1985). Just below the upper transition T_{f1}, the order parameter $q(x)$ has only the first stage of the Parisi symmetry breaking, just as for the random-energy model

discussed above, and the interpretation of the overlaps between states is the same as was given there. At a lower transition T_{f2}, the order-parameter function changes to the form shown in Fig. 68. Physically this means that each of the thermodynamic states has split into an infinite number of partially correlated states. For $p=3,4$ the transitions are continuous, so, for example, $q(x>x_1)$ goes to zero as $T\to T_{f1}^-$. However, in the case of $p>4$ it is first order in the sense that $q(x>x_1)$ jumps discontinuously to zero at T_{f1} but, as for the random-energy model, only second derivatives of the free energy are discontinuous and there is no latent heat. The free energies of the Potts glass and paramagnetic states are sketched in Fig. 69. They are of course equal at T_{f1}, and below T_{f1} the Potts glass solution has a *higher* free energy than the paramagnetic phase. For a small range of temperatures below T_{f1} the paramagnetic state (as well as the Potts glass state) is locally stable; one would naively expect it to be the correct solution because it has the lower free energy (see the discussion at the end of Sec. IV.A). However, it is thermodynamically impossible to get from the paramagnetic solution to the Potts glass state for $T<T_{f1}$ because the free energies are different. Hence the transition *must* occur at T_{f1}, and there has to be some sort of "nonperturbative" instability that destroys the paramagnetic phase below T_{f1}. Numerical tests of these striking predictions would be most desirable. Gross *et al.* (1985) make similar predictions for quadrupolar glasses.

We conclude this long chapter on infinite-range models with a couple of problems from outside the spin glass field but which turn out to have spin-glass-like features. The models of neural networks proposed by Little (1974) and Hopfield (1982) have recently been studied in detail by Amit *et al.* (1985) and Peretto (1984). The long time

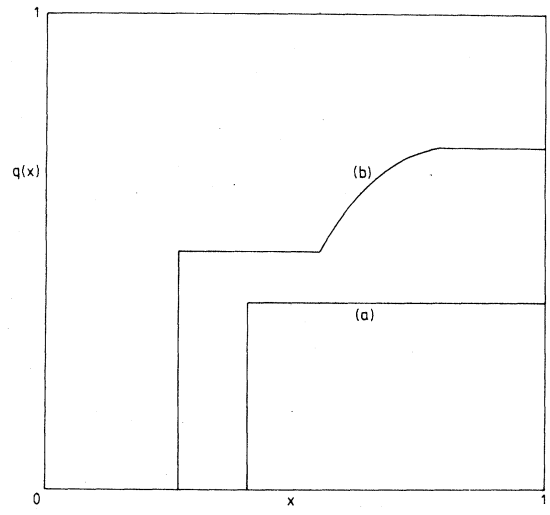

FIG. 68. Order-parameter function $q(x)$ for a Potts glass: (a) $T_{f_2}<T<T_{f_1}$; (b) $T<T_{f_2}$. From Gross *et al.* (1985).

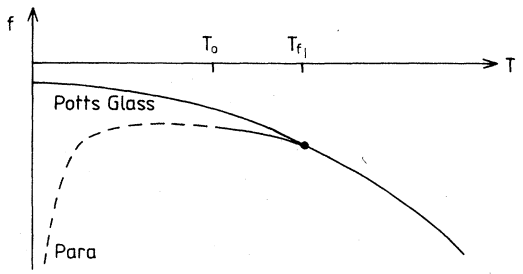

FIG. 69. A sketch of the free energy per spin f against temperature for a Potts glass, following Gross *et al.* (1985). Solid lines indicate locally stable solutions, and the dashed curve is the continuation of the paramagnetic solution into the region for $T<T_0$ where it is locally unstable. For $T_0<T<T_{f_1}$ there are two locally stable solutions and, surprisingly, thermodynamics forces us to take the one of higher free energy, as discussed in the text.

limit of these two models is very similar and related to the statistical mechanics of an infinite-range Ising spin glass model, essentially the one proposed by Provost and Vallée (1983). The memorized configurations are just the spin states in the various valleys and, remarkably, these are determined *in a simple way* by the interactions. We suspect that analogies between spin glasses and memory models will be developed further in the future.

Finally Kirkpatrick and Toulouse (1985) have studied a special traveling salesman problem which, they argue, may be related to infinite-range spin glass models. They find evidence for freezing, due to frustration, an overlap distribution $P(q)$, and an ultrametric structure of configuration space.

I. Conclusions for the mean-field theory

A simple theoretical picture of the SK model has emerged. Below the AT line there are many thermodynamic states whose free energy has distribution (4.195). However, a very small number dominates the statistical sum, which leads to lack of self-averaging. Since there is not a unique state with all the statistical weight, one obtains a nontrivial averaged order-parameter function $P(q)$, which is equal to dx/dq of the Parisi theory. The order parameter in a single thermodynamic state, $q_{\rm EA}$, is the largest value of the Parisi function, i.e., $q(x=1)$. Sample-to-sample fluctuations in the cumulative probability distribution are characterized by the set of functions $\Pi_y(Y)$ and are universal in that they do not depend upon the form of $q(x)$. The thermodynamic states have a hierarchical or "ultrametric" structure. One can now derive the whole theory without replicas. It is, however, unclear at present whether the novel features of the SK model, such as lack of self-averaging, ultrametricity, and an order-parameter distribution, are artifacts of the infinite-range interaction or whether they would also

occur for more realistic models, at least in some range of space dimension d.

V. SHORT-RANGE EDWARDS-ANDERSON MODELS

In this section, we discuss a subject that is in flux, and it is doubtful whether at this time one can give a coherent picture of the many conflicting ideas. Consequently we do not emphasize a particular point of view, and try to mention most of the work on the subject at least briefly, as a final judgment cannot yet be given. There is a general emphasis, however, on computer simulations; this approach has seemed to be most useful recently, though it is also hampered by various problems, which will be discussed.

A. Renormalization-group approaches

The renormalization group (Fisher, 1974a; Ma, 1976; Domb and Green, 1977)[1] has been outstandingly successful in describing the critical behavior of ferromagnetic systems. It was therefore natural to apply those ideas to the spin problem to go beyond the Edwards-Anderson mean-field theory described in Sec. IV. There are two formalisms that can be used. First of all the momentum-space approach starts with a "soft-spin" Ginzburg-Landau-Wilson effective Hamiltonian for long-wavelength fluctuations. This is described by Eqs. (3.49)—(3.54) for the spin glass problem (Chen and Lubensky, 1977). The technique leads to a systematic expansion of the exponents in powers of $d_u - d$, where d_u, the upper critical dimension, is equal to six for spin glasses. Exponents are calculated by studying the system at and above T_f, so one can avoid the problems of replica symmetry breaking (Sec. IV.B) if one is interested only in exponents, though one does run into this problem in calculations of the equation of state below T_f (Pytte and Rudnick, 1979). The alternative approach is to work in real space with fixed-length spins. The main disadvantage to this is that one normally has to make uncontrolled approximations, but, on the other hand, one can obtain global features of the phase diagram, which is not possible from momentum-space techniques, and the replica trick is avoided by rescaling the probability distribution for the random variables (Lubensky, 1975; Young and Stinchcombe, 1975). We discuss the application of each of these techniques in turn to the spin glass problem.

1. Expansions near six space dimensions

The momentum-space approach was first applied to the paramagnetic-spin glass transition by Harris *et al.* (1976). They studied only the part F_q of the effective Hamiltonian in Eq. (3.51). In ferromagnetic systems the lowest-order interaction is of fourth order in the fields, and dimensional analysis then gives $d_u = 4$. However, spin glasses have the $\mathrm{Tr}q^3$ term in Eq. (3.53a), and the same dimensional analysis for a cubic interaction yields $d_u = 6$. To lowest order in $\varepsilon = 6 - d$, Harris *et al.* find the correlation exponents are given by

$$\nu = \frac{1}{2} + \frac{5m\varepsilon}{12(2m-1)} , \tag{5.1a}$$

$$\eta = \frac{-m\varepsilon}{3(2m-1)} , \tag{5.1b}$$

where m is the spin dimensionality and η and ν are defined in Sec. IV.G. Spin glass exponents to first order in ε have also been calculated without the replica trick [used in the derivation of (3.53a)], and the results agree with Eq. (5.1) (Feigelman and Tsvelik, 1979; Zippelius, 1984). For $d \geq 6$, ν and η have their mean-field values, $\nu = \frac{1}{2}, \eta = 0$ [see Eqs. (4.227) and (4.228)]. The thermodynamic exponents, α, β, γ, etc. (see Sec. IV.A), can be obtained from η, ν by scaling laws, e.g., $\gamma = (2-\eta)\nu$, $\beta = \frac{1}{2}(d-2+\eta)\nu$, $\alpha + 2\beta + \gamma = 2$, where the "hyperscaling" relation involving the dimensionality d is only valid for $d \leq 6$ (see Fisher and Sompolinsky, 1985, for a discussion of this and related points). The ε expansion has been extended to ε^2 by Elderfield and McKane (1978) and to ε^3 by Green (1985). For the Ising $m = 1$, case, Green (1985) finds

$$\begin{aligned} \eta &= -0.3333\varepsilon + 1.2593\varepsilon^2 + 2.5376\varepsilon^3 , \\ \nu^{-1} - 2 + \eta &= -2\varepsilon + 9.2778\varepsilon^2 + 4.2336\varepsilon^3 , \end{aligned} \tag{5.2}$$

while for the Heisenberg $m = 3$ model

$$\begin{aligned} \eta &= -0.2\varepsilon + 7.7333\times 10^{-2}\varepsilon^2 - 7.8127\times 10^{-2}\varepsilon^3 , \\ \nu^{-1} - 2 + \eta &= -1.2\varepsilon + 1.164\varepsilon^2 - 1.4735\varepsilon^3 . \end{aligned} \tag{5.3}$$

Note that Eq. (5.1a) and the scaling law $d\nu = 2 - \alpha$ predict $\alpha < -1$, so the temperature derivative of the specific heat is continuous, consistent with there being no observable singularity.

Chen and Lubensky (1977) have investigated competition between spin glass and ferromagnetic orderings by including the F_M and F_{Mq} terms in the effective Hamiltonian (3.51). In particular, they calculated exponents associated with the multicritical point where spin glass, ferromagnetic, and paramagnetic phases meet (see Fig. 49 for the mean-field phase diagram). Surprisingly, the exponents are complex for $m = 2,3$ and $d < 6$, which is completely unphysical.

This last remark leads us to discuss the significance of the ε expansion for behavior in any dimension below 6, especially $d = 3$. Several problems can arise. For instance, some qualitative change in behavior, such as the occurrence of the lower critical dimension, could occur between 6 and 3 dimensions. Thus the ε expansion may give meaningful results for $d_l < d < d_u$, but if $d_l > 3$ it could not be applied for $d = 3$. Other problems arise be-

[1]The work edited by Domb and Green (1977) contains a good collection of articles on the renormalization group.

cause the expansion is at best only asymptotic (Brézin *et al.*, 1977; Lipatov, 1977). Typically the coefficient of ε^K varies as $a^K K!$ for large K. If $a<0$, causing the series to alternate, one can use Padé-Borel methods (Brézin *et al.*, 1977) to sum the series and obtain a finite answer. On the other hand, if $a>0$ the series is not summable. Even if $a<0$ one has to work harder to show that the Padé-Borel sum is the *correct* answer and does not differ from the correct result by typically exponentially small terms (this means showing that the series is uniquely summable). We shall ignore this last problem and just discuss here whether the series alternates. Alternation appears to occur for the Heisenberg case [Eq. (5.3)] but not for the first few terms of the series [Eq. (5.2)], though it is possible the alternation may set in at higher orders. A general study of the alternation of the series, perhaps along the lines of Houghton *et al.* (1978) and McKane (1986) for the related percolation problem, would be highly desirable. The multicritical exponents for $m=2,3$ are obviously unphysical, and it remains to be seen whether the same is true in general for expansions of spin glass critical exponents (the unphysicality could show up in non-Borel summability even when the coefficients are real) or whether this is a special case.

We have noted that $d_u=6$ for spin glasses because of the q^3 term in the effective Hamiltonian. Higher-order interactions are irrelevant in the renormalization-group sense. However, we noted in Sec. IV.D that we could not forget about one of the quartic terms $Q^4_{\alpha B}$ in Eq. (4.72), below T_f even in mean-field theory, because it is responsible for replica symmetry breaking. Hence the $Q^4_{\alpha B}$ is "dangerously irrelevant" (Fisher, 1974b) in the sense that its renormalized value tends to zero at long length scales, but singularities develop in this limit. Fisher and Sompolinsky (1985) have shown that the presence of the dangerous irrelevant value causes exponents involved with replica symmetry breaking to deviate from their mean-field values for $d<8$. As an example consider the Almeida-Thouless line (Sec. IV.B) and the Gabay-Toulouse line (Sec. IV.H), which, for small field H, vary as $\delta T \propto H^{\psi_{\rm AT}}$, $\delta T \propto H^{\psi_{\rm GT}}$, respectively, where $\psi_{\rm AT}=\frac{2}{3}$ and $\psi_{\rm GT}=2$ in mean-field theory. For $6<d<8$, Fisher and Sompolinsky find $\psi_{\rm AT}=4/(d-2)$ (see also Green *et al.*, 1983) and $\psi_{\rm GT}=4/(10-d)$, so that in $d=6$, $\psi_{\rm AT}=\psi_{\rm GT}=\Delta^{-1}=1$ where $\Delta=(\gamma+\beta)/2$ is the expected crossover exponent. The fact that $\psi_{\rm AT}\neq\psi_{\rm GT}\neq\Delta^{-1}$ in mean-field theory violates expected scaling properties, but we see that scaling is *restored* in $d=6$. Hence "below $d=6$ all scaling laws should be valid, implying that the critical properties of *real* spin glasses should be simpler than in mean-field theory" (Fisher and Sompolinsky, 1985).

One can also use the dynamical formalism discussed in Sec. IV.D to investigate critical properties (Zippelius, 1984). In addition to obtaining static exponents without the replica trick, one can also determine the dynamical exponent z. The standard picture of "critical slowing down" (Hohenberg and Halperin, 1977) says that most of the divergence of the relaxation time comes from a divergent susceptibility, but there may also be a (generally weaker) singularity in the kinetic coefficient. The Van Hove theory assumes that the kinetic coefficient is regular and gives for ferromagnetic systems with relaxational dynamics $z=2-\eta$. Zippelius (1984) shows that for spin glasses the Van Hove result for relaxational dynamics is

$$z=2(2-\eta)\ . \tag{5.4}$$

In mean-field theory $\eta=0$, so we recover Eq. (4.126). Zippelius also shows that there are no corrections to Eq. (5.4) to first order in ε. In fact, the results of recent computer simulations in $d=3$, which give $\nu\simeq 1.4$, $\eta\simeq -0.28$ (Bhatt and Young, 1985), and $z\nu\simeq 6\pm 1$ (Ogielski and Morgenstern, 1985), are consistent with Eq. (5.4), so that corrections to it may be small in any dimension.

Most ε-expansion work has been for Ising and Heisenberg models, but Goldschmidt (1985) has applied the approach to Potts spin glasses.

Ueno and Okamato (1981) and Chang and Sak (1984) have studied spin glasses with a long-range interaction $P(J_{ij})\propto \exp[-J_{ij}^2/2(\Delta J_{ij})^2]$, where $\Delta J_{ij}\propto r_{ij}^{-\sigma}$, $r_{ij}=|\mathbf{r}_i-\mathbf{r}_j|$. They find that the upper critical dimension is lowered to $d_u=3\sigma/2$ (for $0<2\sigma-d<2$) and find that near this dimension there is another stable fixed point. Ueno and Okamoto (1981) estimate the boundary (near $d=6$) between the stability region of this fixed point and the fixed point characteristic of short-range forces. They argue that the RKKY interaction which corresponds to $\sigma=d$ can be considered as effectively short range for $d=3$. Kotliar *et al.* (1983) have investigated this model in detail for $d=1$. The interesting region, where transitions occur, is $\frac{1}{2}<\sigma\leq 1$; these transitions are predicted to have mean-field exponents for $\frac{1}{2}<\sigma\leq\frac{2}{3}$ and non-mean-field behavior for $\frac{2}{3}<\sigma\leq 1$. Expansions in $\sigma-\frac{2}{3}$ can be obtained, so $\sigma=\frac{2}{3}$ plays the role of d_u here. Kotliar *et al.* (1983) argue that $\nu\to\infty$ as $\sigma\to 1$ and suggest that for $\sigma=1$ a Kosterlitz-Thouless (1973) transition occurs with no order below T_f [see, however, the discussion in Bhatt and Young (1986) and Sec. V.D]. Consequently $\sigma=1$ plays the role of the lower critical dimension d_l. Interestingly, η is predicted to have its naive value, $\eta=2\sigma-d$, even in the non-mean-field region. Because of this, Bray *et al.* (1986) have argued that the RKKY systems are effectively short range only for $d>d_l$, where d_l is the lower critical dimension of the short-range model. In this case, the transition is at finite temperature. However, in contrast to Ueno and Okamoto (1981), they argue that RKKY models have different behavior from short-range systems for $d<d_l$. In this situation the short-range system has a $T=0$ transition with a power-law divergence of $\chi_{\rm SG}$ and $\xi_{\rm SG}$ as $T\to 0$. However, Bray *et al.* (1986) claim that the corresponding RKKY model, with $\sigma=d$, will be *at*, rather than below its lower critical dimension, so either the transition is at $T=0$, with an exponential divergence as $T\to 0$, or T_f is finite and the transition is rather similar to that in the

two-dimensional XY model (Kosterlitz and Thouless, 1973) or to the one-dimensional Ising model with $1/r^2$ interaction (Anderson and Yuval, 1971). By contrast, Chakrabarti and Dasgupta (1986) find that the RKKY $m=3$ model acts as if it were well below its lower critical dimension, but the sizes used in their simulations may be too small to see the asymptotic behavior.

2. Real-space renormalization

In the real-space approach one works directly on a lattice, with fixed-length spins, usually Ising, though there are also applications to Heisenberg systems, as we shall discuss. Unlike the momentum-space technique, discussed in Sec. V.A.1, which works best near $d=d_u$, the real-space method is most simply applied in low dimensions, so there has been a lot of work for $d=2$, although $d=3$ has also been studied.

Different approximation schemes all have to truncate in some way, the number of interactions generated by the transformation, usually by considering only a small portion of the lattice. The scheme due to Migdal (1975); see also Kadanoff, 1976 is one of the simplest, but has proved very successful when used carefully. It considers only the nearest-neighbor interaction and can be applied without difficulty in any dimension, though it appears most successful in low dimensions. For random problems one has to find the transformation for the probability distribution of the bonds (Lubensky, 1975; Young and Stinchcombe, 1975). In the Edwards-Anderson model the bonds are statistically independent and remain so after rescaling in the Migdal approach, though correlations are generated in more complicated schemes (Southern and Young, 1977). Hence for Ising spins the transformation has the effect

$$P(J_{ij}) \rightarrow P'(J'_{ij}) , \tag{5.5}$$

and critical points are located at fixed-point distributions, i.e., $P=P'$. Exponents can be found by linearizing the distribution about its fixed-point form. The shape of the distribution may be followed (essentially exactly) by numerical methods (Young and Stinchcombe, 1976; Kirkpatrick, 1977; Southern and Young, 1977) or by approximating the distribution at each stage either by a Gaussian (Southern and Young, 1977) or a set of delta functions (Jayaprakash *et al.*, 1977; Benyoussef and Boccara, 1981, 1982, 1983, 1984). Following the distribution in full, one finds a transition at $T_f=0$ for the $d=2$ square lattice Ising model but a finite T_f on a simple cubic lattice in $d=3$. Furthermore in $d=3$, $T_f \simeq \frac{1}{2} T_f^{\mathrm{MF}}$ (Southern and Young, 1977), where $T_f^{\mathrm{MF}}=z^{1/2}\Delta J$ is the mean-field transition temperature, z is the coordination number (6 here), and ΔJ is the width of nearest-neighbor bond distribution. This estimate agrees rather well with recent Monte Carlo simulations (Bhatt and Young, 1985; Ogielski and Morgenstern, 1985) on a $\pm J$ Ising model and domain-wall energy calculations (Bray and Moore, 1985b; McMillan, 1985a) for a Gaussian distribution. The correlation length exponent ν $(=y_T^{-1}$ where y_T is the thermal exponent) is found to be 2.8 (Southern and Young, 1977) if one approximates the distribution by a Gaussian. This agrees fairly well with the result of Bray and Moore (1984a), who get $\nu=3.3\pm0.6$, but is significantly larger than the figures found by Bhatt and Young (1985), Ogielski and Morgenstern (1985), and McMillan (1985a).

Bray and Moore (private communication) used the Migdal approximation to estimate the exponent ν for the zero-temperature transition in $d=2$, obtaining good agreement with their more systematic studies of domain-wall energies (Bray and Moore, 1984b; see also McMillan, 1984c), where $\nu=3.4\pm0.1$. Furthermore the Migdal approach is sufficiently simple that they can calculate an effective exponent $\nu_{\mathrm{eff}}(T)$ which only reaches its asymptotic values at rather low temperatures and is significantly lower than this at higher temperatures, where Monte Carlo simulations are feasible. They suggest, very plausibly, that the smaller values obtained by Young (1983a, 1984) and McMillan (1983) by Monte Carlo methods are really an effective exponent, and that the asymptotic regime, $T\rightarrow0$, was not reached.

Although the Migdal approximation seems to work fairly well for Ising spin glasses, the approximations made are not understood, and other approaches, which also seem reasonable, give quite different results (Kinzel and Fischer, 1978; Tatsumi, 1978c). One therefore needs independent calculations to check out the results obtained.

It is also not completely obvious that valid renormalization schemes can be set up without a more detailed understanding of the low-temperature phase of spin glasses [remember, for example, that in block transformations for antiferromagnets particular symmetry-preserving blocks are required (van Leeuwen, 1975)].

Relatively little has been done for Heisenberg spin glasses, though Benyoussef and Boccara (1983) and Morris *et al.* (1986) have applied the Migdal approximation and found $T_f=0$ in $d=3$ but $T_f\neq0$ in $d=4$. [Benyoussef and Boccara (1983) actually find $T_f=0$ for $d=3$ with Ising spins too, but we believe this is due to the truncation made in approximating the distribution at each stage. There is no doubt that following the full distribution numerically, as discussed above, one finds $T_f\neq0$.] Anderson and Pond (1978) have approximated the equations in the Migdal method and obtained $d=3$ for the lower critical dimension (see Sec. V.E) of vector spins. As for the Ising case, we feel that independent calculations are needed to check the predictions of real-space renormalization-group calculations for Heisenberg spin glasses. We shall see in Sec. V.E that results from the untruncated Migdal equations do agree well with other methods.

McKay *et al.* (1982) and McKay and Berker (1984) have carried out the Migdal renormalization-group transformation for frustrated but nonrandom Ising systems on a hierarchical pseudolattice for which the transformation is exact. They find a region where the renormalization-group trajectories are chaotic (i.e., the fixed points are a "strange attractor") and consequently "at successively longer distances, strong and weak spin

correlations are encountered in a chaotic sequence." It is unclear at present whether this feature persists in a random system on a real lattice, but it does not yet appear to have been seen in numerical simulations.

B. Monte Carlo calculations

1. General considerations

As we have seen in previous sections, the theory of spin glasses is very difficult even in the mean-field limit, and very little information about the properties of short-range spin glasses has been established analytically. In this situation, it is very useful to resort to numerical results provided by the Monte Carlo computer simulation method. Thus it is no surprise that shortly after the introduction of the model of Edwards and Anderson (1975) the first Monte Carlo studies of it appeared (Binder and Schröder, 1976a, 1976b); we shall see below that, while this approach yields much valuable insight, there are still many fundamental questions to be answered, and thus the study of this model continues to be a challenging problem, to which much recent activity has been devoted. Since the Monte Carlo method has yielded useful results also for the infinite-range model (Sec. IV) and more realistic site-disorder models (Sec. VI.C), we start here by giving a brief outline of the foundations of the Monte Carlo method in statistical mechanics (for more details see Binder, 1976, 1979, 1984b); this also serves to introduce some of the basic difficulties by which this approach is plagued.

In a Monte Carlo simulation one studies a finite lattice of linear dimensions $L_1, L_2, \ldots, L_d$ in d dimensions; usually one takes $L_1 = L_2 = \cdots = L_d = L$ and periodic boundary conditions. The Monte Carlo method now aims at a numerical estimate of canonical Gibbs ensemble averages for the considered model. While in an exact treatment of an Ising spin glass one would have to calculate the Boltzmann weight for 2^N states ($N = L_1 L_2 \cdots L_3$), the idea behind the Monte Carlo method is to replace the exact average over all the states by an approximate one, where one averages only over a sample of M states, which are generated by a stochastic process. In principle, there are many possible ways to generate the configurations $\mathbf{X} \equiv \{S_1, S_2, \ldots, S_N\}$ of the Ising spins $S_i = \pm 1$. By the so-called "simple random sampling" one would choose the various states $\mathbf{X}_\nu$ included in the sample randomly, independent of each other, with uniform *a priori* probability. However, since for large N the Boltzmann weight $\exp[-\mathscr{H}(X)/k_B T]$ varies over so many orders of magnitude, the average over such a sample of states does not converge for any reasonably small number of states M. The way round this difficulty is the so-called "importance sampling" (Metropolis *et al.*, 1953): rather than choosing the states $\mathbf{X}_\nu$ of the sample independently of each other, we generate them recursively one from the other by a chosen transition probability $W(\mathbf{X} \to \mathbf{X}')$, where the move $\mathbf{X} \to \mathbf{X}'$ is only a small local change of the configuration (e.g., a single spin flip at a randomly selected lattice site). Usually one then requires that $W(\mathbf{X} \to \mathbf{X}')$ satisfy the "detailed balance condition" with the canonical equilibrium probability $P_{\rm eq}(\mathbf{X})$,

$$P_{\rm eq}(\mathbf{X}) = \exp[-\mathscr{H}(\mathbf{X})/k_B T]/Z \ , \tag{5.6}$$

namely,

$$P_{\rm eq}(\mathbf{X}) W(\mathbf{X} \to \mathbf{X}') = P_{\rm eq}(\mathbf{X}) W(\mathbf{X}' \to \mathbf{X}) \ . \tag{5.7}$$

Equations (5.6) and (5.7) do not specify $W(\mathbf{X} \to \mathbf{X}')$ uniquely; but a simple choice of $W(\mathbf{X} \to \mathbf{X}')$ that is frequently used is that of the kinetic Ising model introduced by Glauber (1963),

$$W(\mathbf{X} \to \mathbf{X}') = (\tfrac{1}{2}\tau_0)(1 - \tanh\{[\mathscr{H}(\mathbf{X}') - \mathscr{H}(\mathbf{X})]/2k_B T\}) \ . \tag{5.8}$$

Equation (5.8) involves an arbitrary parameter τ_0 setting a time scale so that $W(\mathbf{X} \to \mathbf{X}')$ acquires the meaning of a transition probability per unit time. In a Monte Carlo context, this time unit is eliminated by choosing 1 Monte Carlo step (MCS) per spin as the unit of the Monte Carlo "time." The reason why it is necessary to take this kinetic point of view at all is that the Monte Carlo process must be viewed as a numerical realization of a Markow process, described by the following master equation for the probability $P(\mathbf{X}, t)$ that state $\mathbf{X}$ occurs at time t,

$$\frac{d}{dt} P(\mathbf{X}, t) = -\sum_{\mathbf{X}'} W(\mathbf{X} \to \mathbf{X}') P(\mathbf{X}, t) + \sum_{\mathbf{X}'} W(\mathbf{X}' \to \mathbf{X}) P(\mathbf{X}', t) \ . \tag{5.9}$$

[A special case of this description has already been encountered in Eq. (3.138).]

Because of the detailed balance condition, Eq. (5.7), the equilibrium distribution $P_{\rm eq}(\mathbf{X})$ is obviously a stationary solution [$dP(\mathbf{X}, t)/dt \equiv 0$] of the master equation; in fact, for finite systems one can show under fairly general conditions, which apply for the models of interest, that Eq. (5.6) really is the asymptotic distribution of states to which the process $\mathbf{X}_1 \to \mathbf{X}_2 \cdots \mathbf{X}_M$ tends in the limit of large M, irrespective of the arbitrarily chosen initial configuration $\mathbf{X}_1$. As a result, we obtain an approximation $\overline{A}$ for the Gibbs average $\langle A(\mathbf{X}) \rangle_T$ [Eq. (3.14)] which is simply an arithmetic average over the generated states, or—equivalently—a time average,

$$\overline{A} = \frac{1}{M - M_0} \sum_{\nu = M_0 + 1}^{M} A(\mathbf{X}_\nu) \ , \tag{5.10a}$$

$$\overline{A} = \frac{1}{t - t_0} \int_{t_0}^{t} dt' A(\mathbf{X}, t') \ . \tag{5.10b}$$

In this equation, we have already considered that in practice it is advantageous to omit from the average the first M_0 configurations, which are not yet fully characteristic of the final equilibrium due to the influence of the initial condition $\mathbf{X}_1$. The associated "time" is $t_0 = M_0/N$ in our convention, and similarly $t = M/N$. At this point we

have to make contact with our discussion of ergodicity, Sec. III.B. Although we know that the dynamic model defined by Eqs. (5.8) and (5.9) [and a choice of the Hamiltonian, e.g., Eq. (3.27)] is ergodic, we expect a problem: if there is a phase transition, we expect in the ordered phase that the "ergodic time" τ_e will increase exponentially with the size of the system, N. Only if $t_0 \gg \tau_e$ is the influence of the initial condition lost, and the time interval which is averaged over must exceed τ_e (i.e., $t - t_0 \gg \tau_e$) in order that $\bar{A}$ be a reasonable estimate of the canonical Gibbs average $\langle A \rangle_T$.

In principle, these statements apply to any model undergoing a phase transition, including the simple example of an Ising ferromagnet with uniform nonrandom exchange interaction J. Below T_c the probability distribution $P_{\rm eq}(\mathbf{X})$ is sharply peaked at two states, corresponding to states having a magnetization of about $+m_{\rm sp}$ or $-m_{\rm sp}$, where $m_{\rm sp}$ is the spontaneous magnetization appearing for $N \to \infty$. The minimum of $P_{\rm eq}(\mathbf{X})$ in between these ordered states, for a system with all linear dimensions equal and with periodic boundary conditions, is dominated by a "mixed-phase" configuration, in which the system is spilt into two domains separated by two straight ($d=2$) or planar ($d=3$) walls. (With free boundary conditions there would be one wall only, of course.) Consequently, the ratio of the probability $P_{\rm max}$, where $P_{\rm eq}(\mathbf{X})$ has its maximum, to the probability $P_{\rm min}$ of this mixed-phase state is for periodic boundary conditions of the order of

$$P_{\rm max}/P_{\rm min} \propto \exp(2 f_{\rm int} N^{1-1/d}/k_B T) , \tag{5.11}$$

where $f_{\rm int}$ is the interfacial free energy per spin between the phases having magnetization $\pm m_{\rm sp}$ (Binder, 1982c). Since for a single spin-flip kinetic Ising model one can only pass from $+m_{\rm sp}$ to $-m_{\rm sp}$ by moving through this minimum, the ergodic time is estimated as

$$\tau_e \propto P_{\rm max}/P_{\rm min} \propto \exp(2 f_{\rm int} N^{1-1/d}/k_B T) . \tag{5.12}$$

This ergodic time for large N is thus well separated from the spectrum of "intrinsic" relaxation times τ possessed by a (kinetic) Ising ferromagnetic model. Even in the thermodynamic limit, away from T_c all these intrinsic relaxation times are finite for a pure (ideal) Ising ferromagnet; approaching T_c the upper bound of this spectrum $\tau^{\rm max}$ diverges,

$$\tau^{\rm max} \propto \xi^z \propto |1 - T/T_c|^{-z\nu} , \tag{5.13}$$

where ν is the critical exponent of the correlation length ξ and z is the dynamic exponent (Hohenberg and Halperin, 1977). Of course, in a finite system, this divergence is rounded off to a finite value, which from finite size scaling theory (Fisher 1971; Barber, 1983) is estimated as

$$\tau^{\rm max}(T = T_c) \propto L^z = N^{z/d} . \tag{5.14}$$

Thus the N dependence of intrinsic times is always much weaker than that of τ_e. This is no longer true, however, for models with higher spin dimensionality (XY, Heisenberg spin, etc.), in which $f_{\rm int} \equiv 0$ but the term $f_{\rm int} N^{1-1/d}$ has to be replaced by const $\times \Gamma N^{1-2/d}$, for $d > 2$, Γ being the helicity modulus (Fisher *et al.*, 1973), and a power-law divergence of the maximum intrinsic relaxation time [similar to Eq. (5.14)] occurs at all temperatures $T \leq T_c$. These isotropic systems, do not need to pass through the states related to $P_{\rm min}$ in order to explore all their ordered states: rather the direction of the order parameter may undergo a sort of rotational diffusion (Stauffer and Binder, 1978, 1981). The ergodic time τ_e needed for the system to reach a state with $-\mathbf{m}_{\rm sp}$ after a start from $+\mathbf{m}_{\rm sp}$ hence is not given by Eq. (5.14); rather we expect a power-law relation, $\ln \tau_e \propto \ln N$. Note, however, that here we have tacitly assumed a finite-range interaction: for an infinite-range ferromagnet one would have $\ln \tau_e \propto N$ (Griffiths *et al.*, 1966; Binder, 1973) instead of Eq. (5.12). Now the enormous size of the ergodic time τ_e is not at all a practical limitation of Monte Carlo studies of Ising ferromagnets [apart from the case where one wishes to study the ratio $P_{\rm max}/P_{\rm min}$ itself to extract information on the interface free energy $f_{\rm int}$ (Binder, 1982c)]. For $T < T_c$ and $L \gg \xi$ we may choose times t, t_0 such that

$$\tau^{\rm max} \ll t_0, t - t_0 \ll \tau_e , \tag{5.15}$$

and then the simulation will basically yield the partial equilibrium average $\langle A \rangle_T^{(l)}$ [Eq. (3.17)], where the phase (l) is either the state corresponding to $+m_{\rm sp}$ or that corresponding to $-m_{\rm sp}$. Which of the two states if chosen is decided by the initial condition, which has to be chosen such that it is close to one of these states (this can also be an ordered state with all spins up or down, respectively). If one chooses an inappropriate initial condition, e.g., a disordered spin configuration at $T < T_c$, the lower limit of the inequality (5.15), $\tau^{\rm max}$, must be replaced by the nonequilibrium time $\tau_{\rm ne}$. This nonequilibrium time is then determined by the time it takes for the system, which quickly forms small ordered domains of opposite magnetization, to develop towards a monodomain configuration. Since characteristic linear dimensions $l_{\rm dom}$ of the domain grow with time according to a power law involving some exponent a (Lifshitz, 1962),

$$l_{\rm dom}(t) \propto t^a , \tag{5.16}$$

we roughly estimate $\tau_{\rm ne}$ from the condition

$$l_{\rm dom}(\tau_{\rm ne}) \propto L, \quad \tau_{\rm ne} \propto L^{1/a} = N^{1/(da)} . \tag{5.17}$$

For the pure single spin-flip Ising ferromagnet, and thus the associated Mattis spin glass (Sec. III.E.1), all theories (e.g., Lifshitz, 1962; Allen and Cahn, 1979; Ohta, Kawasaki, and Jasnow, 1982) agree that $a = \frac{1}{2}$; but for other cases, smaller values of the exponent a, and hence larger $\tau_{\rm nl}$, may occur (e.g., Sadiq and Binder, 1983; Sahni *et al.*, 1983; Mazenko and Zannetti, 1984). In systems with quenched randomness, however, domain walls are no longer expected to be able to diffuse more or less freely: rather the walls may get pinned in local free-energy minima created by the disorder. In fact, the free energy of an interface as a function of interface position may behave similarly to the free energy qualitatively sketched in Fig.

4. This domain-wall pinning may lead to a much slower rate of domain growth. For Ising ferromagnets exposed to a quenched random magnetic field one expects that (Grinstein and Fernandez, 1984; Villain, 1984)

$$l_{\rm dom}(t) \propto \ln t \ , \tag{5.18}$$

in which case one obtains an exponential variation for $\tau_{\rm nl}$ instead of Eq. (5.17),

$$\ln \tau_{\rm ne} \propto L = N^{1/d} \ . \tag{5.19}$$

In such a situation, simulations cannot satisfy the condition $\tau_{\rm ne} \ll t_0$ for large N (Stauffer *et al.*, 1984), and the choice of an ordered initial state is required to simulate equilibrium properties at system dimensionalities d exceeding the lower critical dimensionality d_l [believed by most workers in this field at present to be $d_l = 2$, for the random-field Ising model (e.g., Grinstein and Ma, 1982; Villain, 1982)]. Conversely, in isotropic systems (XY and Heisenberg magnets) the condition $t - t_0 \ll \tau_e$ is usually not met, and hence one cannot record $\mathbf{m}_{\rm sp}$ but only a sort of root-mean-square order parameter.

What is the consequence of these considerations for simulations of spin glass models? If we work with Ising systems at $d > d_l$, we also expect that there will be an exponential variation of the ergodic time with system size. However, its precise form is not known: in the pure Ising system this variation can be related to the lower critical dimensionality d_l as

$$\ln \tau_e \propto N^{1-d_l/d} \ ; \tag{5.20}$$

whether this law also holds for an Ising spin glass is not known. As a practical proof of ergodic behavior in the Monte Carlo sampling, one often takes the criterion that a system reaches the "inverse state" of a state $\{S_i\}$ reached at earlier time, i.e., the state $\{-S_i\}$ where all spins are flipped over respective to the previous one (Morgenstern and Binder, 1980a; Mackenzie and Young, 1982, 1983; Ogielski and Morgenstern, 1985). However, in a finite system with a particular random-bond configuration $\{J_{ij}\}$ it is conceivable that it is possible even to go from $\{S_i\}$ to $\{-S_i\}$ crossing lower free-energy barriers than would be needed to reach some other states $\{S_i'\}$. Therefore it is very difficult to establish ergodic behavior for spin glass simulations.

A second difficulty is that the ordered configurations of these systems, unlike the case of simple ferromagnets, are not known explicitly. Only for small two-dimensional Ising spin glasses, where explicit ground-state configurations have been obtained from numerical transfer matrix methods (Morgenstern and Binder, 1979, 1980a), can Monte Carlo simulations be performed where the system begins in a true ground-state configuration. In other cases it is hard to distinguish low-lying metastable states from the true ground state.

The third difficulty is that, just as in real spin glass systems, the spectrum of the intrinsic relaxation times τ_i obviously becomes very broad at temperatures far above the freezing temperature (if there is one; Binder, 1977b; Stauffer and Binder, 1978, 1979; Kinzel and Binder, 1983, 1984; McMillan, 1983; Young, 1983a, 1983c, 1984). In fact, it has even been suggested (Randeria *et al.*, 1985) that in random systems the maximum relaxation time $\tau^{\rm max}$ may diverge at the transition temperature of the corresponding pure system [this is a dynamic counterpart of the "Griffiths singularities" (Griffiths, 1969) for static critical phenomena]. We shall return to this problem in more detail below. More interesting are the "characteristic time" τ and "average relaxation time" $\tau_{\rm av}$ (see Ogielski, 1985 and Sec. IV.D.1), which diverge at the spin glass transition. Even for $d < d_l$ one expects a divergence of the average relaxation time $\tau_{\rm av}$, as $T \to 0$, which has an Arrhenius form (McMillan, 1983; Morgenstern, 1983a, 1983b) or is even stronger (Kinzel, 1982a, 1982b; Binder and Kinzel, 1983a, 1983b; Kinzel and Binder, 1983, 1984),

$$\ln \tau_{\rm av} \propto T^{-\nu z_{\rm av}}, \quad T \to 0 \ . \tag{5.21}$$

Here the exponent $\nu z_{\rm av}$ has been chosen for a $T=0$ transition in analogy with Eq. (5.13), but of course it has a different numerical value. Because $\tau^{\rm max}$ diverges at the transition temperature of the pure system, which is well above the spin glass freezing temperature, it is impossible to satisfy Eq. (5.15) in the temperature range of interest. In practice one makes t_0 and $t - t_0$ larger than some value $\bar{\tau}$ chosen so that fluctuations on timescales between $\bar{\tau}$ and $\tau^{\rm max}$ (which are not included correctly) give negligible error. Thus even if there is a transition only at $T=0$, the relaxation times could become so large over a broad regime of nonzero temperatures that it would be impossible—or at least very hard—to meet this condition, namely

$$\bar{\tau} \ll t_0, \ t - t_0 \ . \tag{5.22}$$

In such a case one can never rule out the possibility that a phase transition might occur at a nonzero temperature distinctly lower than the minimum temperature at which the simulation still comes close to equilibrium, defined by $\bar{\tau}(T_{\rm min}) = t_0$. Conversely, any behavior consistent with Eq. (5.13) but giving an estimate for T_f distinctly lower than $T_{\rm min}$ may well be inconclusive, since for $T \lesssim T_{\rm min}$ a crossover to a different behavior occurs, with a true value for T_f much lower—or even at $T=0$. An example which shows that this warning is not academic has been provided by Kirkpatrick (1980), who studied the average relaxation time $\tau_{\rm av}$ defined by

$$\tau_{\rm av} = \int_0^\infty [\langle S_i(0) S_i(t) \rangle_T]_{\rm av} dt \tag{5.23}$$

for the $\pm J$ model in three dimensions at temperatures $k_B T/J \gtrsim 2.1$ [Fig. 70(a)]. He concluded that there was probably a transition described by Eq. (5.13), with $k_B T_f/J \approx 2.0$ and $z_{\rm av}\nu \approx 1$. More recent extensive work performed on a spin glass special-purpose processor (Ogielski and Morgenstern, 1985) reveals that this conclusion definitely was incorrect: data for the same model in the range $1.3 \lesssim k_B T/J \lesssim 2.0$ clearly are still in the disordered phase; they can be fitted to Eq. (5.13), but now

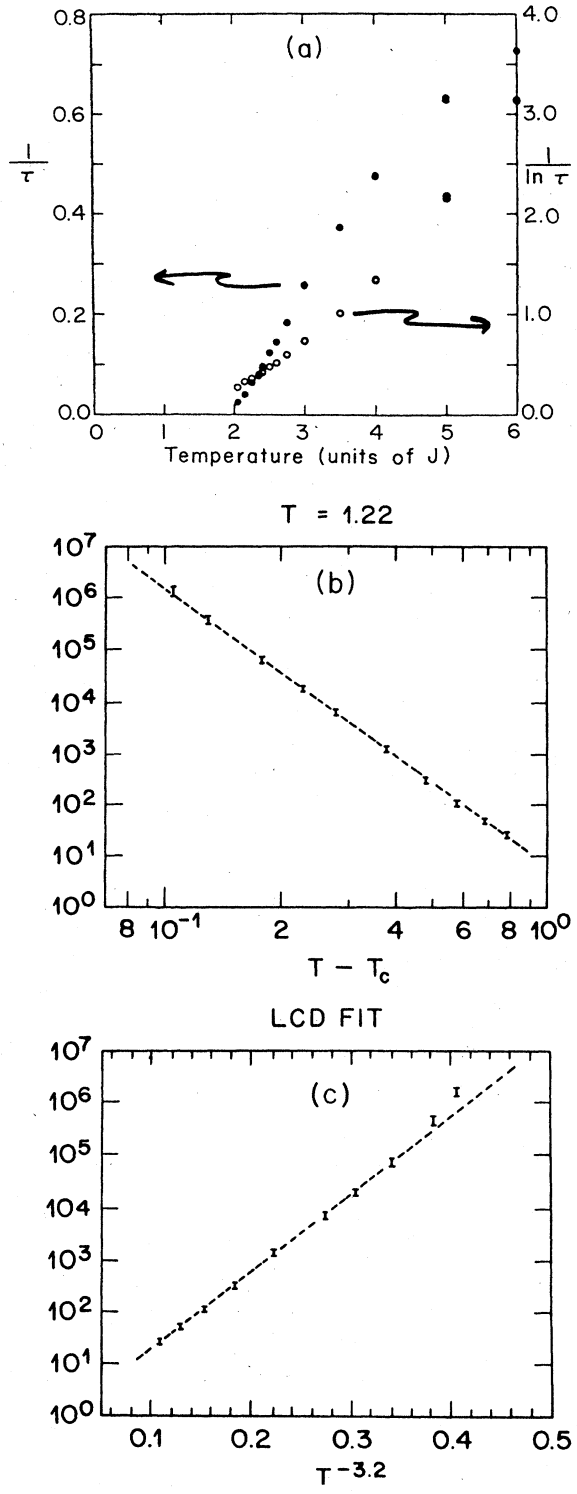

FIG. 70. Relaxation time τ_{av} of the three-dimensional $\pm J$ model, according to Monte Carlo simulations of (a) $30\times30\times30$ lattices (Kirkpatrick, 1980) and (b),(c) $64\times64\times64$ lattices (Ogielski and Morgenstern, 1985). Temperatures are shown in units of k_BJ. In (b) a log-log plot of τ_{av} vs $T-T_c$ is shown, choosing $k_BT_c/J=1.22$.

$k_BT_f/J\approx1.22$ and $z_{av}\nu\approx5.6$ [Fig. 70(b)]; the data could also be fitted to $\ln\tau\propto T^{-z_{av}\nu}$ [i.e., Eq. (5.21)] with $z_{av}\nu\approx3.2$ [Fig. 70(c)]. Because of the systematic deviation of the last point (at the lowest temperature), Ogielski and Morgenstern (1985) strongly favor Eq. (5.13) over Eq. (5.21). Since Eq. (5.21), if it holds at all, should be an asymptotic expression for $k_BT/J\to0$ only, a region not at all probed by the data, we feel sceptical about drawing very strong conclusions about such fits or misfits.

In any case, the discussion of these intrinsic difficulties was intended to elucidate the sources of ambiguity in interpretation of the computer experiments. We shall encounter such interpretive problems frequently in the following sections, as we did with the experiments on real materials.

Finally we comment on the method of equilibrating the system by slow cooling, as done, for example, by Kinzel and Binder (1983, 1984). The above estimates for the nonequilibrium time τ_{ne} refer to the case in which the system is discontinuously quenched from a distant state (where the system was in equilibrium), and thus are not directly applicable to such a continuous change of state. There is various evidence that slow cooling is the most efficient way to come at least close to thermal equilibrium at low temperatures. Some quantities, like the internal energy or the magnetization (in a nonzero field) tend to constant nonzero values as $T\to0$ and then show little temperature variation at low temperatures. Even if the system falls out of equilibrium during the cooling in that temperature regime, these quantities are then less affected by nonequilibrium effects. Cooling rate dependencies of the energy are identified by Grest *et al.* (1986); of course, if the cooling rate is not slow enough, one is hampered by the same difficulty as in the slow-cooling iterative mean-field method (Soukoulis *et al.*, 1982, 1983a, 1983b), which was pointed out by Reger *et al.* (1984): During cooling the effective free-energy hypersurface develops more and more structure, and the system stays near that minimum which develops out of the minimum of the high-temperature phase (Fig. 71). In general, this state is a metastable state only, rather than the equilibrium state which corresponds to the absolute minima of the free energy. Hence at low temperatures this state is not reached easily by cooling; rather barrier hopping is again required and hence equilibrium is obtained only if the cooling is extremely slow. We feel that this picture applies to the experimental situation as well: the field-cooled states are not really in thermal equilibrium as often claimed, but rather metastable; however, these states correspond to rather deep valleys of the free-energy surface and have properties similar to the deepest valleys, and thus are not easily distinguished experimentally from true equilibrium states.

2. Two- and three-dimensional Ising Edwards-Anderson models

The first Monte Carlo work studied Ising Edwards-Anderson spin glass models (Binder and Schröder, 1976a,

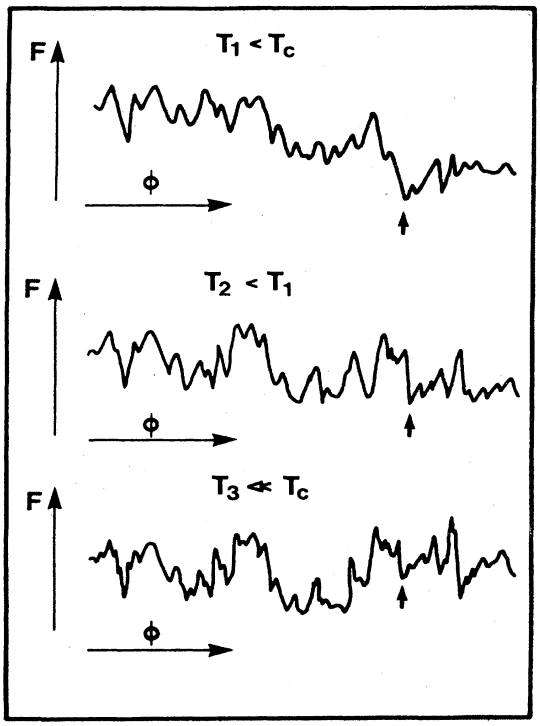

FIG. 71. Schematic plot of the evolution of part of the free-energy "hypersurface" (shown here as a function of one phase-space variable only) with decreasing temperature. Arrow denotes deepest minimum. From Reger *et al.* (1984).

1976b; Binder, 1977a, 1977b, 1977c) with symmetrical nearest-neighbor Gaussian interactions on a square lattice. This model has been studied further by Stauffer and Binder (1978), Kinzel (1978, 1979, 1982a, 1984), Dasgupta *et al.* (1979), Morgenstern and Binder (1980a), Takayama and Takase (1981), Binder (1982b), Binder and Kinzel (1983a, 1983b), Jäckle and Kinzel (1983), Kinzel and Binder (1983, 1984), Nemoto and Takayama (1983), Takayama *et al.* (1983b), and McMillan (1984a).

Similarly, the symmetrical $\pm J$ square lattice [Eq. (3.38a)] was simulated by Ono (1976), Bray and Moore (1977), Kirkpatrick (1977), Sakata *et al.* (1977), Rapaport (1978), Morgenstern and Binder (1979, 1980a), Nemoto *et al.* (1982), Takayama *et al.* (1982, 1983a), McMillan (1983), and Young (1983a).

Higher-dimensional models were studied by Binder and Stauffer (1976c), Binder (1978), Bray *et al.* (1978), Rapaport (1978), Stauffer and Binder (1979), Kirkpatrick (1980), Morgenstern and Binder (1980b), Young (1983c, 1984); Binder and Young (1984), Sourlas (1984), Bhatt and Young (1985), Ogielski (1985), Ogielski and Morgenstern (1985). Asymmetric bond distributions were considered by Sakata *et al.* (1977), Rapaport (1978), Takase and Takayama (1981), and Kinzel (1984).

In spite of this widespread activity—and the above list of authors is certainly incomplete—the properties of these models still are far from being fully understood. Hence we restrict ourselves to describing some typical results and then summarizing the main points; note that properties at $T=0$ are deferred to Sec. V.C and the question of the existence of phase transitions is taken up in Sec. V.E.

Figure 72 shows some of the early data for the susceptibility of these models and the specific heat, which is calculated from energy fluctuations via

$$C/k_B=(\overline{\mathscr{H}^2}-\overline{\mathscr{H}}^2)/[N(k_BT)^2]\ . \tag{5.24}$$

All these data are based on observation times of the order of $t_{\rm obs}\approx 2.10^3$ MCS/spin. It is seen that the susceptibility as calculated from magnetization fluctuations [Eq. (3.78)] increases with decreasing temperature following the Curie law, until the freezing temperature $T_f(t_{\rm obs})$ is reached. For $T<T_f(t_{\rm obs})$ χ decreases again. For the $\pm J$ model one finds another Curie-type increase at very low temperature, due to the fact that even in the "frozen" state of this model the internal effective field acting on the spins, due to the interaction with their nearest neighbors, sometimes exactly cancels. As a result, these "loose" spins behave as if they were free. From these early studies, freezing temperatures were estimated as

$$\frac{k_BT_f(t_{\rm obs})}{J}\approx 1.3\ \ (\pm J)\ \text{model},\ d=2\ , \tag{5.25a}$$

$$\frac{k_BT_f(t_{\rm obs})}{\Delta J}\approx 1.0\ \ (\text{Gaussian model},\ d=2)\ ,$$

$$\frac{k_BT_f(t_{\rm obs})}{\Delta J}\approx 1.5\ \ (\text{Gaussian model},\ d=3)\ . \tag{5.25b}$$

Here we have added the argument $(t_{\rm obs})$ to these estimates, since it is now established that all these data shown in Fig. 72 reflect only dynamic freezing, on the considered time scale (see Sec. V.E). In fact, there are various indications in these data that equilibrium has not been fully reached: specific heats calculated from energy fluctuations [Eq. (5.24)] and from the temperature derivative of the energy [$C/k_B=(\partial\mathscr{H}/\partial T)/N$] disagree at low temperatures; the susceptibility calculated from magnetization fluctuations [Eq. (3.78)] is much smaller than $\chi=\partial\overline{m}/\partial H$ for $H\to 0$ at $T\lesssim T_f(t_{\rm obs})$ and hence is not the equilibrium susceptibility.

While in the $\pm J$ model the peaks of χ and of C occur at about the same temperature, for this choice of $t_{\rm obs}$, for the Gaussian model the peak of χ occurs at a lower temperature than that of C, which furthermore is much broader. This behavior is qualitatively similar to the experimental findings, in contrast to the mean-field case (Sec. IV) in which both χ and C have a cusp at T_f. Thus it was concluded that fluctuation effects characteristic of short-range spin glasses are important in accounting for the real systems. This conclusion, of course, is not at all surprising, since close examination of other phase transition phenomena in nature has revealed in almost all cases that mean-field theory is quantitatively inaccurate, and short-range models provide a better description. Another feature of Fig. 72(b) is that the susceptibility peak is quickly rounded off by a rather small field; in contrast

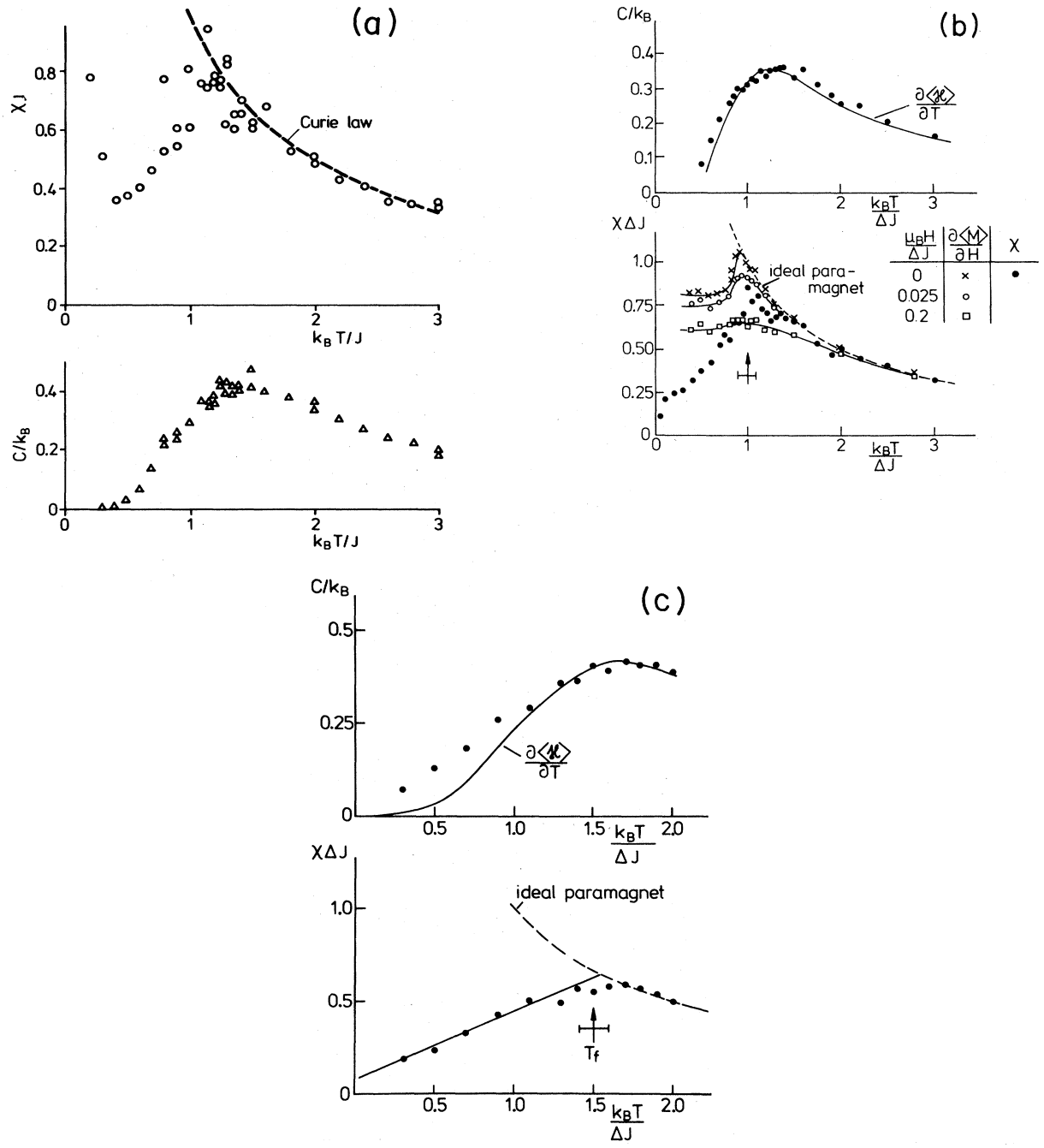

FIG. 72. Susceptibility χ and specific heat C plotted vs temperature: (a) the two-dimensional $\pm J$ model, with lattice size 80×80. From Kirkpatrick (1977); (b) Gaussian model, with lattice size 34×34 up to 80×80. From Binder and Schröder (1976a); (c) three-dimensional Gaussian model; with lattice size 13^3. From Binder (1978). Periodic boundary conditions are used throughout. Dashed curves indicate the Curie law, $\chi=(k_BT)^{-1}$, of the ideal paramagnet. The zero-field susceptibility χ is computed from the fluctuation relation, Eq. (3.78), and C from the analogous relation, Eq. (5.22). Solid curves through points for χ computed from $\partial\langle M\rangle/\partial H$ are drawn only to guide the eye.

the specific heat was found to be affected very little by magnetic field. As we shall see later, the equilibrium behavior of χ is very different from these results, which are valid only for small observation times: for $d=2$, χ is a Curie law, since $T_f(t_{\rm obs}\to\infty)=0$, while for $d=3$ there is probably a nonzero $T_f(t_{\rm obs}\to\infty)$, but this is distinctly lower than Eq. (5.25b).

Another quantity of interest because of its use in some formulations of mean-field theory and because it is accessible experimentally from the interpretation of Mössbauer

and NMR spectra (Sec. II), at least in principle, is the distribution of local internal effective fields $P(H_{\rm eff})$. Here the internal field $H_{\rm eff}(i)$ is defined for the Ising Hamiltonian as

$$H_{\rm eff}(i)=\sum_{j(\neq i)} J_{ij}S_j \ ,$$

and hence it is straightforward to record a histogram representing $P(H_{\rm eff})$ during the simulation. Figure 73 shows that $P(H_{\rm eff})$ resembles a Gaussian at $T \gg T_f$ but develops a minimum at $T < T_f$ for $H_{\rm eff}=0$ rather than a maximum. For $T=0$ this minimum gets deeper with increasing dimensionality, but must retain a finite nonzero value $P(H_{\rm eff}=0,T=0)$ for $d < \infty$ in the short-range case. This value can be approximately related to the slow (logarithmic) relaxation observed at intermediate time scales at low temperatures [Eqs. (3.140)—(3.142)] and to the specific heat and susceptibility at low temperatures [Eqs. (3.143) and (3.144)].

The distinction between the zero-field susceptibility $(\partial m/\partial H)_{H=0}$ and the same quantity observed from magnetization fluctuations [Fig. 72(b)] already indicated irreversible behavior for $T < T_f(t_{\rm obs})$. In fact, switching off a large magnetic field at $t=0$ (so that one starts from a fully aligned spin configuration), one observes that the system quickly relaxes to a state with remanent magnetization σ_r (Fig. 74). With increasing temperature σ_r decreases roughly linearly and vanishes at about the freezing

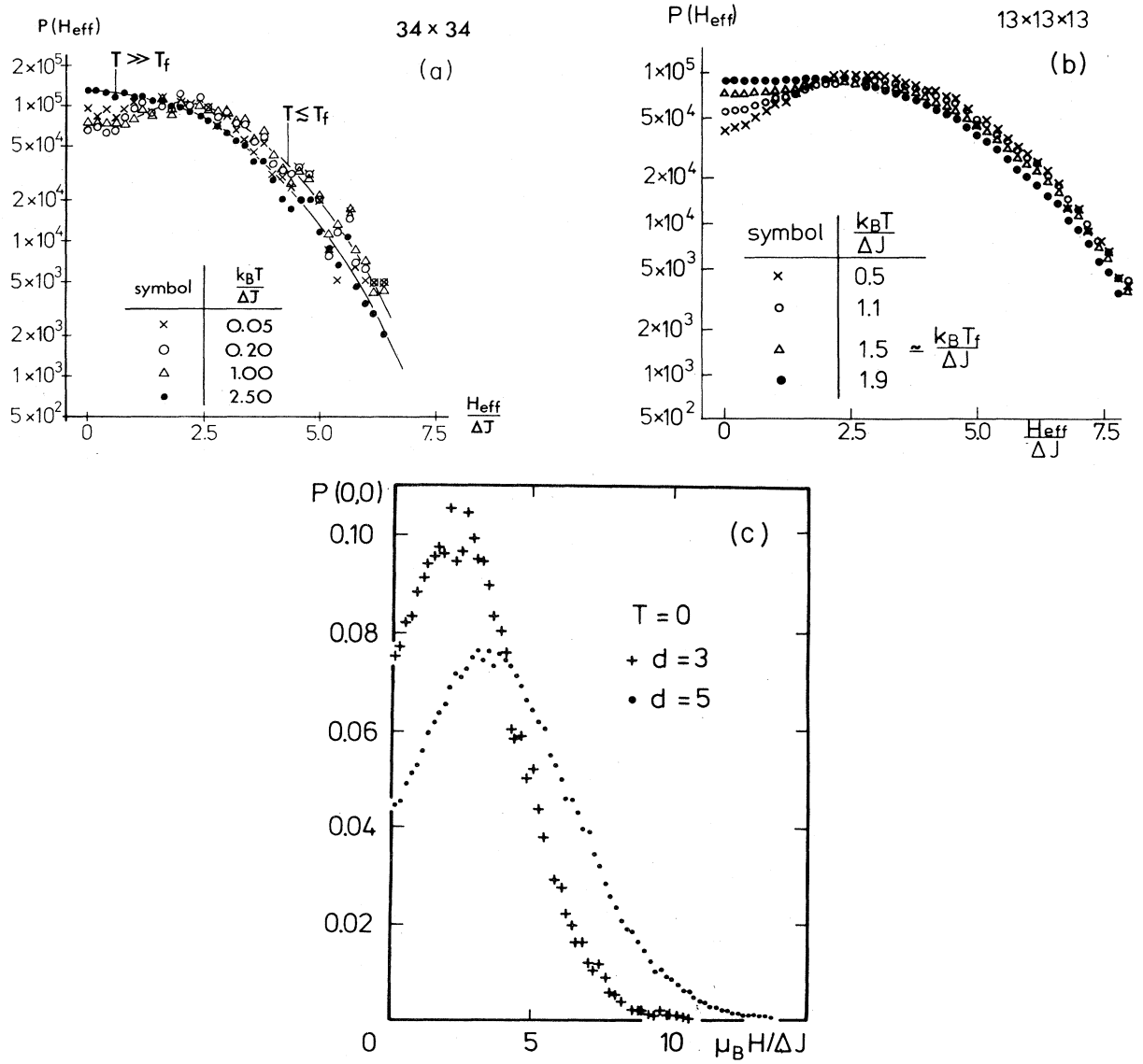

FIG. 73. Distribution function of effective internal fields $P(H_{\rm eff})$ plotted vs $H_{\rm eff}$ at various temperatures for (a) the nearest-neighbor symmetric Gaussian model and $d=2$; (b) $d=3$ (Binder, 1978); (c) the same distribution at zero temperature for $d=3$ and $d=5$. From Stauffer and Binder (1979).

temperature (Binder and Schröder, 1976a, 1976b). While in a spin glass a state with a remanent magnetization clearly should be a metastable state only, it was found to have about the same energy as a state reached by quenching from an initially random spin configuration. Due to the transfer matrix calculations (Sec. V.E.2) we now know, however, that the latter state is also only metastable, and the true equilibrium states have distinctly lower energies at the considered temperatures.

A remanent magnetization of similar magnitude [Fig. 75(a)] is also observed in magnetic field cycles, which again have a qualitative resemblance to some of the experimental data (Fig. 18). The temperature dependence of the short-time remanent magnetization (Fig. 74) likewise has its experimental counterparts, and the same is true with respect to its decay with time on larger time scales [compare Fig. 75(b) to Fig. 19(b)].

In contrast, on these time scales the $\pm J$ model behaves less like experiment, exhibiting a decay of the remanent magnetization with time according to a $1/\sqrt{t}$ law (Kirkpatrick, 1977). While many experimental workers initially took the nearly logarithmic time behavior of the remanence as strong evidence in favor of the Néel model of superparamagnetic clusters (Sec. III.G), these data (Fig. 75) first showed that similar behavior resulted from Edwards-Anderson models as well. In fact, Kinzel (1978, 1979) showed in subsequent careful work, imitating the experimental procedures to obtain thermo-remanence and isothermal remanence (see Sec. II.C.1) in the computer simulation, that the model reproduces most of the experimental systematics on temperature dependence, field dependence, and time dependence. An example of this was already shown in Fig. 17(c) in conjunction with experimental data [Figs. 17(a) and 17(b)].

Slow time dependence is seen not only in the decay of the remanent magnetization (Fig. 75), which reflects relaxation far from thermal equilibrium, but also in the

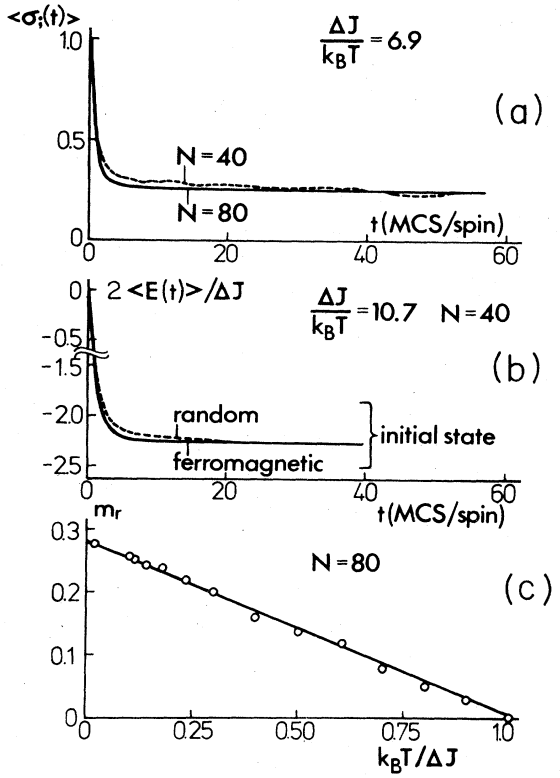

FIG. 74. Time dependence of (a) the remanent magnetization and (b) energy at short time scales, and (c) the temperature dependence of the remanent magnetization reached at these short time scales. From Binder and Schröder (1976a).

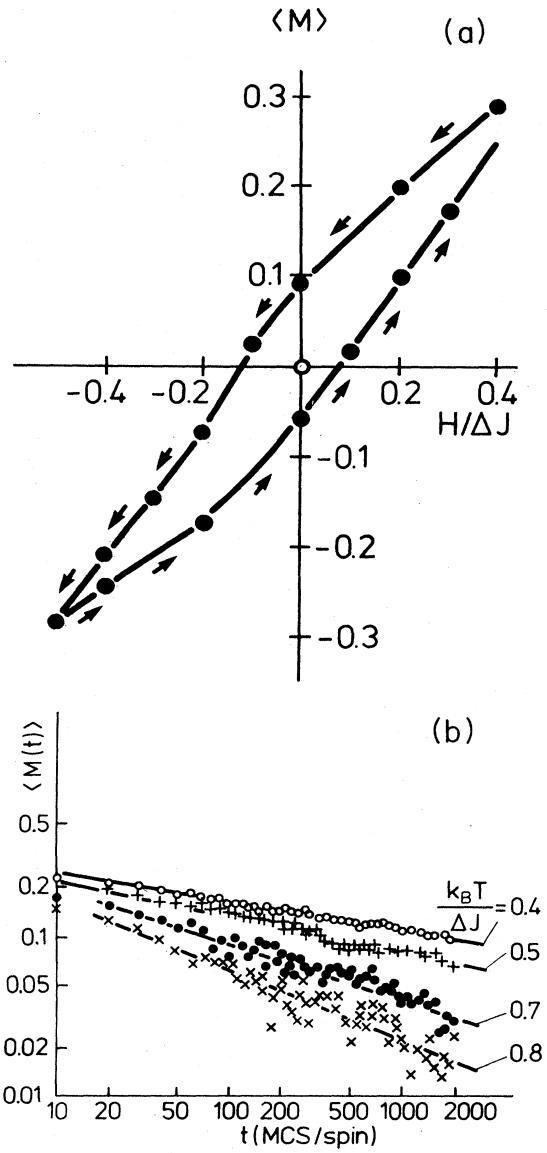

FIG. 75. (a) Magnetization process at $k_BT/\Delta J = 0.7$ recorded in a magnetic field cycle in a two-dimensional Gaussian Ising square lattice of size 80×80. From Binder (1977c). (b) Log-log plot of the remanent magnetization in zero field vs time. From Binder and Schröder (1976a).

self-correlation function $[\langle S_i(0)S_i(t)\rangle_T]_{av}$ of the spins, which reflects the relaxation of fluctuations in thermal equilibrium (Fig. 76). Even rough early data [Figs. 76(a) and 76(b)] indicated a strongly nonexponential decay, nearly consistent with a logarithmic law, $\langle S_i(0)S_i(t)\rangle_T \propto \text{const} - \ln t$, reminiscent of experimental data (Fig. 12), and consistent with some theoretical expectations [Eqs. (3.140)–(3.142)]. Of course, this crude work did not suffice to clarify the precise analytic form of the decay law.

Ogielski (1985) has made an extensive pioneering study of the spin self-correlation function in the three-dimensional $\pm J$ model, using the special-purpose computer of AT&T Bell Laboratories. His data [Fig. 76(c)] give evidence that for $k_BT/J < k_BT_c^F/J \approx 4.51$, the Curie temperature of the corresponding unfrustrated nearest-neighbor Ising ferromagnet, the relaxation function is nonexponential at late times: the empirical ansatz $[\langle S_i(0)S_j(t)\rangle]_T \propto t^{-x}\exp[-(t/\tau)^y]$ describes the data well, with temperature-dependent exponents $x(T), y(T)$. For $T > T_c^F$, $y(T)$ is unity and then decreases and varies approximately linearly with T for $1.3 \lesssim T/J \lesssim 3.5$. For $T/J \lesssim 1.3$, the exponential factor can no longer be observed, due to a dramatic increase of the relaxation time τ in this formula. Ogielski (1985) interprets his results in terms of a phase transition occurring at $k_BT_f/J \approx 1.18$. But even for $T < T_f$ the factor $t^{-x(T)}$ still seems to be there, implying that the self-correlation function may decay to zero. If so there is no spin glass order parameter at any nonzero temperature. The exponent $x(T)$ is found to be $\frac{1}{2}$ for $T > T_c^F$, and then decreases and vanishes as $T \to 0$ (but with a distinctly larger than linear power of T). Near T_f the variation of $x(T)$ is completely smooth, and hence the question could be raised whether this work really can rule out the existence of the factor $\exp[-(t/\tau)^y]$ at all temperatures, but τ is so astronomically large at $T < T_f$ that one could not distinguish $1/\tau$ from zero in the fit. We also note that while the relaxation starts to behave singularly for $T < T_c^F$, in agreement with the suggestions of Randeria *et al.* (1985), their quantitative predictions differ from the results of Ogielski (1985) and probably only dominate for much larger times. Ogielski's suggestion that for $0 < T < T_f$ one has a phase with algebraic decay of correlations rather than a nonzero order parameter can be questioned on the grounds that he follows the decay of the spin-autocorrelation function only to values of about $\frac{1}{3}$ for $T < T_f$; hence a nonzero small order parameter cannot yet be ruled out either.

We shall return below to a more detailed discussion of the relaxation seen in Fig. 76; at this point we only note that these data imply, at least over intermediate observation times t_{obs}, that there is a spin glass order parameter [cf. Eqs. (3.66), (3.67), and (3.77)]. Figure 77 shows both $q(t_{obs})$ [Eq. (3.77)] and the order parameter Ψ, which is the projection of a state onto a ground state [Eq. (3.113)] for various lattice dimensionalities (Stauffer and Binder, 1979). It is seen that on this (short) time scale of $t_{obs} = 2000$ MCS/spin, both $q(t_{obs})$ and ψ behave similarly for all dimensionalities: $q(t_{obs})$ and ψ vanish at about

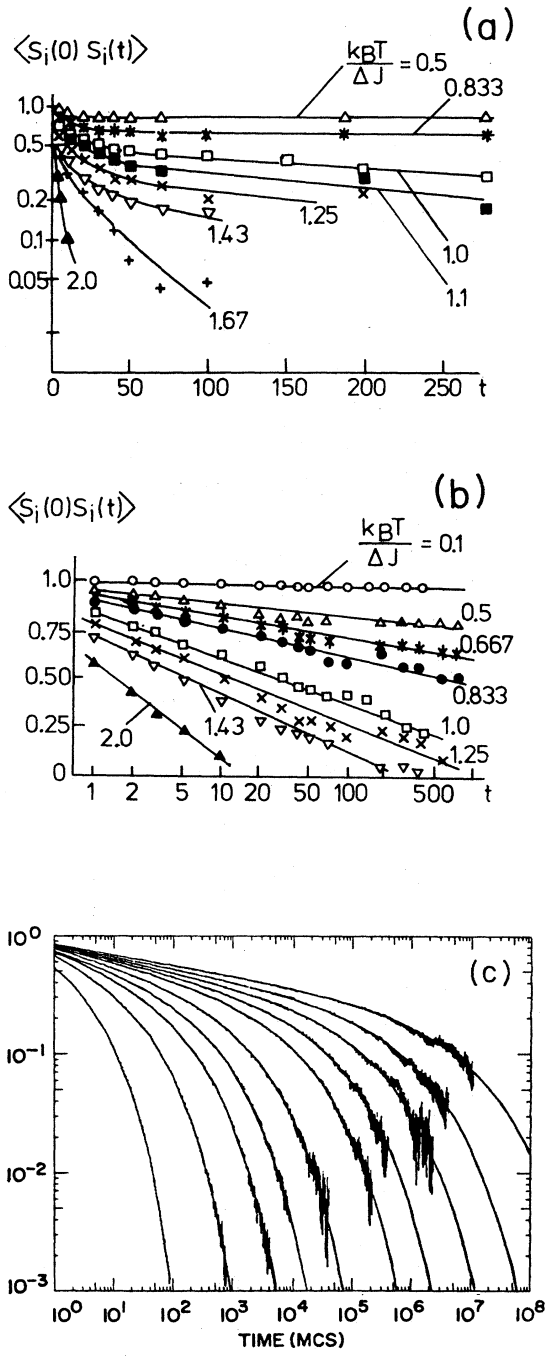

FIG. 76. (a) Semilog plot of the spin-spin self-correlation function vs time, for the two-dimensional nearest-neighbor Gaussian Ising spin glass on a square lattice of size 24×24. (b) Same data replotted with the time axis, rather than the ordinate axis, logarithmic. Various temperatures are shown as indicated. Curves are drawn only to guide the eye. From Binder and Schröder (1976a). (c) Self-correlation function of the three-dimensional $\pm J$ model plotted vs time for lattice size 64^3 and temperatures (from left to right) $T/J = 2.5$, 2.0, 1.8, 1.7, 1.6, 1.5, 1.45, 1.40, 1.35, and 1.3. Solid curves represent fit of the data to the formula $Ct^{-x}\exp[-(t/\tau)^y]$; see text. From Ogielski (1985).

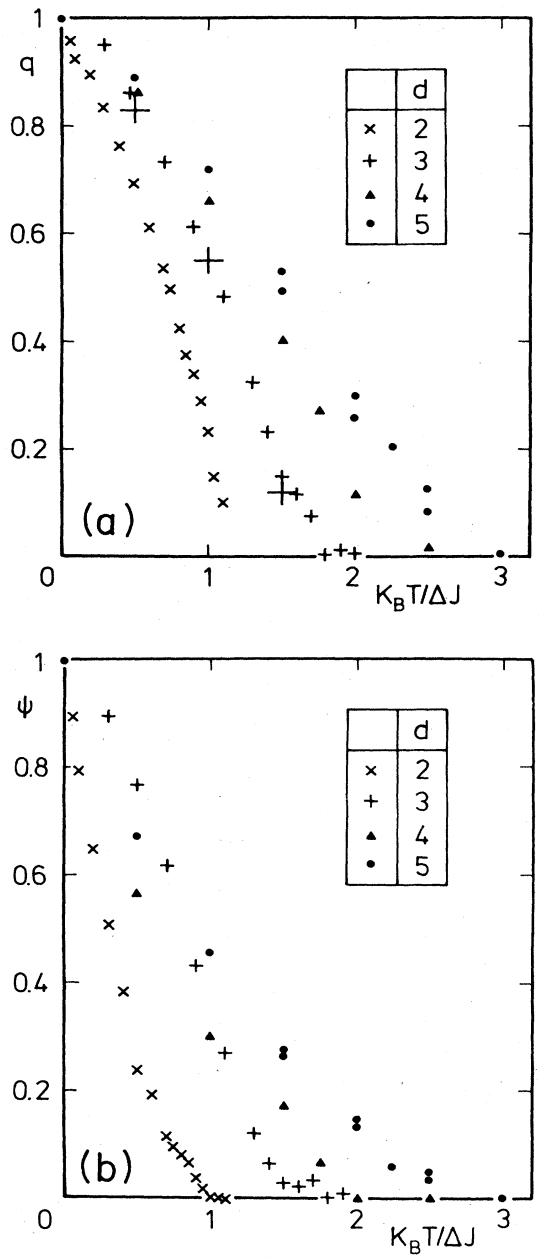

FIG. 77. (a) Edwards-Anderson order parameter $q(t_{obs})$ after an observation time of $t_{obs}=2000$ MCS/spin plotted vs temperature for nearest-neighbor Ising spin glasses with symmetric Gaussian interaction at various dimensionalities: ×, 160×160 ($d=2$); +, 4×4×4; +, 20×20×20; ▲, 9^4; ●, 6^5. (b) Order parameter ψ plotted vs temperature, for dimensionalities d ranging from $d=2$ to 5. From Stauffer and Binder (1979).

the same $T_f(t_{obs})\approx d\Delta J/(k_B)$, and $\psi \le q(t_{obs})$, in contrast to the Mattis spin glass (Sec. III.E.1), in which $\psi\equiv\sqrt{q}$ and hence $\psi \ge q$. It is also remarkable that there is little size dependence of this freezing "transition": in $d=2$, lattice sizes from 10^2 and 160^2 give roughly identical results, and in $d=3$ lattice sizes from 4^3 to 20^3 give roughly the same $q(t_{obs})$, as shown in Fig. 77. As a consequence, it was concluded that the "transition" seen in this figure is not a phase transition of the Edwards-Anderson type, in which a correlation length ξ_{SG} [Eq. (3.89)] diverges, and hence one would have appreciable finite size effects. Rather this "transition" is of mainly kinetic origin, small clusters of spins becoming so slow in their reorientation that the spins belonging to them appear to be frozen on the considered time scale. Growth of correlations would only be important at lower temperatures and larger time scales [see Ogielski (1985)]. The more extensive results on the relaxation time anticipated in Fig. 70 corroborate this interpretation.

The conclusion hence emerges that one must study the nature of this dynamic freezing process in more detail. Pioneering work in this direction has been done by Kinzel (1982a, 1982b) and Takayama *et al.* (1982), who analyzed the behavior of individual spins. The correlation function $\langle S_i(0)S_i(t)\rangle_T$ of individual spins was fitted to a double-exponential relaxation, and from this a distribution of relaxation times was extracted (Nemoto and Takayama, 1983). It appears that this distribution strongly broadens as one lowers the temperature, but nothing particular happens for $k_BT_f(t_{obs}=2000)/\Delta J\approx 1$ at the square lattice. The average relaxation time can be fitted to a Vogel-Fulcher law, Eq. (2.14), with a temperature T_0 where it would diverge for $d=2$ given by $T_0/(k_BJ)\approx 0.5$ ($\pm J$ model) and $T_0/(k_B\Delta J)\approx 0.4$ (Gaussian model), respectively (Takayama *et al.*, 1983a, 1983b). These temperatures are far below the "freezing temperatures" quoted above. An interesting point, which still is not fully understood, is the correlation between these relaxation times and the pattern of frustrated plaquettes in the model (Takayama and Takase, 1981; Nemoto *et al.*, 1982; Takayama *et al.*, 1982).

Particularly illuminating is a study of the distribution function $P(|\langle S_i\rangle|)$ of the local magnetization $|\langle S_i\rangle|$ obtained from a time average over some specified time scale (Kinzel, 1982a). As an example, Fig. 78 shows results for $t_{obs}=400$ MCS/spin. At $T/\Delta J \gg 1$, $P(\langle S_i\rangle)$ would be a rather narrow Gaussian centered on $\langle S_i\rangle=0$, since then all relaxation times are much shorter than t_{obs}, every spin has flipped many times, and the magnetization at every site has averaged out to nearly zero. As the temperature is lowered, some spins have flipped only a few times, and hence the distribution broadens. The phenomenon of freezing then means that a nonzero fraction of spins with $|\langle S_i\rangle|=1$ appears, i.e., spins that have not yet flipped at all during t_{obs}. As the temperature is lowered further, this fraction of frozen spins increases continuously. While near $k_BT_f(t_{obs})/\Delta J\approx 1.0$ the frozen spins appear in small clusters, well isolated from each other, as they must be so long as the fraction of frozen spins is very small, the size of the regions of frozen spins grows as the temperature is lowered, until at about $k_BT/\Delta J\approx 0.5$ they start to form a percolating network. Kinzel (1982a) suggests, however, that this percolation temperature should depend somewhat on the arbitrarily chosen time scale t_{obs}, and hence is not related to any

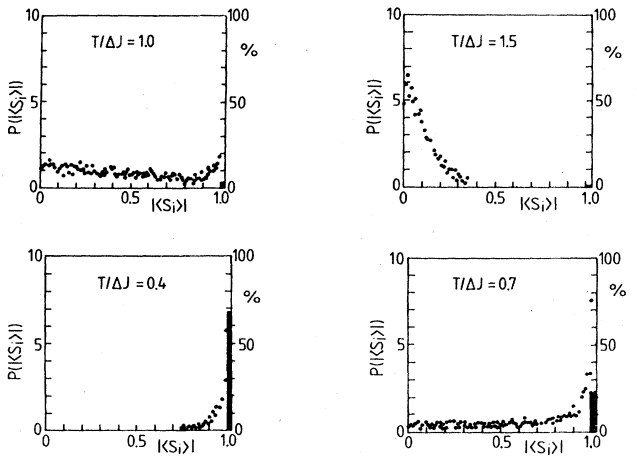

FIG. 78. Distribution of local moments for different temperatures in a two-dimensional nearest-neighbor Gaussian Ising spin glass (lattice size 60×60). The bar gives the fraction of spins that are completely frozen over 400 MCS/spin. From Kinzel (1982a).

physical phase transition phenomenon. We shall return to this problem in Sec. VI.D below.

Most of the Monte Carlo data on relaxation phenomena presented so far refer to dynamical behavior in zero field. Figure 79 now presents results for nonzero magnetic fields, where critical magnetic fields $H_c(t)$ in the H-T plane were estimated (Kinzel and Binder, 1983; Young, 1983a, 1984) in analogy with experimental procedures. When one slowly cools the system in a constant field H, the magnetization first increases and then at a temperature $T(H)$ crosses over to a temperature-independent constant value. The inverse function $H_c^{eq}(T)$ of this onset temperature $T(H)$ of the flat "plateaus" in Fig. 79(a) defines a static critical field. Dynamic critical fields are now defined by cooling the model in zero field to the desired temperature and then applying a field for a given time period t. The magnetization reached during this time agrees with the field-cooled magnetization at high temperature, but distinctly falls below it at low temperatures. From the points where the field-cooled and zero-field-cooled magnetizations start to deviate from each other one can now define a dynamic critical field $H_c(t)$.

Figure 79(b) shows the outcome of this procedure for the nearest-neighbor Gaussian Ising spin glass on the square lattice. The curve $T(H)$ start out for $H\to0$ at the point $T=0, H=0$, reaches a maximum temperature for $H/\Delta J\approx1$, and then bends back to lower temperatures for larger fields. In contrast, the dynamic critical field curves start out at a maximum temperature $[T_f(t)]$ for $H=0$ and then decrease monotonically with increasing field. The shape of these dynamic critical field curves is very much reminiscent of critical field curves found experimentally (Fig. 24). Since the experimental data have been interpreted as evidence for the Almeida-Thouless instability line, Eq. (2.32), it is interesting to compare the results

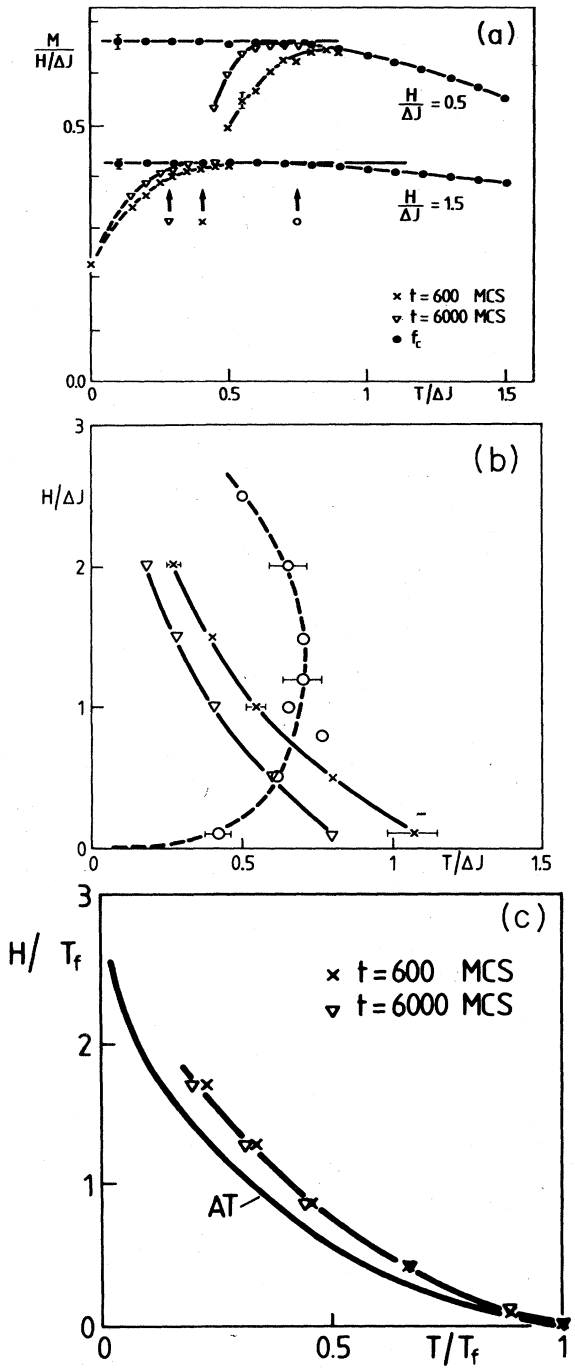

FIG. 79. (a) Magnetization of 60×60 Ising square lattices with nearest-neighbor Gaussian interaction in a magnetic field plotted vs temperature (for two choices of the field as indicated): ●, field-cooled magnetization; ▽, magnetization after a field is applied for $t=600$ MCS/spin to the system in a zero-field cooled state; ×, magnetization after a field is applied for $t=6000$ MCS/spin. Arrows show how the various critical fields are estimated (see text). (b) Critical field curves in the H-T plane, as found from (a): ○, static critical fields; ×, $H_c(t)$ for $t=600$ MCS/spin; ▽, $t=6000$ MCS/spin. Curves are drawn only to guide the eye. (c) Rescaled dynamic critical fields in the H-T plane. Solid curve marked AT is the Almeida-Thouless instability line. From Kinzel and Binder (1983).

of the simulation to Eq. (2.32) as well. This is done in Fig. 79(c), where both temperature and field are rescaled with the freezing temperature $T_f(t)$ (as defined above). The similarity with the AT line is in fact striking.

It would, however, be completely wrong to count this coincidence as another success of the mean-field theory of spin glasses: as will be discussed below (Sec. V.E), the two-dimensional model has no phase transition at nonzero temperature, and hence neither onset of replica symmetry breaking at any line [other than $T(H)\equiv 0$] nor any other singular behavior of any kind at a critical line in the H-T plane. As far as our present understanding of this two-dimensional case goes, the AT line has no significance for it whatsoever. Thus the close agreement seen in Fig. 79(c) is a coincidence. This will be demonstrated using the more extensive data of Young (1983a) on $H_c(t)$ curves in Sec. V.E. Surprising coincidences of this sort may also make the interpretation of experimental data even more difficult.

Similar problems are encountered in an analysis of the static magnetization process (Fig. 80). Again the raw data of the simulation [Fig. 80(a)] look strikingly similar

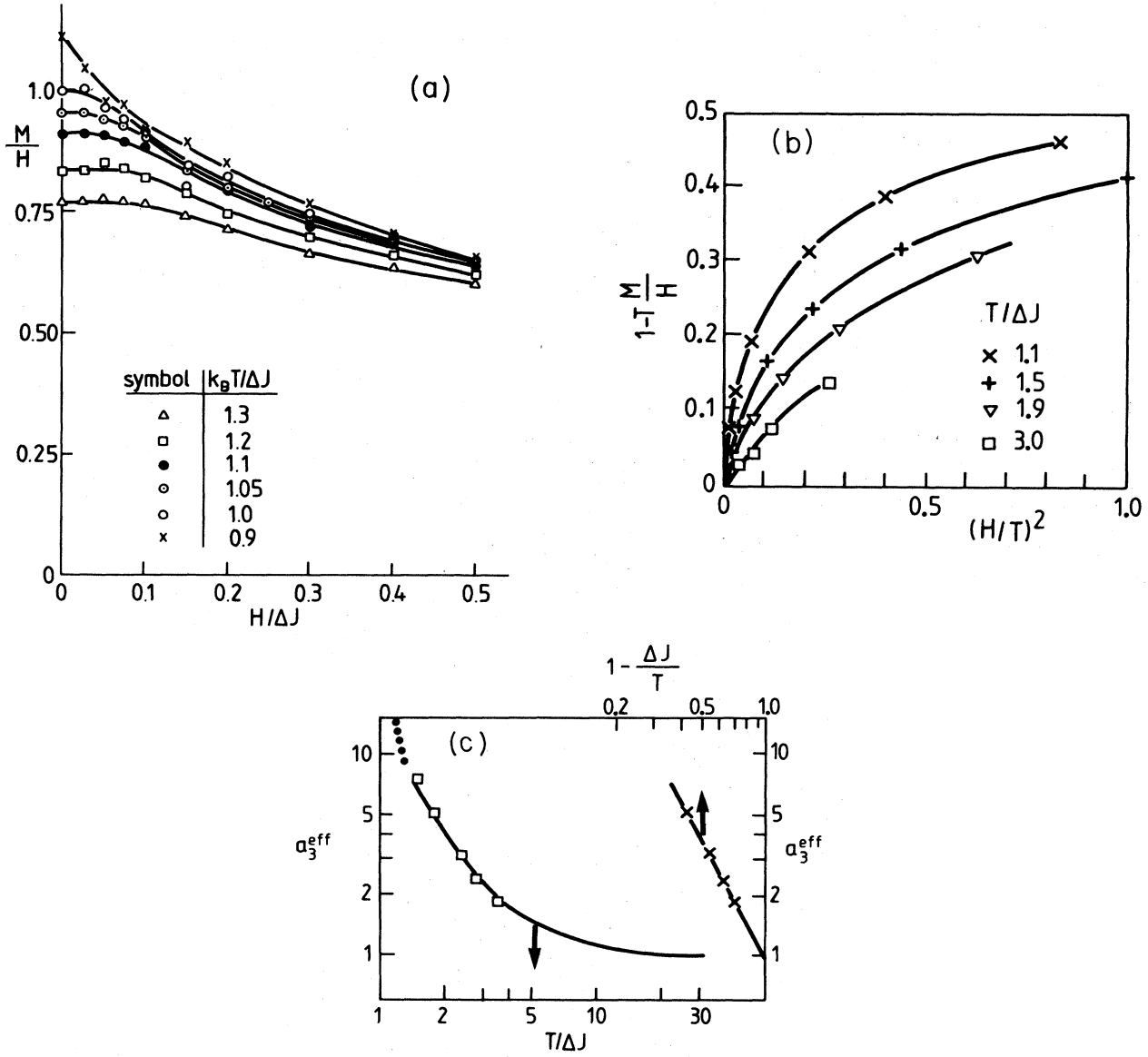

FIG. 80. (a) Magnetization of 80×80 Ising square lattices with nearest-neighbor Gaussian interaction plotted vs magnetic field at various temperatures. From Binder (1982b). (b) Nonlinear part of the magnetization plotted vs $(H/T)^2$. From Kinzel and Binder (1984). (c) Coefficient $a_3^{\rm eff}(T)$ obtained from fitting a straight line to the initial part of the curves in part (b) plotted vs temperature. Left part is a log-log plot vs $T/\Delta J$ (appropriate if a transition occurs at $T_f=0$); right part shows the same data as a log-log plot vs $1-\Delta J/(k_BT)$ (appropriate if a transition occurs at $k_BT_f/\Delta J=1$). ●, data for $a_3^{\rm eff}(T)$ which have been obtained from the data of (a). From Kinzel and Binder (1984).

to corresponding experimental plots [Fig. 26(a)]; over a broad regime of temperatures [but temperatures *not* close to $T_f(t)$] the data for the coefficient a_3 related to the nonlinear susceptibility [Eqs. (3.106) and (3.108)] are consistent with a critical divergence at $T_f(t)$; conversely, a log-log plot of a_3 vs T does not yield the straight-line behavior one expects as an asymptotic law for a phase transition at $T=0$, $a_3 \propto T^{-\gamma}$. The curvature seen on the corresponding log-log plot [Fig. 80(c)] in the present case does not imply, however, that the concept of a transition at $T=0$ does not apply to this model: it simply means that the temperature region where this law holds, which only is valid asymptotically as $T\to 0$, has not yet been reached. Fortunately, more extensive Monte Carlo data (McMillan, 1983; Young, 1983a; Kinzel and Binder, 1984) and transfer matrix calculations (Sec. V.E.) show more convincingly that for $d=2$ the hypothesis of a transition at $T=0$ applies.

We have here emphasized the behavior of the $d=2$ symmetric Gaussian Edwards-Anderson spin glass for two reasons: (i) it has a transition at $T=0$ only, yet many of its properties resemble experiment or even mean-field theory; (ii) it is the model studied most extensively—e.g., for its counterpart with $d=3$ no such data as shown in Figs. 79 and 80 are as yet available. We shall return to the most recent work on this model in our discussion of the problem of the lower critical dimension (Sec. V.E). Other work on ground-state properties will be briefly reviewed in Sec. V.C. At this point, we only mention very briefly work on asymmetric $\pm J$ and Gaussian models by Sakata *et al.* (1977), Rapaport (1978), and Kinzel (1984): all these papers show that with increasing mean value J of the interaction [Eqs. (3.30a) and (3.35)] the spin glass state is replaced by ferromagnetic order. On the other hand, these studies are by far too imprecise to propose a reliable phase diagram, and hence cannot address questions such as whether the ferromagnetic phase boundary is reentrant (Fig. 29), whether mixed ferro—spin-glass states occur, etc.

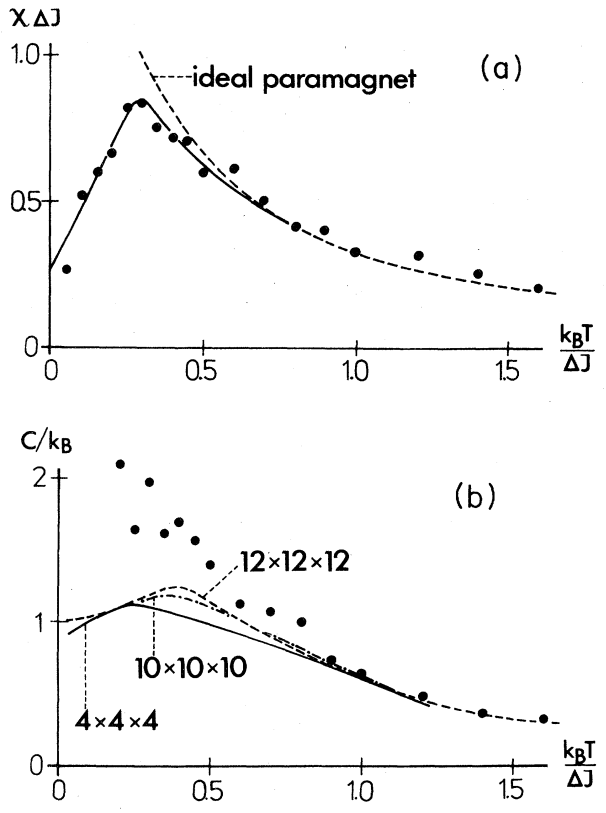

FIG. 81. (a) Susceptibility of the nearest-neighbor classical Heisenberg spin glass with symmetric Gaussian interaction on the simple cubic lattice (size 12^3) plotted vs temperature. Broken curve shows the Curie law. (b) Specific heat of the same model plotted vs temperature. Dots are from energy fluctuations [Eq. (5.22)], curves from numerical differentiation of the energy. Results for lattice sizes 4^3 and 10^3 are taken from Ching and Huber (1977a,1977b). From Binder (1977b).

3. Continuous spin models

Early attempts to study classical Heisenberg spin glasses for $d=3$ are due to Binder (1977b) and Ching and Huber (1977a, 1977b). In the susceptibility (recorded again from magnetization fluctuations) a peak occurs around $k_BT/\Delta J \approx 0.3$ [Fig. 81(a)], but at this rather low temperature there are always serious equilibration problems: the specific heat recorded from energy fluctuations distinctly exceeds that found from differentiating energy versus temperature curves [Fig. 81(b)]. This fact already indicates that the peak of χ is simply a nonequilibrium phenomenon, due to insufficient observation time (which again was $t_{\rm obs}=2000$ MCS/spin). A further drawback of this classical model, of course, is that its specific heat at $T\to 0$ goes to $C/k_B \to 1$, due to the equipartition theorem; any meaningful comparison of the specific heat to that of real systems hence is impossible.

Already in this early work it was concluded that this model probably has no transition at $T\neq 0$, because $q(t_{\rm obs})$ was decreasing rather steadily with increasing observation time [Figs. 82(a) and 82(b)], and the system was obviously not locked in a valley corresponding to a ground state: the projection order parameter ψ was always very small. In an attempt to elucidate questions about lower critical dimensionalities, this work then was extended to study both spin dimensionalities $m=2$ (XY model) and $m=3$ (Heisenberg model) for space dimensionalities from $d=2$ to $d=6$ (Stauffer and Binder, 1981). For short observation times (i.e., $t_{\rm obs}=2000$ MCS/spin) results similar to Fig. 82(b) were obtained for all combinations of (m,d) studied; when the time dependence of $q(t_{\rm obs})$ was studied, a decay towards zero was found, even for $d>4$ where one does expect a phase transition at finite temperatures [Fig. 83(a)]. However, since a small but systematic increase of $q(t_{\rm obs})$ with increasing system size was found for $d=5,6$ (but not for $d=2,3,4$), the data do not allow a definite

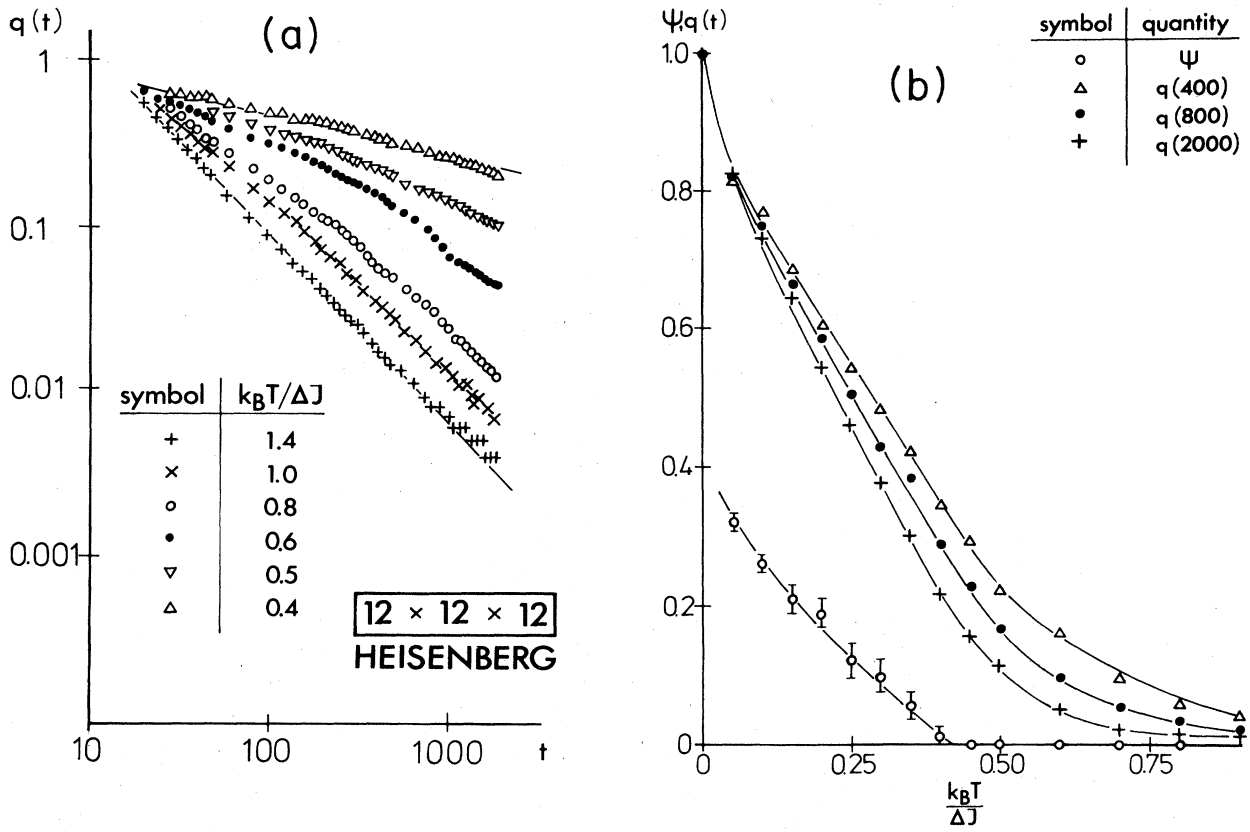

FIG. 82. (a) Log-log plot of the order parameter $q(t)=\sum_i \mathbf{m}_i\cdot\mathbf{m}_i/N$, where $\mathbf{m}_i$ is a time average of $\mathbf{S}_i(t')$ over the time t', vs t, for a three-dimensional classical Heisenberg spin glass (lattice size 12^3), with Gaussian bond distribution. Various temperatures are shown. (b) Temperature dependence of $q(t)$ in this model for three observation times. The projection order parameter ψ [average of $\mathbf{S}_i(0)\cdot\mathbf{S}_i(t)$ over all lattice sites and the time range of 2000 MCS] is also included. From Binder (1977b).

conclusion whether a nonzero order parameter exists at these dimensionalities. On the other hand, the absence of size effects for $d=3$ suggests that this decay is an intrinsic effect. Data for the relaxation time

$$\tau_{\rm av}=\int_0^\infty dt'[\langle \mathbf{S}_i(0)\mathbf{S}_i(t')\rangle]_{\rm av}$$

[see Fig. 83(b)] are consistent with this picture of a transition at $T=0$, but with a divergence of $\tau_{\rm av}$ at least as strong as an Arrhenius law. More accurate recent work (Jain, 1984; Jain and Young, 1986) suggests that for $m=2,d=2$ and probably for $m=2,d=3$ $\tau_{\rm av}$ does in fact follow a power law, $\tau_{\rm av}\propto T^{-z_{\rm av}\nu}$, with $z_{\rm av}\nu\approx 5.16\pm 0.18$ in $d=2$.

Other recent interesting work probing the sensitivity of the system to changes of boundary conditions has bearing on the questions of the lower critical dimension and will be mentioned in Sec. V.E, while computer searches for "two-level systems" and other defect configurations will be discussed in the context of ground-state properties in the next section. Here we turn to the work of Morris and Bray (1984), who performed simulations for the cases $m=2,d=2$ and $m=3,d=3$, including a uniaxial anisotropy D, the exchange being in both cases a nearest-neighbor Gaussian distribution. Longitudinal and transverse order parameters and susceptibilities were calculated. The freezing transitions were found to be distinctly time dependent (no equilibrium transitions), but phase diagrams plotting $T/T_f(t)$, with $t=10^3$ MCS, were in qualitative agreement with mean-field phase diagrams (Fig. 84): Phases occurred with longitudinal ordering, transverse ordering, and both orderings simultaneously.

C. Ground-state properties of spin glasses

Since the pioneering work of Toulouse (1977) pointing out the crucial role that "frustrated plaquettes" (see Sec. III.F) should have for the ground state of the $\pm J$ model [Eq. (3.38)], much work has been devoted to a more quantitative understanding of this problem, which we now briefly summarize.

While the bonds (of the square lattice) are randomly distributed, there is a nontrivial correlation in the distribution of frustrated plaquettes. If the nonfrustrated plaquettes percolate, which happens if the concentration p of

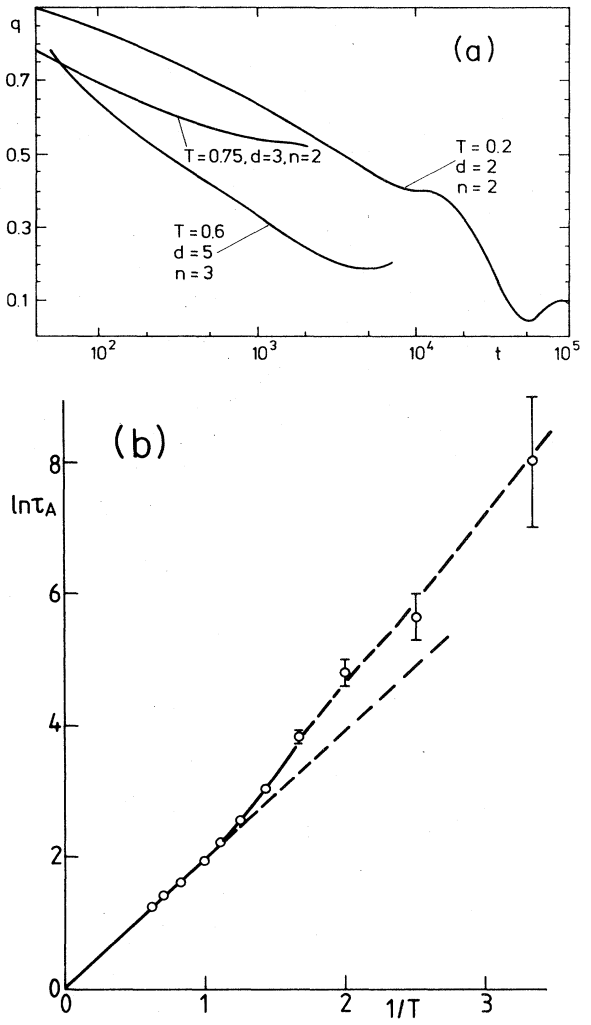

FIG. 83. (a) Semilog plot of the order parameter $q(t)$ vs observation time t at various space and spin dimensionalities (the latter being denoted by n in the figure). Temperatures are quoted in units of $\Delta J/k_B$. Lattice sizes are 56^2 ($d=12$), 15^3 ($d=3$), and 5^5 ($d=5$). At very late times (included for $d=2$) huge fluctuations from run to run occur, and an irregular behavior results due to insufficient statistics. (b) Logarithm of the autocorrelation time of the spins in the three-dimensional classical Heisenberg spin glass plotted vs inverse temperture. From Stauffer and Binder (1981).

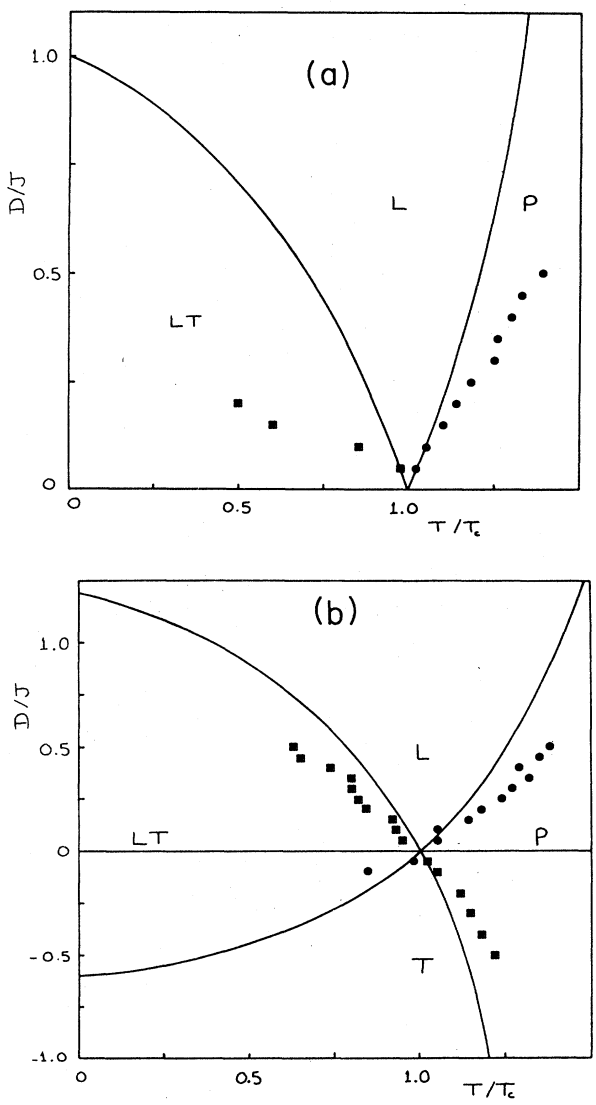

FIG. 84. (a) Phase diagram for an XY spin glass ($m=2$) in two dimensions with uniaxial anisotropy ($-D\Sigma S_{i_1}^2$). The curves represent the predictions of the mean-field theory, including corrections due to replica symmetry breaking. Circles and squares indicate onset of longitudinal (L) and transverse (T) order, P is the paramagnetic phase. The phase diagram for $D<0$ follows from the symmetry $D\leftrightarrow -D, L\leftrightarrow T$. (b) Same as (a) but for a three-dimensional Heisenberg spin glass. From Morris and Bray (1984).

positive bonds exceeds some critical value p_c, one might expect the ground state to be ferromagnetic (Domany, 1979). Monte Carlo estimates for the concentration p_f at which ferromagnetic order sets in are 0.85 (Ono, 1976), 0.85—0.90 (Kirkpatrick, 1977), 0.88—0.91 (Vannimenus *et al.*, 1979), 0.9 (Vannimenus and De Séze, 1979), and 0.85 (Jaggi, 1980); estimates obtained from other methods are $p_f \approx 0.91$ (constructions of ground states "by hand" due to Vannimenus and Toulouse, 1977), 0.88 ± 0.02 (transfer matrix techniques of Morgenstern and Binder, 1980a), 0.9 or 0.91 (extrapolations of exact series in $1-p$ due to Gabay and Garel, 1978, or Grinstein *et al.*, 1978), 0.95 or 0.92 (real-space renormalization-group calculations due to Jayaprakash *et al.*, 1977, or Southern *et al.*, 1979). The most accurate estimate probably results from the generation of ground states via a matching method of graph theory (Bieche *et al.*, 1980), $p_f \simeq 0.855$. On the other hand, p_c was estimated as 0.84 (Domany, 1979) and

0.86 ± 0.02 (Sadiq *et al.*, 1981). Thus, very roughly, indeed $p_c = p_f$, but the accuracy of all these estimates is by far too crude to make any statement as to whether these two numbers should coincide exactly. The situation is complicated by the fact that it is not so clear what phase one is entering when one leaves the ferromagnetic phase: for the symmetric case $p = \frac{1}{2}$, transfer matrix analysis (Morgenstern and Binder, 1980a); Morgenstern and Horner, 1982) suggests at $T=0$ a spin glass phase but with $q_{EA}=0$; this phase differs from a paramagnetic phase, since the spin glass correlation function $g_{SG}(\mathbf{R})$ [Eq. (3.88)] decays to zero as a power law rather than exponentially, and hence $\xi_{SG}=\infty$ [Eq. (3.89)]. However, neither this numerical work nor other considerations (Miyashita and Suzuki, 1981) can yield reliable quantitative estimates for the exponent η describing this decay:

$$[\langle S_0 S_R \rangle^2_{T=0}]_{av} \propto R^{-(d-2+\eta)} \quad (\pm J \text{ model}, d=2) . \tag{5.26}$$

Moreover it is not obvious whether this state persists between $p=\frac{1}{2}$ and $p=p_f$. In fact, studying the ground state by a special polynomial algorithm, Barahona *et al.* (1982) suggest a new type of phase for $0.85 \lesssim p \lesssim 0.9$ at the square lattice, which they term the "random antiphase state": it is made up of ferromagnetic domains separated by erratic domain walls. If this state exists distinct from the above state with decaying correlations, it could be a sort of spin glass similar to the Mattis model, with nonzero order parameter q. However, the existence of this phase is disputed by Morgenstern (1982), who suggests, from transfer matrix calculations, that Eq. (5.26) holds right up to p_f. Of course, one would expect that the ferromagnetic correlation length $\xi_F \to \infty$ as $p \to p_f$, and thus very large ferromagnetic domains of order ξ_F are expected near p_f. Hence this issue is difficult to settle numerically.

There has also been some confusion about another transition at $p_c' = 1/\sqrt{2} = 0.707$. Schuster (1979), using duality transformations, showed that the pair correlation function between two frustrated plaquettes undergoes a "pair dissociation transition" at p_c', if one imposes the (artificial) constraint that all other plaquettes be unfrustrated. Omitting this unphysical constraint, Kolan and Palmer (1980) suggested from numerical work that the average minimal distance between frustrated plaquettes indeed shows an anomaly at p_c', but this was ruled out by later analytical work (Liebmann and Schuster, 1980).

The transfer matrix calculations (Morgenstern and Binder, 1980a; the latest, most accurate values are due to Cheung and McMillan, 1983a) also yielded estimates for the ground-state energy $E(0)$ and entropy $S(0)$ at $p=\frac{1}{2}$, namely $E(0)/J \approx -1.4024 \pm 0.0012$ and $S(0)/k_B \approx 0.0701 \pm 0.0005$. The latter number is in reasonable agreement with the estimates of Vannimenus and Toulouse (1977) but distinctly smaller than a Monte Carlo result [$S(0)/k_B \approx 0.10$] of Kirkpatrick (1977). This larger value can be interpreted as a dynamic effect, resulting in a "frozen-in entropy" or "rest entropy," a concept familiar from ordinary glasses (Jäckle, 1981). Indeed, such a "rest entropy" is also found in simulations of the Gaussian spin glass (Jäckle and Kinzel, 1983), for which $S(0)=0$. The most recent study of $S(0)$ for the square $\pm J$ model yielded $S(0)k_b \approx 0.08$ (Kolan and Palmer, 1982).

Even less is known about the ground-state properties of the $\pm J$ model in higher dimensionality [see also Bachas (1984) for a discussion of why this problem is so difficult]. Again one expects $S(0)>0$ (for analytical arguments on this point, see Schwartz, 1985b). Again there is a similar discrepancy between a transfer matrix estimate [$S(0)/k_B \approx 0.04 \pm 0.01$; Morgenstern and Binder (1980b)] and Monte Carlo work [$S(0)/k_B \approx 0.062$; Kirkpatrick (1977)]. The ground-state energy was estimated as $E(0)/J = -1.76 \pm 0.02$ [$\pm J$ model, Morgenstern and Binder (1980b)], while in the Gaussian model $E(0)/\Delta J = -1.7 \pm 0.03$ ($d=3$) and $E(0)/\Delta J = -1.31 \pm 0.01$ ($d=2$), which is significantly lower than early Monte Carlo estimates (Binder and Schröder, 1976a, 1976b; Binder and Stauffer, 1976c) and thus demonstrates that in these studies the system was trapped in low-lying metastable states rather than the true ground states. Cheung and McMillan (1983b) have studied a model with exchange uniformly distributed between $-J$ and $+J$ and find that there is a continuous distribution of two-level states with a finite density at zero energy, yielding a specific heat varying linearly with temperature. While $q = q_{EA} = 1$ for the Gaussian spin glass at $T=0$, since the ground state of any finite system is only doubly degenerate for practically every bond configuration $\{J_{ij}\}$ if the distribution of bonds is continuous, nothing is as yet known about the order parameter of the $\pm J$ model at $T=0$ for dimensionality $d=3$ and higher. Grest *et al.* (1986) recently studied the cooling rate dependence of the apparent zero-temperature properties of spin glasses obtained from Monte Carlo methods and showed that $E(0)/J \approx -1.791$ ($d=3$). They found that for $d=3$ $E(0)$ depends on the cooling rate only logarithmically, while for $d=2$ there is a power-law dependence.

If one considers the diluted $\pm J$ model [Eq. (3.38b)] one gets a paramagnetic phase at $T=0$ at sufficient dilution and can study a transition from the disordered phase to the spin glass phase as the concentration of nonzero bonds increases (Aharony, 1978; Aharony and Pfeuty, 1979; Aharony and Binder, 1980). Since the replica method in which one takes the limit $T \to 0$ before the limit $n \to 0$ (Aharony, 1978) is inconclusive, having eliminated frustration from the problem (Aharony and Pfeuty, 1979), one either has to combine the replica method with dual representations to make progress (De Dominicis and Stephen, 1978; Fradkin *et al.*, 1978) or to resort to the extrapolation of exact series expansions in the concentration of positive and negative bonds p_1, p_2 (Aharony and Binder, 1980). Only the latter approach has yielded quantitative results: it is found that χ_{SG} diverges when $p = p_1 + p_2$ approaches a critical concentration p_{SG}. For $p_2 \to 0$, p_{SG} agrees with the bond percolation concentration p_c, while for finite nonzero p_1/p_2, p_{SG} exceeds p_c. This means

that for $p_c < p < p_{SG}$ there is an infinite cluster of spins connected by nearest-neighbor bonds, but the ground state of this cluster is still paramagnetic (χ_{SG} and ξ_{SG} being finite). The critical exponent γ_0 describing the divergence of χ_{SG} as $\chi_{SG} \propto (p_{SG}-p)^{-\gamma_0}$ at $T=0$ was estimated to be around $\gamma_0 \approx 2.7$ for $d=2$, $\gamma_0 \approx 2.0$ for $d=3$, $\gamma_0 \approx 1.4$ for $d=4$, and $\gamma_0 \approx 1.05$ for $d=6$. As for percolation and the paramagnet–spin-glass transition driven by temperature, one expects $d=6$ to be the upper critical dimension for this problem, and thus the classical exponent $\gamma_0=1$ should hold for $d \geq 6$ (Aharony and Binder, 1980). Of course, due to the shortness of the series analyzed, the accuracy of the above estimates for γ_0 is rather uncertain.

This method based on calculating χ_{SG} as a systematic series in p_1 and p_2 was also applied to the triangular lattice, which has no ferro-antiferromagnetic symmetry. In fact, the pure triangular antiferromagnet is a "fully frustrated" system (see Sec. VI.E); it stays paramagnetic at all $T>0$ and exhibits a power-law decay of correlations at $T=0$ (Stephenson, 1970). Thus it is interesting that even in the diluted antiferromagnet ($p_1=0$) a transition was found at $p_c \approx 0.42$–0.45 (Aharony and Binder, 1980), implying that there is a spin glass phase, although there are only antiferromagnetic and no ferromagnetic bonds, consistent with suggestions of de Séze (1977) and Grest and Gabl (1979). Figure 85(b) shows the phase diagram at $T=0$, Fig. 85(a) part of the diagrams on which this series extrapolation (Aharony and Binder, 1980) was based. Since the ground state of the $\pm J$ model is so degenerate, it is also very important to investigate the energy barriers separating the various ground states from each other. This question was addressed by Morgenstern and Horner (1982) and Morgenstern (1983a, 1983b) for the two-dimensional case. They analyzed the groups of spins that can be flipped in a ground state without energy cost. The contour around such a group of spins is called a "zero-energy loop." Part of these are "trivial loops," surrounding only single "loose spins" or small clusters. The concentration of these trivial loops decreases rapidly with increasing size of the loop. In addition, there is an irregular network of large loops, the typical diameter of which is estimated from transfer matrix calculations as about 13 lattice spacings (Morgenstern, 1983a, 1983b). Due to entropy effects, correlations arise on distances much larger than this value, and Morgenstern and Horner (1982) suggest that this picture is consistent with a power-law decay of the correlations [Eq. (5.26)]. Morgenstern (1983a, 1983b) then suggests that the largest energy barriers separating various ground states ("valleys" in phase space, Fig. 4) can be understood by analyzing the energy cost of overturning the spins contained within *one* nontrivial loop. Since the minimum energy cost is obtained if we move a wall through the loop, we expect an energy barrier of the loop diameter itself. This estimate is corroborated by numerical work (Morgenstern, 1983a, 1983b), which yields energy barriers of the order of $15J$. This estimate is also consistent with some Monte Carlo results on the temperature dependence of the relaxation time (McMillan,

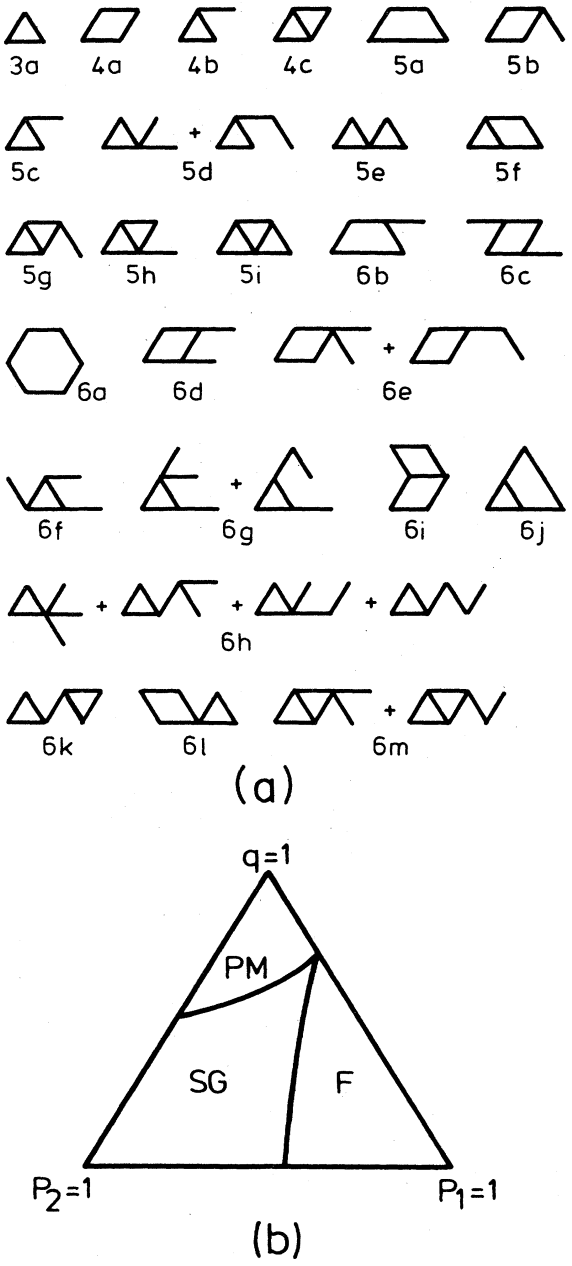

FIG. 85. (a) Clusters anlayzed for a study of

$$\chi_{SG} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} p_1^m p_2^{l-m} \sum_c g_c \chi_{SG}^c$$

on the triangular lattice with concentrations p_1, p_2 of ferro- and antiferromagnetic bonds, where c is a cluster, g_c is the multiplicity of a cluster with l bonds, m of which are ferromagnetic, and χ_{SG}^c is its spin glass susceptibility. Note that only clusters with $l \geq 3$ are shown, and for $l=6$ only a subset of the possible diagrams is displayed here. (b) Conjecture for the resulting phase diagram at $T=0$, displaying paramagnetic (PM), ferromagnetic (F), and spin glass (SG) phases. Note that the point $p_2=1$ is special (fully frustrated triangular lattice). Only the line PM-SG has been located quantitatively from the series extrapolation. From Aharony and Binder (1980).

1983) and with high-temperature series extrapolations (Reger and Zippelius, 1985); these latter results, however, seem to disagree with other Monte Carlo results (Young, 1983a), which imply a divergent *free-energy* barrier as $T \to 0$. We shall return briefly to this problem in Sec. V.E.

Of course, it is a nontrivial matter to relate the *energy* barrier between "valleys" at $T=0$ to the *free-energy* barrier between valleys at $T>0$. One would need to have information of the configurational entropy associated with states close to the saddle-point configuration in phase space (the height of this saddle yields the energy barrier). A crude estimate of this entropy was attempted by Morgenstern (1983a; 1983b). He assumed that the entropy of the group of spins contained within one nontrivial loop stays about the same when one introduces a wall such that one-half of the spins within the loop are overturned (this configuration should be close to a saddle-point configuration). The entropy then is entirely due to the "trivial" loops mentioned above. This entropy was also estimated from exact partition function calculations as $S(0)/k_B = 0.0685 \pm 0.005$ (normalized per spin); multiplying this estimates with a typical loop size of 13×13, Morgenstern (1983a, 1983b) estimated the free-energy barrier as $\Delta F \approx 15J - 11.6 k_B T$. An order-of-magnitude estimate of $T_f(t)$ for *short* observation times was finally obtained by putting $\Delta F \approx 0$, which yielded $T_f(t)/k_B J \approx 1.3$. Although this number is in fair agreement with Eq. (5.25a), this approach should only be considered as an interesting qualitative approach towards understanding the temperature dependence of free-energy barriers in spin glasses. Of course, a simple linear variation of ΔF with T still implies an Arrhenius behavior for $T_f(\omega)$ [Eq. (2.13)], which is not consistent with experiment (see Sec. II.B.1).

When we consider the Gaussian spin glass model instead of the $\pm J$ model, there are no longer any exact degeneracies between various states (apart from the trivial one in which all spins are flipped together). The "trivial" loops of the $\pm J$ model now have their counterparts in small clusters of spins which can be overturned with small energy cost (Fig. 43) rather than $\varepsilon = 0$. Dasgupta *et al.* (1979) studied 20×20 square lattices to obtain the distribution of barrier heights for clusters of two and three spins [Figs. 86(a) and 86(b)]. Using these results in Eq. (3.137), Dasgupta *et al.* (1979) studied the dependence of the exponents α appearing in the decay of the remanent magnetization and energy. The dependence of these exponents on temperature and magnetic field [Fig. 86(c)] was in reasonable qualitative agreement with direct Monte Carlo simulations of the corresponding quantities (Kinzel, 1979).

Ma (1980) suggested that some features of this description could be carried over to the case of Heisenberg spin glasses (with quantum-mechanical spins, $s = \frac{1}{2}$); particularly interesting is the fact that for strongly coupled clusters of two or more spins the "frozen-in" background in the environment of the cluster leads again to a picture like that in Fig. 43. Thus, even if the Hamiltonian is fully iso-

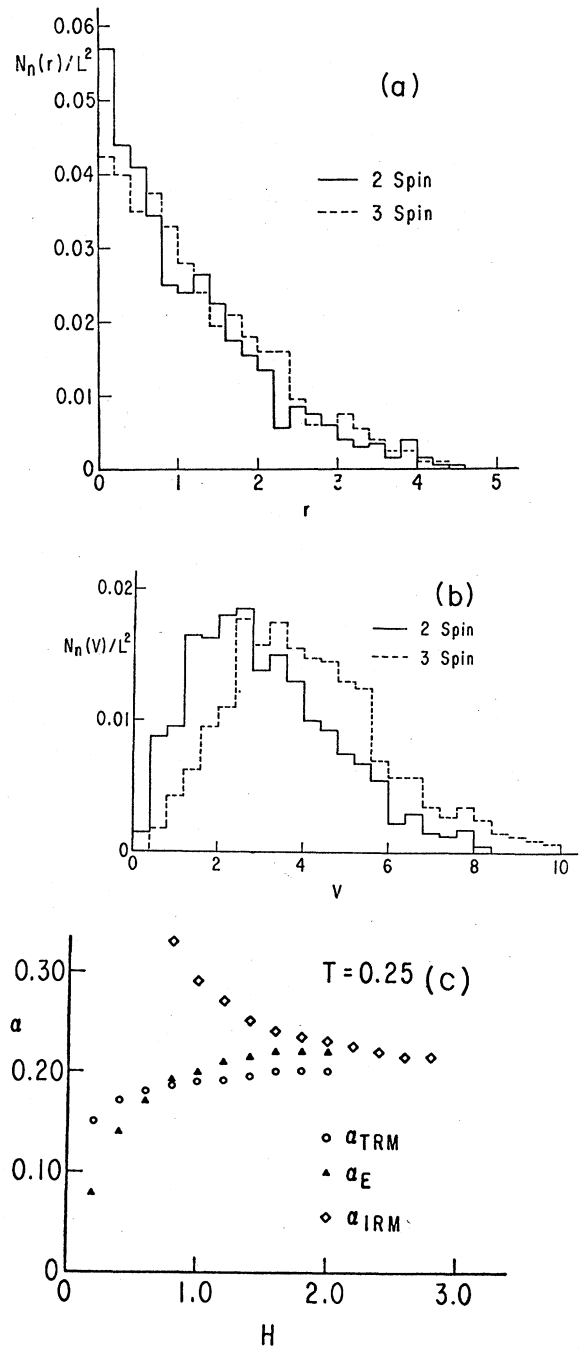

FIG. 86. (a) Distribution of barrier heights r of clusters containing two or three spins, respectively, for a Gaussian Ising square lattice of linear dimension $L=20$; r denotes the energy difference between the barrier and the higher of the two energy minima in Fig. 43. (b) Same as above but for the quantity V (= energy difference between the barrier and the lower of the two energy minima in Fig. 43). (c) Exponent α of the power laws for energy $[E(t)-E(\infty) \propto t^{-\alpha_E}]$ and remanent magnetization $[\sigma_{TRM} \propto t^{-\alpha_{TRM}}, \sigma_{IRM} \propto t^{-\alpha_{IRM}}$, for the thermo-remanent case (TRM) and the isothermal remanent magnetization (IRM), respectively] plotted vs magnetic field (in units of ΔJ) at $T = 0.25 \Delta J$ using the data of parts (a) and (b) of this figure and equations such as Eq. (3.137). From Dasgupta *et al.* (1979).

tropic, the clusters experience an "effective anisotropy energy," which is responsible for the barrier between their energy minima. However, there are not yet any numerical studies aimed at checking and substantiating these results. So far, numerical work has been restricted to a search for "two-level systems" in the classical limit ($S\to\infty$) [see Reed (1979a) and Henley (1984a, 1984b) for the case of Heisenberg spins, and Grzonka and Moore (1983, 1984) for XY spins]. Reed (1979a) studied systems of size 14^3 by an energy minimization method and suggested that localized clusters occur (containing about 125 spins) and may exist in more than one local equilibrium configuration. The similar but much more systematic and extensive study of Henley (1984a, 1984b), using 12^3 lattices, also revealed that the size of clusters that are locked together in rigid rotations is about 100—200 spins. Correlations of local rotation matrices have also been studied and found to be rather long. With regard to the nature of the "relative defects" distinguishing the spin configuration of one local energy minimum from that of another, the picture still is rather unclear: while disclination lines and twists are found to exist, domains in which the spins undergo a reflection are probably more important. An interesting finding is that the energy difference $|\varepsilon|$ between two minima is less than ΔJ in this nearest-neighbor model, in spite of the large size of the clusters of spins involved. We note also that the size of this object is consistent with the number of metastable states estimated analytically for this model by Bray and Moore (1981b).

The work of Henley (1984a, 1984b) clearly is a pioneering first step, particularly important due to its implementation of the idea that defects relative to the $O(m)$ orthogonal matrix $\mathbf{R}(\mathbf{r})$ represent a rotation of one configuration relative to another, which is taken as the "order parameter" for describing low-temperature configurations of m-vector spin glasses. An intriguing problem is to relate these findings to the results of Bray and Moore (1982a), who point out that a typical energy minimum in the (high-dimensional) configuration space has a high density of directions for which the energy surface is locally flat. This implies for dynamic relaxation phenomena that the system stays so long in the vicinity of a particular local minimum that a logarithmic relaxation of $q(t_{\rm obs})$ results,

$$1-q(t_{\rm obs})\propto\frac{k_BT}{\Delta J}\ln t_{\rm obs}\ , \qquad 1\lesssim\ln t_{\rm obs}\lesssim(k_BT/\Delta J)^{-1}\ , \tag{5.27}$$

where the proportionality constant can be estimated from a harmonic expansion of the energy,

$$\delta E\approx\tfrac{1}{2}\sum_\lambda\lambda(\delta S_\lambda)^2\ ,$$

δS_λ being the "normal coordinate" and λ the associated eigenvalue. The density $g(\lambda)$ of the eigenvalues is found to behave as $q(\lambda)\propto\lambda^\alpha$, with $\alpha=0$ for $d=2$ [i.e., $g(0)\neq0$] and $\alpha\approx0.1\pm0.1$ for $d=3$. The proportionality constant in Eq. (5.27) is related to $g(0)$ [if $g(0)=0$, one finds a decay $q(t_{\rm obs})\propto t_{\rm obs}^{-\alpha}$]. If correct, these findings would imply that localized modes are much more important than hydrodynamic excitations (spin waves), which have a much smaller density of states as $\lambda\to0$ (see Sec. VI.B).

The self-consistency of this picture is established by numerically solving the Langevin equation for an XY spin glass in $d=2$ and $d=3$ in order to obtain $q(t_{\rm obs})$ directly. Figure 87 shows that the time variation of q is indeed consistent with Eq. (5.27). Furthermore, the numerical diagonalization of the Hessian matrix for the spin configurations that belong to the corresponding local energy minima yields $g(0)$, and hence the proportionality constant in Eq. (5.27) is in quantitative agreement with the directly observed $q(t_{\rm obs})$. Nevertheless, the work of Grzonka and Moore (1983) implies $\varphi(\lambda)\propto\lambda^{1/2}$ instead of the above result, for $d=3$, using larger lattices and somewhat different numerical techniques. While the exponent $\alpha=\frac{1}{2}$ would be consistent with a spin-wave picture, Grzonka and Moore (1983) argue that only 40% of the prefactor in the relation $g(\lambda)\propto\lambda^{1/2}$ can be accounted for by spin waves. Grzonka and Moore (1983) also study the associated eigenvectors and find that for $d=2$ all states are localized. As a criterion for localization the "inverse participation ratio" (IPR) is computed, defined as

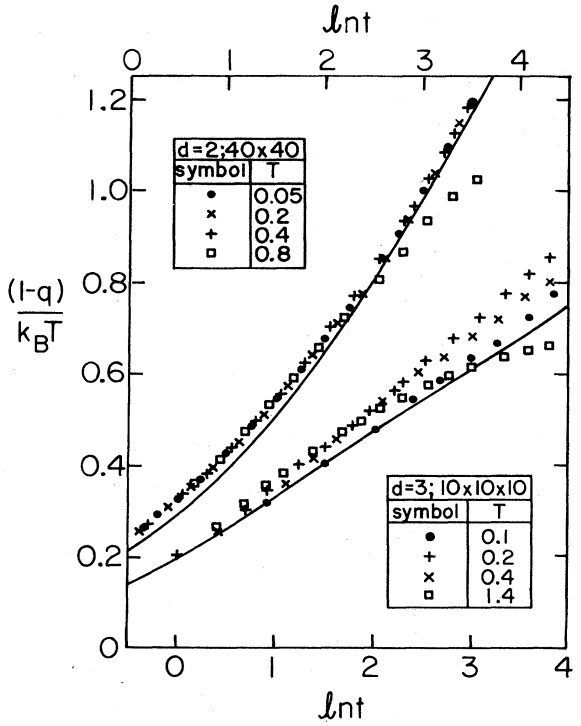

FIG. 87. Plot of $(1-q)/k_BT$ vs $\ln t$ for various temperatures ($\Delta J\equiv1$) in the two- and three-dimensional planar spin glass with nearest-neighbor Gaussian interaction. Solid curves are the theoretical prediction based on the numerical analysis of the density of states $g(\lambda)$, as discussed in the text. From Bray and Moore (1982a).

IPR $=\sum_i |\langle i|\lambda\rangle|^4$ for a normalized eigenvector $|\lambda\rangle$; it is a measure of the number of sites on which the eigenvector has a significant amplitude (for extended eigenvectors the IPR is expected to be of order N^{-1}). For $d=3$ one finds a rather sharp transition from extended to localized states at a critical value of λ. The addition of a small uniform magnetic field or uniaxial anisotropy also produces localized states at small λ for $d=3$, together with a larger reduction in $g(\lambda)$ as $\lambda\to 0$.

If barrier hopping did not contribute at all, the relaxation of the system would be entirely determined by $g(\lambda)$; for instance, the autocorrelation of the spins could be written as

$$[\langle S_i(0)S_i(t)\rangle_T]_{\rm av}=1-k_BT\int_0^\infty \frac{d\lambda}{\lambda}g(\lambda)[1-\exp(-\lambda t)] \tag{5.28}$$

[which leads to Eq. (5.27) if $g(0)\neq 0$, as stated above]. In order to examine the importance of barrier-hopping processes, Grzonka and Moore (1984) have tried to analyze the saddle-point configurations separating two distinct local minima. So far this ambitious program has only been carried through for lattice sizes from 4^3 to 6^3. Many low-energy barriers were found, and the change of the spin configuration to a saddle point is highly localized (again using the IPR as a localization criterion). Unfortunately the saddle-point search procedures applied by Grzonka and Moore (1984) are probably biased, and hence cannot be used to deduce the distribution $g(\varepsilon,v)$ discussed previously for the Ising spin glass. While analytic theories for XY spin glasses have emphasized, for $d=2$, the role of vortices and "half vortices" (Villain, 1977b, 1978, 1980; José, 1978a, 1978b, 1979; see Sec. III.F.2), this aspect of the theory has not received much attention in numerical work. We also note that Eq. (5.27) seems to find numerical support (Stauffer and Binder, 1981) for m-vector spin glasses irrespective of space and spin dimensionality, although the nature of the defect configurations depends distinctly on both m and d.

Other attempts to understand the nature of low-lying states in spin glasses are based on studies of "defect energies" (Reed *et al.*, 1978) and interface energies (Derrida and Vannimenus, 1983). Reed *et al.* (1978) started from a ground state of an Ising spin glass with Gaussian interaction, reversed all the spins in the right part of the square (cubic or hypercubic) sample, and then let the system relax while keeping the spins on both right and left end lines (planes or hyperplanes) fixed. The energy difference between the "defect state" so generated and the original state is the "defect energy" $E_{\rm def}$. In the ferromagnetic state, $E_{\rm def}$ should be of order $E_{\rm def}\propto N^{1-1/d}$, for a sample containing N spins. Reed *et al.* (1978) found that for Gaussian distributions of bonds of width ΔJ centered at $\bar{J}$, the normalized defect energy $E_{\rm def}/N^{1-1/d}$, vanished at about $\bar{J}/\Delta J\approx 0.6$ for $d=2$, 0.35 for $d=3$, and zero for $d=4$. Reed *et al.* (1978) took this as evidence that $d=4$ is a special dimensionality, i.e., the lower critical dimensionality of Ising spin glasses. However, this conclusion is at variance with more recent related work (McMillan, 1984b, 1985a, 1985b; Bray and Moore, 1985a) in which a phenomenological renormalization treatment of the defect energy is attempted and taken as evidence for a finite temperature phase transition for $d=3$. This work and related work investigating the sensitivity of the system to boundary conditions (Banavar and Cieplak, 1982a, 1982b, 1984; Cieplak and Banavar, 1983) will be discussed further in Sec. V.E.

D. One-dimensional models

One-dimensional models of spin glasses have a direct experimental application only for rather special uniaxial substances, in which magnetic interactions are practically restricted to one lattice direction. When such systems are diluted suitably, spin-glass-like behavior has in fact been observed (Tippie and Clark, 1981; Cheikhrouhou *et al.*, 1983). The main interest in one-dimensional models, however, is due to two other facts.

(i) In one dimension one can obtain exact results for a variety of models. Such exact results include models with long-range interactions in which finite-temperature phase transitions occur (Orland *et al.*, 1981; Kotliar *et al.*, 1983) and short-range models with transitions at $T=0$. Although nearest-neighbor models in one dimension and zero field are not frustrated and hence a study of their static behavior is only of pedagogical interest (one can perform exact renormalization-group treatments in spite of the randomness; see Grinstein *et al.*, 1976), a rather interesting and nontrivial behavior occurs in nonzero magnetic field (see, for example, Derrida *et al.*, 1978; Chen and Ma, 1982; Derrida and Hilhorst, 1983; Gardner and Derrida, 1985). For the $\pm J$ Ising chain in zero field, dynamical properties in the framework of a master equation description [Eqs. (5.8) and (5.9)] can also be found (e.g., Bray *et al.*, 1978; Hentschel, 1980a, 1980b; Reger and Binder, 1985; Colborne, 1986). For certain Heisenberg spin glasses, one can show the nonexistence of metastable states (Kaplan, 1981). The spin glass behavior of the random anisotropy model can also be analyzed explicitly (Thomas, 1980).

(ii) In one dimension a variety of numerical techniques can be applied (e.g., Morgenstern *et al.*, 1978; Puma and Fernandez, 1978; Kumar and Stein, 1980; Ariosa *et al.*, 1982; Reger and Binder, 1985) that are not applicable (or at least impractical) at higher dimensionality.

We start by briefly discussing the long-range models (some of which are of the site-disorder type). The Hamiltonian studied by Orland *et al.* (1981) is

$$\mathcal{H}=\sum_{i,j} J(x_i,x_j)S_iS_j$$

with

$$J(x,y)=J\sin(k\,|x-y\,|)\exp(-\gamma\,|x-y\,|)$$

for $\gamma\ll k$ and spins occurring with concentration ρ. A transition is found to occur at $T_c=2J\rho/k$, but the low-

temperature phase does not have an order of the type occurring in mean-field theory (Sec. IV); rather the system "condenses" into solitonlike structures, which can be interpreted as clusters of strongly correlated spins.

Kotliar *et al.* (1983) study the Ising chain with Hamiltonian

$$\mathcal{H}=\sum_{i<j}(\varepsilon_{ij}S_iS_j)/(a\,|\,i-j\,|\,)^{\sigma}\ ,$$

where the ε_{ij} are independent random variables, a is the lattice spacing, and a phase transition occurs if the exponent σ lies in the range $\frac{1}{2}<\sigma<1$. For $\frac{1}{2}<\sigma<\frac{2}{3}$ the exponents exhibit mean-field behavior, while for $\sigma=\frac{1}{2}$ the thermodynamic limit of the free energy no longer exists. For $\sigma>1$, fluctuations destroy any type of ordering, while for $\frac{2}{3}<\sigma<1$ there is Edwards-Anderson-type order but with nonclassical exponents, as renormalization-group expansions show. In particular, near $\sigma=1$ one finds for the correlation length exponent $1/\nu=1.1\sqrt{2(1-\sigma)}$, and hence $\nu\to\infty$, $\alpha=2-\nu\to-\infty$ at this borderline case. Moore (1986) suggests that the model is replica symmetric in the non-mean-field region.

Recently Bhatt and Young (1986) have performed Monte Carlo simulations on this model. Analyzing the results by finite-size scaling, they find that for $\sigma=0.69$ a spin glass transition occurs, with nonzero order parameter below T_f. However, for $\sigma=\frac{3}{4}$ there is evidence for a finite T_f with no order at lower temperatures, reminiscent of the Kosterlitz-Thouless (1973) theory of the two-dimensional XY model. Note that varying σ is analogous to varying d in a short-range model, and a rough argument (Young and Bhatt, 1985) that neglects the exponent η of the short-range model (see Sec. V.A.1) gives the equivalence $d=2/(2\sigma-1)$. Since $\sigma=1$ is the borderline value beyond which no transition occurs, this relation predicts the lower critical dimension $d_l=2$ for short-range systems. However, the equivalence is not exact, so one should not take this result too seriously (see Sec. V.E for a discussion of d_l). The results of Bhatt and Young (1986) could be interpreted as evidence for Kosterlitz-Thouless behavior in the range $0.75\lesssim\sigma\leq 1$, but they may well be affected by corrections to finite-size scaling, so this interpretation is very speculative. Note that Kotliar *et al.* (1983) suggest that Kosterlitz-Thouless behavior occurs but only for $\sigma=1$. Further discussions of models with long- but not infinite-range interactions are given in Sec. V.A.1.

Ariosa *et al.* (1982) study a model in which spins are at random positions x_i along a chain and interact via exchange constants corresponding to a one-dimensional analog of the RKKY interaction,

$$J_{ij}=J_0\cos(\alpha\,|\,x_i-x_j\,|\,)/|\,x_i-x_j\,|\ ,$$

the constant α being chosen arbitrarily as $\alpha=7\pi$. For $N=8,12,16$ the partition function and magnetic susceptibility are found exactly for given sets $\{x_i\}$ of the spin positions (and averages over at least ten configurations $\{x_i\}$ are taken). From the size dependence of these results, Ariosa *et al.* (1982) conclude that the model has a phase transition with properties similar to the Sherrington-Kirkpatrick (1975) model. A drawback of the model, as it stands, is that it does not have a sensible thermodynamic limit for $J_0=0$; rather one must require that $J_0\to 0$ as $N\to\infty$ in a suitable way, which however is not yet known analytically.

Next we turn to the static behavior of the Ising spin glass with nearest-neighbor interaction, where it is clear that a phase transition occurs at $T=0$ only. Chen and Ma (1982) and Gardner and Derrida (1985) consider a probability distribution $P(J_{ij})$ of the nearest-neighbor bonds, which for $|\,J_{ij}\,|\to 0$ behaves as $P(J_{ij})\propto|\,J_{ij}\,|^{k}$, and show that at $T=0$ the magnetization in a field is nonanalytic. It behaves as $M=CH^{(k+1)/(k+3)}$ [see Gardner and Derrida (1985) for an exact expression for the constant C]. Qualitatively, this behavior can be understood as follows: In a weak magnetic field, the magnetization is due to the flip of clusters that are delimited by two weak bonds. The typical distance D between two bonds weaker than $|\,J_0\,|$ behaves like

$$D=\left[\int_{-|J_0|}^{|J_0|}P(J_{ij})J_{ij}\right]^{-1}\propto|\,J_0\,|^{-(k+1)}\ .$$

The typical magnetization at $T=0$ of a cluster of length D is $\sqrt{D}\propto|\,J_0\,|^{-(k+1)/2}$. Therefore the typical cluster of length D delimited by two weak bonds flips for a field $H\approx|\,J_0\,|/\sqrt{D}\propto|\,J_0\,|^{(k+3)/2}$. The typical magnetization per spin of the chain due to the flip of these clusters is therefore $M\approx D^{1/2}/D\propto|\,J_0\,|^{(k+1)/2}$, and hence $M\propto H^{(k+1)/(k+3)}$. For the Gaussian distribution, $k=0$ and hence $M\propto H^{1/3}$; the $\pm$ distribution formally corresponds to the limit $k\to\infty$ and hence $M\propto H$. There is also some evidence that this nonuniversal critical behavior at the $T=0$ transition, which depends on details of the distribution $P(J_{ij})$, also persists for two-dimensional lattices [see Cheung and McMillan (1983a, 1983b)]. In the $\pm J$ chain where the susceptibility at $H\to 0$ is finite, singularities occur at a set of finite values of the field, namely $H/J=2/m$, with m integer (Derrida *et al.*, 1978; Puma and Fernandez, 1978).

We now turn to dynamical properties of the Glauber (1963) Ising chain with random bonds. For the $\pm J$ model, the eigenvalues of the Liouville operator of a finite ring of N spins can be found exactly (Reger and Binder, 1985),

$$\lambda_n=1-[\tanh(2J/k_BT)]\cos\frac{2\pi}{N}\left[n+\frac{1-\Phi}{4}\right]\ ,$$
$$n=0,1,\ldots,N-1\ , \qquad (5.29)$$

where $\Phi=1$ if the ring is not frustrated, $\Phi=-1$ if it is frustrated. For the $\pm J$ model, it is then simple to obtain the time dependence of the remanent magnetization, the spin-spin autocorrelation function, and the dynamic susceptibility. For $N\to\infty$ the contribution of frustrated rings is negligible, and one finds for the remanent magnetization (Hentschel, 1980a)

$$M(t)=M(0)e^{-t}I_0(\gamma t),\quad \gamma=\tanh(2J/k_BT)\ . \tag{5.30a}$$

For the autocorrelation function (Bray *et al.*, 1978),

$$[\langle S_j(0)S_k(t)\rangle_T]_{\rm av}=\delta_{jk}e^{-t} \times\sum_{l=-\infty}^{+\infty}[\tanh(J/k_BT)]^{|l|}I_l(\gamma t)\ ; \tag{5.30b}$$

here I_l denotes modified Bessel functions. The dynamic susceptibility becomes (Hentschel, 1980a, 1980b)

$$\chi(\omega)=\frac{1}{T}\frac{1-\tanh^2(J/k_BT)}{1+\tanh^2(J/k_BT)} \times[(1+\gamma-i\omega)(1-\gamma-i\omega)]^{-1/2}\ , \tag{5.30c}$$

the real part of which exhibits a peak at $T_f(\omega)$ given by

$$JT_f^{-1}(\omega)\approx(\ln 2-\ln\omega)/4\ . \tag{5.31}$$

This simple Arrhenius behavior for a chain really is expected, since the highest energy barrier one can encounter is $2J$.

For the Gaussian model, exact analytical results are no longer available, but one can obtain eigenvalues and eigenvectors of the Liouville operator for small N ($N\leq 10$; typically $N=7$) numerically (Reger and Binder, 1985). Such results are expected to be useful, since for finite $\pm J$ models the lowest eigenvalue for finite N is the same as for $N\to\infty$ [Eq. (5.29)]. Moreover $\chi(\omega)$ is given by Eq. (5.30c) is very well approximated by results found for finite N, for $T\gtrsim\frac{1}{2}T_f(\omega)$. As an example, Fig. 88(a) shows the decay of the remanent magnetization at low temperatures, obtained from this calculation; the results are more accurate than corresponding ordinary Monte Carlo data [Fig. 88(b)] obtained for a chain of $N=1000$ spins by Fernandez and Medina (1979). This calculation exhibits the superposition principle that all curves coincide on a "master curve" if one chooses $T\ln t$ as abscissa [Fig. 89(a)] in the same way that experimental data do [Fig. 89(b)]. The lesson to be learned from this comparison is the following: the $T\ln t$ scaling simply tells us that one probes thermally activated relaxation over a broad distribution of barrier heights independent of temperature. It has no bearing on the question of whether static phase transitions exist (note that $T_f=0$ in this one-dimensional model).

Colborne (1986) shows that at later times the remanent magnetization decays as $M(t)\approx\frac{1}{3}\exp(-\sqrt{t/\tau})$, τ being some relaxation time.

The dynamic susceptibility [Fig. 90(a)] is also qualitatively similar to experimental data [Fig. 90(b)]. As in the experimental procedure, one can define various relaxation times (Fig. 91) and finds that the spectrum of relaxation times dramatically broadens as one lowers the temperature [see Fig. 11 for comparable data on the cobalt-aluminosilicate spin glass (Wenger, 1983)]. However, while the experiment reveals that the maximum relaxation time upon cooling increases faster than an Arrhenius law,

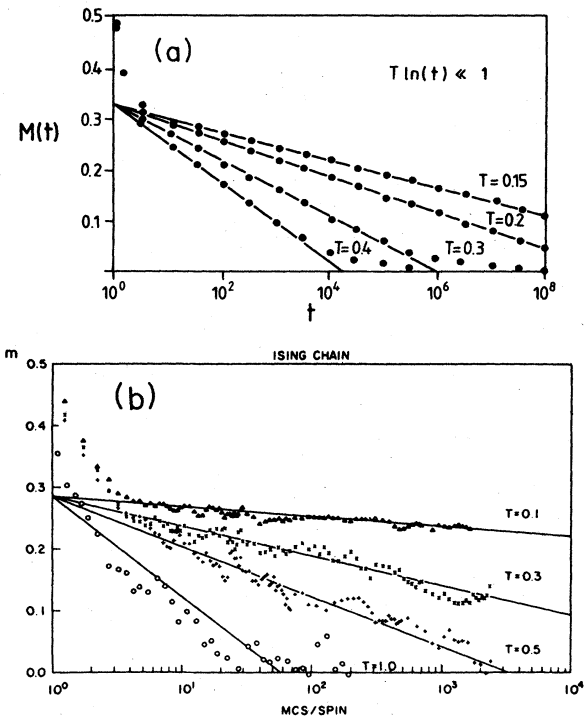

FIG. 88. (a) Decay of the remanent magnetization of an ensemble of clusters of $N=4$ spins each in the Gaussian model. Four temperatures are shown (temperture being measured in units of ΔJ). Straight lines indicate a law $M(t)\cong 0.320-0.075T\ln t$. From Reger and Binder (1985). (b) Same as (a) but from dynamic Monte Carlo simulations of a chain of $N=1000$ spins. From Fernandez and Medina (1979).

here everything is consistent with simple Arrhenius behavior. This fact is also true for the frequency dependence of the freezing temperature, which is found to be $\Delta JT_f^{-1}(\omega)=0.332-0.254\ln\omega$ (Reger and Binder, 1985). It is really a surprise that even "zero-dimensional spin glasses" (that is, ensembles of clusters containing only a small number of spins; see also Banavar *et al.*, 1985) and one-dimensional spin glasses have so much qualitative similarity to real spin glasses. Thus one must study the *deviations* of the frequency dependence of $T_f(\omega)$ from simple Arrhenius behavior very carefully and quantitatively, to ascertain the collective behavior associated with spin freezing in three dimensions.

E. What is the lower critical dimension?

As was discussed in Sec. V.A, mean-field theory is expected to be "qualitatively correct" for short-range systems if the dimensionality d exceeds the upper critical dimension d_u. "Qualitatively correct" means that critical exponents (and other "universal" properties) are predicted correctly, though nonuniversal properties (such as the location of T_f) are predicted inaccurately. For spin glasses, $d_u=6$ for most properties, but certain exponents differ

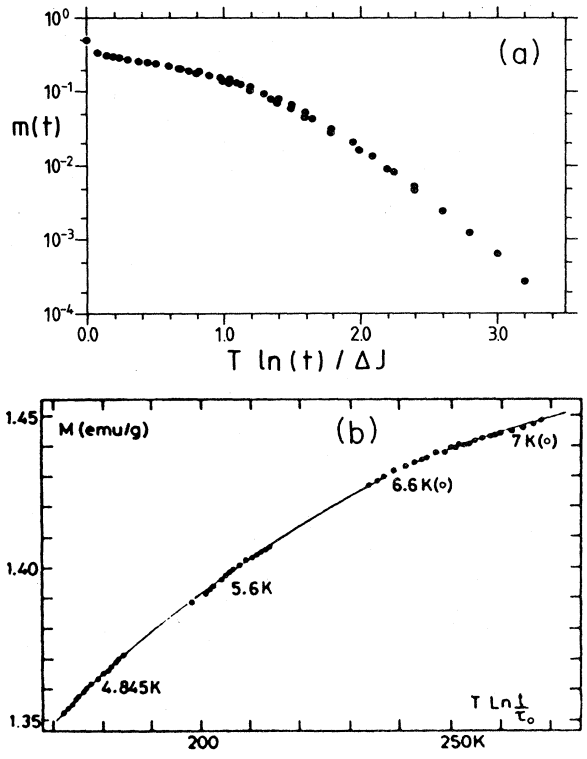

FIG. 89. (a) Relaxation of the magnetization $M(t)$ of an ensemble of clusters of $N=4$ spins each, coupled together by random Gaussian bonds, after at $T=0$ an infinite magnetic field is switched off, at $T/\Delta J=0.4.0.3,0.2,0.15$ [see Fig. 88(a)] plotted vs $T\ln t$. From Reger and Binder (1985). (b) Relaxation of the magnetization $M(t)$ towards equilibrium for CuMn with 5 at % Mn cooled to the considered temperature in zero field, after application of a field $H=10$ kOe at $t=0$, plotted vs $T\ln t$ for various temperatures, as indicated on the figure ($T_f\approx 27.5$ K). From Omari *et al.* (1984).

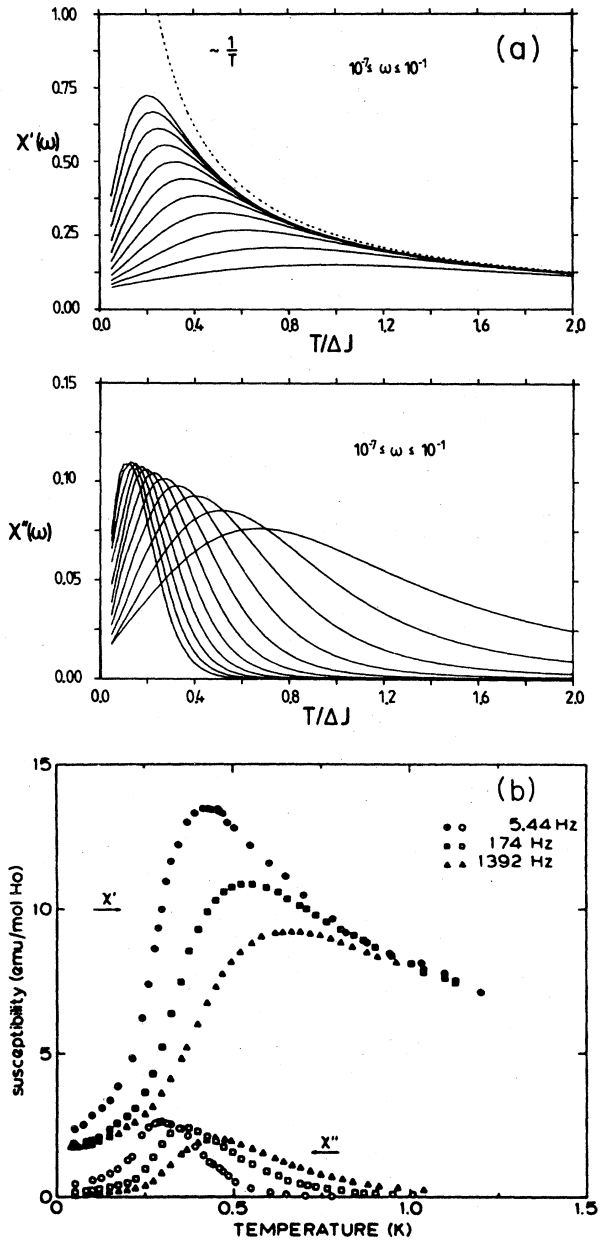

FIG. 90. (a) Real and imaginary parts of the dynamic susceptibility, $\chi(\omega)=\chi'(\omega)+i\chi''(\omega)$ for the Gaussian Ising spin glass with $N=4$, for a wide range of frequencies. Dotted curve indicates the Curie law for the static susceptibility. From Reger and Binder (1985). (b) Same as (a) but for the $(Ho_2O_3)_{0.08}(B_2O_3)_{0.92}$ spin glass. From Wenger (1983).

from their mean-field values when $d<8$ (Fisher and Sompolinsky, 1985). One should also note a possible distinction between behavior at the critical point and in the ordered phase: e.g., Fisher and Huse (1986) argue that mean-field theory for $T<T_f$ is never correct for short-range spin glasses. For dimensions between the upper and lower critical dimensions, $d_l<d<d_u$, mean-field theory correctly predicts the existence of a finite-temperature transition, but gives incorrect values for universal properties. It is not, however, clear to us whether the complicated spin glass state below T_f obtained in mean-field theory (Sec. IV), with many "valleys," breakdown of ergodicity, etc.), persists for short-range models in this range of dimensionality. In particular, one cannot, at this stage, rule out the possibility of another dimension $d_{l'}$ ($<d_u$) such that for $d_l<d<d_{l'}$ there is a finite-temperature transition where χ_{SG} [Eq. (3.102)] diverges but the spin glass order parameter $q^{(2)}$ [Eq. (3.99)] is zero for $T<T_f$ (Anderson, 1979; Young, 1985). The canonical example of this behavior is the Kosterlitz-Thouless (1973) transition in the two-dimensional XY ferromagnet. Young (1985) has pointed out that a finite difference between d_l and $d_{l'}$ is not unreasonable because fluctuations about the mean-field solution diverge more strongly below T_f (when the system orders; see Sec. IV.G) than at T_f itself. Of course, this does not mean that such a difference be-

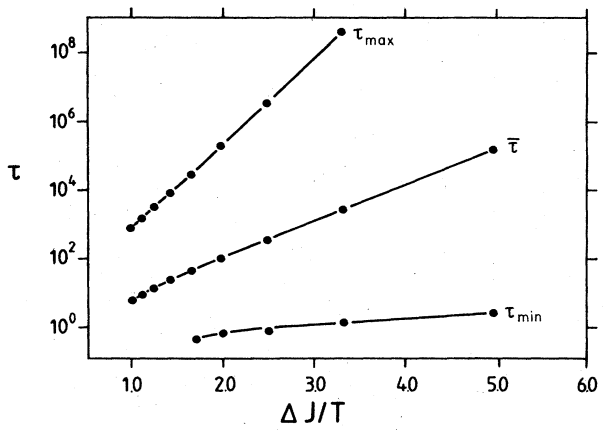

FIG. 91. Logarithm of various relaxation times (maximum relaxation time τ_{max}, average relaxation time $\bar{\tau}$, minimum relaxation time τ_{min}) extracted from an analysis of the dynamic susceptibility $\chi(\omega)$ of an ensemble of clusters of $N=4$ spins each, coupled together by random bonds with Gaussian distribution. From Reger and Binder (1985).

tween d_l' and d_l necessarily occurs—so far this suggestion is only a speculation. For $d<d_l$ the transition is at $T_f=0$. There is a puzzling similarity between some features of real spin glasses and mean-field theory (Sec. IV), for which one knows there is a transition at $T_f\neq 0$, and between other features of real spin glasses and short-range one-dimensional models (Sec. V.D), for which one knows that $T_f=0$. For a more detailed understanding of the freezing transition it is therefore necessary to establish the value of d_l. This question has been very controversial. For Ising spin glasses the suggestions range from $d_l=2$ (Southern and Young, 1977; Anderson and Pond, 1978) to $d_l=4$ (Fisch and Harris, 1977). There exist various pieces of evidence, due to a variety of approaches; unfortunately none of them is completely convincing. We shall discuss in turn the various methods by which a solution of this problem has been attempted.

1. High-temperature series expansion

For standard phase transitions (without quenched disorder) the high-temperature series expansion method has been a very valuable tool (Domb and Green, 1974); for the three-dimensional Ising model, for instance, the accuracy of the resulting exponent predictions is competitive with the best results due to other methods (Nickel, 1981). In brief, a high-temperature expansion is based on the idea of using $1/k_BT$ as a small parameter. Thus in thermal averages the Boltzmann weight can be expanded as

$$\exp(-\mathscr{H}/k_BT)=\sum_k(-1/k_BT)^k\mathscr{H}^k/k!\ ;$$

what one is left with is the calculation of traces at infinite temperature in order to obtain the coefficients in the resulting series explicitly. This can be done—at least in low enough order—because at infinite temperature the spins are not correlated, and hence the calculation of the coefficients is basically a combinational problem (for more details on the method in general, see Domb and Green, 1974).

In practice it is often convenient to choose $w=\tanh^2 J/k_BT$ as the expansion variable, so that (Fisch and Harris, 1977, 1981; Cherry and Domb, 1978; Palmer and Bantilan, 1985)

$$\chi_{SG}=\sum_{n=0}^{n_{max}}a_n w^n\ . \tag{5.32a}$$

For more general symmetric distributions one expands in terms of the moments w_m (Ditzian and Kadanoff, 1979),

$$w_m=\int dJP(J)[\tanh(J/k_BT)]^m\ , \tag{5.32b}$$

$$\chi_{SG}=1+\sum_{q,m_2,m_4,m_6}a^q_{m_2,m_4,m_6,\ldots}\begin{bmatrix}d\\q\end{bmatrix}w_2^{m_2}w_4^{m_4}w_6^{m_6}\cdots\ ,$$

where d is the dimensionality, and q ($q\leq d$) and the m's are positive integers. The coefficients a_n (or $a^q_{m_2m_4m_6}$, respectively) can be precisely defined (and computed) either by formally deriving the linked cluster expansion, which then is directly averaged term by term (Fisch and Harris, 1977; Ditzian and Kadanoff, 1979), or by using the replica Hamiltonian [Eq. (3.29)], which then is expanded in the standard way. Both approaches have been shown to yield identical results (Ditzian and Kadanoff, 1979), giving another justification for the replica method, at least for the regime of the disordered phase. Note that one must not use $[(J_{ij}/k_BT)^2]_{av}$ as an expansion variable for a Gaussian or any other unbounded distribution because the series has zero radius of convergence. To see this, it suffices to note that the expansion of

$$w_2=[\{\tanh(J_{ij}/k_BT)\}^2]_{av}=\sum_m b_m\{[(J_{ij}/k_BT)^2]_{av}\}^m$$

has zero radius of convergence.

Of course, the method is also applicable to asymmetric $P(J_{ij})$; in fact, series for the asymmetric $\pm J$ model have been considered by Domb (1976), Rapaport (1977), and Rajan and Riseborough (1983). Such series have been used to estimate the region of ferromagnetic or antiferromagnetic order in the phase diagrams of these models, and will not be discussed further here. Neither shall we discuss any series for site-disorder models (Binder *et al.*, 1979).

Fisch and Harris (1977) obtained the coefficients a_n up to an order $n_{max}=10$, while Ditzian and Kadanoff (1979) extended their work to arbitrary symmetrical distributions, Eq. (5.32b), to the same order; they also obtained an analogous series for the free energy (up to order $n_{max}=11$). All these authors concluded that there was a spin glass transition for $d>4$ (note that the series are for-

mulated for continuous dimensionality d), but could not locate any transition for $d<4$. Fisch and Harris (1977), concluded that the exponent γ of χ_{SG} was $\gamma=1$ for $d\geq 6$, while γ increased monotonically with decreasing d, and $\gamma\to\infty$ as $d\to 4$, which implies that $d=4$ is the lower critical dimension. This claim seemed to receive support from certain replica symmetry-breaking schemes (Bray and Moore, 1979c), from transfer-matrix-type calculations (Morgenstern and Binder, 1979; Banavar and Cieplak, 1982a; Cieplak and Banavar, 1983), and from analysis of fluctuations around the mean-field solution (Sompolinsky and Zippelius, 1983). Thus, for a while $d_l=4$ was rather generally accepted; more recent work, however, has raised serious doubts as to the validity of this conclusion.

First of all, it was found (Fisch and Harris, 1981; Palmer and Bantilan, 1985) that the original series of Fisch and Harris (1977) contained some errors; in addition, the series is very ill-behaved for $d\leq 4$ and the analysis of it is not at all clear-cut.

As an example, Fig. 92 presents a graph of χ_{SG}^{-1} versus temperature, according to various Padé approximants calculated from the series by Palmer and Bantilan (1985). One can see that there is no good consensus among the various Padé approximants: only the [6,3], [4,5], [5,4] approximants agree fairly well with each other, with the raw series, and with the Monte Carlo data, and hence are credible. Two of these approximants predict no transitions, however, while the third predicts a transition at $T_c/J\approx 0.89$, which is far below all estimates resulting from recent Monte Carlo work. Surprisingly, the work of Palmer and Bantilan (1985) seems to imply that the lower critical dimension depends on dilution; e.g., for a fraction of $c=\frac{2}{3}$ zero bonds, a transition for $d=3$ at $T_c/J\approx 0.8$ is found, with $\gamma\approx 0.78$. However, the contours of estimated critical temperatures in the (c,d) plane are so erratic that serious doubts as to the usefulness of this series analysis must be raised. Probably a much longer series[2] would be needed for a more definitive analysis. At least for now, the authors of this review adhere to the prejudice that d_l should not depend on dilution in a continuous manner, in contrast to the prediction of Palmer and Bantilan (1985).

For the m-vector spin glass, the situation is even more unsatisfactory: Reed (1978) obtained the coefficients of χ_{SG} up to fifth order in $(J/T)^2$ for a Gaussian distribution and located a transition for $4\leq d\leq 6$, again with the finding that γ gets very large for $d\to 4$. However, again we feel that these series are by far too short to warrant any reliable conclusion. Furthermore, the expansion in $(J/T)^2$ is invalid for a Gaussian distribution, as noted above.

Very recently, Reger and Zippelius (1985) presented

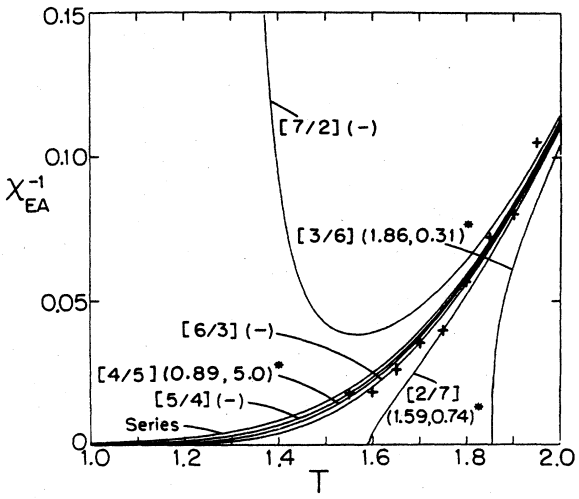

FIG. 92. χ_{SG}^{-1} vs T (in units of J) for the three-dimensional $\pm J$ model, as predicted by the raw series and by various approximants. (T_c,γ) is shown for each approximant with a divergence of χ_{SG}; $(-)$ signifies no divergence. An asterisk signifies that other poles are in the disk $|z|<z^*$. The crosses are Monte Carlo data of Young (1984). From Palmer and Bantilan (1985).

high-temperature series expansions for the relaxation times

$$\tau_{av}=\int_0^\infty dt[\langle S_i(t)S_i(0)\rangle_T]_{av}$$

and τ_{SG} defined by

$$\tau_{SG}=-i\chi_{SG}(\omega=0)\left.\frac{\partial\chi_{SG}^{-1}(\omega)}{\partial\omega}\right|_{\omega=0},$$
$$\chi_{SG}(\omega)=\frac{1}{N}\sum_{ijk}[\chi_{ij}(\omega)\chi_{ik}(\omega)]\ . \tag{5.33}$$

So far the analysis of these times has been restricted to the $\pm J$ model for $d=2$, in which case it was found that as $T\to 0$, $\tau_{SG}\propto\exp(\Delta E/T)$, with $\Delta E\approx 20J$.

2. Exact partition function calculations of small systems

It was pointed out in Sec. V.B that Monte Carlo data, based on short observation times of the order of 10^3 Monte Carlo steps per spin, did not show any qualitative distinction even between dimensionalities as far apart as $d=2$ and $d=5$. As a matter of fact, it was soon suspected (Bray and Moore, 1977; Bray *et al.*, 1978) that one was observing freezing behavior of purely dynamic origin, due to the fact that relaxation times start to exceed the observation time significantly. For $d=2$, this idea could be proven by a complementary method that yields direct estimates of static quantities, unhampered by any observation time effects. In this method, introduced by Morgenstern and Binder (1979, 1980a, 1980b), one calculates recursively the partition function $Z\{J_{ij}\}$ for a chosen

[2]This has recently been obtained by Singh and Chakravarty (1986); see note added in proof.

realization of the random bonds $\{J_{ij}\}$ on a lattice of linear dimension $L\times M$ ($d=2$) or $L\times L\times M$ ($d=3$), respectively. Choosing free boundaries in the direction where one has M rows (or planes, respectively), and applying periodic boundary conditions elsewhere, one has to keep 2^L states ($d=2$) or $2^{L\times L}$ states ($d=3$) at each step of the recursive calculation. This fact severely limits the lattice sizes that can be treated: the largest systems to have been studied were $L=18$ ($d=2$) and $L=4$ ($d=3$), which implies that the method is mainly useful for two dimensions only. A further difficulty is that in order to approximate the disorder average $[\ ]_{\rm av}$ with reasonable accuracy one must repeat the calculations for at least 30–100 realizations $\{J_{ij}\}$ of the random-bond configuration. But in spite of these drawbacks, fairly definitive conclusions about the absence of Edwards-Anderson order for $d=2$ did emerge from this approach, which will be discussed in detail.

In an exact calculation of a finite system without any external field, there is no spontaneous symmetry breaking, of course, and thus $\langle S_i\rangle_T\equiv 0$. But order parameters can be estimated nevertheless, if one uses relations such as $|\psi(l)|^2=\lim_{N\to\infty} k_BT\chi_{ll}/N$ [Eq. (3.114)], in which the thermodynamic limit is approached in a smooth fashion. This is shown in Fig. 93, for the Mattis spin glass (equivalent to an Ising ferromagnet), for which the exact answer is known: for $N\to\infty$, the system is ordered for $T<T_c/J\approx 2.27$, and disordered for $T>T_c$. The results for small lattices converge to the limiting behavior quite rapidly—apart from the region near T_c where the correlation length exceeds L. From the inflection points of these

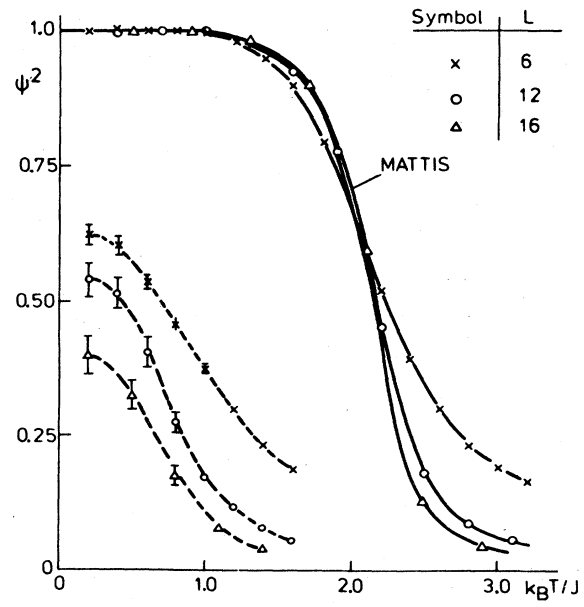

FIG. 93. Projection order parameter of the Edwards-Anderson $\pm J$ model on the square lattice (dashed curves) as compared to the Mattis model (solid curves), for $L\times L$ lattices. From Morgenstern and Binder (1979).

ψ^2 vs T curves one can locate T_c within an accuracy of a few percent. Of course, we are not advocating this method for a precise analysis of the critical behavior—the method only serves to verify that the model has a transition from disorder to order and to locate roughly where this happens.

For the Edwards-Anderson $\pm J$ spin glass the results are clearly rather different: $|\psi|^2$ decreases with increasing L at all temperatures, and the inflection points of the curves shift slowly to lower temperature. These data do not indicate convergence towards a nonzero order parameter, not even in the ground state. Thus Morgenstern and Binder (1979, 1980a) suggested that a phase transition occurs for $T=0$ only. For the Gaussian model, on the other hand, the ground state is perfectly ordered, but again no order was found at nonzero temperatures.

Of course, one must make sure that this lack of order is not an artifact of the lattices' being too small. Thus dynamic Monte Carlo simulations were performed on *precisely the same lattices* (same $\{J_{ij}\}$), Fig. 94(a). These simulations did show nonzero $q(t)$, with the usual slight dependence on observation time; moreover, $q(t)$ agreed well with results for much larger lattices. Hence these small lattices do exhibit spin glass ordering in dynamic observations of finite times. For these times, $\psi^2(t)$ is also much larger than its thermal average. What happens is that the system gets trapped for a long time in a "valley" in phase space (Fig. 4). Observing the time dependence of $\psi^2(t)$ and $\chi(t)$, one sometimes can observe a transition from one valley to another [see Fig. 94(b)]. These transitions where the system has to jump over a saddle point in configuration space occur fairly seldom, as the associated energy barriers are rather high. Therefore for short observation times one samples properties of a single valley—or a few valleys—only, and hence the ordering tendency of the system is grossly overestimated.

The most convincing evidence that at $d=2$ there is no order comes from an analysis of the correlation function $[\langle S_iS_j\rangle_T^2]_{\rm av}$ (see Fig. 95). In the Gaussian model, the data are consistent with an exponential decay at all temperatures $T>0$, while at $T=0$ one has full order. Here $k_BT_f(t=10^3\ \text{MCS})/\Delta J\approx 1.0$, and hence even at $t\approx\frac{1}{2}T_f$ the correlation length is only about 7 lattice spacings, while at $T\approx T_f$ it is only about 2–3 lattice spacings. Due to this smallness of ξ at T_f, it is understandable that there is so little size dependence in the results of the dynamic Monte Carlo simulations.

The data on the temperature dependence of $\xi_{\rm SG}$ indicate that it increases more slowly with decreasing temperature than in the one-dimensional case, both for the $\pm J$ model and for the Gaussian model. This fact indicates that frustration is very effective in reducing the correlation in these models. The data in Fig. 95 are consistent with a behavior $\xi_{\rm SG}\propto T^{-\nu}$, with $\nu\approx 2$ (Binder and Morgenstern, 1983) as $T\to 0$. Of course, for $T/\Delta J\ll 1$ (or $T/J\ll 1$, respectively, for the $\pm J$ model) the correlation length is certainly comparable to L or even larger than L, in which case it can no longer be estimated reliably from this

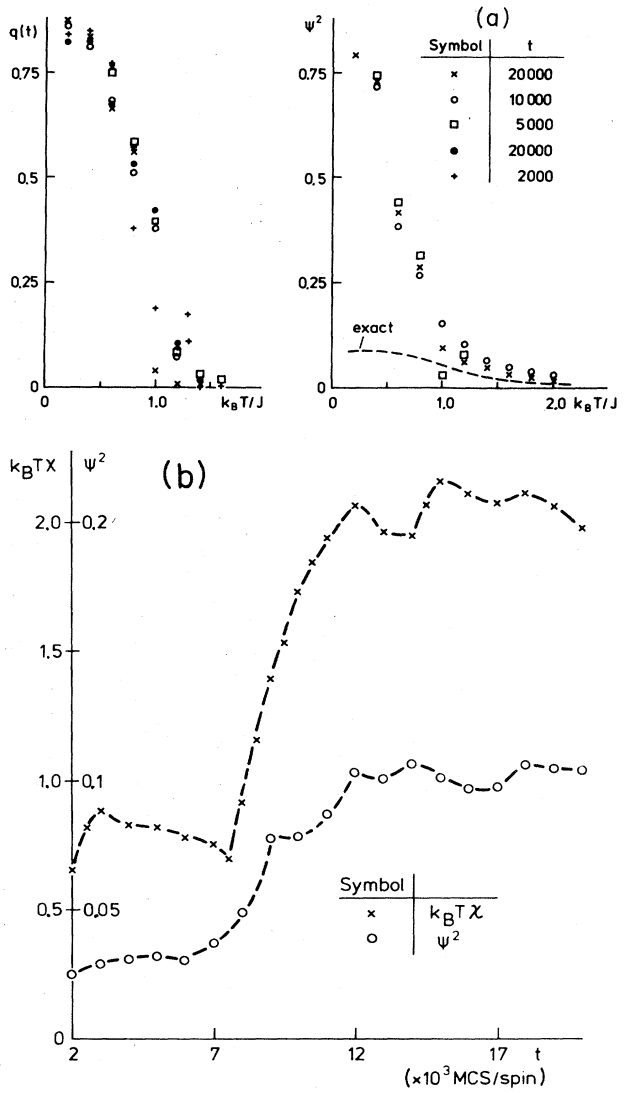

FIG. 94. (a) Time-dependent Edwards-Anderson order parameter $q(t)$ (left part) and $\psi^2(t)$ (right part) plotted vs temperature in 16×16 square $\pm J$ lattices, for different observation times [data for $t=2000$ are due to a Monte Carlo simulation of an 80×80 lattice (Kirkpatrick, 1977)]. Dashed curve is the result of the "exact" calculation for the same $\{J_{ij}\}$. (b) Typical run for a 16×16 lattice at $k_BT/J=1.0$, showing time evolution of the coarse-grained susceptibility, $\chi(t)$ and $|\psi^{(1)}(t)|^2$. From Morgenstern and Binder (1980a).

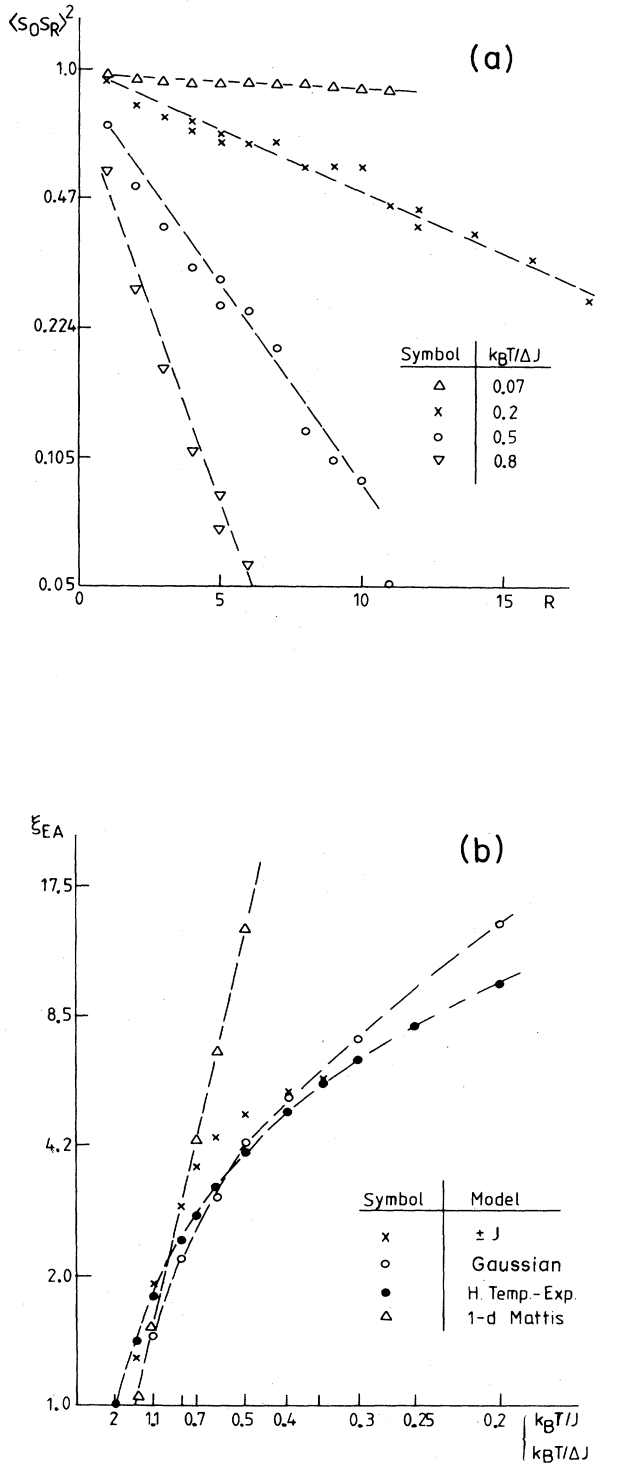

FIG. 95. (a) Averaged square correlation function of the two-dimensional Gaussian model plotted vs distance at four temperatures below the "dynamic freezing temperature" $k_BT_f(t=10^3\ \text{MCS})/\Delta J\approx1.0$. (b) Correlation length plotted vs inverse temperature, for both the Gaussian and the $\pm J$ model. The corresponding first term of the high-temperature expansion (●) and the one-dimensional Mattis spin glass (△) are also shown. From Morgenstern and Binder (1980a).

method. Therefore the above estimate for ν is very uncertain, and this work could not indicate whether ν is the same for the Gaussian and the $\pm J$ model. Later careful Monte Carlo work by Young (1983a, 1984) suggested $\nu\approx2.75$ for the $\pm J$ model, and the raw data for ξ_{SG} at $T/J\lesssim1$ are somewhat larger than those shown in Fig. 95. McMillan (1983) found $\nu\approx2.64\pm0.23$ by fitting Monte Carlo data for the $\pm J$ model in the temperature range $0.86\leq T\leq1.5$, in reasonable agreement with his transfer

matrix calculations, $\nu=2.59\pm0.13$ (Cheung and McMillan, 1983a, 1983b). However, using a transfer matrix technique right at $T=0$, Bray and Moore (1984b) found $\nu=3.4\pm0.1$ from the rescaling of the effective couplings with size, starting from an initial Gaussian distribution, while McMillan (1984c) found $\nu\approx3.56$ from a similar method but using Monte Carlo data.

In these studies the energy of defect lines was studied as a function of system size, and ν was extracted from fitting the data to $E_{\rm def}\propto L^{-1/\nu}$. The discrepancy between these two types of methods for estimating ν may mean that (i) extrapolations of $\xi_{\rm SG}$ from the range $T/J\approx1$ yield an effective exponent $\nu_{\rm eff}$ significantly smaller than the asymptotic value reached only for $T/J\to0$ and/or (ii) the exponent of the size dependence of the defect energy differs from the exponent describing $\xi_{\rm SG}$. In fact, these exponents can be equated by a simple scaling argument (Bray and Moore, 1984b): suppose the initial distribution of the exchange has width ΔJ and we divide the system into blocks of linear dimension L. If the energy of the effective couplings between blocks is $\Delta J(L)$, then the block correlation length $\xi'_{\rm SG}$ diverges as $T\to0$ according to

$$\left[\frac{\Delta J(L)}{T}\right]^{\nu}\propto\xi'_{\rm SG}=\frac{\xi_{\rm SG}}{L}\propto\frac{1}{L}\left[\frac{\Delta J}{T}\right]^{\nu},\tag{5.34}$$

which yields $\Delta J(L)\propto\Delta J L^{-1/\nu}$. Thus, if $1/\nu>0$, one has a transition at $T=0$, while for $1/\nu<0$ the system iterates towards strong coupling and one expects a finite-temperature transition. This scaling theory will be discussed further in the next section. Bray and Moore (1984b) claim that $1/\nu<0$ happens for $d=3$, but there their transfer matrix approach is based on extremely small linear dimensions, $2\le L\le4$, and therefore it is doubtful whether the data are already in the asymptotic regime in which Eq. (5.34) applies. For $d=2$, linear dimensions $2\le L\le12$ are used, and thus the analysis is more credible, but still there is the question whether the defect energy really measures $\Delta J(L)$: in a system with frustration and large ground-state degeneracy the significance of this defect energy is not completely clear.

Very recently Huse and Morgenstern (1985) have used the transfer matrix technique to calculate the correlation length of very long strips of up to 10^6 spins. This eliminates the effect of the free boundaries that are necessarily present in one direction, but Huse and Morgenstern had to use strips of width $w\le8$ lattice spacing in most parts of their analysis. They concluded that $\nu=4.2\pm0.5$. The lower bound of this estimate is close to the value of Bray and Moore (1984b) and McMillan (1984c). However, there are strong corrections to scaling, so they cannot rule out the possibility that ν is significantly larger than 4.2.

In the two-dimensional $\pm J$ model there is no order even at $T=0$, and one expects a power-law decay of correlations. Morgenstern and Binder (1980a) noted that their data at the lowest temperature were consistent with an exponent $\eta=0.4\pm0.1$; but since the lattice sizes were rather small, and a systematic distortion of the data—due to this finite size—was expected, the accuracy of this estimate is completely uncertain. Later McMillan (1983) extracted $\eta\approx0.28\pm0.04$ from his data; but again there was the uncertainty of extrapolating data from $T/J\approx1$ to zero temperature. In two-dimensional periodically frustrated Ising systems undergoing phase transitions at $T=0$, one always finds $\eta=\frac{1}{2}$ (Wolff and Zittartz, 1983b; see Sec. VI.E). The possibility that the same value applies to the $\pm J$ model, too, has not really been ruled out.

Morgenstern and Binder (1980b) also attempted to study the correlation function of three-dimensional spin glasses, as they had in their work in two dimensions (Fig. 95). While they could show that the freezing transition found in the short Monte Carlo runs [Eq. (5.25b) or $k_BT/J\approx1.9$; see Fig. 70(a)] is a purely dynamic phenomenon, and the system has a correlation length of only about two lattice spacings there, this study is certainly not conclusive about the behavior at low temperatures. At $k_BT/J\approx1.5$, the correlation length of the three-dimensional $\pm J$ model is already about 4 lattice spacings (Ogielski and Morgenstern, 1985), and hence the data obtained from $4\times4\times10$ lattices are no longer reliable for $k_BT/J\lesssim1.5$.

3. Monte Carlo evidence

Stauffer and Binder (1979) attempted to clarify the question of the lower critical dimensionality by a comparative Monte Carlo study of Gaussian Ising spin glasses with lattice dimensionalities $d=2$, $d=3$, and $d=5$. In fact, there is little doubt that a transition occurs for $d=5$, and T_f has been located with series methods for the $\pm J$ model by Fisch and Harris (1977) and for the Gaussian model by Ditzian and Kadanoff (1979). The short-observation-time Monte Carlo data yielded $k_BT_f(t=2000)/\Delta J\approx2.4$, in reasonable agreement with the series expansion. The specific-heat peak (Fig. 96) was seen to become somewhat sharper with increasing dimension (and for $d=5$ its location could no longer be distinguished from T_f). However, studies of $q(t)$ for large observation time t at temperatures $T\approx0.6T_f(t=2000)$ failed to give a clear-cut answer as to whether there exists a stable nonzero Edwards-Anderson order parameter (Fig. 97). For both $d=3$ and $d=5$ a behavior $dq(t)/d\ln t=-k_BTP(0,0)$ [Eq. (3.142)] was observed for times up to about 10^3 MCS/spin. Since this behavior is basically accounted for by single spin flips in the effective field due to a "frozen-in" environment, it is clear that one must go to substantially longer times to make sure that the system reaches equilibrium and that one observes the asymptotic behavior of $q(t)$ as $t\to\infty$. For $t>10^3$ MCS, data such as Fig. 97 indicate that the relaxation is even slower than the initial logarithmic law for all values of d studied; but due to strong fluctuations in this late time regime one cannot judge from this work whether $q(t\to\infty)$ is nonzero. It is clear that a substantially larger statistical effort is required to obtain meaningful Monte Carlo data on equilibrium properties of spin glasses. This conclusion is also supported by work of Medina *et al.* (1980), who

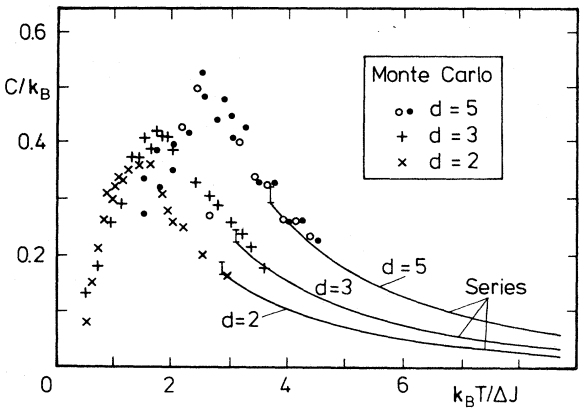

FIG. 96. Specific heat plotted vs temperature for Ising spin glasses with symmetric Gaussian distribution of exchange, for lattice sizes 60^2, 20^3, and 6^5, respectively. For $d=5$, data obtained from energy fluctuations (●) and from derivatives of energy vs temperature curves (○) are shown. Solid curves indicate the results of the (truncated) series of Ditzian and Kadanoff (1979), which has 11 terms (the error bar where the curve ends shows the contribution of the eleventh term alone). From Stauffer and Binder (1979).

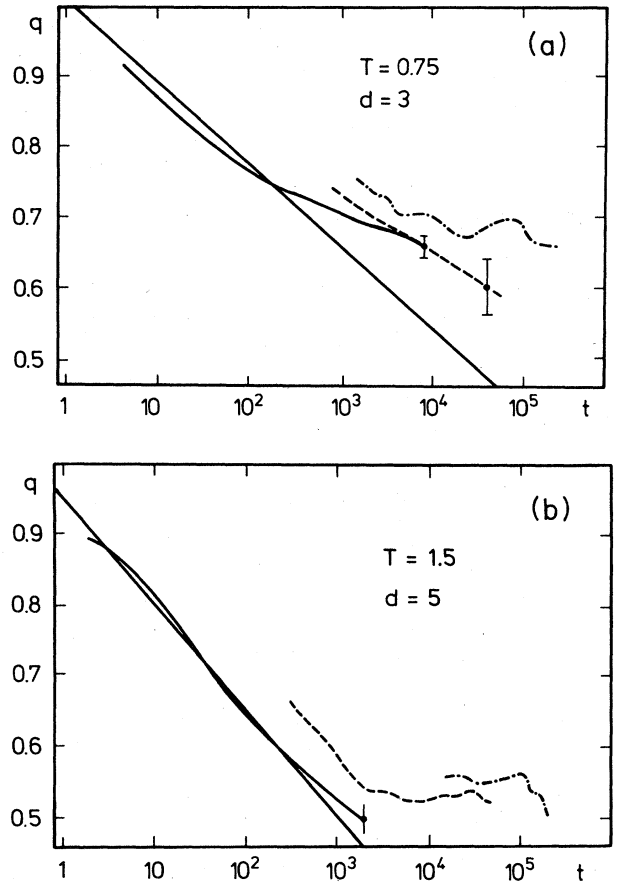

FIG. 97. Edwards-Anderson order parameter $q(t)$ plotted vs observation time t for $d=3$ (a) and $d=5$ (b). Straight lines represent Eq. (3.142); solid curves represent the average of 6 runs for a 20^3 system and 6 runs for a 6^5 system; dashed curves represent the average of 13 runs for a 10^3 system (a) and a 4^5 system (b); while dash-dotted curves are the output of single runs. Temperature is quoted in units of $\Delta J/k_B$. From Stauffer and Binder (1979).

demonstrated that an initial slow logarithmic relaxation of $q(t)$ occurs even in an unfrustrated Mattis-type spin glass with Gaussian bonds.

We have already mentioned in the previous section that d_l has been estimated from the variation of defect energies with system size at $T=0$ using transfer matrix calculations (Bray and Moore, 1984b) and Monte Carlo techniques (McMillan, 1984b, 1984c). The defect energy is the difference in energies (free energies if one allows for nonzero temperatures) of the system with periodic and antiperiodic boundary conditions. We now discuss the scaling theory of this quantity in more detail. One may define (see, for example, Banavar and Cieplak, 1982a; Cieplak and Banavar, 1983; Caflisch *et al.*, 1985)

$$\Delta F = F_{AP} - F_P, \quad \gamma_m = [\Delta F]_{\rm av} , \qquad \gamma_w = ([\Delta F^2]_{\rm av} - [\Delta F]_{\rm av}^2)^{1/2} . \tag{5.35}$$

In ferro- or antiferromagnets, the characteristic free-energy scale of the sensitivity of the system to changes in boundary conditions is measured by the quantity γ_m. In "symmetric" spin glasses, the periodic boundary conditions do not yield a lower (or higher) free energy on average than the antiperiodic ones, so $\gamma_m=0$. This leaves γ_w as the fundamental free-energy scale in the problem; it is often called an effective coupling $J_{\rm eff}$. One expects that in the equilibrium spin glass phase, if such a phase indeed exists, γ_w varies as an algebraic function of the length L and area A, where the boundary conditions are changed for the spins on the surface with area A, i.e.,

$$\gamma_w = \delta A^{1-r} L^{1-p}, \quad A \to \infty, \; L \to \infty , \tag{5.36}$$

where the exponents r and p depend on the dimensionality only, while the coefficient δ depends on temperature and vanishes at T_f (for $T > T_f$, γ_w should decay exponentially with L). For a cubic lattice, $A = L^{d-1}$, so $\gamma_w \propto L^{-\lambda_1}$ where $\lambda_1 = d(r-1)+p-r$. Following Anderson and Pond (1978), Banavar and Cieplak (1982a), Bray and Moore (1984b), and McMillan (1984b, 1984c) identified the lower critical dimensionality by the condition that the exponent λ_1 vanish, i.e., $d_l = (p-r)$. Furthermore they assumed that for $d < d_l$ the correlation length $\xi_{\rm SG}$ diverges as $T \to 0$ like $T^{-\nu}$, with $\nu = \lambda_1^{-1}$ (Bray and Moore, 1984b), as discussed in the last section. Banavar and Cieplak (1982a, 1982b) studied classical Heisenberg spins with nearest-neighbor exchange on a simple cubic lattice, varying A from 4×4 to 12×12, and L from 4 to 8. Typically the configurational average $[\,]_{\rm av}$ in Eq. (5.35) was approximated by an average over 30 samples, and $T=0$ was chosen such that one had simply to study

energy rather than free-energy differences. It was found that the data were consistent with $p=3$ and $r=\frac{1}{2}$, implying that $\gamma_w \propto A^{1/2}L^{-2} \propto L^{-1}$. This finding implies that $d=3<d_l$, in contrast to the results of Anderson and Pond (1978), who found from Migdal approximations and additional assumptions that the "effective coupling" $\gamma_w \propto A^{1/2}L^{-1} \propto L^0$ for the three-dimensional vector spin glasses, implying $d_l=3$. Morris *et al.* (1986), however, were able to obtain $d_l=4$ using Migdal-Kadanoff approximations with no extra assumptions.

While this work is rather suggestive, one might object that it could contain systematic errors due to the failure of reaching true ground states in such simulations—it may be that the exponents r and p are higher for metastable states than for the true ground states (remember that a metastable state could be a mixture of two or more ground states with "walls" in between them). In addition, the linear dimensions quoted above may be too small for the asymptotic regime considered in Eq. (5.36). Cieplak and Banavar (1983) applied this method as well to a two-dimensional Ising spin glass and found $r=\frac{1}{2}, p=\frac{5}{2}$, which implies $\gamma_w \propto L^{-1}$ again, i.e., no order. Assuming that p is independent of dimensionality, they concluded that $\gamma_w \propto L^{(d-4)/2}$, i.e., $d_l=4$. Of course, this conclusion is rather speculative, as p might well be dimensionality dependent (although this does not seem to happen in the Heisenberg case).

McMillan (1984b, 1984c, 1985a, 1985b) generalized this idea by setting up a phenomenological renormalization method ("domain-wall renormalization group") by the equation

$$\gamma_w(L,K)=\gamma_w(L',K'), \quad K=\Delta J/k_B T, \quad L,L' \to \infty \ . \tag{5.37a}$$

This is an implicit relation from which K' may be calculated given K. Critical points correspond to fixed points of this transformation, i.e., $K=K'=K^*$, say. Linearizing about K^* one writes $\delta K' = b^{\lambda_1}\delta K$, where $b=L'/L$ and $\lambda_1=1/\nu$, so λ_1 is given by

$$\frac{1}{\nu}=\lambda_1=\ln[\gamma_w'(L,K^*)/\gamma_w'(L',K^*)]/\ln(L'/L) \ , \tag{5.37b}$$

where the prime on γ_w denotes a derivative with respect to K. If the transition is at $T=0$, then the linearized transformation is $K'=b^{\lambda_1}K$. For $T \to 0$, $\gamma_w = KTu(L)$, on dimensional grounds, where T is a constant (not rescaled) and $u(L)$ is some function of L. From Eq. (5.37b) one immediately sees that $u(L) \propto L^{-\lambda_1}$ in agreement with the discussion following Eq. (5.36). McMillan used very small L ($L,L' \leq 6$) but statistical samples many orders of magnitude larger than those of Banavar and Cieplak (1982a). For the two-dimensional Gaussian Ising spin glass, McMillan (1984c) studied $\gamma_w(L,K)$ in the range from $0.2 \lesssim k_B T/\Delta J \lesssim 0.7$ and extrapolated the data to zero temperature ($K \to \infty$). The results imply $\gamma_w(L,\infty) \propto L^{-\lambda_1}$, with $\lambda_1=0.281\pm0.005$. This finding can be interpreted as implying a phase transition at zero temperature, consistent with all other investigations; the resulting exponent ν (spin glass correlation length $\xi_{SG} \propto T^{-\nu}$ as $T \to 0$) would be $\nu=1/\lambda_1=3.56\pm0.06$.

In three-dimensional Ising spin glasses, however, McMillan (1984b) found $\lambda_1=-0.306\pm0.015$, and hence concluded that the system iterates towards strong coupling, i.e., the system should order at low temperatures. Assuming that the eigenvalue λ_1 depends linearly on dimensionality, the lower critical dimensionality is identified from the condition $\lambda_1=0$, which yields $d=2.64\pm0.10$. McMillan (1985a) applied this method for $d=3$ to finite temperature and concluded that a static phase transition occurs at $k_B T_f/\Delta J=1.0\pm0.2$, with an exponent $\nu=1.8\pm0.5$. Of course, all these conclusions rely on the hope that Eq. (5.37), yields essentially correct results for L as small as $L=3$–6.

Bray and Moore (1984b, 1985a) carried out an analysis in essentially the same spirit, but used transfer matrix methods instead of Monte Carlo methods: this ensured that true ground-state properties were obtained. For $d=2$, they used $2 \leq L \leq 12$ and found $\lambda_1 \simeq 0.296$; they concluded that $\nu=1/\lambda_1=3.4\pm0.1$. Unfortunately for $d=3$ only $L=2,3,4$ was available, which yielded $\lambda_1 \simeq -0.2$, suggesting again a phase transition at finite temperature. Applying the finite-temperature phenomenological renormalization, Bray and Moore (1985a) located this (inverse) transition temperature K from the point where the effective couplings $K_L(K), K_{L'}(K)$ intersect (Fig. 98). $K_L(K)$ is just $(\Delta J)^{-1}K\gamma_w(L,K)$ in our notation, with ΔJ fixed. This yields $K_c=\Delta J/k_B T_c=1.2\pm0.1$, $\nu=3.3\pm0.6$. However, even for $K_L > K$ the data for $L=3,4$ stay so close together that the sceptical observer may doubt whether for larger L there remains a true intersection at all. (Related finite-size scaling analyses have frequently yielded spurious intersections for Kosterlitz-Thouless transitions, for instance.)

McMillan (1985b) also studied Heisenberg spin glasses with his domain-wall renormalization-group method, for $d=2$, 3, and 4, and concluded that $\lambda_1=1.40\pm0.03$ ($d=2$), $\lambda_1=0.65\pm0.08$ ($d=3$), and $\lambda_1=0.16\pm0.07$ ($d=4$). Thus he concluded, in agreement with Banavar and Cieplak (1982a), that there is no transition in the three-dimensional Heisenberg spin glass; he also suggested that for $d=4$ the correct value is probably $\lambda_1=0$; i.e., $d_l=4$.

Morris *et al.* (1986) have also studied vector spin glasses using domain-wall renormalization-group and other approaches. They too found $T_f=0$ in $d=2,3$ and possibly $d_l=4$. Morris *et al.* (1986) also estimated how T_f increases from zero in the presence of anisotropy. Even if the anisotropy comes from Dzyaloshinskii-Moriya or dipolar interactions, which couple different spin directions, the transition will still be Ising-type (Bray and Moore, 1982b). Evidence that three is *below* the lower critical dimension of isotropic Heisenberg spin glasses has also been obtained by finite-temperature Monte Carlo simulations (Olive *et al.*, 1986). Thus there is evidence for a large distinction between the values of d_l for Ising spin glasses

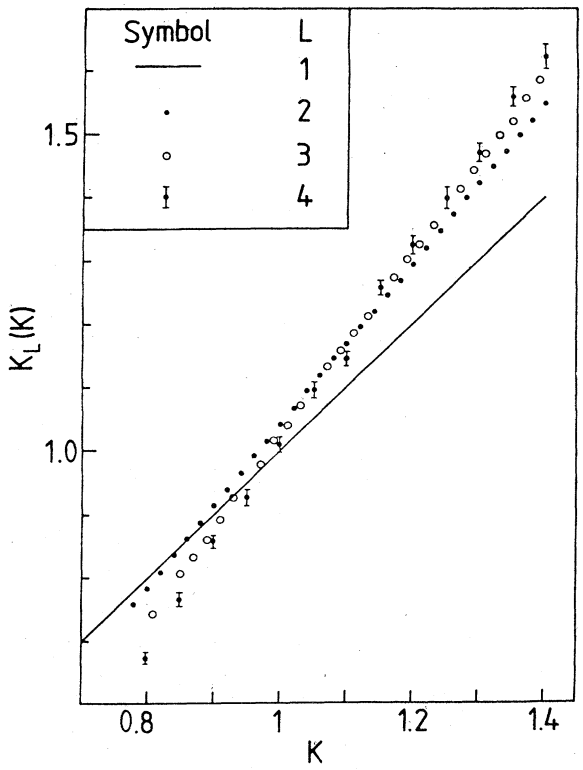

FIG. 98. Effective coupling $K_L(K)$ between cells of size L^3 for the simple cubic nearest-neighbor Gaussian Ising spin glass, for $1 \leq L \leq 4$. 600 000 samples for $L=2$, 30 000 for $L=3$, and 1,600 samples for $L=4$ were used, $K_L = \langle K'^2 \rangle_L^{1/2}$, where $K' = \frac{1}{2}\ln(Z_{\uparrow\uparrow}/Z_{\uparrow\downarrow})$, and arrows indicate fixed-spin boundary conditions on the two free boundaries of the sample, for which the partition functions $Z_{\uparrow\uparrow}, Z_{\uparrow\downarrow}$ are computed exactly. From Bray and Moore (1984a).

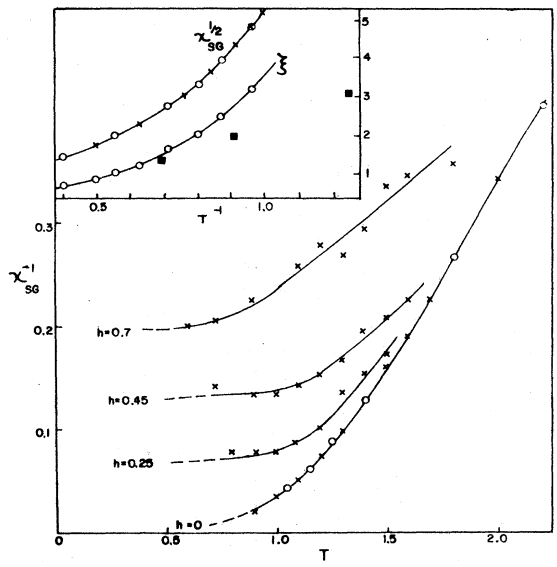

FIG. 99. Inverse spin glass susceptibility χ_{SG}^{-1} plotted against temperature T (in units of J) for the two-dimensional $\pm J$ model and several values of $h = H/J$. Inset shows $\chi_{SG}^{1/2}$ and ξ_{SG}, for $h=0$. (■, data from Morgenstern and Binder, 1980a.) Lattice sizes used are 64^2 (○) and 128^2 (×). From Young (1983a).

and for Heisenberg spin glasses, in contrast to some speculative arguments (Villain, 1979b).

A serious drawback of all these studies discussed so far is the smallness of the lattices studied; in addition, one may have doubts as to the applicability of the renormalization concepts used. An alternative approach not subject to this criticism is the direct Monte Carlo simulation of very large lattices over very large observation times; this is possible nowadays with the help of powerful vector processors, as used by Young (1983a, 1983c, 1984; Bhatt and Young, 1985), or special-purpose computers, as used by McMillan (1983) and Ogielski and Morgenstern (1985). Figure 99 shows data for the spin glass susceptibility and correlation length of the two-dimensional $\pm J$ model (Young, 1983a). One can see that χ_{SG} (for $H=0$) clearly is finite at the apparent freezing temperature of the Monte Carlo work using small observation times [Eq. (5.25a)] and the data are consistent with a power-law divergence at $T=0$, $\chi_{SG} \propto T^{-\gamma}$.

The correlation length is consistent with a behavior $\xi_{SG}^2 \propto \chi_{SG}$. For $H \neq 0$ one finds that χ_{SG} tends to finite asymptotes for $T \to 0$. While Young (1983) originally suggested that the data were consistent with rough estimates quoted by Binder (1982b), $\gamma \approx 4$, $\nu \approx 2$, a later fit (Young, 1984) in the temperature regime $1 \leq T/J \leq 2$ yielded $\gamma \approx 4.1$, $\nu \approx 2.75$. From similar data McMillan (1983) estimated $\nu \approx 2.64 \pm 0.23$, and the exponent η describing the decay of the correlation function at $T=0$, $\eta \approx 0.28 \pm 0.04$; hence $\gamma = (2-\eta)$, $\nu \approx 4.54$. The temperature regime in this last fit was $0.86 \leq T/J \leq 1.5$. Note that the exponents estimated from these fits are reasonably consistent with each other but significantly smaller than the estimates due to the domain-wall renormalization group quoted above. The reason for this discrepancy is not yet clear; of course, one may argue that the Monte Carlo data are not yet in the asymptotic regime of the zero-temperature transition: in order to have a simple power-law description, $\xi_{SG} \propto T^{-\nu}$, $\chi_{SG} \propto T^{-\gamma}$, where any correction terms are neglected, one must have $T/J \ll 1$. This regime cannot be reached by the Monte Carlo simulations, since there the relaxation times are by far too long.

Both Young (1983a) and McMillan (1983) have made careful studies of the relaxation times of the two-dimensional $\pm J$ model. As an example, Fig. 100 shows contours of constant relaxation time τ_{av} $(= \int_0^\infty dt[\langle S_i(t)S_i(0)\rangle]_{av})$ in the H-T plane. These curves have a striking similarity to experimental data on critical magnetic fields $H_c(\omega)$ (see Fig. 25). We shall turn to a more detailed analysis of these data in the next section. At this point, we note that there is still disagreement concerning the behavior at $T=0$: while Young

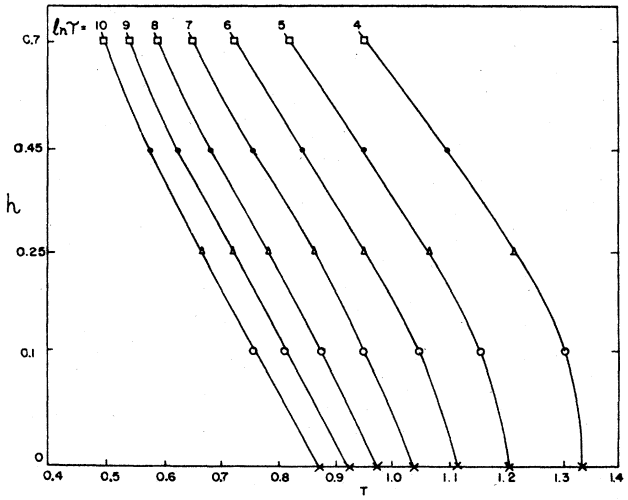

FIG. 100. Contours of constant relaxation time τ in the h-T plane, for the two-dimensional $\pm J$ model. Note that the vertical axis is linear in $h^{2/3}$, and that temperature is measured in units of J. From Young (1983a).

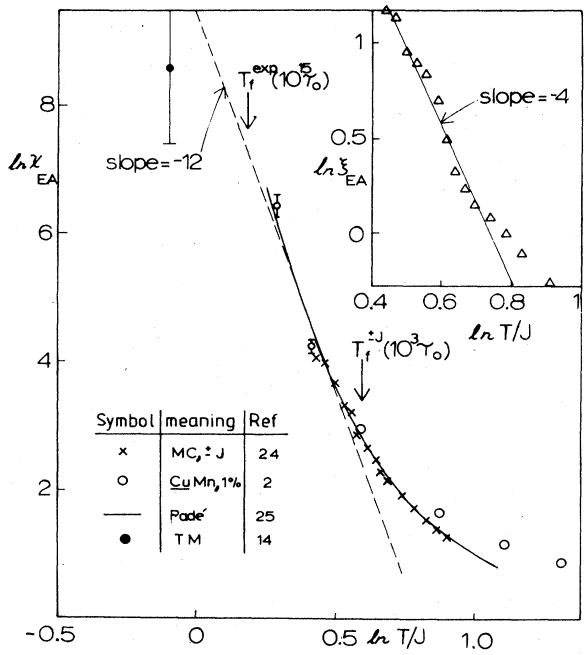

FIG. 101. Log-log plot of χ_{SG} vs T/J for the $d=3$ $\pm J$ Ising spin glass: $\times$, the Monte Carlo results of Young (1984); solid curve, a Padé analysis of the high-temperature series of Palmer and Bantilan (1985); ●, a transfer matrix (TM) calculation (Morgenstern and Binder, 1980b), obtained from $\chi_{EA}\approx 8\Pi\xi_{EA}^3$ with $\xi_{EA}=6\pm 2$. Due to the smallness of the lattice ($4\times 4\times 10$) this estimate is only considered as a lower bound. An arrow marks the dynamic freezing temperature $T_f(t_{obs})$ observed by Bray *et al.* (1978) for $t_{obs}=10^3$ MCS/spin $=10^3\tau_0$. ○, data from Omari *et al.* (1983) for *Cu*Mn with 1 at% Mn with the temperature axes scaled by taking $J=8.39$ K [Our χ_{SG} is $(a_3+2)/3$ in the notation of Omari *et al.* (1983)]. The "static" transition temperature of 10.03 K, quoted by these authors, is also shown. Taking "static" to mean $\omega\approx 10^{-3}$ Hz and assuming $\tau_0\approx 10^{-12}$ sec, one could identify this with $T_f(t_{obs}=10^{15}\tau_0)$ at this temperature. The inset shows ξ_{SG} against T/J on a log-log plot for the $d=3$ Ising model. From Binder and Young (1984).

(1983a, 1984) found that his data could be fit better to a law involving a divergent free-energy barrier, $\ln\tau_{av}\propto T^{-z_{av}\nu}$ with $z_{av}\nu\approx 2$, McMillan (1983) found instead $\ln\tau_{av}\propto 1/T$, i.e., at low enough T one has a simple Arrhenius-type form with a finite free-energy barrier $\Delta F=\Delta E-T\Delta S(T)$, $\tau_{av}\propto\exp(\Delta F/T)$. This latter behavior, originally suggested by Morgenstern (1983a, 1983b), seems to find support in recent high-temperature series results (Reger and Zippelius, 1985). Another very interesting finding is the suggestion of McMillan (1983) that the "noise power spectrum" contributing in the relaxation functions decays with frequency f similar to "$1/f$-noise," that is, as $1/f^{1.28}$.

Young (1983c, 1984) was the first to present a large-scale simulation of the three-dimensional $\pm J$ spin glass. His data, taken on 64^3 lattices using observation times of up to 400 000 MCS/spin (10^5 MCS/spin were dropped off for equilibrating the system) are restricted to temperatures $T/J\geq 1.55$, due to the rapid increase of relaxation time with decreasing temperature. Unfortunately, the data cannot distinguish between a transition at zero temperature and a transition with finite T_c. This fact is already obvious from a comparison of his results for χ_{SG} with the Padé approximates of Palmer and Bantilan (1985; Fig. 92). Binder and Young (1984) assumed as a working hypothesis a zero-temperature transition and tried to extract estimates of the corresponding exponents from the Monte Carlo data. As an example, Fig. 101 shows log-log plots of spin glass susceptibility χ_{SG} and correlation length ξ_{SG} versus temperature. *If $T_c=0$ the data should fall on straight lines, in the asymptotic critical region of this transition, $T/J\ll 1$,* if the exponents $\gamma,\nu<\infty$. Since the *available data* are not from this region, but rather satisfy $T/J\geq 1.55$, there really is no reason to expect straight lines. If one assumes that the corrections to the critical behavior are not very large, however, one may hope that the critical behavior could be seen at least roughly for $T/J\approx 1$. In this spirit, in Fig. 101 a linear extrapolation of the available data to lower temperatures is attempted, with $\nu\approx 4$, $\gamma\approx 12$. Of course, it was already noted by Binder and Young (1984) that "the asymptotic critical region may not yet have been reached. In this case the effective exponents would increase further as the temperature decreases."

Ogielski and Morgenstern (1985) studied this problem further by use of a fast special-purpose computer (Figs. 70 and 102), simulating a lattice of size 32^3 (and sometimes 64^3) with Monte Carlo runs of up to 8×10^7 MCS/spin. Both the data on the relaxation time (Fig. 70) and those on the correlation length (Fig. 102) are con-

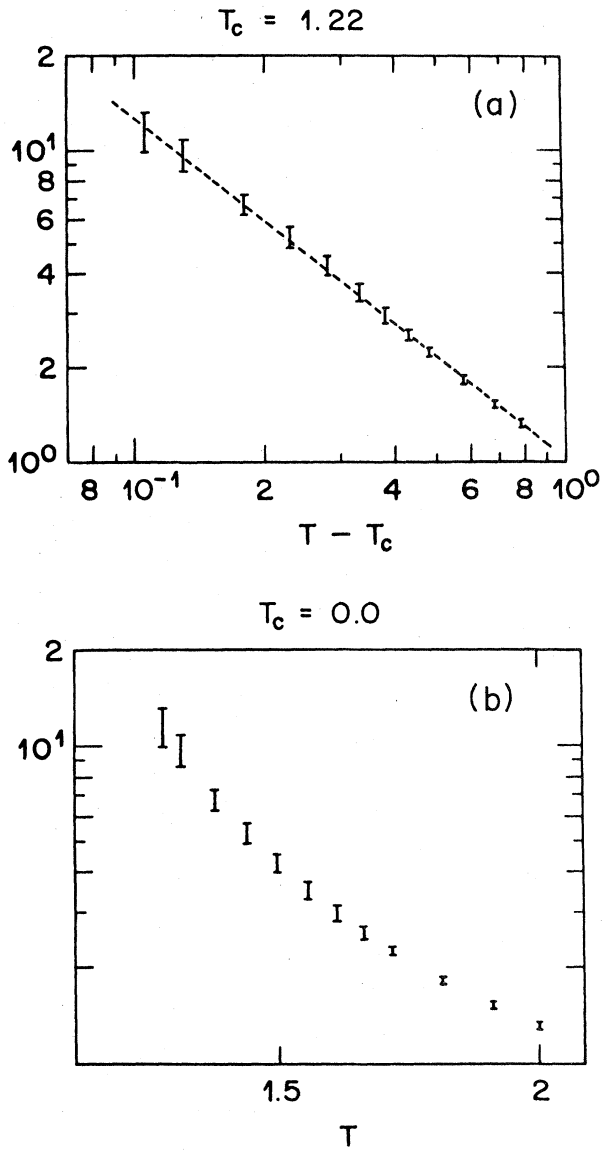

FIG. 102. Temperature dependence of the spin glass correlation length ξ_{SG} of the three-dimensional $\pm J$ model. Top shows a log-log plot vs $T-T_c$, assuming $T_c=1.22$ (temperature in units of J); bottom assumes $T=0$. From Ogielski and Morgenstern (1985).

sistent with a phase transition at $T_f/J\approx 1.22\pm 0.04$, with exponents $\nu\approx 1.12\pm 0.12$, $z\nu=6\pm 1$. If one instead assumes $T_f=0$, the log-log plot shows pronounced curvature in the analyzed temperature region, $1.3\lesssim k_BT/J\lesssim 2$. These data thus clearly show that if $T_f=0$ then the above estimates $\nu\approx 4$, $\gamma\approx 12$ underestimate the true exponents significantly. Ogielski and Morgenstern (1985) conclude that their evidence clearly favors a transition at the nonzero T_f quoted above. However, since all the data fitted to simple power laws are rather far off from the transition ($0.1\lesssim |T/T_f-1|\lesssim 0.7$), it is not so clear whether the evidence really is conclusive. As has been shown in Fig. 80, even some of the two-dimensional Monte Carlo data exhibit curvature on log-log plots where $T_f=0$ is assumed, and can be better fitted using a nonzero T_f as an additional adjustable parameter, but in this case this is clearly an artifact of the fit. In fact, Ogielski and Morgenstern (1985) support their claims by results for $[\langle S_iS_j\rangle_T^2]_{av}$, which tend rapidly to a finite value with increasing $|\mathbf{R}_i-\mathbf{R}_j|$ at $T/J=1.1$. They interpret this as evidence for a nonzero spin glass order parameter. Related to this observation they find a rather distinct change of behavior between the high-temperature region ($T/J\approx 1.3$) and "low" temperatures such as $T/J=1.1$, where they find that the lattice flips over coherently and thus behaves like a well-ordered phase. This observation, in the $T_f=0$ hypothesis, could be interpreted by the fact that the correlation length ξ has grown to a size distinctly larger than the lattice size $L=32$ at that temperature. Obviously, with Monte Carlo methods there is always a difficulty in principle of distinguishing between such possibilities; one can expect only a hint of what happens, and not a final proof.

Further evidence that suggests there may be a phase transition in the $\pm J$ model near $T_f/J=1.2$ comes from the finite-size scaling study of Bhatt and Young (1985). They study the probability distribution function $P_L(q)$ of the spin glass order parameter q defined by Eq. (3.84c) and its moments $\chi_{SG}=L^d\langle q^2\rangle$ and the ratio $g_L=(3-\langle q^4\rangle/\langle q^2\rangle^2)/2$, for sizes L ranging from $L=3$ to $L=20$. Now finite-size scaling implies (Binder, 1981)

$$g_L=\tilde{g}[L^{1/\nu}(T-T_f)],\quad L\to\infty,\ T\to T_c\ . \tag{5.38}$$

Thus plotting g_L vs T for various L should yield a family of curves intersecting in a common point at T_f. Locating such a point of common intersection enables one to find T_f without the need of fitting various parameters simultaneously. This approach works nicely for Ising ferromagnets (Binder, 1981) and for the Sherrington-Kirkpatrick model [Fig. 103(a)]. Of course, due to corrections to finite-size scaling the various intersection temperatures do not coincide precisely, but converge quickly to the exact T_f as the sizes of the systems used are increased. For $T<T_f$ we have $g_L>g_{L'}$ for $L>L'$, which is an indication of a nonzero order parameter. For the $\pm J$ model in $d=3$, on the other hand, the curves g_L for various L seem to merge at $T_f\approx 1.2$ and stay together at lower temperatures rather than truly intersect, at least within statistical error. In this spirit of Eq. (5.38), all temperatures $T<1.2$ behave as critical points, just as in the case of Kosterlitz-Thouless (1973) transitions in the two-dimensional ferromagnetic XY model. In this latter model, however, $\nu=\infty$, while in the present case the data seem to scale with $\nu=1.4$ [Fig. 103(c)]. The distribution $P_L(q)$ itself scales at $T_f=1.2$ with an exponent $\beta/\nu=0.36$. The behavior of Fig. 103(b) could be taken as evidence that $d=3$ is the lower critical dimension. Of course, one may again argue that the sizes L may be too small to show asymptotic behavior, or that with the present accuracy it is not yet resolved whether the g_L

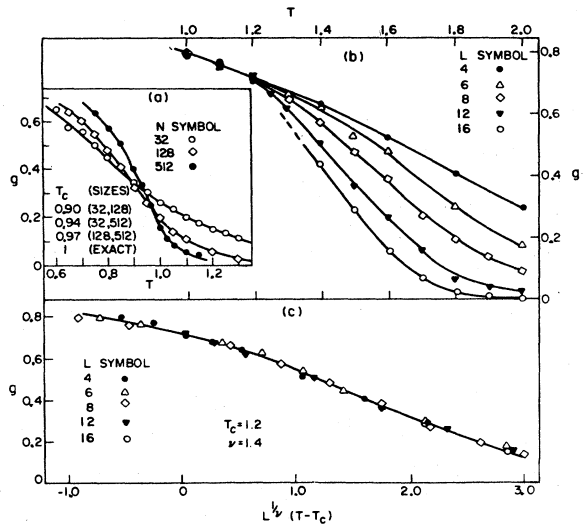

FIG. 103. (a) $g=(3-\langle q^4\rangle/\langle q^2\rangle^2)/2$ vs temperature for the Sherrington-Kirkpatrick infinite-range model for $N=32$, 128, and 512 spins. $0\le g\le 1$ and $g\to 0$ for $T>T_c=1$ as $N\to\infty$. Estimating T_c by intersection of pairs of curves gives results shown. (b) g for the short-range spin glass ($d=3$). At $T=1.2$, g is practically independent of size L (for $L=4,6,8,12$) and remains independent of L ($L=4,6,8$) at $T=1.1$ and 1.0. (c) scaling plot for g testing Eq. (5.38) with $\nu=1.4$. From Bhatt and Young (1985a).

curves slightly intersect (implying $d_l<3$) or slightly fail to merge at $T=1.2$ and merge only at $T_f=0$ (implying $d_l\ge 3$). But in any case a safe conclusion is that for $T\le 1.2$ the correlation length ξ_{SG} is at least very large, if not infinite, and so for practical purposes the system behaves in an ordered fashion for $T\le 1.2$. An interesting suggestion also is that "the lower critical dimension is close to three" (Bhatt and Young, 1985). In this context, it also is amusing to note that a $T_f=1.2$ in Fig. 101 practically coincides with the experimental freezing temperature shown there.

4. Scaling theory for spin glass transitions at zero temperature; and experiments revisited

In this section we briefly discuss the critical behavior that one expects to observe when one approaches a spin glass transition with $T_f=0$ (Binder, 1982b; Kinzel and Binder, 1983, 1984). This approach is appropriate for Ising spin glasses for $d=2$, and probably in $d=3$ for isotropic Heisenberg spin glasses. Even for $d=3$ Ising spin glasses it has been used as a working hypothesis (Binder and Young, 1984), and even though in the light of present evidence it may seem a less likely scenario, this possibility is not yet rigorously ruled out.

Now static scaling implies for the critical part of the magnetization of a spin glass in a magnetic field near its zero-temperature transition

$$\tilde{\Delta}(H,T)\equiv 1-TM/H=T^{\beta}\tilde{C}(HT^{-\Delta}),\quad H\to 0,\ T\to 0\ , \tag{5.39}$$

where β,Δ are two exponents. In order to observe temperature-independent plateaus of M/H as $T\to 0$ (Fig. 79), one must require that the order parameter exponent $\beta=0$. This result is not unexpected for the Gaussian model, of course, where the order parameter jumps from zero to unity as $T\to 0$; for the $\pm J$ model in two dimensions, however, there is no order even at $T=0$, and hence formally $\beta\neq 0$ must be kept.

A direct consequence of Eq. (5.39) is that $M(T\to 0)/H\propto H^{-1/\Delta}$. The Monte Carlo data are consistent with this behavior (Fig. 104), implying $1/\Delta\approx 0.28$. If one tries to interpret the experimentally observed plateaus (Fig. 21) in a way similar to that used in Fig. 104, one finds a rather different result, $1/\Delta\approx 0.03$. This may well be taken as evidence that $d_l<3$, in which case the plateaus should settle at a finite susceptibility $\chi(T\to 0)=$const as $H\to 0$, or that $d_l=3$ (in which case one expects $1/\Delta=0$). Another consequence for $d=2$ is that the "critical field" curve $H_{eq}(T)$, which separates the region of plateaus from the region of nonconstant magnetization, must behave as $H_{eq}(T)\propto T^{\Delta}$, with $\Delta\approx 3.5$. This behavior is also consistent with the data (Fig. 79). Binder and Kinzel (1983b) suggested that experimental data such as Fig. 23 could possibly be interpreted similarly—but a fit then implies a value $\Delta\approx 10^2$ [see Barbara and Malozemoff (1983)], which means either $T_f>0$, or $d_l\approx 3$, as noted already by Binder and Kinzel (1983b).

Next we turn to the direct test of the scaling behavior implied by Eq. (5.39) (see Fig. 105), which indeed seems

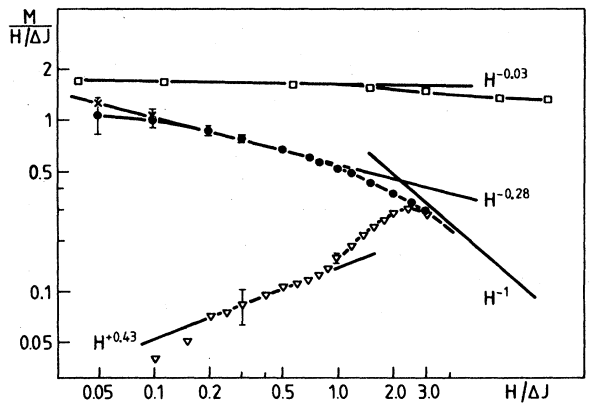

FIG. 104. Log-log plot of "susceptibility" $M(T\to 0)/(H/\Delta J)$ plotted vs field, for the two-dimensional nearest-neighbor Gaussian spin glass: ●, field-cooled magnetization for $|dT/dt|=2.5\times 10^{-4}\Delta J/k_B$ MCS; ×, magnetization for $|dT/dt|=6.25$ and $1.5\times 10^{-5}\Delta J/k_B$ MCS; □, experimental data of Monod and Bouchiat (1982) for *Ag*Mn with 10.6 at. % Mn (see Fig. 21), on arbitrary scales; △, the magnetization obtained from systems cooled to $t=0$ without a field. From Kinzel and Binder (1983).

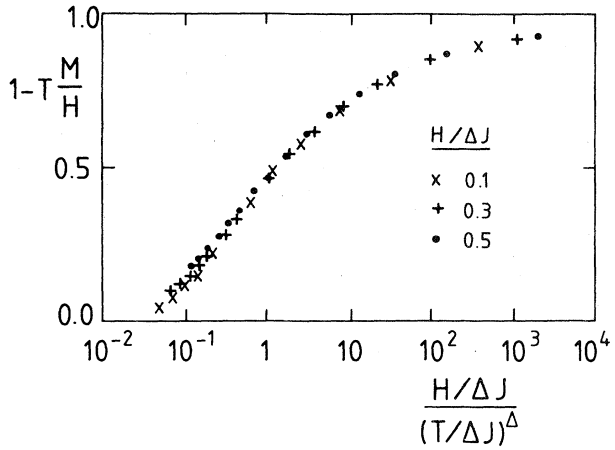

FIG. 105. Nonlinear part of the magnetization, $1-TM/H$, of the two-dimensional Gaussian Ising spin glass plotted as a function of field H, scaled by temperature T^{Δ} with $\Delta=3.5$. The data are from the range $0.1\geq H/\Delta J\leq 0.5$ and $0.1\leq T/\Delta J\leq 1$.

to be reasonably fulfilled. It turns out, however, that these Monte Carlo data of the two-dimensional Ising spin glass are not really very distinctive between a $T=0$ transition and a finite-temperature transition: Fig. 106 shows that nice scaling plots are also obtained (with different choices of the exponents β,γ, of course) if a nonzero T_f is

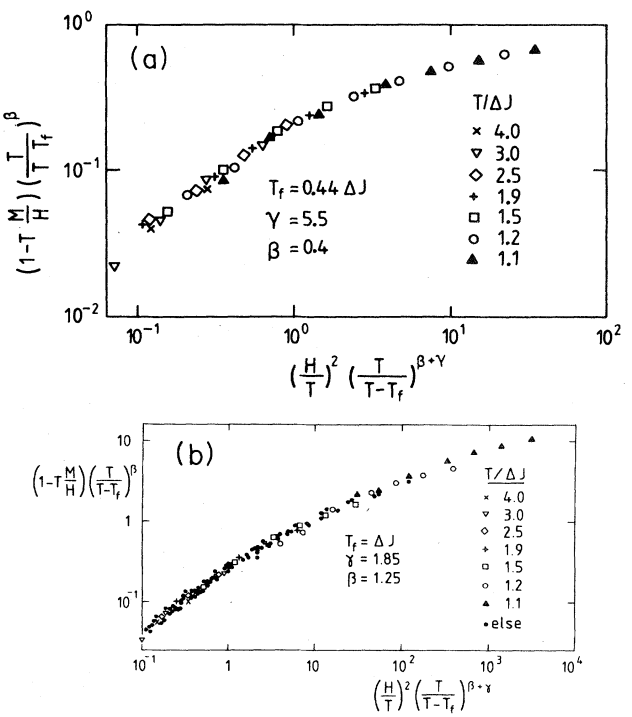

FIG. 106. Magnetization data of the two-dimensional Gaussian Ising spin glass (cf. Fig. 80) in a scaling representation appropriate for finite-temperature transitions, choosing arbitrarily (a) $T_f=0.44\Delta J$ and (b) $T_f=\Delta J$. The respective exponents γ,β resulted from least-squares fits. From Kinzel and Binder (1984).

assumed. For any T_f in the range $0\lesssim k_BT_f/\Delta J\leq 1$ a reasonable scaling fit is obtained, similar to experimental findings (Fig. 28). This has raised the suspicion (Binder and Kinzel, 1983b) that the experimental success of scaling (Fig. 28) might also be an artifact of fitting. On the other hand, there are no experimental data for which a fit to a zero-temperature scaling theory works as convincingly as in Fig. 105.

Finally, we discuss the nonlinear susceptibility χ_{SG} that follows from the relations

$$\tilde{\Delta}(H,T)=\frac{a_3}{3}(H/T)^2+O(H/T)^4\,,$$

$$a_3=3\chi_{SG}(T,H=0)-2\,,$$

and Eq. (5.39). This is $\chi_{SG}\propto T^{2-2\Delta}\approx T^{-5}$. Using $\chi_{SG}\propto\xi_{SG}^{2-\eta}\propto T^{-\nu(2-\eta)}=T^{-\gamma}$ we have the usual scaling relation between the susceptibility and correlation length exponents, $\gamma=\nu(2-\eta)$. Note that with the further scaling relation $\beta=(\nu/2)(d-2+\eta)$ the condition that $\beta=0$ for the two-dimensional Gaussian model also implies $\eta=0$, i.e., then we simply have $2(\Delta-1)=\gamma=2\nu$, $\nu=\Delta-1\approx 2.5$. This value is in reasonable agreement with the estimates from Monte Carlo and transfer matrix calculations, but distinctly smaller than the estimate from the domain-wall renormalization group, $\nu\approx 3.5$, discussed in the previous section. If this last estimate is correct, it would also imply $\Delta\approx 4.5$, and then the exponents used in Figs. 104 and 105 would have the character of "effective exponents" only, which would change when one considers smaller fields and temperatures than those so far available. In spite of this uncertainty about the values of the exponents, this static scaling phenomenology of $T=0$ phase transitions, which also can be derived from real-space renormalization-group arguments (McMillan, 1984d), seems generally accepted.

More controversial, however, is the extension of this description to dynamics. Binder and Kinzel (1983b) and Kinzel and Binder (1984) have suggested that a typical relaxation time in the system is controlled by the collective reorientation of spins in a region the size of which is given by the correlation length ξ_{SG}. Near a finite-temperature transition the various ordered states in which a correlated region of size ξ_{SG}^d can exist are mutually accessible by small-amplitude fluctuations of a coarse-grained local order parameter, and hence the relaxation time exhibits "ordinary" critical slowing down, $\tau\propto\xi_{SG}^z\propto(T-T_f)^{-\nu z}$, z being a dynamic exponent (Hohenberg and Halperin, 1977). Near a transition at $T=0$, on the other hand, the local spin alignment is strong, and fluctuations relating the various locally ordered states in anisotropic systems are "walls"; nucleating these walls requires thermal activation, and hence it is natural to expect that the free-energy barrier ΔF is also controlled by ξ_{SG},

$$\ln(\tau/\tau_0)\propto\Delta F(\xi_{SG})/T\propto\xi_{SG}^{z-1/\nu}/T\propto T^{-z\nu}\,. \qquad (5.40)$$

The extension of this idea to include a magnetic field is straightforward, since ξ_{SG} scales as the nonlinear magnet-

ization, Eq. (5.39), namely $\xi_{SG}(T,H)=T^{-\nu}\xi(HT^{-\Delta})$, and hence

$$\ln[\tau(T,H)/\tau_0]=T^{-z\nu}f(HT^{-\Delta}) , \tag{5.41}$$

where f is some scaling function. Kinzel and Binder (1984) have made phenomenological attempts to relate z again to static exponents and to estimate f, but the validity of this attempt is rather uncertain. Equation (5.40) is not generally accepted for $T=0$ transitions: from a real-space renormalization-group procedure McMillan (1984d) found that for $T\to 0$ and $d<d_l$ the relaxation time follows simple Arrhenius behavior. Only for $d=d_l$ did he find $\ln\xi_{SG}\propto T^{-2}$ (i.e., $\nu=\infty$) and then also $\ln(\tau/\tau_0)\propto\ln\xi_{SG}/T\propto T^{-3}$. In Eqs. (5.40) and (5.41) (which describe the situation $d<d_l$), this Arrhenius behavior means $z\nu=1$. Equation (5.40) has important consequences for the frequency dependence of the dynamic freezing temperature $T_f(\omega)$, which from the condition $\omega\tau[T_f(\omega),H=0]\equiv 1$ is predicted to be

$$T_f(\omega)\propto|\ln(\omega\tau_0)|^{-1/z\nu} . \tag{5.42}$$

While the Arrhenius law ($z\nu=1$) would predict a linear variation in a plot of T_f^{-1} vs $\ln(\omega\tau_0)$, Eq. (5.42) predicts a nonlinear variation of such a plot, as experimentally observed (Fig. 9). Of course, alternative interpretations of the observed curvature (Fig. 9) would be the familiar Vogel-Fulcher law [Eqs. (2.14) and (2.15a)] or a crossover from Arrhenius behavior to standard critical slowing down [$T\propto(T-T_f)^{-\nu z}$] associated with an equilibrium phase transition. Figure 107 shows that Eq. (5.42) is compatible with the data of Fig. 9 if one chooses $z\nu\approx 4$, a value consistent with the Monte Carlo simulations presented by Binder and Young (1984). [The more precise recent results of Ogielski and Morgenstern (1985), Fig. 70, are more compatible with Eq. (5.40) with $z\nu\approx 3.2$, however, or with an ordinary critical slowing down with $T_f/J\cong 1.2$, as discussed above.]

Very recently Feigelman and Ioffe (1985) suggested an exponential divergence of the maximum relaxation time of an Ising spin glass at a finite transition temperature, $\tau_{max}\propto(T-T_f)^6\exp[C(T-T_f)^{-1/4}]$, applying concepts from localization theory.

A serious difficulty with the interpretation of the frequency dependence of $T_f(\omega)$ via Eq. (5.42) arises, however, when one considers the logarithmic derivative defined in Eq. (2.9), or, equivalently,

$$\frac{1}{T_f}\frac{\Delta T_f}{\Delta\log_{10}\omega}=\frac{1}{z\nu}\frac{1}{\log_{10}(1/\omega\tau_0)} . \tag{5.43}$$

For RKKY spin glasses this quantity seems to be nonuniversal, varying from about $\frac{1}{50}$ in *Ni*Mn to around $\frac{1}{200}$ in *Cu*Mn, *Ag*Mn, and *Au*Mn, as discussed in Sec. II. With reasonable values of $\omega\tau_0$, e.g., $\omega\tau_0\approx 10^{-10}$, one would need $z\nu\approx 20$ for *Cu*Mn, in serious disagreement with the above estimates for $z\nu$.

The scaling behavior of the relaxation time as a function of the magnetic field [Eq. (5.41)] has also been checked, using the two-dimensional Ising data of Young

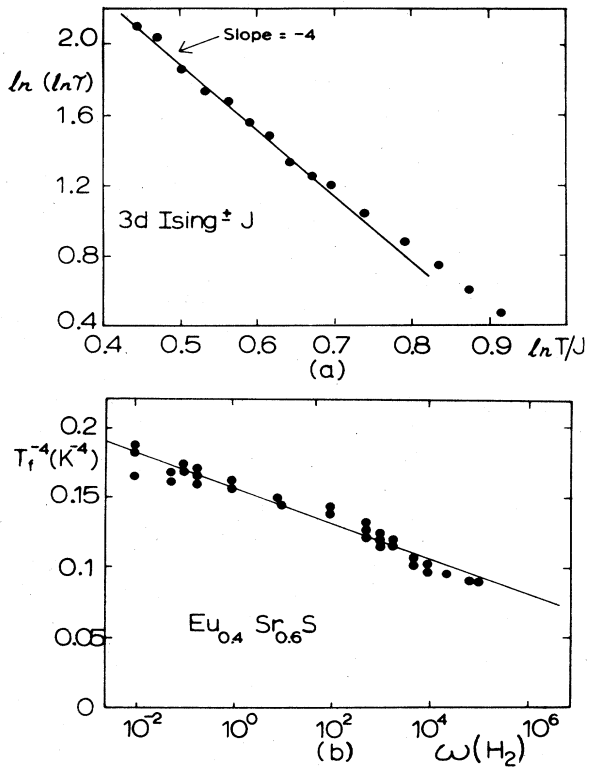

FIG. 107. (a) Log-log plot of $\ln\tau(T,H=0)$ vs T/J for the three-dimensional $\pm J$ Ising spin glass, using Monte Carlo data of Young (1984). (b) Experimental data on $Eu_{0.4}Sr_{0.6}S$ due to Ferré *et al.* (1981; see Fig. 9), replotted as $[T_f(\omega)]^{-4}$ vs $\ln\omega$.

(1983a) displayed in Fig. 100 and $\Delta=\gamma/2+1=1+\nu(1-\eta/2)$ with $\nu\approx 2$–2.5, $\eta\approx 0.4$, i.e., $\Delta\approx 2.8$ (Fig. 108). The data indeed scale reasonably well. Since Fig. 100 looks qualitatively similar to experimental data (Fig. 25), it is tempting to try a similar scaling with the experimental data, and this is also tried in Fig. 108, using $\nu\approx 4$, $\eta=-1$ (this means for the correlation function $[\langle S_iS_j\rangle_T^2]_{av}\propto\exp(-r/\xi_{SG})$, i.e., a simple exponential decay and no power-law prefactor near $T\to 0$) and hence $\Delta=7$. There is considerable scatter of the data points left, however, and thus it is not clear that Eq. (5.41) works for three-dimensional real systems. This problem was carefully studied further by Bontemps *et al.* (1984). They also studied $T_f(H,\omega)$ lines for $Eu_{0.4}Sr_{0.6}S$ for a larger range of frequencies, from $\omega=10^5$ Hz to $\omega=10^{-2}$ Hz, and interpreted the data both with Eqs. (5.40) and (5.41) and with a corresponding dynamic scaling hypothesis appropriate for finite-temperature transitions,

$$\tau/\tau_0\propto\left[\frac{T-T_f}{T_f}\right]^{-z\nu}\tilde{f}\left[\frac{H}{T}\left[\frac{T-T_f}{T_f}\right]^{-\Delta}\right] , \tag{5.44}$$

where $\tilde{f}$ is another scaling function. Obviously, if an equilibrium transition exists, it is Eq. (5.44) by which data for $T_f(H,\omega)$ should be analyzed, rather than the traditional approach to scaling the data as $T_f(H,\omega)/T_f(0,\omega)$

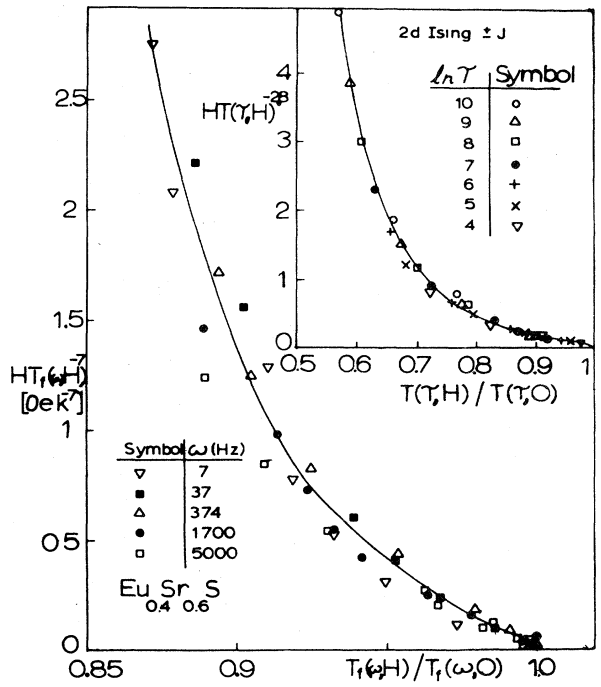

FIG. 108. Scaling plot of the curves in the H-T plane where freezing occurs at a fixed frequency ω for $Eu_{0.4}Sr_{0.6}S$ [data due to Paulsen *et al.* (1984), as presented in Fig. 25]. The inset shows a similar scaling plot of lines of constant relaxation time for the $d=2$, $\pm J$ model due to Young (1983a; see Fig. 100). $T(\tau,H)$ is the temperature at which the relaxation time reaches a given value τ for the chosen field H. From Binder and Young (1984).

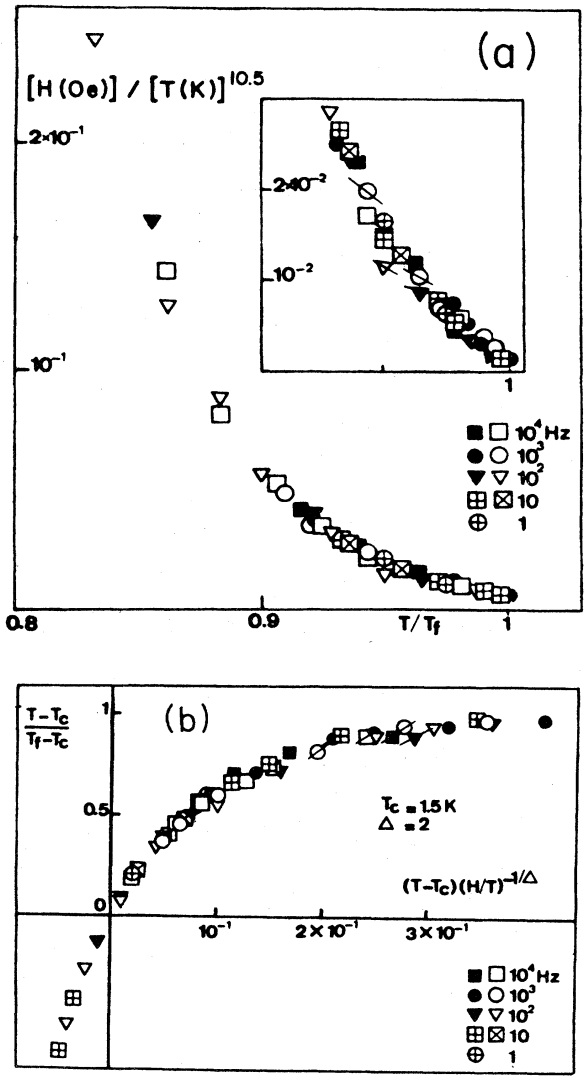

FIG. 109. (a) Scaling of field-temperature $T_f(H,\omega)$ lines for the case $T_c=0$. The inset shows the low-field points with an extended scale. A typical experimental error bar is indicated on a few points. (b) Scaling of the field-temperature lines for the case $T_c=1.5$ K, with $\Delta=2$. From Bontemps *et al.* (1984).

and taking the resulting curve as an estimate of the Almeida-Thouless (1978) line (Salamon and Tholence, 1982; Bontemps *et al.*, 1983). This work, in our opinion, is a very valuable effort to distinguish experimentally between the two conflicting views of a zero-temperature transition and a finite-temperature transition. Unfortunately, the data are consistent with both views: if $T_f=0$ then (Bontemps *et al.*, 1984)

$$\tau_0 \simeq 10^{-5} \text{ sec}, \quad z\nu = 8.0 \pm 0.5, \quad \Delta = 10.5 \pm 2.5 , \tag{5.45a}$$

while otherwise

$$T_f = 1.5 \text{ K}, \quad \tau_0 \approx 2.10^{-7} \text{ sec} ,$$
$$z\nu \approx 7.2 \pm 0.5, \quad \Delta = 2.0 \pm 0.2 . \tag{5.45b}$$

The fits are of comparable quality (Fig. 109), but the value of τ_0 in the case of $T_f=1.5$ K is physically more reasonable. Bontemps *et al.* (1984) take this fact as the main indication that Eq. (5.44) rather than Eq. (5.41) is valid. If $T_f=0$, then γ (and hence Δ) as well as $z\nu$ must be significantly larger than the values tentatively suggested by Binder and Young (1984). While the Monte Carlo data of Ogielski and Morgenstern (1985) also would imply a value of (at least) $\nu \approx 7$ if $T_f=0$, their value of $z\nu$ would strongly disagree with Eq. (5.45a). On the other hand, if $T_f \neq 0$, their result for $z\nu$ ($z\nu \approx 6 \pm 1$) is consistent with Eq. (5.45b). Furthermore, the data of Bhatt and Young (1985) which imply $\gamma \approx 3.2$, $\beta \approx 0.5$, are consistent with the value of $\Delta=(\gamma+\beta)/2$ in Eq. (5.45b). Ogielski and Morgenstern (1985) give exponent values from which one finds $\Delta \approx 1.4$; given the uncertainties, this is not incompatible with Eq. (5.45b).

It is also interesting to note that several careful studies of the nonlinear susceptibility on different systems (Malozemoff *et al.*, 1982; Omari *et al.*, 1983; Beauvillain, Chappert, and Renard, 1984; see also the summary of results in Tholence, 1984) find γ fairly close to three. Therefore critical exponents obtained from Monte Carlo simulations for both static and dynamic quantities agree

reasonably well with experiment, if one assumes a finite T_f. It is rather surprising that highly simplified Ising models give such a good account of real systems which are essentially Heisenberg but with some anisotropy, Dzyaloshinskii-Moriya, dipole-dipole, etc. Isotropic Heisenberg models probably have $T_f=0$ in $d=3$ (Banavar and Cieplak, 1982a; McMillan, 1985b; Morris *et al.*, 1986) so (generally weak) anisotropy is probably necessary to get the apparent finite T_f. We must emphasize, however, that both simulations and experiment may only be determining *effective* exponents and that the true T_f may be lower than the value in the fit, and just conceivably $T_f=0$. We also note that the Monte Carlo data of Bhatt and Young (1985) and Ogielski and Morgenstern (1985) provide a plausible explanation of why the observed freezing lines in the H-T plane [$\delta T_f \propto H^{\psi}$ for $H\to 0$ (Chamberlin *et al.*, 1982; Yeshurun *et al.*, 1982; Bontemps *et al.*, 1983)] give values of ψ fairly close to what is predicted for the de Almeida—Thouless (1978) transition line in mean-field theory, where $\psi_{\rm AT}=\frac{2}{3}$. The AT value is only correct for short-range models if $d>8$ (Green *et al.*, 1983; Fisher and Sompolinsky, 1985; see Sec. V.A.1), and for $6<d<8$ one has $\psi_{\rm AT}=4/(d-2)$, so in $d=6$, $\psi_{\rm AT}=1=\Delta^{-1}$. For $d<6$, the relation $\psi=\Delta^{-1}$ should remain valid (Fisher and Sompolinsky; 1985; see also Malozemoff *et al.*, 1983), and the exponents of Bhatt and Young (1985) and Ogielski and Morgenstern (1985) give $\psi\approx 0.54$ and 0.71, respectively, which are consistent with experiment, given the uncertainties. We therefore see that ψ having roughly the AT value is a coincidence arising because of cancellation from two completely different mechanisms. Note also that the recent careful work of Bouchiat (1986) on AgMn spin glasses, in which data closer to T_f are taken, yielded exponent estimates $\gamma\approx 2.2$, $\beta\approx 1.0$ significantly different from the numbers quoted above.

To sum up this section, we see that the zero-temperature scaling theory does not fit the data as well as a theory with finite T_f, but this might possibly be due to the inaccessibility of the asymptotic scaling regime if $T_f=0$.

5. Other approaches and discussion

We have tried to summarize, in the last few subsections, most of the theoretical work on the lower critical dimension. In addition, Sec. V.A discussed results from renormalization group techniques, and Sec. IV.G looked at fluctuations about mean-field theory. We shall also refer to approaches based on Anderson localization in Sec. VI.E.2 and just note here that two such calculations (Feigelman and Ioffe, 1984, 1985; Hertz, 1985) have argued for $d_l=2$. Section VI.A will refer to calculations by Dzyaloshinskii and Volovik (1978) which predict that $d_l=4$ for Heisenberg models with short-range interactions but that $d_l=3$ if the interactions are RKKY. We also note that Schuster (1980) has found $d_l=4$ for short-range vector spin systems by arguments based on Bogoliubov's inequality.

Since we define d_l by the lowest dimension in which $\chi_{\rm SG}$ diverges at a finite T_f, and since, at T_f,

$$[\langle S_i S_j\rangle_T^2]_{\rm av}\propto |\mathbf{R}_i-\mathbf{R}_j|^{-(d-2+\eta)} \tag{5.46}$$

a necessary condition for a transition is

$$d-2+\eta(d)\geq 0\,, \tag{5.47}$$

in order that the correlations do not increase with distance. If one knew $\eta(d)$ reliably one could presumably estimate d_l by finding where Eq. (5.47) is satisfied as an equality, but with only three terms in an expansion about $d=6$ (Green, 1985) and doubts about the summability of the series (Sec. V.A.1) this is not yet feasible. Note that we do not feel it is necessary to solve the problem of fluctuations about the Parisi solution *below* T_f (see Sec. IV.G) to estimate d_l. Knowledge of this tells one whether the low-temperature phase has the Parisi replica symmetry-breaking form but is not necessary, we believe, to estimate whether or not T_f is finite.

It has already been pointed out at the beginning of this section that the possibility of a dimension, or even a range of dimensions ($d_l<d<d_{l'}$), where there is a phase transition without spin glass order cannot be ruled out at present. Feigelman and Ioffe (1984) and Shapir (1984) had previously suggested there may be an intermediate dimension $d_{l'}$, where $d_l<d_{l'}<d_u$ such that a different type of phase occurs for $d_l<d<d_{l'}$. Shapir (1984) presents a number of arguments, which we consider too speculative to be described here, that $d_l=2, d_{l'}=4$, and $\nu=1$ for $d=3$. While the basic idea that each correlated region has many low-lying states is in common with many other approaches (e.g., Kinzel and Binder, 1984), the work of Shapir (1984) relies heavily on the assumption that the exponents of this "entropy-dominated transition" are identical to those of some pure model at a different dimensionality $\bar{d}$, an assumption for which we do not see much physical justification.

All approaches are united in predicting a zero-temperature transition in two-dimensional Ising systems. Probably $\xi_{\rm SG}$ diverges with a power as $T\to 0$, which would imply that $d_l>2$, but we agree with Huse and Morgenstern (1985) in noting that the asymptotic critical region $T\to 0$ (or $L\to\infty$) has probably not been reached, and hence the possibility that $\xi_{\rm SG}$ diverges faster than any power of T cannot be totally excluded. This latter scenario would imply $d_l=2$.

We have seen that the consensus of many recent experiments, Monte Carlo simulations, real-space renormalization-group methods, and domain-wall energy calculations is that the three-dimensional Ising spin glass is more likely to have a transition with a finite temperature than a zero-temperature transition. The nature of the singularity at T_f, as well as the nature of the low-temperature phase, is not yet well established: none of the results really probe the asymptotic critical region [very small values of $(T-T_f)/T_f$, H, ω, or $1/L$ in the finite-

size analysis]. In any case the implication of the results is that d_l is somewhat below 3. The once widely held belief that $d_l=4$ now seems improbable for Ising spin glasses.

For isotropic Heisenberg systems, Banavar and Cieplak (1982a, 1982b), McMillan (1985b), Morris *et al.* (1986), and Olive *et al.* (1986) have argued that d_l is greater than three. In fact it seems probable that $d_l \approx 4$ for vector spin glasses with short-range interactions, though it is suggested (Dzyaloshinskii and Volovik, 1978; Bray *et al.*, 1986) that $d_l=3$ with RKKY couplings. However, there is moderate agreement over a wide temperature range between the effective critical exponents for γ, Δ, and $z_{\rm av}\nu$ observed in experiments, which are generally on *Heisenberg* systems, with those observed in Monte Carlo simulations on *Ising* models. This may be due to the (relatively small) amount of anisotropy in these predominantly Heisenberg systems; but then one has to understand why the agreement is over such a wide range of temperature above T_f, instead of only a very narrow region of $T-T_f$ where crossover from Heisenberg to anisotropic behavior occurs, as expected if anisotropy is needed to get a transition.

In this subsection we have discussed the value of d_l for the transition in zero field. It is also interesting to ask about the AT transition line in a magnetic field, which separates the region of the H-T plane with many states (see Sec. IV.E) from the region with a single state (two states in zero field). This line will disappear if the spin glass has only a single thermodynamic state (at least in nonzero field). It has been suggested (Bray and Moore, 1984a; McMillan, 1984d) that the critical dimension where this happens is higher than d_l and perhaps as large as 6 (Bray and Roberts, 1980; Moore and Bray, 1985). In fact, Fisher and Huse (1986) propose that the AT line is an artifact of the infinite-range interactions in the SK model and may not occur for any short-range system. However, so far this question has been less carefully studied than the value of d_l in zero field.

VI. OTHER THEORETICAL APPROACHES

In this section we discuss a collection of miscellaneous topics. Our treatment will be less systematic than in Secs. III—V and confined to general ideas, omitting details and technicalities.

A. Gauge theories of spin glasses

In Sec. III.F.2 the importance of frustration (Toulouse, 1977) was emphasized. Frustration is a "serious" disorder, which is invariant under the "gauge transformation," Eq. (3.122) and can be characterized by the frustration function ϕ_f in Eq. (3.125b) for a system with nearest-neighbor interactions in which $J_{ij}=\pm J$. If $\phi_f=-1$ for some closed paths in the lattice, then one has frustration and hence serious disorder. Further discussion of frustration is given in Toulouse (1977, 1979), Fradkin *et al.* (1978), and Sec. III.F.2.

It must be emphasized that the gauge transformations in Eq. (3.122), though formally rather similar to gauge transformations in quantum electrodynamics (see, for example, Sec. III.F.2) and other gauge theories, leads to different predictions because the J_{ij}'s are quenched random variables. In gauge field theories, the gauge variables analogous to the J_{ij} are statistical variables that can fluctuate, and gauge invariance indicates a great degeneracy of the states. By contrast, the gauge transformation in spin glasses relates the partition function and correlation functions of *different* systems, but does not directly give information about the degeneracy of any given system. Nonetheless, as we shall now see, there have been attempts to reexpress the spin glass problem as a genuine gauge theory in which both spins and gauge variables may fluctuate.

Clearly much of the difficulty in the spin glass problem stems from having quenched disorder as well as frustration, and one might ask whether the essential physics is captured by frustration alone. One approach to this problem, periodic frustrated models, is discussed in Sec. VI.E. Here, as our first example of a spin glass gauge theory, we shall discuss another nonrandom frustrated model, an annealed spin glass (Toulouse and Vannimenus, 1980).

Consider a nearest-neighbor model in which $J_{ij}=\pm 1$, say, in suitable units, with equal probability on a d-dimensional simple cubic lattice. Clearly one has

$$[\phi_f]_{\rm av}=[J_{ij}J_{jk}J_{kl}J_{li}]_{\rm av}=0\ , \tag{6.1}$$

where the J's form the four sides of an elementary square or "plaquette." In the annealed model one allows the J's to fluctuate and applies constraint (6.1) on average, in order to imitate a spin glass as closely as possible, so that

$$\langle J_{ij}J_{jk}J_{kl}J_{li}\rangle_T=0\ . \tag{6.2}$$

To ensure that Eq. (6.2) is satisfied, one adds a Lagrange multiplier to the Hamiltonian, so

$$-\beta\mathscr{H}_{ann}=\beta\sum_{\langle i,j\rangle}J_{ij}S_iS_j+\beta_P\sum_P JJJJ\ , \tag{6.3}$$

where the last term runs over plaquettes. The partition function is given by (Toulouse and Vannimenus, 1980)

$$Z_{ann}=\frac{1}{2^{N_L}}\sum_{\{J_{ij}=\pm1\}}\sum_{\{S_i=\pm1\}}e^{-\beta\mathscr{H}_{ann}}\ , \tag{6.4}$$

where N_L is the number of bonds in the system. The frustration constraint (6.3) is satisfied along a line in the β-β_P phase diagram where β_p is *negative*. The annealed spin glass model defined by Eqs. (6.3) and (6.4) is a Z_2 lattice gauge theory, studied in the context of lattice field theories (e.g., Balian *et al.*, 1974), though generally for positive β_p. Note that Eq. (6.3) does indeed describe a gauge theory because $\mathscr{H}_{ann}$ is invariant under the gauge transformation, Eq. (3.122).

In this approach a spin glass transition occurs if the line in the β-β_P plane where Eq. (6.2) is satisfied crosses a transition line. However, Monte Carlo simulations show that this does not happen in $d=2,3,4$ (Bhanot and

Creutz, 1980) or $d=6$ (Aeppli and Bhanot, 1981), and Aeppli and Bhanot have speculated that the same may be true in any dimension. Such behavior is different from quenched models in which there is quite probably a transition in $d=3$ (see Sec. V.E) and where beyond reasonable doubt one occurs in $d=6$. Physically the difference between the quenched and this annealed model is that the latter has correlations between the J_{ij}'s and between frustrated plaquettes, whereas the J_{ij}'s are independent in the quenched case. Although this difference may appear slight, it apparently leads to different behavior.

A gauge theory has considerable degeneracy. For instance, in the Z_2 theory described by Eq. (6.3), one can invert a single spin, $S_i \rightarrow -S_i$, in one of the ground states to obtain another ground state, provided the bonds J_{ij} connected to site i also change sign. Such a high degeneracy does not exist in spin glasses, where the J_{ij} are quenched and so cannot relax. Nonetheless it is clear that substantial degeneracy does occur, so that if we do not invert just a single spin but (thinking now of a Heisenberg model) gradually invert the spin direction over a large distance, the expected leading contribution to the energy change may vanish. Indeed Sompolinsky *et al.* (1984) have shown that the equilibrium exchange stiffness of a Heisenberg spin glass is zero in mean-field theory. This line of reasoning led Dzyaloshinskii and Volovik (1978) and Dzyaloshinskii (1980) to propose that *on large length scales* a spin glass may behave like a gauge theory, and they investigate the consequences of this assumption in some detail for a Heisenberg model. The gauge symmetry is SO(3), so a non-Abelian gauge theory is set up (Yang and Mills, 1954). The basic constituents are a spin field $\mathbf{S}(\mathbf{r})$ of unit length and the "density of dislocations" $\rho_i(\mathbf{r})$, $i=x$, y, or z. If one computes the energy needed to impose a gradual twist on the $\mathbf{S}$ field, then the contribution with the lowest powers of the gradients (the exchange stiffness) "can be made equal to zero by an appropriate rotation of the disclination network" (Dzyaloshinskii and Volovik, 1978). Hence the energy of a twist of wave vector q does not vary as q^2 but as higher power of q. Dzyaloshinskii and Volovik (1978) argue that for short-range interactions the energy goes like q^4, but they obtain a different result, q^3, for RKKY systems, this being the Fourier transform of the square of the RKKY interaction, $J_{ij}^2 \propto |\mathbf{R}_i-\mathbf{R}_j|^{-6}$. Standard spin-wave-like arguments predict that the power of q is just the lowest dimension that can sustain spin glass order. Hence Dzyaloshinskii and Volovik (1980) find that this lower critical dimension is four for short-range Heisenberg spin glasses but three for RKKY models. This conclusion is in contrast to that of Ueno and Okamoto (1981), who argue that the RKKY interaction is effectively short range, but is supported by recent scaling arguments of Bray *et al.* (1986). From the gauge theory approach one can obtain equations of motion whose normal mode spectrum includes (Dzyaloshinskii and Volovik, 1978) not only hydrodynamic spin-wave excitations (Halperin and Saslow, 1977), discussed in the next section, but also "localized zero-gap degrees of freedom connected with a system of uniformly distributed disclinations" (Volovik and Dzyaloshinskii, 1978). Quenched disorder does not appear *explicitly* in this approach but is there *implicitly* because both disorder and frustration are needed to get the degeneracy that makes possible an analogy with gauge theories.

A different gauge theory of spin glasses which *does* include quenched disorder has been proposed by Hertz (1978). To see the motivation for his model we consider a Ginzburg-Landau Hamiltonian for an XY ferromagnet or antiferromagnet, i.e.,

$$\mathscr{H}_{\text{eff}}=\tfrac{1}{2}\int d^dx\{r_0|\phi(x)|^2+u|\phi(x)|^4 + |[\nabla-i\mathbf{Q}(\mathbf{x})]\phi(\mathbf{x})|^2\}, \tag{6.5}$$

where $\phi(x)$ is a complex field and $\mathbf{Q}$ is the wave vector of the ordered state, so $\mathbf{Q}=0$ for a ferromagnet and $\mathbf{Q}\neq 0$ for an antiferromagnet. Now a spin glass is a random mixture of ferromagnetic and antiferromagnetic interactions, so Hertz proposed that this could be represented by a random choice for $\mathbf{Q}(x)$. Since no value of $\mathbf{Q}$ is to be preferred over any other, the distribution of $\mathbf{Q}$ depends only on its space derivatives. One can show that the longitudinal components [i.e., those in the Fourier decomposition where $\mathbf{Q}(\mathbf{k})\propto\mathbf{k}$] give rise to trivial Mattis-type disorder, which can be gauged away (Hertz, 1978). Defining the gauge-invariant quantity

$$F_{\mu\nu}(x)=\partial_\mu Q_\nu(x)-\partial_\nu Q_\mu(x), \tag{6.6}$$

Hertz takes the distribution of $Q(x)$ to be

$$P[\mathbf{Q}]\propto\exp\left[\frac{-1}{2f}\int dx\sum_{\mu\nu}F_{\mu\nu}^2(x)\right]. \tag{6.7}$$

The parameter f measures the degree of frustration in the system. If f is small, there is a weak amount of frustration everywhere. This contrasts with a $\pm J$ lattice model, in which there is a small fraction of negative bonds where the frustration is strong on a small concentration of squares and zero on the rest. The notation in Eqs. (6.6) and (6.7) is borrowed from electromagnetism, and indeed the model is almost the Ginzburg-Landau model of a superconductor, except that the gauge field (electromagnetic field) is quenched. Hertz (1978) also shows that the equivalent Heisenberg model is an SO(3) Yang-Mills (1954) gauge theory with quenched gauge fields.

Having set up the model, one can apply renormalization-group techniques (see Sec. V.A.1), assuming f is small. For $d<4$, f is a relevant variable (Hertz, 1978), so it changes the critical behavior, but it has not been possible to determine the new critical properties or even whether there is a transition. It is unfortunate that so little is known about what seems to us a very interesting model.

B. Hydrodynamic theory of spin waves

A hydrodynamic theory (Halperin and Hohenberg, 1969) has been very successful in determining the frequency and damping of spin waves in ferromagnets, antiferromagnets, and other ordered magnetic states with a *collinear* spin structure. One uses symmetry arguments and conservation laws to determine the form of the low-frequency long-wavelength excitations, without recourse to a detailed microscopic analysis. Given the difficulties with microscopic theories of spin glasses, it is very tempting to extend the hydrodynamic approach to Heisenberg spin glasses, which of course form a *noncollinear* spin structure at low temperatures. We first discuss the simple case of a completely isotropic spin glass, then include a nonzero magnetization, and finally describe a spin glass with weak anisotropy, which leads to interesting predictions for torque measurements and ESR frequencies.

The basic ideas of a spin-wave theory of hydrodynamics were first discussed by Halperin and Saslow (1977) and Andreev (1978). If the system is cooled well below the freezing temperature T_f, the system will be found in one of the many valleys or metastable states that all have equivalent macroscopic properties. At sufficiently low temperatures, one assumes, the system will not fluctuate over a barrier to a different valley on the time scale of an experiment, so one considers only small deviations from the metastable state. In a large system with isotropic Heisenberg interactions, there appear to be the same number of spins pointing to every direction on the unit sphere (e.g., Walker and Walstedt, 1977), so on average the state is isotropic in spin space. Starting from a given state, one obtains another metastable state with the same free energy by a rotation of the spins. It is important to note the difference from a collinear spin structure in which a particular ordered state is specified by the direction of a unit vector, which needs two parameters. For a spin glass, and other systems with a noncollinear ground state, one specifies the state by a rigid body rotation away from an arbitrarily chosen state, and this requires three parameters. Alternatively, one can say that the state is characterized by a *triad* of three orthonormal vectors. We shall see that, as a result, there are three spin-wave branches for spin glasses, as opposed to two for collinear spin structures (Halperin and Hohenberg, 1969). Toulouse (1979) has classified the defect structures that occur when the "manifold of internal states" is specified by a rigid body rotation. Henley (1984a, 1984b) has discussed these defect structures further and studied them numerically. We denote by θ the magnitude and direction of such a uniform rotation. How one defines θ microscopically is discussed, for example, by Halperin and Saslow (1977) for the case of small rotations.

It is of great interest to discuss nonequilibrium states in which the rotation varies slowly in space. This will increase the rotation free energy, even for an isotropic Heisenberg system, by an amount assumed to be proportional to $(\nabla\theta)^2$ at long wavelengths, i.e.,

$$\Delta F[\theta]=\tfrac{1}{2}\rho_s\int(\nabla\theta)^2 d^3r\ . \tag{6.8}$$

We shall see that long-wavelength fluctuations of θ take place at low frequencies because the energy cost is small. In the hydrodynamic approach one must include *all* slow modes and therefore ask whether other degrees of freedom possess low-energy excitations. For an isotropic Heisenberg model, fluctuations of the total magnetization density $\mathbf{M}(\mathbf{r})$ will be at low frequency and must be included. Note that $\mathbf{M}$ and θ are slow modes for different reasons: $\mathbf{M}$ because it is a conserved quantity, and θ because the particular metastable state breaks (locally) rotational invariance of the Hamiltonian. Including both variables, the change in free energy is given by

$$\Delta F[\theta,\mathbf{M}]=\tfrac{1}{2}\int[\chi^{-1}\mathbf{M}^2+\rho_s(\nabla\theta)^2]d^3r\ , \tag{6.9}$$

where χ is the susceptibility.

The equations of motion have two types of terms. The first, which describes reversible motion, comes from the underlying microscopic dynamics and is of the form (Dzyaloshinskii and Volovik, 1980)

$$\frac{\partial A_i}{\partial t}=\{A_i,\Delta F\}\ , \tag{6.10}$$

where $\{\ \}$ denotes a Poisson bracket [which is $(ih)^{-1}$ times the commutator] and A_i denotes one of the variables. Equation (6.10) can be rewritten

$$\frac{\partial A_i}{\partial t}=\sum\{A_i,A_j\}\frac{\partial\Delta F}{\partial A_j}\ , \tag{6.11}$$

so the problem boils down to finding the Poisson bracket relations. For a spin glass the relevant one is (Halperin and Saslow, 1977; Dzyaloshinskii and Volovik, 1980)

$$\{\theta_i(\mathbf{r}),M_j(\mathbf{r}')\}=\gamma\delta_{ij}\delta(\mathbf{r}-\mathbf{r})\ , \tag{6.12}$$

where $\gamma=g\mu_B/h$ is the hydromagnetic ratio and i,j denote Cartesian components. The standard relation $\{M_i(\mathbf{r}),M_j(\mathbf{r}')\}=\gamma\varepsilon_{ijh}M_k(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')$ does not play a role because $\mathbf{M}(\mathbf{r})$ averages to zero in this coarse-grained description. The second type of term in the equations of motion is dissipative and arises from coupling to fast modes not explicitly treated in the hydrodynamic theory. These give rise to spin-wave damping, which turns out to be of higher order in the wave vector k than the real part of the frequency, so we shall not display these terms explicitly here.

From Eqs. (6.9)–(6.12) the equations of motion are found to be

$$\frac{\partial\theta_\alpha(\mathbf{r})}{\partial t}=\gamma\frac{\delta(\Delta F)}{\delta(M_\alpha(\mathbf{r}))}=\gamma M_\alpha(\mathbf{r})\chi^{-1}\ , \tag{6.13}$$

$$\frac{\partial M_\alpha(\mathbf{r})}{\partial t}=-\gamma\frac{\delta(\Delta F)}{\delta(\theta_\alpha(\mathbf{r}))}=\gamma\rho_s\nabla^2\theta(\mathbf{r})\ . \tag{6.14}$$

At a wave vector $\mathbf{k}$ these equations have solutions at frequencies

$$\omega=\pm ck\ , \tag{6.15}$$

where

$$c=\gamma(\rho_s/\chi)^{1/2}\ , \tag{6.16}$$

corresponding to the positive and negative parts of a spin wave with one of three possible polarizations α ($=x$, y, or z). The additional dissipative terms mentioned above give damping $\propto k^2$.

The central assumption in the hydrodynamic approach is that the long-wavelength deviations from equilibrium are determined by long-wavelength fluctuations in the conserved densities, together with other variables needed to describe degrees of freedom associated with any continuous broken symmetry of the Hamiltonian. This is valid provided the frequency of the long-wavelength mode is small compared with relaxation rates of microscopic degrees of freedom. In a spin glass one expects (Walker and Walstedt, 1977, 1980) that there will be a broad range of these rates extending down to very low frequencies, and the question is the size of the coupling between the hydrodynamic mode, of frequency ω, and short-wavelength modes with relaxation times comparable to ω^{-1}. In ordinary glasses phonon damping is larger than expected from hydrodynamics, due to scattering from localized excitations (two-level systems) but, nonetheless, phonon mean free paths are still large compared with the wavelength, and phonon frequencies agree fairly well with what is expected from elastic constants of the glass.

Also implicit in the hydrodynamic theory of spin glasses is the assumption that the spin stiffness constant ρ_s is nonzero. Microscopic calculations at the mean-field level (Sompolinsky *et al.*, 1984) do find a nonzero ρ_s at finite time scales (where the system stays in a single valley), which is the relevant limit for hydrodynamics, though $\rho_s=0$ in a full statistical mechanics calculation where barrier hopping to other valleys is allowed as the nonuniform rotation is applied. By "mean-field level" we refer here to systems with finite-range interactions at high enough dimensionalities ($d\geq 6$) that critical exponents of mean-field theory apply. Numerical calculations by Reed (1979b) and Walstedt (1981) also find a finite ρ_s. However, a microscopic calculation by Feigelman and Tsvelik (1979) gives $\rho_s=0$, and a vanishing spin stiffness constant is assumed in the gauge theory approach of Dzyaloshinskii and Volovik (1978); see Sec. VI.A.

There have been a number of attempts to derive the linear spin-wave spectrum from a microscopic calculation (Ginzburg, 1978l; Takayama, 1978; Barnes, 1981a, 1981b, 1981c; Becker, 1982a, 1982b). In principle these give microscopic expressions for the parameters in the hydrodynamic theory, such as ρ_s. However, in practice one has to make rather uncontrolled approximations in performing the bond average, so these parameters are hard to evaluate precisely.

It is of interest to search for the linear spin-wave spectrum by numerical methods because this could check the assumptions of the hydrodynamic theory. Unfortunately, calculations of the excitation spectrum about a metastable state for the Edwards-Anderson model (Ching *et al.*, 1977, 1981), a model with RKKY interactions appropriate for CuMn (Walker and Walstedt, 1977, 1980), and a model of $Eu_xSr_{1-x}S$ (Ching *et al.*, 1980; Krey, 1980, 1981, 1982) have failed to detect the spin-wave modes. (We shall refer again to these computations in Sec. VI.C). Similarly, neutron scattering experiments (e.g., Fincher *et al.*, 1980) have failed to detect spin waves in the spin glass regime. This may be because the excitations are only sharp at much longer wavelengths than can be probed by neutron scattering and numerical simulation or, perhaps, because the excitations carry very little weight in the structure factor. Clearly, more work needs to be done to understand the dynamic structure factor of an isotropic spin glass. We shall see below that the $k=0$ mode has a nonzero frequency in the presence of anisotropy, and this *has* been seen in resonance experiments. We also note that a linear spin-wave spectrum is also predicted for a planar spin glass (Edwards and Anderson, 1976), and this has been found numerically (Huber *et al.*, 1979; Huber and Ching, 1980).

Next we discuss how the spin-wave frequencies are altered by a magnetic field H (Andreev, 1978; Fischer, 1980; Saslow, 1980, 1981; Schultz *et al.*, 1980; see also Halperin and Saslow, 1977). The threefold degeneracy of the modes given by Eq. (6.15) is broken, and one has a longitudinal branch of unchanged frequency,

$$\omega_l=ck\ , \tag{6.17}$$

and two transverse branches,

$$\omega_+=\gamma H+O(k^2) \tag{6.18}$$

$$\omega_-=Dk^2\ , \tag{6.19}$$

where

$$D=\gamma\rho_s/\chi H\ . \tag{6.20}$$

This result assumes that the magnetization responds reversibly to the field, so that, at equilibrium, $\mathbf{M}=\chi\mathbf{H}$. However, depending on the preparation of the sample, one might have a nonzero remanent magnetization $\boldsymbol{\sigma}_R$, so that at short times where the system responds reversibly one has

$$\mathbf{M}=\boldsymbol{\sigma}_R+\chi\mathbf{H} \tag{6.21}$$

for $\mathbf{H}$ small. If $\boldsymbol{\sigma}_R$ is parallel to $\mathbf{H}$ one finds that the frequencies ω_l and ω_+ are unchanged, but ω_- becomes finite as $k\rightarrow 0$ and

$$\omega_-=\gamma\sigma_R/\chi\ . \tag{6.22}$$

We have assumed that χ and ρ_s are isotropic, even where $\boldsymbol{\sigma}_R$ and $\mathbf{H}$ are nonzero. This seems reasonable provided the field and magnetization are not too big, but it is straightforward to generalize these results to allow for anisotropic susceptibility and spin stiffness tensors (Fischer, 1980; Saslow, 1980, 1981; Schultz *et al.*, 1980).

All real spin glass systems have some anisotropy, and it is of great interest to see how this affects the long-

wavelength spin-wave frequencies. We shall discuss only anisotropy which, on average, does not prefer any spin direction (see the discussion in Sec. II.A.2). In metallic spin glasses the most important mechanism seems to be Dzyaloshinskii-Moriya (DM) anisotropy (Fert and Levy, 1980, 1981). Dipole-dipole interactions occur in both insulating and metallic systems.

In the presence of anisotropy the free energy ΔF in Eq. (6.9) now has terms depending on θ as well as its gradients. Since a rotation by 2π brings back the same state, we can write the anisotropy energy as a function of $\cos\theta$. Expanding up to second order one has

$$E_{\text{anis}}(\theta)=-K_1\cos\theta-\tfrac{1}{2}K_2\cos^2\theta \tag{6.23}$$

(see also Secs. II.A.2 and II.C.2), which defines the anisotropy coefficients K_1 and K_2. Henley *et al.* (1982) and Levy *et al.* (1982) have shown that DM anisotropy gives $K_2=0$, i.e., the anisotropy is unidirectional. Other types of anisotropy give both K_1 and K_2 nonzero (Henley *et al.*, 1982). Because of the assumed isotropy of the spin glass state, the energy depends only on the magnitude of the rotation, not on the direction of the rotation axis. There is evidence (Hippert *et al.*, 1982) that K_1 is the dominant term in metallic spin glasses. Note that a rotation by π changes the energy if $K_1\neq 0$. This can occur only in a noncollinear spin system because if all the spins are parallel a rotation by π is equivalent to an inversion, which costs no energy in the absence of a magnetic field.

Hydrodynamic theories including anisotropy have been developed by Andreev (1978), Saslow (1980, 1981, 1982), Schultz *et al.* (1980), and, in greatest detail, by Henley *et al.* (1982). Let us follow Henley *et al.* and attempt to explain a typical experiment in which the system is cooled below the freezing temperature in a field $\mathbf{H}_c$ and gets trapped in a metastable state. The field may then be changed to $\mathbf{H}$, so that the remanent magnetization $\boldsymbol{\sigma}_R$ is not, in general, parallel to either $\mathbf{H}$ or $\mathbf{H}_c$. Changing the field from $\mathbf{H}_c$ will rotate the system away from its metastable state, which will cost some anisotropy energy. We assume that this energy is isotropic (i.e., depends only on $|\theta|$), even when $\boldsymbol{\sigma}_R$ is nonzero, provided that during cooling $\boldsymbol{\sigma}_R\cdot\mathbf{H}_c$ is much smaller than the exchange energy. It is further assumed that the state of the system is described by a *rigid* rotation of the original metastable state, which should be a good approximation if the exchange energy is larger than the anisotropy of field energy. However, we shall see below that this assumption seems to break down for large θ.

Including the effect of the magnetic field and allowing only for uniform values of θ and $\mathbf{M}$, the free energy can be written [see Eq. (2.27)]

$$\Delta F[\theta,\mathbf{M}]=\frac{1}{2\chi}(\mathbf{M}-\boldsymbol{\sigma}_R)^2-\mathbf{M}\cdot\mathbf{H}+E_{\text{anis}}(\theta)\ , \tag{6.24}$$

where the susceptibility χ is assumed to be isotropic, and $E_{\text{anis}}(\theta)$, the anisotropy energy, is given by Eq. (6.23). Note that $\boldsymbol{\sigma}_R$ is not an independent parameter, but is obtained by reversibly rotating the remanent magnetization after cooling (which is parallel to $\mathbf{H}_c$) by the equilibrium rotation angle θ_0 in the plane between $\mathbf{M}_c$ and $\mathbf{H}$(see Fig. 110). If the angle between $\mathbf{H}$ and $\mathbf{H}_c$ is θ_H, then minimization of Eq. (6.24) with respect to θ and $\mathbf{M}$ gives (for $K_2=0$, which we assume from now on)

$$\mathbf{M}_0=\chi\mathbf{H}+\boldsymbol{\sigma}_R\ , \tag{6.25}$$

$$\sin\theta_0=(H_R H/H_1^2)\sin(\theta_H-\theta_0)\ , \tag{6.26}$$

where $H_R=\sigma_R/\chi$, $H_1=(K_1/\chi)^{1/2}$, and $\mathbf{M}_0$ is the equilibrium magnetization. Note that we are assuming the temperature to be sufficiently low that the irreversible change of the remanent magnetization (see Sec. II.C.1) can be neglected.

Equations of motion can be set up along the lines discussed above. One finds (Henley *et al.*, 1982) that for general values of the angles θ_0 and θ_H the longitudinal and transverse modes are coupled, and frequencies are given by solutions of a cubic equation. However, the longitudinal and transverse modes decouple when $\boldsymbol{\sigma}_R$ is parallel (or antiparallel) to $\mathbf{H}$, or $\boldsymbol{\sigma}_R$ is zero. The case $\boldsymbol{\sigma}_R$ parallel to $\mathbf{H}$ occurs when $\mathbf{M}$ is parallel to $\mathbf{H}_c$ and the resonance frequencies are (at $k=0$)

$$\omega_l=\gamma H_1\ , \tag{6.27}$$

$$\omega_\pm/\gamma=\pm\tfrac{1}{2}(H-K_R)+\tfrac{1}{2}[(H+H_R)^2+4H_1^2]^{1/2} \tag{6.28}$$

(Saslow, 1981; Henley *et al.*, 1982). The special case of $H_R=0$ was derived earlier by Andreev (1978). Schultz *et al.* (1980) obtained Eq. (6.28) [but not (6.27)] with $H_R=0$ from a vector model, which differs from the triad model discussed here in having only two (instead of three) modes. Figure (111) plots the frequencies against H for the case of $H_R=0$. Note that with $H_R=0$ and $H\to 0$ one has

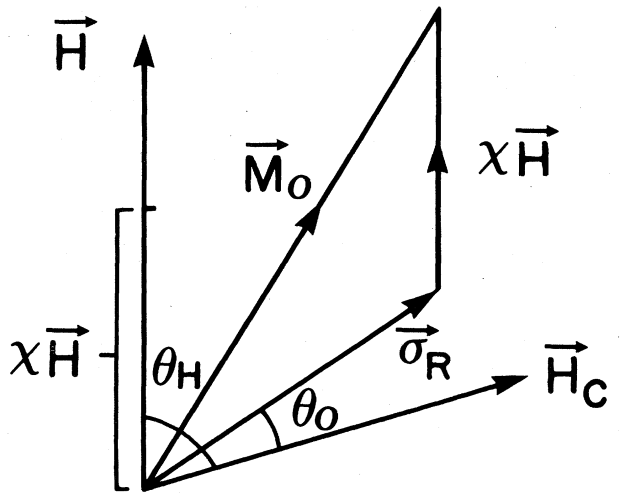

FIG. 110. $\mathbf{H}_c$ is the direction of the cooling field, and $\mathbf{H}$, the applied field, is at an angle θ_H to this. The remanent magnetization $\boldsymbol{\sigma}_R$ is at an angle θ_0 to $\mathbf{H}$, and the equilibrium magnetization is shown by $\mathbf{M}_0$.

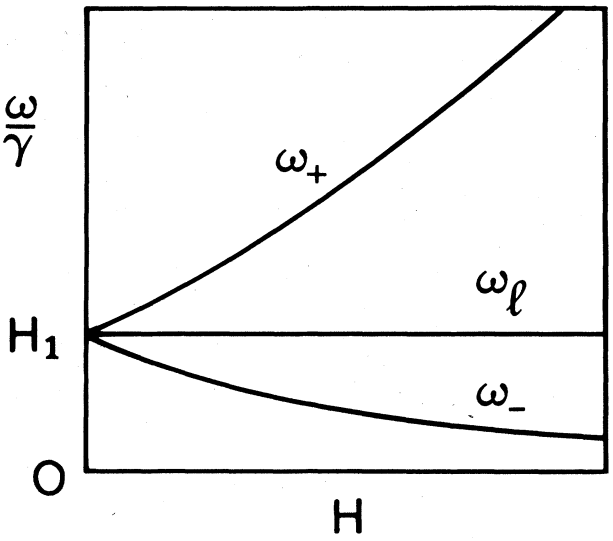

FIG. 111. A plot of the three resonance modes as a function of H for zero remanence.

$$\omega_{\pm}/\gamma = H_1 \pm \tfrac{1}{2}H \ . \tag{6.29}$$

The variation with H in Eq. (6.29) has been convincingly seen for ω_+ by Schultz *et al.* (1980; see also Monod and Berthier, 1980). Schultz *et al.* also found the ω_- mode. The longitudinal mode is difficult to see because it is independent of H, and the experiment is performed by scanning H at fixed resonant frequency. However, with $\theta_H \neq 0$ the longitudinal and transverse modes are coupled, so there is "level repulsion" where the modes cross as H is varied; this has been seen by Gullikson *et al.* (1983). However, Gullikson *et al.* (1983) found that the triad model breaks down for large rotations, presumably because the spins no longer rotate rigidly. Numerical simulations by Morgan-Pond (1983) also found that the triad model works well for small anisotropy and breaks down when the anisotropy becomes large.

To conclude, the triad model with unidirectional anisotropy coming from DM interactions explains fairly well the resonant frequencies observed in metallic spin glasses for small rotation angles, but seems to break down for large rotations.

C. Simulations of realistic models

While enormous effort has been devoted to understanding both the infinite-range Sherrington-Kirkpatrick (1975) model (Sec. IV) and the nearest-neighbor Edwards-Anderson (1975) model (Sec. V), neither of these models should be compared to experimental data quantitatively. Since a study of the short-range Edwards-Anderson model relies heavily on Monte Carlo simulations (Secs. V.B—V.E), it is tempting to apply this technique to more realistic models of real materials as well. So far this has been attempted for a classical RKKY model of CuMn (Walker and Walstedt, 1977, 1980, 1983; Ching and Huber, 1978; Walstedt and Walker, 1981, 1982; Walstedt, 1981, 1982, 1983), as a prototype of metallic spin glasses, and for a diluted classical Heisenberg system with competing ferro- and antiferromagnetic interactions which models $Eu_xSr_{1-x}S$ and related materials (Binder *et al.*, 1979; Krey, 1980, 1981, 1982; Kinzel and Binder, 1981; Binder and Kinzel, 1983a). Comparatively little work has been devoted to other materials, such as Fe-Al alloys (Grest, 1980) or materials with Dzyaloshinskii-Moriya anisotropies (Morgan-Pond, 1981, 1983; Dasgupta and Yao, 1984).

Walker and Walstedt (1977, 1980) and Krey (1980, 1981, 1982) focused attention on the low-temperature excitations in these systems. They first tried to find a ground state, or low-lying excited state, applying Monte Carlo methods that treat the Heisenberg spins as classical unit vectors. Walker and Walstedt (1977, 1980) used systems of no more than 96, 172, and 324 spins, and linearized the quasiclassical equations of motion for the spins, which were then numerically diagonalized to find the excitation spectrum. They found a rather high density of low-lying modes, which—appropriately quantized—gave a good account of the specific heat experimentally observed (Wenger and Keesom, 1975) in *Cu* with 0.88% Mn (Fig. 112). Note that this specific heat at low temperatures cannot be obtained from direct Monte Carlo simulation of the classical Heisenberg spin glass as done by Ching and Huber (1978)—there $C/k_B(T\to 0)\to 1$ since the system is classical.

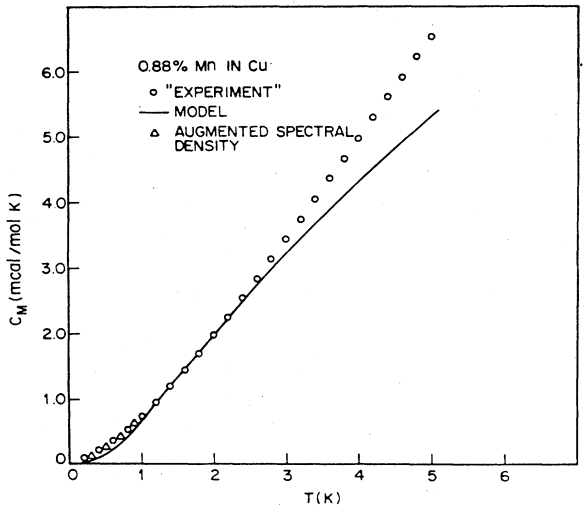

FIG. 112. Plot of experimental specific-heat data, taken from Wenger and Keesom (1975) and model calculations for *Cu*Mn with 0.88% Mn, where the spin-wave frequencies ω, for 172 spins interacting via the Ruderman-Kittel interaction, Eq. (2.1), with $J_0 = 1.02\times 10^{-36}$ erg cm^3 and k_F the value known for bulk Cu, are obtained numerically, and C is obtained as a temperature derivative of $\sum \hbar\omega_i/[\exp(-\hbar\omega_i/k_BT)-1]$. Points marked as augmented spectral density represent results with an *ad hoc* correction for finite size. From Walker and Walstedt (1980).

Krey (1980, 1981, 1982) used ground states of the classical Heisenberg model of $Eu_xSr_{1-x}S$ obtained by Binder *et al.* (1979), with a lattice size of 16^3 fcc unit cells. By applying a continued fraction algorithm utilizing typically 42 moments of the eigenfrequency spectrum, which are calculated exactly for these ground states, Krey could obtain both the density of states and the dynamic structure factor $\chi''(Q,E)$ [Krey, 1982; see Fig. 113(a)]. The peaks seen represent a remnant of the ferromagnetic magnons, still seen at large Q because there is ferromagnetic short-range order in the spin glass in this system; at small Q there is just a central peak. These results agree qualitatively, but not quantitatively, with inelastic neutron scattering experiments (Maletta *et al.*, 1981). It is not clear whether this discrepancy reflects finite-temperature effects or nonlinearities (all this work is based on linearized equations of motion) or inadequacy of the model Hamiltonian for $Eu_xSr_{1-x}S$. In any case, the success of the fit to the specific heat (Fig. 112), which also accounts for its magnetic field dependence [Wosnitza *et al.*, 1986; see Fig. 113(b)], indicates that most of the density of low-lying states in real spin glasses is due to such linear (oscillatory) excitations and not due to two-level systems, as is thought to be the case for ordinary glasses (Phillips, 1972; Anderson *et al.*, 1972).

Recently it has become possible to study the thermal properties of RKKY spin glasses by Monte Carlo simulation of samples containing typically 500 or 960 spins (Walstedt and Walker, 1981, 1982; Walstedt, 1983; Walker and Walstedt, 1983). Interestingly, no spin glass transition is found if one works with purely isotropic exchange, even for very short observation times of $t_{obs}=5\times10^3$ MCS/spin: the time-dependent Edwards-Anderson order parameter $q(t_{obs})\approx0$ down to the lowest temperatures (Fig. 114). However, a transition is found if a pseudodipolar anisotropy is added. When one chooses the strength of this anisotropy such that the freezing temperature seen in the simulation fits the experimental one, the anisotropy seems by far too large. When one studies T_f as a function of sample size in the range from 500 to 4928 spins, it shows a pronounced size effect. Particularly for small values of D, T_f distinctly increases with size. Thus it was not clear whether the behavior seen in Fig. 114 survives in the thermodynamic limit. However, more extensive recent simulations of Chakrabarti and Dasgupta (1986) imply that the transition temperature is zero if there is no anisotropy.

Experimentally, there is no evidence that the amount of anisotropy has a strong effect on the location of T_f (one can vary the anisotropy of *Cu*Mn experimentally by adding Au impurities via the Dzyaloshinskii-Moriya mechanism—see, for example, Prejean *et al.*, 1980). This problem calls for further study. Walstedt and Walker (1982) have also studied the onset of transverse spin glass order in finite magnetic fields (Fig. 115). The transition to transverse ordering seems to remain sharp in the presence of a field, a fact that is qualitatively consistent with the mean-field predictions (Sec. IV).

Walstedt (1983) has also started an investigation of the

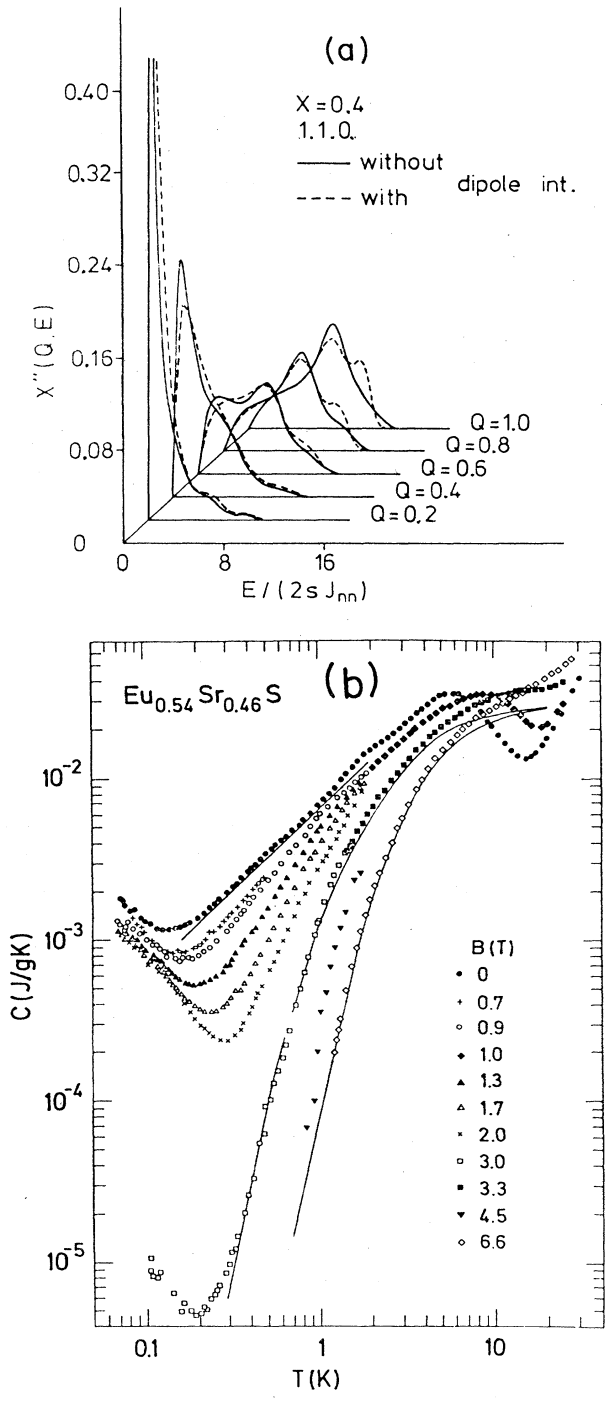

FIG. 113. (a) Dynamic structure factor $\chi''(Q,E)$ for inelastic neutron scattering under wave vector Q and energy transfer E at $T=0$ K, for $Eu_{0.4}Sr_{0.6}S$, for Q vectors in the (1,1,0) direction (Brillouin zone boundary is normalized at $Q=1$). $J_{NN}=0.22$ K, $J_{NNN}=-0.11$ K. Dashed curves denote results including the dipolar interaction; solid curves include exchange interactions only. (b) Log-log plot of the specific heat of $Eu_{0.54}Sr_{0.46}S$ vs temperature for various magnetic fields. Solid curves are theoretical predictions, based on numerical data on the density of states obtained in the same way as $\chi''(Q,E)$ in (a). From Wosnitza *et al.* (1986).

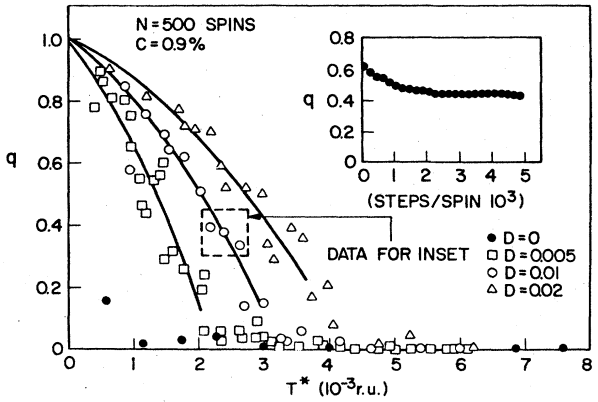

FIG. 114. Edwards-Anderson order parameter $q(t_{obs})$ for the RKKY model of *Cu*Mn spin glasses with 0.9 at. % Mn plotted against reduced temperature T^* for several choices of the strength D of a pseudodipolar anisotropy $[T^*=k_BTa^3/2\sqrt{2}J_0S(S+1)$, where a is the fcc lattice constant]. Inset shows a typical time evolution of q. From Walstedt and Walker (1981).

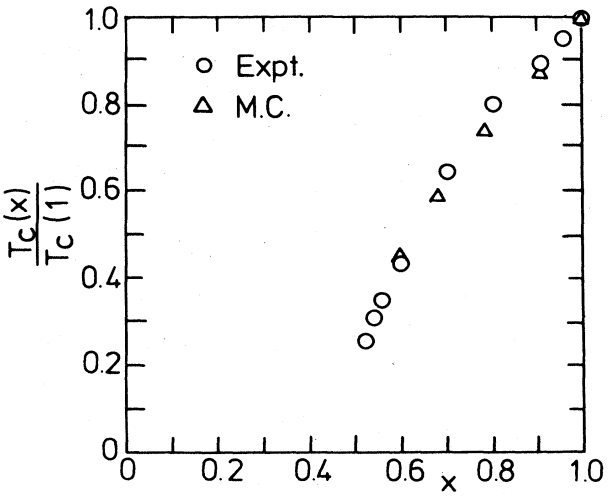

FIG. 116. Ferromagnetic critical temperature plotted vs concentration for $Eu_xSr_{1-x}S$ (Maletta and Felsch, 1979) as compared to Monte Carlo results for a diluted classical Heisenberg fcc ferromagnet with nearest (J_{NN}) and next-nearest (J_{NNN}) exchange, $J_{NNN}=-\frac{1}{2}J_{NN}$. From Binder *et al.* (1979).

mutual projections of the various ground states of this model into each other, and discusses the energy barriers separating these minima. He finds that there are states related to each other by a change of spin orientation in a small "defect" region. This work clearly constitutes an interesting and promising first step towards the investigation of phase-space topology for a realistic model of a spin glass.

We now turn briefly to the simulation studies of the $Eu_xSr_{1-x}S$ model (Binder *et al.*, 1979; Binder and Kinzel, 1983a). Figure 116 shows that the model can account quantitatively, for the quick decrease of the ferromagnetic transition temperature $T_c(x)$ with increasing SR content. For $x<0.5$, however, where the experiment (Maletta and Felsch, 1979) shows a spin glass phase, the simulation (using isotropic exchange only) does not show a stable spin glass order down to the lowest temperatures. Adding the dipolar anisotropy between nearest and next nearest neighbors does stabilize the spin glass order (Binder and Kinzel, 1983), but q diminishes with temperature in a much too gradual fashion. Clearly, a simulation including the untruncated dipolar interaction would be very desirable, but such a simulation has not yet been done.

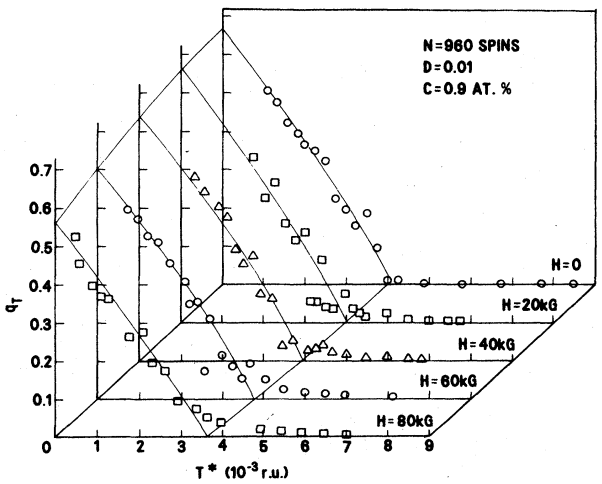

FIG. 115. The transverse Edwards-Anderson order parameter $q_T(t_{obs})$ for the RKKY model of *Cu*Mn spin glasses with 0.9 at. % Mn plotted against reduced temperature T^* for several choices of the magnetic field (for notation, see Fig. 114). The sample contains 960 spins, and the strength of the pseudodipolar anisotropy is $D=0.01$. From Walstedt and Walker (1982).

Although it is clear from the present discussion that simulations of realistic models of spin glasses are somewhat limited, because even the fastest present-day computers do not allow simulations of much larger systems or larger observation times, work along similar lines clearly promises to be fruitful in the future when more powerful machines will have become available.

Finally we mention a first attempt at a "realistic" simulation of the dielectric glass $Rb_{1-x}(ND_4)_xD_2PO_4$, in which a molecular dynamics study of a (two-dimensional) model involving 8741 atoms was tried (Parlinski and Grimm, 1986). Using interaction parameters such that the ferroelectric and antiferroelectric order of RbD_2PO_4 and $ND_4D_2O_4$ was reproduced, Parlinski and Grimm did indeed find in the mixed crystals a glassy phase with many local minima into which the system was locked, for $0.22\leq x\leq 0.75$. The scattering function was obtained in qualitative agreement with experiment.

D. Spin glass transition as a percolation problem

When confronted with a difficult problem, such as the nature of spin glass freezing, it is natural to ask whether

insight can be gained from parallels with familiar and well understood concepts. One such proposal, due to Smith (1974, 1975), is that the spin glass problem is very similar to percolation (for a review, see Stauffer, 1979; Essam, 1980). Smith considered an RKKY interaction, given by Eq. (2.1) and made the approximation that two spins, at sites i and j, will be rigidly coupled if $|J_{ij}| > k_B T$ and completely uncorrelated if $|J_{ij}| < k_B T$. Hence spins are locked to a spin at site i if they lie within a sphere, centered at i, of radius R given by

$$J_0/(k_F R)^3 = k_B T \ , \tag{6.30}$$

where J_0 and k_F are as in Eq. (2.1). As the temperature is lowered, R increases, so eventually these spheres percolate, forming an infinite cluster. Spins in the infinite cluster do not fluctuate and so do not contribute to the susceptibility. Hence one obtains the usual result $k_B T\chi = 1-q$, where q is now the fraction of spins in the infinite cluster. This leads to a cusp in χ at the percolation threshold. Further discussion of the results is given in Smith (1974, 1975), Mookerjee and Chowdhury (1983), and Chowdhury and Mookerjee (1983).

The idea is appealingly simple but unfortunately has several defects. First, the RKKY interaction does not really fall off fast enough to justify the sharp division into rigidly coupled and completely uncoupled spins. There are a large number of spin pairs for which $J_{ij} = k_B T$ and which are therefore partially correlated. Second, and most seriously, the treatment neglects frustration, which gives cancellations and hence the possibility of weak correlations even if the interactions are much larger than $k_B T$.

Abrikosov and Moukhin (1978) and Abrikosov (1978, 1980) have pointed out that the first criticism does not apply if the mean free path is much smaller than the typical spin separation, which can be achieved by adding nonmagnetic impurities. In this case only the interactions between a spin and its one or two nearest neighbors are important, and these can range over many decades depending on separation. However, frustration is still neglected.

It seems to us that frustration *must* feature in any satisfactory spin glass theory. For instance, the percolation model predicts that the lower critical dimension is $d_l = 1$, whereas spin glasses have larger values than this (see Sec. V.E).

Cyrot (1981) has proposed a modified percolation model in which there are blocks of rigidly coupled spins, whose size, which diverges at T_f, allows for the frustration effect. However, the separation into noninteracting blocks does not seem very realistic. It is more natural, in our opinion, to replace this block size by the spin glass correlation length ξ_{SG}, which, incidentally, diverges at T_f with an exponent quite different from that of the percolation correlation length.

To conclude, we have argued that frustration and percolation are very different, so that percolation models of spin glasses are inadequate.

E. Periodic frustrated systems

As discussed earlier, the necessary ingredients for a spin glass appear to be randomness and frustration. However, disorder greatly complicates the problem, and Villain (1977a) suggested that one might obtain a reasonable spin glass model with frustration but no disorder. This can be achieved by making the frustration function ϕ_f, defined in Eq. (3.125b), periodic. In this section we shall discuss systems in which all interactions have the same magnitude, so called "pure frustration models," because the concept of frustration is simplest in this case. Interesting examples are "fully frustrated" (FF) models, where one frustrates *all* elementary polygons or "plaquettes" on the lattice (e.g., squares for square and simple cubic lattices; triangles for triangular and face-centered cubic lattices). Triangular plaquettes are frustrated by negative interactions. Other lattices can be fully frustrated by a periodic array of positive and negative bonds. This is illustrated in Fig. 117 for the square lattice. We shall see in this section that FF and other periodic frustrated models have interesting properties but are rather different from spin glasses.

First of all we discuss Ising models in $d = 2$. The antiferromagnetic nearest-neighbor model on a triangular lattice is the longest studied FF model. Wannier (1950) showed that $T_c = 0$ and computed the ground-state entropy per spin, which is finite. There is a critical point at zero temperature where correlations fall off with distance r, as $r^{-\eta}$, where $\eta = \frac{1}{2}$ (Stephenson, 1967).

Villain (1977a) seems to have been the first to discuss the FF square lattice shown in Fig. 117. He was able to show that the Ising model has $T_c = 0$ and that the ground-state entropy per spin is finite. Subsequently

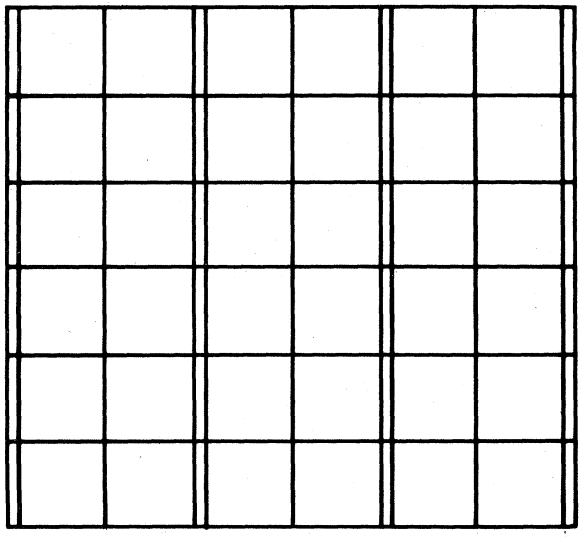

FIG. 117. Part of a fully frustrated square lattice. The double lines represent negative interactions, and the single lines represent positive interactions of the same magnitude.

Southern *et al.* (1980) mapped this model into an 8-vertex model and, as a result, Forgacs (1980) demonstrated that correlations at $T=0$ decay as $r^{-1/2}$, just as for the triangular antiferromagnet. The same value of $\eta=\frac{1}{2}$ has also been found for other models with $T_c=0$ (Wolff and Zittartz, 1983a, 1983b) and seems to be universal for all periodic two-dimensional Ising-type systems where $T_c=0$ and the ground-state entropy per spin is finite. Note that one obtains a different exponent, $\eta=\frac{1}{4}$, if T_c is finite.

A variety of periodically frustrated $d=2$ Ising models has been studied by Wolff and Zittartz (1982, 1983b, and references therein). They find that there is no simple connection between the occurrence of a transition at finite temperature and the occurrence of a finite ground-state entropy per spin. All combinations, transition or no transition, presence or absence of ground-state entropy, are possible. In all cases studied with a finite T_c the ordered state was ferromagnetic or antiferromagnetic, never a spin glass.

This last conclusion was also reached by André *et al.* (1979). They considered a number of periodically frustrated Ising models on a square lattice. One of these is a generalization of the FF square lattice model in Fig. 117, in which the negative interactions J' are stronger than the remaining positive couplings J, i.e., $J'<-J$. In the ground state the J' bonds are all satisfied but the J bonds may or may not be frustrated. Consider a vertical string of spins connected by J bonds in between two antiferromagnetically ordered strings (see Fig. 118). The states where the antiferromagnetic strings are in phase and out of phase have the same energy. One might therefore expect the ground state to have no long-range order. However, this is not so because there are just two states of the even string if the antiferromagnetic strings are out of phase, but there are of order α^L states when the antiferromagnetic strings are in phase, where L is the size of the lattice in the vertical direction and α turns out to be $[(3+\sqrt{5})/2]^{1/2}$. Hence virtually all ground states have the vertical strings in phase (see Fig. 118), and one can define a staggered magnetization on the antiferromagnetic strings that is saturated, i.e., tends to unity as $T\to 0$. The staggered magnetization on the remaining spins, i.e., those on the ferromagnetic vertical strings, is nonzero, but it is not saturated (André *et al.*, 1979). The ground-state entropy per spin is finite, because of the degeneracy on the ferromagnetic strings, and the transition is at a finite temperature. Note that the long-range order in the ground state is an entropy effect. The energy to create out-of-phase antiferromagnetic strings is zero, but the entropy lost diverges in the thermodynamic limit. The relative statistical weight of two states is $\exp(-\Delta F/k_BT)$, where ΔF is the free-energy difference. This can of course be written as $\exp(\Delta S-\Delta U/k_BT)$, where ΔS and ΔU are the entropy and energy differences. The free energy to create a vertical domain wall in this model has been calculated at finite temperature by Derrida *et al.* (1978) and is nonzero for $0<T<T_c$.

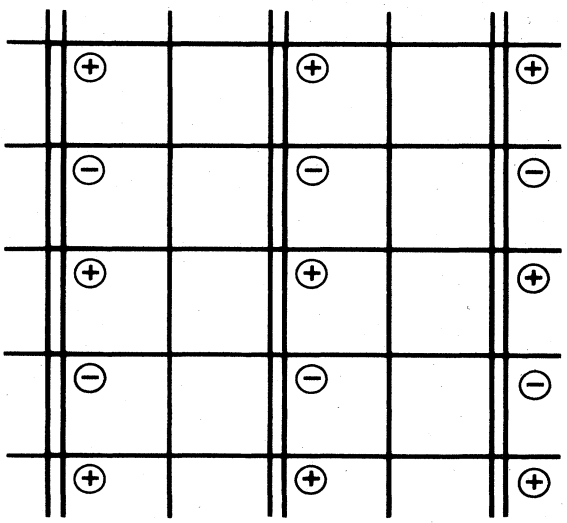

FIG. 118. Same as for Fig. 117, but now the negative bonds are larger in magnitude than the positive ones, so the spins along the double lines have antiferromagnetic order in all ground states. The circled symbols show spin directions in a ground state. Although there are ground states with different antiferromagnetic chains out of phase, virtually all ground states have these chains in phase as shown.

We now go on to discuss periodic frustrated Ising models in three dimensions. The face-centered cubic lattice is fully frustrated by making all interactions negative. This model has a ground-state degeneracy of order $\exp(\alpha N^{1/3})$ (Danielian, 1964), and most of these ground states are nonperiodic, which makes an analogy with spin glasses rather tempting. However, at finite temperature this degeneracy is removed, and a simple periodic structure (Slawny, 1979; Mackenzie and Young, 1981) has the lowest *free energy*. Analogous behavior in a two-dimensional model has been found by Villain *et al.* (1980). As they point out, thermal fluctuations remove the degeneracy and so tend to make the system *more* ordered. For the fcc model the transition to the paramagnetic state is first order (Phani *et al.*, 1980; Liu *et al.*, 1982; Polgreen, 1984; Styer, 1985), which is also very unlike spin glasses. However, the free energies per spin of the stable and metastable phases are very similar, so if the system were rapidly quenched to a metastable state it would take a very long time to reach equilibrium. The dynamics of this model may therefore have some similarity with spin glasses and would be worth investigating. Adding dilution also apparently changes the transition from first to second order (Grest and Gabl, 1979).

The FF simple cubic lattice can be obtained by periodically continuing the arrangement of positive and negative bonds shown in Fig. 119. There are of order $\exp(\alpha N^{2/3})$ ground states (Villain, 1977a; Kirkpatrick, 1981; Chui *et al.*, 1982) obtained from each other by turning over chains of spins. Apart from inversion of all the spins, 12 of these have the periodicity of an elementary cube. The energy barrier between ground states is finite. This is in

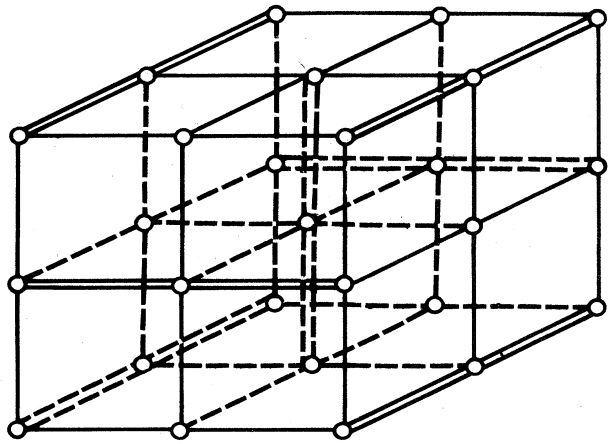

FIG. 119. Construction of a fully frustrated simple cubic lattice. The double lines represent negative interactions.

contrast to the fcc antiferromagnet, in which different ground states are obtained by turning over planes of spins (Danielian, 1964) and the energy barrier between them diverges in the thermodynamic limit. Because there is only a finite energy barrier to overturning the chains at $T=0$, it is widely assumed (e.g., Villain, 1977a) that spins on these chains are disordered as $T\to 0$. This may well be so, but we emphasize that it is really necessary to compute free-energy barriers at a finite temperature to be sure of this result. In addition to free chains there is an infinite network of rigidly coupled spins, so T_c is finite. The region near T_c has been studied by Blankschtein *et al.* (1984) using Ginzburg-Landau and renormalization-group approaches. They find that the order parameter is four dimensional and that there are 8 possible ordered states (apart from total spin reversal) just below T_c. No stable fixed point is found within an ε expansion which is interpreted as a fluctuation-driven first-order transition. The 8 ordered states just below T_c are incompatible with the 12 simple periodic ground states noted above. Monte Carlo simulations (Kirkpatrick, 1981; Diep *et al.*, 1985; Grest, 1985) show a large specific-heat peak at the transition temperature where $T_c \approx 1.35$ in units of the nearest-neighbor interactions. Both Kirkpatrick (1981) and Diep *et al.* (1985) interpret their data as evidence for a second-order transition, in contrast to the results of Blankschtein *et al.* (1984), though Grest (1985), who uses larger sizes, suggests that a very weak first-order transition is more likely. Diep *et al.* (1985) also argue that there is a second transition, below T_c, at which the ordered state changes to one of the 12 simple periodic ground states.

Fully frustrated models in higher dimensions have been studied by Derrida *et al.* (1979) and Alexander and Pincus (1980).

What can we learn from the study of periodic frustrated Ising models? The main result is that, although many of the models have nonperiodic ground states, the state at finite temperature is always a simple periodic magnetic structure. No spin glass phases are observed. However, as we have noted above, there can be long-lived nonperiodic metastable states, so the dynamics of these models may be more spin-glass-like than the statics. There also seems to be rather little universality: for example, FF models on different lattices seem to have rather different properties (e.g., Villain, 1977a). Particularly in two dimensions, fluctuations may drive T_c to zero. However, the precise criterion for T_c to be finite is not fully clear. It is unrelated to the existence or nonexistence of a finite entropy per spin (Wolff and Zittartz, 1982, 1983b). Presumably one has to show that the *free energy* of an interface between two of the assumed broken symmetry states at finite temperature diverges in the thermodynamic limit. The assumed broken symmetry states must, of course, have the same free energy. This calculation is generally rather difficult. A study of the ground states is easier but may be insufficient.

Although periodic frustrated models are different from spin glasses, one can make them into spin glasses by adding disorder. This has been investigated by Grest and Gabl (1979) for the triangular and fcc antiferromagnets, by Anderico *et al.* (1982) for the triangular lattice, and by Kirkpatrick (1981) for the FF simple cubic lattice.

Periodically frustrated models with vector spins have also been studied. One of the most interesting is the FF XY (planar) model on a square lattice. It is easy to see that the ground state of a single frustrated square is doubly degenerate (see Fig. 120) apart from overall rotations (Villain, 1977a, 1977b). One can therefore assign an Ising-type variable or "chirality" to each frustrated square. Chiralities have a logarithmic interaction like the vortices that play a prominent role in the Kosterlitz-Thouless (1973) theory of the pure two-dimensional XY model. At $T=0$ the chiralities form a regular antiferromagnetic array, which suggests that there may be an Ising transition in this system at finite temperatures (Villain, 1977b; Fradkin *et al.*, 1978). This has been seen in Monte Carlo simulations by Teitel and Jayaprakash (1983), who also note that the model is relevant for the understanding of two-dimensional arrays of coupled Josephson junctions.

In his pioneering papers Villain (1977a, 1977b) studied XY and Heisenberg models on a variety of lattices. One

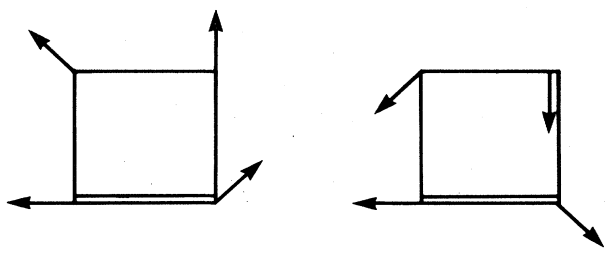

FIG. 120. The two ground states of a single frustrated square with XY (planar) spins that are unrelated by an overall rotation of the spins. The double line represents a negative interaction.

that is particularly important and has been further studied is the Heisenberg fcc antiferromagnet. Apart from interest motivated by spin glasses, this is also a good model for real fcc antiferromagnets if one includes second-neighbor coupling. The nearest-neighbor model has a high ground-state degeneracy like its Ising counterpart, and, as a result, Villain (1977a) and Alexander and Pincus (1980) suggested there would be no order at finite temperature. However, as we have noted above, the degeneracy is removed at finite temperatures, and indeed Fernandez *et al.* (1983) find evidence for a second-order transition at finite T_c from Monte Carlo simulations. They also see that $T_c=0$ for a sufficiently dilute system, consistent with currently expected behavior for a $d=3$ Heisenberg spin glass (see Sec. V). Giebultowicz and Fardyna (1985) have shown that with a small amount of antiferromagnetic second-neighbor coupling, which is a model for type-III antiferromagnets, the transition becomes first order.

As for Ising systems, there is a lack of universality among periodic frustrated vector spin models. They are interesting in their own right, but seem to be rather different from spin glasses.

F. Relations between spin glasses and other topics in the theory of disordered systems

1. Systems with random fields and random anisotropies

If there are random fields or random anisotropies, then even systems with uniform interactions possess the two ingredients of randomness and frustration that are necessary for spin glass behavior. It is therefore useful to compare such systems with conventional spin glasses.

The Hamiltonian of a simple random-field model with Ising spins is given by

$$\mathcal{H}=-\sum_{\langle i,j\rangle} J_{ij}S_iS_j-\sum_i H_iS_i\ , \tag{6.31}$$

where the J_{ij} are nonrandom, positive interactions, typically between nearest neighbors, and the fields H_i are independent random variables with mean and variance given by

$$[H_i]_{\rm av}=0,\quad [H_i^2]_{\rm av}=H_R^2\ . \tag{6.32}$$

The phase diagram is sketched in Fig. 121. There is a transition line separating a ferromagnetic phase F from a "paramagnetic" phase P. Actually the P phase has spin glass order, since $q=[\langle S_i\rangle_T^2]_{\rm av}>0$, induced by the random fields. Nonetheless, it seems appropriate to call this phase paramagnetic and use the term spin glass in situations where a nonzero q arises from *cooperative* effects.

In a pioneering paper, Imry and Ma (1975) gave a simple argument showing that for Ising spins the ground state, fully aligned if $H_R=0$, breaks up into domains, so there is no average magnetization, in the presence of a small random field for dimension d less than 2. This argument has been refined since then, and there is now a rigorous proof of order in the ground state for weak random fields in $d=3$ (Imbrie, 1984). It therefore seems probable that the lower critical dimension d_l is 2 and the phase diagram in Fig. 121 should be qualitatively correct for $d>2$.

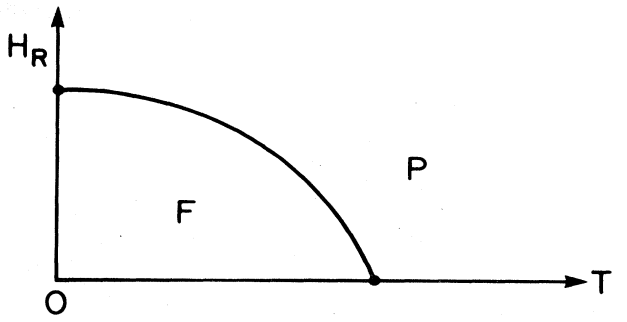

FIG. 121. Sketch of the mean-field phase diagram for the random-field Ising model. Here H_R is the strength of the random field. The different phases are paramagnetic (P) and ferromagnetic (F).

Competition between the exchange and random-field terms in Eq. (6.31) gives rise to many minima in phase space. These are somewhat easier to categorize than for spin glasses and comprise domains such that spins within a domain are parallel. Cooling in the random field traps domains, and one does not observe the expected long-range order experimentally (Birgeneau *et al.*, 1984). However, cooling below the phase boundary in zero field gives a ferromagnetic state (Belanger *et al.*, 1985). Hence there is a difference between field cooling and zero-field cooling as in spin glasses. The dynamics of trapped domains has been studied theoretically (Villain, 1984; Grinstein and Fernandez, 1984; Pytte and Fernandez, 1985). Both this behavior and the properties in the vicinity of the transition temperature (Young and Nauenberg,

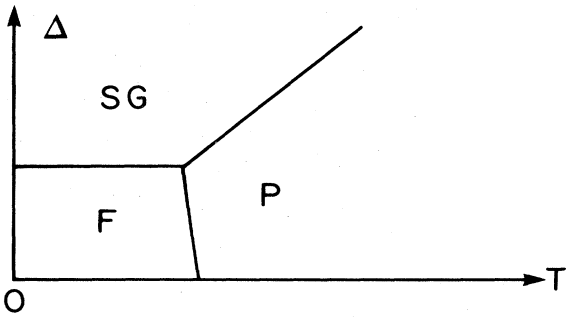

FIG. 122. Sketch of a proposed phase diagram for $d>4$ for the random anisotropy model whose Hamiltonian is given by Eq. (6.33). Δ is proportional to the variance of the components of the random vectors $\mathbf{a}_i$. The different phases are paramagnetic (P), ferromagnetic (F), and spin glass (SG).

1985, Ogielski and Huse, 1986) may need further clarification.

In some respects the random-field problem is somewhat simpler than spin glasses. For example, the order parameter is just the magnetization, which is much less complicated than for spin glasses. Furthermore, the simple Imry-Ma (1975) argument appears to give the lower critical dimension correctly. Unfortunately, there does not seem to be an analogous argument for spin glasses.

Systems with random anisotropy are often described by the random-axis model of Harris *et al.* (1973). One needs vector spins with m components ($m \geq 2$), and the Hamiltonian is given by

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{i,j} \mathbf{S}_i \cdot \mathbf{S}_j - \sum_i (\mathbf{a}_i \cdot \mathbf{S}_i)^2 , \tag{6.33}$$

where the $\mathbf{a}_i$ are vectors whose components are independent random variables with a Gaussian distribution. Physically the model gives each spin an easy axis whose direction is random. Arguments along the lines of Imry and Ma (1975), as well as microscopic calculations (Pelcovits *et al.*, 1978), predict that there is no ferromagnetism in $d < 4$. However, it has been suggested (Pelcovits *et al.*, 1978; Ginzburg, 1981; Goldschmidt, 1983) that a spin glass phase may occur, and a phase diagram similar to Fig. 122 has been proposed for $d > 4$. These calculations rely mainly on results for $m = \infty$ for which a replica-symmetric theory is possible. However, it appears that replica symmetry breaking is needed for finite m (Goldschmidt, 1984; Khurana *et al.*, 1984). Recently it has been shown by Sompolinsky and Fisher (1985) and Bray and Moore (1985b) that the $m = \infty$ limit is very special, in that the spin glass transition occurs even for a single site. Hence it is unclear at present when the random-axis model has a spin glass state. Even in cases where the random-axis model does not have an equilibrium spin glass phase, we expect that, like random-field systems, it will exhibit spin-glass-like characteristics in dynamics (slowly decaying metastable states, remanent magnetization, etc.).

2. Anderson localization

In this section we shall not attempt a review of Anderson localization *per se*, but only mention its relation to spin glasses. A general review of localization can be found in Lee and Ramakrishnan (1985).

One of the earliest theoretical papers on the spin glass problem (Anderson, 1970) asked the question "what is the real nature of the difference between these materials and others which exhibit ferromagnetic or other kinds of long-range ordering?" For a pure ferromagnetic system, the exchange matrix J_{ij} has eigenvalues $J_{\mathbf{q}}$, characterized by a wave vector $\mathbf{q}$, and eigenvectors that are "extended" in the sense of Anderson (1958). That is to say, that have an amplitude of order $N^{-1/2}$ at each of the N lattice sites. According to mean-field theory, a transition occurs when $\chi_{ii}^0 J_{\max} = 1$ where χ_{ii}^0 is the noninteracting local susceptibility and $J_{\max}$ is the largest eigenvalue. A ferromagnetic system has a maximum $J(\mathbf{q})$ at $\mathbf{q}=0$. In a random system one can still formally diagonalize the exchange matrix to get eigenvalues J_λ, but now some or all of the eigenvectors may be "localized" (Anderson, 1958), i.e., some may have an amplitude of order unity on a finite number of sites and zero elsewhere. Anderson (1970) suggested that the real difference between spin glass and ferromagnetic systems is that the former have only localized eigenvectors. Although mean-field theory still predicts ordering when $\chi_{ii}^0 J_{\max} = 1$, this is now qualitatively wrong because a small cluster of spins cannot undergo a sharp transition. If all states are localized, it appears that no transition into any mode can occur; this was in agreement with experiments at that time (Anderson, 1970). Even if there are some extended states, these will be in the middle of the band, so Anderson (1970) suggested that they may not order either, because nonlinear couplings to the localized modes would reduce the effective χ_{ii}.

Hertz *et al.* (1979) developed these ideas further. By going beyond mean-field theory and using the self-consistent spherical (Hartree) approximation, they showed explicitly that ordering does not occur into a localized mode. However, a transition to the first extended state at the "mobility edge" can occur and is interpreted as the spin glass transition. Hertz *et al.* (1979) also argued that the density of states of the inverse susceptibility matrix is finite at the mobility edge and that, as a result, spin glass order below T_f will not occur for an isotropic vector spin model in any dimension because of fluctuations associated with Goldstone modes.

However, Bray and Moore (1982c) noted that even above T_f one should work in the basis of eigenstates of the susceptibility matrix χ_{ij}, rather than the exchange matrix J_{ij}. These are different because of nonlinear couplings between spin fluctuations diagonal in the J_{ij} basis. The transition temperature is where an eigenvalue of the inverse susceptibility matrix first vanishes. If T_f is nonzero, Bray and Moore (1982c) show that the corresponding eigenvector must be extended, in agreement with Anderson (1970) and Hertz *et al.* (1979). However, they also prove that the density of states vanishes at zero eigenvalue, which casts doubts on the conjecture of Hertz *et al.* (1979) that Goldstone modes destroy order in vector spin models for any dimension. The elements of χ_{ij} are not only random, but also correlated up to distances where spins are correlated, i.e., ξ_{SG}, which diverges at T_f. Thus there is an analogy between spin glasses and an unusual localization problem in which correlations between the random elements extend over large distances. This analogy can be made precise for an m-component vector model in the limit that $m \to \infty$ (Bray and Moore, 1982c). We therefore feel it is unlikely that traditional folklore on the standard (uncorrelated) localization problem can give correct values for critical exponents or critical dimensions of the spin glass problem.

The effect of nonlinear interactions between spin fluc-

tuations diagonal in the J_{ij} basis has been discussed in the renormalization-group framework by Hertz (1985), who gives arguments for a phase transition in dimension $d>2$. A rather similar "hierarchical" picture has been given by Feigelman and Ioffe (1984), who make the interesting prediction that there are two critical dimensions for spin glasses, in addition to the upper critical dimension d_u. For $d>4$, there is a phase transition with condensation into a delocalized mode. However, for $2<d<4$ there is a phase transition "into a strongly nonergodic state" at a finite temperature, but "this transition does not have anything in common with macroscopic condensation into one delocalized mode and it is not governed by any scaling laws." The nonlinear susceptibility "can be approximated with equal success by any power functions with exponents in the range 4.75—5."

Although the localized modes near the band edge cannot undergo a sharp transition, they will be frozen over very long times once the temperature is well below their mean-field transition temperature. Hertz (1983c) has developed a theory for spin glass dynamics along these lines, allowing for the possibility that a mode about to freeze out in the mean-field approximation may strongly overlap with an already condensed mode, in which case freezing is suppressed. Relaxation times are found to vary as $\exp[\text{const}/(T-T_f)^{d\nu-1}]$ in d dimensions where ν is the correlation length exponent of the localization problem. This theory gives a very plausible description of the initial stages of freezing at temperatures still some way above T_f, where frozen spins do indeed form isolated clusters (Kinzel, 1982a). However, it is not clear to us that the approximations Hertz (1983c) makes are adequate to describe the strong coupling between modes which occur close to T_f. The localized modes that freeze out at highest temperatures may be concrete realizations of the "clusters" discussed by Tholence and Tournier (1974; see also Soukoulis and Levin, 1977).

There have also been treatments of the SK model (see Sec. IV) which use the basis in which the J_{ij} matrix is diagonal (Thouless *et al.*, 1977; Sompolinsky, 1981b; Dasgupta and Sompolinsky, 1983; Ueno, 1983a, 1983b). A number of differences from short-range models are found. All the eigenvectors are extended, because the interactions are infinite range, and furthermore, for $T \geq T_f$, the susceptibility matrix is diagonal in the representation that diagonalizes J_{ij}. However, below T_f the system does not order with a finite amplitude into any of these modes, because of strong coupling between them (Sompolinsky, 1981b; Dasgupta and Sompolinsky, 1983). Rather *each* mode orders an infinitesimal amount.

One of the difficulties with spin glasses, at least in mean-field theory, is that one does not know how to form a symmetry-breaking field that projects out a single thermodynamic state. One interesting attempt to do this was by Bray (1982), who solved the SK model (see Sec. IV) in the high-temperature phase, including a magnetic field proportional to the eigenvector with the largest eigenvalue of the interaction matrix. The effect was to strongly suppress but not remove the Almeida-Thouless instability, which signals the appearance of many thermodynamic states as discussed in Sec. IV.E.

Localization concepts have also proven useful for a characterization of excited states of vector spin glasses at low temperatures: extended excited states are spin-wave-like, but due to the randomness of the couplings there are also many localized excited states (see Secs. V and VI.C for more details).

So far we have discussed attempts to understand spin glasses from our knowledge of the localization problem. Proceeding in the opposite direction, an analogy has been made (Davies *et al.*, 1982) between spin glasses and the problem of highly localized electrons interacting via the Coulomb potential (Efros and Shklovskii, 1975; Efros, 1976).

This system, known as an "electron glass," maps onto an Ising model in a random field with antiferromagnetic interactions varying as r^{-1} and with conserved total spin. At $T=0$ there is a "Coulomb gap"; the density of states vanishes at the Fermi energy, somewhat reminiscent of the vanishing of the internal field distribution $P(\tilde{H})$, at $\tilde{H}=0$, in the SK model [see Eq. (4.106)]. On cooling from high temperatures Monte Carlo simulations reveal spin-glass-like freezing (Davies *et al.*, 1982). At present there is no convincing evidence as to whether or not this occurs via a sharp transition or gradual freezing. Since a random field is present, the transition, if it occurs, would be analogous to the transition on the AT line (see Sec. IV.B) rather than the conventional zero-field spin glass transition. The AT line is most easily characterized by a divergent spin glass correlation length and relaxation times. It is more complicated to define an order parameter, because the symmetry is broken by the random fields. However, $N^{-1}\chi_{\rm SG}$, where $\chi_{\rm SG}$ is given by Eq. (4.35), is finite below the AT line and zero above it, so it could serve as an order parameter. Alternatively, one could use

$$N^{-2}\sum_{i,j}[\langle S_i S_j\rangle_T^2]_{\rm av}-q^2\ ,$$

which is a little easier to compute.

Aharony and Imry (1977b) used spin glass analogies in an approach to the noninteracting localization problem. They noted that the Green's function related to the conductivity in the localization problem is somewhat analogous to the spin glass susceptibility $\chi_{\rm SG}$. Hence, they argued, one should set up a Ginzburg-Landau Hamiltonian for localization with spin-glass-like variables $q_{\alpha\beta}$. The fundamental work on localization by Wegner (1979) proceeded along similar lines but recognized a crucial symmetry that had not been noted before. In fact, one sees from Wegner (1979) that the analogy with spin glasses is not very strong. The symmetries of the problems are different, since the tensor order parameter in the localization problem involves diagonal elements in replica space, whereas only off-diagonal components occur in spin glass theories, and there is no replica symmetry breaking for localization. This reinforces our remark above that the crit-

ical behavior of the two problems is quite different. However, we believe that some qualitative understanding of spin glasses can be obtained from our knowledge of the localization problem.

3. "Ordinary" glasses

The idea that spin glasses have important features in common with "ordinary" glasses like window glass, vitreous silica, or other amorphous materials is a very old one, and has been responsible for giving spin glasses their name. Despite the technological importance of these "ordinary" glasses, there has been relatively little theoretical work on them, in contrast to the case of spin glasses. In fact, the theory of "ordinary" glasses still may suffer from the lack of a commonly accepted simplified model analogous to the Edwards-Anderson model of spin glasses. Thus we shall have to discuss various concepts of rather different nature here.

Now it is doubtful what can be gained by relating spin glasses, systems that are still incompletely understood, to ordinary glasses, system that are even less well understood. Hence we aim at presenting a rather superficial and qualitative discussion only, emphasizing those properties for which common features are established experimentally, and briefly summarize the theoretical concepts, emphasizing these aspects of the theory on which spin glass theory has had a stimulating influence. Thorough general reviews are to be found in Doremus (1973), Ziman (1979), Zallen (1983), and Jäckle (1986).

There is again an enormous diversity of amorphous materials—a few example are covalently bonded glasses like vitreous silica, the structure of which is modeled by a "continuous random network" of bonds (Zachariasen, 1932); metallic glasses bonded by isotropic pair potentials, whose structure is thought of as a dense random packing of spheres; and amorphous polymers, whose structure is conceived of as an arrangement of interpenetrating random-walk-like coils, strongly entangled with each other. In spite of the diversity in chemical nature, bonding, and geometric structure, one finds again certain rather universal properties—particularly the anomalous behavior of low-energy excitations (see, for example, Phillips, 1981, and von Löhneysen, 1981, for reviews), and the anomalous relaxation when one approaches the glass transition from the fluid side (see, for example, Ramakrishnan, 1983; similarities and differences between relaxations in spin glasses and ordinary glasses are also discussed by Ngai and Wright, 1984).

The anomalous low-energy excitations show up in a "linear" behavior of the specific heat for $T \lesssim 1$ K observed for one or two decades in temperature (actually often a law $C \propto T^x$ with $x \approx 1.2-1.3$ is a more accurate description; see, for example, Zeller and Pohl, 1971), rather than the Debye behavior $C \propto T^3$ that one expects due to phonons in insulators. So, additional low-lying excitations in an amorphous structure must be present. They also show up in the thermal conductivity K, which behaves as $K \propto T^y$ with $1.8 \leq y \leq 2$ instead of $K \propto T^3$ as in crystals, and in the ultrasound absorption and dispersion (Hunklinger and Arnold, 1976).

Anderson *et al.* (1972) and Phillips (1972) showed that a convincing explanation of all these anomalies was possible if one assumed a distribution of asymmetric double-well potentials in the amorphous solid; the two levels differ in energy by ε and are separated by a potential barrier V. The low-energy excitation then is the tunneling motion of the considered degree of freedom in this potential well. Although there is now ample evidence that this description of a glass at low temperatures in terms of two-level systems is correct, and that the excitations are indeed quantum-mechanical tunneling modes and not classical thermally activated barrier hopping, an identification of the tunneling degree of freedom in terms of microscopic structural variables is still lacking. There is, moreover, no description in terms of a more basic Hamiltonian from which one could derive any properties entering this model of two-level systems. The problem of which entity these two-level systems are is rather hard, since one is searching for a rather rare degree of freedom (e.g., at $T = 1$ K, of the order of 10^{-4} such two-level systems per atom are excited).

The general theoretical description of the topologically disordered glassy state is also a challenging problem (Kleman and Sadoc, 1979; Rivier, 1979, 1983; Nelson, 1983a, 1983b; Sethna, 1983; Kleinert, 1984; Nelson and Widom, 1984; Sachdev and Nelson, 1984). Rivier (1979, 1983) focuses on the "odd lines," i.e., closed lines threading through the odd rings in the random network. The number of these odd lines, their entanglement structure, etc. are invariant under distortions of the network that do not break any bonds. This invariance is expressed in terms of a gauge theory, qualitatively similar to what is done for spin glasses (Sec. VI.A). Nelson (198a, 1983b) aims rather at an understanding of the structure of metallic glasses, considering the idealized situation (also studied by simulations, e.g., Nagel *et al.*, 1981, Steinhardt *et al.*, 1981, 1983) of identical particles (actual metallic glasses are formed from two species of atoms having distinctly different size, e.g, Fe and B, in the right concentration regime). The ground state of just four identical particles that interact with isotropic pair potentials is clearly a perfect tetrahedron, with particles at the vertices. Nelson (1983a, 1983b) considers the problem of filling space with such tetrahedra. However, five perfect tetrahedra wrapped around a common bond must leave a gap, with a dihedral angle of about 7.4°. The atom near the gap is "frustrated" because it cannot simultaneously sit in the minima provided by the pair potentials of its near neighbors. Because of this frustration, which is reminiscent of the frustration in spin glasses (Sec. III.F), there is no regular lattice of perfect tetrahedra filling ordinary three-dimensional space. Closed-packed lattices contain octahedra as well as tetrahedra; the octahedra are necessary for the periodic structure, though they do not minimize the energy locally.

Recent computer simulations on supercooled fluids (Steinhardt *et al.*, 1981, 1983) have shown the relevance of such packing considerations by giving evidence for extended icosahedral correlations (20 tetrahedra combine, with only slight distortions, to form a regular thirteen-particle icosahedron). Now a perfect packing of tetrahedra would be possible if space were not flat but curved. Of course, it is not admissible physically to have a model with curved space, and so one removes the average curvature by putting into the structure an appropriate number of disclination lines. If these disclination lines were arranged periodically—which would correspond to a periodic arrangement of frustration—one would arrive at the so-called Frank-Kasper phases (Frank and Kasper, 1958, 1959). Thus a disordered network of disclination lines is the proposed model for the structure of the metallic glass. The slow relaxation near the glass transition is attributed to entanglement of these defect lines. Nelson and Widom (1984), as well as Sethna (1983), who takes a related continuum elasticity approach, propose an effective free-energy functional in terms of an order parameter (which would be uniform and nonzero if the glass had a Frank-Kasper-type regular structure) that has SO(4) symmetry. From this Sachdev and Nelson (1984) derive an explicit expression for the structure factor of (nonatomic) metallic glasses, which is in reasonable agreement with experimental data for amorphous cobalt.

We next turn to theories on the glass transition. Although in practice its location and some physical properties of the glass phase depend on the cooling velocity, it is much debated whether there is an underlying equilibrium phase transition to the glassy state. Such a transition would be reached in the hypothetical case when nucleation of the crystal from the metastable undercooled liquid could be turned off, and so one could let the cooling velocity approach zero.

A phenomenological treatment of the thermodynamics of the glass transition and the glassy state, which is applicable even if there is no equilibrium transition, has been proposed by Jäckle (1981). He assumes that at the glass transition temperature T_f the change of relevant time scales is so abrupt that for $T > T_f$ the thermodynamic variables of interest essentially have the metastable equilibrium values of the supercooled fluids, while for $T < T_f$ all slow variables (describing structural relaxation of the glass, etc.) are essentially quenched in, and so the glass is locked into one particular configuration (apart from vibrations; see also Edwards, 1976). With this assumption, the rest entropy of the glass at $T=0$ can be related to properties at the glass transition temperature T_f itself. This treatment implies a discontinuous jump of the specific heat at T_f, roughly consistent with experimental observation.

The first microscopic theory yielding an equilibrium glass transition was proposed for polymers by Gibbs and DiMarzio (1958). They considered a lattice model in which random walks of N steps were put onto the lattice, as well as empty sites, and allowed for both a bending energy (where a walk makes an angle at the lattice) and a van der Waals bond between neighboring occupied segments. From a mean-field-type calculation they found a temperature T_f at which the configurational entropy would turn negative. They interpreted this behavior as an indication that at T_f a second-order equilibrium phase transition to the glass state occurs, which in this treatment appears as a state of zero configurational entropy. In this treatment, the (experimentally established) rest entropy would be a phenomenon of purely kinetic origin, not intrinsically related to the equilibrium glass transition. The theory of Gibbs and DiMarzio (1958) explicitly bears out the so-called "Kauzmann paradox" (Kauzmann, 1948): if one extrapolates the metastable phase from the region above the glass transition smoothly to low temperatures (i.e., without involving an equilibrium transition), absurd results such as negative entropies are obtained.

While in the theory of Gibbs and DiMarzio (1958) the glass transition appears as a second-order transition, the free-volume theory (Cohen and Turnbull, 1959; Cohen and Grest, 1979, 1981, 1982; Grest and Cohen, 1980, 1982) implies a first-order equilibrium phase transition underlying the glass transition. The free-volume model (Cohen and Turnbull, 1959) makes a number of phenomenological assumptions: (i) It is possible to associate a local volume v with each molecule. (ii) When v exceeds a critical value v_c, the excess $v - v_c$ contributes to the free volume. (iii) Molecular transport occurs only when voids having a volume greater than some critical value v^* form by the redistribution of the free volume. (iv) No local free energy is required for free-volume redistribution. Therefore the statistics of liquidlike cells becomes a problem analogous to the percolation problem: a liquidlike cell is in a liquidlike cluster only if it has z neighbors that are also liquidlike (Cohen and Grest, 1979). In order to have a glass, the liquidlike clusters must be well separated from each other, while in the fluid, they form an infinite percolating net. In the purely geometric percolation problem the transition between these two states is of second order, but inclusion of a suitable free energy $f(v)$ depending on the cell volume makes this equilibrium transition underlying the glass transition a first-order transition. Hence the transition from the glass to the fluid is something like the "condensation" of free volume. Although a percolation threshold is not reached, the size distribution of liquidlike cells (the only kind that support diffusion) is responsible for a relaxation proportional to $\exp[-(t/\tau)^{\zeta}]$, where τ is some relaxation time and ζ some exponent. This behavior is consistent with appropriate experiments, according to the discussion given by Cohen and Grest (1981, 1982). Although this approach clearly is very phenomenological, it has had remarkable success in "fitting" a variety of experimental data, including the dramatic increase in viscosity of the supercooled liquid as it approaches the glass transition [typically the viscosity follows the Vogel-Fulcher law (Vogel, 1921; Fulcher, 1925) for a broad range of temperatures, but deviates from it very close to T_f]. Dynamic Monte Carlo simulations of glass transitions in dense

polymer states (de la Batie *et al.*, 1984) might also be interpreted by this theory.

However, in spite of this, there is no agreement that this approach is a reasonable framework, and very different ideas are also discussed in the literature. For example, Chui, Williams, and Frisch (1982) draw a parallel between the glass transition and the roughening transition of grain boundaries. Thorpe (1983) and Phillips and Thorpe (1985) relate the glass transition not to ordinary ("scalar") percolation but to "vector percolation" (at the threshold for vector percolation, the elastic restoring force of a lattice vanishes). Phillips (1979, 1981, 1983) has introduced concepts such as "constraint counting arguments" to explain whether a material should have a glass transition. Then there are a number of proposals that glass transitions can occur as singularities appearing in kinetic quantities only (Bengtzelius *et al.*, 1984; Fredrickson and Andersen, 1984; Leutheusser, 1984; Das *et al.*, 1985; Leutheusser and Yip, 1985; Edwards and Vilgis, 1986). Static quantities at this transition remain regular. For example, Edwards and Vilgis (1986) suggest that the diffusion constant of a system of hard rods in solution vanishes according to a power law when a critical concentration is reached (while a Vogel-Fulcher-type regime exists as a precursor effect). Closer to real glasses are the theories of Leutheusser (1984) and Leutheusser and Yip (1985), who study the kinetic theory of hard spheres, and of Das *et al.* (1985), who start from hydrodynamic equations and find a transition where the viscosity diverges as a power law.

The idea that glass transitions might be purely kinetic has a certain attractiveness; however, when one applies similar concepts to spin glasses (Götze and Sjögren, 1984) one arrives at variants of dynamic mean-field theories. There, the transition shows up in the divergence of a static quantity as well, the nonlinear susceptibility. Thus it is conceivable that in the above theories there is a corresponding appropriate static quantity, which would also show the transition, has not been looked at yet. According to the ideas of Nelson (1983a, 1983b) and Steinhardt *et al.* (1981, 1983), one would need a quantity related to the quasi-icosahedral order of Frank-Kasper phases in the same way as the Edwards-Anderson order parameter is related to the uniform magnetization. We emphasize that our comments on this point obviously are rather speculative, and both lack of space and lack of expertise prevent us from giving a more careful assessment of the merits of the various approaches to the glass transition.

VII. CONCLUSIONS: TO WHAT EXTENT ARE SPIN GLASSES NOW UNDERSTOOD?

The freezing phenomena observed in spin glasses have proven difficult to understand despite an enormous number of papers on the subject, many of which we have been unable to mention in this review.

One of the problems is that phase space has a complicated landscape of valleys. Barriers between valleys grow as the temperature is reduced, so eventually the system behaves nonergodically, being trapped in a single valley for the duration of an experiment. One may ask, what is the generalization of statistical mechanics that describes this situation? Unfortunately, little progress has been made in applying nonequilibrium statistical mechanics to spin glasses, so, to our knowledge, there is no detailed theory of slowly relaxing metastable states, for example. In mean-field theory, discussed in Sec. IV, the situation is a little better. Barriers diverge in the thermodynamic limit, and one can calculate the properties of a single thermodynamic state (otherwise called "phase," "thermodynamic state," "pure state," or "ergodic component") of a given free energy. For example, the spin glass order parameter for a state of minimum free energy is given in the Parisi theory by $q(x=1)$. Such quantities are of interest because one might hope to prepare the system in a state of near-minimum free energy by slow cooling. However, there does not appear to be a satisfactory theory, even at the mean-field level, which can describe relaxation following a change, for example, in the magnetic field. Note also that it is not clear to what extent mean-field theory is relevant to the real world.

Another difficulty is that we are dealing with a random system for which some averaging over the disorder should be performed. This is particularly troublesome for spin glasses where, at least in mean-field theory, certain quantities of interest do not have the usual self-averaging property (see Secs. III.A and IV.F). Averaging is usually carried out by means of the replica trick (see Sec. III.C), which rather obscures the physics. However, very recently considerable progress has been made on deriving the mean-field theory without replicas (Mézard *et al.*, 1986a, 1986b) (see Sec. IV.F), so we finally seem to be entering the "post-replica" age.

In fact, the mean-field theory, by which we denote the exact solution of the Sherrington-Kirkpatrick (1975) model, is now rather well understood. Parisi's solution seems to be exact, and its physical significance, in terms of the many-valley structure of phase space, is firmly established (see Sec. IV.E).

The question of order parameters has aroused a lot of discussion, so we now summarize what we think is the current position *within mean-field theory.* In general, order parameters are a set of quantities that describe how the low-temperature ordered state differs from the high-temperature disordered state, where, by definition, all order parameters are zero. For the mean-field theory of spin glasses one has to specify whether the term "low-temperature ordered state" refers to a single thermodynamic state or to the set of states in the statistical sum. In the former case the order parameter is very simple. Ordering in a single state of minimum free energy is described by the Edwards-Anderson order parameter q_{EA}, Eq. (3.63), which is given by $q(x=1)$ in the Parisi theory. It is the same for all states of minimum free energy and for all bond configurations. On the other hand, the order parameters describing the set of all phases in the statistical sum are much more complicated because (a) one has

to include overlaps between different states (see Sec. IV.E) and (b) these overlaps are not self-averaging (see Sec. IV.F). Hence one defines an overlap distribution for a single bond configuration $P_J(q)$, given by Eq. (3.86), and the order parameter is then the distribution among different bond configurations of this distribution. A more convenient description has been given by Mézard *et al.* (1984a, 1984b) in terms of the set of distributions $\Pi_y(Y)$, where Y is an abbreviation for

$$Y_J(q)=\int_q^1 P_J(q)dq \ , \tag{7.1}$$

y denotes $y(q)$, which is the average of Y, i.e., $y(q)=[Y_J(q)]_{\rm av}$, and $\Pi_y(Y)$ is the probability among different bond configurations of obtaining a given value for Y at the q value where the average of Y is y. The set of functions $\Pi_y(Y)$ together with a knowledge of $y(q)$ [or equivalently, the average distribution $P(q)$] form the order parameter. This is a useful formulation because the functions $\Pi_y(Y)$ are smooth and are universal in that they do not depend upon the order-parameter function $q(x)$.

The above discussion of order parameters is for mean-field theory. As mentioned below, there may be only a single thermodynamic state for real systems in three dimensions, in which case the order parameter would be just a single number describing the ordering in this state.

Another topic on which we wish to comment is the question of broken symmetry in spin glasses. In zero magnetic field the system is invariant under inversion of the spins. The spin glass state where $\langle S_i \rangle_T \neq 0$ breaks this symmetry. The symmetry tells us that for every thermodynamic state there is another one of the same free energy with spins inverted. In mean-field theory we know that there are additional states, unrelated by symmetry, with virtually the same free energy. We call this phenomenon "ergodicity breaking." In addition, there is a transition in a finite field for Ising spins, at least in mean-field theory. This Almeida-Thouless (AT) line has no symmetry breaking associated with it because spin inversion symmetry is broken in any nonzero field. Hence ergodicity breaking, but not symmetry breaking, occurs on the AT line. In this sense one could say that it is a "purer" spin glass phenomenon than the zero-field transition, which additionally has the standard feature of symmetry breaking. Vector spin glasses in a uniform field have a Gabay-Toulouse (GT) line at which symmetry breaking in the transverse directions take place. An AT-like transition without symmetry breaking would probably occur for a vector spin glass in a random staggered field that pointed with equal probability in all directions.

If mean-field theory is well understood, less is known about the behavior of systems with short-range interactions. It is clear that spin glass freezing involves strong cooperative effects, and both the spin glass correlation length $\xi_{\rm SG}$ and the relaxation times grow as $T \to T_f^+$. Whether they diverge at or near the apparent freezing temperature is still not 100% clear. The most recent numerical calculations on models with Ising spins (Bhatt and Young, 1985; Bray and Moore, 1985a; McMillan, 1985a; Ogielski and Morgenstern, 1985) find reasonably good evidence for a transition (where $\xi_{\rm SG}$ diverges) at a finite temperature in dimension $d=3$. They are consistent with the lower critical dimension d_l (see Sec. V.E), being between 2 and 3. However, because of difficulties in working with very large systems in the temperature region of interest, we feel there is still a faint possibility that $T_f=0$ in $d=3$. If one assumes T_f is finite, one can obtain rough estimates for critical exponents by fitting the data (most of which are far above T_f) to the expected power-law form. For static exponents, Bhatt and Young (1985) obtain

$$\gamma \approx 3.2, \quad \beta \approx 0.5, \quad \nu = 1.4 \ , \tag{7.2}$$

while for dynamics the most accurate estimate is due to Ogielski (1985) who found

$$z\nu \approx 7 \ . \tag{7.3}$$

By contrast to these findings for the Ising model, all approaches for isotropic Heisenberg models (see Banavar and Cieplak, 1982a, 1982b; McMillan, 1985b; Morris *et al.*, 1986; Olive *et al.*, 1986) find $T_f=0$ and probably $d_l=4$. Furthermore, simulations for site-disorder Heisenberg models with RKKY interactions suggest $T_f=0$ in $d=3$ if there is no anisotropy (Walstedt and Walker, 1981, 1982; Chakrabarti and Dasgupta, 1986). Bray *et al.* (1986) have proposed that Heisenberg systems with RKKY interactions behave differently from short-range models and are at, rather than below, the lower critical dimension, though this is still compatible with $T_f=0$.

Although one would think that the RKKY Heisenberg model is most closely related to real systems, there are a number of experiments on different systems that find a nonzero T_f and, for the exponents, even quote numbers close to the above Ising values, Eqs. (7.2) and (7.3). For γ, see, for example, Omari *et al.* (1983), Beauvillain, Dupas, *et al.* (1984), Beauvillain, Chappert, *et al.* (1984), and Barbara and Malozemoff (1983). For Δ, which is $(\beta+\gamma)/2$ according to scaling, see, for example, Bontemps *et al.* (1984) and Hamida and Williamson (1985). For $z\nu$, see Bontemps *et al.* (1984) and Hamida and Williamson (1985). It is not clear whether one should conclude that the critical properties of real spin glasses above T_f agree rather well with the numerical results on Ising models: first of all, the accuracy of the numbers quoted above is very uncertain—even if it is correct that a transition at $T_f>0$ exists, it is not established that the critical region is wide enough for the above fits to be meaningful [remember that a few years ago it was claimed $z\nu \approx 1$ (Kirkpatrick, 1980)]. Second, a similar reservation must be made about the experiments quoted—most of the data are really outside of the region one would identify as the critical region for traditional phase transitions. In fact, a recent attempt to measure exponents closer to T_f has resulted in rather different values, e.g., $\gamma \approx 2.2$ (Bouchiat,

1986). Third, most experimental systems have a predominantly isotropic exchange interaction with relatively weak anisotropy, and current theories indicate $T_f=0$ for a completely isotropic system. If this is so, the anisotropy, though generally weak, must be playing an important role; one should expect that the freezing temperature is affected if the anisotropy is changed by addition of suitable impurities. Moreover, a crossover should take place between Heisenberg behavior well away from T_f and Ising behavior close to T_f. To our knowledge, there is no experimental evidence for this. A complete theory for $T > T_f$, involving both RKKY interactions and anisotropy, would be very desirable.

Let us suppose it is correct that T_f is finite in $d=3$ for Ising spin glasses or Heisenberg spin glasses with anisotropy. Then one would have to ask whether the low-temperature state is similar to that of mean-field theory, where there is (a) spin glass ordering and (b) a complicated ordered state in which several thermodynamic states contribute to the statistical sum, so $P(q)$ is nontrivial. It has been suggested (Bhatt and Young, 1985; Ogielski, 1985) that there might be no ordering in the low-temperature state, but that correlations decay to zero with a power of the distance. More recently Fisher and Huse (1986) have argued that there is spin glass order, and correlations approach this finite limit at large distances with a power-law decay. This also appears to be compatible with the numerical data of Bhatt and Young (1985) and Ogielski (1985). Arguments have been given that there is only one thermodynamic state in $d=3$ (McMillan, 1984d; Moore and Bray, 1985; Fisher and Huse, 1986). Moore and Bray (1985) have proposed that this occurs even for all $d<6$, while Fisher and Huse (1986) suggest that *any* short-range model has only one thermodynamic state. Interestingly, if there is just one state in $d=3$, then real spin glasses are *simpler* in some respects than mean-field theory. This conclusion is also corroborated by scaling and renormalization-group arguments for the critical behavior (Fisher and Sompolinsky, 1985). If there is only a single thermodynamic state, and hence a trivial $P(q)$, there is no AT line in a magnetic field, because this separates a region of the mean-field phase diagram with many states from the region with a single state. Hence the best experimental test of whether a mean-field description of the low-temperature phase is correct would be to look for an AT line. The clearest signature of this would be divergent relaxation times in a field. We feel that experiments done so far which identify characteristic lines in the H-T plane as AT lines (or GT lines) are not definitive because they do not show convincingly that the relaxation times actually diverge. Furthermore, AT and GT lines would (if they both existed) have a similar shape at low fields (Fisher and Sompolinsky, 1985) for $d<6$, in contrast to mean-field theory. If there is only a single thermodynamic state, and therefore mean-field theory is wrong below T_f, what is the correct description? A first step towards answering this question has been taken by Fisher and Huse (1986), but more work is clearly needed.

Progress has been made in understanding some properties of spin glasses well below the observed T_f where, independent of whether at T_f an equilibrium transition occurs, the system is trapped in a metastable state. The triad model (e.g., Henley *et al.*, 1982) involving Dzyaloshinskii-Moriya (DM) anisotropy (Fert and Levy, 1980, 1981; Levy *et al.*, 1982) seems to explain ESR (Schultz *et al.*, 1980; Gullikson *et al.*, 1983) and torque measurements (e.g., Hippert *et al.*, 1982) fairly well. The DM anisotropy may also account for displaced hysteresis loops (Prejean *et al.*, 1980). However, the assumption of rigid spin rotations made in the triad model appears to break down when the spins are rotated by a large angle away from their original directions.

Of course, there is still the problem of understanding the dynamic response of spin glasses well below T_f after changes of the magnetic field. Although some features of the behavior are reproduced qualitatively by Monte Carlo simulations (e.g., Binder, 1977a, 1977b, 1977c; Kinzel, 1979; Kinzel and Binder, 1983), a more quantitative understanding clearly is lacking. Particularly interesting is the observation of "stretched exponential decay" ($\propto \exp[-(t/\tau)^y]$, with $y<1$) of the remanent magnetization (e.g., Chamberlin *et al.*, 1984). There are several "explanations" of such a law, but it is not obvious which of them, if any, is the correct one for real spin glasses. It should also be remarked that the recent simulations of Ogielski (1985) find that a decay of this type also occurs in the spin-autocorrelation function even well above T_f, and that this stretched exponential decay is familiar from relaxation in "ordinary" glasses.

It is interesting to ask whether there are other analogies between spin glasses and real glasses (see Sec. VI.F.3). So far the analogy does not seem to have been pushed much beyond the fairly obvious facts that both systems have degeneracy and slowly relaxing metastable states. It would be very interesting to establish whether there is a characteristic length in the glass problem, too, analogous to the spin glass correlation length, which grows as the glass temperature is approached from above. On the other hand, features that seem similar when superficially observed may not have the same origin: while for glasses there is agreement that the (nearly) linear temperature variations of the specific heat are due to two-level systems, for spin glasses a linear specific heat is accounted for by linear (oscillatory) spin excitations. While a glass obviously has well-defined long-wavelength acoustic photons, it is not so clear that magnon-type excitations become undamped in spin glasses in the long-wavelength limit.

Relatively little is known when we leave the canonical spin glasses and consider systems where ferromagnetic and spin glass orderings compete: are there phases with both types of ordering simultaneously present? Are there additional types of phases (random domain states, or ferromagnets with power-law decay of correlations, etc.)? The behavior near reentrant ferromagnetic phase boundaries needs further experimental clarification, as well as corresponding theoretical work. For example,

what are the necessary ingredients of a model if it is to have a reentrant phase boundary? Another class of systems that deserves careful further study is that of the quadrupolar glasses; again the question of similarities and differences with spin glasses comes up.

For the future we anticipate that more experimental work will be carried out to clarify the form of the slow relaxation, both above and below T_f, as well as to better characterize the critical singularities near T_f. On the theoretical side, we expect a continuation of numerical calculations to further elucidate the nature of spin glass freezing in three dimensions. A breakthrough in analytic approaches for $d=3$ would be highly desirable, in order to clarify our understanding of lower critical dimensions, universality classes for spin glasses, etc.; unfortunately it is not clear from which direction such a breakthrough is likely to come. Since thorough numerical studies of two-dimensional models have been undertaken, and these have $T_f=0$, but still a very interesting dynamical freezing, it would be interesting if one could find real two-dimensional systems experimentally.

Finally, we expect that insight gained from spin glasses will continue to be fruitfully applied to other diverse areas of science, such as neural networks (Little, 1974; Hopfield, 1982), models for evolution (Anderson, 1983), and optimization problems (Kirkpatrick *et al.*, 1983).

Note added in proof. Since the main text of this article was completed, the enormous activity in this field has continued. In this note, we do not attempt to fully cover all this very recent material; rather we mention briefly only a number of papers which we feel are particularly relevant to the topics discussed in the main text.

Experimental work addressing the question of whether there occurs an equilibrium phase transition has been presented by Beauvillain *et al.* (1986), Courtenay *et al.* (1986) and by Vincent *et al.* (1986). Beauvillain *et al.* (1986) analyze the equation of state above T_f in a magnetic field for the amorphous manganese aluminosilicate (which has $T_f=2.95$ K) and the amorphous manganese fluorophosphate spin glass (which has $T_f=2.80$ K). While the exponent γ is about 3.5 for both materials, the scaling fit yields $\beta=1.4\pm0.1$ for the former and $\beta=0.8\pm0.1$ for the latter system. Vincent *et al.* (1986) analyze the dynamic susceptibility of the $CdCr_{2\times0.85}In_{2\times0.15}S_4$ spin glass in a way completely analogous to the work by Bontemps *et al.* (1984, 1986) described at length in the main text. They obtain rather similar conclusions: the data can be fitted either with $T_c=0$ or with $T_c=16.7$ K; but in the former case a very large exponent ($z\nu\approx11$) in the formula of Binder and Young (1984) would be needed, and a somewhat unphysical prefactor. On the other hand, with $T_c=16.7$ K an exponent $z\nu\approx6.9\pm0.9$ results, similar to Bontemps *et al.* (1984, 1986) and to the simulations of Ogielski (1985). Hence, these experiments are also in favor of a finite-temperature transition, but questions about the precise values of the critical exponents remain, see, e.g., Levy and Ogielski (1986) and Souletie and Tholence (1985, 1986). Particularly interesting in this context is the work of Courtenay *et al.* (1986) who study the effect on critical exponents, of changes in the anisotropy.

Additional experimental work is devoted to a study of the anomalous slow relaxation of spin glasses below T_f [e.g., Nordblad, Lundgren, and Sandlund (1986); Nordblad *et al.* (1986); Reim *et al.* (1986); Ocio, Alba, and Hammann (1985); Ocio, Bouchiat, and Monod (1985, 1986); Alba *et al.* (1986); Ferre *et al.* (1986); Carre *et al.* (1986); Tholence *et al.* (1986)]. For example, Ocio *et al.* (1985a) and Alba *et al.* (1986) study the temporal scaling behavior of the aging behavior in $CsNiFeF_6$ spin glasses and in *Ag*Mn 2.6%. In both cases they find that the time decay of the thermoremanent magnetization can be accounted for by a phenomenological theory of the aging of amorphous polymers due to Struik (1978), for a wide range of both waiting times and observation times. The data are very similar to related experiments of Hoogerbeets *et al.* (1985, 1986) and Nordblad, Lundgren, and Sandlund (1986); Nordblad *et al.* (1986); but the analysis and interpretation is somewhat different. Unlike the polymer case, the "master curves" resulting from the present approach distinctly depend on temperature. These "master curves" can be represented by products of a power law with a stretched exponential; moreover, this behavior is consistent with analyses of the magnetic noise spectrum in spin glasses (Ocio *et al.*, 1986). An interesting analysis of the magnetic noise in $Eu_{0.4}Sr_{0.6}S$ is given by Reim *et al.* (1986); the authors suggest that their data show the validity of the fluctuation-dissipation theorem both above and below T_f. The power law decay of the noise spectrum is again related to the stretched exponential form of the decay of the remanent magnetization in this material (Ferre *et al.*, 1986). We consider the proper explanation of these stretched exponential decay laws (see also Continentino and Malozemoff, 1985; Campbell, 1986) as an interesting challenge for future theoretical work.

Aeppli *et al.* (1986) find from neutron scattering studies of $Eu_{0.54}Sr_{0.46}S$ that not only the remanent magnetization but also the magnetic short-range order on length scales less than 75 Å depend on the cooling history of the system. This implies that also on a rather small scale the system can be frozen on several distinct "valleys." Finally, we draw attention to recent measurements of the complete temperature-concentration phase diagrams of quadrupolar glasses (Elschner *et al.*, 1985), and to a recent review of experimental work on spin glasses emphasizing materials containing rare earths (Maletta and Zinn, 1986).

Turning now to the theory, we emphasize that recent studies (Singh and Chakravarty, 1986; Ogielski, 1986; Morgenstern, 1986; Reger and Zippelius, 1986) have continued to find evidence for a finite transition temperature for Ising systems in three dimensions, while a novel type of Monte Carlo simulation which greatly reduces the relaxation times in comparison to standard methods strengthens the evidence that in two dimensions the transition is at $T=0$, with $x_{SG}\propto T^{-5.3}$ (Swendsen and Wang, 1986). Singh and Chakravarty (1986) have greatly extended the high-temperature series for the $\pm J$ model in three

and four dimensions and find results in good agreement with Monte Carlo simulations of Bhatt and Young (1985a) and Ogielski and Morgenstern (1985). Ogielski (1986) studied for the $\pm J$ model the domain-wall free-energy distribution [see also Caflisch *et al.* (1985b) for a discussion of the scaling behavior of the domain-wall free energy], while Morgenstern (1986) investigated the model with Gaussian bonds by a transfer matrix technique. The critical temperatures and exponents found by these two approaches agree reasonably well with the earlier Monte Carlo data.

Statics and dynamics of the three-dimensional $\pm J$ model are studied by Reger and Zippelius (1986) with high-temperature series expansions. Although these series are rather short, they obtain the phase diagram with a paramagnetic phase, a ferromagnetic phase, and a spin glass phase. By studying the moments of the frequency-dependent order parameter susceptibility they find that dynamic scaling holds at the spin glass transition where the relaxation time diverges with a power law and the exponent is similar to that given by Ogielski (1985). The multicritical points seem to lie on the Nishimori (1980, 1981, 1986) line where the energy is nonsingular, and which is given, in general, by the equation $P(J_{ij})/P(-J_{ij})=\exp(2J_{ij}/k_BT)$. This behavior is rather similar to the infinite-range model (Georges *et al.*, 1985). Furthermore, the rather curious fixed point on the ferromagnetic-paramagnetic phase boundary found for the two-dimensional Ising problem by Southern and Young (1977) and McMillan (1984c) also lies very close to the Nishimori line in the more accurate work of McMillan (1984c). The physical significance of the Nishimori line and its relation to the multicritical point needs further clarification.

In this review, we have not discussed reentrant spin glass behavior very much. Many experimental systems undergo the sequence of phases paramagnet-ferromagnet-spin glass for a range of concentration but this behavior is difficult to reproduce theoretically and does not occur in mean-field theory (Sec. IV) either for Ising or Heisenberg systems. Recently, Saslow and Parker (1986) have investigated this problem for an XY spin glass on the square lattice by solving iteratively the local mean-field equations. They find the magnetization initially increases on lowering the temperature, but then decreases due to freezing of the transverse spin components. Clearly, more work still needs to be done on understanding the basic mechanism for reentrant behavior, which also needs to be studied more extensively by experiment (for recent work see, e.g., Mirabeau *et al.*, 1986 and Campbell *et al.*, 1986).

When this review was written, available numerical results (e.g., Olive *et al.*, 1986; Jain and Young, 1986; Morris *et al.*, 1986; Banavar and Cieplak, 1982a, 1982b; Cieplak and Banavar, 1984; McMillan, 1985b) suggested that isotropic vector spin glasses are very different from Ising systems and do not have a transition in $d=2$ or $d=3$. However, Kawamura and Tanemura (1985) have found evidence for a transition in Villain's (1977a) chirality variables in a two-dimensional XY spin glass by Monte Carlo methods. They do, however, agree with earlier Monte Carlo work that there is no Edwards-Anderson order. While there is some doubt whether the results at the lowest temperatures were fully equilibrated, this interesting result suggests that more remains to be understood about isotropic vector systems in two and three dimensions, Kawamura and Tanemura (1985) suggest that their results could possibly account for spin glass behavior in the quasi-two-dimensional system $Rb_2Mn_{(1-x)}Cr_xCl_4$ (Katsumata *et al.* 1984).

Derrida *et al.* (1986) present a method by which one-dimensional random models can be constructed which are exactly soluble at a particular temperature, and review other exact results on random systems in one dimension. Colborne (1986b) shows that the remanent magnetization in the one-dimensional Gaussian spin glass decays as $M(t)\propto\exp[-(t/\tau)^{1/2}]$ where τ is a relaxation time.

Athanasiu *et al.* (1986) show that the ultrametric structure of spin glass states is invariant under redefinitions of the metric (i.e., if the distance between states is measured not by the order-parameter overlap but by the overlap of other quantities, such as, coarse-grained magnetization, energy density, etc.).

Thouless (1986) extends the solution of an Ising spin glass on the Cayley tree obtained by Bowman and Levin (1982) to a now zero magnetic field, and shows that an AT line occurs. It should, however, be emphasized that frustration occurs on a Cayley tree because of competition between the bonds and the boundary conditions on the surface where a finite fraction of the spins lie. This is different from the way it occurs in a conventional lattice where one has closed loops of bonds. It is therefore still unclear whether an AT line would occur for a short-range system in some large (but finite) dimensions. While Fisher and Huse (1986), on the basis of droplet-model-type arguments, have suggested an AT line only occurs in the infinite-range model, their reasoning is criticized by Villain (1986), who shows that this result sensitively depends on assumptions about the hierarchical nesting of droplets inside of droplets. In addition, Bray and Moore (1986) extend the scaling considerations mentioned in Sec. V to develop a detailed one-parameter-scaling theory of the ordered phase of spin glasses in low dimension, which also addresses this problem. In terms of the exponent y, which describes the growth of the characteristic coupling with length scale L at zero temperature, $J(L)\propto JL^y$, the condition for the absence of an AT line is $y<d/2$ (see also McMillan, 1984d)]. Their theory implies that the Parisi overlap function $P(q)$ is trivially a δ function, and that one has a nonzero order parameter for $T<T_f$, while at the same time the connected correlation function $[\langle S_0S_R\rangle_T^2]_{\rm av}-[\langle S_R\rangle_T^2]_{\rm av}$ has a power-law decay with distance R, and the magnetic equation of state responds in a singular fashion to the field $H, m_{\rm sing}\propto h^{d/(d-2y)}$. While y is estimated as $y\approx 0.2$ for $d=3$, not much is known on y for higher dimensions. Another interesting prediction of Bray and Moore (1986) is that there is an effective dynamic AT line, describing the fact that observation time $t_{\rm obs}$ and relaxation time become comparable

(similar in spirit to the discussion in Sec. V.E but adapted to a situation with $d > d_l$); this line is given by the equation

$$(H/T_c)^2 \propto [\ln(\tau_{\rm obs}/t_0)]^{-(d-2y)/\psi}(1-T/T_c)^{\gamma+\beta} ,$$

where t_0 is a microscopic time and ψ describes how free-energy barriers scale with length ($\Delta F \propto L^{\psi}$). The question is still open, however, to identify the range of dimensionalities for which this one-parameter-scaling theory holds.

Apart from this work which mostly addresses the short-range spin glasses, there have also been considerable recent efforts to extend mean-field theory to other cases: quadrupolar glasses (Kanter and Sompolinsky, 1986; Goldbart and Sherrington, 1985); quantum problems (Nishimori *et al.*, 1986; Usadei *et al.*, 1983; Brieskorn and Usadel, 1986; Ishii and Yamamoto, 1985; Usadel, 1986), etc. Also systems with random anisotropy axes find increasing attention (e.g., Jagannathan *et al.*, 1986, Fischer and Zippelius, 1985, 1986). However, the extension of concepts of spin glass research to optimization problems and to brain modeling is probably the field growing most quickly, as is evidenced by the papers collected in van Hemmen and Morgenstern (1986).

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