The semiclassical theory of laser cooling

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This paper reviews the basic theory of the mechanical action of light in resonant interaction with atoms. At present the main application is laser cooling, but the approach is applicable to a broader range of phenomena. It is based on an adiabatic elimination philosophy, which turns out to give the lowest-order quantum corrections to the behavior found when the photon momentum goes to zero. Hence it is called a semiclassical theory. In this manner a subjective but consistent approach can be presented; other treatments are incorporated or mentioned at the appropriate places. Both the classical and the quantum-mechanical approach are discussed. Those readers who wish to obtain only a heuristic overview of the phenomena can concentrate on Sec. III, which treats both the photon momentum effects and their connection with photon counting statistics. The detailed theoretical treatment utilizes Wigner functions and Fokker-Planck techniques. The ensuing theory is applied both to the cooling of free particles and trapped ones. The paper ends with an extensive bibliography, where the author lists most papers of interest for research into the mechanical manifestations of light. For completeness, many papers are included that are not explicitly mentioned in the text.

CONTENTS

I. INTRODUCTION

The classical Maxwell theory showed quite early that the radiation field carries with it momentum. There were some early experiments in Moscow (Lebedev, 1901,1910a,1910b) and in Hannover (Nichols and Hull, 1901,1903), but at that time these small effects were hard to observe. After Einstein introduced the concept of a photon, he showed (Einstein, 1917) that momentum conservation was an important aspect of the equilibrium between matter and radiation, and in the full quantum theory the conservation of energy and momentum in each microscopic event became an important issue. This was investigated experimentally by Compton (1925), Compton and Simon (1925), and Bothe and Geiger (1925).

The discussion of the recoil momentum in radiative processes was intimately tied up with the question of quantizing the radiation. It does, however, occur also in the semiclassical theory, because the momentum change of matter is enforced by the spatial variation of the radiation field. This is not affected by the quantization, which only concerns the dynamical time evolution of the field.

A periodic variation of the electromagnetic field in space will cause scattering between particle states of different momenta, and this fact was stressed by Schrödinger in 1927, when he showed that one can obtain both energy and momentum conservation in the Compton effect without quantization of the radiation field. A similar scattering process from a standing wave gives rise to the Kapitza-Dirac effect, which is essentially Bragg scattering by umklapp processes from the periodic potential.

There are many ways of observing mechanical forces caused by light fields, but the simplest one is to observe the scattering from the spatial pattern created by periodicity at the wavelength of the light. This changes the atomic momentum by the wave vector, and the change can hence be considered as caused by the momentum of the individual photons constituting the radiation field. It is the strongest force achievable in a light wave—there can be no steeper change of the energy density—and hence it is the one utilized to cool the motion of atomic particles. In this review we shall concentrate on the manifestations and theoretical treatments of this force. Other forces exist, and they may be discussed in connection with various applications, but in this article they will only be mentioned.

Before we enter into a detailed discussion of the subject matter, we review briefly in this introduction the history and the various aspects of the topic of light-induced forces. The introduction ends with an overview of the material presented in the main body of the article.

The recoil momentum of light was observed by Frisch in 1933. The question of photon recoil momentum was regarded as settled in 1950, according to Cross and Ramsey (1950). To obtain better results with thermal light sources was difficult, and only occasional investigations

tried to do this, e.g., Picque and Vialle (1972). With laser light the experiments became much easier to perform, and the first successful ones were reported in the early 1970s (Schieder *et al.*, 1972; Jacquinot *et al.*, 1973; Bernhardt *et al.*, 1974).

Another area where photon recoil had been found to affect the measurements was high-resolution spectroscopy. As pointed out in Kol'chenko *et al.* (1969), the spectral lines of saturation spectroscopy become split; these effects were first seen in Boulder (Hall *et al.*, 1976) and in Novosibirsk (Baklanov *et al.*, 1981). A strong signal theory for these cases was first formulated by Stenholm (1974).

The forces acting on an atom in a field consist of gradient forces, discussed by Askar'yan (1962), and resonance light pressure, discussed by Ashkin (1970b). The effects of these forces were early considered by Kazantsev (1973) and Krasnov and Shaparev (1975).

The spreading due to spontaneous emission was also considered fairly early (Kazantsev, 1975; Pusep, 1976). The first derivations of Fokker-Planck descriptions are given in Kazantsev (1975) and in some detail in Baklanov and Dubetskii (1976). The derivation was made systematic for a random-phase field by Javanainen and Stenholm (1980a,1980b), and the coherent contributions were obtained by Gordon and Ashkin (1980), Cook (1980a), and Minogin (1980b).

The use of lasers for cooling was suggested by Hänsch and Schawlow (1975) and discussed theoretically by Baklanov and Dubetskii (1976) and Javanainen and Stenholm (1980b). So far no cooling of thermal atoms has been achieved in this way, but the longitudinal velocity distribution of an atomic beam has been changed by Balykin *et al.* (1979,1980), Phillips and Metcalf (1982), Phillips *et al.* (1983), and Prodan *et al.* (1982).

In a standing wave there appears a spreading of the atomic velocity distribution. This was first observed by Arimondo *et al.* (1979a,1979b). When incoherent spreading is dominant, the process is similar to a diffusion, but when it stays coherent it is diffractive. This has been observed by Moskowitz *et al.* (1983), and a theoretical treatment of the transition between the two regimes is given in Tanguy *et al.* (1984).

The trapping of particles utilizing radiation fields was discussed by Gaponov and Miller (1958), Letokhov (1968), and Ashkin (1970a). Related discussions appear in Kazantsev (1973). The present understanding is that no trap based on steady resonance forces is possible (Letokhov and Minogin, 1978a; Ashkin and Gordon, 1983); for detailed discussions and suggested traps, see Phillips (1984). When the atom is placed in an inhomogeneous magnetic field, the configuration can trap atoms. At the time of finishing this article, I saw the first report of successful magnetic trapping of neutral atoms (Migdall *et al.*, 1985). By the time this article appears, more successful experiments may well have been performed in this active field; see Chu *et al.* (1985).

The use of laser cooling acquired new interest when it was reported (Neuhauser *et al.*, 1978a; Wineland *et al.*, 1978a) that trapped ions had been cooled by light. This work was developed from Dehmelt's earlier investigations of single-particle traps (Dehmelt, 1967). The theory for these experiments was first developed by the experimental groups, Neuhauser *et al.* (1978a), Wineland and Itano (1979), and Itano and Wineland (1981). A strong-signal approach has been pursued by Javanainen and Stenholm (1980c,1981a,1981b), and its final results are reported by Lindberg and Stenholm (1984), Javanainen (1985), and Stenholm (1985). This approach promises many interesting spectroscopic applications.

Mandel (1979a,1979b) first pointed out the relation between the atomic momentum distribution and photon statistics. The connection with the Fokker-Planck theory was investigated by Cook (1980b,1981) and Stenholm (1983). The relationship rests on the conservation of momentum for free particles. Each outgoing photon leaves a permanent recoil record on the atomic velocity, and hence the statistical properties of motion and photons are related. In resonance fluorescence the occurrence of sub-Poissonian statistics mirrors the presence of coherence effects in the momentum spreading. Such statistics have recently been seen by Short and Mandel (1983); so far the relation to atomic momentum lacks direct verification.

This introduction has tried to trace the history of photon recoil effects and the main aspects of laser cooling. In each area only the first papers and some of the most recent ones have been mentioned. More detailed accounts of the various developments are found in connection with the discussions in this paper. In addition, the reference section lists most papers of relevance to the topics treated, including many not explicitly mentioned in the text. Even here a subjective selection has been necessary. For instance, questions concerning recoil effects on high-resolution spectroscopy are referred to only through a few papers. Moreover, the fields of photon statistics and the free-electron response form areas of research on their own, and no complete listing of references has been attempted here. From a few key references the relevant developments can be traced.

In this review I shall concentrate on laser cooling and phenomena connected with it. Many related areas of study will receive only brief discussions. In addition a certain approach to the theory will be pursued. For reasons that will become clear later this is called a semiclassical approach. It is a development of the adiabatic elimination technique, which has been found so useful in many areas of quantum electronics. I choose to use this approach partly because it has been utilized in our own work, but also because it offers a general point of view that gives a certain unity to the various treatments, and it has been arrived at by different workers in the field starting from a variety of points of view. Its physical interpretation is simple, because the classical forces and the classical conservation laws form the starting point around which quantum corrections are calculated. Formally it is an expansion in Planck's constant $\hbar$, but for the different cases this has to be combined with the physical variables to give a dimensionless expansion parameter. For

features not covered in this review the reader is urged to consult the earlier reviews of Kazantsev (1978), Ashkin (1980), Letokhov and Minogin (1981b), and for trapped particles Toschek (1984) and Stenholm (1985). A general overview of the contemporary situation can be obtained from Phillips (1984).

The organization of the present review is as follows: In Sec. II the light pressure force is derived from a completely classical model, which helps us to decide which force expression to choose; quantum theory can be obtained by calculating an expectation value of the corresponding operator (see, for example, Stenholm, 1978b, Sec. 6.2). In the semiclassical approach of the present paper the same force expression emerges from the Fokker-Planck equation.

In Sec. III the physical aspects of laser cooling of free particles are discussed. Qualitative estimates are obtained and their physical meaning is discussed. The same problem is put on a firm theoretical footing in Sec. IV, where the exact solution and its physical implications are given. References to related material are given throughout the treatment.

In Sec. V the trapped-particle case is discussed. The physical picture is introduced and its formalization within quantum theory is presented. In the semiclassical limit a master equation is obtained and its structure and physical implications are discussed in detail.

Section VI reviews the present experimental situation. It is brief, and the reader is urged to consult the references for details. Just now there is much interest in laser cooling experiments, and many new results are expected in the near future. Any detailed description of present experiments would only become rapidly obsolete, whereas it is hoped that the main parts of this review will remain valid, at least as the foundation for future work.

The paper is concluded by a reasonably complete bibliography on the mechanical effects of radiation on atomic matter. It includes references to papers on aspects of the subject not directly discussed in this review. It is hoped that the bibliography will prove useful to all research workers in this promising and inspiring field of laser applications.

II. THE CLASSICAL LIGHT PRESSURE FORCE

When a polarizable particle is situated in a classical field it will acquire a dipole moment, by which the particle tries to minimize its energy in the external field, and hence the particle experiences a force. This is one way to look at the emergence of radiation pressure, and it is of some interest to pursue this discussion because it is not obvious what the force expression is like. It differs from the force on a permanent dipole in the field, and also from the instantaneous force on the charges, which obviously is in the direction of the field vector. The net force in a homogeneous radiation field must be in the direction of the propagation vector, as follows immediately from a consideration of photon momentum.

To be able to follow through the derivation we consider a dipole consisting of two particles of charge Q and $-Q$, respectively, situated in the radiation field $\mathbf{E}(\mathbf{r},t)$, $\mathbf{B}(\mathbf{r},t)$. Their equations of motion are

$$m_1\ddot{\mathbf{r}}_1=Q[\mathbf{E}(\mathbf{r}_1,t)+\dot{\mathbf{r}}_1\times\mathbf{B}(\mathbf{r}_1,t)]+\mathbf{F}\ , \tag{2.1}$$

$$m_2\ddot{\mathbf{r}}_2=-Q[\mathbf{E}(\mathbf{r}_2,t)+\dot{\mathbf{r}}_2\times\mathbf{B}(\mathbf{r}_2,t)]-\mathbf{F}\ . \tag{2.2}$$

Here $\mathbf{F}$ is the mechanical binding force between the two particles; it is a function of their relative separation

$$\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2 \tag{2.3}$$

only. Using this and the center-of-mass coordinate

$$\begin{aligned}\mathbf{R}&=\frac{m_1}{M}\mathbf{r}_1+\frac{m_2}{M}\mathbf{r}_2\ ,\\ M&=m_1+m_2\ ,\end{aligned} \tag{2.4}$$

we can separate the internal motion of the dipole from its center-of-mass motion. We take their distance $|\mathbf{r}|$ to be much less than the wavelength λ of the radiation field and expand in $\mathbf{r}$. This is no restriction, as all our considerations will stay well within the dipole approximation. We find

$$m_1\ddot{\mathbf{R}}+m_r\ddot{\mathbf{r}}=Q\left[\mathbf{E}(\mathbf{R},t)+\frac{m_2}{M}\mathbf{r}\cdot\nabla\mathbf{E}(\mathbf{R},t)+\left[\dot{\mathbf{R}}+\frac{m_2}{M}\dot{\mathbf{r}}\right]\times\left[\mathbf{B}(\mathbf{R},t)+\frac{m_2}{M}\mathbf{r}\cdot\nabla\mathbf{B}(\mathbf{R},t)\right]\right]+\mathbf{F}+O(r^2)\ , \tag{2.5}$$

$$m_2\ddot{\mathbf{R}}-m_r\ddot{\mathbf{r}}=Q\left[\mathbf{E}(\mathbf{R},t)-\frac{m_1}{M}\mathbf{r}\cdot\nabla\mathbf{E}(\mathbf{R},t)+\left[\dot{\mathbf{R}}-\frac{m_1}{M}\dot{\mathbf{r}}\right]\times\left[\mathbf{B}(\mathbf{R},t)-\frac{m_1}{M}\mathbf{r}\cdot\nabla\mathbf{B}(\mathbf{R},t)\right]\right]-\mathbf{F}+O(r^2)\ , \tag{2.6}$$

where the reduced mass is

$$m_r=\frac{m_1m_2}{M}\ . \tag{2.7}$$

Adding Eqs. (2.5) and (2.6) we find for the center-of-mass motion

$$\begin{aligned}M\ddot{\mathbf{R}}=&\,Q[\mathbf{r}\cdot\nabla\mathbf{E}(\mathbf{R},t)+\dot{\mathbf{r}}\times\mathbf{B}(\mathbf{R},t)]\\&+\mathbf{v}\times[Q\mathbf{r}\cdot\nabla\mathbf{B}(\mathbf{R},t)]\ ,\end{aligned} \tag{2.8}$$

where we have written for the velocity of the dipole $\dot{\mathbf{R}}=\mathbf{v}$.

The dipole moment is

$$\boldsymbol{\mu}=Q\mathbf{r}\ , \tag{2.9}$$

and the first term is its ordinary energy expression in an electric field. Here we have, however, two additional magnetic terms.

Subtracting the two equations (2.5) and (2.6), we obtain the equation describing the internal motion of the dipole, but we do not need this here. In most cases the internal time evolution has to be considered quantum mechanically, and only the center of mass behaves in an essentially classical way. We introduce the dipole moment from Eq. (2.9) and rewrite

$$M\ddot{\mathbf{R}}=(\boldsymbol{\mu}\cdot\nabla)[\mathbf{E}(\mathbf{R},t)+\mathbf{v}\times\mathbf{B}(\mathbf{R},t)]+\dot{\boldsymbol{\mu}}\times\mathbf{B}(\mathbf{R},t)\ . \tag{2.10}$$

The last term can be rewritten as

$$\begin{aligned}\dot{\boldsymbol{\mu}}\times\mathbf{B}&=\frac{d}{dt}(\boldsymbol{\mu}\times\mathbf{B})-\boldsymbol{\mu}\times\frac{d\mathbf{B}}{dt}\\&=\frac{d}{dt}(\boldsymbol{\mu}\times\mathbf{B})-\boldsymbol{\mu}\times\left[\frac{\partial\mathbf{B}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{B}\right]\ .\end{aligned} \tag{2.11}$$

Using the Maxwell equation

$$\frac{\partial\mathbf{B}}{\partial t}=-\nabla\times\mathbf{E}\ , \tag{2.12}$$

we rewrite this as

$$\dot{\boldsymbol{\mu}}\times\mathbf{B}=\frac{d}{dt}(\boldsymbol{\mu}\times\mathbf{B})+\boldsymbol{\mu}\times(\nabla\times\mathbf{E})-(\mathbf{v}\cdot\nabla)(\boldsymbol{\mu}\times\mathbf{B})\ . \tag{2.13}$$

The last term can be combined with the second term of Eq. (2.10) to give

$$\begin{aligned}(\boldsymbol{\mu}\cdot\nabla)(\mathbf{v}\times\mathbf{B})-(\mathbf{v}\cdot\nabla)(\boldsymbol{\mu}\times\mathbf{B})&=[(\boldsymbol{\mu}\times\mathbf{v})\times\nabla)]\times\mathbf{B}\\&=\nabla(\boldsymbol{\mu}\times\mathbf{v}\cdot\mathbf{B})-(\boldsymbol{\mu}\times\mathbf{v})\nabla\cdot\mathbf{B}\\&=[\nabla(\mathbf{v}\times\mathbf{B})]\cdot\boldsymbol{\mu}\ .\end{aligned} \tag{2.14}$$

When Eqs. (2.13) and (2.14) are introduced into Eq. (2.10) we obtain

$$\begin{aligned}M\ddot{\mathbf{R}}&=(\boldsymbol{\mu}\cdot\nabla)\mathbf{E}+\frac{d}{dt}(\boldsymbol{\mu}\times\mathbf{B})+(\nabla\mathbf{E})\cdot\boldsymbol{\mu}-(\boldsymbol{\mu}\cdot\nabla)\mathbf{E}\\&\quad+[\nabla(\mathbf{v}\times\mathbf{B})]\cdot\boldsymbol{\mu}\\&=[\nabla(\mathbf{E}+\mathbf{v}\times\mathbf{B})]\cdot\boldsymbol{\mu}+\frac{d}{dt}(\boldsymbol{\mu}\times\mathbf{B})\ .\end{aligned} \tag{2.15}$$

If we consider the slow drift over a long time period, the total time derivative of the strictly periodic function $\boldsymbol{\mu}\times\mathbf{B}$ does not contribute to the drift of the center-of-mass motion. To disentangle the vector form of the first term on the right-hand side of Eq. (2.15) we write it in components, as

$$M\ddot{\mathbf{R}}_i=\sum_j\mu_j\frac{\partial}{\partial x_i}[E_j+(\mathbf{v}\times\mathbf{B})_j]\ . \tag{2.16}$$

Here the term with $\mathbf{v}\times\mathbf{B}$ is the ordinary correction to the electromagnetic force, which in a radiation field is of the order (v/c) and negligible. The main force term $\mu_j\partial E_j/\partial x_i$ was derived by Letokhov and Minogin (1981b) using a more formal argument. Its form differs, of course, from that of the first term in Eq. (2.10). If the electric field is in the form of a plane wave, $E\propto e^{i\mathbf{q}\cdot\mathbf{r}}$, the force will be in the direction of $\mathbf{q}$, as a simple photon momentum argument suggests.

To obtain an idea of the change of energy, we multiply Eq. (2.16) by the velocity v and make the order-of-magnitude estimate $\partial E/\partial x\sim qE$. Then we have

$$\frac{dK}{dt}\sim qv\mu E\ , \tag{2.17}$$

where K is the kinetic energy of the particle. We now need to estimate the magnitude of the average interaction energy $\overline{\mu E}$. This is not clear from our classical argument, but a quantum argument tells us that the transition is nearly optimized when the flipping rate between the quantum levels $(\mu E/\hbar)$ is close to the decay rate Γ, which is the natural linewidth. For an atom moving with velocity v, the Doppler shift is qv, and this should not exceed the linewidth Γ during the final stages of the cooling. Hence for the approach to the final stage we obtain

$$\mu E\sim\hbar\Gamma\sim\hbar qv\ , \tag{2.18}$$

giving from Eq. (2.17) the decay equation for the atom's mechanical energy as

$$\frac{dK}{dt}=-\varepsilon K\ , \tag{2.19}$$

where the recoil energy (in frequency units) is

$$\varepsilon=\frac{\hbar q^2}{2M}\ . \tag{2.20}$$

The cooling time constant

$$T_{\text{cool}}=\varepsilon^{-1} \tag{2.21}$$

is of the order 10^{-3} s for atoms cooled by optical radiation. It provides the slow time scales in the problem; the fast ones are the internal transition rates of the atom. These are given by the flipping rate between the levels $(\mu E/\hbar)$ and the decay rate Γ, both of the same order of magnitude ($\sim10^8$ s^{-1}). One small dimensionless parameter in the problem is hence

$$\xi=\frac{\varepsilon}{\Gamma}\sim10^{-5}\ . \tag{2.22}$$

This parameter, which can be used for expansions, was identified by Javanainen and Stenholm (1980a,1980b) and Letokhov and Minogin (1981b). For free particles, it is the important expansion parameter.

III. THE PHYSICS OF COOLING BY LIGHT

A. The basic phenomenon

In this section we present an argument which gives a detailed picture of the mechanical manifestations of the interaction of optical radiation with matter. The argu-

ment must be based on a quantum-mechanical approach. It will be presented in a formally more adequate way in Sec. IV; here the argument will be based on simple physical considerations.

We consider a simplified atomic system consisting of two levels, separated by the energy $\hbar\omega$ (see Fig. 1). This is acted upon by an ideally monochromatic laser field of angular frequency Ω traveling along the z direction. The light causes resonant transfer of population between the levels when the atomic velocity component along the z axis causes a Doppler shift that compensates the detuning of the laser from the atomic resonance, viz., when

$$v_z=\frac{\Omega-\omega}{q}\ , \tag{3.1}$$

where $q=\Omega/c$ is the wave vector of the light. When the atom absorbs the energy quantum $\hbar\Omega$ from the light field, it must also compensate for the loss of momentum by the field. Thus the atom acquires an additional momentum $\hbar q$ along the direction of propagation of the light. At a subsequent induced emission process this momentum has to be returned to the field and the atomic velocity before absorption reoccurs. When, on the other hand, the excitation is discharged through spontaneous emission, the outgoing photon possesses no memory of the direction of the laser beam, and the momentum recoil experienced by the atom occurs in a random direction. After the emission process the final velocity component has a probability distribution spread over the interval $(Mv_0, Mv_0+2\hbar q)$, where M is the atomic mass. The single-photon momentum-exchange cycle is illustrated in Fig. 2.

From the classical point of view, the spontaneous emission process is fully random, and even a single excitation event followed by spontaneous emission implies a description in terms of an ensemble of atoms. On the average the atom has gained a momentum $\hbar q$, and its velocity has changed by

$$v_r=\frac{\hbar q}{M} \tag{3.2}$$

along the z axis, but the individual atom may end up anywhere within a range of magnitude $2v_r$. This same uncertainty is imposed upon the direction transverse to the laser beam. Thus the basic process of interaction between light and matter can be utilized to effect a change of the average velocity of an atomic system, but as an inescapable adjunct we have a broadening of the uncertainty in the velocity values, which forces us to use a statistical description in terms of an ensemble of systems. Usually this is not a serious drawback, because measurements are carried out on atomic assemblies, which constitute a realization of the ensemble occurring in the theory.

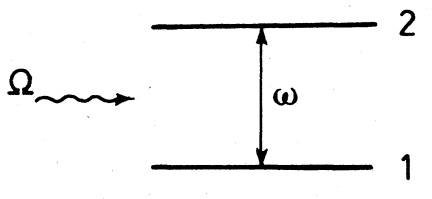

FIG. 1. Our basic model of the atom with the energy $\hbar\omega$ separating the states 1 and 2, and the laser photon with frequency Ω inducing dipole transitions between the levels.

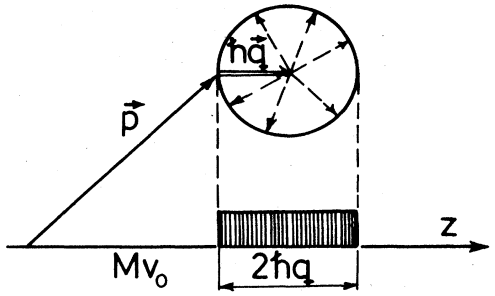

FIG. 2. An atom with initial momentum $\mathbf{p}$ absorbs a laser photon with momentum $\hbar\mathbf{q}$, which later is spontaneously emitted in a random direction. The final atomic momentum ends up anywhere within a range of $2\hbar q$ in the z direction.

For optical fields the recoil velocity v_r is small compared with the velocity range over which the resonant condition (3.1) is nearly satisfied. Its width is given by the natural linewidth Γ, and each single-photon event causes a shift of the Doppler tuning

$$qv_r=\frac{\hbar q^2}{M}\simeq 2\varepsilon\ , \tag{3.3}$$

which is much smaller than Γ, as found already in Eq. (2.22). Thus we need to accumulate many basic processes to achieve an observationally significant change in the atomic velocity. The problem is thus highly nonperturbative, because we must collect a large number of photon momenta. The individual photon contribution can be considered small, and we can assume the exchange of momentum to be a continuous process. Then the discrete nature of the interactions becomes smeared, the quantum features can no longer be discerned, and a classical description of the motion is possible. The two-level system of the atom must, however, be described by quantum mechanics. Hence the picture that emerges can suitably be called *semiclassical.*

To be able to accumulate many units of photon momentum on the same atom, we must be able to reexcite it repeatedly. Hence its spontaneous decay must take place to the lower level 1 only, and this level must be the ground state, or at least a sufficiently metastable state. Otherwise the atoms will escape the cycle we wish to impose on them too soon. If the upper level decays to states other than the lower one, special arrangements may be needed to return them to the cycle. In our model we assume the two-level system to be closed; the probability is conserved as all decay of the upper level ends up on the lower one.

Let us now proceed to make a perturbation estimate of the effect of photon momentum deriving from one single traveling wave. In each single process, the energy must be

conserved within the uncertainty $\hbar\Gamma$, as Γ^{-1} is the average lifetime of the upper level. We choose to enforce this by the dimensionless Lorentzian

$$L(\Delta+vq)=\frac{\Gamma^2}{(\Delta+vq)^2+\Gamma^2}\ , \tag{3.4}$$

where $\Delta=\omega-\Omega$ is the detuning and vq is the Doppler shift.

The number of induced absorption processes is, according to perturbation theory, proportional to the square of the transition dipole μ times the field amplitude E. Here we choose the dimensionless parameter

$$I=\left[\frac{\mu E}{\hbar\Gamma}\right]^2\ , \tag{3.5}$$

which, according to the argument in Sec. II, is the squared ratio of the induced flipping rate to the spontaneous decay rate. It gives the degree of saturation of the light-induced processes. For small values essentially no transfer takes place; for values exceeding unity the efficiency starts to decrease because of saturation. The optimum is approximately $I\approx 1$. The number of spontaneous decay processes per unit time is fixed by the decay rate Γ, and when the laser field is large enough this determines the rate of momentum transfer. Each basic process of an induced absorption followed by spontaneous emission transfers, on the average, a momentum of $\hbar q$ to the atom. The average force can now be estimated from

$$F=\frac{\text{momentum transfer}}{\text{time}}=\hbar qIL(\Delta+vq)\Gamma =\hbar qI\Gamma\frac{\Gamma^2}{(\Delta+vq)^2+\Gamma^2}\ . \tag{3.6}$$

This is the induced light pressure force derived by Ashkin (1970a). When the force (3.6) is used to cool atoms the velocity decreases, and towards the end of the cooling we can expand the force in the remaining velocity v,

$$F=F_0-\beta v+O(v^2)\ , \tag{3.7}$$

where

$$F_0=\hbar q\Gamma I\frac{\Gamma^2}{\Gamma^2+\Delta^2}\ , \tag{3.8}$$

$$\beta=\frac{4MI}{[1+(\Delta/\Gamma)^2]^2}\left[\frac{\varepsilon}{\Gamma}\right]\Delta\ . \tag{3.9}$$

The force F_0 is a steady deflection force, and the part $-\beta v$ is clearly a frictional damping of the motion, when Δ is positive, i.e., $\omega>\Omega$. In that case the transition needs to extract kinetic energy, and there is a damping force directed against the motion. The maximum friction occurs at the detuning $\Delta=\Gamma/\sqrt{3}$, which near $I\simeq 1$ gives the expressions

$$F_0\simeq\hbar q\Gamma\ , \tag{3.10}$$

$$\beta\simeq M\varepsilon=\tfrac{1}{2}\hbar q^2\ . \tag{3.11}$$

The first equation just tells us that the photon momentum $\hbar q$ is extracted at a rate given by Γ; the second gives from Newton's equation,

$$M\frac{dv}{dt}=-\beta v\ , \tag{3.12}$$

the result

$$\frac{dK}{dt}=-\beta v^2=-2\varepsilon K \tag{3.13}$$

in agreement with the estimate (2.19).

The randomness of the spontaneous emission process can be simulated by adding a random force MF to the equation of motion. The ensuing Langevin equation is then

$$M\dot{v}=F_0-\beta v+MF(t)\ . \tag{3.14}$$

To obtain the effects of the spontaneous emission, we must estimate the diffusion constant of the velocity due to the Langevin force $F(t)$. The number of spontaneous emission processes from Eqs. (3.4)—(3.6) is

$$N=I\Gamma L(\Delta+vq)t\ , \tag{3.15}$$

and each one causes the velocity to change by a random step of size v_r. Using an argument familiar from Brownian motion, we can estimate the ensuing velocity spread by setting

$$\langle v(t)^2\rangle=2Dt=v_r^2N\ . \tag{3.16}$$

For $I=1$ and $\Delta\simeq 0$ we obtain

$$D=\tfrac{1}{2}v_r^2\Gamma=\frac{\hbar^2q^2\Gamma}{2M^2}=\frac{\varepsilon\hbar\Gamma}{M}\ . \tag{3.17}$$

The random-walk problem characterized by Eqs. (3.14)—(3.17) can be described by the Fokker-Planck equation

$$\frac{\partial P}{\partial t}-\frac{F_0}{M}\frac{\partial P}{\partial v}+v\frac{\partial P}{\partial z}=\frac{\partial}{\partial v}\left[\frac{\beta v}{M}P+D\frac{\partial P}{\partial v}\right]\ . \tag{3.18}$$

The origin and properties of this equation were recently discussed in detail by Risken (1984). The function $P(z,v,t)$ describes the probability distribution of atoms in the phase space of position and velocity (z,v). Here only the z component is considered; the motion in the transverse directions is a simple diffusive spreading.

To be able to see the general features of this physical situation, let us assume that the friction β can be neglected and no spatial inhomogeneity needs to be considered. Then the solution of Eq. (3.18) is

$$P(v,t)\propto\exp\left[-\frac{(Mv-F_0t)^2}{Dt}\right]\ , \tag{3.19}$$

which describes an ensemble of particles continuously deflected by a constant force, but spreading in velocity in a diffusive manner. This is a simplified version of the beam deflection experiment used for detection purposes and isotope separation (see Fig. 3). Such experiments were first reported by Picque and Vialle (1972), Schieder

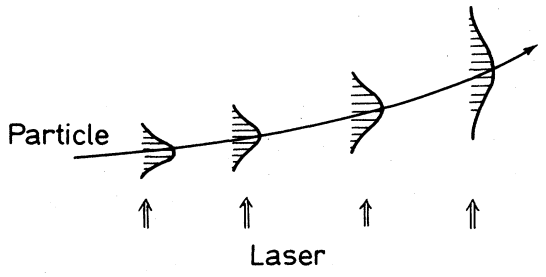

FIG. 3. A beam of atomic particles is defected by laser light pressure, but owing to spontaneous emission the beam profile inevitably spreads diffusively.

et al. (1972), and Jacquinot *et al.* (1973). Recently laser deflection has been explored in detail by Bjorkholm *et al.* (1981).

The deflection process serves to introduce a new time scale $T_{\rm defl}$ for transferring a particle out of resonant interaction by the ensuing Doppler shift. A particle originally at resonance is shifted out of resonance when its velocity has changed by the amount

$$\Delta v_D = \Gamma/q \ . \tag{3.20}$$

From the equation of motion (3.14) we estimate

$$\Delta v_D = F_0 T_{\rm defl}/M \ . \tag{3.21}$$

When this is combined with Eq. (3.20) we find

$$T_{\rm defl} = \frac{M}{\hbar q^2} = (2\varepsilon)^{-1} \ . \tag{3.22}$$

Consequently we find that the deflection time scale $T_{\rm defl}$ is of the same order as the cooling time scale $T_{\rm cool}$ as given by Eqs. (3.13) and (2.19). This is a remarkable result because one time scale derives from the friction term β and the other from the force F_0.

The averaged light pressure force [Eq. (3.6)] was derived under the assumption that the individual photon impacts were small enough to lead to a continuously acting average force. In the Brownian motion argument of Eqs. (3.16) and (3.17) we had to use the discrete step size explicitly to obtain the diffusion coefficient; hence the diffusion process is an indication of the quantized nature of the process. Thus the diffusive spreading of velocity is the lowest-order quantum correction to the semiclassical force picture. Looking at the force expressions (3.10) and (3.11) and the diffusion (3.17), we find that they do, indeed, form consecutive terms in an expansion in $\hbar$. As this, however, is a dimensional quantity it is ill advised to call it small. The corresponding dimensionless variable is determined by the number of single photon recoil momenta needed to take a particle across the velocity range of the interaction, as given by Eq. (3.20). We denote this by

$$\xi^{-1} = 2\frac{M\Delta v_D}{\hbar q} \ , \tag{3.23}$$

which gives the parameter already found in Eq. (2.22),

$$\xi = \hbar\frac{q^2}{2M\Gamma} = \frac{\varepsilon}{\Gamma} \ . \tag{3.24}$$

This is indeed proportional to $\hbar$. Because spontaneous decay is a pure quantum process, Γ depends on $\hbar$ also, but the mechanism for spontaneous decay is totally independent of the photon recoil effects considered here. Consequently we regard the actual value of Γ as accidental and do not require it to follow when we imagine $\hbar$ to go to zero.

B. Cooling in a standing wave

The standing laser wave can be decomposed into a pair of oppositely traveling waves. In perturbation theory we can assume that these two act on the atoms independently, and the resulting force is hence the direct superposition of two waves as described in the previous section. Because the second wave travels in the opposite direction, its wave vector points backwards. One consequence is that the sign of the Doppler shift is reversed, and another is that the momentum transfer between the field and the atom occurs in the opposite direction. Adding the two forces together we obtain, as in Eq. (3.6),

$$F = \hbar q I\Gamma\left[\frac{\Gamma^2}{(\Delta+vq)^2+\Gamma^2} - \frac{\Gamma^2}{(\Delta-vq)^2+\Gamma^2}\right] . \tag{3.25}$$

At zero velocity, this is zero; it is an antisymmetric function of velocity for all values of v, and when Δ is positive its sign is the opposite to that of the velocity. *The standing wave hence provides cooling for all velocities.*

When the velocity is small, the force (3.25) assumes the friction form

$$F = -\beta v + O(v^3) \ ; \tag{3.26}$$

here

$$\beta = \frac{4\hbar q^2\Gamma^3 I}{(\Delta^2+\Gamma^2)^2}\Delta \approx \hbar q^2 \ , \tag{3.27}$$

where we again use the estimates $I \approx 1$ and $\Delta \sim \Gamma/\sqrt{3}$; see Eqs. (3.9) and (3.11). The spontaneous emission remains unaffected by the presence of the fields, but as each traveling wave serves to excite the atom, the diffusion coefficient may be taken to be twice the result of Eq. (3.17).

The Fokker-Planck equation of the system can now be written in velocity space alone, because no average force is exerted by the light. We have

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v}\left[\frac{\beta v}{M}P + D\frac{\partial P}{\partial v}\right] . \tag{3.28}$$

This describes how an initially narrow velocity distribution, centered at an arbitrary initial velocity v_0, is forced to move its average to lower velocities by the friction term, but at the same time the diffusion process broadens the velocity distribution. The chain of evolution is illustrated in Fig. 4. When the broadening reaches the magnitude of the remaining velocity a steady state ensues. In

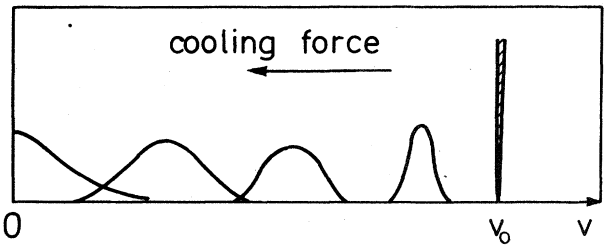

FIG. 4. When an atomic assembly is cooled from an initial velocity v_0 by a standing wave, a steady state occurs when the diffusive spreading exactly compensates the residual cooling force.

contrast to the deflection caused by a traveling wave, the standing wave cools the particles to an ultimate steady state.

The average velocity decays according to the exponential law

$$v=v_0e^{-\beta t}, \tag{3.29}$$

but at the same time the velocity spreads according to

$$\Delta v^2\sim 2Dt . \tag{3.30}$$

From the equation of motion, the decrease in velocity during a time interval Δt is given by

$$\Delta v=-\beta v_\infty \Delta t/M . \tag{3.31}$$

Towards the end of the cooling process, these velocities are of the same order of magnitude as the spreading given by Eq. (3.30); thus identifying Δv and v_∞ we obtain from Eqs. (3.31) and (3.30) the final velocity

$$v_\infty\sim\left[\frac{2DM}{\beta}\right]^{1/2} . \tag{3.32}$$

This same result can be obtained from the final steady-state distribution implied by Eq. (3.28). We find

$$P\sim\exp\left[-\frac{\beta v^2}{2MD}\right], \tag{3.33}$$

which gives a velocity in agreement with Eq. (3.32). The Fokker-Planck equation (3.28) can also be solved for the approach to equilibrium if needed.

The final mechanical energy of the atomic particle is given by Eq. (3.32) as

$$\varepsilon_{\rm fin}=\tfrac{1}{2}Mv_\infty^2\approx\frac{DM^2}{\beta} . \tag{3.34}$$

Introducing the estimates (3.17) and (3.27), we find the result

$$\varepsilon_{\rm fin}\sim\hbar\Gamma . \tag{3.35}$$

This result is of rather general validity. The ultimate physical limit of the cooling is given by the estimate (3.35). It can be understood if we remember that there is an uncertainty in the spontaneous emission time of the order of Γ^{-1}, and the energy of the state of the atom cannot be less than the energy uncertainty implied by this. The resulting lower limit for the energy is precisely Eq. (3.35). Another way to look at the result emerges if we consider the spontaneous emission to be caused by the vacuum fluctuations of the field. Owing to the equipartition theorem, the equilibrium energy must be determined by the average fluctuation energy of the dissipating system. The spontaneous decay rate Γ is clearly a measure of its spectral strength at the frequency of the transition, which again implies the estimate (3.35).

In our calculations we have found a final Maxwellian distribution (3.33), which may be characterized by a temperature T_f. In the atomic case we have Γ of the order 10^8 s^{-1} and then $T_f\sim 10^{-3}$ K. Thus we find that the quantum noise implied by spontaneous emission does not set a very serious limitation on the cooling by light pressure.

If we go to longer wavelength, the cooling limit decreases because in Eq. (3.35) we have

$$\Gamma\propto\lambda^{-3} . \tag{3.36}$$

Because, however, the cooling time scale from (2.21) grows like

$$T_{\rm cool}\propto\lambda^2 , \tag{3.37}$$

the efficiency and practicality of the process rapidly decreases.

The cooling speed can be increased by going to shorter wavelength, when each photon carries away a larger chunk of momentum. The ultimate cooling limit deteriorates rapidly, however, due to Eq. (3.36). When fewer processes are needed to cool, the corresponding noise grows. For small enough λ, the photon momentum $\hbar q$ can never be regarded as small, and our semiclassical approach breaks down. This situation is discussed by Letokhov and Minogin (1981b).

When we consider only the diffusive process, we can introduce a time scale $T_{\rm diff}$, which makes the velocity diffuse over the width [Eq. (3.20)], Δv_D, which is the range of velocities interacting with the radiation. From Eq. (3.30) we obtain

$$2DT_{\rm diff}=\Delta v_D^2 , \tag{3.38}$$

which with (3.20) and (3.17) gives

$$T_{\rm diff}=\frac{1}{2}\left[\frac{\Gamma}{\varepsilon}\right]T_{\rm cool} . \tag{3.39}$$

Because $\Gamma\gg\varepsilon$, the diffusion process is much slower than the cooling and deflection, which takes place over times of the order of $T_{\rm cool}$. Except for providing the estimate for the ultimate cooling limit [Eq. (3.35)], the diffusive motion may often be neglected in the evaluation of cooling schemes. Mathematically this is due to the fact that the light pressure force and the diffusion are consecutive terms in a systematic expansion in the small parameter ξ of Eq. (2.22).

When we tune our laser exactly to resonance, $\Delta=0$, the

friction coefficient β in Eq. (3.27) goes to zero, and only the diffusive spreading remains. This can be observed when a resonant laser beam is made to cross an atomic beam in the manner shown in Fig. 5. When the interaction time t_i is so short that no incoherent relaxation processes have time to act, the pattern observed corresponds to the Bragg scattering of the atomic wave from the periodic potential of the standing electromagnetic wave. In this limit a comb of discrete peaks is expected, owing to the interplay between photons emitted and absorbed from the two traveling waves. Such an effect has been observed by Moskowitz *et al.* (1983).

When the interaction time is such that spontaneous emission can act, the distribution after the scattering approaches a diffusive process. This was the case in the experiment of Arimondo *et al.* (1979a,1979b). A naive application of the diffusion results (3.30) and (3.15)—(3.17) would give for the width of the transverse distribution

$$\langle v_\perp^2 \rangle = 2Dt_i = \frac{\hbar^2 q^2 \Gamma I}{M^2} t_i \propto E^2 t_i \ . \tag{3.40}$$

In the experiment by Arimondo *et al.*, the dependence on the interaction time t_i was not observed, but the deflection was measured as a function of the laser intensity E^2. The result was not in agreement with the calculation (3.40) giving $v \propto E$.

The explanation seems to be that for long enough interaction times the atoms are deflected enough to feel the detuning due to the Doppler effect. This sets a limit to the transverse velocity $v_\perp$ less than that given by Eq. (3.40). For large transverse velocities we can use Eqs. (3.15)—(3.17) to write

$$\langle v_\perp^2 \rangle = \frac{\hbar^2 q^2 \Gamma^3 I}{M^2 (q^2 v_\perp^2 + \Gamma^2)} t_i \ . \tag{3.41}$$

When the transverse velocity becomes large enough to affect the result of the diffusion process, $qv_\perp \sim \Gamma$ and it follows from Eq. (3.41) that

$$v_\perp \propto (It_i)^{1/4} \propto E^{1/2} \ . \tag{3.42}$$

This estimate is in qualitative agreement with the observed dependence on laser intensity. A more detailed investigation of this effect is presented in Arimondo *et al.* (1981). Unfortunately there are no experiments varying the interaction time t_i, which could verify the estimate (3.42), $v_\perp \propto t_i^{1/4}$. The experiment of Grinchuk *et al.* (1981a,1981b) does not provide any detailed analysis of the dependence on the parameters.

The transition between a coherent scattering from the periodic potential to a diffusive regime is calculated in detail by Tanguy *et al.* (1984). They also predict a hitherto unobserved splitting of the deflection profile (Tanguy *et al.*, 1983). The theory is further discussed by Dalibard and Cohen-Tannoudji (1985a).

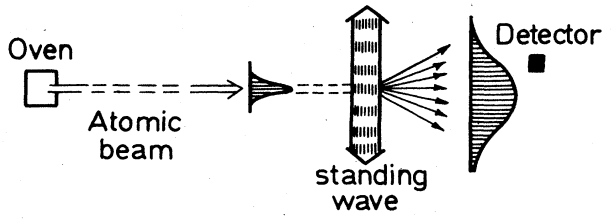

FIG. 5. An atomic beam is made to scatter from a resonantly interacting standing laser wave. The detector sees an interference structure similar to Bragg scattering because of the quantum nature of the single-photon scattering events in the standing wave.

C. Connection with photon statistics

The basic interaction under consideration is the exchange of energy and momentum between a radiation field and an atomic system. Each process emits or absorbs one unit of momentum $\hbar q$ from the field and changes the photon number by one unit. The accumulated change of atomic momentum thus corresponds exactly to the accumulated change in the quantum number characterizing the field. We start the process in the initial state $|p_0,n_0,0\rangle$ with initial atomic momentum p_0, the laser photon number n_0, and the state of spontaneously emitted photons empty. After an interaction period the state must be of the form

$$|\psi\rangle = \sum_l c_l |p_0 + l\hbar q, n_0 - l, l\rangle \ , \tag{3.43}$$

where l is any integer; the situation is illustrated in Fig. 6. The state $|l\rangle$ really stands for a state with l photons in different outgoing modes, as no coherence effects between them are allowed. The laser state is usually much more complicated than is assumed in Eq. (3.43), but, on the other hand, it is not materially affected by the absorption of a few photons. In the following discussion we shall make a semiclassical approximation and neglect the changes in the state of the laser field.

The square of the coefficients, $|c_l|^2$, determines the probability of the momentum change $l\hbar q$ or, alternatively, the probability for the spontaneous emission of l photons. Hence these two observable quantities will have the same statistical properties. The variable l counts the integer number of events required to reach the state (3.43). Hence the momentum distribution caused by the mechanical manifestations of light mirrors exactly the photon distribution created by the process. As the former can be

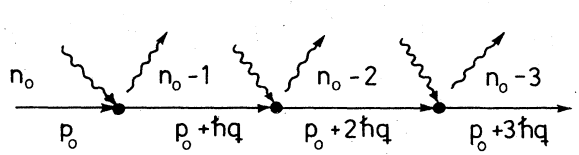

FIG. 6. A series of interaction events experienced by an atom with initial momentum p_0, which enters a laser field with n_0 photons. The conservation of momentum implies that we can either count the decrease in the atomic momentum or the number of spontaneously emitted photons; this shows the reason for the connection between the atomic momentum distribution and the photon statistics.

obtained from a semiclassical theory, so can the latter. This relationship was first pointed out by Mandel (1979a,1979b).

The situation is well known in another area of optical physics, the free-electron laser. There the gain of the laser photons is the quantity of interest, but instead the change of electron momentum is calculated. This is particularly clear from the approach in Bambini and Stenholm (1979). For a review of this field with further references the reader is referred to the article of Stenholm and Bambini (1981).

For the atomic case, Cook (1980b) has connected the momentum statistics with photon statistics in the following manner.

Consider Fig. 2 and write the z component of the atomic momentum after n absorptions followed by spontaneous emissions as

$$p=p_0+n\hbar q+\sum_{l=1}^{n}\hbar q\cos\theta_l\ , \tag{3.44}$$

where $\hbar q\cos\theta_l$ is the momentum projection of the spontaneously emitted photon. The average of the sum in Eq. (3.44) is zero, but

$$\langle p\rangle=p_0+\hbar q\langle n\rangle\ . \tag{3.45}$$

Here, however, both the number n in a given time and the directions of the outgoing photons are random variables, and hence for the dispersion

$$\langle\Delta p^2\rangle=\langle(\Delta p)_0^2\rangle+\hbar^2q^2\left[\langle\Delta n^2\rangle+\sum_{lm}\langle\cos\theta_l\cos\theta_m\rangle\right]\ . \tag{3.46}$$

Because subsequent photon emissions are uncorrelated, we have

$$\sum_{lm}\langle\cos\theta_l\cos\theta_m\rangle=\sum_{l}\langle\cos^2\theta_l\rangle=\alpha\langle n\rangle\ , \tag{3.47}$$

where α is a purely geometric factor. We return to this in Sec. IV.A. We obtain

$$\langle\Delta p^2\rangle=\langle(\Delta p)_0^2\rangle+\hbar^2q^2(\langle\Delta n^2\rangle+\alpha\langle n\rangle)\ . \tag{3.48}$$

From the calculation of the diffusion coefficient we shall find that there is a contribution proportional to α owing to the spreading in each spontaneous emission event. Writing the diffusion coefficient as

$$D=D_0(1+Q+\alpha)\ , \tag{3.49}$$

we may use this to write Eq. (3.46) as

$$\langle\Delta p^2\rangle=\langle(\Delta p)_0^2\rangle+2M^2D_0(1+Q+\alpha)t \tag{3.50}$$

and identify

$$\langle n\rangle=\frac{2M^2}{\hbar^2q^2}D_0t\ , \tag{3.51}$$

$$\langle\Delta n^2\rangle=\frac{2M^2}{\hbar^2q^2}D_0(1+Q)t\ . \tag{3.52}$$

Using Eq. (3.45) we can obtain

$$\frac{d}{dt}\langle p\rangle=\frac{2M^2}{\hbar q}D_0\ , \tag{3.53}$$

which together with Eq. (3.14) implies the relation

$$D_0=\frac{\hbar q}{2M^2}F_0\ , \tag{3.54}$$

which indeed agrees with Eqs. (3.10) and (3.17). The present derivation is, however, more general and should be valid for any theory of the type considered.

The correction to D_0 has been denoted here by Q, and it is the task of a more detailed theory to evaluate it. To see its significance we consider the parameter introduced by Mandel (1979b). For an ideal Poisson distribution we would have $\langle\Delta n^2\rangle=\langle n\rangle$, and hence any deviation from this can be measured through the parameter

$$\frac{\langle\Delta n^2\rangle-\langle n\rangle}{\langle n\rangle}=Q\ , \tag{3.55}$$

as follows from Eqs. (3.51) and (3.52). The parameter Q thus measures the deviations from Poissonian statistics. When this is negative, the photons appear more regularly than in a totally random emission process. This is sub-Poissonian statistics, and the photons appear antibunched. For positive values of Q we have super-Poissonian statistics, with large fluctuations in the photon distributions. The physical significance of the parameter Q is thus to keep track of any coherence between the spontaneously emitted photons. In the present case this is, of course, caused by the presence of the strong laser field acting on the emitting atom.

Cook (1981) used the relation between the momentum distribution and the photon statistics to evaluate the latter completely for some special limiting cases.

D. Discussion and conclusion

From Eqs. (3.8)–(3.11) and (3.25)–(3.27) we find that the light pressure force is proportional to Planck's constant $\hbar$. This dependence derives from the recoil momentum $Mv_r=\hbar q$ transmitted in each discrete quantum event, and it is hence a true manifestation of the quantum nature of the process. In the limit of a small velocity, we expand in the Doppler shift qv, which provides a second factor q making the result proportional to ε. The Doppler shift measures the distance traveled against the wavelength of light, and it is hence a purely classical effect deriving from the wave aspects of light. In contrast, the factor $\hbar q$ is a photon recoil effect.

The quantum-mechanical formulation of the problem attributes the exchange of momentum between radiation and matter to a series of individual events of photon absorptions and emissions. Such events occur at random and appear as a stochastic process. The average value gives the light pressure force, but the statistical spread will be seen as a spread in the ensuing momentum distribution. In Sec. III.A we saw that such a spread is an un-

avoidable consequence of the momentum exchange process.

In Sec. III.A we discussed the discreteness in terms of a Brownian motion. For N individual events, with N given in Eq. (3.15), the expected spread is proportional to $\sqrt{N}$, which is assumed to obtain Eq. (3.17). The diffusion coefficient can be written as

$$D=\frac{\hbar^2 q^2\Gamma^3 I}{2M^2[\Gamma^2+(\Delta+qv)^2]} . \tag{3.56}$$

Except for a factor of proportionality, this is the result of a more detailed theory for the factor D_0 of Eq. (3.49). As we see, this is proportional to $\hbar^2$, and can hence be considered as a manifestation of the quantized nature of the basic interaction process. When the velocity goes to zero, both the diffusion constant and the friction coefficient β of Eq. (3.11) are proportional to q^2, but they depend on $\hbar$ to different powers. Using a consistent expansion in q, Dalibard and Cohen-Tannoudji (1985b) have shown that they are connected by an exact fluctuation-dissipation relation.

In our work we consider the expansion in $\hbar$, and this can hence be regarded as an expansion in the quantum nature of the basic interaction processes. This is the basis for our calling it a *semiclassical expansion.* It is assumed to be convergent, and its rate of convergence is determined by the magnitude of the small dimensionless parameter used. Because the diffusion terms appear in a higher order than the drift terms in the Fokker-Planck equation, the diffusive spreading is assumed to appear as a smearing only around the classical trajectories. In this way the theory also avoids any ambiguity of the kind associated with the names Itô and Stratonovich (see the discussion in Gardiner, 1983, Chap. 4). As this is mainly concerned with the position of the diffusion coefficient with respect to the derivatives, a brief discussion may be illuminating.

In the straightforward mathematical derivation, the diffusion coefficient emerges as a sum of different parts, which occur at various positions with respect to the derivatives. In the final results of the derivation (see Sec. IV) these contributions are moved through the derivatives at will, and the corrections to the drift coefficients are neglected. In the chain

$$\begin{aligned} D(v)\frac{\partial^2}{\partial v^2} &= \frac{\partial}{\partial v}D(v)\frac{\partial}{\partial v}-\left[\frac{\partial D(v)}{\partial v}\right]\frac{\partial}{\partial v} \\ &= \frac{\partial^2}{\partial v^2}D(v)-\left[\frac{\partial D(v)}{\partial v}\right]\frac{\partial}{\partial v} \\ &\quad -\frac{\partial}{\partial v}\left[\frac{\partial D(v)}{\partial v}\right] \end{aligned} \tag{3.57}$$

the first-order derivative terms ought to be combined with the drift term, but they are of second order in the expansion parameter and hence assumed smaller than its main terms, which are of first order. Moreover, if they are added, they provide a contribution $\propto\hbar^2$ to the force expression, which is of order $\propto\hbar$. This constitutes a quantum correction to the classical force, and hence such terms must be omitted if the force is to be identified with the force in Newtonian mechanics. The present argument was developed in Javanainen and Stenholm (1980a) and leads us to agree with the conclusion by van Kampen (1981) that the physics should by itself remove any ambiguity of this subtle kind.

The Fokker-Planck equation is an exact one for strictly Gaussian processes. For others it must be an approximation where successive terms carry higher and higher powers of some small parameter, here Planck's constant $\hbar$. The higher-order terms are negligible because of the smallness of this parameter. In our case we have, to lowest order, an equation of the type

$$m\dot{v}=F\propto\hbar , \tag{3.58}$$

and hence the solution $V(t)$ must be proportional to $\hbar$. If δv denotes the quantum fluctuations around this classical path we find

$$\begin{aligned} \langle v(t)^2\rangle &= \langle[V(t)+\delta v]^2\rangle \\ &= V(t)^2+\langle\delta v^2\rangle \\ &= V(t)^2+2Dt\propto\hbar^2 . \end{aligned} \tag{3.59}$$

Thus we can see that the accuracy $O(\hbar)$ in Eq. (3.58) is sufficient and necessary to assure consistency with the Brownian motion approximation, which implies $D\propto\hbar^2$. The inclusion of terms of order $\hbar^2$ in the force of Eq. (3.58) is, in principle, inconsistent with the truncation required to obtain the Fokker-Planck equation. If, on the other hand, these higher-order terms are included, there is the theorem by Pawula (1967), which states that they may give rise to negative probabilities. This agrees with the conclusion from our semiclassical expansion. Either we use a classical light pressure force with a spreading induced by quantum mechanics or we solve the full quantum problem. No intermediate procedure can be given a physical interpretation, but it can at best serve only as a numerical approximation to the exact quantum-mechanical calculation (see, for example, Risken, 1984, Sec. 4.6).

IV. THE THEORETICAL APPROACH

A. Formulation of the problem

We are going to consider the problem of a two-level atom interacting with a few high-intensity (laser) modes and decaying by interaction with the continuum of vacuum modes. For notational simplicity we choose to confine our system to a large (but finite) box of volume V. In some cases this may be a real cavity, but then the spontaneous decay may be affected too. In our considerations the volume V is assumed to go to infinity, but the strong modes are taken to retain their intensity because they originate from a distant laser source.

The Hamiltonian for the laser field is given by

$$H/\hbar=\sum_q \Omega_q b_q^\dagger b_q \ , \tag{4.1}$$

where $b_q^\dagger$ is the Boson operator creating a photon in the mode q. The polarization indices are suppressed, but they can be added whenever needed.

In the dipole approximation the interaction operator can be written as

$$H_{\rm int}=-\boldsymbol{\mu}\cdot\mathbf{E} \ ; \tag{4.2}$$

it is a valid expression at least when the atomic particles occupy a volume much less than λ^3 and only transitions between bound states are considered.

A considerable amount of controversy has arisen in connection with this choice of interaction operator. The exact quantum-mechanical operator uses the vector potential in accordance with the minimal coupling prescription. In the dipole approximation the Hamiltonian (4.2) can be derived by a canonical transformation representing a change of classical gauge (see Power and Zienau, 1959). Even if the choice of gauge should have no observable consequences, it is difficult to ascertain the validity of the result in approximate calculations. The controversy concerning this question is reviewed by Schlicher *et al.* (1984). The whole question of gauge invariance is clarified by Cohen-Tannoudji *et al.* (1977); a pedagogical presentation is given by Savolainen and Stenholm (1972).

We denote the states of the atom by $|k,\alpha\rangle$, when its translational momentum is $\hbar k$ and its internal state is α $(=1,2)$. For optical transitions only those that approximately conserve energy are of importance. The emission of a photon must thus be accompanied by a transition from an upper level to a lower one, and an upward transition must occur with the absorption of a photon. Consequently only these terms need be retained in the Hamiltonian in atomic physics. Borrowing a term from magnetic resonance physics, we call this the rotating-wave approximation (RWA). Implementing these arguments, we write Eq. (4.2) in the form

$$H_{\rm int}=-\hbar\sum_{kq} g_{21}(q)(\,|k+q,2\rangle\langle k,1|\,b_q + |k-q,2\rangle\langle k,1|\,b_q^\dagger)\ , \tag{4.3}$$

where

$$|g_{21}(q)|^2=\frac{\Omega}{2\varepsilon_0 V\hbar}\mu^2 \ . \tag{4.4}$$

From the q sum over radiation modes in Eq. (4.3) we separate the strong modes by going to a special "interaction picture" with their Hamiltonians $\hbar\Omega_q b_q^\dagger b_q$ as the "unperturbed part." For the field operators we then assume that they can be replaced by c numbers in the spirit of the semiclassical approach. We have

$$\begin{aligned} b_q(t)&=\exp(i\Omega_q b_q^\dagger b_q t)b_q\exp(-i\Omega_q b_q^\dagger b_q t)\\ &\to\beta_q\exp(-i\Omega_q t)\ ,\\ b_q^\dagger(t)&\to\beta_q^*\exp(i\Omega_q t)\ . \end{aligned} \tag{4.5}$$

With these replacements in the interaction operator (4.3) we find its component deriving from the classical fields

$$H_{\rm cl}=-\tfrac{1}{2}\sum_{kq}\mu E_q[\,|k+q,2\rangle\exp(-i\Omega_q t)\langle k,1| + |k-q,2\rangle\exp(i\Omega_q t)\langle k,1|\,]\ , \tag{4.6}$$

where

$$E_q^2=\left[\frac{2\hbar\Omega}{\varepsilon_0 V}\right]|\beta_q|^2 \ . \tag{4.7}$$

The phase of β has been incorporated into the choice of the initial time $t=0$. The interaction operator (4.3) still retains its form, but the q sum now excludes the strong modes. In the continuum limit this implies no consequences except the neglect of quantum fluctuations in the strong modes.

With the RWA, choice (4.7) corresponds to a field

$$E(\mathbf{r},t)=E_q\cos(\mathbf{r}\cdot\mathbf{q}-\Omega t) \ . \tag{4.8}$$

For a radiation field, the energy density is given by

$$\varepsilon_{\rm cl}=\tfrac{1}{2}\varepsilon_0\bar{E}^2+\frac{1}{2\mu_0}\bar{B}^2=\varepsilon_0\bar{E}^2\ , \tag{4.9}$$

which with Eqs. (4.8) and (4.7) gives

$$\varepsilon_{\rm cl}=\tfrac{1}{2}\varepsilon_0 E_q^2=\frac{\hbar\Omega_q\beta_q^*\beta_q}{V} \ . \tag{4.10}$$

From this we can see that $\beta_q^*\beta_q$ is the photon occupation number of the mode q.

When the atom is excited to its upper level $|2\rangle$, it will spontaneously emit a photon, which is irretrievably lost into the continuum of vacuum modes near the transition frequency $\omega=(E_2-E_1)/\hbar$. In this way the quantum fluctuations in the vacuum field act as a bath that determines the spontaneous decay rate Γ. Eliminating the continuum of the vacuum modes by the method of Cohen-Tannoudji (1977), but retaining the momentum dependence of the atomic states, one finds for the density matrix the relaxation terms

$$\frac{d}{dt}\langle k,2|\rho|k,2\rangle=-\Gamma\langle k,2|\rho|k,2\rangle \ , \tag{4.11a}$$

$$\frac{d}{dt}\langle k,1|\rho|k,1\rangle=\sum_q G(k;q)\langle k+q,2|\rho|k+q,2\rangle \ , \tag{4.11b}$$

$$\frac{d}{dt}\langle k,2|\rho|k,1\rangle=(i\Delta-\tfrac{1}{2}\Gamma)\langle k,2|\rho|k,1\rangle \ , \tag{4.11c}$$

where

$$G(k;q)=2\pi|g(q)|^2\delta(\Omega_q-\omega-\varepsilon_{k+q}+\varepsilon_k) \tag{4.12}$$

and

$$\varepsilon_k=\frac{\hbar k^2}{2M} \ . \tag{4.13}$$

The decay rate is given by

$$\Gamma=\sum_q G(k;q) . \tag{4.14}$$

Its dependence on k is so slight that it can be neglected. In the limit of $V\to\infty$ we can replace all q sums by integrals,

$$\sum_q \to \frac{V}{(2\pi)^3}\int d\mathbf{q} . \tag{4.15}$$

We reduce the dependence on momentum to the one direction singled out by the laser momentum $\hbar\mathbf{q}$. The result in Eq. (4.11b) becomes

$$\int d\mathbf{q}\, G(\mathbf{k},\mathbf{q})\rho(\mathbf{k}+\mathbf{q})\to\frac{\Gamma}{2}\int_{-1}^{+1} dx\, W(x)\rho(k+qx) , \tag{4.16}$$

where

$$x=\frac{\mathbf{q}\cdot\mathbf{k}}{kq} . \tag{4.17}$$

The function $W(x)$ is the dipolar angular dependence of the spontaneous emission expressed in terms of the cosine between the momentum of laser photons and that of the outgoing ones. It depends on the geometric configuration of the experiment.

For later use in Fokker-Planck equations we expand in the variable q in Eq. (4.16) and obtain

$$\tfrac{1}{2}\Gamma\int_{-1}^{+1} dx\, W(x)\left[\rho(k)+qx\frac{\partial\rho}{\partial k}+\tfrac{1}{2}q^2x^2\frac{\partial^2\rho}{\partial k^2}+\cdots\right] \tag{4.18}$$

$$=\Gamma\rho(k)+\tfrac{1}{2}\alpha\Gamma q^2\frac{\partial^2\rho}{\partial k^2} . \tag{4.19}$$

Here we have used the normalization

$$\tfrac{1}{2}\int_{-1}^{+1} W(x)dx=1 \tag{4.20}$$

and introduced the second moment

$$\alpha=\tfrac{1}{2}\int_{-1}^{+1} x^2W(x)dx . \tag{4.21}$$

This is a measure of the spreading introduced in each spontaneous emission event; see Fig. 2.

If we make the contrafactual assumption that isotropic spontaneous emission between two levels exists, we have $W(x)=1$, giving from Eq. (4.21)

$$\alpha=\tfrac{1}{3} . \tag{4.22}$$

This case would require a transition between s states in an atom, but as this is dipole forbidden it remains a hypothetical case. It is, however, often convenient to neglect the angular dependence of $W(x)$ in purely theoretical calculations. The ensuing error is never of any great consequence.

For a dipole transition in an ordinary geometry [e.g., $(L=0)\leftrightarrow(L=1)$], we have

$$W(x)=\tfrac{3}{4}(1+x^2) , \tag{4.23}$$

as follows from ordinary quantum theory (see, for example, Corney, 1977, Sec. 5.2.2).

The normalization agrees with Eq. (4.20), and from (4.21) we find

$$\alpha=\tfrac{2}{5} . \tag{4.24}$$

This is the value to use along the direction of the laser beam. The transverse emission pattern is slightly more complicated.

The present approach follows closely our earlier work (Stenholm, 1974,1978a,1978b; Stenholm and Javanainen, 1978). The spread due to the angular distribution of outgoing photons is discussed by Pusep (1976) and Pusep *et al.* (1977); for an arbitrary geometry see Javanainen and Stenholm (1980a), and for a simple but complete derivation see Mandel (1979a).

For a free particle we need only its momentum to label its state, and then there is no information about the spatial dependence of the atomic particle distribution. In many cases, however, it is of interest to follow the spatial development, and then the position dependence of the density matrix becomes important. This is necessary when the particle sits in an external potential or when questions concerned with trapping are important.

In the position representation we have, for one particle,

$$i\hbar\frac{\partial}{\partial t}\langle r_1|\rho|r_2\rangle=-\frac{\hbar^2}{2M}\left[\frac{\partial^2}{\partial r_1^2}-\frac{\partial^2}{\partial r_2^2}\right]\langle r_1|\rho|r_2\rangle + V(r_1)\langle r_1|\rho|r_2\rangle - \langle r_1|\rho|r_2\rangle V(r_2) . \tag{4.25}$$

In addition to the spatial labels displayed explicitly, there may be labels for the internal states of the atom. The potential $V(r)$ can also be an operator on these indices.

The effects of photon recoil are, however, displayed most clearly in the momentum representation, and to retain some measure of this feature one can represent the information contained in the density matrix by the so-called Wigner function,

$$\rho(\mathbf{R},\mathbf{P})=\frac{1}{\sqrt{V}}\int e^{-i\mathbf{P}\cdot\mathbf{r}/\hbar}\langle\mathbf{R}+\tfrac{1}{2}\mathbf{r}|\rho|\mathbf{R}-\tfrac{1}{2}\mathbf{r}\rangle d\mathbf{r} , \tag{4.26}$$

which was introduced by Wigner (1932). Applying this transformation to Eq. (4.25), we obtain

$$\frac{\partial}{\partial t}\rho(\mathbf{R},\mathbf{P})+\frac{\mathbf{P}}{M}\cdot\frac{\partial}{\partial\mathbf{R}}\rho(\mathbf{R},\mathbf{P})=\frac{1}{i\hbar\sqrt{V}}\int e^{-i\mathbf{P}\cdot\mathbf{r}/\hbar}[V(\mathbf{R}+\tfrac{1}{2}\mathbf{r})\rho(\mathbf{R},\mathbf{r})-\rho(\mathbf{R},\mathbf{r})V(\mathbf{R}-\tfrac{1}{2}\mathbf{r})]d\mathbf{r}$$
$$=\frac{1}{i\hbar}\left[V\left[\mathbf{R}+\frac{i\hbar}{2}\frac{\partial}{\partial P}\right]\rho(\mathbf{R},\mathbf{P})-\rho(\mathbf{R},\mathbf{P})V\left[\mathbf{R}-\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial\mathbf{P}}\right]\right] , \tag{4.27}$$

where the derivative $\overleftarrow{\partial/\partial \mathbf{P}}$ acts on the function to its left. When Planck's constant $\hbar$ goes to zero, we find the equation

$$\frac{\partial \rho}{\partial t}+\frac{\mathbf{P}}{M}\cdot\frac{\partial \rho}{\partial \mathbf{R}}-\frac{\partial V}{\partial \mathbf{R}}\cdot\frac{\partial \rho}{\partial \mathbf{P}}=-\frac{\hbar^2}{24}\frac{\partial^3 V}{\partial \mathbf{R}^3}\frac{\partial^3 \rho}{\partial \mathbf{P}^3}+\cdots . \tag{4.28}$$

The left-hand side here is just the classical transport derivative in a potential $V(\mathbf{R})$, and the right-hand side gives its quantum corrections. For potentials of the form $V\propto R^s$, where $s=0$, 1, or 2, the dynamic evolution gives no quantum corrections. The only quantum feature is found in the initial distribution, which cannot be too localized in the $(\mathbf{R},\mathbf{P})$-phase plane owing to the uncertainty principle.

The Wigner representation is not the only useful one in quantum electronics. For a more detailed discussion see Stenholm, 1978b, Sec. 3.2.

When the potential is due to a strong semiclassical field of the type (4.8), we have

$$V=V_q\exp(i\mathbf{q}\cdot\mathbf{R})+\text{H.c.} \tag{4.29}$$

and we find from Eq. (4.27)

$$\exp\left[i\mathbf{q}\cdot\left[\mathbf{R}\pm i\frac{\hbar}{2}\frac{\partial}{\partial \mathbf{P}}\right]\right]\rho(\mathbf{R},\mathbf{P})=e^{i\mathbf{q}\cdot\mathbf{R}}\rho(\mathbf{R},\mathbf{P}+\tfrac{1}{2}\hbar\mathbf{q}) . \tag{4.30}$$

This is then used in the equation of motion for the density matrix, and we restrict the description of one dimension. From Eqs. (4.11) we obtain

$$\left[\frac{\partial}{\partial t}+\frac{P}{M}\frac{\partial}{\partial Z}-\frac{\partial V_{\rm ex}}{\partial Z}\frac{\partial}{\partial P}\right]\rho_{22}(Z,P+\hbar q)=-\Gamma\rho_{22}(Z,P+\hbar q)+i\kappa\left[\rho_{12}\left[Z,P+\frac{\hbar}{2}q\right]-\rho_{21}\left[Z,P+\frac{\hbar}{2}q\right]\right] , \tag{4.31a}$$

$$\left[\frac{\partial}{\partial t}+\frac{P}{M}\frac{\partial}{\partial Z}-\frac{\partial V_{\rm ex}}{\partial Z}\frac{\partial}{\partial P}\right]\rho_{11}(Z,P)+\frac{\Gamma}{2}\int_{-1}^{+1}W(x)\rho_{22}(Z,P+x\hbar q)dx+i\kappa\left[\rho_{21}\left[Z,P+\frac{\hbar}{2}q\right]-\rho_{12}\left[Z,P+\frac{\hbar}{2}q\right]\right] , \tag{4.31b}$$

$$\left[\frac{\partial}{\partial t}+\frac{P}{M}\frac{\partial}{\partial Z}-\frac{\partial V_{\rm ex}}{\partial Z}\frac{\partial}{\partial P}\right]\rho_{21}\left[Z,P+\frac{\hbar}{2}q\right]=-i(\Delta+qP/M-i\gamma)\rho_{21}\left[Z,P+\frac{\hbar}{2}q\right]+i\kappa\left[\rho_{11}(Z,P)-\rho_{22}\left[Z,P+\frac{\hbar}{2}q\right]\right] , \tag{4.31c}$$

where the Doppler shift is included.

Here the coupling constant is given by

$$\kappa=\frac{\mu E}{2\hbar} , \tag{4.32}$$

which is half the Rabi flipping frequency between the two levels in the field (4.8). The decay rate γ of the off-diagonal elements is equal to $\Gamma/2$ if no phase-perturbing processes act in addition to the spontaneous decay. In all cases we have

$$\gamma\geq\tfrac{1}{2}\Gamma . \tag{4.33}$$

It is possible to simplify the equations slightly by introducing the reduced distribution function

$$F(Z,P)=\mathrm{Tr}_{at}\rho=\rho_{22}(Z,P+\hbar q)+\rho_{11}(Z,P) \tag{4.34}$$

and a three-component optical Bloch vector R with components

$$R_1(Z,P)=\rho_{21}\left[Z,P+\frac{\hbar}{2}q\right]+\rho_{12}\left[Z,P+\frac{\hbar}{2}q\right] , \tag{4.35a}$$

$$R_2(Z,P)=i\left[\rho_{21}\left[Z,P+\frac{\hbar}{2}q\right]-\rho_{12}\left[Z,P+\frac{\hbar}{2}q\right]\right] , \tag{4.35b}$$

$$R_3(Z,P)=\rho_{22}(Z,P+\hbar q)-\rho_{11}(Z,P) . \tag{4.35c}$$

The corresponding equations are straightforwardly obtained from Eqs. (4.31). For later reference we write down the case with no external potential ($V_{\rm ex}=0$) and a velocity small enough that the drift term ($\propto\partial/\partial Z$) is negligible. In this case it is simpler to label the state by the atomic wave vector

$$k=P/\hbar \tag{4.36}$$

instead of the momentum, and we find

$$\frac{\partial F}{\partial t}=\frac{1}{2}\Gamma\left[\exp\left[-q\frac{\partial}{\partial k}\right]-1\right](F+R_3)+\tfrac{1}{4}\Gamma\int_{-1}^{+1}W(x)\left[\exp\left[xq\frac{\partial}{\partial k}\right]-1\right]dx(F+R_3) , \tag{4.37}$$

$$\frac{\partial R_1}{\partial t}=-\gamma R_1+(\Delta+v_r k)R_2\ , \tag{4.38a}$$

$$\frac{\partial R_2}{\partial t}=-\gamma R_2+2\kappa R_3-(\Delta+v_r k)R_1\ , \tag{4.38b}$$

$$\frac{\partial R_3}{\partial t}=-\frac{1}{2}\Gamma\left[\exp\left[-q\frac{\partial}{\partial k}\right]+1\right](R_3+F)-2\kappa R_2 -\tfrac{1}{4}\Gamma\int_{-1}^{+1}W(x)\left[\exp\left[xq\frac{\partial}{\partial k}\right]+1\right]dx(F+R_3)\ . \tag{4.38c}$$

By expanding this formally in $q\partial/\partial x$, we generate partial differential equations for F and R_i.

The dependence on k in the density matrix can be no stronger than that given by the Doppler-shifted Lorentzians deriving from Eqs. (4.38a) and (4.38b). Hence the largest value of the expansion parameter is given by

$$q\frac{\partial}{\partial k}\frac{\Gamma^2}{(\Delta-v_r k)^2+\Gamma^2}=2qv_r\frac{(\Delta-v_r k)\Gamma^2}{[(\Delta-v_r k)^2+\Gamma^2]^2} \propto\frac{\varepsilon}{\Gamma}=\xi\ , \tag{4.39}$$

where the detuning is assumed to be of the order Γ, and the expansion parameter is found to be the same ξ as we introduced earlier.

The properties of the Wigner functions have been reviewed by Moyal (1949), Tatarskii (1983), and Hillary *et al.* (1984). The particular equations presented here are derived in Javanainen and Stenholm (1980c). For details the readers should consult these references.

B. A model problem

The basic observation to make regarding Eqs. (4.37) and (4.38) is that when expanding to second order in q we find

$$\frac{\partial F}{\partial t}=\frac{1}{2}\Gamma\left[-q\frac{\partial}{\partial k}+\tfrac{1}{2}q^2(1+\alpha)\frac{\partial^2}{\partial k^2}\right](F+R_3)\ , \tag{4.40}$$

where we have introduced α from Eq. (4.21). From the estimate (4.39) we find that the function F can change only at the rate $\xi\Gamma$, which is assumed small. The third component of the Bloch vector R_3 denotes the population difference between the two levels. For large light intensities these populations become equal and R_3 tends to zero. In this limit, Eq. (4.40) becomes a Fokker-Planck equation for F alone, of exactly the form (3.28). We can directly calculate

$$\hbar\frac{d}{dt}\langle k\rangle=\tfrac{1}{2}\hbar q\Gamma=F_0\ , \tag{4.41}$$

where we recognize the classical force from Eq. (3.10). The present estimate is superior, because it derives from the exact Eq. (4.40). The factor $\frac{1}{2}$ indicates the fact that in saturation only half the population is at the upper level from which spontaneous decay processes may originate.

The velocity diffusion coefficient in Eq. (4.40) is given by

$$D=\frac{\hbar^2q^2}{4M^2}\Gamma(1+\alpha)\ , \tag{4.42}$$

where we can immediately verify the relations (3.54), and from the form (3.49) we can see that the coefficient Q is equal to zero. Hence, for a strong enough laser field, the Mandel parameter (3.55) disappears, and the photon emission obeys Poisson statistics. Because the field flips from the atom back and forth at a very large rate, there is no trace of this seen in the spontaneously emitted photons; in practice they can leave the upper level at any time.

Is it possible to obtain anything else, or will Q remain zero for even more detailed solutions? In order to see how this works out, we shall consider a model case, which at the same time gives us an opportunity to present the various mathematical methods used in laser cooling calculations.

We consider the case of resonant tuning, $\Delta=0$, and neglect the spreading during one photon emission, i.e., we set $\alpha=0$. We also neglect the Doppler shift. In addition, we assume a strongly perturbed phase, so that we can set

$$\gamma\gg\Gamma\ . \tag{4.43}$$

In this case the dipole moment is forced to follow the population difference adiabatically, and from Eqs. (4.38a) and (4.38b) we find

$$R_2=\frac{2\kappa}{\gamma}R_3\ . \tag{4.44}$$

Inserted into Eqs. (4.37) and (4.38c) this gives

$$\frac{\partial F}{\partial t}=\frac{1}{2}\Gamma\left[-q\frac{\partial}{\partial k}+\frac{1}{2}q^2\frac{\partial^2}{\partial k^2}\right](F+R_3)\ , \tag{4.45}$$

$$\frac{\partial R_3}{\partial t}=-\Gamma(1+I)R_3-\Gamma F -\frac{1}{2}\Gamma\left[q\frac{\partial}{\partial k}-\frac{1}{2}q^2\frac{\partial^2}{\partial k^2}\right](F+R_3)\ , \tag{4.46}$$

where we have introduced the saturation parameter

$$I=\frac{4\kappa^2}{\gamma\Gamma}\ . \tag{4.47}$$

It turns out that we do not need the second derivative term in Eq. (4.46).

When there is no recoil ($q\to0$), the function F is conserved and can be normalized to unity. The variable R_3 will change at the rate Γ due to the first two terms on the right-hand side of Eq. (4.46). The change of F will occur at the rate $\xi\Gamma$, which is much slower than Γ, and hence we have identified two basic time scales in our problem. This is a general feature of the physical problem. The internal states will change at the rates Γ, κ, or Δ, which are roughly equal. The population summed will change at a rate that is slower by the factor ξ.

If we introduce the Fourier transform of the function F according to

$$\tilde{F}(x)=\int e^{-ixk/q}F(k)dk\ , \tag{4.48}$$

we can see that this generates the moments of the distribution $F(k)$ according to

$$m_s=\langle k^s\rangle=\left[\left[-iq\frac{\partial}{\partial x}\right]^s\tilde{F}(x)\right]_{x=0}\ . \tag{4.49}$$

When the spreading due to spontaneous emission is neglected, the function $F(k)$ becomes a series of delta functions that denotes the net absorption of n laser photons followed by spontaneous emission [cf. Eq. (3.43)]. We then have

$$F(k)=\sum_{n=0}^{\infty}f_n\delta(k-nq)\ . \tag{4.50}$$

Its Fourier transform (4.48) becomes

$$\tilde{F}(x)=\sum_{n=0}^{\infty}e^{inx}f_n\ , \tag{4.51}$$

where f_n is the probability of having n spontaneous emissions, and it gives the probability distribution of both the emerging photons and the ensuing momentum distribution. Even if $\tilde{F}(x)$ is not the form (4.51), its Fourier transform is the atomic momentum distribution as well as the distribution of the outgoing photons.

To get an idea about the distribution we use Eqs. (4.45) and (4.46) to calculate its moments. According to our normalization we get from (4.48)

$$\tilde{F}(0)=1\ . \tag{4.52}$$

If we define a similar Fourier transform of $R_3(k)$, we obtain directly from Eq. (4.46)

$$\frac{\partial}{\partial t}\tilde{R}_3(0)=-\Gamma(1+I)\tilde{R}_3(0)-\Gamma \tag{4.53}$$

because

$$-iq\frac{\partial}{\partial k}\to x\to 0\ . \tag{4.54}$$

The solution to Eq. (4.53) is

$$\tilde{R}_3(t)=\frac{1}{1+I}(e^{-\Gamma(1+I)t}-1)+R_0e^{-(1+I)\Gamma t}\ , \tag{4.55}$$

where R_0 is the initial condition. The equations to be generated follow from the transformed Eqs. (4.45) and (4.46) in the form

$$\frac{\partial\tilde{F}}{dt}=\frac{\Gamma}{2}(ix-\tfrac{1}{2}x^2)(\tilde{R}_3+\tilde{F})\ , \tag{4.56}$$

$$\frac{\partial\tilde{R}_3}{\partial t}=-(1+I)\Gamma\tilde{R}_3-\Gamma\tilde{F}-\frac{\Gamma}{2}(ix-\tfrac{1}{2}x^2)(\tilde{R}_3+\tilde{F})\ . \tag{4.57}$$

Taking the first derivative of (4.56), setting $x=0$, and using the definition (4.49), we find

$$\dot{m}_1=-iq\left[\frac{\partial\tilde{F}}{\partial x}\right]_{x=0}=\frac{q}{2}[1+\tilde{R}_3(t)]\ . \tag{4.58}$$

Inserting the result (4.55) and integrating, we find

$$m_1(r,t)=q\frac{I\tau+\frac{1}{1+I}(1+e^{-(1+I)\Gamma t})+R_0(1-e^{-(1+I)\Gamma t})}{2(1+I)}\ . \tag{4.59}$$

This process can be continued because if we set

$$\tilde{F}^{(n)}=\left[\frac{\partial^n}{\partial x}\tilde{F}(x)\right]_{x=0}\ , \tag{4.60}$$

$$\tilde{R}_3^{(n)}=\left[\frac{\partial}{\partial x}\tilde{R}_3(x)\right]_{x=0}\ , \tag{4.61}$$

we obtain from Eqs. (4.56) and (4.57) the relations

$$\frac{\partial}{\partial t}\tilde{R}_3^{(1)}=-(1+I)\Gamma\tilde{R}_3^{(1)}-\Gamma\tilde{F}^{(1)}-\frac{i\Gamma}{2}(\tilde{R}_3^{(0)}+1)\ , \tag{4.62}$$

from which we can integrate $\tilde{R}_3^{(1)}(t)$ when (4.55) and (4.59) are given. We then obtain from (4.56)

$$\frac{\partial\tilde{F}^{(2)}}{\partial t}=\Gamma[i(\tilde{F}^{(1)}+\tilde{R}_3^{(1)})-\tfrac{1}{2}(\tilde{R}_3^{(0)}+\tilde{F}^{(0)})]\ . \tag{4.63}$$

This gives the second moment, but the expressions are unwieldy and not very transparent. If we do not truncate the expansions of the exponents in Eqs. (4.37) and (4.38c), we can, in fact, obtain all moments successively in the way indicated. Here we are satisfied to investigate the first moment (4.59). Its behavior for various values of R_0 is shown in Fig. 7. We can see that after an initial period of the order of Γ^{-1} the evolution settles to a linear one with the behavior

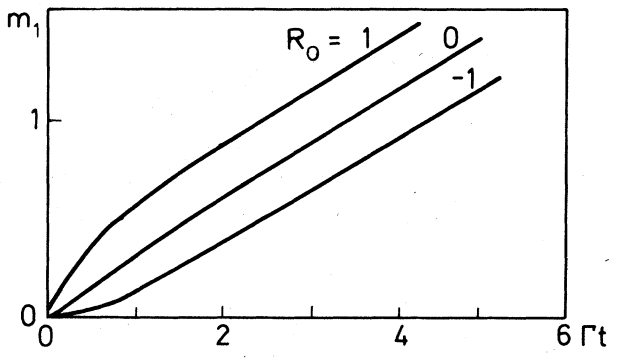

FIG. 7. The emergence of the linear behavior of the first moment m_1 of the atomic velocity distribution. Asymptotically the relationship $m_1\propto t$ holds, independently of the initial state of the atom (R_0), after a transient time of the order of the spontaneous decay lifetime Γ^{-1}. This holds true in the general case even if the transient behavior is more complicated than in this model problem.

$$m_1(t)=\frac{qI}{2(1+I)}\Gamma t+\frac{q}{2(1+I)}\left[\frac{1}{1+I}+R_0\right]. \tag{4.64}$$

After long enough times, the initial value is forgotten, the linear term dominates, and the result is in agreement with Eq. (3.51). The light pressure force can be determined from

$$M\hbar\frac{dm_1(t)}{dt}=\frac{I\hbar q}{2(1+I)}\Gamma, \tag{4.65}$$

in agreement with Eq. (4.41), when the intensity I goes to infinity.

This shows how we approach the diffusive limit after an initial transient of the order of Γ^{-1}. The intercept between the asymptotically linear dependence in Eq. (4.64) for $t=0$ is determined by the initial value R_0, but for long times this lacks importance (see Fig. 7). For small times we have

$$m_1(t)=\frac{\Gamma q}{2}(1+R_0)t-\frac{q}{4}\left[R_0+\frac{1}{1+I}\right]\Gamma^2t^2+\cdots. \tag{4.66}$$

At $R_0=-1$ the curve starts quadratically.

The limiting behavior for large times can be described by a Fokker-Planck equation. It can be seen that the spread in the function $F(k)$ becomes large and the function $\tilde{F}(x)$ is determined by small values of x only. We choose to expand its logarithm by setting

$$\tilde{F}(x)=\exp[ixA-\tfrac{1}{2}Bx^2+O(x^3)] \tag{4.67}$$

and from the definition (4.48) we find

$$F(k)=\frac{1}{2\pi}\int e^{-ikx/q}e^{ixA-Bx^2/2}dx$$

$$=\frac{1}{\sqrt{2\pi B}}\exp\left[-\frac{(k-Aq)^2}{2Bq^2}\right]. \tag{4.68}$$

It turns out that both A and B are asymptotically linear in time, and from Eq. (4.68) we find

$$\langle k\rangle=qA\equiv qS_1t=q\langle n\rangle, \tag{4.69}$$

$$\langle(k-\langle k\rangle)^2\rangle=q^2B\equiv 2q^2S_2t=q^2\langle\Delta n^2\rangle. \tag{4.70}$$

This solution thus corresponds to the solution of the Fokker-Planck equation

$$\frac{\partial F}{\partial t}=\frac{\partial}{\partial k}\left[\left[-S_1+S_2\frac{\partial}{\partial k}\right]F\right]. \tag{4.71}$$

In order to determine the Fokker-Planck equation of the problem or the equivalent first two moments, we need only obtain the coefficients A and B as defined in Eq. (4.67).

This we can do in a systematic way from Eqs. (4.56) and (4.57). We make the ansatz

$$R=(r^0+xr^1+x^2r^2+\cdots)F \tag{4.72}$$

and require to all orders that

$$\frac{dr^i}{dt}=0. \tag{4.73}$$

To zeroth order we have

$$\tilde{F}=\text{const}=1, \tag{4.74}$$

$$\tilde{R}_3=-\frac{1}{1+I}\tilde{F}, \tag{4.75}$$

giving $r^0=-(1+I)^{-1}$. To continue we now need, to zeroth order,

$$\tilde{R}_3+\tilde{F}=\frac{I}{1+I}\tilde{F}, \tag{4.76}$$

which is inserted into (4.56) to give the first-order result

$$\left[\frac{\partial\tilde{F}}{\partial t}\right]_1=ix\frac{I\Gamma}{2(1+I)}\tilde{F}. \tag{4.77}$$

From this we can immediately identify

$$A=\frac{I\Gamma}{2(1+I)}t\equiv S_1t, \tag{4.78}$$

which from Eq. (4.69) gives directly Eq. (4.65). To obtain r^1 we must use Eq. (4.72) in (4.57), but to this order the time derivative can no longer be neglected. We find

$$\frac{\partial\tilde{R}}{\partial t}=\frac{\partial\tilde{R}}{\partial\tilde{F}}\dot{\tilde{F}}=-(1+I)\Gamma r^1x\tilde{F}-\frac{ix\Gamma}{2}(r_0+1)\tilde{F}. \tag{4.79}$$

Introducing $\dot{\tilde{F}}$ to first order from Eq. (4.77), we obtain directly

$$r^1=-i\frac{I^2}{2(1+I)^3}. \tag{4.80}$$

When this result is inserted into Eq. (4.56) we obtain the result to second order in x

$$\left[\frac{\partial\tilde{F}}{\partial t}\right]_2=\Gamma\left[\frac{ix^2}{2}r^1-\frac{x^2}{4}(1+r_0)\right]\tilde{F}$$

$$=-\frac{x^2}{2}\frac{I\Gamma}{2(1+I)}\left[1-\frac{I}{(1+I)^2}\right]\tilde{F}. \tag{4.81}$$

Then the coefficient B becomes

$$B=\frac{I\Gamma}{2(1+I)}\left[1-\frac{I}{(1+I)^2}\right]t. \tag{4.82}$$

The diffusion coefficient in the Fokker-Planck equation is now

$$S_2=\frac{B}{2t}=\frac{I\Gamma}{4(1+I)}\left[1-\frac{I}{(1+I)^2}\right]. \tag{4.83}$$

The final contribution in the large parentheses is an anomalous diffusion contribution. It decreases the diffusion, but it cannot, of course, make it negative. Identifying S_1 in Eq. (4.69) with the factor in (3.51), we have

$$D_0=\frac{\hbar^2q^2}{2M^2}S_1=\frac{\hbar^2q^2}{4M^2}\frac{I\Gamma}{(1+I)}, \tag{4.84}$$

and from Eqs. (3.52) and (4.70) we have

$$D_0(1+Q)=\frac{\hbar^2q^2}{M^2}S_2=\frac{\hbar^2q^2I\Gamma}{4M^2(1+I)}\left[1-\frac{I}{(1+I)^2}\right]$$
$$=D_0\left[1-\frac{I}{(1+I)^2}\right]. \tag{4.85}$$

The anomalous term turns out to be just the Mandel parameter

$$Q=-\frac{I}{(1+I)^2}, \tag{4.86}$$

which we find to be negative. This implies from Eq. (3.55) that

$$\langle\Delta n^2\rangle<\langle n\rangle, \tag{4.87}$$

which is an indication of photon antibunching, i.e., the photons arrive more regularly than in a Poissonian process. If we return to the derivation we find that the anomalous term follows from the time derivative on the left-hand side of Eq. (4.79). Thus we find that in the adiabatic elimination of the optical Bloch vector R, we must take into account that the center of mass of the atom to which it is attached is carried along by the light pressure force. We indicate this dependence explicitly by writing for the density of the dipole moment $R(k-\langle k(t)\rangle)$, where $\langle k(t)\rangle$ is the solution to Eq. (4.41). From Eqs. (4.79) and (4.71) we find that

$$\frac{\partial R}{\partial t}=\frac{\partial R}{\partial F}\dot{F}=\frac{\partial R}{\partial F}\frac{\partial F}{\partial k}\frac{dk}{dt}=\frac{\partial R}{\partial k}\frac{dk}{dt}, \tag{4.88}$$

just as would follow from a straightforward derivation of $R(k-\langle k(t)\rangle)$ with regard to time. To second order we cannot neglect the first-order deflection of the atomic trajectory. This gives an anomalous contribution to the diffusion; in the present case it turns out to be the only one. In the next section we shall develop a formalism that takes care of this automatically.

We still want to approach the present model problem from one more point of view. The time evolution of a linear system like (4.56) and (4.57) can be inferred from its spectrum in the complex frequency plane. Because we are here interested in actual time evolution properties, it is advantageous to use the Laplace transform, which we denote by a caret, i.e., $\hat{F}$ and $\hat{R}_3$. The definition we choose is

$$\hat{F}(s)=\int_0^\infty e^{-st\Gamma}\tilde{F}(x,t)dt, \tag{4.89}$$

which makes s dimensionless. The system then becomes

$$\begin{bmatrix} s-\frac{ix}{2}+\frac{1}{4}x^2 & -\frac{ix}{2}+\frac{1}{4}x^2 \\ 1+\frac{ix}{2}-\frac{1}{4}x^2 & s+(1+I)+\frac{ix}{2}-\frac{1}{4}x^2 \end{bmatrix}\begin{bmatrix}\hat{F}\\ \hat{R}_3\end{bmatrix}=\begin{bmatrix}F_0\\R_0\end{bmatrix}, \tag{4.90}$$

where F_0 and R_0 are the initial values. The determinant of the matrix is

$$D=s^2+s(1+I)+I\left[\frac{ix}{2}-\tfrac{1}{4}x^2\right]. \tag{4.91}$$

If we denote its zeros by s_1 and s_2, we find that the function F must be of the form

$$\hat{F}=\frac{A}{s-s_1}+\frac{B}{s-s_2}, \tag{4.92}$$

giving the time evolution

$$\tilde{F}=Ae^{s_1\Gamma t}+Be^{s_2\Gamma t}. \tag{4.93}$$

From our earlier discussions we can surmise that the negative real part of one root greatly exceeds that of the other; in this way the transients die out and the asymptotic slow time evolution derives from the other one. This is a restatement of the discussion above concerning two time scales differing by a factor ξ.

From our considerations in connection with Eq. (4.68) we concluded that we need to know $\tilde{F}$ as an expansion in x. Thus we can develop the expansion of the roots of the time evolution in the same way.

For $x=0$ we easily find

$$s_1^0=0, \tag{4.94a}$$

$$s_2^0=-(1+I). \tag{4.94b}$$

These are denoted by crosses in Fig. 8. To obtain the corrections we write

$$s=s^0+ixs^1+x^2s^2. \tag{4.95}$$

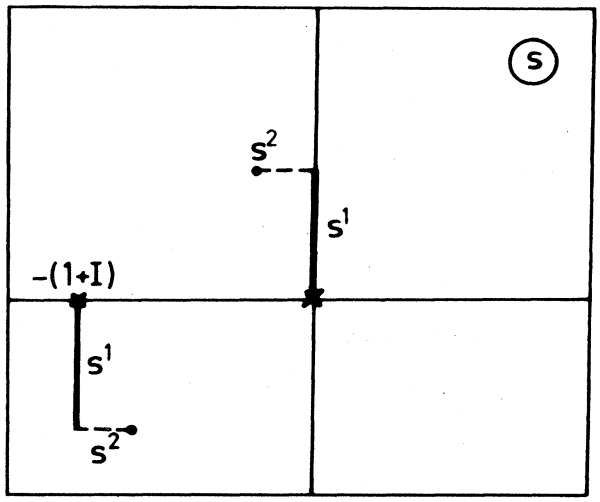

FIG. 8. Behavior of the eigenvalues of the model problem in the complex plane. With no photon recoil there are two poles of the response, one heavily damped [at $-(1+I)$] and one stationary (at zero). To lowest order these acquire only the imaginary parts s^1, which give the light pressure force. To second order the stationary pole acquires the damping s^2 because of the onset of light-induced diffusion. This behavior, found for our simple model problem, shows what happens in the general case too. For bound-particle cooling a similar picture is given in Fig. 18.

Inserting this into Eq. (4.91) and determining the coefficients, we find to first order

$$s_1^1=\frac{I}{2(1+I)}\ , \tag{4.96a}$$

$$s_2^1=-\frac{I}{2(1+I)}\ . \tag{4.96b}$$

These are indicated by solid lines in Fig. 8. They correspond to an imaginary part of the poles and represent the dynamical evolution; comparing Eq. (4.95) with (4.68), we see that s_2^1 indeed gives the correct light pressure force in (4.69). To second order we find

$$s_1^2=-\frac{I}{4(1+I)}\left[1-\frac{I}{(1+I)^2}\right]\ , \tag{4.97a}$$

$$s_2^2=\frac{I}{4(1+I)}\left[1-\frac{I}{(1+I)^2}\right]\ . \tag{4.97b}$$

These are indicated by dotted lines in Fig. 8. The correction to s_2 is of no significance and only slightly affects the rate at which the initial fast transients die out. For the root s_2, however, a major change has appeared. Previously it had a zero real part, signifying a stationary state, which in this order has turned into a slow diffusive-type motion. Equation (4.97a) is seen to give again the result (4.83) when the dimensions are restored by multiplying with Γ. The anomalous contribution is included correctly. From the earlier discussion in connection with Eq. (4.38), we can conclude that the time scale for the force is $\xi\Gamma$, whereas the diffusion emerging to second order must occur over a time scale of order $\xi^2\Gamma$. This is in agreement with the considerations in Eqs. (2.19), (3.22), and (3.39).

Inserting our results into Eq. (4.93) for times much longer than Γ^{-1}, we find

$$F(t)=A\exp\left[ix\frac{I}{2(1+I)}\Gamma t - \frac{x^2}{2}\frac{I}{2(1+I)}\left[1-\frac{I}{(1+I)^2}\right]t\right]\ , \tag{4.98}$$

which is exactly the same result as given in Eqs. (4.77)–(4.81) from the Fokker-Planck treatment.

We have spent a lot of time on our simple-model problem because it illustrates the methods that have been used, gives physically correct results, and provides a general insight into the mechanism and tools of laser cooling calculations. For more complicated systems the manipulations become more involved, but the physics and the theoretical approach remain the same.

When no recoil is considered, there appears a part of the problem which relaxes at a fast rate of the order of Γ. This determines the initial transients which die out soon. The asymptotic behavior for long times is determined by a slow time evolution, which initially, when photon recoil is turned off, does not relax. This will acquire a slow time evolution due to the recoil of the photons, and this evolution will then be described by a Fokker-Planck equation or its equivalent. The zero eigenvalue will then become modified, and the system experiences a slow time evolution which in each order will affect the results of the following one. Depending on the details of the problem, the asymptotic evolution goes on forever or it approaches a stationary state. In the latter case an initially highly degenerate zero eigenvalue will split by the perturbation so that only one single zero eigenvalue remains, which determines the unique steady state approached towards the end of the cooling process. We shall see examples of these statements in the following sections of this review.

C. The adiabatic elimination procedure

In this section we carry out a formal procedure that automatically performs the steps introduced in the previous section. We choose to work with an abstract vector

$$\mathbf{A}=\begin{bmatrix}R\\F\end{bmatrix}\ , \tag{4.99}$$

where the subspace R contains fast relaxation rates of order Γ, and F lies in a subspace that is conserved when photon recoil is neglected. In the previous example R corresponded to the optical Bloch vector and F to the population, but the subspace of F may also have more than one dimension.

We assume that the time evolution of $\mathbf{A}$ is given by a linear equation of the Schrödinger type. In most cases $\mathbf{A}$ will be a density matrix, and we need a superoperator representation to reduce this to a vector. The earliest use of this approach, to my knowledge, was by Klein (1952), and it has been developed further by Zwanzig (1960), Argyres (1966), and Grabert (1982).

The time evolution of $\mathbf{A}$ is expanded in some small parameter ∂ to give

$$\frac{d}{dt}\mathbf{A}=(\mathbf{M}+\mathbf{N}\partial+\mathbf{K}\partial^2)\mathbf{A}\ . \tag{4.100}$$

In the cases discussed so far we have

$$\partial=q\frac{\partial}{\partial\kappa}\sim\xi \tag{4.101}$$

in accordance with Eqs. (4.39). When $\partial=0$ we have a steady state

$$\mathbf{M}\mathbf{r}^0=0\ , \tag{4.102}$$

which is usually highly degenerate with respect to a parameter, which in the case of (4.101) is the momentum $\hbar k$. If $\mathbf{A}(k)$ is a solution for any x, also the state

$$\mathbf{A}(k+xq)=e^{x\partial}\mathbf{A}(k) \tag{4.103}$$

is one. This mirrors the fact that once a particle is given the momentum $\hbar k$, this cannot be altered by interaction with light as long as photon momentum is neglected.

The time evolution operator $\mathbf{M}$ is not self-adjoint, and there exists a left eigenvector $\bar{\mathbf{r}}^0$ which corresponds to the zero eigenvalue in Eq. (4.102), viz.,

$$\bar{\mathbf{r}}^0\mathbf{M}=0 \tag{4.104}$$

and

$$(\bar{\mathbf{r}}^0,\mathbf{r}^0)=1 . \tag{4.105}$$

From these we form the projectors

$$P=\mathbf{r}^0\otimes\bar{\mathbf{r}}^0 , \tag{4.106a}$$

$$Q=1-P , \tag{4.106b}$$

which project onto two subspaces or arbitrary dimensions. We now apply conventional partitioning techniques, using the properties (4.104) and (4.106) to obtain

$$\frac{\partial}{\partial t}P\mathbf{A}=\partial(P\mathbf{N}P\mathbf{A}+P\mathbf{N}Q\mathbf{A}+\partial P\mathbf{K}P\mathbf{A}) . \tag{4.107}$$

Here we have neglected a term $P\mathbf{K}Q$, which will contribute to higher order only. We now introduce the notation

$$\eta=(\bar{\mathbf{r}}^0,P\mathbf{A}) \tag{4.108}$$

and

$$j=(\bar{\mathbf{r}}^0,(P\mathbf{N}P\mathbf{A}+P\mathbf{N}Q\mathbf{A}+\partial P\mathbf{K}P\mathbf{A})) . \tag{4.109}$$

With these, Eq. (4.107) is described by a local conservation law of the form

$$\frac{\partial}{\partial t}\eta=\partial j . \tag{4.110}$$

Thus the degenerate zero eigenvalue corresponds to a conserved quantity η, whose current j is given by Eq. (4.109). When ∂ can be regarded as small, the time evolution of η will remain slow. Equation (4.110) is an exact consequence of our formulation of the problem. It is only when we want to evaluate j that we need further approximations.

To proceed we find

$$\frac{\partial}{\partial t}Q\mathbf{A}=(Q\mathbf{M}Q+\partial Q\mathbf{N}Q)Q\mathbf{A}+\partial Q\mathbf{N}P\mathbf{A}+O(\partial^2) . \tag{4.111}$$

Because the subspace Q is assumed to be rapidly relaxing, the operator $Q\mathbf{M}Q$ must have a negative real part of order Γ. After initial transients, $Q\mathbf{A}$ settles down to its steady-state value, which to order one in ∂ gives from Eq. (4.111)

$$Q\mathbf{A}=-(Q\mathbf{M}Q)^{-1}\partial Q\mathbf{N}P\mathbf{A} . \tag{4.112}$$

Corrections of order ∂^2 can be omitted because the term with $Q\mathbf{A}$ in Eq. (4.107) is already multiplied by ∂, and the expansion in (4.100) does not include higher-order terms than ∂^2. Inserting into Eq. (4.107), we find the time evolution operator

$$\frac{\partial}{\partial t}P\mathbf{A}=\partial\{P\mathbf{N}P+[P\mathbf{K}P-P\mathbf{N}Q(Q\mathbf{M}Q)^{-1}Q\mathbf{N}P]\partial\}P\mathbf{A} . \tag{4.113}$$

This is the desired expression. The operator $Q\mathbf{M}Q$, which has to be inverted, is in the subspace without small eigenvalues, and in principle no problems appear in its evaluation. When ∂ is a differential operator, we see directly from Eq. (4.113) that we have derived an equation of the Fokker-Planck type. We have felt free to transfer the derivative operator ∂ to any place we want without acting on any of the operators in between. This is in agreement with our consistent expansion in ∂, as discussed in Sec. III.D. From the treatment in Sec. IV.B we can see the same property in Eqs. (4.68)—(4.71). The Fokker-Planck treatment is consistent with the result (4.68) only if the factors in S_2 can be moved through the derivative $\partial/\partial k$. There are problems where these considerations may be of importance; they are not discussed here.

The present derivation is based on the work of Stenholm (1984a,1984b), in which it is also shown (1984b) how the present approach follows the steps of the previous section quite closely.

To see this we do not introduce the projection operators, but make an ansatz for the vector

$$\mathbf{A}=\mathbf{r}F=(\mathbf{r}^0+\mathbf{r}^1\partial+\mathbf{r}^2\partial^2+\cdots)F , \tag{4.114}$$

where we require that all time dependence be in the component F, which we here assume to be one dimensional. Then we have

$$\frac{\partial}{\partial t}\mathbf{r}=0 . \tag{4.115}$$

To zeroth order in ∂, it follows directly from Eq. (4.102) that

$$\frac{\partial F^{(0)}}{\partial t}=0 \tag{4.116}$$

as we expected, and the degeneracy property (4.103) applies to $F^{(0)}$. We require the vector r to be normalized according to

$$(\bar{\mathbf{r}}^0,\mathbf{r})=1 , \tag{4.117}$$

and multiplying Eq. (4.100) by $\bar{\mathbf{r}}^0$ and using Eq. (4.104), we find

$$\frac{\partial F}{\partial t}=\partial(\bar{\mathbf{r}}^0,\mathbf{N}\mathbf{r})F+\partial^2(\bar{\mathbf{r}}^0,\mathbf{K}\mathbf{r}^0)F . \tag{4.118}$$

To first order we now find

$$\frac{\partial F^{(1)}}{\partial t}=\partial(\bar{\mathbf{r}}^0,\mathbf{N}\mathbf{r}^0)F^{(1)} , \tag{4.119}$$

which is the linear term in ∂ on the right-hand side of Eq. (4.113).

To go to second order we need to know $\mathbf{r}$ to first order, as seen from Eq. (4.118). We obtain from Eq. (4.100)

$$\frac{d\mathbf{A}}{dt}=\frac{d\mathbf{A}}{dF}\dot{F}=\mathbf{r}\partial(\bar{\mathbf{r}}^0,\mathbf{N}\mathbf{r})F=(\mathbf{M}+\mathbf{N}\partial)\mathbf{r}F . \tag{4.120}$$

Extracting the linear terms, we find

$$\mathbf{N}\mathbf{r}^0+\mathbf{M}\mathbf{r}^1=\mathbf{r}^0(\bar{\mathbf{r}}^0,\mathbf{N}\mathbf{r})=P\mathbf{N}\mathbf{r}^0 \tag{4.121}$$

and have to solve r^1 from

$$\mathbf{M}\mathbf{r}^1=-(1-P)\mathbf{N}\mathbf{r}^0=-Q\mathbf{N}\mathbf{r}^0 . \tag{4.122}$$

Here we have introduced P and Q from Eqs. (4.106). Because the inhomogeneous term in (4.122) lies in the Q subspace, $\mathbf{M}$ can be inverted and we find $\mathbf{r}^1$ in the form

$$\mathbf{r}^1=-\mathbf{M}^{-1}Q\mathbf{N}\mathbf{r}^0 . \tag{4.123}$$

The requirement (4.117) gives

$$(\bar{\mathbf{r}}^0,\mathbf{r})=(\bar{\mathbf{r}}^0,\mathbf{r})+\partial(\bar{\mathbf{r}}^0,\mathbf{r}^1)=1 , \tag{4.124}$$

which with Eq. (4.105) implies

$$(\bar{\mathbf{r}}^0,\mathbf{r}^1)=0 . \tag{4.125}$$

The solution has to be chosen in the subspace orthogonal to $\bar{\mathbf{r}}^0$, i.e., the one selected by Q. Thus we find to this order

$$\mathbf{A}^{(1)}=[1-(Q\mathbf{M}Q)^{-1}Q\mathbf{N}\partial]\mathbf{r}^0F , \tag{4.126}$$

which, inserted into (4.118), gives for the evolution of F to second order

$$\begin{aligned}\frac{\partial F^{(2)}}{\partial t}=&\partial\{(\bar{\mathbf{r}}^0,\mathbf{N}\mathbf{r}^0)\\&+\partial[(\bar{\mathbf{r}}^0,\mathbf{K}\mathbf{r}^0)\\&-(\bar{\mathbf{r}}^0,\mathbf{N}Q(Q\mathbf{M}Q)^{-1}Q\mathbf{N}\mathbf{r}^0)]\}F^{(2)} .\end{aligned} \tag{4.127}$$

It is easy to see that this is exactly the same equation as (4.113). The operator denoted by $(Q\mathbf{M}Q)^{-1}$ is the inverse of $\mathbf{M}$ in the subspace selected by Q. In the present treatment the one Q comes from the modification of the slow time evolution, as seen in Eqs. (4.121) and (4.122), and the other from the normalization requirement, Eqs. (4.124) and (4.125). In the projection formalism both appeared in a totally automatic way. It is easy to see the similarity between the present treatment and that of Eqs. (4.72)—(4.81).

The method presented here brings up quite clearly why we need to project out the slow subspace—there is a zero eigenvalue in the dynamic matrix $\mathbf{M}$. This causes trouble if we try to invert it, which we must do to obtain successive terms in a perturbation expansion. By including the first-order slow time dependence in Eq. (4.120), we find that the singular part is eliminated and the matrix can be inverted as in Eq. (4.122). The trick of using an ansatz of the form (4.114) is a special case of the Chapman-Enskog procedure for treating singular perturbation problems. It has been applied to statistical mechanics problems like ours by Titulaer (1978,1980a,1980b). The first to apply these ideas to laser cooling was Minogin (1980b,1981).

Perturbation theory has long been formulated by projection techniques (Lax, 1950; Löwdin, 1951), and one aim was explicitly the removal of degeneracy. This has been pursued in some detail by Bloch (1958) and Bogoliubov (a suitable reference is Bogoliubov, 1967, but he used the method much earlier). The method as presented here is of interest because it provides a very general scheme for treating statistical mechanical systems where fast microdynamics is eliminated to give much slower macrodynamics.

Adiabatic elimination procedures are, of course, well known in laser physics (see, for example, Risken, 1984, and Haken, 1977). Recently there has been some progress in understanding the adiabatic elimination techniques (Haake and Lewenstein, 1983a,1983b; Casagrande *et al.*, 1984; Gardiner, 1984; Gardiner and Steyn-Ross, 1984). For some other applications of the present approach see Stenholm (1984b). van Kampen (1985) has recently reviewed the technique and its applicability in an interesting way. The adiabatic elimination in a Fokker-Planck system is approached in a way equivalent with the present one by Kaneko (1981); see the discussion in Risken (1984).

D. Cooling of free particles

The smallness of the optical photon momentum $\hbar q$ in comparison with other momentum scales early suggested the use of a continuum theory for the description of the atomic velocity distribution. The resonant light pressure force was derived in the form

$$F_0=\tfrac{1}{2}\hbar q\Gamma\frac{I\gamma^2}{\Delta^2+\gamma^2(1+I)} . \tag{4.128}$$

When we tune to resonance, $\Delta=0$, this agrees with our result in Eq. (4.65). The same result has since been derived by many researchers.

The classical force (4.128) emerges when we regard the scattering of photons as a continuous stream of momentum. When their quantum nature is considered, we obtain a diffusive spreading, as explained in Sec. III.D. This was, to my knowledge, first pointed out by Kazantsev (1974), who also introduced a Fokker-Planck equation to describe the situation (Kazantsev, 1975). As far as I know, the first attempts to derive these from microscopic considerations starting from a density matrix were presented by Krasnov and Shaparev (1975) and Baklanov and Dubetskii (1976). Many of the applications have since used the Fokker-Planck equations, as can be seen in Kazantsev (1978) and Letokhov and Minogin (1981b).

Detailed derivations of a Fokker-Planck equation for free atoms was given in the momentum representation in Javanainen and Stenholm (1980a) and for the Wigner function in Cook (1980a). The broadband approximation in Javanainen and Stenholm (1980a) eliminated the anomalous contributions, which are due to correlation effects. Their presence was first noticed in a paper by Gordon and Ashkin (1980) and were simultaneously found by Minogin (1980b), who used a method based on the Chapman-Enskog procedure. In the work by Cook (1980a) they were also derived by an implicit technique. These terms acquired great interest because Mandel had pointed out (1979b) that they were related to the non-Poissonian character of the spontaneously emitted photons, as discussed in Sec. III.C. Cook (1980b) showed that the anomalous diffusion could be interpreted easily

as a non-Poissonian characteristic which depends on detuning.

These considerations were continued in Stenholm (1983), where the Chapman-Enskog approach was used and some physical interpretation was attempted. The projection operator formalism was applied by Stenholm (1984a), and this method was found to provide the result in a very simple and straightforward way. After the initial separation of the slowly developing subspace, no physical argument is needed along the way. Except for some care in avoiding the singularities in the problem due to the zero eigenvalue, no physical or algebraic complexity is involved.

As a result of these works, the momentum distribution of a free particle in an infinite plane wave is described by the Fokker-Planck equation (4.71), where S_1 is given directly by Eq. (4.128) as $F_0/\hbar$, and we find

$$S_2=\frac{\Gamma}{4}q^2\frac{I\gamma^2}{D}\left[1+\alpha+\frac{I\gamma}{D^2}[\Delta^2(\Gamma-\gamma)-\gamma^2(\Gamma+\gamma)]\right], \tag{4.129}$$

where

$$D=\Delta^2+\gamma^2(1+I)\,. \tag{4.130}$$

If the combination $I\gamma^2$ becomes large, the result approaches our estimate (4.42) when the diffusion in k space is transformed to diffusion in velocity space, $v=\hbar k/M$. If we introduce the case of no phase-perturbing processes, we have

$$\gamma=\tfrac{1}{2}\Gamma\,, \tag{4.131}$$

and the result (4.130) is in agreement with the work by Cook (1980a,1980b).

From the form (4.129) we can directly identify the Mandel Q parameter as in (3.49) to be

$$Q=\frac{I\gamma}{D^2}[\Delta^2(\Gamma-\gamma)-\gamma^2(\Gamma+\gamma)] \tag{4.132a}$$

$$=2\kappa^2\frac{\Delta^2-3\gamma^2}{D^2}\,, \tag{4.132b}$$

where the latter form arises when Eq. (4.131) is used. For large values of the intensity parameter κ this goes to zero as

$$Q\propto\kappa^{-2}\,, \tag{4.133}$$

which shows that the deviations from Poisson statistics disappear, as noted already in Sec. IV.B after Eq. (4.42), where the physical origin of this disappearance was also explained.

When the phase-perturbing processes dominate, we have $\gamma\to\infty$ and we find from Eq. (4.132a) that

$$Q\propto I\propto\gamma^{-1}\,. \tag{4.134}$$

This shows that a large amount of phase incoherence destroys the anomalous diffusion, and Poisson statistics ensue. The atoms cannot remember to carry coherence from one spontaneous emission event to the next. This was the case discussed by Javanainen and Stenholm (1980a,1980b).

When the intensity becomes small,

$$Q\propto\kappa^2\,, \tag{4.135}$$

and the non-Poissonian nature of the statistics is a higher-order effect not obtainable in lowest-order perturbation theory, where $S_2\propto\kappa^2$ only.

When we go to resonance, $\Delta=0$, and let γ become dominantly large, we obtain from Eq. (4.132a) exactly the result (4.86), which shows that our model problem there gave exactly the right answer within its limits of validity. The correction terms are of order (Γ/γ), as the adiabatic assumption (4.43) implies.

At $\Delta=0$, the result (4.132) was calculated by Mandel (1979b). For $\Delta\neq0$ the result was obtained by Cook (1980b) and by Smirnov and Troshin (1979). For $|\Delta|<\sqrt{3}\gamma$ the function (4.132b) gives a negative value and thus implies sub-Poissonian statistics. For larger detunings, a small super-Poissonian distribution is observed. The function Q is plotted as a function of (Δ/γ) in Fig. 9.

For a standing wave there have been some calculations that go beyond low-order perturbation theory: Stenholm *et al.* (1978), Minogin and Serimaa (1979), Minogin (1981), and Kazantsev *et al.* (1985). The cooling is here approaching a steady state, as considered in Sec. III.B. One issue of controversy is the question whether it is possible to trap a neutral particle by a resonant standing wave. The investigations of Letokhov and Minogin (1978a) seem to show that there is no trapping because the quantum-mechanical indeterminism of the atomic position prevents trapping. A simple argument from Stenholm (1978b) can be used to see that the spatial

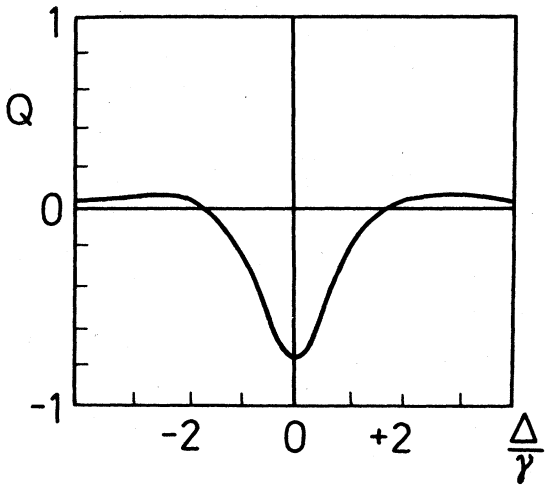

FIG. 9. Behavior of the Mandel Q parameter as a function of laser detuning Δ. When Q is negative, the emitted photons display sub-Poissonian statistics. The parameter is simultaneously a measure of the coherence-induced anomalous diffusion; negative Q indicates a less-than-random spreading.

dependence of the standing-wave pattern cannot be easily observed when spontaneous emission smears the coherences: The particle with velocity v traverses one wavelength in the time

$$\Delta t \sim \frac{\lambda}{v} \sim (qv)^{-1} , \tag{4.136}$$

and the change of velocity achieved by the forces $\hbar q\Gamma$ in this time is approximately

$$\Delta v \sim \frac{\hbar\Gamma}{Mv} . \tag{4.137}$$

The atoms interacting most strongly have a detuning Δ of the order of the Doppler shift qv, and the velocity change (4.137) affects this by an amount

$$q\Delta v \sim \frac{\hbar q^2\Gamma}{M|\Delta|} \sim \varepsilon \frac{\Gamma}{|\Delta|} \sim \varepsilon \ll \Gamma . \tag{4.138}$$

Thus for the atoms that interact most strongly, the radiation force cannot considerably affect velocity, and hence it is hard to see how trapping could occur. The argument is, however, not completely compelling, and some researchers still express different opinions.

The standing-wave case has not been attacked in a consistent way by the projection operator method described in the preceding section. Such a systematic approach may answer some of the question finally. The closest approach is given by Minogin (1981), and from this we can see the complexity of the problem. There are no reasons to doubt the results of this approach, but the complicated manipulations make it difficult to decide conclusively that nothing important is omitted.

Recent work by Dalibard and Cohen-Tannoudji (1985a,1985b) gives a thorough discussion of these questions. The relationship between quantum and classical treatments becomes especially clear in their formulation. They also prove that there exists a relationship between the damping factor and the diffusion coefficient, similar to the well-known fluctuation-dissipation theorem.

When we reject the simplifying assumption of a plane-wave radiation field, additional force terms appear in the direction of the gradient of the intensity. The transverse terms have been included in the calculation in several works. A discussion is found in Letokhov and Minogin (1981b). For their inclusion into the general calculation of light pressure forces, see, for example, Gordon and Ashkin (1980) and Minogin (1980b). These are usually small terms, of the order of (λ/a_0) where a_0 is the beam waist, which usually greatly exceeds the optical wavelength λ. This holds for near-resonant tuning. When, however, the detuning becomes large

$$|\Delta| \gg \gamma , \tag{4.139}$$

the gradient force can be seen to dominate over the light pressure force (see the discussion in Letokhov and Minogin, 1981b, Sec. 5.2). There is, on the other hand, a decrease in both forces due to the lack of resonant interaction, and trapping solely by the gradient force requires exceedingly large laser intensities of the order of 10^8 W/m^2. There has been some additional discussion about trapping by pure light fields, but it has centered on technical questions. Some recent proposals can be found in Dalibard *et al.* (1983,1984) and Ashkin (1984). At the time of this writing it is still an open question whether light trapping is possible or not. This question may find an answer in the near future because the interest in it is great; see Phillips, 1984.

When laser cooling takes place over extended ranges of position and velocity, the theory can be taken one step further. The kinetic Fokker-Planck equation can be replaced by a hydrodynamic approximation similar to that of ordinary kinetic theory. One then uses a local velocity $\langle v(z,t)\rangle$, a local particle density $n(z,t)$, and a local velocity square, which defines a local temperature by

$$\langle v(z,t)^2\rangle \approx \frac{2T(z,t)}{M} . \tag{4.140}$$

This approach has been used by Zueva *et al.* (1981) to describe the atomic beam cooling experiments (Letokhov and Minogin, 1981a). The validity of the Fokker-Planck equation is based on the assumption that we consider times such that $\Gamma t \gg 1$, but the hydrodynamic regime requires an even more coarse-grained view of the process.

V. COOLING OF TRAPPED PARTICLES

A. The physical quantities

One of the fundamental limits on spectroscopic measurements is the Doppler shift owing to particle motion. Laser spectroscopy has developed several methods to overcome its component linear in the velocity, but the quadratic term in the velocity still remains. To achieve the highest accuracy one has to investigate particles that are definitely brought to a standstill. Because we also want long interaction times, this implies the use of trapped particles confined to a limited region of space. If we want to eliminate the interaction between particles, we must turn to extremely low densities, and in the limiting case we consider traps containing only one single particle at a time. This is the ideal configuration for super-high-resolution spectroscopy.

In Sec. IV.D we presented suggestions for trapping particles in pure light fields, but no experimental evidence exists in this area. The experimental progress in trapping atomic particles goes back to early efforts by Dehmelt (1967,1981a) and others to investigate free electrons. It has been possible to trap and cool singly ionized atoms as originally suggested by Wineland and Dehmelt (1975). There cannot exist any static electric traps for charged particles (Wing, 1985); this is usually referred to as the classical Earnshaw theorem. One must consequently find other means. The work by Toschek and his collaborators in Heidelberg and Hamburg uses a dynamic trap, which operates through a radio-frequency quadrupole field (Neuhauser *et al.*, 1978a; Dehmelt, 1981a,1981b; Toschek, 1984). Wineland's group (Wineland *et al.*,

1978a,1978b) uses a Penning trap. Here electric and magnetic forces are combined to keep the ionic orbits confined to a finite volume. In either case the actual motion is rather complicated. The radio-frequency trapping, however, tends to confine the particle to a well-defined position more efficiently than the Penning trap and hence appears to be closer to the simplified models we discuss below. In both cases, however, the trapping fields perturb the spectra of the particles in a less controlled way, and there are difficulties in applying the techniques to super-high-resolution measurements. Not even here can we achieve the theoretical ideal of a free particle at rest in an empty universe.

As our model for a trapped particle interacting resonantly with laser radiation, we take a harmonically bound two-level system, as shown in Fig. 10. We assume an ideal one-dimensional trapping potential of the form

$$V(R)=\tfrac{1}{2}M\nu^2R^2\ , \tag{5.1}$$

where ν is the angular frequency of oscillations in the trap and R is the coordinate of the center of mass of the particle. The complications due to the radio-frequency micromotion and the three-dimensionality of the trap are ignored.

When the motion of the particle is taken to be classical, its velocity in the potential (5.1) obeys

$$v(t)=v_0\cos\nu t\ . \tag{5.2}$$

In a traveling wave the radiation pressure force depends on velocity according to Eq. (3.6), as

$$F\propto\frac{\Gamma^2}{[\Delta-qv(t)]^2+\Gamma^2}\ , \tag{5.3}$$

and at every instant when the Doppler shift compensates the detuning, $qv=\Delta$, the atom can be excited; this may be followed by a subsequent spontaneous decay. If the detuning Δ is such that transitions can occur only when the motion is towards the source of the radiation, i.e., $\Delta>0$, the particle loses the recoil velocity ($\hbar q/M$) for every complete cycle, just as does the free particle. If the orbital period is long compared with the spontaneous emission lifetime, $\nu\ll\Gamma$, the process takes place at almost constant velocity and position, and the velocity is damped towards small values in the fashion illustrated in Fig. 11. Each time the particle crosses the resonance it drops one or several photon recoil velocities. In this case the particle velocity enters mainly as a parameter, and the situation differs in no way from the case of a free particle treated earlier. Because a heavy particle oscillates more slowly, that is, has a small ν, we have called this the heavy-particle limit. It is treated by Gordon and Ashkin (1980) and Javanainen and Stenholm (1980c). In order to assure that the particle changes its velocity and position as little as possible during the spontaneous emission lifetime Γ^{-1} we must have, first, a long oscillational period as compared with the emission lifetime, giving

$$\nu\ll\Gamma\ . \tag{5.4}$$

During the oscillational period $\sim\nu^{-1}$, the particle goes through its range of velocity, which is v_0 given in Eq. (5.2). It sweeps through its Doppler shifts at a rate of the order (v_0q/ν^{-1}), and it thus takes the time $(\Gamma/\nu v_0q)$ to sweep through an interaction region of width Γ. If we want the spontaneous emission to take place at fixed velocity, we must require its time Γ^{-1} to be much shorter than the interaction time estimated above. This gives the condition

$$\nu v_0q\ll\Gamma^2\ , \tag{5.5}$$

which turns out to be a restriction on the energy of the harmonic motion. Squaring Eq. (5.5), we obtain the condition

$$\left[\frac{\nu}{\Gamma}\right]^2\frac{\varepsilon}{\Gamma}E_{\rm harm}\ll1 \tag{5.6}$$

and find that, because $\varepsilon\ll\Gamma$ also here, and we have assumed Eq. (5.4), the condition (5.6) is no essential restriction. This condition is, in fact, the one that allows us to neglect the influence of the harmonic potential on the cooling process; it is called the heavy-particle limit in Javanainen and Stenholm (1980c). This reference gives a Fokker-Planck treatment along the lines developed earlier.

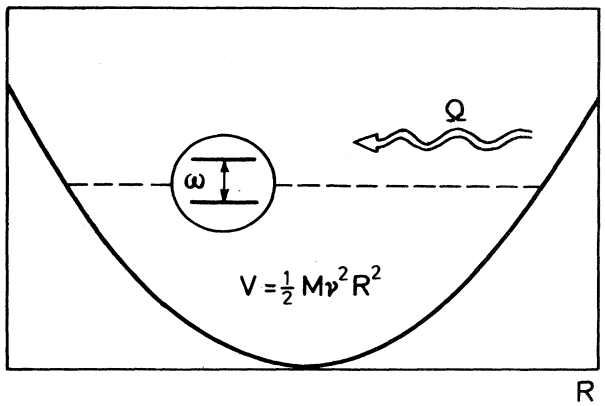

FIG. 10. The basic model for a bound particle. The two-level atom of Fig. 1 is placed in the harmonic potential with frequency ν. As a function of the center-of-mass coordinate R, the atomic potential $V(R)$ is assumed to be purely harmonic.

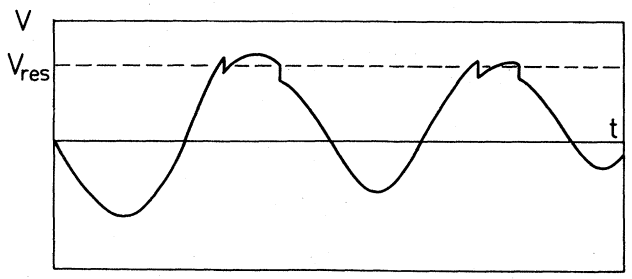

FIG. 11. When the bound two-level atom carries out periodic oscillations it can absorb laser photons and be cooled every time the velocity passes the value $v_{\rm res}$ needed to Doppler-compensate the detuning.

An incoherent approximation is used there, and hence the anomalous terms are missing; the results would hold for a broadband light source.

When the orbital motion becomes fast, $\nu \geq \Gamma$, the emission process takes place over an extended portion of the particle orbit. Then it becomes more instructive to look at the process in the frequency space instead of time. The two-level particle with its frequency ω is oscillated periodically with the velocity $v(t)$ in Eq. (5.2). In the laboratory its resonance frequency appears modulated by the Doppler shift $qv(t)$, and the atomic dipole moment obeys the evolution equation

$$i\dot{\mu} = [\omega + qv(t)]\mu + \cdots , \tag{5.7}$$

which integrates to

$$\mu(t) \propto \exp\left[-i\left[\omega t + \frac{qv_0}{\nu}\sin\nu t\right]\right]$$
$$= e^{-i\omega t}\sum_{k=-\infty}^{+\infty} J_k\left[\frac{qv_0}{\nu}\right]e^{-ik\nu t} . \tag{5.8}$$

Thus the atomic spectrum acquires sidebands spaced by the oscillational frequency ν. In this limit, $\nu \gg \Gamma$, the sidebands are well separated, and each one in its turn can be tuned to interact with the radiation frequency Ω. The spectrum of the atomic transition is shown in Fig. 12. The strength of the sidebands is given by $|J_k(qv_0/\nu)|^2$. If we tune the laser below resonance, $\Omega < \omega$, transitions can be achieved only when the missing energy is supplied by the orbital motion. The photon absorption is aided by the absorption of one or a few quanta from the oscillational motion in the trap. Because spontaneous emission preferably occurs at the atomic frequency ω, cooling takes place on the average. This is termed "sideband cooling" by Dehmelt and his collaborators (Neuhauser *et al.*, 1978a). When the harmonic motion is described by harmonic-oscillator wave functions, the change of their quantum number is given by

$$\nu\Delta n = \omega - \Omega . \tag{5.9}$$

When this is multiplied by Planck's constant $\hbar$, it manifests the energy conservation condition that each individual quantum process has to obey.

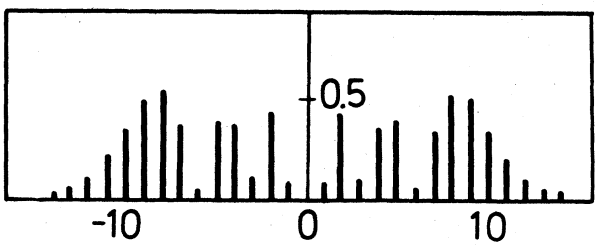

FIG. 12. The periodically oscillating atom displays a series of sideband frequencies to the observer in the laboratory. Their strengths are given by Bessel functions. To cool the atom, the laser should be tuned below the resonance, and the absorption of a laser photon can then occur when aided by the oscillational quanta of the motion.

For the cooling of free particles, the conservation of momentum in each individual process is of equal importance. It is because of this fact that we can use the momentum distribution to obtain the photon distribution, as explained in Sec. III.C. Any momentum change imparted to the particle will be retained and observed in the distribution function. For particles in a trap, the momentum is no longer a constant of the motion, and one might expect that no trace remains of the transfer of the photon momentum to the particle. This is, however, wrong; we can follow the process by looking at the matrix element of a plane-wave field of the type (4.8). The field has to be evaluated at the position R of the particle, and the transition matrix element between the harmonic-oscillator eigenstates $|n\rangle$ will contain the factor

$$U_{nn'} = \langle n | e^{-iqR} | n' \rangle . \tag{5.10}$$

In the general case, the motion of the particle is not confined to within one wavelength of the light, and we cannot expand the exponent $qR \sim R/\lambda$. If we use the momentum representation of the harmonic-oscillator wave functions in the trap, the position operator R goes over into the derivative $i\hbar\partial/\partial p$, and the matrix element (5.10) becomes

$$U_{nn'} = \int dp\, \psi_n^*(p) e^{-ipR}\psi_{n'}(p)$$
$$= \int dp\, \psi_n^*(p)\psi_{n'}(p + \hbar q) . \tag{5.11}$$

The rate of transfer between quantum numbers n and n' depends on the overlap of two momentum wave functions, one of which is displaced by the photon momentum $\hbar q$. The shape of the momentum wave functions of the harmonic oscillator equals the shape of the position wave functions, and hence the picture is as shown in Fig. 13. The argument is very similar to that giving a Franck-Condon factor for the molecular transitions, only here it operates in momentum space.

We know that, when the oscillator is excited, the wave functions peak near the classical turning points. Thus the transition matrix element tends to favor transitions near the maximum classical momentum $p_{\hat{n}}$ of the state with energy $\hbar\nu n$, assuming that energy can be conserved. The maximum momenta are then related by

$$p_{\hat{n}} = p_{\hat{n}'} \pm \hbar q . \tag{5.12}$$

In the classical description the maximum momentum occurs near the origin, where the velocity is nearly uniform. Hence the transitions occur preferably at those locations where the particle motion is nearly that of a free particle and both energy and momentum conservation can be satisfied in the individual quantum process. Thus, even if the momentum conservation is no longer exact, it still plays an essential role in the interaction events.

When the state of excitation is high, that is the fast-particle limit, n is large, it can be shown (see Javanainen and Stenholm, 1981a, Appendix A) that

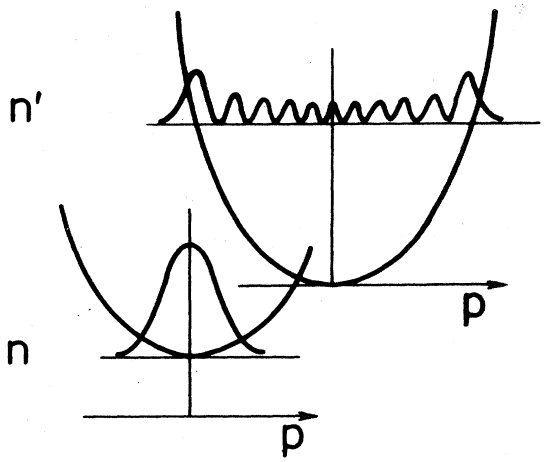

FIG. 13. The harmonic-oscillator wave functions in the momentum representation look exactly like those in the position representation. Transitions between oscillational quantum numbers n,n' are most likely when the peaks of the probability distributions have maximum overlap. This leads to approximate conservation of momentum $p \rightarrow p \pm \hbar q$, by a factor analogous to the Franck-Condon factor of molecular physics.

$$| U_{n,n+\Delta n} | = \left| J_{\Delta n} \left[q \left[\frac{2\hbar n}{M\nu} \right]^{1/2} \right] \right| . \tag{5.13}$$

The relation between the average quantum number n and the maximum velocity v_0 is

$$\tfrac{1}{2} M v_0^2 = \hbar \nu n , \tag{5.14}$$

and hence the result (5.13) for the strength of the transition with the energy change given by (5.9) agrees with the result in Eq. (5.8).

When the oscillational energy is large, the argument of the Bessel function is large, and as a function of Δn the matrix element peaks at

$$\Delta n \sim q \left[\frac{2\hbar n}{M\nu} \right]^{1/2} = \frac{q v_0}{\nu} . \tag{5.15}$$

For this limit the energy change is mainly taken up by the kinetic energy, as follows from

$$\hbar \nu \Delta n \approx \tfrac{1}{2} M \left[\left[v_0 + \frac{\hbar q}{M} \right]^2 - v_0^2 \right] \approx \hbar q v_0 + \cdots , \tag{5.16}$$

which agrees with the result (5.15). These considerations demonstrate the correspondence between a classical description and a quantum description when the excitation of the oscillation is high. In this case we expect the cooling to be described by a classical equation of motion for the center-of-mass coordinate

$$M\ddot{R} = -M\nu^2 R + F_0 - \beta \dot{R} , \tag{5.17}$$

where F_0 and β are of the form given in Eqs. (3.10) and (3.11). The velocity approaches zero which, for the harmonic motion, implies that the displacement approaches zero. In contrast to the free particle, the trapped particle is cooled towards a final stationary state. The center of the motion is shifted to

$$R_0 = \frac{F_0}{M\nu^2} . \tag{5.18}$$

As discussed in Sec. III, we must add a Langevin force to Eq. (5.17) to simulate the randomness of spontaneous emission. When the diffusion coefficient D is taken from Eq. (3.17), we write the Fokker-Planck equation

$$\frac{\partial P}{\partial t} + v \frac{\partial P}{\partial R} + \left[\frac{F_0}{M} - \nu^2 R \right] \frac{\partial P}{\partial v} = \frac{\partial}{\partial v} \left[\frac{\beta v}{M} P + D \frac{\partial P}{\partial v} \right] . \tag{5.19}$$

The steady state of this can be found when we note that the drift terms in phase space conserve the Hamiltonian

$$H = \tfrac{1}{2} M [v^2 + \nu^2 (R - R_0)^2] \tag{5.20}$$

because

$$\left[v \frac{\partial}{\partial R} + \nu^2 (R_0 - R) \frac{\partial}{\partial v} \right] H(R,v) = 0 , \tag{5.21}$$

and hence any $P(H)$ eliminates these terms. Then the right-hand side of Eq. (5.19) gives

$$\frac{\partial P}{\partial E} = -\frac{\beta}{DM^2} P , \tag{5.22}$$

which integrates to

$$P \propto e^{-\beta E / DM^2}$$

$$= \exp\{ -\beta [v^2 + \nu^2 (R - R_0)^2] / 2DM \} . \tag{5.23}$$

The final energy of the cooling process is found to be of the order

$$E_{\text{fin}} = \frac{DM^2}{\beta} = \hbar \Gamma , \tag{5.24}$$

where Eqs. (3.17) and (3.11) have been used. Thus the cooling of a trapped particle is also subject to the same basic limitation as that in a standing wave [see Eq. (3.35)]. The final energy is now found in the vibrational motion of the particle. In addition, there is energy in the constant deflection R_0 of the center of mass. If the laser is switched off suddenly, this energy is released and undoes some of the cooling. For an adiabatic removal of the laser, the solution (5.23) is expected to shift slowly over into the equilibrium position, and the particle stays cooled.

When the particle is cooled close to the ground state of the oscillator, the discrete nature of the oscillator quantum states becomes important, but to treat that case we need not impose any restriction on the ratio ν/Γ. We express the center-of-mass coordinate by the rising and lowering operators, $a^\dagger$ and a, in the harmonic potential as

$$R=\left[\frac{\hbar}{2M\nu}\right]^{1/2}(a^{\dagger}+a)\,, \tag{5.25}$$

and from Eq. (5.10) we obtain

$$\begin{aligned}U_{nn'}&=\langle n\mid e^{-i\eta(a^{\dagger}+a)}\mid n'\rangle\\&=\delta_{nn'}[1-\tfrac{1}{2}\eta^2(2n+1)]\\&\quad -i\eta(\sqrt{n'}\,\delta_{n',n+1}+\sqrt{n}\,\delta_{n',n-1})+\cdots\,.\end{aligned} \tag{5.26}$$

Here we have expanded in the assumedly small parameter η given by

$$\eta^2=\frac{\varepsilon}{\nu}=\left[\frac{\pi a_0}{\lambda}\right]^2, \tag{5.27}$$

where $a_0=(2\hbar/M\nu)^{1/2}$ is the size of the oscillator ground-state wave function. The parameter η is small when the final state of the cooling is extended much less than the wavelength. Because we can estimate the cooling time scale to be ε^{-1} from the free-particle result (2.19), this limit implies that it takes many oscillational periods to reach the cooled final states. In the discussion that follows we call this the Lamb-Dicke limit and define it as the mathematical limit that prevails when η goes to zero. In the next section we shall develop the theory for this case further.

B. The master equation for cooling

Because the damping rate for the trapped particle can be taken to be roughly equal to that of the free particle from Eq. (2.19), we write

$$\frac{dE}{dt}=-\varepsilon E\,. \tag{5.28}$$

If we define a new time scale for the harmonic oscillator $T_{\rm osc}$, which is the time it takes the damping mechanism to decrease the energy by one oscillational quantum $\hbar\nu$, we find from Eq. (5.28)

$$T_{\rm osc}=\frac{\hbar\nu}{\varepsilon\bar{E}}\,, \tag{5.29}$$

where $\bar{E}$ is some typical energy. Towards the end of the cooling process we can set this equal to $\hbar\Gamma$ from Eq. (5.24), and we obtain

$$T_{\rm osc}=\eta^{-2}\Gamma^{-1}\,, \tag{5.30}$$

where Eq. (5.27) has been used. The cooling time scale for the oscillator is thus slower than the time characterizing spontaneous emission by a factor η^{-2}. In the Lamb-Dicke limit we can hence use the parameter η to separate the fast internal processes taking place at the rate Γ from the slow cooling occurring over times of the order of $T_{\rm osc}$. For such times a master equation describes the cooling processes.

The treatment can actually be dressed in the form of a very general method for trapped particles. In the traps the constants of the classical motion are given by the action variables, because all dynamics is periodic and hence can be referred to angle variables. If the action variables are denoted by $\{I_i\}$ we find that any function

$$P=P[I_i] \tag{5.31}$$

remains conserved in time. If we choose to discuss the time evolution in the action-angle phase plane $\{I_i,\varphi_i\}$, the stationary distribution is, according to Eq. (5.31), independent of the variables $\{\varphi_i\}$. Instead of the ordinary closed curves characterizing periodic motion in the phase plane, any initial state stays unchanged in the action-angle plane; this is illustrated in Figs. 14(a) and 14(b).

When the particle becomes coupled to the radiation field, the conserved quantities begin to change owing to momentum exchange. Suitably arranged, the fields will start to force the distribution toward the region of phase space with lower energy; cooling occurs. At the same time the sharp distribution is starting to spread, as indicated in Figs. 14(c) and 14(d). The cooling stops when the spreading just compensates the cooling in every time interval, as shown in Sec. III.B. The cooling can now be described by the master equation

$$\frac{\partial P}{\partial t}=\theta P\,, \tag{5.32}$$

where θ is a linear operator that derives from the photon momentum. It can usually be calculated approximately by identifying some small parameter of the problem like the ξ of Sec. III.

For the one-dimensional trapped particle of the preceding section, the only constant of the motion is energy. Hence our procedure in solving Eq. (5.19) is an instance of the present argument. Its right-hand side is like the operator θ in Eq. (5.32). If we transform the variables to the energy plane we obtain an equation of the type (5.32).

For the harmonic motion, the proper action variable is

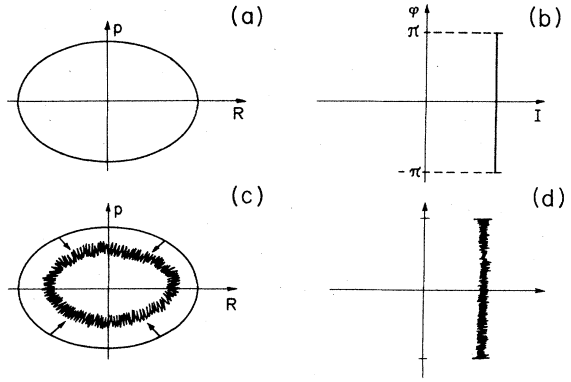

FIG. 14. In the phase plane (R,p) the harmonic motion is restricted to an ellipse when no cooling occurs (a). In the action-angle phase plane (I,φ), the same process appears as a line, when no phase information is given (b). When cooling sets in, the ellipse is seen to shrink and diffuse (c), but in the action-angle plane the same process is seen as a simple decrease of I accompanied by spreading (d). The adiabatic elimination procedure is based on the picture presented in case (d).

$$n=\frac{H}{\hbar\nu}\ , \tag{5.33}$$

which shows that the classical limit of the oscillator occupation number operator $a^\dagger a$ is just the action. It is hence advantageous to use the quantum n representation for the oscillator and write the master equation in this space. There are problems connected with the limit when $\langle n\rangle$ is very large (see the discussion in Javanainen and Stenholm, 1981a), but in the Lamb-Dicke limit and toward the end of the cooling, when the expansion (5.26) holds, the process can be carried out systematically (Javanainen and Stenholm, 1981b; Lindberg and Stenholm, 1984). The restriction to the late stages of the cooling forces us to retain the discrete nature of the quantum number n and describe the time evolution as a series of quantum transitions between these states.

The starting point for the derivation is the density matrix in the n representation, $\langle n\,|\rho|\,n'\rangle$. Without the interaction with radiation, the n occurs only in the oscillator Hamiltonian, and this leads to a time dependence

$$\langle n\,|\rho|\,n+k\rangle\propto\exp(-ivk)\ . \tag{5.34}$$

This oscillation at frequency vk is thus highly degenerate because it is independent of n. In the internal atomic indices ρ is a 2×2 matrix, but of the four eigenvalues corresponding to this, one is zero (compare the argument in Sec. IV.C). If we concentrate on this one, it corresponds to an infinitely degenerate set according to Eq. (5.34). This degeneracy is the same as that following from Eq. (5.21) and that discussed in connection with Eqs. (4.100)–(4.127). It can be lifted by using a version of the method explained in Sec. IV.C (Lindberg and Stenholm, 1984). The degeneracy can be removed in second order by diagonalizing an effective Hamiltonian operator within each degenerate subspace. This operator also describes the time evolution and thus gives the master equation for the problem.

We do not want to reproduce any of the details of this derivation. We use only a heuristic perturbation argument to obtain the rates coupling the populations for different n values; this corresponds to the $k=0$ degenerate set. As we see from Eq. (5.26), a given n only couples to $n+1$ and $n-1$ in the lowest order in η. For oscillator states these have the factors $(n+1)$ and n, respectively, and we obtain the picture in Fig. 15. In Fig. 16 we separate the individual processes. The terms proportional to the factor α are those coupling the levels by spontaneous emission, and α is the spread factor in spontaneous emission. The induced rate in perturbation theory is of the form

$$P(\Delta\omega)=\tfrac{1}{2}I\frac{\gamma^2}{\gamma^2+\Delta\omega^2}\ , \tag{5.35}$$

where I is the dimensionless intensity of Eq. (4.47). The Lorentzian takes care of energy conservation for each transition, so that $\hbar\Delta\omega\approx0$. For $n\to n\pm1$ we find $\Delta\omega=\Delta\pm\nu$. From Figs. 16(a) and 16(b) we read off the spontaneous and induced lowest-order processes,

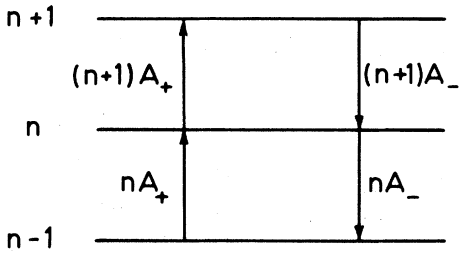

FIG. 15. The processes increasing the oscillator quantum number n are affected by A_+ and those decreasing n by A_-. The transitions are associated with the proper harmonic-oscillator matrix elements. The stationary state emerges when detailed balance is imposed between any two levels; the number of transitions up must equal the number of transitions down.

$$A_+=\Gamma[\alpha P(\Delta)+P(\Delta+\nu)]\ , \tag{5.36a}$$

$$A_-=\Gamma[\alpha P(\Delta)+P(\Delta-\nu)]\ . \tag{5.36b}$$

This result agrees with that derived by Neuhauser *et al.* (1978a). Its corrections require at least two laser photons and are thus of higher order in I. These processes and the saturation behavior were discussed by Javanainen and Stenholm (1981b); Lindberg and Stenholm (1984) derive the complete expressions for A_+ and A_-.

The rates (5.36) differ only in their second term, and if we tune to $\Delta=\nu$ we find

$$A_->A_+\ , \tag{5.37}$$

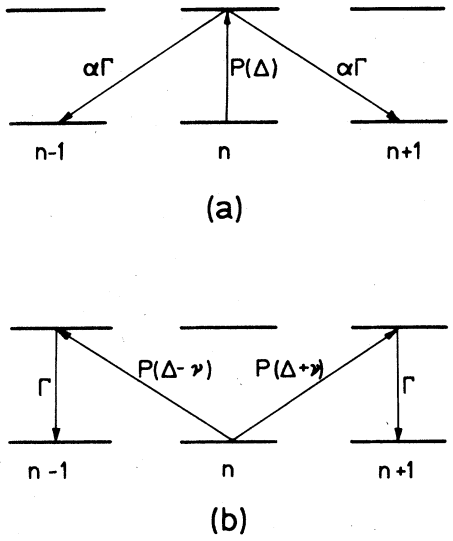

FIG. 16. The processes leading to cooling in lowest order of the Lamb-Dicke theory. They can be regarded as optical pumping between different n values through the upper atomic state. In (a) we excite by the resonant laser light [$\propto P(\Delta)$] and spontaneous decay at rate Γ deposits a fraction α of all excited atoms on the neighboring quantum levels $n\pm1$. In (b) we utilize the oscillational frequency ν to excite to the neighboring levels with probabilities $P(\Delta\pm\nu)$, and these subsequently decay with rates Γ. The cooling occurs because the detuning makes $P(\Delta+\nu)<P(\Delta-\nu)$.

which implies cooling. If we for simplicity consider the limit of well-resolved resonances, we find

$$\gamma \ll \nu = |\Delta| \tag{5.38}$$

and estimate

$$P(\Delta-\nu)=\tfrac{1}{2}I \gg \tfrac{1}{2}\tilde{I}=P(\Delta)=4P(\Delta+\nu)\ , \tag{5.39}$$

where the reduced intensity

$$\tilde{I}=\frac{I\gamma^2}{\nu^2} \ll I\ . \tag{5.40}$$

From Eqs. (5.36) we obtain

$$A_+=\tfrac{1}{8}\tilde{I}\Gamma+\tfrac{1}{2}\tilde{I}\Gamma\alpha\ , \tag{5.41a}$$

$$A_-=\tfrac{1}{2}I\Gamma+\tfrac{1}{2}\tilde{I}\Gamma\alpha\ . \tag{5.41b}$$

The cooling is due to the first term, which, according to Eq. (5.40), implies Eq. (5.37); the second constant term gives the diffusive spreading, as can be guessed from the presence of the spontaneous emission factor α. The additive term $\tilde{I}/2$ derives from a spread in induced processes.

The master equation is now written in the form

$$\dot{P}(n)=\eta^2\{(n+1)A_-P(n+1)-[(n+1)A_++nA_-]P(n) +nA_+P(n-1)\}\ . \tag{5.42}$$

The various terms can easily be interpreted from the processes in Fig. 15. From this result we can see that the time evolution occurs at rates given by $\eta^2\Gamma$, as already suggested by Eq. (5.30).

Equation (5.42) is the basic equation governing the cooling process in the Lamb-Dicke limit. It is accurate to order η^2, and from Eq. (5.27) we can see that this is of order $\hbar$. Hence the processes described by the perturbation terms can properly be considered as quantum corrections to the classically conserved quantities.

C. The final stages of the cooling

We now consider the behavior of the solution of the master equation (5.42) for long times, meaning the asymptotic steady-state condition. In this ultimate limit there can be no flow of probability between the various oscillator levels, and the rates up can be equated with rates down in Fig. 15. This is detailed balance, which gives

$$A_-P(n+1)=A_+P(n) \tag{5.43}$$

with the normalized solution

$$P(n)=(1-s)s^n\ , \tag{5.44}$$

where

$$s=\frac{A_+}{A_-}\ . \tag{5.45}$$

As long as we have the inequality (5.37), i.e., cooling, this is an acceptable solution similar to Planck's blackbody distribution.

With Eq. (5.44) we obtain the final energy of the trapped particle

$$E_{\rm fin}=\hbar\nu(\langle n\rangle+\tfrac{1}{2})=\hbar\nu\left[\frac{s}{1-s}+\frac{1}{2}\right]\ . \tag{5.46}$$

If we introduce a final temperature T_f by setting

$$s=e^{-\hbar\nu/k_BT_f}\ , \tag{5.47}$$

we find from Eq. (5.46) the correct expression

$$E_{\rm fin}=\hbar\nu\left[\frac{1}{e^{\hbar\nu/k_BT_f}-1}+\frac{1}{2}\right] \tag{5.48}$$

for a harmonic oscillator.

If we introduce Eqs. (5.36) into (5.46), we obtain the result first derived by Neuhauser *et al.* (1978a),

$$E_{\rm fin}=\hbar\nu\left[\frac{A_+}{A_--A_+}+\frac{1}{2}\right] =\left[\frac{P(\Delta+\nu)+\alpha P(\Delta)}{P(\Delta-\nu)-P(\Delta+\nu)}+\frac{1}{2}\right]\ . \tag{5.49}$$

In the limit (5.41) we find

$$E_{\rm fin}=\hbar\nu\left[\left[\frac{\gamma}{\nu}\right]^2(\alpha+\tfrac{1}{4})+\tfrac{1}{2}\right]\ . \tag{5.50}$$

If we introduce $\alpha=\frac{1}{3}$, we obtain the factor $\frac{7}{12}$ derived earlier by Javanainen and Stenholm (1981b). For the more realistic value $\alpha=\frac{2}{5}$ we find the factor $\frac{13}{20}$. Because the result is derived in the limit $\gamma \ll \nu$, the excitation energy is

$$E_1=(\alpha+\tfrac{1}{4})\left[\frac{\gamma}{\nu}\right]\hbar\gamma \ll \hbar\gamma\ , \tag{5.51}$$

in contrast to the general results found earlier. The total final energy, Eq. (5.50), is, however, dominated by the zero-point energy

$$E_0=\tfrac{1}{2}\hbar\nu \gg \hbar\gamma \tag{5.52}$$

according to our assumptions.

In the quantum regime the relationship between temperature and energy is given by Eq. (5.47), and we find

$$k_BT_f=\frac{\hbar\nu}{\ln(1/s)}\ , \tag{5.53}$$

which in the limit (5.41) gives

$$k_BT_f=\frac{\hbar\nu}{2}\frac{1}{\ln(\nu/\gamma)}>\frac{\hbar\gamma}{2}\ . \tag{5.54}$$

Hence, for the final temperature, the general cooling limit given by the spontaneous emission rate $\gamma \propto \Gamma$ holds, as we have always found before.

When saturation corrections to Eq. (5.50) are evaluated, one finds (Lindberg, 1984) in the limit (5.38)

$$\Delta E_{\rm sat}=\hbar\nu\frac{\Gamma}{\gamma}I\left[\frac{\gamma}{\nu}\right]^2\left[\frac{2\gamma}{\Gamma}-1\right]. \tag{5.55}$$

The first term here derives from the incoherent rate processes. The second, negative term is due to the coherent processes analogous to the anomalous diffusion terms encountered for free particles. These were omitted in Javanainen and Stenholm (1981b), and hence only the first contribution was obtained. The interesting observation is that in the absence of phase-perturbing processes, $\gamma=\Gamma/2$, the correction (5.55) disappears exactly. Thus there are no saturation corrections to the final energy to first order in the intensity I.

In the opposite limit of broadly overlapping resonances

$$\gamma\gg\nu \tag{5.56}$$

we can approximate

$$P(\Delta)\approx P(\Delta+\nu)=\tfrac{1}{2}I\frac{\gamma^2}{\gamma^2+\Delta^2}, \tag{5.57a}$$

$$P(\Delta-\nu)-P(\Delta+\nu)=2\Gamma\frac{\gamma^2\Delta\nu}{(\gamma^2+\Delta^2)^2}, \tag{5.57b}$$

which inserted into Eq. (5.49) give

$$E_{\rm fin}=\hbar\gamma\left[\frac{1+\alpha}{4}\right]\left[\frac{\gamma}{\Delta}+\frac{\Delta}{\gamma}\right] \tag{5.58}$$

which agrees with results derived for free particles (Letokhov *et al.*, 1977; Javanainen and Stenholm, 1980b) and for a trapped particle (Javanainen and Stenholm, 1981b) when we set $\alpha=\frac{1}{3}$. The optimum detuning is, in this case, found to be

$$\Delta=\gamma\gg\nu. \tag{5.59}$$

In this limit the zero-point energy is negligible, and the classical result of the form (5.24) applies.

Multiplying the rate equation by n and summing over all $n\geq 0$, we find from the master equation (5.42) the result

$$\frac{d}{dt}\langle n\rangle=\eta^2\left[A_-\sum_{n=0}^{\infty}[n(n+1)P(n+1)-n^2P(n)]-A_+\sum_{n=0}^{\infty}[n(n+1)P(n)-n^2P(n-1)]\right]$$
$$=-\eta^2[A_-\langle n\rangle-A_+(\langle n\rangle+1)]. \tag{5.60}$$

If we neglect the spontaneous emission contribution, i.e., assume $\langle n\rangle\gg 1$, we find the rate equation

$$\frac{d}{dt}\langle n\rangle=-W\langle n\rangle \tag{5.61}$$

with the solution

$$\langle n(t)\rangle=\langle n(0)\rangle e^{-Wt}+\langle n(\infty)\rangle(1-e^{-Wt}), \tag{5.62}$$

where $\langle n(\infty)\rangle$ is given by the solution (5.46) and the cooling rate is

$$W=\eta^2(A_--A_+)=\eta^2A_-(1-s). \tag{5.63}$$

Thus the time scale of the problem is determined by the factor

$$\eta^2A_-\sim\eta^2\Gamma, \tag{5.64}$$

as seen from Eqs. (5.35)—(5.41). This agrees with our original estimate in Eq. (5.30).

It is possible to make an ansatz of the form (5.44) and let the Planck ratio s be time dependent. This is the cooling of a field which remains of the blackbody type during the whole cooling process. This proves to be an exact solution of the master equation (Stenholm, 1985) with the parameter

$$s(t)=\frac{A_+[s(0)-1]+[A_+-A_-s(0)]e^{-Wt}}{A_-[s(0)-1]+[A_+-A_-s(0)]e^{-Wt}} \tag{5.65}$$

when the initial condition is denoted by $s(0)$. It is also easy to see that the result approaches the correct limiting value (5.45) for large times. With this solution the result (5.62) takes the simple form

$$\langle n(t)\rangle=\frac{A_+}{A_--A_+}(1-e^{-Wt})+e^{-Wt}\frac{s(0)}{1-s(0)}. \tag{5.66}$$

The exact solution (Lindberg, 1984; Javanainen *et al.*, 1984; Javanainen, 1985), like Eq. (5.58), is of the form

$$E_{\rm fin}\propto\hbar\nu\frac{\gamma}{\Delta}. \tag{5.67}$$

The singularity at exact resonance is physical and is a precursor for the heating instability that emerges when the tuning is turned to the wrong side of the resonance. In the limit $\nu\to 0$, Eq. (5.67) can be shown to agree with the free-particle result (4.129), as we expect in this, the heavy-particle limit. A result of the type (5.49) was first derived by Neuhauser *et al.* (1978a), and the result (5.58) agrees with earlier results for free particles. For the bound case it was first obtained by Javanainen and Stenholm (1980c), who considered the case $\alpha=\frac{1}{3}$. In the general form of Eq. (5.58), a heavy-particle estimate was given by Itano and Wineland (1982). Under somewhat different conditions these authors derive similar estimates in Wineland and Itano (1979).

The absence of linear saturation for $\gamma=\frac{1}{2}\Gamma$ as in Eq. (5.55) was first found by Javanainen (1981a,1981b). The correct form of the final energy (5.67) was first obtained by Lindberg (1984), using a moment expansion method. This was later verified by two independent methods (Lindberg and Stenholm, 1984; Javanainen, 1985), and the various approximations are discussed in Javanainen *et al.* (1984). A perturbation generalization to three dimensions has been carried out by Javanainen (1980a). In a three-dimensional trap there are three principal axes of oscillation. Unless the radiation impinges exactly along one of these, it is sufficient to have one wave to cool all three modes. If, however, these are nearly degenerate, correlations appear between the motions, and the cooling process

breaks down. Physically this derives from the fact that for exact degeneracy there are no principal axes, and the transverse heating can find an eigenmode that is not damped by the impinging laser radiation. This lies in the plane orthogonal to the photon momentum.

D. The theoretical description of cooling

The theory of cooling in a trap comprises three characteristic time scales. The first is the spontaneous decay lifetime Γ^{-1}, which is the relaxation time of the internal levels. The second is given by ε^{-1}, which is the characteristic time scale of the light pressure force acting on the particles, including the cooling due to the friction force. The third is the period of oscillational motion in the trap ν^{-1}. From these we have formed two dimensionless variables,

$$\eta^2=\frac{\varepsilon}{\nu},\quad \xi=\frac{\varepsilon}{\Gamma}\ . \tag{5.68}$$

The third ratio (ν/Γ) has not been given a name. Special limits appear whenever one of these ratios is small and an expansion is possible. The possible asymptotic regions are indicated in Fig. 17. For slow oscillational motion we saw that we can treat the instantaneous velocity as a parameter, expand adiabatically in ξ, and derive ordinary Fokker-Planck equations. Toward the end of the cooling we can often use the Lamb-Dicke expansion in η^2, as we did in the preceding section. For large values of (ν/Γ) the sideband resonances are well resolved, and cooling can be achieved by the use of one sideband only. The overlap between this regime and the Lamb-Dicke regime has repeatedly been utilized in this paper. For small values of the ratio (ν/Γ) we enter the heavy-particle regime, and this can be treated as a special case of the Fokker-Planck description. We have also discussed the fast-particle limit, but this is essentially the requirement that the average oscillator quantum number $\langle n\rangle$ be much greater than unity. We have not found this limit very useful (Javanainen and Stenholm, 1981a).

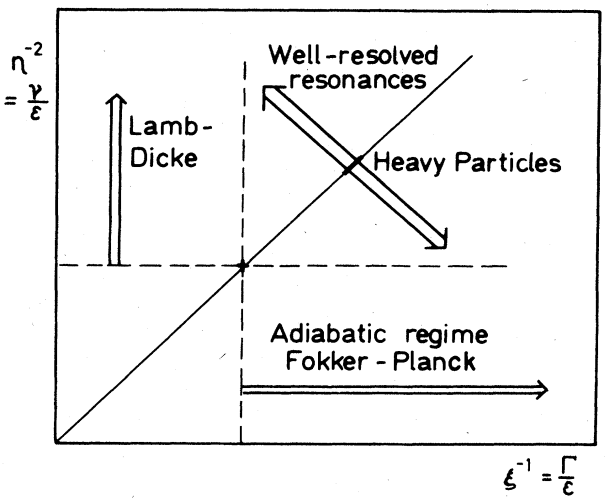

FIG. 17. In the trapped-particle case there are three basic energy parameters: the recoil energy ε, the spontaneous decay rate Γ, and the oscillational frequency ν. This $(\nu/\varepsilon,\Gamma/\varepsilon)$ plane shows the various asymptotic regions treated. The basic limits occur when ε may be regarded as small, but the limits $\nu/\Gamma\to\infty$ and $\nu/\Gamma\to 0$ have also been given names; they have overlaps with the two basic limits.

The derivation of the master equation was based on a diagonalization and the removal of the degeneracy with respect to n of the states $\langle n|\rho|n+k\rangle$. This was carried out in the Lamb-Dicke limit to second order in the parameter η. Within each subspace defined by a fixed k there appeared an effective time evolution operator. We used the subspace with $k=0$ to find the evolution of population probabilities $\langle n|\rho|n\rangle$. The method (Lindberg and Stenholm, 1984) does, however, work for an arbitrary k, and the general master equation is of the form

$$\begin{aligned}\frac{d}{dt}\langle n|\rho|n+k\rangle=&ik\nu\langle n|\rho|n+k\rangle+\eta^2\{[(n+1)(n+1+k)]^{1/2}A_-\langle n+1|\rho|n+1+k\rangle\\&-(n+1+\tfrac{1}{2}k)A_+\langle n|\rho|n+k\rangle-(n+\tfrac{1}{2}k)A_-\langle n|\rho|n+k\rangle\\&+[n(n+k)]^{1/2}A_+\langle n-1|\rho|n-1+k\rangle\}\ .\end{aligned} \tag{5.69}$$

Here the coefficients A_+,A_- are the same as for the diagonal equation (5.42), which is obtained by setting k equal to zero.

To treat Eq. (5.69) further, we introduce a generating function (Javanainen *et al.*, 1984)

$$G_k(z,t)=e^{-i\nu kt}\sum_{n=0}^{\infty}z^n\left[\frac{(n+k)!}{n!}\right]^{1/2}\langle n|\rho|n+k\rangle\ . \tag{5.70}$$

To see the utility of this we define for $n\geq 0$

$$\theta(n,k)=e^{-i\nu kt}\left[\frac{(n+k)!}{n!}\right]^{1/2}\langle n|\rho|n+k\rangle\ , \tag{5.71}$$

which, inserted into Eq. (5.69), gives

$$\begin{aligned}\frac{d}{dt}\theta(n,k)=\eta^2[&(n+1)A_-\theta(n+1,k)\\&-(n+1+\tfrac{1}{2}k)A_+\theta(n,k)\\&-(n+\tfrac{1}{2}k)A_-\theta(n,k)\\&+(n+k)A_+\theta(n-1,k)]\ .\end{aligned} \tag{5.72}$$

Writing Eq. (5.70) in the form

$$G_k(z,t)=\sum_{z=0}^{\infty} z^n\theta(n,k)\ , \tag{5.73}$$

we can now use the relations

$$\sum_{n=0}^{\infty} nz^n\theta(n,k)=z\frac{\partial}{\partial z}G_k(z,t)\ , \tag{5.74a}$$

$$\sum_{n=0}^{\infty}(n+1)z^n\theta(n+1,k)=\frac{\partial}{\partial z}G_k(z,t)\ , \tag{5.74b}$$

$$\begin{aligned}\sum_{n=0}^{\infty} nz^n\theta(n-1,k)&=z\sum_{n=1}^{\infty}(n-1)z^{n-1}\theta(n-1,k)\\&\quad+z\sum_{n=1}^{\infty} z^{n-1}\theta(n-1,k)\\&=z^2\frac{\partial}{\partial z}G_k(z,t)+zG_k(z,t)\ ,\end{aligned} \tag{5.74c}$$

$$\sum_{n=0}^{\infty} z^n\theta(n-1,k)=zG_k\ . \tag{5.74d}$$

When these are inserted into Eq. (5.72), we obtain

$$\eta^2A_-\left[(1-sz)(1-z)\frac{\partial G_k}{\partial z}-\left[\frac{k}{2}(1-s)+s(1+k)(1-z)\right]\right]G_k=\frac{\partial G_k}{\partial t}\ , \tag{5.75}$$

where, as before,

$$s=\frac{A_+}{A_-}\ . \tag{5.76}$$

Following Javanainen *et al.* (1984), we look for eigenvalues of the time evolution in the form

$$G_k(z,t)=\exp(-\eta^2A_-\lambda t)g_k(z)\ . \tag{5.77}$$

When this is inserted into Eq. (5.75) we obtain a simple first-order differential equation for $g_k(z)$, which has the solution

$$g_k(z)=C(1-sz)^{-(k+1+\sigma)}(1-z)^{\sigma}\ , \tag{5.78}$$

where

$$\sigma=-\tfrac{1}{2}k+\frac{\lambda}{1-s}\ . \tag{5.79}$$

We require that all derivatives of the function G_k exist at $z=1$. This is possible only if σ in Eq. (5.78) is a nonnegative integer N. Because $s<1$, the other factor in Eq. (5.78) always behaves properly. Setting $\sigma=N$, we obtain from Eq. (5.79) the eigenvalue

$$\lambda=(1-s)(\tfrac{1}{2}k+N)\ . \tag{5.80}$$

Thus we see that for all k and N the time evolution is determined by the rate $\eta^2A_-(1-s)$, in agreement with Eq. (5.63).

The lifting of the degeneracy of the eigenvalues

$$\lambda_0=-ik\nu \tag{5.81}$$

is shown in Fig. 18. When no cooling is taken into account, we have a series of denumerably infinitely degenerate eigenvalues stretching down to $-\infty$ when $k\to+\infty$ as shown in Fig. 18(a). (A similar discussion can be carried out for negative k.)

When the perturbation sets in, each eigenvalue splits according to the quantum number N in Eq. (5.80), and the oscillations (5.81) acquire a negative real part (damping), which stretches down to $-\infty$, as shown in Fig. 18(b). For a given k the slowest damping rate is given by $\frac{1}{2}Wk$, and hence the larger k values (those with more off diagonality) decay faster, but on the same time scale given by W. Only for $k=0$ do we get an undamped state, which of course is the unique steady state remaining after the lifting of the degeneracy.

For $k=0$ and the steady-state solution $N=0$, the generating function (5.78) becomes

$$g_k=\frac{C}{1-sz}\ , \tag{5.82}$$

which after normalization ($C=1-s$) agrees with the result given by Eq. (5.44).

The picture that emerges is thus that, over times of the order of Γ^{-1}, the internal variables relax to their instantaneous values. Then the different k values have decou-

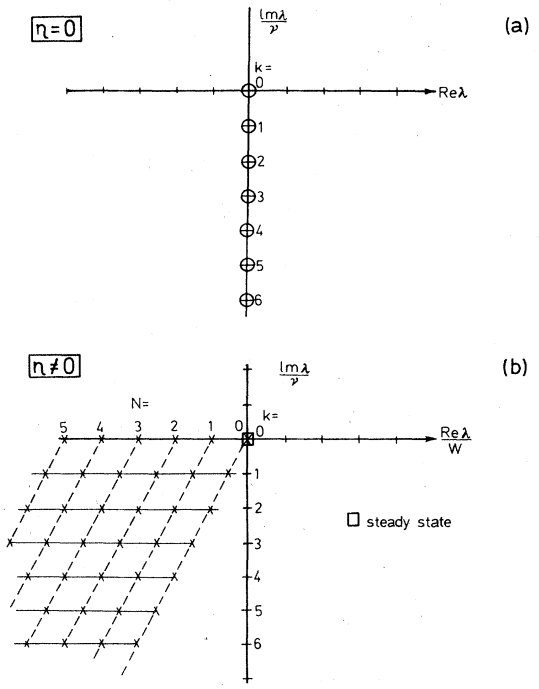

FIG. 18 (a) When recoil is neglected ($\eta=0$) the eigenvalue spectrum is denumerably infinitely degenerate and consists of pure oscillations (purely imaginary). (b) When recoil is included ($\eta\neq0$) the degeneracy is lifted, and all poles acquire a negative real part (i.e., damping) except the $k=0$ pole, which retains a nondegenerate zero, giving the unique steady state. Compare this picture with the simpler case in Fig. 8.

pled, and on a time scale given by W^{-1}, the degenerate subspaces reorganize towards their internal equilibrium; in addition, the off-diagonal elements decay by an overall rate $\frac{1}{2}Wk$. Finally only the asymptotically stable, diagonal steady state remains. These two processes, the approach to steady state and the decay of off-diagonal correlations, occur here on the same time scale, determined by W. There is no possibility of separating the relaxation of the populations from the disappearance of the coherences, as in many problems of nonequilibrium statistical mechanics.

It is also possible to solve the time-dependent equation (5.75). For this solution consult Stenholm (1985). Distributions differing from the thermal one (5.65) change shape during their evolution towards the final distribution.

For $k \neq 0$ the master equation describes the time evolution of coherences. These may be of interest, when one needs to evaluate the spectrum of scattered light from the trapped particle. This interesting subject has been discussed only in perturbation theory so far (Javanainen, 1980c), and more work is required. The corresponding problem of evaluating the resonance fluorescence spectrum for a free particle provided much interesting insight into the physics of spontaneous emission. A complete solution to the much more complicated problem of resonance fluorescence from a harmonically trapped particle is hence of great physical interest.

Finally, one word about the applicability of our results. The quantum-mechanical calculations can refer only to an assembly of trapped particles, and the master equation describes the time evolution of an ensemble of classical ions. We want, however, to apply these results to a single trapped ion, which cannot easily be prepared repeatedly in identical initial states. For long time averages, which determine the steady-state properties, we can assume that ergodic conditions of some sort prevail; in time even the single particle goes through most of the states necessary to simulate the behavior of the ensemble. When real transient behavior is measured, the system does indeed provide the chance to observe one single microscopic system for a long time. It thus may offer some interesting opportunities to test basic questions in quantum mechanics. The details of such possibilities are still largely obscure.

VI. THE EXPERIMENTAL SITUATION

First we must conclude that the theoretical situation is, in many ways, well in hand at the present moment. We have generally agreed on the formulations of the experimental situations. Particular limits have been treated and their physical implications elucidated. There exist specific computational philosophies like the one presented here, and in specific examples (Dalibard and Cohen-Tannoudji, 1985a) the transition between fully coherent quantum evolution and incoherent Brownian-motion-like behavior has been clarified in detail. Computational complexities remain, and there are unresolved questions of real physical significance. We do, however, believe that these can be solved numerically or analytically by the efforts now being directed to these questions.

On the experimental side the situation is much less satisfactory. I would say that there are no experiments at the present time accurate enough to investigate the details of the manifestations of photon momentum. All experiments until now have served more to verify the existence of the phenomenon than to provide a detailed fit to the actual theoretical results; only the main qualitative features have been seen. One reason for this is the interference of physically irrelevant factors like atomic collisions, optical pumping, stray fields, or gravitation, which fortunately can be excluded from the theoretical analysis, but which confuse the experimental observations. In spite of these, much progress has been made, and for the moment there appears to be great interest in these questions, which indicates that the situation may be rapidly improving (see, for example, the workshop edited by Phillips, 1984). In this section I shall briefly outline the experimental evidence available at the time of writing. This may be expected to become out of date soon, and hence I do not go into any details; the interested reader may consult the original works referred to.

The existence of light pressure was seen early in experiments by Lebedev, Nichols, and Hull, but only with the introduction of coherent laser sources did interest in this subject grow. The early work was mainly concerned with beam deflection (Schieder *et al.*, 1972; Jacquinot *et al.*, 1973; Bernhardt *et al.*, 1974,1976a). Applications to isotope separation were investigated in detail by Bernhardt (1976). The details of the momentum exchange were investigated by Bjorkholm *et al.* (1980), who saw heating due to quantum fluctuations. The beam deflection process was investigated by Bjorkholm *et al.* (1981) in considerable detail.

The gradient force was used to provide focusing or defocusing of an atomic beam by Bjorkholm *et al.* (1978). Other fields involve the motion of dielectric spheres under the influence of strong field gradients (see Ashkin, 1980), but as this case does not involve resonant interactions we do not enter into its details.

Recoil effects are seen in modern ultra-high-resolution spectroscopy, but are not really manifestations of photon momentum. Hence I only mention a few references: Hall *et al.*, 1976; Bordé *et al.*, 1979; Baklanov *et al.*, 1981; Hemmer *et al.*, 1981. Further references may be traced from these and from reviews on frequency standards (Ramsey, 1983), where the highest spectral resolution is required.

The deflection of an atomic beam from a standing-wave laser field is an investigation of fundamental importance for the basic interaction process. It generalizes the Kapitza-Dirac effect and depends on the properties of the atoms. The pioneering work was carried out by Arimondo, Lew, and Oka (1979a,1979b). The interaction time in these experiments was such that it allowed about ten spontaneous emission events, and the results were nearly at the diffusion limit (Arimondo *et al.*, 1981). To overcome this problem, experiments were carried out by a

pulsed source (Grinchuk *et al.*, 1981a,1981b), but in these experiments no parameter variation was considered; see, however, Kazantsev *et al.* (1981a,1981b,1981c). The experiment has been improved by Prichard's group (Moskowitz *et al.*, 1983; Gould *et al.*, 1986), and it now clearly shows the Bragg peak structure of the individual photon scattering events. This proves conclusively the existence of atomic wave packets of the size of at least about ten optical wavelengths. For particles as heavy as sodium this is an exciting achievement in itself.

The focus of experimental activity at the present time seems to be on laser cooling of the longitudinal motion of an atomic beam. The first experiments were carried out by Balykin and his collaborators (Balykin *et al.*, 1979,1980; Andreev *et al.*, 1981,1982; Balykin *et al.*, 1984). In these the laser frequency was swept to push the atoms towards lower velocities. Phillips and his collaborators have used Zeeman tuning to keep the decelerating atoms at resonance (Phillips and Metcalf, 1982), but they have also investigated frequency-sweeping techniques (Phillips *et al.*, 1983). Similar work is being carried out by Hall and collaborators (Ertmer *et al.*, 1984,1985). More results are expected to appear in the near future.

Research into light-induced forces is also concerned with trapped particles. Here both purely optical traps and ionic ones have been discussed (Phillips, 1984). Quite recently the first reports of stable trapping of neutral atoms have been published (Migdall *et al.*, 1985; Chu *et al.*, 1985). Several suggestions for purely optical traps exist (Ashkin, 1978; Minogin, 1982a; Minogin and Javanainen, 1982), but there are fundamental reasons why this may prove difficult (Ashkin and Gordon, 1983). Other trapping configurations have been suggested (Wing, 1980; Dalibard *et al.*, 1983; Prichard, 1983; Ashkin, 1984), but so far no experimental results exist to my knowledge.

Ions, on the other hand, have been successfully trapped and cooled by laser radiation both in Penning traps (Wineland *et al.* 1978a,1978b; Drullinger and Wineland, 1979; Drullinger *et al.*, 1980) and in radio-frequency traps (Neuhauser *et al.*, 1978a,1978b,1980,1981; Toschek and Neuhauser, 1980,1981). These experiments provide hours of trapping of a single ion, which is the first time an individual atomic particle has been singled out, confined, and kept under surveillance for such a long time. The ensuing photographs of a single ion are the first direct views of an atom that have been reported. These experiments provide exciting possibilities for tests of basic quantum-mechanical phenomena. For a review of spectroscopy of trapped particles consult Wineland *et al.*, 1983.

The photon statistics of light scattered from a particle were connected to the momentum distribution theoretically (Mandel, 1979a,1979b; Cook, 1981a,1981b; Lenstra, 1982; Stenholm, 1983). As far as I know there exists only one observational investigation of this phenomenon, by Short and Mandel (1983), and this concerns only the photon statistics. There has been no direct way to correlate these experiments with the mechanical manifestations of the photon momentum.

As we have seen, all the experiments referred to have been performed quite recently. The amount of interest is large, and we may expect new results to become available in the near future. These will be motivated by applications, but when accuracy increases they will eventually be able to investigate the details of the interplay between the momentum of radiation and atomic matter.

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