Scale factors $R(t)$ and critical values of the cosmological constant Λ in Friedmann universes

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The authors review the equations, notational choices, and confusing terminology of the Friedmann (zero-pressure) and Lemaître cosmological models, retaining cgs units as far as practical and in particular retaining units cm^{-2} for the present Gaussian curvature K_0 of three-space. They integrate the Friedmann equation numerically, requiring solutions to match the present Hubble parameter H_0 and mass-density ("closure") parameter Ω_0 at present time $t_0=0$, and generate families of curves showing the scale factor $R(\tau)$ (with $R_0=1$) vs τ (time in units H_0^{-1}) for fixed Ω_0 and various values of the cosmological constant Λ (in units H_0^2). These unusual graphs show the continuity of the solutions and the physical significance of Λ. Families for several values of Ω_0 exhibit known but unfamiliar features. The authors also show the family of "standard models" ($\Lambda=0$) and the family satisfying the "inflationary constraint" ($K_0=0$). They obtain new and simple formulas for the critical value $\Lambda_s(H_0,\Omega_0)$, which separates models with a big bang from those without. Their definition of Λ_s at fixed H_0 and Ω_0 differs from usual practice but proves useful. These formulas also give the quasistatic scale factor R_s and redshift z_s for the corresponding Eddington-Lemaître model, and give R_s and z_s approximately for the neighboring "Lemaître coasting models," which have $\Lambda<\Lambda_s$. The conventional wisdom that $\Lambda=\Lambda_c(1+\varepsilon)$ for the coasting models applies to a different characteristic value Λ_c. A quasistatic state in the future, with a second critical value Λ_{s2}, is possible if $\Omega_0>1$. The parameters Ω_0, Λ/H_0^2, Λ_s/H_0^2, and Λ_{s2}/H_0^2 can be used to classify the Friedmann models.

CONTENTS

I. INTRODUCTION

If the geometry of spacetime is Riemannian, the family of Lemaître universes (Lemaître, 1927), with arbitrary cosmological constant Λ, comprises all homogeneous isotropic world models satisfying Einstein's field equations of general relativity. The Friedmann (zero-pressure) universes (Friedmann, 1922,1924) are an important subset of these, with application to our observed universe, which at present probably satisfies the conditions of homogeneity, isotropy, and negligible pressure to a good approximation. Solutions for the scale factor $R(t)$ in Friedmann models are therefore familiar in textbooks, but often appear there in an abstract presentation which obscures the family relationships among the various solutions and fails to reveal to the reader the full range and continuity of the solutions and the simple physical content of the results.

In this paper we aim to fill a gap in the literature by displaying the Friedmann solutions $R(t)$ graphically in a way that shows clearly the relationships among them, and shows in particular the physical significance of the cosmological constant Λ. These graphs are not readily obtainable elsewhere. The reader may find them a useful supplement to the figures more commonly found in textbooks. We approach the problem of $R(t)$ from the point of view of the present epoch t_0 (rather than from the big bang or from some other epoch), requiring that $R(t)$ first of all match present conditions. We then use the Friedmann differential equation to extend $R(t)$ into the past and future. Proceeding from the known into the unknown seems appropriate.

We also review briefly the basic equations of Lemaître and Friedmann, primarily to offer some clarifying comments on the variety of terminology, notation, and conventions now in use. We make choices so as to stay as close to everyday physics as possible, and we use cgs units for the most part. Finally we give new and simple formulas for the "critical values" Λ_s and Λ_{s2} of the cosmological constant in terms of the present mass-density parameter Ω_0 and Hubble parameter H_0. The critical values are those for which the Friedmann universe reaches an asymptotically static state in the infinite past or future.

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In particular, the principal value Λ_s corresponds to an Eddington-Lemaître (EL) model, quasistatic in the past, and separates models with a big bang in the past from those without. The intermittently popular "Lemaître coasting models" are often said to have $\Lambda = \Lambda_c(1+\varepsilon)$, where $0 < \varepsilon \ll 1$. Our curves show that this statement raises a paradox, resolved only by distinguishing between Λ_s and a second characteristic value Λ_c. Finding the critical values Λ_s involves a simple but interesting physical application of the celebrated "irreducible case" of Cardan's method for the solution of a cubic equation.

II. LEMAÎTRE AND FRIEDMANN UNIVERSES

Proofs, further discussion, and history of the following basic equations may be found in standard textbooks (Peebles, 1971; Weinberg, 1972; Landsberg and Evans, 1977; Rindler, 1977; Raychaudhuri, 1979). If spacetime has a Riemannian metric (a fairly weak assumption) and if three-space is homogeneous and isotropic (the cosmological principle), then the line element dl in any three-space can be written

$$dl^2 = R^2(t)[dr^2(1-K_0r^2)^{-1} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)] , \tag{1}$$

where $R(t)$ is an arbitrary function of time, usually called the scale factor, and K_0 is an arbitrary constant, positive, negative, or zero. We measure dl in cm. Equation (1) is the Robertson-Walker metric. The angles θ and φ are ordinary spherical coordinates, with θ the polar angle. The third spatial coordinate r is a radial coordinate whose interpretation we will give shortly. Any fundamental particle P has fixed coordinates (r,θ,φ), attached forever. The three-space with metric (1) has intrinsic Gaussian curvature (units of cm^{-2})

$$K(t) = K_0R^{-2}(t) ; \tag{2}$$

K is uniform over the three-space but varies in time. If K_0 is positive, $K^{-1/2}$ is called the radius of curvature or "radius of the universe." The universe is then closed, having finite volume

$$V = 2\pi^2(K^{-1/2})^3 . \tag{3}$$

This result (Weinberg, 1972, Secs. 6.7, 13.2, and 13.3) holds for the usual ("spherical") model. The coefficient is sometimes quoted incorrectly (Coquereaux and Grossmann, 1982). The coefficient can also be changed by topological identifications (Tolman, 1934, Sec. 138; Rindler, 1977, Sec. 9.5; cf. Lemaître, 1927, and de Sitter, 1930).

Thus a closed universe, by definition, is one with positive curvature K_0. Closure by itself does not tell us whether or not the universe will recollapse. In Sec. III we will show that in relativistic cosmology some open models recollapse and some closed ones do not.

Many authors use a coordinate rescaling of the form $u^2 = \pm K_0r^2$ to transform away the K_0 in Eqs. (1) and (2), replacing it by a dimensionless "curvature index" k that is ± 1 or zero. We refrain from this because we wish to retain physical units (cm) for r. Therefore we use this freedom of a proportional transformation of the radial coordinate to require instead that at the *present* time $t_0 \equiv 0$, a radial dl $(d\theta = d\varphi = 0)$ be $\approx dr$ near the origin (i.e., for $r \ll |K_0|^{-1/2}$). This makes $R(t)$ dimensionless and implies that

$$R(0) = 1 \quad \text{and} \quad K(0) = K_0 . \tag{4}$$

Thus in these coordinates the arbitrary constant K_0 is the present Gaussian curvature of three-space. [The radial coordinate r then also has a simple interpretation. By setting $dr = d\varphi = 0$ and integrating θ from 0 to 2π in Eq. (1), one sees that the coordinate r for any fundamental particle P is $(2\pi)^{-1}$ times the *present* circumference of a circle through P with center at the origin 0.][1] We prefer this dimensionless scale factor $R(t)$, normalized to unity at present. Authors who need the notation R for tensors often use the notation $a(t)$ for $R(t)$. The scale factor $R(t)$ is sometimes called the "radius of the universe" because, if k is used instead of K_0 as described above, one sees readily by analogy with Eq. (2) that $R(t)$ then takes on dimensions in centimeters, and that it becomes the radius of curvature in the case of closed space $(k = +1)$.

Universes with the metric of Eq. (1) should be called Robertson-Walker universes. Equation (1) follows strictly from kinematics and contains no dynamical information; that is why $R(t)$ is a perfectly arbitrary function. If the dynamics is given by general relativity, then application of Einstein's field equations to Eq. (1) yields two important equations:

$$(\dot{R}/R)^2 = \tfrac{8}{3}\pi G\rho(t) - c^2K_0R^{-2} + \tfrac{1}{3}\Lambda , \tag{5}$$

$$\dot{\rho} = -3[\rho(t) + p(t)c^{-2}]\dot{R}/R . \tag{6}$$

Here, the overdot is the time derivative; $\rho(t)$ is the uniform mass density, including the mass equivalent of any energy present; $p(t)$ is pressure; and Λ is the "cosmological constant" added by Einstein to his field equations (Einstein, 1917). Actually Einstein's original λ had dimensions in cm^{-2}. Our Λ above has dimensions in $\sec^{-2}$, and $\Lambda = c^2\lambda$. The notation Λ is sometimes used today in either sense. Note that the third term of Eq. (5) is $-c^2K(t)$.

Equations (5) and (6), together with some equation of state $p = p(\rho)$, determine the behavior of $R(t)$ in a homogeneous isotropic relativistic universe (Harrison, 1967). Equations (5) and (6) were derived by Lemaître (1927), who pointed out their relevance to the Hubble expansion. They should be called the Lemaître equations, and the resulting family of models should be called Lemaître models (Peebles, 1971). Lemaître models with $\Lambda = 0$ may be called "standard models."

The behavior of the equation of state $p(\rho)$ in the very early universe, the question of whether Eqs. (5) and (6) re-

[1]Of course r is not equal to the distance $\overline{OP}$ unless $K = 0$. $\overline{OP}$ is obtained by integrating Eq. (1) from 0 to r with $d\theta = d\varphi = 0$.

quire any modification, and the possibility that a "natural value" of Λ might be identified by studying the physics of that epoch are topics of discussion in the modern inflationary cosmologies (Linde, 1984; Brandenberger, 1985). Obviously radiation pressure is important in the early hot universe. Observation suggests, however, that $p(t)$ is negligible now (Weinberg, 1972, Sec. 15.2)[2] and remains negligible for a long time into the past and future. If $p=0$ (more precisely, if $p \ll \rho c^2$), Eq. (6) has the solution $\rho(t)=\rho_0/R^3$. Putting this into Eq. (5) gives

$$(\dot{R}/R)^2 = CR^{-3} - c^2K_0R^{-2} + \tfrac{1}{3}\Lambda \, , \tag{7}$$

where the constant C is $\frac{8}{3}\pi G\rho_0$. This simple case of Eq. (5) was derived by Friedmann (1922) before Lemaître. Friedmann thereby saw the possibility of a homologous expansion before it was observed by Hubble—a remarkable achievement. His two classic papers have recently been translated (Bernstein and Feinberg, 1986, pp. 49—65). Equation (7) is called the Friedmann equation, and its solutions, i.e., the Lemaître models with zero pressure, are called Friedmann models (Robertson, 1933).

This classical terminology is clear enough. Unfortunately a variety of confusing terminology has been introduced recently. Equation (5), one of the Lemaître equations, is sometimes called the Einstein equation, although Einstein (1917), who was looking for static solutions, did not derive it. Many authors use the term "Lemaître model" to denote a famous special case, namely, the "Lemaître coasting model" with a quasistatic phase. We shall see examples in the graphs. Some authors narrow the term "Friedmann model" by using it to imply Eq. (7) with $\Lambda=0$—an injustice to Friedmann, who included Λ. Others expand the term to include all the Lemaître models, that is, those with $p\neq 0$, satisfying Eqs. (5) and (6)—an injustice to Lemaître. The standard models, those with $\Lambda=0$ and $p=p(\rho)$, are often called Friedmann-Robertson-Walker (FRW) models. This term is particularly bad, being both too specific and too diffuse: Models with $p\neq 0$ are *not* Friedmann models; Friedmann did *not* assume $\Lambda=0$; and the Robertson-Walker models, which prescribe no dynamics, include all the Lemaître (relativistic) models, with or without Λ, and an infinity of other homogeneous isotropic models besides.

By definition, $\dot{R}/R$ on the left-hand side of Eq. (5) is the Hubble parameter $H(t)$, with units $\sec^{-1}$ (or $\mathrm{km\,sec^{-1}\,Mpc^{-1}}$). At any time, Eq. (5) reads

$$c^2K(t) = \tfrac{8}{3}\pi G\rho(t) + \tfrac{1}{3}\Lambda - H^2(t) \, . \tag{8}$$

This pleasant equation, valid for all Lemaître models, shows what endows three-space with Gaussian curvature K: The mass density ρ (≥ 0) and the cosmological constant Λ (if positive) give positive curvature; motion (H) gives negative curvature. The sign of the motion (expansion or contraction of the universe) does not matter, for H enters only as H^2.

Formally, the Friedmann equation (7) is a first-order differential equation, so the solution $R(t)$ is determined uniquely by one boundary condition if the constants C, K_0, and Λ are known. But our choice of coordinate r in Eq. (1) guaranteed that $R(0)=1$. Therefore the constants C, K_0, and Λ determine $R(t)$ completely. We may apply Eq. (8) at $t=0$ to replace any one of these three free parameters by the present Hubble parameter H_0, which is not completely free in our universe, having been measured to within about a factor of 2 (Aaronson and Mould, 1983; Buta and de Vaucouleurs, 1983; Branch *et al.*, 1983; Sandage and Tammann, 1984; Bartel *et al.*, 1985).[3] Going further, and straying from cgs units a little, we may introduce a dimensionless time

$$\tau \equiv H_0 t \tag{9}$$

(time measured in units of the "Hubble time" H_0^{-1}). We also introduce a dimensionless density

$$\Omega(t) \equiv \tfrac{8}{3}\pi G\rho(t)H^{-2}(t) = C\rho(t)\rho_0^{-1}H^{-2}(t) \, . \tag{10}$$

[This is often called the closure parameter, because Eq. (8) shows that the universe is closed ($K>0$) if $\Omega > 1-\frac{1}{3}\Lambda/H^2$. Note that, unlike some current authors, we do not absorb a Λ term into our definitions of Ω, ρ, and p.] Use of τ and Ω and Eq. (8) reduces the Friedmann equation (7) to the form

$$\frac{1}{R^2}\left[\frac{dR}{d\tau}\right]^2 = \Omega_0\frac{1-R}{R^3} + \frac{1}{R^2} + \frac{1}{3}\left[\frac{\Lambda}{H_0^2}\right]\frac{R^2-1}{R^2} \, . \tag{11}$$

Once again we have $R(\tau)=1$ at $\tau=0$, so the solutions $R(\tau)$ for the dimensionless scale factor as a function of the dimensionless time τ are unique and are a *two*-parameter family of curves, with dimensionless parameters $\Omega_0 \equiv \Omega(0)$ and Λ/H_0^2.

Some authors, particularly in older papers, prefer to describe Friedmann models in terms of the density parameter $\sigma_0 \equiv \frac{1}{2}\Omega_0$ and the deceleration parameter $q_0 \equiv -(R\ddot{R}/\dot{R}^2)_0$ rather than in terms of Ω_0 and Λ. (The deceleration parameter in particular has wide application in more general models.) For ease of comparison with other authors, we give here the relationship (Rindler, 1977, Sec. 9.11)

$$q_0 = \sigma_0 - \tfrac{1}{3}\Lambda/H_0^2 = \tfrac{1}{2}\Omega_0 - \tfrac{1}{3}\Lambda/H_0^2 \, , \tag{12}$$

valid in any Friedmann (zero-pressure) model.

III. GRAPHS OF FRIEDMANN SCALE FACTORS $R(\tau)$

The best way to reveal to the reader the physical content of the Friedmann equation (11) is to graph its solu-

[2] This opinion is not unanimous, and models with large $p(0)$ are sometimes discussed (Turner, 1985).

[3] H_0 is roughly 50—100 $\mathrm{km\,sec\,Mpc^{-1}}$, and the Hubble time H_0^{-1} is (10—20)$\times 10^9$ yr.

tions. We have done this by computer, setting $R(\tau)=1$ at the present epoch $\tau=0$ and solving Eq. (11) numerically to push $R(\tau)$ into the past and future. It is not mandatory to integrate the Friedmann equation by computer; the solutions are expressible in terms of tabulated (elliptic) integrals (Rindler, 1977, Sec. 9.10). However, the expressions (Agnese, La Camera, and Wataghin, 1970; Edwards, 1972) are cumbersome and necessarily contain errors, usually discovered only through computation (Campusano, Heidmann, and Nieto, 1975).

The numerical integration of Eq. (11) is straightforward (Rindler, 1977, Sec. 9.10). Starting at the boundary $R_0=1$ (the present time $\tau=0$), the scale factor R_n at time step n can be approximated by the Taylor expansion

$$R_n=R_{n-1}+(dR/d\tau)_{n-1}\Delta\tau+\tfrac{1}{2}(d^2R/d\tau^2)_{n-1}(\Delta\tau)^2 \, . \tag{13}$$

The coefficient of $\Delta\tau$ is given by Eq. (11), and the coefficient of $(\Delta\tau)^2$ may be obtained by differentiating Eq. (11). This is important because without it Eq. (13) would not carry the integration correctly through a value of R_n where $dR/d\tau$ is small or zero. Higher-order terms, however, are not needed; further differentiation shows that they are negligible for small $\Delta\tau$.

We ran the program in each direction of time from $\tau=0$ to ± 5 Hubble times, or until $R=0$ (a big bang or big crunch) was encountered. Experimentation showed that a step size $\Delta\tau=0.0025$ (400 steps per Hubble time) is small enough. We can verify certain critical values of Λ/H_0^2 (see Sec. IV) to seven significant figures. In cases where a big bang is encountered in the past, the τ intercept represents the model age in Hubble times, so that published age calculations provide another check of the program (Glanfield, 1966; Stabell and Refsdal, 1966; Refsdal, Stabell, and de Lange, 1967; Agnese, La Camera, and Wataghin, 1970; Campusano, Heidmann, and Nieto, 1975). (Readers wishing more accurate ages, or ages for models not shown on our graphs, should consult these papers, which also contain a variety of useful graphs and tables for the Friedmann models.)

Figures 1(a)–1(c) show three sample families of solutions[4] for three fixed values of the present closure (density) parameter Ω_0. The free parameter is Λ/H_0^2. Figure 1(a) shows "low-density" models with $\Omega_0=0.1$, typical of models based on astronomical data. These data suggest that $0.01\lesssim\Omega_0\lesssim 0.3$ (Peebles, 1984; Felten, 1985). The family relationships among the curves of Fig. 1(a) make clear the physics of $R(t)$ and the significance of Λ. Several points are noteworthy. All curves pass through the point (0,1) with slope unity, so all are tangent there, but no two curves ever cross one another. Positive Λ acts

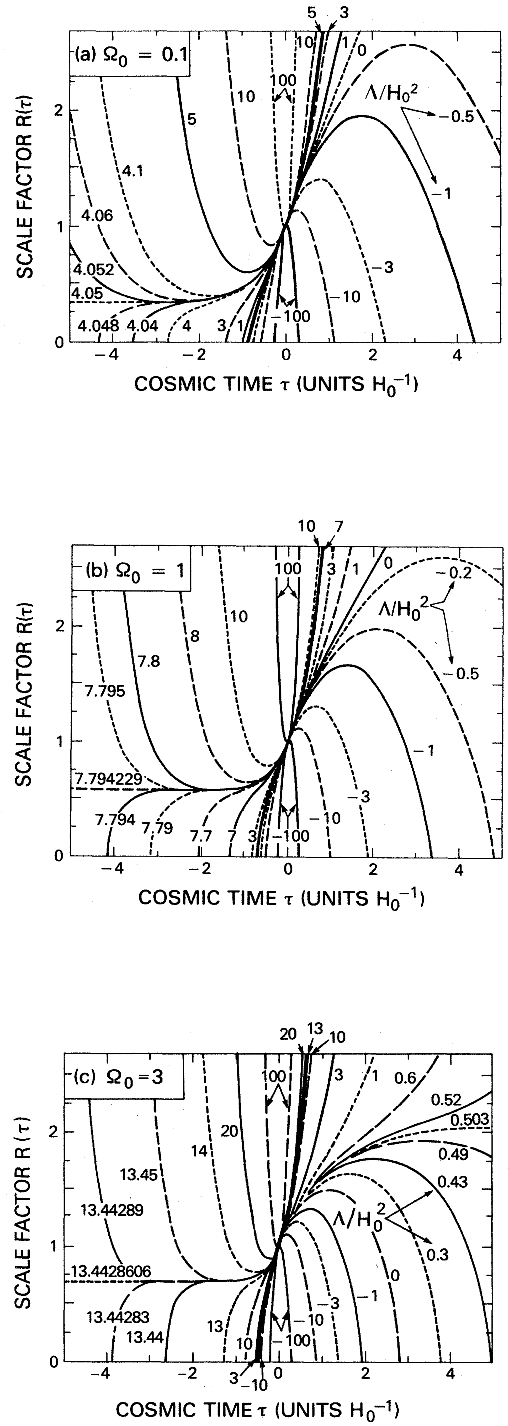

FIG. 1. Solutions of the Friedmann equation. Three families of scale factors $R(\tau)$ for Friedmann (zero-pressure) universes, with three fixed values of the present density parameter Ω_0: (a) $\Omega_0=0.1$; (b) $\Omega_0=1$; (c) $\Omega_0=3$. The free parameter, shown on the curves, is the cosmological constant Λ in units of H_0^2, where H_0 is the present Hubble parameter. The time τ is measured in units of the Hubble time H_0^{-1} and is taken $=0$ at present. The scale factor $R(\tau)$ is normalized to unity at present: $R_0=1$. For further discussion see the text.

[4]The only similar figure we have found in the literature is a sketch by Silk (1980).

like a repulsive force (Petrosian, 1974; Rindler, 1977, Secs. 9.2 and 9.9), so when we go from one curve to a neighbor with, say, a slightly larger Λ, the second curve is more concave upward at every τ, and the two curves can never cross.

Looking first at negative Λ, we see that for $\Lambda<0$ all models recollapse. The attractive force produced by negative Λ is constant in time. Unlike gravity, it is not diluted by expansion and does not drop off as $R(t)$ grows. Therefore even a small negative Λ slows the universe and turns it around eventually. One sees from Eq. (8) that these models with $\Omega_0<1$ and $\Lambda<0$ all have $K_0<0$, illustrating that, with $\Lambda<0$, open (infinite) universes recollapse. The model with $\Lambda=0$ (and $\Omega_0<1$) is, of course, a member of the family of "standard models"; it has $K_0<0$ and expands forever. The models with $\Lambda>0$ and $\Omega_0<1$ also expand forever. Considering positive Λ, and letting Λ increase, we see that the universe starts to develop a "hernia" in the past. (See $\Lambda/H_0^2=3$ and 4.) Equation (8) shows that, for $\Lambda/H_0^2>3(1-\Omega_0)\approx 3$, the universe is closed (finite), so the curves illustrate that some closed models expand forever.

The case $\Lambda/H_0^2=4$ is an example of a "Lemaître coasting model," with a quasistatic epoch, and the cases $\Lambda/H_0^2=4.04$ and 4.048 are more extreme examples. They rise from a big bang, "coast" for a time at a quasistatic value of R, then rise again. These models enjoy popularity from time to time when data seem to require them (Kardashev, 1967; Petrosian, Salpeter, and Szekeres, 1967; Shklovsky, 1967; Petrosian and Salpeter, 1968; Rowan-Robinson, 1968; Brecher and Silk, 1969; Petrosian, 1969; Petrosian, 1974; Gunn and Tinsley, 1975; Tinsley, 1977; Zel'dovich and Syunyaev, 1980). The main advantage of coasting models is that the age of the universe, given by the τ intercept, can be several Hubble times or more, in fact as large as needed. In such cases, objects of great age, e.g., globular clusters, can be accommodated (Sandage, 1982; Janes and Demarque, 1983; Thielemann, Metzinger, and Klapdor, 1983; VandenBerg, 1983; Sandage and Tammann, 1984; Klapdor and Grotz, 1986). These models do dive to a big bang somewhere on the negative τ axis. But if we increase Λ/H_0^2 further, to a critical value Λ_s/H_0^2, equal to 4.050000 in the present case $\Omega_0=0.1$, the universe loses this big bang in the past, and we obtain an Eddington-Lemaître (EL) model, asymptotic to Einstein's static model in the infinite past. This critical Λ_s separates models with a big bang in the past ($\Lambda<\Lambda_s$) from those without ($\Lambda\geq\Lambda_s$). Thus, for example, if $\Omega_0=0.1$ and $H_0=100h$ km sec Mpc^{-1}, a Friedmann model with a big bang must have $\Lambda<4.05H_0^2$ $=4.25\times10^{-35}h^2$ sec^{-2}. The critical Λ_s/H_0^2 for a given value of Ω_0 may easily be evaluated by computer trials, but in Sec. IV we shall derive analytical formulas for it.

For $\Lambda>\Lambda_s$ we obtain models that collapse from infinity, reach a minimum value of $R(t)$, and then expand again. Rindler (1977) calls these the "catenary universes," although we can see that the curves are not true catenaries. These are seldom spoken of today, because the 2.7-K background radiation seems to require a big bang in the past.[5]

Readers should note that the catenary universes, like the recollapsing models with large negative Λ, are time symmetric about their extrema in R, where $dR/d\tau=0$. This is expected, because the Friedmann equation (11) is manifestly time symmetric; if $R(\tau)$ is a solution, $R(-\tau)$ is also. If we reflect the curves of Fig. 1, reversing left and right about the vertical line $\tau=0$, we preserve the boundary condition $R(0)=1$, and we obtain a whole new set of solutions. These solutions have $(dR/d\tau)_0=-1$ instead of $+1$ and might be thought of as universes collapsing at present. [Strictly speaking, however, they are merely unphysical, because we mandated $(dR/d\tau)_0=+1$ as a convention when we defined $\tau\equiv H_0 t$; expanding or collapsing universes ($H_0\gtrless 0$) automatically have the positive sign.] The situation exemplifies a broken symmetry: Eq. (11) is time symmetric, but the symmetry is broken by the boundary condition $R(0)=1$ and the convention $(dR/d\tau)_0=+1$.

The coasting models have $\Lambda=\Lambda_s(1-\varepsilon)$, where $0<\varepsilon\ll 1$. Some readers may be puzzled by this, for in the extensive literature on coasting models (Kardashev, 1967; Petrosian and Salpeter, 1968; Rowan-Robinson, 1968; Brecher and Silk, 1969; Petrosian, 1969; Petrosian, 1974) it is repeatedly stated that $\Lambda=\Lambda_c(1+\varepsilon)$. By the discussion above, our result clearly is physically correct. We shall resolve this paradox in Sec. V by showing that the Λ_s defined above is not the same as Λ_c defined in earlier papers.

Figure 1(b), which shows "critical-density" models ($\Omega_0=1$), has the same qualitative features as Fig. 1(a). For critical density, $R(t)$ still escapes (barely) to infinity if $\Lambda=0$, but it recollapses if $\Lambda<0$. The critical Λ_s/H_0^2 has increased to 7.794 229.

In Fig. 1(c), for $\Omega_0=3$, an interesting new feature appears: There is the possibility of a model that becomes quasistatic in the *future*, with a second critical value, which we may call Λ_{s2}/H_0^2—0.502 977 in this case. One sees that this behavior is physically correct for models with $\Omega_0>1$. Such a model will recollapse if $\Lambda=0$. Therefore a small positive $\Lambda=\Lambda_{s2}$ (repulsive force) can provide equilibrium and prevent recollapse. This model with a quasistatic state in the future may be no more than a curiosity, for no feature of the observed universe seems to require it. In Fig. 1(c), only models with $\Lambda/H_0^2<-6$ have negative curvature. This underscores the point that, for high density, the curvature is positive for a wide range of models, both escaping and recollapsing.

Figure 2 shows the family of models satisfying the "inflationary constraint" (Peebles, 1984). In the modern inflationary theories (Linde, 1984; Brandenberger, 1985), the three-space curvature $K(t)$ should be zero at present, essentially because the exponential increase in R at very

[5]For a contrary view, see Segal (1983).

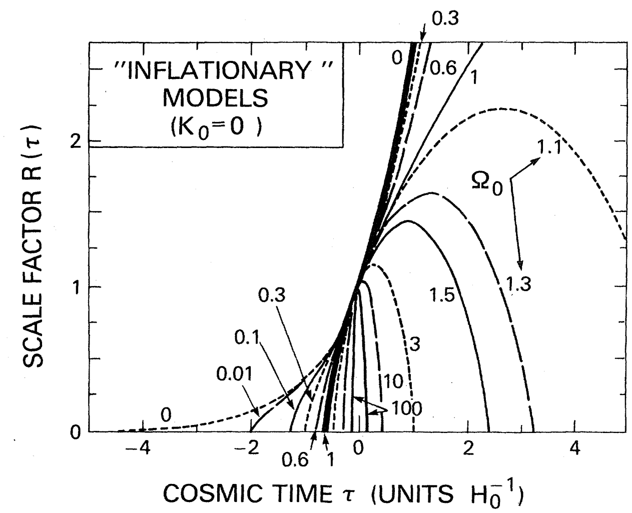

FIG. 2. "Inflationary" Friedmann models. The family of scale factors $R(\tau)$ for models satisfying the "inflationary constraint" (three-space curvature $K_0=0$). The free parameter, shown on the curves, is Ω_0. The cosmological constant Λ is determined from Ω_0 by Eq. (14).

early times stretches K out so that it is negligibly small at later times. Putting $K_0=0$ in Eq. (8) and using Eq. (10) yields

$$\Lambda=3(1-\Omega_0)H_0^2 \tag{14}$$

as the condition that must be satisfied by an inflationary model in the subsequent (present) state when the Friedmann or the Lemaître equation is satisfied. Thus Λ is no longer free in inflationary models; it is determined by H_0 and Ω_0. We show in Fig. 2 the family of these models, with Ω_0 as the parameter.

If $\Lambda=0$, Eq. (14) shows that we require $\Omega_0=1$ in the inflationary theories. Discussions of inflationary models (Linde, 1984; Brandenberger, 1985; Turner, 1985) tend to concentrate on this "standard case," also known as the Einstein–de Sitter model. The model age is then unique and is known analytically to be $\frac{2}{3}H_0^{-1}$ (cf. the τ intercept for $\Omega_0=1$ in Fig. 2), and the age becomes even shorter in non-Friedmann models including effects of pressure (Turner, 1985). This age for the standard inflationary model is embarrassingly short if $H_0\geq 50$ km sec^{-1} Mpc^{-1} ($H_0^{-1}\leq 20\times 10^9$ yr), because the globular clusters (ages $\geq 16\times 10^9$ yr; Sandage, 1982; Janes and Demarque, 1983; VandenBerg, 1983; Sandage and Tammann, 1984) cannot then be accommodated (Klapdor and Grotz, 1986).[6] Peebles (1984) and Turner, Steigman, and Krauss (1984) have therefore directed attention to the alternative models of Fig. 2 with $\Lambda\neq 0$. With $H_0\approx 80$ km sec^{-1} Mpc^{-1}, for example, we could have reasonable consistency with $\Omega_0\approx 0.1$ and a model age $\approx 1.3H_0^{-1}$.

We see that the inflationary family has analogs of the Lemaître coasting models. The limiting case ($\Omega_0=0$, $\Lambda/H_0^2=3$), shown in Fig. 2, is known as the empty de Sitter model; its equation is $R=\exp\tau$. It is in fact an EL model, with infinite age, which becomes quasistatic at $R=0$. Figure 2 shows that coasting models exist for $\Omega_0\ll 1$; they peel off the curve $R=\exp\tau$ at various model ages. But in contrast to Figs. 1(b) and 1(c), ages larger than H_0^{-1} can be obtained only for $\Omega_0\ll 1$ when the inflationary constraint is in force.

Finally Fig. 3 shows the familiar family of "standard models" ($\Lambda=0$), with Ω_0 as the parameter. These, as the simplest, are much beloved by theorists. In the empty case $\Omega_0=0$, $R(\tau)$ is a straight line because there is no force to give acceleration. This model has age H_0^{-1}, the longest obtainable among the standard models. This is still short enough to cause trouble (Sandage and Tammann, 1986) if H_0 is near 100 km sec^{-1} Mpc^{-1}. The higher-density models have progressively shorter ages, which can be read roughly from the τ intercepts. For $\Omega_0\leq 1$ the standard model is open and expands forever; for $\Omega_0>1$ it is closed and recollapses.

The reader should understand, with respect to all of these models, that the Friedmann equation cannot remain valid all the way down to $R=0$. Close to the big bang, pressure must be important. But standard texts show that if the present pressure is mainly that due to the 2.7-K background radiation, then at present we have $p_0\lesssim 10^{-4}\rho_0 c^2$, and p will remain negligible, and the Friedmann solution good, until R gets down to $\lesssim 10^{-3}$. This is small on the scale of Figs. 1–3, so the non-Friedmann effects are only tiny perturbations on the tails

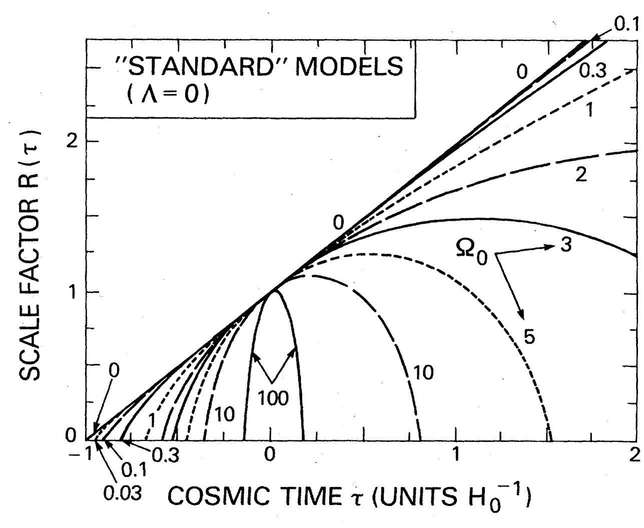

FIG. 3. "Standard" Friedmann models. The family of scale factors $R(\tau)$ for the "standard models" ($\Lambda=0$). The free parameter, shown on the curves, is Ω_0. As shown by the τ intercepts, all models have ages ≤ 1 ($\leq H_0^{-1}$ yr).

[6]For a recent review of these constraints see Sandage and Tammann (1986).

of these curves near $R=0$. The same is true of the more complicated effects caused by phase transitions, particle interactions, and decays at very early epochs. None of these will compromise the validity of the Friedmann equation for deducing gross properties such as the model age or for predicting observable properties at red shifts $z=R^{-1}-1\lesssim 10^3$. On the other hand, if the *present* p_0 *were* appreciable ($\sim\rho_0 c^2$), due, for example, to a high density of relativistic particles, we would be thrown back to the Lemaître equations (4) and (5) throughout the range of τ, and changes in $R(\tau)$ would be noticeable (Turner, 1985). One of the principal effects would be to shorten the model ages further. The ages of some popular models are already uncomfortably short.

IV. CRITICAL VALUES OF Λ

It is of interest to obtain analytical expressions for the critical values Λ_s noted in Sec. III. Expressions already in the literature are not particularly transparent and are not written in terms of the parameters (H_0,Ω_0) popular today. To do this we note that a critical (EL) model is asymptotic to a static Einstein model in the infinite past, so that in that state it must satisfy the familiar pair of equations satisfied by an Einstein model. In our notation these equations are[7]

$$c^2K_0R_s^{-2}=\Lambda_s=\tfrac{3}{2}CR_s^{-3}\ . \qquad (15)$$

The Gaussian curvature K_0 and the density parameter C ($\equiv\frac{8}{3}\pi G\rho_0$) refer to the present epoch; R_s is the scale factor (<1) at which the model becomes quasistatic in the past. Recall from Eqs. (8) and (10) that $C=\Omega_0H_0^2$ and

$$c^2K_0=H_0^2(\Omega_0-1+\tfrac{1}{3}\Lambda_s/H_0^2)\ , \qquad (16)$$

if Λ_s is the value of Λ for this EL model of given H_0 and Ω_0.

We may eliminate R_s from Eq. (15) to obtain

$$\Lambda_s=\tfrac{4}{9}(c^2K_0)^3C^{-2}=\tfrac{4}{9}(c^2K_0)^3(\Omega_0H_0^2)^{-2}\ . \qquad (17)$$

This expression is familiar [Landsberg and Evans, 1977, Eq. (7.1); Rindler, 1977, Eq. (9.89)]. Equation (16) shows that, for fixed H_0 and Ω_0, Eq. (17) is a cubic for the critical $\Lambda_s(H_0,\Omega_0)$. We are interested only in positive roots, for there can be no equilibrium with $\Lambda<0$. We set

$$x\equiv(\Lambda_s/12\Omega_0H_0^2)^{1/3} \qquad (18)$$

in Eqs. (16) and (17) and obtain a dimensionless cubic

$$f(x)\equiv x^3-\tfrac{3}{4}x+\tfrac{1}{4}(\Omega_0-1)/\Omega_0=0\quad(\Omega_0>0)\ . \qquad (19)$$

We seek positive roots. "Cardan's method" (invented by Tartaglia and stolen by Cardan; Burnside and Panton, 1960, Sec. 56) may be used. This cubic is already in the "reduced form," and its discriminant[8] Δ is

$$\Delta\equiv a^{-2}(G^2+4H^3)=\tfrac{1}{16}(1-2\Omega_0)\Omega_0^{-2}\ . \qquad (20)$$

The familiar Cardan solution can be written

$$x=p^{1/3}+\tfrac{1}{4}p^{-1/3}\ , \qquad (21)$$

where

$$p\equiv\tfrac{1}{2}[-G+(G^2+4H^3)^{1/2}]=\tfrac{1}{2}[\tfrac{1}{4}(1-\Omega_0)\Omega_0^{-1}+\Delta^{1/2}]\ . \qquad (22)$$

Equation (21) yields three roots x corresponding to the three (real or complex) cube roots of p. The *positive* roots x may be counted by standard analysis using the value of Δ and Descartes's strong rule of signs (Hart, 1947; Burnside and Panton, 1960, Sec. 43). If $0\le\Omega_0\le\frac{1}{2}$, then p is positive, and the principal cube root of p gives one positive root x—the only one. But for $\Omega_0>\frac{1}{2}$, we have $\Delta<0$, and p is complex. Nevertheless all three roots x are real and distinct. This is the celebrated "irreducible case," in which "... from a real cubic three real roots cannot be extracted by Cardan's algebraic formula without a circuitous passage into, and out of, the domain of complex numbers" (Turnbull, 1952, Secs. 49ff). In this case it is convenient to use the alternative trigonometric solution for the three real roots (Turnbull, 1952). Only one root is positive for $\Omega_0<1$, but two are positive for $\Omega_0>1$. Finally, although Eqs. (20)—(22) are adequate for the positive root x in the case $\Omega_0\le\frac{1}{2}$, there is a hyperbolic form homologous to the trigonometric solution which is more attractive (Turnbull, 1952).

We can write the results as follows: If x is a positive root of Eq. (19), the critical value Λ_s is given by

$$\Lambda_s(H_0,\Omega_0)=12\Omega_0H_0^2x^3\ , \qquad (23)$$

and it is easily shown from the last member of Eq. (15) that the corresponding quasistatic scale factor is

$$R_s=\tfrac{1}{2}x^{-1}\ . \qquad (24)$$

For the positive root or roots x used in Eqs. (23) and (24), we must distinguish three cases.

Case 1: $0<\Omega_0\le\frac{1}{2}$. The positive root is

$$x=\cosh\left[\tfrac{1}{3}\cosh^{-1}\frac{1-\Omega_0}{\Omega_0}\right]\ . \qquad (25a)$$

Case 2: $\frac{1}{2}\le\Omega_0\le 1$. The positive root is

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[7]These equations may be derived by setting $p=\dot R=\ddot R=0$ in Eqs. (9.72) and (9.73) of Rindler (1977). This yields Rindler's Eq. (9.83), where he has taken $c=1$; the last member of Eq. (9.83) actually reads $4\pi G\rho c^{-2}$. Conversion to our notation, and use of the Friedmann density dependence $\rho_s=\rho_0R_s^{-3}$, yield Eq. (15).

[8]Various authors disagree on the numerical factor attached to Δ; we use the definition of Burnside and Panton (1960, Sec. 42).

$$x=\cos\left[\tfrac{1}{3}\cos^{-1}\frac{1-\Omega_0}{\Omega_0}\right]. \tag{25b}$$

Case 3: $\Omega_0>1$. Equation (25b) gives the positive root x which yields Λ_s, and there is a second and smaller positive root,

$$x_2=\cos\left[\tfrac{1}{3}\cos^{-1}\frac{1-\Omega_0}{\Omega_0}+\tfrac{4}{3}\pi\right]. \tag{26}$$

(This root corresponds to a quasistatic state in the future, yielding a second critical value Λ_{s2}, with $R_{s2}=\frac{1}{2}x_2^{-1}>1$, and confirms the result noted in numerical integrations in Sec. III.) Readers whose pocket calculators lack the cosh and $\cosh^{-1}$ functions may easily use Eqs. (20)–(22) in case 1. Note that R_s and Λ_s/H_0^2 are functions only of Ω_0. Figure 4 shows these functions, together with asymptotic forms for $\Omega_0 \ll 1$ and $\Omega_0 \gg 1$.

Surprisingly, these pleasant expressions for Λ_s and R_s seem to to be new in the cosmology literature. [Blome and Priester (1985) derived a more complicated expression, restricted to case 1 only. In the preprint version of their paper, the radical covering the right-hand side of their Eq. (18) was omitted.] Our work was, however, anticipated by Glanfield (1966), who did not graph $R(t)$ or obtain Eqs. (23)–(26), but did write a cubic equivalent to Eq. (19) (his $R\equiv\Omega_0$ and his $L\equiv\frac{1}{3}\Lambda/H_0^2$), obtain roots numerically, and give an excellent discussion of the solutions $R(t)$. His fine paper has fallen into obscurity. Similarly, Bludman (1984) wrote an implicit cubic and obtained a few roots numerically. Expressions like Eqs. (25) and (26) are surprisingly unfamiliar to physicists, although similar expressions are obtainable for the real roots of any real cubic. Apparently they are unfamiliar because relatively few physical problems involve roots of a cubic.

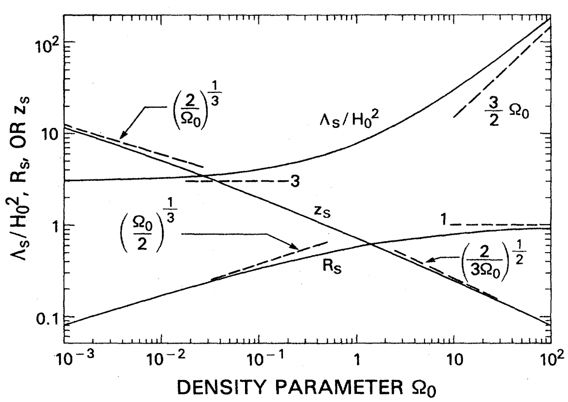

FIG. 4. The critical value Λ_s/H_0^2 [Eqs. (23) and (25)], the corresponding quasistatic scale factor R_s [Eqs. (24) and (25)], and the red shift $z_s(=R_s^{-1}-1)$ of the quasistatic epoch, all shown as functions of the present density parameter Ω_0 in a Friedmann model. Their asymptotic forms for $\Omega_0 \ll 1$ and $\Omega_0 \gg 1$ are also shown.

Since the red shift z at any epoch is given by $R^{-1}-1$, we can obtain Λ_s and Ω_0 in terms of the red shift z_s at the quasistatic epoch. We have $z_s=2x-1$, and from Eq. (23) we find

$$\Lambda_s=\tfrac{3}{2}\Omega_0H_0^2(z_s+1)^3 . \tag{27}$$

Using Eqs. (24) and (25) and the multiple-angle formulas for $\cos 3\theta$ and $\cosh 3\theta$, we easily find

$$\Omega_0=2[(z_s+1)^3-3(z_s+1)+2]^{-1} . \tag{28}$$

Thus Λ_s/H_0^2 and Ω_0 are uniquely determined by z_s. Equations (27) and (28) are exact for an EL model but evidently apply approximately to a coasting model as well. They are implicit or explicit in several earlier papers. During the period 1966–1969 it was thought that quasar red-shift data required a coasting model with $z_s\approx 1.95$ (Burbidge, 1967; Shklovsky, 1967). By Eqs. (27) and (28), this would imply $\Omega_0\approx 0.106$ and $\Lambda\approx 4.09H_0^2$. We show $z_s(\Omega_0)$ and its asymptotic forms in Fig. 4.

V. PARADOX AND DISCUSSION

We wish to resolve the paradox encountered in Sec. III. Is the cosmological constant Λ for a coasting model greater than or less than the EL value Λ_s? It is helpful to visualize this geometrically. The Friedmann models, determined by three parameters (H_0,Ω_0,Λ), can be represented as points in a three-dimensional parameter space, with H_0, Ω_0, and Λ as x, y, and z coordinates. The equation $\Lambda=\Lambda_s$, where Λ_s is given by Eq. (23), defines the EL models, which occupy a two-dimensional surface in this space. At fixed H_0 and Ω_0, the coasting models clearly occupy a short line segment with values of $\Lambda=\Lambda_s(1-\varepsilon)$. This is clear from the physical discussion of Sec. III. Thus the coasting models occupy a thin layer below the EL surface. (The *entire* half-space below the surface comprises the big bang models.) If we start at a given coasting model with parameters H_0,Ω_0,Λ, and move in parameter space in some arbitrary direction, we shall arrive at the EL surface by moving in any direction within almost 2π sr of solid angle. Thus there is a two-dimensional infinity of EL models (not just one) near any given coasting model. It is therefore misleading in this context to speak, as some authors do, of *the* value of Λ for *the* static Einstein universe. As the surface is not horizontal, some of these models will have cosmological constant lower than the value Λ for the given coasting model.

This picture clarifies the following remarks. For *any* given model H_0,Ω_0,Λ (in particular, for a coasting model), we define a characteristic value Λ_s by Eq. (23). Then Λ_s may be described as the value of the cosmological constant for the EL model having the same H_0,Ω_0 as the given model. We would reach this EL model by moving vertically upward (at constant H_0 and Ω_0) from the given model. We find Λ_s a useful number in categorizing models; for example, if $\Lambda<\Lambda_s$ the given model has a big bang, while if $\Lambda=\Lambda_s$ the given model *is* an EL model. But we could also define a second characteristic value for the given model (call it Λ_c) by

$$\Lambda_c \equiv \tfrac{4}{9}(c^2K_0)^3(\Omega_0 H_0^2)^{-2} = \tfrac{4}{9}(c^2K_0)^3 C^{-2}\ , \quad (29)$$

where K_0, Ω_0, and H_0 are properties of the given model. Equation (17) shows that (if K_0 is positive) Λ_c may be described as the value of the cosmological constant for the EL model having the *same Gaussian curvature* K_0 *and the same product* $\Omega_0 H_0^2$ as the given model. In fact, any EL model with the same ratio $(c^2K_0)^3/C^2$ will has the same value $\Lambda = \Lambda_c$. We would reach these models by moving in parameter space in a one-parameter family of directions, but not vertically; none has the same H_0, Ω_0 as the given model. Equation (29) is the characteristic value used by authors who assert that $\Lambda > \Lambda_c$ for a coasting model.

The physical meaning of Eq. (29) becomes clearer if we cast it into an alternative form. If $K_0 > 0$, the volume V is finite, and the total mass of the universe is $M = \rho_0 V_0$. From Eqs. (3), (10), and (29), we find

$$\Lambda_c = \tfrac{1}{4}\pi^2 c^6 (GM)^{-2}\ . \quad (30)$$

This appears correctly but in slightly different form in textbooks [Landsberg and Evans, 1977, Eq. (7.1); Rindler, 1977, Eq. (9.89)]. Thus the characteristic value Λ_c for a given model depends only on the mass M. In comparing Λ with Λ_c, we are comparing Λ for the given model with the value of Λ for an EL model having *the same mass* as the given model. This is attractive theoretically but has some drawbacks in applications (Priester, 1985). The mass M of any model depends implicitly on Λ, which is the hardest of the cosmological parameters to measure in our universe. If three-space is nearly flat, then V and M are large and uncertain, and Λ_c cannot be calculated even to within an order of magnitude. Furthermore M is ambiguous because of the topological uncertainties mentioned below Eq. (3). On the other hand, Λ_s / H_0^2 is a well-behaved function of Ω_0 only, and there are methods of measuring H_0 and Ω_0, at least in principle. The merits of Λ_s are clear.

If the given model is an EL model, then $\Lambda = \Lambda_c = \Lambda_s$; this is obvious from their definitions. But for an arbitrary given model, and in particular for a coasting model, Λ, Λ_c, and Λ_s are all unequal. It is easy to verify that $\Lambda > \Lambda_c$ for the coasting models. Differentiating Eq. (29) at constant H_0 and Ω_0 with the help of Eq. (8), we obtain

$$\left[\frac{\partial \Lambda_c}{\partial \Lambda}\right]_{H_0,\Omega_0} = \frac{4}{9}\frac{(c^2K_0)^2}{(\Omega_0 H_0^2)^2} = \left[\frac{2}{3\Omega_0}\frac{\Lambda_c}{H_0^2}\right]^{2/3}. \quad (31)$$

But at the EL surface, where $\Lambda = \Lambda_c = \Lambda_s$, Eq. (27) or Fig. 4 shows that the last form of Eq. (31) must be > 1. It follows that for the coasting models (which lie at $d\Lambda < 0$, just below the EL surface) we must have $\Lambda > \Lambda_c$. So the conventional wisdom does hold for the coasting models. However, $\Lambda > \Lambda_c$ does not hold for all big bang models; for example, for $\Omega_0 > 1$, we have $K_0 = 0$ and $\Lambda_c = 0$ when $\Lambda = -3H_0^2(\Omega_0 - 1)$. On the other hand, $\Lambda < \Lambda_s$ does hold for all big bang models. The comparison of Λ and Λ_s shows conclusively whether or not a model has a big bang. This is another advantage of Λ_s.

Finally, readers familiar with the subject may note that the parameters Ω_0 and Λ/H_0^2, together with Λ_s/H_0^2 and Λ_{s2}/H_0^2 [the two functions of Ω_0 given by Eqs. (23), (25), and (26)], provide the elements of a scheme for classifying the Friedmann universes (Glanfield, 1966). This is an alternative to Robertson's (1933) famous classification scheme, still often quoted. Consider, for example, the model in Fig. 1(c) labeled 0.52, which coasts in the future. This model has $\Lambda < \Lambda_s$ and therefore has a big bang, but it also has $\Lambda > \Lambda_{s2}$ and therefore escapes to infinity eventually. It is of Robertson's type M_1. For a second example, any model with $\Lambda > \Lambda_s$ is a catenary model. Robertson, following Friedmann (1922), called the catenary models "monotonic worlds of the second kind" (M_2 models), by which they meant that, once in expansion, these models continue to expand. But no one looking at Fig. 1 today would be likely to think of these as monotonic.

In a forthcoming paper (Felten and Isaacman, 1986) we shall discuss the role of Λ_s as an upper limit on Λ and extend our calculations of Λ_s to more general Lemaître models having nonzero pressure. We shall also calculate model ages for a wide variety of such models.

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REFERENCES

Aaronson, M., and J. Mould, 1983, Astrophys. J. **265**, 1.

Agnese, A., M. La Camera, and A. Wataghin, 1970, Nuovo Cimento B **66**, 202.

Bartel, N., A. E. E. Rogers, I. I. Shapiro, M. V. Gorenstein, C. R. Gwinn, J. M. Marcaide, and K. W. Weiler, 1985, Nature (London) **318**, 25.

Bernstein, J., and G. Feinberg, 1986, Eds., *Cosmological Constants: Papers in Modern Cosmology* (Columbia University Press, New York).

Blome, H.-J., and W. Priester, 1985, Astrophys. Space Sci. **117**, 327.

Bludman, S. A., 1984, Nature (London) **308**, 319.

Branch, D., C. H. Lacy, M. L. McCall, P. G. Sutherland, A. Uomoto, J. C. Wheeler, and B. J. Wills, 1983, Astrophys. J.

270, 123.
Brandenberger, R. H., 1985, Rev. Mod. Phys. **57**, 1.
Brecher, K., and J. Silk, 1969, Astrophys. J. **158**, 91.
Burbidge, G. R., 1967, Astrophys. J. **147**, 851.
Burnside, W. S., and A. W. Panton, 1960, *The Theory of Equations* (Dover, New York), Vol. 1.
Buta, R., and G. de Vaucouleurs, 1983, Astrophys. J. **266**, 1.
Campusano, L., J. Heidmann, and J. L. Nieto, 1975, Astron. Astrophys. **41**, 229.
Coquereaux, R., and A. Grossmann, 1982, Ann. Phys. (N.Y.) **143**, 296.
de Sitter, W., 1930, Bull. Astron. Inst. Neth. **5**, 211.
Edwards, D., 1972, Mon. Not. R. Astron. Soc. **159**, 51.
Einstein, A., 1917, Sitzungsber. Akad. Wiss. (Berlin), p. 142 [H. A. Lorentz *et al.*, *Principle of Relativity* (Dover, New York, 1952), pp. 175].
Felten, J. E., 1985, Comments Astrophys. **11**, 53.
Felten, J. E., and R. Isaacman, 1986, in preparation.
Friedmann, A., 1922, Z. Phys. **10**, 377.
Friedmann, A., 1924, Z. Phys. **21**, 326.
Glanfield, J. R., 1966, Mon. Not. R. Astron. Soc. **131**, 271.
Gunn, J. E., and B. M. Tinsley, 1975, Nature (London) **257**, 454.
Harrison, E. R., 1967, Mon. Not. R. Astron. Soc. **137**, 69.
Hart, W. L., 1947, *College Algebra*, 3rd ed. (Heath, Boston).
Janes, K., and P. Demarque, 1983, Astrophys. J. **264**, 206.
Kardashev, N., 1967, Astrophys. J. Lett. **150**, L135.
Klapdor, H. V., and K. Grotz, 1986, Astrophys. J. Lett. **301**, L39.
Landsberg, P. T., and D. A. Evans, 1977, *Mathematical Cosmology* (Clarendon, Oxford).
Lemaître, G., 1927, Ann. Soc. Sci. Bruxelles **A47**, 49 [Mon. Not. R. Astron. Soc. **91**, 483 (1931)].
Linde, A. D., 1984, Rep. Prog. Phys. **47**, 925.
Peebles, P. J. E., 1971, *Physical Cosmology* (Princeton University Press, Princeton, N.J.).
Peebles, P. J. E., 1984, Astrophys. J. **284**, 439.
Petrosian, V., 1969, Astrophys. J. **155**, 1029.
Petrosian, V., 1974, in *Confrontation of Cosmological Theories with Observational Data*, IAU Symposium No. 63, Cracow, September, 1973, edited by M. S. Longair (Reidel, Dordrecht), p. 31.
Petrosian, V., and E. E. Salpeter, 1968, Astrophys. J. **151**, 411.
Petrosian, V., E. Salpeter, and P. Szekeres, 1967, Astrophys. J. **147**, 1222
Priester, W., 1985, private communication.
Raychaudhuri, A. K., 1979, *Theoretical Cosmology* (Clarendon, Oxford).
Refsdal, S., R. Stabell, and F. G. de Lange, 1967, Mem. R. Astron. Soc. **71**, 143.
Rindler, W., 1977, *Essential Relativity* (Springer, New York).
Robertson, H. P., 1933, Rev. Mod. Phys. **5**, 62.
Rowan-Robinson, M., 1968, Mon. Not. R. Astron. Soc. **141**, 445.
Sandage, A., 1982, Astrophys. J. **252**, 553.
Sandage, A., and G. A. Tammann, 1984, Nature (London) **307**, 326.
Sandage, A., and G. A. Tammann, 1986, in *Inner Space/Outer Space: The Interface Between Cosmology and Particle Physics*, edited by E. W. Kolb *et al.* (University of Chicago Press, Chicago), p. 41.
Segal, I. E., 1983, Phys. Rev. D **28**, 2393.
Shklovsky, J., 1967, Astrophys. J. Lett. **150**, L1.
Silk, J., 1980, *The Big Bang* (Freeman, San Francisco), p. 95.
Stabell, R., and S. Refsdal, 1966, Mon. Not. R. Astron. Soc. **132**, 379.
Thielemann, F.-K., J. Metzinger, and H. V. Klapdor, 1983, Z. Phys. A **309**, 301.
Tinsley, B. M., 1977, Phys. Today **30** (6), 32.
Tolman, R. C., 1934, *Relativity, Thermodynamics, and Cosmology* (Clarendon, Oxford).
Turnbull, H. W., 1952, *Theory of Equations*, 5th ed. (Oliver and Boyd, Edinburgh).
Turner, M. S., 1985, Phys. Rev. D **31**, 1212.
Turner, M. S., G. Steigman, and L. M. Krauss, 1984, Phys. Rev. Lett. **52**, 2090
VandenBerg, D. A., 1983, Astrophys. J. Suppl. **51**, 29.
Weinberg, S., 1972, *Gravitation and Cosmology* (Wiley, New York).
Zel'dovich, Ya. B., and R. A. Syunyaev, 1980, Pis'ma Astron. Zh. **6**, 451 [Sov. Astron. Lett. **6**, 249 (1980)].