# On waves in gases. Part I: Acoustics of jets, turbulence, and ducts

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This review on some aspects of waves in gases concentrates first (Part I) on modern research in the acoustics of fluids at rest or in steady or turbulent motion, in free space, in the presence of obstacles, or in ducts. The study of sound, for which the sole restoring force is pressure, will be extended in a later paper (Part II) to include the other three restoring forces, namely, gravity, electromagnetic, and Coriolis forces, leading to current research on internal, magnetic, and inertial waves and their couplings. The Introduction at the beginning of Part I, and the discussion at the end of Part II, concern all four types of waves in gases, and their relevance in physics and engineering. In Part I, the following areas of acoustics are addressed: the generation of noise by turbulence, inhomogeneities or bubbles, in natural and engineering flows, e.g., wind or jets; the scattering of sound by interfaces and diffraction by turbulence, and their effects on spectral and directional redistribution of energy; propagation in ducts, without or with mean flow, e.g., the horns of musical instruments and loudspeakers, and inlets and exhausts of engines; the effects of dissipation and nonlinearity on waves, e.g., in laboratory and engineering shock tubes, and in geophysical and astrophysical conditions. Underlying these topics is the interaction of acoustics with mankind, ranging from the processes of human hearing and speech to the reproduction of desirable sounds (music) and reduction of undesirable sounds (noise).

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## I. INTRODUCTION

The subject of the present review is waves, in the usual sense of disturbances in a medium, which depend on space and time and can propagate, at a finite velocity, from one region to another; a wide variety of waves is studied in different branches of physics, from electromagnetic waves *in vacuo* (Maxwell, 1873; Bateman, 1915; Stratton, 1941; Reitz and Milford, 1967) to probability waves in quantum mechanics (Dirac, 1927; Heisenberg, 1930; Schiff, 1949), and from gravitational waves in general relativity (Eddington, 1923; Tolman, 1934; Møller, 1952; Synge, 1960; Misner, Thorne, and Wheeler, 1973) to waves in continuous media (Rayleigh, 1889; Brekhovskikh, 1960), such as vibrations in solids (Love, 1927; Achenbach, 1973; Hudson, 1978) and oscillations in fluids (Whitham, 1974; Lighthill, 1978a). We shall restrict ourselves to waves in gases, thus excluding both plasmas and liquids; this implies the use throughout of the continuum description of matter, assuming that the number of particles per unit volume is sufficiently large. This is sometimes referred to as the cold plasma (Spitzer, 1956; Stix, 1962). The exclusion of waves in liquids (Stoker, 1953; Philips, 1960a), is not essential, since most properties, *not* depending on the form of the equation of state, are similar to gases.

### A. Hearing, music, audio, and noise

Acoustics (Mason, 1964—1973) interfaces with the biological and medical sciences in the areas of speech and hearing, with the arts in connection with music, and with technology both in these traditional areas and in the current concern with noise reduction. Acoustics has historically accompanied the evolution of science, from the speculations on the relation of sound to music in classical and ancient civilizations, to the dawn of the age of modern, quantitative science (Newton, 1686; Euler, 1772), and it remains today the suitable introduction (Kinsler and Frey, 1950; Beranek, 1954; Morse and Ingard, 1968; Levine, 1978; Pierce, 1981; Dowling and Ffowcs-Williams, 1983) to the various active research areas of waves in gases.

#### 1. Physics of music

The relationship between sound and music was clearly established at the beginning of the nineteenth century (Euler, 1818) and has remained a topic of consistent interest since then (Jeans, 1937; Woods, 1944), to which modern technology has added high-quality sound reproduction (Olson and Massa, 1934; Olson, 1940,1952,1972; Moir, 1961), or audio for short. In spite of this long history, the acoustics of musical instruments (Olson, 1952; Benade, 1976,1980; Berg and Storck, 1982) is only partially understood, e.g., performing techniques and the construction of instruments are based mainly on empirical experience, with some progress from qualitative to quantitative modeling (Brindley, 1973; Howe, 1975a,1981; Kergomard, 1981; Caussé, Kergomard, and Lurton, 1984).

### 2. Hearing and speech

The perception of sound and the intonation of speech relate music to physiological acoustics (Wever and Lawrence, 1954). Physical aspects of hearing (Bekesy, 1960; Viergeuer, 1980) have been the subject of recent research, e.g., as concerns the energy flow in the cochlea (Lighthill, 1981), the determination of the shape of the outer canal (Hudde, 1983), and the modeling of the ear reflex (Stevin, 1984). While the outer ear canal is a nearly straight tube of varying cross section, the vocal tract is curved as well (Mermelstein, 1966; Schroeder, 1967), with branches off toward the lungs (Ishizaka, Matsudaira, and Kaneko, 1976; Jackson, Butler, and Pyle, 1978).

### 3. Horns and nozzles

Ducts of varying cross section are used in certain types of loudspeakers (McLachlan, 1934b,1935,1936; Jordan, 1963) and have analogs in other technological areas, such as (i) the "solid" horns, or tapering bars, used as displacement amplifiers (Merkulov, 1957; Eisner, 1963) in power tools; (ii) the "water hammer" in the hydraulics of tubes (Paynter and Ezekiel, 1958); (iii) electromagnetic horns (Stevenson, 1951a) and nonuniform transmission lines (Schwartz, 1974). The extensive literature on the acoustics of horns (Eisner, 1966; Campos, 1984a) which dates back to the beginnings of wave theory (Truesdell, 1955,1960), is extended by modern research on sound in nozzles (Nayfeh, Kaiser, and Telionis, 1975a; Campos, 1985a), i.e., ducts carrying a mean flow (Lighthill, 1972; Swinbanks, 1975; Campos, 1984b).

### 4. Aircraft and engines

The propagation of sound in nozzles is relevant to the reduction of the noise of jet engines and aircraft, which has been an important motivation for the study of aerodynamic acoustics. The analysis of the noise of jets logically starts with the modeling of sound sources, and the "acoustic analogy" (Lighthill, 1952,1954) has been the subject of substantial research, including estimates of radiation intensity (Curle, 1955; Lighthill, 1964; Crighton and Ffowcs-Williams, 1969; Howe, 1975a; Campos, 1978a; Adam, 1982). This model approach to the generation of sound has been regularly summarized in reviews (Lighthill, 1961,1963; Ffowcs-Williams, 1969,1984a; Crighton, 1975,1981; Campos, 1983a; Möhring, Müller, and Obermeier, 1983), and books (Goldstein, 1976; Lighthill, 1978a; Dowling and Ffowcs-Williams, 1983). The inverse process, of flow induced by sound, is known as acoustic "streaming" (Lighthill, 1978b).

### 5. Signal "clutter"

The sound generated by a source in a jet is modified, both in directivity and spectrum (Candel, Guédel, and Julienne, 1976; Munt, 1977; Beyer and Korman, 1980; Campos, 1984c), by propagation within the flow, which is generally turbulent, and by transmission across the interface separating the jet from the atmosphere. The scattering of waves by interfaces, generally irregular in nature, causes the contamination of a signal, e.g., the cases of "clutter" in radar echoes (Sholnik, 1962), electromagnetic waves incident on rough surfaces (Beckmann and Spizzichino, 1963), sonar waves reflected by the sea bottom (Clarke, 1973) or surface (Essen, 1974; Gazanhes and Léandre, 1974), radio waves used to scan glaciers (Berry, 1973), or sound transmitted across impedance layers (Howe, 1976b) and shear layers (Campos, 1978c). Broadly similar effects occur for waves propagating in random media (Uscinski, 1977; Ishimaru, 1978), such as electromagnetic waves in a perturbed atmosphere (Tatarski, 1965), light in glass with optical impurities (Chernov, 1967), and sound in turbulence, which may be, in the mean, at rest (Lighthill, 1953), in free convection (Campos, 1978b), or confined within a pipe (Howe, 1984b).

### 6. Sonar and ultrasonics

Ordinary sound, even at the threshold of pain to the human ear (110 dB), involves rather little energy, and may be considered a linear disturbance of the atmosphere. Nonlinear effects can occur near the apex of an acoustic horn (Goldstein and McLachlan 1935), where the cross section is small, and near the drivers of high-power sonar arrays (Westervelt, 1963), emitting underwater sound (Brekhovskikh and Lysanov, 1982). Noise can reach nonlinear levels in the interior of jet and reciprocating engines or near them, and, in the extreme case of rocket engines that power satellite launchers and the space shuttle, can cause structural damage. Acoustics does have positive applications in materials science, viz., in ultrasonic inspection and nondestructive testing, and similar scanning techniques are used medically to observe the interior of the human body; since ultrasound easily reaches nonlinear levels, (Blackstock, 1972; Bjørnø, 1974), small amplitudes are used. Nonlinear acoustics lie close (Campos, 1985b) to the subject of unsteady, high-speed gas dynamics (Howarth, 1953; Shapiro, 1954; Emmons, 1958; Von Mises, 1958; Miles, 1959), which is relevant to aircraft aerodynamics (Carafoli, 1969; Krasnov, 1971; Schlichting and Truckenbrodt, 1979).

### 7. "Sonic boom"

Acoustic waves of large amplitude tend to steepen their wave fronts (Riemann, 1860), leading to the formation of discontinuities or shocks, which actually have a small but finite thickness, determined by dissipation effects (Taylor, 1910); shocks can be demonstrated in laboratory tubes (Stollery, Gaydon, and Owen, 1971) and are observed as the "sonic boom" of high-speed aircraft (Hayes, 1973). Dissipation plays a major role in delaying the formation of shocks (Lighthill, 1956; Campos, 1984d) and in providing for their ultimate decay (Beyer, 1974; Rudenko and Soluyan, 1975). One form of the growth of wave amplitudes from linear to nonlinear is propagation in a rarefied medium, e.g., upward in an atmosphere for which the density decreases with height (Yeh and Liu, 1974; Campos, 1983b); the wave can become nonlinear (Yanowitch, 1969; Crighton, 1979), or the amplitude can be limited by dissipation (Yanowitch, 1967a; Campos, 1983c).

We summarize in Table I(a) some of the applications of modern acoustics.

## B. Restoring forces and types of waves

Acoustic waves were the first to be studied, since sound is a commonplace human experience and relevant to our immediate environment. As man's knowledge extended to the study of the oceans, atmosphere, and interior of the Earth, and to astronomical bodies, and as laboratory technology and industrial capability advanced, other types of waves in fluids became the subjects of increasing attention, viz., internal, inertial, and magnetic modes. To each of the four restoring forces that apply in classical (nonquantum, nonrelativistic) fluids, namely, pressure gradients, gravity, Coriolis force, and magnetic (or electric) forces, corresponds one type of wave, respectively, acoustic, internal (or gravity), inertial (or "Kelvin"), and magnetic (or Alfvén); all these waves result from a balance between inertia and restoring forces, and if more than one of the latter is present, then wave coupling results, leading to the modification of the basic modes and the possible appearance of new ones.

### 1. Stratified media

A stratified medium, e.g., a fluid of nonuniform density, under a gravity field, has two possible conditions, separated by the state of marginal stability: (a) if it is unstable, e.g., a heavy liquid on top of a light one, a small disturbance can trigger a large change in the mean state, viz., inversion of the two fluids; (b) if it is stable, e.g., an atmosphere with entropy increasing with height (Landau and Lifshitz, 1953), then a small disturbance will not change the mean state, but may persist as an oscillation. Thus instabilities and waves are two effects of buoyancy, in an inhomogeneous fluid under gravity. Internal waves (Rayleigh, 1890) are produced in a stably stratified fluid when a fluid parcel is disturbed relative to its position of equilibrium, if it is moved upward (downward), it finds itself in less (more) dense surroundings, and its weight (buoyancy) causes it to sink (rise), back to the mean position, generally overshooting due to inertia, so that an oscillation results. Internal waves are a common observation in stratified fluids (Eckart, 1960; Yih, 1965; Turner, 1973), e.g., the oceans (Philips, 1960a; Kraus, 1977) and the atmosphere (Beer, 1974; Gossard and Hooke, 1975).

### 2. Rotating globe

Waves for which the sole restoring force is associated with rotation (Greenspan, 1968) are designated inertial or "Kelvin" (1880) modes. In the case of nearly uniform rotation, as applies to the Earth, of the two force components, the centrifugal term can be incorporated as a modification of the fluid pressure gradient, and the Coriolis term acts as the restoring force. Inertial waves are important for phenomena with periods comparable to or exceeding the period of rotation, i.e., one day. Inertial modes can be visualized as large-scale tidal waves (Miles, 1972), which are scattered by continents (Haines, 1981), producing currents parallel to the coastline and along depth discontinuities (Longuet-Higgins, 1968). If the waves are small on the scale of the Earth, and not too close to the poles, the Coriolis parameter may be taken as a constant; otherwise, its variation with latitude becomes an important effect for large-scale or near-polar inertial waves (Rossby, 1939; Longuet-Higgins, 1964).

TABLE I. Outline of waves in fluids: (a) modern acoustics: simplified diagram showing main applications and motivations for its study; (b) list of four restoring forces and of their respective wave types, which are single-mode when only one effect is present and three-mode interactions when only one effect is absent.

(a) Applications of acoustics

| | | |
|---|---|---|
| Acoustics | Physiological | Hearing, Speech |
| | Physical | Music, Audio |
| | Engineering | Noise, Detection |

(b) Four restoring forces

| Medium: Effect | Medium: Restoring force | Wave: Present | Wave: Absent |
|---|---|---|---|
| Compressibility | Pressure | Acoustic | Magneto-gravity-inertial |
| Stratification | Buoyancy | Gravity | Magneto-acoustic-inertial |
| Rotation | Coriolis | Inertial | Magneto-acoustic-gravity |
| Ionization | Magnetic | Magnetic | Acoustic-gravity-inertial |

### 3. Ionized fluids

The simplest wave motion in an ionized, perfectly conducting fluid is a balance between the fluid inertia (always present) and the magnetic restoring force. These magnetic waves were predicted theoretically (Alfvén, 1942) before being observed experimentally (Lundquist, 1949), and became subsequently common observations in astrophysical plasmas, e.g., in the solar wind (Belcher and Davis, 1971; Denskat and Burlaga, 1977) and atmosphere (Giovanelli and Beckers, 1982; Campos, 1983d). Magnetic waves (Alfvén, 1948) appear as oscillations, traveling along magnetic field lines, similar to the transverse oscillations of stretched elastic strings. The "elastic" tension is replaced by the "magnetic" tension, and the transverse displacement (or velocity) is associated with a magnetic field perturbation. Thus purely magnetic waves are one-dimensional or unidirectional oscillations, even for emission from a point source, in a three-dimensional space.

### 4. Weather and climate

Having considered all four types of uncoupled waves, obtained by balancing inertia force against each of the restoring forces in isolation, we turn to the inverse situation, viz., the four cases of three-wave interactions, obtained [Table I(b)] by excluding one restoring force and allowing for the presence of the other three. In the bodies of natural fluids closest to mankind, the oceans and low atmosphere, the magnetic field plays a minor role, and thus the main restoring forces in geophysics (Pedlosky, 1960) are buoyancy, Coriolis force, and compressibility, leading to (Tolstoy, 1963) acoustic-gravity-inertial waves. These tend to separate into acoustic-gravity waves for periods small relative to one day and gravity-inertial waves for periods comparable to or larger than one day; these waves affect, respectively, local and global weather, through the transport of mass, energy, and linear or angular momentum (Massey, 1980).

### 5. Conducting atmospheres

Outside a layer surrounding the surface of the Earth, i.e., apart from the Earth's crust, oceans, and low atmosphere, matter is mostly ionized, viz., in the core and the magnetosphere of the Earth. In the magnetosphere, ionized particles in the solar wind are trapped by the Earth's magnetic field. These particles bear witness to the fact that the sun and other stars (Schwarzschild, 1958; Chandrasekhar, 1983) are self-gravitating, ionized gas masses, to which classical physics may be applied, except in radiative and thermonuclear cores and at late stages of evolution (Chandrasekhar, 1984). Thus the problems of mass and energy transfer by waves in solar and stellar atmospheres (Bray and Loughhead, 1974; Athay, 1976; Bruzek and Durrant, 1977; Campos, 1984e) involve compressibility, gravity, and magnetic field as the restoring forces, leading to the study of magneto-acoustic-gravity waves (Priest, 1982; Campos, 1983e; Thomas, 1983; Spruit and Roberts, 1984).

### 6. Earth's magnetic field

The existence of the Earth's magnetic field raises the question of its origin. The temperature of the interior of the Earth is above the ferromagnetic point, so that "permanent" magnetization is not possible; a primordial magnetic field, dating from the formation of the Earth, would have "leaked" through the crust, leaving no significant remnant. Thus the Earth's magnetic field must be continuously regenerated, a suitable mechanism being the dynamo effect (Roberts, 1971; Moffatt, 1976,1978; Parker, 1979; Cowling, 1981) coupling rotation and magnetic fields in the molten inner core of the Earth. The two-scale mean-field electrodynamic approach to the dynamo effect requires small-scale motions, such as turbulence and waves, to feed energy to the large-scale magnetic field; suitable wave motions, under the influence of Coriolis and magnetic forces, in an inhomogeneous fluid, are magneto-inertial-gravity waves; these can be simplified, for motions on a scale small compared to that of stratification, to magneto-inertial waves.

### 7. Solar and stellar dynamos

The Earth and the sun may be considered as typical of planets and stars, respectively, and this larger sample of celestial bodies suggests that faster rotation is associated with stronger magnetic fields. The Earth's and the sun's magnetic fields differ significantly (Akasofu and Chapman, 1972), in the greater strength and variability of the latter (Golub, Rosner, Vaiana, and Weiss, 1981; Labonte and Howard, 1982) compared with the former (Barraclough, Harwood, Leaton, and Malin, 1975). An important physical difference is that the Earth's core is basically a liquid, molten metal, and hence practically incompressible, i.e., acoustic time scales are short compared with those of the dynamo; in the solar case, dynamo action takes place in a rotating, ionized gas, so that, neglecting the effects of stratification, for small-scale motions, the relevant waves are magneto-acoustic-inertial, i.e., compressibility cannot be reasonably neglected.

## C. Two-, three-, and four-wave couplings

We have considered the four basic types of waves to be found in the interior of fluids (i.e., excluding surface or interfacial modes), namely, acoustic (Sec. I.A) gravity (Sec. I.B.1), inertial (Sec. I.B.2), and magnetic (Sec. I.B.3), corresponding to the presence of one restoring force in isolation. The exclusion of one restoring force, and allowance for the presence of the other three, leads to four three-wave couplings, namely, acoustic-gravity-inertial (Sec. I.B.4), magneto-acoustic-gravity (Sec. I.B.5), magneto-gravity-inertial (Sec. I.B.6), and magneto-acoustic-inertial (Sec. I.B.7). Besides these cases (indicated in Fig. 1), in order to complete the outline survey, we have to consider $\binom{4}{2}=6$ cases of two-wave interactions, namely, acoustic-inertial (Sec. I.C.1), gravity-inertial (Sec. I.C.2), acoustic-gravity (Sec. I.C.3), magneto-acoustic (Sec. I.C.4) magneto-inertial (Sec. I.C.5), and magneto-gravity (Sec. I.C.6) waves, concluding with the most general case, the single four-wave coupling, viz., magneto-acoustic-gravity-inertial waves (Sec. I.C.7).

### 1. Dissimilar periods

The four basic types of waves in fluids, can be ordered according to their propagation properties: (i) acoustic waves are isotropic and nondispersive; (ii) magnetic waves are nondispersive but anisotropic; (iii) inertial and gravity waves are both dispersive and anisotropic. In the coupling of several waves, the anisotropy and dispersion may be expected to predominate, viz., (a) the coupling of isotropic and anisotropic waves is anisotropic—there are no isotropic two-wave couplings in fluids; (b) the coupling of dispersive and nondispersive waves is dispersive—the only nondispersive two-wave couplings are magneto-acoustic waves, which are anisotropic. All of the six two-wave couplings occur more or less widely in nature and engineering. Perhaps the least common are acoustic-inertial modes, since on the Earth, sound has periods of minutes or less, and inertial waves have periods of a day or more. This coupling occurs for particulate disks, e.g., Saturn's rings, and can be produced by spinning rapidly a vessel containing a compressible fluid.

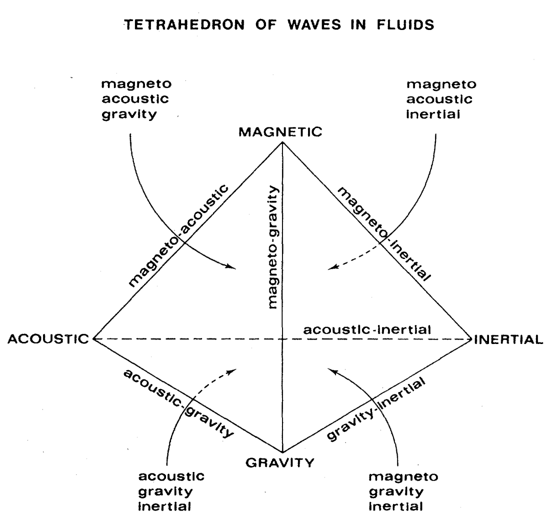


FIG. 1. The tetrahedron of waves in fluids: four single-wave modes at the vertices, six two-wave couplings along the edges, four three-wave couplings on the faces, and one four-wave coupling in the interior.

### 2. Large-scale circulation

Acoustic waves are the only isotropic mode, because pressure is independent of direction, whereas the other three modes are anisotropic, since they have a preferred direction, determined by the force of gravity, the axis of rotation, and the magnetic field, respectively, for gravity, inertial, and magnetic waves; to be more precise, the wave fronts, for emission from a point source, are spherical for sound, plane for magnetic waves, and biconical for inertial and gravity waves, the latter corresponding, in the plane case, to a "Saint Andrew's cross" (Mowbray and Rarity, 1967). Acoustic and magnetic waves are nondispersive, i.e., different wavelengths propagate at the same speed and arrive at an observer at the same time, if emitted simultaneously; in the case of sound, this explains why speech and music, consisting of various frequencies, remain intelligible regardless of distance from the source (apart from dissipation effects). Gravity and inertial waves are dispersive, and thus a "packet" of different wavelengths spreads out as it propagates, distorting the "signal" and allowing the sequence of reception to differ from that of emission. The coupling of the two dispersive "basic" modes as inertial-gravity waves is important for Kelvin and Rossby waves in stratified fluids, an additional effect being wind or shear flow (Varley, Kazakya, and Blythe, 1977; Ahmed and Eltayeb, 1980); these waves affect the large-scale circulation patterns in the atmosphere and oceans, which play a major role in determining the weather (Lighthill, 1969; McIntyre and Palmer, 1983; Peixoto and Oort, 1984).

### 3. Atmospheric phenomena

Acoustic and magnetic waves are nondispersive, which implies that there can be no filtering, i.e., waves of all frequencies, however small or large, can propagate. This reasoning does not apply to gravity and inertial waves, which are dispersive, and in fact have cutoff frequencies at the Brunt-Vaisala and the rotation frequencies, respectively. Gravity and inertial waves are both anisotropic, e.g., gravity waves cannot propagate in the direction perpendicular to gravity, and inertial waves cannot propagate along the rotation axis. Gravity waves, propagating or standing, are commonly observed in the interior of the oceans, e.g., scattering sound or sonar waves (Uscinski, 1980) and exerting forces on offshore structures (Osborne and Burch, 1980). Equally common in the atmosphere are acoustic-gravity waves, which can scatter and otherwise affect electromagnetic signals in the high atmosphere (Delloue and Halley, 1972; Hines, 1974); in low atmosphere, these internal waves are associated with clear air turbulence (Pao and Goldburg, 1969), which causes velocity shears affecting aircraft flight (Campos, 1984f).

### 4. Cold plasmas

The only nondispersive two-wave couplings in fluids are magneto-acoustic waves in ionized gases, which are a generalization of Alfvén waves (Herlofson, 1950; Banos, 1955; Lighthill, 1960; Campos, 1977), discussed in most books of magnetohydrodynamics (MHD) (Landau and Lifshitz, 1956; Cowling, 1957; Alfvén and Falthammar, 1962; Ferraro and Plumpton, 1963; Jeffrey and Taniuti, 1964; Cabannes, 1970). They are a particular case of plasma waves, corresponding to high particle densities and a near balance of ions and electrons; this "cold plasma" model does not apply to the important problem of nuclear fusion (Bruecker and Jorna, 1974; Ribe, 1975), although MHD can be used to study the basic types (Jeffrey and Taniuti, 1966) of instabilities in "pinches" (ionized fluids confined by magnetic fields, e.g., in toroidal geometries such as tokamaks), which have been limiting progress in this area for over three decades. MHD is adequate to describe (Hunt and Shercliff, 1971) many interesting engineering processes (Shercliff, 1965) involving ionized fluids, ranging from energy generators (Shermann and Sutton, 1965) to metallurgical applications (Moffat and Proctor, 1982), and including special processes such as ionic propulsion for artificial satellites, which require low thrust for long periods in extended space travel.

### 5. Magnetic field generation

The coupling of two basic types of waves in fluids can result in (a) their separation, (b) their interaction, or (c) creation of new modes. As an example of (a), we give acoustic-gravity waves (Moore and Spiegel, 1964; Campos, 1982), which have two modes, namely, (i) gravity modes modified by compressibility, below the Brunt-Vaisala frequency and (ii) acoustic modes modified by gravity above another, higher cutoff, leaving a spectral band of evanescent waves to separate the two modes. At the opposite extreme (c) are magneto-acoustic waves, which have three modes, namely, (i) Alfvén waves, as in the incompressible case, uncoupled from the (ii) slow modes and (iii) fast modes, which are acoustic waves modified by the magnetic field and propagating at speeds lower and higher, respectively, than the speed of sound. An intermediate case (b) is that of magneto-inertial waves (Lehnert, 1954,1955), which have two modes, due to splitting of Alfvén waves by rotation, and are relevant to processes in the interior of planets and stars (Acheson and Hide, 1973), e.g., generation of the Earth's and the sun's magnetic fields.

### 6. Magnetic flux tubes

The most interesting of the six two-wave couplings in fluids are magneto-gravity waves, viz., Alfvén waves in a stratified, ionized fluid (Ferraro and Plumpton, 1958; Hollweg, 1972; Leroy, 1980,1982; Campos, 1983f). The reason lies in the dependence of the propagation (or Alfvén) speed on the magnetic field strength and mass density, leaving as the "simplest" possible cases two alternatives: (a) the Alfvén speed is constant if the gas and magnetic pressure balance, i.e., if the fluid is magnetically structured (Rae and Roberts, 1983; Heyvaerts and Priest, 1983); (b) the propagation speed varies rapidly with altitude, and waves are not sinusoidal, if the magnetic field is constant (Zhugzhda and Dzhalilov, 1981; Campos, 1985c). Both of these cases correlate with recent observations of the structure of magnetic fields in solar and stellar atmospheres: (a) in the lower layers, i.e., photospheres and chromospheres, the magnetic flux is concentrated in narrow magnetic flux tubes (Stenflo, 1982), filling a small fraction of the disk; (b) these flux tubes fan out with height, and the magnetic field becomes nearly uniform as they merge in the upper layers, i.e., transition regions and coronas (Gabriel, 1976). Magnetic structures such as "holes" and "arches" are also visible in satellite observations of the corona (Bonnet and Dupree, 1980).

### 7. General waves in fluids

The features of magneto-gravity waves, such as variable propagation speeds and damping rates (Campos, 1983g), are inherited by magneto-acoustic-gravity and magneto-inertial-gravity waves, as well as by the most general waves in fluids, viz., magneto-acoustic-gravity-inertial. The latter are seldom, if ever, considered in physical and engineering applications, since acoustic and inertial modes are often decoupled (Sec. I.C.1), implying that, at most, only three-wave couplings need be considered, and often two-wave or single-wave models are used with success. On the other hand, magneto-acoustic-gravity-inertial waves do hold an interest, as a fundamental study of all four restoring forces and their couplings in fluids. At present, the substantial literature on the various types of waves in fluids, in spite of some broad surveys (Tolstoy, 1963; Lighthill, 1978a), is mostly scattered through the vast literature on physics and engineering, according to area of application, e.g., physiological acoustics, musical sound, noise reduction, oceanography, atmospheric physics, other aspects of geophysics, astrophysics in general, etc., plus diverse technological processes. The aim of the present review is to stress the fundamental unity of all waves in fluids and to point out the peculiar features of each of the four modes, so as to understand the various possible interactions, which become ever more important as modern physics tends to blend formerly separate areas into interdisciplinary fields.

On a purely illustrative level, we note that the various types and couplings of waves in fluids may be represented (Fig. 1) on a "tetrahedron of waves in fluids," as follows: (i) the four vertices represent the basic waves ($A$—acoustic, $G$—gravity, $I$—inertial, $M$—magnetic); (ii) the six edges joining pairs of vertices represent two-wave interactions ($AG$, $AI$, $MA$, $MG$, $MI$, $GI$); (iii) the four triangular faces, each limited by three edges, and with three vertices, represent three-wave couplings (*AGI*, *MAG*, *MAI*, *MGI*); (iv) the interior limited by the four faces represents the most general four-wave coupling (*MAGI*). In addition, we could use the position of a point in the tetrahedron, relative to vertices, edges, or faces, to indicate the proximity to single-, two-, or three-wave modes, i.e., to compare the importance of compressibility, buoyancy, rotation, and magnetism, in a particular application.

### 8. Acoustics of jets, turbulence, and ducts

When considering an acoustic or "noise" problem, the first issue to be dealt with is often that of generation, i.e., the modeling of the sources of sound; thus we start our account of some aspects of modern acoustics, with a discussion of the "acoustic analogy" for sound generation in media at rest, and in low- and high-speed jets, in the presence of turbulence and inhomogeneities. The modeling of acoustic sources, although an essential first step, may not be sufficient if "masking" occurs, e.g., if the sound is scattered by interfaces, possibly irregular and/or in motion, or propagates through turbulence; these effects (Sec. III) can change significantly the directional and spectral distribution of acoustic energy received by an observer, relative to that emitted by the source, due to scattering and diffraction in the medium separating them. In many applications, the source of sound is not in free space, but rather at the end of a duct, e.g., the "driver" of a loudspeaker horn, the blowing of a musical instrument, or the noise coming from the inlet or exhaust of an engine; the effects of ducting of waves are important (Sec. IV) in the acoustics of horns, and are coupled to convection by the mean flow in the case of nozzles. We conclude (Sec. V) with a brief mention of the interaction of sound with the supporting medium, through dissipation, which extracts energy from the wave and deposits it in the fluid, and through nonlinear effects associated with perturbations of large amplitude, which can change the mean state; the effects of dissipation and nonlinearity are often competitive, e.g., in the growth of atmospheric oscillations and in the formation of shock waves. Since the range of subjects to be covered is extensive, we shall, in each section, mention by means of references some of the current research areas, before selecting a fundamental topic for more detailed consideration.

## II. SOUND PRODUCTION BY TURBULENCE AND INHOMOGENEITIES

Sound can be produced "artificially" by man-made devices, such as sirens, whistles, musical instruments, vibrating bodies, etc., and it is also generated "spontaneously" in natural and engineering flows, e.g., the whistling of the wind, the rumble in aerodynamic tunnels, the noise of jets. Aeronautics provides a number of examples of "noise" sources, such as jet exhausts (Laufer and Yen, 1983; Long and Arndt, 1984; Whitaker and Morrison, 1984; Seiner and Yu, 1984), and of efforts to reduce the acoustic "signature" (Nagel, Denham, and Papathanasiou, 1983; Norum, 1984); other subjects of current research on aircraft noise (Yeow, 1984) include the generation of sound by airfoils (Arbey and Bataille, 1983), corners and flaps (Meecham, 1983), turbulent wakes (Hardin and Lamkin, 1984; Johnson and Loehrke, 1984), turbomachinery (Schulten, 1984), and rotors (Aggarwal, 1984). A central consideration in these and other problems is the modeling of the sources of sound, which allows classical wave theory to be applied to the calculation of the radiation field, e.g., in the presence of convection (Dowling, 1976) and walls, viz., compliant (Dowling, 1983), solid (Hoop and Hijden, 1984), or with a bump (Rabinovich, Reutov, and Rybushkina, 1984).

### A. Generation processes and multipole sources

The modeling of the generation of sound by flows has been the subject of an "acoustic analogy" (Lighthill, 1952,1954); this important concept has been reviewed regularly (Lighthill, 1961,1978a; Ffowcs-Williams, 1969, 1984; Crighton, 1975,1981; Goldstein, 1976; Dowling and Ffowcs-Williams, 1983; Campos, 1983a; Möhring, Müller, and Obermeier, 1983), both in its original form and in extensions to include scattering by solid (Curle, 1955; Howe, 1984a) and fluid (Philips, 1960b; Ffowcs-Williams, 1964) boundaries, and convection of sources and mean flow effects (Ffowcs-Williams, 1963; Ffowcs-Williams and Hawkins, 1968; Dowling, Ffowcs-Williams, and Goldstein, 1978). The inverse problem, to the generation of sound by flows, that is, the flow induced by sound (Pickering and Sozou, 1982), is also a subject of reviews (Lighthill, 1978b). Another example of a dual problem is the generation of sound by flames (Jones, 1979) and the use of acoustics as a diagnostic of combustion (Ramachandra and Strahle, 1983). Acoustic methods have also been used to determine fluid properties, such as the gas constant (Quinn, Collough, and Chandler, 1976). Another set of related problems is the detection, by acoustic methods, of the presence of gas bubbles in a liquid (Gazanhes, Arzeliés, and Léandre, 1984), possibly associated with cavitation (Trevena, 1984), and the oscillations of gas bubbles (Fanelli, Prosperetti, and Reali, 1984; Francescutto and Nabergoj, 1984), acting as monopole sources (Sornette and Lagier, 1984) of sound in two-phase flow (Crighton and Ffowcs-Williams, 1969; Whitfield and Howe, 1976). We shall now consider the "wave analogy" in a form that applies to the original acoustic problem (Lighthill, 1952,1954) and allows extensions to account for the effects of inhomogeneous mean flow (Howe, 1975a,1975b; Campos, 1978a; Sec. II.B and II.C) and the presence of restoring forces other than gas pressure (Stein, 1967; Campos, 1977; Sec. VI.B in Part II).

#### 1. The wave analogy

In classical acoustics (Rayleigh, 1879), the sources of sound, such as strings, membranes, sirens, etc., were identified *a priori* as distinct from the medium of propagation and radiating into it. Modern acoustics has also considered the generation of sound by flows, in which case the sources and medium of propagation are not *a priori* distinct, and physical modeling of the noise production mechanism is necessary. A method of addressing this question is the "acoustic analogy" (Lighthill, 1952), which may be applied to other types of waves in fluids and thus may be formulated generally as a "wave analogy" (Campos, 1977), in the following conceptual framework: (i) we start from the fundamental equations of fluid mechanics, including all effects (e.g., presence of turbulence, inhomogeneous flow, external forces) to be considered; (ii) the total fluid variables $T=M+P$ are split into a mean state $M$ plus a perturbation $P$, the latter *not* necessarily small everywhere; (iii) when substituting into the fundamental equations, we can subtract out the terms involving the mean state $M$, leaving only terms linear $LP$ and nonlinear $NP$ in the perturbation, so that the equations can be written in the symbolic form

$$\text{linear terms} = -\text{nonlinear terms} ; \tag{1}$$

(iv) suppose that the large perturbations, for which the nonlinear terms are important, are concentrated in a small region $D$; then (v) outside $D$, Eq. (1) reduces to

$$\text{linear terms} = 0 , \tag{2}$$

which describes the propagation of waves of small amplitude throughout the fluid; (vi) we can now interpret the nonlinear terms, which are "forcing" the wave equation (1), as modeling the "sources" of waves contained in $D$.

## 2. Monopoles, dipoles, and quadrupoles

The paradigm of the preceding general procedure is the "acoustic analogy," for which the starting equations are those (Batchelor, 1967) of an homogeneous, viscous fluid, in a mean state of rest. The equation of continuity, stating the conservation of mass, is

$$\partial\rho/\partial t + \partial(\rho v_i)/\partial x_i = Q , \tag{3}$$

where $\rho$ is the mass density and $v_i$ the velocity, and $Q$ is the output (per unit time) of mass sources. The momentum equation can be written in the form

$$\partial(\rho v_i)/\partial t + c_0^2 \partial\rho/\partial x_i + F_i + \partial T_{ij}/\partial x_j = 0 , \tag{4}$$

where $c_0 \equiv (\partial p/\partial\rho)_s$ denotes the adiabatic sound speed, and we have replaced the pressure gradient $\partial p/\partial x_i$ linearly by $c_0^2 \partial\rho/\partial x_i$, including all nonlinear corrections and other terms in the force $F_i$ and stress $T_{ij}$. Eliminating between Eqs. (3) and (4), we obtain

$$\partial^2\rho/\partial t^2 - c_0^2 \partial^2\rho/\partial x_i^2 = \partial Q/\partial t + \partial F_i/\partial x_i + \partial^2 T_{ij}/\partial x_i \partial x_j , \tag{5}$$

where the left-hand side is the classical wave equation for a medium at rest, and the "forcing" terms on the right-hand side are interpreted as modeling the sources of sound: (i) the mass flux $Q$, which is a scalar, acts as a monopole source, if it varies in time, i.e., is unsteady $\partial Q/\partial t \neq 0$; (ii) the force $F_i$, which is a vector, acts as a dipole source, if it is nonuniform and not divergence free $\nabla\cdot\mathbf{F} \neq 0$; (iii) the stress, which is a tensor, acts as a quadrupole, if its double divergence does not vanish $\partial^2 T_{ij}/\partial x_i \partial x_j \neq 0$.

## 3. Lighthill's stress tensor

The original example of the acoustic analogy is based on modeling the sound sources by a quadrupole term. The viscous momentum equation, in the absence of external forces, can be written (Landau and Lifshitz, 1953)

$$\partial(\rho v_i)/\partial t + \partial(\rho v_i v_j)/\partial x_j + \partial p/\partial x_i = \partial\sigma_{ij}/\partial x_j , \tag{6}$$

where $\sigma_{ij}$ are the viscous stress. Equation (6) can be written in the form (4), with $F_i = 0$, and $T_{ij}$ given by

$$T_{ij} = \rho v_i v_j + (p - c_0^2\rho)\delta_{ij} - \sigma_{ij} ; \tag{7}$$

if there are no mass sources, $Q=0$ in Eq. (3), and Eq. (5) simplifies to

$$\Box^2\rho = \partial^2 T_{ij}/\partial x_i \partial x_j , \tag{8a}$$

$$\Box \equiv \partial^2/\partial t^2 - c_0^2 \partial^2/\partial x_i^2 , \tag{8b}$$

where $\Box$ is the acoustic wave operator (8b), and according to (8a), the model source of sound is the Lighthill (1952) stress tensor $T_{ij}$ [Eq. (7)]. It consists of the following terms: (i) the leading contribution is the Reynolds's stress $\rho v_i v_j$ (or convective momentum flux), modeling the generation of sound by turbulence; (ii) the second term would vanish for a linear, homentropic perturbation, since the equation of state $p = p(\rho,s)$ implies $\nabla p - c_0^2 \nabla\rho = 0$ with $c_0^2 \equiv (\partial p/\partial\rho)_s$, so that, in general, it models the generation of sound by fluid inhomogeneities and has a dipole character, since

$$\partial[(p - c_0^2\rho)\delta_{ij}]/\partial x_j = \partial(p - c_0^2\rho)/\partial x_i ; \tag{9}$$

(iii) the last term, representing the viscous stresses,

$$\sigma_{ij} = \eta_1(\partial v_i/\partial x_j + \partial v_j/\partial x_i) + \eta_2(\partial v_k/\partial x_k)\delta_{ij} , \tag{10}$$

where $\eta_1, \eta_2$ denote, respectively, the incompressible and compressible kinematic viscosities, is linear in the rates of deformation, and models the dissipation of sound.

## 4. Radiation field

The solution of Eq. (8a) is given by the classical Kirchhoff integral:

$$\rho(\mathbf{x},t) = (4\pi c_0^2)^{-1} \partial^2/\partial x_i \partial x_j \times \int_D |\mathbf{x}-\mathbf{y}|^{-1} T_{ij}(\mathbf{y}, t - |\mathbf{x}-\mathbf{y}|/c_0) d^3y , \tag{11}$$

which specifies the density perturbation $\rho$, observed at a position $\mathbf{x}$, due to a distribution of model quadrupole sources $T_{ij}(\mathbf{y},\tau)$, at positions $\mathbf{y}$ in a region $D$; note that the sources are evaluated at the "retarded" time of emission $\tau \equiv t - |\mathbf{x}-\mathbf{y}|/c_0$, differing from the instant of reception $t$ by the time $|\mathbf{x}-\mathbf{y}|/c_0$ taken by sound to travel, at speed $c_0$, the distance $|\mathbf{x}-\mathbf{y}|$ from source $\mathbf{y}$ to observer $\mathbf{x}$. For an observer in the far field, the derivatives $\partial^2/\partial x_i \partial x_j$ may be applied only to the source $T_{ij}$ (and *not* to $|\mathbf{x}-\mathbf{y}|^{-1}$, which would lead to asymptotically negligible terms like $|\mathbf{x}-\mathbf{y}|^{-2}$), so that we obtain

$$\rho(\mathbf{x},t) \sim (4\pi c_0^4)^{-1} x_i x_j r^{-3} \times \int_D [\partial^2 T_{ij}(\mathbf{y}, t - |\mathbf{x}-\mathbf{y}|/c_0)/\partial t^2] d^3 y \;, \tag{12}$$

where we have replaced $x_i - y_i$ and $|\mathbf{x}-\mathbf{y}|$ by $x_i$ and $r \equiv |\mathbf{x}|$, respectively. From Eq. (12) we can deduce a dimensional scaling law, by noting that $T_{ij} \sim \rho_0 v_0^2$, where $\rho_0, v_0$ are the mean density and velocity, the integration over a "compact" source region is equivalent to multiplication by its volume $l^3$ (where $l$ is the length scale), and derivation with regard to time corresponds to multiplication by the Strouhal (1878) frequency $\partial/\partial t \sim \omega \sim v_0/l$. Thus the acoustic density perturbation due to a quadrupole source scales as the fourth power of velocity:

$$\rho_Q(\mathbf{x},t) \sim \rho_0 (l/r)(v_0^4/c_0^4) \sim \rho_0 (l/r) M_0^4 \;, \tag{13}$$

through the Mach (1926) number $M_0 \equiv v_0/c_0$, i.e., its ratio to the sound speed. The intensity of acoustic radiation,

$$W_Q \sim (c_0^3/\rho_0)|\rho(\mathbf{x},t)|^2 \sim \rho_0 (l/r)^2 M_0^5 v_0^3 \sim \rho_0 (l/r)^2 c_0^{-5} v_0^8 \;, \tag{14}$$

satisfies, for quadrupoles, the famous (Lighthill, 1954) eighth-power law on mean turbulent velocity.

## 5. Effect of solid boundaries

The presence of solid boundaries can substantially affect the generation of sound by turbulence; for example, the sound from flow in a wind tunnel is considerably amplified if there is a loose panel, to the extent that the latter may produce more noise than the flow itself. In the presence of reflecting boundaries, the Kirchhoff solution has, in addition to the volume integral [Eq. (11)], a surface integral (Curle, 1955; Lighthill, 1961); for the purpose of comparison with the case of turbulence in free space, viz.,

$$\rho(\mathbf{x},t) \sim (4\pi c_0^2)^{-1} \partial/\partial x_j \int_D |\mathbf{x}-\mathbf{y}|^{-1} [\partial T_{ij}(\mathbf{y}, t - |\mathbf{x}-\mathbf{y}|/c_0)/\partial y_j] d^3\mathbf{y} \;, \tag{15}$$

we note that, in the presence of a boundary surface $S$, application of the divergence theorem to Eq. (15) leads to the surface integral

$$\rho(\mathbf{x},t) \sim (4\pi c_0^2)^{-1} \partial/\partial x_i \int_S |\mathbf{x}-\mathbf{y}|^{-1} [n_j T_{ij}(\mathbf{y}, t - |\mathbf{x}-\mathbf{y}|/c_0)] dS \;, \tag{16}$$

where $\mathbf{n}$ is the normal to $S$. Comparing Eq. (16) with the case of a dipole source, $\partial F_i/\partial x_i$ in Eq. (5), we conclude that

$$P_i(\mathbf{y},\tau)|_S = n_j T_{ij}(\mathbf{y},\tau) \;, \tag{17a}$$

$$\tau \equiv t - |\mathbf{x}-\mathbf{y}|/c_0 \;. \tag{17b}$$

The turbulent stresses $T_{ij}$ induce on the solid surface a stress force $P_i$, which acts as a dipole source of sound. Applying to Eq. (16) a dimensional scaling similar to the quadrupole case [Eqs. (13) and (14)], we conclude that for a dipole, the acoustic density perturbation (18a) scales as the cube of the Mach number:

$$\rho_D(\mathbf{x},t) \sim \rho_0 (l/r) M_0^3 \;, \tag{18a}$$

$$W_D \sim \rho_0 (l/r)^2 M_0^3 v_0^3 \sim \rho_l (l/r)^2 c_0^{-3} v_0^6 \;, \tag{18b}$$

and the energy flux (18b) as the *sixth* power of velocity.

## 6. Two-phase flow

The presence of gas bubbles in a liquid can significantly increase the generation of sound, e.g., when water flows out of tap, there is a louder noise if there is a significant fraction of entrained air. Two-phase flow can radiate noise as a monopole, due to the change in the volume occupied by one of the phases. The acoustic field due to a monopole $Q$ is given by the Kirchhoff integral

$$\rho(\mathbf{x},t) = (4\pi c_0^2)^{-1} \partial/\partial t \int_D |\mathbf{x}-\mathbf{y}|^{-1} Q(\mathbf{y},\tau) d^3 y \;, \tag{19}$$

which simplifies, for an observer in the far field, to

$$\rho(\mathbf{x},t) = (4\pi c_0^2)^{-1} r^{-1} \int_D [\partial Q(\mathbf{y},\tau)/\partial \tau] d^3 y \;. \tag{20}$$

Bearing in mind that the mass flux scales as $Q \sim \rho_0 v_0/l$, the acoustic density perturbation (21a) due to a monopole scales as the square of the Mach number,

$$\rho_M(\mathbf{x},t) \sim \rho_0 (l/r) M_0^2 \;, \tag{21a}$$

$$W_M \sim \rho_0 (l/r)^2 M_0 V_0^3 \sim \rho_0 (l/r)^2 c_0^{-3} v_0^4 \;, \tag{21b}$$

and the radiation intensity (21b) as the *fourth* power of velocity. Comparing monopoles [Eqs. (21a) and (21b)] with dipoles [Eqs. (18a) and (18b)] and quadrupoles [Eqs. (13) and (14)], it follows that

$$\rho_Q(\mathbf{x},t) \sim M_0 \rho_D(\mathbf{x},t) \sim M_0^2 \rho_M(\mathbf{x},t) \;, \tag{22}$$

$$W_Q(\mathbf{x},t) \sim M_0^2 W_D(\mathbf{x},t) \sim M_0^4 W_M(\mathbf{x},t) \;, \tag{23}$$

for the generation of sound by disturbances of low Mach number $M_0^2 \ll 1$; the acoustic fields become weaker by a factor of $M_0$, and the intensity by a factor $M_0^2$.

### 7. A hierarchy of sources

Equation (23) establishes a hierarchy of sound sources: (i) the most effective is the monopole, e.g., cavitation noise, which is equivalent to a volume change, like a sphere pulsating radially; (ii) if there are no monopoles, the most effective radiators are dipoles, e.g., forces acting upon inhomogeneous patches of fluid, in analogy with a sphere building out one hemisphere and contracting the other, so that there is no net volume change, but a nonzero force results; (iii) if there are no monopoles or dipoles, i.e., volume is conserved and forces balance, then the sources are quadrupoles, associated with internal stresses, e.g., turbulence, in analogy with a sphere pulsating in opposition in alternate quarters, so as to produce no volume change or force, leaving only stresses. This hierarchy, for sound sources in free space, is modified by the presence of solid boundaries, which always radiate as dipoles associated with the induced force. Thus the following cases arise: (i) a monopole source is *not* affected by the presence of boundaries, as the latter are dipoles which are negligible, by comparison, in the far field; (ii) for a dipole source, the presence of a solid boundary adds another dipole, i.e., gives a comparable contribution; (iii) for a quadrupole source, the introduction of a boundary enhances sound emission to dipole level, so that the far-field noise is due to the "induced" dipoles on the boundary, and direct source emission is negligible by comparison.

### 8. Combustion and jet noise

We compare the predictions of the theory of aerodynamic sound with experiments, for (i) monopoles and (ii) quadrupoles, leaving for more detailed consideration later (Sec. II.C) the case of dipole sources. For the quadrupole case, the radiation law [Eq. (14)] involves the eighth power of the flow velocity and is consistent [Fig. 2(a)] with the total acoustic power output measured for small (half-inch) air jets (Waterhouse and Berendt, 1958). For the monopole case, the calculation of the noise emitted by a flame, modeled as an assemblage of monopoles expanding in accordance with the rate of burning of the gas, also shows good agreement with experimental measurements [Fig. 2(b)].

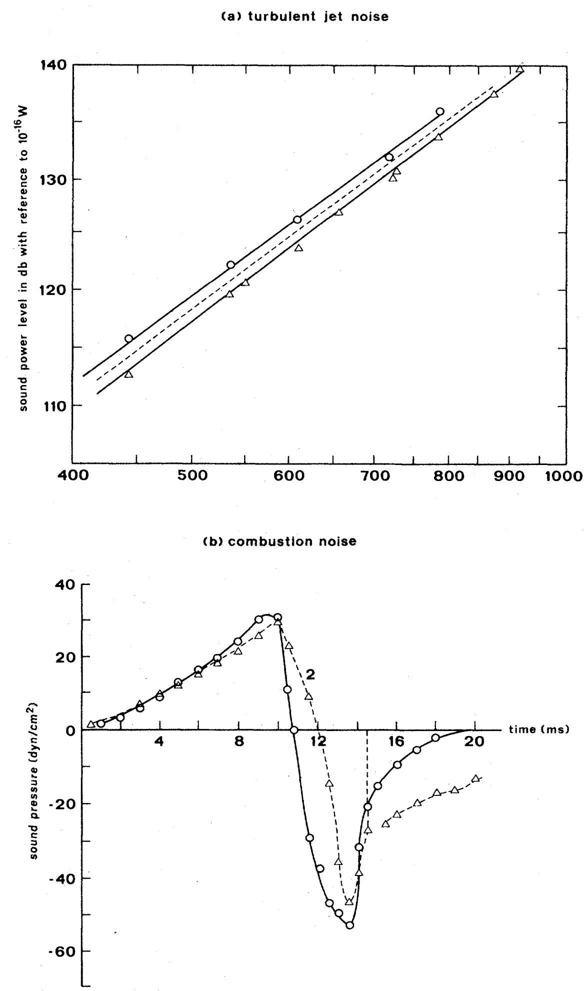


FIG. 2. Comparison of theory and experiment on aerodynamic sound: (a) comparison (Lighthill, 1961) of the eighth-power law (Lighthill, 1952) of velocity [Eq. (14)] for the total acoustic intensity of a quadrupole source (straight line) with noise measurements (Waterhouse and Berendt, 1958) for air jets of half-inch diameter, equal thrusts, and two types of nozzle: ○, cylindrical and △, oblong; (b) comparison of the acoustic pressure, plotted as a function of time, as measured for combustion noise (dotted line), with the prediction (solid line), based on modeling the flame as an assembly of monopoles, with volume changes determined by the rate of burning of gas (Hurle, Price, Sugden, and Thomas, 1968). For dipole sources, see Fig. 3.

## B. Wave equations in moving media

The original form of the "acoustic analogy" is based on the classical wave equation, with forcing terms modeling the generation of sound by dynamic disturbances, contained in a region $D$, in a fluid otherwise at rest, e.g., the noise of a flame in a quiescent medium. There are other situations, also of practical interest, in which sound is generated in a medium moving nonuniformly, e.g., noise production in a jet; in these cases the classical wave equation needs to be generalized to account for the acoustic effects of the presence of a mean flow, generally nonuniform. The generalization of the classical wave equation for three-dimensional sound, in a medium at rest (Poisson, 1807), to a flow of low (Taylor, 1978) or high Mach number (Howe, 1975a; Campos, 1978a) leads to the convected and high-speed wave equations; these are deduced for an inhomogeneous fluid, and thus include, as particular cases, the wave equations for quasi-one-dimensional sound propagation in nonuniform ducts, either horns (Rayleigh, 1916; Webster, 1919), or low-speed (Campos, 1984b) or high-speed (Huerre and Karamcheti, 1973; Lumsdaine and Ragab, 1977) nozzles, which are progressive generalizations of the one-dimensional wave equation (d'Alembert, 1747). All these linear wave equations can be deduced from an acoustic variational principle (Campos, 1985b), which extends those of classical acoustics in free space (Levine, 1978) and in horns (Weibel, 1955) to include the effects of potential mean flow, of arbitrary velocity.

### 1. Acoustic Lagrangian

A variational principle can be formulated for vortical flows (Seeliger and Whitham, 1968) in terms of Clebsch (1857) potentials, which are integral, nonlocal properties of the flow; since the acoustics of vortical flows has been the subject of a recent review (Möhring, Müller, and Obermeier, 1983), we shall concentrate on potential flows, for which the variational principle (Bryan, 1918; Bateman, 1929) takes a local form. We choose as variable the potential $\varphi(\mathbf{x},t)$, in terms of which the acoustic velocity $\mathbf{v}$ and pressure $p$ are given, respectively, by

$$\mathbf{v}=\nabla\varphi \,, \tag{24a}$$

$$p=-\rho_0 d\varphi/dt \,. \tag{24b}$$

Equation (24a) for the velocity is a consequence of assuming irrotational (mean and acoustic) flow $\nabla\times\mathbf{v}=0$; from the equation of momentum (e.g., in Bernoulli's form), it follows (24b) that the pressure is proportional to the material derivative of the potential:

$$d\varphi/dt\equiv\partial\varphi/\partial t+\mathbf{v}_0\cdot\partial\varphi/\partial\mathbf{x}\equiv\dot{\varphi}+\mathbf{v}\cdot\nabla\varphi \,. \tag{25}$$

Thus the kinetic $E_v$ and compression $E_p$ energies, per unit volume, are given, respectively, by

$$E_v=\tfrac{1}{2}\rho_0 v^2=\tfrac{1}{2}\rho_0(\nabla\varphi)^2 \,, \tag{26a}$$

$$E_p=p^2/2\rho_0 c_0^2=(\rho_0/2c_0^2)(d\varphi/dt)^2 \,, \tag{26b}$$

for a linear acoustic wave, leading to an acoustic Lagrangian

$$\begin{aligned}\mathscr{L}(\dot{\varphi},\nabla\varphi;\mathbf{x})&=E_v-E_p\\&=\tfrac{1}{2}\rho_0[(\nabla\varphi)^2-c_0^{-2}(d\varphi/dt)^2] \,,\end{aligned} \tag{27}$$

which is bilinear in the temporal $\dot{\varphi}\equiv\partial\varphi/\partial t$ and spatial $\nabla\varphi\equiv\partial\varphi/\partial\mathbf{x}$ derivatives of the potential $\varphi$, and may depend explicitly on position $\mathbf{x}$, for a fluid of nonuniform density $\rho_0(\mathbf{x})$ or sound speed $c_0(\mathbf{x})$, or a nonuniform mean flow of velocity $v_0(\mathbf{x})$, appearing in the material derivative (25).

### 2. Inhomogeneous classical equation

The variational principle applies, in general (Bolza, 1904; Caratheodory, 1935), to a Lagrangian that may depend on the potential $\varphi$ and its space $\nabla\varphi$ and time $\dot{\varphi}$ derivatives, and also, explicitly, on position $\mathbf{x}$ and time $t$; it requires that the acoustic action, defined as the integral of the Lagrangian over space-time, be stationary,

$$d\int\mathscr{L}(\varphi,\dot{\varphi},\nabla\varphi;\mathbf{x},t)d^3x\,dt=0 \,, \tag{28}$$

and leads (Forsyth, 1926; Pars, 1960; Esgolts, 1970) to the Euler-Lagrange equation, in the form

$$\partial(\partial\mathscr{L}/\partial\dot{\varphi})/\partial t+\nabla\cdot[\partial\mathscr{L}/\partial(\nabla\varphi)]=0 \,, \tag{29}$$

which, on substitution of the Lagrangian, becomes the wave equation. A simple example is the case of a medium at rest, for which the acoustic Lagrangian

$$\mathscr{L}_0(\dot{\varphi},\nabla\varphi;\mathbf{x})=\tfrac{1}{2}\rho_0[(\nabla\varphi)^2-c_0^{-2}\dot{\varphi}^2] \tag{30}$$

leads, through Eq. (29), to the wave equation

$$c_0^2\nabla^2\varphi-\ddot{\varphi}+c_0^2\nabla\varphi\cdot\nabla\ln\rho_0=O(M_0,(\partial\varphi)^2) \,, \tag{31}$$

where the right-hand side reminds us that nonlinear terms $[(\partial\varphi)^2\equiv\dot{\varphi}^2,(\nabla\varphi)^2,\dot{\varphi}\,|\,\nabla\varphi\,|$, etc.] have been neglected and mean-flow effects omitted $O(M_0)$. The linear wave equation in a medium at rest [Eq. (31)] coincides with the classical wave equation (first two terms) for a fluid of constant density $\rho_0\sim$const, and adds an extra term (the third) in the presence of density stratification $\nabla\rho_0\neq 0$.

### 3. Horn wave equation

The inhomogeneous term [last on the left-hand side of Eq. (31)] will be considered subsequently (Sec. V.A) in connection with sound propagation in an atmosphere; we illustrate the meaning of this term by considering quasi-one-dimensional sound propagation in a duct of varying cross section $A(x)$, i.e., the fundamental, longitudinal acoustic mode. This is described in the absence of mean flow, i.e., for a *horn*, by the one-dimensional form of Eq. (3.1), viz., $\nabla^2\varphi$ becomes $\varphi''\equiv\partial^2\varphi/\partial x^2$, where the density per unit volume $\rho_0$ is replaced by the density per unit length $\rho_0 A$, so that $\nabla\ln\rho_0$ becomes $[\ln(\rho_0 A)]'=A'/A$, for a homogeneous fluid $\rho_0\sim$const; thus Eq. (31) includes, as particular cases for fluids of constant density, both the classical wave equation in free space,

$$c_0^2\nabla^2\varphi-\ddot{\varphi}=O(M_0,(\partial\varphi)^2,\nabla\ln\rho_0) \,, \tag{32}$$

and the horn wave equation (Rayleigh, 1916; Webster, 1919),

$$c_0^2[\varphi''+(A'/A)\varphi']-\ddot{\varphi}=O(M_0,(\partial\varphi)^2) \,. \tag{33}$$

Equation (33) can also be deduced from the one-dimensional Euler-Lagrange equation (29), viz.,

$$\partial(\partial\mathscr{L}^*/\partial\dot{\varphi})/\partial t+\partial(\partial\mathscr{L}^*/\partial\varphi')/\partial x=0 \,, \tag{34}$$

using the duct Lagrangian

$$\mathscr{L}_0^*(\dot{\varphi},\varphi')=\tfrac{1}{2}\rho_0 A(\varphi'^2-c_0^{-2}\dot{\varphi}^2)\ . \tag{35}$$

The latter $\mathscr{L}^*$ is the Lagrangian per unit length of duct, and thus equals $\mathscr{L}^*=A\mathscr{L}$, viz., the cross-sectional area $A(x)$ times the Lagrangian $\mathscr{L}$ per unit volume [Eq. (30)].

### 4. Convected wave equation

Consider the effects of a nonuniform mean flow of velocity $v_0(\mathbf{x})$ on three-dimensional sound, in the case of low Mach number $M_0^2\equiv v_0^2/c_0^2 \ll 1$, when the Lagrangian [Eqs. (27) and (25)] simplifies to

$$\mathscr{L}_1(\dot{\varphi},\nabla\varphi;\mathbf{x})=\mathscr{L}_0(\dot{\varphi},\nabla\varphi;\mathbf{x})-\rho_0 c_0^{-2}\dot{\varphi}(\mathbf{v}_0\cdot\nabla\varphi)\ . \tag{36}$$

This case differs from the case $\mathscr{L}_0$ of a fluid at rest [Eq. (30)] by the presence of the last term on the right-hand side of Eq. (36), which is small $O(M_0)$ but *not* negligible. Substituting in the Euler-Lagrange equation (29), we obtain the wave equation for linear sound in an inhomogeneous, low-Mach-number potential flow:

$$c_0^2\nabla^2\varphi-\ddot{\varphi}-2v_0\cdot\nabla\dot{\varphi}+c_0^2\nabla\varphi\cdot\nabla\ln\rho_0 +\dot{\varphi}\mathbf{v}_0\cdot\nabla\ln c_0^2=O((M_0+\partial\varphi)^2)\ . \tag{37}$$

The homogeneous terms (first three) consist of the classical wave equation [first two, Eq. (32)], plus an effect $-2\mathbf{v}_0\cdot\nabla\dot{\varphi}$ of convection by the mean flow; in the case of an inhomogeneous fluid there are two extra terms, a static one (fourth term) associated with nonuniform density $\rho_0$ and a convected term (fifth) involving a nonuniform sound speed $c_0$. If the fluid is homogeneous, both $\rho_0$ and $c_0$ are constant, for a nonuniform flow $\mathbf{v}_0(\mathbf{x})$ of low Mach number, and Eq. (37) reduces to the first three terms, which may be written as the convected wave equation

$$c_0^2\nabla^2\varphi-d^2\varphi/dt^2=O((M_0+\partial\varphi)^2,\nabla\ln\rho_0,\nabla\ln c_0)\ . \tag{38}$$

This equation is similar to the classical wave equation (32), replacing local time derivatives $\ddot{\varphi}\equiv\partial^2\varphi/\partial t^2$ by material derivatives $d^3\varphi/dt^2$, which include [Eq. (25)] the effect of linear convection of sound by the mean flow $\mathbf{v}_0\cdot\nabla\varphi$.

### 5. Duct wave operator

For the fundamental, longitudinal acoustic mode in a low-Mach-number nozzle, i.e., a duct of varying cross section $A(x)$ containing the low-Mach-number flow of an homogeneous fluid, we may, with equivalent results, either transform the three-dimensional wave equation (37) or apply the one-dimensional Euler-Lagrange equation (34) to the duct Lagrangian

$$\mathscr{L}_1^*(\dot{\varphi},\varphi')=\mathscr{L}_0(\dot{\varphi},\varphi')-A\rho_0 c_0^{-2}v_0\dot{\varphi}\varphi'\ , \tag{39}$$

which differs from the case $\mathscr{L}_0^*$ of a horn (35), by the ad dition of a low-Mach-number convection effect [last term in Eq. (39)]. The wave equation for linear sound in a low-Mach-number nozzle,

$$c_0^2\varphi''-\ddot{\varphi}-2v_0\dot{\varphi}'+c_0^2(A'/A)\varphi'=O((\partial\varphi+M_0)^2)\ , \tag{40}$$

consists of the classical wave operator (first two terms) for a medium at rest [Eqs. (32)], with a mean flow effect (third term), as in the convected wave equation (37), and a scattering term due to changes in cross-sectional area (third term), as for a horn [Eq. (33)]. The low-Mach-number nozzle wave equation (40) can be written in the compact form

$$c_0^2A^{-1}(A\varphi')'-d^2\varphi/dt^2=O((M_0+\partial\varphi)^2)\ , \tag{41}$$

which can be obtained by transforming the classical wave equation (32) as follows: (i) in the first term, the Laplacian $\nabla^2\varphi$ for three-dimensional sound in free space is replaced by the duct operator $A^{-1}(A\varphi')'$ for quasi-one-dimensional propagation in a horn of varying cross section $A'\neq 0$; (ii) in the second term, the local time derivatives $\ddot{\varphi}\equiv\partial^2\varphi/\partial t^2$ for a medium at rest are replaced by material derivatives $d^2\varphi/dt^2$, including the effect of linear convection by the mean flow.

### 6. High-speed wave equation

In the case of a high-speed flow, i.e., of Mach number of order unity, the mass density $\rho_0$ and sound speed $c_0$ depend on the flow velocity $\mathbf{v}_0$ and thus are *not* constant in a nonuniform flow $\mathbf{v}_0(\mathbf{x})$, even if the fluid is homogeneous at rest. The complete acoustic Lagrangian [Eqs. (27) and (25)],

$$\mathscr{L}(\dot{\varphi},\nabla\varphi;\mathbf{x})=\mathscr{L}_1(\dot{\varphi},\nabla\varphi;\mathbf{x})-\tfrac{1}{2}\rho_0c_0^{-2}(\mathbf{v}_0\cdot\nabla)^2\varphi\ , \tag{42}$$

adds to the low-Mach-number $\mathscr{L}_1$ form (36) an extra term [the last on the right-hand side of Eq. (42)], of $O(M_0^2)$. Substitution of Eq. (42) into Eq. (29) yields the wave equation for linear sound in a steady potential flow of arbitrary velocity:

$$c_0^2\nabla^2\varphi-\ddot{\varphi}-2\mathbf{v}_0\cdot\nabla\dot{\varphi}-(\mathbf{v}_0\cdot\nabla)^2\varphi+c_0^2\nabla\varphi\cdot\nabla\ln\rho_0 +(\dot{\varphi}+\mathbf{v}_0\cdot\nabla\varphi)\mathbf{v}_0\cdot\nabla\ln c_0^2=O((\partial\varphi)^2)\ , \tag{43}$$

which differs from the low-Mach-number case (37) only by the addition of two terms, $-(\mathbf{v}_0\cdot\nabla)^2\varphi$ and $(\mathbf{v}_0\cdot\nabla\varphi)\mathbf{v}_0\cdot\nabla\ln c_0^2$, both of order $M_0^2$. The high-speed wave equation (43) may be written in the compact form

$$\nabla^2\varphi-d(c_0^{-2}d\varphi/dt)/dt+\nabla\varphi\cdot\nabla\ln\rho_0=O((\partial\varphi)^2)\ , \tag{44}$$

which coincides with the classical wave equation (32) in the first term only. In the second term, $c_0^{-2}\partial^2\varphi/\partial t^2$, local derivatives are replaced by material derivatives (25), as for low-Mach-number convection (38), with the additional difference that the sound speed $c_0(\mathbf{x})$ is *not* uniform, and $c_0^{-2}$ appears *between* the material derivatives; there is, in addition, a new term [the last in (44)], which for a homogeneous fluid is $O(M_0^2)$, and which only appears for high-speed flow.

### 7. High-Mach-number nozzle

In the case of linear sound in a nozzle containing irrotational flow of arbitrary velocity, the complete duct Lagrangian,

$$\mathscr{L}^*(\dot{\varphi},\varphi';x)=\mathscr{L}_1^*(\dot{\varphi},\varphi')-\tfrac{1}{2}\rho_0 c_0^{-2}(v_0^2\varphi''+v_0 v_0'\varphi') , \quad (45)$$

differs from $\mathscr{L}_1^*$, the low-Mach-number version [Eq. (39)], in having two extra $O(M_0^2)$ terms, one involving the mean flow velocity $v_0$ only, and the other the rate of strain $v_0'$, which does not vanish, because changes in cross-sectional area imply a nonuniform velocity. Substitution of Eq. (45) into Eq. (34) yields

$$c_0^2\varphi''-\ddot{\varphi}-2v_0\dot{\varphi}'-v_0^2\varphi''-2v_0v_0'\varphi'+c_0^2(A'/A)\varphi' +2M_0c_0'(\dot{\varphi}+v_0\varphi')=O((\partial\varphi)^2) , \quad (46)$$

where the new, high-Mach-number terms, not appearing in Eq. (40), are the third, fourth, and sixth. The high-speed-nozzle wave equation (46) can be written [compare with Eq. (44)] in the compact form

$$A^{-1}(A\varphi')'-d(c_0^{-2}d\varphi/dt)/dt+\varphi'(\rho_0'/\rho_0)=O((\partial\varphi)^2) , \quad (47)$$

which includes all the modifications, relative to the classical wave equation (32), that may be needed for linear sound in steady potential flows, viz., from left to right, (i) replacing the Laplacian $\nabla^2\varphi$ by the duct operator $A^{-1}(A\varphi')=\varphi''+(A'/A)\varphi'$; (ii) substituting local $\partial/\partial t$ by material $d/dt$ derivatives (25), and putting $c_0^{-2}$ between them, if the sound speed $c_0$ is nonuniform; (iii) adding a term $\varphi'(\rho_0'/\rho_0)$ in the case of either a nonhomogeneous fluid (even at rest), or a high-speed nonuniform flow (even if the fluid is homogeneous).

### 8. Alternate wave equations

We give in Table II(b) a list of the various forms of the linear wave equation (Sec. II.B), to be compared below (Sec. V.B) with Table III, the nonlinear case. We also indicate in Table II(a) the formulas for the bilinear acoustic Lagrangian [Eqs. (25) and (27)]. These are given for two cases: (i) for three-dimensional sound, in an inhomogeneous fluid, in free space, and (ii) for quasi-one-dimensional sound, in a homogeneous fluid, in a duct of varying cross section. In each case a Lagrangian is shown for three types of medium—steady potential flow of arbitrary velocity, low-Mach-number convection, and medium at rest—making in all a total of six combinations.

## C. Analogy of hydrodynamic and electromagnetic forces

In order to apply the wave analogy in a reliable manner, it is necessary (i) to know a linear equation describing the propagation of waves in the medium under consideration; (ii) to establish an *exact* equation, coinciding with the wave equation (i) in the linear terms, whose nonlinear terms may be interpreted as model sources forcing the generation of waves. The original "acoustic analogy" met these requirements by using (i) the classical wave equation for sound in a homogeneous fluid at rest, forced by (ii) the Lighthill tensor, whose leading term is nonlinear. In order to consider the generation of sound in an inhomogeneous flow, we take the same two steps in a generalized form: (i) the wave equation, for linear sound in an inhomogeneous potential flow of arbitrary velocity, has been derived using a variational principle (Sec. II.B); (ii) using an independent method, viz., elimination between the equations of motion, we establish an exact equation, coinciding with the high-speed wave operator in the linear terms, with extra forcing terms representing the generation of sound by vorticity (Powell, 1961,1964; Howe, 1976a) and inhomogeneities (Morfey, 1973; Ffowcs-Williams and Howe, 1975; Marble and Candel, 1977). These mechanisms are relevant to the noise of nozzle flows (Howe, 1979a) and of vorticity in free space (Möhring, 1978a), or shed from a sharp body (Howe, 1978) or blunt bodies (Blevins, 1984) or slots (Howe, 1979b). If, in addition to pressure forces, electric and

TABLE II. Linear acoustics of media at rest, in low-Mach-number convection and in high-speed jets, both for three-dimensional sound in inhomogeneous flow and for quasi-one-dimensional sound in a duct of varying cross section. For each of these six combinations, we indicate (a) the quadratic acoustic Lagrangian for linear sound; (b) the linear wave equation, describing sound of small amplitude.

| Case | Three-dimensional sound in inhomogeneous fluid | Quasi-one-dimensional sound in variable-area duct |
|---|---|---|
| | (a) | |
| Medium | | |
| At rest | $\mathscr{L}_0$, Eq. (30) | $\mathscr{L}_0^*$, Eq. (35) |
| Low-Mach-number convection | $\mathscr{L}_1$, Eq. (36) | $\mathscr{L}_1^*$, Eq. (39) |
| High-speed, steady potential flow | $\mathscr{L}$, Eq. (42) | $\mathscr{L}^*$, Eq. (45) |
| | (b) | |
| Medium | | |
| At rest | Eq. (31) | Eqs. (33) |
| Low-Mach-number convection | Eqs. (37) and (38) | Eqs. (40) and (41) |
| High-speed, steady potential flow | Eqs. (43) and (44) | Eqs. (46) and (47) |

magnetic fields are included, the "wave analogy" leads to a close correspondence (Campos, 1978a) between the hydrodynamic and electromagnetic forces, appearing as dipole sources of sound.

### 1. Stagnation enthalpy

In order to describe the generation of sound in a steady flow, we need to choose a wave variable $q(\mathbf{x},t)$ that has a simple acoustic meaning in the far field; also, since fluid inhomogeneities appear as nonuniformities in the distribution of energy in the flow, the wave variable should be related to the energy balance. The energy equation in a fluid (Landau and Lifshitz, 1953) involves, in the convected flux, the stagnation enthalpy $q$, which is generally (Landau and Lifshitz, 1967b) given by

$$dq=\mathbf{V}\cdot d\mathbf{V}+\Gamma^{-1}dP+T\,ds+\mu\,dN+(\mathbf{E}\cdot d\mathbf{D}+\mathbf{B}\cdot dH)/\Gamma\ , \tag{48}$$

where the first term is the kinetic energy per unit mass $\frac{1}{2}V^2$, with $\mathbf{V}$ the total (mean flow plus acoustic) velocity, and the remaining terms specify the enthalpy, as follows: (i) $\Gamma^{-1}dP$, with $\Gamma,P$ the total mass density and pressure, respectively, is the Legendre transform of the work; (ii) $T\,ds$, with $T$ the temperature and $s$ the entropy density, is the heat; (iii) $\mu\,dN$, with $N$ the mole number and $\mu$ the chemical affinity, is the chemical energy; (iv) $\mathbf{E}\cdot d\mathbf{D}$, $\mathbf{B}\cdot d\mathbf{H}$ are, respectively, the electric and magnetic energy (per unit volume), with $\mathbf{E},\mathbf{D}$ the electric field and displacement, and $\mathbf{B},\mathbf{H}$ the magnetic induction and field. We have used upper-case letters to designate total fluid variables, i.e., mean flow plus acoustic perturbation, and lower-case letters to designate the acoustic perturbation alone [see Eq. (254)]. We may divide the domain of the flow into two regions, namely, (i) a source region $D$, occupied by ionized inhomogeneities, with composition, density, and temperature possibly distinct from the fluid's and acted upon by electromagnetic forces; (ii) outside $D$, an irrotational, homogeneous, nonionized flow, hence homentropic, for which the Bernoulli equation (Curle and Davies, 1968), holds

$$\text{const}\equiv\tfrac{1}{2}V^2+\dot{\Phi}+\int\Gamma^{-1}dP=q+\dot{\varphi}\ , \tag{48'}$$

showing that the stagnation enthalpy $q$ scales as minus the time derivative of the total potential $\Phi$, and that for a steady mean flow $\dot{\Phi}=\dot{\varphi}$, where $\varphi$ is the acoustic potential. Since $q$ is associated with the presence of inhomogeneities in the source region $D$, and reduces to an acoustic variable outside $D$, we choose it as the wave variable.

### 2. Forced wave equation

Rewriting the fundamental equations of fluid mechanics in terms of the velocity $\mathbf{V}$ and stagnation enthalpy $q$ (instead of density $\Gamma$ or pressure $P$), and eliminating for the latter, leads to the exact equation (Howe, 1975a; Campos, 1978a),

$$\begin{aligned}d(c_0^{-2}dq/dt)/dt-(\rho_0c_0^2)^{-1}\nabla P\cdot\nabla q-\nabla^2 q\\=(\rho_0c_0^2)^{-1}\nabla P\cdot\mathbf{A}-\nabla\cdot\mathbf{A}+\Lambda+d(\mathbf{F}\cdot\mathbf{v}/\rho_0c_0^2)/dt\ ,\end{aligned} \tag{49}$$

where all the terms on the right-hand side vanish outside $D$; exterior to $D$ $q=-\partial\varphi/\partial t$ and

$$(\rho_0c_0^2)^{-1}\nabla p_0=\rho_0^{-1}\nabla\rho_0=\nabla(\ln\rho_0)\ ,$$

and the left-hand side of Eq. (49) coincides with the high-speed wave equation (44). Thus we may interpret the terms on the right-hand side as the sources of waves, which are concentrated in fluid inhomogeneities (blobs) and consist of (i) a monopole term

$$\Lambda\equiv(\gamma_0^{-1}-\gamma_1^{-1})[d(\ln p_0)/dt](\mathbf{n}\cdot\mathbf{v})\ , \tag{50}$$

corresponding to blobs of ratio of specific heats $\gamma_1$, distinct from the fluid's $\gamma_0$, when subjected to a mean flow pressure gradient $\nabla p_0$, which causes a variation in the normal velocity $\mathbf{n}\cdot\mathbf{v}$ (positive/negative, respectively, for an expansion/contraction); (ii) a dipole term

$$\mathbf{A}=\mathbf{F}/\rho_1+\mathbf{G}/\rho_0\ , \tag{51}$$

which is the sum of the electromagnetic force $\mathbf{F}$ per unit mass of a blob of density $\rho_1$, and the hydrodynamic force $\mathbf{G}$ per unit mass of displaced fluid of density $\rho_0$.

### 3. Electric and displacement forces

In a low-Mach-number mean flow, the high-speed wave operator (44) reduces to the convected wave operator (38), and the sole remaining source is the total force dipole (51), so that the wave equation

$$\nabla^2q-c_0^{-2}d^2q/dt^2=\nabla\cdot\mathbf{A}\ , \tag{52}$$

emphasizes the similar roles played by the two dipole sources, namely, the electromagnetic $\mathbf{F}$ and hydrodynamic $\mathbf{G}$ forces. It is well known in the classical theory of electricity (Jeans, 1908) and irrotational flow (Milne-Thomson, 1958) that the electrostatic field $\mathbf{E}$ due to electric charges $q_e$, in a dielectric of permeability $\varepsilon$,

$$\nabla\times\mathbf{E}=0\ , \tag{53a}$$

$$\nabla\cdot\mathbf{E}=4\pi q_e/\varepsilon\ , \tag{53b}$$

and the velocity $\mathbf{v}$, in an irrotational flow, due to volume sources $q_m$,

$$\nabla\times\mathbf{v}=0\ , \tag{54a}$$

$$\nabla\cdot\mathbf{v}=4\pi q_m\ , \tag{54b}$$

satisfy similar equations, i.e., the electric field lines are congruent to streamlines. The electric-to-dynamic analogy is extended from fields to forces by comparing the electric $\mathbf{F}_1$ with the displacement $\mathbf{G}_1$ forces

$$\mathbf{F}_1=q_e\mathbf{E}\ , \tag{55a}$$

$$\mathbf{G}_1=(\rho_0/\rho_1-1)\nabla p_0\ . \tag{55b}$$

The pressure gradient $\nabla p_0$ in Eq. (55b) is replaced in (55a) by the electric field $\mathbf{E}$, which is, by (53a), also a gradient, $\mathbf{E}=\nabla\varphi_e$, of the electrostatic potential $\varphi_e$; the electric charge $q_e$ is replaced by a dimensionless ratio of the densities of fluid $\rho_0$ and blob $\rho_1$, so that a positive $q_e>0$ (negative $q_e<0$) charge corresponds to a blob less dense $\rho_1<\rho_0$ (more dense $\rho_1>\rho_0$) than the fluid, i.e., the blob is acted upon by a force along (opposite to) the pressure gradient, as a positive (negative) charge moves along (opposite to) electric field lines. The analogy between electrostatics and irrotational flow extends from the fundamental equations to boundary conditions, e.g., an insulator corresponds to a rigid wall, and a moving wall with normal velocity $v_n$ corresponds to a conductor with surface charge density $4\pi\eta$. In the other classical analogy, between magnetostatics and incompressible flow, the correspondence of boundary conditions, e.g., the surface current on a conductor $(4\pi\mu/c_*)\,\mathbf{j}$ and a surface vorticity $\omega$, is less satisfactory, since it is not usually possible in a flow to specify the vorticity at the walls *a priori*; the specification of normal velocity is a different boundary condition from the tangential current.

## 4. Magnetic and vortical forces

The magnetic induction $\mathbf{B}$ due to electric currents $\mathbf{J}$, in a conductor of magnetic permeability $\mu$ (the speed of light *in vacuo* is denoted by $c_*$),

$$\nabla\times\mathbf{B}=(4\pi\mu/c_*)\mathbf{J}\,, \tag{56a}$$

$$\nabla\cdot\mathbf{B}=0\,, \tag{56b}$$

is analogous to the velocity $\mathbf{v}$ in an incompressible flow, due to vorticity $\boldsymbol{\omega}$,

$$\nabla\times\mathbf{v}=\boldsymbol{\omega}\,, \tag{57a}$$

$$\nabla\cdot\mathbf{v}=0\,, \tag{57b}$$

i.e., the magnetic induction lines curl around the electric currents, in shapes congruent with the streamlines of eddies. This analogy is extended from fields to forces, comparing the magnetic $\mathbf{F}_2$ and vortical $\mathbf{G}_2$ forces:

$$\mathbf{F}_2=(1/c_*)\mathbf{J}\times\mathbf{B}\,, \tag{58a}$$

$$\mathbf{G}_2=\rho_1\boldsymbol{\omega}\times\mathbf{v}\,, \tag{58b}$$

the latter being also known as Lamb's (1879) vector. The velocity $\mathbf{v}$ corresponds to the magnetic induction $\mathbf{B}$, and the vorticity $\boldsymbol{\omega}=\nabla\times\mathbf{v}$ to the electric current $\mathbf{J}$, which is also a curl [by the induction equation (56a)]. The mass density $\rho_1$ of the blob corresponds to $\mu/4\pi$, where $\mu$ is the magnetic permeability, in agreement with the analogy of dynamic $p_v$ and magnetic $p_m$ pressures

$$p_v=\tfrac{1}{2}\rho_1 v^2\,, \tag{59a}$$

$$p_m=\mu B^2/8\pi\,. \tag{59b}$$

Besides this last analogy, we have established a correspondence between the two terms [(55a) and (58a)] of the electromagnetic (or Laplace-Lorentz) $\mathbf{F}$ and the two terms [(55b) and (58b)] of the hydrodynamic (or Lamb) $\mathbf{G}$ forces.

## 5. Generalized Kirchhoff integral

Since the dipole sources of sound are forces, radiation occurs only when the blobs move across nonuniform fields. That is, (i) vorticity emits sound [Eq. (58)] only when eddies move across streamlines, viz., as a body is acted upon by a magnetic force when it cuts induction lines; (ii) a blob emits sound [Eq. (55a)] if its density is distinct from the surrounding fluid's and it is in the presence of a pressure gradient, viz., as an electric charge is acted upon by a force in the presence of an external electric field. The pressure gradients may be due to the deflection of flow around an obstacle, which may also shed vorticity, and if it contains electric charges and/or currents, also create electromagnetic fields. The frequency $\omega$ of the sound radiated by the dipole sources is then determined by the Strouhal (1878) number $\mathrm{St}=\omega l/v_0$, on the basis of mean flow velocity $v_0$ and body scale $l$; the corresponding wavelength $\lambda=2\pi c_0/\omega\sim 2\pi l/M_0$ is much larger than the body scale $l$, for low-Mach-number flow, i.e., the obstacle is a compact scatterer of sound, of dipole character as the blobs or vorticity. The solution of the forced, convected wave equation (52) in the compactness limit is given by the generalized Kirchhoff integral,

$$q(\mathbf{x},t)=(1/4\pi)\int_D |\mathbf{X}-\mathbf{Y}|^{-1}[\nabla\cdot\mathbf{A}(\mathbf{Y},\tau)]d^3y\,, \tag{60}$$

which differs from the free-space form in two related aspects: (i) the positions of the observer $\mathbf{x}$ and source $\mathbf{y}$ are corrected to $\mathbf{X},\mathbf{Y}$, in agreement with the unit flow potential $\boldsymbol{\psi}$,

$$\mathbf{X}=\mathbf{x}+\boldsymbol{\psi}(\mathbf{x})\,, \tag{61a}$$

$$\mathbf{v}_0=\nabla(\mathbf{v}_\infty\cdot\mathbf{X})\,, \tag{61b}$$

where $\mathbf{v}_\infty$ is the velocity of the uniform incident flow at infinity, i.e., the presence of the scattering body is equivalent to changing the space geometry; (ii) the retarded time is changed from that of Eq. (17b) to

$$\tau=t-|\mathbf{X}-\mathbf{Y}|/U=t-|\mathbf{X}-\mathbf{Y}|/c_0+\mathbf{M}_0\cdot(\mathbf{X}-\mathbf{Y})\,, \tag{62a}$$

$$U=c_0+\mathbf{v}_0\cdot\mathbf{m}=c_0[1+\mathbf{M}_0\cdot(\mathbf{X}-\mathbf{Y})/|\mathbf{X}-\mathbf{Y}|]\,, \tag{62b}$$

since the group velocity $U$ of propagation of sound is the sound speed $c_0$ plus the component of the mean flow velocity $\mathbf{v}_0$ in the direction $\mathbf{m}$ from observer to source.

## 6. Reciprocity theorem and flow reversal

The generalized Kirchhoff integral (60) applies to the acoustics of low-Mach-number potential flows in the presence of scattering bodies, provided that the observer be in the far field and the source in the near field, or vice versa, since the reciprocity theorem holds. In order to formulate the latter, it is convenient to consider the Green's function $G$, defined as the sound field due to a point source, at position $\mathbf{y}$ and time $\tau$,

$$\{\partial^2/\partial\mathbf{x}^2 - c_0^{-2}[\partial/\partial t + v_0(\mathbf{x})\cdot\partial/\partial\mathbf{x}]^2\} \times G(\mathbf{x},\mathbf{y};t,\tau) = \delta(\mathbf{x}-\mathbf{y})\delta(t-\tau) , \quad (63)$$

where $\delta$ denotes Dirac's (1927) delta function. Substituting the right-hand side of Eq. (63) into (60) and (62a), we obtain the explicit form of the Green's function:

$$G(\mathbf{x},\mathbf{y};t,\tau) = (4\pi\,|\,\mathbf{X}-\mathbf{Y}\,|\,)^{-1} \times \delta(t-\tau-|\,\mathbf{X}-\mathbf{Y}\,|\,/c_0 + \mathbf{M}_0\cdot(\mathbf{X}-\mathbf{Y})) . \quad (64)$$

In the case $\mathbf{M}_0=0$, $\mathbf{X}=\mathbf{x}$, $\mathbf{Y}=\mathbf{y}$, of a point source in free space at rest, Eq. (64) simplifies to the Green's function for the classical wave equation,

$$G_0(\mathbf{x},\mathbf{y};t;\tau) = (4\pi\,|\,\mathbf{x}-\mathbf{y}\,|\,)^{-1}\delta(t-\tau-|\,\mathbf{x}-\mathbf{y}\,|\,/c_0) , \quad (65)$$

whose symmetry in $(\mathbf{x},\mathbf{y})$ implies the reciprocity theorem: the sound field $G_0(\mathbf{x},\mathbf{y})$ observed at $\mathbf{x}$ due to a source at $\mathbf{y}$ is equal to the sound field $G_0(\mathbf{y},\mathbf{x})$ observed at $\mathbf{y}$ due to the source at $\mathbf{x}$. In the presence of a low-Mach-number mean flow, the Green's function (64) is symmetric in $\mathbf{X},\mathbf{Y}$ if the mean flow direction $\mathbf{M}_0$ is reversed, i.e., the reciprocity theorem is stated $G_{\mathbf{M}_0}(\mathbf{x},\mathbf{y}) = G_{-\mathbf{M}_0}(\mathbf{y},\mathbf{x})$, i.e., the positions of source $\mathbf{y}$ and observer $\mathbf{x}$ can be interchanged $(\mathbf{x},\mathbf{y})\to(\mathbf{y},\mathbf{x})$, provided that the direction of the mean flow be reversed $\mathbf{M}_0\to-\mathbf{M}_0$. The reason for the latter requirement is that the convection of sound by the mean flow affects the speed of propagation [Eq. (62b)] and thus the retarded time [Eq. (62a)]; if the flow convects sound from source to observer, when the latter are interchanged the mean flow velocity must be reversed for the propagation velocity and retarded time to be the same.

### 7. Emission due to activity of forces

The sound field $q(\mathbf{x},t)$ due to an arbitrary source, say $\nabla\cdot\mathbf{A}(\mathbf{y},\tau)$, is given by the convolution in space-time with the Green's functions (64),

$$q(\mathbf{x},t) = \int [\nabla\cdot\mathbf{A}(\mathbf{y},\tau)]G(\mathbf{x},\mathbf{y};t,\tau)d^3y\,d\tau . \quad (66)$$

The stagnation enthalpy $q\sim-\dot{\varphi}\sim[\rho_0(1+\mathbf{M}_0\cdot\mathbf{m})]^{-1}p$ scales as the acoustic pressure [Eqs. (24b) and (25)] in the far field, and in the case of a small blob of volume $v$, it is given by Eqs. (66) and (64),

$$p(\mathbf{x},t) \sim [4\pi c_0^2(\,|\,\mathbf{x}\,|\,+\mathbf{M}_0\cdot\mathbf{x})]^{-1} \times [(\partial/\partial\tau + \mathbf{u}_*\cdot\partial/\partial\mathbf{y})\rho_0 v W] ; \quad (67)$$

the acoustic pressure decays as the inverse of the distance $|\,\mathbf{x}\,|^{-1}$, as a radiation field, involves an inverse Doppler factor $1+\mathbf{M}_0\cdot\mathbf{m}$, and is proportional to the rate change, along the blob's path $\mathbf{u}_* = d\mathbf{y}/d\tau$, of the mass of fluid $\rho_0$ displaced by the blob's volume $v$, multiplied by

$$W = \mathbf{F}/\rho_1 + \mathbf{G}/\rho_0)\cdot\mathbf{w} , \quad (68a)$$

$$\mathbf{w} = c_0\partial[\,|\,\mathbf{X}-\mathbf{Y}\,|\,+\mathbf{M}_0\cdot(\mathbf{X}-\mathbf{Y})]/\partial\mathbf{x} , \quad (68b)$$

which is the activity or work per unit time [Eq. (68a)] of the total force per unit volume [Eq. (51)] at the propagation velocity [Eq. (68b)]. Thus, if the blob moves far from the bodies, where the mean flow and force fields are nearly constant, the activity of forces is not changed, and *no* sound is emitted, i.e., it is convected silently; as the blob is swept in a pressure gradient past a body, or acted upon by nonuniform forces, the change in their activity leads to an excess (or default) of energy, which, in the absence of dissipation, must be liberated (absorbed) by emitting a compression (rarefaction) wave, i.e., sound. Thus the passage of a blob near an obstacle or a field source is testified to by the emission of a sound pulse.

### 8. Signals originating from a blob

As an example, we illustrate in Fig. 3 the case of a potential flow rendered nonuniform by the presence of a spherical obstacle; in Fig. 3(a) a blob, i.e., a patch of fluid of density different from the fluid's, is convected past the sphere, along (b) the streamline of impact distance one-half the radius. As the blob moves past the sphere it emits a sound pulse, which is symmetric (c) for an observer in the sideline direction, i.e., in the far field, in the direction perpendicular to the flow through the center of the sphere. The shape of the sound pulse may be interpreted as follows: as the blob approaches and moves away from (is near) the point closest to the sphere, the streamlines diverge (converge), the pressure reduces (increases), the blob expands (contracts) and emits a compression (rarefaction), thus yielding a pulse consisting of two compression waves separated by a weak rarefaction wave. An observer downstream in the far field (d) receives, from the same blob, a distorted pulse, no longer symmetric, due to the different scattering effects on the two (approach and retreat) sides of the sphere; the comparison of (c) and (d) points to the importance of sound scattering effects, to which we now turn.

## III. SPECTRAL AND DIRECTIONAL BROADENING OF SOUND

The modeling of noise sources may not be sufficient to calculate the acoustic radiation field in those cases when source "masking" occurs, that is, when the directivity and spectrum are changed by scattering and diffraction (e.g., turbulence and interfaces) in the medium separating observer from source. An example is the problem of relating (i) static sound measurements on a test bench with (ii) acoustic experiments in a wind tunnel and (iii) noise records of aircraft in flight; the discrepancies between these three sets of data (Michalke and Michel, 1979;

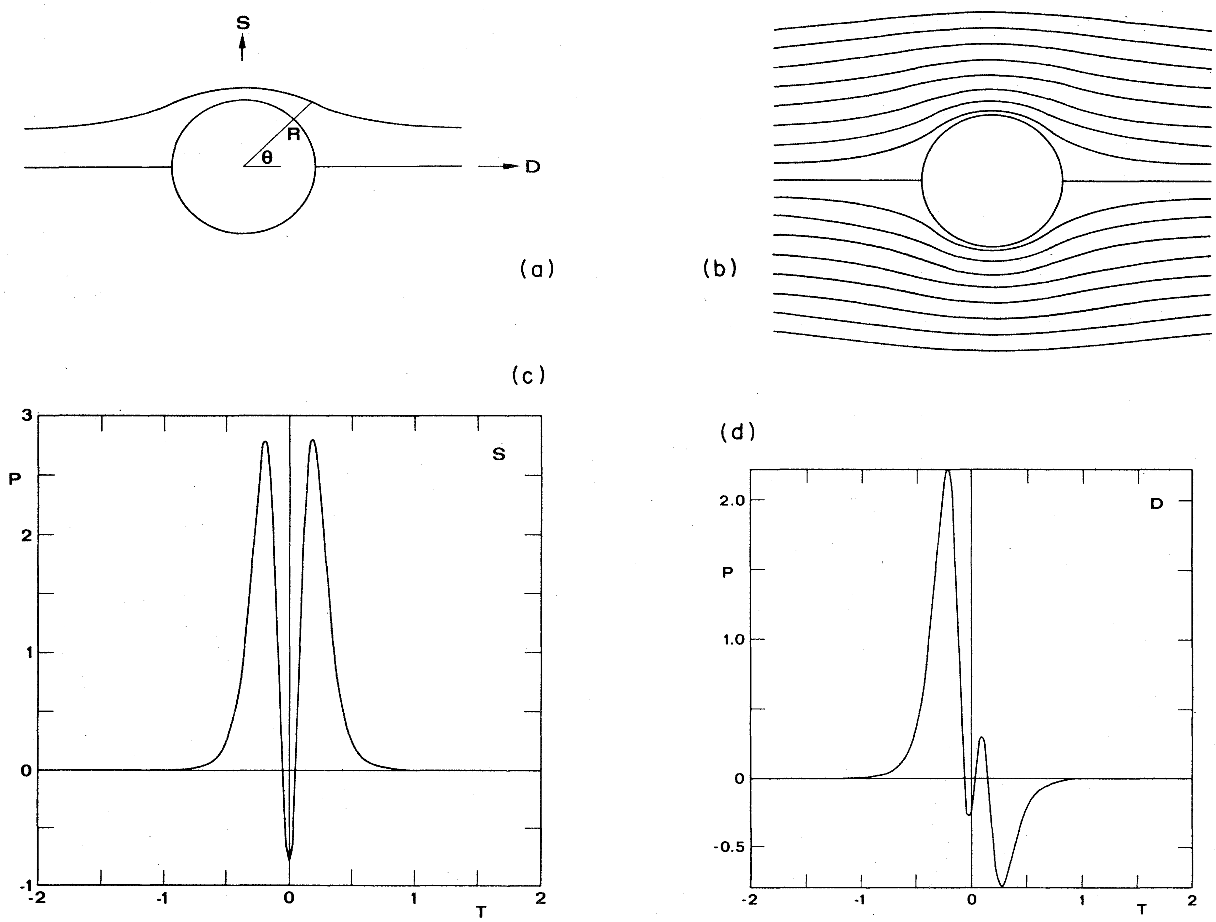


FIG. 3. Sound pulse emitted (Campos, 1978a) by a fluid inhomogeneity, convected (b) by a potential flow past a sphere, (a) along a streamline of impact distance one-half the radius, as received by an observer in the far field, in the (c) sideline and (d) downstream directions.

McGowan and Larson, 1984) can be attributed to differences in the modeling of sources and effects of scattering by the mean flow. The acoustic analogy has been modified to apply to rotational mean flows (Lilley, 1973; Tester and Burrin, 1974) by using a forced third-order wave equation, coupling sound and vorticity; this equation should be relevant to the experimental demonstration (Bechert and Pfizenmaier, 1975; Moore, 1979) that the noise of jets can be substantially increased by discrete acoustic excitation of unstable modes. This phenomenon is not yet fully understood and is probably related to the interaction between acoustic and instability waves; the latter (Schlichting, 1951; Lin, 1955; Chandrasekhar, 1961; Drazin and Reid, 1981) are described in the inviscid case by the Rayleigh (1887) equation, and in the viscous case by the Orr (1907) -Sommerfeld (1908) equation, which is usually solved by expansion procedures (Heisenberg, 1924) or in a truncated form (Tollmien, 1929; Schlichting, 1933). The Lilley equation has been used (Mani, 1976a,1976b; Balsa, 1976a,1976b) to calculate the directivities of jet noise, using some *ad hoc* assumptions. We adopt an alternative approach, which retains the modeling of sound sources (Sec. II) by the original acoustic analogy (Lighthill, 1963; Howe, 1975a; Campos, 1978a) and uses scattering and diffraction theory (Sec. III) to obtain directivity patterns and spectra (Lighthill, 1953; Howe, 1976b; Campos, 1978b); the combination of these two procedures leads to a modeling of jet noise (Munt, 1977; Campos, 1978c; Cargill, 1982) consistent with observations, as regards both directivities (Bechert, Michel, and Pfizenmaier, 1977; Plumblee and Dean, 1983) and spectra (Candel, Julienne, and Julliand, 1975; Candel, Guédel, and Julienne, 1976).

### A. Directivity for transmission across vortex sheets

The simplest example of scattering is sound incident on a plane interface, separating two media, which, in the case of a jet and an ambient medium at rest, is a surface of discontinuity not only of mass density and sound speed, but also of tangential velocity, i.e., a vortex sheet. Thus the basic model of sound transmission from the interior of a jet considers scattering by a vortex sheet (Miles, 1957,1958; Howe, 1970,1975c). This is an instance of the scattering of sound by vorticity (Fabrikant, 1983; Howe, 1983a), and just as sound can destabilize vortices (Broadbent and Moore, 1979), acoustic waves incident on a vortex sheet can trigger instabilities (Jones and Morgan, 1972,1974). A model more elaborate than the vortex sheet, across which the velocity has a discontinuity, is the simple layer (Jones, 1977), i.e., a slab of a small but finite width, in which the velocity of the jet is matched smoothly to the atmosphere, leading to the problem of sound propagation through a shear flow (Lighthill, 1972; MacGregor, Ribner, and Lam, 1973; Candel, 1983; Hansen, 1984), which has been reviewed elsewhere (Nayfeh, Kaiser, and Telionis, 1975a; Möhring, Müller, and Obermeier, 1983). Jets issue from pipes, and thus there is a contribution to the radiation field due to diffraction by the solid surface (Leppington, 1972; Jones, 1973; Crighton and Leppington, 1973; Levine, 1975; Rienstra, 1981; Leppington, Broadbent, and Heron, 1984); this an extension of the classical diffraction problem (Sommerfeld, 1896), to which the Wiener-Hopf technique (Noble, 1958) is well suited, both for plane and cylindrical geometries. The directivity of the noise of jets has been modeled successfully by considering diffraction from a semi-infinite cylindrical pipe, with a trailing vortex sheet (Munt, 1977; Cargill, 1982) and internal shock structure (Howe and Ffowcs-Williams, 1978).

### 1. Rayleigh-Born approximation

We consider the transmission of sound across an interface separating media moving with uniform velocity. A Galilean frame can be chosen so that the "upper" medium of transmission or "ambient" is at rest, and the "lower" medium of incidence or "jet" moves at the relative velocity $\mathbf{v}_0$. Thus we have sound incident from a jet, of mass density $\rho_0$ and sound speed $c_0$, moving at Mach vector $\mathbf{M}_0=\mathbf{v}_0/c_0$, and transmitted to the ambient at rest, with mass density $\rho_1$ and sound speed $c_1$. If we denote by $x=(\mathbf{x}_{\|},z)$ the position vector, where $\mathbf{x}_{\|}$ is the horizontal projection and $z$ the vertical coordinate, the position of the interface is given by $z=\xi(\mathbf{x}_{\|},t)$, where $\xi$ is a function of $\mathbf{x}_{\|}$ for an irregular interface and of $t$ for an unsteady one; if the interface is convected at velocity $\mathbf{u}$, then $z=\xi(\mathbf{x}_{\|}-\mathbf{u}t)$ if it is irregular, and $z=\xi(\mathbf{x}_{\|}-\mathbf{u}t,t)$ if it is unsteady as well. The simplest approach to the scattering of sound by irregular, unsteady, or moving interfaces is the Rayleigh-Born (Rayleigh, 1889; Born and Wolf, 1959) approximation, that (i) the radius of curvature $R$ is much larger than the wavelength $\lambda$, and thus the interface is locally flat $R \gg \lambda$; (ii) the slope $\nabla\xi$ of the interface has zero mean $\langle \nabla\xi \rangle=0$, and its variance $\langle (\nabla\xi)^2 \rangle$ is negligible, so that it is locally horizontal. These assumptions are equivalent to replacing the irregular interface by an assembly of flat, horizontal facets, displaced from the mean position by $\xi(\mathbf{x}_{\|}-\mathbf{u}t,t)$, i.e., at the correct height for each horizontal location $\mathbf{x}_{\|}$ and time $t$, and convected at velocity $\mathbf{u}$.

### 2. Choice of boundary conditions

If we consider a plane wave incident on such an interface, it gives rise to transmitted and reflected waves with the same horizontal wave vector $\mathbf{k}_{\|}$ and frequency $\omega$, and distinct vertical wave numbers, viz., $k_\perp$ for the incident, $-k_\perp$ for the reflected, and $K_\perp$ for the transmitted wave. Omitting the common factor $\exp[i(\mathbf{k}_{\|}\cdot\mathbf{x}_{\|}-\omega t)]$, the condition of continuity of acoustic pressure reads

$$\exp(ik_\perp\xi)+R\exp(=ik_\perp\xi)=T\exp(iK_\perp\xi)\ , \tag{69}$$

where we have assumed amplitude unity for the incident wave, so that the amplitudes of the reflected and transmitted waves specify, respectively, the reflection $R$ and transmission $T$ factors. The second boundary condition at the interface, which, together with Eq. (69), determines $R,T$, applies to the acoustic velocity $v_n \equiv \mathbf{v}\cdot\mathbf{n}$ or displacement $\zeta_n=\boldsymbol{\zeta}\cdot\mathbf{n}$ the normal direction $\mathbf{n}$; the equation of momentum, relating acoustic pressure $p$ and velocity $\mathbf{v}$ or displacement $\zeta$, reads

$$-\rho_0^{-1}/\nabla p = d\mathbf{v}/dt = d^2\boldsymbol{\zeta}/dt^2\ , \tag{70}$$

where $d/dt$ denotes the material derivative [Eq. (25)]; for a plane, monochromatic wave $\sim\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]$, it implies the following relation between acoustic pressure $p$, normal velocity $v_n$, and displacement $\zeta_n$ on the interface, e.g., for the incident wave,

$$(p/\rho_0)k_\perp=(\omega-\mathbf{k}_{\|}\cdot\mathbf{v}_0)v_n=-i(\omega-\mathbf{k}_{\|}\cdot\mathbf{v}_0)^2\zeta_n \tag{71}$$

and similarly for the reflected and transmitted waves. For a medium at rest $\mathbf{v}_0=0$, bearing in mind that the frequency $\omega$ is continuous across the interface, it is immaterial whether the continuity of normal velocity $v_n$ or normal displacement $\zeta_n$ is chosen as the second boundary condition; not so for a jet, since the convection effect in Eq. (71), essentially a Doppler factor, appears to the first power for the velocity and to the second power for the displacement, so that they cannot both be continuous.

### 3. Continuity of velocity or displacement?

The literature on the acoustics of jets is divided on the matter of the boundary condition to be applied, with early references using the continuity of normal velocity (Rayleigh, 1879; Esclangon, 1917; Rudnick, 1946; Franken and Ingard, 1956), and, more recently, a preference for the continuity of displacement (Miles, 1957; Ribner, 1957; Ingard, 1959; Gottlieb, 1959). The controversy on this issue has been fueled by mathematical proofs that are claimed to lead to the continuity of velocity (Keller, 1955; Myers, 1980) or to the continuity of displacement (Mungur and Plumblee, 1969; Poirée, 1982), the two results being incompatible. From a physical point of view, an acoustic wave requires a material medium to support it, i.e., the fluid particles must remain attached to the interface, so that the normal component of displacement must be continuous; it follows that the normal component of acoustic velocity will be discontinuous, if the convection effect is different on the two sides of the interface, e.g., for a jet in still air. Although the issue of the correct acoustic boundary condition at an interface between moving media was considered to be in doubt (Nayfeh, Kaiser, and Telionis, 1975a) until recently, the good agreement with observation of later theories of jet noise (Munt, 1977; Campos, 1978b,1978c; Cargill, 1982) substantiates the choice of continuity of normal component of displacement $\zeta_n$. From Eq. (71) this implies

$$[\exp(ik_\perp\xi)-R\exp(-ik_\perp\xi)](k_\perp/\rho_0)(1-\mathbf{M}_0\cdot\mathbf{n})^{-2} = T(K_\perp/\rho_1)\exp(iK_\perp\xi)\,, \quad (72)$$

where $\mathbf{n}=\mathbf{k}/k$ is the direction normal to the incident wave front.

### 4. Reflection and transmission coefficients

The angles of incidence $\theta$ and transmission $\Theta$ (with the mean position of the interface) are related by the continuity of the horizontal component of the wave number $\mathbf{k}_{\parallel}$:

$$(c_1/\omega)k_{\parallel}=\cos\Theta=(c_1/c_0)M_r\cos\theta\,, \quad (73a)$$

$$M_r\equiv 1-M_0\cos\theta\cos\psi\,, \quad (73b)$$

where $M_r$ is the Doppler factor associated with the jet Mach number $M_0$, and $\psi$ is the angle of the flow velocity with the plane of refraction [Fig. 4(a)]. The normal components of the incident $k_\perp$ and transmitted $K_\perp$ wave vectors are given, respectively, by

$$K_\perp=(\omega/c_1)\sin\Theta\,, \quad (74a)$$

$$k_\perp=(\omega/c_0)\sin\theta M_r^{-1}\,. \quad (74b)$$

Thus all quantities appearing in Eqs. (69) and (72) can be expressed in terms of the angle of incidence $\theta$ and of the jet velocity $\psi$; for a plane interface $\zeta=0$, solving Eqs. (69) and (72) we obtain the reflection $R_0$ and transmission $T_0$ coefficients,

$$R_0=(1-B)/(1+B)\,, \quad (75a)$$

$$T_0=2/(1+B)\,, \quad (75b)$$

where $B$ is given by Eq. (76a) for a medium at rest and by (76b) for a jet:

$$B_0=(\rho_0/\rho_1)(K_\perp/k_\perp)\,, \quad (76a)$$

$$B=B_0M_r^2\,. \quad (76b)$$

Note that $1+R_0=T_0$, so that energy is conserved during scattering by an interface at rest [Eq. (76a)] or a vortex sheet [Eq. (76b)].

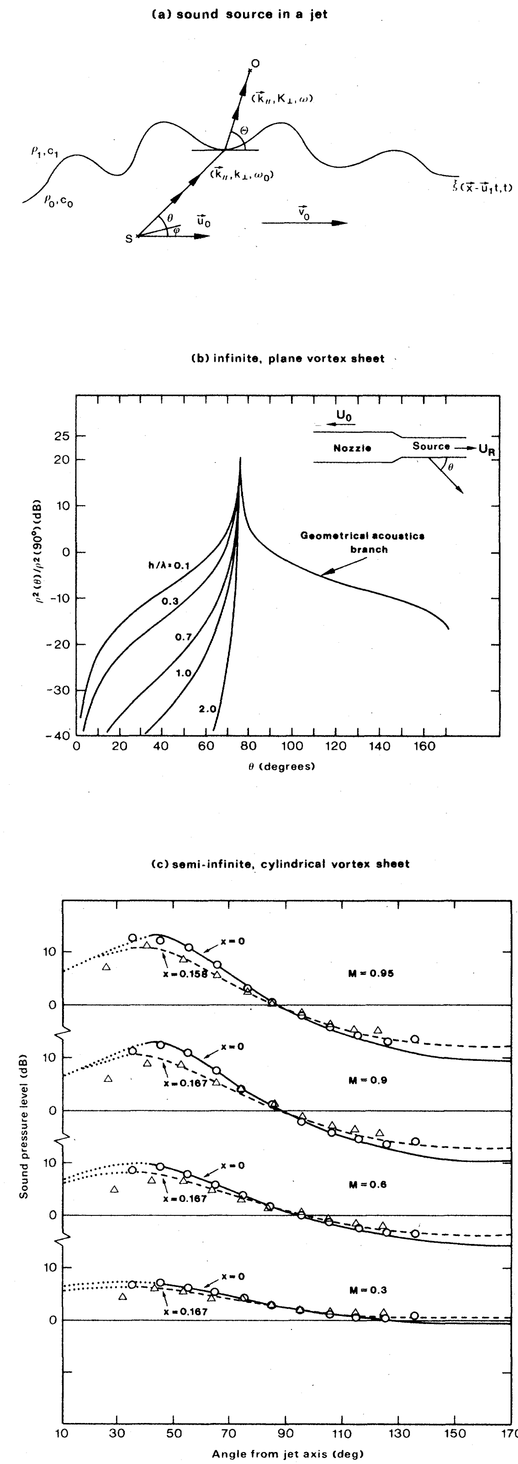


FIG. 4. Sound emitted by (a) a source moving at speed $\mathbf{u}_0$ in a jet of velocity $\mathbf{v}_0$ and received by an observer outside; the directivity is plotted for two interface models: (b) a plane, infinite vortex sheet (Howe, 1975c); (c) a cylindrical, semi-infinite vortex sheet (Munt, 1977). The latter theory (○, dotted line) is compared with experimental measurements (△, solid line; Pinker and Bryce, 1976) for jets of Mach numbers ranging from subsonic to near sonic.

### 5. Moving sound sources

If we consider a homogeneous jet of uniform velocity $\mathbf{v}_0$, the classical wave equation [first two terms in Eq. (32)] is valid in a frame of reference moving with the fluid; returning to a frame at rest corresponds to the transformation $\mathbf{x}\to\mathbf{x}-\mathbf{v}_0 t$, $t\to t$, so that spatial derivatives are unchanged $\partial/\partial\mathbf{x}\to\partial/\partial\mathbf{x}$, but local time derivatives $\partial/\partial t\to\partial/\partial t+\mathbf{v}_0\cdot\nabla\equiv d/dt$ become material derivatives [Eq. (25)], and we obtain the convected wave equation (38). Thus we conclude that the convected wave equation (38) describes the linear propagation of sound in homogeneous fluids in two cases; (a) nonuniform potential flow, of low Mach number, as proved before (in Sec. II.B.4); (b) uniform flow of arbitrary Mach number, as proved here (in Sec. III.A.5). We consider the convected wave equation

$$[c_0^{-2}(\partial/\partial t+\mathbf{v}_0\cdot\nabla)^2-\nabla^2]p(\mathbf{x},t)=S(\partial/\partial\mathbf{x},\partial/\partial t)\exp(-i\omega_0 t)\delta(\mathbf{x}-\mathbf{x}_0-\mathbf{u}_0 t)\ , \tag{77}$$

forced by a term with arbitrary multipolar character, to model the various sources of sound (Sec. II.A), e.g., a monopole $S_0=Q\partial/\partial t$, a dipole $S_1=F_i\partial/\partial x_i=\mathbf{F}\cdot\nabla$, a quadrupole $S_2=T_{ij}\partial^2/\partial x_i\partial x_j$, etc. The delta function in Eq. (77) shows that we consider a point source following the trajectory $\mathbf{x}=\mathbf{x}_0+\mathbf{u}_0 t$, i.e., moving uniformly at velocity $\mathbf{u}_0$, from an initial position $\mathbf{x}_0$; the source is also assumed to be harmonic, with frequency $\omega_0$, so that the solution of Eq. (77) is a harmonic Green's function $p(\mathbf{x},t)=G(\mathbf{x},t;\mathbf{y},\omega_0)$ with $\mathbf{y}=\mathbf{x}_0+\mathbf{u}_0 t$. In the general case of $(\mathbf{y},\omega)$ of a source distribution $D$ in space $\mathbf{y}$, with frequency spectrum $\omega$, the acoustic pressure is given by a Fourier integral in $\omega$ and convolution in $\mathbf{y}$:

$$p(\mathbf{x},t)=\int_{-\infty}^{+\infty}\int_D G(\mathbf{x},t;\mathbf{y},\omega)f(\mathbf{y},\omega)e^{-i\omega t}d\mathbf{y}\,d\omega\ . \tag{78}$$

Thus the method applies to arbitrary sources, in spectrum, spatial distribution, and multipolar character.

### 6. Harmonic Green's function

Since the wave equation (77) has constant coefficients, namely, the sound speed $c_0$ and flow velocity $\mathbf{v}_0$, it can be solved by Fourier analysis:

$$p(\mathbf{x},t)=\int_{-\infty}^{+\infty}p(\mathbf{k},\omega)\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]d^3\mathbf{k}\,d\omega\ , \tag{79}$$

where we denote (Crighton, 1975) the pressure perturbation $p(\mathbf{x},t)$ and spectrum $p(\mathbf{k},\omega)$ by the same symbol $p$, distinguishing them by the arguments. Bearing in mind that space and time derivatives applied to Eq. (79) are equivalent to multiplication, respectively, by the wave vector $\mathbf{k}$ and frequency $\omega$, viz., $(\partial/\partial x,\partial/\partial t)\to i(\mathbf{k},-\omega)$, the acoustic pressure spectrum is given by

$$p_1(\mathbf{k},\omega)=(2\pi)^{-3}S(i\mathbf{k},-i\omega)\exp(-i\mathbf{k}\cdot\mathbf{x}_0)\times[(\omega-\mathbf{k}\cdot\mathbf{v}_0)^2/c_0^2-k^2]^{-1}\delta(\omega-\omega_0-\mathbf{k}\cdot\mathbf{u}_0)\ . \tag{80}$$

When we substitute Eq. (80) into (79), Dirac's delta function performs the $d\omega$ integration, setting the frequency equal to

$$\omega_s=\omega_0/M_s\ , \tag{81a}$$

the source's $\omega_0$, with a Doppler factor

$$M_s\equiv 1-(\mathbf{u}_0\cdot\mathbf{n})/c_0\ , \tag{81b}$$

due to its motion. The factor in square brackets in Eq. (80) vanishes for $\omega=\pm c_0 k-\mathbf{k}\cdot\mathbf{v}_0$, which is the dispersion relation for acoustic waves in a moving medium; this corresponds to a pole in Eq. (79), so that the $dk_\perp$ integral can be evaluated by residues, and we are left with

$$p_1(\mathbf{x},t)=(8\pi^2)^{-1}\int_{-\infty}^{+\infty}(k_\perp)^{-1}S(i\mathbf{k}_{\|},ik_\perp,-i\omega_s)\cdot\exp[i\mathbf{k}_{\|}\cdot(\mathbf{x}-\mathbf{x}_0)-i\omega_s t)]d^2k_{\|}\ , \tag{82}$$

an integration over the horizontal wave vector $\mathbf{k}_{\|}$, specifying the harmonic Green's function for an arbitrary multipole point source of frequency $\omega_0$, moving at a speed $\mathbf{u}_0$, in a jet of velocity $\mathbf{v}_0$.

### 7. Zone(s) of silence

If we consider the sound field [Eq. (82)] in the jet, incident upon a vortex sheet, the transmitted field $p_2$ in the ambient is obtained multiplying by the transmission factor $T$ [Eqs. (75b), (76a) and (76b)]. The acoustic energy flux $W$, radiated across a horizontal plane above the vortex sheet $z\equiv\text{const}>0$, is defined by

$$W\equiv\int_{-\infty}^{+\infty}p_2(\mathbf{x},t)v_{2z}(\mathbf{x},t)d^2x_{\|}\ , \tag{83}$$

where the vertical component of the acoustic velocity $v_{2z}$ is related to the transmitted pressure $p_2$ by the momentum equation (70), viz., $p_2(\mathbf{x},t)=\rho_1 c_0 M_s v_{2z}(\mathbf{x},t)$; thus the acoustic energy flux is given by

$$W=(32\pi^2\rho_1 c_1 M_s)^{-1}\int_{-\infty}^{+\infty}(T_0/k_\perp)^2\,|\,S(i\mathbf{k}_{\|},ik_\perp,-i\omega_s)\,|^2\,|\exp[2iK_\perp(z-b)]\,|\,d^2\mathbf{k}_{\|}\ , \tag{84}$$

where the integration over the horizontal wave vector $d^2k_{||}$ has two parts: (i) for $K_\perp$ real, we have propagating waves, and the energy flux is independent of height $z$ above the shear layer; (ii) for evanescent waves, $K_\perp$ is imaginary $K_\perp \equiv ik_*$, and the energy flux decays exponentially like $\exp[-2k_*(z-h)]$, on the distance $z-h$ from the source, faster for shorter vertical wavelength $\lambda = 2\pi/k_*$. For an observer distant from the source $k_*(z-h) \gg 1$, no sound is received from the directions for which $k_\perp$ is imaginary, and thus a "zone of silence" is formed. The vertical transmitted wave number $K_\perp$ is given by [Eqs. (74a), (73a) and (73b)],

$$(c_1K_\perp/\omega)^2 = 1-(c_1/c_0)^2[\cos^2\theta/(1-M_0\cos\theta\cos\psi)^2] \; , \tag{85}$$

and it changes from real to imaginary when Eq. (85) vanishes, thus specifying the limits of the zone of silence,

$$\theta_\pm = \sec^{-1}(M_0 \pm c_1/c_0) \; , \tag{86}$$

for jet velocity in the plane of refraction $\psi=0$. For a cold air jet $c_1 = c_0$, a "zone of silence" $0<\theta<\theta_+$ always exists in the downstream arc; it also exists in the upstream arc $\theta_- < \theta < \pi$ if the jet speed is over bisonic $M_0>2$, when the "zone of silence" in the forward arc extends to $\theta_+ = 70.5°$.

### 8. Radiation pattern of a jet

The directivity, i.e., acoustic power per unit solid angle, is plotted in Fig. 4(b) for the sound (Howe, 1975c) of a monopole point source radiated across a vortex sheet, with a relative velocity $\mathbf{U}_R = \mathbf{u}_0 - \mathbf{v}_0$. The intensity of radiation is independent of distance outside the zone of silence ($\theta > \theta_+ = 75°$ in this case), but within the zone of silence the sound field is negligible unless the source is within a fraction of a wavelength from the shear layer $h/\lambda \ll 1$. A sharp peak in directivity at the edge of the zone of silence is a feature of the vortex sheet model (Ffowcs-Williams, 1964); it is *not* observed experimentally in jet noise, and it is absent from more elaborate models, such as (a) a semi-infinite vortex sheet, issuing from a nozzle lip (Munt, 1977); (b) an irregular interface, i.e., partially curled-up vortex sheet (Howe, 1976b); (c) a shear layer entraining turbulence (Campos, 1978c). Since we shall discuss models (b) and (c) in more detail later (in Secs. III.B and III.C, respectively), we conclude by illustrating model (a) in Fig. 4(c); this shows the good agreement with measured directivities (Pinker and Bryce, 1976) of the noise of jets, for Mach numbers ranging from subsonic $M_0 = 0.3$ to near sonic $M_0 = 0.95$, obtained by using a model (Munt, 1977) consisting of a semi-infinite vortex sheet issuing from a cylindrical pipe.

## B. Scattering by irregular and moving interfaces

Although the vortex sheet model, together with edge diffraction, describes accurately the directivity of jet noise, it is unable to account for changes in the spectrum, since it preserves the source frequency, apart from constant Doppler shifts. The noise of jets shows irrefutable evidence of changes in the spectrum received in a given direction, e.g., a monochromatic test source placed inside a jet (Candel, Guédel, and Julienne, 1976) is heard outside over a band of frequencies *not* present in the source. This phenomenon of spectral broadening is due to random Doppler shifts, which can occur in association with either scattering by an irregular interface in motion or diffraction by a turbulent velocity field. We shall defer discussion of diffraction until later (Sec. III.C), and consider here scattering by rough surfaces, at rest and in motion. As we have seen (Sec. I.A.5), scattering by static, randomly irregular interfaces (Greenwood, 1984) is well known to give rise to signal "clutter" from studies of various types of waves, e.g., electromagnetic (Beckmann and Spizzichino, 1963), such as radar (Sholnik, 1962), optical (Chernov, 1967), radio (Berry, 1973), and sonar signals hitting the sea bottom (Clarke, 1973) or surface (Gazanhes and Léandre, 1974; Essen, 1974). Scattering by rough surfaces (Watson and Keller, 1984), e.g., sinusoidal (Rayleigh, 1879), is modified in the presence of convection, since distinct emission lobes may have different Doppler shifts (Campos, 1978c), leading to spectral broadening. Multiple scattering (Howe, 1973a,1973b) is also important, in particular as concerns a static "slab" (Howe, 1976b) or a "shielding jet" (Campos, 1978b) with two irregular interfaces, since modern turbofan engines have coaxial jet exhausts (Chen, 1977; Stone, Groesbeck, and Zola, 1983), which help to reduce noise levels.

### 1. Phase shifts

In the case of a rough interface $\xi(\mathbf{x}_{||},t)$, generally irregular and unsteady, the solution of Eqs. (69) and (72) yields reflection $R$ and transmission $T$ coefficients

$$R_1 = R_0 \exp(i2k_\perp\xi) \; , \tag{87a}$$

$$T_1 = T_0 \exp[i(K_\perp - k_\perp)\xi] \; , \tag{87b}$$

which have the same amplitude as for a plane interface [Eqs. (75a), (75b), (76a), and (76b)] and hence satisfy conservation of energy $1+|R|-|T| = 1+R_0-T_0 = 0$; the roughness of the interface introduces phase shifts, which do not vanish, except for the grazing direction $k_\perp = 0$ for the incident wave and undeflected scattering $k_\perp = K_\perp$ for the transmitted wave. The acoustic pressure, incident on the interface from a point multipole source [Eq. (82)], is transmitted as

$$p_s(\mathbf{x},t) = (8\pi^2)^{-1}\int_{-\infty}^{+\infty}(T_0/k_\perp)s(i\mathbf{k}_{||},ik_\perp,-i\omega_s)\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_0)-i\omega_s t]\exp[i(k_\perp - K_\perp)\xi(\mathbf{x}_{||},t)]d^2\mathbf{k}_{||} \; , \tag{88}$$

just above the interface; we assume that the pressure distribution (88) on the interface $z=\xi(\mathbf{x}_{\parallel},t)$ can be displaced to its mean position $z=0$, for the purpose of calculating the acoustic field $p_2(\mathbf{x},t)$ radiated to the observer in the ambient, which has the spectrum

$$p_2(\mathbf{K},\omega)=(2\pi)^{-3}\int_{-\infty}^{+\infty} p_s(\mathbf{x}_{\parallel},t)\exp[-i(K_{\parallel}\cdot\mathbf{x}_{\parallel}-\omega t)]\times d^2x_{\parallel}dt\ . \tag{89}$$

The scattered field just above the interface (88) and the spectrum radiated to the observer (89) both involve a phase shift $\exp[i(k_{\perp}-K_{\perp})\xi]$, depending on the shape of the interface $\xi$; if the latter is irregular and unsteady $\xi(\mathbf{x}_{\parallel},t)$, the spatial $\mathbf{x}_{\parallel}$ and time $t$ dependence are equivalent to changes in wave vector $\mathbf{k}_{\parallel}$ and frequency $\omega$, i.e., imply directional and spectral redistribution of acoustic energy.

### 2. Spectral directivity

The acoustic energy flux [Eq. (83)], averaged over time $\overline{W}$ (Campos, 1978b), can be expressed in terms of Eq. (79), the pressure spectrum $p_2(\mathbf{K},\omega)$, and Eq. (70), the velocity spectrum $v_2(\mathbf{K},\omega)=(\omega/\rho_1)p_2(\mathbf{K},\omega)$, in the ambient, as

$$\overline{W}=(2\pi^2/\rho_1M_s)\int(K_{\perp}/\omega)p_2(\mathbf{K}_{\parallel},\omega)p_2^*(\mathbf{K}_{\parallel},\omega')\times\delta_{\omega\omega'}d^2\mathbf{K}_{\parallel}d\omega\,d\omega'\ , \tag{90}$$

where the asterisk denotes the complex conjugate, and $\delta_{\omega\omega'}$ is the continuous Kronecker delta (equal to zero for $\omega\neq\omega'$, and unity for $\omega=\omega'$). If we write the transmitted wave vector $\mathbf{K}$ in spherical polar coordinates,

$$\mathbf{K}\equiv(\mathbf{K}_{\parallel},K_{\perp})=(\omega/c_1)(\cos\Theta,\sin\Theta\sin\psi,\sin\Theta\cos\psi)\ , \tag{91}$$

the $d^2K_{\parallel}$ integration in Eq. (90) involves the solid angle $d0$,

$$d^2K_{\parallel}=(\omega/c_1)\sin\Theta\cos\psi\,d0\ , \tag{92a}$$

$$d0\equiv\sin\Theta\,d\Theta\,d\psi\ , \tag{92b}$$

so that Eq. (90) implies

$$I(\Theta,\psi,\omega)=(2\pi^2/\rho_1M_sc_1^2)\sin\Theta\cos\psi\times\int(\omega K_{\perp})\cdot|p_2(\mathbf{K}_{\parallel},\omega)|^2d\omega\ , \tag{93}$$

where the spectral directivity $I$ is defined as the mean acoustic energy flux per unit solid angle $d0$ and unit frequency band $d\omega$,

$$d\overline{W}=I(\Theta,\psi,\omega)d0\,d\omega\ . \tag{94}$$

The spectral directivity specifies completely the distribution of acoustic energy, for any received pressure spectrum $p_2(\mathbf{K}_{\parallel},\omega)$; in particular, it yields the directivity $D$ and spectrum $H$,

$$D(\Theta,\psi)=\int_0^{\infty}I(\Theta,\psi,\omega)d\omega\ , \tag{95a}$$

$$H(\omega)=\int_0^{2\pi}\int_0^{\pi}I(\Theta,\psi,\omega)\sin\Theta\,d\Theta\,d\psi\ , \tag{95b}$$

by integration over the frequency (95a) and solid angle (95b), respectively.

### 3. Interference function

In the case of a vortex sheet, the transmitted spectrum $p_2(\mathbf{K}_{\parallel},\omega)$ coincides with the incident spectrum [Eq. (80)], with the term in square brackets replaced by $k_{\perp}^{-1}$ and multiplied by the transmission factor $T_0$ [Eq. (75b)]; substitution in Eq. (93) and the use of well-known integration properties [Eqs. (99a) and (99b)] show that

$$I_0(\Theta,\psi,\omega)=D_0(\Theta,\psi)\delta(\omega-\omega_s)\ , \tag{96a}$$

i.e., all sound is emitted at the source's frequency $\omega_s$ [Eqs. (81a) and (81b)], and the directivity is given by

$$D_0(\Theta,\psi)\equiv D(\Theta,\psi,\omega_s)\equiv(32\pi^2\rho_1c_1M_s)^{-1}(K_{\perp}T_0/k_{\perp})^2\times|S(i\mathbf{k}_{\parallel},ik_{\perp},-i\omega_s)|^2\ . \tag{96b}$$

Equation (96b) for the directivity of the plane vortex sheet can be shown to agree $D_0=dW/d0$ with Eq. (84), by using Eqs. (92a) and (92b); we have in (96b) considered only propagating waves, for which $K_{\perp}$ is real. In the more general case of an irregular and unsteady interface, the received pressure spectrum is given by Eqs. (88) and (89), and thus the spectral directivity [Eq. (93)] by

$$I(\Theta,\psi,\omega)=(256\pi^2\rho_1c_1M_s)^{-1}\sin^2\Theta\cos^2\psi\int_{-\infty}^{+\infty}(\omega T_0s/k_{\perp})^2\exp[i(\mathbf{k}_{\parallel}-\mathbf{K}_{\parallel})\cdot\mathbf{s}-i(\omega-\omega_s)\tau]\times C(\mathbf{s},\tau)d^2\mathbf{k}_{\parallel}d^2\mathbf{s}\,d\tau\ , \tag{97}$$

where $C$ denotes the interference function (97) between the phases of two waves,

$$C(\mathbf{s},\tau)=\exp\{i(k_{\perp}-K_{\perp})[\xi(\mathbf{x}_{\parallel},t)-\xi(\mathbf{x}_{\parallel}+\mathbf{s},t+\tau)]\}\ , \tag{98}$$

which we assume to depend only on the spatial $\mathbf{s}$ and temporal $\tau$ separation of the corresponding scattering elements. In the case of a vortex sheet $\xi=0$, there is no interference $C=1$, and Eq. (97) simplifies to Eqs. (96a) and (96b), in contrast to the case of an irregular and/or unsteady interface $\xi\neq0$, for which wave interference affects the directivity and spectrum.

### 4. Convected undulations

For simplicity we consider one-dimensional irregularities in the $x$ direction, and assume the interface to be smooth in the $y$ direction, so that the interference function $C(x,\tau)$ does not depend on $y$. Thus $y,k_y$ appear only in terms of the form

$$\int_{-\infty}^{+\infty} \exp[i(k-K)y]dy = 2\pi\delta(k-K) , \tag{99a}$$

$$\int_{-\infty}^{+\infty} f(k)\delta(k-K)dk = f(K) . \tag{99b}$$

The properties (99a) and (99b) of Dirac's (1927) delta function are proved in the theory of generalized functions (Schwartz, 1949; Lighthill, 1958; Guelfand and Chilov, 1962—1970; Jones, 1966) and imply the Fourier integral theorem,

$$\int_{-\infty}^{+\infty} \int f(k) \exp[i(k-K)y]dy\, dk = 2\pi f(K) , \tag{100}$$

which is fundamental in the theory of integral transforms (Wiener, 1933; Titchmarsch, 1937; Sneddon, 1972). We have used the properties of Eqs. (99a), (99b), and (100) repeatedly in Secs. III.A.6—III.B.3, and apply them once more to the $dy\, dk_y$ integrations in Eq. (97),

$$I(\Theta,\omega) = (128\pi^4 \rho_1 c_1^3 M_s)^{-1} \sin^2\Theta \int_{-\infty}^{+\infty} (\omega T_0 s/k_\perp)^2 \exp[i(k_{\parallel} - K_{\parallel})x + i(\omega-\omega_s)\tau] C(x,\tau) dk_{\parallel} dx\, d\tau , \tag{101}$$

where the observer lies in the "fly-over" plane $\psi=0$, and we use the notations $k_{\parallel} \equiv k_{\parallel x}$, $K_{\parallel} \equiv K_{\parallel x}$, and $s_x = x$. The interference function is given by

$$C(x,\tau) = \exp[i(k_\perp - K_\perp)f(s)] , \tag{102a}$$

$$s \equiv x - u_1\tau , \tag{102b}$$

for an interface with undulations of shape $f(s)$, steady in a frame (102b) convected at velocity $u_1$.

### 5. Spectral-directional lobes

Assuming that the undulations $f(s/L)$ are deterministic, with a period $L$, the interference function (102a) can be expanded in a Fourier series:

$$C(x,\tau) = \sum_{n=-\infty}^{+\infty} c_n \exp[i2\pi n(x-u_1\tau)/L] , \tag{103a}$$

with coefficients

$$c_n \equiv L^{-1} \int_0^L \exp[i(k_\perp - K_\perp)f(s/L) - i2n\pi s/L]ds ; \tag{103b}$$

as an example, consider sinusoidal undulations of height $a$ and length $L$,

$$f(s) = a \sin[2\pi(x-u_1\tau)/L] . \tag{104a}$$

The coefficients are (McLachlan, 1934a; Watson, 1944) Bessel functions:

$$c_n \equiv \pi^{-1} \int_0^{\pi} \cos[(k_\perp - K_\perp)a \sin(2\pi\eta) - in\eta]d\eta = J_n((k_\perp - K_\perp)a) . \tag{104b}$$

Each term of the Fourier series (103a) yields, when the $d\tau$ integration in (101) is performed, a delta function $\delta(\omega-\omega_n)$, with frequency

$$\omega_n = \omega_s + 2\pi n u_1/L = \omega_0 + k_{\parallel} u_0 - (k_{\parallel} - K_{\parallel})u_1 , \tag{105a}$$

equal to the source's $\omega_s$ [Eq. (81)], plus a multiple of the convection effect $2\pi u_1/L$; each "convection harmonic" (105a) has a directivity

$$D_n(\Theta) \equiv (32\pi^2 \rho_1 c_1^3 M_s)^{-1} \times \sin^2\Theta (\omega_n^2 T^2 S^2/k_\perp^2)_{K_{\parallel} = k_{\parallel} + 2\pi n/L} \tag{105b}$$

similar to that of a vortex sheet [Eq. (96b)], but at frequency $\omega_n$ (105a) and horizontal transmitted wave number $K_{\parallel} = k_{\parallel} + 2\pi n/L$ embodying the scattering effect. Thus the spectral directivity

$$I_*(\Theta,\omega) = \sum_{n=-\infty}^{\infty} c_n D_n(\Theta) \delta(\omega-\omega_n) \tag{106}$$

consists of a series of convection harmonics (105a), each corresponding to a directivity lobe (105b). In the case of static undulations (Rayleigh, 1879), the lobes $D_n$ all have the same frequency $\omega_s$, and for a sinusoidal shape

$$c_n = J_n((k_\perp - K_\perp)a) \sim O(((k_\perp - K_\perp)a)^n) \sim O((a/\lambda)^n) ,$$

their amplitude relative to the fundamental lobe $D_0$ decays as the ratio of the height of undulations to wavelength. In the limit of a flat interface $a/\lambda \to 0$, only the $n=0$ term remains in Eq. (106), which reduces to Eqs. (96a) and (96b) for a vortex sheet.

### 6. Formation of spectral broadband

The preceding case, of a spectrum consisting of "spikes," each with its own directivity lobe, has been observed for sound reflected from the sea surface (Gazanhes and Léandre, 1974); it corresponds to a deterministic interface, with regular undulations, of height smaller than the wavelength, i.e., low-frequency scattering. In the opposite limit, of diffraction of high-frequency waves by an interface with random irregularities, the interference function (98) is determined statistically. For a Gaussian random process, it will be shown (in Sec. III.B.7) to reduce to

$$C(x,\tau)=\exp[-b(x-u_1\tau)^2/L^2]\ , \tag{107}$$

showing that, beyond a correlation scale $L$, interference cancels most of the acoustic energy. Only a fraction $\exp(-b)$ remains, where $b$ is a constant of order unity. The formula for the spectral directivity (101) becomes, for the interference function (107), introducing the convected separation $s$ (102b),

$$I(\Theta,\omega)=(128\pi^4\rho_1c_1^3M_s)^{-1}\sin^2\Theta\int_{-\infty}^{+\infty}(\omega T_0S/k_\perp)^2\exp\{i[\omega-\omega_0+k_{\|}u_0-(k_{\|}-K_{\|})u_1]\tau\}\times\exp[i(k_{\|}-K_{\|})s-bs^2/L^2]dk_{\|}ds\,d\tau\ , \tag{108}$$

where (i) the integral in $ds$ is of Gaussian type,

$$\int_{-\infty}^{+\infty}\exp[i(k_{\|}-K_{\|})s-bs^2/L^2]ds=L\sqrt{\pi/b}\,\exp[-(k_{\|}-K_{\|})^2L^2/4b]\ , \tag{109}$$

and shows that the spectrum has a "hump" shape, with maximum for the frequency of undeflected transmission $k_{\|}=K_{\|}$; and (ii) the $d\tau\,dk_{\|}$ integrations use properties (99a) and (99b) of the delta functions,

$$\int_{-\infty}^{+\infty}\int f(\omega)\exp\{i[\omega-\omega_0+k_{\|}u_0-(k_{\|}-K_{\|})u_1]\tau\}d\tau\,dk_{\|}=[2\pi/(u_1-u_0)]f(\omega_1)\ , \tag{110a}$$

showing that spectral broadening occurs only if $u_1\neq u_0$, i.e., if the irregular interface $u_1$ is in motion relative to the source $u_0$ and the relevant frequency

$$\omega_1=\omega_0-k_{\|}u_0+(k_{\|}-K_{\|})u_1=\omega_s+(k_{\|}-K_{\|})u_1 \tag{110b}$$

is the source frequency (81), with a convection effect (105a). Thus the sound from a monochromatic source of frequency $\omega_0$, in motion at velocity $u_0$, is scattered by a randomly irregular interface, in relative motion at velocity $u_1\neq u_0$, into a spectral broadband:

$$I_2(\Theta,\omega)=(64\pi^3\rho_1c_1M_s)^{-1}\sqrt{\pi/b}\,(K_\perp/k_\perp)^2[T_0^2S^2/(u_1-u_0)]\exp[-(k_{\|}-K_{\|})^2L^2/4b]\ , \tag{111}$$

which is broader the smaller the wavelength $\lambda$ relative to the correlation scale, $(k_{\|}-K_{\|})^2L^2\sim\lambda^2/L^2$, i.e., for high-frequency waves.

### 7. Correlations of bivariant process

It is clear that the transmission of a monochromatic, high-frequency tone as a spectral broadband is due to diffraction by the random irregularities of the interface, causing Doppler shifts and hence interference between wave components. The statistics of this random process is entirely contained in the interference function (98), which, for a stationary aleatory process, depends only on the spatial $\mathbf{s}$ and temporal $\tau$ separation of the irregularities. For example, in the convection case,

$$C(s)=\langle\exp\{i(k_\perp-K_\perp)[\xi(s_0)-\xi(s_0+s)]\}\rangle\ , \tag{112}$$

where we have taken the mean $\langle\cdots\rangle$ over all possible realizations of the interface. Thus we have a random process (Kolmogorov, 1933; Von Mises, 1964) in two variables, viz., the heights $\xi_1\equiv\xi(s_0)$ and $\xi_2\equiv\xi(s_0+s)$, or phase shifts $\Lambda_{1,2}=(k_\perp-K_\perp)\xi_{1,2}$, and Eq. (112) is the joint characteristic function

$$c(\mu_1,\mu_2)=\langle\exp[i(\mu_1\xi_1-\mu_2\xi_2)]\rangle=\sum_{n,m=1}^{\infty}(-)^m(i^{n+m}/n!m!)M_{n,m}\ , \tag{113a}$$

whose expansion in power series specifies all the moments $M_{n,m}$ of the aleatory process

$$M_{n,m}\equiv\langle\xi_1^n\xi_2^m\rangle=\langle[\xi(s_0)]^n[\xi(s_0+s)]^m\rangle\ , \tag{113b}$$

such as the variance $\sigma^2$ and correlation $E$,

$$\sigma^2\equiv M_{2,0}=M_{0,2}=\langle\xi^2\rangle\ , \tag{114a}$$

$$E\equiv M_{1,1}/M_{0,2}=\sigma^{-2}\langle\xi_1\xi_2\rangle\ . \tag{114b}$$

If we consider a sequence of sound fields at time intervals longer than the correlation time, they are uncorrelated, and by the ergodic theorem (Khinchin, 1948), we can use averages over realizations instead of over time. By the central limit theorem (Lindeberg, 1922), after a long sequence, the random process becomes Gaussian. An irregular, convected interface entrains turbulence, and the Gaussian distribution of phase shifts [Fig. 5(a)] has been observed (Schmidt and Tillmann, 1970) for sound crossing jets and wakes.

### 8. Statistics of sound in turbulence

For a bivariant Gaussian process, the characteristic function, in the terminology of the theory of probability and statistics, is

$$C(s)=\exp\{-\sigma_1^2[1-E(s)]\}\ , \tag{115a}$$

$$\sigma_1^2\equiv(k_\perp-K_\perp)^2\overline{\xi^2}\ , \tag{115b}$$

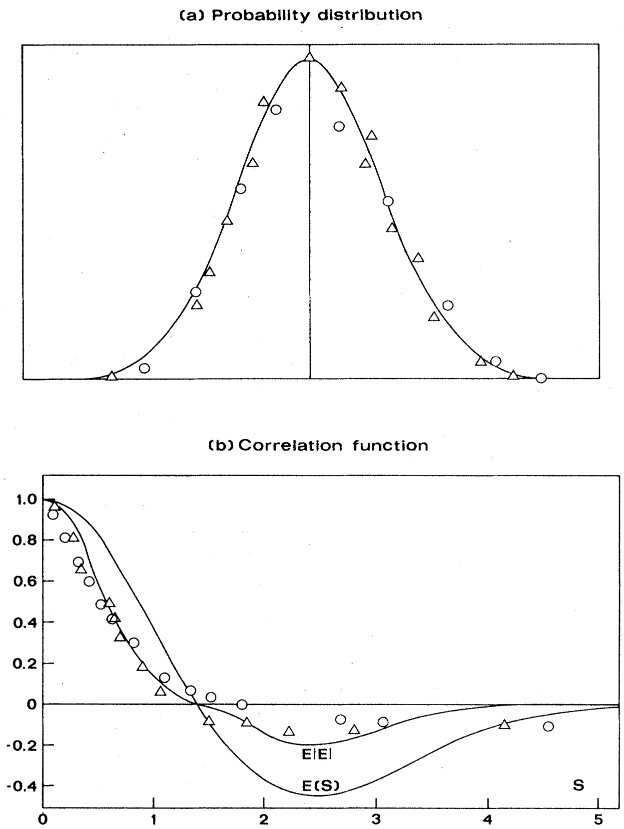


FIG. 5. Comparison of theory (solid lines) and experiments ($\bigcirc,\triangle$) for the statistics of phase shifts of sound transversing turbulent jets and interfaces: (a) Gaussian probability distribution vs measured acoustic phase shifts (Schmidt and Tillmann, 1970) for weak ($\bigcirc$) and strong ($\triangle$) turbulence; (b) correlation function of the acoustic phase shifts of sound (Ho and Kovasznay, 1976a,1976b) of frequency 10 kHz ($\bigcirc$) and 20 kHz ($\triangle$) transmitted across a double-sided jet (from Campos, 1978b).

where the mean deviation $\sigma_1$ is related to ratio of the rms height of irregularities to wavelength $\lambda$, viz., $\sigma_1^2 \sim \overline{\xi^2}/\lambda^2$; the function (115a) can be designated an interference function in wave theory, since it shows that strictly correlated waves $E(s)=1$, i.e., in phase, suffer no attenuation, $C(0)=1$, but uncorrelated components $E(s) \simeq 0$ have a decay factor $\exp(-\sigma_1^2)$. The shape of the spectrum is determined mainly by the correlation coefficient, usually of the form

$$E(s)=p(s/L)\exp(-s^2/L^2) , \tag{116}$$

showing that the correlation becomes negligible beyond a correlation scale, i.e., $E \ll 1$ for $s \gg L$. The polynomial factor adapts to special properties. For example, the interface of a jet should conserve fluid volume, so that the displacement $\xi(s)$ has zero integral, and so does the correlation:

$$\int_{-\infty}^{+\infty} E(s)ds = 0 , \tag{117a}$$

$$p(s/L)=1-2s^2/L^2 . \tag{117b}$$

Equation (117b) is the symmetric polynomial of lowest degree, which gives the correlation function (116) the integral property (117a). We note in passing that for small $s$ we have $E(s)=1-3s^2/L^2$, and thus the asymptotic form of the interference function (115a) is $C(s)=\exp(-3\sigma_1^2 s^2/L^2)$, which agrees with the form used earlier (107), with $b=3\sigma_1^2$. The property of volume conservation is thus met by the correlation function [(116) and (117b)] for a single interface, or $E(s)|E(s)|$ for sound crossing a jet with two interfaces. A type $E(s)$ correlation function (Campos, 1978b) has been observed for the phase shifts of electromagnetic waves in atmospheric turbulence (Tatarski, 1971), and the correlation function $E(s)|E(s)|$ is consistent [Fig. 5(b)] with the phase shifts measured for sound crossing a double-sided jet (Ho and Kovasznay, 1976a,1976b).

### C. Diffraction and interference in random media

We use the designation scattering to describe the propagation of long waves, in homogeneous media, or in the presence of obstacles, of scale shorter than the wavelength, e.g., sound transmission across an interface of thickness small on a wavelength scale; scattering can substantially alter the directivity pattern (Sec. III.A), but not so much the spectrum (Sec. III.B), e.g., sound from a monochromatic source is received as spike(s), with Doppler shifts in the presence of convection. The case of diffraction of high-frequency sound by an irregular interface demonstrates spectral broadening, which is associated with wave interference and aleatory phase shifts in random media (Campos, 1978a,1978b; Wentzel, 1980; Bjørnø and Larsen, 1984); another example is the propagation of sound in turbulence (Lighthill, 1953; Sunyach, Juvé, and Comte-Bellot, 1982), which, in the diffraction limit of wavelength short compared with the length scale of change in the flow, can be studied by the "ray" approximation. Random phase shifts have been observed extensively for high-frequency sound propagating in turbulence (Schmidt and Tillmann, 1970; Ho and Kovasznay, 1976a,1976b; Blanc-Benon and Juvé, 1981a,1981b,1982a, 1982b); for intermediate frequencies, spikes and broadband coexist (Candel, Julienne, and Julliand, 1975; Candel, Guédel, and Julienne, 1976; Beyer and Korman, 1980). The effect is similar to the random Doppler shifts, due to the motion of gas molecules, that widen the emission lines of the Bunsen burner (Rayleigh, 1873,1889, 1915), and to the broadening of line profiles of radiation from the interior and atmospheres of stars such as the sun (Athay, 1976; Bonnet and Dupree, 1980). In aeroacoustics, the theories of propagation of sound in jets with "deterministic" velocity profiles (Atvars, Schubert, and Ribner, 1964; Liu and Maestrello, 1975; Durbin, 1983a,1983b), when compared with wind tunnel measurements (Grosche, Stiewitt, and Binder, 1977; Ross 1981), may require correction for the spectral broadening effects that are observed to occur during transmission across an irregular and turbulent shear layer.

### 1. Convected eikonal equation

A jet of velocity $\mathbf{v}_0$, e.g., a shear layer, usually contains turbulence, with nonuniform, unsteady velocity $\mathbf{v}_1(\mathbf{x},t)$, whose typical rms value is (Barrat, Davies, and Fisher, 1963) about $\alpha_1 \sim 0.15$ of the jet speed; thus, even in a bisonic jet $M_0 = v_0/c_0 = 2$, the local turbulence is incompressible, $M_1^2 = v_1^2/c_0^2 \sim \alpha_1^2 M_0^2 \sim 0.1$, implying that the propagation of sound is described by the convected wave equation,

$$\{[\partial/\partial t + (\mathbf{v}_0 + \mathbf{v}_1)\nabla]^2 - c_0^2\nabla^2\} p(\mathbf{x},t) = 0 , \tag{118}$$

since (i) the mean flow may be of high Mach number, but is uniform (Sec. III.A.5); (ii) the turbulent velocity $\mathbf{v}_1$ is nonuniform, but of low Mach number. The acoustic pressure may be represented by

$$p(\mathbf{x},t) = p_*(\mathbf{x},t)\exp[i\Lambda(\mathbf{x},t)] , \tag{119}$$

where, in the ray approximation, the amplitude $p_*$ varies on a scale $L$ much larger than the wavelength, $L^2 \gg \lambda^2$. In this case the phase $\Lambda$ satisfies the convected eikonal equation

$$[\partial\Lambda/\partial t + (\mathbf{v}_0 + \mathbf{v}_1)\cdot\partial\Lambda/\partial\mathbf{x}]^2 - c_0^2(\partial\Lambda/\partial\mathbf{x})^2 = 0(\lambda^2/L^2) . \tag{120}$$

Defining the wave vector $\mathbf{k}$ and (minus) the frequency $\omega$ as the space and time derivatives, respectively, of the phase,

$$\mathbf{k} \equiv \partial\Lambda/\partial\mathbf{x} , \tag{121a}$$

$$\omega \equiv -\partial\Lambda/\partial t , \tag{121b}$$

yields the wave conservation equation

$$\partial\mathbf{k}/\partial t + \partial\omega/\partial\mathbf{x} = 0 , \tag{121c}$$

which corresponds to the identity $\partial^2\Lambda/\partial\times\partial t = \partial^2\Lambda/\partial t\partial\mathbf{x}$, valid if the phase $\Lambda$ has continuous derivatives of second order.

### 2. Sound rays in turbulence

Substituting (121a) and (121b) into the eikonal equation (120) leads to the dispersion relation

$$\omega(\mathbf{k}) = c_0 k + \mathbf{v}_0\cdot\mathbf{k} + \mathbf{v}_1\cdot\mathbf{k} , \tag{122}$$

showing that the group velocity

$$d\mathbf{x}/dt = \mathbf{w} \equiv \partial\omega/\partial\mathbf{k} = c_0\mathbf{n} + \mathbf{v}_0 + \mathbf{v}_1(\mathbf{x},t) \tag{123}$$

consists of the sound speed $c_0$ in the direction normal to the wave front $\mathbf{n} \equiv \mathbf{k}/k$, plus two convection effects, uniform by the mean flow $\mathbf{v}_0$, and nonuniform and unsteady by the turbulent velocity $\mathbf{v}_1(\mathbf{x},t)$. The sound rays are straight lines for sound in a uniform mean flow $\mathbf{w}_0 = c_0\mathbf{n} + \mathbf{v}_0$, but the component of the turbulent velocity transverse to the wave normal $\mathbf{v}_{1\perp} = \mathbf{n}\times(\mathbf{n}\times\mathbf{v}_1)$ causes "crinkling" of the ray paths; the uniform mean flow causes a constant phase change $\mathbf{k}\cdot\mathbf{v}_0$, relative to sound in a medium at rest, to which the component of the turbulent velocity along the ray $v_{1\|} = \mathbf{v}_1\cdot\mathbf{n}$ adds a random phase shift $\mathbf{k}\cdot\mathbf{v}_1 = kv_{1\|}$. This phase shift implies early (late) arrival of the wave for $\mathbf{k}\cdot\mathbf{v}_1 > 0$ ($\mathbf{k}\cdot\mathbf{v}_1 < 0$), i.e., a phase lead (lag) $\mathbf{k}\cdot\mathbf{v}_1$ per unit time, equivalent to $O((v_1/c_0)^2)$, to a phase shift $\mathbf{k}\cdot\mathbf{v}_1/c_0 = \mathbf{k}\cdot\mathbf{M}_1$ per unit length, or to a total phase change

$$\Lambda_2(\mathbf{x},t) = -\int_R [\mathbf{k}\cdot\mathbf{M}_1(\mathbf{x},t)]ds , \tag{124}$$

along a ray path. Since it can be shown (Campos, 1978b) that sound does not exchange energy with low-Mach-number turbulence, it follows that the main acoustic effect of such turbulence is to transform coherent into "incoherent" beams by causing random phase shifts given by (124).

### 3. Integral transmission operator

The shear layer separating a jet from a medium at rest may be modeled as (i) consisting of an irregular interface, convected at a fraction $\alpha_0 \sim 0.6$ of the jet velocity, across which mass density and sound speed may change, leading to a transmission factor (87b), consisting of amplitude and phase changes; (ii) entraining a layer of turbulence, of rms velocity a fraction $\alpha_1 \sim 0.15$ of the jet velocity, which causes no mean amplitude change, but adds a phase shift (124), so that the total transmission factor is given by

$$T = T_0 \exp[i(\Lambda_1 + \Lambda_2)] , \tag{125}$$

$$\Lambda_1(\mathbf{x},t) = (k_\perp - K_\perp)\xi(\mathbf{x},t) , \tag{126}$$

and consists of (a) an amplitude factor $T_0$ similar [Eq. (75b)] to that of a vortex sheet; (b) a phase shift $\Lambda_1$ due [Eq. (126)] to scattering by the irregularities $\xi$ of the interface; (c) an additional phase shift $\Lambda_2$ due [Eq. (124)] to diffraction of sound rays in turbulence. The total transmission factor (125) may be used to calculate Eqs. (88) and (89), the acoustic pressure spectrum $p_2(\mathbf{K},\omega)$ transmitted across a shear layer, due to a source $S$ in the jet (77):

$$p_2(\mathbf{K},\omega) = \mathscr{T}[S(\mathbf{k},\omega_s)] \equiv (64\pi^s)^{-1}\int_{-\infty}^{+\infty} (T_0/k_\perp) S(i\mathbf{k}_\|, ik_\perp, -i\omega_s)\exp[i(\mathbf{K}_\| - \mathbf{k}_\|)\cdot\mathbf{x}_\| + i(\omega-\omega_s)t]\exp[i\Lambda(\mathbf{x},t)]d^2\mathbf{k}_\| d^2\mathbf{x}_\| dt . \tag{127}$$

This equation may be taken as the definition of the integral transmission operator $\mathscr{T}$. In the case of a vortex sheet, there are no random phase shifts,

$$\Lambda(\mathbf{x},t)\equiv\Lambda_1+\Lambda_2=0 , \tag{128}$$

$$p_2(\mathbf{K},\omega)=(8\pi^2)^{-1}(T_0/k_\perp)S(i\mathbf{K},-i\omega_s)\times\delta(\omega-\omega_s) , \tag{129}$$

and the transmission operator $\mathscr{T}$ reduces to multiplication by the amplitude factor $T_0$; otherwise it is its generalization to turbulent and/or irregular shear layers.

### 4. Multiple refraction series

The integral operators, like Eq. (127), describing refraction of sound by shear layers, and including scattering of low-frequency waves by interfaces and diffraction of high-frequency waves by turbulence, can also be applied to more complex systems of multiple shear layers. An example of practical use as a sound attenuator is the "shielding jet," with two shear layers separating source and observer. The sound field $p_2$ received by the observer "above" the jet is related to the sound field $p_1$ emitted by the source "below" (i.e., on the other side of the jet), through the sound field in the jet, which we decompose into upward $p_+$ and downward $p_-$ propagating components. The four sound fields $p_1,p_2,p_+,p_-$ are related by integral operators like (127) as follows.

(i) The sound field propagating upward in the jet $p_+$ is due to transmission $\mathscr{T}_-$, across the lower shear layer, of sound $p_1$ from the source, plus reflection $\mathscr{R}_-$, at the lower shear layer, of the downward propagating fields $p_-$,

$$p_+=\mathscr{T}_-\{p_1\}+\mathscr{R}_-\{p_-\} . \tag{130}$$

(ii) The sound field $p_+$ propagating upward in the jet, incident on the upper shear layer, is reflected $\mathscr{R}_+$ into the downward-propagating field $p_-$ in the jet and transmitted $\mathscr{T}_+$ to be received $p_2$ by the observer outside the jet:

$$p_-=\mathscr{R}_+\{p_+\} , \tag{131}$$

$$p_2=\mathscr{T}_+\{p_+\} . \tag{132}$$

The system of coupled integral equations [(130)—(132)] can be solved (Campos, 1978b) to express the sound field $p_2$ received by the observer in terms of that $p_1$ emitted by the source,

$$p_2=(\mathscr{T}_+\mathscr{T}_-+\mathscr{T}_+\mathscr{R}_-\mathscr{R}_+\mathscr{T}_-+\mathscr{T}_+\mathscr{R}_-\mathscr{R}_+\mathscr{R}_-\mathscr{R}_+\mathscr{T}_-+\cdots)\{p_1\} , \tag{133}$$

by means of the refraction series (in curly brackets), whose terms can be interpreted as follows: (a) the zeroth-order term $\mathscr{T}_+\mathscr{T}_-$ represents "direct" transmission from source to observer across the lower $\mathscr{T}_-$ and upper $\mathscr{T}_+$ shear layers; (b) the $n$th-order term $\mathscr{T}_+\mathscr{D}^n\mathscr{T}_-$ involves $n$ intermediate double reflections $\mathscr{D}\equiv\mathscr{R}_-\mathscr{R}_+$ in the jet, at the upper $\mathscr{R}_+$ and lower $\mathscr{R}_-$ shear layers, after transmission from the source $\mathscr{T}_-$ and before transmission $\mathscr{T}_+$ to the observer.

### 5. Attenuation factor

Having shown that the method of integral refraction operators can be applied to any system of shear layers, we return to the single-layer case to calculate the spectral directivity [Eq. (97)] or acoustic energy radiated per unit solid angle and frequency band. Since the spectral directivity is quadratic in the acoustic fields, it depends on the interference function (98), which is the mean value, over all realizations of the shear layer, of the total phase shift [Eq. (128)], due to irregularities [Eq. (126)] and turbulence [Eq. (124)], for two wave components a distance $s$ and time $\tau$ apart:

$$C(\mathbf{s},\tau)=\langle \exp[i\Lambda(x,t)-i\Lambda(x+s,t+\tau)]\rangle . \tag{134}$$

For a stationary Gaussian process, Eq. (115a) is given by

$$C(s)=\exp\{-\sigma^2[1-E(s)]\} , \tag{135a}$$

$$\sigma^2\equiv\sigma_1^2+\sigma_2^2 , \tag{135b}$$

where the variance of phases $\sigma^2$ is the sum of those due to scattering by irregularities $\sigma_1^2$ and diffraction by turbulence $\sigma_2^2$, assuming the two processes to be statistically independent. The variance $\sigma_2^2$ scales (124) as

$$\sigma_2^2=\langle[\Lambda_2(\mathbf{x},t)]^2\rangle=(k^2l^2/c_0^2)\overline{v_1^2} , \tag{136}$$

on the squares of wave number $k$ and sound speed $c_0$, the length of the ray $l$ in the turbulent region, and the rms turbulent velocity. The sum of Eqs. (136) and (115b) is the effective attenuation factor $\sigma^2$ for uncorrelated sound waves $E(s)\simeq 0$ transmitted through the shear layer, since in this case the interference function (135a) reduces to a constant $C=\exp(-\sigma^2)$. Note that the mean phase shifts due to scattering [Eq. (126)] and diffraction [Eq. (124)] are zero, $\langle\Lambda\rangle=0$, because the irregularities of the interface $\xi$ and turbulent velocity $\mathbf{v}_1$ have zero mean $\langle\xi\rangle=0=\langle\mathbf{v}_1\rangle$; the variance of the phase shifts is not zero, and specifies the attenuation factor $\sigma^2$, which has two contributions due to (i) scattering by the interface [Eq. (115b)], which vanishes only for the direction of undeflected transmission $k_\perp=K_\perp$ and increases away from it, scaling as the square of the ratio of rms height to wavelength $\sim\overline{\xi^2}/\lambda^2$; (ii) diffraction by turbulence [Eq. (136)], which does not vanish in any direction and increases away from the vertical, as the ray paths in turbulence become longer $l\sim\csc\theta$, scaling as the squares of the Mach number of turbulence $\sim\overline{v_1^2}/c_0^2$ and ray length divided by wavelength $\sim l^2/\lambda^2$.

### 6. Correlation scale

The variance $\sigma_2$, as well as the correlation scale $L$ appearing in the correlation function $C$ [Eq. (116)] can be calculated explicitly from the statistical theory of turbulence (Batchelor, 1953; Townsend, 1956; Hinze, 1975;

Deissler, 1984). We denote by $M_{ij} \equiv \langle M_i M_j \rangle$ the autocorrelation of the turbulent Mach number $M_i$, and by $N_{ij}(\boldsymbol{\kappa},\chi)$ the turbulence spectrum of wave vector $\boldsymbol{\kappa}$ and frequency $\chi$; sound waves are affected by the reduced spectrum,

$$N(\boldsymbol{\kappa}) \equiv n_i n_j N_{ij}(\boldsymbol{\kappa}, c_0 \mathbf{n}\cdot\boldsymbol{\kappa}) , \tag{137}$$

which is the projection, upon the wave normal direction $n_i$, of the spectrum of turbulence $N_{ij}$, with frequency $\chi = c_0 \mathbf{n}\cdot\boldsymbol{\kappa}$ determined by the sound speed $c_0$, and projection of the wave vector of turbulence $\boldsymbol{\kappa}$ on the acoustic wave normal. For example, in the case of isotropic turbulence it is given by

$$N(\boldsymbol{\kappa}) = N_0(\kappa)[(\boldsymbol{\kappa}\cdot\mathbf{m})^2 - \kappa^2] , \tag{138}$$

where $N_0(k)$ is the turbulence spectrum function. The variance of acoustic phase shift due to scattering by turbulence is (Campos, 1978b) the integral of the reduced spectrum, over turbulence wave vector space $W$, and along the ray $\mathscr{R}$:

$$\sigma_2^2 = \int_R \int_W N(\boldsymbol{\kappa}) d^3\boldsymbol{\kappa}\, ds , \tag{139}$$

i.e., the variance is larger for longer ray paths in stronger turbulence; the correlation scale is

$$L^{-2} = \sigma_2^{-2} \int_R \int_W \kappa^2 N(\kappa) d^3\boldsymbol{\kappa}\, ds , \tag{140}$$

shorter for large wave number $\kappa$ (i.e., flow disturbances concentrated on small scales) and longer for small wave number (i.e., flow disturbances on large scales). The interference function (135a) takes the form

$$C(s) = e^{-\sigma^2} \sum_{n=0}^{\infty} (\sigma^{2n}/n!)(1-2s^2/L^2)^n e^{-ns^2/L^2} , \tag{141}$$

where (i) the first term $e^{-\sigma^2}$ is an attenuation due to the variance of phase shifts for uncorrelated wave components; (ii) the remaining terms $(\sigma^2 E)^n/n!$ of all orders reduce this attenuation, in agreement with Eqs. (116) and (117b), the correlation functions for wave components separated less than length scale $s < L$. Since the interference function is less than unity, $C(s) < 1$, an irregular and turbulent shear layer always transmits less acoustic energy than a vortex sheet between the same media.

## 7. Spike and sidebands

A turbulent and irregular shear layer also distributes the acoustic energy over a wider range of directions and frequencies than a vortex sheet, as can be shown by calculating the spectral directivity [Eq. (101)],

$$I(\Theta,\omega) = (128\pi^4 \rho_1 c_1^3 M_s)^{-1} \sin^2\Theta \int_{-\infty}^{+\infty} (\omega T_0 S/k_\perp)^2 \exp\{i(k_\parallel - K_\parallel)s + i[\omega - \omega_s - (k_\parallel - K_\parallel)u_0]\tau\} C(s) dk_\parallel ds\, d\tau , \tag{142}$$

with the interference function [Eq. (141)]. The radiated power at frequency $\omega$, in the direction $\Theta$, is given by

$$I(\Theta,\omega) = e^{\sigma^2} D_0(\Theta)\delta(\omega - \omega_s) + \sum_{n=1}^{\infty} I_n(\Theta,\omega) , \tag{143}$$

where (i) the zeroth-order term, corresponding to the first term in Eq. (141), is the directivity of the plane vortex sheet (96b), at the source's frequency (81), with attenuation (135b) due to irregularities (115b) and turbulence (136); (ii) all the remaining terms are spectral broadbands, whose energy totals less than the attenuated from the "spike" at source's frequency. The $n$th broadband is given by

$$I_n(\Theta,\omega) = (64\pi^3 \rho_1 c_1^3 M_s)^{-1} \times \sin^2\Theta[(\omega_1 T_0 S/k_\perp)^2 e^{-\sigma^2}/(u_1 - u_0)]\Pi_n , \tag{144}$$

where the frequency $\omega_1$ [Eq. (110b)] includes the relative motion of the shear layer past the source $u_1 \neq u_0$, and where the shape function (Campos, 1984c)

$$\Pi_n \equiv \int_{-\infty}^{+\infty} (1-2s^2/L^2)^n e^{-ns^2/L^2} \exp[i(k_\parallel - K_\parallel)s] ds = L\sqrt{\pi} \exp(-\Omega^2) \sum_{m=0}^{n} [(2n)^m (n-m)! n!]^{-1} H_{2m}(\Omega) \tag{145}$$

is a Gaussian hump, modified by Hermite polynomials $H_{2m}$, with the dimensionless frequency $\Omega \equiv (k_\parallel - K_\parallel)L/2\sqrt{n}$ as variable. $\Omega = 0$ at the source's frequency $\omega = \omega_1$, and $\Omega > 0$ otherwise. The spectrum decays far from the source's frequency; near to the source frequency (small $\Omega$), the shape of the spectrum is determined by the Hermite (1864) polynomials $H_0(\Omega) = 1$ and $H_2(\Omega) = 4\Omega^2 - 2$ (Courant and Hilbert, 1953), so that the first sideband $\Pi_1(\Omega) = H_0 - H_2/2 = 2\Omega^2$ has a dip, partially "filled" at higher orders $n = 2, 3, \ldots$ . Thus the total spectrum consists of an attenuated spike, at the source frequency, plus a series of sidebands, leaving a dip near the spike.

## 8. Experimental and aircraft noise

This kind of spectrum, consisting of a spike plus sidebands (Campos, 1978c), has been observed (Candel, Guédel, and Julienne, 1976) for a sound received from a monochromatic test source placed inside a low-speed air jet. There is good agreement between the theory and observation, as shown in Fig. 6. It can be seen from this figure that spectral broadening increases with jet velocity and source frequency, the latter also leading to more significant attenuation of the spike. The simulation of the transmission of sound for the hot, high-speed jet exhausts

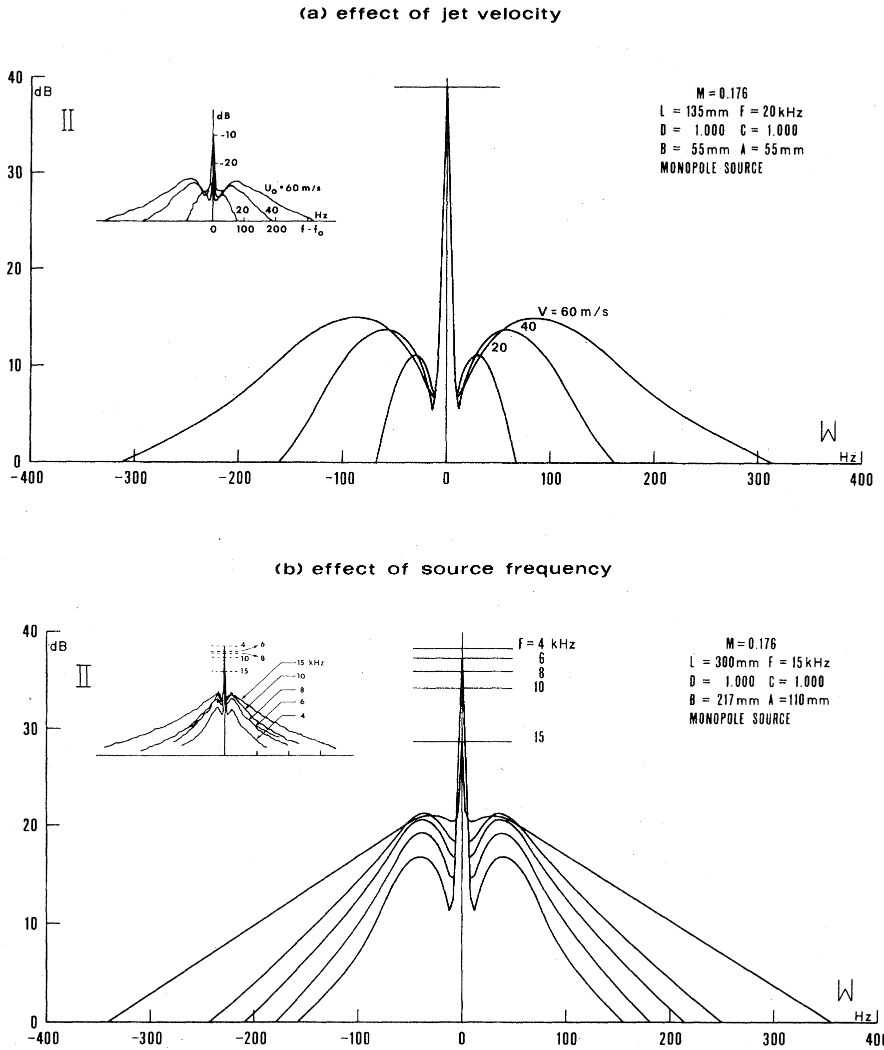


FIG. 6. Comparison of theory (large plots) with measurements (inset plots) of the spectra (Campos, 1978c) received from a monochromatic test source placed inside a cold air jet (Candel, Guédel, and Julienne, 1976). The effect of increasing the jet velocity (a) is to enhance the sidebands; increasing the source frequency (b) also reinforces spectral broadening and accentuates the attenuation of the spike.

of Concorde in takeoff conditions is presented in Fig. 7, for a monopole source with a turbine tone frequency of 8 kHz. The spectra, given at 15° intervals, extend more to the high frequencies in the downstream arc, and more to the low frequencies in the upstream direction, as observed for low-speed shielding jets (Candel, Julliand, and Julienne, 1975) in a less marked form. The turbulent and irregular shear layer transmits a broadband into the zone of silence ($\Theta < 73°$) of a vortex sheet, and a spike and sideband into other directions; the spike could be absorbed into the broadband in all directions (second set of curves) by doubling the thickness of the shear layer, a procedure roughly equivalent to using a shielding jet with two shear layers. The plot of directivity (at the bottom) shows that the acoustic emission for a vortex sheet (solid line) is spread over a wider range of directions for a single shear layer (○) and further reduced in intensity by a double shear layer (△).

## IV. PROPAGATION IN NONUNIFORM HORNS AND NOZZLES

The propagation of waves in nonuniform media is characterized by a dimensionless compactness parameter $\varepsilon = kL \equiv 2\pi L/\lambda$, which is essentially (apart from the factor $2\pi$) the ratio of the length scale of nonuniformities $L$ to the wavelength $\lambda$. The limiting cases of small and

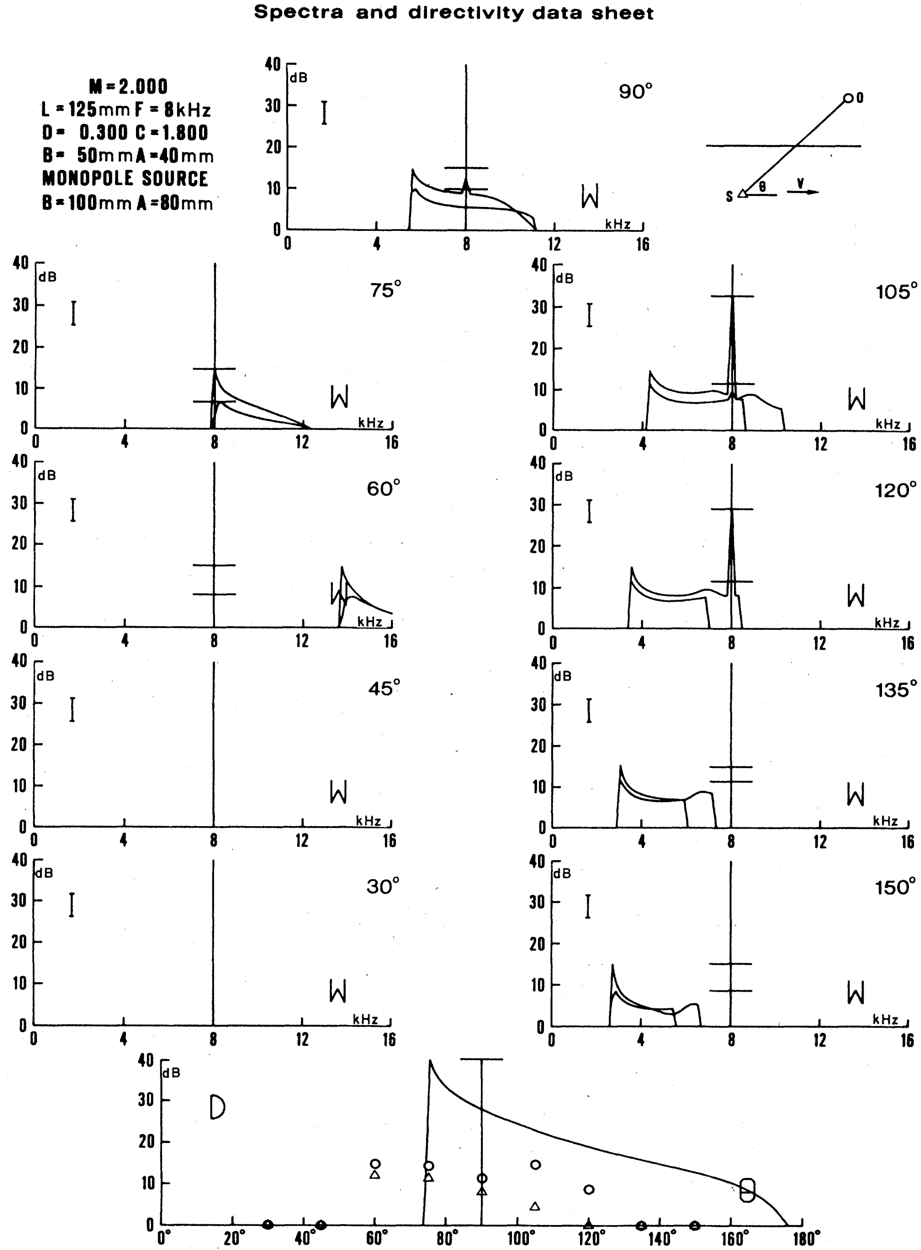


FIG. 7. Simulation of the noise radiation from a source (e.g., combustion noise) at the frequency of the turbine tone 8 kHz, in the hot, high-speed jet exhaust of Concorde, in the takeoff configuration. Spectra are given, in nine directions, at 15° intervals, for "actual" single-shear-layer and a possible double-shear-layer configuration. The directivity plot, at the bottom, compares the single (○) and double (△) shear layers with the vortex sheet model (solid line).

large compactness have both been considered, respectively, (i) $\varepsilon \ll 1$ for scattering of sound by "compact" bodies (Sec. II.C) and "thin" vortex sheets (Sec. III.A), and (ii) $\varepsilon^2 \gg 1$ for the diffraction of rays by irregularities (Sec. III.B) and turbulence (Sec. III.C). The strongest interaction between waves and the nonuniformity of the medium occurs in case (iii) for a wavelength of the order of the length scale $\lambda \sim L$, or compactness of order unity $\varepsilon \sim 1$, which requires exact solutions of the wave equations. A simple example of all three cases [(i), (ii), (iii) above] is the propagation of the fundamental longitudinal acoustic mode in a duct of varying cross section. The acoustics of horns (Lagrange, 1760; Euler, 1772) together with the vibrations of tapering strings (Euler, 1764,1766; Bernoulli, 1767), provide the prototype problem of waves in nonuniform media, following the first studies of waves in "uniform" media, viz., the vibrations of uniform strings (D'Alembert, 1747). These early researches bear witness to the analogies between waves in fluids (Truesdell, 1955) and in solids (Truesdell, 1960); a similar historic process of extension, from waves in uniform to nonuniform media, also occurred for other types, e.g., electromagnetic waves in free space (Maxwell, 1873) and in transmission lines (Heaviside, 1882). Thus the acoustics of horns is a suitable introduction to the general properties of waves in nonuniform media. For simplicity, we concentrate on quasi-one-dimensional models.

### A. General properties of acoustic ducts

The theory of waves in nonuniform media is analogous (Eisner, 1966; Campos, 1984a) for a variety of modes, wider than the preceding brief historical survey would suggest. The following related problems are discussed in the literature: (i) propagation of sound in fluid-filled horns, e.g., air ducts of varying cross section (Rayleigh, 1916; Webster, 1919); (ii) vertical oscillations of a compressible fluid under gravity, e.g., an atmosphere (Rayleigh, 1890; Lamb, 1910); (iii) water waves in tapering channels (Green, 1837) and the "water hammer" in hydraulics (Paynter and Ezekiel, 1958); (iv) longitudinal vibrations of "solid" horns, i.e., tapering elastic bars (Merkulov, 1957; Eisner, 1963); (v) transverse vibrations of tapering strings (Morse and Ingard, 1968) and torsional oscillations of tapering bars (Pyle, 1967); (vi) electromagnetic waves in nonuniform transmission lines (Stevenson, 1951a; Schwartz, 1974). We note in passing that the ray theory (Hamilton, 1827—1832), sometimes referred to as the WKB approximation, after its first users in quantum mechanics, Wentzel (1926), Kramers (1926), and Brillouin (1926), or WKBJ to include a somewhat earlier account in applied mathematics by Jeffreys (1924), was used much earlier (Green, 1837) for water waves in narrow channels tapering gradually. From a conceptual point of view, there is an analogy between classical mechanics (Whittaker, 1904; Landau and Lifshitz, 1949) and the ray approximation, with particle trajectories corresponding to sound rays; the transition from classical to quantum mechanics (Pauling and Wilson, 1935; Landau and Lifshitz, 1966) corresponds to the extension of the ray approximation to the exact theory of waves in nonuniform media, to which we now turn.

#### 1. Alternative wave equations

The acoustics of ducts is described generally by the three-dimensional wave equations (Secs. II.B.2, II.B.4, and II.B.6), with appropriate boundary conditions at the walls. For example, an impedance $Z=p/v$ condition, specifying the ratio of acoustic pressure $p$ to velocity $v$, is often used to model acoustic liners in the locally reacting approximation. The boundary condition is not needed, i.e., is implied in the quasi-one-dimensional form of the wave equation (Secs. II.B.3, II.B.5, and II.B.7) for the fundamental longitudinal acoustic mode. The horn wave equation (41) for the acoustic potential becomes, for the pressure $p=\rho_0\partial\varphi/\partial t$ [see Eq. (24b)] and the velocity $v=\partial\varphi/\partial x$ [see Eq. (24a)]

$$(\partial^2/\partial t^2-c_0^2A^{-1}\partial/\partial x A\,\partial/\partial x)p(x,t)=0\ , \tag{146a}$$

$$(\partial^2/\partial t^2-c_0^2\partial/\partial x A^{-1}\partial/\partial x A)v(x,t)=0\ , \tag{146b}$$

where $A(x)$ is the cross-sectional area, which, for a rigid-walled horn, depends only on the longitudinal coordinate $x$. The wave equation is the same for all variables in a homogeneous medium, but may take different forms for different variables in nonuniform media. As an instance of this we note that the horn wave equations for the pressure (146a) and velocity (146b) coincide only in the case

$$(A'/A)'=0\ , \tag{147a}$$

$$A/A'\equiv L=\text{const}\ , \tag{147b}$$

$$A(x)=A(0)e^{x/L}\ , \tag{147c}$$

of an exponential horn (147c), with constant length scale $L$ (147b), for variations in cross-sectional area $A'\equiv dA/dx$; the limiting case of infinite length scale $L\to\infty$ is the uniform duct $A(x)\sim$const, for which the horn wave equations (146a) and (146b) not only coincide, but also reduce to the classical wave equation.

#### 2. Equipartition of energy

Before proceeding to derive general properties of the acoustics of horns, we recall the conditions of validity of Eqs. (146a) and (146b), which are discussed in more detail elsewhere (McLachlan, 1935b; Campos, 1985a), namely (i) that the duct have a straight axis, hard, smooth walls, and no internal obstacles; and (ii) that the sound waves be of small amplitude and propagate a plane wave front perpendicular to the duct axis. The latter condition is satisfied if the wavelength is larger than the transverse dimensions of the duct, i.e., if only the fundamental longitudinal acoustic mode can exist, and if acoustic quantities are averaged over the cross section (Stevenson, 1951b). Note that for a spherical wave in a conical duct, the ratio of the area of the wave front to the cross section is a constant, and for other nonuniform ducts its variation along the duct axis is neglected. With reference to (a) plane waves in uniform tubes and (b) spherical waves in conical ducts, we recall that the former (a) satisfy the equipartition of kinetic and compression energies everywhere, whereas the latter (b) do not comply with equipartition, except asymptotically, at large distance $r$, as the wave fronts become flat on a wavelength scale $\lambda$, viz., $kr=2\pi r/\lambda\gg 1$. This remark raises the issue of which are the horn shapes for which equipartition of energy holds exactly at all stations; to answer this question we note that the kinetic [Eq. (26a)] and compression [Eq. (26b)] energies are equal everywhere if the acoustic velocity $v$ and pressure $p$ are related by

$$p(x,t)=\rho_0 c_0 v(x,t) \tag{148}$$

all along the duct. Assuming this is true initially at $x=0$, it will remain so for all $x\neq 0$, if $p,v$ satisfy the same wave equation, i.e., if Eqs. (146a) and (146b) coincide, implying that Eqs. (147a), (147b), and (147c) are satisfied. Thus there is only one nonuniform acoustic duct satisfying equipartition of kinetic and compression energies exactly, at all stations $x$ for all frequencies $\omega$, namely, the exponential horn.

### 3. Duality principle

Another exclusive property of the exponential horn is that, if we define dual ducts as those having inverse cross sections $A(x),1/A(x)$, the exponential is the only self-dual shape; that is, the dual of an exponential horn [Eq. (147)] is another exponential horn $1/A_0(x) = [(1/A_0)]e^{-x/L}$, one being convergent if the other is divergent, i.e., the length scales are $L,-L$. In order to demonstrate the properties of dual ducts $A,1/A$ in general, it is convenient to apply the operators $A\partial/\partial x$ to Eq. (146a) and $\partial/\partial xA$ to Eq. (146b), leading, respectively, to

$$[\partial^2/\partial t^2-c^2(1/A)^{-1}\partial/\partial x(1/A)\partial/\partial x]A\partial p/\partial x=0\ , \tag{149a}$$

$$[\partial^2/\partial t^2-c^2\partial/\partial x(1/A)^{-1}\partial/\partial x(1/A)]\partial(vA)/\partial x=0\ . \tag{149b}$$

Comparing the three pairs of equations (146a),(146b); (146a),(149a); (146b),(149b), we obtain three equivalent statements of the duality principle:

$$A(x)p(x,t)\leftrightarrow 1/A(x),A(x)v(x,t)\ , \tag{150a}$$

$$A(x),p(x,t)\leftrightarrow 1/A(x),A(x)\partial p(x,t)/\partial x\ , \tag{150b}$$

$$A(x),v(x,t)\leftrightarrow 1/A(x),\partial[A(x)v(x,t)]/\partial x\ . \tag{150c}$$

Equation (150a) is the original form (Pyle, 1965), stating that the acoustic pressure $p$ in a duct $A$ coincides with the volume velocity (cross section times velocity) $Av$ in its dual $1/A$. Equations (150b) and (150c) are the alternative forms (Campos, 1985a), involving only pressure or velocity, stating that if $p$ ($v$) is the acoustic pressure (velocity) in a duct $A$, then $A\partial p/\partial x$ [$\partial(Av)/\partial x$] is the acoustic pressure (velocity) in the dual duct $1/A$.

### 4. Constant pressure cutoff

We introduce the reduced acoustic variables $p,v(x;\omega)$ defined by

$$v,p(x,t)\equiv[A(x)]^{-1/2}e^{-i\omega t}v,p(x;\omega)\ , \tag{151}$$

for a wave of frequency $\omega$; it follows that the horn wave equation for the pressure [Eq. (146a)] takes a Schrödinger form:

$$[d^2/dx^2+J(x)]v,p(x;\omega)=0\ , \tag{152}$$

$$J_p(x)\equiv\omega^2/c_0^2-A''/2A+(A'/2A)^2\ , \tag{153}$$

where the wave invariant [Eq. (153)], written in the form

$$J=(\omega^2-\omega_*^2)/c_0^2\ , \tag{154a}$$

$$\omega_*=c_0/2l\ , \tag{154b}$$

specifies the cutoff frequency $\omega_p$ for the acoustic pressure, since for lower (higher) frequencies $\omega<\omega_p$ ($\omega>\omega_p$) we have $J_p<0$ ($J_p>0$), implying that Eq. (152) has only standing-mode (also propagating-wave) solutions. The horns of constant cutoff frequency for the pressure [Eq. (153)] have cross sections satisfying

$$2AA''-A'^2-A^2/l^2=0\ , \tag{155}$$

where $l$ is a constant length scale; the solution of Eq. (155) is

$$A_2(x)=A_2(0)[\cosh(x/2l)+\beta_2\sinh(x/2l)]^2\ , \tag{156}$$

i.e., the hypex family of horns, which are the only shapes having a constant cutoff frequency for the acoustic pressure.

### 5. Constant velocity cutoff

If we use the duality principle (150a), the preceding result implies that the only ducts with constant cutoff frequency for the acoustic velocity are the inverse hypex family:

$$A_3(x)=A_3(0)[\operatorname{sech}(x/2l)+\beta_3\operatorname{csch}(x/2l)]^2\ . \tag{157}$$

This result can be proved directly by noting that the transformation (151) applied to the horn wave equation for the velocity (146b) also reduces it to the Schrödinger form (152), with an invariant

$$J_v(x)=\omega^2/c_0^2+A''/2A-3(A'/2A)^2 \tag{158}$$

for the reduced velocity $v(x;\omega)$, which is similar to that [Eq. (153)] for the reduced pressure $p(x;\omega)$, replacing $A$ by $1/A$. It can be checked that the condition (154a) of a constant cutoff frequency (154b), leads to the differential equation for the cross section

$$2AA''-3A'^2+A^2/l^2=0\ , \tag{159}$$

whose solution is Eq. (157). The common member of the hypex [Eq. (156)] and inverse [Eq. (157)] families of ducts is the exponential horn [Eq. (147c)], which can be obtained setting $\beta_2=1$, $L=l$ in Eq. (156). Thus the exponential horn is the only duct with constant cutoff frequencies for *both* the acoustic velocity and the pressure, which are identical, $\omega_*=c/2L$. The fact that the exponential duct (147c) has a constant cutoff frequency is well known (Lighthill, 1978a; Dowling and Ffowcs-Williams, 1983), but we have proved the stronger result that (i) *no* other duct has the same constant cutoff frequency for both the acoustic velocity and pressure, and that (ii) if we require a constant cutoff for pressure (velocity) *alone*, then the wider additional class of shapes meeting it is the hypex (inverse) family [Eqs. (156) and (157), respectively].

### 6. Existence of elementary solutions

If we replace, in the hypex family [Eq. (156)], the real length scale $l$ by an imaginary one $il$, we obtain the sinusoidal family,

$$A_4(x)=A_4(0)[\cos(x/2l)+\beta_4\sin(x/2l)]^2 \;, \qquad (160)$$

whose cross section satisfies the differential equation

$$2AA''-A'^2+A^2/l^2=0 \;. \qquad (161)$$

The cutoff frequency [Eq. (154b)] becomes "imaginary" $\omega_*=-ic_0/2l$, i.e., there is no real cutoff, and the sinusoidal family of ducts is acoustically "transparent," that is, waves of all frequencies can propagate, since the wave invariant in Eq. (152) is always positive:

$$J=(\omega^2+\omega_*^2)/c_0^2 \;. \qquad (162)$$

The wave invariant takes the same form [Eq. (162)] in the wave equation for the reduced velocity (instead of pressure), if we consider the inverse sinusoidal family

$$A_5(x)=A_5(0)[\sec(x/2l)+\beta_5\csc(x/2l)]^2 \;, \qquad (163)$$

whose cross section satisfies the differential equation

$$2AA''-3A'^2-A^2/l^2=0 \;, \qquad (164)$$

which is the dual of Eq. (161), i.e., it follows from Eq. (161) by the transformation $A\rightarrow 1/A$, and from Eq. (159) by the change $l\rightarrow il$. The reduced wave equation (152) has elementary solutions, expressible in finite terms, using only exponential circular and hyperbolic functions, if the wave invariant $J$ is a constant. Since it is real, the only possible cases are positive [Eq. (162)] and negative or zero [Eq. (154a)]; thus we conclude that the horn wave equation [(146a) and (146b)] for the acoustic pressure/velocity has exact elementary solutions only for the sinusoidal (160) and hypex (156) families/inverse shapes [Eqs. (163) and (157)].

### 7. Exclusive families of shapes

We have used the acoustics of ducts to identify gradually wider families of horns, according to the properties they satisfy:

Exact equipartition of kinetic and compression energies: exponential horn;

Constant cutoff frequencies both for acoustic velocity and pressure: exponential horn;

Constant cutoff frequency for acoustic pressure (velocity) alone: hypex (inverse) family;

Existence of exact elementary solutions of the horn wave equation for the acoustic pressure (velocity): hypex and sinusoidal (inverse) families.

### 8. Musical instruments

The exponential horn (147c) is a single-parameter $L$ family, including the uniform duct for $L\rightarrow\infty$; the hypex [Eq. (156)], sinusoidal [Eq. (160)], and inverse [Eqs. (157) and (163)] horns are two-parameter $\beta,l$ families, with the hypex family reducing to the exponential for $\beta=1,l=L$. The importance of these exclusive families is not so much that these horns have the stated properties, but rather that all others do not; thus the properties stated, which are taken for granted for waves in homogeneous media, hold only exceptionally for waves in nonhomogeneous media. We may conclude, for example, that exact solutions of the horn wave equations for any shapes other than hypex, sinusoidal, and inverse require the use of special functions, in finite or infinite form. The simple shapes of horn considered are not without practical relevance, since they have been used in various devices long before being analyzed theoretically; examples include the "hearing trumpet" of exponential shape and the mouths of musical instruments (Fig. 8), e.g., sinusoidal for the English (Nagarkar and Finch, 1971) and power law for the French (Benade, 1976) horns, which have been in use for over two centuries.

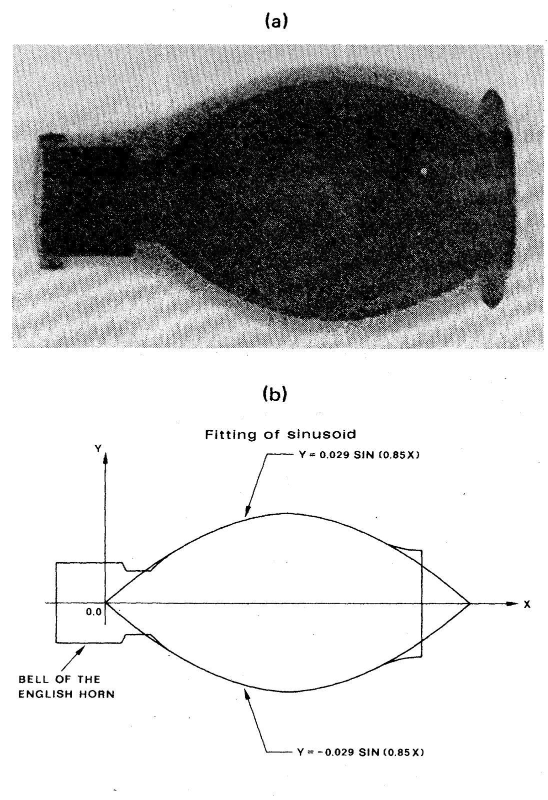


FIG. 8. Cross section of the mouth of an English horn (Nagarkar and Finch, 1972), (a) as seen in a gamma-ray photograph, compared with (b) a sinusoid fitting. Other simple duct shapes include the power law for the French horn and the exponential for the "ear trumpet."

### B. Exact solutions for specific profiles

The preceding account of the general properties of acoustic horns gives, by implication, exact elementary expressions for the sound fields for five families of shapes, namely, (i) the exponential horn (Hanna and Slepian, 1924); (ii) the hypex family (Salmon, 1946a); (iii) the sinusoidal family (Nagarkar and Finch, 1971); (iv) the inverse catenoidal (Campos, 1984a), and, as a consequence, (v) inverse sinusoidal shapes. Exact solutions for other shapes involve special functions, in finite or infinite form, e.g., Hermite functions (Bies, 1962) for the Gaussian duct (Parodi, 1945) and Bessel functions (Ballantine, 1927) for the power-law ducts (Lagrange, 1760), the latter including, as particular cases, the conical horn (Stewart, 1920) containing a spherical wave (Euler, 1759), and the parabolic (Olson and Wolff, 1930) and hyperbolic (Freehafer, 1940) shapes. More elaborate shapes include the tractrix horn (Lambert, 1954) and parametric families (Mawardi, 1949; Molloy, 1975), obtained by transformations of simple shapes; another possibility is the matching of different ducts (Poisson, 1817) to obtain more desirable impedance characteristics (Olson, 1938; Merkulov and Kharitonov, 1959) or as a means of approximating numerically arbitrary shapes (Zamorski and Wyrzykowski, 1981), a related topic being the effects of cross-sectional changes in wave guides (Miles, 1981; Hasegawa, 1983; Grigoryan, 1984; Thomson, 1984). Other topics in the acoustics of horns include transient (McLachlan and McKay, 1936) and finite-amplitude (Goldstein and McLachlan, 1935; Nayfeh, 1975b) effects, amplitude, and phase laws (Salmon, 1946b), resonance (Thiessen, 1950), radiation (Benade and Jansson, 1974; Jansson and Benade 1974), cutoff frequencies (Kergomard, 1981), finite-length effects (Wang and Tszeng, 1984), random scattering (Macaskill and Uscinski, 1981), effects of elastic walls (Barclay, Moodie, and Haddow, 1977; Sinai, 1981), undulated walls (Nayfeh, 1975a; Bostrom, 1983), or collapsible walls (Lighthill, 1975; Pedley, 1980), internal gradients of density (Shaw, 1970) or temperature (Cole, 1979), thermal dissipation (Keefe, 1984) and viscous dissipation (Kergomard, 1981) at the walls. Extensions include two-dimensional horns (Yeow, 1974) and baffles (Cho, 1980); three-dimensional waves in tubes of a simple shape have been a subject of longstanding interest (Duhamel, 1839; Pochhammer, 1876; Barton, 1908; Hoersch, 1925).

#### 1. Filtering function

We begin our consideration of some exact solutions of the horn wave equation with the simplest shape, namely, the exponential duct, which has a constant length scale,

$$L(x) \equiv A(x)/A' = \{d[\ln A(x)]/dx\}^{-1} = l \ . \tag{165}$$

The horn wave equation is identical, in this case, for velocity and pressure, which are given (151) by

$$v_1, p_1(x;\omega) = e^{-x/2l} e^{-i\omega t} f(x;\omega) \ , \tag{166}$$

where $f(x;\omega)$ is the filtering function, satisfying Eqs. (152) and (154a),

$$d^2 f/dx^2 + [(\omega^2 - \omega_*^2)/c_0^2] f = 0 \ , \tag{167}$$

which is the equation for a second-order system with a single, constant cutoff frequency. The solution of Eq. (167) is

$$f(x;\omega) = \begin{cases} C_1 = \exp(K_1 x/2l) + C_2 \exp(-K_1 x/2l) \quad \text{for } \omega < \omega_* & (168a) \\ C_1 x + C_2 \quad \text{for } \omega = \omega_* \equiv c_0/2l & (168b) \\ C_1 \exp(iK_0 x) + C_2 \exp(-iK_0 x) \quad \text{for } \omega > \omega_* \ , & (168c) \end{cases}$$

viz., a linear function (168b) at the cutoff frequency $\omega = \omega_* \equiv c_0/2l$, a standing pattern (168a) below $\omega < \omega_*$, and propagating waves (168c) above $\omega > \omega_*$. The arbitrary constants of integration $C_1, C_2$ are determined by two independent boundary conditions at the horn entrance and/or exit.

#### 2. Effective wave number

The parameter $K_1$ in the standing-wave pattern (168a), below the cutoff frequency $\omega = \omega_*$, is a real quantity given by

$$K_1 \equiv (c_0/2l)(\omega_*^2 - \omega^2)^{1/2} = (1 - \omega^2/\omega_*^2)^{1/2} \ , \tag{169}$$

so that it varies between zero at the cutoff ($K_1 = 0$ as $\omega = \omega_*$) and unity as the frequency tends to zero ($K_1 \to 1$ as $\omega \to 0$); in the latter case, the first term in Eq. (168a), together with Eq. (166), yields a constant amplitude, and the second a steady flow, viz., $C_1 + C_2 e^{-x/l}$. For propagating waves [Eq. (168c)] above the cutoff frequency $\omega > \omega_*$, the effective wave number $K_0$ is a real quantity given by

$$K_0 \equiv (\omega^2 - \omega_*^2)^{1/2} c_0 = (\omega/c_0)(1 - \omega_*^2/\omega^2)^{1/2} \ , \tag{170}$$

which (i) coincides with the ordinary wave number $K_0 \sim \omega/c_0 \equiv k_0$, in the ray approximation, for frequencies much higher than the cutoff $\omega^2 \gg \omega_*^2$; (ii) is smaller than the ordinary wave number $K_0 < \omega/c_0$ at intermediate frequencies $\omega \gtrsim \omega_*$; (iii) vanishes at the cutoff frequency ($K_0 = 0$ for $\omega = \omega_*$), when propagation becomes impossible. The acoustic velocity $v$ and pressure $p$, for a wave of frequency $\omega$ in a horn of cross section $A(x)$, are related by

$$v(x,t)=-(i/\rho_0\omega)\partial p/\partial x \ , \tag{171a}$$

$$p(x,t)=-i(\rho_0 c_0^2/\omega)(\partial v/\partial x+v/L) \ , \tag{171b}$$

so that, for an exponential duct (constant $L$), these two acoustic variables have the same spatial dependence; for all other shapes, the horn wave equations for pressure (146a) and velocity (146b) are different; it is usually easier to solve one of them, to determine the respective acoustic variable, and then to use Eq. (171a) or (171b) to determine the other variable.

### 3. Smooth matching of ducts

For the hypex family of ducts [Eq. (156)], the length scale

$$L_2(x)\equiv\frac{l[1+\beta_2\tanh(x/2l)]}{\beta_2+\tanh(x/2l)} \tag{172}$$

is not constant, except in the case $L=l$ of the exponential duct $\beta_2=1$. Since there is a constant cutoff frequency for the acoustic pressure (Sec. IV.A.4.), the latter is given [Eq. (151)] by

$$p_2(x,t)=[\cosh(x/2l)+\beta_2\sinh(x/2l)]^{-1}e^{-i\omega t}f(x;\omega) \ , \tag{173}$$

where the filtering function satisfies Eq. (167), i.e., is the same as before. For the inverse hypex family of ducts [Eq. (157)], since $A_2(x)A_3(x)=0$ and hence $L_2(x)+L_3(x)=0$, the length scale is minus [Eq. (172)]; the constant cutoff frequency now applies (Sec. IV.A.5) to the acoustic velocity [Eq. (151)],

$$v_3(x,t)=[\operatorname{sech}(x/2l)+\beta_3\operatorname{csch}(x/2l)]^{-1}e^{-i\omega t}f(x;\omega) \ . \tag{174}$$

The acoustic velocity for the hypex family is obtained from Eqs. (171a) and (173), and the acoustic pressure for the inverse hypex family from Eqs. (171b) and (174); both variables require more complicated expressions, with two terms, and not of the constant cutoff type. The families considered here are all smooth matchings of exponential ducts: (i) the hypex family [Eq. (156)] matches exponentially diverging ducts

$$A_2(x)\sim\tfrac{1}{4}A_2(0)(1+\beta_2)^2\exp(|x|/l) \quad \text{for } |x|\gg l \ ,$$

through a cross section at the origin which is finite for the catenoidal (or $\cosh^2$) shape, and zero for the hyperboloidal (or $\sinh^2$) shape; (ii) the inverse hypex family [Eq. (157)] matches exponentially converging ducts

$$A_3(x)\sim 4A_3(0)(1+\beta_3)^2\exp(-|x|/l) \quad \text{for } |x|\gg l \ ,$$

through a "hump" near $x=0$ for the "solitary" wave shape $\sim\operatorname{sech}^2$, or with a baffle (infinite cross section) for the $\operatorname{csch}^2$ shape.

### 4. Transparency function

The exponential, hypex, and inverse families all have constant cutoffs, and thus involve the filtering function [Eq. (168)], satisfying Eq. (167). For the sinusoidal and inverse families, the filtering function is replaced by Eqs. (152) and (162),

$$d^2f_*/dx^2+[(\omega^2+\omega_*^2)/c_0^2]f_*=0 \ , \tag{175}$$

whose solution is a transparency function,

$$f_*(x;\omega)=C_1\exp(iK_2x)+C_2\exp(-iK_2x) \ , \tag{176}$$

which allows propagating waves for all frequencies (no cutoff), with effective wave number

$$K_2=(\omega^2+\omega_*^2)^{1/2}/c_0=(\omega/c_0)(1+\omega_*^2/\omega^2)^{1/2} \ , \tag{177}$$

which always exceeds the ordinary wave number $k_0\equiv\omega/c_0$. The length scale for the sinusoidal family [Eq. (160)], is

$$L_4(x)=l[1+\beta_4\tan(x/2l)]/[\beta_4-\tan(x/2l)] \ , \tag{178}$$

and $L_5(x)=-L_4(x)$ for the inverse sinusoidal family. The acoustic pressure in the sinusoidal and velocity in the inverse ducts are given, respectively, by

$$p_4(x,t)=[\cos(x/2l)+\beta_4\sin(x/2l)]^{-1}e^{-i\omega t}f_*(x;\omega) \ , \tag{179}$$

$$v_5(x,t)=[\sec(x/2l)+\beta_5\csc(x/2l)]^{-1}e^{-i\omega t}f_*(x;\omega) \ , \tag{180}$$

where $f_*$ is the transparency function [Eqs. (176) and (177)]. The acoustic fields have been given exactly, in terms of elementary functions, for the exponential (Secs. IV.B.1 and IV.B.2), hypex (Sec. IV.B.3), sinusoidal (Sec. IV.B.4), and inverse families of ducts; this is *not* possible (Sec. IV.A.7) for any other duct shapes, for which the horn wave equation has solutions only in terms of special functions.

### 5. Convergent and diffuser horns

As an example of the above statement we consider the power-law family of ducts:

$$A_6(x)=A_6(x_0)(x/x_0)^{2n} \ , \tag{181a}$$

$$L_6(x)=x/2n \ , \tag{181b}$$

whose length scale [Eq. (181b)] increases (in modulus) without bound, from zero at the origin. The power-law family [Eq. (181a)] includes the uniform tube $A_6(x)=A_6(0)$, as the trivial case $n=0$, separating divergent ($n>0$) from convergent ($n<0$) horns. All the "conic" ducts are particular cases of Eq. (181a), viz., the conical $n=1$, parabolic $n=\frac{1}{2}$, and hyperbolic $n=-\frac{1}{2}$ horns. For the power-law family [Eqs. (181a) and (181b)], the horn wave equation for the acoustic pressure (146a) becomes

$$\partial^2 p/\partial x^2+(2n/x)\partial p/\partial x - c_0^{-2}\partial^2 p/\partial t^2=0\ , \tag{182}$$

which reduces, by means of the substitution

$$p_6(x,t)=e^{-i\omega t}(x/x_0)^{1/2-n} j(x;\omega) \tag{183}$$

to a Bessel equation (Jeffreys and Jeffreys, 1946):

$$j''+xj'+[(\omega^2 x/c_0)^2-(n-\tfrac{1}{2})^2]j=0\ , \tag{184}$$

of variable $\omega x/c_0$ and order $n-\frac{1}{2}$. The transformation (183) is quite predictable, since it considers a wave of frequency $\omega$, and allows for the fact that, in the ray (or high-frequency approximation) the amplitude scales as $A^{-1/2}\sim x^{-n}$, with an additional factor $x^{1/2}$, since the Bessel function scales as $x^{-1/2}$ for large variable. Thus the acoustic pressure in the power-law ducts of arbitrary exponent $n$ is given exactly by Eq. (183), where $j(x;\omega)$ is a linear combination of Hankel functions,

$$j(x;\omega)=C_+ H^{(1)}_{n-1/2}(\omega x/c_0)+C_- H^{(2)}_{n-1/2}(\omega x/c_0)\ , \tag{185}$$

with $H^{(1)},H^{(2)}$ representing waves propagating in the positive and negative $x$ directions, respectively, and their amplitudes $C_+,C_-$ determined from the boundary conditions at the duct ends.

### 6. Generalized spherical waves

For the power-law ducts [Eq. (181a)] with nonintegral exponent $n\neq 0,\pm 1,\pm 2,\ldots$, the acoustic fields [Eqs. (183) and (185)] have no exact expression in finite terms, using only elementary functions. For example, for the parabolic $n=\frac{1}{2}$ and hyperbolic $n=-\frac{1}{2}$ they are specified by Hankel functions of orders zero $H_0^{(1,2)}$ and unity $H_{-1}^{(1,2)}=-H_1^{(1,2)}$, respectively. The Hankel functions of any order $n$ can be expressed (Ross, 1975; Lavoie, Osler, and Tremblay, 1976) as derivatives of complex order of elementary functions, by

$$H^{(1/2)}_{n-1/2}(\chi)=\sqrt{2/\pi}\, e^{i\pi(n-1/2)}\chi^{n-1/2} \times(\chi^{-1}d/d\chi)^{n-1}\{\chi^{-1}e^{\pm i\chi}\}\ ; \tag{186}$$

the derivatives of complex order $n$ reduce (Oldham and Spanier, 1974; Campos, 1984g) to ordinary derivatives for positive integer order $n=1,2,3,\ldots$, i.e., the Hankel functions are finite expansions in the "spherical" case of orders $\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots$. From Eqs. (183) and (186), the acoustic fields are given exactly, in finite terms, by

$$p(x,t)=C_\pm e^{-i\omega t}(x^{-1}d/dx)^{n-1} \times[(x/x_0)^{-1}e^{\pm i\omega(x-x_0)/c_0}]\ , \tag{187}$$

for waves propagating in the positive/negative $x$ direction, in ducts of cross section $x^{2n}$, with $n$ a positive integer. The simplest case, $n=1$, corresponds to a spherical wave $p\sim x^{-1}\exp[i\omega(\pm x/c_0-t)]$ in a conical duct of cross section $\sim x^2$; the remaining positive integer values of the exponent $n=2,3,4,\ldots$, represent "generalized" spherical waves [Eq. (187)], propagating in divergent ducts of cross section $\sim x^4,x^6,x^8,\ldots$.

### 7. Linear displacement amplifier

Just as fluid-filled convergent horns can act as sound concentrators, a tapering bar or "solid horn" in longitudinal oscillation can act as a displacement amplifier; the displacement $\xi(z,t)$ satisfies the horn equation (146a), with the speed of longitudinal elastic waves $c_0=\sqrt{E/\rho_0}$ specified by the Young modulus $E$ and mass density $\rho_0$ (Landau and Lifshitz, 1967a). In the design of displacement amplifiers, also called sonic transformers, arises the so-called "inverse problem," viz., if we specify the displacement law

$$\xi(x,t)=h(x)\exp(-i\omega_a t) \tag{188}$$

at a frequency $\omega_a$, which shape of "solid horn" $A(x)$ will meet this requirement? The answer lies in substituting Eq. (188) into the horn wave equation and solving for the cross section,

$$1/L\equiv d[\ln A(x)]/dx = -[h''+(\omega_a/c_0)^2 h]/h'\ , \tag{189}$$

where the prime denotes derivative with regard to $x$. Note that the shape of the horn depends on the frequency, i.e., it will be possible to achieve the specified displacement law $h(x)$ at one frequency $\omega_a$, the "design" frequency; at other frequencies $\omega\neq\omega_a$, the solution of the horn wave equation

$$\partial^2\xi/\partial x^2+L^{-1}\partial\xi/\partial x+(\omega/c_0)^2\xi=0\ , \tag{190}$$

with the length scale $L$ given by Eq. (189), will *not* be Eq. (188), i.e., the displacement law will be different.

### 8. Power tools with uniform stress

Displacement amplifiers, such as those used in power tools, are subject to breakage; in order to reduce the risks of fracture, it is desirable to have a uniform longitudinal stress, which corresponds to a constant strain $|\partial\xi/\partial x|=a$, i.e., to a linear displacement $h(x)=ax$ in Eq. (188); from Eq. (189), it follows that this is achieved by a "Gaussian" horn,

$$A_7(x)=A_7(0)\exp(-b^2x^2)\ , \tag{191a}$$

$$b\equiv(\omega_a/c_0)/\sqrt{2}\ , \tag{191b}$$

whose "variance" $1/b^2=2c_0^2/\omega_a^2=2/k_a^2$ is determined [Eq. (191b)] by the "design" wave number $k_a\equiv\omega_a/c_0$. The length scale is $L=-1/(2b^2x)$, and so the displacement $\xi$, at an arbitrary frequency $\omega$, satisfies

$$\xi''-2b^2x\xi'+(\omega/c_0)^2\xi=0\ , \tag{192}$$

which is (Morse and Feshbach, 1953) a Hermite equation, as for the harmonic oscillator in quantum mechanics (Pauling and Wilson, 1935). The solution of Eq. (192) is

$$\xi(x,t)=e^{-i\omega t}[C_1 H_\nu(bx)+C_2 H_{-\nu}(bx)] , \tag{193}$$

where $H_{\pm\nu}(bx)$ are Hermite functions of variable $bx \equiv k_a x/\sqrt{2}$ and order $\nu \equiv \omega^2/2b^2c_0^2=(\omega/\omega_a)^2$. If the order is a positive integer $n$, i.e., at the resonant frequencies $\omega_n=\omega_a\sqrt{n}$, the displacement of the Gaussian horn is specified by Hermite polynomials, as the wave function of the stationary states of the harmonic oscillator. The first resonant frequency $n=1$ is the design frequency $\omega_1=\omega_a$, implying a linear displacement law $H_1(bx)=2bx=\sqrt{2}(\omega_a x/c_0)$, which has been demonstrated experimentally (Fig. 9; Bies, 1962), using a Gaussian horn.

### C. Convection by accelerated or decelerated flows

An important extension of the study of acoustic ducts is the case in which there is a mean flow. The acoustics of tubes of uniform cross section, has been considered for uniform flow (Morfey, 1971; Jacques, 1975; Perulli, 1979; Prasad and Crocker, 1984; Rienstra, 1984), and sheared velocity profiles (Pridmore-Brown, 1959; Swinbanks, 1975; Möhring and Rahman, 1976; Mani, 1980), both for unblocked ducts and in the presence of obstacles (Leppington and Levine, 1980; Namba, Notomi, and Fujimoto, 1984; Welsh, Stokes, and Parker, 1984; Quinn and Howe, 1984). Engine inlet and exhaust ducts often have varying cross sections, whose effects are modeled most simply for quasi-one-dimensional flow and sound (Huerre and Karamcheti, 1973; Lumsdaine and Ragab, 1977; Mani, 1981; Miles, 1981; Salikuddin and Mungur, 1983), corresponding to the fundamental longitudinal mode; transverse modes have also been considered (Tester, 1973a,1973b; Cho and Ingard, 1983; Baxter and Morfey, 1983; Silcox, 1984; Myers and Chuang, 1984; Uenishi and Myers, 1984), and popular technique is the use of perturbation expansions (Nayfeh, 1973,1975a,1975b; Nayfeh, Kaiser, and Telionis, 1975a,1975b; Kaiser and Nayfeh, 1977; Nayfeh, Shaker, and Kaiser, 1980; Nayfeh, Kelly, and Watson, 1982; Kelly, Nayfeh, and Watson, 1982), which apply if the change in cross section is gradual. The muffling of noise transmitted through ducts has motivated extensive research on wall effects (Crighton, 1980,1984; Namba and Fukushige, 1980; Sobolev, 1982; Fuller, 1982; Howe, 1983b; Koch and Möhring, 1983; Page and Mee, 1984), such as the use of perforated liners (Howe, 1979a,1979b,1980; Yoshida, 1981); there is a substantial collection of experimental data on acoustic liners (Plumblee, Dean, Wynne, and Burrin, 1973; Nayfeh, Kaiser, Marshall, and Hurst, 1980; Silcox and Lester, 1982; Cummings, Parrett, and Astley, 1982; Baumeister, Eversman, Astley, and White, 1984; Fuller and Silcox, 1984; Watson, 1984), which is suitable for comparison with theories of sound attenuation. Another area of extensive research is the effect of mean flow on the acoustic energy balance (Blokhintsev, 1946; Cantrell and Hart, 1964; Garrett, 1967; Bretherton, 1968; Bretherton and Garrett, 1969; Hayes, 1968; Candel, 1975; Möhring, 1971,1978b,1980; Hayes, 1980), which is relevant to the technique of active noise control (Ffowcs-Williams, 1984b), i.e., cancellation of sound by sound (Swinbanks, 1973; Kempton, 1976; Ford, 1984). A variety of other topics has been considered in the acoustics of ducted flows (Davies, Coelho, and Bhattacharya, 1980; Bull and Norton, 1980; Davies, 1981; Hasan, Islam, and Hussain, 1984; Vaydia, 1984), including sound in near-sonic flows (Myers and Callegari, 1972; Hariharan and Lester, 1984). Unlike the case of horns, for which there are numerous exact solutions (Sec. IV.B), there are relatively few cases of exact solutions of the acoustic equations of nonuniform nozzle flows; instances in the literature include an iterative technique (Powell, 1959,1960), transformations for throated ducts carrying high-speed flows (Tsien, 1952; Crocco and Cheng, 1967; Eisenberg and Kao, 1969), and solutions in terms of special functions, for low-Mach-number convergent and divergent nozzles (Campos, 1984b,1985d).

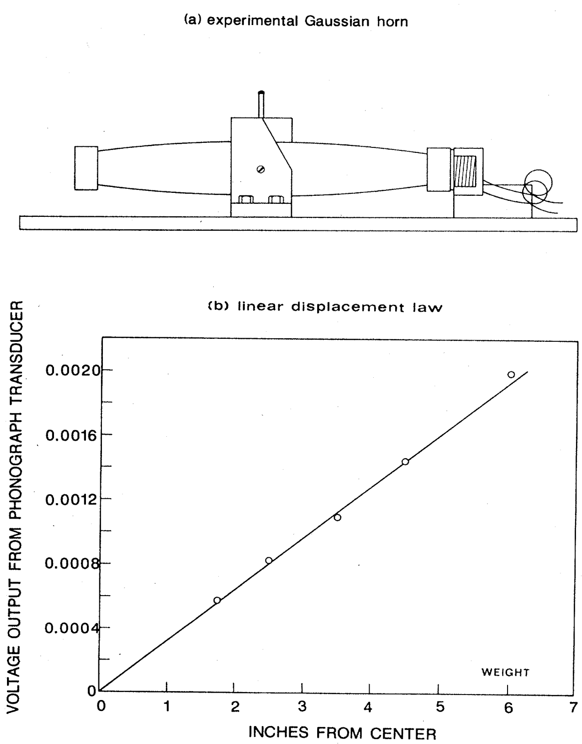


FIG. 9. Diagram of a Gaussian horn (a) used by Bies (1962) to demonstrate the linear displacement law (b), when driven electrically to oscillate longitudinally at the "design" frequency, i.e., the harmonic corresponding to the Hermite polynomial of order one: ○, experiment, solid lines, theory.

#### 1. Nonuniform mean flow

The generalization of the acoustics of horns to nozzles is associated with the presence of a mean flow, which is generally nonuniform, i.e., accelerated or decelerated axially, as a consequence of changes in cross-sectional area;

in the case of an incompressible mean flow, the conservation of the mass flux implies that the volume flux is constant, and thus the velocity varies inversely with the cross-sectional area $v_0(x)A(x) \sim \text{const}$, where, in the quasi-one-dimensional approximation (Sec. IV.A.2), all quantities are averaged over the cross section and thus depend only on the axial coordinate $x$, for the mean state, and also on time $t$, for the sound field. The low-Mach-number nozzle wave equation (41) is different for the potential $\varphi$ and velocity $v=\partial\varphi/\partial x$,

$$[(\partial/\partial t+v_0\partial/\partial x)^2-c_0^2A^{-1}\partial/\partial x A\partial/\partial x]\varphi(x,t)=0\ , \tag{194a}$$

$$[(\partial/\partial t+\partial/\partial x v_0)^2-c_0^2\partial/\partial x A^{-1}\partial/\partial x A]v(x,t)=0\ , \tag{194b}$$

in that the latter replaces the material derivative $d/dt$ by the operator

$$\partial/\partial t+\partial/\partial x v_0=d/dt+dv_0/dx\ , \tag{195a}$$

$$d/dt\equiv\partial/\partial t+v_0\partial/\partial x\ , \tag{195b}$$

which never coincides with $d/dt$, for a nonuniform flow, i.e., in a nozzle of varying cross section. The wave equation for the acoustic velocity (194b) in low-Mach-number $(v_0^2 \ll c_0^2)$ nozzles,

$$(\partial^2/\partial t+2\partial/\partial x v_0\partial/\partial t-c_0^2\partial/\partial x A^{-1}\partial/\partial x A)v(x,t)=0\ , \tag{196}$$

differs from the horn wave equation (146b) on account of the presence of the second term, representing nonuniform convection by the mean flow. The first and third terms in Eq. (196), which coincide with Eq. (146b), satisfy the duality principle (150c). Bearing in mind that $v_0 \sim 1/A$, the condition that the second term also satisfies duality, is the vanishing of

$$A\partial/\partial x A^{-1}-A^{-1}\partial/\partial x A=-2A'/A=-2/L\ , \tag{197}$$

which implies a uniform cross section. Thus the duality principle, for nonuniform ducts of inverse cross sections $A(x)$, $1/A(x)$, holds for all horns [Eqs. (150a)–(150c)] and is invalid for all nozzles in the presence of accelerated or decelerated mean flow.

### 2. Amplitude and phase of rays

We represent the reduced acoustic velocity $v(x;\omega)$ by

$$v(x,t)=e^{-i\omega t}\{A(x)\}^{-1/2} \times\exp\left[-ik_0\int^x M_0(\xi)d\xi\right]v(x;\omega)\ , \tag{198}$$

where $k_0\equiv\omega/c_0$ is the wave number, and $M_0(x)=v_0(x)/c_0$ the Mach number; then the low-Mach-number nozzle wave equation (196) transforms to the Schrödinger form [Eq. (152)], with complex invariant

$$J_v(x)=k_0^2+A''/2A-3(A'/2A)^2+i2k_0M_0'\ , \tag{199}$$

whose real part is the same as for horns [Eq. (158)] and whose imaginary part is associated with nonuniform mean flow $M_0'\equiv dM_0/dx\neq0$. The imaginary part is a constant for $M_0'=a_0\equiv\text{const.}$, i.e., a linearly accelerated flow $v_0(x)=a_0x+a_1$ in a hyperbolic nozzle $A(x)\sim1/(a_0x+a_1)$; this shape of duct is not included in the exponential, hypex, sinusoidal, and inverse families, for which the real part of the invariant is a constant and the horn wave equation has exact elementary solutions (Sec. IV.A.6). Thus the real and imaginary parts of the wave invariant [Eq. (199)] are never exactly constant simultaneously, and thus the nozzle wave equation has no elementary exact solutions for any shape. The wave invariant will reduce to a constant,

$$J_v(x)=k_0{}^2[1+O((1/k_0L+M_0)^2)]\ , \tag{200}$$

in the combined $M_0/k_0L\ll1$ ray $k_0^2L^2\gg1$ and low-Mach-number $M_0^2\ll1$ approximations, for which the solution of Eq. (152) is a plane wave $v(x;\omega)\sim\exp(\pm ik_0x)$. Thus the acoustic velocity in a low-Mach-number nozzle is given, in the ray approximation, by

$$v_a(x,t)\sim\exp(\pm ik_0x-i\omega t)\{A(x)\}^{-1/2} \times\exp\left[-ik_0\int^x M_0(\xi)d\xi\right]\ , \tag{201}$$

as a plane wave propagating in the positive/negative $x$ direction, with amplitude scaling as the inverse square root of the cross section, as for horns [Eq. (151)], and a phase shift due to convection by the nonuniform mean flow, as for sound in turbulence [Eq. (124)]. Thus, for the fundamental longitudinal acoustic mode, a nozzle of varying cross section acts as if it were a "ray tube."

### 3. Acoustic pressure and velocity

From the linearized momentum and continuity equations, it follows (Campos, 1985a) that the acoustic pressure $p$ and velocity $v$ are related by

$$\partial p/\partial t=\rho_0v_0\partial v/\partial t-\rho_0c_0^2A^{-1}\partial(Av)/\partial x\ , \tag{202}$$

where the second term corresponds to horns, and the first adds convection by the mean flow. In the ray approximation [Eq. (201)], the relation (202) between acoustic pressure and velocity simplifies to

$$p(x,t)=\rho_0c_0v(x,t)[\pm1+i/(2k_0L)]\ ; \tag{203}$$

thus the acoustic pressure $p=p_\alpha+p_\beta$ consists of two components, a primary field $p_\alpha\sim\rho_0c_0v$, in phase with the velocity and of comparable amplitude $O(1)$, and a secondary field $p_\beta=i\rho_0c_0v\varepsilon/1$, out of phase by $\pi/2$ and of small magnitude $O(\varepsilon)$, with $\varepsilon\equiv1/k_0L$ such that $\varepsilon^2\ll1$. The mean-square pressure $\overline{p^2}$, averaged over a period, is determined by the primary field $\overline{p_\alpha^2}$, since $\overline{p_\alpha p_\beta}=0$ because the secondary field is out of phase with the primary, and $\overline{p_\beta^2}=O(\varepsilon^2)$ is negligible,

$$|p(x,t)|=\rho_0 c_0\,|v(x,t)|\ . \tag{204}$$

It follows that there is equipartition of kinetic [Eq. (26a)] and compression [Eq. (26b)] energies, i.e., the total energy is twice either of them:

$$E \equiv E_v+E_p=2E_v=\rho_0 A v^2=2E_p=p^2/\rho_0 c_0^2 \tag{205}$$

in the ray approximation $k_0^2L^2 \gg 1$, i.e., for high frequencies $\omega^2 \gg L^2/c_0^2$. Equation (204) does not extend to lower frequencies, since Eq. (202) does not reduce exactly to Eq. (148) for any duct of varying area, on account of the nonuniformity of the mean flow velocity $v_0(x)$. Thus, whereas equipartition of kinetic and compression energies hold exactly, at all frequencies and stations, for one horn, the exponential (Sec. IV.A.3), it fails for all nozzles, including the last-mentioned shape.

### 4. Energy density and flux

The coupling of reflections from the walls, as in horns, with nonuniform convection in nozzles renders the acoustics of the latter more complex than those of the former. None of the general properties of horns concerning (i) equipartition of energy, (ii) duality principle, or (iii) existence of elementary solutions extend to nozzles. Simple results can be obtained in the acoustics of nozzles, however, in the ray approximation, which neglects reflections from the walls and assumes nearly uniform convection on a wavelength scale. The ray law [Eq. (201)] implies conservation of the total energy density $E$ and flux $F$,

$$E=\rho_0(x)A(x)\,|v_a(x,t)|^2\ , \tag{206a}$$

$$F=c_0E\ . \tag{206b}$$

The energy then, Eq. (206b), is equal to the sound speed $c_0$ times Eq. (206a), for the energy flux in a reference frame moving with the fluid. The energy density $E$ and flux $F$ satisfy the conservation equation

$$0=dE/dt+\partial F/\partial x=(d/dt+c_0\partial/\partial x)E\ , \tag{207}$$

showing that the acoustic energy propagates at a sound speed relative to the flow, viz., $d/dt$ is Eq. (195b) the material derivative. The ray approximation for low-Mach-number flow is based on the smallness $\varepsilon^2 \ll 1$ of the parameters $\varepsilon \equiv 1/k_0L, M_0$. If we multiply the acoustic velocity by $1+O(\varepsilon)$, the Schrödinger equation (152) gains two "new" terms $\varepsilon''v+2\varepsilon'v'=O(\varepsilon^2)$, which are negligible. A factor $1+O(\varepsilon)$ in the acoustic velocity (201) does not affect Eq. (204) for the pressure, so that equipartition of energies [Eq. (205)] still holds to $O(\varepsilon^2)$, even though the energies gain new $O(\varepsilon)$ terms; the $1+O(\varepsilon)$ in the energy density $E$ and flux $F$ does not change the energy equation (207), since $\partial\varepsilon/\partial t=0$ and $v_0\partial\varepsilon/\partial x \sim O(\varepsilon^2)$. In conclusion, the combined ray $\varepsilon \equiv 1/k_0L$ and low-Mach-number $\varepsilon \equiv M_0$ approximations $\varepsilon^2 \ll 1$ involve two levels of accuracy: (i) to $O(\varepsilon)$ for the acoustic variables (velocity and pressure) and energy (density and flux); (ii) to $O(\varepsilon^2)$ for the wave equation, equipartition of energies, and energy balance.

### 5. Wave action

As an example of an $O(\varepsilon)$ correction to the acoustic velocity [Eq. (201)] in the ray approximation, we consider

$$v_b(x,t) \sim [1 \mp M_0(x)/2]v_a(x,t)\ , \tag{208}$$

which implies the conservation of

$$B_\pm=\rho_0 A(x)\,|v_b(x,t)|^2[c_0 \pm v_0(x)]\ . \tag{209}$$

This is the energy flux [(206a) and (206b)] for a frame at rest, for which the propagation speed $u_\pm$ is the sound speed $c_0$, plus or minus the mean flow velocity $v_0$, respectively, for propagation downstream/upstream:

$$B_\pm=E(x)u_\pm(x)\ , \tag{210a}$$

$$u_\pm(x)=c_0 \pm v_0(x)\ . \tag{210b}$$

The energy equation (207) can be written

$$\begin{aligned}0&=[\partial/\partial t+(c_0 \pm v_0)\partial/\partial x]E\\ &=\partial E/\partial t+u_\pm \partial E/\partial x\ ,\end{aligned} \tag{211}$$

showing that the group velocity (210b) is the velocity of energy propagation [cf. Eq. (123)]. The conservation of the Blokhintsev invariant [Eq. (209)] is equivalent to the conservation of wave action,

$$\mathscr{L}_\pm \equiv B_\pm/c_0\omega=E(x)/\omega_\pm\ , \tag{212a}$$

$$\omega_\pm \equiv \omega/[1 \pm M_0(x)]\ , \tag{212b}$$

where $E(x)$ is the energy density and $\omega_\pm$ the wave frequency $\omega$ with a Doppler factor to account for mean flow effects. The conservation of wave action implies that the energy density and wave amplitude increase or decrease for sound propagating upstream or downstream, respectively, corresponding to the lower or the upper sign in Eqs. (208), (209), (210b), (211), and (212b). The conservation of wave action [Eq. (212a)], like the equipartition of energies (205), fails to extend beyond the ray approximation in the acoustics of nozzles, when the wavelength is comparable to the length scale for changes in cross section or flow velocity; in the latter case it can be shown (Campos, 1985d) that on the approach to a blockage, the kinetic energy predominates over the compression energy and the wave action is mainly kinetic, whereas on the approach to an opening, compression energy dominates and the wave action is mainly compressive.

### 6. Convergent and diffuser nozzles

The latter result can be proved from exact solutions of the nozzle wave equation, which are never elementary but can be obtained in terms of special functions, for certain shapes, e.g., the parabolic and hyperbolic nozzles, of cross section, respectively,

$$A_8(x)=A_8(x_0)(x/x_0)\ , \tag{213a}$$

$$A_9(x)=A_9(x_0)(x_0/x)\ , \tag{213b}$$

the exact acoustic velocity is given, respectively, by

$$v_8(x,t)=v_8(x_0,0)(x/x_0)^{-ik_0x_0M_{00}} \times e^{-i\omega t}[H_{\nu_1}^{(1,2)}(k_0x)/H_{\nu_1}^{(1,2)}(k_0x_0)] , \tag{214a}$$

$$v_9(x,t)=v_9(x_0,0)(x/x_0)\exp[(i/2)k_0M_{00}(x_0-x^2/x_0)] \times e^{-i\omega t}[H_0^{(1,2)}(\nu_2k_0x)/H_0^{(1,2)}(\nu_2k_0x_0)] , \tag{214b}$$

where $M_{00}\equiv M_0(x_0)$ is the Mach number at axial station $x_0$; both solutions involve (Bateman, 1932) Hankel functions $H^{(1)}$ $(H^{(2)})$ for waves propagating in the positive (negative) $x$ direction, with complex order $\nu_1$ for the parabolic nozzle [Eq. (214a)], and with complex variable $\nu_2k_0x_0$ for the hyperbolic nozzle [Eq. (214b)], with $\nu_1,\nu_2$ given, respectively, by

$$\nu_1\equiv 1+k_0x_0M_{00} , \tag{215a}$$

$$\nu_2\equiv(1+2iM_{00}/k_0x_0)^{1/2} . \tag{215b}$$

In the case $M_0=0$ when the mean flow is absent, Eqs. (215a) and (215b) simplify to $\nu_1=1=\nu_2$, and Eqs. (214a) and (214b) yield the acoustic velocity in the parabolic and hyperbolic horns, respectively,

$$v_8(x,t)=v_8(x_0,0)e^{-i\omega t}[H_1^{(1,2)}(k_0x)/H_1(k_0x_0)] , \tag{216a}$$

$$v_9(x,t)=v_9(x_0,0)e^{-i\omega t}(x/x_0) \times[H_0^{(1,2)}(k_0x)/H_0^{(1,2)}(k_0x_0)] . \tag{216b}$$

Equations (216a) and (216b) for the acoustic velocity $v$ are consistent with Eqs. (183) and (185) for the acoustic pressure $p$, bearing in mind that $v\sim\partial p/\partial x$ in the cases $n=\frac{1}{2}$ (parabolic) and $n=-\frac{1}{2}$ (hyperbolic) horns. The acoustic velocity in the parabolic and hyperbolic horns [Eqs. (213a) and (213b)] satisfies [Eqs. (216a) and (216b)] the relations $v_8\sim\partial(A_9v_9)/\partial x$ and $v_9\sim\partial(A_8v_8)/\partial x$, in agreement with the duality principle [Eq. (150c)]; these relations do *not* hold for the acoustic velocity in nozzles [Eqs. (214a) and (214b)], because the duality principle fails in the presence of nonuniform mean flow. Since there are (Erdelyi, 1953) no known relations between Hankel (or Bessel) functions of complex variable (214b) and complex order (214a), the example of the parabolic $A_8$ and hyperbolic $A_9$ dual ducts $A_8A_9\sim$const in Eqs. (213a) and (213b) shows that there exists no simple extension of the duality principle from horns to nozzles.

## 7. Correction factor

The similarities and differences between horns and nozzles can be illustrated by considering the simplest of nonuniform ducts, the exponential shape; the cross-sectional area and mean flow Mach number are given, respectively, by

$$A_1(x)=A_{00}e^{\pm x/L} , \tag{217a}$$

$$M_1(x)=M_{00}e^{\mp x/L} , \tag{217b}$$

where the constant length scale is assumed to be positive $L>0$, the upper sign applies to decelerated flow in a diffuser nozzle, and the lower sign applies to accelerated flow in a convergent nozzle. The exact acoustic velocity is given by

$$v_1(x,t)=v_1(0,0)\exp[i(K_0x-\omega t)]e^{\mp x/2L} \times[B(x)/B(0)] , \tag{218}$$

where the first three factors are the same [Eqs. (166) and (168a)] as for waves propagating in the positive $x$ direction in the exponential horn, with effective wave number $K_0$ given by Eq. (170); the effects of mean flow are concentrated in the factor $B(x)$, denoting the confluent hypergeometric function,

$$B(x)\equiv F[\tfrac{3}{2}\mp iK_0L;1\mp 2iK_0L;i2k_0LM_1(x)] , \tag{219}$$

which reduces to unity in the absence of mean flow ($B=1$ for $M_1=0$) and otherwise is complex, i.e., introduces amplitude and phase changes. We can compare the exact solution [Eq. (219)] with the ray approximation (201),

$$v_1(x,t)=A_{\mp}(x)v_1(0,0)\exp[i(K_0x-\omega t)] \times e^{\mp x/2L}\exp[\pm iK_0LM_{00}(e^{\mp x/L}-1)] , \tag{220}$$

where we have replaced ordinary $k_0\equiv\omega/c_0$ by effective wave number $K_0$. The correction factor $A_{\pm}(x)$ can be determined so that the "modified ray formula" (220) coincides with the exact solution [Eqs. (218) and (219)], i.e., it measures the accuracy of the ray approximation when applied outside its domain of validity $k_0^2L^2\gg 1$ to lower frequencies.

## 8. Acoustics of flow nozzles

The ray correction function is plotted in Fig. 10 for the exponential diffuser $A_-$ (right-hand side) and convergent $A_+$ (right-hand side) nozzles, with Mach number 0.3 at the narrowest section; since the correction function $A_{\pm}(x)$ is generally complex, the modulus $|A_{\pm}|$ or amplitude factor and argument $\arg(A_{\pm})$ or phase shift are plotted separately (at the top and bottom, respectively), versus dimensionless axial distance $x/L$. Five sets of curves are given for values of the compactness $k_0L=\omega L/c_0=\omega/2\omega_*=1,2,5,10,20$, ranging from wave frequencies $\omega$ close to the cutoff $\omega_*$ [Eq. (154b)], to much larger. The amplitude correction is greater (smaller) than unity for the diffuser (convergent) nozzle, showing that ray theory overestimates amplitude effects, i.e., the local sound level decreases in a diffuser and increases in a convergent nozzle less than ray theory would predict; the ray approximation also overestimates phase leads (lags), due to propagation downstream (upstream) in nonuniform flows, the maximum error being about 25° for the phase and 20% for the amplitude of acoustic variables. The amplitude and phase corrections become constant in regions of very small flow velocity $M_0\ll 1$ and, conversely,

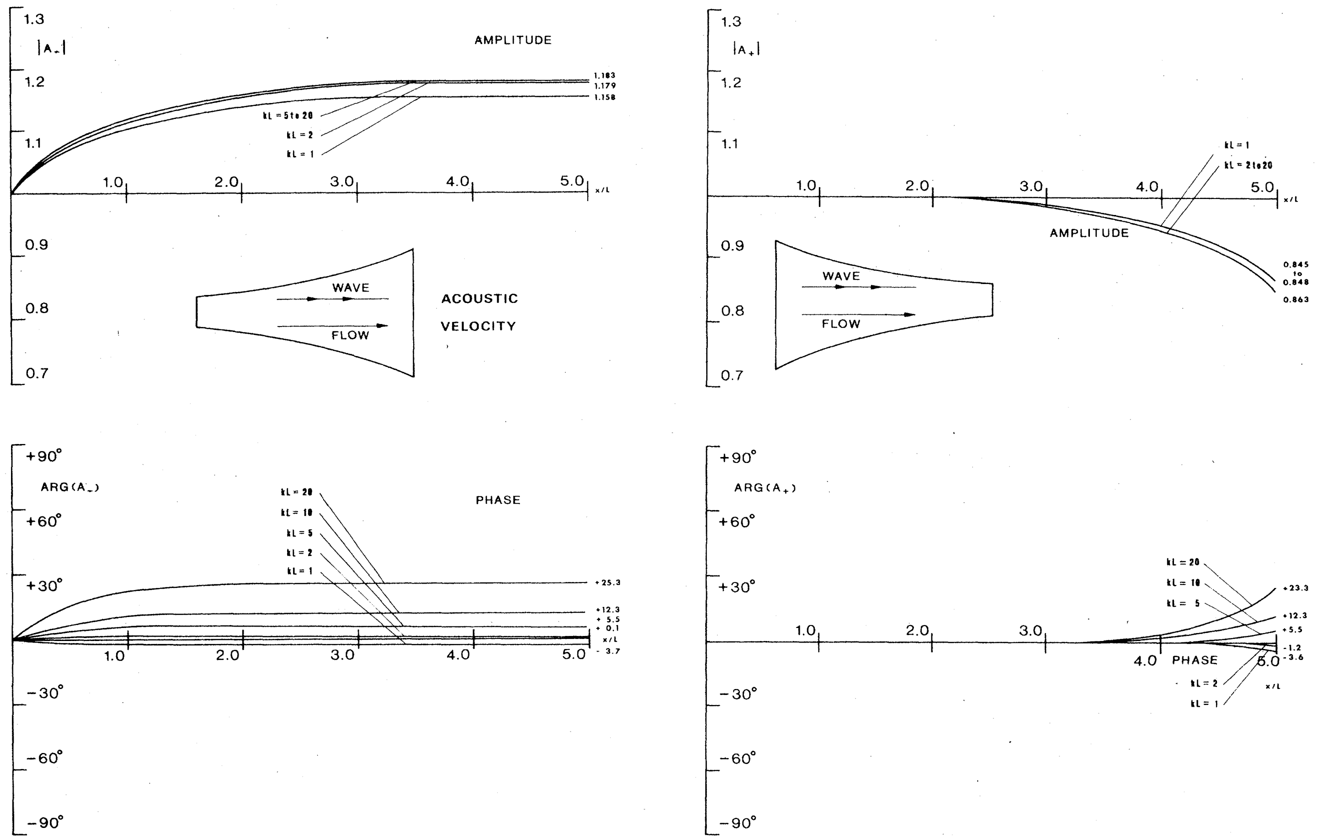


FIG. 10. Amplitude (top) and phase (bottom) of correction factor (Campos, 1984b) of exact acoustic velocity, relative to ray approximation, for exponential diffuser nozzles (right) and convergent nozzles (left), plotted as a function of dimensionless axial distance $x/L$, for five values of the compactness $k_0L = 1.2,5,10,20$. The sound wave is assumed to propagate downstream (insets) in both cases, and the jet Mach number to be 0.3 at the narrowest section.

vary more rapidly near the entrance (exit) for diffuser (convergent) nozzles, i.e., where the cross section is smaller, and the mean flow deceleration (acceleration) is more noticeable.

## V. EFFECTS OF NONLINEARITY AND VISCOUS DISSIPATION

We have so far discussed sound of small amplitude, which does not change the mean properties of the medium and which is described by linear wave equations. There are a number of situations in which, say, the pressure perturbation in an acoustic wave becomes comparable to the mean state pressure, i.e., sound is a large perturbation of the medium, described by nonlinear equations. Examples include resonance (Rayleigh, 1868) and forced oscillations (Mortell and Seymour, 1979) in closed (Seymour and Mortell, 1980) and open tubes (Disselhorst and Van Wijngaarden, 1980; Chester and Moser, 1982) and semi-infinite uniform (Aranha, Yue, and Mei, 1982) and nonuniform ducts (Peube and Chasseriaux, 1973; Nayfeh, 1975b), and oscillations in cavities (Keller, 1984) and near orifices (Cummings, 1984); the linear theory applies if the sound amplitude is initially small and does not grow, and nonlinear methods are needed for large-amplitude driving or in cases of wave growth. The latter occurs not only under forcing or resonance conditions, but also for free oscillations in stratified media, e.g., due to the decay of density with altitude, in an atmosphere. The nonlinear acoustics of inhomogeneous media has been studied by variational methods (Fornberg and Whitham, 1978; Peregrine and Smith, 1979) and by direct analysis of the equations of motion (Varley and Cumberbatch, 1970). Sound waves of large amplitude tend to steepen into shocks (Riemann, 1860; Blackstock, 1965; Hayes, 1973; Glimm, Marshall, and Plohr, 1984), a process that is opposed by dissipation mechanisms (Lighthill, 1956; Crighton, 1979; Crighton and Scott, 1979; Scott, 1981,1982; Campos, 1984d); diffusion ultimately causes the decay of shocks and, together with the shedding of vorticity (van Dyke, 1972), is the main mechanism of sound absorption (Howe, 1984b), for all amplitudes (large or small).

### A. Wave growth and damping in atmospheres

The processes of (i) wave growth due to nonuniformity of the medium and (ii) amplitude limitation by diffusion mechanisms are well illustrated by acoustic-gravity waves in atmospheres, respectively, (i) nondissipative (Rayleigh, 1890; Lamb, 1910; Groen, 1948; Moore and Spiegel, 1964; Thorpe, 1968; Lindzen, 1970; Yeh and Liu, 1974; Campos, 1983b) and (ii) dissipative (Yanowitch, 1967a,1967b,1969; Lyons and Yanowitch, 1974; Campos, 1983c). Similar phenomena occur for purely acoustic waves in various nonhomogeneous and diffusive media (Bergmann, 1946; Mihalas and Mihalas, 1983; Moorhem and Landheim, 1984), but the case of linear acoustic-gravity waves, propagating vertically in an atmosphere, provides the closest analogy with the acoustics of ducts. For example, the decay of mass density with altitude, in an atmosphere, can be simulated in a tapering duct, by choosing the cross section so as to enclose the same mass per unit length; from this analogy, it follows that a wave propagating upward (downward) in an atmosphere, increases (decreases) in amplitude, as for sound in a converging (diverging) duct. The wave growth, for upward propagation in an atmosphere at rest, implies an increasing amplitude for a constant frequency, i.e., gradually steeper gradient of the waveform; this wave growth is opposed by dissipation, e.g., by viscosity, which limits wave amplitude and phase.

#### 1. The scale height

For a fluid in hydrostatic equilibrium, the mean-state pressure $p_0$ gradient $dp_0/dz$ with altitude $z$ balances the weight of fluid per unit height, i.e., the acceleration of gravity $g$ times the mass density $\rho_0$,

$$dp_0/dz = -\rho_0 g \ , \tag{221a}$$

$$p_0 = \rho_0 R T_0 \ , \tag{221b}$$

is the equation of state for a perfect gas, for which, in hydrostatic equilibrium [Eq. (221a)], the pressure decays

$$dp_0/dz = -p_0/L \ , \tag{222a}$$

on the scale height $L$, defined by

$$L \equiv T_0 R/g \ . \tag{222b}$$

In the case of an isothermal atmosphere, under constant gravity, the scale height is a constant, and both the gas pressure and mass density decay exponentially,

$$p_0(z)/p_0(0) = \rho_0(z)/\rho_0(0) = \exp(-z/L) \ , \tag{223}$$

so that the scale height is the distance over which there is a decay by a factor $e^{-1}$. From Eq. (223) it follows that

$$m_0 \equiv \int_0^\infty \rho_0(z)dz = L\rho_0(0) \ , \tag{224a}$$

$$z_G = m_0^{-1} \int_0^\infty z\rho_0(z)dz = L \ , \tag{224b}$$

which may be interpreted as (a) the total mass $m_0$ of the atmosphere, above the level $z=0$, is equal to the mass of a cylinder, of uniform density equal to the "ground level" value $\rho_0(0)$, and height equal to the scale height $L$; (b) the "center of gravity" $z_G$ of the atmosphere is located one scale height $L$ above the level $z=0$. Thus, for a cold isothermal atmosphere, the scale height is small, the density decays rapidly with height, and the center of gravity is at low altitude; for a hot isothermal atmosphere, the scale height is large, the density decays slowly with height, and the center of gravity is at high altitude. For the same initial mass density, the atmosphere has a greater (lesser) total mass for larger (smaller) scale height $L$.

#### 2. Viscous acoustic-gravity waves

The vertical momentum equation for a viscous fluid under gravity can be linearized,

$$\rho_0 \partial v/\partial t + \partial p/\partial z + \rho g = \eta_s \partial^2 v/\partial z^2 \ , \tag{225}$$

where we have subtracted the mean-state stratification [Eq. (221a)], $v,p,\rho$ denote the velocity, pressure, and density perturbations, respectively, and $\eta_s = \eta_{2s} + \frac{4}{3}\eta_{1s}$ the total static viscosity (where $\eta_{1s}$ and $\eta_{2s}$ are the first and second viscosities, respectively). The density perturbation is specified by the equation of continuity,

$$\partial \rho/\partial t = -\partial(\rho_0 v)/\partial z \ . \tag{226}$$

It is related to the pressure by the adiabatic condition, in a convected frame,

$$\partial p/\partial t + v\, dp_0/dz = c_0^2(\partial \rho/\partial t + v\, d\rho_0/dz) \ , \tag{227}$$

where $c_0$ is the adiabatic sound speed, viz., $c_0^2 \equiv (\partial p_0/\partial \rho_0)_s$. Using Eqs. (221a) and (226) we may rewrite Eq. (227)

$$\partial p/\partial t = \rho_0 g v - \rho_0 c_0^2 \partial v/\partial z \ . \tag{228}$$

Taking $\partial/\partial t$ of Eq. (225) and replacing $\partial\rho/\partial t, \partial p/\partial t$ from Eqs. (226) and (228), respectively, we obtain the vertical, viscous acoustic-gravity wave equation,

$$\partial^2 v/\partial t^2 - c_0^2 \partial^2 v/\partial z^2 + \gamma g \partial v/\partial t = \eta \partial^3 v/\partial z^2 \partial t \ , \tag{229}$$

where $\eta \equiv \eta_s/\rho_0$ denotes the kinematic viscosity.

#### 3. Analogy with horns

We consider first $\eta=0$, i.e., nondissipative, vertical acoustic-gravity waves, which satisfy

$$\partial^2 v/\partial t^2 - c_0^2 \partial^2 v/\partial z^2 + (c_0^2/L)\partial v/\partial z = 0 \tag{230}$$

in an isothermal atmosphere of length scale given by [Eq. (222b)]

$$L = c_0^2/\gamma g \ , \tag{231a}$$

$$c_0^2 = \gamma R T_0 \ , \tag{231b}$$

in terms of the sound speed $c_0$; since all coefficients of Eq. (230) are constant, the wave equation is the same for all acoustic variables, such as velocity $v$, displacement $v \equiv \partial\xi/\partial t$, and potential $v = \partial\varphi/\partial z$, and we can readily compare with other forms of the wave equation derived before. Thus (i) the acoustic wave equation for an inhomogeneous fluid at rest [Eq. (31)] coincides, in a one-dimensional form $\nabla^2 \equiv \partial^2/\partial z^2$, with Eq. (230), since $c_0^2 \nabla(\ln\rho_0)\nabla = -(c_0^2/L)\partial/\partial z$, where $L \equiv -(\rho_0^{-1} d\rho_0/dz)^{-1}$ is the positive length scale of decay of density with altitude; (ii) the horn wave equations (146a) and (146b) both coincide with Eq. (230) in the case of constant length scale $L \equiv A/A'$, i.e., for the exponential duct [Eqs. (147a)—(147c)], with the sign of $L$ reversed compared with $L \equiv -\rho_0/\rho_0'$. The analogy between the vertical acoustic-gravity waves in an isothermal atmosphere $\rho_0 \sim e^{-z/L}$ and the fundamental mode in an exponential horn $1/A \sim e^{-x/L}$ [Eq. (147a)] is that the mass of fluid, per unit length or altitude, decays exponentially with distance, on the scale height $L$. Using this analogy, it follows that vertical acoustic-gravity waves have a cutoff frequency [Eq. (154b)]

$$\omega_* = c_0/2L = \tfrac{1}{2}(\gamma g/L)^{1/2} = \tfrac{1}{2}(\gamma/RT_0)^{1/2} , \qquad (232)$$

in an isothermal atmosphere; the wave fields are given by Eqs. (166) and (168), e.g.,

$$v(x,t) = C_{\pm} e^{z/2L} \exp(\pm iK_0 z - i\omega t) , \qquad (233)$$

respectively, for an upward-downward (plus/minus sign) propagating wave, of frequency $\omega$ above the cutoff $\omega > \omega_*$, with effective wave number $K_0$ given by Eq. (170).

## 4. Two altitude ranges

The present atmospheric wave problem has two dimensionless parameters besides $z/L$ (altitude divided by scale height), namely,

$$\Omega \equiv \omega/\omega_* = 2\omega L/c_0 , \qquad (234a)$$

$$\Delta \equiv \omega\eta/c_0^2 . \qquad (234b)$$

The dimensionless frequency $\Omega$, defined as [Eq. (234a)] the ratio of wave $\omega$ to cutoff frequency $\omega_*$, is the only parameter for nondissipative waves, and the dissipation parameter (234b) accounts for the effects of viscosity. In an isothermal atmosphere, the scale height [Eq. (222b)], sound speed [Eq. (231b)], cutoff [Eq. (232)], and dimensionless frequency [Eq. (234a)] are all constant; the static viscosity $\eta_s$, which depends mainly on temperature, is also constant, but the kinematic viscosity $\eta \equiv \eta_s/\rho_0(z)$ increases exponentially [Eq. (223)] on the scale height,

$$\Delta(z) = \delta e^{z/L} , \qquad (235a)$$

$$\delta \equiv \omega\eta(0)/c_0^2 = \omega\eta_s/\rho_0 c_0^2 , \qquad (235b)$$

from the value $\delta$ at zero altitude [Eq. (235b)]. The viscous acoustic-gravity wave equation (229) is written, for a wave frequency $\omega$, in an atmosphere isothermal or not,

$$[(1-i\Delta)L^2 d^2/dz^2 - L\, d/dz + (\Omega^2/4)]v(z)e^{-i\omega t} = 0 . \qquad (236)$$

In a general, nonisothermal atmosphere, the wave equation (236) has a singularity $\Delta(z_*) = 1$, i.e., at the altitude $z = z_*$ where the effects of viscosity and compressibility balance,

$$\eta(z_*)\omega = [c_0(z_*)]^2 . \qquad (237a)$$

For the isothermal case [Eq. (235a)], the altitude of the singular level (237a) is

$$z_* = L \ln(c_0^2/\omega\delta) . \qquad (237b)$$

The singular level $z = z_*$ divides the atmosphere into two altitude ranges: (i) in the low-altitude range $z < z_*$, viscosity is weak $\Delta(z) < 1$, and appears as a damping of acoustic-gravity waves; (ii) in the high-altitude region $z > z_*$, viscosity dominates "compressibility" and ultimately determines the "character" of the "wave."

## 5. Three singularities

The two altitude ranges, low $z_1 < z < z_*$ and high $z_* < z < z_2$, are bounded by three possible singularities $z_1, z_2, z_*$ of the viscous acoustic-gravity wave equation (236): (i) the solution in powers of $\Delta$, for small viscosity, around the lower singularity $\Delta = 0$ or $z_1 \equiv -\infty$, corresponds to the initial regime of acoustic-gravity waves, with viscous damping; (ii) the asymptotic solution, in powers of $1/\Delta$, for large viscosity, around the upper singularity $\Delta = \infty$ or $z_2 = \infty$, specifies the regime where compressibility is less important; (iii) the intermediate singularity $\Delta = 1$, at the critical level $z = z_*$, describes the transition from the acoustic-gravity propagation below to the viscous regime above. The simplest second-order linear differential equation, with three regular singularities (Ince, 1926; Poole, 1936; Kamke, 1944) is the hypergeometric type, and indeed Eq. (236) can be transformed into a hypergeometric equation, in the case of an isothermal atmosphere. In order to perform this transformation, we adopt ($-i$ times the inverse of) the dissipation parameter as variable, instead of altitude:

$$\zeta \equiv -i\delta e^{-z/L} , \qquad (238a)$$

$$v(z,t) \equiv e^{-i\omega t} h(\zeta) , \qquad (238b)$$

where the function $h(\zeta)$ satisfies the hypergeometric equation:

$$(1-\zeta)\zeta h'' + (1-2\zeta)h' - (\Omega^2/4)h = 0 . \qquad (239)$$

The solution is (Abramowitz and Stegun, 1964) a linear combination of hypergeometric functions of the first $F$ and second $G$ kinds,

$$v(z,t)=e^{-i\omega t}\{C_1F_1[-(i/\delta)e^{-z/L}] + C_2G_1[-(i/\delta)e^{-z/L}]\}, \qquad (240a)$$

$$F_1,G_1(\zeta)\equiv F,G(\tfrac{1}{2}+iK_0L,\tfrac{1}{2}-iK_0L;1;\zeta), \qquad (240b)$$

where $K_0$ is the effective wave number [Eq. (170)], and $C_1,C_2$ are arbitrary constants of integration; the solution [(240)a) and (240b)] is valid in the "viscous regime" $|\zeta|<1$, corresponding [Eq. (238a)] to the high-altitude range $z>z_*$, above the singular level [Eq. (237b)].

## 6. Dissipation condition

Below the singular level $z<z_*$, we have $|\zeta|>1$, and thus we use the transformation $\zeta\to 1/\zeta$ for hypergeometric functions (Caratheodory, 1950), to obtain the velocity perturbation in the "acoustic regime,"

$$v(z,t)=e^{z/2L-i\omega t}[C_+e^{iK_0z}F_+(i\delta e^{z/L}) + C_-e^{-iK_0z}F_-(i\delta e^{z/L})], \qquad (241a)$$

$$F_\pm(1/\zeta)\equiv F(\tfrac{1}{2}\pm iK_0L,\tfrac{1}{2}\pm iK_0L;1\pm 2iK_0L;1/\zeta), \qquad (241b)$$

$$C_\pm\equiv(-i\delta)^{1/2\pm iK_0L}\{C_1+2[\psi(1)-\psi(\tfrac{1}{2}\mp iK_0L) + i\pi/2]C_2\}, \qquad (241c)$$

which corresponds [Eq. (241a)] to a superposition of upward/downward propagating (plus/minus sign) acoustic-gravity waves [Eq. (233)], modified by viscosity according to the hypergeometric functions (241b), with amplitudes $C_\pm$ given by Eq. (241c) in terms of $C_1,C_2$ [Eq. (240a)], where $\psi(\cdots)$ denotes the digamma function of the argument. The viscous acoustic-gravity waves must satisfy the following constraint, which may be designated (Yanowitch, 1967a,1967b) the dissipation condition:

$$\dot{E}_\eta\equiv(\eta/2)\int_0^\infty |\partial v(z,t)/\partial z|^2dz<\infty\ . \qquad (242)$$

This condition states that the energy dissipated over an infinite column of fluid, per unit time, must be finite. In order to illustrate the relevance of this condition, suppose we select a purely upward-propagating viscous acoustic-gravity wave, by setting $C_-=0$ in Eq. (241a), so that generally $C_1\neq 0\neq C_2$ in Eq. (241c). The wave amplitude would grow exponentially on twice the scale height, and the phase would vary linearly with altitude, as in Eq. (233), with modifications due to viscosity [Eq. (241b)] in the low-altitude range; in the high-altitude range, as $z\to\infty$, far above the critical level, $\zeta\to 0$, the first term of (240a) yields a finite amplitude and phase $F_1(\zeta)\to 1$, and the second $G_1(\zeta)\sim\ln\zeta\sim z/L$ leads to a linear growth. The latter corresponds to a rate of strain $\partial v/\partial z\sim$const that does *not* vanish at high altitude and thus leads to an infinite rate of energy dissipation, violating Eq. (242). Thus we conclude that if we solve the viscous acoustic-gravity wave equation (229) with $\eta\neq 0$, and then let viscosity vanish $\eta\to 0$, we *do not* obtain the solution of the inviscid acoustic-gravity wave equation (230), i.e., the solution of the wave equation and the limit $\eta\to 0$ are noncommutative.

## 7. Reflecting layer

This result implies that the effect of viscosity upon an acoustic-gravity wave is not merely damping, since the latter would vanish as $\eta\to 0$. The important mathematical difference is that the inviscid wave equation (230) does not have singularities, and its solution is valid everywhere, whereas the equation with viscosity (229) has a singular level [Eq. (237a)], which in the limit of zero viscosity $\delta\to 0$ does not disappear, but recedes to infinity $z_*\to\infty$ in Eq. (237b). In order to interpret this result physically, we apply the dissipation condition (242), requiring that the rate of strain $|\partial v/\partial z|\to 0$ must vanish at high altitude; since this is not met by the hypergeometric function of the second kind $G_1$ [Eq. (240b)], we suppress it by setting $C_2=0$ in Eq. (240a). Thus the "viscous regime" limits the amplitude and phase of the wave to a finite value, as altitude $z\to\infty$. In the low-altitude range $z<z_*$, the condition $C_1\neq 0=C_2$ implies that $C_+\neq 0\neq C_-$, i.e., we have both upward- and downward-propagating waves, viz., the singular level $z=z_*$ acts as a reflecting layer for waves. In conclusion, in the limit of zero viscosity $\eta\to 0$, an upward-propagating viscous acoustic-gravity wave [$C_+$ in Eq. (241a)] simplifies to the inviscid form [Eq. (233)]; the singular level (237b) recedes to infinity $z_*\to\infty$, but it still reflects the waves, producing a downward-propagating wave [$C_-$ in Eq. (241a)], which is *not* present in the inviscid solution.

## 8. Amplitude and phase limiting

Equations (240a),(240b) and (241a)–(241c) specify the wave field above and below, respectively, the reflecting layer $z_*$, in terms of hypergeometric functions of variable $\zeta$ and $1/\zeta$, respectively. These variables are two from the group of six $\zeta,1/\zeta,1-\zeta,1/(1-\zeta),1-1/\zeta,\zeta/(\zeta-1)$, which transforms (Klein, 1933) the hypergeometric equation into itself. The most convenient variable for the present problem is (Campos, 1983c) $1/(1-\zeta)$, because $|1/(1-\zeta)|<1$ for all $z$, by Eq. (238a); transforming Eq. (240a), with $C_2=0$, into a hypergeometric function (Forsyth, 1885) of variable $1/(1-\zeta)$, we obtain the expression for the velocity perturbation of viscous acoustic-gravity waves,

$$v(x,t)=e^{-i\omega t}(C_+[1+(i/\delta)e^{-z/L}]^{1/2+iK_0L}F_+\{[1+(i/\delta)e^{-z/L}]^{-1}\}+\text{interchange}+K_0\text{ with }-K_0)\ , \qquad (243)$$

where $F_{\pm}$ are the hypergeometric functions [Eq. (241b)] and $C_{+}$ a constant of integration. Equation (243) applies over the whole altitude range, including at the reflecting layer $e^{-z_*/L}=\delta$, where the terms in bold parentheses take the value $1+i$. The effects of viscosity upon atmospheric waves are illustrated in Fig. 11, which is based on nonlinear computations (Yanowitch, 1969); it shows that the exponential amplitude growth (a) and linear phase growth (b) for acoustic-gravity waves is sharply limited to a finite value, above the reflection layer, which appears as a "knee" in the curves.

### B. Sound of large amplitude in potential flows

It has been shown (Sec. V.A.) that diffusion processes can limit wave amplitude in atmospheres. For example, if they are strong enough the wave growth may be checked

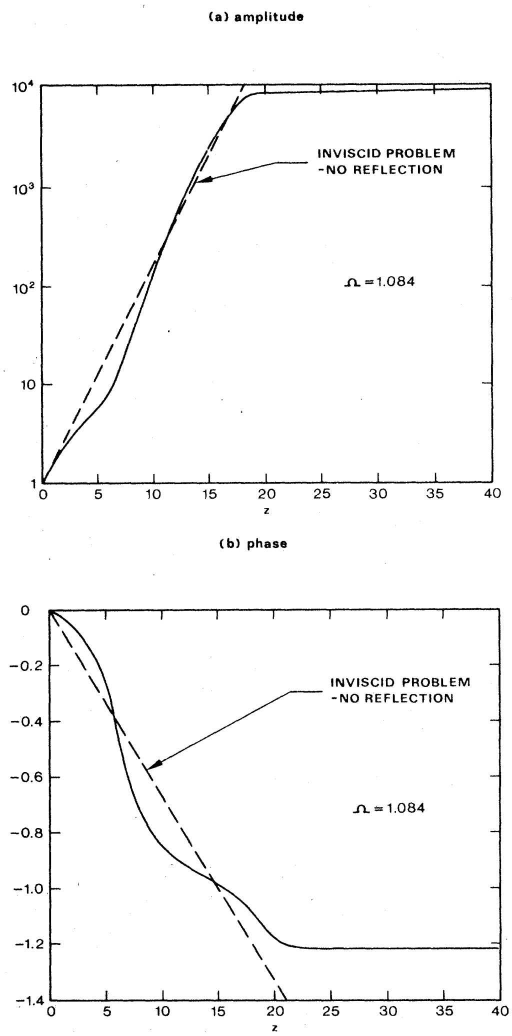


FIG. 11. Logarithm of amplitude (a) and phase (b) as a function of altitude, computed (Yanowitch, 1969) for a vertical viscous acoustic-gravity wave, showing the "knee" near the reflecting layer.

at a linear level. If diffusion is weak, the waves may grow to large amplitude before significant damping occurs, leading to a combined nonlinear and dissipative problem (Sec. V.C). In the initial stages of a disturbance, for times shorter than the typical diffusion time or distances of propagation shorter than the diffusion length, dissipation can be neglected. The wave equations (Lumsdaine and Ragab, 1977; Campos, 1985b) of nondissipative, nonlinear acoustics coincide with the exact equations of high-speed gas dynamics (Bateman, Murnaghan, and Dryden, 1956; Tsien, 1958), which, in the case of a potential flow, can be derived either from the equations of motion (Prandtl and Tietjens, 1934; Milne-Thomson, 1958) or from a variational principle (Bryan, 1918; Bateman, 1929), using local variables. In the case of vortical flow, instead of the equations of motion (Basset, 1888; Milne-Thomson, 1938), it is possible to use a variational principle (Seeliger and Whitham, 1968; Lynden-Bell and Katz, 1981; Katz and Lynden-Bell, 1982; Mobbs, 1982), in terms of Clebsch (1857) potentials (Lamb, 1879), which are integral (i.e., nonlocal) properties of the flow. The consideration of potential flow implies the assumption of homentropic conditions, and studies of waves of finite amplitude *not* relying on this restriction are relatively rare (Varley and Cumberbatch, 1970; Prasad, 1973); the common instance of nonhomentropic flow is the study of shock waves (Courant and Friedrichs, 1948; Raiser and Zel'dovich, 1966), which imply entropy jumps as a consequence of mass, momentum, and energy conservation relations applied across the shock front.

#### 1. Exact potential equations

The momentum equation for the potential flow of an inviscid fluid reduces to the Bernoulli equation,

$$\dot{\Phi}+\tfrac{1}{2}(\nabla\Phi)^2+H=H_* \ , \tag{244a}$$

$$H\equiv\int\Gamma^{-1}dP=\int(C^2/\Gamma)d\Gamma=C^2/(\gamma-1) \ , \tag{244b}$$

where $\Phi$ is the total potential, $\dot{\Phi}\equiv\partial\Phi/\partial t$ its time derivative, $\mathbf{V}=\nabla\Phi$ the flow velocity, and $H,H_*$ the free stream and stagnation enthalpies, respectively. For homentropic conditions, the enthalpy is given by Eq. (244b), where $\Gamma,P$ denote the total density and pressure, respectively, and $C$ the exact adiabatic sound speed $C^2\equiv(\partial P/\partial T)_s$. The exact equation of continuity or mass conservation,

$$0=\dot{\Gamma}+\nabla(\Gamma\mathbf{V})=\dot{\Gamma}+\nabla\Phi\cdot\nabla\Gamma+\Gamma\nabla^2\Phi \ , \tag{245a}$$

for a potential flow $\mathbf{V}=\nabla\Phi$, can be written in terms of the enthalpy [Eq. (244b)]:

$$C^2\nabla^2\Phi+\dot{H}+\nabla\Phi\cdot\nabla H=0 \ . \tag{245b}$$

Substituting Eq. (244a) into (245b) we obtain the exact equation for the unsteady potential,

$$C^2\nabla^2\Phi-\ddot{\Phi}-2\nabla\Phi\cdot\nabla\dot{\Phi}-\nabla\Phi\cdot[(\nabla\Phi\cdot\nabla)\nabla\Phi]=0 \ , \tag{246a}$$

$$C^2=C_*^2-(\gamma-1)[\dot{\Phi}+\tfrac{1}{2}(\nabla\Phi)^2] \ , \tag{246b}$$

where the sound speed in the free stream $C$ is given by Eqs. (244a) and (244b), i.e., in terms of the stagnation value $C_*$, by Eq. (246b).

### 2. Nonlinear local derivative

For nonlinear sound in a medium at rest the total potential $\Phi=\varphi$ coincides with the perturbation potential $\varphi$, which satisfies [Eqs. (246a) and (246b)] the exact wave equation:

$$\begin{aligned}\Box_0\varphi \equiv\; & c_0^2\nabla^2\varphi-\ddot{\varphi}-2\nabla\varphi\cdot\nabla\dot{\varphi}-\nabla\varphi\cdot[(\nabla\varphi\cdot\nabla)\nabla\varphi]\\ & -(\gamma-1)\dot{\varphi}\nabla^2\varphi-[(\gamma-1)/2](\nabla\varphi)^2\nabla^2\varphi\\ =\; & O(M_0)\,. \end{aligned} \tag{247}$$

Here the linear terms (first two) coincide with the classical wave equation (32), where $c_0\equiv C_*$ is the sound speed at rest; the additional, nonlinear terms are of degree two (third and fifth) and three (fourth and sixth), but the equation remains of order two, i.e., two boundary conditions specify its solution uniquely. The nonlinear wave equation (247) may be written in a form similar to the classical wave equation (32),

$$[c_0^2-(\gamma-1)\overline{\delta\varphi/\delta t}]\nabla^2\varphi-\delta(\overline{\delta\varphi/\delta t})/\delta t=O(M_0)\,, \tag{248}$$

by introducing the nonlinear $\delta\varphi/\delta t$ and self-convected $\overline{\delta\varphi/\delta t}$ local time derivatives, defined by

$$\delta\varphi/\delta t\equiv\dot{\varphi}+(\nabla\varphi)^2\,, \tag{249a}$$

$$\overline{\delta\varphi/\delta t}=\dot{\varphi}+\tfrac{1}{2}(\nabla\varphi)^2\,. \tag{249b}$$

Both reduce to the local time derivative $\delta\varphi/\delta t=\partial\varphi/\partial t=\overline{\delta\varphi/\delta t}$ in the linear approximation, and the nonlinear term $\nabla\varphi\cdot\nabla\varphi$ corresponds to "passive convection" of sound by sound for the nonlinear local derivative [Eq. (249a)]; for the self-convected local derivative [Eq. (249b)], the nonlinear term $\frac{1}{2}(\nabla\varphi)^2$ has a factor $\frac{1}{2}$ relative to passive convection, i.e., it corresponds to the kinetic energy (per unit mass) of the disturbance.

### 3. Self-convected local derivative

In the case of quasi-one-dimensional potential flow in a horn of cross section $A(x)$, the Bernoulli equations [(244a) and (244b)] are unchanged, in one-dimensional form, e.g., $\nabla\Phi\to\Phi'\equiv\partial\Phi/\partial x$; in the equation of continuity (245a), the mass per unit volume $\Gamma$ is replaced by mass per unit length $\Gamma A$, i.e.,

$$0=A\dot{\Gamma}+(\Gamma VA)'=A\dot{\Gamma}+\Gamma'A\Phi'+A'\Gamma\Phi'+A\Gamma\Phi''\,, \tag{250a}$$

or, in terms of the enthalpy,

$$C^2[\Phi''+(A'/A)\Phi']+\dot{H}+H'\Phi'=0\,. \tag{250b}$$

Substituting Eq. (244a), the exact equation for the unsteady potential is

$$C^2\Phi''-\ddot{\Phi}+C^2(A'/A)\Phi'-2\Phi'\dot{\Phi}'-\Phi'^2\Phi''=0\,, \tag{251a}$$

$$C^2=C_*^2-(\gamma-1)(\dot{\Phi}+\tfrac{1}{2}\Phi'^2)\,. \tag{251b}$$

Here the sound speed $C$ is given by Eq. (251b), so that the only difference in (251a) from the free-space expression (246a) is the replacement of the Laplacian $\Phi''$ by the duct operator $\Phi''+(A'/A)\Phi'$. Thus the nonlinear horn wave equations (251a) and (251b) with $\Phi\equiv\varphi$ differ from the nonlinear classical wave equation (247) by the term

$$\Box_1\varphi\equiv\Box_0\varphi+[c_0^2-(\gamma-1)(\dot{\varphi}+\tfrac{1}{2}\varphi'^2)](A'/A)\varphi'=0\,; \tag{252}$$

it can be written in compact form

$$[c_0^2-(\gamma-1)\overline{\delta\varphi/\delta t}]A^{-1}(A\varphi')'-\delta(\overline{\delta\varphi/\delta t})/\delta t=O(M_0)\,, \tag{253}$$

which is analogous to the linear horn equation (33) with a nonlinear correction to the sound speed (square brackets in the first term) and replacement of local time derivatives $\ddot{\varphi}\equiv\partial^2\varphi/\partial t^2$ by nonlinear Eq. (249a)] and self-convected [Eq. (249b)] local derivatives.

### 4. Nonlinear material derivative

Nonlinear sound is indistinguishable from unsteady, compressible "mean flow," and in this case the preceding wave equations (Secs. V.B.2 and V.B.3) should be used for the total potential $\Phi$. If there is a steady, nonuniform mean flow, the acoustic perturbation, also nonuniform, can be distinguished by its unsteadiness. Thus the total flow variables (capital letters) can be decomposed in the sum of a steady flow (subscript 0) and an acoustic perturbation (lower-case letters),

$$\begin{aligned}\Phi,\mathbf{V},P,\Gamma,C(\mathbf{x},t)=\;&\varphi_0,\mathbf{v}_0,p_0,\rho_0,c_0(\mathbf{x})\\ &+\varphi,\mathbf{v},p,\rho,c(\mathbf{x},t)\,,\end{aligned} \tag{254}$$

for the potential, velocity, pressure, mass density, and sound speed, respectively. Substituting in Eqs. (246a) and (246b) and retaining all linear and nonlinear terms for sound of large amplitude, we obtain, in a low-Mach-number mean flow for which the density $\rho_0$ and sound speed $c_0\equiv C_*$ are constant, the nonlinear convected wave equation,

$$\begin{aligned}\Box_2\varphi\equiv\;&\Box_0\varphi-2\mathbf{v}_0\cdot\nabla\dot{\varphi}-(\gamma-1)(\mathbf{v}_0\cdot\nabla\varphi)\nabla^2\varphi\\ &-(\nabla\varphi\cdot\nabla)(\mathbf{v}_0\cdot\nabla\varphi)-\nabla\varphi\cdot[(\mathbf{v}_0\cdot\nabla)\nabla\varphi]=0\,,\end{aligned} \tag{255}$$

which differs from the nonlinear classical wave equation (247) by having a number of extra linear (second) and nonlinear (third to fifth) terms, involving the mean flow velocity $\mathbf{v}_0$. The nonlinear convected wave equation (255) and (247) can be written in the form

$$[c_0^2-(\gamma-1)D\varphi/Dt]\nabla^2\varphi-D(\overline{D\varphi/Dt})/Dt=O(M_0^2)\,, \tag{256}$$

which is analogous to the linear case (38), replacing the linear material derivative (25), by the nonlinear $D\varphi/Dt$ and self-convected $\overline{D\varphi/Dt}$ material derivatives,

$$D\varphi/Dt \equiv d\varphi/dt + (\nabla\varphi)^2 , \tag{257a}$$

$$\overline{D\varphi/Dt} = d\varphi/dt + \tfrac{1}{2}(\nabla\varphi)^2 , \tag{257b}$$

which have the nonlinear terms as the local forms [Eqs. (249a) and (249b)].

## 5. Self-convected material derivative

In the case of acoustic waves of finite amplitude, in quasi-one-dimensional steady low-Mach-number potential flow in a nozzle, the substitution of Eq. (254) into Eqs. (251a) and (251b) yields the nonlinear low-speed nozzle wave equation,

$$\Box_3\varphi \equiv \Box_1\varphi - 2v_0\dot{\varphi}' - 3v_0\varphi'\varphi'' + (2-\gamma)v_0\varphi'[\varphi'' + (A'/A)\varphi'] = O(M_0^2) , \tag{258}$$

which differs from the nonlinear horn equation (252) by a number of linear (second) and nonlinear (third and fourth) terms, involving the mean flow velocity. The nonlinear nozzle wave equation can be written in the compact form

$$[c_0^2 - (\gamma-1)\overline{D\varphi/Dt}]A^{-1}(A\varphi')' - D(\overline{D\varphi/Dt})/Dt = O(M_0^2) , \tag{259}$$

which is similar to the linear form (41), replacing linear material derivatives (25) by nonlinear (257a) and self-convected (257b) material derivatives. The nonlinear nozzle (259) and horn (253) wave equations differ in the same way as the nonlinear convected (256) and classical (248) wave equations, namely, they exchange local [Eqs. (249a) and (249b)] and material [Eqs. (257a) and (257b)] derivatives, both nonlinear and self-convected:

$$D\varphi/Dt = \delta\varphi/\delta t + \mathbf{v}_0\cdot\nabla\varphi , \tag{260a}$$

$$\overline{D\varphi/Dt} = \delta\varphi/\delta t + \mathbf{v}_0\cdot\nabla\varphi . \tag{260b}$$

Thus the generalization of the wave equation from a medium at rest to a low-Mach-number flow involves the convection term $\mathbf{v}_0\cdot\nabla\varphi$, for both linear [Eq. (25)] and nonlinear [Eqs. (260a) and (260b)] sound.

## 6. General wave equation

For nonlinear sound in a high-speed, i.e., compressible, steady mean flow, the mass density $\rho_0$ and sound speed $c_0$ are *not* constant. The exact sound speed [Eq. (246b)] is given, using Eq. (254), by

$$C^2 = c_0^2 - (\gamma-1)[\dot{\varphi} + \mathbf{v}_0\cdot\nabla\varphi + \tfrac{1}{2}(\nabla\varphi)^2] , \tag{261a}$$

$$c_0^2 \equiv C_*^2 - [(\gamma-1)/2]v_0^2 , \tag{261b}$$

where $c_0$ is the mean flow sound speed [Eq. (261b)], which is constant and equal to the stagnation value $c_0 = C_*$ only at low Mach number $M_0^2 \equiv v_0^2/c_0^2 \ll 1$. The general, nonlinear wave equation in a high-speed potential flow differs from the low-Mach-number case [Eq. (255)] by terms involving the nonuniformity of mass density $\rho_0$ and sound speed $c_0$,

$$\Box_4\varphi \equiv \Box_2\varphi + c_0^2\nabla\varphi\cdot\nabla\ln\rho_0 + (\dot{\varphi} + \mathbf{v}_0\cdot\nabla\varphi)\mathbf{v}_0\cdot\nabla\ln c_0^2 + [(\gamma-1)/2](\nabla\varphi)^2\mathbf{v}_0\cdot\nabla\ln\rho_0 . \tag{262}$$

The most general wave equation in a potential flow, accounting for nonlinearity [Eq. (247)], convection [Eq. (255)], and nonuniformity [Eq. (262)], can be written in the compact form

$$[c_0^2 - (\gamma-1)\overline{D\varphi/Dt}]\nabla^2\varphi - D(\overline{D\varphi/Dt})/Dt + c_0^2\nabla\varphi\cdot\nabla\ln\rho_0 + (\overline{D\varphi/Dt})\mathbf{v}_0\cdot\nabla\ln c_0^2 = 0 , \tag{263}$$

which is similar to the linear case [Eq. (44)], replacing linear [Eq. (25)] by nonlinear [Eq. (257a)] and self-convected [Eq. (257b)] material derivatives,

$$D\varphi/Dt = \dot{\varphi} + \mathbf{v}_0\cdot\nabla\varphi + (\nabla\varphi)^2 , \tag{264a}$$

$$\overline{D\varphi/Dt} = \dot{\varphi} + \mathbf{v}_0\cdot\nabla\varphi + \tfrac{1}{2}(\nabla\varphi)^2 , \tag{264b}$$

where the three terms correspond to the classical wave equation, linear convection of sound by the mean flow, and nonlinear convection of sound by sound.

## 7. Transformation of wave equations

For nonlinear sound in a quasi-one-dimensional potential flow of arbitrary Mach number in a nozzle, there are, in addition to the terms in the low-speed nonlinear nozzle equation (258), the following:

$$\Box_5\varphi \equiv \Box_3\varphi + 2M_0c_0'(\dot{\varphi} + v_0\varphi') + M_0c_0'\varphi'^2 = 0 . \tag{265}$$

These terms involve the nonuniformity of the sound speed $c_0$ for high-Mach-number mean flow [Eq. (261b)]. The most general nonlinear acoustic wave equation, for quasi-one-dimensional propagation in ducts of varying cross section $A$, including the effects of nonlinearity [Eq. (247)], reflections from the walls [Eq. (252)], low-speed convection [Eq. (258)], and high-speed mean flow [Eq. (265)], can be written in the compact form

$$[c_0^2 - (\gamma-1)\overline{D\varphi/Dt}]A^{-1}(A\varphi')' - D(\overline{D\varphi/Dt})/Dt + 2M_0c_0'(\overline{D\varphi/Dt}) - v_0v_0'\varphi' = 0 , \tag{266}$$

which reduces to Eq. (47) in the linear case. Comparing the classical wave equation (32) with the nonlinear high-speed nozzle wave equation (266), we can list all the transformations needed to write wave equations in potential flows: (i) for nonlinear sound, the exact sound speed $C$ is corrected [Eqs. (261a) and (261b)] relative to the mean flow sound speed $c_0$ by

$$C^2=c_0^2-(\gamma-1)\overline{D\varphi/Dt}\ , \tag{267}$$

where $\overline{D\varphi/Dt}$ is the self-convected material derivative [Eq. (264b)]; (ii) the latter and the nonlinear material derivative $D\varphi/Dt$ [Eq. (264a)] replace the local time derivatives $\ddot{\varphi}$, to account both for nonlinear and linear convection; (iii) reflections from the walls are included by replacing the Laplacian $\nabla^2\varphi$ by the duct operator $A^{-1}(A\varphi')'$; (iv) nonuniformity of the sound speed $c_0$ and mass density $\rho_0$ in homogeneous or high-Mach-number flow add extra terms.

### 8. Temporal and spatial operators

We list in Table III(a) the nonlinear acoustic wave equations, in the same six cases as the linear forms [Table II(b)]. The nonlinear equations were deduced directly from the equations of fluid mechanics (Sec. V.B); since the linear terms coincide with those derived from the acoustic variational principle [Table II(a)], we have an independent check of its validity. The acoustic wave equations in potential flows are all generalizations of the classical wave equation, in three directions: (i) changing from three-dimensional propagation in free space to the quasi-one-dimensional fundamental mode, in a duct of varying cross section, replaces the Laplacian by the duct operator; (ii) passing from a medium at rest to low-Mach-number convection replaces the local by the material derivative, and high-speed mean flow adds inhomogeneous terms; (iii) the extension from linear to nonlinear sound implies taking into account convection of sound by sound in the nonlinear and self-convected derivatives. All these transformations involve space and time derivatives, listed in Table III(b) after the nonlinear wave equations.

## C. Front steepening versus gradient smoothing

Having considered, separately, viscous dissipation of a linear sound (Sec. V.A), and nonlinear, inviscid acoustics (Sec. V.B), we now combine the two problems by discussing the evolution of a large pressure pulse in a viscous fluid. The combination of large amplitude and diffusion is an important topic in modern nonlinear acoustics (Blackstock, 1972; Beyer, 1974; Bjørnø, 1974; Rudenko and Soluyan, 1977), since they have opposing effects, respectively, steepening (Riemann, 1860) and smoothing (Lighthill, 1956) of a waveform, and their relative importance may change during the propagation of a pressure pulse (Blackstock, 1965; Ockendon and Spence, 1969). This kind of evolution can be modeled by considering compressibility effects to second order and dissipation to first order (Crighton, 1979), since this is the lowest level of complexity at which both effects compete at deforming a waveform. The wave equation involves vector and scalar potentials and is of fourth order, if temperature boundary conditions are applied (Blackstock, 1964); it reduces to second order, involving only scalar potentials, if only dynamic boundary conditions are used. The approximate solutions by perturbation (Nayfeh, 1973) and singular perturbation (van Dyke, 1964; Lesser and Crighton, 1975) methods are usually untractable beyond the second or third order. Explicit, analytic solutions have been obtained by special transformations of specific

TABLE III. Nonlinear acoustics in the same six cases as the linearized forms listed in Table II, viz., in media at rest, in low-speed convection, and in high-speed potential flows, for both three-dimensional and quasi-one-dimensional propagation: (a) the acoustic wave equations; (b) the derivatives that transform the wave equations in the twelve preceding cases (six linear and six nonlinear), in space (Laplacian and duct), and time (local, material, nonlinear, and self-convected).

| Case | Three-dimensional sound in free space | Quasi-one-dimensional sound in nonuniform duct |
|---|---|---|
| | (a) | |
| Potential | | |
| Total | Eqs. (246a) and (246b) | Eqs. (251a) and (251b) |
| Perturbation in: | | |
| Medium at rest | Eqs. (247) and (248) | Eqs. (252) and (253) |
| Low-Mach-number convection | Eqs. (255) and (256) | Eqs. (258) and (259) |
| High-speed steady potential flow | Eqs. (262) and (263) | Eqs. (265) and (266) |
| | (b)[a] | |
| Medium: Designation: | Fluid at rest local derivative | Potential flow material derivative |
| Linear | $\partial/\partial t$, Eq. (25) | $d/dt$, Eq. (25) |
| Nonlinear | $\delta/\delta t$, Eq. (249a) | $\overline{\delta/\delta t}$, Eq. (249b) |
| Self-convected | $D/Dt$, Eq. (264a) | $\overline{D/Dt}$, Eq. (264b) |

[a]Space derivatives: Three-dimensional, free space, $\nabla^2\phi$; quasi-one-dimensional duct, $A^{-1}(A\phi')$.

equations. Two of the best-known are (i) the inverse scattering transformation (Gardner, Greene, Kruskal, and Miura, 1967,1974) of the Korteweg–de Vries equation (Boussinesq, 1871; Rayleigh, 1876; Korteweg and de Vries, 1895; Miles, 1981), which leads to soliton-type solutions (Freeman, 1979; Cheng, 1984; Drazin, 1984), that have been observed (Russel, 1845; Osborne and Burch, 1980) and are studied extensively for water (Chang, Melville, and Miles, 1979; Melville, 1980) and internal (Grimshaw, 1981) waves, and more rarely for gas motions (Roberts and Mangeney, 1982), analogous to deep water waves (Benjamin, 1967; Ono, 1975); (ii) the transformation (Burgers, 1948; Cole, 1950; Hopf, 1951) of the nonlinear diffusion equation (Burgers, 1974) into a heat equation (Carslaw and Jaeger, 1946), which has been thoroughly studied in connection with the formation and evolution of nonlinear acoustic and shock waves (Hayes, 1958; Whitham, 1974; Lighthill, 1978a) and has applications in various contexts (Campos, 1983g,1984d; Baxter, 1984; Campos and Leitão, 1985).

### 1. Invariants along characteristics

The exact, nonlinear, one-dimensional equation of momentum for a viscous fluid is

$$\partial V/\partial t+V\partial V/\partial x+C^{-2}\partial\Gamma/\partial x=\eta\partial^2 V/\partial x\ , \tag{268}$$

where $V$ denotes the velocity, the pressure gradient $\partial P/\partial x$ is related to the density gradient $\partial\Gamma/\partial x$ through the adiabatic sound speed $C^2\equiv(\partial P/\partial\Gamma)$, and $\eta\equiv\eta_2+\frac{4}{3}\eta_1$ is the total kinetic viscosity (where $\eta_1,\eta_2$ are the first and second viscosities, respectively). Comparing with Eq. (245a), the exact, nonlinear, one-dimensional equation of continuity,

$$\partial\Gamma/\partial t+V\partial\Gamma/\partial x+\Gamma\partial V/\partial x=0\ , \tag{269}$$

suggests that we multiply Eq. (269) by $\pm C/\Gamma$ to give it the same dimensions as Eq. (268) and then add the two equations, to obtain

$$\{\partial/\partial t+(V\pm C)\partial/\partial x\}J_\pm=\eta\partial^2 V/\partial x^2\ , \tag{270}$$

$$J_\pm(x,t)=V\pm\int(C/\Gamma)d\Gamma=V\pm[2/(\gamma-1)]C\ . \tag{271}$$

The integral in Eq. (271) was calculated in homentropic conditions $C^2=\gamma P/\Gamma$ and $P\sim\Gamma^\gamma$, where $\gamma$ is the adiabatic exponent. Equation (270) states that viscosity causes the decay of the Riemann invariant (271), along the characteristic curves defined by

$$dx/dt=V\pm C\equiv U_\pm \tag{272}$$

as the "paths" of propagation of waves at the "group velocity," i.e., the fluid velocity plus or minus the sound speed. In other words, each of the two directions of propagation $U_\pm$ corresponds to one characteristic curve and one invariant $J_\pm$.

### 2. Deformation of waveform

The pair of equations [(270) and (271)] can be interpreted in gradually more complex stages: (i) linear, nondissipative, (ii) nonlinear, nondissipative, and (iii) nonlinear, dissipative. In case (i), viz., nondissipative $\eta=0$ and linear approximation, the velocity perturbation is neglected compared with the sound speed. The characteristics of Eq. (272) simplify to $dx/dt=\pm c_0$ where $c_0=(\partial p_0/\partial\rho_0)_s$ is the adiabatic sound speed, calculated for the mean state, i.e., neglecting the acoustic perturbation; Eq. (270) simplifies to

$$(\partial/\partial t\pm c_0\partial/\partial x)J_\pm=0\ , \tag{273a}$$

$$J_\pm(x,t)=f(x\mp c_0 t)\ , \tag{273b}$$

showing that the Riemann invariants $J_\pm$ reduce to linear waves, propagating at the sound speed $c_0$, in the positive/negative $(+/-)$ $x$ direction, *without* change in the shape of the waveform. In case (ii), still nondissipative, but now nonlinear, the velocity perturbation $V$ is comparable to the sound speed $C$, and the latter is affected by the presence of the wave as a large perturbation of the mean flow, so that the characteristics $C_\pm$ are given by Eq. (272); Eq. (270) with $\eta=0$ shows that the Riemann invariant [(271)] is conserved along the characteristics

$$(\partial/\partial t+U_\pm\partial/\partial x)J_\pm=0\ , \tag{274a}$$

$$J_\pm(x,t)=f[x-U_\pm(x)t]\ , \tag{274b}$$

and the nonlinear solution differs fundamentally from the linear one, in that $J_\pm$ and $U_\pm$ are mutually dependent, i.e., Eq. (273b) is explicit, whereas Eq. (274) is implicit; unlike the former, the latter allows for the deformation of the waveform. In the compression (rarefaction) phase of wave, the acoustic velocity adds to (subtracts from) the sound speed, so that crests propagate faster than troughs, leading to a steepening of the compression phase and tailing off of the expansion.

### 3. Simple wave

In the absence of dissipation, the steepening of the wave front would continue until it became vertical, and a shock discontinuity would form; in this case, the velocity gradient $\partial v/\partial x\rightarrow\infty$ would increase without bound. It is clear that the presence of dissipation will limit the steepness of the waveform to a finite value. Thus, in case (iii) described above (Sec. V.C.2), dissipation will cause the Riemann invariant [(271)], describing the propagation of nonlinear sound along the characteristics (272), to decay [Eq. (270)]. In order to determine the sound field at any event (at position $x$ and time $t$), we have to identify the two characteristics $C_+$ and $C_-$ passing through $(x,t)$, and follow the evolution of the invariant $J_\pm$ back to a known event, e.g., the generation of the wave; the values of $J_\pm(x,t)$, so derived, completely specify the acoustic field. For example, we may consider a piston advancing into a tube containing a viscous fluid and creating a large compression pulse. Assuming the tube to be uniform and semi-infinite, there is no reflected wave, and the invariant $J_-$ must be constant, viz., we can set it to zero, since a constant is a nonpropagating term. Setting $J_-=0$ in Eq. (271) we obtain the following relations between velocity perturbation $V$, sound speed $C$, and propagation velocity $U \equiv U_+$:

$$V=[2/(\gamma-1)]C , \tag{275a}$$

$$U \equiv V+C=[(\gamma+1)/(\gamma-1)]C=[(\gamma+1)/2]V . \tag{275b}$$

The wave, propagating in the positive direction at total velocity (275b), is completely described by the nonvanishing invariant $J \equiv J_+$:

$$J=V+[2/(\gamma-1)]C=2V=[4/(\gamma-1)]C=[4/(\gamma+1)]U . \tag{276}$$

Substituting in Eq. (270) we obtain

$$\partial V/\partial t+[(\gamma+1)/2]V\partial V/\partial x=(\eta/2)\partial^2 V/\partial x^2 , \tag{277}$$

$$\partial C/\partial t+[(\gamma+1)/(\gamma-1)]C\partial C/\partial x=(\eta/2)\partial^2 C/\partial x^2 , \tag{278}$$

$$\partial U/\partial t+U\partial U/\partial x=(\eta/2)\partial^2 U/\partial x^2 , \tag{279}$$

as the exact equations satisfied, respectively, by the flow $V$, sound $C$, and propagation $U \equiv V+C$ velocities.

## 4. Viscous potential equation

The preceding deduction (Lighthill, 1956) of the Burgers (1948) or nonlinear diffusion equation (279), was based on a one-dimensional, nonlinear, *first*-order differential equation [(270)], describing unidirectional waves. An alternative approach relies on nonlinear, three-dimensional, *second*-order differential equations, which include the possibility of propagation in opposite directions. In order to illustrate the latter approach, for three-dimensional nonlinear dissipative acoustics, we consider the viscous momentum equation,

$$\partial \mathbf{V}/\partial t+\nabla(V^2/2)+\mathbf{V}\wedge(\nabla\wedge\mathbf{V})+\nabla H=\eta_1\nabla^2\mathbf{V}+(\eta_2+\tfrac{1}{3}\eta_1)\nabla(\nabla\cdot\mathbf{V}) , \tag{280}$$

where we have written the enthalpy gradient $\nabla H=\Gamma^{-1}\nabla P$, neglecting the entropy terms $T\nabla s$, where $T$ is temperature. For a potential flow $\nabla\wedge\mathbf{V}=0$, we can integrate Eq. (280) into the viscous Bernoulli equation,

$$\partial\Phi/\partial t+\tfrac{1}{2}(\nabla\Phi)^2-\eta\nabla^2\Phi+H=H_* , \tag{281}$$

satisfied by $\mathbf{V}=\nabla\Phi$ the total potential $\Phi$, where $\eta\equiv\eta_2+\frac{4}{3}\eta_1$ is the total kinematic viscosity. Substituting the enthalpy $H$ from Eq. (281) into the continuity equations [(245b) and (246b)],

$$\dot{H}+\nabla\Phi\cdot\nabla H+\{C_*^2-(\gamma-1)[\partial\Phi/\partial t+\tfrac{1}{2}(\nabla\Phi)^2]\}\nabla^2\Phi=0 , \tag{282}$$

we obtain the exact equation for the total potential in a viscous fluid, in homentropic conditions:

$$C_*^2\nabla^2\Phi-\partial^2\Phi/\partial t^2-2\nabla\Phi\cdot\nabla(\partial\Phi/\partial t)-(\gamma-1)(\partial\Phi/\partial t)\nabla^2\Phi-\nabla\Phi\cdot[(\nabla\Phi\cdot\nabla)\nabla\Phi]$$
$$-[(\gamma-1)/2](\nabla\Phi)^2\nabla^2\Phi+\eta\nabla^2(\partial\Phi/\partial t)+\eta[(\nabla\Phi\cdot\nabla)\nabla^2\Phi]=0 . \tag{283}$$

This equation coincides with the nonlinear, "classical" wave equation [(246a) and (246b)] in nondissipative terms, and adds two viscous terms.

## 5. Nonlinear diffusion equation

The preceding equations should be considered as mathematical models of nonlinear, dissipative acoustics, since the physical assumptions underlying them may not be strictly consistent. For example, a viscous flow is usually rotational $\nabla\wedge\mathbf{V}=0$, and a scalar potential $\mathbf{V}=\nabla\Phi$ does not exist, at least in the boundary layer near a wall. At a nonlinear level, there is dissipation of energy by viscosity, scaling as the square of the rate of strain $(\partial v_i/\partial x_j)^2$, and this invalidates the conservation of entropy. For rotational, nonhomentropic flow, the velocity $\mathbf{v}=\nabla\Phi+\nabla\wedge\boldsymbol{\psi}$ may be expressed in terms of scalar $\Phi$ and vector $\boldsymbol{\psi}$ potentials, and the equations of continuity, momentum, and energy lead to (Blackstock, 1964) fourth-order wave equations, which can meet boundary conditions for velocity and temperature, e.g., in the presence of heat diffusion. We have obtained a second-order equation (283) by omitting temperature but allowing velocity boundary conditions for potential homentropic flow; in the one-dimensional case [Eqs. (277)—(279)], the scalar potential always exists

$$\Phi(x,t)=\int^x v(\xi,t)d\xi ,$$

but the homentropic assumption was also made in (271). Henceforth, we shall also assume the kinematic viscosity $\eta=\eta_s/\Gamma$ to be a constant, although it is the ratio of the static viscosity $\eta_s$ (which depends mainly on temperature $T$) to the mass density $\Gamma$, both of which can vary significantly in a nonlinear wave. All these assumptions are "acceptable" for the simplest model theory of waveform deformation, which considers dissipation *linearly* and compressibility to *second* order [since, to first order, compressibility does not change the waveform (273a) and (273b)]. Thus we simplify Eq. (283) to

$$c_0^2\nabla^2\varphi-\partial^2\Phi/\partial t^2-2\nabla\Phi\cdot\nabla(\partial\Phi/\partial t)-(\gamma-1)(\partial\Phi/\partial t)\nabla^2\Phi+\eta\nabla^2(\partial\Phi/\partial t)=0\ . \qquad (284)$$

In the linear, nondissipative, one-dimensional case Eq. (284) reduces to the first two terms,

$$\partial^2\Phi/\partial t^2-c_0^2\partial^2\Phi/\partial x^2=(\partial/\partial t-c_0\partial/\partial x)(\partial/\partial t+c_0\partial/\partial x)\Phi\ , \qquad (285a)$$

which is a combination of the waves [(273a) and (273b)], propagating in opposite directions, without deformation of waveforms. The first-order approximation $c_0\partial\Phi/\partial x\sim-\partial\Phi/\partial t$, or $\partial\Phi/\partial t-c_0\partial\Phi/\partial x\sim2\partial\Phi/\partial t$, may be used (Crighton, 1979) in the second-order terms of Eq. (284), retaining the same nominal order of accuracy,

$$\partial\Phi/\partial t+c_0\partial\Phi/\partial x-(\eta/2)\partial^2\Phi/\partial x^2+[(\gamma-1)/4](\partial\Phi/\partial x)^2=0\ , \qquad (285b)$$

where we have suppressed throughout a $\partial/\partial t$ factor, which represents a nonpropagating term. Applying $\partial/\partial x$ to the equation for the potential (285b) we obtain, for the velocity $V\equiv\partial\Phi/\partial x$,

$$\partial V/\partial t+\{c_0+[(\gamma+1)/2]V\}\partial V/\partial x=(\eta/2)\partial^2V/\partial x^2\ , \qquad (286)$$

which coincides with Eq. (277), with the change of variable $x\to x-c_0t$ to a frame moving at sound speed $c_0$ for the fluid at rest.

## 6. Linearizing transformation

Of the four equivalent equations, (277), (278), (279), and (286), we proceed with an analysis of Eq. (279), which is similar to the Navier-Stokes equation in one dimension, without pressure gradient, and with viscosity $\eta$ halved; if we take into account thermal conduction, then $\eta$ is replaced by the diffusivity of sound $\bar\eta\equiv\eta+(\gamma-1)\eta_*$, where $\eta_*$ is the thermal diffusivity. Equation (279) is similar to a linear heat equation (first and third terms), with a nonlinear convection term (the second); the latter can be suppressed by a suitable change of dependent variable, that linearizes the equation but leads to nonlinear initial or boundary conditions. In order to find this transformation, we note that the nonlinear diffusion equation can be written in the form

$$-2\partial U/\partial t=\partial[U^2-\eta(\partial U/\partial x)]/\partial x\ , \qquad (287)$$

which is identically satisfied, if a function $\mathcal{X}$ exists, such that

$$U=-\eta\, d(\ln\mathcal{X})/\partial x\ , \qquad (288)$$

$$2\eta\partial(\ln\mathcal{X})/\partial t=U^2-\eta\partial U/\partial x\ . \qquad (289)$$

These equations imply

$$\mathcal{X}(x,t)=\exp\left[-\eta^{-1}\int_{-\infty}^{x}U(\xi,t)d\xi\right]\ , \qquad (290)$$

$$\partial\mathcal{X}/\partial t=(\eta/2)\partial^2\mathcal{X}/\partial x^2\ , \qquad (291)$$

i.e., the nonlinear diffusion equation (279) transforms, through Eqs. (288) and (290), to a linear heat equation (291), of which many solutions are known. An initial value problem, the evolution [Eq. (279)] of an initial velocity pulse $U(x,0)$, may be calculated as follows: (i) the initial condition is transformed, via Eq. (290), to $\mathcal{X}(x,0)$; (ii) this initial condition is used in the solution $\mathcal{X}(x,t)$ of the heat equation (291); (iii) substitution into Eq. (288) yields the velocity pulse $U(x,t)$ at all times.

## 7. Evolution of a pulse

We illustrate this procedure for a simple pulse shape, viz., a single initial hump:

$$U(x,0)=B\delta(x)=\eta\,\mathrm{Re}\delta(x)\ , \qquad (292a)$$

$$\mathrm{Re}\equiv B/\eta=\eta^{-1}\int_{-\infty}^{+\infty}U(\xi,0)d\xi\ . \qquad (292b)$$

Here $\delta(x)$ is Dirac's delta function and $B$ the area under the velocity pulse [Eq. (292b)]. Since it has the same dimensions [$(\text{length})^2\times\text{time}$] as the kinematic viscosity, we can form a dimensionless Reynolds (1883) number $\mathrm{Re}\equiv B/\eta$ with their ratio. In the presence of thermal diffusivity $\eta_*$, the diffusivity of sound is $\bar\eta=\eta+(\gamma-1)\eta_*$ and Eqs. (292a) and (292b) are replaced by the diffusion number

$$De\equiv B/[\eta+(\gamma-1)\eta_*]=1/[1/\mathrm{Re}+(\gamma-1)/\mathrm{Pe}]\ ,$$

where $\mathrm{Pe}\equiv B/\eta_*$ is the Péclet (1843) number. The initial "temperature" [Eq. (290)] corresponding to Eq. (292a) would be

$$\mathcal{X}(x,0)=\begin{cases}1 & \text{if } x<0\\ \exp(-\mathrm{Re}) & \text{if } x>0.\end{cases} \qquad (293)$$

The solution of the heat equation (291) with the initial "temperature" profile (293) is

$$\mathcal{X}(x,t)=(2\pi\eta t)^{-1/2}\int_{-\infty}^{+\infty}\mathcal{X}(\xi,0)\exp[-(\eta-\xi)^2/2\eta t]d\xi=1-[1-\exp(-\mathrm{Re})]\mathrm{erf}[x/(2\eta t)^{1/2}]\ , \qquad (294)$$

where erf denotes the error function (Frank and Von Mises, 1931). From Eq. (288), the corresponding velocity pulse is

$$U(x,t)=-(2\eta/\pi t)^{1/2}\exp(-x^2/2\eta t)/\{[1-\exp(-\mathrm{Re})]^{-1}-\mathrm{erf}[x/(2\eta t)^{1/2}]\}\ , \qquad (295)$$

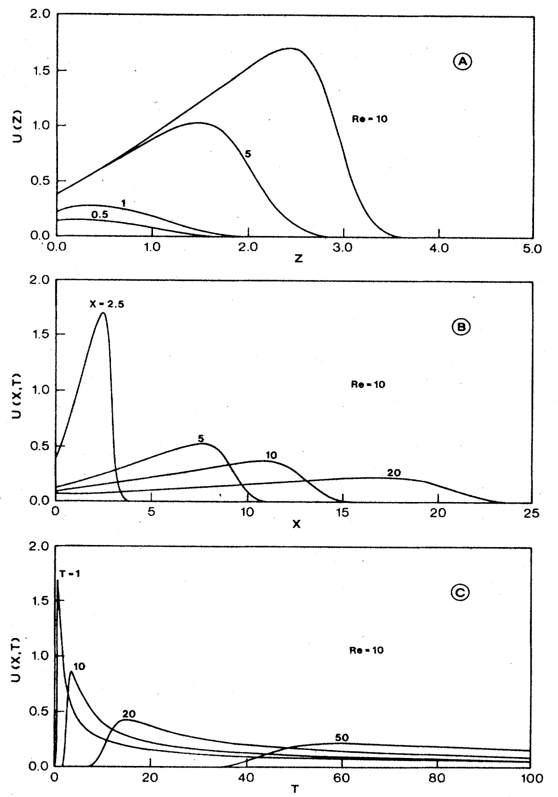


FIG. 12. Nonlinear acoustic pulse propagating in a viscous fluid: (a) near Gaussian decay at low Reynolds number, and shock formation at "high" Reynolds number; (b) "photographs" of the wave at different times, showing the decay of the peak, front steepening, and final dissipation; (c) "records" of the signal at different locations, showing the decay of the peak, the gradually less steep rise, and the lengthening tail.

and at high Reynolds number it takes the shape (Campos, 1983g,1984d)

$$U(x,t) \sim -(2\eta/\pi t)^{1/2}\{\exp(-x^2/2\eta t)/\mathrm{erfc}[x/(2\eta t)^{1/2}]\}\ , \tag{296}$$

of a Gaussian hump modified by a complementary error function $\mathrm{erfc}(\xi) \equiv 1-\mathrm{erf}(\xi)$.

### 8. Wave "front" and "tail"

We have illustrated in Fig. 12(a) the pressure pulse, in dimensionless variables $U(x)=\sqrt{t/\eta}\,U(x,t)$, $X \equiv x/\sqrt{2\eta t}$, for values of the Reynolds number $\mathrm{Re}=0.5, 1.0, 5.0, 10.0$; for small values $\mathrm{Re} \lesssim 1$, diffusion predominates, and the pulse is a slightly modified Gaussian hump [Eq. (296), with erfc = 1], as in ordinary heat conduction from a point source. For larger values of the Reynolds number, nonlinear effects lead to the formation of a shock, with a higher peak and steeper front, the longer the diffusion takes to act, i.e., the smaller the viscosity (or thermal diffusivity); thus for the "high" Reynolds number case $\mathrm{Re}=10$ (actually much lower than usual in ordinary hydrodynamics, where it may reach values of up to $10^6$), the pulse exhibits significant evolution, which may be followed in space or time. The temporal evolution is illustrated by plotting, in Fig. 12(b), the waveform $U(X,T)$ as a function of position $X \equiv x/L$ (where $L$ is a length scale), at fixed times $T = tL^2/\eta = 1, 10, 20, 50$; the peak decays monotonically, but the front first steepens, while nonlinear effects dominate, and then the gradient is limited by diffusion, which ultimately causes a global decay. The spatial evolution is illustrated in Fig. 12(c), where the signal $U(X,T)$ is plotted as a function of time $T$, as received at various positions $X = 2.5, 10, 20$; again the peak decays monotonically, and, as the observer moves farther downstream, there is a longer time delay for the signal to arrive, with a front gradually less steep, and a progressively lengthening tail.

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(a)

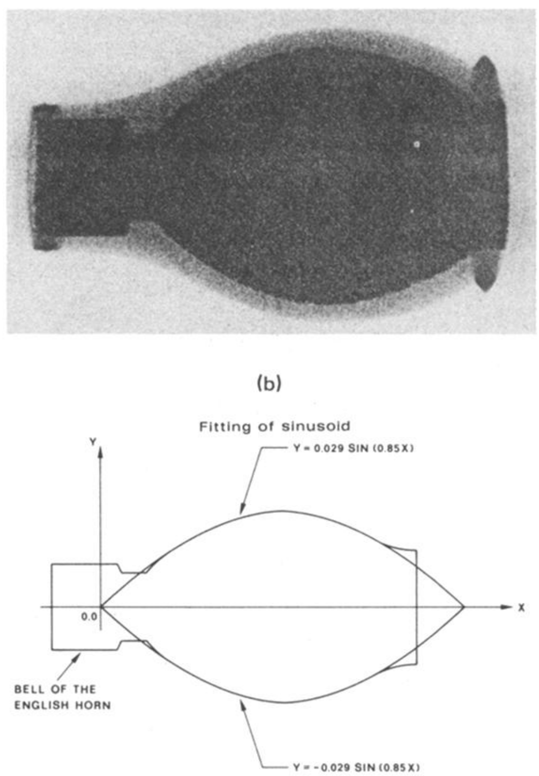


FIG. 8. Cross section of the mouth of an English horn (Nagarkar and Finch, 1972), (a) as seen in a gamma-ray photograph, compared with (b) a sinusoid fitting. Other simple duct shapes include the power law for the French horn and the exponential for the "ear trumpet."