Renormalization theory and ultraviolet stability for scalar fields via renormalization group methods

Giovanni Gallavotti

Dipartimento di Matematica, II Università di Roma, Via O. Raimondo, 00173 Roma, Italy

A self-contained analysis is given of the simplest quantum fields from the renormalization group point of view: multiscale decomposition, general renormalization theory, resummations of renormalized series via equations of the Callan-Symanzik type, asymptotic freedom, and proof of ultraviolet stability for sine-Gordon fields in two dimensions and for other super-renormalizable scalar fields. Renormalization in four dimensions (Hepp's theorem and the De Calan—Rivasseau $n!$ bound) is presented and applications are made to the Coulomb gases in two dimensions and to the convergence of the planar graph expansions in four-dimensional field theories (t' Hooft—Rivasseau theorem).

CONTENTS

ΗΘΕΣΙΑΙ ΠΑΥΟΝΤΑΙ*

I. INTRODUCTION

The aim of this work is to provide a self-contained introduction to field theory illustrating, at the same time, most of the known properties of the simplest fields.

I shall develop some of the ideas and methods of constructive field theory whenever they exist, providing the construction (nonperturbative) of various fields with one of the few methods available (which I consider conceptually the simplest).

While I have no pretension of saying something new, particularly to the theoretical physicists, I hope that this review might be useful, as many mathematical physicists have never worked on field theory and are not familiar with its remarkable problems, and as many physicists have never had any wish or need to look at the rigorous version of a statement that they seem to consider obvious.

In this section I review some of the philosophy behind the setting of quantum field theory, mostly for completeness and with the hope that this might help some beginners.

The special theory of relativity, in spite of its elegance and simplicity, raises a large number of new problems by imposing the rejection of the notion of action at distance to describe interacting mechanical systems.

In fact, the electromagnetic field in the vacuum or the free particles provide simple examples of relativistic systems, but it is difficult to describe a relativistically invariant mutual interaction between particles or between particles and fields.

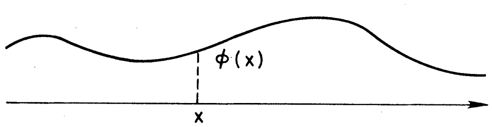
Within the framework of special relativity it is possible, and quite simply so, to describe relativistically invariant interactions between fields; however, in classical mechanics there is only one field, the electromagnetic field (gravitation is not considered here), and a variety of particles which seem to interact only through it, being charged entities. Their interaction with the electromagnetic field is hard to describe in a fundamental way because of the in-

*From the newly discovered Augustus's meridian in Roma.

finite self-energy that it implies.

There has been, and still there is, great hope that the electromagnetic field's quantization, or more generally the quantization of a system of fields, would lead to the unification of the field-particle dualism and to the possibility of a description of relativistic quantum interactions between particles and fields. In the remaining part of this section I summarize the heuristic reasoning behind this hope.

Classically a field describes the configurational state of an elastic body. As a primitive example, consider the case of the one-dimensional vibrating string: describe it through the value $\varphi(x)$ of the transversal deformation in the point of abscissa x; see diagram 1,



. (1)

The string's parameters will be the density μ, its tension μc^2 (with c the wave propagation speed), and the restoring constant $\mu\omega^2$: i.e., the string's Lagrangian is

$$\mathscr{L}=\frac{1}{2}\mu\int_\alpha^\beta\left[\dot\varphi(x)^2-c^2\left[\frac{d\varphi}{dx}(x)\right]^2-\omega^2\varphi(x)^2\right]dx\ , \tag{1.1}$$

where α,β are the points where the string's ends are attached ($\alpha=-\infty$, $\beta=+\infty$ if one wishes relativistic covariance).

The equation of motion is therefore

$$\ddot\varphi(x)-c^2\frac{\partial^2\varphi}{\partial x^2}(x)+\omega^2\varphi(x)=0\ , \tag{1.2}$$

which describes a relativistically invariant field (if $\alpha=-\infty$, $\beta=+\infty$), because if $(x,t)\to\varphi(x,t)$ solves (1.2), so does $(x,t)\to\varphi(R(x,t))$ for any Lorentz transformation R:

$$R=\begin{bmatrix}\cosh y & \sinh y\\ \sinh y & \cosh y\end{bmatrix},\ \ y\geq 0,\ c\equiv 1\ .$$

The solutions of (1.2), with $\alpha=-\infty$, $\beta=+\infty$, can be developed in plane waves:

$$e^{i[kx-\varepsilon(k)t]},\ \ k\in R\ , \tag{1.3}$$

where

$$\varepsilon(k)=\pm(\omega^2+c^2k^2)^{1/2}\ . \tag{1.4}$$

Via the correspondence principle and Bohr's relations

$$p=\hbar k,\ \ E=\hbar\varepsilon\ , \tag{1.5}$$

with $\hbar=(2\pi)^{-1}\times$(Planck's constant), one sees that the quantized vibrating string should describe particles for which the relationship between momentum p and velocity v is

$$v=\frac{d\varepsilon}{dk}=\frac{c^2k}{(\omega^2+c^2k^2)^{1/2}}=\frac{c^2}{\hbar}\frac{p}{\left[\omega^2+\left[\frac{cp}{\hbar}\right]^2\right]^{1/2}}\ ,$$
$$p=v\frac{\omega\hbar/c^2}{(1-v^2/c^2)^{1/2}}\ , \tag{1.6}$$

i.e., the quantized string describes relativistic particles with rest mass

$$m_0=\omega\hbar/c^2\ . \tag{1.7}$$

It would be easy to convince oneself that such particles do not interact mutually.

However, it is easy to introduce an interaction between them which is relativistically invariant. The simplest way is to modify the classical Lagrangian (1.1) by nonquadratic terms and then quantize it. Consider

$$\mathscr{L}=\frac{1}{2}\mu\int_\alpha^\beta\left[\dot\varphi(x)^2-c^2\left[\frac{\partial\varphi}{\partial x}(x)\right]^2-\left[\frac{m_0c^2}{\hbar}\right]^2\varphi(x)^2-I(\varphi(x))\right]dx\ , \tag{1.8}$$

where $I(\varphi)$ is some function of φ.

The nonlinearity of the resulting wave equation produces the result that when two or more wave packets collide they emerge out of the collision quite modified and do not just go through each other as in the case of the linear string, so that their interaction is nontrivial.

It is important to stress one feature of (1.1): in order to describe a particle of mass m_0 it is necessary to consider a string with restoring force constant $\omega^2=m_0c^2/\hbar$. It is this dependence of ω on $\hbar$ which provides that, in the classical limit $\hbar\to 0$, a particle with rest mass m_0 is no longer described by a classical solution of the wave equation. The limit $\hbar\to 0$ has to be discussed with more care because of its very singular nature. The actual discussion leads to the natural picture that the classical waves obtained as limits of quantum states describing a set of freely traveling quantum particles of momenta $p_1,p_2,\ldots$ ("coherent states") are a δ-function wave:

$$\prod_{i=1}^n\delta[x_i-v(p_i)t]\ , \tag{1.9}$$

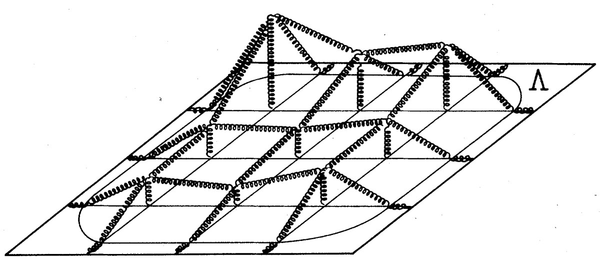
with $v(p)$ given by (1.6) ("point particles").

There is, however, an obvious exception: the case $m_0=0$. This time the limit as $\hbar\to 0$ does not have the same singular character as before and the classical limits of quantum states are generally correctly described by classical fields verifying the wave equation.

The above discussion, which cannot be developed in more detail here, is the basis for the solution of the "wave-particle dualism": the classical waves and particles being in a natural sense the classical limits of quantum fields (respectively, massless or massive).

But one should not think that the quantization of the string or of a more general D-dimensional elastic body

($D=1,2,3$; see diagram 2 for the case $D=2$, with the body being a discrete set of springs oscillating over the region $\Lambda_0 \subset R^D$),



,

(2)

with Hamiltonian

$$\mathscr{H}=\int_{\Lambda_0}\left\{\frac{\pi(\underline{x})^2}{2\mu}+\frac{\mu}{2}\left[c^2\left[\frac{\partial\varphi}{\partial\underline{x}}(\underline{x})\right]^2+\left[\frac{m_0c^2}{\hbar}\right]^2\varphi(\underline{x})^2 + I(\varphi(\underline{x}))\right]\right\}d^D\underline{x} \tag{1.10}$$

is an easy matter; it is in fact the scope of this paper to review the related problems.

I start with the "naive" quantization: the quantum states will be, by the "natural extension of the usual quantization rules," functions of the function φ describing the configurational shape of the elastic deformations; and $\pi(\underline{x})$ will have to be thought of as the operator $i\hbar\delta/\delta\varphi(\underline{x})$. So the Hamiltonian operator acts on the wave function F as

$$(\mathscr{H}F)(\varphi)=\int_{\Lambda_0}\left\{-\frac{\hbar^2}{2\mu}\frac{\delta^2F}{\delta\varphi(x)^2}(\varphi)+\frac{1}{2}\mu\left[c^2\left[\frac{\partial\varphi}{\partial\underline{x}}(\underline{x})\right]^2+\left[\frac{m_0c^2}{\hbar}\right]^2\varphi(\underline{x})^2+I(\varphi(\underline{x}))\right]F(\varphi)\right\}d^D\underline{x}\ , \tag{1.11}$$

and it should be defined in the space L_2("$d\varphi$"), where the scalar product ought to be

$$(F,G)=\int\overline{F(\varphi)}G(\varphi)\text{"}d\varphi\text{"} \tag{1.12}$$

and

$$\text{"}d\varphi\text{"}=\prod_{\underline{x}\in\Lambda_0}d\varphi(\underline{x})\ .$$

Even though by now the mathematical meaning that one should try to attach to expressions like (1.11) and (1.12) as "infinite dimensional elliptic operators" and "functional integrals" is quite well understood, particularly when $I=0$, formulas like the above are still quite shocking for a conservative mathematician, even more so because they turn out to be very useful.

One possible way to give meaning to (1.11) is to go back to first principles and recall the classical interpretation of the vibrating string or elastic body as a system of finitely many oscillators, following the brilliant theory of the discretized wave equation and of the related Fourier series due to Lagrange (see, for instance, Gallavotti, 1983a, pp. 252—283); see diagram 2.

Suppose that the region Λ_0 is a parallelepiped of side size L and, for the sake of simplicity, with periodic boundary conditions; replace it with a square lattice Z_a with bonds of size a such that L/a is an integer. In every point $\underline{n}\,a$ of Z_a put an oscillator with mass μa^D, described by a coordinate $\varphi_{\underline{n}\,a}$ giving the oscillator's elongation over the equilibrium position, and subject to a restoring force with potential energy

$$\tfrac{1}{2}\mu a^D(m_0c^2/\hbar)^2\varphi_{\underline{n}\,a}^2\ ,$$

to a nonlinear restoring force with potential energy $\frac{1}{2}\mu a^D I(\varphi_{\underline{n}\,a})$, and, finally, to a linear elastic tension coupling between nearest neighbors $\underline{n}\,a,\underline{m}\,a$ with potential energy $\mu c^2a^{D-2}(\varphi_{\underline{n}\,a}-\varphi_{\underline{m}\,a})^2/2$.

Therefore, the Lagrangian of the system is

$$\mathscr{L}=\tfrac{1}{2}\mu a^D\sum_{\underline{n}\,a\in\Lambda_0}\left[\dot{\varphi}_{\underline{n}\,a}^2-c^2\sum_{j=1}^{D}(\varphi_{\underline{n}\,a+\underline{e}_ja}-\varphi_{\underline{n}\,a})^2/a^2-(m_0c^2/\hbar)^2\varphi_{\underline{n}\,a}^2-I(\varphi_{\underline{n}\,a})\right], \tag{1.13}$$

where $\underline{e}_i$ are D unit vectors oriented as the lattice's directions; if $\underline{n}\,a+a\underline{e}_i$ is not in Λ_0 but $\underline{n}\,a$ is in Λ_0, then the ith coordinate n_ia is equal to L and $\underline{n}\,a+\underline{e}_ia$ has to be interpreted as the point whose ith coordinate is a—i.e., (1.13) is interpreted with periodic boundary conditions with coordinates identified modulo L.

Of course there is no conceptual problem in quantizing the system described by (1.13); it is described by the operator on $L_2(\prod_{\underline{n}\,a}d\varphi_{\underline{n}\,a})$:

$$H_{\text{quantum}}=-\frac{\hbar^2}{2\mu a^D}\sum_{\underline{n}\,a\in\Lambda_0}\frac{\partial^2}{\partial\varphi_{\underline{n}\,a}}+\mu\frac{a^D}{2}\sum_{\underline{n}\,a\in\Lambda_0}\left[c^2\sum_{j=1}^{D}(\varphi_{\underline{n}\,a+\underline{e}_ja}-\varphi_{\underline{n}\,a})^2/a^2+(m_0c^2/\hbar)^2\varphi_{\underline{n}\,a}^2+I(\varphi_{\underline{n}\,a})\right], \tag{1.14}$$

with domain (of essential self-adjointness) $C_0^\infty(\prod_{\underline{n}\,a\in\Lambda_0}R)$, provided $I(\varphi)$ is assumed bounded below, as it should always be.

The properties of the quantum vibrating string, or body if D is larger than 1, which will be usually interesting will be properties of the Hamiltonian (1.14) holding uniformly in the "ultraviolet cutoff a." In fact, in most applications one is actually interested also in properties holding uniformly in the "infrared cutoff L" as well, with L the size of the box Λ_0.

It will become clear that in studying such "ultraviolet stable properties" it will be necessary to put upon the "interaction" $I(\varphi)$ very stringent requirements to avoid the system's becoming trivial in the limit $a \to 0$, the "ultraviolet limit."

Also, last but not least, it should be clear that the objective of field theory is to formulate a relativistically invariant quantum theory of interacting particles, and it might conceivably happen that the above way of trying to give a meaning to (1.11) and (1.12) based on (1.14) may fail: i.e., in the limit $a \to 0$ one is left with a theory describing only free particles. Such a failure, in principle, would not prove the impossibility of giving a nontrivial meaning to (1.11) but only that the way proposed through (1.14) is not appropriate.

In the next section I shall proceed to give a more complete formulation of the ultraviolet problem in connection with (1.14) ("lattice regularization"). Later, in Sec. III, a different approach naturally emerges which will be the one which will be really investigated in this work ("Feynman regularization")—in the few cases in which the theory can be pushed beyond formal perturbation theory the two approaches turn out to be equivalent.

This does not mean that in the case of questions that are still open the two approaches should be thought equivalent; however, there is no reason why one should be preferred to the other or to any one among many others that one can *a priori* conceive [see Gallavotti and Rivasseau (1984)]: therefore, I shall avoid entering into regularization-dependent questions, and I shall use one well-defined regularization only for definiteness. This will preclude the discussion of some recent deep results based on special regularization assumptions, but the reader is referred to the literature on such questions (Aizenman, 1982; Fröhlich, 1982).

II. FUNCTIONAL INTEGRAL REPRESENTATION OF THE HAMILTONIAN OF A QUANTUM FIELD

A very convenient representation of the Hamiltonian and a tool for the analysis of the ultraviolet limit is the functional integral representation [this seems to be a rather old representation; here I follow Nelson (1966,1973a); see also Symanzik (1966,1969), Wilson (1971,1972) Guerra, Rosen, and Simon (1975), and Glimm and Jaffe (1981)].

Instead of studying the operator H_{quantum} itself, (1.14), introduce the operator on $L_2(\prod_{\underline{n}a} d\varphi_{\underline{n}a})$:

$$T_t = \exp[-(H_{\text{quantum}} - E)t/\hbar], \quad t \geq 0 , \tag{2.1}$$

where E is the ground-state energy of H_{quantum}.

Denoting $\underline{\varphi} = (\varphi_{\underline{n}a})_{\underline{n}a \in \Lambda_0}$ and

$$\begin{aligned}
&e(\underline{\varphi}) = \text{ground-state wave function for } H_{\text{quantum}} ,\\
&e_0(\underline{\varphi}) = \text{ground-state wave function}\\
&\qquad \text{for } (H_{\text{quantum}})_{I=0} = H_0 ,\\
&E_0 = \text{ground-state energy for } H_0 ,\\
&T_t(\underline{\varphi}, \underline{\varphi}') = \text{kernel of } T_t \text{ on } L_2\left[\prod_{\underline{n}a} d\varphi_{\underline{n}a}\right] ,\\
&T_t^0(\underline{\varphi}, \underline{\varphi}') = \text{kernel of } \exp[-t(H_0 - E_0)]\\
&\qquad \text{on } L_2\left[\prod_{\underline{n}a} d\varphi_{\underline{n}a}\right] ,
\end{aligned} \tag{2.2}$$

it is possible to introduce a probability measure on the space of the continuous functions $(t, \underline{n}a) \to \varphi_{\underline{n}a}(t)$ such that the sets

$$E(A; t_1, \ldots, t_n) = \{\varphi \mid (\underline{\varphi}(t_1), \ldots, \underline{\varphi}(t_n)) \in A\} , \tag{2.3}$$

with $A \subset (R^{\Lambda_0})^n$, will have the measure

$$\begin{aligned}
P(E(A; t_1, \ldots, t_n))
&= \int_A e[\underline{\varphi}(t_1)] \prod_{j=1}^{n-1} T_{t_{j+1}-t_j}(\underline{\varphi}(t_j), \underline{\varphi}(t_{j+1}))\\
&\qquad \times e[\underline{\varphi}(t_n)] \prod_{j=1}^{n} d\underline{\varphi}(t_j) ,
\end{aligned} \tag{2.4}$$

where

$$d\underline{\varphi}(t_j) = \prod_{\underline{n}a \in \Lambda_0} d\varphi_{\underline{n}a}(t_j)$$

and $t_1, \ldots, t_n$ play the role of indices.

One readily checks that (2.4) does verify the compatibility conditions necessary to interpret it as a measure on the algebra of sets generated by the sets like (2.3) on the space of the continuous functions $t \to \underline{\varphi}(t) \in R^{\Lambda_0}$, $\underline{\varphi}(t) \equiv [\varphi_{\underline{n}a}(t)]_{\underline{n}a \in \Lambda_0}$ [i.e., $P(E(\ )) \geq 0$, $P(E((R^{\Lambda_0})^n; t_1, \ldots, t_n)) \equiv 1$, and, if for A, A_0 it is $E(A; t_1, \ldots, t_n) = E(A_0; t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n)$—in other words, if the value of $\underline{\varphi}(t_j)$ is irrelevant to decide whether $(\underline{\varphi}(t_2), \ldots, \underline{\varphi}(t_n)) \in A$, then $P(E(A; t_1, \ldots, t_n)) = P(E(A_0; t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n))$].

If $F, G \in L_2[e(\underline{\varphi})^2 d\underline{\varphi}]$ and if U maps $L_2[e(\underline{\varphi})^2 d\underline{\varphi}] \leftrightarrow L_2(d\underline{\varphi})$ and is defined by

$$(UF)(\underline{\varphi}) = e(\underline{\varphi}) F(\underline{\varphi}) , \tag{2.5}$$

and if $\tilde{H} = U^{-1} H_{\text{quantum}} U$, it is, by definitions (2.4) and (2.5):

$$\begin{aligned}
(F, e^{-t(\tilde{H}-E)/\hbar} G)_{L_2[e(\underline{\varphi})^2 d\underline{\varphi}]} = (UF, e^{-tH_{\text{quantum}}/\hbar} UG)_{L_2(d\underline{\varphi})} &= \int e(\underline{\varphi}) F(\underline{\varphi}) T_t(\underline{\varphi}, \underline{\varphi}') e(\underline{\varphi}') G(\underline{\varphi}') d\underline{\varphi}\, d\underline{\varphi}'\\
&= \int F(\underline{\varphi}(0)) G(\underline{\varphi}(t)) P(d\varphi) ,
\end{aligned} \tag{2.6}$$

which shows that the measure P contains all the information needed to study the operator H_{quantum} or its equivalent U image $\widetilde{H}$ [see Nelson (1973)].

In the above formulas the notation φ has been used to denote an element of the space of the continuous functions $t \to \underline{\varphi}(t)$ with values in R^{Λ_0} (while $\underline{\varphi}$ denotes an element of R^{Λ_0}); this notation will be kept consistent.

The object P is of course quite complex and needs, anyway, the theory of H_{quantum} to be really constructed.

It is easy to relate P to the measure P_0, defined as P but with $I(\underline{\varphi})=0$, and to find "explicit" expressions for P_0 itself [see Nelson (1973)].

The measure P_0 is a Gaussian measure because the Green's function $T_t^0(\underline{\varphi},\underline{\varphi}')$, being the heat kernel for the Laplace operator on $L_2(R^{\Lambda_0})$ plus a quadratic potential, is a Gaussian kernel [in fact, the heat kernel for the Laplace operator A is Gaussian, as it is well known, and the addition to A of a quadratic potential B does not change this because of the Trotter's formula $\exp(A+B) = \lim(\exp A/n)(\exp B/n)^n$, and because the composition of several Gaussian kernels is still a Gaussian kernel].

Therefore P_0 can be completely described in terms of its "covariance" or "propagator"; if $\xi=(\underline{n}\,a,t)\in\Lambda_0\times R$ and $\varphi_\xi \equiv \varphi_{\underline{n}\,a}(t)$ and $\eta=(\underline{m}\,a,t')$ the covariance is defined as

$$C_{\xi\eta}=\int_{\mathscr{C}(\Lambda_0 R)}\varphi_\xi\varphi_\eta dP_0 \ , \tag{2.7}$$

where $\mathscr{C}(\Lambda_0\times R)$ is the set of the continuous functions φ on $\Lambda_0\times R$.

A well-known elementary calculation allows us to find an explicit formula for C; let $\xi=(\underline{x},t)$, $\eta=(\underline{y},t')$; then

$$C_{\xi\eta}=\sum_{\underline{n}\in Z^D}\overline{C}_{(\underline{x}+\underline{n}L,t),(\underline{y},t')} \tag{2.8}$$

and

$$\overline{C}_{\xi\eta}=\frac{\hbar}{(2\pi)^{D+1}\mu}\int_{-\pi/a}^{\pi/a}\int_{-\infty}^{+\infty}\frac{e^{ip_0(t-t')}e^{i\underline{p}(\underline{x}-\underline{y})}d^D\underline{p}\,dp_0}{\left[\dfrac{m_0c^2}{\hbar}\right]^2+p_0^2+2C^2\displaystyle\sum_{j=1}^{D}\frac{1-\cos(p_ja)}{a^2}} \ ; \tag{2.9}$$

see Appendix A for a sketchy proof.

Then the measure P is related to P_0 by

$$P(d\varphi)=\lim_{T\to\infty}Z(L,T)^{-1}\left[\exp\left[\frac{-\mu a^D}{2\hbar}\int_{-T/2}^{T/2}I(\underline{\varphi}(\tau))d\tau\right]\right]P_0(d\varphi) \ ,$$
$$Z(L,T)=\int\left[\exp\left[\frac{-\mu a^D}{2\hbar}\int_{-T/2}^{T/2}I(\underline{\varphi}(\tau))d\tau\right]\right]P_0(d\varphi) \ . \tag{2.10}$$

This is the "Feynman-Kac formula" [see Nelson (1966,1973a): recall that L is the infrared cutoff, i.e., the side size of the cube Λ_0 with periodic boundary conditions, above which the oscillators vibrate]. The proof of (2.10) is not hard and its rough sketch can be found in Appendix B.

Rather than using (2.10) to deduce the properties of the measure P as the ultraviolet cutoff a tends to zero, it is convenient to study a more explicit representation for P. This representation is a corollary of (2.9) and (2.10) and it is

$$P(d\varphi)=\lim_{T\to\infty}\lim_{b\to 0}Z^{-1}\left[\exp\left\{-\frac{\mu b a^D}{2\hbar}\sum_{\underline{n}a\in\Lambda_0}\sum_m\left[\left[(\varphi_{\underline{n}\,a,mb}-\varphi_{\underline{n}\,a,mb+b})^2/b^2+c^2\sum_{j=1}^{D}(\varphi_{\underline{n}\,a,mb}-\varphi_{\underline{n}a+\underline{e}_j a,mb})^2/a^2\right.\right.\right.\right.$$
$$\left.\left.\left.\left.+\ (m_0c^2/\hbar)^2\varphi^2_{\underline{n}\,a,mb}+I(\varphi_{\underline{n}\,a,mb})\right]\right]\right\}\right]\prod_{\underline{n},m}d\varphi_{\underline{n}\,a,mb} \ , \tag{2.11}$$

where m is an integer varying between $-T/2b$ and $T/2b$ (supposed integer), the points $\pm T/2b$ are identified ("periodic boundary conditions" in the time direction), and Z is a normalization factor depending on L,T,a,b. The proof of (2.11) is hinted at toward the end of Appendix B.

Call Λ the parallelepiped with sides L,T in $R^{D+1}=R^d$ considered with periodic boundary conditions and call $P_{L,T,a,b}$ the measure under the limit sign in (2.11). The "ultraviolet problem" on the lattice is the problem of the theory of the limit:

$$\lim_{a\to 0}\lim_{b\to 0}P_{L,T,a,b}=P_{L,T} \ . \tag{2.12}$$

Here I shall study only questions related to the existence of this limit, which is a problem typical of field theory, while no attention will be devoted to the other fundamental problem of analyzing the limit

$$\lim_{L,T\to\infty}P_{L,T}=P_\infty \ , \tag{2.13}$$

called the "infrared problem." The latter problem can be considered a "thermodynamic limit" problem typical of statistical mechanics (which does not mean that it is easy).

The existence of the limit (2.12) will be attacked by trying to establish upper and lower bounds ("ultraviolet stability") uniform in a,b, for quantities like

$$\langle e^{\varphi(f)}\rangle \equiv \int P_{L,T,a,b}(d\varphi)\exp\left[a^D b\sum_{\xi\in\Lambda} f(\xi)\varphi_\xi\right], \quad (2.14)$$

where f is a C^∞-smooth function with fixed support in the interior of Λ. Equation (2.14) is usually called the "generating function" for the "Schwinger functions" of the measure $P_{L,T,a,b}$.

For simplicity it is convenient to fix the ratio a/b to be equal to c, the speed of light, $a=bc$; also, I shall choose $L=cT$ so that the measure $P_{L,L/c,a,a/c}$ can be rewritten:

$$P_{L,a}(d\varphi)=\left[\exp\left[-\frac{\mu a^d}{2c\hbar}\sum_{\xi\in\Lambda} I(\varphi_\xi)\right]\right]P^0_{L,a}(d\varphi)/Z, \quad (2.15)$$

where $P^0_{L,a}$ is the Gaussian measure on the finite space R^Λ with covariance, if $p(\xi-\eta)=\underline{p}(\underline{x}-\underline{y})+c(t-t')p_d$:

$$C_{\xi\eta}=\sum_{\underline{n}\in Z^d}\bar{C}_{\xi+\underline{n}L,\eta},$$

$$\bar{C}_{\xi\eta}=\frac{\hbar}{(2\pi)^d\mu c}\int_{-\pi/a}^{\pi/a}\frac{e^{ip(\xi-\eta)}d^dp}{m^2+2\sum_{j=1}^{d}[1-\cos(p_ja)]/a^2}, \quad (2.16)$$

where $m=m_0c/\hbar$ [to understand (2.15) note the symmetric role of the d directions in (2.11) once $a=bc$, $L=cT$], or explicitly

$$P^0_{L,a}(d\varphi)=Z^{-1}\left\{\exp\left[-\frac{\mu a^d}{2\hbar c}\sum_{\xi\in\Lambda}\left[\sum_{j=1}^{d}(\varphi_{\xi+ae_j}-\varphi_\xi)^2/a^2+m^2\varphi_\xi^2\right]\right]\right\}\prod_\xi d\varphi_\xi. \quad (2.17)$$

The measure (2.17) is called the "lattice free field" and if $\underline{\delta}^2_a$ denotes the finite difference Laplace operator on the lattice Z^d_a one sees that $\bar{C}_{\xi\eta}$ in (2.16) is just the kernel of the operator

$$\bar{C}=[\mu c\hbar^{-1}(m^2+\underline{\delta}^2_a)]^{-1} \quad (2.18)$$

(by finite difference Laplace operator one means, here, the nearest-neighbor second difference divided by a^2), while if $\underline{\delta}^2_{a,L}$ denotes the finite difference Laplace operator on the lattice Z^d_a with periodic boundary conditions on the boundary of the cube Λ, it is

$$C=[\mu c\hbar^{-i}(m^2+\underline{\delta}^2_{a,L})]^{-1}, \quad (2.19)$$

i.e., C is the same as $\bar{C}$ apart from the boundary conditions.

The problem of studying the limit as $a\to 0$ of (2.15) is not exactly the same as that of studying the $\lim_{a\to 0}\lim_{b\to 0}$ in (2.12). The really difficult problem being the limit as $a\to 0$, it turns out that setting $b=a/c$ does not make the problem any easier or any harder. All the results that follow could also be obtained if one considered first the limit as $b\to 0$ and then the limit as $a\to 0$.

III. THE FREE FIELD AND ITS MULTISCALE DECOMPOSITIONS

It has become clear that the right way to look at the measures (2.17) (free field) is to consider them as stochastic processes indexed by the points of Λ; thus the free field will be thought of as a Gaussian process.

Furthermore, it is convenient to regard the free field as defined everywhere in Λ and not just on the lattice points $Z^d_a\cap\Lambda$; this can be done easily by observing that $C_{\xi\eta}$ makes sense, by (2.16), for all $\xi,\eta\in R^d$ and therefore we may actually imagine that it describes a family of Gaussian random variables indexed by $\xi\in\Lambda$, whose distribution is still denoted $P^0_{L,a}$.

Since $C_{\xi\eta}$ is infinitely smooth, it follows from the general theory of Gaussian processes that the "sample fields" φ_ξ will be, with probability one with respect to $P^0_{L,a}$, C^∞ functions of ξ. However, this does not really imply that they are smooth in a physical sense: in fact, the expected values of φ_ξ^2, $(\partial\varphi_\xi)^2,\ldots$, all diverge as $a\to 0$ (if, as will be always supposed, $d\geq 2$).

This means that the fields φ_ξ are indeed smooth but to see that they are, one has to look at them on a scale as small as a. An easy calculation shows that in fact

$$\int(\partial^k\varphi_\xi)^2P^0_{L,a}(d\varphi)=\begin{cases}O(a^{-2k}a^{-(d-2)}) & d>2\\ O(a^{-2k}\ln a^{-1}) & d=2,\end{cases} \quad (3.1)$$

where ∂^k is any kth-order derivative of φ_ξ.

The relation (3.1) tells us that φ_ξ has to be regarded as a smooth function which can be as large as $a^{-(d-2)/2}$, if $d>2$, or as $(\ln a^{-1})^{1/2}$ if $d=2$, and which can have a kth-order derivative larger by a factor a^{-k}, i.e., the field looks smooth only on scale a.

In general, in understanding the structure of a stochastic field, two main scales have to be specified: the scale on which the field is smooth and the scale on which the field is without correlations, i.e., the scales on which the field can be regarded as a constant and those on which the values that it takes can be regarded as independent random variables.

In our case it would be easy to show that

$$|C_{\xi\eta}|\leq Me^{-\kappa|\xi-\eta|},\quad \forall\xi,\eta\in Z^d_a,\ \kappa^{-1}\leq|\xi-\eta|<\frac{L}{2}, \quad (3.2)$$

where $M,\kappa>0$ are a independent; this can be interpreted as saying that the field φ_ξ has an independence scale of $O(\kappa^{-1})$.

If one calls "regular" the random fields with identical smoothness and independence scales, it is clear that the fields of interest here (free fields) are not regular; and this is the distinctive feature of field theory with respect to statistical mechanics of weakly interacting systems (i.e., away from the critical point). It introduces the new problems of ultraviolet stability characteristic of field theory.

In fact, a regular field is hardly different from a lattice spin system of essentially independent spins located on a lattice with spacing equal to the unique scale of regularity and independence.

Since the techniques for studying lattice spin systems have been well developed in statistical mechanics, at least in some easy cases, the idea arose (Wilson, 1971,1972,1974,1983) of trying to represent irregular fields as decomposed into sums of regular ones. This leads to the "multiscale decompositions" of the free field which are discussed below and which will be the basic tool in analyzing the non-Gaussian fields in which we are interested and which are small perturbations of the Gaussian field $P^0_{L,a}$.

It is, in fact, possible to write the field φ as

$$\varphi_\xi=\sum_{k=0}^{\infty}\varphi_\xi^{(k)}\ , \tag{3.3}$$

where $\varphi_\xi^{(k)}$ are independently distributed over the index k and are regular on scale $\gamma^{-k}m^{-1}$, if $\gamma>1$ is an *arbitrarily preassigned* number.

The decomposition (3.3) can be done in different ways and with different requirements on $\varphi^{(k)}$.

In general, one desires that if $\gamma^{-k}m^{-1}\gtrsim a$, i.e., if the length scale of the field $\varphi^{(k)}$ is larger than the ultraviolet scale a, then the samples of $\varphi^{(k)}$ should be smooth on scale $\gamma^{-k}m^{-1}$, with the kth derivatives being of the order of γ^k times the size of the field itself [see (3.1)], for $k\leq p$. Such a decomposition will be called a "class C^p multiscale decomposition" of φ into regular random fields.

There is a simple algorithm to construct multiscale decompositions of the Gaussian random field with covariance operator (2.18). It is based on the following trivial identities:

$$\begin{aligned}\frac{1}{m^2+\varepsilon^2}&\equiv\sum_{k=0}^{\infty}\left[\frac{1}{m^2\gamma^{2k}+\varepsilon^2}-\frac{1}{m^2\gamma^{2k+2}+\varepsilon^2}\right]\\
&=\sum_{k=0}^{\infty}\frac{m^2(\gamma^2-1)\gamma^{2k}}{m^4\gamma^{4k+2}+m^2(\gamma^2+1)\gamma^{2k}\varepsilon^2+\varepsilon^4}\\
&\equiv\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\left[\frac{m^2(\gamma^2-1)\gamma^{2k}}{m^4\gamma^{4k+2}\gamma^{4h}+m^2(\gamma^2+1)\gamma^{2k}\varepsilon^2+\varepsilon^4}-\frac{m^2(\gamma^2-1)\gamma^{2k}}{m^4\gamma^{4k+2}\gamma^{4h+4}+m^2(\gamma^2+1)\gamma^{2k}\varepsilon^2+\varepsilon^4}\right]\\
&=\sum_{k=0}^{\infty}\sum_{h=0}^{k}\frac{m^2(\gamma^2-1)\gamma^2m^4(\gamma^4-1)\gamma^{6k}\gamma^{-2h}}{[m^4\gamma^2\gamma^{4k}+m^2(\gamma^2+1)\varepsilon^2\gamma^{2k-2h}+\varepsilon^4][m^4\gamma^{4k}\gamma^6+m^2(\gamma^2+1)\gamma^{2k-2h}\varepsilon^2+\varepsilon^4]}\equiv\cdots\ ,\end{aligned}\tag{3.4}$$

where in the last equality a change of variables is made, changing $k+h$ into k.

The way to read (3.4) is the following: $(m^2+\varepsilon^2)^{-1}$ can be written as a sum of reciprocals of fourth-order polynomials in ε, or as a sum of reciprocals of eighth-order polynomials, or of sixteenth-order, etc., by the algorithm displayed self-explanatorily in (3.4).

Then to each such decomposition one can associate a decomposition of the random field φ like (3.3): if

$$\varepsilon_a(p)^2=2\sum_{i=1}^{d}[1-\cos(ap_i)]/a^2$$

and if one defines

$$\overline{C}^{(k)}_{\xi\eta}=\frac{\hbar}{(2\pi)^d\mu c}\int_{-\pi/a}^{\pi/a}\frac{m^2(\gamma^2-1)\gamma^{2k}e^{ip(\xi-\eta)}d^4p}{m^4\gamma^{4k+2}+m^2\gamma^{2k}(\gamma^2+1)\varepsilon_a(p)^2+\varepsilon_a(p)^4}\ , \tag{3.5}$$

and if $C^{(k)}$ is defined as in (2.16), with $\overline{C}$ replaced by $\overline{C}^{(k)}$, one realizes that by the first identity (3.4) the field φ has the same distribution as the sum of a sequence of fields $\varphi^{(k)}$ with covariances given by $C^{(k)}$.

Similarly, using the last identity in (3.4) and setting

$$\begin{aligned}\overline{C}^{(k)}_{\xi\eta}=\frac{\hbar}{(2\pi)^d\mu c}\sum_{h=0}^{k}\int_{-\pi/a}^{\pi/a}&m^6\gamma^2(\gamma^2-1)(\gamma^4-1)\gamma^{6k-2h}e^{ip(\xi-\eta)}d^dp\\
&\times\{[m^4\gamma^2\gamma^{4k}+m^2(\gamma^2+1)\gamma^{2k-2h}\varepsilon_a(p)^2+\varepsilon_a(p)^4]\\
&\times[m^4\gamma^6\gamma^{4k}+m^2(\gamma^2+1)\gamma^{2k-2h}\varepsilon_a(p)^2+\varepsilon_a(p)^4]\}^{-1}\end{aligned}\tag{3.6}$$

and if $C^{(k)}$ is defined as in (2.16) with $\overline{C}$ replaced by $\overline{C}^{(k)}$, one again finds that the field φ has the same distribution as the sum of a sequence of fields $\varphi^{(k)}$ with covariance given by $C^{(k)}$, etc.

The fields $\varphi^{(k)}$ with covariance $C^{(k)}$ related to (3.5) or to (3.6) or to the "higher-order generalizations" of them, are regular fields for all values of k such that $\gamma^{-k}m^{-1}\gtrsim a$ and are basically independent fields, when restricted to the lattice points, for the larger values of k. Furthermore, if $\gamma^{-k}m^{-1}\gtrsim a$, the fields $\varphi^{(k)}$ have essentially the same distribution up to trivial scalings.

To see the above properties of $\varphi^{(k)}$ for $\gamma^{-k}m^{-1}>a$ one can, heuristically, fix k and let $a\to 0$ (so that $\gamma^{-k}m^{-1}\gg a$).

Then (3.5) becomes

$$\overline{C}^{(k)}_{\xi\eta}\equiv\gamma^{(d-2)k}\overline{C}^{(0)}_{\gamma^k\xi\gamma^k\eta},\quad C^{(k)}_{\xi\eta}=\sum_{\underline{n}\in Z^d}\overline{C}^{(k)}_{\xi+\underline{n}L,\eta}\,,$$
$$\overline{C}^{(0)}_{\xi\eta}=\frac{\hbar}{(2\pi)^d\mu c}\int_{-\infty}^{+\infty}\frac{m^2(\gamma^2-1)e^{ip(\xi-\eta)}d^dp}{m^4\gamma^2+m^2(\gamma^2+1)p^2+p^4}\,, \tag{3.7}$$

and it is easy to see that $\overline{C}^{(0)}_{\xi\eta}$ is well defined and smooth with its derivatives of order $2(1-\varepsilon)$ if $d=2$ and of order $2(\frac{1}{2}-\varepsilon)$ if $d=3$ and it decays exponentially for $|\xi-\eta|$ large on scale $O(m^{-1})$. This means that the field $\varphi^{(k)}$, with covariance $C^{(k)}$ in (3.7), is Hölder continuous on scale $\gamma^{-k}m^{-1}$ with exponent less than 1 if $d=2$ or less than $\frac{1}{2}$ if $d=3$ (here $\varepsilon>0$ is arbitrary); it is, however, still irregular if $d\geq 4$.

If one uses the second decomposition of φ introduced above, associated with (3.6), one finds

$$\overline{C}^{(k)}_{\xi\eta}=\gamma^{(d-2)k}\sum_{h=0}^{k}\gamma^{-2h}\overline{C}^{(0,h)}_{\gamma^k\xi\gamma^k\eta}\,,$$
$$C^{(k)}_{\xi\eta}=\sum_{\underline{n}\in Z^d}C^{(k)}_{\xi-\underline{n}L,\eta}\,, \tag{3.8}$$
$$\overline{C}^{(0,h)}_{\xi\eta}=\frac{\hbar}{(2\pi)^d\mu c}\int_{-\infty}^{+\infty}\frac{m^6\gamma^2(\gamma^2-1)(\gamma^4-1)e^{ip(\xi-\eta)}d^dp}{[m^4\gamma^2+m^2(\gamma^2+1)\gamma^{-2h}p^2+p^4][m^4\gamma^6+m^2(\gamma^2+1)\gamma^{-2h}p^2+p^4]}\,,$$

and it is easy to see that $\overline{C}^{(k)}_{\xi\eta}$ has the same qualitative properties of $\gamma^{k(d-2)}C^{(0,0)}_{\gamma^k\xi,\gamma^k\eta}$ and $C^{(0,0)}$ is well defined and smooth together with its derivatives of order $2(3-\varepsilon)$ if $d=2$, or order $2(\frac{5}{2}-\varepsilon)$ if $d=3$, of order $2(2-\varepsilon)$ if $d=4$ (ε being an arbitrary positive number).

This means that the field $\varphi^{(k)}$ has second derivatives which are Hölder continuous with exponent $(1-\varepsilon)$ for $d=2$, $(\frac{1}{2}-\varepsilon)$ for $d=3$, and first derivatives which are Hölder continuous with exponent $(1-\varepsilon)$ if $d=4$ [for $d=5$ the first derivatives would be Hölder continuous with exponent $(\frac{1}{2}-\varepsilon)$ while for $d=6,7$ the field itself would only be Hölder continuous with exponent $(1-\varepsilon)$ or $(\frac{1}{2}-\varepsilon)$, respectively, and for $d\geq 8$ it would be irregular].

The above statements can be made more quantitative (see below); their proof is essentially a repetition, adapted to the circumstances, of the famous proof of Wiener asserting the Hölder continuity of the sample paths of the Brownian motion and it will not be repeated here [one can use the classical method of Wiener as in Colella and Lanford (1973); the cases (3.7) and (3.8) as well as the others arising from the higher-order identities in (3.4) are specifically treated in Benfatto *et al.* (1980b), as a part of a general theory of a class of Markovian Gaussian random fields].

The above discussion on the fields $\varphi^{(k)}$ suggests yet another approach to the ultraviolet stability which will be the one really followed in the upcoming sections.

Namely, choose $\varphi^{(k)}$ to be the random fields with covariance (3.7) [or (3.8) or any other associated with the higher-order identities in (3.4)] and define

$$\varphi^{(\leq N)}_\xi=\sum_{k=0}^{N}\varphi^{(k)}_\xi\,. \tag{3.9}$$

Then the measure (2.15) can be regarded as obtained by integrating over the $\varphi^{(k)}$'s the measure

$$Z^{-1}\left[\exp\left[-\frac{\mu a^d}{2c\hbar}\sum_{\xi\in\Lambda}I(\varphi_\xi)\right]\right]\prod_{k=0}^{\infty}P(d\varphi^{(k)})=\lim_{N\to\infty}Z^{-1}_{N,a}\left[\exp\left[-\frac{\mu a^d}{2c\hbar}\sum_{\xi\in\Lambda}I(\varphi_\xi)\right]\right]\prod_{k=0}^{N}P(d\varphi^{(k)}) \tag{3.10}$$

under the condition that

$$\varphi=\sum_{k=0}^{\infty}\varphi^{(k)}$$

is held fixed; the Z's normalize to one the measures in (3.10).

Now there will be a change in point of view and the fields $\varphi^{(k)}$ will no longer be regarded just as auxiliary fields but as objects interesting in their own right: the stability problem will be extended to the problem of showing that not only φ but also $\varphi^{(k)}$, for each k, have a well-defined limit distribution as $a\to 0$, if they are given, for $a>0$, the distribution (3.10).

The plan is to attack the ultraviolet stability problem by studying the measure

$$P^{(\leq N)}(d\varphi)=Z^{-1}_{N,a}\left[\exp\left[-\frac{\mu a^d}{2c\hbar}\sum_{\xi}I(\varphi^{(\leq N)}_\xi)\right]\right]\times\prod_{j=0}^{N}P(d\varphi^{(j)}) \tag{3.11}$$

uniformly in a,N, $\gamma^{-N}m^{-1} \ll a$, allowing $I(\varphi)$ to depend on a,N, and on the derivatives of φ, if this becomes necessary in order to ensure the existence of an interesting limit as $a \to 0$ after letting $N \to \infty$.

Ultimately "interesting" should mean a field theory susceptible to a physical interpretation as a theory of interacting particles: it should verify various properties among which the possibility of defining an operator formally equal to $\mathscr{H}$ in (1.11). For instance, one could require that the field φ verifies the Nelson axioms or the Osterwalder-Schrader axioms or that it should lead in some way or another to the construction of Wightman fields (which undoubtedly is the minimal requirement thought so far) [for a critical discussion and a review of the axioms of various type and their relations see Simon (1974); see also Nelson (1973a,1973b,1973c), Osterwalder and Schrader (1973), and Wightman (1956)].

In other words, one is free to change the rules of the game provided one eventually succeeds in constructing a Wightman field theory describing nontrivial interactions; see also the comments at the end of Sec. I and in Sec. XXII.

Of course, the more one changes the rules of the game, the more one has to work at a later stage. For instance, in passing from the lattice regularized-continuous time approach of Sec. II to the problem of taking the $N \to \infty$ limit in (3.11), we lose the "unitary character" of the theory, because it is no longer clear (and in fact it is not true) that the process $P^{(\leq N)}(d\varphi)$ can be generated by a Hamiltonian as was the case in Sec. II, i.e., before starting the chain of "slight" changes leading to (3.11). So once the limit as $N \to \infty$ will have been taken, we shall have to worry whether it has the properties which would allow us to interpret it as generated by a Hamiltonian operator, i.e., whether a formula like (2.6) holds for some operator H_{quantum}.

In constructing a field theory, it may sometimes be convenient to give up temporarily some of the properties of the final theory; note, on the other hand, that the continuous time lattice regularization, although "unitary," is not translationally invariant [a property holding at $b=0$ for (2.11), which is not unitary].

At this point, in view of the above remark, it is very tempting to simplify the problem by interchanging the limits on a and on N and let $a \to 0$ while keeping N fixed and then let $N \to \infty$. This leads to the measures

$$P^{(\leq N)}(d\varphi) = Z^{-1}\left[\exp\left[-\frac{\mu}{2c\hbar}\int_\Lambda I(\varphi_\xi^{(\leq N)})d\xi\right]\right] \times \prod_{j=0}^{N} P(d\varphi^{(j)}) , \tag{3.12}$$

where now $P(d\varphi^{(j)})$ denotes the distribution of the field $\varphi^{(j)}$ with covariance associated with (3.7) or, alternatively, (3.8) or any higher-order version of them, and $I(\varphi)$ depends on N, possibly.

The advantage of studying (3.12) is that it is obviously easier, in some respects, than (3.11), because the fields $\varphi^{(j)}$ are now really related by simple scalings, as the first of (3.7) or (3.8) shows (i.e., $\varphi_\xi^{(j)}$ has roughly the same distribution as $\gamma^{k(d-2)/2}\varphi^{(0)}_{\gamma^k\xi}$); note, however, that even in case (3.7) there are small corrections, because, although in this simple case $\overline{C}^{(k)}$ scales exactly, the covariance $C^{(k)}$ does not do so because of the imposition of the periodic boundary conditions.

Furthermore, one does not have to distinguish between the cases $\gamma^{-j}m^{-1} \gtrsim a$ and $\gamma^{-j}m^{-1} \lesssim a$. However, notice that the fields $\varphi^{(j)}$ with $\gamma^{-j}m^{-1} \lesssim a$ are somewhat trivial (i.e., they are approximately independently distributed on the lattice points) and thus one heuristically thinks that the limits of (3.12), as $N \to \infty$, should lead to the same measures as the limit, as $N \to \infty$ first and $a \to 0$ second, of (3.11).

This remark could in fact be made more precise to the extent that it can become "all the results discussed in this paper and concerning the existence of non-Gaussian limits of (3.12) as $N \to \infty$, or the existence of formal perturbation expansions of various quantities, could also be obtained considering the limits $\lim_{a\to 0}\lim_{N\to\infty}$ of (3.11)"; this statement should emerge quite clearly from what follows, but it will not be explicitly proved (to limit the material presented here).

The theory of the limits as $N \to \infty$ of (3.12), is already complex and interesting enough, and studying (3.11), as far as the problems discussed here are concerned, does not lead to any new ideas but only to rather trivial technical digressions.

Therefore, from now on I shall concentrate on the discussion of (3.12) with $\varphi^{(k)}$ being defined by (3.7) or (3.8) or their higher-order analogs, depending on the models, the aim being to obtain a limit as $N \to \infty$ in which all the variables $\varphi^{(j)}$ are well defined, although they are not described as Gaussian variables.

One says that the approach to field theory based on (3.11) is a "nearest-neighbor lattice regularization" approach, while the one adopted here, via (3.12), is a "Pauli-Villars regularization" approach of some order; more appropriately it should be called "Feynman regularization"—see Pauli and Villars (1949). Both approaches are widely used in the literature: see for some examples Bogoliubov and Shirkov (1959), Callan (1976), Park (1977), Aizenman (1982), Fröhlich (1982), and Brydges *et al.* (1983).

Before starting the analysis of (3.12) it is important to stress once more at the cost of being repetitious and to avoid hiding important issues, that while the theories of (3.11) and (3.12) are equivalent up to technicalities as far as the results presented in this work are concerned, it is by no means clear that they are equally suitable for pursuing the quest of the results that we should like to obtain, first among them showing the existence or the nonexistence of a nontrivial φ^4 field in dimension $d=4$. Furthermore, there are other possible approaches most of which give the same results as the one presented here, if applied to the solved problems presented here, and which might be better suited for the study of the hard open problems—see comments at the end of Sec. I and in Sec. XXII [see also

Gallavotti and Rivasseau (1984)].

The fields $\varphi^{(k)}$ with covariance (3.7) will be called "first-order Pauli-Villars fields" of frequency index k, while those with covariance (3.8) will be "second-order Pauli-Villars fields" with frequency index k (shortly "with frequency k"); similarly one can define the mth-order Pauli-Villars fields via the use of the higher-order identities in (3.4) and with $\varepsilon = p^2$—see below.

Formula (3.12) will define the mth-order regularized interacting measure if $\varphi^{(k)}$ has the meaning of an mth-order Pauli-Villars field—of course only functions $I(\varphi)$ will be allowed that are such that $I(\varphi^{(\leq N)})$ has a meaning, at least with probability 1, with respect to the measure $\prod_{j=0}^{N} P(d\varphi^{(j)})$.

The latter remark is very important: in fact, it shows that

$$I(\varphi) \propto \varphi^4 \tag{3.13}$$

is not admissible for $d \geq 4$ if one uses the Pauli-Villars first-order field [because the expected value of $\varphi_\xi^{(\leq N)2}$ is infinite if $d \geq 4$, by the second of (3.7)]. However,

$$I(\varphi_\xi) = \lambda\varphi_\xi^4 + \mu\varphi_\xi^2 + \alpha(\underline{\partial}\varphi_\xi)^2 \tag{3.14}$$

is meaningful if one uses in (3.12) the second-order Pauli-Villars regularization even for $d = 4$, because the expected values of $(\varphi_\xi^{(\leq N)})^2$ and $(\underline{\partial}\varphi_\xi^{(\leq N)})^2$ are finite if computed using (3.8) rather than (3.7).

This section will be concluded by listing a more quantitative meaning to be given to the regularity statements about the fields $\varphi^{(k)}$ made after (3.7) and (3.8).

Let $\varphi^{(k)}$ be a sample of a Gaussian random field distributed with covariance $C_{\xi\eta}^{(k)}$ in (3.7). Then if Λ is imagined paved by a lattice Q_k of square boxes Δ with side size $\gamma^{-k}m^{-1}$, one finds that, for $d = 2,3$ and for all choices of $B_\Delta > 0$,

$$|\varphi_\xi^{(k)}| \leq B_\Delta \gamma^{k(d-2)/2}, \quad \forall \xi \in \Delta \in Q_k \ , \tag{3.15}$$

$$|\varphi_\xi^{(k)} - \varphi_\eta^{(k)}| \leq B_\Delta \gamma^{k(d-2)/2} (\gamma^k |\xi - \eta|)^{(4-d)/2-\varepsilon} \ , \quad \forall \xi \in \Delta, \ |\xi - \eta| < \gamma^{-k}m^{-1} \ , \tag{3.16}$$

hold with probability bounded below by

$$\prod_{\Delta \in Q_k} (1 - \overline{A}e^{-\overline{\alpha}B_\Delta^2}) \tag{3.17}$$

if $\overline{A}, \overline{\alpha} > 0$ are suitable constants depending on the choice of the arbitrary parameter $\varepsilon > 0$ but k independent. Of course one assumes here that the side of Λ is divisible by $\gamma^{-k}m^{-1}$ for all $k \geq 0$; this assumption could easily be released for the study of the problems considered in this paper—see, however, the comments in Sec. XXII.

More generally the chain (3.4) can be continued to express $(m^2 + p^2)^{-1}$ as a sum of reciprocals of polynomials of degree 2^{n+1} in p^2, $n = 0,1,2,\ldots$. In this way one can define a field

$$\varphi^{(\leq N)} = \sum_{k=0}^{N} \varphi^{(k)} \ , \tag{3.18}$$

where $\varphi^{(\leq N)}$ is a very smooth Gaussian field which is decomposed into regular independent fields with covariances $C^{(k)}$ defined by "periodizing" a covariance $\overline{C}^{(k)}$ via the second relation in (3.7) [or (3.8)] and with $\overline{C}^{(k)}$, verifying

$$\begin{aligned} &|\partial^j \overline{C}_{\xi\eta}^{(k)}| \leq A_0 \gamma^{kj} \gamma^{k(d-2)} e^{-\kappa_0 \gamma^k |\xi - \eta|} \ , \\ &\qquad j = 0,1,\ldots,j_0 - 1 \ , \\ &|\partial^{j_0-1}\overline{C}_{\xi\eta}^{(k)} - \partial^{j_0-1}\overline{C}_{\xi'\eta}^{(k)}| \leq A_\varepsilon \gamma^{k(j_0-1)} \gamma^{k(d-2)} \\ &\qquad \times (\gamma^k |\xi - \xi'|)^{1-\varepsilon} \\ &\qquad \times e^{-\kappa_0 \gamma^k |\xi - \eta|} \ , \end{aligned} \tag{3.19}$$

where A_0, A, κ_0 are suitable constants and $\varepsilon > 0$ is arbitrary, and where $j_0 = 2^{n+1} - d$. For instance, the case $n = 2$ is worked out explicitly in (3.8). The nth-order Pauli-Villars fields defined by $C^{(k)}$ verify, with probability bounded below by (3.17),

$$\begin{aligned} &|\partial^j \varphi_\xi^{(k)}| \leq B_\Delta \gamma^{jk} \gamma^{k(d-2)/2}, \quad 2j < j_0 \ , \\ &\left| \varphi_\xi^{(k)} - \sum_{2|\underline{a}| < j_0} \frac{1}{\underline{a}!} \frac{\partial^{\underline{a}} \varphi_\xi^{(k)}}{\partial \xi^{\underline{a}}} (\xi - \eta)^{\underline{a}} \right| \\ &\qquad \leq B_\Delta \gamma^{k(d-2)/2} (\gamma^k |\xi - \eta|)^{(j_0/2) - \varepsilon} \ , \end{aligned} \tag{3.20}$$

where ∂^j denotes any jth-order derivative and

$$\underline{a}! = \prod_{i=1}^{d} a_i!, \quad |\underline{a}| = \sum_{i=1}^{d} a_i \ ,$$

$\partial^{\underline{a}}$ is the derivative of order $a_1 + \cdots + a_d$ of order a_1 with respect to the first component, etc., so that the second of (3.20) is an estimate for the remainder of a Taylor series.

For instance if $n = 3$, $d = 4$, the field $\varphi^{(k)}$ admits five derivatives (and the fifth is Hölder continuous with exponent less than $\frac{1}{2}$) and $C^{(k)}$ admits 11 derivatives.

Since periodic boundary conditions are being used unless explicitly stated otherwise, here as well as in the rest of the paper, $(\xi_2 - \xi_1)$ will be a symbolic notation for a periodic function on $\Lambda \times \Lambda$ equal to the vector from ξ_1 to ξ_2 when the distance between ξ_1 and ξ_2 on Λ is small and, for larger distances, equal to $(\xi_2 - \xi_1)\chi(|\xi_2 - \xi_1|^2)$ with $\chi \in C^\infty$ and $\chi(r) = 0$, if $r > 1$, $\chi(r) = 1$ if $r < \frac{1}{2}$, and $|\xi_2 - \xi_1|$ the distance of the torus Λ.

The inequalities (3.19) are elementary consequences of the analysis of the asymptotic behavior of the integrals in (3.7) and (3.8) and of their generalizations to order n. Whereas the inequalities (3.20) and (3.16) follow, as mentioned above, via the classical idea of Wiener (for the Hölder continuity of the sample paths of the Brownian motion) from the regularity properties of the covariances $C^{(k)}$ expressed by (3.19) [see Colella and Lanford (1973) and Benfatto, Gallavotti, and Nicolò (1982b)] plus the fact that the $\varphi^{(k)}$ form a Markov process.

In the literature other regularizations are also used which produce infinitely smooth fields $\varphi^{(k)}$ by using "nonpolynomial" decompositions like

$$\frac{1}{1+p^2}=\frac{\chi_0(p)}{1+p^2}+\sum_{k=1}^{\infty}\frac{\chi_1(\gamma^k p)}{1+p^2}\ , \tag{3.21}$$

where

$$\chi_0(p)+\sum_{k=1}^{\infty}\chi_1(\gamma^k p)\equiv 1 \tag{3.22}$$

and χ_0,χ_1 are C^∞ functions positive such that χ_1 has support for $1\le p^2\le\gamma$. Such decompositions produce $\overline{C}^{(k)}$'s which verify (3.19) with j_0 arbitrarily prefixed but with the modification that the exponential decay factor is replaced by $(1+\gamma^k|\xi-\eta|)^{-w}$ with w arbitrarily prefixed [i.e., the decoupling takes place on the same scale as in (3.19), i.e., γ^{-k}, but is slower than exponential, although still faster than any power].

IV. PERTURBATION THEORY AND ULTRAVIOLET STABILITY

I shall try to be very general, not for love of generality, but, rather, because perturbation theory is conceptually very simple, and if one discusses it in the few examples in which one is really interested, one makes it appear more complex, as all the fine details peculiar to each model become most inextricably mixed up with its structure.

The first thing to fix is the interaction $I(\varphi)$: choose $I(\varphi)$ to have the form

$$V(\varphi^{(\le N)},\underline{\lambda},N)=\sum_{\alpha=1}^{t}\lambda^{(\alpha)}\int_\Lambda v_N^{(\alpha)}(\varphi_\xi^{(\le N)},\partial\varphi_\xi^{(\le N)})d\xi$$
$$\equiv I(\varphi)\ . \tag{4.1}$$

If $\varphi\equiv\varphi^{(\le N)}$, the function $I(\varphi)$ spans a finite dimensional linear space $\mathscr{I}_N$ as $\underline{\lambda}=(\lambda^{(\alpha)})_{\alpha=1,\dots,t}$ spans R^t or a linear subspace of R^t, fixed *a priori;* one can regard $\mathscr{I}_N$ as a subspace of

$$L_2\left[\prod_{j=0}^{N}P(d\varphi^{(j)})\right]\ .$$

It is convenient to assume that the spaces $\mathscr{I}_N$, $N=0,1,\dots,$ are so related that for all $N'\le N$ it is

$$\int_\Lambda v_{N'}^{(\alpha)}(\varphi_\xi^{(\le N')},\partial\varphi_\xi^{(\le N')})d\xi$$
$$=\int P(d\varphi^{(N'+1)})\cdots P(d\varphi^{(N)})\times\int_\Lambda v_N^{(\alpha)}(\varphi_\xi^{(\le N)},\partial\varphi_\xi^{(\le N)})d\xi\ , \tag{4.2}$$

or, in other words, $v_{N'}^{(\alpha)}$ is the projection on $\mathscr{I}_{N'}$ of $v_N^{(\alpha)}\in\mathscr{I}_N$ executed by using as projection operator the integration with respect to the field components of frequency higher than N'. This property, which is verified automatically in all the models that are considered here (since every model will be written in Wick-ordered form—see below), is very convenient for the exhibition of general structural properties of perturbation theory. In probability it is known as a "martingale" property.

The sequence $\mathscr{I}=(\mathscr{I}_k)_{k=0,\dots,\infty}$ of linear spaces verifying the martingale relation (4.2) will be called an "interaction."

Of course the choice of the free fields $\varphi^{(j)}$, i.e., of the regularization's order, will always have to be such that the integrals in (4.1) make sense [for instance, if $v^{(\alpha)}$ is really depending on $\partial\varphi_\xi$ we shall use at least a second-order regularization for $d\le 4$; if $v^{(\alpha)}$ depends only on φ_ξ then we could also use the first-order regularization, provided $d<4$—see (3.20)].

A field theory with interaction $\mathscr{I}$ can be defined in two, usually nonequivalent, ways: nonperturbatively as a probability measure which is the limit as $N\to\infty$ of measures defined by

$$P_{\mathscr{I},N}(d\varphi^{(\le N)})=\{Z^{-1}\exp[V(\varphi^{(\le N)},\underline{\lambda}_N,N)]\}\times\prod_{j=0}^{N}P(d\varphi^{(j)})\ , \tag{4.3}$$

where $\underline{\lambda}_N$ is a given sequence of coupling constants called "bare couplings," for which

$$\exp[V(\ )]\in L_2\left[\prod_{j=0}^{N}P(d\varphi^{(j)})\right]\ ,$$

or perturbatively—this definition is based on the following idea (Schwinger, 1949; Feynman, 1949; Dyson, 1949). Consider the following formal power series in the parameters $\underline{\lambda}=(\lambda_1,\dots,\lambda_t)\in R^t$:

$$\underline{\lambda}_N(\underline{\lambda})=\sum_{m_1,\dots,m_t}\underline{l}_N(\underline{m})\lambda_1^{m_1}\cdots\lambda_t^{m_t}\equiv\sum_{\underline{m}}\underline{l}_N(\underline{m})\underline{\lambda}^{\underline{m}}\ , \tag{4.4}$$

where $\underline{l}_N(\underline{m})\in R^t$. Then compute

$$\int e^{\varphi^{(\le N)}(f)}P_{\mathscr{I},N}(d\varphi)=\frac{\int e^{\varphi^{(\le N)}(f)}e^{V(\varphi^{(\le N)},\underline{\lambda}_N(\underline{\lambda}),N)}\prod_{j=1}^{N}P(d\varphi^{(j)})}{\int e^{V(\varphi^{(\le N)},\underline{\lambda}_N(\underline{\lambda}),N)}\prod_{j=1}^{N}P(d\varphi^{(j)})} \tag{4.5}$$

formally by expanding all the exponentials in powers and then using (4.4) to express the results as a power series in $\underline{\lambda}$ by collecting terms with equal powers:

$$\int e^{\varphi^{(\le N)}(f)}P_{\mathscr{I},N}(d\varphi)\equiv\langle e^{\varphi^{(\le N)}(f)}\rangle=\sum_{\underline{m}}S(\underline{m},N,f)\underline{\lambda}^{\underline{m}}\ . \tag{4.6}$$

Then the perturbative field theory with interaction $\mathscr{I}$ and bare constants $\underline{\lambda}_N(\underline{\lambda})$ given by (4.4) is well defined if the limits

$$S(\underline{m},f)=\lim_{N\to\infty}S(\underline{m},N,f) \tag{4.7}$$

exist for all smooth test functions f and for all $\underline{m}$.

The theory will be called "perturbatively trivial" if the power series

$$\sum_{\underline{m}} S(\underline{m},f)\underline{\lambda}^{\underline{m}} \tag{4.8}$$

formally converges to the exponential of a quadratic form in f (Gaussian theory) for $|\underline{\lambda}|$ small.

Similarly if the limits of (4.3) are Gaussian measures for all possible choices of $\underline{\lambda}_N$, one says that the theory is trivial.

If it is impossible to find a formal power series (4.4) such that the limits (4.7) exist, one says that $\mathscr{J}$ is a "nonrenormalizable" theory.

The power series (4.8) is called the "renormalized series for $\mathscr{J}$," and the parameters $\underline{\lambda}$ in it are called the "renormalized couplings," while the corresponding formal series (4.4) define the perturbative bare couplings [note that the formal power series (4.4) do not necessarily converge].

It is perhaps worth stressing again that the real objects that one is trying to find are more complex than a probability measure P limit of (4.3) (in a perturbative or nonperturbative sense), so after such limits are constructed, one still has to see if they have the right properties to allow their interpretation as relativistically invariant quantum field theories.

However, in the few cases in which the measures P have been constructed as limits for $N\to\infty$ of (4.3) the understanding of the problems remaining before a full interpretation of the results, as relativistic quantum fields, has been carried out without excessive difficulties [after the basic techniques and tools to deal with this question were developed in the basic papers—see Nelson (1966,1973a,1973b,1973c), Glimm (1968), Glimm and Jaffe (1968,1970a,1970b,1973), Glimm, Jaffe, and Spencer (1975), Osterwalder and Schrader (1972), and Guerra (1972)], so I shall not develop this question further here, after warning the reader of its paramount importance, on the grounds that it should not properly be thought of as a part of the main subject of this paper, i.e., of the ultraviolet limit problem.

Perturbation theory plays a major role even in the so-called nonperturbative approaches (Glimm, 1968; Glimm and Jaffe, 1968,1970a,1970b,1973; Magnen and Seneor, 1976; Feldman and Osterwalder, 1976; Benfatto *et al.*, 1978; Benfatto, Cassandro *et al.*, 1980; Benfatto, Gallavotti, and Nicolò, 1980; Gallavotti, 1978,1979,1980; Gawedski and Kupiainen, 1980,1983,1984; Balaban, 1981,1983; Federbush and Battle, 1982,1983; Brydges *et al.*, 1984).

Here perturbation theory will be treated from the point of view of the renormalization group, expanding the ideas developed and used in Benfatto *et al.* in the just-cited papers. I shall follow the theory presented in Gallavotti and Nicolò (1984), with some modifications here or there. The first to treat completely, to all orders, perturbation theory by literally applying the renormalization group methods has been Polchinskii (1984), who adopts a method slightly different from the one presented here (obtaining weaker results—e.g., the $n!$ bounds are not treated in his work, at least not explicitly).

The renormalization group approach to field theory grew out of several earlier works [e.g., Kadanoff (1966), Wilson (1965,1971,1972), and DiCastro and Jona-Lasinio (1969,1971); for reviews see Wilson and Kogut (1974), Jona-Lasinio (1975), Ma (1976), and Wilson (1983)].

In this work applying the renormalization group method will mean that one regards the fields $\varphi^{(0)},\ldots,\varphi^{(k)},\ldots,$ as real entities describing phenomena taking place on their own length scale $\gamma^{-k}m^{-1}$, and one defines the effective interaction on scale $\gamma^{-k}m^{-1}$ as

$$e^{V^{(k)}(\varphi^{(\le k)})}=\int e^{V(\varphi^{(\le N)},\underline{\lambda}_N(\underline{\lambda}),N)} \times P(d\varphi^{(N)})\cdots P(d\varphi^{(k+1)})\;. \tag{4.9}$$

In perturbation theory one fixes the formal power series $\underline{\lambda}_N(\underline{\lambda})$ in such a way that $V^{(k)}$ turns out to be given by a formal power series in $\underline{\lambda}$, which, order by order, has a limit for $N\to\infty$ if $\varphi^{(0)},\ldots,\varphi^{(k)}$ verify (3.20) (n being the order of the chosen regularization), and the limit has a short-range structure allowing the interpretation of $V^{(k)}$ as a statistical mechanics interaction between spins (the $\varphi^{(k)}$'s) which are located on a lattice of mesh $\gamma^{-k}m^{-1}$ (recall that the fields $\varphi^{(j)}$ are regular and therefore can be thought of as lattice fields on a lattice of mesh $\gamma^{-j}m^{-1}$ rather than as continuous fields—see Sec. III).

One might be worried that the fields $\varphi^{(j)}$ do not really have a physical meaning (yet) and that knowing that they are well-defined objects even in the presence of interaction does not really tell anything about their sum $\varphi^{(\le N)}$, which is the object with physical meaning; one could repair this objection by imagining that the last term in (4.1) has the form

$$\lambda^{(t)}V_N^{(t)}(\varphi_\xi^{(\le N)},\partial\varphi_\xi^{(\le N)})\equiv\lambda^{(t)}f(\xi)\varphi_\xi^{(\le N)} \tag{4.10}$$

and show that the effective potentials are well defined with a choice of $\underline{l}_N(\underline{m})$ leading to $\lambda_N^{(t)}\equiv\lambda^{(t)}$ and to an expression of the other bare coupling constants $\lambda_N^{(1)},\ldots,\lambda_N^{(t-1)}$ involving only $\lambda^{(1)},\ldots,\lambda^{(t-1)}$ (and not $\lambda^{(t)}$); then the effective potentials and the coefficients $S(\underline{m},N,f)$ would be simply related and the problem of proving the existence and ultraviolet stability of the effective potentials would be, in principle, harder than proving that of the limit (4.7) (although it will be, in fact, an equivalent problem in the cases studied later).

Alternatively one could decide to worry about this problem after completing the theory of the effective potentials: in fact, the formal connection between the effective potentials and the Schwinger functions will be briefly discussed in Sec. X.

V. EFFECTIVE POTENTIALS: THE ALGORITHM FOR THEIR CONSTRUCTION

Given an interaction $\mathscr{J}$ as defined in Sec. IV [see (4.1) and (4.2)], let

$$V(\varphi^{(\le N)})=\sum_{\alpha=1}^{t}\lambda^{(\alpha)}\int_\Lambda v_N^{(\alpha)}(\varphi_\xi^{(\le N)};\partial\varphi_\xi^{(\le N)})d\xi\;. \tag{5.1}$$

The effective interaction on the length scale $\gamma^{-k}m^{-1}$ is defined by

$$\exp[V^{(k)}(\varphi^{(\leq k)})]=\int \{\exp[V(\varphi^{(\leq N)})]\} \times P(d\varphi^{(N)})\cdots P(d\varphi^{(k+1)}) . \qquad (5.2)$$

To be slightly more concrete it is convenient to list the cases which will be treated here or which can be treated easily with the methods reviewed in this paper.

(1) Polynomial fields in two dimensions:

$$v_N^{(\alpha)}(\varphi_\xi^{(\leq N)})=:(\varphi_\xi^{(\leq N)})^\alpha: , \qquad (5.3)$$

where the dots denote the Wick polynomials.

In this case, as well as in the cases below, the property (4.2) is trivially a consequence of the properties of the Wick polynomials. Such properties are remarkable and the reader will be supposed familiar with them. For ease of reference the definitions, their main properties, and the ideas from which they are proved are provided in Appendix C.

The notion of the Wick monomial will not be needed in this section nor in the following sections, VI—X, where everything is worked out without referring to Wick ordering or Wick monomials.

(2) Sine-Gordon field in two dimensions:

$$V(\varphi^{(\leq N)})=\sum_{\sigma=\pm 1}\frac{\lambda}{2}\int_\Lambda :e^{i\sigma\alpha\varphi_\xi^{(\leq N)}}:+\nu\int_\Lambda d\xi$$

$$=\lambda\int_\Lambda :\cos(\alpha\varphi_\xi^{(\leq N)}):d\xi+\nu\int_\Lambda d\xi,\quad \alpha>0 . \qquad (5.4)$$

(3) Exponential field, $d\geq 2$:

$$V(\varphi^{(\leq N)})=-\lambda\int_\Lambda :e^{\alpha\varphi_\xi^{(\leq N)}}:d\xi+\nu\int_\Lambda d\xi . \qquad (5.5)$$

(4) φ^4 field in three dimensions:

$$V(\varphi^{(\leq N)})=-\int_\Lambda [\lambda:(\varphi_\xi^{(\leq N)})^4:+\mu:(\varphi_\xi^{(\leq N)})^2:+\nu]d\xi . \qquad (5.6)$$

(5) φ^4 field with wave function renormalization for $d\leq 4$:

$$V(\varphi^{(\leq N)})=-\int_\Lambda [\lambda:(\varphi_\xi^{(\leq N)})^4:+\mu:(\varphi_\xi^{(\leq N)})^2: +\alpha:(\underline{\partial}\varphi_\xi^{(\leq N)})^2:+\nu]d\xi . \qquad (5.7)$$

(6) φ^6 field with wave-function renormalization for $d\leq 3$:

$$V(\varphi^{(\leq N)})=-\int_\Lambda [\sigma:(\varphi_\xi^{(\leq N)})^6:+\lambda:(\varphi_\xi^{(\leq N)})^4: +\mu:(\varphi_\xi^{(\leq N)})^2:+\alpha:(\underline{\partial}\varphi_\xi^{(\leq N)})^2:+\nu]d\xi . \qquad (5.8)$$

All the above cases are examples of interactions $\mathscr{I}$ in the sense of (4.1) and (4.2) (see Appendix C for the properties of the Wick monomials).

In view of the above ambitious models one might think that it would be very hard to find reasonable expressions for $V^{(k)}$; this is not really the case, as the algorithm below proves, the reason being that the construction of $V^{(k)}$ can be carried out in general, without using the detailed structures (5.5)—(5.8) or the Wick ordering properties, starting from (5.1), (4.1), and (4.2).

The mathematical basis for the algorithm is a trivial Taylor series. To define it introduce the notations

$$\mathscr{E}(\ )=\text{expectation value with respect to a probability measure ,}$$
$$\mathscr{E}_k(\ )=\text{expectation value with respect to the Gaussian measure } P(d\varphi^{(k)}) , \qquad (5.9)$$

and in general, given p random variables $x_1,\ldots,x_p$ and p positive integers $n_1,\ldots,n_p$, one defines the truncated expectations of $x_1,\ldots,x_p$ of orders $n_1,\ldots,n_p$ as

$$\mathscr{E}^T(x_1,\ldots,x_p;n_1,\ldots,n_p) =\left.\frac{\partial^{n_1+\cdots+n_p}}{\partial\lambda_1^{n_1}\cdots\partial\lambda_p^{n_p}}\ln\mathscr{E}(e^{\lambda_1x_1+\cdots+\lambda_px_p})\right|_{\lambda_1=\cdots=\lambda_p=0} . \qquad (5.10)$$

The symbol $\mathscr{E}_k^T$ will therefore have a well-defined meaning if $x_1,\ldots,x_p$ are p functions depending on $\varphi^{(k)}$. It is easy to prove by induction the Leibniz rule: $\forall\omega_1,\ldots,\omega_p\in R$:

$$\mathscr{E}^T(\omega_1x_1+\cdots+\omega_px_p;n) =\sum_{\substack{n_1,\ldots,n_p\\ n_1+\cdots+n_p=n}}\frac{n!\omega_1^{n_1}\cdots\omega_p^{n_p}}{n_1!\cdots n_p!} \times\mathscr{E}^T(x_1,\ldots,x_p;n_1,\ldots,n_p) , \qquad (5.11)$$

and if $n=n_1+\cdots+n_p$,

$$\mathscr{E}^T(x;1)\equiv\mathscr{E}(x),\quad \mathscr{E}^T(x;0)\equiv 0 ,$$
$$\mathscr{E}^T(x,x,\ldots,x;n_1,\ldots,n_p)\equiv\mathscr{E}^T(x;n) . \qquad (5.12)$$

Then the following Taylor expansion holds formally ("cumulant expansion"):

$$\mathscr{E}(e^x)=\exp\left[\sum_{p=1}^\infty\frac{\mathscr{E}^T(x;p)}{p!}\right] \qquad (5.13)$$

for any bounded random variable x.

Hence modulo convergence problems:

$$\int P(d\varphi^{(N)})e^V\equiv\exp\left[\sum_{n=1}^\infty\frac{\mathscr{E}_N^T(V;n)}{n!}\right] =\exp(V^{(N-1)}) \qquad (5.14)$$

and

$$\int P(d\varphi^{(N-1)})P(d\varphi^{(N)})e^{V}=\exp\left[\sum_{h=1}^{\infty}\frac{\mathscr{E}_{N-1}^{T}(V^{(N-1)};h)}{h!}\right]$$

$$=\exp\left[\sum_{h=1}^{\infty}\sum_{n_1+n_2+\cdots=h}\frac{1}{n_1!n_2!\cdots}\frac{1}{1!^{n_1}2!^{n_2}3!^{n_3}\cdots}\times\mathscr{E}_{N-1}^{T}(\mathscr{E}_N^T(V;1),\mathscr{E}_N^T(V;2),\mathscr{E}_N^T(V;3),\ldots;n_1,n_2,n_3,\ldots)\right], \quad (5.15)$$

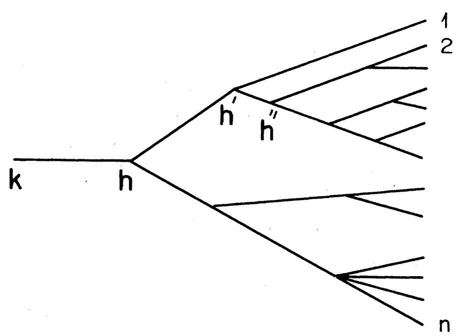
once we have applied the Leibnitz rule (5.11).

It is clear that by combining (5.9) and (5.13) one can easily find a formal expression for $V^{(k)}$ of the type (5.15): its structure will be elucidated in Sec. VI by means of a graphical interpretation of the general term arising in the iteration of the above expansions.

VI. A GRAPHICAL EXPRESSION FOR THE EFFECTIVE INTERACTIONS

The structure of $V^{(k)}$, as obtained from V by doing successively the integrations over the fields of increasing length scale, can be described easily in terms of a certain family of planar graphs, actually trees.

Draw n points $1,2,\ldots,n$ (diagram 3)



(3)

and imagine that they are the end points of a tree γ whose vertices v bear an index h_v, with $k\le h_v\le N$ and $h_v<h_{v'}$, if $v<v'$ in the tree's order; the lowest vertex r of γ, called the "root," bears the index k, denoted $k(\gamma)$, and out of it emerges one branch only. All the other vertices $v>r$ are branching points with at least two branches. The tree's end points are not regarded as vertices.

Two trees will be regarded as identical if they can be superposed, together with the labels appended to their vertices, up to a permutation of the end point labels $(1,2,\ldots,n)$ and up to a change in the lengths of the branches and the location of the vertices which does not alter the tree's topological structure. In drawing trees we shall agree to think that they are drawn in some standard fashion which always leads to the construction of a given representative in each class.

The number of end points of γ (n in diagram 3) will be called the "degree of γ." A tree of degree 1 will be called trivial, and it will contain only one line, from the root r to the end point 1.

The first vertex after r will be called v_0; it exists if and only if the tree γ is not trivial.

Given a nontrivial tree γ, let $\gamma_1,\gamma_2,\ldots,\gamma_s$ be the trees which bifurcate in γ from v_0, i.e., from the first nontrivial vertex (in diagram 3 it is $s=2$). The s trees can be divided into q classes of trees which are identical up to the end point labelings, and let $\bar\gamma_1,\bar\gamma_2,\ldots,\bar\gamma_q$ be the representatives of each class. Let $p_1,p_2,\ldots,p_q$ be the number of elements in each class. Define a "combinatorial factor" $n(\gamma)$ inductively as

$$n(\gamma)=\prod_{i=1}^{q}p_i!\,n(\bar\gamma_i)^{p_i}\ , \quad (6.1)$$

setting $n(\gamma)=1$ if γ is the trivial tree.

The index h_v associated with each vertex of γ will be called a frequency index or the frequency of v.

If one stares, for a conveniently long time, at (5.13) and (5.15) it becomes clear that

$$V^{(k)}=\sum_{\gamma:k(\gamma)=k}\frac{V(\gamma)}{n(\gamma)}\ , \quad (6.2)$$

where the sum runs over the trees with root at frequency k and with frequency indices $h_v\le N$; $V(\gamma)$ is a function of the field $\varphi^{(\le k)}$, which, although it could be explicitly written, is more conveniently defined by induction. If γ is trivial, let

$$V(\gamma)=\mathscr{E}_{k+1}\cdots\mathscr{E}_N(V)\ , \quad (6.3)$$

and if γ bifurcates on the first vertex v_0 following its root r into $\gamma_1,\ldots,\gamma_s$ at frequency $h_{v_0}=h$, let

$$V(\gamma)=\mathscr{E}_{k+1}\cdots\mathscr{E}_{h-1}\mathscr{E}_h^T(V(\gamma_1),\ldots,V(\gamma_s);1,\ldots,1)\ . \quad (6.4)$$

As a result of (6.3) and (6.4) one sees that each vertex of γ with index p corresponds to $\mathscr{E}_p^T$, while each line of γ joining two vertices $v<v'$ corresponds to

$$\mathscr{E}_{h_v+1}\cdots\mathscr{E}_{h_{v'}-1}\ , \quad (6.5)$$

the lines joining a vertex v to an end point correspond to

$$\mathscr{E}_{h_v+1}\cdots\mathscr{E}_N\ , \quad (6.6)$$

and, finally, each end point corresponds to a function V.

The proof of (6.4) is obtained by combining (6.2) and (6.3) with (5.9)—(5.14): one gets (6.4) immediately by induction on the degree of the tree.

The above algorithm can be modified to obtain more explicit expressions for $V^{(k)}$.

Let

$$V=\sum_{\alpha=1}^{t}\lambda^{(\alpha)}\int_{\Lambda}v_N^{(\alpha)}(\varphi_\xi^{(\le N)},\partial\varphi_\xi^{(\le N)})d\xi\ , \tag{6.7}$$

which is the case of interest here, and introduce what will be called a "decorated tree," which is a tree whose end points bear labels $(\xi_1,\alpha_1),\ldots,(\xi_n,\alpha_n)$ instead of $1,2,\ldots,n$, with $\xi_j\in R^d$ and $\alpha_j\in(1,2,\ldots,t)$.

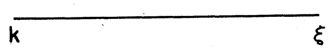
Then (6.2) and (6.4) imply that

$$V^{(k)}=\sum_n\int\sum_{\alpha_1,\ldots,\alpha_n}d\xi_1\cdots d\xi_n\sum_{\substack{\gamma:\deg\gamma=n\\ k(\gamma)=k\\ \theta(\gamma)=((\xi_1,\alpha_1),(\xi_2,\alpha_2),\ldots)}}\frac{V(\gamma)}{n(\gamma)}\equiv\sum_{k(\gamma)=k}\frac{V(\gamma)}{n(\gamma)}\ , \tag{6.8}$$

where the sum runs over all the decorated trees γ with root frequency k and with vertex frequencies h_v with $k<h_v\le N$ for $v>r$, and the value $V(\gamma)$ will have to be computed by using (6.3) and (6.4) except that V has to be replaced in the evaluation of the trivial tree's contribution by

$$\lambda^{(\alpha)}V_N^{(\alpha)}(\varphi_\xi^{(\le N)},\partial\varphi_\xi^{(\le N)})$$

if the trivial tree is



. (4)

The third sum in (6.8) is performed by keeping fixed the decoration $\theta(\gamma)=((\xi_1,\alpha_1),(\xi_2,\alpha_2),\ldots,(\xi_n,\alpha_n))$. Finally, the combinatorial factor $n(\gamma)$ is defined as identical with the combinatorial factor of the undecorated tree $\tilde{\gamma}$ obtained by stripping γ of its decorations.

In other words, one can say that the rule for evaluating a decorated tree is the same as that for evaluating an undecorated tree but with a different interpretation of the end points, which depends on the decorating indices.

For later use it is convenient to define a tree shape which is a tree of the above types once stripped of all its indices and decorations, except the indices α attached to the end points, which will be called type indices.

This completes the discussion of the basic graphical algorithm used to build $V^{(k)}$ for $k\ge 0$. However, for reasons that will become clearer later, it is convenient to define $V^{(-1)}$, also.

For this purpose one thinks of $\varphi^{(\le N)}$ as being given by

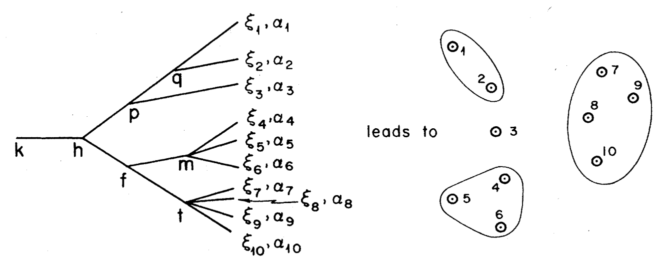
$$\varphi^{(\le N)}=\varphi^{(-1)}+\varphi^{(0)}+\cdots+\varphi^{(N)}\ , \tag{6.9}$$

where the field $\varphi^{(-1)}$ is distributed independently relative to the other $\varphi^{(j)}$, $j\ge 0$, and it has its own covariance $C_{\xi\eta}^{(-1)}$, which need not be specified, because it will eventually be taken to be identically zero whenever it appears in some interesting formula. The introduction of $V^{(-1)}$ allows meaning to be given to some expressions that will be met so that the case $k=0$ can be treated on the same grounds as the cases $k>0$, and $V^{(-1)}$ should be thought as void of any physical meaning or mathematical relevance other than the just-mentioned one. $V^{(-1)}$ will be described by trees with root frequency $k=-1$ via (6.8); see (6.9).

The following interpretation of a decorated tree is interesting and important for later applications.

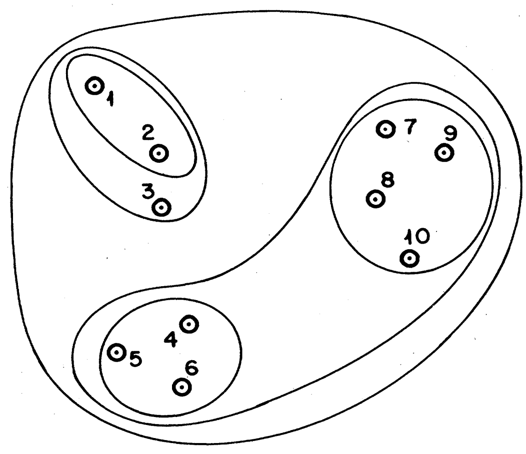
Each vertex v of γ can be interpreted as a cluster of the end points' "positions," and the tree γ provides an organization, into a hierarchy of clusters, of the points $\xi_1,\ldots,\xi_n$, which are the position labels of the tree's end points.

To get a picture of such clusters first draw a box around each point $\xi_1,\ldots,\xi_n$; then consider a vertex v highest on the tree: out of it emerge s lines with labels $(\xi_{j_1},\alpha_{j_1}),\ldots,(\xi_{j_s},\alpha_{j_s})$ at the other end point. Draw a box enclosing $\xi_{j_1},\ldots,\xi_{j_s}$ and do this for all the other highest vertices. For instance,



. (5)

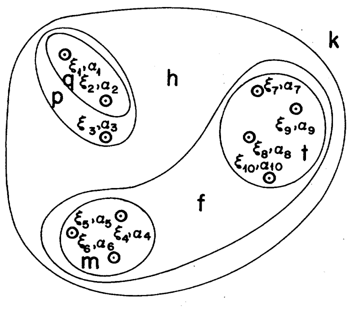
Then consider the next generation vertices and draw boxes around all the end points that can be reached from each of them, climbing the tree, etc.,



. (6)

Actually, the above cluster representation of γ becomes completely equivalent to the description of γ if inside each box one writes the frequency h_v of the vertex v corresponding to it [(1) attribute, conventionally, index $N+1$, or better no index at all, to the innermost boxes enclosing only single points, (2) append to the jth innermost box the index α_j, and (3) attribute to the outside of the outermost box the index k of the root of γ].

For instance, in the case of diagram 5 one gets



, (7)

where the frequencies $N+1$ have not been marked (being obvious).

Therefore, to each decorated tree one will be able to associate with each vertex a cluster of "points" and to associate to each cluster a frequency index in the above manner; furthermore, each point is a position label of γ, and a type label can be attached to it in the manner described above and exemplified in the above pictures.

The "order" of a vertex v will be the number of points in the cluster corresponding to it: it coincides with the number of end points that can be reached from v by climbing the tree. So the degree of the tree coincides with the order of its root vertex as well as with the order of the first nontrivial vertex v_0 (if present).

VII. RENORMALIZATION AND RENORMALIZABILITY TO SECOND ORDER

Consider an interaction $\mathscr{I}$, as defined in Sec. IV, (4.1) and (4.2), and a formal power series like (4.4):

$$\underline{\lambda}_N(\underline{\lambda})=\underline{\lambda}+\sum_{|\underline{m}|\geq 2}\underline{l}_N(\underline{m})\underline{\lambda}^{\underline{m}} \tag{7.1}$$

and define—see (4.4) and (4.1):

$$V_{j,N}^{(\alpha)}=\left[\sum_{|\underline{m}|=j} l_N^{(\alpha)}(\underline{m})\underline{\lambda}^{\underline{m}}\right]\times\int v_N^{(\alpha)}(\varphi_\xi^{(\leq N)},\underline{\partial}\varphi_\xi^{(\leq N)})d\xi\ ,$$

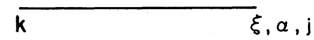
$$V=\sum_{j=1}^{\infty}\sum_{\alpha=1}^{t}V_{j,N}^{(\alpha)}\equiv V(\varphi^{(\leq N)};\underline{\lambda}_N(\underline{\lambda}),N)\ . \tag{7.2}$$

From the general theory of the preceding section it is easy to find the rule to compute the effective potential $V^{(k)}$ corresponding to the V in (7.2). The reader who finds the discussion below too abstract for a first reading can compare the abstract steps described here with the concrete corresponding steps done in studying the specific model φ^4, as described in Sec. XVII, or the sine-Gordon field, in Sec. XII.

One merely allows trees whose end points are decorated by

$$(\xi,\alpha,j),\ \ \xi\in R^d,\ \alpha=1,\ldots,t,\ j=1,2,3\ldots \tag{7.3}$$

Then if the trivial tree



(8)

is interpreted as [see also (4.2)]

$$\mu_j^\alpha(\underline{\lambda})\mathscr{E}_{k+1}\cdots\mathscr{E}_N[v_N^{(\alpha)}(\varphi_\xi^{(\leq N)};\underline{\partial}\varphi_\xi^{(\leq N)})]\equiv\mu_j^\alpha(\underline{\lambda})v_k^{(\alpha)}(\varphi_\xi^{(\leq k)};\underline{\partial}\varphi_\xi^{(\leq k)})\ , \tag{7.4}$$

$$\mu_j^\alpha(\underline{\lambda})\equiv\sum_{|\underline{m}|=j}l_N^\alpha(\underline{m})\underline{\lambda}^{\underline{m}}\ ,$$

it follows [see Sec. VI] that

$$V^{(k)}=\sum_{n=1}^{\infty}\int d\xi_1\cdots d\xi_n\sum_{\alpha_1\cdots\alpha_n}\sum_{j_1\cdots j_n}\sum_{\substack{\gamma:k(\gamma)=k\\ \text{degree }\gamma=n}}\frac{V(\gamma)}{n(\gamma)}\ , \tag{7.5}$$

which expresses $V^{(k)}$ as a power series in $\underline{\lambda}$: the pth-order term being obtained by selecting in (7.5) the contributions such that $j_1+j_2+\cdots+j_n=|\underline{j}\,|=p$.

If one defines, given a tree γ with decorations $(\xi_1,\alpha_1,j_1),\ldots,(\xi_n,\alpha_n,j_n)$, the degree $D(\gamma)$ of γ as

$$D(\gamma)=j_1+\cdots+j_n=|\underline{j}\,|\ , \tag{7.6}$$

then the contribution to $V^{(k)}$ of order p is obtained by restricting the sum in (7.5) to the trees γ with $D(\gamma)=p$. If we denote it $V^{(k),p}$, it is

$$V^{(k),p}=\int d\underline{\xi}\sum_{\underline{\alpha}}\sum_{\underline{j}}\sum_{\substack{k(\gamma)=k\\ D(\gamma)=p}}\frac{V(\gamma)}{n(\gamma)}\ . \tag{7.7}$$

Define

$$\overline{V}^{(k),p}(\varphi^{(\leq k)})=\lim_{N\to\infty}V^{(k),p}(\varphi^{(\leq k)})\ , \tag{7.8}$$

where

$$\varphi^{(\leq k)}=\sum_{j=0}^{k}\varphi^{(j)}$$

is supposed such that each $\varphi^{(j)}$ verifies the smoothness properties (3.15), (3.16), or (3.20), depending on the regularization used for the free field.

The existence of the limits (7.8) clearly depends upon the choice of the coefficients $\underline{l}_N(\underline{m})$ in (7.1). According to the discussion of Sec. IV, the theory is renormalizable if there is a choice of the constants $\underline{l}_N(\underline{m})$ such that the limit (7.8) exists.

It is worth pointing out here a trivial property of the renormalized series: if $\underline{\lambda}$ is expressed as a formal power series with N-independent coefficients in terms of new parameters $\underline{\lambda}'$, then $\underline{\lambda}_N(\underline{\lambda})$—see (7.1)—becomes a new formal power series in $\underline{\lambda}'$ with new coefficients $\underline{l}'_N(\underline{m})$; it should be clear that if the power series (7.5) in $\underline{\lambda}$ is renormalized, i.e., if the limits (7.8) exist, then also the power series in $\underline{\lambda}'$ is renormalized in the same sense (provided the series expressing $\underline{\lambda}$ in terms of $\underline{\lambda}'$ has no constant terms, of course). This shows that the coefficients $\underline{l}_N(\underline{m})$ cannot be uniquely determined by the requirement that the theory is renormalized [i.e., that the limits (7.8) exist].

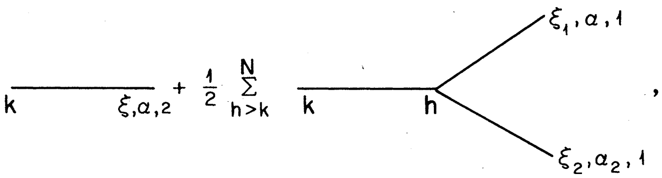
Now the problem is to decide whether a theory is renormalizable and, if so, to find a choice of the coefficients $\underline{l}_N(\underline{m})$ and to estimate in some way the size of $V^{(k),p}$ and, if possible, of $V^{(k)}$ itself.

It is easy to find a general renormalizability criterion and general renormalization rules [i.e., rules to build the coefficients $\underline{l}_N(\underline{m})$ in (7.1)]. The whole theory stems from the simple examples considered below.

Clearly $V^{(k),1}(\varphi^{(\leq k)})$ will always admit a limit as $N\to\infty$, being N independent, because of property (4.2) of $\mathscr{J}$.

Therefore, the requirement of existence of the limit (7.8) can put nontrivial restrictions only on $V_{2,N}, V_{3,N},\ldots$, and one can start by looking at the conditions on $V_{2,N}$ [i.e., on $\underline{l}_N(\underline{m})$ with $|\underline{m}|=2$] imposed by the requirement of existence of the limit (7.8) for $p=2$: it should be clear that if the theory is renormalizable it must be possible to fix $V_{2,N}$ so that $\overline{V}^{(k),2}$ exists, simply because $V_{3,N}, V_{4,N},\ldots,$ do not contribute to $V^{(k),2}$.

Clearly $V^{(k),2}$ is determined by the sum of the contributions of the second-order trees, i.e., graphically



(9)

where the summation over the α indices and the integration over the ξ indices is understood.

In formulas, diagram 9 becomes

$$\mathscr{E}_{k+1}\cdots\mathscr{E}_N(V_{2,N})+\frac{1}{2!}\sum_{h>k}^{N}\mathscr{E}_{k+1}\cdots\mathscr{E}_{h-1}\mathscr{E}_h^T(\mathscr{E}_{>h}(V_1),\mathscr{E}_{>h}(V_1))\,, \tag{7.9}$$

where $\mathscr{E}_{>h}=\mathscr{E}_{h+1}\cdots\mathscr{E}_N$ and $\mathscr{E}^T(x_1,\ldots,x_q)=\mathscr{E}^T(x_1,\ldots,x_q;1,\ldots,1)$.

Two cases arise: (1) the second term in (7.9) converges to a limit as $N\to\infty$ [for $\varphi^{(\leq k)}$'s satisfying the smoothness mentioned above, see (3.20)]; or, (2) this does not happen.

In the first case one can take $V_{2,N}=0$ without affecting the finiteness of the theory to second order.

In the second case one must choose $V_{2,N}$ conveniently, if possible at all, to compensate the divergence present in the second term.

Since $V_{2,N}(\varphi^{(\leq N)})$ *will always have to be* in the interaction space $\mathscr{J}_N$, the divergence of the second term in (7.9) can be compensated by a suitably chosen $V_{2,N}$ only if such divergence arises because the second term in (7.9) has some very large component on $\mathscr{J}_k$.

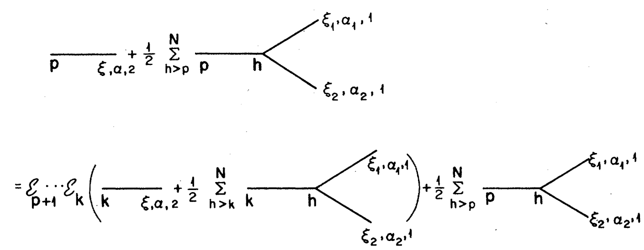
It is, however, unclear how to define, in an abstract context, the component to be considered: for the time being, and to remain on very general grounds, one can just say that there should be an operation $\mathscr{L}_k$ with range in $\mathscr{J}_k$ such that the two expressions

$$\frac{1}{2!}\sum_{h>k}(1-\mathscr{L}_k)\mathscr{E}_{k+1}\cdots\mathscr{E}_{h-1}\mathscr{E}_h^T(\mathscr{E}_{>h}(V_1),\mathscr{E}_{>h}(V_1))\,, \tag{7.10}$$

$$\mathscr{E}_{k+1}\cdots\mathscr{E}_N(V_{2,N})+\frac{1}{2!}\sum_{h>k}^{N}\mathscr{L}_k\mathscr{E}_{k+1}\cdots\mathscr{E}_{h-1}\times\mathscr{E}_h^T(\mathscr{E}_{>h}(V_1),\mathscr{E}_{>h}(V_1))\,, \tag{7.11}$$

are convergent as $N\to\infty$, if $\varphi^{(\leq k)}$ is smooth [in the sense of (3.20)].

If such an $\mathscr{L}_k$ exists for each k it is clear that it must depend on k in a special way, because one can compute, for $p<k$, the effective potential $V^{(p),2}$ in two necessarily equivalent ways, as graphical relation of diagram 10 explains (summations over ξ,α indices understood):



(10)

where the right-hand side (rhs) is obtained by integrating (to second order in $\underline{\lambda}$) the exponential of the expression in (7.9), using (5.13).

Since the convergence of (7.10) and (7.11) should imply convergence of both sides of the equation in diagram 10 for fixed k,p, $k>p$, one finds after a brief calculation that

$$\frac{1}{2!}\sum_{h>k}^{N}(\mathscr{L}_p\mathscr{E}_{p+1}\cdots\mathscr{E}_k-\mathscr{E}_{p+1}\cdots\mathscr{E}_k\mathscr{L}_k)\times\mathscr{E}_{k+1}\cdots\mathscr{E}_{h-1}\mathscr{E}_h^T(\mathscr{E}_{>h}(V_1),\mathscr{E}_{>h}(V_1)) \tag{7.12}$$

should admit a limit as $N\to\infty$.

A simple way to enforce such a property is, of course, to require that for all $p<k$

$$\mathscr{L}_p\mathscr{E}_{p+1}\cdots\mathscr{E}_k=\mathscr{E}_{p+1}\cdots\mathscr{E}_k\mathscr{L}_k\,. \tag{7.13}$$

This leads to the conclusion that one would like $\mathscr{L}_k$ to be defined so that (7.13) holds. Then, proceeding heuristically, note that the limits of (7.11) as $N\to\infty$ exist for all fixed k if they exist for just one k, as the above argument implies. One can thus determine $V_{2,N}$ and $\mathscr{L}_k$ by imposing the existence of the limit as $N\to\infty$ of (7.11) for $k=-1$ and, at the same time, imposing the requirement that $\mathscr{L}_k$ make (7.10) convergent as $N\to\infty$.

For instance, one can require that

$$(\mathscr{E}_0\cdots\mathscr{E}_N V_{2,N})(\varphi^{(-1)})+\frac{1}{2!}\left[\sum_{h=0}^{N}[\mathscr{L}_{-1}\mathscr{E}_0\cdots\mathscr{E}_{h-1}\mathscr{E}_h^T(\mathscr{E}_{>h}(V_1),\mathscr{E}_{>h}(V_1))]\right](\varphi^{(-1)})=0 . \tag{7.14}$$

To continue in great generality, suppose that there is a way of defining the operation $\mathscr{L}_k$ verifying (7.13) and (7.14) and making (7.10) convergent: its existence, or nonexistence, will turn out to be a very easy question in the concrete models to be examined later. Once such a sequence of operations $\mathscr{L}_k$ is found one can produce a new sequence with the same property by setting

$$\mathscr{L}'_kF=\mathscr{L}_kF+\sum_{\alpha=1}^{t}\left[\sum_{|\underline{m}|=2}\tilde{l}^{(\alpha)}(\underline{m})\underline{\lambda}^{\underline{m}}\right]\times\int v_k^{(\alpha)}(\varphi_\xi^{(\le k)}\underline{\partial}\varphi_\xi^{(\le k)})d\xi , \tag{7.15}$$

where the coefficients $\tilde{l}^{(\alpha)}(\underline{m})$ are arbitrarily chosen; note that $\mathscr{L}'_k$ verifies (7.13), if $\mathscr{L}_k$ does, because of (4.2).

The remarkable and interesting fact to be pointed out now is that if the initial interaction V_1 is changed to $V_1+V_{2,N}$, there are very simple graphical rules that allow one to compute the effective interaction generated by $V=V_1+V_{2,N}$ to any order in terms of new types of trees: "partially dressed trees."

The idea of defining such trees comes from computing $V_{2,N}$ defined by (7.14), thought of as a (trivial) linear equation for the t coefficients [see (7.4)]

$$\mu_2^{(\alpha)}=\sum_{|\underline{m}|=2}l_N^{(\alpha)}(\underline{m})\underline{\lambda}^{\underline{m}}$$

in $V_{2,N}(\varphi^{-1})$.

Using (4.2) and (7.13) we find (7.14) becomes

$$\begin{aligned}0&=\sum_{\alpha=1}^{t}\left[\sum_{|\underline{m}|=2}l_N^{(\alpha)}(\underline{m})\underline{\lambda}^{\underline{m}}\right]\int v_{-1}^{(\alpha)}(\varphi_\xi^{(-1)},\underline{\partial}\varphi_\xi^{(-1)})d\xi+\frac{1}{2!}\sum_{h>-1}^{N}\mathscr{L}_{-1}\mathscr{E}_0\cdots\mathscr{E}_{h-1}\mathscr{E}_h^T(\mathscr{E}_{>h}(V_L),\mathscr{E}_{>h}(V_1))\\&=\sum_{\alpha=1}^{t}\left[\sum_{|\underline{m}|=2}l_N^{(\alpha)}(\underline{m})\underline{\lambda}^{\underline{m}}\right]\int v_{-1}^{(\alpha)}(\varphi_\xi^{(-1)},\underline{\partial}\varphi_\xi^{(-1)})d\xi+\frac{1}{2!}\sum_{h=0}^{k}\mathscr{L}_{-1}\mathscr{E}_0\cdots\mathscr{E}_{h-1}(\mathscr{E}_{>h}(V_1),\mathscr{E}_{>h}(V_1))\\&\quad+\frac{1}{2!}\sum_{h>k}^{N}\mathscr{E}_0\cdots\mathscr{E}_k\mathscr{L}_k\mathscr{E}_{k+1}\cdots\mathscr{E}_{h-1}\mathscr{E}_h^T(\mathscr{E}_{>h}(V_1),\mathscr{E}_{>h}(V_1)) .\end{aligned} \tag{7.16}$$

By (7.14) the second term in the rhs is $-\mathscr{E}_0\cdots\mathscr{E}_k(V_{2,k})$ which implies, using again (4.2) and paying some attention to the following important identity, crucially dependent on the definitions

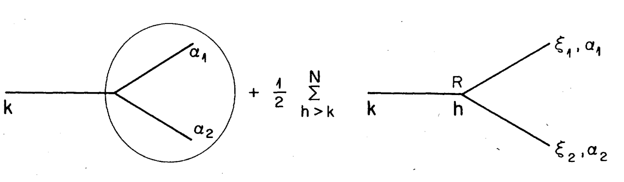
$$\begin{aligned}\mathscr{E}_{k+1}\cdots\mathscr{E}_NV_{2,N}&\equiv\sum_{\alpha=1}^{t}\left[\sum_{|\underline{m}|=2}l_N^{(\alpha)}(\underline{m})\right]\int v_k^{(\alpha)}(\varphi_\xi^{(<k)},\underline{\partial}\varphi_\xi^{(<k)})d\xi\\&\equiv V_{2,k}-\frac{1}{2!}\sum_{h>k}^{N}\mathscr{L}_k\mathscr{E}_{k+1}\cdots\mathscr{E}_{h-1}\mathscr{E}_h^T(\mathscr{E}_{>h}(V_1),\mathscr{E}_{>h}(V_1)) .\end{aligned} \tag{7.17}$$

Then, inserting (7.17) in (7.9), one finds Eq. (7.9) equal to

$$V_{2,k}+\frac{1}{2!}\sum_{h>k}[(1-\mathscr{L}_k)\mathscr{E}_{k+1}\cdots\mathscr{E}_{h-1}\mathscr{E}_h^T\times(\mathscr{E}_{>h}(V_1),\mathscr{E}_{>h}(V_1))] , \tag{7.18}$$

with $V_{2,k}$ defined by (7.14), with k replacing N.

Relation (7.18) together with the graphical representation in diagram 9 suggests the representation of (7.18) as



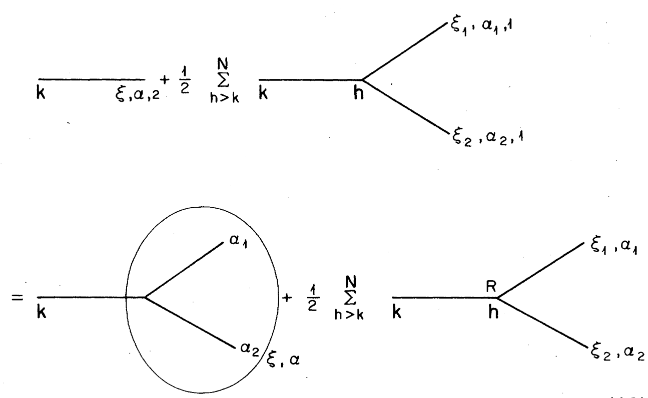
(11)

where the summation over the ξ,α indices is understood, the first graph represents symbolically

$$\left[\sum_{|\underline{m}|=2}l_k^{(\alpha)}(\underline{m})\underline{\lambda}^{\underline{m}}\right]v_k^{(\alpha)}(\varphi_\xi^{(\le k)},\underline{\partial}\varphi_\xi^{(\le k)}) , \tag{7.19}$$

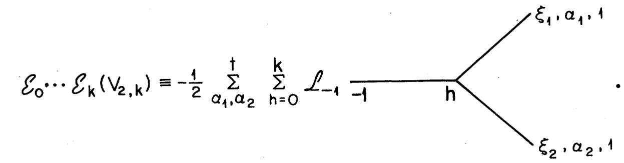
and the second graph represents symbolically the second term of (7.18).

Relation (7.18) becomes the graphical identity



(12)

where in the rhs two new indices α_1,α_2 appear inside the frame, reminding us that $V_{2,k}$ is naturally defined as a sum of t^2 terms indexed by α_1,α_2 and the symbols in diagram 13 allow us to identify them:

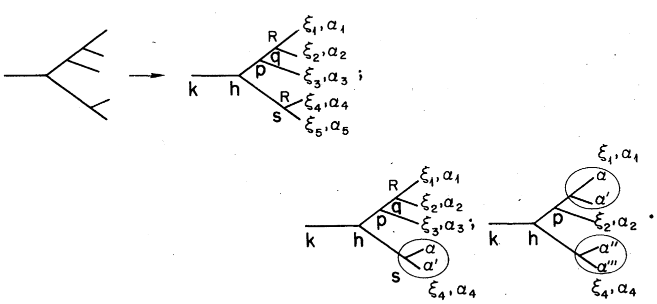


(13)

To explain how to compute the higher-order effective potentials generated by $V=V_1+V_{2,N}$ the identity in diagram 12 above suggests introducing the notion of decorated trees "dressed to second order."

These are objects constructed from an ordinary tree with no appended indices: first, one considers all the vertices out of which bifurcate exactly two branches ending in end points and one either appends a decorating index R or encloses the vertex together with the branches emerging from it into a frame; second, to each framed end point one appends an index $\alpha=1,\ldots,t$; furthermore, to each frame and to each unframed end point a pair $\xi\in R^d$ and $\alpha=1,\ldots,t$ is appended; finally, all the unframed vertices receive a frequency index h_v.

For instance, diagram 14 illustrates three trees dressed to second order:

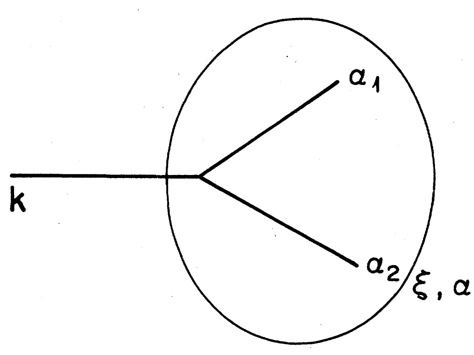


(14)

Given a partially dressed tree γ, dressed to second order, the $V(\gamma)$ will be defined so that

$$V^{(k)}=\int d\underline{\xi}\sum_{\underline{\alpha}}\sum_{\gamma:k(\gamma)=k}\frac{V(\gamma)}{n(\gamma)}\,, \tag{7.20}$$

where the rule to compute $V(\gamma)$ is simply the same as the one used so far, except that the final lines of the form



have to be interpreted as representing the contributions to $V_{2,k}$ described in connection with diagram 13.

Also, the R over a vertex has to be interpreted as saying that the rule to combine the two V_1's in the computation of the truncated expectations $\mathscr{E}_h^T(\mathscr{E}_{>h}(V_1),\mathscr{E}_{>h}(V_1))$ has to be modified and produces, instead, the term in square brackets of (7.18).

The factor $n(\gamma)$ in (7.20) is now defined by defining it as identical to the combinatorial factor $n(\tilde{\gamma})$ of the tree $\tilde{\gamma}$ obtained from γ by stripping it of all its frames and their contents as well as of all its α decorations.

The above discussion is rather long, although conceptually simple; however, it has the advantage of suggesting the procedure for construction of the higher-order counterterms and for describing the results of their presence in the effective potentials.

VIII. COUNTERTERMS, EFFECTIVE INTERACTION, AND RENORMALIZATION IN A GRAPHICAL REPRESENTATION (ARBITRARY ORDER)

The discussion of the preceding section can be extended naturally to provide an algorithm to build $V_{3,N},V_{4,N},\ldots$, i.e., the formal series (7.1).

Again, if the reader finds the discussion below too abstract for a first reading, he can compare the abstract steps described here with the corresponding ones worked out in Sec. XVIII for the φ^4 model.

The basic objects are the "dressed trees" and the "trees dressed to order p."

A tree dressed to order p will be an object obtained by considering a tree with no labels appended on it.

(1) To each end point append an index $\alpha\in(1,\ldots,t)$.

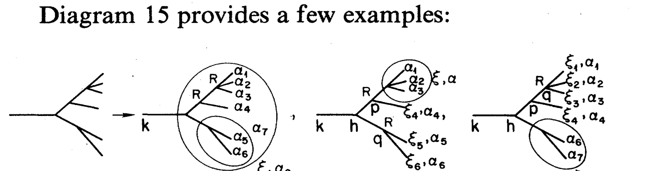
(2) To each vertex different from the root and of order $\le p$ (i.e., followed eventually, though not necessarily immediately, by $\le p$ end points) append an index R or, alternatively, enclose it in a frame together with the part of the tree following it, excluding the preceding vertices.

(3) Append to each frame an index $\alpha\in(1,\ldots,t)$.

(4) Append to each outer frame (note that, in fact, some frames may be inside others) and to each unframed end point an index $\xi\in R^d$.

(5) Append to the unframed vertices a frequency index, increasing along the tree.

Diagram 15 provides a few examples:



(15)

where the first is a tree dressed to order 6, the second to order 4 (or 5), the third to order 3. The above notion is the obvious extension to $p\ge 3$ of the $p=2$ case met in Sec. VII.

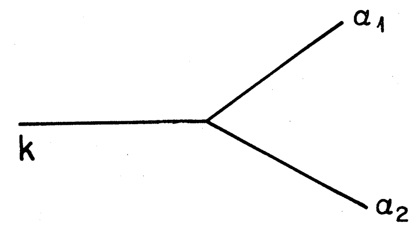
To each partially dressed tree γ one associates a function $V(\gamma)$ so that

$$V^{(k)}=\int \sum_{\underline{\alpha}} d\underline{\xi} \sum_{\substack{\gamma:k(\gamma)=k \\ \gamma \text{ dressed to order } p}} \frac{V(\gamma)}{n(\gamma)} \tag{8.1}$$

would be the effective potential for $\varphi^{(\leq k)}$ obtained by starting from $V_1+V_{2,N}+V_{3,N}+\cdots+V_{p,N}$.

The definition of $V_{3,N},\ldots,$ is inductive and so built that the last statement is true. Having already constructed $V_{2,N}$ in Sec. VII, one has to explain how $V_{p+1,N}$ is obtained from $V_1,V_{2,N},\ldots,V_{p,N}$ so that (8.1) holds once $V(\gamma)$ is appropriately defined.

Call γ_0 a shape of a degree-two tree



and call $\mathscr{L}_k^{(\gamma_0)}$ the sequence of operations introduced in Sec. VII and called there simply $\mathscr{L}_k$.

In general, one looks for a sequence $\mathscr{L}_k^{(\sigma)}$ of operations indexed by the shapes σ of the trees dressed up to order $p=$(degree of $\sigma-1$) (a "shape" of a tree γ dressed to order p is the tree obtained by stripping γ of all the frequency labels as well as of the ξ labels, leaving only the frames, the R labels, and the α labels). The operation $\mathscr{L}_k^{(\sigma)}$ is meant to define the divergent contribution to the effective potential due to the trees with shape σ and arbitrary frequency labels.

The operation $\mathscr{L}_k^{(\sigma)}$ will be subject to the following requirements (see Sec. VII).

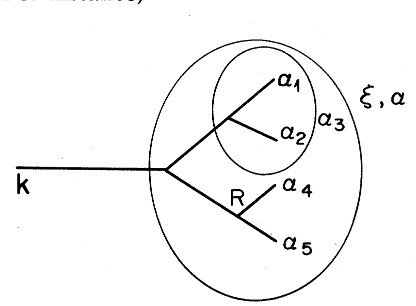
(i) $\mathscr{L}_k^{(\sigma)}$ acts on certain functions of the field $\varphi^{(\leq k)}$ and has range in the interaction space $\mathscr{I}_k$. Also, if F is in the domain of $\mathscr{L}_k^{(\sigma)}$, then $\mathscr{C}_{q+1}\cdots\mathscr{C}_k F$ is in the domain of $\mathscr{L}_q^{(\sigma)}$, for $q<k$.

(ii) The following extension of (7.13) holds:

$$\mathscr{L}_k^{(\sigma)}\mathscr{C}_k\cdots\mathscr{C}_h\equiv\mathscr{C}_k\cdots\mathscr{C}_h\mathscr{L}_h^{(\sigma)} . \tag{8.2}$$

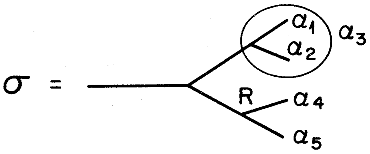
To evaluate the function $V(\gamma)$ associated with a partially dressed tree one will have to interpret a branch of the tree emerging from a vertex with frequency label k and ending in a frame containing a shape σ and carrying frame labels ξ,α [see conditions (3) and (4) above] as representing a function which, if integrated over ξ, is in $\mathscr{I}_k$.

For instance,



(16)

encloses a shape



and counts in the evaluation of $V(\gamma)$ as

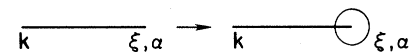
$$l_{N,\sigma}^{(\alpha)}(k)\underline{\lambda}^{\sigma}v_k^{(\alpha)}(\varphi_{\xi}^{(\leq k)},\underline{\partial}\varphi_{\xi}^{(\leq k)}) , \tag{8.3}$$

where $l_{N,\sigma}^{(\alpha)}(k)$ are certain coefficients, "form factors of shape σ," to be defined later and $\underline{\lambda}^{\sigma}=\prod\lambda^{(\tilde{\alpha})}$, with the product ranging over the indices $\tilde{\alpha}$ appended to the end points of σ (in diagram 16 they are $\alpha_1,\alpha_2,\alpha_4,\alpha_5$).

In other words, once the coefficients in (8.3) are defined and the meaning of R is specified, the meaning of $V(\gamma)$ is essentially the same as would be attributed to a decorated tree (with decorations which are more complicated, as they can be framed shapes of trees).

With the meaning of the framed shapes explained, it remains to explain the meaning of the R superscripts and the rule to determine the coefficients $l_{N,\sigma}^{(\alpha)}(k)$ in (8.3).

For uniformity of notation it is convenient, in this section, to consider the unframed end points of a partially dressed tree as framed end points containing the trivial shape, i.e.,



(17)

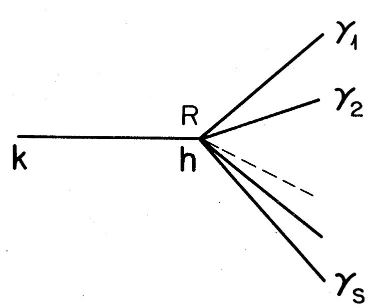
In this way a partially dressed tree γ can be regarded as always ending in "endframes" containing tree shapes; the name end point will be reserved for the end points of the tree obtained from γ by deleting all its frame.

The "bare degree" $\delta(\gamma)$ of γ is the number of end points of γ, while the "dressed degree" of γ will be the number of external endframes.

The meaning of the R superscripts, as well as the construction of the coefficients $l_{N,\sigma}^{(\alpha)}(k)$ and of the counterterms $V_{p,N}$, is described in terms of the operations $\mathscr{L}_k^{(\sigma)}$.

The definition of $l_{N,\sigma}(k)$ and of $\mathscr{L}_k^{(\sigma)}$ is inductive on the (bare) degree of σ. Since these objects have been already defined for σ of degree two, suppose that they have been defined also for arbitrary shapes σ of degree $\leq p$.

Let γ be a tree dressed to order $p+1$ and with degree $p+1$. Suppose that its first nontrivial vertex v_0 carries a superscript R and is the origin of an s-fold bifurcation into s dressed trees; suppose that the frequency index of v_0 is h; the situation is described in diagram 18:



(18)

Then if σ is the shape of γ one interprets diagram 17 as representing

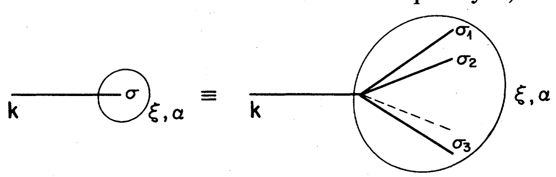
$$V(\gamma)=\mathcal{E}_{k+1}\cdots\mathcal{E}_{h-1}[(1-\mathcal{L}_{h-1}^{(\sigma)})] \times\mathcal{E}_h^T(V(\gamma_1),\ldots,V(\gamma_s)) , \tag{8.4}$$

where $\mathcal{L}_h^{(\sigma)}$ has to be defined verifying requirements (i) and (ii) above and so that the summation of (8.4) over all the trees γ with the same shape σ and root frequency k is ultraviolet finite (i.e., has a limit when $N\to\infty$ if $\varphi^{(\leq k)}$ is smooth).

In the present general context one cannot discuss the existence or nonexistence of such $\mathcal{L}_h^{(\sigma)}$, although in each model considered in the following sections this will be a very easy problem; here, to continue, assume that at least one such $\mathcal{L}_h^{(\sigma)}$ exists. Of course, as already remarked, the operation $\mathcal{L}_h^{(\sigma)}$, which basically isolates the "divergent" part of (8.4), will not be uniquely determined, if existing at all.

This completes the definition of the meaning of the R superscripts.

The next step is to define the coefficients $l_{N,\sigma}(k)$ in (8.3). This will be done via the following prescription. Consider the tree shape σ of degree $p+1$ dressed to order p and enclosed in a frame attached at frequency k;

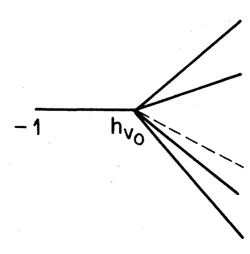


(19)

and assume that σ bifurcates at its first framed vertex v_0 into s completely dressed shapes $\sigma_1,\ldots,\sigma_s$ (a "completely dressed" tree is one dressed to an order equal at least to its degree).

As said above, the framed shape in diagram 18 represents a function of the field $\varphi^{(\leq k)}$ of the form (8.3); to define it, the procedure of the preceding section is followed, in a rather natural sense, as described below.

Delete the outer frame enclosing σ and insert frequency indices at all the unframed vertices of σ as well as pairs ξ,α at all the new external outer frames (formerly next to the outer frame); the root of σ is given frequency -1 and the indices (ξ,α) attached to the deleted frame are also deleted (compare diagram 20 below with diagram 13):



(20)

(no R superscript is above v_0 because σ was supposed dressed only to order p and of degree $p+1$).

Since one is supposing inductively that $V(\gamma_1),\ldots,V(\gamma_s)$ are already defined, one can evaluate the tree in diagram 20 above giving the usual interpretation of truncated expectation to the vertex v_0, which carries no R superscripts. Then one can define the coefficient $l_{N,\sigma}(k)$ in (8.3) in terms of the value of the tree in diagram 20 (see also diagram 13):

$$\sum_{\alpha=1}^{t}\int \underline{\lambda}^\sigma l_{N,\sigma}(k)\,v_{-1}^{(\alpha)}(\varphi_\xi^{(-1)},\underline{\partial}\varphi_\xi^{(-1)})\,d\xi=\int d\underline{\xi}\sum_{h_{v_0}=0}^{k}\sum_{\mathbf{h}}\frac{1}{n(\gamma)}\mathcal{L}_{-1}^{(\sigma)}[\mathcal{E}_0\cdots\mathcal{E}_{h_{v_0}-1}\mathcal{E}^T_{h_{v_0}}(V(\gamma_1),\ldots,V(\gamma_s))] , \tag{8.5}$$

where the second sum runs over the frequency assignments to the other vertices of the tree γ.

Finally, again in analogy with the second-order case, the "counterterms" $V_{p,N}$ of order p will be a sum of contributions $V_{p,N,\sigma}$ each coming from a tree shape σ of degree p:

$$V_{p,N,\sigma}=\int \underline{\lambda}^\sigma l_{N,\sigma}(N)\,v_N^{(\alpha)}(\varphi_\xi^{(\leq N)},\underline{\partial}\varphi_\xi^{(\leq N)})\,d\xi , \tag{8.6}$$

where $\underline{\lambda}^\sigma$ has the meaning described after (8.3).

Proceeding exactly as in Sec. VII, one proves that using the above rules to interpret diagrams 15 and 19 one determines, via (8.1), the effective potential corresponding to $V_1+V_{2,N}+\cdots+V_{p,N}$ simply by interpreting a partially dressed tree of arbitrary degree as computed using the above rules starting from the highest vertices and interpreting the lower vertices with no R superscripts as simply representing the truncated expectation of the functions defined by the s-ple of trees branching out of a vertex. The proof is by induction, once more, and it is left to the reader with the warning that the definitions above have been conceived with the aim of making possible this inductive proof.

It remains to define $\mathcal{L}_k^{(\sigma)}$ in a more concrete way in each model.

As already remarked elsewhere, the ambiguity in the coefficients of the counterterms [and therefore in the definition of the operations $\mathcal{L}_k^{(\sigma)}$ of identification of the divergent parts] has its deep origin in the trivial fact that if

$$\underline{\lambda}=\underline{\lambda}'+\underline{L}(\underline{\lambda}') , \tag{8.7}$$

and $\underline{L}$ is analytic near the origin with a second-order zero, then inserting (8.7) into (7.1) and rearranging that formal power series in $\underline{\lambda}$ into a formal power series in $\underline{\lambda}'$ one necessarily obtains another power series which will enjoy the same properties as the former one as far as the stability as $N\to\infty$ is concerned.

This situation is very much reminiscent of the state of perturbation theory in classical mechanics where there are formal power series for various objects, which are ambiguously defined for trivial reasons and even "divergent" and which can be "renormalized" by suitable prescriptions (Gallavotti, 1983b).[1]

An interaction $\mathscr{J}$ for which the operations $\mathscr{L}_k^{(\sigma)}$ can be taken identically zero for trees of bare degree $>p_0$ will be called "super-renormalizable."

The basic idea of the above construction of the counterterms is from Zimmerman (1969), who introduces the notation of "forest" (here called tree); however, here the notion of tree is independent of the notion of Feynman graph, not yet introduced, while in the literature the forests are always associated with given Feynman graphs. It seems conceptually simplifying and practically advantageous to be able to introduce the notion of forest without any reference to Feynman graphs.

That perturbation theory can be perhaps done in a neater way by avoiding as much as possible the use of Feynman diagrams has been clearly pointed out by Polchinskii (1983), who presents a method quite similar to the one introduced here to deal with perturbation theory using multiscale properties and effective potentials working in the momentum space (here configuration space is used, instead). The method outlined here has been used in various super-renormalizable cases already in Benfatto *et al.* (1978), Gallavotti (1978,1979), Benfatto, Cassandro *et al.* (1980), and Benfatto, Gallavotti, and Nicoló (1980). In the latter papers, however, the super-renormalizability masks the power of the method [which becomes clearer in Benfatto *et al.* and in Nicoló (1982,1983), even though the theories treated are still super-renormalizable].

IX. RESUMMATIONS: FORM FACTORS AND BETA FUNCTION

Before starting the real work, i.e., the analysis of concrete models, there are still quite a few remarkable abstract considerations that can be made.

If an interaction $\mathscr{J}$ is super-renormalizable, the renormalization leads only to a slightly more complex structure of the trees (which have to be dressed up to a finite order p_0 if the subtraction operators are chosen to be zero when the degree of the trees is larger than the convergence threshold p_0—see Sec. VIII) and there is little to discuss about them.

But if $\mathscr{J}$ is only renormalizable or even if it is super-renormalizable and one chooses to define $\mathscr{L}_k^{(\sigma)}$ to be nonzero for σ's of large degree, i.e., if one "oversubtracts," the graphical representation of $V^{(k)}$ is enormously more complex and one wishes to simplify it as much as possible by collecting together as many terms as possible without losing control of what may be going on.

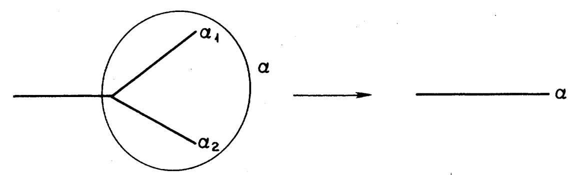
The trouble is that one would naturally like to collect together infinitely many trees but, as will become sadly clear, there are no chances that the resulting series will converge in a naive sense. Nevertheless it is possible to devise a simple "summation rule" permitting us to give a meaning to important resummations.

A concrete example on the abstract and general discussion below is in Sec. XX, where the reader who finds too abstract, on first reading, the contents of this section can see the same ideas worked out concretely in the case of φ^4.

The idea leading to such developments can be best illustrated via an example in which it is even rigorous: the well-known "resummation of the leading divergences"—see Landau (1955), Landau and Pomeramchuk (1955), 't Hooft (1982,1984), and Rivasseau (1984).

One defines a "pruning operation" on the dressed trees, consisting of isolating the final bifurcations of a tree γ which have the form of diagram 21, called a "most-divergent branch" or a "most-divergent endframe."

The pruning will just delete the "most divergent endframes" of γ and their contents, as diagram 21 shows,



(21)

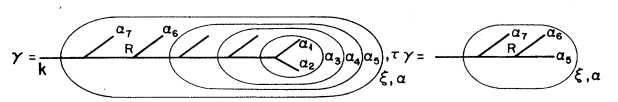
but this will not be all, because after the deletion of the most-divergent endframes of γ new most-divergent endframes may appear in what is left of γ: then the pruning will be pursued until no most divergent branches are left. This defines a "pruning mapping" $\gamma \to \tau\gamma$.

The idea is to define

$$V^R(\gamma)=\sum_{\gamma':\tau\gamma'=\gamma} V(\gamma') \,. \tag{9.1}$$

where γ is a tree with no most-divergent branches (i.e., $\tau\gamma=\gamma$); clearly in (9.1) the sum runs over infinitely many trees (*even* if the ultraviolet cutoff N is finite).

For instance,



(22)

However, it is also clear that the result of the resummation in (9.1), if convergent in any sense, cannot lead to anything other than the conclusion that $V^R(\gamma)$ is evaluated by "slightly modifying" the rules to build $V(\gamma)$: this follows from observing that the sum (9.1) leads to a change in the meaning of the lines reaching the end points with index ξ,α of a pruned tree γ (i.e., such that $\tau\gamma=\gamma$) and representing, according to the usual rules, the function

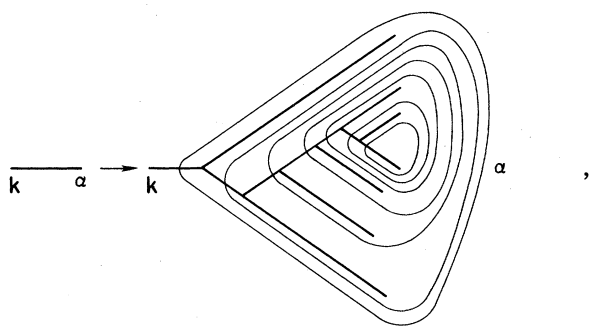
$$\lambda^{(\alpha)} v_k^{(\alpha)}(\varphi_\xi^{(\le k)}, \underline{\partial}\varphi_\xi^{(\le k)}) \,. \tag{9.2}$$

The modification is explained below.

Consider a tree γ which is pruned: $\tau\gamma=\gamma$. Then all

[1]Although the main result of this paper has been previously obtained by Rüssman (1967), the connection with renormalization theory is somewhat new and relevant as a reference here.

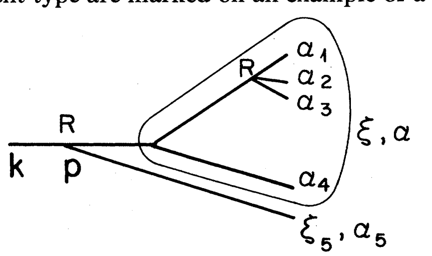
the trees γ' such that $\tau\gamma'=\gamma$ are obtained from γ simply by considering each end point of γ with index α and growing on it a tree of arbitrary size with simple bifurcations in two branches at each vertex and then drawing a frame around every new vertex, as in diagram 23,



(23)

attributing to each vertex and to each new frame indices $\alpha',\alpha'',\ldots$ (not drawn in diagram 23).

An end point of γ can be either "framed," bearing an index α (and no index ξ), or it can be "free," bearing a pair of indices (ξ,α); in diagram 24 end points of different type are marked on an example of a pruned tree:



(24)

They are the end points with labels α_1 (say), and α_5.

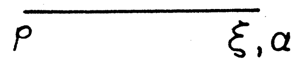
Consider first the case of an end point which is free and attached by a tree branch to a vertex of frequency index p.

We shall now assume, throughout this section and in the sections following Sec. XVI (where applications of the following considerations are presented), that the operations $\mathscr{L}^{(\sigma)}$ depend, for a general σ (not necessarily a most-divergent one), only on what remains of σ after deleting all the frames that it may contain as well as their contents. This property is very convenient and natural, but it has not been assumed since the beginning in order to develop a formalism flexible enough to permit the simultaneous analysis of the super-renormalizable and the just renormalizable cases. What follows here is not relevant for super-renormalizable theories, unless one is interested in studying the effects of "oversubtractions" (as will become clear).

Since a frame with index α attached to a vertex of frequency index p and enclosing a shape σ (whether most divergent, as of interest here, or not) represents, by the general theory of Sec. VIII [see (8.5)], the function

$$\underline{\lambda}^{(\sigma)} l_{N,\sigma}^{(\alpha)}(p) v_p^{(\alpha)}(\varphi_\xi^{(\le p)},\underline{\partial}\varphi_\xi^{(\le p)}) \,. \tag{9.3}$$

It is clear that summing over all the γ', with $\tau\gamma'=\gamma$ and obtained by adding to each free vertex of γ any framed most-divergent shape σ just means interpreting the end-branches which are like the

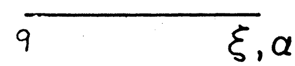


as meaning

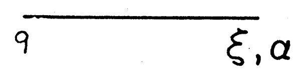
$$\left[\sum_\sigma{}^{*} \lambda^\sigma l_{N,\sigma}^{(\alpha)}(p)\right] v^{(\alpha)}(\varphi_\xi^{(\le p)},\underline{\partial}\varphi_\xi^{(\le p)}) \equiv \lambda^{(\alpha)}(p) v_p^{(\alpha)}(\varphi_\xi^{(\le p)},\underline{\partial}\varphi_\xi^{(\le p)}) \,, \tag{9.4}$$

where the sum runs over all the shapes σ that can be attached to the end point and are most divergent.

Similarly consider a framed end point of γ with index α (like the end point with index α_1 in diagram 24. In this case the addition of a most-divergent tree shape enclosed in a frame and attached to the considered end point just modifies the meaning of the frames of γ as follows. Recall that the form factor $\lambda^{\hat\sigma} l_{N,\hat\sigma}(p)$ corresponding to a framed shape $\hat\sigma$ is evaluated (see Sec. VIII, diagram 19) recursively by reducing, eventually, oneself to the evaluation of the function representing the simple trees

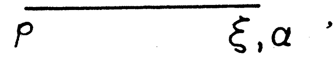


corresponding to the end points of $\hat\sigma$ (once, in the evaluation process, they become unframed) and having the meaning of $\lambda^{(\alpha)} v_q^{(\alpha)}$. If to each end point of $\hat\sigma$ is added a most-divergent framed shape σ and one performs the summation over all possible such σ's, it is clear that one simply gets the same result that would be obtained by interpreting



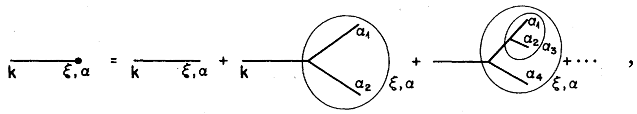
as meaning again (9.4).

In other words, one may consider, in computing the effective potentials, only the trees γ such that $\tau\gamma=\gamma$, provided one interprets the end points of γ attached to a vertex with frequency index p as having the meaning (9.4): this meaning has to be kept, for consistency, even when the end points of γ are inside frames (as in diagram 24). This means that when one computes the form factors for the framed parts of γ and, in doing so, eventually reduces oneself to the case of the tree



one interprets it as meaning (9.4) instead of simply $\lambda^{(\alpha)} v_p^{(\alpha)}(\varphi_\xi^{(\le p)},\underline{\partial}\varphi_\xi^{(\le p)})$.

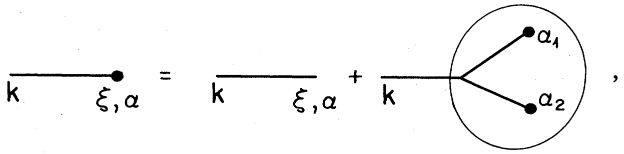
If we give to the end points of a tree $\gamma=\tau\gamma$ the new interpretation and represent this by using heavy dots at the end points of γ, it is clear from the above discussion that the form factors $\lambda^{(\alpha)}(k)$ verify the graphical relation of diagram 25,



(25)

where the left-hand side (lhs) can be taken as a symbolic representation of (9.4) and where the rhs has a summation over the indices $\alpha_1, \alpha_2, \ldots$, understood, in each term.

The equation in diagram 25 can be written pictorially,



(26)

which is, actually, a simple recursive relation for the form factors $\underline{\lambda}(k)$: its iterative solution leads to expressing $\underline{\lambda}(k)$ as a power series in $\underline{\lambda}$. This power series, once substituted into the $V^R(\gamma)$, defined, as explained above, interpreting the end points of γ as bearing a heavy dot and as meaning that in the evaluation of $V(\gamma)$ a line

has to be interpreted as in (9.4), yields the representation

$$V^{(k)} = \sum{}^{*} \frac{V^R(\gamma)}{n(\gamma)}\,, \tag{9.5}$$

where the sum runs over the trees $\gamma = \tau\gamma$ (i.e., over the pruned trees) only.

On the other hand, it might happen that the relation in diagram 26 thought of as an equation for the form factors admits true solutions, not just formal solutions in the form of power series generated by iterating it: then one can use this solution to define the "summation rule" that the sum (9.1) is by definition the expression $V^R(\gamma)$ computed with the same rules as $V(\gamma)$ but interpreting the end points as bearing heavy dots, which means that they must be interpreted as in the rhs of (9.4), with $\underline{\lambda}(k)$ defined by the given solution of the equation represented by diagram 26.

In other words, the equation in diagram 26 has two different well-defined possible uses. One is to generate by iteration the various terms graphically represented in diagram 25 [i.e., the formal power series for the form factors $\underline{\lambda}(k)$ in (9.4)]. The other is to provide a nonperturbative meaning to the sum of the series in diagram 25 for the form factors, i.e., a summation rule for the most-divergent graphs. The first use is also quite interesting, being equivalent to the direct definition of the various trees in diagram 25 described in Sec. VIII; this is, manifestly, a conceptually simpler way to build the coefficients $\lambda^{(\alpha)}(k)$, although, as is clear from the principle of conservation of difficulties, this does not really save any work if one wishes to perform a real calculation (the point being, as will be explicitly illustrated in the case of the models considered later, that diagram 26 can be converted into an analytic equation only at the price of doing all the calculations necessary to evaluate the trees in diagram 25, i.e., the formal power series for the form factors).

Following the rules of Secs. VII and VIII for the evaluation of the coefficients of the element of $\mathscr{I}_k$ associated with a frame (see diagram 19), one gets, for $0 \le k \le N$

$$\lambda^{(\alpha)}(k) = \lambda^{(\alpha)} + \sum_{h=0}^{k} \sum_{\alpha_1,\alpha_2=1}^{t} B^{(\alpha)}_{\alpha_1\alpha_2}(h)\lambda^{(\alpha_1)}(h)\lambda^{(\alpha_2)}(h)\,, \tag{9.6}$$

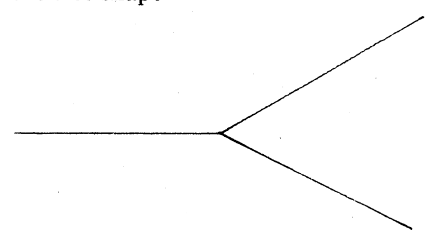
where $B^{(\alpha)}_{\alpha_1\alpha_2}(h)\lambda^{(\alpha_1)}\lambda^{(\alpha_2)}$ is the coefficient of

$$\int_\Lambda v^{(\alpha)}_{-i}(\varphi_\xi^{(-1)}, \underline{\partial}\varphi_\xi^{(-1)})d\xi$$

in

$$-\tfrac{1}{2}\int \mathscr{L}^{(\sigma_0)}_{-1}\mathscr{E}_0 \cdots \mathscr{E}_{h-1}\mathscr{E}^T_h(v_h^{(\alpha_1)}(\varphi_\xi^{(\le h)}, \underline{\partial}\varphi_\xi^{(\le h)})\,, \\ v_h^{(\alpha_2)}(\varphi_\eta^{(\le h)}, \underline{\partial}\varphi_\eta^{(\le h)}))d\xi\, d\eta\,. \tag{9.7}$$

The factor $\frac{1}{2}$ comes from the combinatorial factor associated with the tree shape

$\sigma_0=$  .

The coefficients B are manifestly N independent.

To proceed any further one needs explicit expressions for the B's: the ideal situation arises when the B's have a structure allowing one to conclude that, possibly adjusting the initial values $\lambda^{(\alpha)}$, there is a solution to (9.6) such that

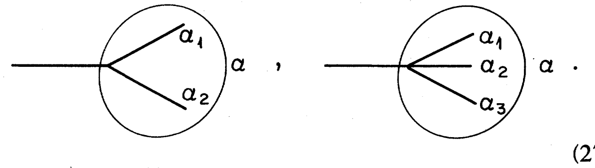
$$\gamma^{-\nu(\alpha)k}\lambda^{(\alpha)}(k) \underset{k\to\infty}{\longrightarrow} 0\,, \tag{9.8}$$

where $\nu(\alpha)$ is some "dimension" suggested by each concrete model (in fact, as will be seen in the models treated later, one can expect to have a k dependence of the form factors which goes exponentially at a rate characteristic of each form factor and, usually, the order-by-order behavior of the perturbative coefficients of the form factors is estimated to be much worse than the *a priori* guessed exponential).

When this is the case, and this depends upon the interaction $\mathscr{I}$, it is clear that the above simple resummation can produce a great gain in the expressions of $V(\gamma)$ and in their estimates, because it may introduce a damping in the contributions from the trees having in them bifurcations at too high frequencies. Furthermore, this damping results as a consequence of summing, by a well-defined summation rule, a series which might be divergent (as in fact happens in the simplest applications described later).

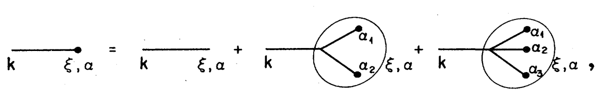
The above method to build resummation rules can be extended to cover more complicated sets of trees by modifying conveniently the pruning operation.

The pruning operation can be extended by prescribing the pruning of a given set of shapes; for instance one can prune all the framed endbranches like



(27)

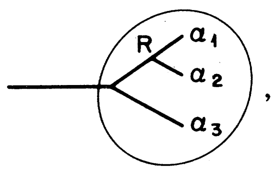
In this case the equation in diagram 26 is modified into



(28)

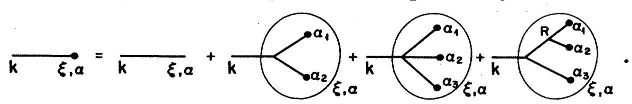
where the summation over the indices $\alpha_1,\alpha_2,\alpha_3$ is understood in each term.

Or if one prunes out also the endbranches like



(29)

then the equation in diagram 28 is replaced by

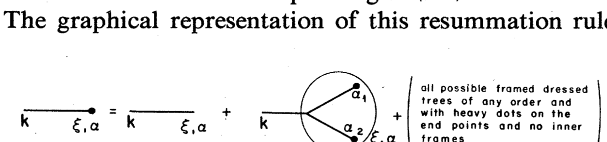


(30)

In the case of the equations in digrams 28 or 30 Eq. (9.6) is replaced by a similar one in which the rhs also contains cubic terms; their coefficients are still N independent. However, the N dependence is implicit through the fact that the frequencies are bound to vary between 0 and N. It should also be clear that, if the cutoff is N, only trees with at most $N+1$ vertices between any two successive frames are possible (and therefore can be considered in the resummations; for instance the resummation in diagram 30 makes sense only if $N\geq 1$, while the other two are meaningful even for $N=0$). The N dependence of the B's will not be explicitly marked except when necessary in Secs. XX and XXII; note that each B becomes N independent for N large enough.

The ultimate greatest resummation can be associated with the "total pruning operation" whereby all the frames are pruned and one is left just with dressed trees *without* frames in the formula corresponding to (9.1).

The graphical representation of this resummation rule is



(31)

where again all the summations over the indices $\alpha_1,\alpha_2,\alpha_3,\dots,$ are understood.

The equation in diagram 31 becomes explicitly, for $k\leq N$,

$$\lambda^{(\alpha)}(k)=\lambda^{(\alpha)}+\sum_{h=0}^{k}\sum_{n=2}^{\infty}\sum_{\substack{h_i\geq h\\ \alpha_1,\dots,\alpha_r}}^{N} B^{(\alpha)}_{\alpha_1\cdots\alpha_r}(h;h_1,\dots,h_r)\times\lambda^{(\alpha_1)}(h_1)\cdots\lambda^{(\alpha_r)}(h_r)\,, \tag{9.9}$$

where the coefficients B must be computed according to the rules of Sec. VIII—see diagram 19—and are expressed as sums of the coefficients of

$$\lambda^{(\alpha_1)}\cdots\lambda^{(\alpha_r)}\int v^{(\alpha)}_{-1}(\varphi^{(-1)}_{\xi},\underline{\partial}\varphi^{(-1)}_{\xi})d\xi$$

in $\mathscr{L}_{-1}V^{(-1)}(\sigma)/n(\sigma)$, σ being one of the trees with r end points in diagram 31 deprived of the first frame and bearing no R superscript on the first vertex v_0 and with frequency indices h appended to v_0 and $h_1,h_2,\dots,h_r$ appended to the vertices out of which emerge the r-final branches of σ.

By the assumption of renormalizability and of existence of the operators $\mathscr{L}_k$ it follows that the coefficients in Eq. (9.9) will be such that if the form factors $\lambda^{(\alpha_i)}(h_i)$ are replaced by constants $\lambda^{(\alpha_i)}$ then the summation at fixed r converges uniformly in N.

The problem is that, as will appear in the concrete case of φ^4, the sum over r is not well controlled.

Introduce the functionals $\mathscr{B}_N,\mathscr{B}$ acting on the space of the sequences $\underline{\underline{\lambda}}$ of functions $\underline{\lambda}^{(\alpha)}(k)$ are defined formally as

$$(\mathscr{B}_N\underline{\underline{\lambda}})^{(\alpha)}(k)=\sum_{h=0}^{k}\sum_{r=2}^{\infty}\sum_{\substack{h_i\geq h\\ \alpha_i=1,\dots,t}}^{N} B^{(\alpha)}_{\alpha_1,\dots,\alpha_r}(h;h_1,\dots,h_r)\times\lambda^{(\alpha_1)}(h_1)\cdots\lambda^{(\alpha_r)}(h_r) \tag{9.10}$$

and

$$(\mathscr{B}\underline{\underline{\lambda}})^{(\alpha)}(k)=\sum_{h=0}^{k}\sum_{r=2}^{\infty}\sum_{\substack{h_i\geq h\\ \alpha_i=1,\dots t}}^{\infty} B^{(\alpha)}_{\alpha_1,\dots,\alpha_r}(h;h_1,\dots,h_r)\times\lambda^{(\alpha_1)}(h_1)\cdots\lambda^{(\alpha_r)}(h_r) \tag{9.11}$$

and rewrite (9.9) as

$$\underline{\lambda}(k)=\underline{\lambda}+(\mathscr{B}_N\underline{\underline{\lambda}})(k),\quad 0\leq k\leq N\,. \tag{9.12}$$

The difference with the preceding "rigorous" resummation schemes is that the rhs of (9.12) does not really make sense other than as a formal power series in $\lambda^{(\alpha)}(h)$, because, as mentioned above, there is no control over the summations over r in (9.10) or (9.11).

Therefore, the only use, already very interesting, of (9.12) is that it can produce, by a formal solution by iteration, a well-defined power series in $\underline{\lambda}$, obtaining a formal power-series expression of the form factors associated with the resummation. Furthermore, as $N\to\infty$ the coefficients of a given order in such power series for the form-factor solutions of (9.11) converge to the corresponding coefficients of the formal power series obtained by iterating

$$\underline{\lambda}(k)=\underline{\lambda}+(\mathscr{B}\underline{\underline{\lambda}})(k) \ . \tag{9.13}$$

The result of the above discussion is quite nontrivial: through the knowledge of all the coefficients $B_{\underline{\alpha}}^{(\alpha)}(h;\underline{h})$ one can compute the form factors $\underline{\lambda}(k)$ to any order desired in the renormalized couplings and then reduce the computation of $V^{(k)}$ to the computation of $V(\gamma)$ for all the trivially dressed trees, i.e., for the trees with only R superscripts on the vertices and no frames at all, provided their end points are interpreted as meaning the rhs of (9.4) with $\underline{\lambda}(k)$ being now a form factor defined by (9.12) to any order in perturbation theory.

In other words, the knowledge of the coefficients in Eqs. (9.12) and (9.13) allows one to reduce the calculation of $V^{(k)}$ to essentially the same calculations that would be necessary in the absence of renormalization. The economy of thought gained in using this approach in computing perturbation theory coefficients is obvious. However, it is worth stressing that, as already remarked, in practice the calculation of the B coefficients is exactly equivalent to the evaluation of the trees with frames; it is perhaps better to regard Eqs. (9.12) and (9.13) as a convenient way to organize the calculations of perturbation theory by separating the "true calculations" (corresponding to the trees with no frames) from the form-factor calculations.

Unfortunately, unlike the simple cases of the "moderate" resummations described by diagrams 26, 28, or 30 or, more generally, involving a finite number of "pruned shapes," no rigorous use of (9.12) and (9.13) can be made to prescribe resummation rules, because no information is available on the nonperturbative meaning to be attached to the rhs of (9.12) and (9.13): one can say only that if on the rhs of (9.12) and (9.13) the second order "dominates," then $\underline{\lambda}(k)$ should behave for $k\to\infty$ in the same way as the $\underline{\lambda}(k)$ that would be obtained from the most divergent resummation [i.e., from Eq. (9.6)].

Many triviality arguments for φ^4 are based on this assumption (domination of the most divergent graphs), and this point will be discussed in more detail in Secs. XIX, XX, and XXII below.

It is customary to write (9.12) and (9.13) as difference equations obtained by "writing them for k and $k+1$ and subtracting"

$$\begin{aligned}&\underline{\lambda}(k+1)=\underline{\lambda}(k)+(B_N\underline{\underline{\lambda}})(k), \quad 0\le k+1\le N \ ,\\ &\underline{\lambda}(k+1)=\underline{\lambda}(k)+(B\underline{\lambda})(k), \quad 0\le k \ ,\end{aligned} \tag{9.14}$$

where $\underline{\lambda}(-1)\equiv\underline{\lambda}$, and

$$\begin{aligned}&(B_N\underline{\lambda})(k)=\sum_{r=2}^{\infty}\sum_{h_i\ge k+1}^{N}\sum_{\alpha_i}B^{(\alpha)}_{\alpha_1,\dots,\alpha_e}(k+1;h_1,\dots,h_r)\lambda^{(\alpha_1)}(h_1)\cdots\lambda^{(\alpha_r)}(h_r), \quad 0\le k+1\le N \ ,\\ &(B\underline{\lambda})(k)=\sum_{r=2}^{\infty}\sum_{h_i\ge k+1}^{\infty}\sum_{\alpha_i}B^{(\alpha)}_{\alpha_1,\dots,\alpha_r}(k+1;h_1,\dots,h_r)\lambda^{(\alpha_1)}(h_1)\cdots\lambda^{(\alpha_r)}(h_r), \quad 0\le k+1\end{aligned} \tag{9.15}$$

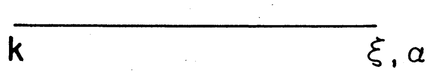
and B is basically the "beta function" [see Callan (1970,1975), Symanzik (1970,1973)], which therefore can be used to simplify conceptually the perturbation theory in the sense explained above.

Usually Eqs. (9.14) are more homogeneous if written for other form factors trivially related to the ones just discussed by

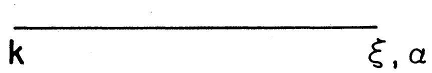
$$\lambda^{(\alpha)}(k)=\gamma^{\nu(\alpha)k}\lambda_k^{(\alpha)} \ , \tag{9.16}$$

where $\nu(\alpha)$ is a suitable dimension (this will become clear in the treatment of the concrete model φ_4^4).

To conclude this section it is useful to point out that the constants $\underline{\lambda}(k)$ verifying the first of (9.14) can be naturally called also "effective coupling constants at frequency γ^k," because they represent the trivial trees



in the same sense in which the renormalized couplings represent the trivial trees



By definition it is true that $\underline{\lambda}(N)$ is precisely the "bare coupling" (7.1): note that it is the formal power series (for $k=N$) generated by the recursive solution of the first of (9.14) starting from the zeroth-order approximation $\underline{\lambda}(k)=\underline{\lambda}$.

It is convenient to label the formal power-series solution of (9.12) and (9.13) [or (9.14)], by the symbols $\underline{\lambda}(k;N)$, $k\le N$, or, respectively, $\underline{\lambda}(k;\infty)$, to avoid any confusion between them.

Clearly the bare couplings are, in this notation, $\underline{\lambda}(N;N)$, and they should not be confused with $\underline{\lambda}(N;\infty)$; note also that, while $\underline{\lambda}(k;\infty)$ can be (and will be in the cases treated later) regularization independent, the form factors $\underline{\lambda}(k,N)$ may be strongly dependent on the regularization used.

The latter statement requires some more detailed explanations, since the use of a different regularization seems to yield results which just are not comparable with the ones coming from another regularization. Therefore, to illustrate the above statement it is convenient to "compare" the results of the Pauli-Villars regularization at a given order n and the corresponding results for a radically different regularization, e.g., the lattice regularization (see Secs. II and III). The comparison of the two approaches can be made by thinking that the lattice free fields are also decomposed into a sum of independent fields associated with a hierarchy of scales γ^k, $k=0,1,2,\ldots$, via the identities (3.4) of order n using

$$\varepsilon(p)^2=\varepsilon_a(p)^2=2\sum_{i=1}^{d}[1-\cos(ap_i)]/a^2$$

rather than $\varepsilon(p)=p^2$.

Then one proceeds, exactly as in the Pauli-Villars case, to study the effective potentials for the fields $\varphi^{(\le k)}$. Their effective potentials will be described by Eq. (9.13) with the B coefficients depending on the cutoff a (here $N=\infty$ from the beginning, because one does not need $N<\infty$ for regularization purposes when one is assuming $a>0$): such coefficients converge to the coefficients of (9.14) as $a\to 0$ term by term, but for $a>0$ they depend on a and for large r [see (9.11)] their dependence on a itself is strong.

It is even conceivable that $\underline{\lambda}(k;\infty)$ could be defined as nonperturbative solution of the second of (9.14) [or (9.13)] while $\underline{\lambda}(k;N)$ could admit interesting nonperturbative solutions only for suitably chosen regularizations [because the terms of (9.12) are regularization dependent on the sense above, while those of (9.13) are not]. This question will be discussed in more detail in Sec. XXII.

X. SCHWINGER FUNCTIONS AND EFFECTIVE POTENTIALS

If f is a smooth test function, one considers the following formal chain of identities:

$$\begin{aligned}
S(f;p)&\equiv\mathcal{E}^T_{\text{int}}(\varphi(f);p)\equiv\frac{\partial^p}{\partial\theta^p}\ln\mathcal{E}_{\text{int}}(e^{\theta\varphi(t)})\,|_{\theta=0}\\
&\equiv\sum_{k_1,\ldots,k_p}\mathcal{E}^T_{\text{int}}(\varphi^{(k_1)}(f),\ldots,\varphi^{(k_p)}(f);1,\ldots,1)\\
&\equiv\sum_{q=1}^{p}\sum_{k_1<\cdots<k_q}^{0,\infty}\sum_{\substack{m_1,\ldots,m_q\\ m_1+\cdots+m_q=p}}\frac{p!}{m_1!\cdots m_q!}\mathcal{E}^T_{\text{int}}(\varphi^{(k_1)}(f),\ldots,\varphi^{(k_q)}(f);m_1,\ldots,m_q)\\
&\equiv\sum_{q=1}^{p}\sum_{k_1<\cdots<k_q}\mathcal{E}^T_{\text{int}}(\varphi^{(k_1)}(f)+\cdots+\varphi^{(k_q)}(f);p)\\
&\equiv\sum_{q=1}^{p}\sum_{k_1<\cdots<k_q}^{0,\infty}\frac{\partial^p}{\partial\theta^p}\ln\left.\frac{\mathcal{E}_{\ge 0}(e^{\theta[\varphi^{(k_1)}(f)+\cdots+\varphi^{(k_q)}(f)]}e^{V})}{\mathcal{E}_{\ge 0}(e^{V})}\right|_{\theta=0}\\
&\equiv\sum_{q=1}^{p}\sum_{k_1<\cdots<k_q}^{0,\infty}\frac{\partial^p}{\partial\theta^p}\ln\left.\frac{\mathcal{E}_0\cdots\mathcal{E}_{k_q}(e^{\theta[\varphi^{(k_1)}(f)+\cdots+\varphi^{(k_q)}(f)]}e^{V^{(k_q)}})}{\mathcal{E}_0\cdots\mathcal{E}_{k_q}(e^{V^{(k_q)}})}\right|_{\theta=0}\\
&\equiv\sum_{q=1}^{p}\sum_{k_1<\cdots<k_q}^{0,\infty}\sum_{s=1}^{\infty}\frac{1}{s!}\mathcal{E}^T_{(0,k_q)}(\varphi^{(k_1)}(f)+\cdots+\varphi^{(k_q)}(f),V^{(k_q)};p,s)\,,
\end{aligned}\tag{10.1}$$

where $\mathcal{E}_{\text{int}}$ is the expectation with respect to the "interaction measure"

$$(\exp V)\prod_{j=1}^{N}P(d\varphi^{(j)})$$

and

$$\mathcal{E}_{(0,k)}\equiv\mathcal{E}_0\mathcal{E}_1\cdots\mathcal{E}_k\,,$$

$$\varphi(f)=\int\varphi_\xi f(\xi)d\xi\,.$$

In some sense, the crucial step in (10.1) is the identity preceding the last, where V is replaced by the effective potential.

The functions $S(f;p)$ are called the truncated Schwinger functions of order p for the interacting measure: they are trivially related to the nontruncated Schwinger functions of Sec. IV. The relevance of (10.1) is to show that the Schwinger functions can be expressed in terms of the effective potentials [and, as can be easily seen from (10.1), at least formally, vice versa].

Even though (10.1) might present convergence problems *a priori* it will be easy to check that, in fact, the rhs of the series in (10.1) will converge, order by order, in perturbation theory in the renormalized constants $\underline{\lambda}$: this will be so in the cases which will be encountered in this paper, provided convergence problems do not arise already in the perturbative definitions of the effective potentials themselves.

Sometimes one wishes to study more complex "observables" like

$$\rho(f)\equiv \int_\Lambda :\cos(\alpha\varphi_\xi):f(\xi)d\xi \tag{10.2}$$

through their average values and the average values of their powers with respect to the interaction measure.

A way to analyze such quantities via the effective potentials technique, which in particular can also be applied to the Schwinger functions, is to include $\rho(f)$ in the interaction potential and to try to show that if $V=V_1+V_{2,N}+\cdots$ yields a well-defined ultraviolet stable effective potential, then so does $\rho(f)+V$.

Examples of how this could be done are provided by the theory of the sine-Gordon interaction.

However, for reasons of space I shall not dedicate much time to questions of the above type.

It is worth stressing that the convergence of the Schwinger functions of a theory with cutoff N to their limit values as $N\to\infty$ need not be pointwise but might take place only in the sense of distributions or even worse, at least if one expresses the results in terms of $S(f;p)$, i.e., of smoothed expressions involving the truncated averages,

$$\mathscr{E}^T_{\rm int}(\varphi_{\xi_1},\ldots,\varphi_{\xi_p};1,\ldots,1)\ . \tag{10.3}$$

It is probably important to avoid putting any specific convergence requirements on how the expectations (10.3) should approach their limits as $N\to\infty$; in the absence of physical reasons to prefer one type of convergence to other types, one should leave this question aside, allowing for any type of convergence which will *a posteriori* be subject to only one constraint, namely, that of leading to a probability measure $P_{\rm int}$ on the space of the fields in a sense suitable to infer the existence of, say, a Wightman field.

XI. THE COSINE INTERACTION MODEL IN TWO DIMENSIONS, PERTURBATION THEORY AND MULTIPOLE EXPANSION

The ideas and methods of the preceding sections can now be applied to the actual theory of the simplest fields. If

$$\varphi^{(\le N)}=\sum_{j=-1}^{N}\varphi^{(j)}$$

denotes a regularized free field as defined in Sec. III via a first-order Pauli-Villars regularization [see (3.3) and (3.7)], consider the interaction $\mathscr{I}_N$:

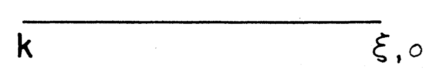
$$V_1(\varphi)=\int_\Lambda\left[\frac{\lambda}{2}\sum_{\sigma=\pm1}:e^{i\sigma\alpha\varphi_\xi^{(\le N)}}:+\nu\right]d\xi$$

$$\equiv\int_\Lambda[\lambda:\cos(\alpha\varphi_\xi^{(\le N)}):+\nu]d\xi\ , \tag{11.1}$$

which will be called the "cosine field" or the "massive sine-Gordon field" with "open boundary conditions": the latter specification refers to the fact that in Secs. XI–XV the field $\varphi^{(k)}$ will be supposed to have covariance given by $\overline{C}^{(k)}$ in (3.7) and not by its periodized version denoted in (3.7) by $C^{(k)}$ ("nonperiodic open boundary conditions"). Nevertheless, to avoid complicating the notations we shall denote simply by $C^{(k)}$ and not by $\overline{C}^{(k)}$ the covariance of $\varphi^{(k)}$, since there is no possibility of confusion, in Secs. XI–XV. We shall also set $\hbar=c=\mu=1$.

It will turn out that the interaction $\mathscr{I}$ in (11.1) is renormalizable (actually, trivially super-renormalizable (in the sense defined at the end of Sec. VIII), for $\alpha^2<4\pi$ and slightly less trivially for $\alpha^2\in[4\pi,8\pi]$).

By the general theory of Sec. VI the effective interaction $V^{(k)}$, as given by (6.8), will be described in terms of trees with end points bearing, besides the position index $\xi\in R^2$, the index $\alpha=+1$, -1, or 0 representing, respectively, the three terms in the intermediate term of (11.1). Since $\alpha=0$ represents a constant and the trees represent truncated expectations, the index $\alpha=0$ can appear only in the trivial tree



The indexes $\alpha=\pm1$ will be denoted σ, and they will be called "charges."

Using the cluster interpretation of the trees (see diagram 7), one can interpret each vertex v of a tree as a cluster and can define the "charge" Q_v of the vertex v as the sum of the indices σ associated with the points in the cluster defined by v.

Given a tree γ, let v be one of its vertices with frequency label h_v, which, if thought of as a cluster, contains the points $\xi_{j_1},\ldots,\xi_{j_s}$ with indices $\sigma_{j_1},\ldots,\sigma_{j_s}$; then one sets

$$\varphi_v^{(\ )}\equiv\sum\sigma_{j_k}\varphi^{(\ )}_{\xi_{j_k}},\quad\text{"cluster field,"}$$

$$Q_v\equiv\sum\sigma_{j_k},\quad\text{"cluster charge."} \tag{11.2}$$

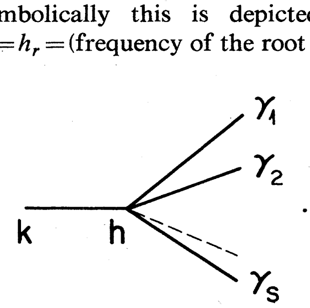
When $v=r=$(root of the tree γ), the φ_v,Q_v will also be denoted $\varphi(\gamma)$and $Q(\gamma)$. Given any $h\ge -1$, it makes sense to consider the fields $\varphi_v^{(\le h)}$ and $\varphi_v^{(h)}$.

To find the rules for the computation of $V(\gamma)$, one proceeds empirically trying to find an appropriate ansatz. After a while, it clearly emerges that a reasonable ansatz is that the contribution to the effective potential of the tree γ is

$$\overline{V}(\gamma):e^{i\alpha\varphi(\gamma)}:\ , \tag{11.3}$$

where $\overline{V}(\gamma)$ is a suitable function of the tree γ.

Let $\gamma_1,\ldots,\gamma_s$ be the s subtrees, with root v_0 equal to the first nontrivial vertex of γ branching out of v_0 in γ; symbolically this is depicted in diagram 32, where $k=h_r=$(frequency of the root of γ) and $h=h_{v_0}$:



(32)

Then, combining (11.3) with the general recursion relation (6.4), one finds the following relation between the various $\overline{V}(\gamma)$, for $k<h$:

$$\overline{V}(\gamma){:}e^{i\varphi^{(\leq k)}(\gamma)}{:}=\overline{V}(\gamma_1)\cdots\overline{V}(\gamma_s)\mathscr{E}_{k+1}\cdots\mathscr{E}_{h-1}\mathscr{E}_h^T({:}e^{i\alpha\varphi^{(\leq h)}(\gamma_1)}{:},\ldots,{:}e^{i\alpha\varphi^{(\leq h)}(\gamma_s)}{:})\ , \tag{11.4}$$

which, using the rules on the Wick monomials [see (C15) and (C16) in Appendix C], yields (note that the term in large square brackets below is φ independent) for $k<h$

$$\overline{V}(\gamma)=\overline{V}(\gamma_1)\cdots\overline{V}(\gamma_s)\left[\left[\prod_{j=1}^{s}{:}e^{i\alpha\varphi^{(\leq h-1)}(\gamma_i)}{:}\right]\Big/({:}e^{i\alpha\varphi^{(\leq h-1)}(\gamma)}{:})\right]\sum_{\tau\in\mathscr{T}^*}\prod_{\lambda\in\tau}(e^{-\alpha^2C_\lambda^{(h)}}-1)\ , \tag{11.5}$$

where $\mathscr{T}^*$ is the set of simple graphs connecting the symbolic objects $\gamma_1,\ldots,\gamma_s$ (i.e., graphs with no repeated bonds and such that for any two γ_i,γ_j there is a path of bonds connecting them), which may be regarded as the clusters of points determined by the vertices $v_1,\ldots,v_s$ following v_0 in γ; furthermore, if $\lambda=(\gamma_i,\gamma_j)$, one means

$$\begin{aligned}C_\lambda^{(h)}&=\mathscr{E}_h(\varphi^{(h)}(\gamma_i),\varphi^{(h)}(\gamma_j))\equiv C_{\gamma_i\gamma_j}^{(h)}\\&\equiv\sum_{\substack{\xi\in\gamma_i\\ \eta\in\gamma_j}}\sigma_\xi\sigma_\eta C_{\xi\eta}^{(h)}\ ,\\ C_\lambda^{(\leq h)}&=\mathscr{E}_0\cdots\mathscr{E}_h(\varphi^{(\leq h)}(\gamma_i),\varphi^{(\leq h)}(\gamma_j))\equiv C_{\gamma_i\gamma_j}^{(\leq h)}\\&\equiv\sum_{\substack{\xi\in\gamma_i\\ \eta\in\gamma_j}}\sigma_\xi\sigma_\eta C_{\xi\eta}^{(\leq h)}\ .\end{aligned} \tag{11.6}$$

Notice that the two relations in (11.6) have the interpretation of electrostatic potential between the charged clusters γ_i and γ_j relative to the electric potential $C_{\xi\eta}^{(\ )}$.

If we use the definition

$${:}e^{i\alpha\sigma\varphi}{:}=e^{(\alpha^2/2)\mathscr{E}(\varphi^2)}e^{i\alpha\sigma\varphi}$$

(see Appendix C), Eq. (11.5) becomes

$$\begin{aligned}\overline{V}(\gamma)=\ &\overline{V}(\gamma_1)\cdots\overline{V}(\gamma_s)\exp\left[-\alpha^2\sum_{i<j}C_{\gamma_i\gamma_j}^{(\leq h-1)}\right]\\&\times\sum_{\tau\in\mathscr{T}^*}\prod_{\lambda\in\tau}(e^{-\alpha^2C_\lambda^{(h)}}-1)\ ,\end{aligned} \tag{11.7}$$

which, considered together with

$$\overline{V}(\gamma)=\frac{\lambda}{2}$$

for $\gamma=$ k ———— ξ,σ ,

$$\overline{V}(\gamma)=\nu \tag{11.8}$$

for $\gamma=$ k ———— $\xi,0$,

provides a recursive definition of $\overline{V}(\gamma)$ and proves ansatz (11.3).

The effective potential then has the form

$$\begin{aligned}V^{(k)}=&\sum_{n=1}^{\infty}\sum_{\sigma_1,\ldots,\sigma_n}\int d\xi_1\cdots d\xi_n\\&\times\sum_{\gamma:k(\gamma)=k}^{*}\frac{\overline{V}(\gamma)}{n(\gamma)}{:}e^{i\alpha\varphi^{(\leq k)}(\gamma)}{:}\ ,\end{aligned} \tag{11.9}$$

where the third summation runs over the trees γ with n end points (i.e., of degree n) carrying the end point labels $\xi_1,\sigma_1,\ldots,\xi_n,\sigma_n$ and root frequency k.

Expression (11.9) will be called the "multipole expansion" for the effective interaction on scale γ^{-k}. This name comes from the following simple and interesting argument.

Consider the quantity Z below and compute it by expanding the exponential in powers and using the properties of the Gaussian integrals (see Appendix C):

$$\begin{aligned}Z&\equiv\int\{\exp[V^{(k)}(\varphi^{(\leq k)})]\}P(d\varphi^{(\leq k)})\\&\equiv\sum_{p=0}^{\infty}\int\frac{V^{(k)}(\varphi^{(\leq k)})^p}{p!}P(d\varphi^{(\leq k)})\\&\equiv\sum_{p=0}^{\infty}\frac{1}{p!}\int d_{\underline{\sigma}_1}X_1\cdots d_{\underline{\sigma}_p}X_p w(X_1,\underline{\sigma}_2)\cdots w(X_p,\underline{\sigma}_p)\\&\qquad\times\exp\left[-\alpha^2\sum_{i<j}V_{\underline{\sigma}_i\underline{\sigma}_j}(X_i,X_j)\right]\ ,\end{aligned} \tag{11.10}$$

where

$$\begin{aligned}&\int d_{\underline{\sigma}}X\equiv\sum_{n=0}^{\infty}\sum_{\sigma_1\cdots\sigma_n}\int d\xi_1\cdots d\xi_n\ ,\\&X\equiv(\xi_1,\ldots,\xi_n),\quad \underline{\sigma}\equiv(\sigma_1,\ldots,\sigma_n)\ ,\\&w(X,\underline{\sigma})=\sum_{\substack{\text{degree }\gamma=n\\ \underline{\sigma}(\gamma)=\underline{\sigma}}}\frac{\overline{V}(\gamma)}{n(\gamma)}\ ,\\&V_{\underline{\sigma}\underline{\sigma}'}(X,X')=\sum_{\substack{\xi\in X\\ \xi'\in X'}}\sigma_\xi\sigma_{\xi'}C_{\xi\xi'}^{(\leq k)}\ ,\end{aligned} \tag{11.11}$$

i.e., Z in (11.10) is indeed, formally [i.e., modulo convergence problems in (11.10)], the partition function of a multipole gas in which the multipole with charges $\sigma_1,\ldots,\sigma_n$ located in the volume elements $d\xi_1\cdots d\xi_n$ has activity

$$w(\xi_1,\ldots,\xi_n;\sigma_1,\ldots,\sigma_n)d\xi_1\cdots d\xi_n\equiv w(X,\underline{\sigma})dX\ . \tag{11.12}$$

To complete the analysis of perturbation theory for the cosine interaction one has to show that the theory is ultraviolet finite. This is indeed the case for $\alpha^2<4\pi$, but if $\alpha^2\geq 4\pi$, this is so only for $\alpha^2<8\pi$ and, perhaps, for $\alpha^2=8\pi$. This problem is studied in Sec. XII below.

XII. ULTRAVIOLET STABILITY FOR THE COSINE INTERACTION AND RENORMALIZABILITY FOR α^2 UP TO 8π

Let

$$\varphi^{(\leq k)}=\sum_{j=-1}^{k}\varphi^{(j)}$$

be a sample field in which $\varphi^{(j)}$ verifies (3.15) and (3.16) and let the covariances $C^{(j)}$, $j=-1,0,\ldots,N$, verify (3.19) ($j_0=1$ in the present case) being defined by $\overline{C}^{(j)}$ in (3.7) [see comment following (11.1)].

To study the ultraviolet stability of the effective potentials $V^{(k)}(\varphi^{(\leq k)})$ one bounds [see (11.3)] the quantity

$$M(\Delta_1,\ldots,\Delta_n;\hat{\gamma})\equiv\sum_{\substack{\gamma:s(\gamma)=\hat{\gamma}\\ k(\gamma)=k,\underline{\sigma}(\gamma)=\underline{\sigma}}}\int_{\Delta_1,\ldots,\Delta_n}\frac{|\overline{V}(\gamma)|}{n(\gamma)}\times e^{(\alpha^2/2)C_{\gamma\gamma}^{(\leq k)}}d\xi_1\cdots d\xi_n\,, \tag{12.1}$$

having estimated the Wick-ordered exponentials

$$:e^{i\alpha\varphi^{(\leq k)}(\gamma)}:=e^{(\alpha^2/2)C_{\gamma\gamma}^{(\leq k)}}e^{i\alpha\varphi^{(\leq k)}(\gamma)}$$

by $\exp[(\alpha^2/2)C_{\gamma\gamma}^{(\leq k)}]$; and the sum runs over all trees with fixed shape $s(\gamma)=\hat{\gamma}$ [to avoid confusion the shape is here denoted $s(\gamma)$ rather than $\sigma(\gamma)$], fixed root frequency index k, and fixed charge labels $\underline{\sigma}=(\sigma_1,\ldots,\sigma_n)$ and n end points. Hence the sum runs over the frequency labels h_v that can be assigned to the nontrivial vertices $v>r$ of the shape $\hat{\gamma}$. Finally, $\Delta_1,\ldots,\Delta_n$ are n cubes extracted from a pavement of Λ with cubes of side size γ^{-k}; we shall denote this pavement Q_k, supposing the side of Λ divisible by γ^{-k}.

Of course one looks for bounds uniform in N and uniformly summable over the choices of $\Delta_2,\ldots,\Delta_n$ in Q_k. In fact, this is motivated by the observation that the contribution to $V^{(k)}$ from the trees with shape $\hat{\gamma}$ and charges $\underline{\sigma}$ can be written

$$\sum_{\Delta_1,\ldots,\Delta_n}\sum_{\substack{\gamma:s(\gamma)=\hat{\gamma}\\ k(\gamma)=k,\underline{\sigma}(\gamma)=\underline{\sigma}}}\int_{\Delta_1,\ldots,\Delta_n}\frac{\overline{V}(\gamma)}{n(\gamma)}e^{(\alpha^2/2)C_{\gamma\gamma}^{(\leq k)}}\times e^{i\alpha\varphi^{(\leq k)}(\gamma)}d\xi_1\cdots d\xi_n\,, \tag{12.2}$$

so that a bound, valid for all N and all $k\leq N$, like

$$M(\Delta_1,\ldots,\Delta_n;\hat{\gamma})\leq m(\hat{\gamma})e^{-\kappa\gamma^k d(\Delta_1,\ldots,\Delta_n)}\times\gamma^{\{[(\alpha^2/4\pi)-2](n-1)+(\alpha^2/4\pi)\}k}\,, \tag{12.3}$$

where $m(\hat{\gamma})$ is a suitable constant depending only on the shape of $\hat{\gamma}$, would be sufficient to show that the effective potentials are well defined order by order in perturbation theory, so that they converge to limits as $N\to\infty$ on subsequences; actually, it will be very clear that one could also easily prove plain convergence without need of subsequences.

The estimate (12.3) shows more, as it shows that the effective potential has a strong "short-range" property on the scale γ^{-k} naturally associated with the frequency k; the short-range property is expected to play an important role in the infrared stability, but, as will become clear later, also play a role in the ultraviolet stability.

In trying to prove (12.3) it is convenient to rewrite the recursive relation (11.7) as

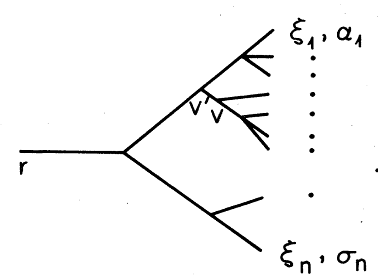
$$\overline{V}(\gamma)e^{(\alpha^2/2)C_{\gamma\gamma}^{(\leq k-1)}}=e^{-(\alpha^2/2)(C_{\gamma\gamma}^{(\leq h-1)}-C_{\gamma\gamma}^{(\leq k-1)})}\left[\prod_{i=1}^{s}\overline{V}(\gamma_i)e^{(\alpha^2/2)C_{\gamma_i\gamma_i}^{(\leq h-1)}}\right]\sum_{\tau\in\mathcal{T}^*}\prod_{\lambda\in\tau}(e^{-\alpha^2C_\lambda^{(h)}}-1)\quad\text{for } k<h\,, \tag{12.4}$$

where the relation

$$\sum_{i,j}C_{\gamma_i\gamma_j}^{(\leq h-1)}\equiv C_{\gamma\gamma}^{(\leq h-1)}$$

is used and $C^{(-1)}\equiv 0$.

Let $v>r$ be any vertex of γ and denote v' the vertex of γ preceding v; denote γ_v the subtree of γ with root at v' and first vertex v; for instance, in diagram 33 γ_v is the tree consisting in all the branches of γ that can be reached climbing the tree starting from v' and passing through v:



(33)

Call $\xi_1,\ldots,\xi_n,\sigma_1,\ldots,\sigma_n$ the end point labels for the positions and, respectively, the charges.

Equation (12.4) implies [see below]

where v_i is the tree's vertex directly connected to the end point ξ_i, $\mathcal{N}_{\hat{\gamma}}$, $\kappa_0>0$ are constants, and

$$d^*(X_v)=\text{graph distance of the points of } X_v \text{ modulo the clusters inside } v\ , \tag{12.6}$$

i.e., $d^*(X_v)$ is obtained by drawing lines connecting points in the clusters X_v and belonging to distinct maximal subclusters of X_v (which are the clusters corresponding to the vertices of γ following v immediately; see diagram 7) in such a way that any subcluster can be reached from any other by walking on such lines and jumping inside the subclusters: $d^*(X_v)$ is the minimum of the sum of the lengths of the above lines over the possible ways of drawing them.

The exponential factor in (12.5) requires an explanation; it arises from a bound on the last product in (12.4) and from the exponential decay of $C^{(h)}_{\xi\eta}=C^{(0)}_{\gamma^h\xi\gamma^h\eta}$ [see (3.19)] for some $\kappa>0$:

$$\left|\prod_{\lambda\in\tau}(e^{-\alpha^2C^{(h)}_\lambda}-1)\right|\le\prod_{\lambda\in\tau}(e^{\alpha^2|C^{(h)}_\lambda|}\alpha^2|C^{(h)}_\lambda|)\le\prod_{\lambda\in\tau}\left[\alpha^2e^{\alpha^2n_v^2C^{(0)}_{00}}\sum_{\xi,\eta}{}^*Ae^{-\kappa\gamma^h|\xi-\eta|}\right], \tag{12.7}$$

where n_v is the numbers of vertices in X_v ($n_v\le n$), and the sum runs over the pairs ξ,η in the subclusters joined by λ, whose number is bounded by $n_v^2\le n^2$. Since $|\xi-\eta|$ is larger or equal to the minimum distance between the two subclusters, (12.5) follows, with $\mathcal{N}_{\hat{\gamma}}$ being a coefficient depending only on the family of numbers n_v, i.e., on the shape $\hat{\gamma}=s(\gamma)$ of γ only.

To proceed one has to find a reasonable bound on the first product on (12.5).

Let $C^{(\ )}_{\bar{\gamma}_v\bar{\gamma}_v}$ denote the same expression as $C^{(\ )}_{\gamma_v\gamma_v}$ when all the points in the cluster corresponding to v are collapsed in one of them; it is

$$C^{(\cdot)}_{\bar{\gamma}_v\bar{\gamma}_v}\equiv C^{(\ )}_{00}Q_v^2\ , \tag{12.8}$$

where Q_v is the charge of the cluster v.

Then the first product in (12.5) can be written

$$\left[\prod_v e^{-(\alpha^2/2)(C_{00}^{(\le h_{v'}-1)}-C_{00}^{(\le h_v-1)})Q_v^2}\right]\left[\prod_v\{\exp[-(\alpha^2/2)(C^{(\le h_v-1)}_{\gamma_v\gamma_v}-C^{(\le h_v-1)}_{\bar{\gamma}_v\bar{\gamma}_v})]\right.$$
$$\left.\times\exp[(\alpha^2/2)(C^{(\le h_{v'}-1)}_{\gamma_v\gamma_v}-C^{(\le h_{v'}-1)}_{\bar{\gamma}_v\bar{\gamma}_v})]\}\right], \tag{12.9}$$

and the term in curly brackets can be bounded by using (3.19) and

$$\sum_{\xi,\eta\in X_v}(|C_{\xi\eta}^{(\le h_v-1)}-C_{00}^{(\le h_v-1)}|+|C_{\xi\eta}^{(\le h_{v'}-1)}-C_{00}^{(\le h_{v'}-1)}|)\le 2n_v^2A_{1/2}(1-\gamma^{-1/2})^{-1}[\gamma^{h_v}d(X_v)]^{1/2}$$
$$=\tilde{A}[\gamma^{h_v}d(X_v)]^{1/2}\ , \tag{12.10}$$

where

$d(X_v)$=length of the shortest path connecting all the points of the cluster X_v .

In the last step of (12.10) use has been made of (3.19) via

$$|C_{\xi\eta}^{(\le h)}-C_{00}^{(\le h)}|\le\sum_{p=0}^{h}|C^{(0)}_{\gamma^p\xi\gamma^p\eta}-C^{(0)}_{00}|$$
$$\le A_{1/2}\sum_{p=0}^{h}(\gamma^p|\xi-\eta|)^{1/2}$$
$$\le A_{1/2}(1-\gamma^{-1/2})(\gamma^h|\xi-\eta|)^{1/2}\ . \tag{12.11}$$

Hence using the easily proved inequality

$$\sum_v\gamma^{h_v}d^*(X_v)\ge\sum_v\gamma^{h_v}d(X_v)/n_v^2\ . \tag{12.12}$$

[hint: (12.12) does not hold "without the sums"; see (18.15) for a similar but deeper inequality], one finds that

$$-\frac{\kappa}{2}\sum_v\gamma^{h_v}d^*(X_v)+\tilde{A}\sum_v[\gamma^{h_v}d(X_v)]^{1/2}\le A(\hat{\gamma})<+\infty\ , \tag{12.13}$$

and one can bound (12.5) as

$$| \bar{V}(\gamma) | e^{(\alpha^2/2)C^{(\le k-1)}_{\gamma\gamma}} \le \mathcal{N}'_{\hat{\gamma}} \left[\prod_{v>r} e^{-(\alpha^2/2)Q_v^2(C_{00}^{(\le h_v-1)} - C_{00}^{(\le h_{v'}-1)})} e^{-(\kappa/2)\gamma^{h_v} d^*(X_v)} \right] \prod_{i=1}^{n} e^{(\alpha^2/2)C^{(\le h_{v_i}-1)}_{\xi_i\xi_i}} \tag{12.14}$$

The integral (12.1) can now be estimated using (see Appendix D)

$$\int_{R^2\times\cdots\times R^2} d\xi_2 d\xi_3 \cdots d\xi_n \prod_v e^{-(\kappa/4)\gamma^{h_v} d^*(X_v)} \le B_n \prod_{v>r} \gamma^{-2h_v(s_v-1)} , \tag{12.15}$$

if s_v = number of branches emerging from the vertex v in γ and if B_n is some constant.

Using also $C_{00}^{(\le h)} = (h+1)C_{00}^{(0)}$, we see that

$$M(\Delta_1,\dots,\Delta_n;\hat{\gamma}) \le \mathcal{N}''_{\hat{\gamma}} e^{-(\kappa/4)d(\Delta_1,\dots,\Delta_n)} \sum_{\underline{h}} \left[\prod_{i=1}^{n} e^{(\alpha^2/2)h_{v_i} C_{00}^{(0)}} \right] \prod_{v>r} \gamma^{-2h_v(s_v-1)} e^{-(\alpha^2/2)Q_v^2(h_v-h_{v'})C_{00}^{(0)}} , \tag{12.16}$$

where the sum runs over the frequency labelings of the shape $\hat{\gamma}$ such that $k(\gamma)=k$.

Taking into account the relation between the number s_v of branches emerging from v in γ and the number n_v of points in the cluster X_v corresponding to v

$$\sum_{v>w} (s_v-1) = n_w - 1 , \tag{12.17}$$

one easily checks, denoting $C = C_{00}^{(0)} \equiv (\ln\gamma)/2\pi$:

$$\sum_i \frac{\alpha^2}{2} C(h_{v_i}-k) - \ln\gamma \sum_{v>r} \left[2(h_v-k)(s_v-1) + \frac{\alpha^2}{2} C Q_v^2 (h_v-h_{v'}) \right]$$
$$\equiv \ln\gamma \sum_{v>r} \left[\left[\frac{\alpha^2}{4\pi} - 2 \right] (n_v-1) + \frac{\alpha^2}{4\pi} - \frac{\alpha^2}{4\pi} Q_v^2 \right] (h_v-h_{v'}) , \tag{12.18}$$

so that, using again (12.17) and (12.16),

$$M(\Delta_1,\dots,\Delta_n;\hat{\gamma}) \le \mathcal{N}'''_{\hat{\gamma}} e^{-(\kappa/4)d(\Delta_1,\dots,\Delta_n)\gamma^k} (\gamma^{[(\alpha^2/4\pi-2)(n-1)-\alpha^2/4\pi]k}) \sum_{\underline{h}} \sum_{v>r} \gamma^{-\rho_v(h_v-h_{v'})} , \tag{12.19}$$

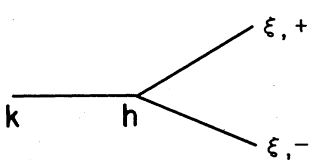
with

$$\rho_v = -\left[\frac{\alpha^2}{4\pi} - 2 \right] (n_v-1) - \frac{\alpha^2}{4\pi} + \frac{\alpha^2}{4\pi} Q_v^2 . \tag{12.20}$$

The summation in (12.19) is over the frequency labelings of $\hat{\gamma}$ and therefore over the h's such that $N \ge h_v - h_{v'} \ge 1$.

Clearly (12.3) follows, provided $\rho_v > 0$ for all v. In fact, since $v>r$ implies $n_v \ge 2$ and $|Q_v| \ge 0$, it is clear that $\rho_v \ge -\alpha^2/2\pi + 2$—i.e., $\rho_v > 0$ if $\alpha^2 < 4\pi$.

This proves (12.3) and the ultraviolet stability for $\alpha^2 < 4\pi$. Since one can easily check, as the bounds (12.19) and (12.20) hint, that for $\alpha^2 \ge 4\pi$ the contribution to $V^{(k)}$ from the trees



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is in fact divergent as $N \to \infty$, the problem for $\alpha^2 \ge 4\pi$ has to be reexamined—i.e., renormalization is necessary.

The key observation for studying the case $\alpha^2 \ge 4\pi$ is that the bounds (12.19) and (12.20) can be directly improved.

In fact, let v be a "zero charge" or "neutral" vertex of γ: $Q_v = 0$. Let v' be the vertex preceding v, and let $h_v, h_{v'}$ be their frequency labels.

Then in the evaluation of $V(\gamma)$ the subtree γ_v of γ with root v' and containing v and all the following vertices has the meaning, according to the general theory of the tree expansion in Secs. IV and V,

$$\mathcal{E}^T_{h_{v'}}(:e^{i\alpha\varphi^{(h_{v'})}(\gamma_v)}:, \dots, :e^{i\alpha\varphi^{(h_{v'})}(\gamma_{v_s})}:) , \tag{12.21}$$

if $v_1, \dots, v_s$ are the other vertices following v immediately.

However, when all the points of the cluster X_v coincide, it is $\varphi(\gamma_v) = 0$, because $Q_v = 0$, and it is clear that (12.21), being a truncated expectation, must vanish [in fact, the first argument becomes identically 1 and $\mathcal{E}^T(1,\dots) \equiv 0$].

Therefore, (12.21) will be equal to $\exp[i\alpha\varphi^{(h_{v'}-1)}(\gamma_{v'})]$ times a factor which will be proportional [given $\varepsilon \in (0,1)$ arbitrary; see (3.16)]

$$\left[\gamma^{h_{v'}} \sum_{\xi,\eta \in X_v} |\xi - \eta| \right]^{1-\varepsilon} . \tag{12.22}$$

However, if one collects together the contributions to $V^{(k)}$ from the trees having the same shape up to the charge indices and having fixed clusters of zero charge, then it is easy to realize that this improves the estimate producing a result which is a finite sum of terms which can all be bounded by the same bound that can be put on the "worst" among them, namely, the one obtained by replacing $:\exp[i\alpha\varphi^{(\le h_{v'})}(\gamma_v)]:$ by $:\cos[\alpha\varphi^{(\le h_{v'})}(\gamma_v)]:$, which in turn will introduce in the evaluation of the expressions analogous to (12.21) a factor proportional to

$$(\gamma^{h_{v'}}|\xi-\eta|)^{2(1-\varepsilon)}, \tag{12.23}$$

if $Q_{v'}=0$ (because the cosine differs from 1 by a second-order infinitesimal). The details will not be discussed here, as a much more complicated similar analysis will be presented and treated in Sec. XVIII. This leads, via some simple algebra, to replacing (12.3) by a bound on

$$\widetilde{M}(\Delta_1,\ldots,\Delta_n)=\sum_{\underline{\sigma}}\int_{\Delta_1,\ldots,\Delta_n}\sum_{\substack{s(\gamma)=\hat{\gamma}\\ \underline{\sigma}(\gamma)=\underline{\sigma}}}\bar{V}(\gamma)e^{(\alpha^2/2)C^{(\le k)}_{\gamma\gamma}}e^{i\alpha\varphi^{(\le k)}(\gamma)}d\xi_2\cdots d\xi_n\,, \tag{12.24}$$

which, if v_0 denotes the first vertex of $\hat{\gamma}$ following the root r, is estimated by

$$|\widetilde{M}(\Delta_1,\ldots,\Delta_n)|\le\widetilde{\mathcal{N}}_{\hat{\gamma}}e^{-(\kappa/4)d(\Delta_1,\ldots,\Delta_n)}(\gamma^{(\alpha^2/4\pi-2)n+\alpha^2/4\pi})^k\sum_{\underline{h}}\left[\prod_{v>v_0>r}\gamma^{-(\rho_v+2-\varepsilon)(h_v-h_{v'})}\right]\gamma^{-\rho_{v_0}(h_{v_0}-k)}\,, \tag{12.25}$$

because in the intermediate steps the integral (12.15) will be replaced by

$$\int d\xi_2\cdots d\xi_n\left[\prod_{v>r}e^{-(\kappa/4)\gamma^{h_v}d^*(X_v)}\right]\prod_{\substack{v:\rho_v=0\\ v>v_0}}\left\{\gamma^{h_{v'}}\sum_{\xi,\eta\in X_v}|\xi-\eta|\right\}^{2-\varepsilon} \tag{12.26}$$

by using the remarks leading to (12.23): the first nontrivial vertex of $\hat{\gamma}$, v_0, plays a special role, because if $Q_{v_0}=0$ the expression $:\cos[\alpha\varphi^{(\le k)}(\gamma_{v_0})]:$ will be proportional to the result obtained after the last truncation and no further truncation will be done at frequency k. Therefore, no factor like (12.23) can be contributed by the first vertex v_0.

The integral (12.26) obviously leads to an extra factor in (12.15) of the form

$$\bar{B}^n\prod_{v>v_0}\gamma^{-2(1-\varepsilon)(h_v-h_{v'})}\,;$$

in fact, the product of exponentials in (12.26) forces the points in the cluster v to be within a distance γ^{-h_v}; hence (12.23) can be replaced in the integral (12.26) by

$$(\gamma^{h_{v'}}\gamma^{-h_v})^{2(1-\varepsilon)}h_v^2(B')^{n_v}\,,$$

provided $\kappa/4$ is replaced by $\kappa/8$ and B' is conveniently chosen (see Appendix D).

The bound (12.25) proves that if one collects together several trees of the same shape and if use is made of the charge symmetry, then all the trees with nonzero charge $Q_{v_0}\neq 0$ (note that this implies $|Q_{v_0}|\ge 1$) yield $\rho_{v_0}>0$ and $\rho_v+2-2\varepsilon>0$ if ε is taken small enough, for all $\alpha<8\pi$. Hence (12.25) proves that the ultraviolet stability can be violated, for $\alpha^2<8\pi$, only by the trees which have zero charge: $Q_{v_0}=0$.

For $\alpha^2<8\pi$ not all the neutral trees have ultraviolet stability problems, only the neutral ones with n end points, $n=n_{v_0}$, such that [see (12.20)]

$$\left[\frac{\alpha^2}{4\pi}-2\right](n-1)+\frac{\alpha^2}{4\pi}<0\,. \tag{12.27}$$

So, for $\alpha^2<8\pi$, there is a sequence of thresholds obtained setting the lhs of (12.27) equal to zero:

$$\frac{\alpha_n^2}{4\pi}=2\frac{2n-1}{2n}=1,\tfrac{3}{2},\tfrac{5}{3},\ldots\,. \tag{12.28}$$

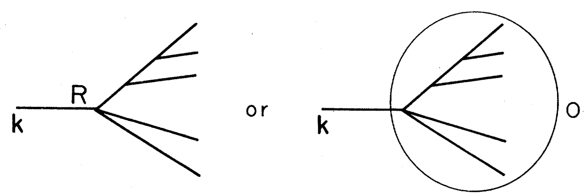
As α^2 reaches α_n^2 and beyond it, the trees with n vertices and zero change "become ultraviolet unstable"—i.e., their contribution to the effective potentials are not convergent as $N\to\infty$.

However, the reason for the instability is somewhat trivial and it is due manifestly to the fact that the first nontrivial vertex v_0 of γ, when γ is neutral, gives a contribution to $V(\gamma)$ of the form $:\exp[i\alpha\varphi^{(\le k)}(\gamma)]:$, which does not vanish when the position labels $\xi_1,\ldots,\xi_n$ for the end points of γ become identical.

But if one defines $\mathscr{L}_k^{(\gamma)}=0$, unless $Q_{v_0}=0$ or $\alpha^2<\alpha_n^2$ and

$$\mathscr{L}_k^{(\gamma)}\int :e^{i\alpha\varphi^{(\le k)}(\gamma)}:\bar{V}(\gamma)d\underline{\xi}\equiv\int\bar{V}(\gamma)d\underline{\xi}\,, \tag{12.29}$$

if $Q_{v_0}=0$ and $\alpha^2<\alpha_n^2$, and, if one collects together the trees γ of the same shape up to the charge indices and with the same frequency indices and the same vertices of zero charge, one sees that the operators $\mathscr{L}_k^{(\gamma)}$ define renormalization operations, according to the general theory of Secs VI—IX, such that the dressed graphs have two types:



(35)

Either they contain an index R as a superscript on the first vertex v_0 after the root r or they are entirely contained in a single frame with an index $\sigma=0$ appended to the frame [meaning that they contribute a constant to the effective potential, because $\mathscr{L}_k^{(\gamma)}$ takes values in the space of the constants, by (12.29)].

A tree with an R over the first vertex will mean a contribution to the effective potential which is equal to the one that would be given by the tree without the R but with $:\exp[i\alpha\varphi^{(\leq k)}(\gamma)]:$ replaced by $:[\exp(i\alpha\varphi^{(\leq k)}(\gamma)]-1:$.

Collecting again the contributions to the effective potential from all the trees with given shape, up to the charge indices, and summing their contributions over all the possible frequency labels and charge labels at fixed neutral vertices, one sees that the contribution to the effective potential sums up to the same quantity (12.24), with

$$:e^{i\alpha\varphi^{(\leq k)}(\gamma)}: \text{ replaced by } :\cos[\alpha\varphi^{(\leq k)}(\gamma)]-1: , \tag{12.30}$$

and the latter expression vanishes when the points $\xi_1,\ldots,\xi_n$ collapse into one single point and the zero is of the order of the square of the zero of $\varphi^{(\leq k)}(\gamma)$. The latter can be evaluated by recalling the basic smoothness properties of $\varphi^{(\leq k)}$ described by (3.16) (recall that the space dimension is here $d=2$): it is of the order of

$$B^2\left[\gamma^2\sum_{\xi,\eta}|\xi-\eta|\right]^{2(1-\varepsilon)}, \tag{12.31}$$

if $B=\sup B_\Delta$, and $\varepsilon>0$ is prefixed arbitrarily.

This improves the bound (12.25) by replacing also ρ_{v_0} by $\tilde{\rho}_{v_0}=\rho_{v_0}+2-2\varepsilon$.

The arbitrariness of ε implies that, if $\alpha^2<8\pi$, ε can be chosen so that $\tilde{\rho}_{v_0}>0$, and, therefore, all the unframed dressed trees are ultraviolet finite in the sense that, collecting together the contributions from the trees with given shape, up to the charge indices, one obtains a total contribution to the effective potential which is ultraviolet finite.

The framed trees contribute only to the constant part of the effective potential and therefore need not be studied. However, their theory would also be simple, and they turn out to be ultraviolet finite: in fact, the sum of the contributions to the effective potential coming from the neutral trees of a given degree is a constant which can be written as $\int \nu_k d\xi$, and, from (12.25) and the general theory, one can find

$$|\nu_k|\leq \mathscr{N}(\hat{\gamma})(\gamma^{(\alpha^2/4\pi-2)(n-1)+\alpha^2/4\pi})^k\lambda_n . \tag{12.32}$$

Since, given $\alpha^2<8\pi$, $\mathscr{L}_k^{(\gamma)}\equiv 0$ if n, the number of end points ("degree") of γ, is large enough, it follows that the cosine interaction is super-renormalizable in the sense of Sec. VIII (see final comments of Sec. VIII).

Exercise: study the exponential interaction (5.5) and show that it is ultraviolet finite up to $\alpha^2<4\pi$. Show that it is not renormalizable for $\alpha^2\geq 4\pi$ (hint: just repeat the same steps and estimates used for the cosine case).

XIII. BEYOND PERTURBATION THEORY IN THE COSINE INTERACTION CASE: ASYMPTOTIC FREEDOM AND SCALE INVARIANCE

Having completed the perturbative analysis for the cosine field theory in terms of formal power series with no control on convergence one wonders what it really means to study an interacting field theory.

The simplest type of result that one can think to try to prove for the interacting measures $P_{\rm int}$ is the following.

"There exist (infinitely many inequivalent) one parameter families P_λ of measures on $\mathscr{S}'(R^2)$, the space of the distributions on R^2, whose Schwinger functions admit an asymptotic expansion in the parameter λ near $\lambda=0$ coinciding with the formal perturbation theory expansion" of the cosine interaction discussed in Secs. XI and XII (with $\nu=0$).

Super-renormalizability is the deep property behind the methods so far known to obtain a proof of the above proposition in the cosine interaction case as well as in the proof of its version for many other super-renormalizable field theories (e.g., $-\lambda{:}\varphi^4{:}$ in two dimensions or $-\lambda{:}\varphi^4{:}-\mu{:}\varphi^2{:}-\nu$ in three dimensions; in fact, the ideas and methods involved do not distinguish between the above theories).

The first idea is to try to build P_λ as limit of measures of the form

$$Z^{-1}\left[\prod_{j=0}^{N}\chi_j(\varphi^{(j)})\right]\exp[V(\varphi^{(\leq N)})]\prod_{j=0}^{N}P(d\varphi^{(j)}) , \tag{13.1}$$

where χ_j are characteristic functions selecting fields having so large a probability that

$$1\geq \int\sum_{j=0}^{\infty}[\chi_j(\varphi^{(j)})P(d\varphi^{(j)})]$$
$$\geq \exp[-\varepsilon(\lambda)|\Lambda|] , \tag{13.2}$$

with

$$\varepsilon(\lambda)\underset{\lambda\to 0}{\longrightarrow} 0$$

faster than any power and, of course (see Sec. XI) (11.1) with $\nu=0$,

$$V(\varphi^{(\leq N)})=\int_\Lambda[\lambda{:}\cos(\alpha\varphi_\xi^{(\leq N)}){:}+\nu_N(\lambda)]d\xi , \tag{13.3}$$

where $\nu_N(\lambda)$ is the sum of the counterterms due to the renormalization described in Sec. XII [see (12.31)], if any (i.e., if $4\pi\leq\alpha^2<8\pi$).

The characteristic functions χ_j will be so chosen to allow one to treat "naively" the fields $\varphi^{(j)}$ when $\chi_j(\varphi^{(j)})=1$; i.e., χ_j will be the characteristic function of the set:

$$\{\varphi\,|\,\sin(\alpha/2)(\varphi_\xi^{(i)}-\varphi_\eta^{(j)})\,|<B_j(\gamma^j|\xi-\eta|)^{1-\varepsilon} ,$$
$$\forall\xi,\eta\in\Lambda\} , \tag{13.4}$$

where

$$B_j=B(1+j)^a\ln(e+j+\lambda^{-1})$$

for some $B>0,\varepsilon>0,a\geq\frac{1}{2}$, if $|\xi-\eta|$ denotes the distance.

The probability of the above event is bounded below, by using (3.17), for all $a\geq\frac{1}{2}$ by

$$\prod_{\Delta\subset\Lambda}\{1-\bar{A}\exp[-\bar{\alpha}B^2(1+j)][\ln(e+\lambda+j)]^2\}\geq\{1-\bar{A}\exp[-\bar{\alpha}B^2(1+j)]\ln(e+\lambda+j)^2\}^{\gamma^{2j}|\Lambda|}\ , \tag{13.5}$$

where the product on the lhs runs over the cubes of the pavement Q_j of Λ with cubes of side size γ^{-j}, whose number is $\gamma^{2j}|\Lambda|$.

Clearly in (13.2) one can take, when χ_j is chosen as described in (13.4),

$$\varepsilon(\lambda)=\sum_{j=0}^{\infty}\gamma^{2j}\ln[1-\bar{A}(1+j+\lambda^{-1})^{-\bar{\alpha}B^2(1+j)\ln[1-\bar{A}(1+j+\lambda^{-1})]}]=O(\lambda^{\infty})\ , \tag{13.6}$$

i.e., $\varepsilon(\lambda)\to 0$ as $\lambda\to 0$ faster than any power of λ.

Since the amount of phase space thrown away by the insertion of the characteristic functions in (13.1) is, if measured with the free-field measure, very negligible, see (13.2) and (13.4), it is quite clear that the perturbation-theory expansion for the Schwinger functions of the measure (13.1) and those of the measure obtained by taking away from (13.1) the characteristic functions are identical uniformly in N.

Therefore, if one succeeds in showing that the measure (13.1) has a limit as $N\to\infty$ (possibly only on subsequences), the one-parameter family claimed to exist in the above proposition is constructed.

This is in fact true and it is the way which will be followed in proving the proposition at the beginning of this section.

Of course, since the construction clearly depends on the arbitrary parameter B in (13.4), one must expect that the family P_λ of measures obtained as limits of (13.1) is B dependent.

The measure (13.1) will be called a "restricted cosine field": it is an object of limited physical interest even in the limit $N\to\infty$. Its importance lies only in the fact that its understanding is preliminary to the understanding of the interesting case (essentially obtained by letting $B\to\infty$).

Before we continue, it is important to make the following remark: the restrictions (13.4) *do not* imply that the field $\varphi^{(\leq N)}$ is constrained to be smooth for large N; actually, a simple computation shows that the cutoff on rough or large fields imposed by the inequalities (13.4) is such that $\varphi_\xi^{(\leq N)}-\varphi_\eta^{(\leq N)}$ have essentially the same covariance, and hence the same average size, with respect to the free Gaussian measure and in the free restricted Gaussian measure [i.e., the Gaussian measure restricted to the ensemble of fields described by (13.4)]. This means that the problem of taking the limit as $N\to\infty$ of (13.1) is still nontrivial and that some new idea is necessary for its solution.

Arguments on field theory are often given, in the literature, treating the fields as if they verified (13.4). And the problem of controlling what happens when the field violates the conditions imposed in (13.4), i.e., the problem of controlling the large fluctuations, is often solved by handwaving methods, saying that the large fluctuations "are depressed" by the "positivity of the action."

In fact, it will be clear that, in practice, many real problems arise in trying to give a rigorous meaning to such arguments; in my understanding the situation is, in general, very subtle and I cannot see the actual solution of the above large-fluctuation problem (in the cases where it is known how to handle it on a mathematical rigorous basis) as just a refined way of rephrasing the mentioned argument based on the positivity of the action. Furthermore, this is a case in which it makes no sense to appeal to "physical arguments," because the issue is precisely whether field theory has anything to do with physics.

In any event, the problem of the relevance of the large fluctuations seems to have been clearly perceived as a deep one, even in field theories with a formally positive action, in constructive field theory, and it should be regarded as one of its contributions; see Nelson (1966,1973), Glimm and Jaffee (1968,1969,1973), Osterwalder and Feldman (1976), Magnen and Seneor (1976), Gallavotti (1978), Benfatto *et al.* (1978,1982), Benfatto, Cassandro *et al.* (1980), Benfatto, Gallavotti, and Nicoló (1980), and Nicoló (1983).

The new ideas needed to deal with the problem of proving the existence of the limit of (13.1) as $N\to\infty$, at fixed B, are two: (i) asymptotic freedom and (ii) scale invariance. Their role and interplay in field theory seems to have been clearly realized as early as 1969 by Wilson (1971). It turns out that they are best illustrated in the theory of the cosine field.

Suppose that one wishes to study the distribution of the low-frequency fields $\varphi^{(0)},\dots,\varphi^{(p)}$ in the restricted ensemble. Then the function

$$F^{(N)}(\varphi^{(0)},\dots,\varphi^{(N)})\equiv\left[\prod_{j=0}^{p}\chi_j(\varphi^{(j)})\right]\times\int\left[\prod_{j=p+1}^{N}\chi_j(\varphi^{(j)})\right]e^{V(\varphi^{(\leq N)})}\times\prod_{j=p+1}^{N}P(d\varphi^{(j)}) \tag{13.7}$$

is the density of the distribution of $\varphi^{(0)},\dots,\varphi^{(p)}$ with respect to the measure $P(d\varphi^{(0)})\cdots P(d\varphi^{(p)})$.

The first step is to show that the integral in (13.7) is an integrable function of $\varphi^{(0)},\dots,\varphi^{(p)}$ with respect to $P(d\varphi^{(0)})\cdots P(d\varphi^{(p)})$, and, furthermore, that the integrable function can be bounded uniformly in the ultraviolet limit, $N\to\infty$.

From the above discussion on the relevance of the phase space "neglected" [see (13.2) and (13.6)], it is natural to think that the result of the integration in (13.7) should simply be

$$\exp[V^{(k)}(\varphi^{(\leq k)})] \tag{13.8}$$

up to corrections negligible as $\lambda \to 0$ and due to the presence of the characteristic functions in (13.1).

However, this does not really make sense, because the theory of the preceding sections provides an asymptotic expansion in λ for $V^{(k)}$ which has little chance of being convergent.

The next best guess is that instead of (13.8) one gets, for any integer $t \geq 0$

$$\exp\{[V^{(k)}(\varphi^{(\leq k)})]^{[t]}+\lambda^{t+1}R_t(\varphi^{(\leq k)};\lambda)\} , \tag{13.9}$$

where $[\ ]^{[t]}$ denotes the truncation of a power series in λ to order t and R_t represents a "remainder."

Therefore, $[V^{(k)}]^{[t]}$ will be given just by the perturbation theory developed in the preceding section counting only the trees with at most t end points; the choice of t in (13.9) is arbitrary, provided that the remainder can be well estimated for the chosen t.

The validity of a result like (13.9) means that the integral of $\exp[V(\varphi^{(\leq N)})]$ over $\varphi^{(N)}, \ldots, \varphi^{(p+1)}$ can be performed successively by using perturbation theory; therefore in order to have any hope of proving (13.9) with reasonable bounds on the remainder it is necessary that $V^{(q)}(\varphi^{(\leq q)})$ regarded as a "potential" on $\varphi^{(q)}$ at fixed $\varphi^{(q-1)}, \ldots, \varphi^{(0)}$ has a very small size, at least on the restricted ensemble (13.4); actually, not only should its size be small, but it should even go to zero as $q \to \infty$ ("asymptotic freedom"), if $N=\infty$.

In order that the above property hold for all $q \leq N$ it must of course hold for $q=N$. Hence the check of the property of asymptotic freedom starts with a check of its validity for $q=N$.

To explain what the above words mean concretely one considers the field $\varphi^{(N)}$ and observes that, as discussed in Sec. III, it can be regarded as smooth and essentially constant on cubes Δ of size γ^{-N}, which will be thought of as extracted from a pavement Q_N of Λ with cubic tesserae of side length γ^{-N}. Furthermore, the values of $\varphi^{(N)}$ on different tesserae are almost independent because of the exponential decay on scale γ^{-N} of the covariance of $\varphi^{(N)}$.

This suggests writing the nonconstant (i.e., nontrivial) part of the interaction as a sum of contributions each coming from a given $\Delta \in Q_N$, i.e., as

$$\sum_{\Delta \in Q_N} \lambda \int_\Delta :\cos[\alpha(\varphi_\xi^{(\leq N-1)}+\varphi_\xi^{(N)})]:d\xi = \sum_{\Delta \in Q_N} (\lambda e^{(\alpha^2/2)C_{00}^{(\leq N)}}|\Delta|)\left[\frac{1}{|\Delta|}\int_\Delta \cos[\alpha(\varphi_\xi^{(\leq N-1)}+\varphi_\xi^{(N)})]d\xi\right]$$
$$\equiv \sum_{\Delta \in Q_N} \lambda\gamma^{(\alpha^2/4\gamma-2)N}S_\Delta , \tag{13.10}$$

where use has been made of $C_{00}^{(\leq N)}=(N+1)(\ln\gamma)/2\pi$, $|\Delta|=\gamma^{-2N}$, and by the preceding arguments one regards the variables S_Δ (which have "order 1," because they are averages of a cosine) as random variables of the field $\varphi^{(N)}$ parametrized by the field $\varphi^{(\leq N-1)}$, and $\varphi^{(j)}$, $j \leq N-1$, are supposed to be in the set defined by (13.4).

The variables S_Δ can be thought of as continuous spins sitting on the lattice Q_N and the calculation of the integral

$$\int \chi_N(\varphi^{(N)})\exp\left[\sum_\Delta \lambda\gamma^{(\alpha^2/4\pi-2)N}S_\Delta\right]P(d\varphi^{(N)}) \tag{13.11}$$

can be thought of as the problem of evaluating the partition function of a spin system, on the lattice Q_N, which is a perturbation by an energy

$$\sum_\Delta \lambda\gamma^{(\alpha^2/4\pi-2)N}\bar{S}_\Delta$$

of the "free measure":

$$P\left[\prod_\Delta d\bar{S}_\Delta\right]=\int\prod_\Delta \delta(S_\Delta-\bar{S}_\Delta)\chi_N(\varphi^{(N)})P(d\varphi^{(N)}) , \tag{13.12}$$

which, intuitively, can be thought of as an almost-factorized measure with respect to the variables $\bar{S}_\Delta$.

So the problem of computing the integral (13.11) in terms of its value for $\lambda=0$ can be interpreted as a statistical mechanics problem problem for a spin system of bounded uncorrelated spins with a local perturbation whose size is

$$\lambda\gamma^{(\alpha^2/4\pi-2)N} . \tag{13.13}$$

If $\alpha^2<8\pi$, one sees that the "effective coupling on the fields with frequency N" is (13.13) and it goes to zero as $N\to\infty$, which means that the spin system is at "very high temperature" for large N, and one can very reasonably hope to use the high-temperature expansion techniques of statistical mechanics to estimate perturbatively the integral (13.11): the result of such estimates is in general that

$$\int \exp(\lambda W\,dp)=\exp\left[\sum_{p=0}^{t}\frac{\lambda^p}{p!}\mathscr{E}^T(W;p)+R_\varepsilon\right] \tag{13.14}$$

and

$$|R_t| \leq \lambda^{t+1}\times(\text{system's volume in lattice spacing units})\times\text{const.} \tag{13.15}$$

It is therefore clear that the result of the integral (13.14) gives rise to a very complex new function of $\varphi^{(\leq N-1)}$.

For this reason one does not say that a theory is asymptotically free just if the computation of the effective coupling constant for $\varphi^{(N)}$ gives a result tending to zero with $N\to\infty$, as in (13.13).

The correct definition of asymptotic freedom is set up by considering the main term of (13.9) and by interpreting it as a potential for $\varphi^{(k)}$ parametrized by $\varphi^{(\leq k-1)}$; one then computes the "effective coupling constant $\lambda_N(k)$" and says that the interaction is asymptotically free if

$$\lim_{k\to\infty}\lim_{N\to\infty}\lambda_N(k)=0 . \tag{13.16}$$

The self-consistent nature of this condition being clear, one can hope to be really able to check (13.16) and use it, afterwards, to obtain good estimates for R_t.

Although the calculation, or estimates, of $\lambda_N(k)$ looks *a priori* much harder than the evaluation of $\lambda_N(N)$ performed above [see (13.13)], it turns out that one can easily estimate $\lambda_N(k)$ by using the general theory of perturbations developed in the previous sections.

To obtain an estimate of $\lambda_N(k)$ one first needs its precise definition: in fact, $[V^{(k)}(\varphi^{(\leq k)})]^{[t]}$ no longer depends on a single constant, which, as done above when $k=N$, can be naturally related to N, but it is rather a "many-body" nonlocal interaction being a finite sum, over the trees with $\leq t$ end points, of terms like [see (11.9)] (12.31):

$$\int \sum_{\underline{\sigma}} \sum_{\substack{\hat{\sigma} \\ \underline{\sigma}(\hat{\gamma})=\underline{\sigma}}} V(\xi_1,\ldots,\xi_n;\hat{\gamma}):\{\cos[\alpha\varphi^{(\leq k)}(\gamma)]-\delta_{Q_{\hat{\gamma}}0}\}: \times d\xi_1\cdots d\xi_n\ , \tag{13.17}$$

where $\hat{\gamma}$ denotes a tree shape of degree n (i.e., with n end points), $\underline{\sigma}$ are the charges at the end points of $\hat{\gamma}$, and

$$Q_{\hat{\gamma}}=\sum_i \sigma_i$$

is the total charge of $\hat{\gamma}$.

To interpret (13.17) as a spin-spin interaction for a lattice spin system one has to recall the main property of $\varphi^{(k)}$ of being approximately constant and smooth on the scale γ^{-k} and of being independently distributed on the same scale (approximately, of course).

Therefore, following the same philosophical principles already used above, one splits (13.17) into a sum over all possible n-tuples of tesserae $\Delta_1,\ldots,\Delta_n\in Q_k$ of terms like

$$\int_{\Delta_1,\ldots,\Delta_n} V(\xi_1,\ldots,\xi_n;\hat{\gamma}):\{\cos[\varphi^{(\leq k)}(\hat{\gamma})]-\delta_{Q_{\hat{\gamma}}0}\}: \times d\xi_1\cdots d\gamma_n\ . \tag{13.18}$$

Then one will interpret (13.18) as a many-body interaction between the spins $(S_{\Delta_1},\ldots,S_{\Delta_n})\cong(\varphi^{(k)}_{\xi_1},\ldots,\varphi^{(k)}_{\xi_n})$ and check that (13.18) is bounded by

$$C_t\lambda_N(k)^n e^{-\kappa\gamma^k d(\Delta_1,\ldots,\Delta_p)} \tag{13.19}$$

uniformly in N and with $\lambda_N(k)$ and κ independent on the particular term like (13.18) contributing to the effective potential, and also independent of the considered expansion order t, $C_t,\kappa>0$ being suitable constants.

Then the constant $\lambda_N(k)$ will naturally be called the "effective coupling constant" for the field $\varphi^{(k)}$: the interaction (13.18) becomes susceptible to the very same interpretation as (13.10) in terms of continuous lattice spin systems.

All the technical work necessary to study bounds (13.19) in the cosine field case has already been done in the proof of its renormalizability: in fact, estimate (12.25) with ρ_{v_0} replaced by $\rho_{v_0}+2-2\varepsilon$, as explained after (12.31), immediately yields a bound on (13.18) of the form (13.19) with

$$C_t=\widetilde{C}_tB^2\ ,$$
$$\lambda_N(k)=\lambda\gamma^{(\alpha^2/4\pi-2)k}(1+k)^{2a}(\ln e+k+\lambda^{-1})^4\ , \tag{13.20}$$

and $\kappa=\kappa_0/4>0$, for all N, provided ε in (13.4) is chosen so that $\rho_{v_0}+2-2\varepsilon>0$—i.e., $\varepsilon\ll(2-\alpha^2/4\pi)$.

Therefore, the cosine interaction is asymptotically free for $\alpha^2\in[0,8\pi)$, provided it is correctly renormalized for $\alpha^2\in[4\pi,8\pi)$.

Notice, however, that there is a deep difference between the cases $\alpha^2<4\pi$ and $\alpha^2\in[4\pi,8\pi)$: in the first case conditions (13.4) are not necessary to obtain (13.20) because bounds (12.19) and (12.25) can be used and because they had been obtained without using any smoothness (or boundedness) property of $\varphi^{(\leq k)}$.

Such properties are necessary to obtain the improvement on (12.25) (i.e., $\rho_{v_0}\to\sim\rho_{v_0}+2$) needed to have the ultraviolet stability for $\alpha^2<8\pi$. Recall that the improvement follows, after renormalization, only if $:\cos[\alpha\varphi^{(\leq k)}(\gamma)]-1:$ is bounded by (12.31) and this is possible only if the smoothness condition in (13.4) holds [the boundedness condition in (13.4) is not really necessary and one could proceed without it].

Actually, this remark shows that (13.20) can be improved by replacing C_t by a B-independent constant, if $\alpha^2<4\pi$. Also, one can observe that while the proof of the proposition at the beginning of Sec. XIII on the existence of the P's easily implies, by the arbitrariness of B in (13.4), the complete construction of the cosine theory in the ultraviolet limit for $\alpha^2<4\pi$, this is no longer so for $\alpha^2\geq 4\pi$, when the presence of the field cutoff introduced by the characteristic functions in (13.1) is really essential to have asymptotic freedom. In the latter case, $\alpha^2\in[4\pi,8\pi)$, new ideas are necessary to control the $B\to\infty$ limit.

The discussion of the $B\to\infty$ limit will be postponed to Sec. XIV and in this section no more differences will arise between the cases $\alpha^2<4\pi$ and $\alpha^2\in[4\pi,8\pi)$.

Having proved the asymptotic freedom for the cosine interaction, one realizes that the partial solution of the ultraviolet problem provided by the proposition stated at the beginning of this section still requires the analysis of many statistical mechanics problems of weakly coupled continuous lattice spin systems.

One can, in fact, regard as such the problem of performing the successive integrations over $\varphi^{(p)}$ of

$$\chi(\varphi^{(\leq p)})\exp[V^{(p)}(\varphi^{(\leq p)})]^{[t]} \tag{13.21}$$

for $p=N,N-1,\ldots,k+1$.

The reason behind the feasibility of the above feat is the second important idea on the problem: the fields $\varphi^{(0)},\ldots,\varphi^{(N)}$ are identically distributed up to trivial scaling (see Sec. III).

This means that, whatever p is, the integral (13.21) can be regarded as the computation of the partition function of the same spin system on a fixed lattice affected by a perturbation which is p dependent and which, by the asymptotic freedom property, has a p dependence becoming weaker as p becomes larger.

Therefore, as a matter of fact, one can perform the integral of (13.21) over $\varphi^{(p)}$ by trying to use the naive formula

$$\mathscr{E}_p(\chi_p e^{[V^{(p)}]^{[t]}})=\exp\left\{\left[\sum_{j=1}^{t}\frac{1}{j!}\mathscr{E}_p^T([V^{(p)}]^{[t]};j)\right]^{[t]} + \theta\lambda_N(p)^{t+1}\overline{R}_t\gamma^{2p}|\Lambda|\right\} \tag{13.22}$$

[see (13.14)], where $|\theta|\le 1$ and $\overline{R}_t$ is a positive constant depending on C_t and κ [see (3.19)]; the factor γ^{2p} in front of the volume $|\Lambda|$ comes from the fact that the volume has to be measured on the scale on which lives the field $\varphi^{(p)}$ (see below).

The validity of (13.22) rests on the following lemma.

Lemma 1. Formula (13.22) is valid for $p=0$, replacing $[V^{(0)}]^{[t]}$ by a finite linear combination of expressions like (13.17), with $k=0$, such that the integrals (13.18) are bounded by (13.19), with $p=0$ and $\lambda_N(0)$ replaced by a free parameter λ.

By the scale invariance of the multiscale decomposition such a lemma would then imply (13.22) for arbitrary p.

Accepting the above lemma, and hence (13.22), one observes that

$$\left[\sum_{j=0}^{t}\frac{1}{j!}\mathscr{E}_p^T([V^{(p)}]^{[t]};j)\right]^{[t]}\equiv[V^{(p-1)}]^{[t]}, \tag{13.23}$$

which is evident if one recalls the definition of the formal power series in λ for $V^{(p-1)}$ in terms of that for $V^{(p)}$ [see (5.13) and (5.14) and the relations following them in Sec. V].

Then, since $[V^{(p-1)}]^{[t]}$ verifies the bound (13.19) with $p-1$ replacing p, provided $\varphi^{(0)},\ldots,\varphi^{(p-1)}$ verify (13.4), it follows that the integral (13.7) is, recursively, estimated by

$$F^{(k)}(\varphi^{(0)},\ldots,\varphi^{(N)})=\left[\prod_{j=0}^{k}\chi_j(\varphi^{(j)})\right]\exp\left[[V^{(k)}]^{[t]}+\sum_{p=k+1}^{N}\theta\overline{R}_t\gamma^{2p}|\Lambda|\lambda_N(p)^{t+1}\right], \tag{13.24}$$

where $|\theta|\le 1$ and the remainder is simply the sum of the remainders produced by successively integrating the fields $\varphi^{(N)},\ldots,\varphi^{(k+1)}$ using (13.22), i.e., Lemma 1 above.

So the remainder in (13.24) is bounded by

$$\overline{\overline{R}}_k|\Lambda|=|\Lambda|\overline{R}_t\sum_{p=k}^{\infty}[\lambda\gamma^{(\alpha^2/4\pi-2)p}(1+p)^{2a}(\ln e+k+\lambda^{-1})^4]^{t+1}\gamma^{2p}. \tag{13.25}$$

This proves that $F^{(k)}$ [see (13.7)] is well defined and bounded uniformly in N if $\alpha^2<8\pi$: in fact, it is enough to choose in (13.24) and (13.25) the arbitrary integer $t\ge 0$ to be not smaller than t_0, where t_0 is the first integer such that $(\alpha^2/4\pi-2)(t_0+1)+2<0$, so that $t_0=1$ if $\alpha^2<4\pi$, $t_0=2$ if $\alpha^2\in[4\pi,16\pi/3)$, $t_0=3$ if $\alpha^2\in[16\pi/3,6\pi)$, $t_0=4$ if $\alpha^2\in[6\pi,32\pi/5)$, etc.

If $F^{(k)}$ is well defined and bounded in N, it follows from abstract analysis that there is a subsequence of the sequence of measures (13.1) which converges "weakly" to a limit P_λ as $N\to\infty$ for all values of $\lambda\in R$; any such one-parameter family will verify the properties in the proposition stated at the beginning of the section [there are many sequences of measures (13.1), since one can change the parameter B in (13.4), or, more generally, since one can modify the choice of the characteristic functions]. I shall not discuss the details of such an analysis, since I consider it not too relevant to the heart of the matter treated here.

So the discussion of the proposition at the beginning of this section is complete for the cosine interaction and rests on the above technical lemma; this lemma will not be proved here (although to do so is not particularly difficult, since it is a "mean field theory bound" in its statistical mechanical interpretation, as the reader familiar with statistical mechanics can convince himself). The relevance of Lemma 1 for the ultraviolet problem from the constructive field theory point of view has been pointed out in Gallavotti (1978,1979,1980), Benfatto *et al.* (1978, 1982), Benfatto, Cassandro *et al.* (1980), and Benfatto, Gallavotti, and Nicolò (1980), and then used by many workers who have often built it in as an important ingredient necessary in the development of new more daring and deep ideas [see Gawedski and Kupiainien (1982,1983) and Balaban (1982,1983,1984); see Westwater (1980) for related ideas]. Some of the methods in Gallavotti (1978) had been previously introduced in the brilliant papers on the hierarchical model in statistical mechanics [see Bleher and Sinai (1974,1975) and Collet and Eckmann (1978)] [these are the methods used to attack a model similar to the model called the hierarchical field in Gallavotti (1978,1979)]; in some sense the role of the application of such methods to field theory was to point out the path to follow to apply the renormalization group in constructive field theory using techniques already developed in statistical mechanics and taking almost literally the ideas introduced in statistical mechanics and field theory by Wilson (1969).

The proof of Lemma 1 can be found in a rudimentary form in Gallavotti (1978,1979) and in a complete form in Benfatto *et al.* (1978,1982), Benfatto, Cassandro *et al.* (1980), and Benfatto, Gallavotti, and Nicolò (1980), where a much stronger version (see Lemma 2 of Sec. XIV, of this paper) is derived; in Gallavotti (1979) Lemma 1 is obtained by literally reducing it to a classical statistical mechanics problem of high-temperature expansions for a system of weakly coupled spins, using the techniques of Kunz (1976) and Sylvester (1977), and later improved in Cammarota (1982) [see Seiler (1982) for a review].

The proof in Benfatto, Cassandro *et al.* (1982), Benfatto, Gallavotti, and Nicolò (1980), and Benfatto *et al.* (1982) has been criticized as complex and unnecessarily so, being based on "delicate" properties of higher-order elliptic boundary value problems; I do not think that this criticism is justified. While it is true that one relies on properties of PDE's, interesting in themselves but technically involved, it should be stressed that the proof proposed in the above reference is conceptually very simple and intuitive and also provides a nice general technique for the theory of Markov fields. The basic ideas behind the proof are explained in a simple form and in simple cases in Gallavotti (1980). A simpler account on the other earlier ideas can be found in Gallavotti (1979). The detailed proofs of Lemma 1 presented in the above-cited papers should not mislead the reader into believing that they are much more than technical developments of a very simple probabilistic idea. I also believe that the so-called simpler proofs are either weaker or equivalently difficult, not surprisingly so by the well-known law of conservation of difficulties.

Field theory is a technical domain, and I believe that all proofs there are equivalently hard and equivalent to the first proofs ever given; it is useful to devise new ones, because they can lead to the more efficient organization of the proofs and to the intuition behind them, which seems essential for further progress.

XIV. LARGE DEVIATIONS: THEIR CONTROL AND THE COMPLETE CONSTRUCTION OF THE COSINE FIELD BEYOND $\alpha^2=4\pi$

The work done in Sec. XIII solves in some sense the problem of the ultraviolet stability when the random fields into which one decomposes the free field are constrained to fluctuate by a finite amount, albeit large compared to their average fluctuation. The amount of the allowed fluctuations is described by the parameter B in (13.4).

One cannot easily take the limit $B\to\infty$ because (see Sec. XIII) the error estimates in (13.24) diverge with B in general ($R_t \to_{B\to\infty} \infty$).

Actually, this is the case for $\alpha^2\in[4\pi,8\pi)$, while for $\alpha^2<4\pi$, as already mentioned in Sec. XIII, the properties (13.4) are not necessary to obtain bounds on the effective potentials and the error term in (13.24) is uniform in B [because in (13.20) the constant C_t can be taken independent of B; see the remark after (13.20)].

For $\alpha^2<4\pi$ it is therefore easy to let $B\to\infty$ and build a family P_λ, $\lambda\in R$, of probability measures on the fields on R^2, which verifies the properties of the proposition at the beginning of Sec. XIII but which is not concentrated on an ensemble of fields restricted by (13.4); this is a family of measures that can naturally be taken as defining the interacting cosine field for $\alpha^2<4\pi$; with some extra work it could also be proved that the limit as $N\to\infty$ of the interaction measure (13.1) with $B=+\infty$ exists without any need of passing to subsequences, and hence no nonuniqueness problems arise.

A complete theory of the cosine interaction for $\alpha^2<4\pi$ has been first worked out in Fröhlich (1976), where the infrared limit is also studied.

Much more interesting, as a field theory problem, is the case $\alpha^2\in[4\pi,8\pi)$. So far the possibility of removing the "field cutoff" B has been really proved only in the interval $\alpha^2\in[4\pi,32\pi/5)\subset[4\pi,8\pi)$; the values $\alpha^2\in[32\pi/5,8\pi)$ have not yet been reached, because, as will become clear soon, one has to find some suitable positivity property of the effective potential, and in Benfatto *et al.* (1982) and Nicolò (1983) the positivity has been checked "by hands" rather than on the basis of a general algorithm; since the positivity requirements become stronger and stronger as $\alpha^2\to 8\pi$, it is impossible to take α^2 too close to 8π unless one understands in a simpler way why things seem to adjust to produce the right signs at the right moments.

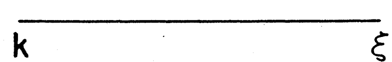
I shall first discuss in some detail the mechanism which allows one to remove the field cutoff ($B\to\infty$) for $\alpha^2\in[4\pi,16\pi/3)$: this is the case in which the minimum value that can be given to t in (13.24) is $t=2$, as discussed in Sec. XIII.

Since t is so small, it is easy to write explicitly $[V^{(k)}(\varphi^{(\le k)})]^{[t]}$ in terms of the graphically eloquent tree language or as a plain old-fashioned formula.

In the trees' picture one has

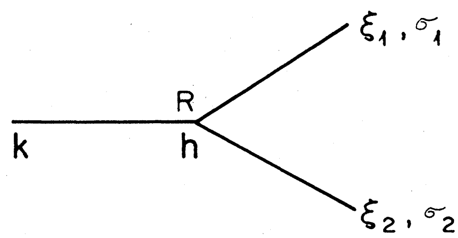
$$[V^{(k)}(\varphi^{(\le k)})]^{[2]}=\int \sum_{\substack{\gamma,k(\gamma)=k\\ \text{degree }\gamma\le 2}} \frac{V(\gamma)}{n(\gamma)}d\underline{\xi}\,, \tag{14.1}$$

and the $V(\gamma)$ are represented, if $\sigma=\pm1$, by



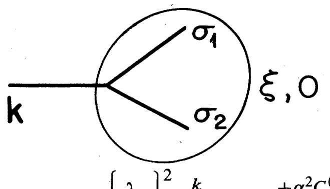
$$=\frac{\lambda}{2}:e^{i\alpha\sigma\varphi_\xi^{(\le k)}}:\,, \tag{14.2}$$

$$=\nu\,, \tag{14.3}$$



$$=\left[\frac{\lambda}{2}\right]^2\left(e^{-\alpha^2C^{(h)}_{\xi_1\xi_2}\sigma_1\sigma_2}-1\right)e^{-\alpha^2\sigma_1\sigma_2C^{(\le h-1)}_{\xi_1\xi_2}}$$

$$\times\;:\left(e^{i\alpha(\sigma_1\varphi^{(\le k)}_{\xi_1}+\sigma_2\varphi^{(\le k)}_{\xi_2})}-\delta_{\sigma_1+\sigma_2,0}\right):\,, \tag{14.4}$$



$$= -\left[\frac{\lambda}{2}\right]^2 \sum_{h=0}^{k} \int (e^{+\alpha^2 C^{(h)}_{\xi_1\xi}} - 1) e^{+\alpha^2 C^{(\le h-1)}_{\xi_1\xi}} \times \delta_{\sigma_1+\sigma_2,0} d\xi_1 , \qquad (14.5)$$

and in (14.4) and (14.5) the subtraction affects only the zero-charge trees ($\sigma_1+\sigma_2=0$) as expressed by $\delta_{\sigma_1+\sigma_2,0}$; the combinatorial factor $n(\gamma)$ is 1 for (14.2) and (14.3) and 2! for (14.4) and (14.5).

If we sum over the frequencies and the charges, the following analytic representation for $[V^{(k)}]^{[2]}$ emerges:

$$[V^{(k)}]^{[2]} \equiv \int_\Lambda [\lambda{:}\cos(\alpha\varphi_\xi^{(\le k)}){:} + \nu] d\xi + \left[\frac{\lambda}{2}\right]^2 \int_\Lambda (e^{-\alpha^2 C^{(\le N)}_{\xi\eta}} - e^{-\alpha^2 C^{(\le k)}_{\xi\eta}}){:}\cos[\alpha(\varphi_\xi^{(\le k)} + \varphi_\xi^{(\le k)})]{:} d\xi\, d\eta$$
$$+ \left[\frac{\lambda}{2}\right]^2 \int_\Lambda (e^{+\alpha^2 C^{(\le N)}_{\xi\eta}} - e^{+\alpha^2 C^{(\le k)}_{\xi\eta}}){:}\{\cos[\alpha(\varphi_\xi^{(\le k)} - \varphi_\xi^{(\le k)}) - 1]\}{:} d\xi\, d\eta - \left[\frac{\lambda}{2}\right]^2 \int_\Lambda (e^{+\alpha^2 C^{(\le k)}_{\xi\eta}} - 1) d\xi\, d\eta . \qquad (14.6)$$

In this special case one recognizes the features of the general cases discussed in Sec. XIII: the only "dangerous term" is the third, big because of the $+\alpha^2$ in the exponential. However, using the ideas of the preceding section, one can see (as already seen in general in Sec. XIII) that its contribution to the effective coupling is

$$\sim \left[\frac{\lambda}{2}\right]^2 \int_{\Delta^2} |\xi-\eta|^{-\alpha^2/2\pi} \alpha^2 B_\kappa^2 (\gamma^k |\xi-\eta|)^{2-2\varepsilon} d\xi\, d\eta = \lambda_N(k)^2 , \qquad (14.7)$$

where $\exp(\alpha^2 C^{(\le N)}_{\xi\eta})$ has been bounded, uniformly in N, by $C^{(\le N)}_{\xi\eta} \le (2\pi)^{-1} \ln |\xi-\eta|^{-1}$ and it has been assumed [see (3.16)] that

$$|\sin(\alpha/2)(\varphi_\xi^{(\le k)} - \varphi_\eta^{(\le k)})| \le B_k (\gamma^k |\xi-\eta|)^{1-\varepsilon} , \quad \xi,\eta \in \Delta \qquad (14.8)$$

for some B_k; Δ is a cube of the pavement Q_k of Λ by cubes of side length γ^{-k}. Then the integral (14.7) is easily evaluated by a scale transformation of Δ to a unit box and, in conformity with the general bounds of Sec. XIII, yields

$$\lambda_N(k)^2 \equiv \lambda^2 \gamma^{-4k} \gamma^{(\alpha^2/2\pi)k} B_k^2 \cdot \text{const}$$
$$\simeq (\lambda\gamma^{(\alpha^2/4\pi - 2)k})^2 B_k^2 , \qquad (14.9)$$

expressing the asymptotic freedom of the second-order contribution to $V^{(k)}$, for $\alpha^2 < 8\pi$.

The problem of going beyond the formal perturbation theory is that one cannot neglect the regions where (14.8) does not hold with B_k given by

$$B_k = B[\ln(e + k + \lambda^{-1})](1+k)^a , \qquad (14.10)$$

as one would like to do on the grounds that, for $a \ge \frac{1}{2}$, the probability of field fluctuations' violating (14.8) is exceedingly small, as described by the phase-space estimates (13.6).

In fact, although such fluctuations are irrelevant in the description of the free field, they might be enhanced in the interacting field case, because the potential $[V^{(k)}(\varphi^{(\le k)})]^{[2]}$ becomes very large (and, worse, its size is even N dependent, even for k small) in the regions $(\xi,\eta) \in \Lambda^2$ where (14.8) is violated.

At this point one is usually confronted with the statement "well, the free field $\varphi^{(\le k)}$ will have a distribution which depresses the phase-space regions where the free-field measure contains, among other things, a term like $\exp[-\frac{1}{2}\int_\Lambda (\partial\varphi_\xi)^2 d\xi]$."

More precisely, one refers here to the possibility of bounding the third term in (14.6) by using the inequalities $(1-\cos x) \le x^2/2$ and

$$(e^{\alpha^2 C^{(\le N)}_{\xi\eta}} - e^{\alpha^2 C^{(\le k)}_{\xi\eta}}) \le \text{const} \times \frac{e^{-\kappa\gamma^k |\xi-\eta|}}{|\xi-\eta|^{\alpha^2/2\pi}} , \qquad (14.11)$$

which follows from the properties of $C^{(0)}_{\xi\eta}$. One finds the bound

$$\left| \lambda^2 \int \{1 - \cos[\alpha(\varphi_\xi^{(\le k)} - \varphi_\eta^{(\le k)})]\} (e^{\alpha^2 C^{(\le N)}_{\xi\eta}} - e^{\alpha^2 C^{(\le k)}_{\xi\eta}}) d\xi\, d\eta \right|$$
$$\le \text{const} \times \lambda^2 \alpha^2 \int (\varphi_\xi^{(\le k)} - \varphi_\eta^{(\le k)})^2 |\xi-\eta|^{-\alpha^2/2\pi} e^{-\gamma^k \kappa |\xi-\eta|} d\xi\, d\eta$$
$$\le (\text{use here Lagrange's theorem}) \le \text{const} \times \lambda^2 \alpha^2 \gamma^{2(\alpha^2/4\pi - 2)k} \int |\partial\varphi_\xi^{(\le k)}|^2 d\xi \qquad (14.12)$$

[the Wick ordering in the third term in (14.6) has been neglected, as it is not very important, since the term comes from a zero-charge tree], expressing the notion that the "bad term" in (14.6) is dominated by $\int (\partial\varphi_\xi^{(\leq k)})^2 d\xi$ times a small constant, if k is large.

However, the proof that follows controls the large part of (14.6) by a method *not reducible* just to the inequality (14.12) and making use of more detailed properties of the expression (14.6); this seems to be the reason why the proof below cannot be immediately extended to cover the whole range $\alpha^2\in[4\pi,8\pi)$; of course, this does not mean that a proof based just on the validity of the inequality (14.12) is not possible—and, in fact, one should look for it.

In fact, one can see that the region of the φ fields where (13.4) fails gives only a very small correction in the computation of the error terms via the following argument.

Fix $B>1$ in (13.4) once and for all [see below] and $a>\frac{1}{2}$ large (say, $a=\frac{3}{2}$; this parameter could probably be taken even equal to $\frac{1}{2}$ by suitably refining the estimates below).

Given $\varphi^{(0)},\ldots,\varphi^{(k-1)},\varphi^{(k)}$, define

$$\mathscr{D}_k=\{\xi,\eta\mid \xi,\eta\in\Lambda \text{ and } |\sin(\alpha/2)(\varphi_\xi^{(\leq k)}-\varphi_\eta^{(\leq k)})|>B_k(\gamma^k|\xi-\eta|)^{1-\varepsilon}\}\ , \tag{14.13}$$

where $\varepsilon>0$ is the number in (13.4), fixed so that (13.20) holds [i.e., $\varepsilon \ll (2-\alpha^2/4\pi)$]. Let $\mathscr{D}_{-1}=\varnothing$.

Define also $R_{-1}=\varnothing$ and

$$R_k=\{\Delta\mid\Delta\in Q_k,\ \exists\,\xi,\eta \text{ with } \xi\in\Delta,\ \gamma^k|\xi-\eta|<1, \text{ and } |\sin(\alpha/2)(\varphi_\xi^{(k)}-\varphi_\xi^{(k)})|>(B_k/\sigma)(\gamma^k|\xi-\eta|)^{1-\varepsilon}\}\ , \tag{14.14}$$

where $\sigma>1$ is conveniently chosen later.

Then for $k\geq 0$

$$\mathscr{D}_k\subset\mathscr{D}_{k-1}\cup(R_k\times R_k)\ . \tag{14.15}$$

In fact, let $(\xi,\eta)\in\mathscr{D}_k$ and $\xi\in\Delta,\eta\in\Delta'$. Suppose that $(\xi,\eta)\notin\mathscr{D}_{k-1}\cup(R_k\times R_k)$; then $(\gamma^k|\xi-\eta|)<B_k^{1/(1-\varepsilon)}<B_k^{-1}<1$, and

$$\begin{aligned}&|\sin(\alpha/2)(\varphi_\xi^{(k)}-\varphi_\eta^{(k)})|<\frac{B_k}{\sigma}(\gamma^k|\xi-\eta|)^{i-\varepsilon}\ ,\\ &|\sin(\alpha/2)(\varphi_\xi^{(\leq k-1)}-\varphi_\eta^{(\leq k-1)})|<B_{k-1}(\gamma^{k-1}|\xi-\eta|)^{1-\varepsilon}\ ;\end{aligned} \tag{14.16}$$

otherwise $\Delta\in R_k$ and $\Delta'\in R_k$ hence $(\xi,\eta)\in R_k\times R_k$ or $(\xi,\eta)\in\mathscr{D}_{k-1}$.

But (14.16) implies, for $k\geq 1$, the contradiction with $(\xi,\eta)\in\mathscr{D}_k$:

$$\begin{aligned}|\sin(\alpha/2)(\varphi_\xi^{(\leq k)}-\varphi_\eta^{(\leq k)})|&\equiv|\sin(\alpha/2)(\varphi_\xi^{(\leq k-1)}-\varphi_\eta^{(\leq k-1)})\cos(\alpha/2)(\varphi_\xi^{(k)}-\varphi_\eta^{(k)})\\ &\qquad+\cos(\alpha/2)(\varphi_\xi^{(\leq k-1)}-\varphi_\eta^{(\leq k-1)})\sin(\alpha/2)(\varphi_\xi^{(k)}-\varphi_\eta^{(k)})|\\ &\leq B_{k-1}(\gamma^{k-1}|\xi-\eta|)^{1-\varepsilon}+\frac{B_k}{\sigma}(\gamma^k|\xi-\eta|)^{1-\varepsilon}\\ &\leq(B_{k-1}\gamma^{-(1-\varepsilon)}+B_k/\sigma)(\gamma^k|\xi-\eta|)^{1-\varepsilon}\\ &\leq B_k(\gamma^k|\xi-\eta|)^{1-\varepsilon}(\gamma^{-(1-\varepsilon)}+1/\sigma)\leq B_k(\gamma^k|\xi-\eta|)^{1-\varepsilon}\end{aligned} \tag{14.17}$$

provided σ is chosen so large, as it can and will be, that

$$\gamma^{-(1-\varepsilon)}+\sigma^{-1}\leq\theta<1,\quad \forall k\geq 1\ . \tag{14.18}$$

The case $k=0$ is analogous, if $\varphi^{(-1)}\equiv 0$.

Coming back to (14.15), assume, inductively, that it has been possible to prove that

$$\int e^{V(\varphi^{(\leq N)})}P(d\varphi^{(N)})\cdots P(d\varphi^{(k+1)})\leq e^{[\hat V_\Lambda^{(k)}+R_+(k)|\Lambda|]}\ , \tag{14.19}$$

where [see (14.6)],

$$\begin{aligned}\hat V_\Lambda^{(k)}\equiv{}&\lambda\int_\Lambda :\cos(\alpha\varphi_\xi^{(\leq k)}):d\xi+\left[\frac{\lambda}{2}\right]^2\int_{\Lambda^2}(e^{-\alpha^2C_{\xi\eta}^{(\leq N)}}-e^{-\alpha^2C_{\xi\eta}^{(\leq k)}}):\cos[\alpha(\varphi_\xi^{(\leq k)}+\varphi_\eta^{(\leq k)})]:d\xi\,d\eta\\ &+\left[\frac{\lambda}{2}\right]^2\int_{\Lambda^2/\mathscr{D}_k}(e^{\alpha^2C_{\xi\eta}^{(\leq N)}}-e^{\alpha^2C_{\xi\eta}^{(\leq k)}}):\cos[\alpha(\varphi_\xi^{(\leq k)}-\varphi_\xi^{(\leq k)})-1]:d\xi\,d\eta\\ &-\left[\frac{\lambda}{2}\right]^2\int_{\Lambda^2}(e^{\alpha^2C_{\xi\eta}^{(\leq k)}}-1)d\xi\,d\eta\ ,\end{aligned} \tag{14.20}$$

i.e., one assumes that the part of the interaction which caused the worst problem in (14.6) is actually missing in (14.20) [and of course one will also assume a good bound $R_+(k)$; see below].

The reason this is not a terrible approximation is related to a special property of $\widetilde{V}^{(k)} \equiv [V^{(k)}(\varphi^{(\le k)})]^{[2]}$, whereby such bad terms, if present, would be very negative, and therefore they could be really thrown out of the integration of the exponential of (14.20), because one is interested only in upper bounds (the lower bounds having been discussed in Sec. XIII).

The negativity of $\widetilde{V}^{(k)} - \widehat{V}_\Lambda^{(k)}$, i.e., of

$$\int_{\mathscr{D}_k} (e^{\alpha^2 C_{\xi\eta}^{(\le N)}} - e^{\alpha^2 C_{\xi\eta}^{(\le k)}}):\cos[\alpha(\varphi_\xi^{(\le k)} - \varphi_\eta^{(\le k)}) - 1]:d\xi\, d\eta$$

$$\equiv \int_{\mathscr{D}_k} (e^{\alpha^2 C_{\xi\eta}^{(\le N)}} - e^{\alpha^2 C_{\xi\eta}^{(\le k)}}) e^{\alpha^2 (C_{00}^{(\le k)} - C_{\xi\eta}^{(\le k)})} (\{\cos[\alpha(\varphi_\xi^{(\le k)} - \varphi_\eta^{(\le k)}) - 1]\} + (1 - e^{-\alpha^2 (C_{00}^{(\le k)} - C_{\xi\eta}^{(\le k)})}))d\xi\, d\eta \tag{14.21}$$

holds, because in $\mathscr{D}_k$ it is

$$-2\sin^2(\alpha/2)(\varphi_\xi^{(\le k)} - \varphi_\eta^{(\le k)}) + \alpha^2 (C_{00}^{(\le k)} - C_{\xi\eta}^{(\le k)}) \le -2B_k^2 (\gamma^k |\xi - \eta|)^{2-2\varepsilon} + \alpha^2 \overline{C} |\gamma^k (\xi - \eta)|^{2-2\varepsilon} , \tag{14.22}$$

where

$$C_{00}^{(\le k)} - C_{\xi\eta}^{(\le k)} = \sum_{j=0}^{k} (C_{00}^{(j)} - C_{\xi\eta}^{(j)}) \le \sum_{j=0}^{k} \widetilde{C}_\varepsilon (|\xi - \eta| \gamma^j)^{2-2\varepsilon} \le \overline{C} (\gamma^k |\xi - \eta|)^{2-2\varepsilon} \tag{14.23}$$

($\widetilde{C}_\varepsilon$ and $\overline{C}$ being suitable constraints) has been used.

If B is supposed to be so chosen that $B_0^2 > \alpha^2 \overline{C}$, it follows that the rhs of Eq. (14.22) is bounded by

$$-B_k^2 (\gamma^k |\xi - \eta|)^{2-2\varepsilon} < 0 . \tag{14.24}$$

This observation makes it possible to neglect the interaction, or at least its bad part, in the regions where the field is rough and one can use the free-field properties to prove this via rigorous bounds.

The precise way in which one uses the above ideas to study the integral

$$\int e^{\widehat{V}_\Lambda^{(k)}} P(d\varphi^{(k)}) \tag{14.25}$$

is the following.

The first step in estimating (14.25) is to replace $\widehat{V}_\Lambda^{(k)}$ by a simpler function, at least as far as the functional dependence on $\varphi^{(k)}$ is concerned; note that the $\varphi^{(k)}$ dependence of (14.20) is neither polynomial nor trigonometrical, since $\varphi^{(k)}$ enters in a most complex way into the integration domains.

To find the simpler form that is sought, think that Λ in (14.20) is replaced by an arbitrary set J and call $\widehat{V}_J$ the resulting expression.

Then for a suitably chosen $A, \widetilde{A}(\lambda)$:

$$\begin{aligned} \widehat{V}_J^{(k)} &\le \widehat{V}_{J/R_k} + \mathscr{N}(R_k)(\lambda \gamma^{(\alpha^2/4\pi - 2)k} \\ &\qquad + \lambda^2 \gamma^{2(\alpha^2/4\pi - 2)k} B_k^2) A \\ &= \widehat{V}_{J/R_k} + \mathscr{N}(R_k) \widetilde{A}(\lambda) \gamma^{(\alpha^2/4\pi - 2)k} B^2 , \end{aligned} \tag{14.26}$$

which follows immediately from the asymptotic freedom bounds (13.20), which in turn hold because $\varphi_\xi^{(\le k)} - \varphi_\eta^{(\le k)}$ is considered only in the region $J^2/\mathscr{D}_k$; $\mathscr{N}(R_k)$ is just the number of boxes composing R_k.

Therefore, (14.25) can be bounded above by

$$\sum_{R_k} \int e^{\widehat{V}_{\Lambda/R_k}} \chi(R_k) P(d\varphi^{(k)}) e^{\widetilde{A}(\lambda) B^2 \gamma^{(\alpha^2/4\pi - 2)k} \mathscr{N}(R_k)} , \tag{14.27}$$

where χ recalls that $\varphi^{(k)}$ is constrained to be such that the rhs of (14.14) is precisely R_k.

Then call H_J the expression obtained from $\widehat{V}_J$ by replacing $\mathscr{D}_k$ by $\mathscr{D}_{k-1}$; it is therefore, by definition,

$$\begin{aligned} H_{\Lambda/R_k} \equiv{}& \lambda \int_{\Lambda/R_k} :\cos(\alpha \varphi_\xi^{(\le k)}):d\xi \\ &+ \left[\frac{\lambda}{2}\right]^2 \int_{(\Lambda/R_k)^2} (e^{-\alpha^2 C_{\xi\eta}^{(\le N)}} - e^{-\alpha^2 C_{\xi\eta}^{(\le k)}}):\cos[\alpha(\varphi_\xi^{(\le k)} + \varphi_\eta^{(\le k)})]:d\xi\, d\eta \\ &+ \left[\frac{\lambda}{2}\right]^2 \int_{(\Lambda/R_k)^2/\mathscr{D}_{k-1}} (e^{\alpha^2 C_{\xi\eta}^{(\le N)}} - e^{\alpha^2 C_{\xi\eta}^{(\le k)}}):\cos[\alpha(\varphi_\xi^{(\le k)} - \varphi_\xi^{(\le k)}) - 1]:d\xi\, d\eta \\ &- \left[\frac{\lambda}{2}\right]^2 \int_{(\Lambda/R_k)^2} (e^{\alpha C_{\xi\eta}^{(\le k)}} - 1) d\xi\, d\eta . \end{aligned} \tag{14.28}$$

It is easy to check that

$$\widehat{V}_{\Lambda/R_k} \equiv H_{\Lambda/R_k} + \left[\frac{\lambda}{2}\right]^2 \int_{S_k} (e^{\alpha^2 C_{\xi\eta}^{(\le N)}} - e^{\alpha^2 C_{\xi\eta}^{(\le k)}}):\cos[\alpha(\varphi_\xi^{(\le k)} - \varphi_\eta^{(\le k)}) - 1]:d\xi\, d\eta , \tag{14.29}$$

where $S_k=(\Lambda/R_k)^2\cap(\mathscr{D}_{k-1}/\mathscr{D}_k)$ as

$$[(\Lambda/R_k)^2/\mathscr{D}_k]=\{[(\Lambda/R_k)^2/\mathscr{D}_{k-1}]\cup S_k\}/[(\Lambda/R_k)^2\cap\mathscr{D}_k/\mathscr{D}_{k-1}]\,, \tag{14.30}$$

and the set $(\Lambda/R_k)^2\cap\mathscr{D}_k/\mathscr{D}_{k-1}$ is empty because of $\mathscr{D}_k\subset\mathscr{D}_{k-1}\cup(R_k\times R_k)$ [see (14.15)].

Let $(\xi,\eta)\in S_k\subset\mathscr{D}_{k-1}\cap(\Lambda/R_k)^2$, $k\geq 1$; then it is

$$(\gamma^{k-1}|\xi-\eta|)^{1-\varepsilon}B_{k-1}<1,\ \text{ i.e., } (\gamma^k|\xi-\eta|)^{1-\varepsilon}<\gamma^{1-\varepsilon}B_{k-1}^{-1} \tag{14.31}$$

(because the sine is bounded by 1).

Hence for all $k\geq 1$ and $(\xi,\eta)\in S_k$

$$\begin{aligned}
&|\sin(\alpha/2)(\varphi_\xi^{(\leq k)}-\varphi_\eta^{(\leq k)})|\\
&\quad\geq|\sin(\alpha/2)(\varphi_\xi^{(\leq k-1)}-\varphi_\eta^{(\leq k-1)})|\,|\cos(\alpha/2)(\varphi_\xi^{(k)}-\varphi_\eta^{(k)})|-|\sin(\alpha/2)(\varphi_\xi^{(k)}-\varphi_\eta^{(k)})|\\
&\quad\geq B_{k-1}(\gamma^{k-1}|\xi-\eta|)^{1-\varepsilon}\left[1-\left[\frac{B_k}{\sigma}(\gamma^k|\xi-\eta|)^{1-\varepsilon}\right]^2\right]^{1/2}-\frac{B_k}{\sigma}\gamma^{1-\varepsilon}(\gamma^{k-1}|\xi-\eta|)^{1-\varepsilon}\\
&\quad\geq B_k\left[\frac{B_{k-1}}{B_k}\left[1-\frac{B_k^2}{B_{k-1}^2}\frac{1}{\sigma^2}\right]^{1/2}-\frac{\gamma^{1-\varepsilon}}{\sigma}\right](\gamma^{k-1}|\xi-\eta|)^{1-\varepsilon}\geq B_k\theta(\gamma^{k-1}|\xi-\eta|)^{1-\varepsilon}\,,
\end{aligned} \tag{14.32}$$

and, if we suppose (as we can) that σ is large enough and use (14.11), θ is

$$\theta=\min_{k\geq 1}\left[\frac{B_{k-1}}{B_k}\left[1-\frac{B_k^2}{B_{k-1}^2}\frac{1}{\sigma^2}\right]^{1/2}-\frac{\gamma^{1-\varepsilon}}{\sigma}\right]>0\,. \tag{14.33}$$

The inequality between the first and the last terms can be checked, also, for $k=0$, directly.

Therefore, the integral in (14.29) is for all $k\geq 0$ nonpositive, provided [see also (14.22) and (14.23)]

$$-2\sin^2(\alpha/2)(\varphi_\xi^{(\leq k)}-\varphi_\eta^{(\leq k)})+\alpha^2(C_{00}^{(\leq k)}-C_{\xi\eta}^{(\leq k)})\leq(-2\theta^2B_k^2\gamma^{-2(1-\varepsilon)}+\alpha^2\overline{C})(\gamma^k|\xi-\eta|)^{2(1-\varepsilon)}<0\,, \tag{14.34}$$

i.e., if B is supposed large enough, as is possible. Hence for all $k\geq 0$

$$\hat{V}_{\Lambda/R_k}\leq H_{\Lambda/R_k}\,, \tag{14.35}$$

which implies, if we go back to (14.27),

$$\int e^{\hat{V}_\Lambda^{(k)}}P(d\varphi^{(k)})\leq\sum_{R_k}\left[\left[\int\chi(R_k)e^{H_{\Lambda/R_k}}P(d\varphi^{(k)})\right]\exp[\mathscr{N}(R_k)\widetilde{A}(\lambda)\gamma^{(\alpha^2/4\pi-2)k}]\right]. \tag{14.36}$$

The advantage of replacing $\hat{V}_{\Lambda/R_k}^{(k)}$ in (14.27) by H_{Λ/R_k} is that the function H_{Λ/R_k} is a "simple trigonometrical expression" in the fields $\varphi^{(k)}$ [see (14.28)], and no dependence is any more present on the very complicated set $\mathscr{D}_k$; of course there is a dependence on R_k, but R_k is a union of cubes and therefore this dependence is not so bad—besides, one wishes to keep it fixed, as the integral (14.36) is performed at fixed R_k (because of the presence of the χ functions).

At this point one needs a way of estimating integrals like the one in (14.36).

What is known about the integrand is that H_J can be written as

$$\begin{aligned}
H_J=&\sum_{\Delta\in Q_k}\lambda\int_{\Delta\cap J}h(\xi_1)\cos(\alpha\varphi_{\xi_i}^{(\leq k)})\frac{d\xi_1}{|\Delta|}+\sum_{\substack{\Delta_1,\Delta_2\in Q_k\\ \sigma_1,\sigma_2=\pm1}}\lambda^2\int_{(\Delta_1\times\Delta_2)\cap J^2}h_{\sigma_1\sigma_2}^{(2)}(\xi_1,\xi_2)\cos[\alpha(\sigma_1\varphi_{\xi_1}^{(\leq k)}+\sigma_2\varphi_{\xi_2}^{(\leq k)})]\frac{d\xi_1}{|\Delta|_2}\frac{d\xi_2}{|\Delta_2|}\\
&+\sum_{\Delta_1,\Delta_2,\in Q_k}\lambda^2\int_{\Delta_1\times\Delta_2\cap J^2}h^{(2)}(\xi_1,\xi_2)\frac{1-\cos[\alpha(\varphi_{\xi_1}^{(\leq k)}-\varphi_{\xi_2}^{(\leq k)})]}{(\gamma^k|\xi_1-\xi_2|)^{1-\varepsilon}}\frac{d\xi_1}{|\Delta_1|}\frac{d\xi_2}{|\Delta_2|}\\
&+\sum_{\Delta_1,\Delta_2,\in Q_k}\lambda^2\int_{\Delta_1\times\Delta_2\cap J^2}h^{(0)}(\xi_1,\xi_2)\frac{d\xi_1}{|\Delta_1|}\frac{d\xi_2}{|\Delta_2|}\,,
\end{aligned} \tag{14.37}$$

where the $h^{(\,)}$ functions are λ independent and where the $\mathscr{D}_{k-1}$ dependence can be thought as included in the h functions. Furthermore, the theory of the preceding sections or the explicit expressions for the h functions [see (14.28)] imply that

$$\int_\Delta |\lambda|\,|h(\xi_1)|\,d\xi_1 \le \widetilde{A}\lambda\gamma^{(\alpha^2/4\pi-2)k} \le \overline{H}_k\ , \tag{14.38}$$

$$\int_{\Delta_1\times\Delta_2} \lambda^2\,|h^{(\ )}(\xi_1,\xi_2)|\,d\xi_1 d\xi_2 \le (\lambda\gamma^{(\alpha^2/4\pi-2)k})^2\overline{A}B_k^2 e^{-\kappa\gamma^k d(\Delta_1,\Delta_2)} \le \overline{H}_k\ , \tag{14.39}$$

where $\widetilde{A},\overline{A},\overline{H}_k$ are suitably chosen constants.

In other words at fixed R_k the integral in (14.36) looks like the partition function of a classical spin system on the lattice Q_k.

The reason the estimates (14.38) and (14.39) do not depend on $\varphi^{(\le k-1)}$ is that in the "bad terms" of H_Λ *no pair* $(\xi,\eta)\in\mathscr{D}_{k-1}$ appears, so that (14.39) is obtained by the same estimates leading to the proof of asymptotic freedom (and actually follows from them) in Sec. XIII.

It is possible to formulate a rather general version of the Mayer expansion allowing one to estimate naively the integral (14.36).

Let χ_Δ^b, $\mathring{\chi}_\Delta^b \equiv 1-\chi_\Delta^b$ be the characteristic functions of the events on $\varphi^{(k)}$, given $\Delta\in Q_k$,

$$\{\varphi^{(k)}\,|\,\exists\,\xi,\eta\in\Delta,\ |\varphi_\xi^{(k)}-\varphi_\eta|<b(\gamma^k|\xi-\eta|)^{1-\varepsilon}\} \tag{14.40}$$

and its complement.

Let R be a subset of Λ pavable by Q_k, i.e., union of Δ's in Q_k, and let R^c be its complement; denote

$$\chi_R = \prod_{\Delta\subset R} \chi_\Delta\ ,$$

if R is the disjoint union of the cubes $\Delta\subset R$; then the following lemma closely related to Lemma 1, Sec. XIII, holds.

Lemma 2. Given $t\ge 0$ integer, there exist constants G,g,g',b^* depending only on t (and on the parameters $\gamma,\varepsilon,\kappa$) such that if H_Λ verifies (14.38)—(14.40) then

$$\int P(d\varphi^{(k)})\chi_{R^c}^b \mathring{\chi}_R^b e^{H_\Lambda/R} \le \left[\int \mathring{\chi}_R^b P(d\varphi^{(k)})\right]^{1/2} \left\{\exp\left[\left[\sum_{p=1}^t \mathscr{E}_k^T(H_\Lambda;p)p!^{-1}\right]^{[t]} + [\delta(b,\overline{H}_k)\gamma^{2k}|\Lambda| + \delta'(b,\overline{H}_k)\mathscr{N}(R)]\right]\right\}, \tag{14.41}$$

where the errors have a value close to the one which would be naively expected from the point of view of statistical mechanics:

$$\delta(b,\overline{H}_k) \le G[(\overline{H}_k b^g e^{\overline{H}_k b^g})^{t+1} + e^{-g'b^2+g\overline{H}_k b^g}]\ ,$$
$$\delta'(b,\overline{H}_k) \le G\overline{H}_k b^g\ , \tag{14.42}$$

and $\mathscr{N}(R)$ is the number of cubes Δ in R. Furthermore, if b is large enough, $b>b^*$:

$$\int P(d\varphi^{(k)})\chi_\Lambda^b e^{H_\Lambda} \ge \exp\left[\left[\sum_{p=1}^t \frac{\mathscr{E}_k^T(H_\Lambda;p)}{p!}\right]^{[t]} - \delta(b,\overline{H}_k)\gamma^{2k}|\Lambda|\right], \tag{14.43}$$

and finally, for suitably chosen, k independent, constants α_0,β_0:

$$\int P(d\varphi^{(k)})\mathring{\chi}_R^b \le (\alpha_0 e^{-\beta_0 b^2})^{\mathscr{N}(R)}\ . \tag{14.44}$$

The k dependence of the constants is trivially due to the scaling properties of the field. The first bound in (14.42) could be easily improved: here it is given in the form in which it had been found in Benfatto *et al.* (1978,1982), Benfatto, Cassandro *et al.* (1980), and Benfatto, Gallavotti, and Nicolò (1980) where Lemma 2 is proved under the extra assumption γ close to 1 (an assumption which can be easily released but which is anyway sufficient for our purposes, since γ is restricted only to be $\gamma>1$).

Clearly (14.43) implies as a special case Lemma 1 of Sec. XIII. Lemmas 1 and 2 will not be proved here, because their statistical mechanics character makes them somewhat foreigners to field theory; also, a detailed proof would be very long in spite of its conceptual simplicity; the reader can find this proof in the references given above.

At this point it is easy to conclude all the estimates, if one observes that in the present case

$$\left[\sum_{p=1}^2 \frac{1}{p!}\mathscr{E}_k^T(H_\Lambda;p)\right]^{[2]} \le \hat{V}^{(k-1)}(\varphi^{(\le k-1)})\ . \tag{14.45}$$

This is because, the lhs being a Gaussian integral of simple trigonometric functions, one can explicitly compute the lhs; after a simple calculation one finds that the difference between the rhs and lhs is given exactly by

$$\int_{\mathscr{D}_{k-1}} (e^{\alpha^2 C_{\xi\eta}^{(\le k)}} - e^{\alpha^2 C_{\xi\eta}^{(\le k-1)}}) \times :\cos[\alpha(\varphi_\xi^{(\le k-1)}-\varphi_\eta^{(\le k-1)})-1]:d\xi\,d\eta\ , \tag{14.46}$$

which is not positive for the same reasons (14.21) and (14.22) were not positive.

Therefore one applies Lemma 2 to evaluate the integral (14.36), choosing $t=2$ and $b=B_k$; the result, using also (14.45) and (14.46), is

$$\int e^{\hat{V}^{(k)}}P(d\varphi^{(k)}) \le \sum_{R_k} \{ e^{\hat{V}^{(k-1)}+\gamma^{2k}\delta(B_k,\overline{H}_k)} (\alpha_0 e^{-\beta_0 B_k^2})^{\mathcal{N}(R_k)/2} \exp[\mathcal{N}(R_k)\delta'(B_k,\overline{H}_k)] \}$$

$$\equiv e^{\hat{V}^{(k-1)}} e^{\gamma^{2k}\delta(B_k,\overline{H}_k)} (1+\sqrt{\alpha_0} e^{-(\beta_0/2)B_k^2} e^{\delta'(B_k,\overline{H}_k)})^{|\Lambda|\gamma^{2k}} = e^{\hat{V}^{(k-1)}+\varepsilon(k)|\Lambda|} , \qquad (14.47)$$

and, by (14.42), (14.39), and (14.38), $\sum_{k=0}^{\infty}\varepsilon(k)=O(\lambda^t)$.

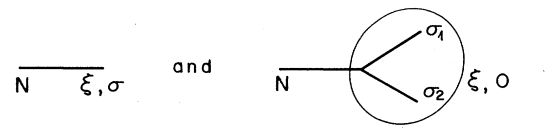
This means that if one assumes (14.19) for $k=N-1$ and (14.29) holds for all $k \ge K(\lambda)$, with

$$R_+(k)=R_+(k+1)+\varepsilon(k) . \qquad (14.48)$$

Hence the ultraviolet stability will be proved as soon as one is able to check (14.29) for $k=N-1$—i.e., one is able to estimate

$$\int e^{V(\varphi^{(\le N)})}P(d\varphi^{(N)}) . \qquad (14.49)$$

In this case the $V(\varphi^{(\le N)})$ is just the sum of the trees



(36)

i.e.,

$$\lambda \int_\Lambda :\cos(\alpha\varphi_\xi^{(\le N)}):d\xi + \nu \int_\Lambda d\xi - \left[\frac{\lambda}{2}\right]^2 \int_\Lambda (e^{\alpha^2 C_{\xi\eta}^{(\le N)}}-1)d\xi\, d\eta$$

$$\equiv \sum_{\Delta\in Q_N} \lambda\gamma^{(\alpha^2/4\pi-2)N} \int_\Delta \cos(\alpha\varphi^{(\le N)})\frac{d\xi}{|\Delta|} + \nu\sum_\Delta \gamma^{-2N} - \sum_\Delta h(\lambda\gamma^{(\alpha^2/4\pi-2)/N})^2 , \qquad (14.50)$$

where

$$h=\int_{\Delta\times\Lambda}\gamma^{-(\alpha^2/2\pi)N} e^{\alpha^2 C_{\xi\eta}^{(\le N)}}(1-e^{-\alpha^2 C_{\xi\eta}^{(\le N)}}) \times \frac{d\xi}{|\Delta|}d\eta \le \text{const} . \qquad (14.51)$$

Hence in the first step one does not have to worry about $\mathscr{D}_N,\mathscr{D}_{N-1}$, because there is no obstacle in using Lemma 2 in evaluating the integral (14.49): assumptions (14.38) and (14.39) are verified with $H_\Lambda = V^{(N)} \equiv V$, by (14.50) and (14.51); i.e., in the first step there is no need to worry about the smoothness of $\varphi^{(N)}$ in order to get the asymptotic freedom bounds, as explicitly remarked in Sec. XIII [see comments following (13.13)].

The validity of the inductive hypothesis for $k=N-1$ is completed by checking that

$$\left[\sum_{p=1}^{2} \mathscr{E}_N^T(V^{(N)};p)\frac{1}{p!}\right]^{[2]} \le \hat{V}^{(N-1)} , \qquad (14.52)$$

which is proved as (14.45) by (14.21) and (14.22) written for $k=N-1$.

This completes the proof of the ultraviolet stability for $\alpha^2 < 16\pi/3$.

There would be no problem in applying the above techniques to evaluate the integral of $\exp(V^{(N)})$ to an arbitrarily fixed order $t>2$.

If $\alpha^2\in[16\pi/3,6\pi)$, still nothing changes, basically, in the above scheme of proof except the series of the errors, both in the upper and in the lower bounds, will converge only if $t\ge 3$; the positivity of the "bad terms" was used in an essential way in the above proof in two steps and now it can be used in the same way. In fact, the two steps were first to remove the region $\mathscr{D}_{k-1}$ from $\hat{V}_{\Lambda/R_k}$ [done in (14.35)], and second to reintroduce it to "rebuild" $\hat{V}^{(k-1)}$ [done in (14.45) and (14.52)].

The just-mentioned two "positivity steps" are now, for $\alpha^2\in[16\pi/3,6\pi)$, carried through in the same way, because it turns out that no new positivity property is needed on $\hat{V}^{(k)}$ besides the one, already pointed out and amply used, present in the second-order part of $\hat{V}^{(k)}$: the second-order dominates in the inequalities necessary to control the third-order terms and its positivity properties are enough for the estimates.

The situation changes for $\alpha^2\ge 6\pi$: now the second-order dominates only in the inequalities necessary to carry out the first of the two steps of the proof where the positivity is needed. In the second step it is not known whether it dominates; in fact, the proof has been carried through in the interval $[6\pi,(\sqrt{17}-1)\pi)$ and later up to $32\pi/5$ by using other ideas, slightly improving on the above ones, based on detailed properties of the effective interaction to fourth order in Benfatto *et al.* (1982) and Nicolò (1983).

In order to obtain ultraviolet stability up to $\alpha^2<8\pi$ some new idea seems necessary, and the paper by Nicolò (1983) seems to go in the right direction; see also the comments after (14.12) above.

The above difficulties are also an indirect consequence of the fact that the large-fluctuation problem has not been solved in the naive way, by free-field domination [see comments after (14.12)], and a better understanding of this point seems important and desirable.

The techniques used for the sine-Gordon equation can be used also to treat the exponential interaction (5.5) for $\alpha^2<4\pi$ [see Fröhlich (1976)]; the exponential interaction can be treated also for $\alpha^2 \gg 4\pi$, for $d=2$, and for $d\ge 3$, which are cases in which it can be proved to be trivial [see Albeverio *et al.* (1979)].

XV. THE COSINE FIELD AND THE SCREENING PHENOMENA IN THE TWO-DIMENSIONAL COULOMB GAS AND IN RELATED STATISTICAL MECHANICAL SYSTEMS

Before studying the φ^4 fields it is appropriate to conclude the theory of the cosine fields by pointing out their "surprising" connection with the two-dimensional classical statistical mechanics of Coulomb systems and Yukawa gases.

The "neutral Coulomb gas" and the "charged Yukawa gas" describe charged particles of charge ± 1, presenting, for some values of the temperature and of the density, very interesting and nontrivial "screening phenomena."

In general, a system of charged particles interacting via a potential C_{xy} will be defined by the grand canonical partition function

$$Z^1(\Lambda,\beta,\lambda)=\sum_{n=0}^{\infty}\left[\frac{\lambda}{2}\right]^n\frac{1}{n!}\sum_{\sigma_1,\ldots,\sigma_n}\int_{\Lambda^n}\exp\left[-\beta\sum_{i<j}\sigma_i\sigma_j C_{x_ix_j}\right]dx_1\cdots dx_n\ , \tag{15.1}$$

where $\sigma_j=\pm 1$; or, in the case of *a priori* neutral systems:

$$Z^0(\Lambda,\beta,\lambda)=\sum_{n=0}^{\infty}\left[\frac{\lambda}{2}\right]^n\frac{1}{n!}\sum_{\substack{\sigma_1,\ldots,\sigma_n\\ \sum_i\sigma_i=0}}\int_{\Lambda}\exp\left[-\beta\sum_{i<j}\sigma_i\sigma_j C_{x_ix_j}\right]dx_1\cdots dx_n\ . \tag{15.2}$$

The cases which can be studied in terms of the cosine interaction are

(a) the regularized Yukawa gas, with C_{xy} given by

$$C_{xy}^{(m_0,M)}=\frac{1}{(2\pi)^2}\int e^{ip(x-y)}\left[\frac{1}{m_0^2+p^2}-\frac{1}{M^2+p^2}\right]dp\ , \tag{15.3}$$

(b) the "regularized Coulomb gas" with C_{xy} given by

$$V_{xy}^{(m_0,M)}=C_{xy}^{(0,M)}-C_{00}^{(0,m_0)}\ , \tag{15.4}$$

where the rhs has to be interpreted as the limit of $C_{xy}^{(m,M)}-C_{00}^{(m,m_0)}$ as $m\to 0$:

$$V_{xj}^{(m_0,M)}=\frac{1}{(2\pi)^2}\int\frac{1}{p^2}\left[\frac{M^2}{M^2+p^2}\cos[p(x-y)]-\frac{m_0^2}{m_0^2+p^2}\right]dp\ . \tag{15.5}$$

Note that when the regularization parameter M is let to $+\infty$ it is

$$V_{xy}^{(m_0,M)}\underset{M\to\infty}{\longrightarrow}\frac{1}{2\pi}\ln(am_0\,|\,x-y\,|\,)^{-2}\le\frac{1}{2\pi}\ln(\overline{m}_0\,|\,x-y\,|\,)^{-1}\ , \tag{15.6}$$

where $a>0$ is a suitable constant ($a=\ln 2-g$, g being a Euler-Mascheroni constant).

The partition function for the above systems can be easily written in terms of a Gaussian random field $\psi^{(-R,N)}$ sum of $N+R+1$ independent fields:

$$\psi_\xi^{(-R,N)}=\sum_{j=-R}^{N}\psi_\xi^{(j)} \tag{15.7}$$

and in terms of the functions

$$:\cos(\alpha\psi_\xi^{(-R,N)}):_{UV}\equiv e^{(\alpha/2)^2C_{00}^{(m_0,m_0\gamma^{N+1})}}\times\cos(\alpha\psi_\xi^{(-R,N)})\ . \tag{15.8}$$

The covariance $C^{(j)}$ of the random field $\psi^{(j)}$ will have the Fourier transform (see Sec. III) (free field with open boundary conditions)

$$\frac{1}{m_0^2\gamma^{2j}+p^2}-\frac{1}{m_0^2\gamma^{2j+2}+p^2}\ . \tag{15.9}$$

Then it is easy to prove that the regularized Yukawa gas partition function is, if we set $\alpha=\sqrt{\beta}$, $M=m_0\gamma^{N+1}$,

$$Z_{Y(N)}^1(\Lambda,\beta,\lambda)=\int\left[\exp\left[\lambda\int_\Lambda:\cos(\alpha\psi_\xi^{(0,N)}):_{UV}d\xi\right]\right]P(d\psi^{(0)})\cdots P(d\psi^{(N)})\ , \tag{15.10}$$

while the regularized Coulomb gas partition function in the neutral grand canonical ensemble and with potential (15.4) with $M=m_0\gamma^{N+1}$ and m_0 replaced, for notational convenience, by $m_0\gamma^{-1}$, is

$$Z_{C(N)}^0(\Lambda,\beta,\lambda)=\lim_{R\to\infty}\int\left[\exp\left[\lambda\int_\Lambda:\cos(\alpha\psi_\xi^{(-R,N)}):_{UV}d\xi\right]\right]P(d\psi^{(-R)})\cdots P(d\psi^{(N)})\ . \tag{15.11}$$

It is convenient to introduce also the auxiliary partition function

$$Z_{C(R,N)}^1(\Lambda,\beta,\lambda)=\int\left[\exp\left[\lambda\int_\Lambda:\cos(\alpha\psi_\xi^{(-R,N)}):_{UV}d\xi\right]\right]P(d\psi^{(-R)})\cdots P(d\psi^{(N)})\ , \tag{15.12}$$

which will be called an "infrared regularized (non-neutral) Coulomb gas" partition function, and (15.11) can then be written

$$Z^0_{C(N)}(\Lambda,\beta,\lambda)=\lim_{R\to\infty} Z^1_{C(R,N)}(\Lambda,\beta,\lambda)\ . \qquad (15.13)$$

Finally, observe the following relation between the Coulomb gas and the Yukawa gas [see (15.10)]:

$$Z^1_{Y(N)}\equiv Z^1_{C(0,N)}\ . \qquad (15.14)$$

The proof of (15.10)—(15.13) has essentially already been explained in Sec. XI (and called there the "multipole expansion"); however, the interpretation work necessary to derive the present claims from Sec. XI is such that it is simpler to derive the above relations from scratch.

Consider the integral in (15.10) and expand the exponential in powers: calling $C^{(-R,N)}\equiv C^{(m_0\gamma^{-R},\gamma^{N+1}m_0)}$, for simplicity, the covariance of $\psi^{(-R,N)}$, we see that

$$\sum_{p=0}^{\infty}\left[\frac{\lambda}{2}\right]^p\frac{1}{p!}\sum_{\sigma_1,\dots,\sigma_p} e^{(\alpha^2/2)pC^{(0,N)}_{00}}\int dx_1\cdots dx_n\,\mathscr{E}\left[\prod_{j=1}^{p} e^{i\alpha\sigma_j\psi^{(0,N)}_{x_j}}\right]$$
$$\equiv\sum_{p=0}^{\infty}\left[\frac{\lambda}{2}\right]^p\frac{1}{p!}\sum_{\sigma_1,\dots,\sigma_p} e^{(\alpha/2)pC^{(0,N)}_{00}}\int dx_1\cdots dx_n\exp\left[-\frac{\alpha^2}{2}\sum_{i,j=1}^{n}\sigma_i\sigma_jC^{(0,N)}_{x_ix_j}\right]=Z^1_{Y(N)}(\Lambda,\beta,\lambda)\ , \qquad (15.15)$$

because the diagonal terms in $\sum_{i,j=1}^{n}$ ("self-energy terms") are canceled by the exponential factor outside the integral; in the first step of (15.15) the formulas for the Wick ordering of the cosine and for the expectation $\mathscr{E}$ of the exponentials have been used (see Appendix C and Sec. XI).

Recalling that the cancellation of the self-energy terms in (15.15) was due to the exponential factor due to the Wick ordering, we see that the evaluation of the integral (15.12) by the same technique will lead to expression of the rhs of (15.12) as

$$\sum_{p=1}^{\infty}\frac{1}{p!}\left[\frac{\lambda}{2}\right]^p\sum_{\sigma_1,\dots,\sigma_p}\int dx_1\cdots dx_p\exp\left[-\alpha^2\sum_{i<j}\sigma_i\sigma_jC^{(0,N)}_{x_ix_j}-\frac{\alpha^2}{2}\sum_{i,j=1}^{n}C^{(-R,-1)}_{x_jx_j}\sigma_i\sigma_j\right]\ , \qquad (15.16)$$

because $:\,:_{UV}$ in (15.8) is a "partial Wick ordering" and therefore it can produce the cancellation of the diagonal terms of only the "ultraviolet part of the potential", i.e., $C^{(0,N)}_{xy}$.

Expression (15.16) can be rewritten as

$$\sum_{p=0}^{\infty}\frac{1}{p!}\left[\frac{\lambda}{2}\right]^p\sum_{\sigma_1,\dots,\sigma_p}\int dx_1\cdots dx_p\exp\left[-\alpha^2\sum_{i<j}\sigma_i\sigma_jC^{(0,N)}_{x_ix_j}-\frac{\alpha^2}{2}\sum_{i,j=1}^{n}(C^{(-R,-1)}_{x_ix_j}-C^{(-R,-1)}_{00})\sigma_i\sigma_j\right]$$
$$\times\exp\left[-\frac{\alpha^2}{2}\left[\sum_{i=1}^{n}\sigma_i\right]^2C^{(-R,-1)}_{00}\right]$$
$$\equiv\sum_{p=0}^{\infty}\frac{1}{p!}\left[\frac{\lambda}{2}\right]^p\sum_{\sigma_1,\dots,\sigma_p}\int dx_1\cdots dx_p\exp\left[-\alpha^2\sum_{i<j}(C^{(-R,N)}_{x_ix_j}-C^{(-R,-1)}_{00})\sigma_i\sigma_je^{-(\alpha^2/2)Q^2_{\underline{\sigma}}C^{(-R,-1)}_{00}}\right]\ , \qquad (15.17)$$

with $Q_{\underline{\sigma}}=\sum_i\sigma_i$. In this way one obtains an expression for $Z^1_{C(R,N)}$, (15.12), implying (15.11) with Coulomb potential $V^{(\tilde m_0,M)}$ with $M=m_0\gamma^{N+1}$, $\tilde m_0=m_0\gamma^{-1}$ (the latter choice is a matter of notational convenience, γ being fixed).

It is expected that the "neutral Coulomb gas" described by

$$Z^0_C(\Lambda,\beta,\lambda)=\lim_{N\to\infty}Z^0_{C(N)}(\Lambda,\beta,\lambda) \qquad (15.18)$$

is a well-defined thermodynamical system, exhibiting some kind of screening phenomena, for $\alpha^2<4\pi$; basically it should behave as a neutral Yukawa gas with m_0 determined by λ,α (and $M=+\infty$), at least for α^2 small [see Brydges (1978), Fröhlich and Spencer (1981)].

For $\alpha^2\in[4\pi,8\pi)$ one expects that the Coulomb gas "collapses in the ultraviolet," remaining nontrivial in the sense that the collapse produces just a background of multipoles, with infinite density, on which free charges move and interact through nontrivial screening phenomena (note that in two dimensions the Coulomb gas interaction does not go to zero at infinity and, for $\alpha^2\in[4\pi,8\pi)$ it even diverges too fast near zero, making the partition function infinite, because it involves integrating the nonsummable factor $|x-y|^{-\alpha^2/2\pi}$).

The same collapse is expected to happen to the Yukawa gas in the same region of α^2 except that no screening in the infrared is necessary in order for the system to exhibit well-defined thermodynamic behavior (in fact, the potential decays exponentially at infinity as a consequence of the choice $m_0>0$, which gives a meaning to m_0^{-1} as a natural screening length); however, screening phenomena are expected to occur in the ultraviolet region where the Yukawa gas should collapse in the same way as the Coulomb gas, i.e., by producing an infinite density background of multipoles on which free charges move.

In other words, the conjecture is that the Coulomb gas (nonregularized in the infrared and in the ultraviolet) and the Yukawa gas (nonregularized in the ultraviolet) with parameters λ,α,m_0 describe the same physical phenomena, or at least have partially overlapping physical interpretations, if the Yukawa range m_0^{-1} is suitably chosen as a function of λ,α for $\alpha^2\in[0,8\pi)$.

For $\alpha^2>8\pi$ it is believed that the nonregularized Yukawa gas is trivial (i.e., it collapses without hope) and the Coulomb gas no longer exhibits infrared phenomena of any kind; at least not so strong as to produce an exponentially decaying effective interaction or correlations.

The work done in Secs. XI and XII on the cosine interaction allows one to make rigorous some of the above conjectures, though much work remains to be done towards the complete understanding of the whole theory.

In the case of the Yukawa gas the above-mentioned connection (the sine-Gordon transformation) between the Yukawa gas and the cosine field allows one to translate the properties of stability of the cosine field into the properties of stability of the Yukawa gas in the region $\alpha^2\in[0,32\pi/5)$, where the cosine field's stability is under control. And for $\alpha^2\in[4\pi,32\pi/5)$ the above-mentioned interpretation of the Yukawa gas as gas with an infinite density of collapsed dipoles (for $\alpha^2\in[4\pi,6\pi)$) and of dipoles and quadrupoles (for $\alpha^2\in[6\pi,32\pi/5)$), with zero total charge, emerges quite clearly; I do not enter here into the details of this interpretation of the results of the theory of the cosine interaction: the work is begun in Benfatto *et al.* (1982), and Nicolò (1983).

In the case of the Coulomb gas some of the above conjectures also follow as corollaries of the theory of stability of the cosine field, but the connection requires some explanations.

The first remark is that the problem of studying the Coulomb systems with "no infrared cutoff," i.e., with $R=+\infty$ in (15.12), can be reduced to the theory of the cosine interaction in the *ultraviolet* regime by the following chains of identities and arguments.

Rewrite (15.12), using the factorization $:e^{x+y}:=:e^x::e^y:$ for x,y independent Gaussian variables:

$$Z^1_{C(R,N)}(\Lambda,\beta,\lambda)=\int P(d\psi^{(0)})\cdots P(d\psi^{(N)})\left[\int P(d\psi^{(-R)})\cdots P(d\psi^{(-1)})\times\exp\left[\sum_\sigma\int_\Lambda(\tfrac{1}{2}\lambda e^{-(\alpha^2/2)C_{00}^{(-R,-1)}}:e^{i\sigma\alpha\psi_\xi^{(0,N)}}:):e^{i\sigma\alpha\psi_\xi^{(-R,-1)}}:d\xi\right]\right]$$

$$\equiv\int P(d\psi^{(0)})\cdots P(d\psi^{(N)})\left[\int P(d\varphi^{(1)})\cdots P(d\varphi^{(R)})\exp\left[\sum_\sigma\int_{\Lambda\gamma^{-R}}\lambda_{\sigma,R}(\xi'):e^{i\sigma\alpha\varphi_{\xi'}^{(0,R-1)}}:d\xi'\right]\right], \quad (15.19)$$

where $\xi=\xi'\gamma^R$ and $\varphi_{\xi'}^{(0,R-1)}\equiv\psi_{\gamma^R\xi'}^{(-R,-1)}$, so that $\varphi^{(0,R-1)}$ has the same distribution as a sum of independent fields $\varphi^{(j)}$:

$$\varphi_{\xi'}^{(0,R-1)}=\sum_{j=0}^{R-1}\varphi_{\xi'}^{(j)} \quad (15.20)$$

with covariance $C^{(0)}_{\gamma^j\xi\gamma^j\eta}$. This follows immediately from the definitions by computing and comparing covariances; actually one could put $\varphi_{\xi'}^{(j)}=\psi_{\gamma^R\xi'}^{(R-j)}$. Furthermore, in (15.19) $\lambda_{\sigma,R}(\xi')$ means

$$\lambda_{\sigma,R}(\xi')=\tfrac{1}{2}\lambda\gamma^{2R}e^{-(\alpha^2/2)C_{00}^{(-R,-1)}}:e^{i\alpha\sigma\psi_{\gamma^R\xi'}^{(0,N)}}:. \quad (15.21)$$

The interpretation of

$$e^{V_c(\psi^{(0,N)})}\equiv\int P(d\varphi^{(0)})\cdots P(d\varphi^{(R-1)})\left[\exp\left[\sum_\sigma\int_{\Lambda\gamma^{-R}}\lambda_{\sigma,R}(\xi'):e^{i\alpha\sigma\varphi_{\xi'}^{(0,R-1)}}:d\xi'\right]\right] \quad (15.22)$$

is, clearly, that of an effective interaction in the sense used in the preceding sections on field theory; it should describe the Coulomb gas on scales m_0^{-1} (through an equivalent gas of multipoles; see Sec. XIII for this interpretation; see also below).

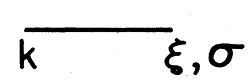
To describe (15.22) one can try to find an expansion for $V_C(\psi^{(0,N)})$ in powers of λ.

The work for such an expansion has already been done in Secs. XI and XII, because the integral (15.22) can be interpreted as an integral of the type studied there.

Using the results and the notations of Secs. XI and XII, one expresses it in terms of trees:

$$V_c=\int d\underline{\xi}\sum_{\underline{\sigma}}\sum_{\substack{\gamma:\underline{\sigma}(\gamma)=\underline{\sigma}\\ \underline{\xi}(\gamma)=\underline{\xi}}}\frac{V(\gamma)}{n(\gamma)}, \quad (15.23)$$

where the $V(\gamma)$ are computed with exactly the same rules of Sec. XI provided that we interpret the elementary trees



(37)

as $\lambda_{\sigma,R}(\xi)$, defined by (15.21), rather than $\lambda/2$; the index σ is ±1, while the index 0 is not allowed, because in the exponential in (15.22) there is no constant term.

All the results and bounds of Sec. XII carry through with essentially no change, besides the mentioned change of interpretation of diagram 37.

One therefore finds that $V(\gamma)$ can be expressed, to a given order in λ, as

$$\left[\prod_{j=1}^{n}\lambda_{\sigma_j,R}(\xi_j)\right]W_\gamma(\xi_1,\ldots,\xi_n)\,, \tag{15.24}$$

and W_γ will verify [see (12.5), (12.8), (12.9), and (12.14)]

$$|W_\gamma(\xi_1,\ldots,\xi_n)|\le\mathcal{N}_{\hat\gamma}\left[\prod_{j=1}^{n}e^{(\alpha^2/2)C_{00}^{(\le h_{v_j}-1)}}\right]\left[\prod_{v>r}e^{-(\alpha^2/2)Q_v^2(h_v-h_{v'})C_{00}^{(0)}}e^{-(\kappa/4)\gamma^{h_v}d^*(\xi_v)}\right], \tag{15.25}$$

where $\mathcal{N}_{\hat\gamma}$ depends only on the shape of γ. Actually, one will be interested only in expressions like (15.24) summed on the indices of γ: in particular, one is interested in the summations of (15.24) over the different indices $\underline{\sigma},\underline{\xi}$ that can be appended to the end points of trees otherwise identical. In this way the charge symmetry is used and some cancellations appear, as explained in Sec. XII, which allow one to improve the bound (15.25) by replacing $Q_v^2/4\pi$ by $[Q_v^2/4\pi+2(1-\varepsilon)\delta_{Q_v,0}]$ if $v>v_0=$(first nontrivial vertex of the tree); this cancellation, in fact, takes place already when one sums only over the charge configurations which attribute the same absolute charge to each vertex v and integrate over ξ.

According to the discussion of Sec. XI the lhs of (15.22) via (15.23), can be interpreted as the Boltzmann-Gibbs factor in a gas of multipoles, each represented by the trees with the same shape up to the charge indices which vary subject to the restriction that the absolute charge $|Q_v|$ of each vertex v is fixed. The activity of the multipole will be defined, quite arbitrarily [see Gallavotti and Nicolò (1984) for a deeper discussion]:

$$\sum_{\underline{\sigma}}^{*}\sum_{\underline{h}}\int_{\gamma^{-R}\Delta\times(\gamma^{-R}\Lambda)^{n-1}}d\xi_1\cdots d\xi_n\left[\prod_{j=1}^{n}\lambda_{\sigma_j,R}(\xi_j)\right]W_\gamma(\xi_1,\ldots,\xi_n)\Big/{:}\exp\left[i\alpha\sum_j\sigma_j\psi^{(0,N)}_{\gamma^R\xi_j}\right]{:}\,, \tag{15.26}$$

where the $\sum^*$ runs over all the charge configurations $\underline{\sigma}$ which attribute given absolute value of the total charge Q_v to each of the clusters associated with the vertices v of γ (called above, simply, vertex charges); the sum $\sum_{\underline{h}}$ runs over all the possible frequency labels that can be appended on the shape of γ and Δ is a fixed unit cube.

The collection of the terms with the same vertex charges is natural for physical reasons (charge symmetry), and mathematically it produces the just-mentioned cancellations.

If we reexpress (15.26), by "going back to scale 1," (15.26) becomes

$$\sum\int_{\gamma^{-R}\Delta\times(\gamma^{-R}\Lambda)^{n-1}}d\xi_1\cdots d\xi_n(\lambda\gamma^{2R-(\alpha^2/4\pi)R})^n\prod_{j=1}^{n}{:}e^{i\alpha\sigma_j\psi^{(0,N)}_{\xi_j\gamma^R}}{:}W_\gamma(\xi_1,\ldots,\xi_n)$$

$$\equiv\sum\gamma^{-2Rn}\int_{\Delta\times\Lambda^{n-1}}d\underline{x}(\lambda\gamma^{-(\alpha^2/4\pi-2)R})^nW_\gamma(\gamma^{-R}x_1,\ldots,\gamma^{-R}x_n)e^{-\alpha^2Y_N(\underline{x},\underline{\sigma})}\,, \tag{15.27}$$

where $\sum\equiv\sum_{\underline{\sigma}}^{*}\sum_{\underline{h}}$ [see (15.26)], and $Y_N(x_1,\ldots,x_n;\sigma_1,\ldots,\sigma_n)$ is the Yukawa potential, with ultraviolet cutoff N, of the charges $\sigma_1,\ldots,\sigma_n$ at positions $x_1,\ldots,x_n$.

Therefore, the activity of the multipole will be bounded by [using (15.24) and (15.25) and the cancellation remarked after (15.25) and recalling that $\varepsilon>0$ is an arbitrary parameter which can be chosen as small as necessary]

$$\sum_{\underline{h}}\gamma^{-2Rn}\int_{\Lambda^{n-1}}(\lambda\gamma^{-(\alpha^2/4\pi-2)R})^n\left|\sum^{*}W_\gamma(\gamma^{-R}x_1,\ldots,\gamma^{-R}x_n)\right|dx_2\cdots dx_n$$

$$\le\mathcal{N}_{\hat\gamma}\gamma^{-2R}\gamma^{-(\alpha^2/4\pi-2)Rn}\sum_{\underline{h}}\left[\prod_{v>v_0}(\gamma^{(\alpha^2/4\pi-2)(n_v-1)+\alpha^2/4\pi-\alpha^2/4\pi Q_v^2+2(1-\varepsilon)\delta_{Q_v,0}})^{h_v-h_{v'}}\right]$$

$$\times(\gamma^{(\alpha^2/4\pi-2)(n-1)+\alpha^2/4\pi-\alpha^2/4\pi Q_{v_0}^2})^{h_{v_0}}U_{N,n}\,, \tag{15.28}$$

where the sum over the frequency indices is of course bounded by the infrared cutoff: $h_v \le R$. For more details see Gallavotti and Nicolò (1984).

Before discussing formula (15.28), let us note that if $N=-1$—i.e., if the Coulomb potential has no ultraviolet part (which is the case usually considered in the literature)—the effective potential becomes a constant and it is no longer a random variable and has the interpretation of (grand canonical) pressure of the gas. Therefore, (15.28) becomes a bound on the Mayer coefficients of the gas (see below for some consequences of this remark).

To examine the remarkable formula (15.28) one distinguishes two cases: either $\alpha^2 \ge 8\pi$ or $\alpha^2 < 8\pi$. Below one uses the arbitrariness of ε by taking it conveniently small.

In the first case the rhs of (15.28) goes to zero as $R\to\infty$, as can be checked elementarily, for $Q_{v_0}\neq 0$: the gas is a gas of "neutral multipoles" [i.e., in the infrared limit one is in a multipole phase; see Fröhlich and Spencer (1981)]. If $Q_{v_0}=0$, then one can check that the rhs of (15.18) is uniformly bounded in R.

In the second case let

$$\rho=\left[\frac{\alpha^2}{4\pi}-2\right]n+2\equiv\left[\frac{\alpha^2}{4\pi}-2\right](n-1)+\frac{\alpha^2}{4\pi}\ ; \qquad (15.29)$$

then either $\rho\le 0$ and the rhs of (15.28) *diverges* in general as $R\to\infty$ or $\rho>0$ and in this case the bound (15.28) is uniformly bounded in R, and tends to zero if $Q_{r_0}\neq 0$.

The conclusions from the above estimates are as follows.

(1) If $\alpha^2>8\pi$ (hence $\rho>0$), the picture of the Coulomb gas as consisting, as far as its properties on scale m_0^{-1} are concerned, of neutral multipoles is consistent, because the activity of such multipoles is finite. This will be called "the multipole theorem" [see also Fröhlich and Spencer (1981)].

(2) If $\alpha^2>\alpha_n^2$, where α_n^2 are the thresholds defined by setting $\rho=0$ in (15.29), then thinking that the gas contains neutral multipoles of p charges with $p\le n$ but that no multipoles with more than n charges can be well-defined entities ("molecules," of course, should be their name) becomes consistent.

(3) It is remarkable that the above thresholds α_n^2, above which the Coulomb gas (with ultraviolet cutoff) generates molecules of p bound atoms, were precisely coinciding with the thresholds α_n^2, where the Yukawa gas charges collapse into clusters of $p\le n$ particles (in the ultraviolet limit) [see Fröhlich (1975)].

This is a confirmation of the above implicitly conjectured "duality" between the infrared properties of the Coulomb gas and the ultraviolet properties of the Yukawa gas for α^2 in $[0,8\pi)$.

If we call $p_C(\lambda,\beta)$ the pressure of the Coulomb gas with ultraviolet cutoff, as a function of the charge activity λ and of the temperature $\beta=\alpha^2$, the above analysis proves, as is easily checked, that if

$$p_C(\lambda,\beta)=\sum_{p\le n}\lambda^p f_C^{(p)}(\beta)+\lambda^n R^{(p)}(\lambda,\beta)\ ,$$
$$R^{(p)}(\lambda,\beta)\underset{\lambda\to 0}{\longrightarrow}0\ , \qquad (15.30)$$

then the coefficients $f_C^{(p)}(\beta)$ can be shown to be uniformly bounded in the infrared limit $R\to\infty$ for $\alpha^2>\alpha_n^2$, and that only the even ones have a nonzero limit.

In other words, the Mayer coefficients of order $\le n$ are formally well defined by convergent integrals for $\alpha^2>\alpha_n^2$.

The latter property follows immediately by considering the case $N=-1$, in which, as remarked above, the effective potential coincides with the grand canonical pressure. It should be obvious that the general case $N\ge -1$ (but finite) can be reduced always, and in a trivial way, to the $N=-1$ case.

This leads to the natural conjecture that $p_C(\lambda,\beta)$ is smoother and smoother in λ at $\lambda=0$ as $\beta=\alpha^2$ grows.

For $\alpha^2<4\pi$ not much can be said about smoothness; for $\alpha^2\in(4\pi,6\pi)$ the function should have two derivatives [actually, three if $\alpha^2\in(16\pi/3,6\pi)$]; for $\alpha^2\in(6\pi,40\pi/6)$ it should have four derivatives [actually, five for $\alpha^2\in(32\pi/5,40\pi/6)$], etc.; for $\alpha^2>8\pi$ the pressure should be infinitely smooth at $\lambda=0$.

By *derivative* one means here that (15.30) holds as an asymptotic formula with $R^{(n)}(\lambda,\beta)$ tending to zero as $\lambda\to 0$.

This conjecture suggests that while α^2 grows (i.e., while the temperature decreases), the Coulomb gas presents an infinite sequence of phase transitions in which it passes from the "plasma phase," small α^2, with Debye screening phenomena, to the "multipole phase," α^2 large, with no screening in the infrared: the Kosterlitz-Thouless regime would be the last stage in a sequence of increasingly complex phase transitions in which bound states ("neutral molecules") of increasing size become possible in thermal equilibrium.

So far the bases of the above conjecture are the estimates of this section (15.28) which imply the finiteness of the coefficients of the Mayer expansion; such estimates have been pointed out in Gallavotti and Nicolò (1984a). Further work towards a full proof of (15.30), i.e., with estimates on the remainder in (15.30), is in progress [Gallavotti and Nicolò (1984b)].

I think that the beautiful properties of the cosine interaction exhibited in this section justify its inclusion in this work, although they are not strictly an example of a problem of field theory: they show that field theory is not just a theory of quantum relativistic objects but that it can be relevant to very different matters—Coulomb gases are only one example out of many more, in solid-state physics and in physics of fluids, for instance.

XVI. NATURE AND CLASSIFICATION OF THE DIVERGENCES FOR φ^4 FIELDS

In order to see how to build the operators $\mathscr{L}_k^{(\sigma)}$ realizing the renormalization of the theory $\mathscr{J}$ defined by

$$V_1=\int[-\lambda{:}\varphi_\xi^{(\le N)4}{:}-\mu{:}\varphi_\xi^{(\le N)2}{:}$$
$$-\alpha{:}(\underline{\partial}\varphi_\xi^{(\le N)})^2{:}-\nu]d\xi\ , \qquad (16.1)$$

φ^4 field, it is useful, albeit not strictly necessary, to have a clear idea of how divergences arise in it and how strong they are.

I shall consider in detail only the four-parameter interaction (16.1) in four dimensions, calling $\lambda^{(\alpha)}$, $\alpha=4,2,2',0$, the parameters $-\lambda,-\mu,-\alpha,-\nu$, respectively.

If $d=2,3$ one could consider theories simpler than (16.1) which in some cases can be constructed as true field theories, going beyond the formal theory of perturbations, by literally repeating the arguments of Secs. XIII and XIV [e.g., if $d=2$ one could consider the interaction (5.3), or, if $d=3$ one could consider the interaction (5.6)]. Some more details on these simple ("super-renormalizable") cases will be presented in Sec. XXI.

Suppose that the field $\varphi^{(\leq N)}$ with ultraviolet cutoff γ^N, $\gamma>1$, is decomposed as a sum of independent fields living on scales γ^{-k}, $k=0,1,\ldots,N$ and verifying (3.17)—(3.20) with $n=3$ (say):

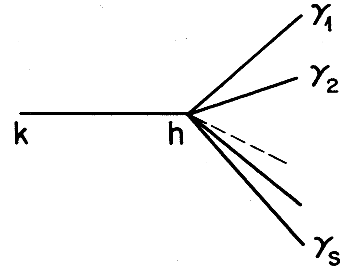
$$\varphi_\xi^{(\leq N)}=\varphi_\xi^{(-1)}+\varphi_\xi^{(0)}+\cdots+\varphi_\xi^{(N)}\;, \tag{16.2}$$

where $\varphi^{(-1)}$ is a degenerate field with covariance $C^{(-1)}$, which will be eventually put equal to zero, so that $\varphi^{(-1)}=0$, and which is introduced only for the purpose of unifying certain notations.

The use of a Pauli-Villars regularization of order $n\geq 2$ is necessary to give a meaning to (16.1) if $d=4$; actually, the third term in (16.1) already requires $n\geq 2$ even for $d=2$, while the first two lose meaning only if $d\geq 4$ when $n=1$. Here the choice $n=3$ is motivated by the fact that in the subtraction algorithm built to renormalize the divergences it will be convenient to be able to say that the fields have two derivatives and therefore can be developed in Taylor series to second order included. It is not impossible that one could perform the work with $n=2$ but it would certainly be harder: in any case, this question acquires importance when one tries to go beyond perturbation theory; when one is dealing only with perturbation expansions it would be even better to have fields so regular to have derivatives of any order, e.g., the ones arising from regularization (3.21).

Fixed $\underline{\lambda}=(-\lambda,-\mu,-\alpha,-\nu)$ in (16.1), the effective potential $V_1^{(k)}$ on "scale k" will be expressed in terms of simple trees as explained in Sec. VI: the end points will be marked by a pair (ξ,α) with $\xi\in R^d$ and $\alpha=0,2,2',4$ expressing which of the four terms in (16.1) is represented by the end point under consideration.

An expression for $V(\gamma)$ can be found by the same technique used in the cosine field case in Secs. XI and XII, namely, let a tree γ bifurcate, at the first nontrivial vertex v_0 after the root r, into s subtrees $\gamma_1,\ldots,\gamma_s$, and let h be the "frequency" of the vertex v



. (38)

As in the case of the cosine interaction, one has to guess first the form of $V(\gamma)$, and an obvious guess is the following very general one:

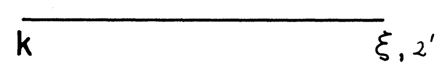
$$V(\gamma)=\sum_P V^{(k)}(\xi_1,\ldots,\xi_n;\gamma,p)P(\varphi^{(\leq k)},\underline{\partial}\varphi^{(\leq k)})\;, \tag{16.3}$$

where the summation runs over all the possible Wick monomials of the form $P=:(\partial\varphi_\xi)^2$: if γ is the trivial tree, or

$$P=:\varphi_{\xi_{i_1}}^{n_1}\cdots\varphi_{\xi_{i_a}}^{n_a}\partial\varphi_{\xi_{i_{q+1}}}\cdots\partial\varphi_{\xi_{i_m}}:\;,$$
$$0\leq q\leq n,\;1\leq n_i\leq 4\;, \tag{16.4}$$

where the derivative ∂ means a derivative with respect to one of the coordinates of the field's argument; the above assumption is made only for the trees having nontrivial vertices.

In fact, (16.4) is not true for the trivial tree



representing $-\alpha:(\underline{\partial}\varphi_x)^2:$; this will be the only (natural) exception.

Assuming (16.3) and (16.4) are true, with the above-mentioned exception, we find the rules for the evaluation of the truncated expectations of products of Wick monomials, see Appendix C, yield the following recursion relation, deduced from diagram 38 after recalling that a tree vertex has the meaning of a truncated expectation (see Secs. VI and VII):

$$V^{(k)}(\xi_1,\ldots,\xi_n;\gamma,P)=\sum_{P_1,\ldots,P_s}\left[\prod_{j=1}^{s}V^{(h)}(\underline{\eta}_j;\gamma_i,P_j)\right]\sum_{\pi\in\mathcal{T}_P}\sum_{\substack{\tau\subset\pi\\ \text{connected}}}\left[\prod_{\substack{\lambda\in\tau\\ \lambda=(a,b)}}C_{ab}^{(h)}\right]\left[\prod_{\substack{\lambda\in\pi/\tau\\ \lambda\in(a,b)}}C_{ab}^{(\leq h-1)}\right]\;, \tag{16.5}$$

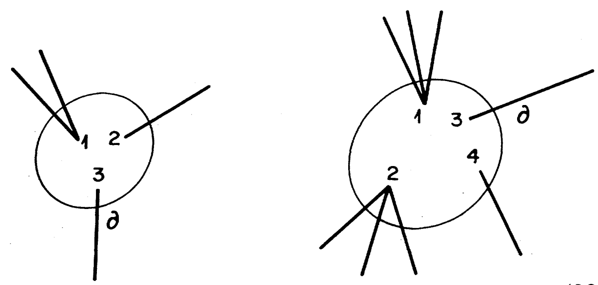
where $\underline{\eta}_1,\ldots,\underline{\eta}_s$ are the s clusters into which the points $\underline{\xi}$ are decomposed by $\gamma_1,\ldots,\gamma_s$, i.e., the s clusters corresponding to the vertices $v_1,\ldots,v_s$ of the tree γ immediately following v_0 and such that $v_j\in\gamma_j$. $\mathcal{T}_P$ is the set of graphs obtained as follows.

Represent a Wick monomial P like (16.4) by drawing q points $\xi_{i_1},\ldots,\xi_{i_q}$ in R^d and $n_1,n_2,\ldots,n_q$ pairwise distinct lines, respectively, emerging from each of them, and m-q points $\xi_{i_{q+1}},\ldots,\xi_{i_m}$ with one line, labeled ∂ emerging from each of them.

It is convenient to think of the points $\xi_{i_1},\ldots,\xi_{i_m}$ as enclosed in a box out of which emerge the lines just defined. For instance, the monomials

$$:\varphi_{\xi_1}^2\varphi_{\xi_2}\partial\varphi_{\xi_3}:,\quad :\varphi_{\xi_1}^3\varphi_{\xi_2}^3\partial\varphi_{\xi_3}\partial\varphi_{\xi_4}:$$

are represented as in diagram 39, where 1,2,..., stand for $\xi_1,\xi_2,\ldots,$:



(39)

and each of the above objects will be called a "Wick cluster."

Then given s Wick monomials $P_1, \ldots, P_s$, the symbol $\mathscr{T}_P$ will denote the set of the graphs obtained by joining pairwise some of the lines associated with the Wick clusters representing $P_1, \ldots, P_s$ in such a way that (explicit examples are worked out in diagrams 40—43 below): (i) two lines emerging from the same cluster cannot be joined together; (ii) there should be enough lines paired so that the lines plus the sets inside the boxes associated with each P_j form a connected set; (iii) the set of the points associated with $P_1, \ldots, P_s$ together with the lines emerging from them and still "free" (i.e., not paired with other lines) represent, once the points from which they emerge are enclosed into a single box, the monomial P.

In the above definitions and constructions, as well as in the upcoming ones, one has to bear always in mind that the lines emerging from each point are regarded as pairwise distinct (and this will eventually give rise to a combinatorial problem).

Furthermore, $\tau \subset \pi$ with the subscript "connected" [see (16.5)], means a subset of the lines of π which still keeps the connection between the boxes. A line λ obtained by pairing ("joining" or "contracting" will be synonyms of "pairing") two lines is identified by its two extreme points together with the field indices (∂ or nothing) which will be kept and appended to the line near the end point from which they emerge (so that it might happen that a line carries two, one, or no indices ∂; if it carries only one it will be appended near the appropriate end point).

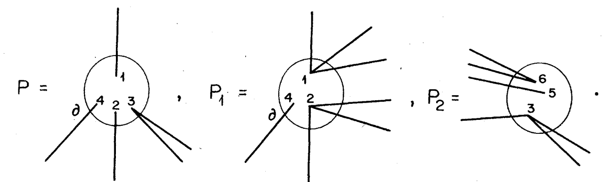
Therefore, $\lambda=(a,b)$ with $a=\xi, b=\xi'$ represents a line obtained by joining two lines without labels emerging from ξ and ξ'; similarly, if $a=(\xi,\partial), b=\xi'$, then $\lambda=(a,b)$ represents the line obtained by joining together a nonlabeled line emerging from ξ' and a labeled one emerging from ξ. The resulting line will be represented by a segment joining ξ with ξ' carrying a label ∂ near ξ (or, equivalently carrying a label ∂_ξ); and similar interpretations are given to the cases $a=\xi, b=(\xi',\partial)$ or $a=(\xi,\partial), b=(\xi',\partial)$.

The symbols $C_\lambda^{(\;)} \equiv C_{ab}^{(\;)}$ denote the appropriate covariances

$$
\begin{aligned}
&C_{ab}^{(\;)}=\mathscr{E}(\varphi_\xi^{(\;)}\varphi_2^{(\;)}), \quad C_{ab}^{(\;)}=\mathscr{E}(\varphi_\xi^{(\;)}\partial_\eta\varphi_\eta^{(\;)}) ,\\
&C_{ab}^{(\;)}=\mathscr{E}(\partial_\xi\varphi_\xi^{(\;)}\varphi_\eta^{(\;)}), \quad C_{ab}^{(\;)}=\mathscr{E}(\partial_\xi\varphi_\xi^{(\;)}\partial_\eta\varphi_\eta^{(\;)}) ,
\end{aligned}
\tag{16.6}
$$

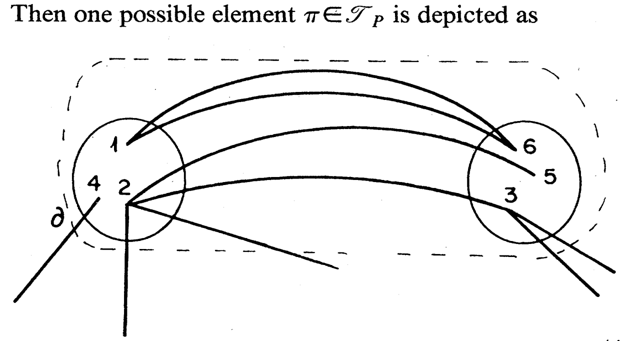
if $(a,b)=(\xi,\eta),(\xi,(\eta,\partial)),((\xi,\partial),\eta),((\xi,\partial),(\eta,\partial))$, respectively. Recall that here ∂ or ∂_ξ means a derivative with respect to some component of ξ whose index is omitted for simplicity of notation.

For instance, consider diagram 40, where the integer j stands for ξ_j:



(40)

Then one possible element $\pi \in \mathscr{T}_P$ is depicted as

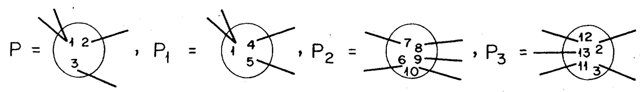


(41)

where the dotted box represents the box corresponding to P.

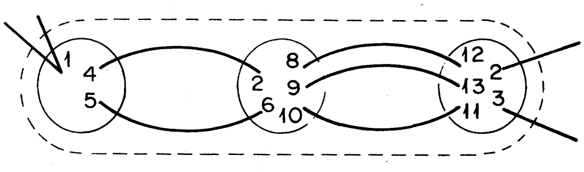
A possible $\tau \subset \pi$ is any nonempty subset of the inner lines, inside the dotted box, diagram 41.

Similarly, if



(42)

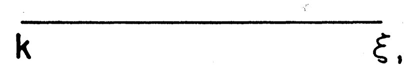
a simple possible π is



(43)

and τ is any subset of the five inner lines which contains at least one of the first two and one of the last three.

Relation (16.5) defines recursively and completely the coefficients $V^{(k)}(\gamma;P)$ once one specifies the meaning of $V^{(k)}(\gamma;P)$ for the elementary trees γ_0:



Of course $V^{(k)}(\gamma_0;P)=0$ unless P is $:\varphi^{(\le k)4}:$, $:\varphi^{(\le k)2}:$, $:(\underline{\partial}\varphi^{(\le k)})^2:$, or 1; and in such cases $V^{(k)}(\gamma_0;P)$ is just $-\lambda, -\mu, -\alpha, -\nu$, respectively, for $\alpha=4,2,2',0$ (no confusion should arise between the renormalized coupling constant α and the end point index carrying the same name).

To find bounds on $V^{(k)}(\gamma;P)$ one can proceed as follows: first by using (16.5) recursively one decomposes this quantity into a (very large) sum; each term of the sum will correspond to a fixed selection S of one index for every possible summation arising by applying recursively (16.5). One has to imagine that one such special selection S has been fixed (say, one special choice of P, of $P_1,\ldots,P_s$, of π, τ, etc., with similar choices made for each of the successive vertices of γ which appear while disassembling $\gamma_1,\gamma_2,\ldots,\gamma_s$, etc.).

The bases for the bound that will be derived shortly are the estimates (3.19), (3.20), and (3.17), and the following notions which have already been introduced in Secs. IV—IX above and in the preceding lines of this section, but which it will be convenient to collect again and organize in the form in which they will be used below.

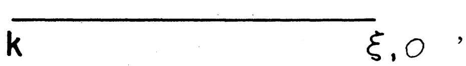
(1) Each vertex v of a tree γ is associated with a cluster of end points of γ; this cluster will be denoted $\underline{\xi}_v$.

(2) The selection S of the summation indices just introduced permits one to associate with each vertex v a monomial P_v which can be thought of as graphically represented by a box containing the points $\underline{\xi}_v$, with lines emerging from them and out of the box itself: some of the lines may bear an index ∂; the lines emerging from the box represent the graphical image of the monomial P_v.

(3) The number n_v^e of lines emerging from the box enclosing the cluster $\underline{\xi}_v$ will be the sum $n_{1,v}^e+n_{0,v}^e$ of the number of labeled lines, $n_{1,v}^e$, and of the number, $n_{0,v}^e$, of unlabeled lines (e: external; 1: labeled; 0: unlabeled).

(4) A selection S of the summation indices leads to graphical representation of the corresponding contributions to $V^{(k)}(\gamma;S)$.

Out of each end point ξ of γ emerge either four unlabeled lines or two unlabeled lines or two labeled lines or no lines at all, depending upon the value of the appended type index $\alpha=4,2,2',0$. The case $\alpha=0$ can appear only in the trivial trees

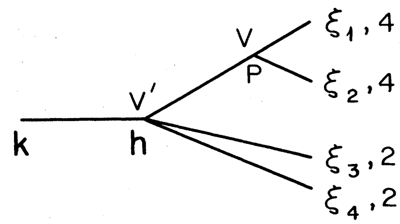


which will be disregarded for the time being; in fact $\alpha=0$ corresponds to a constant $P=1$ and the truncation of the expectations eliminates it unless the tree is trivial, i.e., indicates no truncations.

The structure of γ encloses the end points into a hierarchically arranged sequence of boxes, each corresponding to a tree vertex v, and it is possible to make the convention that the pairings of the lines are drawn in the graphical representation so that the lines contracted between the clusters $v_1,v_2,\ldots,v_s$, representing $P_{v_1},\ldots,P_{v_s}$, to build the monomial P_v corresponding to the cluster v (v being the vertex immediately followed by $v_1,\ldots,v_s$) are all contained inside the box corresponding to v, as in diagrams 40—43 above.

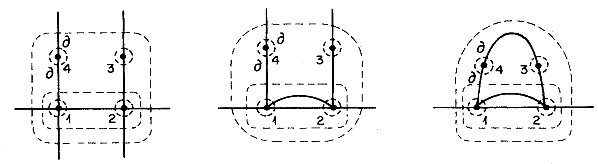
For uniformity of notation it is convenient to imagine that the end points of γ also represent clusters of a single point and that they generate little boxes around it (recall that, however, the end points of a tree are conventionally not regarded as tree vertices).

For instance, three possible selections corresponding to the tree



(44)

(where v,v' are vertex names, p,h,k, are frequency labels) are represented by



(45)

(if $1,2,\ldots$ stand for $\xi_1,\xi_2,\ldots$).

(5) If in S there is a line paired to another, there will be a smallest box containing the contracted line, i.e., the two end points (this is enough by the above drawing convention); if v is the corresponding tree vertex and h_v is its frequency index, then one says that the contracted line has frequency h_v and one attributes the index h_v to each of the two lines giving rise to the contracted line of frequency h_v; the uncontracted lines will be given the frequency index $k=k(\gamma)=$(frequency of the root of the tree); they are called "external."

So to each box one can associate a frequency index which is the frequency index h_v of the vertex v corresponding to the box. As a consequence one can associate to each line in S a frequency index which is the frequency index of the box which first encloses the line. Note that the association of a frequency index to a line depends on S and not just on the tree γ. By convention the box associated with the root r of the tree γ is the whole space.

(6) The above set of indices still does not specify completely the selection S: one has to mark, for this purpose, each line which belongs to the sets called τ in (16.5) by a label—say, θ—recalling its origin (as a line in a set τ): "character label"; lines with the label θ will be called "hard" or "high-frequency" lines.

(7) It is important to stress, again, to avoid combinatorial errors, that in the above construction two lines emerging from the same vertex still have to be regarded as different and distinguishable; to keep track of the combinatorics it is convenient to imagine that the lines emerging from the innermost vertices (i.e., from the end points) are numbered [from 1 to 4 if the vertex represents $-\lambda{:}\varphi^4{:}$, from 1 to 2 if it represents $-{:}\varphi^2{:}$ or $-{:}(\partial\varphi)^2{:}$]. Such labels will be called "identity labels."

Of course, selections S which differ just by the distribution of the identity labels yield identical contributions to $V(\gamma)$.

It is also clear that the number of selections differing just by the identity labels is bounded by 4^n if the tree has n end points.

Before we continue, it is important to stress that, by our definitions, a selection S of summation indices yields a connected graph joining all the end points $\underline{\xi}$ of γ with lines marked by

(a) a frequency index,

(b) a character index or no index per internal line: if the index is missing, the line is "soft"; if it is present, the line is "hard,"

(c) an index ∂ or no index per each end point of the line, and

(d) an identity index per each end point of the line (internal or not).

The frequency indices and the character indices are not random: they are organized by tree γ in such a way that if we draw the boxes corresponding to each vertex of γ, the lines internal to each box form a connected graph and so does their subset formed by the hard ones among them.

The lines which are external, together with the points out of which they emerge and the largest (finite) box, form a graphical representation of the Wick monomial P selected by S.

The reader familiar with Feynman graphs will recognize in such a representation of S something which can be called a "decorated Feynman graph," the decorations being the above collection of labels listed in (a)—(d). One also recognizes the connection between the above decorated graphs and trees and the basic notion of forest in Zimmermann (1969). To proceed to obtain bounds on $V^{(k)}$ one considers the contribution to it by a choice of the summation indices S.

Denoting $V(\gamma;S)P_S$ such a contribution, where P_S denotes the Wick monomial selected by S, and using (3.19) and (3.20), one finds after some meditation the ("good") estimate in terms of $\overline{\varepsilon}=\max(|\lambda|,|\mu|,|\alpha|,|\nu|)$:

$$|V(\gamma;S)|\gamma^{k[(d-2)/2]n^e_{0,v_0}}\gamma^{k\{[(d-2)/2]+1\}n^e_{1,v_0}}\le\overline{\varepsilon}^{\,n}\left[\prod_{l_0}\gamma^{[(d-2)/2]h_{l_0}}\right]\left[\prod_{l_1}\gamma^{\{[(d-2)/2]+1\}h_{l_1}}\right]\left[\prod_{\lambda}Be^{-\kappa\gamma^{h_\lambda}|\lambda|}\right], \tag{16.7}$$

where $d>2$, for simplicity, λ represents an "inner" line of frequency h_λ associated with S, and $|\lambda|$ is the distance between the end points of λ; regarding the contracted lines of S as composed by two half lines, and regarding the external lines also as half lines, we find that the first product in the rhs is over the half lines bearing no ∂ label and the second product is over the half lines bearing a ∂ label. The first nontrivial vertex of γ is denoted by v_0, and $B,\kappa>0$ are suitable constants.

The factor multiplying the lhs, $|V(\gamma;S)|$, has been introduced for convenience [it will be clear shortly that it is a natural multiplier in the lhs of the inequality (16.7)].

Bound (16.7) is really trivial "power counting," once the presence of the exponential factors is understood. It arises from bounding $C^{(h)}_\lambda$ contributed by the hard lines λ in S, with frequency index h.

Recalling (16.6), one sees, for instance, that there is $B>0$ such that if $\lambda=(a,b)$, $a=\xi$, $b=\eta$:

$$|C^{(h)}_{ab}|\le(\gamma^{[(d-2)/2]h})^2Be^{-\kappa\gamma^h|\xi-\eta|} \tag{16.8}$$

or if $a=\xi$, $b=(\xi',\partial)$:

$$|C^{(h)}_{ab}|\le\gamma^{\{[(d-2)/2]+1\}h}\gamma^{[(d-2)/2]h}Be^{-\kappa\gamma^h|\xi-\eta|} \tag{16.9}$$

or if $a=(\xi,\partial)$, $b=(\xi',\partial)$:

$$|C^{(h)}_{ab}|\le(\gamma^{\{[(d-2)/2]+1\}h})^2Be^{-\kappa\gamma^h|\xi-\eta|}, \tag{16.10}$$

while $C^{(\le h-1)}_{ab}$, contributed by the soft lines, can be bounded only by (16.8), (16.9), and (16.10) without the last exponential factor [or, rather, with that factor replaced by $\exp(-\kappa|\xi-\eta|)$, useless], provided $d>2$ (if $d=2$, an extra factor h has to be added in bounding $C^{(\le h-1)}_{ab}$).

One could write the exponential factor in (16.7) as

$$\prod_v\exp[-\kappa\gamma^{h_v}d^*(X_v)],$$

using the notations introduced in Sec. XII [especially Eq. (12.6)] to treat the cosine field; however, this remark has been made only for the sake of comparison and will not be needed in what follows.

It remains for us to cast (16.7) into a more usable form.

Select a vertex $v\in\gamma$ and let $m_{2,v},m_{4,v},m_{2',v}$ be the numbers of vertices in the cluster $\underline{\xi}_v$ associated with v and bearing an index $\alpha=2,4,2'$, respectively: $m_{2,v}+m_{4,v}+m_{2',v}=n_v=$(number of points in $\underline{\xi}_v$), and note that if v is a nontrivial vertex of the tree, $n_v\ge2$ because v represents a truncation operation.

For each v also introduce

$n^{\text{inner}}_{0,v}=$ number of lines without ∂ label before the contractions, contained in the box corresponding to v but not in any smaller one,

$n^{\text{inner}}_{1,v}=$ number of lines with label ∂ before the contractions, contained in the box corresponding to v but not in any smaller one,

the number of lines being counted before the contractions, each inner line of a graph S counting twice in the evaluation of n^{inner}, and

$n^e_{0,v}=$ number of lines without ∂ label before the contractions, emerging from the box corresponding to v,

$n^e_{1,v}=$ number of lines with ∂ label before the contractions, emerging from the box corresponding to v.

Then a simple counting allows us to rewrite (16.7) as

$$\overline{\varepsilon}^{\,n}\gamma^{[(d-2)/2]kn^e_{0,v_0}}\gamma^{(d/2)kn^e_{1,v_0}}\left[\prod_\lambda Be^{-\kappa\gamma^{h_\lambda}|\lambda|}\right]\times\left[\prod_{v>r}\gamma^{h_v[(d-2)/2]n^{\text{inner}}_{0,v}}\gamma^{h_v(d/2)n^{\text{inner}}_{i,v}}\right], \tag{16.11}$$

where r is the root vertex of the tree γ.

Let γ have n end points labeled $\xi_1, \ldots, \xi_n$ and let the external lines of the graph S emerge from the points $\xi_1, \ldots, \xi_p$, as can be assumed without loss of generality. Then one is interested, according to the general ideas developed in Sec. XII in connection with the asymptotic freedom notion and the interpretation of the effective potential as a potential for a continuous spin system, in bounding

$$M_s(\Delta_1, \ldots, \Delta_p) = \int_{\Delta_1 \times \cdots \times \Delta_p \times \Lambda^{n-p}} |V(\gamma;s)| \sup |P_s(\varphi^{(\leq k)}, \underline{\partial}\varphi^{(\leq k)})| \, d\xi_1 \cdots d\xi_n \,, \tag{16.12}$$

where $\Delta_1, \ldots, \Delta_p \in Q_p$ are cubes of side size γ^{-p} in which the points appearing as labels to fields in P_S vary in (16.12): these cubes are extracted from a pavement Q_p of Λ.

The supremum in (16.12) is over the fields

$$\varphi^{(\leq k)} = \sum_{j=0}^{k} \varphi^{(j)} \,,$$

with $\varphi^{(j)}$ verifying (3.20). If one denotes $\widetilde{B} = \sup |B_\Delta|$ [see (3.20)], one finds (setting $n^e_{0,v_0} = n^e_0, n^e_{1,v_0} = n^e_1, n^e = n^e_0 + n^e_1$)

$$\sup |P_s| \leq \gamma^{[(d-2)/2]kn^e_0} \gamma^{(d/2)kn^e_1} \widetilde{B}^{\,n^e} \mathcal{N} \,, \tag{16.13}$$

and the constant $\mathcal{N}$ depends only on the degree of the polynomial P_S [hence it depends neither on k nor on the degree n of γ: in fact $\mathcal{N} = O(n^e!)$].

Inserting (16.13) into (16.12), one finds that (16.12) is estimated by $\widetilde{B}^{\,n^e}\mathcal{N}$ times the integral over $\Delta_1 \times \Delta_2 \times \cdots \times \Delta_p \times \Lambda^{n-p}$ of the rhs of (16.7) [and this explains also why the factor in the rhs of (16.7) is a natural one to consider].

The only term in (16.7) which is not constant is the last factor: its integral over the set indicated in (16.13) has already been considered in Sec. XII [see (12.15)]—see remark following (16.8)—and the result is expressed by

$$B_1^n \left[\prod_{v>r} \gamma^{-dh_v(s_v-1)} \right] \gamma^{-kd} e^{-(\kappa/2)d(\Delta_1, \ldots, \Delta_p)\gamma^k} \,, \tag{16.14}$$

where $B_1 > 0$ is a suitable constant and s_v is the number of branches emerging from v in γ [see Appendix D for a proof of (16.14)].

Using (16.13), (16.7), (16.11), and (16.14), we can bound integral (16.12) by

$$M_s(\Delta_1, \ldots, \Delta_p) \leq \gamma^{-(\kappa/2)\gamma^k d(\Delta_1, \ldots, \Delta_p)} \bar{\varepsilon}^{\,n} \widetilde{B}^{\,n_e} \mathcal{N} \gamma^{-kd} \gamma^{k[(d-2)/2]n^e_0 + k(d/2)n^e_1} \left[\prod_{v>r} \gamma^{hv[(d-2)/2]n^{\text{inner}}_{0,v}} \gamma^{h_v(d/2)n^{\text{inner}}_{1,v}} \gamma^{-h_v d(s_v-1)} \right] . \tag{16.15}$$

The latter estimate can be elaborated using the identity

$$\sum_{v'<v} (s_v - 1) = (n_{v'} - 1)$$

[see (12.17)]. Remembering that the end points of γ are not considered as vertices of γ and denoting simply $m_2, m_4, m_{2'}, n^e_1, n^e_0, n^e$ the $m_{2,v_0}, m_{4,v_0}, \ldots$, if v_0 is the first nontrivial vertex of γ following the root, one finds

$$\begin{aligned} M_s(\Delta_1, \ldots, \Delta_p) &\leq \bar{\varepsilon}^{\,n} \widetilde{B}^{\,n_e} \widetilde{B}_1^{\,n} \gamma^{-dk} \left[\prod_{v>r} \gamma^{-d(h_v-k)(s_v-1)} \gamma^{[(d-2)/2](h_v-k)n^{\text{inner}}_{i,v}} \gamma^{(d/2)(h_v-k)n^{\text{inner}}_{1,v}} \right] \\ &\quad \times \gamma^{-dk\sum_v (s_v-1)} \gamma^{[(d-2)/2]k(2m_1+4m_4-n^e_0)} \gamma^{(d/2)k(2m_{2'}-n^e_1)k} \gamma^{[(d-2)/2]n^e_0 k} \gamma^{(d/2)n^e_1 k} e^{-(\kappa/2)\gamma^k d(\Delta_1, \ldots, \Delta_p)} \mathcal{N} \\ &\equiv \bar{\varepsilon}^{\,n} \widetilde{B}^{\,n_e} B_1^n \mathcal{N} e^{-(\kappa/2)\gamma^k d(\Delta_1, \ldots, \Delta_p)} \gamma^{-k[2m_2+(4-d)m_4]} \\ &\quad \times \prod_{v>r} \gamma^{-d(h_v-k)(s_v-1) + [(d-2)/2]k(h_v-k)n^{\text{inner}}_{0,v} + (d/2)(h_v-k)n^{\text{inner}}_{1,v}} \,. \end{aligned} \tag{16.16}$$

Denoting v' the vertex preceding v in γ and denoting $\widetilde{n}^{\text{inner}}_{j,v}$, $j=0,1$, the number of lines, before contractions (i.e., half lines), inside the box corresponding to v (which is not necessarily the first box containing them; i.e., $\widetilde{n}^{\text{inner}}_{j,v} \geq n^{\text{inner}}_{j,v}$, in general) and using

$$\begin{aligned} &\sum_{v>r} (h_v - k)(s_v - 1) \equiv \sum_{v.r} (h_v - h_{v'})(n_v - 1) \,, \\ &\sum_{v>r} (h_v - k) n^{\text{inner}}_{j,v} \equiv \sum_{v>r} (h_v - h_{v'}) \widetilde{n}^{\text{inner}}_{j,v} \,, \quad j = 0,1 \,, \end{aligned} \tag{16.17}$$

$$\widetilde{n}^{\text{inner}}_{0,v} \equiv 2m_{2,v} + 4m_{4,v} - n^e_{0,v} \,,$$

$$\widetilde{n}^{\text{inner}}_{1,v} \equiv 2m_{2',v} - n^e_{1,v} \,,$$

one realizes from (16.16) that

$$\begin{aligned} M_s(\Delta_1, \ldots, \Delta_p) &\leq \mathcal{N} \bar{\varepsilon}^{\,n} \widetilde{B}^{n^e} B_1^n e^{-(\kappa/2)\gamma^k d(\Delta_1, \ldots, \Delta_p)} \\ &\quad \times \gamma^{-k[2m_2+(4-d)m_4]} \prod_{v>r} \gamma^{-p_v(h_v - h_{v'})} \,, \end{aligned} \tag{16.18}$$

with

$$\rho_v = -d + 2m_{2,v} + (4-d)m_{4,v} + \frac{d-2}{2} n_{0,v}^e + \frac{d}{2} n_{1,v}^e . \tag{16.19}$$

Therefore, recalling that the contribution to $V^{(k)}$ of the trees of given shape is obtained by summing over all the possible choices S and over all the possible frequency assignments to the vertices of the trees (i.e., over all the possible values of $h_v - h_{v'} > 0$, $h_v \le N$), one realizes that the estimate (16.18) and (16.19) for (16.12) implies ultraviolet finiteness if for all S and all tree shapes it is $\rho_v > 0$.

However, clearly, there are plenty of cases with $\rho_v \le 0$ for some v, for $d \ge 2$.

The situation would be slightly better if one had started with a more restrictive interaction—e.g., if $\mathscr{I}$ had been replaced by

$$\int_\Lambda (-\lambda{:}\varphi_\xi^{(\le N)4}{:} - \mu{:}\varphi_\xi^{(\le N)2}{:} - \gamma) d\xi . \tag{16.20}$$

In this case it is easily realized that one has to take in just (16.18) and (16.19), $n_{1,v}^e \equiv 0, m_{2',v} = 0$. This implies that

$$d = 2 \Longrightarrow \rho_v > 0 , \quad \forall v$$

$$d = 3 \Longrightarrow \rho_v > 0$$

unless $n_{0,v}^e = 2$ and $m_{2,v} + m_{4,v} = 2$ or $n_{0,v}^e = 0$ and $m_{2,v} + m_{4,v} = 0$, i.e., the theory (16.20) is ultraviolet finite in dimension $d = 2$. However, if $d = 3$, it is not ultraviolet stable and one has to check whether it is renormalizable.

Going back to (16.1) for $d = 4$, we discover many cases with $\rho_v \leqq 0$; in general it is, however, clear that $\rho_v > 0$ if there are too many external lines to the box corresponding to v, i.e., if $n_{0,v}^e \ge 5$.

The above discussion completes the analysis of the origin of the divergences and of their strength. In the next sections the problem of renormalizing the theory (16.1) will be studied and solved for $d \le 4$.

A final but, as it will turn out, very important remark is that the above method allows producing estimates of (16.12) when the rule to compute $V(\gamma;S)$ is modified by replacing the $\lambda^{(\alpha)}$ contributions from the end points of γ with constants $r^{(\alpha)}(h_j)$ with h_j being the frequency of the vertex to which the jth end point is attached by its tree branch.

Suppose that

$$r^{(\alpha)}(h) = \gamma^{(4-d)h\delta_{\alpha,4} + 2h\delta_{\alpha,2} + dh\delta_{2,0}} \bar{r}^{(\alpha)}(h) \tag{16.21}$$

and repeat the power counting argument leading to the bound (16.18). In this case the result will be, for $n > 1$,

$$M_s(\Delta_1, \ldots, \Delta_p) \le \mathscr{N} \widetilde{B}_1^{n^3} e^{-(\kappa/2)d(\Delta_1, \ldots, \Delta_n)\gamma^k} \times \left[\prod_{v > r} \gamma^{-\rho_v'(h_v - h_{v'})} \right] \times \left[\prod_{j=1}^n \bar{r}^{(\alpha_j)}(h_j) \right] \tag{16.22}$$

and

$$\rho_v' = -d + \frac{d-2}{2} n_{0,v}^e + \frac{d}{2} n_{1,v}^e , \tag{16.23}$$

i.e., the lines coming from vertices of type $\alpha = 2$ acquire the "same dimension" as those coming from the vertices of type $\alpha = 4$.

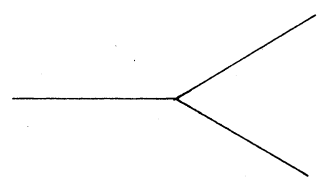
In the bounds (16.22) and (16.23) the values α_j must be nonzero so that the factor $\gamma^{dh\delta_{2,0}}$ plays no role in deducing them. It has been inserted only for later reference.

XVII. RENORMALIZATION TO SECOND ORDER OF THE φ^4 FIELD

The application of the general renormalization theory (see Secs. VII and VIII) to cure the ultraviolet instability pointed out in Sec. XVI follows the same scheme met in the case of the cosine field, in Sec. XII.

It is slightly more complex, because the polynomials do not have nice multiplication properties, not as nice as those of the complex exponentials' multiplication rules which played a (hidden) role in simplifying the algebra in the discussion of the cosine interaction.

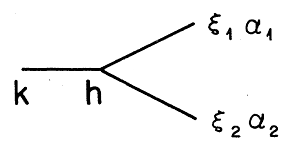
However, it is still true that, to proceed, one has to understand in detail only the renormalization theory to second order, i.e., the definition of the subtraction operators $\mathscr{L}_k^{(\sigma_0)}$ with

$\sigma_0 =$  ,

the other cases being easily understandable in terms of this special one.

However, a detailed understanding of the above simple case is absolutely essential and the inexperienced reader should check the minutest details of the following few straightforward but lengthy calculations, which are the heart of renormalization theory (contrary to what is sometimes asserted about the true difficulties being connected with the "overlapping divergences," a term which will not be even defined here).

To proceed as in Sec. VII one starts by defining the trees dressed to order 1: i.e., they are just the trees considered in Sec. XVI. Then one considers the trees of degree two (i.e., with two end points); actually, they are



$\alpha_1, \alpha_2 = 2, 2', 4$. (46)

They have been estimated in Sec. XVI, but it is easy to compute them explicitly from their expressions in Sec. VI; after integration over the end points position labels ξ_1, ξ_2 they contribute to $V_1^{(k)}$:

$$\frac{1}{2}V_1^{(k,\alpha_1,\alpha_2)}=\frac{1}{2}\int_\Lambda \mathscr{E}_{k+1}\cdots\mathscr{E}_{h-1}\mathscr{E}_h^T(v_h^{(\alpha_1)}(\varphi_{\xi_1}^{(\le h)}),v_h^{(\alpha_2)}(\varphi_{\xi_2}^{(\le h)}))d\xi_1 d\xi_2\,. \tag{17.1}$$

A simple calculation which the reader should perform at least once in his life, in spite of its length (after all, not so bad), gives

$$\begin{aligned}
&(1)\quad V^{(k,2,2)}=\mu^2\begin{bmatrix}2\\1\end{bmatrix}^2\int :\varphi_1\varphi_2:C_{12}^{(h)}d\xi_{12}+\mu^2 2!\int (C_{12}^{(\le h)^2}-C_{12}^{(<h)2})d\xi_{12}\,,\\
&(2)\quad V^{(k,2,2')}=\mu\alpha\begin{bmatrix}2\\1\end{bmatrix}^2\int :\varphi_1\partial\varphi_2:\partial_2 C_{12}^{(h)}d\xi_{12}+\mu\alpha 2!\int [(\partial_1 C_{12}^{(\le h)})^2-(\partial_1 C_{12}^{(<h)})^2]d\xi_{12}\,,\\
&(3)\quad V^{(k,2,4)}=\mu\lambda\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix}\int :\varphi_1\varphi_2^3:C_{12}^{(h)}d\xi_{12}+\mu\lambda 2!\begin{bmatrix}4\\2\end{bmatrix}\int :\varphi_2^2:(C_{12}^{(\le h)2}-C_{12}^{(<h)2})d\xi_{12}\,,\\
&(4)\quad V^{(k,2',2')}=\alpha^2\begin{bmatrix}2\\1\end{bmatrix}^2\int :\partial\varphi_1\partial\varphi_2:\partial_{12}^2 C_{12}^{(h)}d\xi_{12}+\alpha^2 2!\int [(\partial_{12}^2 C_{12}^{(\le h)})^2-(\partial_{12}^2 C_{12}^{(<h)})^2]d\xi_{12}\,,\\
&(5)\quad V^{(k,2',4)}=\alpha\lambda\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix}\int :\partial\varphi_1\varphi_2^3:\partial_1 C_{12}^{(h)}d\xi_{12}+\alpha\lambda\begin{bmatrix}4\\2\end{bmatrix}2!\int :\varphi_2^2:[(\partial_1 C_{12}^{(\le h)})^2-(\partial_1 C_{12}^{(<h)})^2]d\xi_{12}\,,\\
&(6)\quad V^{(k,4,4)}=\lambda^2\begin{bmatrix}4\\1\end{bmatrix}^2 1!\int :\varphi_1^3\varphi_2^3:C_{12}^{(h)}d\xi_{12}+\lambda^2\begin{bmatrix}4\\2\end{bmatrix}^2 2!\int :\varphi_1^2\varphi_2^2:(C_{12}^{(\le h)2}-C_{12}^{(<h)2})d\xi_{12}\\
&\qquad +\lambda^2\begin{bmatrix}4\\3\end{bmatrix}^2 3!\int :\varphi_1\varphi_2:(C_{12}^{(\le h)3}-C_{12}^{(<h)3})d\xi_{12}+\lambda^2\begin{bmatrix}4\\4\end{bmatrix}^2 4!\int (C_{12}^{(\le h)4}-C_{12}^{(<h)4})d\xi_{12}\,,
\end{aligned} \tag{17.2}$$

where φ_1,φ_2 means $\varphi_{\xi_1}^{(\le k)},\varphi_{\xi_2}^{(\le k)}$, and $C_{12}^{(\ )}$ means $C_{\xi_1\xi_2}^{(\ )}$, $d\xi_{12}=d\xi_1 d\xi_2$, $\partial_1\equiv\partial/\partial\xi_1$, $\partial_2=\partial/\partial\xi_2$, $\partial_{12}=\underline{\partial}_1\cdot\underline{\partial}_2$, $(\partial_1 C^{(\ )})^2=\underline{\partial}_1 C^{(\ )}\cdot\underline{\partial}_1 C^{(\ )}$; the symbol $V^{(k,\alpha_1,\alpha_2)}$ denotes $V^{(k),(\gamma)}$ with γ given by diagram 46.

Some of the above integrals are not ultraviolet stable, once appropriately summed over h (i.e., for h in $[k+1,N]$), as is easy to check, using the bounds of Sec. XVI and showing that they are "good bounds" or by direct computation; see the following table:

	first term	second term	third term	fourth term
(1)	stable	unstable for $d=4$		
(2)	stable	unstable for $d\ge 2$		
(3)	stable for $d\ge 2$	unstable for $d=4$		
(4)	unstable for $d\ge 2$	unstable for $d\ge 2$		
(5)	stable	unstable for $d\ge 2$		
(6)	stable for $d\ge 2$	unstable for $d=4$	unstable for $d\ge 3$	unstable for $d\ge 3$

Using (17.2) and proceeding according to the theory of Sec. VII, we can find counterterms $V_{2,N}$ to V_1 so that the effective potentials $V_2^{(k)}$ of $V_1+V_{2,N}$ are ultraviolet finite to second order.

Following the ideas developed in Sec. VII, one can start by trying to define the operations $\mathscr{L}_k$ making (7.10), i.e., $(1-\mathscr{L}_k)$ applied to (17.1) and summed over $h\in[k+1,N]$, finite as $N\to\infty$.

From (17.2) it appears that the divergences arise because some integrals obtained after summing (17.2) over $h\in[k+1,N]$ diverge for $\xi_1=\xi_2$.

Therefore, one can think of defining $\mathscr{L}_k$ by specifying its action on functions F having the form of the rhs of (17.2) with the kernels in front of the Wick monomials replaced by general kernels $w(\xi_1,\xi_2)$:

$$F=\int w(\xi_1,\xi_2)P\,d\xi_1 d\xi_2\,, \tag{17.3}$$

with the restriction that the w kernels are translation invariant on Λ (recall that periodic boundary conditions are imposed on Λ) and rotation covariant with respect to the rotations by $\pi/2$ around the coordinate axes (which are the only meaningful rotations on the torus Λ). The covariance here refers to the fact that the Wick monomials in (17.3) may contain derivatives of the fields: each derivative bears an index denoting to which component it refers and hence will bear corresponding indices—i.e., it will be a tensor; this fact does not explicitly show up in (17.2) or in the upcoming formulas because of the convenient convention used here that suppresses the indices of the derivatives, for simplicity of notation.

Once the action of $\mathscr{L}_k$ is specified on the functions of form (17.3) it will be extended to their linear combinations by linearity, some more comments on $\mathscr{L}_k$ as an operator will be made later after discussing its action on

F's like (17.3).

To produce the cancellation of the divergences which appear once the rhs of (17.2) are summed over h, generating expressions which are linear combinations of expressions like (17.3) diverging for $\xi_1=\xi_2$, the action of $(1-\mathscr{L}_k)$ should result in replacing the monomial P in (17.3) by a new expression RP vanishing as $\xi_2-\xi_1\to 0$ to an order so high that the integrals are no longer divergent.

An examination of the integrals in (17.2) shows that the following choice of RP would produce ultraviolet finite integrals:

$$\begin{aligned}
&R\,1=0\,,\\
&R{:}\varphi_1\varphi_2{:}={:}\varphi_1[\,\varphi_2-\varphi_1-(\xi_2-\xi_1)\cdot\partial\varphi_1\\
&\qquad -\tfrac{1}{2}(\xi_2-\xi_1)\times(\xi_2-\xi_1)\partial^2\varphi_1]{:}\,,\\
&R{:}\varphi_1\partial\varphi_2{:}={:}\varphi_1[\partial\varphi_2-\partial\varphi_1-(\xi_2-\xi_1)\cdot\underline{\partial}\partial\varphi_1]{:}\,,\\
&R{:}\partial\varphi_1\partial\varphi_2{:}={:}\partial\varphi_1(\partial\varphi_2-\partial\varphi_1){:}\,,\\
&R{:}\varphi_1^2\varphi_2^2{:}={:}\varphi_1^2(\varphi_2^2-\varphi_1^2){:}\,,\\
&R{:}\varphi_1\varphi_2^3{:}={:}\varphi_2^3(\varphi_1-\varphi_2){:}\,,\\
&RP=P\quad\text{otherwise}\,,
\end{aligned}\tag{17.4}$$

and using (3.20) (recall that the regularization being used here has $n=3$), one sees that the replacement of P by RP replaces P by a Wick polynomial which has a zero of order, respectively, ∞, third, second, first, first, zero.

Hence if there is an operation $\mathscr{L}_k$ such that $(1-\mathscr{L}_k)$ acting on the integrals in (17.2) just changes P into RP, then $\mathscr{L}_k$ has the property that (7.10) is, in the present case, ultraviolet finite because the above-mentioned orders of zero of RP are sufficient, in the worst cases, to make the expressions (17.4) ultraviolet finite.

From (17.4) one deduces that the operation $\mathscr{L}_k$ "which identifies the divergent parts" of $V_1^{(k)}$ to second order has to act on the integrals in (17.2) or more generally in (17.3) as

$$\begin{aligned}
&\mathscr{L}_k\int w(\xi_1,\xi_2)d\xi_1 d\xi_2=\int w(\xi_1,\xi_2)d\xi_1 d\xi_2\,,\\
&\mathscr{L}_k\int w(\xi_1,\xi_2){:}\varphi_{\xi_1}\varphi_{\xi_2}{:}d\underline{\xi}=\int w(\xi_1,\xi_2){:}\{\varphi_{\xi_1}[\varphi_{\xi_1}+(\xi_2-\xi_1)\partial\varphi_{\xi_1}+\tfrac{1}{2}(\xi_2-\xi_1)^2\times\partial^2\varphi_{\xi_1}]\}{:}d\xi_1 d\xi_2\,,\\
&\mathscr{L}_k\int w(\xi_1,\xi_2){:}\varphi_{\xi_1}\partial\varphi_{\xi_2}{:}d\underline{\xi}=\int w(\xi_1,\xi_2){:}\varphi_{\xi_1}[\partial\varphi_{\xi_1}+(\xi_2-\xi_1)\cdot\underline{\partial}\partial\varphi_{\xi_1}]{:}d\underline{\xi}\,,\\
&\mathscr{L}_k\int w(\xi_1,\xi_2){:}\partial\varphi_{\xi_1}\partial\varphi_{\xi_2}{:}d\underline{\xi}=\int w(\xi_1,\xi_2){:}(\partial\varphi_{\xi_1})^2{:}d\underline{\xi}\,,\\
&\mathscr{L}_k\int w(\xi_1,\xi_2){:}\varphi_{\xi_1}^2\varphi_{\xi_2}^2{:}d\underline{\xi}=\int w(\xi_1,\xi_2){:}\varphi_{\xi_1}^4{:}d\underline{\xi}\,,\\
&\mathscr{L}_k\int w(\xi_1,\xi_2){:}\varphi_{\xi_1}\varphi_{\xi_2}^3{:}d\underline{\xi}=\int w(\xi_1,\xi_2){:}\varphi_{\xi_2}^4{:}d\underline{\xi}\,,\\
&\mathscr{L}_k\equiv 0\quad\text{otherwise}\,,
\end{aligned}\tag{17.5}$$

so that the action of $(1-\mathscr{L}_k)$ on the integrals like (17.3) is precisely obtained by replacing in them P by RP.

One has to check that $\mathscr{L}_k$ takes values in the space of the interactions; this is in fact the basic reason the theory is renormalizable.

Possibly integrating by parts or using the rotation invariance properties of the coefficients $w(1,2)$, one can easily check that the action of $\mathscr{L}_k$ on the integrals in (17.5) is equivalent to the action of the following operator $\overline{\mathscr{L}}$ on the Wick monomials inside the integrals (here $\varphi\equiv\varphi^{(\le k)}$; $\theta,\theta'=1,2,\ldots,d$; $\partial_\theta=\partial/\partial\xi^{(\theta)}$, if $\xi^{(\theta)}$ is the θth component of the point ξ):

$$\begin{aligned}
&\overline{\mathscr{L}}1=1\,,\\
&\overline{\mathscr{L}}{:}\varphi_{\xi_1}\varphi_{\xi_2}{:}={:}\varphi_{\xi_1}^2{:}-\frac{(\xi_2-\xi_1)^2}{2d}{:}(\partial\varphi_{\xi_1})^2{:}\,,\\
&\overline{\mathscr{L}}{:}\varphi_{\xi_1}\partial_\theta\varphi_{\xi_2}{:}=-(\xi_2-\xi_1)_\theta{:}(\partial_\theta\varphi_{\xi_1})^2{:}\,,\\
&\overline{\mathscr{L}}_k{:}\partial_\theta\varphi_{\xi_1}\partial_{\theta'}\varphi_{\xi_2}{:}=\delta_{\theta\theta'}{:}(\partial_\theta\varphi_{\xi_1})^2{:}\,,\\
&\overline{\mathscr{L}}{:}\varphi_{\xi_1}^2\varphi_{\xi_2}^2{:}={:}\varphi_{\xi_1}^4{:}\,,\\
&\overline{\mathscr{L}}{:}\varphi_{\xi_1}\varphi_{\xi_2}^3{:}={:}\varphi_{\xi_2}^4{:}\,,\\
&\overline{\mathscr{L}}=0\quad\text{otherwise}\,.
\end{aligned}\tag{17.6}$$

This proves that the range of $\mathscr{L}_k$ is in the space of the interactions if one takes $\mathscr{L}_k$ to be defined by acting on expressions like (17.3) by replacing P inside by $\overline{\mathscr{L}}P$ [see (17.6)].

The above analysis shows also, that, trivially, the action of $(1-\mathscr{L}_k)$ on expressions like (17.3) is precisely the substitution of P by RP.

It is convenient to stop to point out the following. The operation $\mathscr{L}_k$ defined above is not unambiguously defined as an operator in the sense of functional analysis: to let $\mathscr{L}_k$ act on a function like (17.1) one has, by definition, first to express it as a sum of functions like (17.3) and then to act term by term replacing P by $\overline{\mathscr{L}}P$ [see (17.6)].

However, the expression of (17.1) as a linear combination of expressions like (17.3) is not unique.

Therefore, in order that the above definition of $\mathscr{L}_k$ makes sense one has also to prescribe how one writes (17.1), or more generally a function in the domain of $\mathscr{L}_k$, as a linear combination of terms like (17.3). Expression (17.2) is the prescription used here for the functions of interest. Also, later on we shall have to use a well-defined prescription for the decompositions of the effective potentials as sums of terms on which the action of the higher-order subtraction operations $\mathscr{L}_k^{(\sigma)}$ is defined. The prescription for the decomposition of the effective potential has therefore to be thought of as a part of the definition of $\mathscr{L}_k$.

Taking into account the above comment, we then check relation (7.13) first by verifying that the prescription to decompose the interesting functions (i.e., the effective potentials) as a sum of terms in the domain of the $\mathscr{L}_k$ operations commutes with the expectations $\mathscr{E}_{p+1}\cdots\mathscr{E}_k$ and second by asking whether $\mathscr{L}_k$ also commutes [in the sense of (7.13)] with them.

Both the above checks are very simple in our case; actually, the systematic use of Wick-ordered interactions and of Wick monomials has the basic motivation of making this check an essentially trivial consequence of (4.2) (implied by Wick ordering) and the definition of $\mathscr{L}_k$.

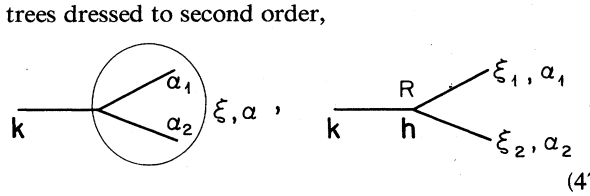
From (17.6) and (17.4) and applying the general theory of Sec. VII [see (7.14) and (7.16)], one finds easily the following expression of the counterterms $V_{2,N}$; if $\varphi\equiv\varphi^{(\le N)}$, it is

$$
\begin{aligned}
V_{2,N}=-\int d\xi_1\sum_{h=0}^{N}\Bigg\{ &:\varphi_{\xi_1}^2:\int\Bigg[\frac{1}{2}\mu^2\begin{bmatrix}2\\1\end{bmatrix}^2 C_{12}^{(h)}+\mu\lambda 2!\begin{bmatrix}4\\2\end{bmatrix}(C_{12}^{(\le h)2}-C_{12}^{(<h)2})+\alpha\lambda\begin{bmatrix}4\\2\end{bmatrix}2![(\partial_1 C_{12}^{(\le h)2})-(\partial_1 C_{12}^{(<h)2})]\\
&\qquad+\frac{\lambda^2}{2}\begin{bmatrix}4\\3\end{bmatrix}^2 3!(C_{12}^{(\le h)3}-C_{12}^{(<h)3})+\mu\alpha\begin{bmatrix}2\\1\end{bmatrix}^2\frac{1}{2}\partial_{12}C_{12}^{(h)}\Bigg]d\xi_2\\
&+:(\partial\varphi_{\xi_1})^2:\int\Bigg[-\mu\alpha\begin{bmatrix}2\\1\end{bmatrix}^2\frac{\xi_2-\xi_1}{d}\partial_2 C_{12}^{(h)}-\frac{\mu^2}{2}\begin{bmatrix}2\\1\end{bmatrix}^2\frac{(\xi_2-\xi_1)^2}{2d}C_{12}^{(h)}\\
&\qquad-\frac{\lambda^2}{2}\begin{bmatrix}4\\3\end{bmatrix}^2 3!\frac{(\xi_2-\xi_1)^2}{2d}(C_{12}^{(\le h)3}-C_{12}^{(<h)3})+\tfrac{1}{2}\alpha^2\partial_{12}C_{12}^{(h)}\Bigg]d\xi_2\\
&+:\varphi_{\xi_1}^4:\int\Bigg[\mu\lambda\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix}C_{12}^{(h)}+\frac{\lambda^2}{2}\begin{bmatrix}4\\2\end{bmatrix}2!(C_{12}^{(\le h)2}-C_{12}^{(<h)2})\Bigg]d\xi_2\\
&+1\int\Bigg[\tfrac{1}{2}\mu^2 2!(C_{12}^{(\le h)2}-C_{12}^{(<h)2})+\mu\alpha 2![(\partial_1 C_{12}^{(\le h)2})-(\partial_1 C_{12}^{(<h)})^2]\\
&\qquad+\frac{\alpha^2}{2}2![(\partial_{12}C_{12}^{(\le h)})^2-(\partial_{12}C_{12}^{(<h)})^2]+\frac{\lambda^2}{2}\begin{bmatrix}4\\4\end{bmatrix}^2 4!(C_{12}^{(\le h)4}-C_{12}^{(<h)4})\Bigg]d\xi_2\Bigg\}\,.
\end{aligned}
\tag{17.7}
$$

It should be stressed that for some terms in (17.2) rule (17.6) produces needless subtractions, as far as the ultraviolet stability is concerned; in fact, rule (17.6) coincides with the "usual rule" in the literature only in the "usual" case $\alpha=\mu=0$, $d=4$; if $d<4$, then (17.6) is oversubtracting even in this case.

Nevertheless, "universal rule" (17.6) will be used for simplicity of exposition; it would probably be not difficult to make the theory of Secs. VII and VIII more flexible so that more refined subtraction methods become possible permitting one to introduce counterterms only for "really divergent" parts of the effective interaction.

It is easy to compute in the above cases the meaning of the trees dressed to second order,



(47)

According to Sec. VII [see (7.19)], the framed tree represents one of the terms in (17.7) with the summations over h ranging from 0 to k (rather than to N) and with φ now meaning $\varphi^{(\le k)}$, up to a factor 2. Precisely select the contribution to (17.7) from the term $V^{(k,\alpha_1,\alpha_2)}$ in (17.2) [or $V^{(k,\alpha_2,\alpha_1)}$, whichever is present in (17.2)] containing $v_N^{(\alpha)}$, $\alpha=4,2,2',0$, i.e., containing $:\varphi^4:, :\varphi^2:, :(\underline{\partial}\varphi)^2:, 1$; then the frame in diagram 47 means

$$r^{(\alpha)}(\sigma,k)v_k^{(\alpha)}(\varphi^{(\le k)},\underline{\partial}\varphi^{(\le k)})\ , \tag{17.8}$$

where the r coefficient is the coefficient of the term in (17.7) just selected but with the summation over h ranging from 0 to k; here σ is a symbol for the tree shape framed in diagram 47. Clearly, $r^{(\alpha)}(\sigma,k)$ is proportional to $\lambda^{(\alpha_1)}\lambda^{(\alpha_2)}$.

The unframed dressed tree in diagram 47 represents, if we follow the rules of Secs. VII and VIII, exactly (17.2) with the replacement induced by (17.4), $P\to RP$ [see (7.18)].

Thus if one introduces the new fields

$$
\begin{aligned}
&D_{\xi_1\xi_2}=\varphi_{\xi_1}-\varphi_{\xi_2}\ ,\\
&D^1_{\xi_1\xi_2}=\partial\varphi_{\xi_1}-\partial\varphi_{\xi_2}\ ,\\
&S^1_{\xi_1\xi_2}=\partial\varphi_{\xi_1}-\partial\varphi_{\xi_2}-(\xi_1-\xi_2)\cdot\underline{\partial}\partial\varphi_{\xi_2}\ ,
\end{aligned}
\tag{17.9}
$$

$$S_{\xi_1\xi_2}=\varphi_{\xi_1}-\varphi_{\xi_2}-(\xi_2-\xi_1)\underline{\partial}\varphi_{\xi_2}\,,$$

$$T_{\xi_1\xi_2}=\varphi_{\xi_1}-\varphi_{\xi_2}-(\xi_2-\xi_1)\cdot\underline{\partial}\varphi_{\xi_2}-\tfrac{1}{2}(\xi_2-\xi_1)^2\times\underline{\partial}^2\varphi_{\xi_2}\,,$$

where $\underline{\delta}^2\times\underline{\partial}^2$ means, if $\underline{\delta}$ is a vector in R^d,

$$\sum_{i,j=1}^{d}\delta_i\delta_j(\partial^2/\partial\xi^{(i)}\partial\xi^{(j)})\,,$$

then the contributions from the unframed tree in diagram 47 to the effective potential $V_2^{(k)}$ due to $V_1+V_{2,N}$ at fixed h,α_1,α_2 are (if $\varphi\equiv\varphi^{(\le k)}$)

$$\frac{1}{2}\mu^2\begin{bmatrix}2\\1\end{bmatrix}^2 :\varphi_{\xi_1}T_{\xi_2\xi_1}:C^{(h)}_{\xi_2\xi_1},\quad \alpha_1=\alpha_2=2\,, \tag{17.10}$$

$$\mu\alpha\begin{bmatrix}2\\1\end{bmatrix}^2 :\varphi_{\xi_1}S^1_{\xi_2\xi_1}:\partial_{\xi_2}C^{(h)}_{\xi_1\xi_2},\quad \alpha_1=2,\ \alpha_2=2'\,, \tag{17.11}$$

$$\mu\lambda\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix} :\varphi^3_{\xi_2}D_{\xi_1\xi_2}:C^{(h)}_{\xi_1\xi_2},\quad \alpha_1=2,\ \alpha_2=4\,, \tag{17.12}$$

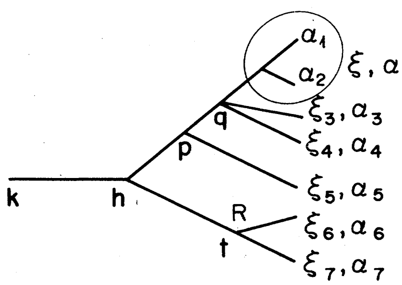
$$\frac{1}{2}\alpha^2\begin{bmatrix}2\\1\end{bmatrix}^2 :\partial\varphi_{\xi_1}D^1_{\xi_2\xi_1}:\partial_{\xi_1\xi_2}C^{(h)}_{\xi_1\xi_2},\quad \alpha_1=\alpha_2=2'\,, \tag{17.13}$$

$$\alpha\lambda\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix} :\partial\varphi_{\xi_1}\varphi^3_{\xi_2}:\partial_{\xi_1}C^{(h)}_{\xi_1\xi_2},\quad \alpha_1=2',\alpha_2=4\ \text{(unchanged)}\,, \tag{17.14}$$

$$\frac{1}{2}\left[\lambda^2\begin{bmatrix}4\\1\end{bmatrix}^2 1!:\varphi^3_{\xi_1}\varphi^3_{\xi_2}:C^{(h)}_{\xi_1\xi_2}+\lambda^2\begin{bmatrix}4\\2\end{bmatrix}^2 2!:\varphi^2_{\xi_1}(\varphi_{\xi_1}+\varphi_{\xi_2})D_{\xi_2\xi_1}:(C^{(\le h)2}_{\xi_1\xi_2}-C^{(<h)2}_{\xi_1\xi_2})+\lambda^2\begin{bmatrix}4\\3\end{bmatrix}^2 3!:\varphi_{\xi_1}T_{\xi_2\xi_1}:(C^{(\le h)3}_{\xi_1\xi_2}-C^{(<h)3}_{\xi_1\xi_2})\right]. \tag{17.15}$$

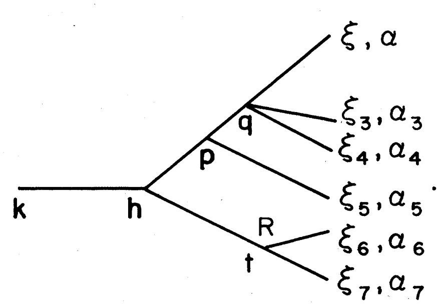
A simple way of describing the construction of (17.10)–(17.15) from (17.2), i.e., to interpret the R over the vertex of the tree in diagram 47, is that the tree in diagram 47 is computed from the values of the same tree without the R followed by the replacement of P by RP in the result.

It is also easy to compute the meaning of the most general tree dressed to order 2 (see Sec. VIII for this notation), as with the example in diagram 48 below:

γ = 

. (48)

According to the general theory of Sec. VIII the first act will be to "trim" the tree γ of the frame and its content (if there are more frames, one trims all of them), obtaining a simpler tree $\bar\gamma$; e.g., in the case of diagram 48 the result would be

$\bar\gamma$ = 

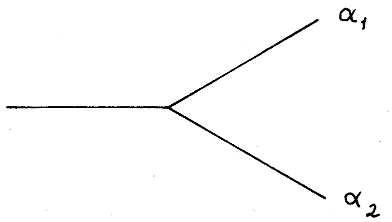
. (49)

Then one proceeds to write the truncated expectation formula for the evaluation of the contribution to $V^{(k)}$ of tree $\bar\gamma$, ignoring the presence of the R superscript (see comments in Sec. VII after diagram 13).

The vertex bearing the R contributes

$$\mathscr{E}^T_t(v^{(\alpha_6)}(\varphi^{(\le t)}_{\xi_6},\underline{\partial}\varphi^{(\le t)}_{\xi_6}),v^{(\alpha_7)}(\varphi^{(\le t)}_{\xi_7},\underline{\partial}\varphi^{(\le t)}_{\xi_7})) \tag{17.16}$$

in the above example and a similar expression in general; then one just replaces in (17.16) the Wick monomials P by RP according to (17.4). Finally one replaces factor $\lambda^{(\alpha)}$ contributed to the effective potential by the end point (ξ,α) with factor $r^{(\alpha)}(\sigma,q)$, σ being the shape



framed, according to (17.8).

This completes the analysis of the second-order renormalization. It justifies calling (17.8) "form factors with structure σ."

It will be clear that a detailed check of all the above formulas is the heart of renormalization theory and therefore the inexperienced reader should proceed only after having well understood the details of the above calculations.

As an exercise the reader can consider the theory of renormalization to second order in the following problems.

(1) Let $\mathscr{I}_N$ be

$$-\lambda\int_\Lambda :\varphi_\xi^{(\leq N)4}:d\xi \tag{17.17}$$

and show that if $d=2$ one can take $\mathscr{L}_k^{(\sigma)}=0$, i.e., no renormalization is necessary.

(2) Let $\mathscr{I}_N$ be

$$\int_\Lambda (-\lambda:\varphi_\xi^{(\leq N)4}:-\mu:\varphi_\xi^{(\leq N)2}:-\nu)d\xi \tag{17.18}$$

and work out the renormalization to second order in the case $d=3$, proving that one could use as a definition of $\mathscr{L}_k$ instead of (17.6) the following:

$$\overline{\mathscr{L}}1=1,\ \overline{\mathscr{L}}:\varphi_{\xi_1}\varphi_{\xi_2}:=:\varphi_{\xi_1}^2: . \tag{17.19}$$

(3) Let $\mathscr{I}_N$ be

$$\int_\Lambda [-\mu:\varphi_\xi^{(\leq N)2}:-\alpha:(\underline{\partial}\varphi_\xi^{(\leq N)})^2:]d\xi\ ; \tag{17.20}$$

work out in detail the renormalization theory showing that, unless $\alpha=0$, one still needs nontrivial renormalization. However, the theory can be rigorously built if μ,α are small, or non-negative.

(4) Show that the theory with interaction (17.18) is not renormalizable if $d=4$, not even to second order, in the sense of Secs. VI—VIII (not identical although trivially related to it to the one usual in the literature).

XVIII. RENORMALIZATION AND ULTRAVIOLET STABILITY TO ANY ORDER FOR φ^4 FIELDS

Section XVII has shown that renormalization to second order suggests representation of the effective potential in terms of Wick monomials more general than the ones used in Sec. XVI (16.4) and precisely as a sum of contributions like

$$\sum_P \int \frac{V(\gamma;P)}{n(\gamma)} P\, d\underline{\xi}\ , \tag{18.1}$$

where P has the form [$\varphi\equiv\varphi^{(\leq k)}$ and see (17.9) for the symbols]

$$P=:\left[\prod_j \varphi_{\rho_j}^{n_j}\right]\left[\prod_j D_{\eta_j\eta_j'}^{m_j}\right]\left[\prod_j D_{\theta_i\theta_j'}^{1n_j}\right] \times\left[\prod_j \partial\varphi_{\sigma_j}^{p_j}\right]\left[\prod_j S_{\xi_j\xi_j'}^{q_j}\right] \times\left[\prod_j S_{\xi_j\xi_j'}^{1r_j}\right]\left[\prod_j T_{\mu_j\mu_j'}^{t_j}\right]: , \tag{18.2}$$

with $n_j\leq 4$, $p_j\leq 2$, $m_j,n_j,q_j,r_j,t_j\leq 2$.

The most naive way to proceed is to define recursively the localization operations $\mathscr{L}_k^{(\sigma)}$ associated with tree shapes of degree $p+1$ (i.e., with $p+1$ end points) partially dressed to order p simply by using again the localization prescription (17.6) and the corresponding renormalization prescriptions for the interpretation of the vertices with R superscripts (17.4), the idea being that, as suggested by (16.18) and (16.19), the divergences are caused by the contributions to $V(\gamma;P)P$ from the vertices v of γ describing a Wick monomial of degree ≤ 4.

However, if P is given a general form (18.2), it is clear that there will be plenty of monomials of order ≤ 4 which do not appear in (17.4) and for which the operations R and $\mathscr{L}$ are not defined yet. The first task is to classify them.

One assumes, inductively, that the renormalized effective potential corresponding to an interaction renormalized to order p:

$$V_p=V_1+V_{2,N}+V_{3,N}+\cdots+V_{p,N} \tag{18.3}$$

is still described in terms of decorated Feynman graphs S as

$$\sum_{n=1}^\infty \int \sum_{\gamma:\text{degree }\gamma=n}\ \sum_S \frac{V(\gamma;S)}{n(\gamma)}d\underline{\xi}\ , \tag{18.4}$$

where now the graphs S will bear more decorations (compared to the cases treated in Sec. XVI) to describe the "effects" of the renormalization.

One checks this inductive assumption in the case $p=2$ first, where it can be checked, because $V_{2,N}$ has already been studied in Sec. XVII.

Let γ be any tree dressed to order 2, e.g., see diagram 48. Trim γ of the endframes and consider one decorated Feynman graph S corresponding to the evaluation of the effective potential due to the trimmed tree $\overline{\gamma}$ but ignoring the superscripts R.

Draw a box B_v around the cluster of position labels of $\overline{\gamma}$ corresponding to the vertex v of $\overline{\gamma}$: the box B_v will be drawn so that the lines of S with frequency index h_v are inner to B_v but not inner to v' if $v'>v$, as in Sec. XVI.

Therefore, out of each box B_v emerge lines of S possibly carrying ∂ labels, as in Sec. XVI; the notations introduced in Sec. XVI will be systematically used below; for instance, $n^e_{1,v}$ and $n^e_{0,v}$ will be, respectively, the number of lines emerging from B_v and carrying or not carrying a ∂ label; n^e_v will be defined to be the sum of the above two numbers.

So each box B_v represents a Wick monomial P_v, as in Sec. XVI. Consider the vertices v bearing in $\overline{\gamma}$ an R: observe that they must correspond to some innermost nontrivial clusters and precisely to those with two points in them.

Let one such v represent, in the given S, a Wick monomial P_v: one replaces it by RP_v [see (17.4)].

If $RP_v=P_v$, nothing has to be said; but if $RP_v\neq P_v$, one has to interpret that the vertex v contributes, via the graph S, RP_v rather than P_v to the evaluation of the truncated expectations corresponding to $\overline{\gamma}$.

If RP_v is a Wick mononial in the fields $\varphi,\partial\varphi,D,D^1,S^1,S,T$ [see (17.9)], then one denotes this operation of substitution of P_v by RP_v by simply adding an index 0 to the box B_v; but in some cases—actually, only one in the cases considered so far—RP_v may not be a Wick monomial in the above fields. In fact, the $R{:}\varphi_\xi^2\varphi_\eta^2{:}$ is, by (17.4),

$$:\varphi_\xi^2(\varphi_\eta^2-\varphi_\xi^2):=:\varphi_\xi^3 D_{\eta\xi}:+:\varphi_\xi^2\varphi_\eta D_{\eta\xi}:$$

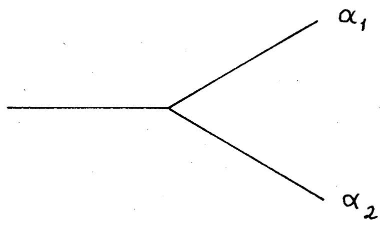
(i.e., a sum of two monomials).

If RP_v is a Wick polynomial sum of various monomials numbered from 0 to m, then one attaches to the box B_v a label $\beta_v=0,\ldots,m$ to recall which term in RP_v one selects in the evaluation of the truncated expectations as a contribution from v.

One takes into account this index β_v by changing accordingly the meaning of the lines of S emerging from the box B_v—e.g., the line representing φ_2 in $:\varphi_1\varphi_2:$ is replaced by T_{21}, that representing $\partial\varphi_2$ in $:\varphi_1\partial\varphi_2:$ is replaced by S^1_{21}, that representing $\partial\varphi_2$ in $:\partial\varphi_1\partial\varphi_2:$ is replaced by D^1_{21}, that representing one of the two φ_2's in $:\varphi_1^2\varphi_2^2:$ is replaced by D_{21} if the index β_v appended to the box B_v (which now takes values 0 or 1) is $\beta_v=1$, while, if $\beta_v=0$, then one of the two lines representing φ_2^2 is replaced by a line representing φ_1 and the other by one representing D_{21} (which is replaced by what is irrelevant—e.g., one can decide on the basis of the identity indices appended to the lines emerging from a point, say, lexicographically); in the $:\varphi_1\varphi_2^3:$ case the line representing φ_1 now represents D_{12}.

Then the evaluation of $V(\overline{\gamma};S)$ proceeds as before with the consequent change of meaning of the covariances associated with the contracted lines (when two lines are contracted, they give rise to the covariance between the two fields that they represent, of course).

Clearly, at the end of the computation one still has to replace the $\lambda^{(\alpha)}$ contributed by the end points of $\overline{\gamma}$ (coming from trimmed end frames bearing inside the shape σ and attached to a vertex of frequency h) by new factors $r^{(\alpha)}(\sigma;h)$, as explained in Sec. XVII; see (17.8) and the related discussion:

$\sigma=$ 

.

The result of the above procedure is a formula like (18.4) for $V_2^{(k)}=$(effective potential due to $V_2=V_1+V_{2,N}$).

Hence the inductive assumption is verified for $p=2$.

Assume (18.4) for $p=2,3,\ldots,p_0$ and let γ be a tree dressed to order p_0 and of degree p_0+1; assume to have already defined operations $\mathcal{L}_k^{(\sigma)}$ for all the shapes of degree $\leq p_0$.

Assume also that the result of the action of such operations leads to a rule of evaluation of the trees with no frames (and possibly with some R indices), which consists in examining successively the boxes B_v corresponding to the vertices of a tree, starting from the innermost ones, and changing successively the monomials P_v, which each of them represents, into a new monomial in the fields $\varphi,\partial\varphi,D,S,T,D^1S^1$ appearing in the polynomial RP_v defined as follows.

If P has one of the forms contemplated in (17.4), then RP is defined as in (17.4)—i.e., if $\varphi_j=\varphi_{\xi_j}^{(\leq k)}$, $\delta_{ij}=\xi_i-\xi_j$,

$$\begin{aligned}
&R1=0\,,\\
&R{:}\varphi_1\varphi_2{:}={:}\varphi_1T_{21}{:}\,,\\
&R{:}\varphi_1\partial\varphi_2{:}={:}\varphi_1S^1_{21}{:}\,,\\
&R{:}\partial\varphi_1\partial\varphi_2{:}={:}\partial\varphi_1D^1_{21}{:}\,,\\
&R{:}\varphi_1^2\varphi_2^2{:}={:}\varphi_1^3D_{21}{:}+{:}\varphi_1^2\varphi_2D_{21}{:}\,,\\
&R{:}\varphi_1\varphi_2^3{:}={:}D_{12}\varphi_2^3{:}\,,
\end{aligned}\tag{18.5}$$

where (18.5) is just a way of rewriting (17.4) in the new notations (17.9) and the D,S,T,D^1,S^1 fields have indices j which mean ξ_j and have also frequency indices which are the same as those of φ and which are not explicitly written.

With the same notations the action of R on other monomials of degree ≤ 4 is defined by

$$\begin{aligned}
&R{:}\varphi_2D_{12}{:}={:}\varphi_2T_{12}{:}\,,\\
&R{:}\varphi_2S_{12}{:}={:}\varphi_2T_{12}{:}\,,\\
&R{:}\varphi_1S_{23}{:}={:}D_{12}S_{23}{:}+R{:}\varphi_2S_{23}{:}\,,\\
&R{:}\varphi_1S_{12}{:}={:}D_{12}S_{12}{:}+{:}D_{21}T_{12}{:}+{:}\varphi_1T_{12}{:}\,,\\
&R{:}\varphi_1D_{32}{:}\equiv R{:}D_{12}D_{32}{:}+R{:}\varphi_2D_{32}{:}\\
&\qquad=-{:}S_{12}S_{32}{:}+{:}S_{12}D_{32}{:}+{:}D_{12}S_{32}{:}\\
&\qquad\quad+{:}D_{21}T_{32}{:}+{:}\varphi_1T_{32}{:}\,,\\
&R{:}D_{12}D_{32}{:}=-{:}S_{12}S_{32}{:}+{:}S_{12}D_{32}{:}+{:}D_{12}S_{32}{:}\,,
\end{aligned}$$

$$RD_{12}D_{34}=-:S_{12}\delta_{34}D^1_{34}:+:D_{12}\delta_{34}\cdot D^1_{24}:$$
$$-:S_{12}S_{34}:+:S_{12}D_{34}:+:D_{12}S_{34}:\,,$$
$$R:\partial\varphi_1\partial\varphi_2:=:\partial\varphi_1 D^1_{21}:\,, \tag{18.6}$$
$$R:\varphi_1\partial\varphi_2:=:\varphi_1 S^1_{21}:\,,$$
$$R:\partial\varphi_1 D_{21}:=:\partial\varphi_1 S_{21}:\,,$$
$$R:\varphi_1 D^1_{21}:=:\varphi_1 S^1_{21}:\,,$$
$$R:\varphi_3 D^1_{21}:=:D_{31}D^1_{21}:+:D_{13}S^1_{21}:+:\varphi_3 S^1_{21}:\,,$$
$$R:\partial\varphi_1 E_{23}:=:D^1_{13}D_{23}:+:\partial\varphi_3 S_{23}:\,,$$
$$R:\varphi_1\varphi_2^2\varphi_3:=:D_{12}\varphi_2^2\varphi_3:+:D_{21}\varphi_2^2 D_{32}:+:\varphi_1\varphi_2^2 D_{32}:\,,$$
$$R:\varphi_1\varphi_2\varphi_3\varphi_4:=:\varphi_1 D_{21}\varphi_3\varphi_4:+:\varphi_1 D_{12}D_{31}\varphi_4:$$
$$+:\varphi_1\varphi_2 D_{31}\varphi_4:+:\varphi_1 D_{12}D_{13}D_{41}:$$
$$+:\varphi_1\varphi_2 D_{13}D_{41}:+:\varphi_1 D_{12}\varphi_3 D_{41}:$$
$$+:\varphi_1\varphi_2\varphi_3 D_{41}:\,.$$

The action of R on the monomials which differ from the ones listed above by a sign (e.g., $\varphi_2 D_{21}$) is that R acts by changing the sign to the rhs; for the remaining monomials one puts $RP=P$.

The basic idea informing the definitions (18.5) and (18.6) is to subtract from each monomial P its "value at coinciding points" (defined below by the $\overline{\mathscr{L}}$ operation) to an order such that RP contains a zero of order:

1	if	degree of $P=4$	and	$n^e_{1,v}=0$
3	if	degree of $P=2$	and	$n^e_{1,v}=0$
2	if	degree of $P=2$	and	$n^e_{1,v}=1$
1	if	degree of $P=2$	and	$n^e_{1,v}=2$
∞	if	degree of $P=0$.		

In other words, if we call $\widetilde{\rho}_v$ the above order of zero, $\widetilde{\rho}_v$ is defined as the smallest integer for which $\widetilde{\rho}_v+\rho'_v>0$ [see (16.23)].

Furthermore, the definition of R is such that each of the monomials on the rhs can be thought of as obtained by substituting for one of the factors in P an "improved" factor climbing the chains $\varphi\to D\to S\to T$ or $D\to S\to T$ or $D\to D^1\to S^1$ or $\partial\varphi\to D^1\to S^1$ or $D^1\to S^1$ or $S\to T$.

In analogy with the second-order case of Sec. XVII it is natural to try to define the operation $\mathscr{L}_k^{(\sigma)}$ on the contribution of the tree γ with shape σ to the effective potential $V^{(k)}_{p_0}$, assuming that the tree has degree p_0+1 but that it is dressed to order p_0 only (this is the situation that has to be considered according to the general theory of Sec. VIII). If this contribution is denoted

$$\sum_S \int V(\gamma;S)P_S d\underline{\xi}\,, \tag{18.7}$$

then, in analogy with Sec. XVII, $\mathscr{L}_k^{(\sigma)}$ should act on (18.7) by just changing P_S into $\overline{\mathscr{L}}P_S$ and $\overline{\mathscr{L}}$ should be defined so that for any kernel w verifying the translation invariance and rotation covariance (for rotations of $\pi/2$ around the coordinate axes only, since Λ is taken with periodic boundary conditions) it should be

$$\int w[(1-\overline{\mathscr{L}})P]d\underline{\xi}=\int wRP\,d\underline{\xi}\,.$$

After some thought one realizes that this aim can be achieved by defining the action of $\overline{\mathscr{L}}$ to be that of replacing a nonlocal Wick monomial P by its Taylor expansion truncated to an order $\widetilde{\rho}_v-1$.

Since P is nonlocal, it will have to be decided which of the points appearing as indices of the fields in P will be made the Taylor expansion. For instance, one could choose any of them and then symmetrize the result on the choice; however, it is notationally and practically simpler to choose the one among them giving the simplest form to the result; sometimes this may still leave some ambiguity; the ambiguity will be solved by arbitrary choices guided by the labels j in the points ξ_j. Of course this implies that one has to be careful in imagining to draw the trees on the plane always in a standard way (a precaution ignored so far), i.e., picking up systematically one representative from each equivalence class and numbering the end points also in a standard way (e.g., from top to bottom).

In the expression below the indices of the fields in P are always supposed to be $\xi_{j_1},\xi_{j_2},\xi_{j_3},\xi_{j_4}$, but the shortened notation $\varphi_j=\varphi^{(\le k)}_{\xi_j}$ will be used as well as $\delta_{ij}=(\xi_{j_i}-\xi_{j_j})$. Also, if θ,θ' are component indices,

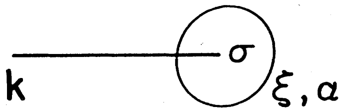
$$\delta^2_{ij}\times\underline{\underline{\partial}}^2\varphi_t\equiv\sum_{\theta,\theta=1}^{d}(\delta_{ij})_\theta(\delta_{ij})_{\theta'}\frac{\partial^2}{(\partial\xi)_\theta(\partial\xi)_{\theta'}}\,.$$

Then, with the above conventions,

$$\overline{\mathscr{L}}1=1\,,$$
$$\overline{\mathscr{L}}:\varphi_1\varphi_2:=:\phi_1(\varphi_1+\delta_{21}\partial\varphi_1+\tfrac{1}{2}\delta^2_{21}\times\underline{\underline{\partial}}^2\varphi_1):\,,$$
$$\overline{\mathscr{L}}:\partial_1\partial\varphi_2:=:\partial\varphi_1\partial\varphi_1:\,,$$
$$\overline{\mathscr{L}}:\varphi_1\partial\varphi_2:=:\varphi_1(\partial\varphi_1+\delta_{21}\cdot\underline{\partial}\partial\varphi_1):\,,$$
$$\overline{\mathscr{L}}:\varphi_1 D_{21}:=:\varphi_1(\delta_{21}\cdot\underline{\partial}\varphi_1+\tfrac{1}{2}\delta^2_{21}\cdot\underline{\underline{\partial}}\varphi_2)\,,$$
$$\overline{\mathscr{L}}:D_{13}D_{23}:=\delta_{13}\cdot\underline{\partial}\varphi_3\delta_{23}\cdot\underline{\partial}\varphi_3\,,$$
$$\overline{\mathscr{L}}:\varphi_1 D_{23}:=:\delta_{13}\cdot\underline{\partial}\varphi_3\delta_{23}\underline{\partial}\varphi_3:$$
$$+:\varphi_3(\delta_{23}\cdot\underline{\partial}\varphi_3+\tfrac{1}{2}\delta^2_{23}\times\underline{\underline{\partial}}^2\varphi_3):\,,$$
$$\overline{\mathscr{L}}:D_{12}D_{34}:=:\delta_{12}\cdot\underline{\partial}\varphi_2\delta_{34}\cdot\underline{\partial}\varphi_2:\,,$$
$$\overline{\mathscr{L}}:\varphi_1 S_{21}:=\tfrac{1}{2}:\varphi_1\delta_{21}\times\underline{\underline{\partial}}^2\varphi_1:\,, \tag{18.8}$$
$$\overline{\mathscr{L}}:\varphi_1 S_{12}:=\tfrac{1}{2}:\varphi_2\delta^2_{12}\times\underline{\underline{\partial}}^2\varphi_2:\,,$$
$$\overline{\mathscr{L}}:\varphi_1 S_{23}:=\tfrac{1}{2}:\varphi_3\delta_{23}\times\underline{\underline{\partial}}^2\varphi_3:\,,$$
$$\overline{\mathscr{L}}:\varphi_1 D^1_{21}:=:\varphi_1\delta_{21}\cdot\underline{\partial}\partial\varphi_1:\,,$$
$$\overline{\mathscr{L}}:\varphi_1 D^1_{23}:=:\varphi_3\delta_{23}\cdot\underline{\partial}\partial\varphi_3:\,,$$
$$\overline{\mathscr{L}}:\partial\varphi_1 D_{21}:=:\partial\varphi_1\delta_{21}\cdot\underline{\partial}\varphi_1:\,,$$
$$\overline{\mathscr{L}}:\partial\varphi_1 D_{23}:=:\partial\varphi_3\delta_{23}\cdot\underline{\partial}\varphi_3:\,,$$
$$\overline{\mathscr{L}}:\varphi_1\varphi_2^3:=:\varphi_2^4:\,,\quad\overline{\mathscr{L}}:\varphi_1^2\varphi_2^2:=:\varphi_1^4:\,,$$
$$\overline{\mathscr{L}}:\varphi_1\varphi_2^2\varphi_3:=:\varphi_2^4:\,,\quad\overline{\mathscr{L}}:\varphi_1\varphi_2\varphi_3\varphi_4:=:\varphi_1^4:\,,$$

$\overline{\mathscr{L}}P=0$ if P does not differ by a sign from one of the above monomials, $\mathscr{L}P=-\overline{\mathscr{L}}-P$ if P differs by a sign from one of the above monomials.

If the above is taken as definition of $\overline{\mathscr{L}}$, one can find $\mathscr{L}_k^{(\sigma)}$ and hence by the general algorithm of Sec. VIII, the counterterms of order p_0+1 as well as the meaning of the tree



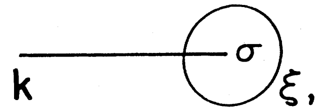
Recall that one is proceeding inductively and the definition of the counterterms (and the meaning of the dressed trees) is supposed known for trees of degree $\leq p_0$. Of course one has first to check that the operation $\mathscr{L}_k^{(\sigma)}$ has range in the space of the interactions (see Secs. VII and VIII). This follows, as in the case of second-order renormalization, by studying the integrals of expressions like (18.8) times kernels verifying the translation invariance and rotation covariance properties mentioned above (and possibly integrating by parts to obtain expressions of the appropriate form).

It is perhaps worth saying why $\mathscr{L}_k^{(\sigma)}$ bears an index σ: in fact, $\overline{\mathscr{L}}$ is defined independently of σ. However, $\mathscr{L}_k^{(\sigma)}$ acts on the functions of the form (18.7) and a function F can be written in several ways in the form (18.6). As discussed in Sec. XVII, the operation $\mathscr{L}_k^{(\sigma)}$ acts on the effective potential written in the form (18.7) as it arises from the prescriptions of calculation to be followed in evaluating the contribution of the graph S to the effective potential once the tree γ is given (such prescriptions are the ones discussed in detail in Sec. XVI): the prescription depends on the shape σ of γ; hence so does $\mathscr{L}_k^{(\sigma)}$. To be more precise, in Sec. XVI the prescriptions for the evaluation of the effective potential in terms of decorated Feynman graphs were given in the absence of renormalization; but renormalization just allows more complex Wick monomials and therefore a possibility of giving to the graphs' lines the meaning of more complex fields and one can still use the same graphical rule to build the evaluation of the expectations via the Wick rules.

Therefore, it will be decided to choose as definition of $\mathscr{L}_k^{(\sigma)}$ on the expressions (18.7) the action of the operations $\overline{\mathscr{L}}$ in the integrands. Then, by the above construction, the action of $(1-\overline{\mathscr{L}})$ generates an interpretation of the R superscripts in the trees dressed to order p_0+1 as meaning that the Wick monomial represented in a given graph S by a vertex v of order p_0+1 has to be replaced by RP defined by (18.5) and (18.6).

This means that the inductive assumption is indeed verified for $p=p_0+1$ and hence for all p. It also means that $\mathscr{L}_k^{(\sigma)}$ depends on σ only through the tree shape $\tau\sigma$ obtained by deleting the frames of σ as well as their contents, a necessary property in order to apply the resummation theory of Sec. IX to the present problem.

For later use it is convenient to recall the meaning of the tree

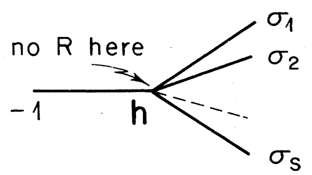


It is obtained by the rules of Sec. XVI; see diagrams 18 and 19 and Eq. (8.5).

One starts by erasing the frame around the shape σ and its labels ξ,α. Then one attributes frequency indices to the vertices of σ which are outside the remaining frames, and one also attributes position indices to the unframed end points of σ and to the endframes of σ: in this way one builds a partially dressed tree $\gamma\equiv(\sigma^{\underline{h}},\underline{\xi})$, because the first vertex of γ bears no superscript R (because before erasing the frame it was enclosed inside it and therefore had no R superscript).

Suppose that the indices $\underline{h}$ are such that the root frequency is -1: $h_r=-1$.

One proceeds by computing, with the rules explained above, the effective potential $V(\gamma;S)$, where S is a decorated Feynman graph,



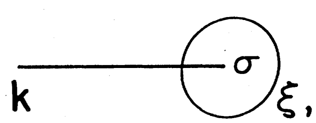
, (50)

with enough decorations on every box B_v to allow recognizing which choice among the monomials of RP_v is made at that vertex (as explained above, this is done by adding an index β_v at each box B_v corresponding to a vertex v bearing a superscript R and β_v can take only a few values [from (18.5) and (18.6) one sees that $\beta_v=0,1,2,3,4,5,6$ are enough in the most complex cases]).

Since the rhs of (18.8) is made up of local expressions in the fields and the coefficients are kernels with translation invariance and rotation covariance (in the sense considered above), it follows that the integrals over the position labels of $V(\gamma;S)P_S$ summed over S can be cast in the form "of an interaction":

$$\int_\Lambda [\, l^{(4)}(\sigma^{\underline{h}}){:}\varphi_\xi^{(-1)4}{:}+l^{(2)}(\underline{\sigma}^{\underline{h}}){:}\varphi_\xi^{(-1)2}{:} +l^{(2')}(\underline{\sigma}^{\underline{h}}){:}(\partial\varphi_\xi^{(-1)})^2{:}+l^{(0)}(\sigma^{\underline{h}})]d\xi\ , \qquad (18.9)$$

and this means that [see (8.5)] the form factor corresponding to



is

$$r^{(\alpha)}(\sigma,k)=\sum_{h=0}^{k}\sum_{\underline{h}'}\frac{l^{(\alpha)}(\sigma^{\underline{h}})}{n(\sigma)}\ , \qquad (18.10)$$

where h denotes the frequency index of the first vertex of $\sigma^{\underline{h}}$ after the root and $\underline{h}'$ are the frequency indices on the higher vertices (and the root frequency is supposed to be -1).

Naturally the N dependence of (18.10) is in the fact that the summation indices over $\underline{h}'$ run with upper bounds equal to N; nevertheless, it will appear that $r^{(\alpha)}(\sigma,k)$ admits a limit as $N\to\infty$, at fixed k.

This completes the inductive description of the counterterms and of their effects on the tree representation of the effective potentials.

The final result is that after complete renormalization

$$V^{(k)}=\sum_{n=1}^{\infty}\int\sum_{\substack{\gamma:k(\gamma)=k\\ \text{degree }\gamma=n,\xi(\gamma)=\xi\\ \gamma\text{ dressed}}}\sum_{S}\frac{V(\gamma;S)}{n(\gamma)}P_S d\underline{\xi}\ , \tag{18.11}$$

where the sum runs over the Feynman graphs S associated with the trimmed tree $\overline{\gamma}$ (i.e., γ deprived of the outer frames and of their contents), decorated by boxes (defining the clusters associated with the vertices v of $\overline{\gamma}$) bearing indices β_v explaining the selection to be made in evaluating RP_v (the index β_v can take at most seven values). Furthermore, the graph S bears all the other decorations already possible in the nonrenormalized cases (i.e., frequency, character, identity, and ∂ indices—see Sec. XVI).

It remains for us to check that, with the above definitions of the subtraction operations, the new theory is ultraviolet finite.

Given a dressed tree $\overline{\gamma}$ with no frames (i.e., with every vertex of γ bearing an index R), one has to study, given $\varphi^{(\le k)}$ verifying (3.20) (with $n=3$), the expression [see the analog (16.12)]

$$M_S(\Delta_1,\dots,\Delta_p)=\int_{\Delta_1\times\cdots\times\Delta_p\times\Lambda^{n-p}}|V(\overline{\gamma};S)|\sup|P_S|\times d\underline{\xi},\ \Delta_j\in Q_k, p\le n\ , \tag{18.12}$$

where S is a given decorated Feynman graph: n is the degree of γ, k is the root frequency.

Clearly the integral (18.12) is evaluated by just the same type of analysis leading to the bounds (16.18) in the case of no renormalization. One has only to replace some covariances with new covariances due to the fact that some lines have the meaning of new fields (D,S,T,D^1,S^1).

However, a few remarkable improvements are generated by such changes.

Call a line of S representing fields like (17.9) a "renormalized line." Below $\overline{\gamma}$ and S will be fixed.

Looking at the graph S, one can see which is the vertex v "causing" the change of meaning of a renormalized line compared to the meaning that the line would have in the graph S_0 obtained from S by erasing all the decorations which allow one to interpret it as a renormalized graph. It must be a vertex v corresponding to a box B_v, which in S_0 would determine a monomial P_v on which R acts nontrivially ($RP_v\ne P_v$) [see (17.4), (18.5), and (18.6)]. The actual meaning of a renormalized line cannot, however, be determined by v alone. In fact, as (18.6) shows, it may happen that its meaning is changed again in correspondence of a vertex $v'<v$ such that $B_{v'}$ contains two external lines.

But the change of meaning cannot take place more than four times, because the meaning of the line keeps "improving" (i.e., the corresponding order of zero in the RP polynomial keeps increasing): a φ line can become a D, S, or T line, a D line can become an S or T line or a $\partial\varphi$ line, an S line can become a T line, a $\partial\varphi$ line can become a D^1 or S^1 line, and a D^1 line can become an S^1 line; and R is the identity when acting on Wick monomials containing S^1 or T fields.

So, given $\overline{\gamma}$, S, and a renormalized line of S, one can define the first vertex v responsible for its change of meaning with respect to the meaning it would have in S_0; one can also define the vertices $v_1,v_2,\dots,$ following v where the line again changes meaning before acquiring its final meaning; from (18.6) and (18.5) one sees that this change of meaning cannot take place more than a fixed number of times (e.g., four). Finally, one can define the vertex w where the line becomes internal to a box B_w for the first time: $w=r=$root of γ if the line is external.

Call ρ_v the parameter associated with the vertex v [see (16.18) and (16.19)] in the graph S_0. Then it is clear that the fact that the line has changed meaning introduces in the basic bound (16.7) an extra factor given, at least, by

$$B_2[\gamma^{h_w}d(\underline{\xi}_v)]^{\delta_v}\equiv B_2\gamma^{-(h_v-h_w)\delta_v}[\gamma^{h_v}d(\underline{\xi}_v)]^{\delta_v}\ , \tag{18.13}$$

where B_2 is a suitable constant and δ_v is the variation of the order of zero of P_v, as $d(\underline{\xi}_v)\to 0$, introduced in P_v by the R operation via the change in meaning of the line under consideration.

Therefore, every time a given line changes meaning at vertices $v_1>v_2>\cdots$ new factors like (18.13) arise in the bound on $M_S(\Delta_1,\dots,\Delta_p)$, and by construction the sum over the lines λ that change meaning and over the vertices v of the quantities δ_v is such that

$$\sum_{\lambda}\sum_{v}\delta_v(h_v-h_w)\ge\sum_{v}\widetilde{\rho}_v(h_v-h_{v'})\ , \tag{18.14}$$

if v' is the vertex immediately preceding v in γ, and $\widetilde{\rho}_v=-\rho_v+1$ if $\rho_v\le 0$ and $\widetilde{\rho}_v=0$ otherwise.

Eventually the bound on $M_S(\Delta_1,\dots,\Delta_p)$ becomes

$$\left\{\int_{\Delta_1\times\cdots\times\Delta_p}\left[\left[\prod_{\lambda}e^{-\kappa\gamma^{h_\lambda}|\lambda|}\right]\left[\prod_{v}\gamma^{-(h_v-h_{v'})\widetilde{\rho}_v}\right]\prod_{v}[\gamma^{h_v}d(\underline{\xi}_v)]^{\widetilde{\theta}_v}\right]d\xi\right\}$$
$$\times\overline{\varepsilon}^{\,n}\mathcal{N}B^{n_e}\widetilde{B}_2^{\,n_e}B_2^{4n}\gamma^{[(d-2)/2]kn^e_{0,v_0}}\gamma^{(d/2)kn^e_{1,v_0}}\prod_{v>r}(\gamma^{h_v[(d-2)/2]n^{\text{inner}}_{0,v}}\gamma^{h_v(d/2)n^{\text{inner}}_{0,v}})\ , \tag{18.15}$$

where it is defined by a formula like (16.13) in which P_S has the new meaning [and the rhs is changed accordingly in the natural way: note that the new rhs will contain, in general, factors like (18.13) when P_S contains renormalized fields and use is made of (3.20) to exhibit the order of zero in the D,S,T,D^1,S^1 fields; the constants $\widetilde{\theta}_v$ can, in principle, be read by comparing (18.13) and (18.15)].

The exponents $\widetilde{\theta}_v$ can be bounded by the maximum of $\widetilde{\rho}_v$ (i.e., three) times the number of times a line can change meaning (i.e., four at most) times the number of lines that do change meaning at the vertex v [by (18.5) and (18.6) at most three: actually, this happens only when P_v looks like $\varphi_1\varphi_2\varphi_3\varphi_4$—see (18.5)]. Call T the above bound ($T=3^2 4$).

Hence for all $\zeta>0$ it is, if $d(\underline{\xi})$ is the graph distance between the points of $\underline{\xi}=(\xi_1,\ldots,\xi_n)$,

$$\prod_v [\gamma^{h_v} d(\underline{\xi}_v)]^{\widetilde{\theta}_v} \le \left[\frac{T!}{\zeta}\right]^{4n} \exp\left[\zeta T \sum_{v>r} \gamma^{h_v} d(\underline{\xi}_v)\right] \tag{18.16}$$

(where the $4n$ arises from the fact that the lines changing meaning at v can become internal at different vertices w: at most 4).

This is used to choose ζ so that $\zeta T<\kappa(1-\gamma^{-1})/4$, which can be used together with the inequality

$$\sum_\lambda \gamma^{h_\lambda} |\lambda| \ge (1-\gamma^{-1}) \sum_{v>r} \gamma^{h_v} d(\xi_v) , \tag{18.17}$$

a consequence of $\gamma^h \ge (1-\gamma^{-1})(1+\gamma^{-1}+\gamma^{-2}+\cdots+\gamma^{-h})\gamma^h$ and of elementary geometry, to bound (18.15) by [see Appendix D for the bound (16.14) on the integral]

$$\int_{\Delta_1\times\cdots\times\Delta_p} \exp\left[-\kappa\sum_\lambda \gamma^{h_\lambda}|\lambda|\right]\prod_v [\gamma^{h_v} d(\underline{\xi}_v)] d\underline{\xi} \le e^{-(\kappa/4)\gamma^k d(\Delta_1,\ldots,\Delta_p)}\left[\frac{T!}{\zeta}\right]^{4n} \times \int_{\Lambda^{n-1}} \gamma^{-kd} \exp\left[-\frac{\kappa}{4}\sum_\lambda \gamma^{h_\lambda}|\lambda|\right] d\xi_2\cdots d\xi_n , \tag{18.18}$$

which, inserted in (18.15) and after the appropriate power counting, becomes

$$M_s(\Delta_1,\ldots,\Delta_p) \le \bar{\varepsilon}^n \mathcal{N} B^{n^e} \widetilde{B}_3^n e^{-(\kappa/4)\gamma^4 d(\Delta_1,\ldots,\Delta_p)} \gamma^{-k[2m_2+(4-d)m_4]} \prod_{v>r} \gamma^{-(\rho_v+\widetilde{\rho}_v)(h_v-h_{v'})} \tag{18.19}$$

for a suitable $\widetilde{B}_3$, if v' denotes the vertex immediately before v in γ and with ρ_v defined in (16.19), using the graph S_0 obtained from S by erasing all the labels referring to the renormalization, and

$$\rho_v+\widetilde{\rho}_v = -d+2m_{2,v}+(4-d)m_{4,v}+\frac{d-2}{2}n^e_{0,v}+\frac{d}{2}n^e_{1,v}+\delta_{n^e_{0,v}4}\delta_{n^e_{1,v}0} + 3\delta_{n^e_{0,v}2}\delta_{n^e_{1,v}0}+2\delta_{n^e_{0,v}1}\delta_{n^e_{1,v}1}+1\delta_{n^e_{0,v}0}\delta_{n^e_{1,v}2} \ge \tfrac{1}{2}=\bar{\rho} \tag{18.20}$$

(here $n^e_v, n^e_{1,v} n^e_{0,v}$ are counted as they appear in S_0).

Actually, for later use, one can observe that if $2m_{2,v}+(4-d)m_{4,v}$ is replaced by 0 in (18.20) one obtains a new expression $\rho'_v+\rho_v$ which, nevertheless, is still larger than $\bar{\rho}=\frac{1}{2}$ [see (16.20)–(16.23)].

Expressions (18.19) and (18.20) prove the ultraviolet finiteness for the trees which are dressed but contain *no* frames.

If γ bears frames enclosing shapes $\sigma_1,\ldots,\sigma_m$, $m\le n=$degree of γ, attached to the trimmed tree $\bar{\gamma}$, obtained from γ by trimming it at the vertices of frequency $h_1,\ldots,h_m$ (allow here the convention that the unframed end points are regarded as framed by a frame containing the trivial shape, as already done in the previous sections), then bound (18.17) is obviously replaced by

$$M_s(\Delta_1,\ldots,\Delta_n) \le \mathcal{N} B^{n^e}\widetilde{B}_3^n e^{-(\kappa/2)\gamma^k d(\Delta_1,\ldots,\Delta_n)} \times\gamma^{-[2m_2+(4-d)m_4]k} \times\prod_{v>r}\gamma^{-(\rho_v+\widetilde{\rho}_v)(h_v-h_{v'})} \times\prod_{j=1}^m |r^{(\alpha_j)}(\sigma_j;h_j)| , \tag{18.21}$$

where the factors $r^{(\alpha)}(\sigma;h)$ are the form factors associated with the shapes σ [see Secs. VIII and XVII, and (17.8)], defined by (18.10), $r^{(\alpha)}(\sigma,h)\equiv\lambda^{(\alpha)}$ if the shape σ enclosed in the frame is trivial.

Consider $d=4$ and suppose that one could prove that

$$|r^{(\alpha)}(\sigma,h)| \le \gamma^{2h\delta_{\alpha,2}}\gamma^{4h\delta_{\alpha,0}} h^s \bar{\varepsilon}^s C_s , \tag{18.22}$$

where s is the degree of the shape σ and

$$\bar{\varepsilon}=\max(|\lambda|,|\mu|,|\alpha|,|\nu|)=\max_\alpha|\lambda^{(\alpha)}| .$$

Then, as already noted in Sec. XVI and after (18.20) above, the factors $\gamma^{2h\delta_{\alpha,2}}$ would affect the bounds (18.19) and (18.20) by replacing $\widetilde{\rho}_v+\rho_v$ by $\rho'_v+\rho_v$ and $2m_{2,v}$ by 0, so that (18.19) becomes

$$M_s(\Delta_1,\ldots,\Delta_p) \le \mathcal{N} B^{n^e}\widetilde{B}_3^n e^{-(\kappa/2)\gamma^k d(\Delta_1,\ldots,\Delta_p)} \times\sum_{\underline{h}}\prod_{v>r}\gamma^{-\bar{\rho}(h_v-h_{v'})}\prod_{i=1}^m h_i^{s_i} C_{s_i} , \tag{18.23}$$

and the ultraviolet finiteness would follow also for the frame-bearing dressed trees.

It is convenient to observe that bound (18.23) can be considerably improved at no cost if one notes that, by the nature of the bounds leading to the $d(\Delta_1,\ldots,\Delta_p)$ in the exponential, one could have obtained instead the quantity $d_S(\Delta_1,\ldots,\Delta_p)$, where this is defined as the sum of the distances between the cubes Δ joined in S by a hard line. It is clear that this is a much better bound for very structured graphs.

It remains for us to prove (18.22); however, in Sec. XIX a much stronger bound compared to (18.22) (easy, as it will appear) will be proved. Therefore, the proof of (18.22) is postponed to Sec. XIX.

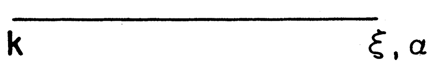
The results of this section basically contain the "Hepp theorem" (Hepp, 1966,1969): this theorem provided the first completely rigorous proof of ultraviolet stability [see also Zimmermann (1969), Speer (1974), and Eckmann and Epstein (1979)].

XIX. "n! BOUNDS" ON THE EFFECTIVE POTENTIAL

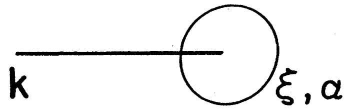
It is now possible to find concrete bounds on the coefficients of the effective potentials. In this section we take $d=4$, for simplicity (the cases $d<4$ are similar and slightly easier).

From the preceding analysis emerges the following organization of the contributions to $V^{(k)}$ of the trees of degree n.

A dressed tree γ will be described by its trimmed part $\overline{\gamma}$, obtained by cutting out of γ all frames and their contents, and by the actual contents of the external frames of γ: one per end point of γ which bears a frame; for uniformity of notation one imagines here that all the end points of the dressed trees are framed so that if



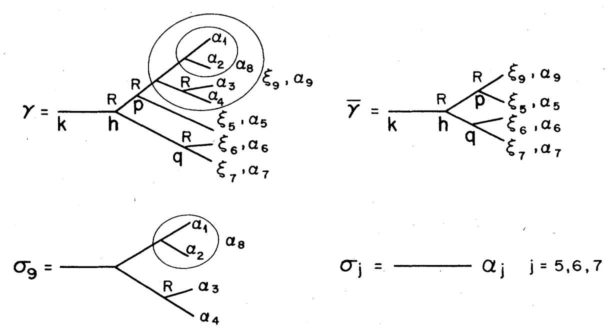
is an endbranch of γ which bears no frames one imagines to transform it into



The degree of γ will in general be larger than or equal to the degree m of $\overline{\gamma}$, which will be called the "renormalized degree of γ."

So a dressed tree γ will be described by $\overline{\gamma}$ and m shapes $\sigma_1,\sigma_2,\ldots,\sigma_m$, which have to be enclosed in frames attached to the end points of $\overline{\gamma}$ to rebuild γ: if γ has degree n and σ_i degree n_i it must be $n=\sum_{i=1}^{m} n_i$.

For instance, the following picture shows a tree γ together with its trimmed part $\overline{\gamma}$ and the shapes $\sigma_1,\sigma_2,\ldots,$



(51)

The number of shapes of degree s can be easily estimated by D_1^s for some D_1 (one can take $D_1=2^4$).

Consider the contribution to $V^{(k)}$ from the trees of degree n:

$$V^{(k),n}=\int \sum_{\substack{k(\gamma)=k\\ \text{degree }\gamma=n\\ \underline{\xi}(\gamma)=\underline{\xi}}} \sum_{S} \frac{V(\gamma;S)}{n(\gamma)} d\underline{\xi}\,, \tag{19.1}$$

where P_S has the form (18.2) and S is a decorated Feynman graph as described in Sec. XVIII.

The aim of this section is to show that if $\xi_1,\ldots,\xi_m$ are the endframe labels of $\overline{\gamma}$, then there are γ-independent constants B,κ,D,b such that if $B=\sup_\Delta B_\Delta$ in (3.20) and $\overline{\varepsilon}=\max_\alpha|\lambda^{(\alpha)}|$ it is

$$\begin{aligned}M(\Delta_1,\ldots,\Delta_p)=&\int_{\mathscr{D}(\Delta_1,\ldots,\Delta_p)} \sum_{\substack{k(\gamma)=k,\underline{\xi}(\gamma)=\underline{\xi}\\ \text{degree }\gamma=n}} \sum_{\substack{S\\ P_S=P}} \frac{|V(\gamma;S)|}{n(\gamma)} \times \sup|P|\,d\underline{\xi}\\ &\le \mathscr{N} B^n \overline{\varepsilon}^{\,n} e^{-\kappa\gamma^k d(\Delta_1,\ldots,\Delta_p)} n! \sum_{j=0}^{n-1} \frac{(bk)^j}{j!}\,,\end{aligned} \tag{19.2}$$

where $\mathscr{D}(\Delta_1,\ldots,\Delta_p)$ is the product $\Delta_1\times\Delta_2\times\cdots\times\Delta_p\times\Lambda\times\cdots\times\Lambda$ or a domain obtained by permuting such factors; $\Delta_j\in Q_k$ and the supremum of P means supremum over the fields

$$\varphi^{(\le k)}=\sum_{j=0}^{k}\varphi^{(j)}$$

with $\varphi^{(j)}$ verifying (3.20); $\mathscr{N}$ depends on the degree of P only: $\mathscr{N}=O(n^e!)$.

Equation (19.2) will be called the n! bound: this bound was obtained in a slightly different form (i.e., as a bound on the Schwinger functions rather than on the effective potentials, and in "momentum space" rather than in "position space") and with a somewhat different method in the remarkable work of De Calan and Rivasseau (1982). The approach presented below follows essentially Gallavotti and Nicolò (1984).

The first problem is to find explicit combinatorial estimates on the number of terms in (19.2).

Since S has the interpretation of a decorated Feynman graph with m vertices, m being the renormalized degree of γ (i.e., the degree of its trimmed part), and since the decorations consist of finitely many indices attached to each line and vertex of the graph (see Sec. XVIII for an explicit description of such indices, each of which can take a number of values which is finite and graph independent, except for the frequency indices), it follows that one can bound the number of terms in $\sum_S$ in (19.2) at fixed γ by a constant of the form D_2^m times the number of Feynman graphs, i.e., connected graphs, which can be built by joining pairwise $(4m_4+2m_2+2m_{2'})$ lines emerging from $m=(m_4+m_2+m_{2'})$ vertices out of m_4 of which emerge four distinct lines, while out of the other $(m_2+m_{2'})$ emerge only two lines, possibly leaving a few lines unpaired. This number is clearly bounded by

$$(2m_4+m_2+m_{2'})!4^{2m_4+m_2+m_{2'}} \le (2m)!2^{4m} ,$$

and this is therefore an estimate of the number of terms in the $\sum_S$.

However, the above number is too big, and it can be replaced by a better bound. This is so because the $(2m)!2^{4m}$ ways described above come from multiplying the $\lesssim m!2^{4m}$ connected graphs built with m unlabeled points ("topologically distinct graphs") times the $m!$ ways of labeling such points by $\xi_1,\ldots,\xi_m$. But the rules of construction of a graph S associated with a tree γ are such that if a graph S is given and can arise in the sum (19.2) for a given γ, i.e., $\bar{\gamma}$, then the same graph with the vertices relabeled does not necessarily arise.

Given a graph G with no labels, one can consider the number N of ways of labeling G compatibly with $\bar{\gamma}$ and with given numbers n_v^e of external lines (of any type) emerging from the subgraph of G associated with the vertices v of $\bar{\gamma}$. Then N is bounded by

$$n(\sigma)C_\varepsilon^n \exp\left[\varepsilon \sum_v n_v^e\right] ,$$

for all $\varepsilon>0$ and suitable C_ε, if σ is the shape of $\bar{\gamma}$ and $n(\sigma)$ is the corresponding combinatorial factor. This bound replaces an incorrect one of a previous version, and I am indebted to G. Felder for pointing out the error and its correction (see Appendix F, by Felder). The bound will be combined with the remark that the summation over $\bar{\gamma}$ can in fact be thought of as a sum over the shapes σ and the frequency labels $\underline{h}$ assigned to the vertices of σ. However, various frequency assignments $\underline{h}$ to the vertices of σ produce the same $\bar{\gamma}=(\sigma,\underline{h})$, because of our convention on the trees' equivalence, and the correct relation between the sum over $\bar{\gamma}$ and that over $(\sigma,\underline{h})$ is

$$\sum_{\bar{\gamma}} n(\bar{\gamma})^{-1} = \sum_\sigma \sum_{\underline{h}} n(\sigma)^{-1} .$$

Let then $\gamma=(\sigma_0,\underline{h},\underline{\sigma},\underline{\xi})$ be the dressed tree obtained by choosing a trimmed shape σ_0, labeling its vertices with frequency indices $\underline{h}$, and then choosing m dressed tree shapes $\sigma_1,\ldots,\sigma_m$, of given degrees $n_1,\ldots,n_m$ such that $\sum n_i=n$, framed inside endframes attached to the end points of σ_0 and bearing position indices $\underline{\xi}=(\xi_1,\ldots,\xi_m)$. Let S be a decorated Feynman graph, compatible with γ, such that P_S is a given P and such that the number of external lines n_v^e emerging from the subgraph of S corresponding to the vertex v of σ_0 are given. Then $M(\ )$ in (19.2) can be obviously bounded, by taking into account the above combinatorial considerations, as

$$\begin{aligned} M(\Delta_1,\ldots,\Delta_p) &\le \sup_{\substack{\sigma_0;\sigma_1,\ldots,\sigma_m\\ S}} m!D_3^n \sum_{\{m_v^e\}} \sum_{\{h_v\}} \exp\left[\varepsilon \sum n_v^e\right] \int_{\mathcal{D}(\Delta_1,\ldots,\Delta_p)} |V^{(k)}(\gamma;s)| \sup|p|\, d\underline{\xi} \\ &\le e^{-\kappa\gamma^k d(\Delta_1,\ldots,\Delta_p)} \mathcal{N} m! D_4^n \gamma^{-2m_{2,v}k} \sum_{\underline{h}} \left[\prod_{v>r} \sum_{n_v^e} e^{\varepsilon n_v^e} \gamma^{-(h_v-h_{v'})(\tilde{\rho}_v+\rho_v)}\right] \prod_j |r^{(d_j)}(\sigma_j,h_j)| , \end{aligned} \qquad (19.3)$$

if D_3,D_4 are suitable constants and the notations of Sec. XVIII are used; furthermore, the summations over n_v^e from 0 to ∞ can be controlled by

$$\sum_{n_v^e=0}^{\infty} e^{\varepsilon n_v^e} \gamma^{-(h_v-h_{v'})(\tilde{\rho}_v+\rho_v)} \le \text{const} \times \gamma^{-(h_v-h_{v'})(\tilde{\rho}+2m_{2,v})} \qquad (19.4)$$

because of (18.18) and $h_v-h_{v'}\ge 1$, if ε (arbitrary so far) is chosen small enough. Here the notations of Sec. XVIII are used: in particular, v' denotes the vertex immediately preceding v in γ.

Therefore, bound (19.4) reduces the problem to that of the coefficients $r^{(\alpha)}(\sigma;h)$, which end up, in this way, playing the central role in the quantitative theory of renormalization.

The theory of the coefficients $r^{(\alpha)}(\sigma;h)$, to a degree of depth allowing the proof of the $n!$ bound, is in fact easy as soon as one makes the right guess as what to prove; the guess has to be made by trial and error methods, and it is pointless to repeat the search here. The result is that one should try to prove that there exist constants $b>0$, $D_5>0$ such that (always for $d=4$)

$$|r^{(\alpha)}(\sigma;k)| \le \bar{\varepsilon}^n D_5^{n-1}(n-1)! \times \sum_{j=0}^{n-1} \frac{(bk)^j}{j!} \gamma^{2k\delta_{\alpha,2}+4k\delta_{\alpha,0}} , \qquad (19.5)$$

where n is the degree of σ and, as usual, $\bar{\varepsilon}=\max_\alpha |\lambda^{(\alpha)}|$.

Before proving (19.5) we shall find it reassuring to check that (19.5) is really what one needs.

In fact, inserting (19.5) in (19.4), one estimates the rhs of (19.4) by using the remarks following (18.20) and leading to (18.21); it follows that

$$M(\Delta_1,\ldots,\Delta_p)\le e^{-\kappa\gamma^k d(\Delta_1,\ldots,\Delta_p)} m!\mathcal{N} D_4^m \bar{\varepsilon}^n D_5^{n-m}\sum_{\underline{h}}\prod_{v>r}\gamma^{-\bar{p}(h_v-h_{v'})}\prod_{j=1}^m\left[(n_j-1)!\sum_{p=0}^{n_j-1}\frac{(bh_j)^p}{p!}\right], \tag{19.6}$$

where n_j is the degree of σ_j:

$$\sum_{j=1}^m n_j=n\ . \tag{19.7}$$

This gives immediately (19.2) via the inequality

$$\sum_{\underline{h}}\prod_{v>r}\gamma^{-\bar{p}(h_v-h_{v'})}\prod_{j=1}^m\left[(n_j-1)!\sum_{p=0}^{n_j-1}\frac{(bh_j)^p}{p!}\right]\le D_6^m(n-m)!\sum_{p=0}^{n-m}\frac{(bk)^p}{p!}, \tag{19.8}$$

valid for suitably chosen b,D_6.

The latter remarkable inequality can be proved by induction on the number of vertices, and its (simple) proof is in Appendix E.

Coming back to the proof of (19.5), one shall again proceed by induction. Consider a shape σ enclosed in a frame f_0 and fix it.

Therefore, the shape σ will have *no R* superscript on the first nontrivial vertex. Let σ_0 be the shape obtained by trimming σ of the outer frames and their contents, let $n\ge m$ be the degrees of σ and σ_0: of course no confusion should arise with the quantities with the same names used in the first part of this section. It is convenient to avoid proliferation of the symbols, but the reader should bear in mind that what follows is the proof of (19.5), quite independent on the first part of the section.

If f is any frame in σ and if m_f denotes the degree of the trimmed tree inside the frame f, it is

$$n-1=\sum_f(m_f-1)\ , \tag{19.9}$$

where the sum runs over the frames of σ *and* on the frame f_0 enclosing σ (so that $m_{f_0}=m$), which one imagines to have erased in setting up the computation of the form factor $r^{(\alpha)}(\sigma;h)$ as prescribed in Sec. XVIII [see (18.10) and the discussion preceding it]. Relation (19.9) is basically the same relation used several times [see, for instance, the comments before (16.16) or (12.17)].

As discussed in Sec. XVIII [(18.9) and (18.10)], it follows from the general theory of Secs. VII and VIII that $r^{(\alpha)}(\sigma;k)$ can be estimated in terms of the coefficients $V(\gamma;S)$ corresponding to the Feynman graphs S such that P_S has degree 4, 2, or 0 and $\gamma=(\sigma^{\underline{h}},\underline{\xi})$ is the tree obtained by attributing to σ frequency labels $\underline{h}$ and endframe position labels $\underline{\xi}$ so that the root of γ receives frequency -1 and the first nontrivial vertex of γ receives frequency index $h\le k$. Note that γ is only partially dressed, because by construction the vertex v_0 bears no R superscript, having been obtained by deleting the frame f_0 originally containing it.

Assuming, inductively, that the r coefficients $r^{(\alpha)}$ verify bounds (19.5) when the degree of γ is less than n (which is trivially true for $n=1$), one sees that (19.4) and (18.21) together with the previous counting estimates imply (if $d=4$ and just applying the definitions)

$$|r^{(\alpha)}(\sigma;k)|\le D_7 m! D_4^m\sum_{h_{v_0}=0}^k\sum_{\underline{h}'}\bar{\varepsilon}^m D_5^{n-m}\left[\prod_{v>v_0}\gamma^{-\bar{p}(h_v-h_{v'})}\right]\gamma^{-\rho_{v_0}h_{v_0}}\prod_{j=1}^m\left[(n_j-1)!\sum_{p=0}^{n_j-1}\frac{(bh_j)^p}{p!}\right], \tag{19.10}$$

where $0<\bar{p}\le\rho'_v+\rho_v$ is fixed and $\rho_{v_0}\geqq -4+n^e_{0,v_0}+2n^e_{1,v_0}=-4\delta_{\alpha0}-2\delta_{\alpha2}$ [see (16.19)], because the first vertex v_0 has no superscript R; hence no improvement on ρ_{v_0} is provided by the renormalization ("no renormalization is operating on v_0"); in (19.10) h_j denotes the frequency of the vertex at which the jth endline of σ_0 is attached in $\sigma_0^{\underline{h}}$.

Using the inequality (19.8), one can easily estimate the sums over $\underline{h}'=\{h_v\}_{v>v_0}$.

Suppose that at v_0 bifurcate $\tilde{m}$ branches, each of degree $\tilde{n}_1,\ldots,\tilde{n}_{\tilde{m}}$ so that $\sum\tilde{n}_j=n$; then by (19.8)'s being applied to each branch

$$|r^{(\alpha)}(\sigma;k)|\le D_7 m! D_4^m\times\sum_{h=0}^k\gamma^{4h\delta_{\alpha0}+2h\delta_{\alpha2}}\bar{\varepsilon}^m D_5^{(n-m)}\times\prod_{s=1}^{\tilde{m}}(\tilde{n}_s-\tilde{m}_s)!\sum_{p=0}^{\tilde{n}_s-\tilde{m}_s}\frac{(bh)^p}{p!}D_6^{\tilde{m}_s}, \tag{19.11}$$

where $\tilde{m}_s$ is the number of end points of the sth branch, after trimming it of its endframes: $\sum_s\tilde{m}_s=m$.

Then one can use the following bound valid for all non-negative integers $a_1,\ldots,a_q$:

$$\prod_{s=1}^q\left[a_s!\sum_{j=0}^{a_s}\frac{(bh)^j}{j!}\right]\equiv\sum_{r=0}^{\sum a_s}\frac{(bh)^r}{r!}\left[\sum_{\substack{j_1=0\\ j_1+\cdots+j_q=r}}^{a_1}\cdots\sum_{j_q=0}^{a_q}\frac{r!}{j_1!\cdots j_q!}a_1!\cdots a_q!\right]\le\left[\left[\sum_s a_s\right]!\right]\sum_{r=0}^{\sum a_s}\frac{(bh)^r}{r!}, \tag{19.12}$$

following from the fact that the large parentheses in the intermediate step is bounded by the square bracket on the rhs; a proof of this elementary combinatorial inequality can be found by induction.

Bound (19.12) can be used in (19.11) to infer

$$| r^{(\alpha)}(\sigma;k) | \le D_7 m! D_4^m \overline{\varepsilon}^m D_5^{n-m} D_6^m \gamma^{4k\delta_{\alpha 0}+2k\delta_{\alpha 2}}(n-m)! \sum_{h=0}^{k} \sum_{r=0}^{n-m} \frac{(bh)^r}{r!} , \tag{19.13}$$

and using

$$\sum_{h=0}^{k} h^r \le k^r + \int_0^k h^r dh = k^r + k^{r+1}/(r+1) , \tag{19.14}$$

implying

$$\sum_{h=0}^{k} \sum_{r=0}^{n-m} \frac{(bh)^r}{r!} \le \frac{b+1}{b} \sum_{r=0}^{n-m+1} \frac{(bk)^r}{r!} , \tag{19.15}$$

one deduces the bound

$$| r^{(\alpha)}(\sigma;k) | \le D_7 \frac{b+1}{b} D_5^n (\overline{\varepsilon} D_6 D_4 D_5^{-1})^m \gamma^{4k\delta_{\alpha 0}+2k\delta_{\alpha 2}} \times (n-1)! m \sum_{r=0}^{n-m+1} \frac{(bh)^r}{r!}, \quad m>1, \tag{19.16}$$

where $D_7 > 0$ is a suitable constant.

Thus, if D_5 is chosen so large that

$$D_5 D_7 \frac{b+1}{b} (D_6 D_4 D_5^{-1})^m m < 1, \quad \forall m > 1 , \tag{19.17}$$

then (19.5) follows by induction from (19.16): in fact, bound (19.5), as already remarked, holds for $m=1$ (trivial shape σ), and the above chain of inequalities proves that the bound holds for trees of degree n, if it holds for trees of lower degree.

The constant b is not arbitrary, because it must be such that (19.8) holds.

The constant D_5 can be taken $\overline{\varepsilon}$ independent.

By repeating the same argument and taking into account that $n-m+1$ can be considerably smaller than $n-1$, one could improve (19.6) by the following inequality:

$$| r^{(\alpha)}(\sigma;k) | \le \overline{\varepsilon} (\overline{\varepsilon} \overline{D})^{n-1} (n-1)! \times \sum_{j=0}^{f} \frac{(bk)^j}{j!} \gamma^{(4\delta_{\alpha 0}+2\delta_{\alpha 2})k} , \tag{19.18}$$

where $f-1$ is the number of frames in σ: this bound shows that the number of frames in σ measures the rate of growth of $r^{(\alpha)}(\sigma;k)$ with k, or at least bounds it.

XX. AN APPLICATION: PLANAR GRAPHS AND CONVERGENCE PROBLEMS—A HEURISTIC APPROACH

Consider the power series for the effective potentials and, given a dressed tree γ, consider the contributions $\int V(\gamma;S) P_S d\underline{\xi}$, associated with γ, to the effective potential coming from a decorated Feynman graph S, as explained in the previous sections.

Most of the graphs S will have a complicated topological structure and it will be impossible to draw them on a plane (without causing line intersections which are not, actually, graph vertices or without enclosing one of the external lines inside a region surrounded by internal lines).

For instance, the graphs below are nonplanar (if the bumpy crossings are not graph vertices):

(52)

The planar φ^4 theory is the set of power series for the effective potentials (as well as for the Schwinger functions) obtained by restricting the summation

$$\int \sum_{\gamma} \sum_{G} \frac{V(\gamma;G)}{n(\gamma)} P_G d\underline{\xi} \tag{20.1}$$

to the planar graphs G only; of course, such a restriction also applies in the graphs arising in the evaluation of the counterterms and of the "form factors" $r^{(\alpha)}(\sigma;h)$ (otherwise, one would lose the ultraviolet stability).

For what concerns the physical as well as the mathematical meaning of such a planar theory perhaps the best interpretation is that of "leading order" in an N^{-1} expansion in a vector $(\varphi^2)^2$ theory, where $\underline{\varphi}$ is a $N\times N$ matrix with $(\underline{\varphi}^2)^2 = \mathrm{Tr}(\underline{\varphi}^* \underline{\varphi})^2$; this interpretation will not, however, be discussed here [see 't Hooft (1982, 1983,1984); Rivasseau (1984)].

Therefore, in this paper the planar field theory for φ^4 will be considered only as a set of formal power series and as a protoype of a situation in which the resummation ideas of Sec. IX can be applied.

The main property of the planar graphs is that the unlabeled planar graphs, "topological planar graphs," are not too many and their number can be bounded by N_0^n, where N_0 is some constant and $n = m_4 + m_2 + m_{2'}$ is the number of vertices. One can take $N_0 = 3^6$ [see Koplik *et al.* (1977)].

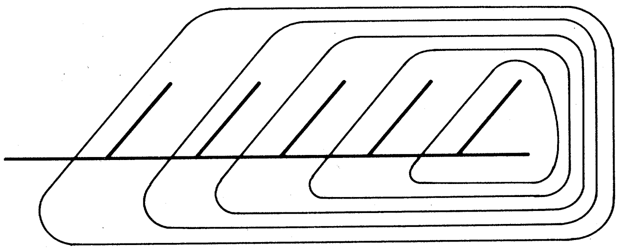
Without our entering once more into the details, it should be quite clear, or at least reasonable, that the whole theory of the preceding sections for the shape form factors $r^{(\alpha)}(\sigma;k)$ remains essentially unchanged, except that factors like $n! n(\gamma)$ estimating the number of graphs relevant for a tree γ with n endframes are now replaced by factors like $N_0^n n(\gamma)$.

The basic bound (19.5) becomes, as is proved in the same way,

$$| r_{\mathrm{planar}}^{(\alpha)}(\sigma;k) | \le \overline{\varepsilon} (\overline{\varepsilon} \widetilde{D})^{n-1} f! \sum_{j=0}^{f} \frac{(bk)^j}{j!} , \tag{20.2}$$

where $(f-1)$ is the number of frames inside σ: in other words, instead of $(n-1)!$ one finds $f!$ (note that $f \le n-1$); compare this with the improved bound (19.18) to understand a little more how this is possible.

The improvement over (19.5) and (19.18) is clearly very strong when $f \ll n$. However, f can be as large as $(n-1)$, and therefore the sums (20.1) still present convergence problems of a major nature being a power series in the renormalized couplings $\underline{\lambda}=(-\lambda,-\mu,-\alpha,-\nu)$ with factorially growing coefficients coming from the trees γ with f of the order of the number of vertices of γ—e.g.,



(53)

To understand better the problem of convergence one can consider the resummation procedures outlined in Sec. IX.

Precisely consider the pruning operation τ (see Sec. IX) cutting out of a tree all the frames.

The resulting resummation equation (9.9) for the fully summed coefficient $r^{(\alpha)}(k)$ [called in Sec. IX $\lambda^{(\alpha)}(k)$] becomes

$$r^{(\alpha)}(k)=\lambda^{(\alpha)}+\sum_{r=2}^{\infty}\sum_{h=0}^{k}\sum_{\substack{h_1,\ldots,h_r\ge h\\ \alpha_1,\ldots,\alpha_r}}\overline{B}^{(\alpha)}_{\alpha_1,\ldots,\alpha_r}(h;h_1,\ldots,h_r)\times\prod_{i=1}^{r}[r^{(\alpha_i)}(h_i)]\ . \tag{20.3}$$

The simplest rigorously correct, interpretation of (20.3) is that it can be used to generate recursively a power-series expansion for the functions $r^{(\alpha)}(k)$ in the renormalized coupling constants.

It is convenient to recall that in the previous sections this expansion was studied in some detail and led to the representation

$$r^{(\alpha)}(k)=\lambda^{(\alpha)}+\sum_{\sigma}r^{(\alpha)}(\sigma;k)\ , \tag{20.4}$$

where σ are all the possible shapes of trees (see diagram 24).

Clearly, (20.4) is a power series in $\underline{\lambda}$ and $r^{(\alpha)}(\sigma;k)$ is part of the polynomial of degree equal to the degree of σ in the expansion of $r^{(\alpha)}(k)$.

From the general theory of Sec. IX it follows that (20.4) must verify, if thought of as a formal power series, relation (20.3) and therefore (20.4) can be generated just by solving recursively (20.3) as an equation for $\underline{r}(k)$ with $\underline{\lambda}$ as input.

Of course it is not surprising that once the coefficients $\overline{B}$ in (20.3) are known one can reduce the problem of computing

$$\sum_{\text{degree }\sigma=m}r^{(\alpha)}(\sigma;k)\equiv r^{(\alpha)}_m(k)$$

to a simple "algebraic" problem, i.e., that of iterating m times (20.3), retaining only the mth-order monomials in $\underline{\lambda}$. From the definitions it is clear that the computation of the coefficients $\overline{B}$ is a necessary prerequisite for the computation of $r^{(\alpha)}(\sigma;h)$, since computing the $\overline{B}$ factors amounts precisely to computing the dressed trees with no frames. In fact, recall that the computation of $r^{(\alpha)}(\sigma;h)$ for general σ is reduced inductively to the no-frame case; on this fact are based the $n!$ estimates of Sec. XIX. But it is quite evident that (20.3) provides a very economic and systematic way of organizing the calculations of the factors $r^{(\alpha)}(\sigma;k)$.

Equation (20.3) is similar to the Callan-Symanzik equations (Callan, 1975; Symanzik, 1973).

From the work of Secs. XVI—XIX the coefficients $\overline{B}$ can be easily computed for small r and estimated for large r, uniformly in the ultraviolet cutoff N (in fact, they are independent on N, as the reader should eventually realize—but they depend on the regularization chosen, as it will be pointed out later).

Coefficients $\overline{B}$ can be bounded following the same procedures used in Secs. XVIII and XIX; one just has to take into account that only planar graphs will ever be considered. The work is a repetition of what was done there, and it will not be reproduced here. The coefficients B arise from the computation of trimmed trees, i.e., of trees with no frames so that $f=1$, by keeping in all the computations only the planar graphs, so that the factorials $m!n(\gamma)$ are replaced by $N_0^m n(\gamma)$. And no factorials arise produced by the frames, so that no factorials arise in the estimates of B. It is [see also Gallavotti and Nicolò (1984)] for some C_1

$$\sum_{\substack{h_i\ge h\\ h\ \text{fixed}}}|B^{(\alpha)}_{\alpha_1,\ldots,\alpha_r}(h;h_1,\ldots,h_r)|\,\gamma^{\sum_i(2\delta_{\alpha_i,2}+4\delta_{\alpha_i,0})h_i}\gamma^{-(2\delta_{\alpha,2}+4\delta_{\alpha,0})h}\le C_1^{r-1}\ . \tag{20.5}$$

For $r=2$ one can perform some explicit easy calculations starting from (17.7):

$$B^{(2)}_{22}(h;h,h)=-\frac{1}{2}\begin{bmatrix}2\\1\end{bmatrix}^2\int C^{(h)}_{12}d\xi_2=-\gamma^{-2h}[\beta^{(2)}_{22}+O(\gamma^{-2h})]\ ,$$

$$
\begin{aligned}
&B_{42}^{(2)}(h;h,h)=-\frac{2!}{2}\begin{bmatrix}4\\2\end{bmatrix}\int (C_{12}^{(\le h)2}-C_{12}^{(<h)2})d\xi_2=-[\beta_{42}^{(2)}+O(\gamma^{-2h})]\ ,\\
&B_{42'}^{(2)}(h;h,h)=-\frac{1}{2}\begin{bmatrix}4\\2\end{bmatrix}2!\int [(\partial C_{12}^{(\le h)})^2-(\partial_1 C_{12}^{(<h)})^2]d\xi_2=-\gamma^{2h}[\beta_{42'}^{(2)}=O(\gamma^{-2h})]\ ,\\
&B_{44}^{(2)}(h;h,h)=-\frac{1}{2}\begin{bmatrix}4\\3\end{bmatrix}^2 3!\int (C_{12}^{(\le h)3}-C_{12}^{(<h)3})d\xi_2=-\gamma^{2h}[\beta_{44}^{(2)}+O(\gamma^{-2h})]\ ,\\
&B_{22'}^{(2')}(h;h,h)=-\frac{1}{2}\begin{bmatrix}2\\1\end{bmatrix}^2\int \frac{\xi_2-\xi_1}{d}-\partial_2 C_{12}^{(h)}d\xi_2=-\gamma^{-2h}[\beta_{22'}^{(2')}+O(\gamma^{-2h})]\ ,\\
&B_{22}^{(2')}(h;h,h)=\frac{1}{2}\begin{bmatrix}2\\1\end{bmatrix}^2\int \frac{(\xi_2-\xi_1)^2}{2d}C_{12}^{(h)}d\xi_2=\gamma^{-4h}[\beta_{22}^{(2')}+O(\gamma^{-2h})]\ ,\\
&B_{44}^{(2')}(h;h,h)=\frac{1}{2}\begin{bmatrix}4\\3\end{bmatrix}^2 3!\int \frac{(\xi_2-\xi_1)^2}{2d}(C_{12}^{(\le h)3}-C_{12}^{(<h)3})d\xi_2=[\beta_{44}^{(2')}+O(\gamma^{-2h})]\ ,\\
&B_{24}^{(4)}(h;h,h)=-\frac{1}{2}\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix}\int C_{12}^{(h)}d\xi_2=-\gamma^{-24}[\beta_{24}^{(4)}+O(\gamma^{-2h})]\ ,\\
&B_{44}^{(4)}(h;h,h)=-\frac{1}{2}\begin{bmatrix}4\\2\end{bmatrix}^2 2!\int (C_{12}^{(\le h)2}-C_{12}^{(<h)2})d\xi_2=-[\beta_{44}^{(4)}+O(\gamma^{-2h})]\ ,\\
&B_{22}^{(0)}(h;h,h)=-\frac{1}{2}2!\int (C_{12}^{(\le h)2}-C_{12}^{(<h)2})d\xi_2=-[\beta_{2}^{(0)}+O(\gamma^{-2h})]\ ,\\
&B_{22'}^{(0)}(h;h,h)=-\frac{1}{2}2!\int [(\partial_1 C_{12}^{(\le h)})^2-(\partial_1 C_{12}^{(<h)})^2]d\xi_2=-\gamma^{2h}[\beta_{22'}^{(0)}+O(\gamma^{-2h})]\ ,\\
&B_{2'2'}^{(0)}(h;h,h)=-\frac{1}{2}2!\int [(\partial_{12} C_{12}^{(\le h)})^2-(\partial_{12} C_{12}^{(<h)})^2]d\xi_2=-\gamma^{4h}[\beta_{2'2'}^{(0)}+O(\gamma^{-2h})]\ ,\\
&B_{44}^{(0)}(h;h,h)=-\frac{1}{2}\begin{bmatrix}4\\4\end{bmatrix}^2 4!\int (C_{12}^{(\le h)4}-C_{12}^{(<h)4})d\xi_2=-\gamma^{4h}[\beta_{44}^{(0)}+O(\gamma^{-2h})]\ ,
\end{aligned}
\tag{20.6}
$$

where $O(\gamma^{-2h})$ denotes something bounded by $h^p\gamma^{-2h}$ for some p; all the other B's with $r=2$ vanish or reduce to the above by $B_{\alpha_1\alpha_2}^{(\alpha)}=B_{\alpha_2\alpha_1}^{(\alpha)}$.

It is convenient to introduce new form factors, more naturally depending on k; they are the "adimensional form factors" defined by

$$r^{(\alpha)}(k)\equiv\lambda^{(\alpha)}(k)\gamma^{(2\delta_{\alpha,2}+4\delta_{\alpha,0})k}\ , \tag{20.7}$$

and one can rewrite (20.3) in terms of new functions $\beta_{\alpha_1,\ldots,\alpha_r}^{(\alpha)}(h;h_1,\ldots,h_r)$

$$\lambda^{(\alpha)}(k)=\lambda^{(\alpha)}\gamma^{-k(2\delta_{\alpha,2}+4\delta_{\alpha,0})}+\sum_{r=2}^{\infty}\sum_{h=0}^{k}\sum_{\substack{h_1,\ldots,h_r\ge h\\ \alpha_1,\ldots,\alpha_r}}\beta_{\alpha_1,\ldots,\alpha_r}^{(\alpha)}(h;h_1,\ldots,h_r)\gamma^{(h-k)(2\delta_{\alpha,2}+4\delta_{\alpha,4})}\prod_{i=1}^{r}\lambda^{(\alpha_i)}(h_i)\ , \tag{20.8}$$

and it can be checked that

$$\lim_{h\to\infty}\beta_{\alpha_1,\ldots,\alpha_r}^{(\alpha)}(h;h_1,\ldots,h_r)=\widetilde{\beta}_{\alpha_1,\ldots,\alpha_r}^{(\alpha)}(h_1-h,\ldots,h_r-h) \tag{20.9}$$

exist if h_i-h are kept constant and the basic bounds of Sec. XVIII imply via (20.5)

$$\sum_{\substack{h_i\ge h\\ h\ \text{fixed}}}|\beta_{\alpha_1,\ldots,\alpha_r}^{(\alpha)}(h;h_1,\ldots,h_p)|\le C_1^{r-1} \tag{20.10}$$

Therefore, if we call

$$
\begin{aligned}
&\lambda(k)=-\lambda^{(4)}(k),\quad \mu(k)=-\lambda^{(2)}(k)\ ,\\
&\alpha(k)=-\lambda^{(2')}(k),\quad \nu(k)=-\lambda^{(0)}(k)\ ,
\end{aligned}
$$

(20.8) can be written explicitly:

$$\begin{aligned}
\lambda(k)&=\lambda+\sum_{h=0}^{k}[\beta_{44}^{(4)}(h)\lambda(h)^2+2\beta_{24}^{(4)}\lambda(h)\mu(h)]+\cdots ,\\
\alpha(k)&=\alpha-\sum_{h=0}^{k}[\beta_{44}^{(2')}(h)\lambda(h)^2-2\beta_{22'}^{(2')}\mu(h)\alpha(h)+\beta_{22}^{(2')}\mu(h)^2]+\cdots ,\\
\mu(k)&=\mu\gamma^{-2k}+\sum_{h=0}^{k}\gamma^{2(h-k)}[\beta_{22}^{(2)}(h)\mu(h)^2+2\beta_{42}^{(2)}(h)\lambda(h)\mu(h)+2\beta_{42'}^{(2)}(h)\lambda(h)\alpha(h)+\beta_{44}^{(2)}(h)\lambda(h)^2]+\cdots ,\\
\nu(k)&=\nu\gamma^{-4k}+\sum_{h=0}^{k}\gamma^{4(h-k)}[\beta_{22}^{(0)}(h)\mu(h)^2+2\beta_{22'}^{(0)}(h)\mu(h)\alpha(h)+\beta_{2'2'}^{(0)}(h)\alpha(h)^2+\beta_{44}^{(0)}(h)\lambda(h)^2]+\cdots ,
\end{aligned}\tag{20.11}$$

and the functions $\beta^{(\alpha)}_{\alpha_1\alpha_2}(h)$ will have a well-defined positive limit as $h\to\infty$, as follows from (20.6); the dots denote the "higher-order terms," $r\geq 2$.

The limits $\beta^{(\alpha)}_{\alpha_1\alpha_2}$ of $\beta^{(\alpha)}_{\alpha_1\alpha_2}(h)$ are reached exponentially fast $[O(\gamma^{-2h})]$ and are not all independent, e.g., $\beta_{44}^{(4)}=8\beta_{22}^{(0)}=3\beta_{42}^{(2)},\ldots$.

Obviously, because of the truncated expectation meaning of the vertices of the trees, it follows that no $\nu(k)$ appears in the first three of (20.11); this means that the fourth equation in (20.11) decouples from the first three and determines $\nu(k)$ completely as soon as $\lambda^{(\alpha)}(h)$ are known for $\alpha=4,2,2'$ [because, also, no $\nu(k)$ appears in the rhs of the fourth term in (20.11)]. For this reason the fourth equation in (20.11) is not too important in setting up the theory of renormalization.

The power series in (20.11) [in the variables $\underline{\lambda}(k)$] can be used, as already mentioned, to generate expressions of $\underline{\lambda}(k)$ as a power series in $\underline{\lambda}$.

As proved in Secs. XVIII and XIX, this power series has coefficients which are uniformly bounded in the ultraviolet cutoff and this also follows directly (but not independently of the theory of Secs. XVIII and XIX) from the bounds (20.10).

However, it is clear that the coefficients one gets must coincide with the ones estimated in Sec. XIX, and which grow with the order n as $O(n!)$, even in the planar case being considered here [because of the contributions that these coefficients receive from the trees with many frames—see (20.2)]. One can convince himself that such estimates are not pessimistic unless some cancellations take place.

In fact, the bounds are reasonable and "optimal" on each individual graph, as one can easily identify graphs (planar) and trees giving contributions to the nth-order coefficients of $\underline{\lambda}(k)$ which are of the order of $n!$; this was pointed out by Lautrup (1977).

However, cancellations between several big terms can take place and in various possible senses.

A way of exhibiting such cancellations is to find a sequence $(\underline{\lambda}(k))_{k=0}^{\infty}\equiv\underline{\underline{\lambda}}$ verifying (20.11). This sequence could then be taken as a definition of the sum of the power series in the $\underline{\lambda}$'s which define perturbatively $\underline{\lambda}(k)$ as a (probably divergent) power series in $\underline{\lambda}$.

To make sense of the rhs of (20.11) it seems natural to impose on the sequence $\underline{\underline{\lambda}}$ a decay condition at $k=\infty$, in apparent contradiction with the bounds (20.2) which are strongly growing with k.

So one introduces

$$|\underline{\underline{\lambda}}|_q=\sup_{k\geq 0}(1+k)^q|\underline{\lambda}(k)| .\tag{20.12}$$

Bounds (20.10) allow one to define an operator $\mathscr{B}$ on the $\underline{\lambda}$'s with $|\underline{\underline{\lambda}}|_q<\infty$ for some $q\geq 0$; in fact, bounds (20.10) imply (recall that they hold in the planar case only) that the operator $\mathscr{B}$,

$$(\mathscr{B}\underline{\underline{\lambda}})^{(\alpha)}(h)\equiv\sum_{r=2}^{\infty}\sum_{\substack{h_1,\ldots,h_r\geq h\\ \alpha_1,\ldots,\alpha_r}}\beta^{(\alpha)}_{\alpha_1,\ldots,\alpha_r}(h;h_1,\ldots,h_r)\times\prod_{i=1}^{r}\lambda^{(\alpha_i)}(h_i)\tag{20.13}$$

has the property

$$|\mathscr{B}\underline{\underline{\lambda}}|_q\leq\frac{1}{C_1}(C_1|\underline{\underline{\lambda}}|_q)^2,\quad q=0,1,\ldots ,\tag{20.14}$$

and therefore $\mathscr{B}$ is well defined on the space (20.12) if $|\underline{\underline{\lambda}}|_q$ is small and C_1 is introduced in (20.10).

More generally $\mathscr{B}$ is well defined if for some $\eta>0$, $B>0$, and a suitably chosen $K_{\eta,B},\delta$

$$|\underline{\underline{\lambda}}|_\eta<B,\quad\sup_{k\leq K_{\eta,B}}|\underline{\lambda}(k)|<\delta ,\tag{20.15}$$

as follows from (20.10).

Equation (20.11) becomes

$$\lambda^{(\alpha)}(k)=\lambda^{(\alpha)}\gamma^{-K(2\delta_{\alpha,2}+4\delta_{\alpha,0})}+\sum_{h=0}^{k}\gamma^{(h-k)(2\delta_{\alpha 2}+4\delta_{\alpha 0})}(\mathscr{B}\underline{\underline{\lambda}})^{(\alpha)}(h) ,\tag{20.16}$$

i.e., if $\lambda^{(\alpha)}(-1)\equiv\lambda^{(\alpha)}\gamma^{(2\delta_{\alpha,2}+4\delta_{\alpha,0})}$:

$$\lambda^{(\alpha)}(k+1)=\gamma^{-(2\delta_{\alpha,2}+4\delta_{\alpha,0})}\lambda^{(\alpha)}(k)+(\mathscr{B}\underline{\underline{\lambda}})^{(\alpha)}(k+1),\quad k\geq -1 ,\tag{20.17}$$

and one looks for a solution $\underline{\underline{\lambda}}$ such that, say,

$$|\underline{\underline{\lambda}}|_1<1\ \text{ and }\ \sup_{k\leq K_{1,1}}|\underline{\lambda}(k)|<\delta .$$

In studying (20.17) one is thus interested in solutions $\underline{\lambda}(k)\to_{k\to\infty}0$; therefore it is natural to replace $\mathscr{B}$ by its second-order part $\mathscr{B}_2$ for the purpose of getting first an approximate solution:

$$(\mathscr{B}_2\underline{\underline{\lambda}})^{(\alpha)}(h)=\sum_{\alpha_1,\alpha_2}\beta^{(\alpha)}_{\alpha_1\alpha_2}(h)\lambda^{(\alpha_1)}(h)\lambda^{(\alpha_2)}(h) \qquad (20.18)$$

[see (20.11)].

In turn, since $\beta^{(\alpha)}_{\alpha_1\alpha_2}(k)\rightarrow_{k\rightarrow\infty}\beta^{(\alpha)}_{\alpha_1\alpha_2}$, it is convenient to study first the relation

$$\lambda^{(\alpha)}(k+1)=\lambda^{(\alpha)}(k)\gamma^{-(2\delta_{\alpha 2}+4\delta_{\alpha 0})}+(\overline{\mathscr{B}}_2\underline{\underline{\lambda}})^{(\alpha)}(k),\quad k\geq 0\ , \qquad (20.19)$$

with $\overline{\mathscr{B}}_2$ defined as (20.18) with $\beta^{(\alpha)}_{\alpha_1\alpha_2}(h)$ replaced by their limits $\beta^{(\alpha)}_{\alpha_1\alpha_2}$ as $h\rightarrow\infty$. Explicitly the last equation is

$$\begin{aligned}
\lambda(k+1)&=\lambda(k)+\beta^{(4)}_{44}\lambda(k)^2+2\beta^{(4)}_{24}\lambda(k)\mu(k)\ ,\\
\alpha(k+1)&=\alpha(k)-\beta^{(2')}_{22}\mu(k)^2-\beta^{(2')}_{44}\lambda(k)^2+2\beta^{(2')}_{22'}\mu(k)\alpha(k)\ ,\\
\mu(k+1)&=\gamma^{-2}\mu(k)+\beta^{(2)}_{22}\mu(k)^2+2\beta^{(2)}_{42}\lambda(h)\mu(k)+2\beta^{(2)}_{42'}\lambda(h)\alpha(k)+\beta^{(2)}_{44}\lambda(k)^2\ ,\\
\nu(k+1)&=\gamma^{-4}\nu(k)+\beta^{(0)}_{22}\mu(k)^2+2\beta^{(0)}_{22'}\mu(k)\alpha(k)+\beta^{(0)}_{2'2'}\alpha(k)^2+\beta^{(0)}_{44}\lambda(k)^2\ .
\end{aligned} \qquad (20.20)$$

This relation can be regarded as an iteration of a map T on R^4 [or R^3 if one disregards the last (decoupled) equation].

One can therefore apply the techniques developed in the general theory of maps to analyze (20.20).

One looks for data $\overline{\lambda},\overline{\mu},\overline{\alpha},\overline{\nu}$ for $\underline{\lambda}(0)$ such that $\underline{\lambda}(k)\rightarrow_{k\rightarrow\infty}0$. Their existence can be proved by using the general theory of the central manifold [see Lanford (1973) or Gallavotti (1983a), Chap. 5, Secs. 6 and 8, and related problems]; there exists a surface Σ, in general nonunique:

$$\begin{aligned}
\mu&=\mu(\alpha,\lambda)=A\alpha^2+L\lambda^2+I\alpha\lambda+\cdots\ ,\\
\nu&=\nu(\alpha,\lambda)=A'\alpha^2+L'\lambda^2+I'\alpha\lambda+\cdots\ ,
\end{aligned} \qquad (20.21)$$

where the dots represent terms of higher order, which is invariant under the map T defined by (20.20) and such that the T images of any point $\overline{\lambda}$ close enough to the origin evolves under repeated iterations of T by approaching exponentially fast the surface Σ as long as they stay close enough to the origin.

A simple exercise ["substitute (20.21) in (20.20) to find $A,A',\ldots$"] yields

$$\begin{aligned}
&I=2\beta^{(2)}_{42'}(1-\gamma^{-2})^{-1},\quad A=-\beta^{(2)}_{2'2'}(1-\gamma^{-2})^{-1}\ ,\\
&L=-\beta^{(2)}_{44}(1-\gamma^{-2})^{-1}\ ,\\
&I'=0,\quad A'=-\beta^{(0)}_{2'2'}(1-\gamma^{-4})^{-1}\ ,\\
&L'=-\beta^{(0)}_{44}(1-\gamma^{-4})^{-1}\ ,
\end{aligned} \qquad (20.22)$$

and the map (20.20) becomes on Σ

$$\begin{aligned}
\lambda(k+1)&=\lambda(k)+\beta^{(4)}_{44}\lambda(k)^2+\cdots\ ,\\
\alpha(k+1)&=\alpha(k)-\beta^{(2')}_{44}\lambda(k)^2+\cdots\ ,
\end{aligned} \qquad (20.23)$$

where the dots represent terms of higher order.

Neglecting the higher-order corrections once more, and setting $\beta=\beta^{(4)}_{44}>0$, $\beta'=\beta^{(2')}_{44}>0$, one considers the relations

$$\begin{aligned}
\lambda(k+1)&=\lambda(k)+\beta\lambda(k)\lambda(k+1)\ ,\\
\alpha(k+1)&=\alpha(k)-\beta'\lambda(k)\lambda(k+1)\ ,
\end{aligned} \qquad (20.24)$$

which admit solutions with data $\overline{\lambda},\overline{\alpha}=-\beta'\beta^{-1}\overline{\lambda}$, $\overline{\lambda}<0$:

$$\lambda(k)=\frac{\overline{\lambda}}{1-\beta k\overline{\lambda}},\quad \alpha(k)=-\beta'\beta^{-1}\lambda(k)\ . \qquad (20.25)$$

From general consideration of stability theory it follows that Eqs. (20.23) also admit a solution behaving as $k\rightarrow\infty$ as (20.25) with initial data $\overline{\lambda}<0$, $\overline{\alpha}=-\beta'\beta^{-1}\overline{\lambda}+O(\overline{\lambda}^2)$ and such that $\underline{\lambda}(k)\rightarrow_{\overline{\lambda}\rightarrow 0}0$ at fixed k.

This means, via (20.21), that (20.20) admits a solution with data $\overline{\lambda},\overline{\alpha}=-\beta'\beta^{-1}\overline{\lambda}+O(\overline{\lambda}^2)$, $\overline{\mu}=O(\overline{\lambda}^2)$, $\overline{\nu}=O(\overline{\lambda}^2)$ which is such that $\lambda(k),\alpha(k)=O(k^{-1})$ and $\mu(k),\nu(k)=O(k^{-2})$ as $k\rightarrow\infty$ and such that $\underline{\lambda}(k)\rightarrow 0$ at fixed k when $\overline{\lambda}\rightarrow 0$.

Hence one finds a solution to (20.20) depending on one parameter $\overline{\lambda}$ such that $|\underline{\lambda}|_1\leq O(\beta^{-1})$ for $\overline{\lambda}$ small and such that $\underline{\lambda}(k)$ is as small as one wishes for any fixed number of k's, say $k\leq K_1$.

Hence such $\underline{\underline{\lambda}}$ is in the domain if the "beta function" $\mathscr{B}$ defined in (20.13) and by some more efforts of abstract perturbation theory it could be proved that there is a solution to (20.17) depending on one parameter $\overline{\lambda}$, with $\lambda(k),\mu(k),\alpha(k),\nu(k)$ given approximately by (20.25) and (20.21).

Such a solution will not be such that $|\underline{\underline{\lambda}}|_1$ is small [rather, the above discussion suggests $|\underline{\underline{\lambda}}|_1=O(\beta^{-1})$], although $\underline{\lambda}(k)$ at fixed k will be small for small $\underline{\lambda}(0)$ or for small values of the parameter $\overline{\lambda}$ on which the solution depends. This "nonuniform smallness" is related to the fact that $\underline{\underline{\lambda}}$ cannot be found perturbatively, although it has, by construction, the correct asymptotic expansion in $\underline{\lambda}$.

Note also that (20.25) shows that (at least the approximating) $\underline{\underline{\lambda}}$ has singularities at points accumulating at $\overline{\lambda}=0$, as a function of $\overline{\lambda}$.

The renormalized couplings are defined by the (convergent) series:

$$\lambda^{(\alpha)}=-\sum_{k=0}^{\infty}(\mathscr{B}\underline{\underline{\lambda}})^{(\alpha)}(k) \qquad (20.26)$$

if $\overline{\lambda}$ is small, obtained by setting $\lambda^{(\alpha)}(+\infty)=0$ in (20.16). Alternatively one can use

$$\lambda^{(\alpha)}(0)=\lambda^{(\alpha)}+(\mathscr{B}\underline{\underline{\lambda}})^{(\alpha)}(0)$$
$$\Longrightarrow \lambda^{(\alpha)}=\lambda^{(\alpha)}(0)-(\mathscr{B}\underline{\underline{\lambda}})^{(\alpha)}(0)\ , \tag{20.27}$$

which determines $\lambda^{(0)},\lambda^{(2)}$, as well.

The family of solutions to (20.17) constructed above is a one-parameter family; however, one could alter the coefficients in front of the few covariances or their mass terms so that one has built a many-parameter family of field theories "like φ_4^4 planar"; however, it does not seem possible to choose $\overline{\alpha}=0$ nor, by (20.27), $\overline{\mu}=0$, because $A\neq 0$ in (20.22) if one wishes that $\alpha(k),\mu(k)\to 0$ as $k\to\infty$.

The meaning of the statement that one has built planar φ_4^4 theory is explained below and can be summarized in the statement "the resummed tree expansions for the effective planar potentials converge for small negative coupling."

The solution to Eq. (20.17) discussed above is roughly like (20.25) and (20.23), i.e., is such that

$$|\underline{\underline{\lambda}}|_0=\sup_k |\underline{\lambda}(k)| =\widetilde{\varepsilon} \underset{\overline{\lambda}\to 0}{\longrightarrow} 0\ . \tag{20.28}$$

Therefore, the effective potential of the planar theory corresponding to the above definition of $\underline{\lambda}(k)$ will be described by dressed trees with no frames but with "heavy end points" contributing to the effective potential the form factor $r^{(\alpha)}(h)=\lambda^{(\alpha)}(h)\gamma^{2h\delta_{\alpha 2}+2h\delta_{\alpha 0}}$ when they are attached to vertices of the tree bearing a frequency label h.

Furthermore, since one is considering only the planar theory, one evaluates the contributions of a tree γ to the effective potential by using the "few"

$$N_0^n M\equiv N_0^n n(\gamma)C_\varepsilon^n \exp\left[\varepsilon\sum_v n_v^e\right]$$

planar or graphs compatible with γ (see Sec. XIX).

This means that, by the theory of Secs. XVIII and XIX, bound (19.2) is replaced, if D_0 is a suitable constant, by

$$\mathscr{N} B^{n^e}\widetilde{\varepsilon}^{\,n} D_0^n e^{-\kappa\gamma^k d(\Delta_1,\ldots,\Delta_p)} N_0^n\ , \tag{20.29}$$

with no $n!$, because $n!$ arose for two reasons: one was that the number of Feynman graphs associated with a tree γ were bounded by $Mn!$, where n is the number of end points in the trimmed tree γ, and the other was the $n!$ in the form factors $r^{(\alpha)}(\sigma;h)$ [see (19.5)] due to the endframes of σ.

However, in the planar theory the graphs are far fewer, and the form factors, still badly dependent on the degree of the shapes (as pointed out at the beginning of this section), are "resummed" to yield new form factors:

$$r^{(\alpha)}(h)=\lambda^{(\alpha)}(h)\gamma^{(4\delta_{\alpha 0}+2\delta_{\alpha 2})h}\ , \tag{20.30}$$

with $\lambda^{(\alpha)}(h)\to_{n\to\infty} 0$ (this quantity was not only not small in perturbation theory, but even divergent with h as $h\to\infty$). And at the same time the resummation leading to the form factors (20.30) eliminates the necessity of considering contributions from trees with frames to $V^{(k)}$: hence (20.29) is really a simple consequence of the estimates of Secs. XIX [(19.19)] and XVIII [(18.21)].

Since for $\widetilde{\varepsilon}$ small the (20.29) can be summed over n, one gets the effect, in the above-considered planar theory, that the resummed series for the effective potentials is really convergent for small $\overline{\lambda}<0$ [i.e., small negative $\underline{\lambda}(0)$—i.e., small negative renormalized coupling]. Therefore, in the planar theory the effective potentials can be defined beyond perturbation theory.

The series defining the effective potential is a power series in the resummed form factors (20.30), the form factors being nonanalytic near $\overline{\lambda}=0$ [in the sense roughly expressed by approximation (20.25)], it is clear that one cannot expect that the effective potentials be analytic in the renormalized coupling constant near zero.

The resummation procedure induced by the beta function allows one to express the effective potentials analytically in the new effective coupling constants or "form factors," (20.30), and provides a well-defined resummation prescription. It seems highly plausible that among the above solutions there is one which is the Borel sum of its perturbative nonresummed series; this was proved in the case $\alpha=\mu=\nu=0$ [not covered here because I have chosen for simplicity, the initial $\overline{\alpha}$ and $\overline{\mu}$ so that $\underline{\lambda}(k)\to 0$ as $k\to\infty$] [see 't Hooft (1983a,1983b) and Rivasseau (1984)].

Another interesting possibility is that the series may converge even for some $\overline{\lambda}>0$: the formula (20.25) allows the possibility that for $\overline{\lambda}>0$ the effective potentials are defined for "most" values of $\overline{\lambda}$. The resemblance with the situation arising in classical mechanics in the Hamiltonian stability problems in connection with the appearance of small denominators seems interesting: maybe here one needs some imagination.

XXI. CONSTRUCTING φ^4 FIELDS IN DIMENSION 2 OR 3

The theory of renormalization in dimension $d=2,3$ can be done in a much simpler way, compared to the $d=4$ case.

Of course there is no problem in repeating word by word the four-dimensional theory in dimension 2 or 3 (and in fact in Secs. XVI—XX one had never really used the case that $d=4$ but only that of $d<5$).

The real simplification arises when one remarks that if $d=2,3$ one can study much simpler theories which lead, or may lead, to nontrivial fields (i.e., the fields with nonquadratic effective potentials $V^{(k)}$) of φ^4 type.

What is more important is that the simpler theories (which would not make sense if $d=4$) can be treated rigorously for "small couplings" and really shown to exist beyond the level of formal perturbation theory.

The theories which make sense if $d=2$ and that are simpler than the ones considered so far are those generated by the interaction $\mathscr{I}_N$:

$$V_1=-\lambda\int_\Lambda :\varphi_x^{(\leq N)4}:d^2x\ , \tag{21.1}$$

while if $d=3$ a theory simpler than the one arising from (16.1) is provided by the interaction $\mathscr{I}_N$:

$$V_1=\int_\Lambda(-\lambda:\varphi_x^{(\leq N)4}:-\mu:\varphi_x^{(\leq n)2}:-\nu)d^3x\ . \qquad (21.2)$$

The main reason (21.1) and (21.2) are much simpler than (16.1) is that no resummations have to be devised to organize the corresponding renormalized perturbative series, because only finitely many trees lead to divergences.

The renormalizability in the above case with $d=2$ follows immediately from the formulas and estimates of Sec. XVI setting $n_{1,v}^e=0, m_{2,v}=m_{2',v}=0$, so that (16.19) becomes, for all $m_{4,v}>1$

$$\rho_v=-2+2m>0 \qquad (21.3)$$

and this not only shows renormalizability of the "pure φ^4 field" but also shows that *no* renormalization is ever necessary (this same remarkable conclusion would hold, when $d=2$, for the most general Wick-ordered polynomial interaction).

If $d=3$ and (21.2) is considered, one can still use the general bounds of Sec. XVI setting $n_{1,v}^e=m_{2',v}=0$ so that

$$\rho_v=-3+2m_{2,v}+m_{4,v}+\tfrac{1}{2}n_{0v}^e>0\ , \qquad (21.4)$$

unless (recall that $m_{2,v}+m_{4,v}\geq 2$ in the nontrivial cases)

$m_{4,v}=1$	$m_{2,v}=1$	$n_v^e=0$	(in fact impossible)
$m_{4,v}=2$	$m_{2,v}=0$	$n_v^e=0,2$	(possible)
$m_{4,v}=3$	$m_{2,v}=0$	$n_v^e=0$	(possible) .

(21.5)

So only trees of degree 2 or 3 need the definition of the $\mathscr{L}^{(\sigma)}$ localization operations, and the only nontrivial case is $m_{4,v}=2$, $n_v^e=2$ ("mass diagrams") yielding $\rho_v=0$, and therefore it can be cured by a simple subtraction:

$$V_{2,N}=\int r_N^{(2)}:\varphi_x^{(\leq N)2}:d^3\underline{x}\ . \qquad (21.6)$$

Formally $\mathscr{L}^{(\sigma)}$ is defined in terms of the action $\overline{\mathscr{L}}$ on Wick monomials, as in Secs. XVII and XVIII:

$$\overline{\mathscr{L}}1=1 \text{ if degree } \gamma\leq 3\ ,$$
$$\overline{\mathscr{L}}:\varphi_x\varphi_y:=:\varphi_x^2: \text{ if degree } \gamma=2\ , \qquad (21.7)$$

which leads to a simple expression for $V_{2,N}, V_{3,N}$, i.e., for the counterterms [note that $V_{3,N}$ is a constant and that the nonconstant part of $V_{2,N}$ must have the form (21.6)]:

$$V_{2,N}=-\frac{\lambda^2}{2}\begin{bmatrix}4\\1\end{bmatrix}^2 3!\int C_{\xi_1\xi_2}^{(\leq N)3}:\varphi_{\xi_1}^{(\leq N)2}:d\xi_1 d\xi_2-\frac{\lambda^2}{2}4!\int C_{\xi_1\xi_2}^{(\leq N)4}d\xi_1 d\xi_2\ ,$$

$$V_{3,N}=-\frac{\lambda^3}{3!}\int \mathscr{E}_{(\leq N)}^T(:\varphi_{\xi_1}^{(\leq N)4}:,:\varphi_{\xi_2}^{(\leq N)4}:,:\varphi_{\xi_3}^{(\leq N)4}:)d\xi_1 d\xi_2 d\xi_3\ . \qquad (21.8)$$

The theory of Secs. XVI—XIX now becomes much simpler and one can prove that the effective potential has the form

$$\int\sum_\gamma\sum_S\frac{V(\gamma;S)}{n(\gamma)}P_S d\underline{\xi}\ , \qquad (21.9)$$

where S represents the decorated Feynman graphs and P_S has the form [if $\varphi\equiv\varphi^{(\leq k)}$, $D\equiv D^{(\leq k)}$]:

$$:\prod_i\varphi_{\eta_i}^{n_i}\prod_j D_{\xi_j\xi_j'}^{m_j}:\ , \qquad (21.10)$$

where, as in Sec. XVII, $D_{\xi\eta}=\varphi_\xi-\varphi_\eta$.

The same techniques of Secs. XVI—XIX (easier now, in practice) yield the bound

$$M(\Delta_1,\ldots,\Delta_p)\leq\mathscr{N}n!\overline{\varepsilon}(\overline{\varepsilon}D)^{n-1}k^2 \times e^{-\kappa\gamma^k d(\Delta_1,\ldots,\Delta_p)}\gamma^{-nk}B^{n_e}\ , \qquad (21.11)$$

with the same notations as in Secs. XVI—XIX, i.e., $n=$(degree of γ), $\overline{\varepsilon}=\max(|\lambda|,|\mu|,|\nu|)$, $k=$(root frequency of the tree),

$$M(\Delta_1,\ldots,\Delta_p)\equiv\int_{\Delta_1\times\cdots\times\Delta_p\times\Lambda\times\cdots\times\Lambda}\sum_{\substack{\gamma\\ \text{degree }\gamma=n}}\sum_{\substack{G\\ P_G=P}}\frac{|V(\gamma;G)|}{n(\gamma)}\sup|P|\,d\underline{\xi}\ , \qquad (21.12)$$

where the supremum of $|P_G|=|P|$ is over the fields $\varphi^{(\leq k)}$ verifying $\varphi^{(\leq k)}=\varphi^{(0)}+\varphi^{(1)}+\cdots+\varphi^{(k)}$ and (3.20) with regularization of order 1, and $B=\sup B_\Delta$. Finally, $\mathscr{N}$ is the "adimensional bound" on P:

$$\sup|P|\leq B^{n^e}\prod_j\gamma^k|(\xi_j-\xi_j')|\gamma^{kn^e/2}\mathscr{N}$$

and $\mathscr{N}$ depends on n^e only because there are only a finite number of Wick monomials P of degree n^e of the type (21.10), apart from the values of the position labels.

The presence of the factor γ^{-nk} in (21.11) proves that the theory is asymptotically free.

In the case $d=2$ one replaces, basically, γ^{-nk} by γ^{-2nk}.

The above bounds were found in special cases and by using techniques of the previous sections, in Benfatto *et al.* (1978), Benfatto, Cassandro *et al.* (1980), and Benfatto, Gallavotti, and Nicolò (1980); for the Schwinger functions expansions analogous to bounds hold and were well known; see, for instance, Glimm and Jaffe (1968,1973,1981).

Since in the approach presented here there is little difference between $:\varphi^4:_2$ and $:\varphi^4:_3$, I shall focus only on the $d=3$ case, (21.2), in this section.

The actual construction of the theory can be easily performed by taking advantage of the asymptotic freedom just pointed out [see the factor γ^{-nk} in (20.11)] and following, basically word for word, the procedure adopted in the cosine interaction case (which is, in fact, equivalently hard). For simplicity of exposition it is convenient to choose the values of the renormalized coupling constants μ and $\nu=0$; of course this does not mean that the three-dimensional space in (21.2) is replaced by the one-dimensional space (21.1) but only that the theory has only one renormalized coupling, namely, λ, but still the counterterms can generate nonzero constant and mass terms (which will be of higher order in λ).

The strength of the asymptotic freedom shows that if the integrals over the "small fields" are computed via the cumulant expansion, i.e., via expressions like (13.22) [see also (5.13)], the expansion must be carried out at least to third order, since only the remainders of order, in λ, larger or equal to four give rise to an error of controllable size; such remainders at order $t+1$ are now estimated by a bound analogous to (13.25)

$$|\Lambda| \sum_{p=0}^{\infty} [\lambda\gamma^{-kp}(1+p)^a \ln(e+p+\lambda^{-1})]^{t+1}\gamma^{3p}$$

convergent for $t\geq 3$ (if $d=2$, one could control errors of order ≥ 2 so that the cumulant expansion could be carried out stopping only at order $t=1$, in practice a good simplification).

The other hand problem is that of the "large deviations" or "large fields."

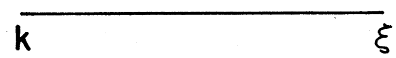
The D factors $(D_{\xi\eta}=\varphi_\xi-\varphi_\eta)$ present in the effective interaction are dangerous much as the $\{1-\cos[\alpha(\varphi_\xi-\varphi_\eta)]\}$ factors were in the cosine field case: they are treated exactly in the same way, because they appear with the right sign (i.e., the corresponding effective potential tends to $-\infty$ when the field φ becomes so rough that $D_{\xi\eta}$ is too large compared to its covariance).

In this case there are also other dangerous terms in the third-order effective potential, namely, *all* the others. In fact, the field φ can be very large and make P itself very large; this was not a problem in the case of the cosine interaction, because there the fields appeared only inside trigonometric functions and therefore in a "bounded form." The large fields have also to be treated by positivity arguments.

The positivity properties needed in the theory can all be extracted from the fact that the effective interaction contains the following two terms:

$$W=-\lambda\int_\Lambda :\varphi_\xi^{(\leq k)4}: d\xi - \frac{\lambda^2}{2}\begin{bmatrix}4\\1\end{bmatrix}^2 3!\int_{\Lambda^2}(C_{\xi\eta}^{(\leq N)3}-C_{\xi\eta}^{(\leq k)3}) :D_{\xi\eta}^{(\leq k)2}: \times d\xi\, d\eta\,, \qquad (21.13)$$

corresponding to contributions from the trees



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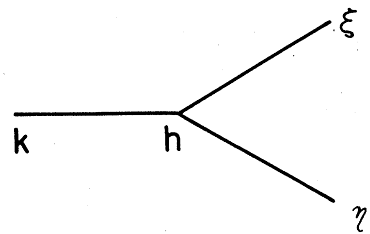
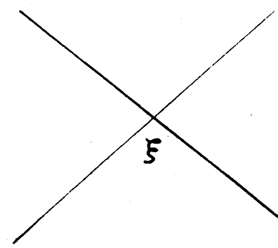


due to the Feynman graphs



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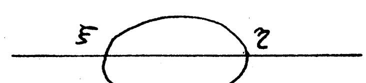


,

the first term being very negative when φ_v is large and the second being very negative when $D_{\xi\eta}=(\varphi_\xi-\varphi_\eta)$ is large compared to $(\gamma^k|\xi-\eta|)^{1/2}$; here one uses $\lambda>0$, $\lambda^2>0$ (which must be a further restriction, although no restriction on the size of λ is necessary).

The details are essentially identical to those explained in the cosine case and they will not be repeated here; and the reader is referred to the literature [see Benfatto *et al.* (1978), Benfatto, Cassandro *et al.* (1980), and Benfatto, Gallavotti, and Nicolò (1980)]. It is, however, important to stress once more that nothing is really different from the case of the cosine field treated in detail in Secs. XIII and XIV, as the reader can check by a glance at the above references.

The result of the analysis is the existence of a constant $E>0$ such that for all $f\in C^\infty$ with support in a set Λ_f it is

$$\int e^{V_1(\varphi^{(\leq N)})+\varphi^{(\leq N)}(f)}P(d\varphi^{(\leq N)})\leq e^{|\Lambda|E+\|f\|_\infty^2|\Lambda_f|}\,, \qquad (21.14)$$

which proves, up to technicalities, the existence of the limit of the "interaction measure" at least on subsequences as $N\to\infty$: i.e., it proves the "nonperturbative ultraviolet stability."

With some extra work and using the same ideas plus abstract arguments one could prove the actual existence of the ultraviolet limit (with no subsequences involved); this is not written explicitly in the literature but, at least for λ small, the result is known also by other methods [see Glimm, Jaffe, and Spencer (1975,1976), Magnen and Seneor (1976), Feldman (1974), Feldman and Osterwalder (1976), Federbush and Battle, (1982)].

As in the case of the cosine interaction the other limit, $\Lambda \to \infty$, the "infrared limit," has to be treated under extra assumptions (like λ small), because, contrary to the ultraviolet limit, *in the cases considered so far,* it may be affected by nonuniqueness phenomena: "(infrared) phase transitions" corresponding to the ordinary phase transitions of statistical mechanics. Such transitions have to be expected here, too, as the main idea of the multiscale approach is that field theory can be reduced to the theory of a spin system on a lattice of scale 1. And such systems are known to exhibit, in general, phase transitions in the infrared limit (also called thermodynamic limit) $\Lambda \to \infty$.

Finally, let me mention that in some cases with $d=2,3$ the theory can be performed completely—i.e., up to the extent of really constructing a field theory verifying the Wightman axioms, hence with the proper interpretation of a physical quantum field theory describing in some of its states, interacting relativistic quantum particles [see Glimm, Jaffe, and Spencer (1975,1976a,1976b), Ma (1976), Koch (1980)]—however, these kinds of questions go beyond the scopes of the present review.

XXII. COMMENTS ON RESUMMATIONS, TRIVIALITY, AND NONTRIVIALITY. SOME APOLOGIES

The reason one cannot perform the resummations, described in the preceding section, in a rigorous way is simply that the coefficients β of the "beta function," (20.13), formally defining the resummed "adimensional form factors" $\lambda^{(\alpha)}(k)$, $\alpha=4,2,2',0$

$$(\mathscr{B}\underline{\underline{\lambda}})^{(\alpha)}(k)=\sum_{r=2}^{\infty}\sum_{\substack{\alpha_1,\ldots,\alpha_r\\ h_1,\ldots,h_r}}\beta^{(\alpha)}_{\alpha_1,\ldots,\alpha_r}(h;h_1,\ldots,h_r)\times\prod_{i=1}^{r}[\gamma^{-\bar{p}(h-h_i)}\lambda^{(\alpha_i)}(h_i)] \tag{22.1}$$

are badly behaved in r as $r\to\infty$: i.e., they are bounded by $r!C^r$ [unless one restricts oneself to the planar theory where (20.10) holds (see Sec. XX)].

This is in conflict with the fact that the idea of using the equation (of "Callan-Symanzik")

$$\lambda^{(\alpha)}(k+1)=\lambda^{(\alpha)}(k)+(\mathscr{B}\underline{\underline{\lambda}})^{(\alpha)}(k+1) \tag{22.2}$$

to define the adimensional form factors in a nonperturbative way requires the existence of a sequence $\underline{\underline{\lambda}}=[\lambda^{(\alpha)}(k)]_{\alpha,k}$ of form factors for which $\mathscr{B}\underline{\underline{\lambda}}$ makes sense and verifies (22.2).

Because of the bad bounds on the β coefficients and because, as emerges from considering only the second-order part of (22.2), a solution to (22.2) cannot tend to zero too fast as $k\to\infty$ [see (20.25)], the only way in which $\mathscr{B}\underline{\underline{\lambda}}$ could make sense for interesting sequences $\underline{\underline{\lambda}}$ is that there are cancellations in the β's (which are sums of many terms of uncontrolled signs) and, possibly, the existence of such cancellations might depend upon the sequences $\underline{\underline{\lambda}}(k)$ chosen in (22.1) and not just on the β coefficients.

In this section I elaborate on what could happen if (22.2) admitted a solution verifying $\underline{\underline{\lambda}}(k)\to_{k\to\infty}0$ and providing the necessary cancellations needed to make sense of the rhs of (22.1) and, consequently, of (22.2).

In this situation one should reasonably expect that the solution of (22.2) behaves as $h\to\infty$ exactly as the solution to an equation like (22.2) but with $\mathscr{B}$ replaced by its second-order part [i.e., by the terms with $r=2$ in (22.1)]; see Coleman and Weinberg (1973).

Such an equation was the basis for the theory of the adimensional form factors in the "planar theory" of Sec. XX and, as discussed there, one expects that it has a solution in which $-\lambda^{(4)}(h)$ behaves as [see (20.25)]

$$\underset{h\to\infty}{\simeq}\bar{\lambda}(1-\beta h\bar{\lambda})^{-1} \tag{22.3}$$

and similarly should behave $\alpha(h)$, while $\mu(h),\nu(h)$ ought to go to zero as the square of (22.3).

Then the following remarks can be made.

(1) In itself a solution to (22.2) behaving like (22.3) does not yet yield a solution to the problem of showing that the effective potentials $V^{(k)}$ are well defined as sums of resummed perturbation series [see 't Hooft (1983)].

In fact, the resummation operation just permits one to describe the effective potentials in terms of dressed trees "with no frames" and with end points (ξ,α) providing an adimensional form factor $\lambda^{(\alpha)}(h)$ rather than $\lambda^{(\alpha)}$, if h is the frequency index of the tree vertex to which they are joined by a branch of the tree.

Although this is a big improvement, as far as the k dependence of $V^{(k)}$ is concerned [it suffices to recall that the nonresummed adimensional form factors were diverging with h as powers of an order depending on their degree of complexity and with no *a priori* bounds—see (19.5) and (19.18)—while the resummed adimensional form factors even go to zero with the frequency h as $h\to\infty$] one is still confronted with the problem of summing the contributions to $V^{(k)}$ of the above "simple" (i.e., frameless) trees.

One finds, in doing so, a power series in the resummed adimensional form factors (coming from the trees of order n) whose nth terms can still be bounded only by $n!$. If we use the bounds of Sec. XIX, the effective potential is now given by an expression like (19.1) with a sum running only over the trees with no frames and such that the contributions from the trees of degree n can be bounded as in (19.2) with the last sum (divergent, *a priori*) replaced by k^{-n}—a rather minor gain as far as the n dependence is concerned.

However, the structures of the beta-function coefficients and those of the $V(\gamma;S)$ coefficients in (19.1) are obviously related, and "basically the same," so that if one is willing to accept the existence of cancellations allowing giving a meaning to $\mathscr{B}(\underline{\lambda})$ one should also accept that the very same mechanism might produce cancellations in the expression of the effective potential in terms of the resummed form factors $\underline{\lambda}$.

However, this cancellation mechanism is totally unclear (as this time the beta function cannot help, as it did in the planar case of Sec. XX, to exhibit such cancellations) and it can only be hoped to exist.

(2) It might be that the parameter γ plays an important role in the theory: for instance, in (22.3) the singularities in $\bar{\lambda}$ are located at γ-dependent positions (in fact, one could check that $\beta/\ln\gamma \to_{\gamma\to 1} \beta_0 > 0$, by explicit calculation).

This leads to the possibility that the theory could be defined for many but not all $\bar{\lambda}$'s near zero, e.g., for the values of the renormalized coupling constant which avoid a suitable set of small measure [union of small neighborhoods of the points $(\beta h)^{-1}$ in the case (22.3)] where the form factors could be singular functions of $\bar{\lambda}$. Such a situation is not uncommon in perturbation theory in classical mechanics and it might appear also in field theory.

(3) The possibility of the existence of cancellations mentioned in remark (1) above is hinted at also by the "triviality proofs," where, via some very special assumptions on the regularization and the form of the counterterms, one shows that the adimensional form factors $\lambda^{(\alpha)}(k;N)$ defined in the presence of an ultraviolet cutoff at length γ^{-N} vanish as $N\to\infty$: $\lambda^{(\alpha)}(k;N)\to_{N\to\infty}0$.

The fact that $\lambda^{(\alpha)}(h;\infty)=0$ is a property that can be proved nonperturbatively under special assumptions [see Aizenman (1982), Fröhlich (1982)] hints at the existence of nontrivial cancellation mechanisms in the summations involved in the construction of the effective potentials and of the beta function. Paradoxically the "triviality arguments" might be interpreted as nontriviality arguments.

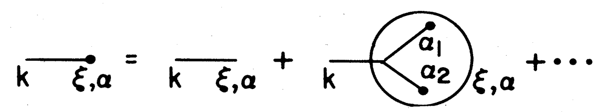
If we go back to a slightly more concrete frame of mind, some comments on the cutoff dependence of the above discussion, brought up in the last remark, as well as on the classical triviality arguments of Landau [see Landau (1955), Thirring (1958), p. 198, Landau and Pomeranchuk (1955), Bogoliubov and Shirkov (1959), p. 528] seem appropriate here. In fact, they hinge upon the just—brought-up question of the cutoff and of the regularization dependence of the whole theory.

The form-factor resummations can be studied with no formal change in the presence of an ultraviolet cutoff γ^N. In the previous sections the N dependence of the form factors was seldom made explicit because one was interested in properties which were uniform in N.

Contrary to what is sometimes stated, fixing N does not make the theory well defined; in fact, one can easily see that there is a simple relation between the form factors of the theory with ultraviolet cutoff N, denoted $\lambda^{(\alpha)}(k;N)$ and the bare coupling constants. Precisely, the bare couplings are $\lambda^{(\alpha)}(N;N)\gamma^{2N\delta_{2,\alpha}+4N\delta_{0,\alpha}}$.

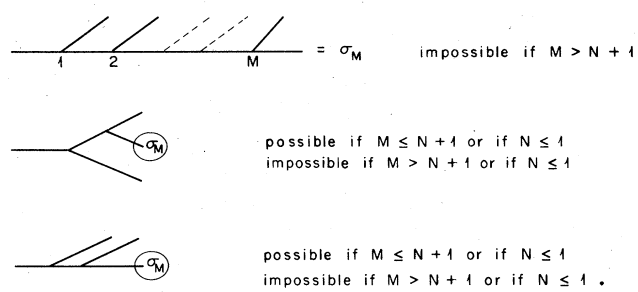
The reason the bare coupling constants are undefined even in a theory with cutoff is simply that $\underline{\lambda}_N \equiv \underline{\lambda}(N;N)$ are still power series in the renormalized couplings with only $n!$ bounds on their coefficients—i.e., they are formal power series, probably divergent.

One can use the resummation ideas of Secs. IX and XIX to try to say something about the bare couplings $\underline{\lambda}_N$; in fact, $\underline{\lambda}(h;N)$ is formally defined by the same recursion relation as $\underline{\lambda}(h)\equiv\underline{\lambda}(h;\infty)$:



(54)

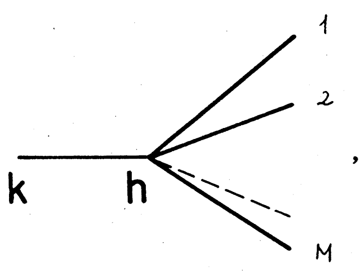
(see also diagram 31) the difference being that $k\le N$ and that everywhere only some tree shapes can appear. Thus, if one fixes a frame and deletes all the inner frames and their contents, the tree shape left inside the selected frame has to be a shape which can arise in computing the effective potentials in the presence of a cutoff γ^N—for instance,



(55)

In diagram 55 the first tree is impossible if $M > N+1$, because one cannot attach allowed frequency labels h_i to the vertices of σ_M with $h_i < h_{i+1}$ and root at -1 (as should have been the case had σ_M been a tree which could have arisen in the presence of a cutoff γ^N).

Note that $\lambda(N;N)$ is a (probably) divergent series, because there are infinitely many trees compatible, even with a finite cutoff N, e.g.,



, $M\ge 1$.

The equation in diagram 54 is very similar to the equations discussed in Secs. XIX and XX, and in fact it coincides with them if one restricts the k and h indices in (22.1) and (22.2) to be $\le N$.

It is therefore clear that in the theory of (22.2) performed in the approximation in which the second order "dominates," i.e., in which (22.2) becomes equivalent to

(20.19)—and hence eventually to (20.20) and (20.24), one can manage to find a solution to (20.2) with

$$\lambda^{(4)}(h;N) \underset{\text{large } h}{\simeq} \bar{\lambda}(1-\beta h\bar{\lambda})^{-1} , \tag{22.4}$$

which would lead to [setting $\lambda^{(4)}(N;N)=\lambda_N=$(bare coupling)] the following surprising relation:

$$\bar{\lambda}=\frac{\lambda_N}{1+\beta N\lambda_N} , \tag{22.5}$$

where $\bar{\lambda}\equiv\lambda(0;N)$ is a "renormalized coupling constant" expressed in terms of the bare coupling λ_N.

Triviality follows from (22.5), which implies that

$$\bar{\lambda} \underset{N\to\infty}{\longrightarrow} 0 \Longrightarrow \lambda(h;\infty)=0 , \tag{22.6}$$

no matter how λ_N behaves, provided $\lambda_N \geq 0$.

On the other hand, $\lambda_N < 0$ is obviously not allowed, as this would make the theory in the presence of a cutoff undefined.

Of course the above argument is based on the identification of $\underline{\lambda}(N;N)$ with $\underline{\lambda}(N;\infty)$ which, to say the least, is not proved (even in an approximate sense).

To understand better the structure of (22.5) one can remark that the bare couplings $\underline{\lambda}(N;N)$ are a formal power series in the renormalized couplings (for simplicity take $\mu=\alpha=\nu=0$ so that there is only one renormalized coupling). The coefficients diverge with N as $N\to\infty$ like powers of N: precisely as N^{n-1} to order n.

The latter statement can be proved by going back to (19.19), which tells us that the bare couplings $\underline{\lambda}(N;N)$ can receive the "most divergent contribution" from the trees γ containing the largest number of frames. Such a number is, if n is the degree of γ, $f-1<n$. Furthermore, the trees which contain the maximal number of frames, $f=n$, really give a contribution to the form factors like $\lambda(\lambda D)^{n-1}(bN)^{n-1}$ to leading order in N.

This can easily be seen by observing that $f=n$ implies that each vertex of σ is framed and gives rise to a bifurcation in just two branches (otherwise, $f<n$).

In other words, the resummation of the most divergent contributions is obtained simply by considering what in Sec. IX was called the resummation of the most divergent graphs. In the language of Sec. XX and of this section this means replacing $\mathscr{B}$ by $\mathscr{B}_2$ in the beta function [so that one also finds the interpretation of the approximation in which $\mathscr{B}$ is replaced by $\mathscr{B}_2$: it just means counting only trees simplest in structure and completely framed, i.e., with no renormalization vertex (no unframed vertex) allowed].

Since, as was explained in Sec. XX, one knows that the well-behaved solutions to

$$\underline{\lambda}(k+1)=\underline{\lambda}(k)+(\mathscr{B}_2\underline{\underline{\lambda}})(k+1) \tag{22.7}$$

behave like (22.4), one sees another interpretation of Landau's result: it leads to triviality if one neglects everything except the most divergent contributions to the (adimensional) form factors.

At the same time it also allows one to compute rigorously the most divergent contributions to the coefficients of the expansion of the bare couplings in terms of the renormalized ones. For example, λ_N has, to order n in the renormalized coupling, a most-divergent contribution exactly equal to

$$\lambda^n(\beta N)^{N-1} , \tag{22.8}$$

while α_N has the contribution:

$$-(\beta'/\beta)\lambda^n(\beta N)^{n-1} \tag{22.9}$$

with the notations of Sec. XX, see (20.24).

A more detailed analysis allows one easily to select the Feynman graphs which, in the evaluation of the most-divergent trees contributions, really give the leading behavior in N: in the language of classical perturbation theory they are the so-called "parquet graphs" and one could find (22.8) and (22.9) also starting from the consideration of such graphs. This involves quite hard work but is very interesting [see the appendix written by Rivasseau in the paper by Gallavotti and Rivasseau (1983)]. This point will not be discussed further here because it involves too many new definitions necessary to establish contact between the formalism developed here and the classical language of the Feynman graphs.

I collect now a few concluding comments to stress some of the ideas and problems already foreshadowed in all the sections of this work.

(a) The assumption that the form factors $\lambda(h;N)$ verify essentially the same equation as the $\lambda(k;\infty)$ seems hard to accept [at least if one wishes to claim from this that the $\lambda(k;N)$ and the $\lambda(k;\infty)$ have the "same" properties] if one accepts that perturbation theory gives correctly the asymptotic expansion for the beta function when the renormalized couplings λ,μ,α,ν are suitably chosen (say, as functions of λ).

In fact, in order that this could be true some important cancellation effects must be present (to compensate the factorially growing coefficients) and the recursion relations for $\underline{\lambda}(k;N)$ being "slightly different" from those of $\underline{\lambda}(k;\infty)$ may just miss the cancellations.

(b) It is clear once an ultraviolet cutoff is specified together with the bare Lagrangian, the coefficients $\underline{\lambda}(h;N)$ are well defined and can be expressed in terms of the bare coupling constants $\underline{\lambda}(N;N)$ both as formal power series or as true functions (of the bare couplings) or as formal power series in the renormalized form factors $\underline{\lambda}(0;N)$ or in the renormalized coupling constants.

On the other hand, the functions $\underline{\lambda}(h;\infty)$ are perturbatively well defined as a formal power series in the renormalized constants and, thought of as a formal power series, are completely independent of the regularization used.

The approach in which one prescribes the bare constants $\underline{\lambda}(N;N)$ and tries to study the renormalized constants $\underline{\lambda}(0;N)$ looks conceptually clearer; however, it suffers from the drawback of necessarily relying on special assumptions on the cutoff and on the regularization and on the bare Lagrangian.

For instance, the well-known lattice approximation in which $\partial\varphi$ is the nearest-neighbor difference and the Lagrangian is taken to be

$$\gamma^{-4N}\sum_{\xi}[-\lambda_N{:}\varphi_\xi^4{:}-\mu_N{:}\varphi_\xi^2{:}-\alpha_N{:}(\partial\varphi_\xi)^2{:}-\nu_N] \qquad (22.10)$$

with the free field defined by

$$\left[\exp\left[-\gamma^{-4N}\sum_{\xi}[(\partial\varphi_\xi)^2+\varphi_\xi^2]\right]\right]\prod_{\xi}d\varphi_\xi \qquad (22.11)$$

has the drawback of making "indistinguishable" the "main" $(\partial\varphi)^2$ term from the similar "counterterm": whether this point is relevant is not known but it is certainly one of the main properties necessary in the existing triviality proofs of the lattice regularization of φ_4^4 [in the sense of (22.10) and (22.11)].

A sign that something might be wrong with the lattice regularization, with respect to the old problem of finding a meaning for the perturbation-theory formal series, is that the most divergent contributions to the expansion of the bare couplings λ_N,α_N in a series of the renormalized couplings λ,μ,α,ν are (when $\mu=\alpha=\nu=0$ for simplicity) all positive for λ_N and all negative for α_N [see Gallavotti and Rivasseau (1983)] hinting at the possibility that in the bare theory the counterterms on $(\partial\varphi)^2$ might be antiferromagnetic and therefore a detailed description of their form [e.g., whether $\alpha_N(\partial\varphi)^2$ is the nearest-neighbor difference or a many-neighbor version of it] might be essential.

This also hints at the possibility that the convergence of the fields φ_ξ on the lattice to the continuum fields might be more complicated than the naive pointwise convergence of the Schwinger functions (even at distinct points).

(c) Expression (22.4) hints at the possibility that $\underline{\lambda}(k;N)$ could be defined only for some values of $\bar{\lambda}$ which accumulate to zero together with other values of $\bar{\lambda}$ for which $\underline{\lambda}(k;N)$ cannot be defined. Such regular and singular values of $\bar{\lambda}$ may depend on γ: i.e., the parameter γ itself may play a nontrivial role in defining the theory. The existence of another relevant parameter is somewhat necessary if one believes that the antiferromagnetic effects discussed above may have some importance: such a parameter should describe on which scale such effects are smoothed out (an event that should happen, since the final Schwinger functions, as defined order by order by perturbation theory, are smooth except at coinciding points).

(d) Of course one cannot even exclude the possibility that $\bar{\lambda}$ should be negative [which might eliminate the singularities in $\bar{\lambda}$ for $\bar{\lambda}$ small, as shown in the approximate formulas (22.10)].

In fact, from the observation that $\underline{\lambda}(N;N)\neq\underline{\lambda}(N;\infty)$, there seems to be little (or no) relation between the signs of $\underline{\lambda}(N;N)=$(bare coupling) and those of the effective form factors $\underline{\lambda}(0;\infty)$ (which for small renormalized coupling should have the same sign as the renormalized coupling itself, called above $\bar{\lambda}$): at least unless special assumptions on the bare interaction Lagrangian are made [see Coleman and Weinberg (1973), who prove that $\lambda(0,N)<0$ implies $\lambda(N,N)<0$ in a class of nonperturbative lattice-regularized φ^4 models with a ferromagnetic kinetic term; see also Aizenman (1982), Fröhlich (1982), for a rigorous version of a similar result].

(e) If $d=2$ or 3 one could still perform the (mostly unnecessary if $d=3$ and totally unnecessary if $d=2$) subtractions that one would perform in the case $d=4$, as described in Secs. XVII and XVIII. Contrary to what is sometimes stated, the problem is far from being easy in spite of the strength of the asymptotic freedom.

The bare couplings are still given by nonconvergent (*a priori*) series and the same happens for the form factors.

The only gain is that the dimensionless form factors are bounded or grow with a power of the frequency index at any fixed order of perturbation theory and the power is a number independent of the order n.

However, the dependence of the perturbation series coefficients for the form factors is, at order n, bounded only by $n!$.

Understanding whether, in spite of this, one can still make sense, beyond perturbation theory, of ${:}\varphi^4{:}$ fields in dimension $d=2$ or 3 with the subtractions of ${:}\varphi^4{:}_4$ would help in understanding the role of the asymptotic freedom in constructive field theory. By "subtractions of ${:}\varphi^4{:}_4$" one means here essentially the usual zero-momentum subtractions "to fourth order for the four-external-lines diagrams and to third order for the two-external-lines diagrams."

This problem, surprisingly, does not seem to have been considered in the literature.

I apologize for this section, which has a somewhat different character from the rest of the work, mostly dealing with open or ill-defined problems. The main reason for including it is to stress a fact that I think is a rather important one, namely, that the problem of the construction of a nontrivial ${:}\varphi^4{:}_4$ field theory, or a proof of its triviality, is still very open and hard.

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APPENDIX A: COVARIANCE OF THE FREE FIELD; HINTS

Let $H_{\text{quantum}}=-\frac{1}{2}A\Delta+V$, where Δ is the Laplace operator on the space $L_2(R^{(L/a)^d})=\mathscr{H}$, $A=\hbar^2/\mu a D$ [see (1.14)].

The operator $H_{\text{quantum}}=H$ has a simple lowest eigenvalue, because $V\to\infty$ at infinity [see (1.14)]; therefore if φ_x denotes also the multiplication operator on $\mathscr{H}$ by φ_x, it is

$$\begin{aligned}C_{\xi\eta}&=(e_0,\varphi_{\bar x}T_t\varphi_{\bar y}e_0)\equiv(e_0,T_\tau\varphi_{\bar x}T_t\varphi_{\bar y}T_{\tau-t}e_0)\\&\equiv\lim_{\tau\to\infty}\operatorname{Tr}(T_\tau\varphi_{\bar x}T_t\varphi_{\bar y}T_{\tau-t})\\&=\lim_{\tau\to\infty}\int T_\tau(\underline{\varphi}_{-\tau},\underline{\varphi}_0)(\underline{\varphi}_0)_{\bar x}T_t(\underline{\varphi}_0,\underline{\varphi}_t)(\underline{\varphi}_t)_{\bar y}T_{\tau-t}(\underline{\varphi}_t,\underline{\varphi}_{-\tau})d\underline{\varphi}_{-\tau}d\underline{\varphi}_0 d\underline{\varphi}_t\,,\end{aligned}\tag{A1}$$

where $\underline{\varphi}=(\varphi_x)_{x\in\Lambda\cap Z^D a}$ and $\xi=(\bar x,0)$, $\eta=(\bar y,t)$.

Using Trotter's formula [see comments before (2.7)], one finds (if $b>0$ and $2\tau/b=N$ is an integer)

$$\begin{aligned}C_{\xi\eta}&=\lim_{\tau\to\infty}\lim_{b\to 0}e^{2E_0\tau/\hbar}\int(e^{(b/2)A\Delta}e^{bV})^{\tau/b}(\underline{\varphi}_{-\tau},\underline{\varphi}_0)(\underline{\varphi}_0)_{\bar x}(e^{(b/2)A\Delta}e^{bV})^{t/b}(\underline{\varphi}_0,\underline{\varphi}_t)\,.\\&\qquad\times(\underline{\varphi}_t)_{\bar y}(e^{(b/2)A\Delta}e^{bV})^{(t-\tau)/b}(\underline{\varphi}_t,\underline{\varphi}_{-\tau})d\underline{\varphi}_{-\tau}d\underline{\varphi}_0 d\underline{\varphi}_t\\&=\lim_{\tau,b}\frac{\int\prod_{j=0}^{N}(e^{(b/2)A\Delta})(\underline{\varphi}_{-\tau+bj},\underline{\varphi}_{-\tau+bj+b})e^{bV(\underline{\varphi}_{bj})}\varphi_{(\bar x,0)}\varphi_{(\bar x,t)}\prod_{j=0}^{N}d\varphi_{-\tau+jb}}{\int\prod_{j=0}^{N}(e^{(b/2)A\Delta})(\underline{\varphi}_{-\tau+bj},\underline{\varphi}_{-\tau+bj+b})e^{bV(\underline{\varphi}_{bj})}1\prod_{j=0}^{N}d\varphi_{-\tau+jb}}\,,\end{aligned}\tag{A2}$$

where one assumes that τ/b is also an integer and the fields in the kernels have been denoted, for reasons which will be soon clear, with a "time index" $-\tau+bj$ rather than by j itself. Also, one writes $(\underline{\varphi}_\theta)_x=\varphi_{(x,\theta)}$ and $\underline{\varphi}=(\varphi_x)_{x\in Z^d a\cap\Lambda}$. The denominator within the last limit is essentially $\exp(-2E_0\tau/\hbar)$, being, after the limit $b\to 0$, equal to the trace of $\exp(-H_{\text{quantum}}2\tau/\hbar)$.

Using the explicit form of the heat equation kernel:

$$\prod_{j=0}^{N}e^{(b/2)A\Delta}(\underline{\varphi}_{-\tau+bj},\underline{\varphi}_{-\tau+bj+b})=\exp\left[-\frac{1}{2}\frac{1}{Ab}\sum_{j=0}^{N}\sum_{x\in\Lambda}(\varphi_{x,-\tau+jb}-\varphi_{x,-\tau+jb+b})^2\right]\tag{A3}$$

and

$$\begin{aligned}\prod_{j=0}^{N}e^{bV(\underline{\varphi}_{bj})}=\exp\Bigg[&-\frac{1}{2}\frac{ba^D\mu}{\hbar}\sum_{x\in\Lambda}\sum_{j=0}^{N}\sum_{i=1}^{D}(\varphi_{x+ae_i,-\tau+jb}-\varphi_{x,-\tau+jb})^2\\&-\frac{1}{2}\frac{ba^D\mu}{\hbar}\left[\frac{m_0c^2}{\hbar^2}\right]^2\sum_{x\in\Lambda}\sum_{j=0}^{N}(\varphi_{x,-\tau+bj})^2\Bigg]\,,\end{aligned}\tag{A4}$$

one finds, if ξ denotes a point on the $d=D+1$ dimensional lattice with spacing a in the first D directions and b in the last one, and if e_j, $j=1,\dots,D,0$, are unit vectors in the lattice directions, one finds that the integral in (A2) has the form

$$\lim_{\tau,b}\times\text{const}\int e^{-(1/2)(Q\varphi,\varphi)}\varphi_\xi\varphi_{\bar\xi}\prod_\omega d\varphi_\omega\,,\tag{A5}$$

where the constant is a normalization constant and $Q=(Q_{\xi\eta})_{\xi,\eta\in\tilde\Lambda}$, where $\tilde\Lambda$ is $[(\Lambda\cap Z^d a)]\times[(-\tau,\tau)\cap Zb]$ with "periodic boundary conditions" and $\xi=(x,0)$, $\eta=(y,t)$ is given by

$$(Q\varphi,\varphi)=\frac{\mu}{\hbar}ba^D\left[\sum_{\xi\in\tilde\Lambda}\left[c^2\sum_{j=1}^{D}\frac{(\varphi_{\xi+e_ja}-\varphi_\xi)^2}{a^2}+\frac{(\varphi_{\xi+e_db}-\varphi_\xi)^2}{b^2}\right]+\sum_{\xi\in\tilde\Lambda}\frac{(m_0c^2)^2}{\hbar^2}\varphi_\xi^2\right]\,.\tag{A6}$$

But the integral (A5) is simply $(Q^{-1})_{\xi\eta}$ and Q^{-1} can be easily found by explicit diagonalization: because of the periodic boundary conditions the eigenvectors of Q are complex exponentials.

In the limit $\Lambda\to\infty$, $\tau\to\infty$ the eigenvalues fill the Brillouin zone and Q^{-2} becomes, if $p=(\underline{p},p_0)\in R^{D+1}$,

$$Q_{\xi\eta}^{-1}=\frac{\hbar}{\mu(2\pi)^d}\int_{-\pi/a}^{\pi/a}d\underline{p}\int_{-\pi/b}^{\pi/b}dp_0\frac{e^{ip(\xi-\eta)}}{\left[\frac{m_0c^2}{\hbar}\right]^2+2c^2\sum_{j=1}^{D}\frac{1-\cos(ap_j)}{a^2}+\frac{2[1-\cos(bp_0)]}{b^2}}\,, \tag{A7}$$

and (2.8) follows from (A7) by letting $b\to 0$.

APPENDIX B: HINT FOR (2.10)

For the proof of (2.10) one proceeds as in Appendix A. Everything is the same up to (A4), where, in the present case, an extra term appears:

$$\exp\left[-\frac{ba^D\mu}{2\hbar}\sum_{\underline{\xi}}I(\varphi_{\underline{\xi}})\right]. \tag{B1}$$

Setting $T=\tau$, one sees that the proof of (2.10) is the proof of the admissibility of an interchange of two limits.

This problem should be studied by the reader as a test of understanding of the theory of Brownian motion. On a heuristic level the reader can accept (2.10) and proceed to see that is done with it.

The identity of the P in (2.10) and (2.11) is a byproduct of the above discussion.

APPENDIX C: WICK MONOMIALS AND THEIR INTEGRALS

Let $x_1,\ldots,x_p$ be Gaussian random variables with covariance matrix

$$C_{ij}=\mathscr{E}(x_ix_j)\,. \tag{C1}$$

One defines, for any of the above variables x,

$$:x^p:=[2\mathscr{E}(x^2)^p]^{1/2}H_p\{x/[2\mathscr{E}(x^2)]^{1/2}\}\,, \tag{C2}$$

where H_p is the pth Hermite polynomial defined by the generating function:

$$\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}H_p(\xi)=e^{-\alpha^2/4+\alpha\xi}\,. \tag{C3}$$

More generally, one defines inductively

$$:x_1^{n_1+1}x_2^{n_2}\cdots x_p^{n_p}:=x_1:x_1^{n_1}x_2^{n_2}\cdots x_p^{n_p}: -\sum_{j=1}^{p}C_{1j}n_j:x_1^{n_1}\cdots x_j^{n_j-1}\cdots x_p^{n_p}:\,, \tag{C4}$$

interpreting the last term as 0 if $n_j=0$ and setting

$$:x_1^0\cdots x_p^0:=1,\quad :x_1^0\cdots x_k^1\cdots x_p^0:=x_k\,. \tag{C5}$$

Expressions (C4) and (C5) are a natural extension of the recursion relation for the Hermite polynomials expressed by [if $C=\mathscr{E}(x^2)$]

$$:x^{n+1}:=x:x^n:-nC:x^{n-1}:\,. \tag{C6}$$

Note that if $x_1=x_2=x$ it is $:x_1^{n_1}x_2^{n_2}:=:x^{n_1+n_2}:$. Expression (C4) implies by induction

$$:\left[\sum_{i=1}^{p}\omega_ix_i\right]^q:=\sum_{q_1+\cdots+q_p=q}\frac{q!}{q_1!\cdots q_p!}\omega_1^{q_1}\cdots\omega_p^{q_p}\times:x_1^{q_1}\cdots x_p^{q_p}:\,, \tag{C7}$$

which is the "Leibnitz rule" for Wick monomials.

The basic property of the Wick-ordered monomials is the "Wick rule" for the expectations of products of Wick monomials.

Let $D_1,\ldots,D_s$ be s subsets of $(1,2,\ldots,n)$ and let

$$:x_{D_j}:=:\prod_{\alpha\in D_j}x_\alpha:\,; \tag{C8}$$

then the integral

$$\mathscr{E}\left[\prod_{j=1}^{s}:x_{D_j}:\right]$$

is computed as follows.

Draw, say, on a plane, s clusters of $|D_1|,\ldots,|D_s|$ points each and arbitrarily label the points in the cluster D_j by the elements in D_j.

Draw one line out of each of the vertices $\alpha\in D_j$ and think of it as representing the variable x_α.

Let $\mathscr{T}$ be the set of the graphs obtained by joining pairwise all such lines in all possible ways so that no lines constituting a pair emerging from the same cluster are ever joined together. Denote (α,β) the elements of $\pi\in\mathscr{T}$ obtained by joining (or "contracting") a line emerging from the vertex α with a line emerging from the vertex β.

Then, denoting (α,β) the lines in π joining α and β, we have

$$\mathscr{E}\left[\prod_{j=1}^{s}:x_{D_j}:\right]=\sum_{\pi\in\mathscr{T}}\prod_{(\alpha,\beta)\in\pi}C_{\alpha\beta}\,. \tag{C9}$$

Equation (C9) is the "Wick rule" and it is easily proved by induction from (C4) and its special case when D_j contains one point for each j.

The latter case is treated directly from the relation

$$\mathscr{E}\left[\sum_{q=0}^{\infty}\frac{1}{q!}\left[\sum_i\omega_ix_i\right]^q\right] =\sum_{n_1+\cdots+n_q}\frac{\omega_1^{n_1}\cdots\omega_q^{n_q}}{n_1!\cdots n_q!}\mathscr{E}(x_1^{n_1}\cdots x_q^{n_q}) =\mathscr{E}\left[\exp\left[\sum_i\omega_ix_i\right]\right]=\exp\left[\tfrac{1}{2}\sum_{i,j}\omega_i\omega_jC_{ij}\right], \tag{C10}$$

where the last integration is the general integration formula of the exponential of a Gaussian variable x: $\mathscr{E}(e^x)=\exp[\mathscr{E}(x^2)/2]$.

The formula that one is seeking follows from (C10) by developing the last exponential in powers and by identifying the coefficients of equal powers in the second and fourth terms of (C10) and then interpreting the result graphically.

But the most remarkable property of the Wick monomials is related to the possibility of simple formulas for the truncated expectations.

In fact,

$$\mathscr{E}^T(:x_{D_1}:,\ldots,:x_{D_p}:;s_1,\ldots,s_p) \tag{C11}$$

can be computed via the following rule: draw s_1 clusters of $|D_1|$ points each, s_2 clusters of $|D_2|$ points each, etc., and label the points in them by the elements of $D_1,D_2,\ldots,D_p$, respectively, plus another index identifying which cluster is being considered among the s_j clusters of $|D_j|$ points.

Then consider all the possible graphs π obtained by joining pairs of such points, avoiding drawing lines joining points belonging to the same cluster and with the property that each graph π would be connected if all the points inside each cluster were considered identical or—the same thing—connected (i.e., π should be connected "modulo the clusters").

Then, if λ is a line in π joining the pair of points $(\alpha,\beta)\equiv\lambda$, it is

$$\mathscr{E}^T(:x_{D_1}:,\ldots,:x_{D_p}:;s_1,\ldots,s_p)=\sum_{\pi}\prod_{\lambda\in\pi}C_{\alpha\beta}\,. \tag{C12}$$

In particular, it is remarkable that $\mathscr{E}^T(\ )\geq 0$ if $C_{\alpha\beta}\geq 0$ (which, however, is a property not necessarily true, because C is constrained only to be a positive definite matrix).

The (C12) can be generalized to the case where $x_i=y_i+z_i$, with y_i and z_i, $i=1,\ldots,p$ being two sets of independent Gaussian random variables with covariances C^0_{ij} and C^1_{ij}, and one considers

$$\mathscr{E}^T_1(:x_{0_1}:,\ldots,:x_{D_p}:;s_1,\ldots,s_p)\,, \tag{C13}$$

where $\mathscr{E}_1$ means expectation (i.e., integration) with respect to the z variables at fixed y.

If $\mathscr{T}$ denotes now the set of the graphs obtained by joining pairs of points of different clusters as before but now allowing that some points stay disconnected from the others provided the set of lines joining the points still makes the sets of clusters connected (if each of them is regarded as connected), let $:x_\pi:$ denote $:\prod^*_\alpha x_\alpha:$, where the product is over the points which in $\pi\in\mathscr{T}$ are left unconnected with other points.

Then

$$\mathscr{E}^T_1(:x_{D_1}:,\ldots,:x_{D_p}:;s_1,\ldots,s_p)$$

$$=\sum_{\pi\in\mathscr{T}}:y_\pi:\sum_{\substack{\tau\subset\pi\\ \tau\in\mathscr{T}}}\prod_{\lambda\in\tau}C^1_{\alpha\beta}\prod_{\lambda\notin\pi/\tau}C^0_{\alpha\beta}\,, \tag{C14}$$

where the second sum runs over the subgraphs of π which are still elements of $\mathscr{T}$ (i.e., which still form a graph connected modulo the identification of the points in each cluster).

One first checks that (C14) is an immediate consequence of (C12) by writing $x_i=y_i+z_i$ and developing the sums using the Leibnitz rule: actually, it is convenient to note from the beginning that (C14) is true in general if it is true for $s_1=s_2=\cdots=s_p=1$. This is seen by using the identity valid for truncated expectations:

$$\mathscr{E}^T(x_1,\ldots,x_1,x_2,\ldots,x_2,\ldots,x_{p'},\ldots,x_{p'};1,1,\ldots,1)$$

$$=\mathscr{E}^T(x_1,\ldots,x_{p'};s_1,\ldots,s_{p'})\,, \tag{C15}$$

if x_j is repeated s_j times in the lhs of (C15).

Then one checks that (C14) follows from (C12) by developing the summations as mentioned above in the case $s_1=\cdots=s_p=1$.

Finally, one checks (C12) as a consequence of another remarkable formula (in the case $s_1=\cdots s_p=1$, which is not restrictive, as noted above):

$$\mathscr{E}^T(:e^{\omega_1x_1}:,\ldots,:e^{\omega_px_p}:;1,1,\ldots,1)$$

$$=\sum_{\pi\in\mathscr{T}^*}\prod_{\substack{\lambda\in\tau\\ \lambda=(\alpha,\beta)}}(e^{\omega_\alpha\omega_\beta C_{\alpha\beta}}-1)\,, \tag{C16}$$

where $\mathscr{T}^*$ is the set of graphs with lines joining p points and forming a connected set in which there are never two lines joining the same pair of vertices. Below, the notation $\lambda=(\alpha,\beta)$ is used to identify a line with its end points; and one also defines

$$:e^{\omega x}:\equiv\sum_{p=0}^{\infty}\frac{(\omega)^p}{p!}:x^p:=e^{(\omega^2/2)C}e^{\omega x}\,, \tag{C17}$$

where the latter equality follows from (C2) and (C3).

More generally, if $x_i=y_i+z_i$, and y,z are independent with covariances C^0,C^1, respectively,

$$\mathscr{E}^T_1(:e^{\omega_1x_1}:,\ldots,:e^{\omega_px_p}:;1,\ldots,1)$$

$$=\left[\prod_{j=1}^{p}:e^{\omega_iy_i}:\right]\sum_{\pi\in\mathscr{T}^*}\prod_{\lambda\in\pi}(e^{\omega_\alpha\omega_\beta C^1_{\alpha\beta}}-1)\,, \tag{C18}$$

as a consequence of (C16) and [see (C17)] of

$$:e^{\alpha x}:=:e^{\alpha y}::e^{\alpha z}:\,. \tag{C19}$$

Equation (C12) follows from (C16) by expanding both sides in powers of ω and identifying equal powers of ω.

Therefore, the only formula that one must prove is (C16). One possible proof of (C16) can be given as follows. Consider

$$Z=\int\left[\exp\left[\sum_{j=1}^{p}\lambda_j:e^{i\omega_jx_j}:\right]\right]P(d\underline{x})\,, \tag{C20}$$

where $P(d\underline{x})$ is the Gaussian distribution of $\underline{x}$ and $\lambda_j>0$, $\omega_j\in R$, and $i=\sqrt{-1}$.

Then, expanding in powers of λ, one finds

$$Z=\sum_{n=0}^{\infty}\int\frac{\left[\sum_{i=1}^{p}\lambda_p{:}e^{i\alpha x_i}{:}\right]^n}{n!}P(d\underline{x})$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{n_1+\cdots+n_p=n}\frac{n!}{n_1!\cdots n_p!}\lambda_1^{n_1}\cdots\lambda_p^{n_p}\mathscr{E}[(:e^{i\omega_1x_1}:)^{n_1}\cdots(:e^{i\omega_px_p}:)^{n_p}]$$

$$=\sum_{n_1,\ldots,n_p}\frac{\lambda_1^{n_1}}{n_1!}\cdots\frac{\lambda_p^{n_p}}{n_p!}\int e^{i(\omega_1n_1x_1+\cdots+\omega_pn_px_p)}e^{(1/2)(\omega_1^2n_1C_{11}+\cdots+\omega_p^2n_pC_{pp})}P(d\underline{x})$$

$$=\sum_{n_1,\ldots,n_p}\left[\prod_{j=1}^{p}\left[\frac{\lambda_j^{n_j}}{n_j!}e^{-(n_j^2-n_j)\omega_j^2 j/2}\right]\right]\exp\left[-\sum_{i<j}n_in_j\omega_i\omega_jC_{ij}\right].\tag{C21}$$

Therefore, one has to study

$$\frac{\partial^p}{\partial\lambda_1\cdots\partial\lambda_p}\ln Z\Big|_{\underline{\lambda}=\underline{0}}$$

$$\equiv\mathscr{E}^T(:e^{i\omega_1x_1}:,\ldots,:e^{i\omega_px_p}:;1,\ldots,1)\,,\tag{C22}$$

and one realizes that, for this purpose, one can replace Z by

$$Z'=\sum_{n_1,\ldots,n_p=0,1}\lambda_1^{n_1}\cdots\lambda_p^{n_p}\exp\left[-\sum_{i<j}\omega_i\omega_jn_in_jC_{ij}\right]$$

$$\equiv\sum_{X\subset\{1,\ldots,p\}}\left[\prod_{\xi\in X}\lambda_\xi\right]\exp\left[-\sum_{(\xi,\eta)\subset X}C_{\xi\eta}\omega_\xi\omega_\eta\right],\tag{C23}$$

where the last sum is over the pairs (ξ,η) in $X=(x_1,\ldots,x_p)$; this fact follows from the last expression of (C21) (because $n_j^2-n_j=0$ if $n_j=1$).

One realizes that Z' in (C23) is the grand canonical partition function for a system of particles with variable activity λ_ξ sitting on a finite set $(1,2,\ldots,p)$ and interacting with a pair potential $C_{\xi\eta}\omega_\xi\omega_\eta$.

The theory of the Mayer expansion teaches that the logarithm of Z' can be expanded in a series of the activities and the coefficients of this series are well known and can be obtained via a graphical algorithm: the coefficient of $\lambda_1,\ldots,\lambda_p$ (which in any event is easy to compute independently of the theory of the Mayer expansion) is precisely

$$\sum_{\pi\in\mathscr{T}^*}\prod_{\lambda\in\pi}(e^{-\omega_\alpha\omega_\beta C_{\alpha\beta}}-1)\,,\tag{C24}$$

which proves (C16) replacing ω_α by $i\omega_\alpha$ [the imaginary unit has been introduced in (C20) to avoid convergence problems in the definition of Z as an integral].

APPENDIX D: PROOF OF (16.14)

One has to show that

$$I(\gamma_0)=\int_{\Lambda^n}\prod_\lambda e^{-(\kappa/4)\gamma^{h_\lambda}|\lambda|}d\xi_1\cdots d\xi_n$$

$$\le|\Lambda|B_1^{n-1}\prod_{v>r}\gamma^{-dh_v(s_v-1)},\quad n>1\,,\tag{D1}$$

which is clearly equivalent to (16.14). Here one imagines to have fixed a tree (with no decorations or frames but just frequency indices $\underline{h}$ and position labels $\xi_1,\ldots,\xi_n$ at the end points). The vertices v of the tree organize the end points into a hierarchy of clusters $\underline{\xi}_v$.

The lines λ are drawn so that the ones which join pairs $\xi,\eta\in\underline{\xi}_v$ which are not both located inside any smaller cluster $\underline{\xi}_w$ with $w>v$ are enough lines to connect all the points in the cluster $\underline{\xi}_v$ modulo the smaller clusters (i.e., imagining that the points inside the smaller clusters are connected): for such lines λ it is $h_\lambda=h_v=$frequency index of the vertex.

Define $I(\gamma_0)=|\Lambda|$ if $n=1$, i.e., if the tree is trivial.

Assume that the tree γ_0 has root frequency k and has a first nontrivial vertex v_0 where it bifurcates into s subtrees $\gamma_1,\ldots,\gamma_s$, $s>1$.

Clearly in proving (D1) it is not restrictive to suppose that the lines connecting the clusters $\underline{\xi}_{v_1},\ldots,\underline{\xi}_{v_s}$, associated with the vertices immediately following v_0 in γ_0, do the connection in a simply connected way; otherwise, one just deletes the extra factors in (D1).

Once this is supposed it is clear that one can perform the integrals in (D1) by keeping first all the points in $\underline{\xi}_{v_2},\ldots,\underline{\xi}_{v_s}$ fixed and the positions of the points in the first cluster fixed relative to the point ξ, which is linked by a line λ to the other clusters; here one is supposing that $\underline{\xi}_{v_1}$ is one of the (at least two) clusters connected to only one other cluster (which is no loss of generality).

The result of the integration, followed by the integration over the remaining coordinates, yields the inequality

$$I(\gamma_0)=\left[\int_\Lambda e^{-(\kappa/4)\gamma^h|\rho|}d\rho\right]I(\gamma')\left[\frac{1}{|\Lambda|}I(\gamma_1)\right]$$

$$\leq B_1\frac{1}{|\Lambda|}I(\gamma_1)I(\gamma')\gamma^{-dh}\ , \qquad \text{(D2)}$$

where γ' is the tree obtained from γ_0 by deleting the branch γ_1 and $B_1\gamma^{-dh}$ is a Λ-independent bound on the integral in (D2); in deducing (D2) the translation invariance of the problem has been used; furthermore, the above inequality holds even if one of the trees in the rhs is trivial, provided one defines, as above, $I=|\Lambda|$ for the trivial tree.

Hence by iteration

$$I(\gamma_0)\leq B_1^{s-1}\frac{1}{|\Lambda|^{s-1}}I(\gamma_1)\cdots I(\gamma_s)\ . \qquad \text{(D3)}$$

Obviously (D3) implies (D1) for γ_0 if (D1) is supposed valid for $\gamma_1,\ldots,\gamma_s$; hence the theorem follows by induction being true, by definition, for $n=1$: note that here the relation used several times,

$$\sum_{v'>v}(s_{v'}-1)=n_v-1\ ,$$

is useful [see (12.17)].

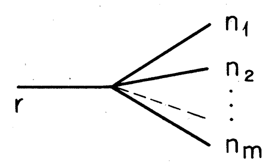
APPENDIX E: PROOF OF (19.8)

Given a tree shape σ without any frames and with $m\geq 2$ final lines each carrying an index n_j, so that $\sum_j n_j=n$, $n_j>0$, consider the sum

$$\sum_{\underline{h}}\left[\prod_{v>r}\gamma^{-\bar{\rho}(h_v-h_{v'})}\right]\prod_{j=1}^{m}\left[(n_j-1)!\sum_{p=0}^{n_j-1}\frac{(bh_j)^p}{p!}\right]\ , \qquad \text{(E1)}$$

where m is the degree of σ and $\underline{h}$ is a frequency assignment to the vertices of σ with root frequency k.

Consider first the case in which σ is



. (56)

In this case one has to study, changing for convenience of notation n_j into n_j+1,

$$\sum_{h>k}\gamma^{-\bar{\rho}(h-k)}\prod_{j=1}^{m}\left[n_j!\sum_{p=0}^{n_j}\frac{(bh)^p}{p!}\right]$$

$$=\sum_{t=1}^{\infty}\gamma^{-\bar{\rho}t}\prod_{j=1}^{m}\left[n_j!\sum_{p=0}^{n_j}\frac{[b(t+k)]^p}{p!}\right]$$

$$=\sum_{t=1}^{\infty}\gamma^{-\bar{\rho}t}\left[\prod_{j=1}^{m}n_j!\right]\sum_{j_x,\ldots,j_m}b^{j_1+\cdots+j_m}\frac{(j_1+\cdots+j_m)!}{j_1!\cdots j_m!}\sum_{r=0}^{j_1+\cdots+j_m}\frac{t^{j_1+\cdots+j_m-r}k^r}{r!(j_1+\cdots+j_m-r)!}$$

$$\leq\left[\prod_{j=1}^{m}n_j!\right]\sum_{j_1=0}^{n_1}\cdots\sum_{j_m=0}^{n_m}b^{j_1+\cdots+j_m}\frac{(j_1+\cdots+j_m)!}{j_1!\cdots j_m!}\sum_{t=1}^{\infty}\sum_{r=0}^{j_1+\cdots+j_m}\frac{\gamma^{-\bar{\rho}t}t^{j_1+\cdots+j_m-r}k^r}{r!(j_1+\cdots+j_m-r)!}\ , \qquad \text{(E2)}$$

and for all $\theta>0$ the rhs of (E2) is bounded by

$$\leq\left[\prod_{j=1}^{m}n_j!\right]\sum_{j_1=0}^{n_1}\cdots\sum_{j_m=0}^{n_m}\left[\frac{b}{\theta}\right]^{j_1+\cdots+j_m}\frac{(j_1+\cdots+j_m)!}{j_1!\cdots j_m!}\sum_{t=1}^{\infty}\gamma^{-\bar{\rho}t}e^{\theta t}\sum_{r=0}^{j_1+\cdots+j_m}\frac{(\theta k)^r}{r!}\ , \qquad \text{(E3)}$$

so that if $\gamma^{-\bar{\rho}}e^{\theta}<1$ and $b<\theta$,

$$\text{Eq. (E3)}\leq\gamma^{-\bar{\rho}}e^{+\theta}\frac{\left[\sum_i n_i\right]!}{1-\gamma^{-\bar{\rho}}e^{\theta}}\sum_{r=0}^{\sum_i n_i}\frac{(\theta k)^r}{r!}\frac{n_1!\cdots n_m!}{(n_1+\cdots+n_m)!}$$

$$\times\left[\sum_{q\geq r}\left[\frac{b}{\theta}\right]^q\left[\sum_{\substack{j_1=0\\ \sum j_i=q}}^{n_1}\cdots\sum_{j_m=0}^{n_m}\frac{(j_1+\cdots+j_m)!}{j_1!\cdots j_m!}\right]\right]$$

$$\leq\frac{\left[\sum_i n_i\right]!\gamma^{-\bar{\rho}}e^{\theta}}{1-\gamma^{-\bar{\rho}}e^{\theta}}\sum_{r=0}^{\sum_i n_i}\frac{(\theta k)^r}{r!}\left[\frac{b}{\theta}\right]^r\frac{1}{1-b/\theta}\leq\frac{\gamma^{-\bar{\rho}}e^{\theta}}{1-\gamma^{-\bar{\rho}}e^{\theta}}\frac{\left[\sum_i n_i\right]!}{1-b/\theta}\sum_{r=0}^{\sum_i n_i}\frac{(bk)^r}{r!}\ , \qquad \text{(E4)}$$

where we have used the inequality (to be proved by induction)

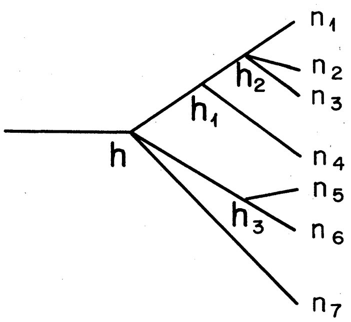
$$\frac{n_1!\cdots n_m!}{(n_1+\cdots+n_m)!}\sum_{j_i=0}^{n_1}\cdots\sum_{\substack{j_m=0\\ j_1+\cdots+j_m=q}}^{n_m}\frac{q!}{j_1!\cdots j_m!}\le 1\,,\qquad \forall n_i,q\,. \tag{E5}$$

Finally, let

$$D_6=\gamma^{-\bar{p}}e^{\theta}(1-\gamma^{-\bar{p}}e^{\theta})^{-1}(1-b/\theta)^{-1}\,, \tag{E6}$$

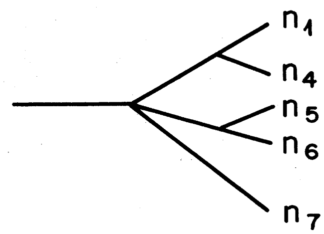
and (9.8) follows with $b=\theta/2$ suitably chosen [e.g., $b=(\bar{p}/4)\ln\gamma$] and with D_6 replacing D_6^m, which is correct in the special case just considered.

Consider next a general tree,



(57)

Using recursively the bound found in the case of diagram 56, one reduces the problem of estimating (19.8) to the problem of a similar estimate for a simpler tree. In fact, summing over h_v, where v is one of the highest vertices of the tree, and if $m_v\ge 2$ is the number of branches emerging out of v, we find the estimate (19.8) to be reduced to that relative to the tree σ' without the vertex v and with the line $v'v$ joining v to the preceding vertex v', being a final line bearing an index $\sum_i n_i$, where the sum is over the end points' indices (of σ) of the end points linked to v by a final branch. For instance, in the case of diagram 57 one gets, if v is taken to be the vertex with frequency h_2,



(58)

Every time the procedure is repeated one gets a factor D_6 and an expression similar to the one to be bounded but for a simpler tree. Since in $m-1$ steps at most one reaches the case drawn in diagram 56, (19.8) is proved.

APPENDIX F: ESTIMATE OF THE NUMBER OF FEYNMAN GRAPHS COMPATIBLE WITH A TREE

This appendix is due to Giovanni Felder, Zürich, who proves the following.

Lemma. Let G be an unlabeled Feynman graph with n vertices, and let γ be a tree with n end points. Then the number $N(G,\gamma,\{n_v^e\}_{v\in\gamma})$ of labelings of G compatible with γ and such that for all vertices v the subgraph of G corresponding to v has n_v^e external lines, is bounded above by $C_\varepsilon^n n(\sigma)\exp\varepsilon(\sum_v n_v^e)$, for all $\varepsilon>0$ and some constant C_ε, if σ is the shape of the tree γ.

Proof. Consider $G,\gamma,\{n_v^e\}_{v\in\gamma}$ fixed. Let γ_v be the subtree of γ with root v, and $N_v(j)$ the number of ways of choosing and labeling a subgraph of G compatible with γ_v and having an external line connected to the vertex j of G. Furthermore, let $v_1,\dots,v_{s_v}$ be the vertices following v in γ. Since the subgraphs $G_{v_1},\dots,G_{v_{s_v}}$ corresponding to $v_1,\dots,v_{s_v}$ have to be connected together, there exists at least one tree diagram T_v with vertices $v_1,\dots,v_{s_v}$ whose lines correspond to propagators connecting $G_{v_1},\dots,G_{v_{s_v}}$. Let d_{v_i} be the number of lines of T_v emerging from v_i. We have the estimate

$$N_v(j)\le\left[\prod_{i=1}^{s_v}\left[\max_{j'\in G}N_{v_i}(j')\right]\right]\sum_{\substack{d_{v_1},\dots,d_{v_{s_v}}\ge 1\\ \sum_i(d_{v_i}-1)=s_v-2}}\left[\prod_{i=1}^{s_v}(n_{v_i}^e)^{d_{v_i}-1}\right]\frac{s_v(s_v-2)!}{\prod_{i=1}^{s_v}(d_{v_i}-1)!}\,, \tag{F1}$$

where the last ratio is the Cayley formula for the number of rooted trees T_v with fixed coordination numbers [see J. W. Moon (1967), *Enumerating Labelled Trees*, in *Graph Theory and Theoretical Physics*, edited by F. Harary (Academic, London)], and

$$\prod_{i=1}^{s_v}(n_{v_i}^e)^{d_{v_i}-1}$$

is a bound on the number of ways of choosing external lines of G_{v_i} corresponding to the lines of T_v. The sum over d_{v_i} can be performed explicitly:

$$N_v(j)\le\left[\prod_{i=1}^{s_v}\left[\max_{j'}N_{v_i}(j')\right]\right]s_v\left[\sum_i n_{v_i}^e\right]^{s_v-2}\,, \tag{F2}$$

and, using $x^u\le k!\varepsilon^{-k}\exp(\varepsilon k)$, $\sum_v(s_v-1)=n-1$, we get

$$N(\gamma,G,\{n_v^e\}_{v\in\gamma})\le\bar{C}_\varepsilon^n\left[\sum_{v\in\gamma}s_v!\right]\exp\left[\varepsilon\sum_{v\in\gamma}n_v^e\right]\,. \tag{F3}$$

But

$$\left[\prod_{v\in\gamma} s_v!\right]\Big/ n(\sigma)=\prod_{v\in\gamma}\left[s_v!\Big/\prod_i t_{i,v}!\right]$$

(where $t_{i,v}$ are the multiplicities of the different tree shapes of the trees that start from v) is just the number of ways of drawing the shape σ by choosing at each vertex how to order the trees starting from it: this number is bounded by the number of ways of drawing all the trees with n end points, which, by the same argument used to count the trees, is bounded by C^n for some constant C.

APPENDIX G: APPLICATION TO THE HIERARCHICAL MODEL

A very simple and particularly interesting example of field theory is the φ^4 hierarchical model.

This model is defined by an interaction like (5.6), i.e., "pure φ^4," but with a different interpretation of the free field $\varphi^{(\le N)}$ and with d being any integer ≤ 4.

In this appendix I discuss very briefly the minor changes necessary to treat this new case; in fact, it will be a useful exercise for the reader to check the statements made below, while reading various parts of this paper.

To define the free fields $\varphi^{(\le N)}$ with cutoffs at scales γ^{-N} one introduces a sequence $Q_0,Q_1,Q_2,\dots,$ of compatible pavements of the unit cube Λ: the pavement Q_j is built with cubes of side size γ^{ij}, where $\gamma>1$ is the "scale parameter."

Each point ξ is in one cube $\Delta\in Q_j$, for $j=0,1,2,3,\dots,$ with the obvious (and trivial) exception of the points on the boundaries of the cubes. Then one defines

$$\varphi_\xi^{(\le k)}=\sum_{j=1}^{k}\gamma^{[(d-2)/2]j}z_{\Delta_j}\equiv\sum_{\substack{\Delta\ni\xi\\ |\Delta|\ge\gamma^{-kd}}}|\Delta|^{(d-2)/2d}z_\Delta\,, \tag{G1}$$

where Δ_j is the cube in Q_j containing ξ and the z_Δ are Gaussian-independent variables with covariance $\frac{1}{2}$ except for one among them, z_{Δ_0}, corresponding to $\Delta_0\equiv\Lambda\in Q_0$, which will be assumed to have covariance $1/2(1-\gamma^{-(d-2)})$, just to simplify some formulas.

The fields $\varphi^{(\le N)}$ behave roughly as the Euclidean free field with cutoff at γ^{-N}.

Hierarchical models in field theory were introduced in the papers of Wilson [see Wilson (1971,1972)[2] and Wilson and Kogut (1974)] as approximations to the Euclidean theory and called, therefore, "approximate recursion formulas." In statistical mechanics they were introduced by Dyson (1969,1971) and studied also by Bleher and Sinai (1973,1975) and by Collet and Eckmann (1978).

Model (G1) is not the one studied in the above-mentioned papers; its relevance and importance for field theory were pointed out in Gallavotti (1978,1979b), and it was applied to constructive field theory for Euclidean fields in a series of papers by Benfatto *et al.* (1978) [see also Benfatto, Cassandro *et al.* (1980)]. It is mentioned earlier in Wilson and Kogut (1974, p. 120, line 11) [see (G3) below for comparison] without comments except perhaps the implication that it may be not too relevant; see Wilson and Kogut (1974, p. 119, line 11). Many of the results that follow would apply as well to the hierarchical models considered in the above-mentioned papers after some obvious changes; for some earlier papers on such "classical" hierarchical models see Gawedski and Kupiainen (1981,1983,1984) and their references.

The theory of the φ^4 field with interaction given by (5.6) for $d=4$ can be pursued exactly as in Secs. XVI—XX with a few remarkable simplifications; the results, and the simplifications just mentioned, are listed below. The reader who has followed Secs. V—IX and XVI—XX will find them very easy to prove; their proof is, however, very instructive, as it shows the true problems of perturbative field theory deprived of most technical complications which accompany them.

(1) Classifying the divergences leads to the same results of Sec. VXI, provided one sets everywhere $m_{2',v}=0$, $n_{1,v}=0$, thus disregarding the $\partial\varphi$ fields (which are not defined in this model and which are absent from the interaction).

(2) The renormalization is also done along the same lines. It is, however, much easier in practice, because the effective interaction (very peculiarly for this model) remains "purely local" on each scale: i.e., the effective potential on scale k has the form

$$V^{(\le k)}=\sum_{n=1}^{\infty}\omega(k,n)\int_\Lambda :\varphi_x^{(\le k)2n}:dx$$
$$\equiv\sum_{\Delta\in Q_k}\Omega_x(X_\Delta)\,, \tag{G2}$$

where $X_\Delta=\varphi_x^{(\le k)}/[2\mathscr{E}(\varphi_x^{(\le k)2})]^{1/2}$ if $x\in\Delta$, and in the second step use has been made of the fact that $\varphi_x^{(\le k)}$ is constant over boxes Δ of side size γ^{-k}. The function $\Omega_k(x)$ is defined implicitly by (G2) as well as the coefficients $\omega(k,n)$. The normalized field is introduced for convenience.

(3) In fact, one can see, independently of perturbation theory, that the functions $\Omega_k(x)$ are related by a recursion formula, namely, it is $\Omega_k=T\Omega_{k+1}$, $k>0$, where T is [see also Gallavotti (1979b)]

$$(T\Omega)(x)=\gamma^d\ln\int\{\exp[\Omega(\alpha z+\beta x)]\}e^{-z^2}\frac{dz}{\sqrt{\pi}}\,, \tag{G3}$$

[2]See in particular footnote 8. This paper introduces a hierarchical model and deals mainly with φ_3^6; other similar hierarchical models had been introduced earlier in Dyson (1969) and later in Baker (1972) in statistical mechanics and in Gallavotti (1978) in field theory—a general theory of the recursion relations associated with certain hierarchical models can be found in Collet and Eckmann (1978), who extend the work initiated by Bleher and Sinai (1973,1975).

with $\alpha=(1-\beta^2)^{1/2}$, $\beta=\gamma^{-(d-2)/2}$.

Therefore, $\Omega_k=T^{N-k}\Omega_N$ if $k>0$, and a simple calculation shows that the interaction

$$\int_\Lambda(\lambda_N:\varphi_x^{(\le N)4}:+\mu_N:\varphi_x^{(\le N)^2}:+\nu_N)dx$$

can be written as (G2), with

$$\Omega_N(x)=[\lambda_N\gamma^{-(4-d)N}C^4H_4(x)+\mu_N\gamma^{-2N}C^2H_2(x)$$

$$+\nu_N\gamma^{-dN}]\ , \qquad \text{(G4)}$$

where $C=(1-\gamma^{-(d-2)})^{-1/2}$. The reason $k=0$ is special is that z_Δ, for $\Delta\in Q_0$, has a slightly different covariance. If z_Δ, $\Delta\in Q_0$, had been taken with covariance $\frac{1}{2}$, too, then α,β in (G3) would, however, have turned out slightly k dependent.

(4) Because of remark (2) the $\mathscr{L}$ and R operations of Sec. XVIII need only to be defined for 1, $:\varphi_1^2:$, $:\varphi_1^4:$, and are simply

$$\begin{aligned}&\mathscr{L}1=1,\quad \mathscr{L}:\varphi_1^2:=:\varphi_1^2:,\quad \mathscr{L}:\varphi_1^4:=:\varphi_1^4:\ ,\\ &\mathscr{L}:\varphi_1^n:=0\quad \text{if } n>4\ ,\\ &R1=0,\quad R:\varphi_1^2:=0,\quad R\,|\,\varphi_1^4:=0\ ,\\ &R:\varphi_1^n:=:\varphi_1^n:\quad \text{if } n>4\ .\end{aligned} \qquad \text{(G5)}$$

No D_{xy} fields arise: because of the locality remark (2), above, x would be equal to y, so that $D_{xy}\equiv 0$.

Since the D_{xy} fields vanish, there is no need to increase the order of subtraction, because D_{xy} "has clearly a zero of infinite order."

Therefore, the above theory is renormalizable, in spite of the absence of $(\partial\varphi)^2$ terms in the interaction. This is my proof of a theorem by Wilson; in fact Wilson (1972, line 26 from bottom of p. 424) proved this result (just by stating it) in φ_3^6 theory (and hence in φ_4^4 theory also, the argument being the same in the two cases). It seems that this deep result of Wilson went almost unnoticed, probably because he failed to stress its interest, very high in my opinion. The difference between the models used here is irrelevant, and the above proof can be repeated verbatim in the "classical" hierarchical models.

(5) Finally, consider the resummations. The equation for the form factors, diagram 32, and formulas (9.9) and (20.8) can be written in terms of the "beta functional"

$$(B\underline{\lambda})^{(\alpha)}(k)=\sum_\sigma\sum_{\underline{h},h_r\ge k+1}\beta_\sigma^{(\alpha)}(k+1;\underline{h},\underline{\alpha}')\times\prod_{\substack{\text{end point}\\ \text{of }\sigma}}\lambda^{(\alpha_i')}(h_i)\ , \qquad \text{(G6)}$$

where we have explicitly exhibited the decomposition of the β coefficients of (20.8) in terms of the contributions from the various trees.

Then, as the reader can easily check, the bounds on β are, if v_0=first vertex of σ,

$$|\beta_\sigma^{(\alpha)}(k+1;\underline{h},\underline{\alpha}')|\le C_0^n n!\gamma^{-\sum_{v>v_0}\hat\rho}\times\prod_{v>v_0}\gamma^{-\hat\rho(h_v-h_{v'}-1)} \qquad \text{(G7)}$$

with the usual notations and with $C_0>0$, $\hat\rho=-d+(6-\varepsilon)(d-2)/2$, $\varepsilon>0$. The 6 in $\hat\rho$ is explained by the fact that the vertices of σ carrying a superscript R generate [because of (G5)] only Feynman graphs with at least six external lines (in fact, the R operation just deletes the contributions from nontrivial Feynman graphs with two or four external lines emerging out of clusters generated by the vertices of the tree σ).

Expression (G7) suggests that the hierarchical model may have a "$1/\gamma$ expansion" for the beta function $\underline{B}$ in (G6). One sees that the right-hand side of (G7) is of order $O(1)$, as $\gamma\to\infty$ only for the trees γ which have only one nontrivial vertex v_0, which we can call the "simple" trees. For the other trees the bound (G7) contains terms of $O(1/\gamma^{\hat\rho})$.

Therefore, it might be of some interest to analyze the equation for the dimensionless form factors (G6) in the approximation in which only "simple" trees are considered in the right-hand side of (G6). This approximation is *not* equivalent to taking an order-by-order dominant term in the γ^{-1} expansion of the β_σ for the σ of given order n, because even for simple trees σ the β_σ depend on γ and have subleading corrections in γ^{-1}.

Therefore, the above approximation has the same character as the "leading log" or "most-divergent graphs" resummation or as the "planar graphs" resummation discussed in Secs. IX and XX. However, it is in some sense to be clarified (one hopes) a deeper resummation, as, unlike the cases of the "most-divergent graphs" or the "planar graphs" resummations, its beta function has an asymptotic expansion which has zero-radius of convergence: the contribution from the tree with n end points being proportional to $n!$ [see (G7) and (G8) below].

But the really interesting aspect of the above resummation is that the beta function can be computed "exactly." In fact, from the graphical interpretation of Eqs. (5.13) and (5.14) in terms of "simple" trees one can easily recognize that the contribution of the simple trees to the right-hand side of (G6) is just the power series expansion of a function $B_g[\underline{\lambda}(k+1)]$ in formal powers of $\underline{\lambda}(k+1)$, and $B_g^{(\alpha)}(\lambda^{(4)},\lambda^{(2)})$ is

$$B_g^\theta(\lambda^{(4)},\lambda^{(2)})=\gamma^d\frac{2^\theta}{\theta!}C^{-\theta}\int H_\theta(x)e^{-x^2}\frac{dx}{\sqrt{\pi}}\left[\ln\int e^{-z^2}\frac{dz}{\sqrt{\pi}}\exp\left[\sum_{\theta'}\lambda^{(\theta')}C^{\theta'}H_{\theta'}(\alpha z+\beta x)\right]\right]\ , \qquad \text{(G8)}$$

where $\theta=2,4$, $C=(1-\gamma^{-(d-2)})^{-1/2}$, $\alpha=(1-\beta^2)^{1/2}$, $\beta=\gamma^{-(d-2)/2}$.

Thus, if we set $\lambda=-\lambda^{(4)}(k+1)C^4$, $\mu=-\lambda^{(2)}(k+1)C^2$, $\lambda'=-\lambda^{(4)}(k)C^4$, $\mu'=-\lambda^{(2)}(k)C^2$, it follows that the dimensionless form factors on scale k are expressed in terms of those on scale $k+1$ simply by

$$\lambda'=-\gamma^d\frac{2^4}{4!}\int H_4(x)e^{-x^2}\frac{dx}{\sqrt{\pi}}\left[\ln\int e^{-\lambda H_4(\alpha z+\beta x)-\mu H_2(\alpha z+\beta x)}e^{-z^2}\frac{dz}{\sqrt{\pi}}\right],$$
$$\mu'=-\gamma^d\frac{2^2}{2!}\int H_2(x)e^{-x^2}\frac{dx}{\sqrt{\pi}}\left[\ln\int e^{-\lambda H_4(\alpha z+\beta x)-\mu H_2(\alpha z+\beta x)}e^{-z^2}\frac{dz}{\sqrt{\pi}}\right]. \tag{G9}$$

This formula [which I derived, with some algebraic errors, later corrected by Nicolò, from the remark that (5.13) and (5.14) imply that B_g can be summed explicitly] gives a recursion relation somewhat interesting in itself. But more interesting would be to see in what sense (if at all) the above resummation provides a good resummation rule up to $O(\gamma^{-1})$. Such problems have not been investigated yet.

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