Disordered electronic systems

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This paper reviews the progress made in the last several years in understanding the properties of disordered electronic systems. Even in the metallic limit, serious deviations from the Boltzmann transport theory and Fermi-liquid theory have been predicted and observed experimentally. There are two important ingredients in this new understanding: the concept of Anderson localization and the effects of interaction between electrons in a disordered medium. This paper emphasizes the theoretical aspect, even though some of the relevant experiments are also examined. The bulk of the paper focuses on the metallic side, but the authors also discuss the metal-to-insulator transition and comment on problems associated with the insulator.

CONTENTS

I. INTRODUCTION

A. Scope of the paper

Crystalline materials have been studied intensively by physicists since the beginning of quantum mechanics. The periodicity of the crystal permits the classification of electronic wave functions as Bloch waves, so that the band structures of rather complicated crystalline materials can be calculated. However, in real life, the crystalline state is the exception rather than the rule. Disorder exists in varying degree, ranging from a few impurities or interstitials in an otherwise perfect crystalline host to the strongly disordered limit of alloys or glassy structures. The weak-disorder limit is traditionally described by the scattering of Bloch waves by impurities. In a metal, this leads to a Boltzmann transport equation for the quasiparticles, so that the low-temperature resistivity has the form

$$\rho(T) = \rho_0 + A T^n . \tag{1.1a}$$

In terms of conductivity, this can be written, at sufficiently low temperature, as

$$\sigma(T) = \sigma_0 - A\sigma_0^2 T^n , \qquad (1.1b)$$

where σ_0 is the residual conductivity due to impurity scattering. As the temperature is raised, the amount of scattering usually increases due to the excitation of phonons or electron-electron collisions, so that A is positive and n is a positive integer usually greater than or equal to two ($n=2$ if electron-electron scattering dominates). On the other hand, if the disorder is strong, as in the case of alloys where two types of atoms randomly occupy lattice sites, the traditional approach is to force some average periodicity on the system and then apply the familiar concepts of ordered systems. The coherent-potential approximation (CPA) (Elliott, Krumhansl, and Leath, 1974) is an example of this approach.

In the past few years there has been a growing realization that disordered materials cannot be understood by evading the issue and forcing them into the mold of ordered systems. Instead, new concepts must be introduced which treat the disorder from the beginning. One consequence of the recent advances is that today we know, both experimentally and theoretically, that even in the weak-disorder limit, basically all aspects of the Boltzmann description of Eq. (1.1) are wrong. The coefficient A may be positive or negative, and n is typically $\frac{1}{2}$ for three-dimensional systems. A certain universality is also emerging in that if the proper questions are asked, the behavior of granular metals or Si-MOSFET (metal-oxide-semiconductor field-effect transistor) inversion layers are the same, even though their electron density may differ by several orders of magnitude.

The new understanding is based on advances in two different areas of the problem. The first is the problem of Anderson localization, which deals with the nature of the wave function of a single electron in the presence of a random potential. A scaling description of the Anderson localization problem is now available that has greatly deepened our understanding. The second aspect of the problem is the interaction among electrons in the presence of a random potential. It turns out that the simple fact that electrons are diffusive instead of freely propagating leads to a profound modification of the traditional view based on the Fermi-liquid theory of metals.

In this paper we shall review the progress made on these two aspects of the problem. The bulk of the paper will deal with the weak-disorder limit, where the theory is on firm ground and quantitative comparison with experiments can be made. We try to emphasize the physical concepts involved, at the expense of technical details and completeness in our references. The strongly disordered regime is discussed qualitatively, with a view towards raising more questions rather than providing answers. Our coverage of the experimental situation is brief, and the reader is referred to a forthcoming article by Bishop and Dynes for a more detailed treatment. Other excellent reviews can be found in the Taniguchi symposium

proceedings edited by Nagaoka and Fukuyama (1982) and in the papers by Fukuyama (1984), Altshuler and Aronov (1984), and Bergmann (1984).

B. Basic concepts of Anderson localization and the mobility edge

In this section we briefly review the basic concept of localization introduced by Anderson in 1958 (Anderson, 1958) and the concept of the mobility edge and metal-insulator transition. Prior to the development of the scaling theory to be described later, a substantial literature had developed on this problem, and there exist excellent reviews by Mott and Davis (1979) and Thouless (1979).

In 1958, Anderson pointed out that the electric wave function in a random potential may be profoundly altered if the randomness is sufficiently strong. The traditional view had been that scattering by the random potential causes the Bloch waves to lose phase coherence on the length scale of the mean free path l. Nevertheless, the wave function remains extended throughout the sample. Anderson pointed out that if the disorder is very strong, the wave function may become localized, in that the envelope of the wave function decays exponentially from some point in space, i.e.,

$$|\psi(\mathbf{r})| \sim \exp(|\mathbf{r} - \mathbf{r}_0|/\xi) , \qquad (1.2)$$

and ξ is the localization length. This is illustrated in Fig. 1. The existence of the localized state is easily understood if we go to the limit of very strong disorder. Then a zeroth-order description of the eigenstate would be a bound state or a localized orbital bound by deep fluctuation in the random potential. We could then consider the admixture between different orbitals as a perturbation. The main point is that such admixtures will not produce an extended state composed of linear combinations of infinitely many localized orbitals. The reason is that orbitals that are nearby in space, so that the wave functions overlap significantly, are in general very different in energy, so that the admixture is small because of the large energy denominator. On the other hand, states that are nearly degenerate are in general very far apart in space, so that the overlap is exponentially small. Thus, in the strongly disordered limit, the wave function will be exponentially localized. Indeed, it is easier to establish the existence of localized states than to establish that of extended ones. For example, in one dimension it can be shown rigorously that all states are localized, no matter
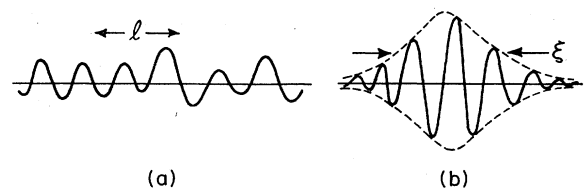

FIG. 1. Typical wave functions of (a) extended state with mean free path l; (b) localized state with localization length ξ.

how weak the disorder (Mott and Twose, 1961; Borland, 1963). On the other hand, the existence or nonexistence of an extended state in two dimensions has been a point of contention for many years and forms an important part of this paper.

Now that we understand the two limits of weak and strong disorder, the interesting question is what happens for intermediate disorder. Instead of varying the amount of disorder, we can also consider varying the energy of the eigenstates. We expect the states deep in the band tails to be localized, since these are states that are formed from localized orbitals bound in deep potential fluctuations. The states in the center of the band have the best chance of remaining extended for a moderately disordered system. Thus, as a function of energy, the states must change their character from being localized to being extended. The critical energy at which this change occurs is called the mobility edge (Mott 1967). It is so named because, if the Fermi energy lies in a region of localized states, the conductivity at zero temperature would vanish, whereas extended states give rise to a finite zero-temperature conductivity. Thus the mobility edge marks the transition between a metal and an insulator. This is illustrated in Fig. 2. The next question is whether the transition is continuous. Mott (1973) has argued for a discontinuous transition based on the idea of Ioffe and Regel (1960) that the lower limit for the mean free path in a metal is the interatomic spacing or k_F^{-1}. Assuming that the Boltzmann transport or weak scattering theory for conductivity is still adequate, one has for an electron density n

$$\sigma = \frac{e^2}{\hbar}\left[\frac{n}{k_F^2}\right](k_F l) \geq \left[\frac{e^2}{\hbar}\right]\left[\frac{n}{k_F^2}\right].$$ (1.3)

Thus the minimum metallic conductivity, or the size of conductivity jump at the disorder-induced metal-insulator transition, is

$$\sigma_{min}^{3D} \simeq \left[\frac{1}{3\pi^2}\right]\left[\frac{e^2}{\hbar}\right]\left[\frac{1}{a}\right],$$ (1.4)

where a is some microscopic length scale in the problem, such as the inverse of the Fermi wave number, $a \approx k_F^{-1}$. For many years, experimental support for the existence of σ_{min} has been found for a large variety of systems, as summarized in Mott and Davis (1979). The concept has

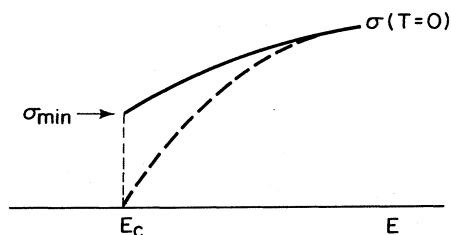FIG. 2. Schematic illustration of the mobility edge E_c, which separates localized and extended states. The two possibilities of a continuous or discontinuous transition with σ_{min} are shown.

also been extended to two dimensions, where

$$\sigma_{min}^{2D} \approx 0.1 e^2/\hbar .$$ (1.5)

The interesting observation is that, just on dimensional grounds, the length parameter a in Eq. (1.4), which varies from material to material, is now absent, and one expects σ_{min} for two dimensions to be universal (Licciardello and Thouless, 1975). The scaling theory of localization to be reviewed in the next section (Sec. II) calls into question the existence of σ_{min} in both three and two dimensions. It predicts instead that the metal-insulator transition is a continuous one in three dimensions, and that all states are localized in two dimensions. The concept of σ_{min} may provide a useful parameter for understanding experimental data taken at relatively high temperatures, but there is increasing experimental evidence that it does not describe the true zero-temperature limit. This will be reviewed in Sec. VI. It is to be emphasized that the above concepts were all developed for noninteracting electrons. It is known that Coulomb correlation in an ordered system may lead to a metal-to-insulator transition as well (Mott, 1949). There have been many qualitative discussions of the interplay between disorder and interaction, but until recently very few quantitative results were known. In the last few years a quantitative theory of the interacting disordered systems has begun to emerge, at least in the weakly disordered limit. This development is reviewed in Sec. III.

II. SCALING THEORY OF LOCALIZATION

A. Early formulation of scaling

In the mid-seventies, Thouless and co-workers, in a series of papers, began to formulate a scaling description of the localization problem. (For a review see Thouless, 1974.) The conceptual framework is important for the subsequent development. Thouless envisioned building a sample of size $(2L)^d$ in d dimensions by putting squares or cubes of size L^d together. It seems reasonable that the nature of the eigenstates of the $(2L)^d$ sample will be dictated by the nature of the states of the L^d samples, but the question is whether the description can be summarized by one or a few parameters. The eigenstate of the $(2L)^d$ sample is a linear combination of the eigenstates of the L^d samples, and the amount of admixture depends on the overlap integral and the energy denominator. The energy denominator is typically the spacing δW between the energy levels in the L^d sample, i.e., $\delta W = (N_0 L^d)^{-1}$ where N_0 is the density of states. To estimate the overlap integral, Thouless observed that if a given L^d sample is repeated in one direction to form an infinite periodic chain, the individual eigenvalue will broaden out to form a band, and the bandwidth will be a good estimate of the overlap integral. The bandwidth just corresponds to the variation in energy ΔE of the eigenstate of the L^d sample subject to periodic or antiperiodic boundary conditions. In particular, if the eigenstate is localized, ΔE will be insensitive to

the boundary condition and therefore exponentially small. In that case $\Delta E / \delta W$ is exponentially small, so that the eigenstate of the $(2L)^d$ sample will be localized mainly in one of the L^d samples. On the other hand, if $\Delta E / \delta W$ is large, the eigenstate of the $(2L)^d$ sample will reside in all the L^d samples and is therefore extended. Thus sensitivity to boundary conditions, or the ratio $\Delta E / \delta W$, appears to be the single parameter that controls the nature of the eigenstate as the system doubles in size.

Thouless further noted that the conductance G (not conductivity) of the L^d sample is a dimensionless quantity when expressed in units of $e^2 / \hbar$. Introducing the dimensionless conductance

$$g = G / (e^2 / \hbar) , \qquad (2.1)$$

Thouless argued that $G / (e^2 / \hbar)$ is linearly related to $\Delta E / \delta W$. While it was subsequently shown (at least in one dimension) that g is proportional to $(\Delta E / \delta W)^2$ (Anderson and Lee, 1981), the essential point is that $G / (e^2 / \hbar)$ is a physically measurable quantity, directly related to $(\Delta E / \delta W)$, and is the single parameter that controls the behavior of the system as it doubles in size.

Wegner (1976) developed the scaling idea further by casting it in the language of the scaling theory of critical phenomena. He found that for a one-parameter scaling theory to hold, the metal-to-insulator transition must be continuous, such that

$$\sigma(T = 0) = (E - E_c)^\mu , \qquad (2.2)$$

where E_c is the mobility edge and μ is the critical exponent. It was also clear that $d = 2$ is a marginal dimension and therefore special because in two dimensions the conductivity has the same dimension as the conductance. These ideas were developed further by combining the scaling idea with perturbation theory (Abrahams, Anderson, Licciardello, and Ramakrishnan, 1979) and will be discussed next.

B. Scaling theory

1. Introduction

In scaling theory one tries to understand localization by considering the behavior of the conductance g as a function of system size L, or of other scale variables. We first describe this idea qualitatively and then construct the scaling function $\beta(g) = d(\ln g) / d(\ln L)$ using asymptotic forms, perturbative corrections, etc. The predictions for conductivity behavior of disordered systems of different dimensionality are then discussed. Finally, assumptions and results of scaling theory are critically reviewed.

Consider an electron moving in a disordered medium. The phase of its wave function changes randomly. The distance over which it fluctuates by about 2π defines the mean free path l. Beyond l, the electron motion is not ballistic but is diffusive, so that, upon averaging over the impurity configurations, the averaged one-electron propagator $\langle G(\mathbf{r}) \rangle$ is $\sim \exp(- |\mathbf{r}| / l)$. The mean free path l is

the microscopic length scale of interest in the localization problem. It is the lower length cutoff for diffusive motion. The conductance g_0 at this length scale is a microscopic measure of disorder, being small if disorder is large, and conversely. We now show that $g(L)$ has two very different asymptotic forms for $L \gg l$ depending on the degree of microscopic disorder.

When the random potential is small, or the scattering concentration low, the electron wave function is extended and is nearly plane-wave-like. The mean free path l between collisions is large compared to atomic spacing or to the Fermi wavelength k_F^{-1}. Conventional transport theory, which relies on weak scattering, i.e., $(k_F l)^{-1} \ll 1$, as an expansion parameter, leads to a conductivity $\sigma = ne^2 \tau / m = ne^2 l / \hbar k_F$, as given in Eq. (1.3). Here n is the electron density, and $\tau = l / v_F$ is the relaxation time. Equation (1.3) is correct to leading order in $(k_F l)^{-1}$. The conductivity σ is an intensive quantity, independent of scale size L, provided the system is large enough to have a well-defined mean free path, i.e., provided $L \gg l$. The conductance of a bigger piece of metal is given by Ohm's law, which for a d-dimensional hypercube of linear extent $L \gg l$ states that

$$g(L) = \sigma L^{d-2} . \qquad (2.3)$$

If states near the Fermi energy ε_F are localized, however, dc transport occurs by an electron's hopping from an occupied state to an unoccupied state of nearly the same energy. As mentioned in the Introduction, localized states very close in energy are very far apart in space, so that the hopping matrix element between them is exponentially small, the relevant length scale being the localization length ξ. The localization length is in general larger than the mean free path l. One expects that in this regime, for $L \gg \xi$,

$$g(L) \propto \exp(-L / \xi) . \qquad (2.4)$$

This is clearly a very non-Ohmic scale dependence.

For a particular disorder, $g(L)$ evolves smoothly from g_0 as L increases beyond l, going over finally to either of the forms Eq. (2.3) or Eq. (2.4). The limiting behavior reached depends on microscopic disorder, i.e., the conductance g_0 at scale l, as well as on dimensionality. The latter is obviously significant, since for example, in one dimension, all states are known to be localized with localization length ξ of the order of the mean free path l (Mott and Twose, 1961). In this case there is no sizable length scale over which Ohm's-law behavior [Eq. (2.3)] is valid, and the only relevant asymptotic form is Eq. (2.4). The objective of a scaling theory is to describe how $g(L)$ changes with L for all $L > l$, in various dimensions.

Abrahams, Anderson, Licciardello, and Ramakrishnan (1979) argued that the logarithmic derivative $\beta(g) = d \ln g / (d \ln L) = (L / g)(dg / dL)$ is a function of conductance g alone. The idea is that the change in effective disorder when the system becomes a little bigger is determined by its value at the previous length scale, the only measure of this effective disorder being the conductance.

Such a picture is suggested by thinking about the energy levels of a block $(2L)^d$ in terms of the energy levels of the 2^d constituent blocks with dimension L. The energy levels of the former differ from those of the latter, due for example to interfacial perturbation caused by putting the blocks together. If this is the principal effect, it then appears plausible that sensitivity to boundary perturbations for the larger block, i.e., the Thouless ratio $(\Delta E /\delta W)_{2L}$, is a function of $(\Delta E /\delta W)_L$. Using the association of this ratio with conductance, a scaling behavior for the latter is indicated. While the initial formulation of the scaling theory relied on this suggestive though tenuous line of argument, it has received further support from perturbation theory (Anderson, Abrahams, and Ramakrishnan, 1979; Abrahams and Ramakrishnan, 1980; Gor'kov, Larkin, and Khmel'nitskii, 1979) as well as the renormalization group analysis of an equivalent field theory (Wegner, 1979).

2. Scaling function

We now discuss the scaling function $\beta(g)$ for various regimes.

a. Large conductance $g \gg g_c$

Here g_c is a characteristic dimensionless conductance that turns out to be of order π^{-2}. In this regime, Ohm's law, i.e., Eq. (2.3), is valid for the conductance. This leads to the asymptotic form

$$\beta(g)=(d-2) \tag{2.5}$$

for $g \gg g_c$. In two dimensions, $\beta(g)$ tends to zero; this reflects the fact that g and σ have the same physical dimension for a planar system, i.e., the conductance of a square does not depend on its size.

b. Small conductance $g \ll g_c$

Electronic states are localized, so that the scale dependence of $g(L)$ is described by Eq. (2.4). This means that $\beta(g)$ is given by

$$\beta(g)=\ln(g/g_c) \tag{2.6}$$

independent of dimensionality. $\beta(g)$ is negative, corresponding to a decrease in g as length scale increases.

c. Perturbative regime

For weak disorder, i.e., for $(k_F l)^{-1} \ll 1$, it is possible to calculate corrections to the Boltzmann transport theory result for σ using diagrammatic perturbation theory (Sec. II.C). It turns out that to higher order in $(k_F l)^{-1}$ there are significant scale-dependent corrections to conductivity arising from singular backscattering. These terms contribute a correction going as g^{-1}, so that for large g

$$\beta(g)=(d-2)-a/g . \tag{2.7}$$

For an electron gas (Fermi gas with spin-$\frac{1}{2}$ particles), $a=g_c=(\pi^{-2})$. $\beta(g)$ is thus always less than its Ohm's-law value, so that conduction in a disordered electronic system is never quite Ohmic. The conductance always increases more slowly with scale size than is suggested by Eq. (2.3).

The scaling curve can be constructed using the form Eq. (2.7) for large g and Eq. (2.6) for small g, and the assumptions that $\beta(g)$ is continuous and monotonic. $\beta(g)$ is expected to be continuous because it describes how the conductance of a *finite* system evolves as a function of scale size. As g decreases, one tends to a more localized behavior, so the conductance should decrease more strongly with increasing scale size. The monotonic behavior appears quite plausible, even though we shall later encounter exceptions to this. The scaling functions $\beta(g)$ constructed this way for $d=3$, 2, and 1 are shown in Fig. 3 as a function of conductance g. Their implications for conductivity behavior of disordered systems at $T=0$ are discussed below.

3. Consequences of scaling theory

a. Three dimensions

Since $\beta(g)$ starts at a positive value equal to unity, moves downwards for large g, and is negative for very small g (localized regime), it must pass through zero at a certain conductance, say g_3. Suppose the state of microscopic disorder in the system is such that the conductance g_0 at the microscopic cutoff length l is larger than g_3. One thus starts somewhere on the positive part of the β curve, the exact location depending on the value of g_0. On slightly increasing the length scale from l, g increases, and one moves up a little on the $\beta(g)$ curve. Continuing this, at asymptotically large length scales the limit $\beta(g)=1$ is reached, i.e., the system is an Ohm's-law con-
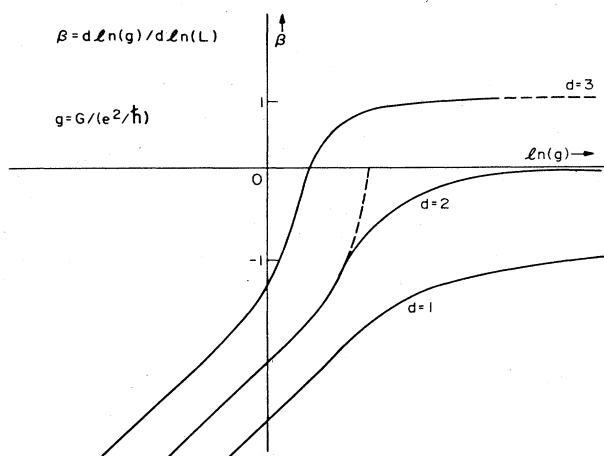

FIG. 3. The scaling function $\beta(g)$ vs the dimensionless conductance g for different dimensions. If $\sigma_{\min}$ exists in 2D, the behavior of β is shown by the dashed lines.

ductor. On the other hand, for $g_0 < g_3$, $\beta(g)$ is negative. Increasing the length scale from l decreases g, so that one moves downwards on the scaling curve. At large enough length scales, $\beta(g)$ corresponds to the scaling function for localized states. A system with conductance $g_0 > g_3$ at the microscopic length scale l is a metal, while one with $g_0 < g_3$ is an insulator.

Critical disorder is characterized by a conductance g_3 at length scale l. The critical point $\beta(g) = 0$ is an unstable fixed point, i.e., for small departures from it, the scaling of conductance takes one asymptotically to qualitatively different regimes. The scaling trajectories move away from the point $\beta(g) = 0$ which marks a change of regime. We identify it with the mobility edge in the following sense. Since current is transported by electrons with Fermi energy ε_F, g_0 refers to the conductance at this energy. Now, for example if disorder is kept fixed and ε_F is varied, g_0 will change smoothly and, for some ε_F equal to the mobility edge energy ε_c, will coincide with the critical value g_3. Thus the conductivity behavior of the system for small deviations of ε_F from ε_c can be described by considering what happens when g_0 deviates proportionally from g_3. One has

$$g_0(\varepsilon_F) = g_0(\varepsilon_c) + (\varepsilon_F - \varepsilon_c)g_0'$$

or

$$(g_0 - g_3) = (\varepsilon_F - \varepsilon_c)g_0' , \tag{2.8}$$

where $g_0' = (dg_0/d\varepsilon_F)$ at $\varepsilon_F = \varepsilon_c$. Thus to study critical behavior near the mobility edge, one considers predictions of scaling theory for g_0 close to g_3. This is done below.

Near the fixed point $\beta(g) = 0$ suppose $\beta(g)$ has a slope $(1/\nu)$, so that

$$\beta(g) = \frac{1}{\nu}\left[\frac{g - g_3}{g_3}\right] = \frac{\delta g}{\nu} \tag{2.9}$$

for $\delta g \ll 1$. Consider first the case of $\delta g > 0$. Using Eq. (2.9) and integrating from $g = g_0$ at l out to $\beta(g) \simeq 1$ at large length scales L, we find that $g(L) = \sigma L$ where

$$\sigma = (Ag_3/l)(\delta g)^\nu \tag{2.10}$$

and where A is a constant of order unity. The conductivity is reduced from its microscopic value (g_0/l) $\simeq (Ag_3/l)$, and there is no minimum metallic conductivity. On approaching the mobility edge, the conductivity goes to zero with a universal exponent ν. The exponent ν can be calculated by perturbation theory, which is accurate for $(2+\varepsilon)$ spatial dimensions with ε small. The result is that $\nu = \varepsilon^{-1}$, so that, extending this to three dimensions, the critical exponent ν is unity. [We discuss below the theory for $(2+\varepsilon)$ dimensions.]

A diverging correlation length ξ can be identified from Eq. (2.10) by writing the conductivity as

$$\sigma = g_3/\xi . \tag{2.11a}$$

Clearly,

$$\xi = \left[\frac{l}{A}\right](\delta g)^{-\nu} \tag{2.11b}$$

so that ξ diverges with the conductivity exponent. In terms of the scaling function, starting from the critical regime $\beta(g) \gtrsim 0$ at length scale l, one gets to the Ohmic regime $\beta(g) \approx d - 2$ on scaling out to a length of order ξ. Conduction is characteristically non-Ohmic for length scales L less than ξ, and Ohmic with a conductivity $\sigma = (g_3/\xi)$ [Eq. (2.11a)] for L larger than ξ. ξ is thus the correlation length as in other critical phenomena.

On the localized side of the fixed point, $\beta(g) < 0$. Starting from the critical regime with small negative δg, and using the form Eq. (2.9) for $\beta(g)$, one crosses at large enough length scales to the exponentially localized regime, where we find

$$g(L) = g_3 \exp(-B \mid \delta g \mid ^\nu L/l) , \tag{2.12}$$

where B is a constant of order unity. The system is thus characterized by a localization length

$$\xi_{\text{loc}} = (l/B)(\delta g)^{-\nu} . \tag{2.13}$$

The localization length diverges at the mobility edge. Further, as in any critical phenomenon with a single correlation length, the conductivity length ξ on the metallic side and the localization length on the insulating side diverge with the same exponent ν.

b. 2+ε dimensions

The perturbative β function describes the localization transition accurately in $(2+\varepsilon)$ dimensions for $\varepsilon \ll 1$. The Ohmic limit given by Eq. (2.3) depends on dimensionality, so that

$$\beta(g) = \varepsilon - \frac{a}{g} . \tag{2.14}$$

The critical disorder [$\beta(g) = 0$] occurs at

$$g^*_{2+\varepsilon} = (a/\varepsilon) , \tag{2.14'}$$

which is a larger conductance for small ε. Since $g^*_{2+\varepsilon}$ is large, the perturbative form Eq. (2.14) is accurate in the critical regime, the slope of $\beta(g)$ there being $\nu^{-1} = \varepsilon$. On the metallic side, using Eq. (2.14) and integrating out to $\beta(g) \simeq \varepsilon$, and combining with Eq. (2.3), one finds that

$$\sigma = (Ag^*_{2+\varepsilon}/l^\varepsilon)(\delta g)^\mu , \tag{2.15}$$

where $\mu = \nu\varepsilon$. To lower order in ε, we find $\mu = 1$. Equation (2.15) corresponds to a critical conductivity of the form

$$\sigma = Ag^*_{2+\varepsilon}/\xi^\varepsilon , \tag{2.16a}$$

where the correlation length ξ diverges as

$$\xi = l(\delta g)^{-\nu} . \tag{2.16b}$$

Extending these results to $d = 3$, i.e., $\varepsilon = 1$, implies $\nu = 1$. However, since $\varepsilon = 1$ is not small, estimates of g_3 and of ν using a $2 + \varepsilon$ expansion can only be approximate.

c. Two dimensions

In two dimensions, $\beta(g) < 0$ always, so that, at large enough length scales, only localized behavior is possible. If the system is weakly disordered, with large conductance g_0 at length l, one starts at the point $g = g_0$ on the scaling curve and moves downwards along it as length scale increases, until asymptotically $\beta(g) \sim \ln(g/g_c)$. Thus one predicts that there are no truly extended states in two dimensions. At a large enough length scale, even for small microscopic disorder, electronic states are localized. An estimate of the localization length can be made as follows. On integrating the perturbation theory result Eq. (2.7), i.e.,

$$(d \ln g / d \ln L) = -a/g \tag{2.17a}$$

between length scales l and L, one has

$$g(L) = g_0 - \frac{e^2}{\hbar \pi^2} \ln \left[\frac{L}{l} \right] , \tag{2.17b}$$

where g_0 is the conductance at the lower cutoff l. In conventional transport theory, $g_0 = (e^2/2\pi\hbar)(k_F l)$. The conductance decreases logarithmically with size, and this scale-dependent reduction becomes comparable to the Boltzmann conductivity for $L = \xi_{\text{loc}}^{(2)}$, which is

$$\xi_{\text{loc}}^{(2)} \simeq l \exp(k\pi^2 g_0/e^2) = l \exp \left[\frac{\pi}{2} k_F l \right] . \tag{2.18}$$

This is the perturbative estimate of the localization length in two dimensions. It depends exponentially on the mean free path, and consequently the localization effects are difficult to observe experimentally for weak disorder. Another striking consequence of the scale dependence equation (2.17b) for g is that a two-dimensional system is non-Ohmic at all length scales. We shall see later that this leads to characteristic "nonmetallic" resistance increases as temperature decreases.

d. One dimension

In one dimension, $\beta(g)$ is less than unity and decreases further with decreasing conductance, so that one rapidly goes over into the localized regime. In perturbation theory, using the form Eq. (2.7) for $\beta(g)$, one finds that in strictly one dimension, the scale-dependence corrections to g become comparable to the Boltzmann transport term at a length scale of order l. This is the perturbative estimate of the localization length. As mentioned earlier, in one dimension, all states are known to be localized due to repeated backscattering (Mott and Twose, 1961; Landauer, 1957,1970), and the localization length is indeed of the order of the backscattering mean free path. A very detailed and illuminating analysis of the conductance of a one-dimensional random system at all length scales has been made recently by Anderson, Thouless, Abrahams, and Fisher (1980) using the Landauer connection (Landauer, 1970) between the conductance and the scattering

properties of the system.

These authors emphasized that since the resistance of a given sample varies exponentially with its length, its value can fluctuate wildly and its distribution function becomes increasingly broad as the length increases. Consequently, the average resistance becomes very different from the typical resistance. Thus the choice of a scaling variable requires some care. Anderson *et al.* showed that the quantity $\ln(1 + g^{-1})$ has a well-behaved distribution as the length goes to infinity. This quantity has an additive mean when two sections of wires are joined together, so that $\langle \ln(1 + g^{-1}) \rangle = \alpha L$ and α has the natural interpretation of the inverse localization length. Furthermore, by writing $g^{-1} = \exp(\alpha L - 1)$, one can compute the β function by simple differentiation to obtain

$$\frac{d \ln g}{d \ln L} = -(1 + g)\ln(1 + g^{-1})$$

$$= -1 - \frac{1}{2g} + \frac{1}{6g^2} + \cdots . \tag{2.19}$$

This has the form postulated in Eq. (2.14), but the coefficient of the g^{-1} term is different from that calculated in a $(2 + \varepsilon)$-dimension expansion using diagrammatic techniques (see Sec. II.C). The difference probably arises from the fact that, in perturbative calculations, it is always the average conductance that is being calculated, which is not the same as the typical conductance.

The picture that emerges is that the conductance in one dimension has a very skewed distribution, and that there exists a small but significant probability of finding conductance close to unity. These highly unlikely conductances will dominate the mean conductance. The important question remains: Under what physical conditions is the measured conductance the mean conductance or the typical conductance? The discussion is brought into sharp focus by the work of Lifshitz and Kirpichenkov (1979) and Azbel (1983), who point out that resonance tunneling is a specific mechanism for producing large conductances. Resonance tunneling requires that the incident energy be resonant with a localized state that happens to be localized near the center of the sample, in which case the transmission coefficient is of order unity [as opposed to $\exp(-2L/\xi)$]. The optical analog of such resonances was observed by Schultz at La Jolla in the late 1960s in experiments in microwave transmission through a long waveguide with randomly placed dielectric slabs inside. DiVincenzo and Azbel (1983) further considered the possibility of thermal activation into these resonance energies. However the effect of inelastic processes present at finite temperature on the resonant tunneling process itself has to be considered (Stone and Lee, 1985). Many interesting questions must be addressed before one can satisfactorily interpret a number of intriguing experimental observations (Fowler, Hartstein, and Webb, 1982; Kwasnick, Kastner, Melngailis, and Lee, 1984; Lee, 1984).

e. Minimum metallic conductivity

The scaling analysis of localization does not bear out the Mott minimum-metallic-conductivity hypothesis (Mott, 1973; Mott and Davis, 1979). In two dimensions there is no metallic state, while in three dimensions the minimum conductivity is zero. However, in three dimensions there is a minimum metallic conductance g_3. This corresponds to a conductivity $(g_3/l) \simeq \sigma_{min}$ at the lower cutoff length scale. Thus σ_{min} marks, microscopically, a change of regime. Experiments performed at relatively high temperatures may never probe length scales much larger than l, and a rapid drop in the conductivity when it falls below σ_{min} may be observed. A true test of the σ_{min} concept requires very-low-temperature data and careful extrapolation to zero temperatures. Such data have become available in the past few years and will be reviewed in Sec. VI. The Mott σ_{min} idea treats quantum interference effects asymmetrically. They are assumed to be absent in the metallic phase, even close to critical disorder, but for disorder slightly larger than critical, they lead to localization with a diverging localization length. In the scaling theory, these effects lead to identically diverging correlation lengths as the transition is approached from either side. If indeed there is a σ_{min} for a metal but a diverging localization length on the insulating side, then within the framework of the one-parameter scaling theory, the corresponding $\beta(g)$ in two dimensions has the form shown in Fig. 3 (dotted lines). In three dimensions a discontinuous jump of the β function will be required. This appears quite implausible, and is not supported by theory. A number of recent experiments, carried out at low enough temperatures actually to probe large effective length scales, confirm in detail the predictions of scaling theory (Sec. VI), interaction effects being equally significant (Sec. III).

C. Perturbation theory

The conductivity of a noninteracting electron gas weakly scattered by rigid random impurities can be calculated from first principles (Kohn and Luttinger, 1956). The conductivity can be expressed as a current-current correlation function (Kubo formula), and this can be evaluated using Feynman diagrams to represent the scattering process (Edwards, 1958; see also Abrikosov, Gor'kov, and Dzhyaloshinskii, 1969). The Kubo formula for static conductivity σ is

$$\sigma_{\gamma\gamma'}(0) = \frac{e^2\hbar}{2\pi L^d} \sum_{p,p'} \left[\frac{p_\gamma}{m}\right] G_{pp'}^{\Pi+-} \left[\frac{p'_{\gamma'}}{m}\right], \qquad (2.20)$$

where $G_{pp'}^{\Pi+-}$ is the two-particle Green's function describing the propagation of an electron-hole excitation at the Fermi level from a momentum state p to a momentum state p'. Three typical diagrams for $G_{pp'}^{\Pi+-}$ are shown in Fig. 4.

The upper electron line has a frequency $(0 + i\eta)$ with respect to the Fermi level with η a positive infinitesimal,
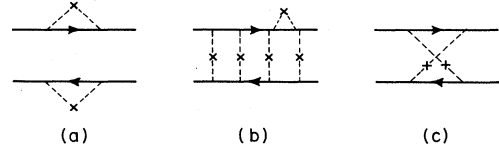

FIG. 4. Examples of diagrams for the particle-hole propagator. Dashed line with cross denotes impurity scattering.

i.e., it is retarded, while the lower electron line has a frequency $(0 - i\eta)$. The cross represents an impurity, and the dotted line represents scattering. On averaging over random positions of impurities, one finds that electron momentum is conserved. Diagrams of the type of Fig. 4(a) modify separately the propagation of the electron and the hole, while Fig. 4(b) describes the interference between their propagation due to impurity scattering. For a zero-range potential, contributions of diagrams in which the scattering [Fig. 4(b)] occurs once or is repeated vanish on averaging over electron momentum, and one has, from diagrams of the type Fig. 4(a)

$$\sigma_{\gamma\gamma'}(0) = \delta_{\gamma\gamma'} \frac{e^2}{2\pi L^d} \sum_p \left[\frac{p_\gamma}{m}\right]^2 G_p(0^+)G_p(0^-) \qquad (2.21a)$$

$$= \frac{ne^2\tau}{m}\delta_{\gamma\gamma'} \qquad (2.21b)$$

where $G_p(0^+)$ $[G_p(0^-)]$ is the retarded (advanced) electron propagator and

$$\frac{\hbar}{\tau} = 2\pi \mid v \mid^2 \rho(\varepsilon_F)n_i \qquad (2.22)$$

to lowest order in the scattering potential v. Here n_i is the density of impurities and $\rho(\varepsilon_F)$ is the density of states.

Langer and Neal (1966), in attempting to go beyond the lowest-order perturbation theory in the impurity density, noticed that the diagram Fig. 4(c), for example, contributes (in three dimensions) a term of order $n_i^3 \ln(n_i)$ to the conductivity. They also noted that a maximally crossed diagram of arbitrary order in n_i, e.g., Fig. 5(a), also contributes a term $n_i^3 \ln n_i$. This clearly means that the entire infinite set of such processes has to be considered together (Anderson, Abrahams, and Ramakrishnan, 1979; Abrahams and Ramakrishnan, 1980; Gor'kov, Larkin, and Khmel'nitskii, 1979). On summing the geometric series of maximally crossed diagrams, the amplitude for this process is seen to be

$$W(\mathbf{p},\mathbf{p}') = \frac{d(n_i v^2)^2}{(\mathbf{p}+\mathbf{p}')^2 v_F^2 \tau^2}. \qquad (2.23)$$
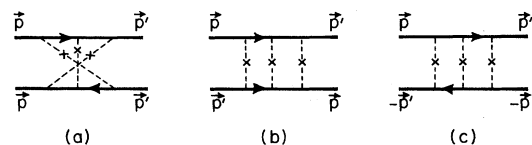

FIG. 5. (a) Example of maximally crossed diagram. (b) Redrawing of (a). (c) A particle-hole propagator derived from (b) using time-reversal symmetry.

We note that the amplitude is divergent for $p+p' \approx 0$, which can be interpreted as a singular backscattering due to the random potential. This is an interference effect, of higher order in the random potential or impurity concentration, and is the crucial localizing process. To understand Eq. (2.23) it is instructive to turn the hole line around in Fig. 5(a); we thereby obtain the ladder diagram contribution to the particle-particle propagation shown in Fig. 5(b). If time-reversal symmetry is satisfied, the value of this diagram is unchanged if the electron line is turned into a hole line with opposite momentum, as shown in Fig. 5(c). This diagram is more familiar because it describes density fluctuations. Due to the conservation of particles, density fluctuations are diffusive, and Fig. 5(c) is known to be proportional to the diffusive form $(-i\omega + D_0 q^2)^{-1}$ where $D_0 = v_F \tau/d$ is the diffusion constant, ω is the frequency, and q is the momentum of the density fluctuation. In Fig. 5(c), $q = p + p'$, $\omega \approx 0$, thus explaining the form of the amplitude given in Eq. (2.23). The connection between the diffusion pole and the particle-particle propagation is discussed in detail by Vollhardt and Wölfle (1980b). Because of this connection, we expect that even in a higher-order calculation, the form Eq. (2.23) will survive, with bare coefficients corrected. Further, perturbations that break time-reversal invariance, e.g., magnetic fields or magnetic impurities, will affect the scaling behavior of conductivity (see Sec. II.E).

It is clear that the singular backscattering Eq. (2.23) will reduce the conductivity. Including only this process, the conductivity σ is

$$\sigma = \frac{ne^2\tau}{m} - \frac{2e^2}{\hbar\pi}\frac{1}{L^d}\sum_Q \left[\frac{1}{Q^2}\right].\tag{2.24}$$

The lower Q cutoff for a system of length L is $\sim(1/L)$, the upper cutoff being $Q_c \sim l^{-1}$. One thus has the following forms for the scale-dependent conductivity:

$$\sigma_{3D}(L) = \sigma_0 - \frac{e^2}{\hbar\pi^3}\left[\frac{1}{l} - \frac{1}{L}\right],\tag{2.25a}$$

$$\sigma_{2D}(L) = \sigma_0 - \frac{e^2}{\hbar\pi^2}\ln\left[\frac{L}{l}\right],\tag{2.25b}$$

$$\sigma_{1D}(L) = \sigma_0 - \frac{e^2}{\hbar\pi}(L - l).\tag{2.25c}$$

The corresponding β function has the form

$$\beta(g) = (d-2) - \frac{a}{g}\tag{2.26}$$

for all dimensions d, with g defined as $\sigma(L\pi)^{d-2}$ $[\sigma(L\pi)^\varepsilon$ for $d = 2 + \varepsilon]$. In order for scaling to hold, the next-order term in $(k_F l)^{-1}$ should not have a part going as $\ln^2$, i.e., for two dimensions, in an expansion

$$\sigma_{2D}(L) = \sigma_0[1 - c_1(k_F l)^{-1}\ln(L/l)$$

$$+ c_2(k_F l)^{-2}\ln^2(L/l) + \cdots]\tag{2.27}$$

the coefficient c_2 should vanish. Otherwise $d\ln g/d\ln L$ will depend explicitly on L. That this is so was first shown by Gor'kov, Larkin, and Khmel'nitskii (1979).

We see explicitly from Eq. (2.25) that, due to quantum interference, there are characteristic and significant deviations from Ohm's law $[\sigma(L) = \sigma_0]$ in disordered electron systems (at absolute zero). The size of these deviations depends on the intrinsic disorder characterized by the Boltzmann transport conductivity σ_0, or equivalently by the dimensionless ratio $(k_F l)$. In one and two dimensions, perturbation theory fails at sufficiently large length scales because the reduction in conductivity due to backscattering grows as L increases. As discussed earlier, the length scales are $\xi_{loc}^{1D} \simeq \pi l$ and $\xi_{loc}^{2D} \simeq l\exp(\pi k_F l/2)$. These are the perturbative estimates of the localization length. In three dimensions, the backscattering reduction is of relative order $(k_F l)^{-2}$ for an infinite system.

Thus, in three dimensions, conventional transport theory is accurate for weak disorder. The leading correction goes as $(1/L)$. This is small but significant, since statistical finite-size fluctuations in the conductivity go as $L^{-3/2}$ and are qualitatively smaller. Mott (1976,1981) has discussed the effect of statistical fluctuations on the conductivity near the mobility edge, in a model where the localization length ξ diverges with an exponent ν on the localized side, while on the metallic side there is minimum metallic conductivity. Suppose a sample is just on the metallic side of the mobility edge ε_c. We have to consider the possibility that statistical fluctuations produce volumes in which the disorder is stronger than average, so that states are localized within that volume. If this happens with high probability, the mobility edge will be smeared. Statistical fluctuations within a volume L^d give rise to local disorder, which can be represented as a local energy fluctuation $\delta\varepsilon = (\varepsilon - \varepsilon_c)/\varepsilon_0$, and typically $\delta\varepsilon \sim L^{-d/2}$. Such fluctuations can produce localized states with localization length $\xi \sim |\delta\varepsilon|^{-\nu}$. Consistency requires $\xi < L$, or $\nu < 2/d$, leading Mott to suggest that if $\nu < 2/d$, the conductivity jump will be smeared into a continuous transition. These arguments are very similar to the Harris criterion (Harris, 1974), which estimates the effect of statistical fluctuations due to randomness on the critical point. Thus, in order that statistical fluctuations be irrelevant, one needs $\nu d/2 > 1$, i.e., $\nu > \frac{2}{3}$ in three dimensions. The perturbative estimate of $\nu = (d-2)^{-1} = 1$ in three dimensions is consistent with this condition. It is interesting to note that, in perturbation theory, since $\nu = (d-2)^{-1}$, finite size fluctuation effects can become important for $d > 4$. This suggests that there could be a change of regime at $d_c = 4$. Suggestions that the upper critical dimension for localization d_c is 4 have been made by Kunz and Souillard (1983).

In the discussion above, disorder is assumed to be on the scale of an electron wavelength, i.e., on a scale of order k_F. A classical percolation model is a widely used idealization for systems in which inhomogeneity is on a much larger length scale. For example, if pieces of metal with linear dimension $b \gg k_F^{-1}$ are randomly removed so that a fraction p remains, the system would go insulating for $p < p_c$, where p_c depends on dimensionality, etc. However, even in the classically metallic regime, $p > p_c$, quantum interference can lead to localization, depending

on p and on the resistance R_b at length scale. An approximate phase diagram in the (R_b,p) plane for $d=3$ has been constructed by Shapiro (1982a,1983) using a real-space renormalization group method (see also Khmel'nitskii, 1981). There is a quantum insulator domain for high R_b, bounded by a curve terminating at the classical percolation limit $R_b=0$, $p=p_c$. The form of the percolation-localization crossover has been discussed by Shapiro. A more restrictive model is the Anderson tight-binding model, in which a fraction p of the bonds are randomly removed (quantum percolation model). Here, there is no well-defined classical regime, quantum interference is the dominant effect, and the localization transition occurs at $p_q=p_{loc}(d,n)>p_c(d)$, where n is the electron density (or band-filling factor). This model has been investigated numerically by Raghavan and Mattis (1981) and analytically by Shapiro, Aharony, and Harris (1982). These authors find that in general $p_q>p_c$, the difference decreasing as dimensionality d increases. In two dimensions, p_q seems to be close to unity, consistent with the absence of a metallic state for a random two-dimensional system. The quantum percolation model on a lattice has an unusual feature, first pointed out by Kirkpatrick and Eggarter (1972), namely that localized and extended states can coexist. This is because some localized states exist on subclusters of the infinite cluster and are therefore orthogonal to extended states. The lack of mixing clearly requires the special symmetry of a lattice, and is not expected to hold in the general percolation-localization problem.

D. Inelastic cutoffs of scaling

The scaling theory discussed so far is applicable at zero temperature and for finite length scales. On the other hand, experiments are carried out at nonzero temperatures and usually for samples of macroscopic size. These two limits need to be connected. Thouless (1977) has argued that inelastic scattering introduces random fluctuations in the time evolution of an electronic state. Such fluctuations limit quantum interference necessary for localization. Suppose an electron in a particular energy eigenstate of the static random potential has a lifetime τ_{in}. If $\tau_{in}\gg\tau$, the elastic scattering time, the electron diffuses a distance

$$L_{Th}=(D\tau_{in})^{1/2} \qquad (2.28)$$

between dephasing inelastic collisions. Here $D=(v_F^2\tau/d)$ is the diffusion constant. Scale-dependent quantum interference or localization effects are cut off beyond L_{Th}. Thus the $T=0$ theory with $L_{Th}(T)$ as cutoff describes the localization effect on conductance at a nonzero temperature T.

The above consideration can also be cast in the form of an energy uncertainty relation. The single-particle energy levels of a block L^d have an energy uncertainty $\Delta E_{in}\simeq(\hbar/\tau_{in})$. If this is larger than the boundary perturbation shift ΔE (Sec. II.A), the block is effectively uncoupled from other such adjacent blocks, and consequently its conductivity is scale independent. This criterion also

leads to Eq. (2.28).

We have implicitly assumed $\tau_{in}(E,T)$ to be the lifetime of an exact eigenstate of the random potential, with energy E and at temperature T. Altshuler, Aronov, and Khmel'nitskii (1981,1982) have emphasized that under certain conditions the inelastic scattering time is not the appropriate time to enter Eq. (2.28). If the energy change ΔE during an individual collision is small compared with τ_{in}^{-1}, the phase change $\Delta E\tau_{in}$ after time τ_{in} is small compared with 2π. It will take many collisions for the phase to drift by 2π. This time is denoted τ_φ and estimated by Altshuler *et al.* to be $\tau_\varphi\sim(\Delta E\tau_{in})^{-2/3}\tau_{in}$. For electron-phonon scattering, $\Delta E\tau_{in}$ is usually not small. However, in one and two dimensions, for electron-electron scattering or for electrons interacting with fluctuating electromagnetic fields, $\Delta E\tau_{in}$ can become small, and special care must be taken. The question is discussed further in Sec. III.

The inelastic scattering time depends on temperature, increasing as temperature decreases. Suppose $\tau_i\propto T^{-p}$ where p is an index depending on scattering mechanism, dimensionality, etc. We then have $L_{Th}=aT^{-p/2}$, so that scale-dependent effects will be more evident at lower temperatures. For example, with this as the length cutoff for the scale-dependent conductivity $\sigma(L)$ of Eq. (2.25), we have

$$\sigma_{3D}(T)=\sigma_0+\frac{e^2}{\hbar\pi^3}\frac{1}{a}T^{p/2}, \qquad (2.29a)$$

$$\sigma_{2D}(T)=\sigma_0+\frac{p}{2}\frac{e^2}{\hbar\pi^2}\ln\left[\frac{T}{T_0}\right], \qquad (2.29b)$$

$$\sigma_{1D}(T)=\sigma_0-\frac{ae^2}{\hbar\pi}T^{-p/2}. \qquad (2.29c)$$

We notice that the conductivity decreases with decreasing temperature. This is a signature of localization; as temperature decreases, the relevant scale size L_{Th} over which quantum interference is effective increases, so that localization behavior is progressively evident. The temperature dependences are characteristically different in different dimensions.

In Fig. 6 we show the first experimental demonstration of a $\ln T$ rise in resistivity in thin metallic films (Dolan and Osheroff, 1979). The order of magnitude of the effect is consistent with Eq. (2.29b). However, it is now known that electron-electron interactions also give rise to a $\ln T$ correction to the conductivity very similar to Eq. (2.29b) (see Sec. III). A fuller discussion of the experimental situation is given in Sec. VI.

The effective dimensionality of the system is the number of dimensions for which the system size is larger than the inelastic length. For example, a wire of cylindrical cross section (radius a) is three dimensional if $L_{Th}<a$; it is one dimensional in the opposite limit. Since L_{Th} is a function of temperature, one can cross over from three- to one-dimensional behavior for a given wire on cooling.

The temperature dependence of the inelastic rate due to electron-phonon collisions depends on whether the

FIG. 7. (a) Maximally crossed diagram which corrects the conductivity, drawn in a spatial representation. (b) A redrawing of (a) to emphasize the particle-particle channel.
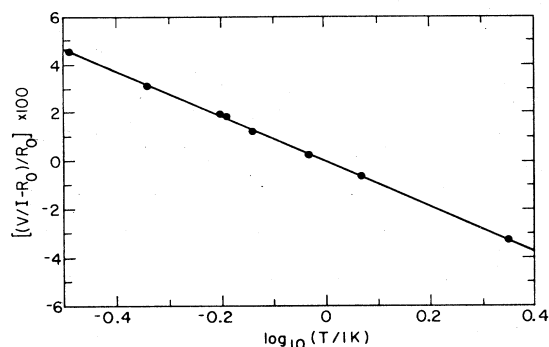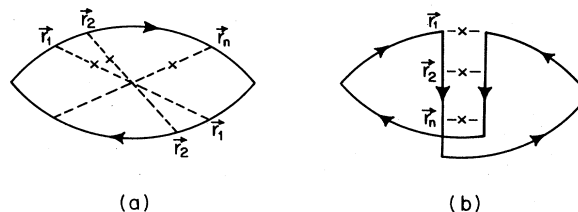

FIG. 6. Resistivity rise plotted vs $\ln T$ for PdAu film (from Dolan and Osheroff, 1979).

thermal phonon wavelengths λ_{Th} are larger than one or more dimensions of the system (phonon dimensionality) and on whether $\lambda_{Th} > l$ (dirty limit) (Thouless, 1977). The former affects the phase space available, while in the latter limit momentum is not conserved in an electron-phonon collision. For example, in a wire of diameter a such that $\lambda_{Th} \gg a$, and $\lambda_{Th} \gg l$, $\tau_i^{ph} \propto T^{-2}$. We note that there is no diffusion enhancement of these rates, i.e., no enhancement due to diffusive electron motion (Schmid, 1973; Sec. III.F). In contrast, as discussed in Sec. III.B., quasiparticle decay rates due to electron-electron collisions are greatly enhanced by disorder. This means that L_{Th}, and therefore the size of localization effects, is greatly reduced.

In experiments on thin films, non-Ohmic behavior is observed at low temperature, so that I/V contains a $\ln V$ component. This can be explained as heating of the electron gas (Anderson, Abrahams, and Ramakrishnan, 1979). The electron-phonon time τ_i^{ph} becomes very long at low temperatures, so that the electron gas gains an average energy $\sim eVL_p$ between collisions with phonons, where $L_p = (D\tau_i^{ph})^{1/2}$. The electron gas temperature then becomes eVL_p, which could be larger than T, thus explaining the $\ln V$ instead of $\ln T$ behavior. The electron-phonon scattering rate can be extracted from the non-Ohmic behavior and is in agreement with theoretical expectations. A number of authors have suggested that a dc electric field can provide a cutoff length without heating the electron gas (Tsuzuki, 1981; Mott and Kaveh, 1981). This conclusion has been disputed by Altshuler, Aronov, and Khmel'nitskii (1981), who show by explicit calculations that only an ac electric field can affect quantum interference. We return to this point later.

E. Relevant perturbations: magnetoresistance and spin-orbit scattering

We have seen in Sec. II.C that a particular set of scattering processes leads to a length-scale-dependent conductivity. It is useful to draw the diagram in spatial representation as in Fig. 7(a), from which we see that the important process is one in which the electron and hole

created by the electromagnetic field are scattered by the same impurities located at r_1 to r_n but in precisely the reverse order.

Furthermore, r_1 and r_n must be within a mean free path of each other because they are near where the particle-hole pairs were created. It is instructive to draw Fig. 7(a) in an alternative way, shown in Fig. 7(b), which shows the correlation between a pair of electrons (made up of the electron and a hole moving backwards in time) that are scattered by the same impurities in sequence, so that they experience the same potential in space, but at different times. This picture is very reminiscent of the theory of superconductivity, even though here we are discussing noninteracting electrons. It is not surprising, then, that perturbations that affect superconductivity (pair-breaking terms) will affect the localization diagram as well. The first example of this was worked out by Lee (1980), who showed that magnetic impurities destroy the coherence, so that on a length scale longer than $L_s = (D\tau_s)^{1/2}$, where τ_s is the spin-flip time, the conductivity is no longer length dependent. This effect apparently has to do with the destruction of time-reversal symmetry by the spin-flip Hamiltonian. A uniform magnetic field also destroys time-reversal symmetry and provides a length cutoff (Altshuler, Khmel'nitskii, Larkin, and Lee, 1980). This is because the electron pair acquires a phase $\Delta\varphi = 2 \int \mathbf{A} \cdot d\mathbf{l}$ upon completion of the contour which equals φ/φ_0, where φ is the enclosed magnetic flux and $\varphi_0 = hc/2e$ is the flux quantum. Since we must average over all possible contours, there will be destructive interference when the typical $\varphi/\varphi_0 \simeq 1$. This occurs when the Thouless length L_{Th} becomes comparable to the Landau orbit size, $L_H = (eH/\hbar c)^{-1/2}$. Since a magnetic field suppresses the localization effect, this picture always predicts a negative magnetoresistance. Furthermore, since L_{Th} can be quite large, the characteristic magnetic field can be very small, of the order of tens of gauss. In two dimensions the following formula for the magnetoresistance is obtained (Altshuler, Khmel'nitskii, Larkin, and Lee, 1980; Hikami, Larkin, and Nagaoka, 1980):

$$\sigma(H,T) - \sigma(0,T) = \frac{e^2}{2\pi^2\hbar}\left[\psi\left[\frac{1}{2} + \frac{1}{x}\right] + \ln x\right], \quad (2.30)$$

where ψ is the digamma function and

$$x = L_{Th}^2 4eH/\hbar c . \quad (2.31)$$

Here L_{Th} is the dephasing length discussed in Sec. II.D. For fields such that $L_H \ll L_{\text{Th}}$, $x \gg 1$, and Eq. (2.30) predicts a $\ln H$ behavior. A similar formula for magnetoresistance was worked out in 3D by Kawabata (1980a,1980b), who found that $\sigma(H,T) - \sigma(0,T)$ goes as $H^{1/2}$ for $x \gg 1$. The magnetoresistance in the bulk is isotropic. For strictly 2D systems, Eq. (2.30) applies only for the field component normal to the phase because the effect comes from the orbital motion of the electron. For a thin film of finite thickness $t \ll L_H$, a negative magnetoresistance also exists for a magnetic field parallel to the plane. However, the characteristic field strength is much stronger, being given by the condition (Altshuler and Aronov, 1981a)

$$\frac{t^2 L_{\text{Th}}^2}{3 L_H^4} \approx 1 \ . \tag{2.32}$$

A similar situation obtains for thin wires of cross-section area A. The conductivity is given by

$$\sigma(H,T) = \frac{e^2}{\pi^2 \hbar} \left[\frac{1}{L_{\text{Th}}^2} + \frac{1}{D\tau_H} \right]^{-1/2}, \tag{2.33}$$

where $D\tau_H = C L_H^4 / A$ and the coefficient C is a number of order 2 or 3 that depends on the orientation of the field (transverse or longitudinal) and on the shape of the cross section.

In the presence of spin-orbit scattering, the spin of an individual electron is no longer a good quantum number, but unlike spin-flip scattering, time-reversal symmetry is preserved. Hikami, Larkin, and Nagaoka (1980) found that spin-orbit scattering leads in perturbation theory to a logarithmic *increase* in conductivity for length scales larger than both l and the spin-orbit scattering diffusion length $L_{\text{SO}} \sim (D\tau_{\text{SO}})^{1/2}$, where τ_{SO} is the spin-orbit scattering time. In 2D this results in a sign change in the β function, so that

$$\beta(g) = +\frac{1}{2g} \ . \tag{2.34}$$

Hikami *et al.* have worked out the magnetoresistance when spin-orbit scattering is present. More detailed formulas and the inclusion of Zeeman spin splitting are given by Maekawa and Fukuyama (1981). There are three length scales now, i.e., the flux quantum length $L_H = \sqrt{hc/2eH}$, the spin-orbit diffusion length $L_{\text{SO}}[\cong (D\tau_{\text{SO}})^{1/2}]$, and the Thouless inelastic length $L_{\text{Th}}[\sim (D\tau_i)^{1/2}]$. The magnetoresistance regime depends on their relative sizes. Both positive and negative magnetoresistance are possible, and the rich variety of behavior can be experimentally probed by changing the magnetic field, temperature, etc. This has been done recently by Bergmann (1982b), who found nonmonotonic behavior of $\sigma(H)$ in quantitative agreement with the Hikami, Larkin, and Nagaoka theory (see Sec. VI).

A somewhat more physical picture can be found in the discussion of Altshuler, Aronov, Khmel'nitskii, and Larkin (1983) and Bergmann (1982a). Instead of considering the averaged conductivity diagrams (Fig. 7), we may consider the unaveraged one-particle Green's function $G(\mathbf{r},\mathbf{r}',E)$, which will contain information on the wave function and therefore localization. (This information is lost upon averaging over impurity configurations.) The Green's function $G(\mathbf{r},\mathbf{r}',E)$ describing the propagation of an electron from $\mathbf{r}$ to $\mathbf{r}'$ can be constructed as a Feynman path integral over all paths connecting $\mathbf{r}$ and $\mathbf{r}'$. Altshuler, Aronov, Khmel'nitskii, and Larkin (1983) point out that in the presence of impurities, most paths will arrive with random phase, with the exception of paths that are self-intersecting. As shown in Fig. 8, for each self-intersecting path, the closed loop can be circumscribed in two opposing directions. In the presence of time-reversal symmetry, these two paths will interfere with each other. This interference is interpreted as the origin of the singularities due to the maximally crossed conductivity diagram. In this picture it is easy to see that a uniform magnetic field will dephase the interference between the two paths. A similar picture was presented by Bergmann (1982a) in momentum space. Bergmann further pointed out that the sign change in the β function in the presence of a spin-orbit interaction can be understood as an overlap factor between the spins as they arrive at $\mathbf{r}'$. Spin-orbit scattering preserves time-reversal symmetry, so that interference is still possible, but the spins are rotated in opposite directions along the two paths in Fig. 8. Bergmann showed that the average overlap between the spins as a result of this rotation is $-\frac{1}{2}$, basically because a rotation by 2π of a spin-$\frac{1}{2}$ state leads to a sign change.

Altshuler, Aronov, and Spivak (1981) proposed an experimental configuration that demonstrates most dramatically the orbital origin of the magnetoresistance. They suggested measuring the conductivity of a ring or a thin cylinder as a function of the magnetic flux φ through the ring or cylinder. In the case of the cylinder, the conductivity is measured along the cylinder axis and parallel to the magnetic field. The diffusing electron pair shown in Fig. 7(b) or the self-intersecting path in Fig. 8 are now constrained to run around the cylinder, so that the phase is periodic in φ/φ_0, where $\varphi_0 = hc/2e$. Under the condition that L_{Th} be large compared with the radius of the ring or cylinder, the conductivity should oscillate with the magnetic field with the period φ/φ_0.

This effect was experimentally observed by Sharvin and Sharvin (1982), who evaporated Mg and Li films on quartz fibers. The oscillation is small ($\approx 10^{-4}$), but that
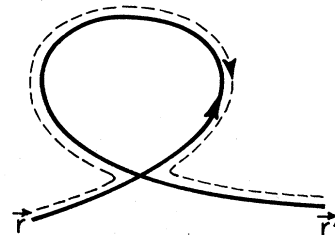

FIG. 8. A self-intersecting Feynman path for an electron to propagate from $\mathbf{r}$ to $\mathbf{r}'$. Propagation along solid and dashed paths can interfere.

is because the film is rather thick (thousands of Å). Experiment on rings instead of cylinders should yield much larger effects. The experiments on cylinders have been confirmed and analyzed in detail by Gijs, Van Hasendonck, and Bruynseraede (1984). Experiments on large arrays of rings have also been reported recently by Pannetier et al. (1984).

Altshuler, Aronov, and Khmel'nitskii (1981,1982) also discussed the effect of an electric field on the conductivity. A dc electric field does not break time-reversal invariance and is equivalent to a static potential gradient. The interference between the two electron loops shown in Fig. 8 should not be affected. This point of view is contrary to the conclusions of Tsuzuki (1981) and Mott and Kaveh (1981), but we find it physically more transparent. Experimentally, Bergmann (1984) has determined that a dc field has no effect apart from heating. On the other hand, in the presence of an ac field $E_0 e^{i\Omega\tau}$, the potential experienced by the electron will depend on time and introduce a random phase, which can be estimated as follows. In a time τ such that $\Omega\tau \ll 1$, the electric field will change by $E_0\Omega\tau$. The electron diffuses a distance $L \sim (D\tau)^{1/2}$, and the energy change is $\Delta E \sim E_0\Omega D\tau^{3/2}$. The phase change is $\Delta\varphi \sim \Delta E_\tau \approx E_0\Omega D\tau^{5/2}$. Equating this with unity gives the characteristic cutoff time (to be compared with τ_{in})

$$\tau_\varphi \sim (E_0\Omega D)^{-2/5} .$$ (2.35)

This time is in agreement with that obtained by an explicit solution of the electron pair propagation in the presence of an electromagnetic field (Altshuler, Aronov, and Khmel'nitskii, 1982).

Finally we mention that the orbital effect of a magnetic field was incorporated into a scaling theory of the metal-insulator transition by Khmel'nitskii and Larkin (1981). They found the crossover exponent for the magnetic field to be $\frac{1}{2}$, which simply expresses the fact that the relevant length scale is the Landau orbit size, which scales as $H^{-1/2}$. As an example, they predicted a magnetoconductivity at the metal-insulator transition of the form $\sigma(H) \sim A_3(e^2/\hbar)(eH/\hbar c)^{1/2}$ in 3D, where A_3 is some universal constant depending only on the symmetry of the system, e.g., the presence of spin-orbit scattering. Furthermore, the mobility edge is shifted by an amount of the order of $H^{1/2\nu}$.

F. Scaling results for other transport properties

1. ac conductivity and the dielectric function

Gor'kov, Larkin, and Khmel'nitskii (1979) showed that the frequency ω can serve as a cutoff instead of the sample L. Considering for $\sigma(\omega)$ the maximally crossed diagram Fig. 7(a), they found that for an infinite system

$$\sigma = \sigma_0 \left[1 + \frac{3\sqrt{6}}{4(k_F l)^2} \sqrt{\omega\tau} \right]$$ (2.36a)

for 3D and

$$\sigma = \sigma_0 \left[1 + \frac{1}{k_F l} \ln |\omega\tau| \right]$$ (2.36b)

for 2D; the correction terms are as usual presumed to be small. We notice that in three dimensions the conductivity initially rises with frequency as $\omega^{1/2}$, a distinctly non-Drude type of behavior. In two dimensions, the conductivity falls logarithmically as frequency decreases, so that this result suggests again that $\sigma \to 0$ as $\omega \to 0$, and that there is a characteristic frequency $\omega_0 \sim (1/\tau)\exp(-\pi k_F l)$ which marks the crossover to the exponentially localized regime. In the perturbative regime, the forms Eq. (2.36) for $\sigma(\omega)$ could be guessed at from the forms (2.25) for $\sigma(L)$, and the relation $\omega^{-1} \sim \tau_L \sim [L^2 D(L)]$ for the time taken to diffuse a length L. If $D(L)$ depends only weakly on length scale L, $\sqrt{\omega}$ scales as L^{-1} and the results [Eq. (2.36)] follow.

Near the metal-insulator transition, the conductivity depends on length scale according to Eq. (2.16a). Thus the diffusion constant $D(L) = \sigma(e^2/\hbar)^{-1}(dn/d\mu)^{-1}$ varies with length scale as

$$D(L) = \frac{(dn/d\mu)^{-1}g^*}{L^{d-2}} ,$$ (2.37)

where g^* is the fixed-point value of conductance given in Eq. (2.14a). The connection between ω and L is now modified to read $\omega \approx D(L)L^{-2} = (dn/d\mu)^{-1}g^*L^{-d}$. We thus have a new length scale $L_\omega = (\omega dn/d\mu)^{-1/d}$. Putting this length scale into the conductivity equation, we obtain the frequency-dependent conductivity in the critical region as

$$\sigma \propto \omega^{(d-2)/d} ,$$ (2.38)

which goes as $\omega^{1/3}$ in three dimensions. This result was first obtained by Wegner (1976) using general scaling arguments and subsequently discussed by a number of authors (Shapiro and Abrahams, 1981b; Shapiro, 1982b; Imry, Gefen, and Bergmann, 1982a,1982b; Vollhardt and Wölfle, 1982). The frequency-dependent conductivity can be cast into a scaling form

$$\sigma(\omega) = \frac{e^2}{\hbar\pi^2}\xi^{2-d}f(\xi/L_\omega) .$$ (2.39)

For $L_\omega \gg \xi$, i.e., low frequency, $\sigma(\omega)$ is independent of ω on the metallic side and $\sigma(\omega) \sim (e^2/\hbar\pi^2)\xi^{2-d}$. The function $f(x)$ must approach unity for $x \ll 1$. In the opposite limit, $\xi \gg L_\omega$, we are in the critical region, and $\sigma(\omega)$ must be independent of ξ. Thus $f(x) \sim x^{d-2}$ for $x \gg 1$, and we recover Eq. (2.38) in this limit.

The above scaling considerations can be extended to include the full q- and ω-dependent conductivity or dielectric function. This is important in the discussion of screening near the metal-insulator transition and also permits a discussion of the critical behavior of the dielectric constant on the insulating side. The polarizability function $\Pi(q,\omega)$ takes the form

$$\Pi(q,\omega) = \frac{dn}{d\mu} \frac{D(q,\omega)q^2}{D(q,\omega)q^2 - i\omega} \; , \tag{2.40}$$

where we have generalized the standard expression for disordered metals to include q- and ω-dependent diffusion. If $q\xi \gg 1$, we are in the critical region, where the diffusion constant is scale dependent according to Eq. (2.37), and we should replace L by q^{-1}. In this limit, the usual diffusion pole is replaced by

$$\frac{1}{Dq^2 - i\omega} \rightarrow \frac{1}{\left[\dfrac{dn}{d\mu}\right]^{-1} g^* q^d - i\omega} \; . \tag{2.41}$$

This result was first obtained by Wegner (1976), who pointed out a very useful analogy of the diffusion pole with the transverse magnetic susceptibility in a system with a continuously broken symmetry,

$$\chi_\perp(q,h) = \frac{1}{\kappa q^2 + |h|} \; , \tag{2.42}$$

where h is an external field and κ is the spin-wave stiffness constant. In the critical region, $\chi_\perp \sim q^{2-\eta}$ defines the critical exponent η. In comparison with Eq. (2.35) this exponent is (Wegner, 1976; McKane and Stone, 1981)

$$\eta = 2 - d \; . \tag{2.43}$$

This unusual diffusive behavior was exploited by Anderson, Muttalib, and Ramakrishnan (1983) in their study of superconductivity (see Sec. VII.C). It was also used by Lee (1982) in his investigation of the role of the Coulomb interaction.

The screening properties on both sides of the transition that follows from the polarization function given in Eq. (2.40) were discussed by McMillan (1981). A lucid discussion was presented by Imry, Gefen, and Bergmann (1982a, 1982b), and a discussion of the scaling function in $(2 + \varepsilon)$ dimensions can be found in Abrahams and Lee (1985). The essential feature is that, on the localized side of the transition, we can follow the scaling away from the fixed point g^* up to a length scale $L = \xi$, which corresponds to the localization length. Inside the length scale we are in the critical region, and the polarization function can be described by Eq. (2.40). The real part of the dielectric constant is given by

$$\varepsilon'(q,\omega) = 1 + \frac{4\pi e^2}{q^2} \mathrm{Re}[\Pi(q,\omega)]$$

$$\approx 4\pi e^2 \frac{dn}{d\mu} q^{-2} \; . \tag{2.44}$$

Setting $q \approx \xi^{-1}$, we obtain an estimate of the static dielectric constant on the insulating side as

$$\varepsilon' = 4\pi e^2 \frac{dn}{d\mu} \xi^2 \; , \tag{2.45}$$

which implies a divergent ε' as $(g - g^*)^{-2\nu}$. Furthermore, comparison with Eq. (2.16a) shows that in three dimensions,

$$\varepsilon' \sigma^2 = \mathrm{const} \; , \tag{2.46}$$

if ε' and σ are measured equidistant from the metal-insulation transition point. Equation (2.46) is consistent with the experimental data of Paalanen, Rosenbaum, Thomas, and Bhatt (1982).

2. Anisotropic systems

The perturbative scaling theory of localization has been recently generalized to the case of anisotropic systems by Wölfle and Bhatt (1984). They show for a general anisotropy leading to differing principal Boltzmann conductivities $\sigma_{\mu\mu}^B$ (where $\mu = x$ and y in 2D) that

$$\sigma_{\mu\mu}(\omega) = \sigma_{\mu\mu}^B - \frac{e^2}{2\pi^2\hbar} \frac{\sigma_{\mu\mu}^B}{\overline{\sigma}^B} \ln\left[\frac{1}{\omega\tau}\right] \; , \tag{2.47}$$

where $\overline{\sigma}^B = (\sigma_{xx}^B \sigma_{yy}^B)^{1/2}$. Thus the logarithmic term has the same anisotropy as the conductivity. This interesting result agrees with the measurements of Bishop, Dynes, Lin, and Tsui (1984) on various Si inversion layer faces. Note that Eq. (2.47) has the following simple interpretation. (See Altshuler, Aronov, Larkin, and Khmel'nitskii, 1981.) According to the dogma that conductance is the only scaling variable, we first rescale the x and y length scales differently so that the *conductance* is isotropic. Since $G_x = \sigma_{xx}^B L_y/L_x$ and $G_y = \sigma_{yy}^B L_x/L_y$, this clearly requires $L_y/L_x = (\sigma_{yy}^B/\sigma_{xx}^B)^{1/2}$. We then perform the usual scaling argument in this anisotropic frame, and the correction to the conductance is the usual universal term $e^2/(2\pi^2\hbar)\ln\omega\tau$. We finally rescale to the original x and y scale, and the correction to the anisotropic conductivity takes the form given in Eq. (2.47). The effect of anisotropy on the correction to the conductivity due to interaction effects (see Sec. III) is discussed by Altshuler and Aronov (1979c).

3. Hall conductivity

The Hall conductivity for weak disorder, considering the leading-order scaling corrections (i.e., the crossed diagram process in perturbation theory), was first calculated by Fukuyama (1980a). He showed, for two dimensions, that to this order the Hall resistance was unchanged. That is, defining $R_H = (E_y/J_x H)$ for small E and H (linear response), one has

$$\delta R_H / R_H^0 = 0 \; . \tag{2.48}$$

This is an important result because, as we shall see (Sec. III), Coulomb interaction effects lead in contrast to a nonvanishing $(\delta R_H/R_H^0)$. The result is true in three dimensions as well.

A general scaling analysis of the Hall conductivity using perturbative exponent estimates and scaling forms has been made by Shapiro and Abrahams (1981a). They find, for example, that near the mobility edge the Hall conductivity approaches zero as $\sigma_H = (E - E_c)^{2\mu}$, where μ is the

conductivity exponent [$\mu = 1$ in perturbation theory, in $(2 + \varepsilon)$ dimensions]. Consequently, the Hall constant $R_H = \sigma_H / \sigma_{xx}^2$ is nonsingular, of order σ_{min}^{-1} as the metal-insulator transition is approached from the metallic side.

4. Thermoelectric power

Ting, Houghton, and Senna (1982) have calculated the thermoelectric power of the two-dimensional system. They found that the minimally crossed diagrams do not introduce any logarithmic anomalies. However, when interaction is taken into account in the way described in Sec. III, logarithmic corrections are found. To our knowledge there is no experimental verification of this effect so far.

G. Beyond lowest-order perturbation theory

The perturbative estimate (Sec. II.C) of the scale-dependent conductivity and of the curve $\beta(g)$, using the scaling idea, is accurate for large conductance or long mean free paths, i.e., for $g \gg 1$ or $k_F l \gg 1$. (In two dimensions, the expansion fails at large enough length scales or low frequencies even for $k_F l \gg 1$.) Since the critical conductance $g_c = (e^2/\hbar\pi^2)\varepsilon^{-1}$ for a disordered electron gas in $(2 + \varepsilon)$ dimensions, perturbation theory is capable of describing the behavior for $\varepsilon \ll 1$. The transition in three dimensions occurs for intermediate coupling, and the exponent estimates are approximate. There have, therefore, been several attempts to go beyond lowest-order perturbation theory, and also to develop a theory that describes both extended and localized regimes. We briefly mention them here.

The next terms in $\beta(g)$ have been calculated by Khmel'nitskii (1980), Efetov, Larkin, and Khmel'nitskii (1980) and by Hikami (1981). They are found to vanish. Hikami (1981,1982) in particular has shown that there are no terms to three-loop order, i.e., to order $(1/g^3)$. In principle, the higher-order terms in the β function depend on the cutoff procedure used. Hikami uses the dimensional regularization methods of Callan and Symanzik. The critical exponents, such as ν, are however universal, and the vanishing of the higher-order terms suggests that there are no higher-order corrections to ν in an ε expansion that is analytic in ε.

Vollhardt and Wölfle (1980a,1980b) have given an approximate self-consistent theory of localization. The same results were obtained by Hikami (1982), who solved the renormalization group equations for the diffusion constant assuming that all higher-order terms in $\beta(g)$ vanish.

Vollhardt and Wölfle consider the density-density correlation function

$$\chi(\mathbf{q},\omega) = \chi^T(q,0)\frac{D(q,\omega)q^2}{-i\omega + D(q,\omega)q^2} .$$ (2.49)

This diffusive low-frequency form defines $D(q,\omega)$. It is more convenient to examine the current relaxation kernel

$M(q,\omega)$, related to D by

$$D(q,\omega) = \frac{in}{m\chi^T}\frac{1}{M(q,\omega)} .$$ (2.50)

$M(q,\omega)$ essentially measures the current relaxation rate. In the metallic limit, it is a constant given in the standard Born or relaxation-time approximation by $M(0,0) = i/\tau$, where τ is given by Eq. (2.22). A general equation relating M to moments of the two-particle scattering amplitude can be obtained. The latter satisfies a scattering or Bethe-Salpeter equation. The backscattering terms, which can be separated using time-reversal invariance, make a singular contribution at low frequencies ω. To lowest order,

$$M(0,\omega) = \frac{i}{\tau} - \frac{2}{\tau}\sum_k \left[\frac{1}{\omega + iD_0 k^2}\right] .$$ (2.51)

In $d \leq 2$, Eq. (2.51) has an infrared singularity, so that D is strongly modified. One should therefore not use the bare D_0 on the right-hand side, but in a self-consistent (dynamic Hartree) theory, use $D(q,\omega)$ instead, so that $M(0,\omega)$ satisfies the equation

$$M(0,\omega) = \frac{i}{\tau} - \frac{2}{\tau}\sum_k \frac{1}{\omega - k^2 D_0 \tau^{-1} M(0,\omega)^{-1}} .$$ (2.52)

Equation (2.52) can be solved for low frequencies, and in $d \leq 2$ gives

$$M(0,\omega) = \frac{i}{\tau} - \frac{\omega_0^2}{\omega} ,$$ (2.53)

which corresponds to an insulator with the frequency-dependent conductivity $\sigma(\omega)$ given by

$$\sigma(\omega) = \left[\frac{ne^2\tau}{m}\right](\omega^2/\omega_0^4) .$$ (2.54)

The frequency ω_0 is the scale at which localization effects become important; it is proportional to $(k_F l)^{-1}$ in one dimension and to $e^{-\pi k_F l}$ in two dimensions. For $\omega \gg \omega_0$, the conductivity is nonzero; it decreases logarithmically (in 2D) as frequency decreases, and then crosses over to that for localized states when $\omega \ll \omega_0$. The localization length can be estimated from the static polarizability $\alpha(0) \sim l_{loc}^{-2} = \xi^{-2}$. In 2D, as expected from perturbation theory, it is exponential in weak disorder. The authors find good agreement with the direct many-body theory results of Abrikosov and Ryzhkin (1978) for a one-dimensional system. A diagrammatic classification leading to Eq. (2.52) as a first approximation has been discussed by Wölfle and Vollhardt (1982b). From their expression for the conductivity Vollhardt and Wölfle (1982a,1982b) have obtained the scaling function implied by their self-consistent theory. It has the correct asymptotic forms for large and small g, the latter being obtained on considering nonlocal response to the applied electric field. In two dimensions, they find

$$\beta(g) = -\frac{1}{g}(1 - e^{-2g}) \quad g \gg 1$$ (2.55a)

$$= \ln g \quad g \ll 1 \ . \tag{2.55b}$$

There is no correction of higher order in $(1/g)$ for $\beta(g)$ which is analytic in that variable [Eq. (2.55a)]. Similar scaling functions for large g have been obtained by Hikami (1982), who solved the equation for the β function.

Historically, the first mode-mode coupling theory for the conductivity correction was given by Götze (1979). However, Götze obtained his singularity from the slow diffusive mode in the density fluctuations and ignored the quantum interference effects. The corrections he found are higher order in $(k_F l)^{-1}$ and disagree with the results of Abrahams et $al.$ (1979). It was pointed out by Vollhardt and Wölfle (1981a,1981b) that, from conservation laws, the coupling to the diffusive mode goes as q^2 for small q, and does not contribute a singular correction in two dimensions. In general, as noted by them and by Maleev and Toperverg (1975), diffusion corrections to the conductivity of a noninteracting system are nonsingular to leading order of $(k_F l)^{-1}$. As discussed by Götze (1979,1981,1983) the mode-mode coupling theory leads to a scale-dependent correction of order $(k_F l)^{-2}$ and a dc conductivity exponent μ of $\frac{1}{2}$ rather than unity, i.e., a scaling function $\beta(g) = (d-2) - (\alpha/g^2)$. Prelovsek (1981) and Belitz, Gold, and Götze (1981) have incorporated quantum interference into the mode-mode coupling theory of Götze (1979,1981,1983); this leads to results identical with scaling theory in the critical regime.

H. The high-magnetic-field limit: The quantum Hall regime

The magnetoresistance calculation reviewed in Sec. II.E was carried out treating the magnetic field semiclassically, which should be valid under the condition $\omega_c \tau \ll 1$, where ω_c is the cyclotron frequency. There is great interest in understanding the opposite regime $\omega_c \tau \gg 1$, particularly in view of the quantum Hall effect observed in this limit. The important issue is whether all states remain localized in the high-field regime. The theoretical understanding of the quantized Hall effect requires the existence of extended states somewhere within the Landau subband (Laughlin, 1981), and Halperin (1982) has provided an argument for the existence of extended states. A quantitative treatment of the problem is difficult because of the absence of a small parameter. The dimensionless resistance g^{-1} is of order unity or larger according to the self-consistent Born approximation (Ando and Uemura, 1974). Thus a perturbative treatment is not possible. Ono (1982) has generalized the self-consistent treatment of Vollhardt and Wölfle to this case, and found that, exactly at the band center, an extended state exists, with the localization length diverging as the band center is approached. This result is in qualitative but not quantitative agreement with numerical simulations (Ando, 1982,1983). However, in view of the absence of an expansion parameter, the selection of diagrams in this approach appears rather ad hoc. The field theory treatment of this problem is briefly

reviewed at the end of Sec. V, and the conclusion again is that extended states exist (Levine, Libby, and Pruisken, 1983). A phenomenological scaling approach by Khmel'nitskii (1983) suggests the existence of a single extended state in the middle of the band. Very recently, Hikami (1984) has extended Wegner's (1983) exact calculation of the density of states at the band center to perform a perturbative expansion of the conductivity in powers of the impurity scattering strength, which he then summed by a Borel-Padé approximation. The results indicate an extended state at the band center.

We mention that the interaction effects to be discussed in the next section have also been calculated in the high-field limit (Girvin, Jonson, and Lee, 1982; Houghton, Senna, and Ying, 1982). These calculations are in reasonable agreement with experimental observation when a few Landau levels are filled (Paalanen, Tsui, and Gossard, 1982).

III. INTERACTION EFFECTS

A. Introduction

The study of the interacting electron gas has a long history. Early studies using perturbation theory in the unscreened Coulomb interaction led to a strong singularity near the Fermi surface. It was then realized that a proper calculation, taking screening into account, removed all the singularities (see Nozières and Pines, 1966). These results can be fitted into the general framework of Landau's Fermi-liquid theory, which concludes that the effects of interaction can be represented by the introduction of a number of Fermi-liquid parameters, describing the renormalization of physical quantities such as specific heat and magnetic susceptibility. While these renormalizations can be large, they are finite. In the early 1960s these studies were extended to the vicinity of the ferromagnetic transition. It was found that low-lying spin fluctuations (paramagnons) lead to strong renormalization of physical quantities and, in addition, introduce singularities near the Fermi surface. However these singularities are typically weak, such as a $T^3 \ln T$ term in the specific heat. The general feeling was that impurity scattering would not lead to any essential modification of the Fermi-liquid theory (see Betbeder-Matibet and Nozières, 1966). Thus it came as quite a surprise when it was shown by Altshuler and Aronov (1979a,1979c) that interactions in a disordered Fermi liquid lead to strong singularities near the Fermi level. For example, a singularity of the form $(\omega - E_F)^{1/2}$ is predicted for the tunneling density of states, and $T^{1/2}$ and $T^{3/2}$ low-temperature corrections are predicted for the conductivity and the specific heat, respectively. Interestingly some of these effects were already anticipated by Brinkman and Engelsberg (1968), who showed that the diffusive nature of spin fluctuations in a disordered system leads to singularities stronger than in a pure system. In particular, the $T^3 \ln T$ term in the specific heat becomes $T^{3/2}$. The Altshuler-Aronov effect

can be understood if we replace spin diffusion in Brinkman-Engelsberg by density diffusion. The irony is that perhaps Fermi-liquid theory was so successful and so entrenched in our thinking that our understanding of the disordered Fermi liquid may have been delayed by more than a decade.

B. Self-energy corrections and lifetime in a disordered Fermi liquid

1. Model calculation of the density of states

Altshuler and Aronov (1979a,1979c) treated the disordered-Fermi-liquid problem by performing a perturbation theory to lowest order in the interaction strength. The disorder is treated by the conventional diagrammatic technique (Abrikosov, Gor'kov, and Dzyaloshinskii, 1963), which should be valid in the limit $k_F l \gg 1$. In this limit crossed impurity lines are considered higher order in $(k_F l)^{-1}$ and are ignored. Thus the localization effects discussed in the preceding section are specifically excluded. For simplicity we shall first discuss a model problem in which the electrons interact via a static interaction $v(q)$,

$$H_1 = \tfrac{1}{2} \sum_q v(q) \rho_q^+ \rho_q , \tag{3.1}$$

where $\rho_q = \sum_k a_{k+q}^+ a_k$. We shall return to discuss the complications of the dynamically screened Coulomb interaction later.

The important ingredient of the interaction theory is that the interaction vertex is dressed by the impurity scattering, as shown in Fig. 9. A straightforward calculation shows that, for small q and ω_m, the vertex correction is given by

$$\Gamma(q,\omega_m,\varepsilon_n) = \begin{cases} (|\omega_m| + Dq^2)^{-1}\tau^{-1}, & \varepsilon_n(\varepsilon_n - \omega_m) < 0 \\ 1 & \text{otherwise}, \end{cases} \tag{3.2}$$

where $\omega_m = 2\pi m kT$ and $\varepsilon_n = \pi(2n+1)kT$ are the Matsubara frequencies. Note that when the electron lines have frequencies with opposite signs, the vertex correction has the diffusion form and becomes singular in the limit of small q and ω_m. The relation with density diffusion is best illustrated by calculating the polarization function $\Pi(q,\omega_m)$ using the diagrams shown in Fig. 10. We obtain

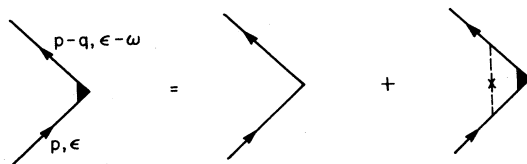$$\Pi(q,\omega_m) = \frac{dn}{d\mu} \left[1 - \frac{|\omega_m|}{|\omega_m| + Dq^2} \right] \tag{3.3a}$$

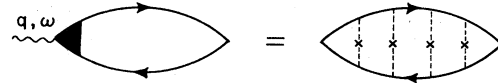FIG. 9. Impurity dressing of the interaction vertex.

FIG. 10. Diagram for the polarization function $\Pi(q,\omega)$.

$$= \frac{dn}{d\mu} \frac{Dq^2}{|\omega_m| + Dq^2} . \tag{3.3b}$$

In Eq. (3.3a) the first (second) term comes from the region where the frequencies of the electron and hole lines have equal (opposite) frequencies. Upon analytic continuation we obtain the well-known result for the diffusive form of the response function,

$$\Pi(q,\omega) = i \int_0^\infty dt \, dx \, \overline{\langle [\rho(\mathbf{x},t),\rho(0,0)] \rangle} e^{-i\mathbf{q}\cdot\mathbf{x}} e^{i\omega t}$$

$$= \frac{dn}{d\mu} \frac{Dq^2}{-i\omega + Dq^2} . \tag{3.4}$$

Equation (3.4) can be derived from more general considerations by making the assumption that density fluctuations satisfy the diffusion equation (Forster, 1975).

To lowest order in the coupling constant, the correction to the Green's function in the momentum representation is given by the diagram shown in Fig. 11. The physical quantity that is independent of representation is the single-electron density of states

$$\delta N(E) = \pi \sum_k \text{Im} G(k,E) . \tag{3.5}$$

This was first calculated by Altshuler and Aronov (1979a,1979c), who showed that the diffusion pole associated with the two vertex corrections in Fig. 11 leads to unexpected singular behavior in the density of states, such as an $E^{1/2}$ behavior in three dimensions. We emphasize that the singularities are unrelated to the long-range nature of the Coulomb interaction. In fact, starting with the short-range model interaction we see that the vertex correction generates an effective long-range and retarded coupling. This expresses the physical idea that since the electron motion is diffusive, the electrons spend a longer time in a given region in space relative to the plane-wave states, and their interaction is enhanced.

We note that the particular diagram Fig. 11 is constructed in a conserving approximation in the sense of Kadanoff and Baym (1962) from the diagram for the free energy shown in Fig. 12 by cutting the fermion line at an arbitrary place. The relation with diffusion can be seen from the fact that Fig. 12 is obtained from the response function Fig. 10 by connecting the external vertices by an interaction line.

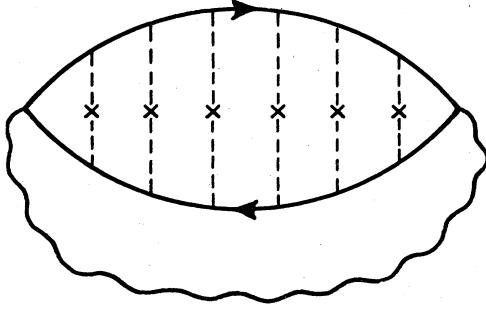FIG. 11. Self-energy correction due to interaction (wavy line).

FIG. 12. First-order correction to the free energy in the presence of impurity scattering.

The diagrammatic treatment is carried out in the basis of momentum eigenstates. The plane wave is very far from being an eigenstate of the disordered Hamiltonian. This is why in a perturbation treatment of the interaction, large vertex corrections are necessary. Furthermore, quantities like self-energy and decay rate are meaningless in the k representations because the spectral representation contains a large width given by the elastic scattering rate τ^{-1}. A more natural basis for the perturbation expansion is the set of exact eigenstates ψ_m with eigenvalue E_m of the noninteracting Hamiltonian (Abrahams et al., 1982; Maldague, 1981). In this basis the single-particle Green's function is given by

$$G_{mm}(\omega) = \langle m \mid (\omega - H)^{-1} \mid m \rangle , \qquad (3.6)$$

and the effect of interaction is included in the self-energy correction $\Sigma_m(\omega)$, so that

$$G_{mm}(\omega) = [\omega - E_m - \Sigma_m(\omega)]^{-1} . \qquad (3.7)$$

We keep only the diagonal terms in the self-energy because we shall find that it is enhanced by wave-function correlation. Writing $\Sigma_m = \Delta_m + i\Gamma_m$, we perform the standard expansion

$$\Delta_m(\omega) = \Delta_m(\widetilde{E}_m) + (\omega - \widetilde{E}_m)\frac{\partial \Delta_m}{\partial \omega} , \qquad (3.8)$$

where

$$\widetilde{E}_m = E_m + \Delta_m(\widetilde{E}_m) . \qquad (3.9)$$

We obtain

$$G_{mm}(\omega) = Z / [(\omega - \widetilde{E}_m) - i\gamma_m] , \qquad (3.10)$$

where

$$Z^{-1} = [1 - \partial \Delta_m / \partial \omega]_{\omega = \widetilde{E}_m} \qquad (3.11)$$

and

$$\gamma_m = Z\Gamma_m(\omega = \widetilde{E}_m) . \qquad (3.12)$$

In Eq. (3.10), $\widetilde{E}_m$ is interpreted as the quasiparticle energy, γ_m as the decay rate of the quasiparticle, and Z as the fractional weight of the quasiparticle excitation.

To discuss the average energy shift and the average decay rate we have to study the impurity average of the self-energy for a fixed E_m, i.e.,

$$\widetilde{\Sigma}_E(\omega) = \frac{1}{N_0}\overline{\sum_m \delta(E - E_m)\Sigma_m(\omega)} , \qquad (3.13)$$

where N_0 is the one-spin density of states and the bar denotes impurity averaging. Writing $\widetilde{\Sigma}_E = \widetilde{\Delta}_E + i\widetilde{\Gamma}_E$, we see that the average density of quasiparticle states is given by

$$N_Q = \frac{N_0}{1 + \delta\widetilde{\Delta}_E / \delta E} , \qquad (3.14)$$

and the average quasiparticle decay rate is given by

$$\widetilde{\gamma}_E = \overline{Z\Gamma} \approx \widetilde{\Gamma}_E , \qquad (3.15)$$

where

$$\widetilde{\Gamma}_E = (1/N_0)\overline{\sum_m \delta(E - E_m)\Gamma_m(\omega = \widetilde{E}_m)} .$$

The model problem of electrons interacting via a static interaction $v(\mathbf{r})$ is particularly simple in that, to first order in v, $Z = 1$ and $\widetilde{\gamma}_E = 0$. In the Hartree-Fock approximation, the four-fermion term in Eq. (3.2) is factorized. Let us focus our attention on the exchange term; the Hartree term can be shown to be small if the potential $v(r)$ has a range larger than the interparticle spacing. We shall see that the diagonal exchange term is enhanced because of wave-function correlation, so that we write $H = \sum_m \varepsilon_m a_m^+ a_m$ where

$$\varepsilon_m = -\sum_{\text{occupied}} \int d\mathbf{r}\, d\mathbf{r}'\psi_m^*(\mathbf{r})\psi_n^*(\mathbf{r}')\psi_m(\mathbf{r}')\psi_n(\mathbf{r})v(\mathbf{r} - \mathbf{r}') .$$
$$\qquad (3.16)$$

Suppose we insert a particle at energy E. Its energy will be shifted on the average by the amount

$$\widetilde{\Sigma}_E = \frac{1}{N_0}\overline{\delta(E - E_m)\varepsilon_m}$$
$$= -\int_{-\infty}^{0} dE' F(E, E'; \mathbf{r}, \mathbf{r}')v(\mathbf{r} - \mathbf{r}') , \qquad (3.17)$$

where

$$F(E, E'; \mathbf{r}, \mathbf{r}') = \sum_{m,n} \overline{\delta(E - E_m)\delta(E' - E_n)\psi_m^*(\mathbf{r})\psi_n^*(\mathbf{r}')\psi_m(\mathbf{r}')\psi_n(\mathbf{r})} . \qquad (3.18)$$

In Eq. (3.18) we need to know the average of the products of four wave functions. Since we do not have an explicit solution of the impurity problem, we seem to be faced with a hopeless task. Fortunately we notice that the combination required in Eq. (3.18) is precisely that which enters in a density-density correlation function defined as

$$A(q,\omega)= \int_{-\infty}^{\infty} dt\, d\mathbf{r}\, d\mathbf{r}' e^{i\omega t}e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}$$

$$\times \langle [\rho(\mathbf{r}'t),\rho(\mathbf{r},0)]\rangle \ . \tag{3.19}$$

This is the spectral function for the density-density response function given in Eq. (3.4). Again, under the assumption of density diffusion above, we obtain

$$A(q,\omega)=\sum_{m,n}{}' \int d\mathbf{r}\, d\mathbf{r}' e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}\psi_m(\mathbf{r})\psi_n^*(\mathbf{r})\psi_m^*(\mathbf{r}')\psi_n(\mathbf{r}')\delta(E_n-E_m-\omega) \ , \tag{3.21}$$

where the sum is restricted to n occupied and m unoccupied. We convert the sum into an energy integration and compare with Eq. (3.18). We have

$$A(q,\omega)= \int_0^{\infty} dE \int_{-\infty}^0 dE',F(E,E',\mathbf{r},\mathbf{r}')e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}$$

$$\times \delta(E-E'-\omega) \ . \tag{3.22}$$

We expect $F(E,E';\mathbf{r},\mathbf{r}')$ to be dependent on $E-E'$ and $\mathbf{r}-\mathbf{r}'$ (the latter because we have translational invariance after impurity averaging). Then $F(E,E';\mathbf{r}-\mathbf{r}')=F(\omega,\mathbf{r}-\mathbf{r}')$, and Eq. (3.22) becomes

$$A(q,\omega)= \int F(\omega,\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}d\mathbf{r} \ . \tag{3.23}$$

Comparing with Eq. (3.20), we have

$$\int F(\omega,\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}d\mathbf{r}=\frac{\partial n}{\partial\mu}\frac{Dq^2}{\omega^2+(Dq^2)^2} \ . \tag{3.24}$$

Equation (3.24) is remarkable because it is divergent in the limit $q,\omega\to0$. Going back to the definition $F(\omega,\mathbf{r})$, we see that the seemingly innocent assumption of diffusion implies that eigenstates that are nearby in energy are also correlated in space. We are now in a position to calculate the self-energy using Eqs. (3.17) and (3.24),

$$\tilde{\Sigma}_E=\frac{\partial n}{\partial\mu}\int_{-\infty}^0 dE' \int \frac{d\mathbf{q}}{(2\pi)^d}\frac{Dq^2}{(E-E')^2+(Dq^2)^2}v(q) \ . \tag{3.25}$$

According to Eq. (3.25), the exchange interaction between the added electron with energy E and the electrons in the Fermi sea depends strongly on the energy separation. As a result the self-energy is also dependent on E. This will give rise to a change in the density of states, given by

$$\frac{\delta N}{N_1}=\delta\tilde{\Sigma}_E/\partial E= \int \frac{d\mathbf{q}}{(2\pi)^d}\frac{Dq^2v(q)(dn/d\mu)}{E^2+(Dq^2)^2} \ . \tag{3.26}$$

The integrand in Eq. (3.26) is singular in the limit $q,\omega\to0$. We can therefore replace $v(q)$ by $v(0)$. Just from power counting we immediately see that δN is logarithmically divergent in 2D and goes as $\sqrt{E}$ in 3D. Equation (3.26) agrees with results given based on summation of diagrams.

In an electron gas it is necessary to introduce a dynamical screened Coulomb interaction,

$$A(q,\omega)=\frac{\partial n}{\partial\mu}\mathrm{Im}\left[\frac{Dq^2}{-i\omega+Dq^2}\right] \ , \tag{3.20}$$

where D is the diffusion coefficient.

To make the connection to Eq. (3.18), we expand the operator ρ in Eq. (3.19) in terms of the exact eigenstates (Abrahams, Anderson, Lee, and Ramakrishnan, 1982). Restricting ourselves to $T=0$ for simplicity, for $\omega>0$ only one ordering of the commutator is nonvanishing, and Eq. (3.19) becomes

$$V_c(q,\omega)=V_B(q)/[1+V_B(q)\Pi(q,\omega)] \ , \tag{3.27}$$

where the polarization function that enters into screening is given by Eq. (3.3), and $V_B(q)$ is the bare Coulomb interaction. In 3D we have

$$V_{C3}(q,\omega_n)=\frac{4\pi e^2}{q^2}\frac{|\omega_n|+Dq^2}{|\omega_n|+DK_3^2} \ , \tag{3.28a}$$

where $K_3^2=4\pi e^2(dn/d\mu)$. In strictly 2D (such as the Si-MOSFET inversion layer), we have

$$V_{C2}(q,\omega_n)=\frac{2e^2}{q}\frac{|\omega_n|+Dq^2}{|\omega_n|+DqK_2+Dq^2} \ , \tag{3.28b}$$

where $K_2=2\pi e^2(dn/d\mu)$. It is interesting to note that in the static and long-wavelength limit ($\omega=0,q\to0$), the dimensionless coupling constant $V_c(dn/d\mu)=1$. The bare coupling constant e^2 is cancelled in the problem.

2. Dynamic screening and electron lifetime

The dynamic aspect of the effective interaction introduces a number of complications in that $\tilde{\Sigma}$ is now complex and depends on ω. The quasiparticle now acquires a lifetime due to the electron-electron interaction contained in the inelastic (imaginary) part of the dynamically screened interaction. If we compute $\tilde{\Sigma}$ using Eq. (3.16) with a complex V given by Eqs. (3.27) and (3.28a), we obtain a decay rate Γ_E which varies as $E^{d/2}$. This result in 3D was first obtained by Schmid (1974) and Altshuler and Aronov (1979b). Note that while the decay rate is enhanced compared with the standard E^2 result, it is still small compared with E, so that the quasiparticle concept is still valid. In 2D a further complication arises because of the existence of logarithmic singularities. The simple argument described here is not sufficient to yield the full ω dependence. Abrahams et al. (1982) showed that it is possible to rewrite Eq. (3.18) in terms of Green's functions and then perform the averaging using standard diagrammatic techniques in the momentum representation. The result is that, at $T=0$,

$$\tilde{\Gamma}_E^{2D}=\frac{e^2K_2}{4}\frac{1}{E_K}\left[\omega+(E-\omega)\ln\left|\frac{E-\omega}{E}\right|\right] \ , \tag{3.29}$$

while at finite T,

$$\tilde{\Gamma}^{2D}=e^2K_2(kT/E_K)\,|\,\ln T/T_1\,|\ ,\tag{3.30}$$

where $E_K=DK_2^2$ and $T_1=DE_K^2/e^4$. These results are for strictly 2D films. In the momentum integral in Eq. (3.26), it is the momentum transfer satisfying $Dq^2<\max(E,kT)$ that is important. In films of thickness t there will be a crossover from 2D to 3D behavior when the temperature or frequency exceeds Ω_c given by $t=(D/\Omega_c)^{1/2}$. In the 2D regime, the screening constant K_2 appearing in Eqs. (3.29) and (3.30) and in the definition of E_k must be replaced by $K_2(k_F t/\pi)$ to account for both the phase-space restriction of the thin-film geometry and the change in screening properties. The crossover from 2D to 3D is discussed in detail by Lopes dos Santos and Abrahams (1984).

The peculiar $\ln T$ dependence in Eq. (3.30) comes about because, in evaluating the decay rate on the mass shell $\omega=E$, a logarithmic divergence is encountered. Abrahams et al. (1982) argue that this divergence is cut off by including the shift in the quasiparticle pole. This leads to the rather peculiar energy scale T_1, in Eq. (3.30). Altshuler, Aronov, and Khmel'nitskii (1982) have questioned the observability of the decay rate given in Eq. (3.30). As discussed in Sec. II.E, they claim that it is the phase relaxation time τ_φ that is the physically relevant quantity, and not the lifetime of an exact eigenstate as calculated here. In one and two dimensions, the important contribution to the lifetime comes from electron scattering with very small energy transfer, so that the distinction between τ_φ and the lifetime becomes significant. Altshuler et al. (1982) further state that the low-energy electron-electron scattering is equivalent to the interaction of an electron with the thermal fluctuations of electromagnetic waves. A physical manifestation of τ_φ is that it serves as a cutoff of the interference effect leading to weak localization, as discussed in Sec. II.E. Altshuler et al. solve explicitly for the interference effect in the presence of a thermal bath of electromagnetic radiation, and they interpret their results in terms of a dephasing time given by

$$\tau_\varphi^{-1}=\frac{T}{2\pi DN_0\hbar}\ln(\pi DN_0\hbar)\ .\tag{3.31}$$

Note that τ_φ^{-1} is smaller than the decay rate given in Eq. (3.30) and does not involve a $\ln T$ enhancement. Recently Fukuyama (1984b) reported diagrammatic calculation of the influence of interference effects in weak localization by electron-electron collision, which corrected an earlier version by Fukuyama and Abrahams (1983a), and the result is in full agreement with Eq. (3.31), indicating that the $\ln(T/T_1)$ term in Eq. (3.30) should not appear in conductivity experiments.

In one dimension, similar calculations of τ_φ by Altshuler et al. (1982) yield

$$\tau_\varphi^{-1}(d=1)=\left[\frac{T}{D^{1/2}N_0}\right]^{2/3}\ .\tag{3.32}$$

This is to be compared with a decay rate Γ_1 analogous to

Eq. (3.30), which goes as $T^{1/2}$. Interestingly, the combination $\tau_\varphi\Gamma_1$ obeys the relationship

$$\tau_\varphi\Gamma_1\sim(T\tau_\varphi)^{-1/2}\ ,\tag{3.33}$$

so that τ_φ and Γ_1 become comparable when $T\tau_\varphi\approx1$. This latter condition gives the temperature above which the quasiparticle concept remains valid. Thus it appears that in the regime where the theory of localization based on a quasiparticle picture is valid (T greater than τ_φ^{-1} or Γ_1), the dominant mechanism for quasiparticle decay is via scattering by thermal radiation, and Eq. (3.32) should be applicable. This predicts a low-temperature rise of the resistivity in quasi-one-dimensional wires that goes as $T^{1/3}$, which is consistent with experiments.

3. Hartree terms

So far we have discussed the exchange contribution to the self-energy. In the limit of a δ-function interaction, it is clear that the Hartree term with parallel spin and the exchange term are equal in magnitude and opposite in sign. Diagrammatically the Hartree version of Fig. 11 is shown in Fig. 13. Physically the Hartree term is the interaction of a given eigenfunction with the nonuniform electron density in the ground state. Singularities appear because wave functions nearby in energy are correlated in space. Several differences from the exchange term should be noted. The Hartree term requires zero-frequency transfer, so that only the static limit of the screened Coulomb interaction is involved. Furthermore the momentum transfer in the interaction is not dominated by small q because, unlike the exchange term, is it not the same as the momentum appearing in the diffusion pole. Thus for a static screened interaction $v(q)$, the Hartree term is reduced relative to the static exchange term by a factor F,

$$F=\frac{\int d\hat{\Omega}\,v[q=2k_F\sin(\theta/2)]}{\int d\hat{\Omega}\,v(0)}\ ,\tag{3.34}$$

which is the average of the interaction on the Fermi surface over the solid angle $\hat{\Omega}$.

C. Specific heat and tunneling density of states

We begin by discussing the model problem with short-range interaction. The correction to the single-particle density of states per spin N_1 which is observable by tun-
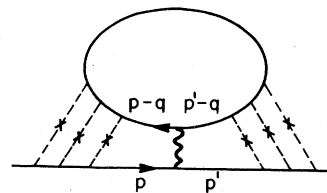

FIG. 13. The Hartree correction to the self-energy.

neling experiments is given by Eq. (3.26). In two dimensions we have

$$\frac{\delta N(\Omega)}{N_1} = \frac{N_1 v(0)}{2\pi\varepsilon_F\tau}(1-2F)\ln|\Omega\tau| \ . \tag{3.35}$$

The term proportional to $2F$ is the Hartree contribution, with the factor 2 coming from the spin sum. Note that the magnitude of the correction increases with disorder. For the short-range problem, the quasiparticle fraction $Z=1$ to first order in $v(0)$, and the correction to the linear term in the specific heat is given by Eq. (3.35) with Ω replaced by T. In three dimensions, the corresponding correction is given by

$$\frac{\delta N(\Omega)}{N_1} = \frac{N_1 v(0)(1-2F)}{4\pi^2(dn/d\mu)D}\left[\frac{T}{2D}\right]^{1/2} \ . \tag{3.36}$$

Next we consider the case of electrons interacting via the Coulomb interaction. This introduces a number of complications. We first consider the exchange-type contribution. The static interaction $v(q)$ in Eq. (3.26) is replaced by the dynamically screened interaction Eq. (3.27). In three dimensions it is legitimate to ignore the term unity in the denominator in Eq. (3.27), and we obtain (Altshuler and Aronov, 1979)

$$\frac{\delta N(\Omega)}{N_1} = \frac{1}{2\pi^2(dn/d\mu)D}\left[\frac{\Omega}{2D}\right]^{1/2} \ . \tag{3.37}$$

However, in two dimensions, a similar approximation leads to a logarithmic singularity in the momentum integration which is not cut off by Ω or T. It is necessary to keep the full screening from Eq. (3.28b). Performing the integral we find that if $DK_2^2 > \tau^{-1}$,

$$\frac{\delta N(\Omega)}{N_1} = \frac{1}{4\pi\varepsilon_F\tau}\ln|\Omega\tau|\ln\frac{\Omega\tau^{-1}}{(DK_2^2)^2} \tag{3.38a}$$

(Altshuler, Aronov, and Lee, 1980), while if $DK_2^2 < \tau^{-1}$,

$$\frac{\delta N(\Omega)}{N_1} = \frac{1}{4\pi\varepsilon_F\tau}\left[\ln\frac{\Omega}{DK_2^2}\right]^2 \tag{3.38b}$$

(Castellani, DiCastro, Lee, and Ma, 1984). This peculiar $\ln^2\Omega$ dependence can be traced to the Z factor of the quasiparticle coming from the dynamic nature of the screened interaction.

Recently Altshuler, Aronov, and Zyuzin (1984) have pointed out that in a tunneling experiment there is always a metallic electrode separated by an oxide barrier of thickness Δ. The image charge converts the Coulomb interaction to a dipolar one at sufficiently long distance, and the $\ln[\Omega\tau^{-1}/(DK_2^2)^2]$ factor in Eq. (3.38a) is replaced by a constant factor $\ln(2K_2\Delta)$. In a typical experimental setup, this logarithmic factor leads to a considerable enhancement, which is needed to bring theory into agreement with the experiment of Imry and Ovadyahu (1982c).

A second complication concerns the treatment of Hartree-type terms coming from the short-range part of the interaction. Initially, the perturbative results [terms proportional to $-2F$ in Eqs. (3.35) and (3.36)] were sim-

ply added to Eqs. (3.37) and (3.38), and it was thought to be a safe procedure if $F \ll 1$. However, this procedure is dangerous because the dynamically screened interaction includes all orders in perturbation theory, and it is not obvious that one can simply add more first-order terms. Recently Finkelshtein (1983) carried out an expansion to lowest order in $(k_F l)^{-1}$ but to all orders in the interaction coupling constant. He arrived at the rather surprising conclusion that, even if the parameter F is small, the Hartree term should be proportional to $-\frac{3}{2}F$ instead of $-2F$. Altshuler and Aronov (1983,1984) provided a very nice explanation of this observation. Essentially the particle-hole scattering should be divided into total spin singlet and triplet channels, and multiple scattering should be allowed between the particle and hole. The singlet channel has the feature that an interaction line can be an intermediate state. Upon summing an infinite series of such diagrams, we have an effective interaction of the form given by Eq. (3.27), but with $V_B(q)$ replaced by $V_B(q)-F/2$. Since $V_B(q)$ is singular for small q, the $F/2$ term is negligible, and we recover the dynamically screened term as given by Eq. (3.28), which in turn contributes the factor 2 in the $(2-2F)$ factor in Eq. (3.38). Thus basically the factor $-2F$ should be decomposed into $-(F/2+\frac{3}{2}F)$, and the $-F/2$ is really the first term in an infinite series that gets absorbed into the exchange contribution. Altshuler and Aronov (1983,1984) found that, in all formulas where $2F$ appears, it should be replaced by $\frac{3}{2}\widetilde{F}$, and $\widetilde{F}$ is a different function of F depending on whether one is calculating density of states or conductivity corrections, etc. However, in all cases $\widetilde{F}\to F$ for $F \ll 1$. For the density of states, the Hartree term, being proportional to $\ln|\Omega\tau|$, is always dominated by the exchange term in two dimensions for long-range interaction. This is not the case for the specific heat correction or for the conductivity. As already mentioned, the $\ln^2\Omega$ singularity arises from the Z factor, which in Fermi-liquid theory does not enter thermodynamic quantities. Indeed, calculation using the standard expression for the free energy,

$$\Delta F = \frac{1}{2}T\sum_{\omega,q}\int_0^1 d\lambda \frac{V_B(q)\Pi(q,\omega)}{1-\lambda V_B(q)\Pi(q,\omega)} \ , \tag{3.39}$$

leads to the following correction to the electronic specific heat:

$$\frac{\delta C}{C} = \frac{1-\frac{3}{2}\widetilde{F}_c}{4\pi^2(dn/d\mu)D}\left[\frac{T}{2D}\right]^{1/2} \tag{3.40a}$$

$$= \frac{1}{\pi\varepsilon_F\tau}(1-\frac{3}{2}\widetilde{F}_c)\ln|T\tau| \tag{3.40b}$$

for 3D and 2D, respectively, where $\widetilde{F}_c = 4[(1+F/2)^{d/2}-1]/d$ in d dimensions (Altshuler and Aronov, 1983,1984).

So far there has been no experimental confirmation of these predictions for the specific heat. An intriguing possibility is to consider small metallic particles that are elec-

trically isolated. If the particle size is small compared with $(D/T)^{1/2}$, this will constitute a collection of zero-dimensional samples, and the specific heat $\delta C/C$ is predicted to show a rather unusual T^{-1} behavior. (The theory is, of course, limited to $\delta C/C$ small.)

D. Conductivity, magnetoresistance, and magnetic susceptibility

The conductivity can be calculated using the Kubo formula to lowest order in the interaction. The appropriate diagrams are shown in Figs. 14(a)—14(e). These are generated from the free-energy diagram shown in Fig. 12 by inserting two external current vertices at all possible positions on the fermion lines. Figures 14(a)—14(c) can easily be shown to cancel each other. The remaining diagrams, combined with the Hartree version, give the following corrections to the conductivity:

$$\delta\sigma_I = -\frac{1}{A}\frac{e^2}{\hbar}\frac{1}{2\pi}(4-\tfrac{3}{2}\widetilde{F}_\sigma)(D/2T)^{1/2} \qquad (3.41a)$$

for 1D,

$$\delta\sigma_I = \frac{e^2}{\hbar}\frac{1}{4\pi^2}(2-\tfrac{3}{2}\widetilde{F}_\sigma)\ln(T\tau) \qquad (3.41b)$$

for 2D,

$$\delta\sigma_I = \frac{e^2}{\hbar}\frac{1}{4\pi^2}\frac{1.3}{\sqrt{2}}(\tfrac{4}{3}-\tfrac{3}{2}\widetilde{F}_\sigma)\sqrt{T/D} \qquad (3.41c)$$

for 3D, where $\widetilde{F}_\sigma = 8(1+F/2)\ln(1+F/2)/F-4$ in 2D, $\widetilde{F}_\sigma = -[32/d(d-2)][1+dF/4-(1+F/2)^{d/2}]F$ in $d\neq2$, and A is the wire cross-section area. We should also point out that the dynamic (imaginary) part of $v(q,\omega)$ makes a contribution equal to the static part in 2D and $\tfrac{1}{3}$ of the static part in 3D. This accounts for the factors 2 and $\tfrac{4}{3}$ in Eqs. (3.41b) and (3.41c), respectively. Details of the calculation can be found in Altshuler and Aronov (1979b) and in Altshuler, Khmel'nitskii, Larkin, and Lee (1980); in Eq. (3.41) we have made the $2F\rightarrow\tfrac{3}{2}\widetilde{F}_\sigma$ correction as discussed in the preceding section.

In Sec. II we discussed how even a weak magnetic field

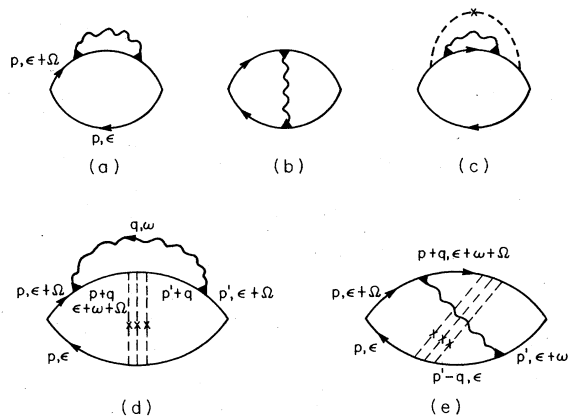(a)   (b)   (c)

(d)   (e)

FIG. 14. Diagrams for the correction to conductivity.

can suppress the localization effect. The origin of that effect is that the particle-particle channel is sensitive to the magnetic flux. The interaction effects we have discussed so far are all based on the particle-hole diffusion channel, so that similar sensitivity to the magnetic field does not occur. We shall return later to a discussion of the contribution of particle-particle scattering to the interaction effects. As far as the particle-hole channel is concerned, the dominant effect is the splitting of the spin-up and spin-down bands (Kawabata, 1981; Lee and Ramakrishnan, 1982; results in these papers need correction due to $-2F\rightarrow-\tfrac{3}{2}\widetilde{F}_\sigma$ which has been made in the results given below). The physical idea is most simply illustrated for the self-energy correction. As discussed in Sec. III.B, the singular correction is due to the correlation between the wave function of the added electron and the wave functions of the occupied electrons that are nearby in energy. In the presence of a magnetic field, the triplet term proportional to $-\tfrac{3}{2}F$ is divided into an $S_z=0$ and two $|S_z|=1$ terms. The exchange (singlet) and the $S_z=0$ triplet terms involve correlation with electrons with the same spin, and are unaffected by the spin splitting. This leaves $S_z=\pm1$ terms, and the spin splitting produces a gap $g\mu_B H$ between the lowest unoccupied spin-up electron and the highest occupied spin-down electron. The singularity of that term is therefore cut off for $g\mu_B H$ greater than kT. In a magnetic field, the correction to the conductivity can be written as a sum of two terms,

$$\delta\sigma_I(H,T)=\delta\sigma_I'(T)+\delta\sigma_I''(H,T) . \qquad (3.42)$$

The first term $\delta\sigma_I'$ is the field-independent exchange and $S_z=0$ Hartree contribution, and is the same as Eq. (3.41) except that the factors $(2-\tfrac{3}{2}\widetilde{F}_\sigma)$ and $(\tfrac{4}{3}-\tfrac{3}{2}\widetilde{F}_\sigma)$ are replaced by $2-F$ and $\tfrac{4}{3}-\tfrac{1}{2}\widetilde{F}_\sigma$, respectively. The second term is the $|S_z|=1$ triplet contribution. Its field dependence is given by

$$\delta\sigma_I''(H,T)-\delta\sigma_I''(0,T)=-\frac{e^2}{\hbar}\frac{\widetilde{F}_\sigma}{4\pi^2}g_2(h) \qquad (3.43a)$$

$$=-\frac{e^2}{\hbar}\frac{\widetilde{F}_\sigma}{4\pi^2}\sqrt{T/2D}\,g_3(h) , \qquad (3.43b)$$

for 2D and 3D, respectively, where

$$g_2(h)=\int_0^\infty d\Omega\frac{d^2}{d\Omega^2}[\Omega N(\Omega)]\ln\left|1-\frac{h^2}{\Omega^2}\right| \qquad (3.44a)$$

in 2D and

$$g_3(h)=\int_0^\infty d\Omega\frac{d^2}{d\Omega^2}[\Omega N(\Omega)]$$

$$\times(\sqrt{\Omega+h}+\sqrt{|\Omega-h|}-2\sqrt{\Omega}) \qquad (3.44b)$$

in 3D, and where $h=g\mu_B H/kT$. The zero-field contribution $\delta\sigma_I''(0,T)$ is the usual one, and is given by Eq. (3.41), with $(2-\tfrac{3}{2}\widetilde{F}_\sigma)$ and $(\tfrac{4}{3}-\tfrac{3}{2}\widetilde{F}_\sigma)$ replaced by $-\widetilde{F}_\sigma$.

The functions g_2 and g_3 can be computed numerically. They have the limiting behavior

$$g_2 = \begin{cases} \ln(h/1.3) & h \gg 1 \\ 0.084h^2, & h \ll 1 \end{cases} \qquad (3.45a)$$

and

$$g_3 = \begin{cases} \sqrt{h} - 1.3, & h \gg 1 \\ 0.053h^2, & h \ll 1. \end{cases} \qquad (3.45b)$$

It is important to point out that the spin-orbit scattering rate τ_{SO}^{-1} or spin-flip rate τ_s^{-1} has been ignored in the above theory. Spin scattering mixes the spin-up and spin-down channels, and we require $g\mu_B H \gg \tau_{SO}^{-1}$ or τ_s^{-1} in addition to $g\mu_B H \gg kT$ before the magnetoresistance due to the spin-splitting mechanism is operational. In heavy metal, such as Pt, τ_{SO}^{-1} can easily be large enough for this to be an important consideration. We also remark that in almost ferromagnetic materials, such as Pd, the internal field that gives rise to spin splitting may be much enhanced, making the magnetoresistance effect more readily observable. These effects and the effects of spin-orbit scattering on the magnetoresistance have been discussed by Millis and Lee (1984).

We now return to a discussion of the particle-particle channel contribution to the interaction effects. For simplicity we discuss the density-of-states corrections. Altshuler, Khmel'nitskii, Larkin, and Lee (1980) noted that Figs. 15(a) and 15(b) are the particle-particle version of Figs. 13 and 11 and yield equal contributions. These diagrams should be sensitive to the orbital effects of the magnetic field. The resulting magnetoresistance has been evaluated by Fukuyama (1980b) and by Altshuler, Aronov, Larkin, and Khmel'nitskii (1981). Fukuyama considered first-order perturbation theory in some coupling constant (his g_2 and g_4), whereas Altshuler et al. pointed out that it is necessary to sum a ladder involving repeated interactions between the electrons. A typical Hartree diagram is shown in Fig. 16. This replaces the coupling λ by the effective coupling

$$\tilde{\lambda} = \frac{\lambda}{1 + \lambda \ln(E_F/T_0)} , \qquad (3.46)$$

where $T_0 = \max(T, D/L_H^2)$ and $L_H = \sqrt{\hbar c/2eH}$ is the Landau orbit size. Equation (3.46) is very similar to the theory of superconductivity, except that for repulsive interaction the coupling constant scales to weak coupling. Indeed, if a phonon-induced attractive coupling λ_p is also present, the λ in Eq. (3.46) should be replaced by
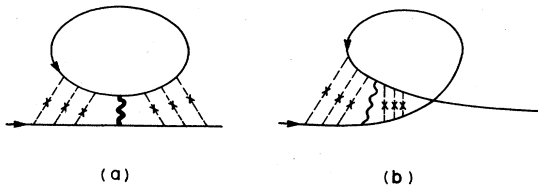

(a)   (b)

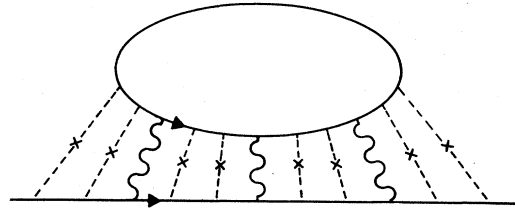FIG. 15. Particle-particle channel version of the self-energy correction.

FIG. 16. Ladder diagram version of Fig. 15(a).

$$\lambda \to \lambda_p + \frac{\mu}{1 + \mu \ln(E_F/\omega_D)} , \qquad (3.47)$$

where μ is the electron-electron interaction, and E_F in Eq. (3.46) should be replaced by ω_D. Equation (3.47) is very familiar in the theory of superconductivity, and for attractive λ_p it is more proper to think of the predicted anomalies as due to superconducting fluctuations. The surprising element is that, even for repulsive interaction, relatively strong temperature-dependent effects are predicted for the density of states and conductivity, even though the overall size of the effect is small compared with that given by Figs. 15(a) and 15(b) because of the renormalization of λ given by Eq. (3.46). Since the renormalization depends only logarithmically on T_0, the ladder sum can be approximated by a phenomenological coupling constant. From this point of view Fukuyama's theory is in basic agreement with that of Altshuler et al. if his g_2 and g_4 are understood to be phenomenological constants smaller than g_3 except when superconducting fluctuations are important.

According to Fukuyama and Altshuler et al., the particle-particle channel leads to positive magnetoresistance when the Landau orbit size becomes comparable to the thermal length $(D/T)^{1/2}$, i.e.,

$$\frac{2eH}{\hbar c} > \frac{kT}{D} . \qquad (3.48a)$$

For $k_F l \gg 1$ this occurs at a smaller field than the requirement for spin splitting discussed earlier,

$$g\mu_B H > kT \qquad (3.48b)$$

for normal values of g. Larkin (1980) has pointed out the importance of a class of diagrams involving the particle-particle channel which is analogous to the Maki-Thompson diagram for superconductivity. This produces a positive magnetoresistance when the field satisfies

$$\frac{2eH}{\hbar c} > \frac{1}{D\tau_{in}} . \qquad (3.49)$$

Usually the inelastic scattering rate τ_{in}^{-1} is smaller than kT, so that this occurs at an even weaker field than Eq. (3.48). In fact this effect takes the same form as the magnetoresistance of noninteracting electrons, due to the suppression of localization, except that the overall magnitude is very small for normal metals, being proportional to $\pi^2 \tilde{\lambda}^2/6$ for $|\tilde{\lambda}| \ll 1$.

The combination of the spin-splitting effect and the lo-

calizing effect leads to very rich behavior in the interacting model, and magnetoresistance is clearly a powerful tool for disentangling the two contributions. This is especially true in 2D, since the orbital contribution is sensitive only to the magnetic field component normal to the plane, whereas the spin-splitting term should be isotropic. Many positive magnetoresistance results can be analyzed using the spin-splitting and the localization terms only, which presumably means that $\tilde{\lambda}$ is small for these systems. There are apparently also other systems where $\tilde{\lambda}$ is not negligible, and it will be very interesting to separate experimentally the three magnetic field regimes discussed in Eqs. (3.48) and (3.49). The experimental situation will be discussed further in Sec. VI.

The spin-splitting effect also gives rise to a correction to the magnetic susceptibility. In a free-electron theory the spin susceptibility is proportional to the density of states, and one might naively expect the density-of-states corrections to appear in the susceptibility as well. However, this is not entirely true. In the presence of a magnetic field, the up-spin and down-spin bands are split, each with its own Fermi energy. The exchange correction involves the interaction between up- and down-spin particles separately. Thus the exchange correction of each spin band can be considered separately and is tied to its own Fermi energy. Consequently, in the exchange correction to the density of states the relative populations of the up and down spins cancel out, and the susceptibility is the same as the free-electron value. A similar situation is well known in the electron-phonon problem, where the density-of-states enhancement does not affect the spin susceptibility. The same consideration applies to the Hartree term involving equal spins. On the other hand, the Hartree interaction between up and down spins can modify the susceptibility. This was first shown by Fukuyama (1981a), who calculated the transverse spin-fluctuation correlation function in a disordered metal using a diagrammatic technique. He found an enhancement of the susceptibility, which moreover depends on scale size in the same way as conductivity. The same result can be obtained by considering the field-dependent part of the Hartree term for the free energy. For a zero-range interaction U, this is just the Stoner term

$$\delta F(H) = U \langle \psi_\uparrow^+(r)\psi_\uparrow(r)\psi_\downarrow^+(r)\psi_\downarrow(r) \rangle . \tag{3.50}$$

This can be rewritten in terms of the equal space, equal time limit of a spin-spin correlation function, i.e.,

$$\delta F(H) = \frac{U}{2}\langle n_\uparrow \rangle \langle n_\downarrow \rangle$$
$$- \lim_{\substack{r \to r' \\ t \to t'}} \left[\frac{U}{2}\langle T\{S^+(r,t)S^-(r',t')\} \rangle + \text{H.c.} \right] .$$
$$\tag{3.51}$$

In addition to the usual Hartree or Stoner term, one has a contribution here due to the diffusive motion of spin fluctuations. The spin splitting due to the magnetic field acts as a low-energy cutoff of the diffusion pole, so that δF is given by

$$\delta F(H) \sim -\frac{2U}{\tau^2}\sum_q \sum_m \frac{|\omega_m|\tau}{(|\omega_m| + Dq^2)^2 + 4\mu^2 H^2} . \tag{3.52}$$

It is clear that there is a susceptibility enhancement $\Delta\chi = -d^2\delta F/dH^2$. Further, in two dimensions the temperature-dependent part goes as

$$\Delta\chi \sim [2UN^2(0)/k_F l]\ln(T\tau)^{-1} , \tag{3.53a}$$

and in three dimensions

$$\Delta\chi \sim -[UN^2(0)/(k_F l)^{3/2}](T/\varepsilon_F)^{1/2} . \tag{3.53b}$$

The correction depends on temperature (or length scale or frequency) and on the diffusion constant in exactly the same way as the conductivity correction due to interactions. If the latter effect becomes large near the mobility edge, one might expect a corresponding susceptibility enhancement and slowing down of spin diffusion.

So far we have considered only the spin-splitting effect. Just as in the case of the density-of-states corrections, we must also consider the orbital effect on the magnetic susceptibility via the particle-particle channel. It turns out that this problem was investigated long ago by Aslamazov and Larkin (1974) in connection with superconducting fluctuations. They found that in the presence of disorder, corrections to the susceptibility persist much above the superconducting T_c, and indeed exist even for normal metals, when the electron-electron interaction is repulsive. The corrections take the same form as Eq. (3.53), except that they are proportional to $\tilde{\lambda}$ instead of U and the magnitude is enhanced by a factor $k_F l$. This is because the scale for the magnetic field is much smaller, being set by Eq. (3.48a) rather than Eq. (3.48b). These results are well summarized in Altshuler, Aronov, and Zyuzin (1983,1984). Up until now we know of no experimental test of this effect.

E. Electron-phonon interaction

The long-wavelength electron-phonon vertex is not enhanced by the diffusive motion of the electron in a random system, in contrast to the effect on the Coulomb vertex discussed in the preceding section. There are two reasons for this, as realized in essence by Pippard in his theory of ultrasonic attenuation in disordered metals (Pippard, 1955). First, the random scattering centers are embedded in the lattice and move with it, so that electron relaxation by scattering is in the moving frame. Second, electron density fluctuations induced by coupling to longitudinal phonons are perfectly screened by the electron gas.

The first microscopic calculation of the electron-phonon vertex in a disordered metal is due to Schmid (1973). He used the method of Tsuneto (1960), in which one transforms to a frame moving with the lattice. The appropriate canonical transformation leads to an interaction between the lattice strain and electron kinetic energy fluctuation or electronic stress. Schmid considered a Coulomb-interacting electron gas, and showed that because of perfect screening the diffusion enhancement of

the electron-longitudinal-phonon vertex is cancelled to order $(k_F l)^{-1}$. Since the transverse phonon does not couple to electron density fluctuations, no diffusion enhancement is expected for it. An alternative direct many-body analysis, which does not make use of the canonical transformation but treats scattering from moving impurities explicitly, has been discussed by Eisenriegler (1973) and by Grünewald and Scharnberg (1974,1975). This work shows in detail how the impurity-motion-dependent part of the phonon self-energy largely cancels the static term, and how screening of all the bare long-range Coulomb interactions (ion-electron, impurity-electron, electron-electron) is crucial.

The decay of a phonon in a metal into an electron-hole pair depends on the effective electron-phonon coupling, i.e., the electron-phonon vertex. The decay rate resulting from the correct unenhanced vertex is that calculated semiclassically by Pippard (1975). For example, the attenuation coefficient for a longitudinal phonon of wave vector q is found to be

$$\alpha(q) = \frac{4}{5} \left[\frac{v_s}{v_F} \right] (qv_s)(ql) ,\qquad (3.54)$$

where v_s is the sound velocity and l the mean free path. This expression is valid for $ql \ll 1$. The attenuation is small because of the mismatch between the sound velocity v_s and the Fermi velocity v_F. It is proportional to q^2, whereas the frequency depends linearly on q, so that long-wavelength phonon modes are well defined. Rather surprisingly, the attenuation decreases with decreasing l, i.e., increasing disorder; this is because stronger impurity scattering makes equilibrium easier. Equation (3.54) is correct to lowest order in impurity scattering and does not therefore include either effects due to incipient localization or effects due to interaction.

The result that the electron-phonon vertex is unenhanced in a disordered metal has implications for superconductivity, as discussed by Keck and Schmid (1975,1976). To leading order in $(k_F l)^{-1}$, the attractive part arising from exchange of phonons with $q \ll l^{-1}$ is unaffected. Clearly, the part due to phonons with $q \gg l^{-1}$ is not changed. The phonon-mediated coupling is of short range $(\sim q_D^{-1})$, so that over most of phase space there is no enhancement. Keck and Schmid find model-dependent corrections from the regime $q \sim l^{-1}$, e.g., an increase in the attractive term from shear modes.

The fact that the long-wavelength electron-phonon vertex is not disorder enhanced means that there are no characteristic effects on conductivity, etc., due to the exchange of phonons, of the sort discussed for Coulomb interactions. However, there is a Hartree-type electron-phonon interaction term (similar to that in Fig. 13), which contributes only when the system is disordered, since in a clean system the process describes the exchange of a $q = 0$ phonon or uniform lattice translation (Ramakrishnan, 1984). In a disordered system, due to scattering, there are local short-range fluctuations in the electron density which produce a lattice distortion. These fluctuations

couple to other electrons. The effect of this polaronic process, to lowest order in electron-phonon coupling, is similar to that of the Hartree-type Coulomb interaction term, except that the sign is opposite. It thus leads to a reduction in the density of states and the scale-dependent conductivity.

F. Scaling theory of the disordered interaction problem

So far we have treated the interaction only in lowest-order perturbation theory and in the weak-impurity scattering regime. One would like to extend the theory to the region of the metal-insulator transition in three dimensions, where the interaction and localization effects are both strong. The hope is that a scaling theory for the combined interaction and disorder problems exists, so that one can obtain a description of the transition region in $(2+\varepsilon)$ dimensions. It is worth noting that the effects of interaction discussed here require the presence of disorder, since the diffusive motion of the electrons plays a crucial role. Thus the interaction-driven metal-insulator transition discussed in this section must not be confused with the Mott-Hubbard transition (see Mott, 1974), which is driven by correlation effects due to Coulomb or short-range interaction, in the absence of disorder.

McMillan (1981) was the first to write down a scaling description of the disordered interaction problem, basically by extrapolating the perturbation expansion from the coupling constant. He proposed that the one-parameter scaling description on the noninteracting problem be expanded to a two-parameter problem, with the dimensionless interaction constant as the new scaling parameter in addition to the conductance. In the development of the theory, McMillan made the assumption that the screening constant and the conductivity are related to the single-particle density of states $N(0)$, i.e., $K^2 = 4\pi e^2 N(0)$ and $\sigma = e^2 N(0)D$, and the singularity in $N(0)$ discussed in Sec. III.B plays an important role in his scaling process. This assumption has been criticized by Lee (1982), who pointed out that K^2 and σ should be proportional to $dn/d\mu$, the change in density with chemical potential, which, unlike $N(0)$, has no singular corrections. Consequently the relations derived by McMillan between various exponents should not be trusted. However, as pointed out by Grest and Lee (1983), many of the features in the McMillan theory are generic to any two-parameter scaling theory, and as such the theory is useful as a starting point for data analysis. For example, the conductivity as a function of temperature T is predicted to take the form

$$\sigma(T) = \sigma(0)[1 + C(T/\Delta)^{(d-2)/2}], \quad E < \Delta , \qquad (3.55a)$$

$$\sigma(T) = E^\alpha, \quad E > \Delta , \qquad (3.55b)$$

where Δ is a characteristic energy scale that vanishes as a power of the distance to the metal-insulator transition, as measured, for example, by $\delta n = n - n_c$, where n_c is the critical dopant concentration in the case of doped semiconductors,

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$$\Delta \approx (\delta n)^p .$$ (3.56)

This energy scale separates the region $T < \Delta$, where the $T^{1/2}$ behavior predicted by perturbation theory [see Eq. (3.41c)] is still valid, from the "critical region" $T > \Delta$, where a new critical exponent appears, as given by Eq. (3.55b). In Eq. (3.55a) $\sigma(0)$ itself vanishes at the metal-insulator transition

$$\sigma(0) = \delta n^\mu .$$ (3.57)

By demanding continuity at $T = \Delta$ in Eq. (3.55), one immediately obtains the scaling relation

$$\mu = \alpha p .$$ (3.58)

General considerations of this kind also apply to the behavior of the density of states in the vicinity of the metal-insulator transition and have been used to analyze data (see Sec. VI).

Various attempts at a microscopic deviation of scaling theory by extending the perturbation theory to higher order have been unsuccessful. The attempt to construct a $1/n$ expansion, where n is the number of orbitals per site, is incomplete (Oppermann, 1982; Ma and Fradkin, 1983). Grest and Lee (1983) attempted a brute force calculation on the perturbation theory to second order in the coupling constant V. First they considered the simpler case, where the maximally crossed diagrams are suppressed by time-reversal-symmetry-breaking fields. They found a perturbation series for various physical quantities like conductivity and magnetic susceptibility of the form

$$\sigma = \sigma_0 [1 + a_1 Vt \ln\omega + a_2 (Vt)^2 \ln^2\omega + \cdots] .$$ (3.59)

By demanding that these physical quantities scale multiplicatively, Grest and Lee constructed scaling equations. However, it was pointed out by Castellani et al. (1983) that certain anomalous diagrams were left out, which give terms of the form $bVt^2\ln^2\omega$ in Eq. (3.59). The number of diagrams of order $V^2t^2\ln^2$ proliferates, and some recent progress was made in computing the corrections to the density of states (Castellani, DiCastro, and Forgacs, 1984).

A more promising approach is via the field theory method (see Sec. V). Finkelshtein (1983) has constructed a field theory for the interacting fermion problem, with the maximally crossed diagrams suppressed. He then constructed a renormalization group treatment of the field theory. Castellani, DiCastro, Lee, and Ma (1984) have rederived his results using conventional diagrammatic techniques. Here we simply summarize a few key results. Due to the necessity of summing over the frequencies of the electrons, which are analogous to external magnetic fields in the field theory, the renormalization group is somewhat different from the usual ones encountered in critical phenomena. As interpreted by Castellani et al. (1984a), the renormalization group is one in which strips in frequency and momentum space given by

$$0 < z \,|\, \omega \,|\, \tau < \lambda, \quad \lambda' < Dq^2\tau < \lambda$$
and

$$\lambda' < z \,|\, \omega \,|\, \tau < \lambda, \quad 0 < Dq^2\tau < \lambda'$$ (3.60)

are successively integrated out. Compared with the noninteracting problem, the new feature is that a frequency scale renormalization denoted by z is introduced.

Finkelshtein treated the long-range Coulomb interaction problem. His only expansion parameter is $(\varepsilon_F\tau)^{-1}$, and he assumed weak disorder and treated the interaction to all orders. His results are surprising in that, in two dimensions, the system scales towards weaker and weaker disorder, so that the conductivity scales towards infinity. In $(2+\varepsilon)$ dimensions, his theory predicts that the system always stays metallic, and a metal-to-insulator transition is impossible. Recently a factor-of-2 error was found in the nonlinear term in Finkelshtein's scaling equation for the interaction coupling constant (Castellani, DiCastro, Lee, Ma, Sorella, and Tabet, 1984b; Finkelshtein, 1984b). The conclusions of the original Finkelshtein solution are changed considerably. The conductivity is still enhanced as temperature decreases, but it now saturates to a constant in the zero-temperature limit. The mechanism for the conductivity enhancement can be seen from perturbation theory. In Eq. (3.41b) the triplet contribution proportional to $\frac{3}{2}\tilde{F}_\sigma$ enhances conductivity. In Finkelshtein's theory, the parameter F is replaced by a scattering amplitude Γ_2. Upon scaling, $\frac{3}{2}\tilde{F}_\sigma$ is replaced by a function of Γ_2/z. According to the modified scaling equations, Γ_2 and z and Γ_2/z all scale to infinity. The triplet term overcomes the exchange term in Eq. (3.41b), leading to a net conductivity enhancement.

The spin triplet diagrams leading to the conductivity enhancement are very much reminiscent of the ladder diagrams in paramagnon theories (see, for example, Brinkman and Engelsberg, 1968). Castellani et al. (1984b) and Finkelshtein (1984b) calculated the spin susceptibility to be

$$\chi = (z + \Gamma_2)\chi_0 ,$$ (3.61)

where χ_0 is the free-electron spin susceptibility. Thus χ scales to infinity at low temperatures. However, unlike the paramagnon theory, in this problem the frequency scale z also scales to infinity, and a finite length scale is generated, so that χ is independent of length scale up to a finite cutoff. This suggests a picture of localized spin fluctuations. The triplet spin-density propagator takes the form $[Dq^2 - i\omega(z + \Gamma_2)]^{-1}$, so that the spin-diffusion constant is given by

$$D_s = \frac{D}{z + \Gamma_2} ,$$ (3.62)

which scales to zero, even though the charge-diffusion constant D scales to a finite value. The vanishing of the spin-diffusion constant reinforces the picture that strong local spin fluctuations develop upon scaling.

The real difficulty with the modified Finkelshtein solution is that scaling is towards strong coupling, so that the model eventually breaks down. In any event, a model that simply suppresses maximally crossed diagrams does not describe the true zero-temperature limit. In a physical sit-

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uation, these diagrams are suppressed with a magnetic field or with spin-flip scattering. In either of these situations, the spin degrees of freedom are affected, and new solutions are expected to appear.

From the above discussion, it is clear that triplet fluctuations are responsible for the scaling towards a conductor. If the triplet fluctuations are modified, it should be possible to produce a theory that scales to an insulator in two dimensions, and, more interestingly, one that describes a metal-to-insulator transition in $(2+\varepsilon)$ dimensions. This possibility has been realized in two different models.

(i) Singlet-only model. The condition for this model is strong spin-flip scattering, $\tau_s^{-1} \gg kT$, or strong spin-orbit scattering, $\tau_{SO}^{-1} \gg kT$, with a small magnetic field which suppresses the maximally crossed diagrams.

(ii) Strong magnetic field, so that spin splitting of the bands occurs. The conditions are $g\mu_B H \gg kT$ and $g\mu_B H \gg \tau_s^{-1}$ or τ_{SO}^{-1}. These conditions are naturally satisfied in ferromagnetic metals, in which case H is the internal field.

Model (i) was first discussed by Altshuler and Aronov (1983), and the scaling theory for both models was worked out by Finkelshtein (1984a) and by Castellani, Di-Castro, Lee, and Ma (1984a). We summarize some of the results below. We note that, unlike the original Finkelshtein model, in both models (i) and (ii) the suppression of the maximally crossed diagrams is a natural consequence of the model. Furthermore, the theory involves an expansion in weak disorder only, and should be accurate in the vicinity of the metal-insulator transition in $(2+\varepsilon)$ dimensions. Thus these results should describe the metal-insulator transition in physically realizable situations.

We first give the results for the long-range Coulomb interaction problem, and we begin with the results for model (ii) because it is simpler. In fact, as far as the conductivity is concerned, its critical behavior is the same as that of the noninteracting theory. In two dimensions, a universal logarithmic correction is predicted,

$$\sigma(T) = \sigma_0 + \frac{e^2}{2\pi^2\hbar}(2 - 2\ln 2)\ln T\tau . \qquad (3.63)$$

Basically, the Hartree-type correction in Eq. (3.41) scales to a constant, and the parameter $\widetilde{F}_\sigma$ disappears from the problem. In $(2+\varepsilon)$ dimensions, near the fixed point, the scaling equation for the conductance g takes the same form as the noninteracting case given in Eq. (2.14), except that the constant a is different. Since the critical exponents are independent of this constant, they are the same as in the noninteracting case. Furthermore, the frequency renormalization factor z scales to a constant in this case, so that the critical behavior in frequency or momentum space is also the same as in the noninteracting case. The conductivity obeys Eq. (3.55), with $\alpha = (d-2)/d$, by the same arguments as given in Eq. (2.38). The exponent $p = 2/\varepsilon + O(1)$ and $\mu = 1 + O(\varepsilon)$ to the accuracy of this calculation. Furthermore, the coefficient of the $T^{\varepsilon/2}$ correction for small T should diverge as $(\delta n)^{p\varepsilon/2 - \mu} \approx (\delta n)^{(d-2)/d}$. These predictions should be

amenable to experimental tests.

The dielectric constant on the insulator side can be estimated from the scaling theory, as done in Sec. II.F, and we find that the relation between the dielectric constant and the conductivity on the metallic side, given by Eq. (2.46), i.e., $\varepsilon'\sigma^2 = $ const, still holds.

The single-particle density of states at the Fermi level observable in tunneling experiments is very different from the noninteracting case, in that it is predicted to vanish at the metal-insulator transition. It is predicted by Finkelshtein (1984a), to vanish as a power law $N(\Omega) \sim \Omega^\beta$, with $\beta^{-1} \approx 2(2 - 2\ln 2)$, whereas Castellani et al. (1984a) predict $N(\Omega) \sim \exp\{-[(d-2)/8(2-2\ln 2)]\ln^2\Omega\}$. The difference between the two predictions has to do with different ways of taking the $\varepsilon \to 0$ limits.

The above results were derived for a dynamically screened Coulomb interaction. One can also imagine experimental situations where a short-range interaction is appropriate. Examples are disordered neutral fermions such as He$_3$ on a disordered substrate, or a two-dimensional electron gas screened by a nearby metallic sheet. The case of short-range interaction was treated by Castellani et al. (1984a). The critical exponents are unchanged for model (ii). The density of states is predicted to vanish as a power law with nonuniversal exponents, and the coefficient of $\ln T$ in Eq. (3.63) is nonuniversal.

We next discuss the results for the singlet-only problem [model (i)]. As first pointed out by Altshuler and Aronov (1983), the spin scattering cuts off the diffusion pole in the triple channel, so that in perturbation theory only the exchange term survives. For long-range interactions, we have in 2D

$$\sigma = \sigma_0 + \frac{e^2}{2\pi^2\hbar}\ln T\tau . \qquad (3.64)$$

This prediction should be easily tested experimentally. For example, one can study a thin film with strong spin-orbit scattering with a small normal magnetic field to suppress weak antilocalization effects. A recent experiment by Nishida et al. (1983) on $Si_{1-x}Au_x$ films appears to confirm this prediction.

In $(2+\varepsilon)$ dimensions, the scaling equation also takes the form of the noninteracting problem, Eq. (2.14). However, there is one important difference, as pointed out by Finkelshtein (1983b). In this model the frequency renormalization factor z in Eq. (3.60) scales to zero. Thus the renormalization procedure differs from the noninteracting case in frequency scale, while Eq. (3.55) continues to hold; the critical exponent α is modified to be $\alpha = \varepsilon/(2 + \varepsilon/2)$, while μ and p remain the same to lowest order in ε. The relation between the dielectric constant ε' and the conductivity measured equidistant from the metal-insulator transition is modified to read

$$\varepsilon'\sigma^2 = (\delta n)^\delta , \qquad (3.65)$$

where the exponent $\delta = p\varepsilon/4 \approx \frac{1}{2} + O(\varepsilon)$. The tunneling density of states is also expected to vanish at the transition with a different behavior, according to Finkelshtein (1984a) and to Castellani et al. (1984a).

Finally it is interesting to note that, for short-range interactions the singlet-only model behaves very differently. Castellani *et al.* (1984a) found that the interaction constant scales to zero, so that the noninteracting theory describe the metal-insulator transition. The density of states remains finite at the transition in this case.

In summary, we now have available, in certain cases, scaling theories for the metal-insulator transition that include interaction effects. In all these theories, the weak localization effects discussed in Sec. II are explicitly suppressed. [We should note that Finkelshtein (1984b) treated the localization effects in lowest order and argued that they are unimportant.] While these special cases may describe certain experiments, the important problem of the full interplay between localization and interaction is not yet addressed. What is clear, however, is that one should expect to find a number of different universality classes, depending on the experimental situation.

IV. NUMERICAL TESTS

It is in principle straightforward to test the predictions of localization theories by performing numerical solutions of the Schrödinger equation on a finite lattice. The brute force diagonalizing of an $N \times N$ matrix required to describe a lattice of N sites requires computer storage proportional to N^2 and processing time proportional to N^3. Furthermore, since we are dealing with a random system, averaging over many lattices is required. Thus the lattice size that can be studied by the brute force method is severely limited. Several different methods have been devised to reduce the storage and time requirements. At the same time there are a number of different ways to make measurements on the lattices and extract results that are related to the conductivity. Some of the approaches are summarized below.

(i) Yoshino and Okazaki (1977) directly observed the eigenstates of random lattices up to 100×100. They obtained a graphic demonstration of localized and extended states. However, since only a few states are studied at a time, this method does not permit quantitative studies of the conductivity.

(ii) Computation of the conductivity using the Kubo formula. This method was used by Stein and Krey (1980) and more recently by Yoshino (1982). A finite energy resolution γ is required for studies of finite systems, so that one effectively averages over transitions between a number of eigenstates. The energy scale γ plays the role of an inelastic scattering rate and effectively limits the sample size to $L_{\mathrm{Th}} = (D/\gamma)^{1/2}$. Careful extrapolation to $\gamma = 0$ has to be made. Even in one dimension it is not simple to reproduce the known results that all states are localized (Czycholl and Kramer, 1980). This problem was clarified in detail by Thouless and Kirkpatrick (1981).

(iii) Licciardello and Thouless (1975) related the conductivity to the sensitivity of the eigenvalues to changes in boundary conditions. The advantage of this technique

is that only eigenvalues, not eigenvectors, need to be computed. When combined with the techniques of diagonalizing sparse matrices (Edwards and Thouless, 1976), this becomes a highly efficient method. The first hints that all states may be localized in two dimensions came from numerical studies using this method.

(iv) Instead of diagonalizing the Hamiltonian, an alternative approach is to follow the time development of a wave packet. If the conductivity is finite, the wave packet should spread according to the diffusion constant. This method was implemented by Prelovsek (1976). However, Prelovsek's choice of the initial wave packet led to oscillations in the time development. This can be minimized by an optimal choice of the initial wave packet (Sher, 1983).

(v) Approximate methods. When the scaling idea for localization was first proposed, attempts were made to implement these ideas numerically. The approximation involves either truncation on the basis idea set every time the lattice dimension is doubled (Lee, 1979) or the introduction of an effective Hamiltonian (Sarker and Domany, 1980,1981). As the lattice size becomes large, the approximation becomes uncontrolled, and the result is either inconclusive or erroneous.

(vi) Localization may be studied by computing the transmission matrix T through a disordered region and relating it to the conductivity via some generalized Landauer formula (Landauer, 1970); the Landauer formula states that the conductivity is given by $(e^2/\hbar) |T|^2/(1 - |T|^2)$ in one dimension. The generalization of this formula to higher dimensions is a subtle problem and depends on assumptions about equilibrium in the wires outside the sample. This problem was discussed by Thouless (1981) and treated in detail by Langreth and Abrahams (1981), and more recently by Büttiker, Imry, Landauer, and Pinhas (1985). The transmission coefficient can be computed quite efficiently. Lee and Fisher (1981) computed the conductivity of 16^2, 32^2, and 64^2 samples, using an approximate form of the generalized Landauer formula (Fisher and Lee, 1981). They found that even for relatively weak disorder, $(W/V = 4)$, the conductivity decreases with increasing sample size in a way consistent with the scaling theory and the perturbation results. These methods are easily extended to include a magnetic field normal to the two-dimensional sample. The conductivity no longer decreases with increasing sample size, again in agreement with the perturbation theory, since a magnetic field destroys time-reversal symmetry. Lee and Fisher (1981) also reported that for exponentially localized states, the localization length is *increased* with the application of a magnetic field.

(vii) Finite-size scaling methods. In this method one calculates the properties of a long chain with finite cross section (width M for a 2D strip and $M \times M$ for a 3D block) and studies how the properties of the chain scale with M (Pichard and Sarma, 1981; MacKinnon and Kramer, 1981). In practice the localization length $\lambda(M, W/V)$ is calculated for the strip or block. In analogy with critical phenomena (Nightingale, 1976), it is pro-

posed that if a scaling theory exists, then $\lambda(M, W/V)$ should scale in the following way:

$$\frac{\lambda(M, W/V)}{M} = f_d[\lambda_\infty(W/V)/M] \qquad (4.1)$$

and $\lambda_\infty(W/V)$ should be the characteristic length scale for a sample of infinite cross section which is identified as the 2D or 3D localization length if the system is localized $[W/V > (W/V)_c]$. The advantage of this scheme is that $\lambda(M, W/V)$ can be calculated efficiently and with arbitrary accuracy. The statistical error can be controlled better than in the direct computation of conductivity. The scaling relation can be tested using the numerical data and found to be satisfactory. The length scale λ_∞ can be extracted. With this method MacKinnon and Kramer (1981) concluded that states are localized in two dimensions down to $W/V = 2$. Furthermore, by making an additional assumption relating the conductivity with λ_M, MacKinnon and Kramer (1983) constructed numerical scaling functions $\beta(g)$ for 1, 2, and 3 dimensions. These are consistent with the scaling picture of Abrahams et al. (1979). While this method involves a number of assumptions, it provides a convincing consistency test of the scaling theory. Recently these calculations were extended to include the presence of a magnetic field (MacKinnon and Kramer, 1983).

V. FIELD THEORY DESCRIPTION OF THE LOCALIZATION PROBLEM

The close relation between the localization problem and problem of critical phenomena suggests that a mapping of the localization problem to a field theory should be possible. This mapping was first accomplished by Wegner (1979) and made precise by Schäfer and Wegner (1980). It has since been elaborated upon and extended by many authors. Much of the insight that led to the mapping was gained from an earlier study of a model with n orbitals per site, and a perturbation expansion was carried out to lowest order in $1/n$ by Oppermann and Wegner (1979). The $1/n$ expansion is very similar to an expansion in weak disorder, i.e., in $D_0^{-1} \sim (\varepsilon_F \tau)^{-1}$, and we shall follow the latter approach. A detailed and lucid derivation of the field theory approach can be found in McKane and Stone (1981), and Wegner (1982) has provided a concise review article. We shall not reproduce the derivation here. Instead we shall describe the resulting field theory and make qualitative comments.

It has long been appreciated that the impurity-averaged Green's function

$$\overline{G}(r,r',E \pm i\eta) = [G(r,r',E \pm i\eta)]_{av}, \qquad (5.1)$$

where $G(r,r',E \pm i\eta) = \langle r \mid E \pm i\eta - H) \mid r' \rangle$, contains no information on the localization properties of the wave function. For example, the density of states

$$N(E) = \frac{1}{\pi} \mathrm{Im}[\overline{G}(r,r,E - i\eta)] \qquad (5.2)$$

is completely smooth as E varies across the mobility edge.

To study localization, it is necessary to consider the impurity average of the two-particle Green's function

$$K(r,r',\eta) = \langle \mid G(r,r',E + i\eta/2) \mid^2 \rangle_{av}, \qquad (5.3)$$

which is related to the density-density correlation function. To represent Eq. (5.3) as a functional integral, it is necessary to introduce field variables φ^1 and φ^2 corresponding to $G(E + i\eta/2)$ and $G(E - i\eta/2)$, respectively. Furthermore, we replicate the fields to n components φ_α^p, $p = 1,2$, $\alpha = 1, \ldots, n$. This permits the impurity averaging of the Green's function, provided the limit $n \to 0$ is taken at the end of the calculation. Suppose the Hamiltonian is given as

$$H = \sum_{r,r'} v_{r,r'} \mid r \rangle \langle r' \mid, \qquad (5.4)$$

where $v_{r,r'}$ is a random tight-binding potential. For real $v_{r,r'}$ we can choose the φ's to be real. On the other hand, if time-reversal symmetry is broken, $v_{r,r'}$ is in general complex, and the φ's are to be complex. In terms of the φ fields, the correlation functions are given as

$$\overline{G}(r,r',E \pm i\eta) = \mp i \langle \varphi_\alpha^p(r)\varphi_\alpha^{p*}(r') \rangle_{\mathscr{H}}, \qquad (5.5)$$

where $p = 1$ (2) for $+$ $(-)$, and

$$K(r,r') = \langle \varphi_\alpha^1(r)\varphi_\alpha^{1*}(r')\varphi_\beta^2(r')\varphi_\beta^{2*}(r) \rangle_{\mathscr{H}}. \qquad (5.6)$$

In Eqs. (5.5) and (5.6), the averaging $\langle \ \rangle_{\mathscr{H}}$ denotes an averaging over the effective Hamiltonian $\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_1$, where

$$\mathscr{H}_0 = -\frac{i}{2} \sum_{rr'} (E\delta_{rr'} - v_{rr'}) \sum_\alpha [\ \varphi_\alpha^1(r)\varphi_\alpha^1(r')$$
$$-\varphi_\alpha^2(r)\varphi_\alpha^2(r')] \qquad (5.7)$$

and

$$\mathscr{H}_1 = -\frac{\eta}{4} \sum_{r,\alpha} [\varphi_\alpha^1(r)]^2 + [\varphi_\alpha^2(r)]^2. \qquad (5.8)$$

The impurity averaging over $v_{rr'}$ can now be performed in Eqs. (5.5) and (5.6) in the usual way. But before we do that, we point out an important symmetry in the effective Hamiltonian. As noted by Wegner, $\mathscr{H}_0$ is invariant under a global transformation, which leaves

$$I = \sum_\alpha (\varphi_\alpha^{1*}\varphi_\alpha^1 - \varphi_\alpha^{2*}\varphi_\alpha^2) \qquad (5.9)$$

invariant. The negative sign in Eq. (5.9) comes from the requirement that the representation of $G(E \pm i\eta)$ in terms of the $\varphi_\alpha^{(1)}$ and $\varphi_\alpha^{(2)}$ fields be convergent. For real fields φ, this means that while $\mathscr{H}_0$ is invariant under an $0(n)$ rotation among the φ^1 and φ^2 fields, it is not invariant under $0(2n)$. Instead the symmetry group is $0(n,n)$. On the other hand, $\mathscr{H}$ is invariant under $0(2n)$ and not $0(n,n)$. Thus $\mathscr{H}$ breaks the $0(n,n)$ symmetry and η plays the role of a symmetry-breaking field, analogous to a small magnitude field in a ferromagnet. The field conjugate to η is

$$\langle \varphi_\alpha^{1*}(r)\varphi_\alpha^1(r) + \varphi_\alpha^{2*}(r)\varphi_\alpha^2(r) \rangle = i[\overline{G}(E + i\eta/2)$$
$$-\overline{G}(E - i\eta/2)]. \qquad (5.10)$$

According to Eq. (5.2), this equals $2\pi N(E)$ in the $\eta \rightarrow 0$ limit. Thus the density of states play the role of an order parameter. We have a peculiar situation where the order parameter is finite in both the localized and extended phases, a situation that is possible only by virtue of the $n \rightarrow 0$ limit of the field theory. The case of complex fields is similar except that the symmetry group is $U(n,n)$.

The invariant Eq. (5.9) is described by Shäfer and Wegner (1980) in terms of pseudounitary transformations, i.e., H_0 is invariant under $\varphi \rightarrow \hat{T}\varphi$ where $\varphi = (\varphi^1, \varphi^2)$ and $\hat{T}$ obeys the relation

$$\hat{T}^+ \hat{T} = \hat{T}\hat{T}^+ = 1 \ . \tag{5.11}$$

Here

$$\hat{T}^+ = \jmath^\dagger \hat{T}^\dagger \jmath \tag{5.12}$$

and

$$\jmath_{\alpha\alpha'}^{pp'} = \jmath_\rho \delta_{pp'} \delta_{\alpha\alpha'} \ , \tag{5.13}$$

where $\jmath_1 = i$ and $\jmath_2 = -i$. In the orthogonal case, $\hat{T}$ is constrained to be a real matrix.

The next step is to average over the impurities and to introduce the composite operator $Q_{\alpha\beta}^{pp'}(\mathbf{r})$, which basically plays the role of $(s_p s_{p'})^{1/2} \varphi_\alpha^{p*}(\mathbf{r})\varphi_\beta^{p'}(\mathbf{r})$. Under the transformation $\varphi \rightarrow \hat{T}\varphi$, we have

$$Q \rightarrow TQT^+ \tag{5.14}$$

where

$$T = \jmath^{1/2}\hat{T}\jmath^{\dagger 1/2} \ . \tag{5.15}$$

For complex φ, $\hat{T}$ is pseudounitary and so is T. For real φ, $\hat{T}$ is real and T^{11}, T^{22} are real whereas T^{12}, T^{21} are imaginary. The latter transformation is referred to as pseudo-orthogonal.

In terms of the Q matrices, the correlation functions are

$$\langle Q_{\alpha\beta}^{pp'} \rangle_{\mathscr{H}} = \delta_{pp'}\delta_{\alpha\beta}G(\mathbf{r},\mathbf{r},E \pm i\eta) \tag{5.16}$$

and

$$K(r,r') \propto \langle Q_{\alpha\beta}^{12}(r)Q_{\beta\alpha}^{21}(r')\rangle_{\mathscr{H}} \ . \tag{5.17}$$

The average in Eqs. (5.16) and (5.17) is over an effective Hamiltonian $\mathscr{H}$, which is a functional of the Q matrices. Since impurity averaging has been performed, $\mathscr{H}$ is no longer a random Hamiltonian. The important point about $\mathscr{H}$ is that it contains a part $\mathscr{H}_0$ that is invariant under the transformation, Eq. (5.14), i.e., $0(n,n)$ or $U(n,n)$ symmetry. There is a second part of $\mathscr{H}$ that is proportional to η, which breaks the $0(n,n)$ or $U(n,n)$ symmetry but which preserves the $0(n) \times 0(n)$ or $U(n) \times U(n)$ symmetry of rotation among the φ^1 or φ^2 components alone. According to Eq. (5.16) there is always a spontaneous breaking of the $0(n,n)$ or $U(n,n)$ symmetry.

From this point on, Wegner proceeded in direct analo-

gy with the well-studied vector-spin $0(n)$ model,

$$H = \sum_{\gamma,\gamma'} J_{rr'} \sum_\alpha s_\alpha(r)s_\alpha(r') \ , \tag{5.18}$$

where s_α has n components $\alpha = 1, \ldots, n$. Equation (5.18) has $0(n)$ symmetry, which is spontaneously broken. The important fluctuations at low temperature are the angular fluctuations of the spin. The amplitude fluctuations are negligible. More precisely, suppose the symmetry is broken in the direction $s_0 = (1,0,\ldots,0)$. The important degrees of freedom are those obtained by operating the rotation operator 0 on s_0, i.e., $s(r) = 0(r)s_0$ such that $0(r)$ is slowly varying in space. Instead of the full Hamiltonian, we can study

$$\mathscr{H}_s = \int d\mathbf{r}\{\tfrac{1}{2}\kappa[\nabla\jmath(r)]^2 + \mathbf{H}\cdot\mathbf{s}\} \ , \tag{5.19}$$

where κ is the spin stiffness constant.

By definition, $\jmath$ is constrained to have unit length $\jmath \cdot \jmath = 1$. Thus we have removed the irrelevant amplitude fluctuations from the original Hamiltonian. Note that any 0 that can be written as $0 = 0_1 0_2$, where 0_2 is an $0(n-1)$ rotation among the $\alpha = 2, \ldots, n$ spin components, generates the same $\jmath$ for any 0_2. Thus $\jmath$ is defined not on the space $0(n)$ but on the coset space $0(n)/0(n-1)$, which identifies all $0(n-1)$ rotations. The analogous procedure for the localization problem is to introduce the Hamiltonian

$$\mathscr{H} = \int d\mathbf{r} \left[\frac{1}{2t}(\nabla\tilde{Q}\nabla\tilde{Q}) + i\eta \sum_\alpha (\tilde{Q}_{\alpha\alpha}^{11} - \tilde{Q}_{\alpha\alpha}^{22}) \right] \ . \tag{5.20}$$

The matrices $\tilde{Q}$ are generated just as in the $0(n)$ spin case. We first consider the $0(n,n)$ symmetry to be spontaneously broken, and we choose a particular direction of the "quadrupole" so that $\tilde{Q}_0 = \jmath$ where the matrix $\jmath$ is defined in Eq. (5.13). Then $\tilde{Q}$ is the field generated by

$$\tilde{Q}(r) = T(r)\jmath T^+(r) \ , \tag{5.21}$$

where T is any pseudo-orthogonal or pseudounitary operator. Again, for any T that can be written as $T_1 T_2$ where T_2 is a unitary rotation in the φ^1 or φ^2 subspace, the resulting $\tilde{Q}$ is independent of T_2. Thus $\tilde{Q}$ is defined not in the space $0(n,n)$ or $U(n,n)$, but in the coset space $0(n,n)/0(n) \times 0(n)$ or $U(n,n)/U(n) \times U(n)$.

From Eq. (5.21) it is clear that the matrices $\tilde{Q}$ satisfy the constraint

$$\tilde{Q}^2 = -I \ , \tag{5.22}$$

and its elements are not independent. For practical calculations it is convenient to parametrize $\tilde{Q}$ in terms of its independent variables, just as in the $0(n)$ spin model it is convenient to parametrize

$$\jmath = \left[\left[1 - \sum_i \pi_i^2 \right]^{1/2}, \pi_i \right], \quad i = 2, \ldots, n \ ,$$

in terms of the $n-1$ independent π fields. There are many possible parametrizations of $\tilde{Q}$, but a convenient one is

$$\tilde{Q}=\begin{bmatrix} i(I+\tilde{Q}^{12}\tilde{Q}^{21})^{1/2} & \tilde{Q}^{12} \\ \tilde{Q}^{21} & -i(I+\tilde{Q}^{21}\tilde{Q}^{12})^{1/2} \end{bmatrix},\qquad (5.23)$$

where

$$\tilde{Q}^{21}=\tilde{Q}^{12\dagger}.\qquad (5.24)$$

In the pseudo-orthogonal case, $\tilde{Q}^{12}$ is an arbitrary real

$n\times n$ matrix, whereas in the pseudounitary case, $\tilde{Q}^{12}$ is an arbitrary complex $n\times n$ matrix. It can be shown (Schäfer and Wegner, 1980) that Eq. (5.23) is equivalent to Eq. (5.21). Equations (5.20) and (5.23) define the field theory mapping of the localization problem.

By combining Eqs. (5.23) and Eq. (5.20) we have a field theory with unconstrained variables:

$$\mathcal{H}_0=\int d\mathbf{r}\left[\frac{1}{2t}(\nabla\tilde{Q}^{12}\nabla\tilde{Q}^{21}+\nabla\tilde{Q}^{21}\nabla\tilde{Q}^{12})-\frac{1}{2t}\nabla(I+\tilde{Q}^{12}\tilde{Q}^{21})^{1/2}\nabla(I+\tilde{Q}^{12}\tilde{Q}^{21})^{1/2}\right.$$
$$\left.-\frac{1}{2t}\nabla(I+\tilde{Q}^{21}\tilde{Q}^{12})^{1/2}\nabla(I+\tilde{Q}^{21}\tilde{Q}^{12})^{1/2}\right].\qquad (5.25)$$

We treat the first term in Eq. (5.25) as the unperturbed Hamiltonian and form a perturbation series by expanding out the square root in the remaining two terms. This expansion was performed by Hikami (1981) and compared with the direct impurity diagrammatic perturbation expansion of the density-density correlation function in powers of D_0^{-1}, the inverse of the bare diffusion constant. He found that the perturbation series agree to the order he studied, which included products of six $\tilde{Q}$'s. In zeroth order the correlation function is given by

$$\int d\mathbf{r}\,e^{i\mathbf{q}\cdot\mathbf{r}}\langle\tilde{Q}^{12}(r)\tilde{Q}^{21}(0)\rangle=\frac{1}{t^{-1}q^2+\eta}.\qquad (5.26)$$

Recalling that this correlation function is related to the density-density correlation function, [i.e., (5.17)] and in the metallic limit, we expect the diffusive behavior

$$\int d\mathbf{r}\,e^{i\mathbf{q}\cdot\mathbf{r}}K(r)=\frac{1}{D_0q^2-i\omega}.\qquad (5.27)$$

We therefore identify the parameter in the field theory t as D_0^{-1}. Furthermore, if time-reversal symmetry is not broken, the field φ_α is real and K can represent either a particle-hole or a particle-particle propagator. Thus the singularity in the particle-particle channel (the maximally crossed diagram) discussed in Sec. II is included. On the other hand, if time-reversal symmetry is broken, K represents only the particle-hole propagator, and the singularity in the maximally crossed diagram is suppressed. In the field theory the situation is analogous to the introduction of anisotropy fields in the spin model, where, for example, a Heisenberg model may cross over to an x-y model. The breaking of time-reversal invariance is represented as a crossover from a $O(n,n)/O(n)\times O(n)$ field theory to a $U(n,n)/U(n)\times U(n)$ one.

Wegner (1982) has pointed out the analogy of this model to the more familiar magnetic system described by Eq. (5.19). In that case we have a spontaneous magnetization M in the low-temperature phase, and the transverse fluctuation is described by the transverse susceptibility $\chi_\perp$ given by

$$\chi_\perp=\frac{M}{\kappa q^2+H}.\qquad (5.28)$$

Comparing Eq. (5.28) with Eq. (5.27), we see that $i\omega$ plays the role of the external field H, and the density of states N [not explicitly written in Eq. (5.27)] plays the role of the magnetization M. The critical exponent corresponding to M or N is β. As mentioned earlier, the localization problem is peculiar in that the order parameter is finite on both sides of the transition. This is possible only if $\beta=0$. In the field theory, β is proportional to n, so that it indeed vanishes in the $n\to 0$ limit. Using the standard scaling relations, $\beta=0$ implies $\eta=2-d$. This means that at criticality the density-density correlation function behaves as

$$K\sim\frac{1}{q^{2-\eta}}=\frac{1}{q^d}.\qquad (5.29)$$

This agrees with the analysis based on general scaling arguments given in Sec. II.F.

We mention here that the critical behavior of higher moments of the quantity $Q^{++}-Q^{--}$ has been reported by Wegner (1982). These moments are related to moments of the wave function and provide information on the fluctuation of wave-function amplitudes near the mobility edge. For example, Wegner found that

$$\rho_k\equiv\sum_i|\psi_i(r)|^{2k}\delta(E-E_i)\sim(E-E_c)^{\pi_k},\qquad (5.30)$$

where $\pi_k=(k-1)(2\varepsilon^{-1}+1-k)+O(\varepsilon)$. This critical behavior indicates strong amplitude fluctuation near the mobility edge.

We should point out that the above mapping of the localization problem to a field theory is not unique. Efetov, Larkin, and Khmel'nitskii (1980) introduced φ fields which are Grassmann (anticommuting) fields instead of c-number fields. This method avoids the convergence problem that forced the introduction of the metric Eq. (5.9) and the resulting noncompact $O(n,n)$ symmetry. The resulting field theory is again given by Eq. (5.20), but the matrix $\tilde{Q}$ is now a $2n\times 2n$ matrix whose entries are quarternions, i.e., they are 2×2 matrices parametrized by $\tilde{Q}_{mn}=\sum_i q_{mn}^i\tau_i$, where

$$\tau_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tau_1 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix},$$

$$\tau_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \tau_3 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix},$$

$$(5.31)$$

and q_{mn} are real numbers. The matrix $\widetilde{Q}$ is subject to the constraint

$$\widetilde{Q}^2 = I, \quad \mathrm{Tr}\widetilde{Q} = 0 . \qquad (5.32)$$

If time-reversal symmetry is preserved, this constraint can be satisfied by

$$\widetilde{Q} = O^\dagger \Lambda O , \qquad (5.33)$$

where Λ is a diagonal matrix with entries 1 and -1 for $m = 1$ to n and $m = n + 1$ to $2n$, respectively, and O is an orthogonal rotation with quarternion entries.

If time-reversal symmetry is broken, $\widetilde{Q}$ becomes simply a complex matrix defined in the space $U(2n)/U(n) \times U(n)$, i.e., it is parametrized by

$$\widetilde{Q} = U^\dagger \Lambda U \qquad (5.34)$$

where U is a unitary $2n \times 2n$ matrix. It is interesting that in this case a field theory defined either on $U(n,n)/U(n) \times U(n)$ [Eq. (5.21)] or on $U(2n)/U(n) \times U(n)$ represents the localization problem in the $n \to 0$ limit. The two field theories have different scaling behavior for finite n, but the β functions are the same for $n \to 0$.

So far we have discussed spin-independent scattering. Introduction of spin-flip scattering of the form $H' = \int d\mathbf{r}\, \mathbf{h}(r)\cdot\boldsymbol{\sigma}(r)$, where $\mathbf{h}(r)$ is a random field, destroys time-reversal symmetry and simply converts the orthogonal symmetry to the unitary symmetry. However, if we introduce spin flip via the spin-orbit scattering, time-reversal symmetry is preserved. The corresponding field theory using real φ fields is given by Eq. (5.20), with $\widetilde{Q}$ defined in the symplectic space $sp(n,n)/sp(n) \times sp(n)$, so that in the parametrization Eq. (5.23) the entries $\widetilde{Q}^{12}$ are quarternions. In the Grassmann φ representation of Efetov et al. (1980), the entries of the $\widetilde{Q}$ matrices become

$$\widetilde{Q} = q_0\tau_0 + iq_1\tau_1 + iq_2\tau_2 + q_3\tau_3 , \qquad (5.35)$$

where q_i are real numbers.

Recently still another field theory mapping was introduced by Efetov (1982), who used a mixed real and Grassmann φ field. In this technique the $n \to 0$ limit is avoided. Essentially closed loops in a diagrammatic expansion are removed, not by $n \to 0$, but by cancellation between fermions and boson loops, which enter with opposite signs. The β function was reproduced by this method, and Efetov (1983) solved the problem of the level statistics in small particles using this technique.

The mapping of localization problems to field theories puts the scaling theory of localization on a firmer basis, in that the renormalization of the field theory can be checked using standard techniques. The β function can

be computed to high orders in an expansion in powers of t (Hikami, 1980). However, the renormalization of the field theories has so far been based on an expansion in powers of t, and leaves open the scaling behavior in the strongly disordered regime.

The problem of the field theory mapping of the localization problem in the presence of a strong magnetic field is particularly intriguing. The initial view was that a magnetic field simply destroys time-reversal symmetry, so that the pseudounitary symmetry case should apply. The β function is given by (Hikami, 1980)

$$\beta(t) = -2t^3 \qquad (5.36)$$

in two dimensions [Efetov et al. (1980) initially favored $\beta(t) = 0$, but this claim was later withdrawn by Efetov (1982) in favor of Eq. (5.36)]. Although the leading $-t^2$ term is missing, Eq. (5.36) still implies that all states are localized. This is in conflict with the quantized Hall effect, which requires for its explanation the existence of extended states (Laughlin, 1981; Halperin, 1982). Recently Pruisken (1983) and Levine, Libby, and Pruisken (1983) showed that an additional surface term is needed in Eq. (5.20). Furthermore, they claim that topological excitations are important in the $U(2n)/U(n) \times U(n)$ field theory, as they introduce nonperturbative corrections to the β function in Eq. (5.36). According to Levine et al., these considerations lead to the presence of extended states for a sufficiently strong magnetic field. It is amusing to note that topological excitations exist in the $U(2n)/U(n) \times U(n)$ but not in the $U(n,n)/U(n) \times U(n)$ formulation of the field theory. The role of topological excitations in the field theory for the localization is a subject that merits further investigation.

VI. EXPERIMENTAL STUDIES OF LOCALIZATION AND INTERACTION EFFECTS

A. Introduction

The observable consequences of electron localization were first explored theoretically by Mott (see, for example, Mott and Davis, 1979). Directly stimulated by his work, a large number of workers studied metal-insulator transitions in disordered systems. Their data and already existing data have been extensively analyzed by Mott, and provide broad support for the ideas of variable-range hopping in the localized regime and minimum conductivity for the metal. In this section we discuss (mostly) recent experiment work on disordered metals at low temperatures. The work bears largely on relatively small (a few percent) but characteristic anomalies in transport properties. These are localization effects predicted by the theories described in the previous sections. The properties studied are resistivity, magnetoresistance, Hall effect, density of states, quasiparticle lifetime, and superconductivity. The systems are effectively one, two, and three dimensional. In three dimensions, there are a few experiments

near the metal-insulator transition boundary, where the effects are large, but much more work remains to be done in the strong-localization regime.

A detailed review of the experimental situation is given by Bishop and Dynes (1984) in a separate article. What follows is a very incomplete selection of recent experimental work, chosen to illustrate the effects observed.

We first summarize briefly work on wires, i.e., effectively one-dimensional systems (Sec. VI.B). Experiments on these were stimulated by the paper of Thouless (1977), who pointed out that as the inelastic cutoff length L_{Th} increases on cooling, a thin wire becomes effectively one dimensional for radius a less than L_{Th}. For example, a wire with radius 250 Å and resistivity $\rho \approx 100 \, \mu\Omega$ cm is expected to become one dimensional at $T \simeq 1$ K and to exhibit anomalous temperature dependence of resistivity below this temperature. In trying to look for these effects, Dolan and Osheroff (1980) uncovered a logarithmic temperature and voltage dependence of resistivity in films, which was associated by Anderson, Abrahams, and Ramakrishnan (1980) with a two-dimensional localization effect. In Sec. VI.C we discuss the rather extensive experimental work on two-dimensional localization, while work on bulk systems is described in Sec. VI.D. Three-dimensional systems have been studied for a long time. There are some characteristic small effects, for example, resistivity saturation, failure of Mathiessen's rule, $\sqrt{T}$-dependent resistivity, and $\sqrt{H}$ magnetoresistance at low temperature. Large effects near the metal-insulator transition, i.e., $\sigma \ll \sigma_{min}$, a giant dip in density of states, etc., have been seen recently.

B. Wires

Early experiments on effectively one-dimensional systems were carried out by Giordano, Gibson, and Prober (1979; see also Giordano, 1980) and by Chaudhuri and Habermeier (1980a,1980b). They showed the presence of anomalous temperature-dependent terms of the right sign, size, and functional form. More recent and extensive work by White, Tinkham, Skocpol, and Flanders (1982), Masden and Giordano (1982), Skocpol, Jackel, Hu, Howard, and Fetter (1982), Wheeler, Choi, Goel, Wisnieff, and Prober (1982), and Santhanam, Wind, and Prober (1984) tests several aspects of localization and interaction predictions. These difficult experiments, in which the well-defined narrow dimension is of order 100–1000 Å, have become possible because of advantages in photolithography, ion beam etching, and other microelectronic techniques.

For a wire of cross section A, we have seen from Eqs. (2.21c) and (3.41a) that the leading perturbative corrections to conductivity are

$$\Delta\sigma = \frac{e^2}{\pi\hbar} \frac{1}{A} [L_{eff}^{(1)} + (2 - \tfrac{3}{4}\widetilde{F}_\sigma)L_{eff}^{(2)}] \,, \qquad (6.1)$$

where $L_{eff}^{(1)}$ and $L_{eff}^{(2)}$ are the Thouless length and the thermal diffusion length $(\hbar D/k_B T)^{1/2}$, respectively. The

sample length L is the cutoff if it is shorter. This quasi-one-dimensional form is valid if $L_{eff}^{(1)}$ and $L_{eff}^{(2)}$ are larger than the narrow dimension $(\sim A^{1/2})$. Otherwise, the system is effectively two or three dimensional, depending on the shape of the cross-sectional area A.

It was established quite early that the anomalous term does indeed scale as the inverse of the cross-sectional area A. The existence of a length cutoff $L_{eff}^{(1),(2)}$ has been verified directly by Masden and Giordano (1982), who compared $\Delta\sigma(T)$ for otherwise identical wires of differing lengths. They found that, for wires shorter than a certain (temperature-dependent) length, resistivity increases with length, whereas for longer wires it does not. This is direct evidence for non-Ohmic behavior due to quantum localization or interaction. However, according to Masden and Giordano (1982), there are difficulties in modeling quantitatively the observed length dependence.

The perturbation theory also makes predictions concerning the temperature and disorder dependence of $\Delta\sigma(T)$. As discussed earlier (Sec. II.D), the most poorly known quantity is the inelastic mean free path, which could have (for a one-dimensional system) a temperature dependence varying from T^{-2} to $T^{-1/2}$ depending on the inelastic mechanism and the degree of disorder. Another complication is possible dimensional crossover for $\Delta\sigma$ when one of the relevant length scales becomes comparable to and smaller than the narrow dimension.

The interaction effect term in $\Delta\sigma(T)$ goes as $T^{-1/2}$. Such a temperature dependence is generally seen, and White et al. (1982) show that it has the size expected from Eq. (6.1) in many wires over a wide range of disorder. They argue that this is the dominant term. On the other hand, Wheeler et al. (1982) find that the magnetoresistance and their high-conductivity narrow inversion layers fits the prediction of Altshuler and Aronov (1981a) for localization effects in a restricted-geometry system. They conclude that both localization and interaction contribute significantly to $\Delta\sigma$ in these systems. A similar conclusion is reached by Skocpol et al. (1982) for much more strongly disordered systems.

While interaction terms are clearly present, it is difficult to assess the size of the localization term. Magnetoresistance has not been exploited sufficiently as a diagnostic tool, especially in metallic wires. Part of the reason is that, unlike the 2D case, the orbital contribution is reduced greatly by finite size effects (Altshuler and Aronov, 1981a), so that the characteristic field for the suppression of localization becomes closer to the spin-splitting field required to affect the interaction term (Sec. III.D).

We should mention one system that does not appear to fit into this scheme. Sacharoff, Westervelt, and Bevk (1982) measured conductivity in platinum wires drawn to a diameter as small as 800 Å. The resistivity rises as $T^{-1/2}$, but the size of the effect is 1 or 2 orders of magnitude larger than predicted by the 1D interaction theory. In any event, the criterion for the 1D interaction effect is not met even in the thinnest wires, and 3D interaction theory would predict an even smaller effect. On the other hand, the magnetoresistance predicted on the basis of lo-

calization theory was not observed (Sacharoff, Westervelt, and Bevk, 1984). The drawn wires are different from the evaporated system in that they are believed to be heavily dislocated. There is at present no explanation of this effect.

C. Films

Vapor-deposited films and inversion layers are two classes of systems largely used to study two-dimensional effects. The former range in thickness from a monolayer to about 200 Å, and in resistance per square from 10 kΩ to 1 Ω. They are pure metals [van den Dries, van Haesendonck, Bruynseraede, and Deutscher, 1981 (Cu); Bergmann, 1982a,1982b,1982c (Mg, Cu, Ag, Au); Markiewicz and Harris, 1981 (Pt)]; alloys [Dolan and Osheroff, 1979 (Au-Pd)], and metal-metal oxide composites [Ovadyahu and Imry, 1981 ($In_2O_{3-\delta}$); Kobayashi, Komori, Ootuka, and Sasaki, 1980 (Cu-CuO)].

The films are two dimensional for localization effects if the length scale L_{Th} up to which electrons diffuse without inelastic collisions is larger than the film thickness t, i.e., $L_{Th} \simeq (L_i L_e)^{1/2} > t$ where $L_i = v_F \tau_i$ and $L_e = v_F \tau$ and τ_i and τ are the inelastic and elastic scattering times, respectively. Since generally L_i decreases as temperature increases, there is a crossover to three-dimensional behavior above a certain temperature. In many films, boundary scattering is the basic elastic scattering mechanism, i.e., $L_e \sim t$. In that case, the condition of two dimensionality is $L_i > t$. This kind of dimensionality crossover has been observed, e.g., in GaAs FET's by Poole, Pepper, Berggren, Hill, and Myron (1982). It is possible for a film to be effectively two dimensional with respect to localization and three dimensional with respect to Coulomb interaction (see, for example, Imry and Ovadyahu, 1982). The condition for the latter is $(D/kT)^{1/2} < t$. In the temperature window specified by $(D/kT)^{1/2} \lesssim t \lesssim [L_i(T)L_e]^{1/2}$, the system is three dimensional for interaction effects and two dimensional for localization effects. Clearly, at low enough temperatures, the first inequality is not satisfied, and the film is two dimensional for all disorder-caused transport anomalies.

The inversion layer system is intrinsically two dimensional, the electrons being confined to the inversion layer. The areal density of electrons, and hence the Fermi energy ε_F, can be varied by changing the gate voltage. For example, when the latter is increased from 8 to 200 V, n increases from $3 \times 10^{11}/cm^2$ ($\varepsilon_F \sim 20$ K) to $5.8 \times 10^{12}/cm^2$ ($\varepsilon_F \sim 360$ K). [The numbers are for the (100) face of a Si-MOSFET.] The effective disorder can be varied independently by applying a substrate bias, which moves the electron wave function away from or close to the Si-oxide interface. Experiments have been done in the resistance-per-square range of 300 Ω to 10 kΩ. Since the electron-lattice coupling in these systems is relatively weak, they are ideal for studying the effects of disorder and of Coulomb interactions in two dimensions.

Most of the recent studies have concentrated on the regime where disordered system anomalies are relatively

small, i.e., the perturbative ($k_F l \gg 1$) results for localization and interaction effects are sufficient. A number of observable characteristic anomalies have been discussed in Secs. II and III. To the lowest significant order, localization and interaction corrections to conductivity $\sigma(T,H)$ are additive. For example, the zero-field conductivity shows a logarithmic increase as temperature decreases, i.e.,

$$\sigma(T) = \sigma(T_0) + \frac{e^2}{2\hbar\pi^2}[\alpha p + (1 - \tfrac{3}{4}\widetilde{F}_\sigma)]\ln\left[\frac{T}{T_0}\right]. \quad (6.2)$$

For an orbitally nondegenerate free-electron gas, $\alpha = 1$ and $p/2$ is the temperature index of the Thouless length L_{Th}. The index p depends on the dominant collision mechanism. The Coloumb term is $(1 - 3\widetilde{F}_\sigma/4)$, where the size of the Hartree part $\widetilde{F}_\sigma$ depends on the screening length [Eq. (3.34)]. The logarithmic increase of Eq. (6.3) has been seen in a large number of systems with a coefficient of order unity, over a wide range of $\sigma(T_0)$ (from 1 to 10^4 Ω per square). Experiments described below enable α, p, and $\widetilde{F}_\sigma$ to be determined independently.

A related anomaly is the logarithmic dependence of σ on applied steady voltage if the latter causes sufficient Joule heating to push the electron temperature up significantly (Sec. II.D). Dolan and Osheroff (1979) and Bishop, Tsui, and Dynes (1980) found such a $\ln V$ term. Its slope relative to that of Eq. (6.3) is predicted to be $2/(2+p')$ where the electron energy relaxation time $\tau_{e\text{-ph}}$ due to scattering from phonons goes as $T^{-p'}$. The experimental result for p' is 3, which is the theoretically expected dependence. This heating effect has been extensively studied in inversion layers by Uren, Davies, and Pepper (1980; see also Pepper, 1981).

The unique magnetoresistance behavior of disordered metal films has been discussed earlier, in Sec. II.E for localization and in Sec. III.D for interaction effects. These sections predict logarithmic and quadratic dependences for strong and weak fields, respectively. However, the characteristic fields are different [see Eqs. (3.48) and (3.49)]; for the localization term it is $H_l \simeq (hc/2e)L_{Th}^{-2}$, whereas for the spin-splitting part of the interaction term it is $H_s \simeq (k_B T/g\mu_B)$. The orbital contribution to the latter has a lower characteristic field $H_0 \simeq (k_B T/g\mu_B)(k_F l)^{-1}$. The magnetoresistance due to localization is negative, and for a thin film purely transverse, whereas the interaction magnetoresistance is positive, being isotropic for spin splitting and transverse for the orbital part. These differences help one to disentangle the two terms and to determine their sizes, etc. In general, $H_l < H_s$ and H_0. Further, except at very low temperatures, it is very hard to reach the limit $H_s \simeq (k_B T/g\mu_B)$. Therefore, the transverse magnetoresistance is almost always negative except at very low temperatures and high fields, where it can be positive if $\widetilde{F}_\sigma$ is large enough. The longitudinal magnetoresistance is positive for a truly two-dimensional film. In the presence of spin-orbit scattering, the behavior is more complex (see below).

Negative magnetoresistance was first observed and

analyzed by Kawaguchi and Kawaji (1980a,1980b) in a Si inversion layer (see also Kawaji and Kawaguchi, 1968 and Eisele and Dorda, 1974 for earlier observations of negative magnetoresistance). Since then the theoretical predictions have been confirmed in great quantitative detail by experiments on Si inversion layers (Wheeler et al., 1981; Davies, Uren, and Pepper, 1981; Kawaguchi and Kawaji, 1982a,1982b; Dynes, 1982; Bishop, Dynes, and Tsui, 1982), Cu films (van Haesendonck et al., 1982), Pt films (Markiewicz and Harris, 1981), Mg films (Bergmann, 1982a), and other systems. From $\sigma(H,T)$ one can determine, in addition to α, the dephasing length L_{Th} and its temperature dependence, as well as the size of the Hartree term $\widetilde{F}_\sigma$. The accurate results on the Si inversion layers, extending down to the millidegree range, have so far been analyzed considering for interaction effect only spin splitting and ignoring the orbital term. This works well and suggests that the coefficients of the latter are temperature renormalized to small values (Sec. III.D).

The Hall coefficient R_H is another quantity that behaves differently for localization and interaction effects. There is no logarithmic correction due to the former, whereas in interaction theory

$$\frac{\delta R_H}{R_H} = 2\frac{\delta R}{R} , \qquad (6.3)$$

where δR is the logarithmic resistivity anomaly (Sec. III.G). This prediction has been verified by Bishop, Tsui, and Dynes (1980) and by Uren, Davies, and Pepper (1980). These authors measured the Hall coefficient in a magnetic field sufficient to suppress the localization effect ($H > H_l$); when the field is decreased, the ratio $[(\delta R_H/R_H)/(\delta R/R)]$ decreases from two, due probably to localization effects.

The effect on magnetoresistance of scattering by magnetic impurities and by impurities with spin-orbit coupling has been demonstrated in a series of beautiful experiments by Bergmann (Bergmann, 1982a,1982b,1982c, 1982d). By covering thin quenched-condensed metallic films with small controlled quantities of a magnetic ion like Fe, or a heavy ion with large spin-orbit coupling like Au, he was able to observe dramatic changes in the size and sign of magnetoresistance as a function of the field. In Fig. 17 we show the magnetoconductance of an Mg film as a function of magnetic field and Au coverage (Bergmann, 1982b). The data clearly show the change of magnetoresistance from negative to positive with increasing spin-orbit scattering, and its curvature back to a negative value for large enough field. The data can be fitted quantitatively by a calculation due to Hikami, Larkin, and Nagaoka (1980), who considered the effect of random magnetic impurity scattering, spin-orbit scattering, magnetic field, and inelastic collisions on the quantum backscattering interference term of Sec. II.C. When these are all present together, the individual length scales appear in characteristic combinations, and their effects can be unravelled by analyzing $\Delta\sigma(T,H)$ for various coverages. One then obtains experimental values for rates of spin-orbit scattering, spin-disorder scattering, and inelastic collisions. For example, from experiments with a 10^{-3} monolayer of Fe on Mg, a magnetic scattering time of 4.7×10^{-12} sec is found. The spin-orbit scattering cross section for an Au atom on Mg is deduced to be 0.5×10^{-16} cm^2. As pointed out by Bergmann, weak localization effects can now be used as an accurate tool to measure basic solid-state properties. Spin-orbit-scattering-related magnetoresistance anomalies have been observed in many pure metal films such as Mg, Cu, Ag, Au (Bergmann, 1982b) and in In-P inversion layers (Poole, Pepper, and Hughes, 1982). The situation in Mg is particularly interesting in that the onset of spin-orbit scattering has been observed as a function of decreasing temperature (White, Dynes, and Garno, 1984).

We now briefly discuss the experimentally determined values of the parameters α, $\widetilde{F}_\sigma$ that appear in Eq. (6.3) and the Thouless length L_{Th}. From detailed measurements of $\sigma(T,H)$ over a resistivity range of about 25, Bishop, Dynes, and Tsui (1982) concluded that $\alpha = 1.0 \pm 0.1$, close to the theoretical estimate. We should note that Larkin (1980) has produced a term in interaction theory which contributes to magnetoresistance in exactly the same way as the localization term, and effectively reduces α from unity [see Eq. (3.49)]. The data show that the correction is small in Si-MOSFET, but interaction effects are apparently observable in GaAs, where α was found to be $\simeq 0.8$ (Lin et al., 1984) or α was analyzed as being field dependent (Nambu et al., 1984). Since τ_{in}^{-1} turns out to be close to T in this system, it is possible that both the Larkin term and the particle-particle interaction term [Eq. (3.48a)] contribute to magnetoresistance.

The size of the Hartree term $\widetilde{F}_\sigma$ has been investigated only in a few cases where there is clear evidence for high-field positive magnetoresistance contributions attributable to it. Bishop, Dynes, and Tsui (1982) measured the magnetoresistance in a magnetic field parallel to the 2D plane. This should probe the spin-dependent part of the magnetoresistance, and should have contributions only from the
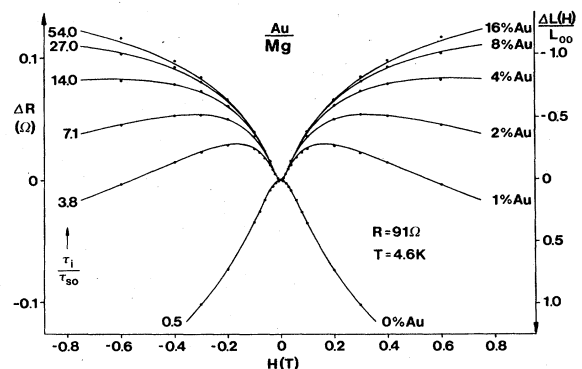

FIG. 17. The magnetoconductance curve of a Mg film with different coverages of Au. [$\Delta L(H)$ is the magnetoconductance, and $L_\infty = e^2/2\pi^2\hbar$.] The coverages shown are in percent of an atomic layer. Increasing Au coverage converts the positive magnetoconductance to negative. Full curves through the data points are fits using the theory of Hikami, Larkin, and Nagaoka (1980). Figure is taken from Bergmann (1982b).

interaction term. A positive magnetoresistance was observed, and by comparison with the perturbation theory [Eq. (3.43)], values for $\widetilde{F}_\sigma$ were extracted as a function of $k_F l$. $\widetilde{F}_\sigma$ was found to increase from unity at $k_F l \approx 10$ to ~ 3 for $k_F l \approx 4$. This large value of $\widetilde{F}_\sigma$ is surprising and unaccounted for by theory. It is possible that the use of the perturbation result is not adequate in this case, as we expect $\widetilde{F}_\sigma$ to be renormalized as discussed in Sec. III.F, and this renormalization is more important for small $k_F l$. This is clearly a problem that deserves further attention. In particular, the parallel field experiment is the ideal geometry for investigating the universal behavior predicted in Eq. (3.62).

The Thouless length L_{Th}, being the effective scale size for quantum interference effects, determines the size of localization anomalies. One of the major initial surprises was the smallness of localization effects; this is most likely due to the disorder-induced enhancement of inelastic decay rates leading to small L_{Th}. Typically, compared to the pure system estimates, the inelastic mean free path L_i ($\approx L_{\mathrm{Th}}^2/L_e$) is less by a factor of 10^2 or 10^3, the discrepancy being larger for more resistive films. For a number of well-characterized systems, L_i has been obtained as a function of temperature from magnetoresistance measurements.

In Cu films of very low resistance per square (van Haesendonck et al., 1982a) and in Mg films (Bergmann, 1982a), L_i^{-1} goes as T^2, indicating that electron-electron collision (in the clean limit) is the dominant inelastic relaxation mechanism. The temperature index decreases to about 1.5 for more resistive Cu films with $R_\square > 50\ \Omega$ (van den Dries et al., 1981), and is 1.65 for noble-metal films with $R_\square > 100\ \Omega$. This temperature dependence is not understood. The films are thin enough for the inelastic process to be two dimensional, in which case the inelastic length due to Coulomb interaction is given by Eq. (3.31). This length seems to be of the right order of magnitude, but has the wrong temperature dependence.

In the intrinsically two-dimensional inversion-layer systems, it appears from the work of Wheeler (1981), Bishop, Dynes, and Tsui (1982), and Poole, Pepper, and Hughes (1982) that the inelastic rate is in good agreement with the predictions of Eq. (3.30), i.e., that τ_i^{-1} is proportional to $e^2 K_2(k_B T/D K_2^2)\ln(T_1/T)$, where K_2^{-1} is the screening length and T_1 a high-temperature cutoff. The disorder and temperature dependences of $\tau_i^{-1}(T)$, as well as its size, all agree with the above form. Poole, Pepper, and Hughes (1982), however, suggest that there is evidence against the presence of the $[\ln(T_1/T)]$ term in InP inversion layers. These results may be more consistent with Eq. (3.31).

Experiments on highly resistive Pd and Pd-Pt films with $R_\square \sim 6000\ \Omega$ (Markiewicz and Harris, 1981) also suggest very short inelastic mean free paths. These are complicated systems, with strong spin-orbit interactions, and Stoner-enhanced spin fluctuations, and more needs to be known about their effects.

One of the predictions of the scaling theory of localization is that, at large enough length scales, any disordered

film is insulating. This has not been directly verified yet, though from the continuous $R_\square(T)$ curves of increasingly disordered Si-MOSFET's, for example, it appears clear that there is no σ_{min} and that the transition to localized behavior is continuous. Since the localization length ξ_{loc} increases exponentially with decreasing disorder, and the Thouless cutoff length increases only as $T^{-1/2}$, one has to go to exponentially low temperatures to probe length scales larger than ξ_{loc}. At such low temperatures and large length scales, electron heating effects become important and limit the lowest effective electron temperature attainable.

D. Bulk systems

We have briefly reviewed (in Sec. I) earlier work on the metal-insulator transition in three-dimensional disordered systems which broadly supports the ideas of mobility edge, minimum metallic conductivity, and variable-range hopping. In some systems, for example, granular metals, there is evidence for the importance of Coulomb interactions. We discuss here recent experiments, mainly those done at low temperatures, where there are characteristic localization as well as interaction effects. The few experiments probing the critical region near the mobility edge are then discussed; it is possible that there is a rich variety of behavior. Further discussion of some of the systems that are less well understood is given in Sec. VII.

1. Systems studied

Two classes of systems, namely doped semiconductors and vapor-deposited thick films (alloys or metal-metal oxide composites), have been investigated in detail near the metal-insulator transition in the past few years. In the former, there is an insulator-metal transition due to shallow impurity-state overlap when the dopant concentration is large enough. For example, in Si:P the critical P concentration is $n_c \sim 3.74 \times 10^{18}\ \mathrm{cm}^{-3}$. The system is intrinsically disordered because the P atom randomly substitutes a Si atom. Experiments include low-temperature studies of resistivity as a function of temperature and magnetic field on the metallic side (Rosenbaum, Andres, Thomas, and Lee, 1981; Rosenbaum, Milligan et al., 1981; Thomas, Kawabata et al., 1981), observation of the absence of σ_{min} (Rosenbaum et al., 1980), stress tuning of the metal-insulator transition (Paalanen et al., 1982), effect of compensation (Thomas et al., 1982a,1982b) and dielectric behavior near n_c on the insulating side (Hess et al., 1982). (See Rosenbaum et al., 1983 for a review, as well as recent results.)

Vapor-deposited thick films of $Al\text{-}Al_2O_3$ (Dynes and Garno, 1980), $In\text{-}In_2O_3$ (Ovadyahu and Imry, 1981), $Au_{1-x}Ge_x$ (McMillan and Mochel, 1981; Dodson et al., 1981), and $Nb_{1-x}Si_x$ (Hertel et al., 1983) have been studied as a function of oxygen content and grain size in the former systems, and as a function of Ge or Si content in the alloys. In addition to resistivity and magnetoresis-

tance, tunneling measurements of the density of states show striking square-root anomalies due to interaction effects.

Recently, a new class of system has been added to the list. Interesting results have been reported in a magnetic semiconductor $Gd_{3-x}S_4$, which can be driven metallic with a magnetic field (von Molnar et al., 1983).

2. Low-temperature conductivity anomalies

As discussed in Secs. II and III, there are characteristic temperature-dependent corrections to conductivity, arising from localization (2.29a) and Coulomb interaction (3.41c). To lowest order, these are additive, so that

$$\Delta\sigma(T) = \frac{e^2}{\hbar}\frac{1}{\pi^2}\frac{T^{p/2}}{a} + \frac{e^2}{4\pi^2\hbar}\frac{1.3}{\sqrt{2}}(\tfrac{4}{3} - \tfrac{3}{2}\widetilde{F}_\sigma)\frac{\sqrt{T}}{D} \ . \quad (6.4)$$

The localization term reduces conductivity as temperature decreases because the scale of quantum interference is set by the inelastic collision length $L_{Th} \sim aT^{-p/2}$ (Sec. II.D), which increases as temperature decreases. The interaction term has a $\sqrt{T}$ dependence, but its sign depends on the relative size of the exchange and Hartree terms, which depends on the screening length. In doped semiconductors, solid-state effects, such as the presence of many degenerate conduction-band minima in k space (valleys), intervalley scattering, and mass anisotropy, have all to be considered if a detailed quantitative comparison is desired. Some of these effects have been considered by Fukuyama (1981b) and by Bhatt and Lee (1983).

At the lowest temperatures, $\Delta\sigma(T)$ is dominated by the interaction term because the index p is greater than 1. (The estimates for p are $\tfrac{3}{2}$, 2, and 3, depending on whether Coulomb interactions in the dirty limit, clean limit, or electron-phonon scattering determine the inelastic scattering rate.) The cusplike $\sqrt{T}$ conductivity behavior has been seen in doped semiconductors (Ootuka et al., 1979; these authors attributed it to the Kondo effect). Rosenbaum, Andres, Thomas, and Lee (1981) observed it in Si:P and successfully explained its sign, size, and dependence on electron density (density of P) using a Thomas-Fermi screening approximation for F. [The perturbative form $(\tfrac{4}{3} - 2F)$ was used in the analysis, instead of the correct form $(\tfrac{4}{3} - \tfrac{3}{2}\widetilde{F}_\sigma)$ given in Eq. (6.5). Thus some quantitative adjustment of the analysis will be necessary, but the qualitative features will not be affected.] The $\sqrt{T}$ coefficient was found to change sign as a function of disorder, a change which can be interpreted as being due to a sign change in $(\tfrac{4}{3} - \tfrac{3}{2}\widetilde{F}_\sigma)$. The effect is sizable, e.g., for $n = 4.5 \times 10^{18}$ cm^3 σ increases from 112 $(\mu\Omega\,cm)^{-1}$ to 125 $(\mu\Omega\,cm)^{-1}$ as temperature decreases from 4 K to 50 mK. No localization effect was considered. Thomas et al. (1982a) have recently measured the temperature-dependent conductivity of Ge:Sb from 10 mK to 1 K and have fitted the results with a more realistic version of Eq. (6.4) that includes anisotropy and many-valley effects. They include a localization term, which contributes a small opposite sign correction, and show that it is neces-
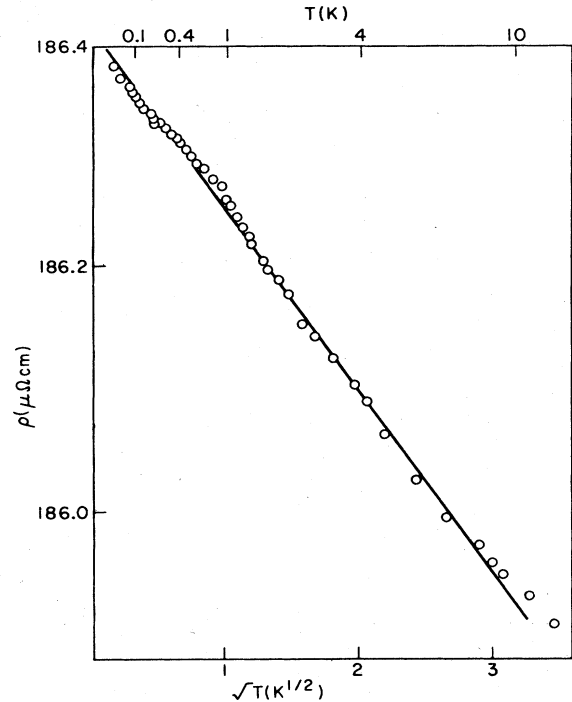

FIG. 18. Conductivity of the amorphous metal $Fe_{40}Ni_{40}P_{14}B_6$ showing $T^{1/2}$ behavior. Data from Rapp, Bhagat, and Gudmundsson (1982).

sary to account for the curvature of $\sigma(T)$ at higher temperatures, i.e., temperatures of order 1 K. A fairly good fit is obtained with $p = 2$, i.e., the exponent characteristic of electron-electron inelastic processes in the pure regime. An evaluation of the importance of intervalley scattering in this system is given by Bhatt and Lee (1983).

Low-temperature $T^{1/2}$ anomalies in the conductivity have also been observed in a number of metallic glasses. Such anomalies have been known in the literature for some time, but they have typically been plotted versus $\ln T$ and interpreted in terms of Kondo-type scattering by structural defects (see, for example, Tsuei, 1976; Rapp, Bhagat, and Johannesson, 1977). Figure 18 shows a replot of existing data that extend down to 30 mK, revealing very nice $T^{1/2}$ behavior (Rapp, Bhagat, and Gudmundsson, 1982). Such $T^{1/2}$ behavior appears to be a common feature in many amorphous alloys (Cochrane and Strom-Olsen, 1984).

3. Magnetoresistance

We have discussed earlier the negative magnetoresistance due to magnetic field suppression of localization (Sec. II.E) and the positive magnetoresistance of an interacting electron gas due to spin splitting and orbital effects (Sec. III.D). To the leading order they are additive, the localization contributions to magnetoconductivity being (Kawabata, 1980a,1980b)

$$\Delta\sigma(H,T) = \frac{e^2}{2\pi^2\hbar}\sqrt{eH/\hbar c}\,f_3(x) \,, \qquad (6.5a)$$

where $x = \hbar c/(4eHL_{\text{Th}}^2)$ and

$$f_3(x) = \sum_{n=0}\left[2(\sqrt{n+x+1}-\sqrt{n+x})-\frac{1}{\sqrt{n+x+1/2}}\right] \,, \qquad (6.5b)$$

with the asymptotic forms

$$f_3(x) = 0.605 \quad \text{for } x \ll 1$$

$$= (x^{-3/2}/48) \quad \text{for } x \gg 1 \,. \qquad (6.5c)$$

The spin-splitting contribution to positive magnetoresistance is given in Eqs. (3.43)–(3.45). The orbital terms are discussed by Fukuyama (1980) and by Altshuler, Aronov, Larkin, and Khmel'nitskii (1981). The asymptotic forms are $\sqrt{H}$ for large fields and H^2 for small fields. The former behavior is a characteristic signature of disordered system anomalies in three dimensions, and is different from all nonsaturating magnetoresistance field dependences known so far.

Kawabata (1980a,1980b) has discussed the negative magnetoresistance of doped semiconductors from this point of view, and has shown that both the field dependence and the size fit localization predictions. At very low temperatures such that $\mu_B H \gtrsim k_B T$ is accessible, the interaction contribution to magnetoresistance becomes more prominent. Experiments on Si:P first clearly established such a term (Rosenbaum et al., 1981). These authors analyzed their results to show that both interaction and localization effects were present with roughly the expected density and disorder dependence. Kawabata (1982d) has included, in addition to spin-splitting, an orbital interaction term in analyzing the earlier data of Ootuka et al. (1979) on Ge:Sb. Low-temperature magnetoresistance experiments have also been done on n-InSb (Morita et al., 1982), a direct band gap semiconductor that has isotropic effective mass and is thus free from many-valley and anisotropy complexities of the other systems. Model calculations for parameters appropriate to these experiments have been reported by Isawa, Hoshino, and Fukuyama (1982).

Negative magnetoresistance going as $\sqrt{H}$ has been observed also in a very different type of system, namely granular aluminum, by Chui, Lindenfeld, McLean, and Mui (1981). These authors find clear evidence for interaction effects in their high-resistivity samples. Data analysis in these systems is complicated by the presence of superconducting fluctuations.

Magnetoresistance in disordered metals and metallic glass has been studied for some time (e.g., Hake et al., 1980). Recently the $\sqrt{H}$ magnetoresistance has also been reported in amorphous alloys. Examples include the Cu-Ti system (Howson and Greig, 1983), La-Al system (Lu and Tsai, 1984), and Cu-Zr system (Bieri et al., 1984).

4. Critical regime

One of the most important questions connected with the localization transition is the critical behavior. The first question is whether there is a minimum metallic conductivity, or whether conductivity goes to zero with a universal exponent. From the insulating side, one is interested in the divergence of the dielectric constant. The behavior of other physical quantities, such as the density of states, is another important problem. The few experimental results in this area suggest a rich variety of behavior.

a. Conductivity

Conductivity measurements close to the metal-insulator transition have been made in Si:P and the metal semiconductor alloys $Au_{1-x}Ge_x$ and $Nb_{1-x}Si_x$. In Si:P, the metal-insulator transition occurs at $n_c = 3.74\times10^{18}/\text{cm}^3$. Rosenbaum et al. (1980) found a few sample specimens whose zero-temperature conductivity was much less than the minimum metallic value, in one instance nearly a thousand times less. However, for most of these the dopant density lies within a percent of n_c, close to the limit of accuracy with which its change can be monitored. Mott (1976,1981) has pointed out that inhomogeneities can lead to very low conductivities even if there is a nonzero $\sigma_{\min}$, in two ways. One is a distribution of dopant density due to preparation conditions. The other is statistical finite size (or $N^{-1/2}$) fluctuation in dopant concentration. This can lead to a rounding of the conductivity transition if the localization length exponent ν is smaller than $2/D$. Mott suggests (1981) that the relatively sharp transition in Si:P is due to such a rounding effect's masking a $\sigma_{\min}$. Even when there is no $\sigma_{\min}$, such a smearing can mask the conductivity exponent if it is less than $(2/d)$, as mentioned in Sec. II.C.

In a recent experiment, Paalanen, Rosenbaum, Thomas, and Bhatt (1982) have made a high-precision study of the transition in Si:P by starting with a slightly insulating sample and applying uniaxial stress to drive the system metallic. This permits a detailed study of the transition region by tuning a continuously variable parameter, the stress. They find, by combining results with their work above, that conductivities in the range $(\sigma_{\min}/4) < \sigma < 10\sigma_{\min}$ [corresponding to $10^{-3} < (n-n_c)/n_c < 1$] scale with an exponent $\mu = 0.55\pm0.1$. Further, this exponent is nearly half that for the divergence of the dielectric constant (Capizzi et al., 1980; Hess et al., 1982). Since the latter is expected to behave as (localization length)^2, the resistivity and localization length diverge with the same exponent. The exponent, $\mu = 0.55$, is, however, rather different from the prediction $\mu = 1$ of localization theory without interactions. At present there is no clear understanding of why this value of μ is different in Si:P from other systems.

An important and relatively unexplored question is the significance of compensation. In uncompensated semi-

conductors such as Si:P described above, there are as many sites as carriers. Correlation effects are maximal in such a half-filled band, and the observed critical behavior may be connected with this. It is then interesting to study samples with compensation where there are more sites than carriers. Recently Thomas, Ootuka, Katsumoto, Kobayashi, and Sasaki (1982b) have shown for Ge:Sb that with increasing compensation σ vs n tends from the form $(n/n_c-1)^{1/2}$ to (n/n_c-1). Recent measurements by Zabrodskii and Zinoveva (1984) on germanium samples as a function of compensation show a conductivity exponent of approximately 0.8, consistent with the work of Thomas et al. (1982b). One way a compensated sample may be different is that local moment formation is more likely, so that spin-flip scattering may be present and turn the system into a different universality class, as discussed in Sec. III.F.

Recent measurements on $Ge_{1-x}Au_x$ (Dodson, McMillan, Mochel, and Dynes, 1981), granular Al (Dynes and Garno, 1981), and Nb_xSi_{1-x} (Hertel et al., 1983) all find a metallic range, i.e., $\sigma(T=0)\neq0$ with $\sigma \ll \sigma_{min}$. In granular Al, there is superconductivity as well as a significant density-of-states effect for $0.05\sigma_{min}<\sigma<\sigma_{min}$. In this class of materials, the most precise determination of the conductivity exponent was made in Nb_xSi_{1-x}, where the critical Nb concentration was $x \sim 0.12$. Graded alloys with a variety of concentrations x were prepared. In the range $0.12<x<0.18$, σ varied from 5 to 150 $(\Omega\,cm)^{-1}$ where σ_M was ~ 20 $(\Omega\,cm)^{-1}$. The conductivity was found to vanish linearly with x near the critical concentration.

Recently the metal-insulator transition in the magnetic semiconductor system $Gd_{1-x}v_xS_3$, where v stands for vacancy, was studied in detail (von Molnar et al., 1983). At zero applied field, the conduction electron is supposed to form a magnetic polaron with the Gd ions, so that the system is insulating. Of course, the presence of disorder due to the vacancies introduces further complications into the picture. Nevertheless, the application of magnetic field reduces the binding of the magnetic polaron and induces an insulator-to-metal transition. The advantage of this system is that the transition occurs as a function of an external field, which can be varied continuously. Conductivity down to mK range was measured in the vicinity of the transition, and the conductivity was found to vanish linearly with applied field. In view of the recent theory reviewed in Sec. III.F, which is applicable to a metal-insulator transition in the presence of a strong external field, this is a particularly interesting system to pursue in greater detail.

b. Density of states

Altshuler and Aronov (1979) showed that the single-particle density of states, as measured by tunneling, exhibits an $E^{1/2}$ dip at the Fermi level. As reviewed in Sec. III.F, McMillan (1981) proposed that this dip should extend to zero at the metal-insulator transition and suggested the following functional form:
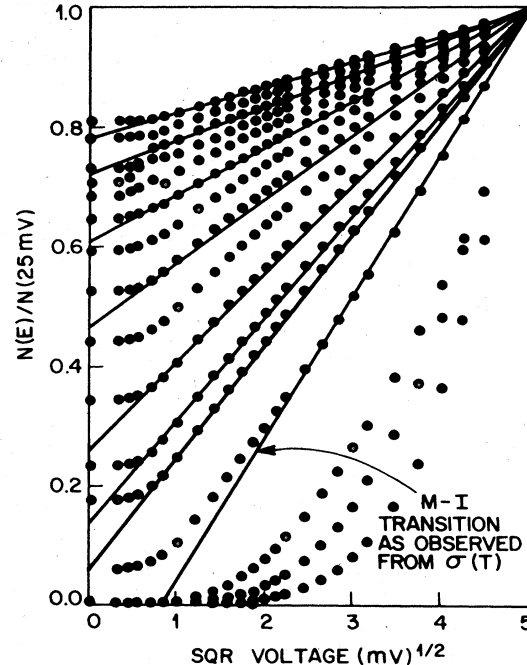

FIG. 19. Single-particle density of states measured by tunneling experiments on Nb_xSi_{1-x}. Different sets of data correspond to different x. The data show $V^{1/2}$ behavior on the metallic side and the vanishing of the density of states in the vicinity of the metal-to-insulator transition. Figure from Hertel et al. (1983).

$$N(E)=N(0)(1+\sqrt{E/\Delta}) \,, \tag{6.6}$$

where $N(0)$ vanishes at the transition and Δ is an energy scale that vanishes as $[\sigma(T=0)]^n$. The power n is 3, according to perturbation theory in the metallic regime, and is modified near the transition. The vanishing of the density of state was observed by McMillan and Mochel (1980) in Au_xGe_{1-x} and by Dynes and Garno (1981) in granular aluminum. A very detailed tunneling study of Nb_xSi_{1-x} was reported by Hertel et al. (1983), who found excellent agreement with Eq. (6.6), with the exponent n measured to be near 2. The tunneling data are shown in Fig. 19. As discussed in Sec. III.F, the functional form given in Eq. (6.6) is a common feature of scaling theories, while the relations between critical exponents given by McMillan require further assumptions that are subject to question. This distinction should be kept in mind in assessing the good agreement reported by Hertel et al. with McMillan's theory.

VII. REMARKS AND OPEN PROBLEMS

Instead of a conclusion section, in this final section we address a number of open problems that are not so well understood. Our remarks are necessarily incomplete and speculative. We are hopeful that the advances reviewed so far may lead to progress in these more difficult problems.

A. High-temperature anomalies

The way resistivity of metals and alloys varies with the temperature depends broadly on resistivity or disorder. When resistivity is small, as in many relatively pure metals and alloys, the resistive scattering from phonons has a well-understood temperature dependence. At high temperatures $(T \gtrsim \theta_D$, the Debye temperature), it is proportional to the mean-square amplitude of lattice vibrations, i.e., to $(k_B T)$. The resistivity is typically in the range $10-15 \, \mu\Omega$ cm. This and the resistivity due to static disorder (residual resistivity) are additive. Strongly disordered metals and alloys, amorphous metals, metallic glasses, etc., show characteristic deviations from this behavior; we mention here three kinds, namely the Mooij correlation, the saturation effect, and breakdown of Matthiessen's rule. These may all be connected. These effects have all associated with them a characteristic resistivity of the same order as the Mott maximum metallic resistivity ρ_{min} (perhaps smaller by a factor of 5 or so). We discuss several largely qualitative explanations, including one in terms of incipient localization.

The size and sign of the temperature coefficient of resistivity (TCR) in many disordered systems correlates well with its residual resistivity ρ_0, as first pointed out by Mooij (1973). For the transition-metal alloys discussed by him, the high-temperature TCR changes from positive to negative around $\rho_0^c \simeq 150 \, \mu\Omega$ cm, an approximate equation for $\rho(T)$ being

$$\rho(T) = \rho_0 + (\rho_0^c - \rho_0)AT \, , \tag{7.1}$$

where A is about 10^{-5} per K. This is a very small coefficient, so that the temperature-dependent part is rather small, of order 10^{-3} to $10^{-2}\rho_0$. The resistivity changes weakly with temperature, and may decrease with increasing temperature for sufficiently resistive systems. This kind of connection between resistivity and its temperature dependence is fairly general, and there are many systems that show a sizable negative TCR.

In $A15$ compounds such as Nb_3Sn (see, for example, Gurvitch et al., 1981) or even in elemental Nb (Allen et al., 1976), the resistivity at high temperatures rises more slowly than the linear dependence predicted by simple electron-phonon scattering theory. It is as if there were a tendency for the resistivity to saturate (a term introduced by Fisk and Webb, 1976). In all such systems the resistivity is rather large $(50-150 \, \mu\Omega$ cm), much longer than typical electron-phonon resistivity $(10-15 \, \mu\Omega$ cm), so that there is a clear connection between resistivity and its saturation.

Systems with saturation behavior exhibit striking deviation from Matthiessen's rule. An example is shown in Fig. 20, where the resistivity of $LuRh_4B_4$ is exhibited as a function of temperature for various values of the residual or low-temperature resistivity ρ_0 obtained by α-particle irradiation. The curves should be parallel, the relative shift being ρ_0, if resistive scattering by static and thermal disorder were additive (Matthiessen's rule). Clearly, this rule is not followed; this is a restatement of the saturation
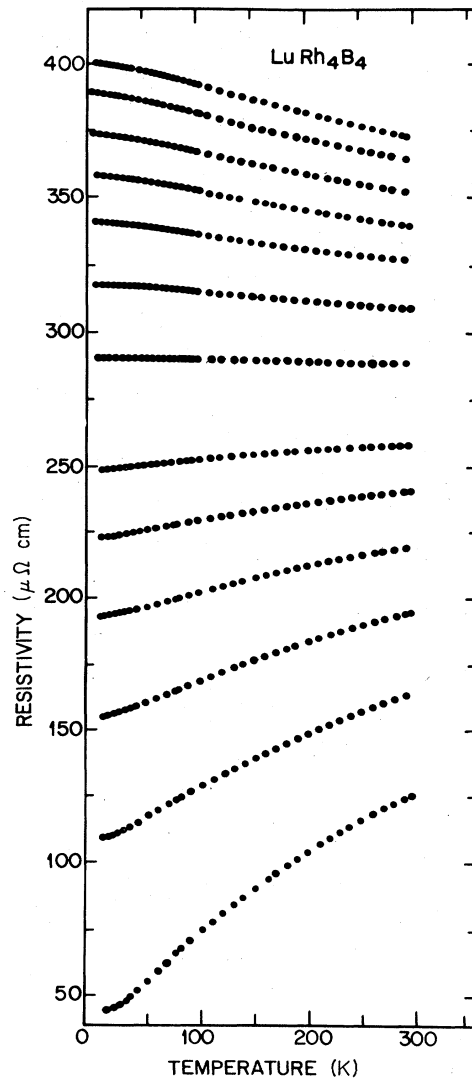

FIG. 20. Resistivity as a function of temperature for $LuRh_4B_4$ at various damage levels. The numbers represent the α-particle dose in units of $10^{16}/cm^2$. From Dynes, Rowell, and Schmidt (1981).

behavior. Instead, the data show a change from positive TCR to negative as the disorder is increased.

A phenomenological model that fits the data on saturation quite well is that of a shunt resistor ρ_{sh} in parallel to the actual system (Wiesmann et al., 1977). One then has

$$\rho^{-1}(T) = \rho_{ideal}(T)^{-1} + \rho_{sh}^{-1} \, , \tag{7.2a}$$

where $\rho_{ideal}(T)$ has the Matthiessen additive form, i.e.,

$$\rho_{ideal}(T) = \rho_0 + \rho_{ph}(T) \, . \tag{7.2b}$$

The quantity ρ_{sh} is a characteristic of a given system, and does not depend on disorder (ρ_0). Typical values of ρ_{sh} are in the range $150-200 \, \mu\Omega$ cm for the $A15$'s. The physical origin of such a shunt resistor is not clear, though it has been suggested that there is an interband conduction channel in addition to the standard transport

channel (Chakraborty and Allen, 1979). However, the argument is quite incomplete. Gurvitch et al. (1980) have argued that the statistical distribution of relaxation times has a lower cutoff related to the Ioffe-Regel criterion, and averaging over such a distribution obtains a parallel resistorlike formula. In a completely different context, namely resistivity of mixed valent compounds such as $CePd_3$, a saturation resistivity appears naturally. The system is crudely described as a lattice of resonant levels. The effect of thermal lattice vibrations and of other disorder is to randomly dephase scattering from different sites. If the dephasing is strong enough, each resonant level scatters independently; this is the saturation limit (Ramakrishnan, 1982). It is possible that the resistivity saturation in $A15$'s is due to a similar mechanism. A shortcoming of the shunt resistor model and of most saturation mechanisms is that they cannot describe the regime of negative TCR; the resistivity is always less than ρ_{sh} and increases towards it as temperature rises.

There is a different class of models in which temperature-dependent conductivity is a sum of two terms, one a normal phonon term that decreases with increasing temperature, and another leading to an increase. The competition can lead to a Mooij-type correlation. Jonson and Girvin (1979; Girvin and Jonson, 1980) suggest that the latter is phonon-induced hopping between slowly diffusing, spatially fluctuating extended electronic states. In the localized regime, this hopping leads to nonzero conductivity increasing with temperature. Jonson and Girvin argue that there are precursor effects in the metallic regime. A model numerical calculation for a tight-binding system shows that TCR changes sign as resistivity increases. A detailed analytical theory with a similar idea, i.e., phonon-induced tunneling, has been recently developed by Götze, Belitz, and Schirmacher (Belitz and Götze, 1982; Belitz and Schirmacher, 1983). This term increases the conductivity and is found to be of order $\lambda_{ph}(k_Fl)^{-1}$ where λ_{ph} is the electron-phonon coupling. Since the temperature-dependent part of the conductivity is small, i.e., $\Delta\rho(T)/\rho(0) \lesssim 10^{-2}$, the lowest-order term in λ_{ph} is adequate. The standard phonon term, again to lowest order in λ_{ph}, goes as $-\lambda_{ph}(k_Fl)^2$. Since the two are of opposite sign, it is argued that for $(k_Fl) \sim 1$, the TCR can go negative. Obviously, such a conclusion can be relied upon only when all terms of relative order up to $(k_Fl)^{-3}$ are collected together and their temperature dependences, signs, etc., are compared.

It has been suggested that the observed negative TCR is a manifestation of incipient localization (Imry, 1980; see also Kaveh and Mott, 1982, for a recent discussion). As discussed in Sec. VI.D, there is evidence for localization and also interaction contribution to the low-temperature conductivity, particularly from magnetoresistance studies (Bieri et al., 1984). Whether these ideas continue to hold at high temperature, $T \approx \theta_D$, when τ_{in}^{-1} becomes large is not all clear. In the most naive estimate, the dominant inelastic process is electron-phonon scattering, so that $\tau_{in}^{-1} \sim T$ and $\Delta\sigma(T) \cong \sigma_0(T/\theta_D)^{1/2}$. There is, in addition, a normal electron-phonon resistivity term

$$\Delta\sigma_{e\text{-ph}}(T) \approx \lambda_{ph} \left[\frac{T}{\theta_D} \right] \frac{\sigma_0}{(k_Fl)^2} .$$

Since the temperature dependences are different, the temperature dependence of the sum of these two terms will in general change sign over some temperature range, and such sign change is not typically observed.

Laughlin (1982) attempted to explain the resistivity saturation by applying the interaction theory in the presence of disorder. He argued that strong disorder is induced by thermal fluctuations, and that exchange corrections would lead to a suppression of the density of states. His theory is a rather phenomenological extrapolation of the perturbation theory of Altshuler and Aronov (1979a) to the strong coupling regime, and has been criticized by Gurvitch (1983) on experimental grounds.

A weak-scattering explanation of small and negative TCR is based on Ziman's theory of electrical resistivity of liquid metals. Here electrons are assumed to scatter weakly off a static temperature-dependent arrangement of atoms. The observed TCR is attributed to the latter (Nagel, 1977,1982) or to "ineffectiveness" of phonons with wave vector $q < l^{-1}$ (Cote and Meisel, 1977,1978). The first explanation is too specific, and the latter is incorrect (see Sec. III.F). More generally, the basic assumption is unsound, since in these systems scattering is not weak, mean free paths being comparable to interatomic spacing. However, an effective medium analysis (Nicholson and Schwartz, 1982) of a structurally disordered metal leads to conclusions similar to that from perturbative Ziman theory.

There are some possibly related transport anomalies, such as in thermopower (Nagel, 1978), T^2 dependence of low-temperature resistivity (e.g., Gurvitch, 1980), and resistivity minimum in amorphous metals, both magnetic and nonmagnetic (see, for example, Tsuei, 1976; Grest and Nagel, 1979). All these are observed in stongly disordered systems with short mean free paths, so that these could be characteristic disorder effects.

B. Electron-phonon interaction and polaronic effects

Some known results about electron-phonon interaction in disordered metals have been mentioned in Sec. III.E, namely, that the interaction vertex is not enhanced by disorder, and that the ultrasonic attenuation has the Pippard form. We mention here several open questions, connected with disordered metals and with the motion of single electron in a disordered deformable medium (a polaron).

There is as yet no analysis of the electron-phonon vertex or of phonon propagation in a disordered metal beyond the Schmid approximation (Sec. III.E), i.e., one which considers processes arising in localization or interaction theories. These are of higher order in $(k_Fl)^{-1}$, but can lead to sizable characteristic anomalies as electron diffusion slows down close to the metal-insulator transition. Sound propagation in metallic glasses, which are strongly disordered, has been extensively studied experi-

mentally (e.g., Guntherodt, 1981). In insulating glasses there is good agreement between the observed temperature and amplitude dependence (saturation behavior) of sound velocity, attenuation, etc., and the two-level or tunneling model of Anderson, Halperin, and Varma (1972) and of Phillips (1972). However in metallic glasses there are many differences, for which explanations are sought in coupling of two-level systems to low-lying electron hole excitations in addition to phonons. (See, for example, Vladar and Zawadowski, 1983, for recent work and an extensive review.) It is possible, however, that in these systems there could be significant and characteristic disorder effects on phonon propagation.

In perturbation theory, the Hartree-type process involving coupling between disorder-induced electron density fluctuations via lattice distortion leads, as mentioned in Sec. III.E, to a reduction in conductivity and in the density of states. For strong electron-phonon coupling and strong disorder, i.e., in the vicinity of the metal-insulator transition, the effect of this kind of process is not known.

An electron coupled to a deformable vibrating lattice (i.e., acoustic or optical phonons) carries the lattice distortion with it. In clean systems, the consequences of this polaronic effect (reduction in ground-state energy, effective mass, mobility, self-trapping for large electron-acoustic phonon coupling, etc.) have been investigated for a long time. (See, for example, Mahan, 1981.) Holstein and particularly Emin (see Emin, 1983, for a recent review) have argued that polaronic effects should be particularly strong in disordered systems, since slow diffusion or localization promotes local lattice distortion. In amorphous semiconductors (elemental and chalcogenide), electron-phonon coupling is strong and Coulomb screening effects are absent. Emin has developed a detailed description of transport properties of amorphous semiconductors, in which the charge carriers are small polarons that hop, the small-polaron formation being due both to large electron-phonon coupling and to the strong disorder.

There is as yet not much theoretical work on the problem of a single electron in a disordered, deformable medium. The medium has both quenched and annealed disorder, and is not static since there are lattice vibrations. In the adiabatic approximation, which ignores dynamics, Cohen, Economou, and Soukoulis (1983) have discussed polaron effects recently and have argued that the conductivity transition becomes discontinuous, the discontinuity being proportional to (electron-phonon coupling)$^{2/3}$. On the localized side of the mobility edge, Anderson (1972) has suggested that the feedback effect of self-trapping can lead to a gap in the density of states. There is at present no theoretical analysis of these questions from a microscopic point of view.

C. Superconductivity and localization

The ground state of an electron gas with phonon-mediated net attraction between electrons is supercon-

ducting. One expects that as disorder increases and electronic states near ε_F localize, superconductive coherence is destroyed, and the system goes insulating. This transition is not understood, since it occurs when the system is strongly disordered. We briefly summarize here the experimental and theoretical work that has touched upon this difficult problem.

An interesting perturbative precursor effect of superconductivity in a disordered metal has been discussed by Larkin (1980). He calculated the vertex enhancement of the fluctuation conductivity caused by impurity scattering. The Maki-Thompson process considered involves the exchange of a single Cooper-pair excitation. The contribution of this process to magnetoresistance depends on field in the same way as the localization term (Sec. II.E), and the coefficient, i.e., the effective coupling, depends strongly on how close one is to the superconducting T_c. Such a strongly temperature-dependent magnetoresistance above T_c has been observed by Gordon, Lobb, and Tinkham (1983), by Bruynseraede et al. (1983), and by Gershenson, Gubankov, and Zhuravlev (1983), the agreement with theory being very good.

The superconducting transition temperature T_c depends rather weakly on disorder in general. Matthias and co-workers showed first in the late 1950s that while a low concentration of magnetic ions in a superconductor depresses their T_c considerably, nonmagnetic impurities have virtually no effect. Anderson (1959) provided a fundamental explanation of this fact by pointing out that Cooper pairs can be formed out of time-reversed exact eigenstates whose state density is not strongly affected by disorder. Since the phonon-mediated coupling is of short range in space, it is not expected to change for weak disorder, i.e., for $l \gg q_D^{-1}$. Thus the superconducting transition is unaffected. Gor'kov (1960) has explicitly shown to leading order in random potential scattering that the two-particle propagator is unaffected. Since the electron-phonon vertex is also unchanged (Sec. III.E), T_c does not decrease. However, the presence of magnetic impurities breaks time-reversal invariance, so that time-reversed Cooper pairs acquire a finite lifetime. This depresses T_c.

The analyses of Anderson and Gor'kov do not consider either the localizing effect of strong disorder (Sec. II) or the interference between interaction and disorder (Sec. III). A perturbative analysis of interaction effects has been carried out by Maekawa and Fukuyama (1981,1982) and by Takagi and Kuroda (1982). They consider the two-dimensional case, where the BCS temperature T_c describes a pseudotransition, and not the onset of superconductive order, which occurs at a lower vortex binding temperature T_v. They find that corrections to pair density of states and to the interaction vertex both affect T_c, which satisfies the equation

$$\ln\left[\frac{T_c}{T_{c0}}\right] = -\frac{(g_1 - 3g_0)N(0)}{4\pi\varepsilon_F\tau}\left[\ln\frac{1}{T_{c0}\tau}\right]^2$$

$$-\frac{(g_0 + g_1)N(0)}{6\pi\varepsilon_F\tau}\left[\ln\frac{1}{T_{c0}\tau}\right]^3, \qquad (7.3)$$

where T_{c0} is the transition temperature in the absence of disorder, g_1 is the electron repulsion, and g_0 is the phonon-mediated coupling. The corrections are of first order in $(\varepsilon_F\tau)^{-1}$. It is worth noting that, starting from the BCS equation, which reads $T_c = \tau^{-1}\exp-[g_0 N(0)-\mu^*]^{-1}$ in this case, a modification of the Coulomb repulsion μ^* by $\delta\mu^*$ would lead to a change in T_c given by $\ln(T_c/T_{c0}) = -\delta\mu^*\ln^2(T_{c0}\tau_0)$. Thus the last term in Eq. (7.3) can be interpreted as a logarithmic correction to the effective Coulomb repulsion μ^*. We also note that in Eq. (7.3) the usual renormalization of the Coulomb repulsion from its bare value to μ^* is not incorporated in the model of Fukuyama and Maekawa, which does not take into account the phonon-mediated nature of the interaction g_0. Using Eq. (7.3) and ignoring g_0 in comparison to g_1, the authors argue that T_c decreases with disorder, the natural scale of the latter being $\varepsilon_F\tau\approx 1$ resistivity. Recent measurements of Graybeal and Beasley (1984) on ultrathin Mo-Ge films find that the pseudotransition temperature decreases with disorder, in good agreement with the prediction of Eq. (7.3). Retardation effects due to phonon-mediated interactions and dynamic screening, as well as extensions of the weak-impurity scattering calculations to three dimensions, were recently given by Fukuyama, Ebisawa, and Maekawa (1984).

In a large number of high-T_c superconductors, e.g., $A15$ compounds such as Nb_3Ge, cluster compounds such as $ErRh_4B_4$, etc., strong disorder reduces T_c drastically. It is an experimental fact that T_c depends on low-temperature resistivity and not on whether disorder is produced by irradiation, alloying, etc. It also does not depend on other quantities such as the resistivity ratio. The reduced T_c vs ρ curves for various systems are similar (see, for example, Fig. 21). T_c falls by a factor of 5 or 6 in many cases and then saturates, as does ρ. A large part of the decrease in T_c occurs when ρ is sizable, greater than 50 $\mu\Omega$ cm or so. There is at least one exception to this behavior, namely Mo_3Ge, whose T_c increases from 1.5 to 4.5 K and then saturates as resistivity ρ increases beyond 100 $\mu\Omega$ cm.

Anderson, Muttalib, and Ramakrishnan (1983) have pointed out that the strongly scale-dependent diffusion characteristic of a system close to critical disorder in three dimensions enhances the repulsive Coulomb pseudopotential and thus decreases T_c. Assuming that the interaction μ is very short range, its effective strength depends on the electron residence probability as a function of time, i.e., on $\varphi(t) = \langle\rho(r=0,t)\rho(r=0,0)\rangle$. The Coulomb kernel is

$$K^c(\omega) = \mu\left[1 + 2N(0)^{-1}\int_\tau^\infty \varphi(t)^{-1}\cos(\omega t)dt\right]. \quad (7.4)$$

For normal diffusion $\varphi(t)\sim t^{-3/2}$, whereas at critical disorder we have $D(L)\sim L^{-1}$ (see Sec. II.C), so that $\varphi(t)\sim t^{-1}$, i.e., electron diffusion is very slow. This enhances the Coulomb kernal. For near-critical disorder, diffusion is nonclassical up to around a distance $\xi\approx l(\rho/\rho_c)$, so that $\varphi(t)\sim t^{-1}$ for $t < (\rho/\rho_c)^3\tau$. The consequent enhancement of effective repulsion reduces T_c;
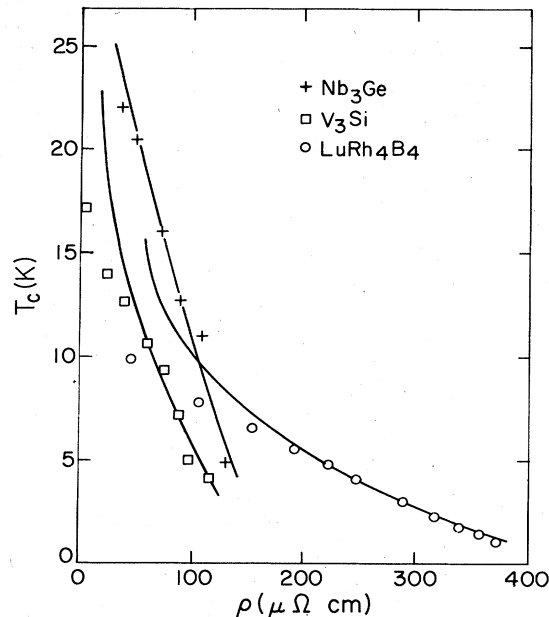

FIG. 21. The reduction of superconducting T_c with increasing high-temperature resistivity for several compounds. The solid lines are theoretical fits and also serve as a guide to the eye. Data are from Rowell and Dynes (unpublished), as reproduced in Anderson, Muttalib, and Ramakrishnan (1983).

this reduction depends on disorder as measured by resisitivity. This is a localization-dependent process whereby superconductivity is suppressed. It is assumed that other effects due to disorder are small. The theory describes the observed "universal" degradation of T_c reasonably well, provided the critical resistivity ρ_c is assumed to be rather low (~ 50 $\mu\Omega$ cm). Some reasons for this are given. There are a number of other explanations for this universal degradation that do not involve localization. A popular one argues that there is a sharp peak in the density of states near the Fermi energy, where smearing by disorder reduces T_c (Testadri and Mattheiss, 1978).

Besides the transition temperature, there are some measurements on the upper critical field H_{c2} of strongly disordered superconductors. There are clear anomalies in both the temperature dependence and the size of H_{c2}. The best studied atomically disordered systems are amorphous metals (Tenhover, Johnson, and Tsuei, 1981; see also Ikebe et al., 1981; Coleman et al., 1983; and for thin films Kobayashi et al., 1983). The shape and size of $H_{c2}(T/T_c)$ cannot be fitted by any variant of dirty superconductor theory. Also, $H_{c2}(T=0)$ is higher than expected. There are a number of similar reports in the literature (see, however, Karkut and Hake, 1983). In many cases, $H_{c2}(T=0)$ exceeds the Clogston-Chandrasekhar or paramagnetic limit ($\mu_B H_{c2} = k_B T_c$) by a factor of 2 or 3, sometimes more. On the other hand, Graybeal and Beasely (1984) report for films that H_{c2} is smaller than the standard Ginzburg-Landau value. In all of these cases, the system is very strongly disordered, with conductivity close to σ_{min}. In this connection, we note

that the Clogston-Chandrasekhar limit for a dirty super-conductor corresponds also to the Ioffe-Regel condition, since from the dirty limit estimate of $\xi^2 = (\hbar v_F / k_B T_c) l$ one has $\mu_B H_{c2} = (k_B T_c)/(k_F l)$. The paramagnetic limit at which spin and orbital effects are comparable thus corresponds to $k_F l \simeq 1$. In this regime both interaction and localization effects are strong, so that the local weak-disorder Ginzburg-Landau equation theory of H_{c2} has to be revised to include effects of scale-dependent diffusion.

Takagi, Souda, and Kuroda (1982) have computed the modification of the Ginzburg-Landau parameters due to interaction effects in the presence of disorder and obtained expressions for the critical field H_{c2}. Maekawa, Ebisawa, and Fukuyama (1983) have generalized the T_c calculation of Maekawa and Fukuyama (1982a) to include an external magnetic field, and also to consider some localization effects. The perturbative calculations are for a two-dimensional system. One of their results is that the H_{c2} vs T curve can become concave due to localization effects. Such a concave curvature has been observed by Kobayashi et al. (1983) for thin Zn films. Coffey, Muttalib, and Levin (1984) compute the temperature dependence of H_{c2} for bulk superconductors, assuming that T_c decreases mainly because of Coulomb pseudopotential enhancement due to localization. Since a magnetic field reduces localization effects (Sec. II.E), $H_{c2}(T)$ is enhanced over the standard result for dirty superconductors.

Existing calculations for T_c, $H_{c2}(T)$, etc., are incomplete in several ways. Disorder effects are either not properly computed, or one or more processes (among many with comparable size and disorder dependence) are neglected, or critical disorder effects are estimated approximately. Experimentally, the reduction in T_c and enhancement of H_{c2} are small ($\approx 10\%$) for moderate disorder. Large effects occur for $A15$'s and near the metal-insulator transition for granular superconductors where the transition becomes broad and T_c rather abruptly drops to zero. The problem of superconductivity in the vicinity of the metal-insulator transition is addressed very recently. Kapitulnik and Kotliar (1985) pointed out that the superconducting coherence length becomes very short, so that T_c is greatly suppressed due to thermal fluctuations. Ma and Lee (1985) suggested that superconductivity may persist into the insulating side, so that a coherent paired state with localized quasiparticle excitations may be possible.

There has been considerable recent work on superconductivity in disordered films, stimulated by the Kosterlitz-Thouless-Berezinskii theory of vortex-unbinding transitions in two-dimensional systems. Below the BCS or mean-field transition temperature, there are thermally excited free vortices, which move under the action of external electromagnetic fields, thus leading to finite resistivity. Below a certain temperature T_{KT}, vortex-antivortex pairs bind, so that the system is rigid against phase fluctuations and is a superconductor. There are a number of characteristic predictions for resistivity versus frequency, temperature, and applied voltage (see, for example, Halperin and Nelson, 1979). Some of these

have been directly probed experimentally (see, for example, Fiory, Hebard, and Glaberson, 1983). The theoretical analyses do not include either localization or interaction effects, and predict for example that

$$T_{KT} = T_c^0 (1 + R_\square g_c) , \tag{7.5}$$

where $g_c^{-1} \simeq 2.5 \times 10^4 \ \Omega$. Thus, while T_{KT} decreases with increasing disorder, it does not vanish no matter how large $R_\square$ is. Experimentally, T_{KT} follows Eq. (7.5) for low $R_\square$ and then drops very rapidly to zero around $R_\square \simeq 3 \times 10^4 \ \Omega$ (Dynes, Garno, and Rowell, 1978).

This is a strong effect, and could be due to the localization length's becoming of the order of the vortex core size, so that the vortex core is an insulator. In that case, the vortex could disintegrate because of large fluctuations in the phase φ of the superconducting order parameter. Since the localization length depends exponentially on disorder, crossover to the nonsuperconducting regime is very rapid, and occurs close to $R_\square \sim 3 \times 10^4 \ \Omega$. Interaction effects are also expected to be strong in this regime. As is well known, electron number (or density) and the phase φ are conjugate variables, so that fluctuations in the two are coupled. With increasing disorder, the effective dynamic interactions between density fluctuations increase, thus promoting phase disorder and suppressing superconductivity. There is as yet no theory of these effects.

The most commonly studied strongly disordered bulk superconductors are granular, consisting of metal grains separated by insulating oxide or by another codeposited nonmetal. (See Deutscher, 1982, and Deutscher et al., 1983 for a concise review.) Metallic, superconducting, and insulating phases are all known in such systems. Recent experiments on these (see, for example, Sec. VI) show large characteristic localization and interaction effects not considered in theoretical models for superconductivity in such systems developed so far (Deutscher, 1982). The models differ, depending on the ratio of a typical grain size or volume to the volume of a Cooper pair. If the former is large enough so that the average intragranular energy-level spacing is smaller than $k_B T_c$, fluctuations in magnitude $|\Delta|$ of the BCS order parameter are small, and the phase φ of the grain is a good dynamical variable. The system is described as a collection of coupled Josephson junctions (disordered planar spin or XY model). Such a system always orders. However, interaction between charge imbalance on the grains leads to quantum fluctuations of the phase, which can destroy phase order. The problem has been considered by Abeles (1977), Simanek (1980), Efetov (1981), and most recently by Doniach (1981), who developed an explicit quantum XY spin model for a two-dimensional system. In the opposite, small-grain limit, both amplitude and phase fluctuations are important. A classical percolative model of "effectively" connected grains has been used to describe this limit (Deutscher et al., 1983).

It is clear that in the small-grain limit the system is an atomically disordered metal, with disorder effects dependent on two dimensionless parameters, namely the mean free path $k_F l$ and the screened Coulomb interaction

$vN(0)$. Some work on this problem has been mentioned earlier (Maekawa and Fukuyama, 1982; Takagi and Kuroda, 1982). In the large-grain regime, the effective coupling between them is scale dependent, due to quantum interference effects characteristic of localization, so that at least close to critical disorder in three dimensions the large-length-scale behavior is the same as that of the atomically disordered model with renormalized parameters.

Finally, consider a system in which electron-phonon coupling is the dominant interaction. As disorder is increased, the system passes from a metal to an insulator. The former has a superconducting ground state, while the later is a negative U insulator with opposite-spin electrons paired locally to take advantage of the lattice distortion. The one-electron excitation spectrum has a gap (see, for example, Anderson, 1975). It is not yet known how this transition from momentum space to real space pairing takes place with increasing disorder for a given electron-phonon coupling.

D. Coulomb effects in the insulator

So far in this paper we have focused our attention on the metallic side of the metal-insulator transition. This is because the metallic side is amenable to a perturbative treatment in the strength of the disorder. The insulating side of the transition is a highly nontrivial problem in itself, the main feature being the competition between randomness and the long-range Coulomb potential. We shall review a number of concepts developed to describe the insulator.

Much of the interesting physics is already contained in a simple model that treats the extremely localized situation, in which the hopping between sites is ignored. The Hamiltonian can be written as

$$H = \sum_i n_i \varphi_i + \frac{1}{2} \sum_{i \neq j} \frac{n_i n_j}{r_{ij}} + H' , \qquad (7.6)$$

where n_i is the number operator for a localized state on site i, r_{ij} is the distance between two sites, and H' describes the Coulomb interaction with some neutralizing background change. For simplicity, we may also assume that double occupation of each site is forbidden by an on-site repulsive term. The energy φ_i is a random on-site potential. Disorder is introduced into the problem by the distribution of φ_i, or by the random distribution of sites, or both. An excellent practical realization of such a system is the impurity band of lightly doped, compensated semiconductor, where the disorder arises from the random distribution of impurities over the host's lattice sites. The carriers remaining in the majority band interact strongly with unscreened Coulomb potentials, and are also subject to a large random field from the ionized minority impurities and the unoccupied majority impurities. These forces are all of long range, unlike quantum-mechanical effects, such as tunneling, which depend exponentially on the separation between sites. A more detailed justification of the purely classical model has been given by Shklovskii and Efros (1980).

The spatial distribution of the electrons in this model is highly nontrivial because the total energy E_i of an electron on site i is given by

$$E_i = \varphi_i + \sum_{j \neq i} n_j / r_{ij} \qquad (7.7)$$

and depends on the occupation of other sites. The competition between the disorder and the Coulomb energy leads to a depletion of the single-particle density of states near the chemical potential known as the Coulomb gap (Pollack, 1970; Srinivasan, 1971; Efros and Shklovskii, 1975). We shall give an argument for the Coulomb gap due to Efros and Shklovskii.

The single-particle density of states N_1 is defined as the distribution of the energy $E_i - \mu$, which is the energy required to add an electron to an empty site i (or minus the energy for adding a hole to an occupied site), holding the rest of the electrons fixed. If an electron is moved from an occupied site i to an empty one j, the change in energy of the system due to this one-electron hop (or particle-hole excitation) is

$$E_{ji} = E_j - E_i - 1/r_{ij} . \qquad (7.8)$$

The last term is the attraction of the electron-hole pair created, and its presence causes the Coulomb gap. From the ground state, all excitation energies like Eq. (7.8) must be positive. This implies a minimum spatial separation between pairs of sites whose single-particle energies lie on either side of the chemical potential; if the states are assumed to be homogeneously distributed through space, there will be a bound on the single-particle density of states $N_1(E)$ of the form

$$N_1(E) \propto |E - \mu|^s , \qquad (7.9)$$

with $s \geq D - 1$ in D dimensions. In this description, the Coulomb gap is necessary to prevent an excitonic collapse. By using a "self-consistent" argument and an approximation in which the stability of the ground state is considered only in terms of particle-hole transitions, Efros (1976) showed that $s = D - 1$ and derived the constant of proportionality. He also obtained a sharper bound for three-dimensional systems by considering many particle-hole excitations in which the surrounding electrons were allowed to relax; his density of states had the form

$$N_1(E) \propto \exp[-|E_0/(E - \mu)|^{1/2}] . \qquad (7.10)$$

This exponential gap arises from the existence of short particle-hole excitations with very low transition energies. Baranovskii, Shklovskii, and Efros (1980) showed that the number of such excitations should go to zero logarithmically as the energy goes to zero, and took this into account to obtain another form for the single-particle density of states:

$$N_1(E) \propto \exp[-\lambda y/(\ln y)^{7/4}] , \qquad (7.11)$$

where $y = E_0/(E - \mu)$. This holds only for very low energies, and so cannot be tested numerically. There is also the possibility that the close pairs with low excitation en-

ergies may be removed by quantum-mechanical tunneling in a real material, and that these exponential forms for the Coulomb gap may therefore be unimportant in practice.

The addition of a single electron is not the lowest elementary excitation in this system. The low-lying excitations include a region of relaxation around the added particle, and therefore have a lower energy than the bare excitation (Mott, 1975). These excitations may be termed "electronic polarons." Efros (1976) argued that the polarization cloud should have a finite radius, beyond which the polaron appears simply as a charged quasiparticle. The argument leading to Eq. (7.9) should be correct, if everywhere a particle is replaced by a polaron, and the density of polaron states $N_p(E)$ should obey Eq. (7.9) and not Eq. (7.10) or Eq. (7.11).

The vanishing of $N_p(E)$ at the chemical potential shows that it is impossible to add an extra particle to the system with an infinitesimal energy increase over the ground state, even if local relaxation is permitted. By contrast, if total relaxation of the system is allowed, the resulting density of states $dn/d\mu$ for adding an extra electron at the chemical potential is not expected to be zero. This is an indication that the system is behaving like a glass, with regions of configuration space inaccessible from the ground state at low temperatures. This glassy state arises from competition between the Coulomb energies and the random site energies. Another way of expressing this dichotomy is that, according to Thomas-Fermi theory, the screening constant is proportional to $dn/d\mu$ and the insulator would screen like a metal. It is only due to the glasslike behavior that this does not happen in any finite time scale.

The existence of the Coulomb gap has been tested numerically by Baranovskii, Efros, Gelmont, and Shklovskii (1979) and by Davies, Lee, and Rice (1982,1984), who also examined the polaron density of states. The results are consistent with the predictions of Eqs. (7.9) and (7.10) for the polaron and single-particle density of states, respectively. Davies *et al.* also carried out the calculation at finite temperature and examined the possibility of a glass transition at some temperature, analogous to the spin-glass transition. The results are suggestive, but not conclusive, partly due to difficulties in defining a spin-glass-type order parameter.

The existence of a Coulomb gap should be tested experimentally by tunneling experiments. There are indications of a parabolic tunneling characteristic on the insulator side of the metal-insulator transition in the experiments of McMillan and Mochel (1981) and Hertel *et al.* (1983). However, in tunneling into an insulator, it is not easy to ascertain that the tunneling is a one-step process which measures the density of states, a problem encountered in earlier tunneling work into doped semiconductors (Wolf *et al.*, 1971,1975). Further experimental studies are clearly desirable.

A second experimental manifestation of the Coulomb gap is in the temperature dependence of the conductivity. Mott's variable-range hopping theory (Mott, 1968) was derived assuming a constant density of states. This predicted $\sigma(T) \sim \exp[-(T/T_0)^{1/(d+1)}]$. The introduction of the Coulomb gap equation (7.9) modifies this to

$$\sigma(T) \sim \exp[-(T/T_c)^{1/2}] \tag{7.12}$$

in all dimensions (Efros, 1976). There are some experimental indications of this type of behavior in the literature (Redfield, 1975).

The Coulomb interaction also manifests itself in the low-frequency conductivity $\sigma(\omega)$ and the dielectric constant $\varepsilon^1(\omega)$. To understand this, it must be emphasized that the presence of the Coulomb gap in $N_1(E)$ does not imply the absence of *particle-hole* excitations with low energies; the only requirement is that all such excitations not have negative transition energies. There are, in fact, many of them, but most involve short hops; only for large separations of the electron and hole is the number of possible excitations with low energy reduced greatly by the Coulomb gap. In a noninteracting system, the particle-hole density of states is linear in energy. This density of states is, in fact, greatly enhanced by Coulomb interaction because the energies of the electron and hole are no longer restricted to be within E of the chemical potential. This affects, for example, the frequency-dependent conductivity, which is given in a noninteracting system at $T=0$ and at low frequencies by (Mott, 1970)

$$\sigma(\omega) \propto \omega^2 |\ln\omega|^4 . \tag{7.13}$$

However, if the Coulomb interaction and a parabolic single-particle density of states for a three-dimensional system are included, the result becomes (Efros and Shklovskii, 1981; Davies, Lee, and Rice, 1984)

$$\sigma(\omega) \propto \omega/|\ln\omega| . \tag{7.14}$$

By the Kramers-Kronig relation, this implies a logarithmically divergent dielectric constant in the limit $\omega \to 0$. This intriguing behavior is presumably related to the glasslike properties of system. A similar behavior has been found at finite temperatures by Efros (1981). The frequency-dependent conductivity and dielectric constant were recently studied by Paalanen, Rosenbaum, Thomas, and Bhatt (1983) in phosphorus-doped silicon just on the insulator side of the metal-insulator transition, and a very slow rise in the dielectric constant that extended down to very low frequencies was observed. The low-frequency conductivity is consistent with Eq. (7.14). Bhatt and Ramakrishnan (1984) have presented arguments for why this should be so, even near the metal-insulator transition.

Finally we make some brief remarks on the spin degrees of freedom. In the strongly localized limit, on-site repulsion will cause states to be singly occupied, and the spins will behave as local moments. The interactions between these spins are antiferromagnetic by the Heisenberg exchange mechanism. The question arises as to the nature of the ground state, whether it is antiferromagnetic, spin-glass, etc. Experimental investigations have been carried out by Geschwind *et al.* (1980) in CdS and by Andres *et al.* (1981) in Si:P. The magnetic susceptibility typically increases like a power law with decreasing tem-

perature, but with an exponent less than the Curie value of unity. Furthermore, there is no evidence of a spin-glass or antiferromagnetic transition. These results were interpreted as the gradual formation of singlet pairs, starting from spins that happen to be nearby, and extending to spins that are far apart as the temperature is decreased. The singlet formation was demonstrated by calculations on a cluster of a few (up to eight) spins by Walstedt et al. (1979), and by a scaling calculation (Bhatt and Lee, 1982).

The presence of low-lying magnetic excitation also appears as a magnetic-field-dependent term in the linear T specific-heat coefficient. In general, because of the presence of low-lying particle-hole excitations, the specific heat of the localized insulator has a linear T term in the specific heat, quite unlike the insulator with a band gap. Part of this linear term is found to be magnetic-field-dependent (Kobayashi et al., 1979). An analysis of these experiments can be found in Takemori and Kamimura (1982).

It is clear from the above brief summary that the Anderson insulator with Coulomb interaction is quite unlike the ordinary insulator with a band gap. Many of the issues raised here will undoubtedly receive more experimental and theoretical attention in the years to come.

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