Addendum: Ergodic theory of chaos and strange attractors [Rev. Mod. Phys. 57, 617 (1985)]

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A. Further references. An early paper on the reconstruction of orbits from their time series is "Geometry from a time series" by N. H. Packard, J. P. Crutchfield, J. D. Farmer, and R. S. Shaw, Phys. Rev. Lett. **45**, 712 (1980). It treats the reconstruction in terms of differences of successive data points.

The notion of *information dimension* was coined by J. D. Farmer in his paper, "Information dimension and the probabilistic structure of chaos," Z. Naturforsch. **37a**, 1304 (1982). This concept of dimension has been known in the earlier literature (Balatoni and Renyi) as the "dimension of a probability distribution." Farmer also points out that it is the information dimension which is most easily accessible in experiments.

B. On the choice of an embedding dimension m for the estimation of the information dimension $\dim_H \rho$. In Sec. V.A we showed how to construct α_m such that $\alpha_m = m$ for $m \leq \dim_H \rho$, and $\alpha_m = \dim_H \rho$ for $m \geq \dim_H \rho$. While this method for obtaining the information dimension is, in principle, correct, it has to be taken with a grain of salt. If we only assume that $m \geq \dim_H \rho$, it may occur that the reconstruction of the attractor be noninjective, i.e., different trajectories may cross each other. In other words, different "sheets" of the attractor may cross. Suppose two sheets cross at $x(i)$, then $C_i^m(r)$ as defined by (5.1) is multiplied by a factor 2. One tends thus to overestimate $C_i^m(r)$ and therefore to underestimate α_m and the information dimension. (A sign of this is visible in Fig. 19 of our paper.) The cure to this problem is to take m large enough (probably $m \geq 2\dim_H \rho + 1$ will do; see Mañé's theorem in Sec. II.G). We thank A. Mullhaupt for drawing our attention to this explanation of the seemingly slow convergence of the dimension in experimental figures such as our Fig. 19.

C. The algorithms proposed in our paper have now been implemented in collaboration with S. Kamphorst Leal da Silva. Diskettes containing the Fortran code are available from the authors.