Strong turbulence of plasma waves*

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This paper reviews recent work related to modulational instability and wave envelope self-focusing in dynamical and statistical systems. After introductory remarks pertinent to nonlinear optics realizations of these effects, the author summarizes the status of the subject in plasma physics, where it has come to be called "strong Langmuir turbulence." The paper treats the historical development of pertinent concepts, analytical theory, numerical simulations, laboratory experiments, and spacecraft observations. The role of self-similar self-focusing Langmuir envelope wave packets is emphasized, both in the Zakharov equation model for the wave dynamics and in a statistical theory based on this dynamical model.

CONTENTS

I. INTRODUCTION

This review will survey developments over the last twelve years concerning certain nonlinear dynamical and statistical models of electron plasma waves in a nonmagnetic plasma. We shall focus on a particular set of phenomena, rich in physics and mathematics, and significant for plasmas, fluids, and nonlinear optics.

Among the dynamical effects we consider are modulational instability (in which a high-frequency wave envelope breaks up into shorter wavelengths) (Vedenov and Rudakov, 1964; Nishikawa, 1968), the related nonlinear spatial self-focusing of localized wave envelopes (Zakharov, 1967,1972), and manifestations of these processes during external driving and dissipation of wave energy (Sudan, 1973,1975; Kim, Stenzel and Wong, 1974; Hafizi *et al.*, 1982).

Statistical treatments of such effects differ from the conventional "weak" turbulence, based on three-wave interactions of randomly phased plane waves. Nonlinear wave structures may be used instead of plane waves as the building blocks of the turbulence (Kingsep, Rudakov, and Sudan, 1973; Galeev, Sagdeev, Sigov, Shapiro, and Shevchenko, 1975), or else one goes beyond the usual random-phase approximation to admit phase-coherence and renormalization effects, as in the application of the "direct-interaction" approximation to this problem (DuBois *et al.*, 1979; DuBois and Rose, 1981). Such theories, together with the dynamical equations upon which they rest, have come to be called "strong" turbulence, to distinguish them from conventional "weak" turbulence.

This review has been prompted in part by intense activity in the Soviet Union during the last decade on the problem of Langmuir turbulence. Much less attention has been paid to this class of problems outside of the Soviet Union, and it is our purpose to try to stimulate interest and research by presenting a summary and interpretation accessible to the general physics community. For this reason we have chosen to suppress derivations and details and concentrate on heuristic explanations of the physics and mathematics wherever possible.

After a few introductory and technical remarks concerning elementary plasma physics (Sec. II), we proceed to develop the fundamental concepts of modulational instability, solitons, and self-focusing phenomena in a general physics context (Sec. III). Our starting point is the notion of a nonlinear index of refraction, which enters into the simplest wave equation describing such phenomena—the nonlinear Schrödinger equation. This simple dynamical model is relevant to optics and fluid dynamics, as well as

*This paper is based on an invited review paper presented at the 25th annual meeting of the Division of Plasma Physics of the American Physical Society, held in Los Angeles, November 7–11, 1983.

to plasmas. The important roles of spatial dimension and of self-similar solutions are stressed. We then draw upon selected examples of relevant experiments from nonlinear optics, hydrodynamics, and astrophysics, which demonstrate the relevance of the basic phenomena (Sec. IV; Garmire *et al.*, 1966; Feir, 1967; Campillo *et al.*, 1973; Yuen and Lake, 1975). This section is designed only to highlight the relevant concepts and is not comprehensive.

In Sec. V, we address the historical development of these ideas in theoretical plasma physics and the role of ponderomotive force, which underlies Langmuir wave-wave interactions in plasmas. Encouraging results are displayed from recent two-dimensional particle-in-cell numerical simulations which demonstrate spatial self-focusing and eventual dissipation of the energy of an initial Langmuir envelope wave packet (Anisimov *et al.*, 1982; Forslund, 1983).

An important dynamical model describes Langmuir wave spatial collapse and other wave-wave interactions—the so-called Zakharov equations (Hasegawa, 1970; Zakharov, 1972). We examine the stability of the Zakharov equations and show that they contain the cubic Schrödinger equation in a certain adiabatic limit, but that they also describe dynamical ion phenomena, such as Langmuir wave scattering off ion-acoustic waves, supersonic collapse, and radiation of sound waves from density cavities. Numerical solutions of an initial-value problem are shown to lead to self-similar collapse (Pereira, Sudan, and Denavit, 1977).

Section VI is devoted to possible applications of these ideas to physical plasmas. A number of experiments are cited which invoke the notions of strong Langmuir turbulence in their interpretations. Special emphasis is placed on observations of electron-beam-excited instabilities in the laboratory (Cheung *et al.*, 1982, Leung *et al.*, 1982; Michel *et al.*, 1982) and in interplanetary space (Gurnett and Frank, 1975; Gurnett and Anderson, 1977; Gurnett *et al.*, 1980,1981), which seem to involve self-focusing and/or modulational instability. Measurements taken in conjunction with so-called Type-III solar radio-wave emission are presented and interpreted (Lin *et al.*, 1981; Goldman, 1983).

In Sec. VII, we modify the two-dimensional Zakharov equations to include an electron-beam driver and present numerical solutions relevant to the Type-III burst problem (Nicholson *et al.*, 1978; Hafizi *et al.*, 1983). These solutions exhibit a (short) "cascade" from beam-resonant Langmuir modes down to long wavelengths, followed by modulational instability and collapse of the long-wavelength "condensate." Criteria are developed for when such wave-wave saturation of a (warm) beam instability dominates the more familiar wave-particle saturation in which a quasilinear plateau is formed on the beam distribution function.

Finally, in Sec. VIII, one particular strong-turbulence statistical theory is described, which is based on self-similar solutions for spatially collapsing wave packets (Galeev *et al.*, 1975,1976,1977a,1977b,1977c; Pelletier, 1983). The theory includes quasilinear dissipation of collapsed wave-packet energy into self-consistently diffused electrons. A set of wave kinetic equations follows. They possess a steady-state power-law solution for the Langmuir wave spectral function and the electron distribution function.

In the conclusion (Sec. IX) we summarize the status of strong Langmuir turbulence, both theoretically and experimentally, and attempt to define the "cutting edge" of current research.

For those readers familiar with plasma physics, it may be useful at the outset to mention that we treat only "fluidlike" theories of strong plasma turbulence, in which the particle trajectories are essentially linear (or, at most, quasilinear). That is, we do not consider particle trapping or particle phase-space "structures" which may follow from a full Vlasov treatment. [Such processes have been considered, for example, in connection with low-frequency turbulence by Dupree (1972), Boutros-Ghali and Dupree (1981), and Berman *et al.*, (1983).] It is also important to note that the nonlinear self-focusing structures we deal with in more than one dimension are intrinsically dynamical and should not be confused with stationary nonlinear "equilibria" such as Bernstein-Greene-Kruskal (BGK) modes (Bernstein *et al.*, 1957).

II. WAVES AND PARTICLES IN PLASMAS

A. Linear theory

In a fully ionized plasma, the dynamical variables are the particle distribution functions and the electric and magnetic fields which are self-consistent with the charge and current densities arising from those distribution functions. When the plasma is sufficiently dilute, close collisions may be neglected, and the particles are only influenced by the long-range self-consistent fields. In a uniform, infinite plasma with no externally imposed fields, the resulting nonlinear equations may be linearized, and there are linear solutions for electrostatic plane waves of two kinds (see the discussion in the basic reference, Krall and Trivelpiece, 1973).

(i) Electron plasma waves, or Langmuir waves, in which electrons oscillate out of phase with neutralizing ions to produce compressional waves near the plasma frequency ω_p:

$$\omega \approx \omega_p[1+(3/2)k^2\lambda_D^2] ,\quad \omega_p=(4\pi ne^2/m)^{1/2} . \tag{2.1}$$

Here n, e, and m are the electron density, charge, and mass, and λ_D is the Debye wave number, equal to v_e/ω_p, where v_e is the electron thermal velocity in one degree of freedom (k_BT/m_e).

(ii) Ion-acoustic waves, in which electrons oscillate almost in phase with ions to produce waves with small charge separation near the frequency

$$\omega \approx c_s k , \tag{2.2}$$

where c_s is the sound speed, $c_s=(T_e/M)^{1/2}$, and M is the ion mass.

These dispersion relations also come out of a reduced "moment" description, in which "fluid" velocity and density variables replace distribution functions as the dynamical variables. (The nonlinear wave-wave interactions we shall be describing are also fluidlike, in the following sense: The trajectories of those particles mediating the wave-wave coupling are taken as linear and nonresonant. A fluid description is therefore appropriate for the nonlinear wave coupling coefficients, as well as for the linear mode frequencies.)

In order to treat the dissipation of Langmuir waves as $k\lambda_D \rightarrow 1$ (and of ion-acoustic waves) a kinetic description is required in terms of the original particle distribution functions. In the resulting theory of linear wave damping it is well understood how a prespecified distribution function, such as a Maxwellian, produces wave dissipation. If the reduced (one-dimensional) electron distribution function in the direction of propagation $\mathbf{k}$ is $F_e(v)$, then the damping rate for a wave of wave vector $\mathbf{k}$ is (Krall and Trivelpiece, 1973)

$$\gamma_k \propto -\partial_v F_e(v)\,|_{v=\omega/k} \ . \tag{2.3a}$$

For a Maxwellian electron velocity distribution function, this linear damping of waves by particle is called "thermal Landau damping" and has the following form:

$$\gamma_k=(\pi/8e^3)^{1/2}(k\lambda_D)^{-3}\exp[-(k\lambda_D)^{-2}/2] \ . \tag{2.3b}$$

For distribution functions with nonthermal features, in which the local slope of F_e can be positive (such as electron beams), negative damping or instability is produced, and wave growth at the rate given by Eq. (2.3a) can be used to "drive" the turbulence. Such beam instabilities are sometimes called "bump-on-tail" instabilities.

For nonthermal distributions F_e, which have negative slopes in certain velocity ranges, there arises a "nonthermal Landau damping" of magnitude given by (2.3a). This is often stronger than (thermal) Landau damping. We shall append such linear damping/growth terms to our nonlinear fluid equations as needed, recognizing that they come from a linear kinetic description.

Ion-acoustic waves are damped by an analogous mechanism. At comparable electron and ion temperatures, there is heavy Landau damping to ions. However, when $T_e \gg T_i$ there is relatively weak Landau damping (by electrons), and the ion-acoustic modes have a long lifetime (Krall and Trivelpiece, 1973).

B. "Quasilinear" theory and "weak" turbulence

There is a well-known nonlinear wave-particle interaction capable of saturating the "bump-on-tail" beam instability. In the so-called "quasilinear" theory (Krall and Trivelpiece, 1973), the unstable Langmuir waves are permitted to react back on the beam to a limited extent and remove its free energy. The mechanism by which this is accomplished is velocity-space electron diffusion from a statistical ensemble of growing Langmuir waves.

Let W_k be a measure of the energy per $\mathbf{k}$-space volume d^3k. The "damping" rate given by Eq. (2.3a) causes the waves resonant with the beam to grow and others to damp. Neglecting spontaneous emission, the kinetic equation for W_k is

$$\partial_t W_k + 2\gamma_k W_k = 0 \ .$$

According to Eq. (2.3a), γ_k depends on $F_e(v)$. In the quasilinear theory, $F_e(v)$ is determined by

$$\partial_t F_e(v) = \partial_v D \partial_v F_e(v) \ ,$$

where $D(v)$ is a diffusion coefficient that is a linear functional of W_k.

Bump-on-tail instabilities are generally thought to saturate when such velocity-space diffusion removes or flattens the positive velocity gradient in $F_e(v)$ which had been the source of the negative sign of γ_k [Eq. (2.3a)]. This is called "plateau" formation (Krall and Trivelpiece, 1973).

We shall argue that bump-on-tail instabilities need not saturate by quasilinear velocity-space "plateau" formation. This is apparently also understood from certain laboratory experiments (Malmberg, 1984). In the proper parameter regions it will be shown that saturation can occur via wave-wave interactions which include the "strong" turbulence effects of Langmuir wave modulational instability and spatial collapse to short scales.

Quasilinear wave-particle interactions are a part of what is called "weak" turbulence in plasma physics, since the spectral function W_k is assumed to be composed of randomly phased plane waves (Kadomtsev, 1965). Another part is the scattering of Langmuir waves off ion-acoustic waves (for the case $T_e \gg T_i$) (Oraeevski and Sagdeev, 1962). Such scattering can take Langmuir wave energy out of resonance with a source of free energy such as an electron beam and lead to a wave cascade (Kadomtsev, 1965; Nicholson and Goldman, 1978). This process is sometimes represented by an additional "stimulated scattering" term in the kinetic equation for W_k. The stimulated scattering term is quadratic in W_k.

C. Smallness parameters for strong and weak turbulence

Our theoretical discussion will be limited to mean Langmuir wave energy densities $\langle |E|^2\rangle 4\pi$, which are small compared to the background particle energy density nk_BT:

$$W=\langle |E|^2\rangle/4\pi n k_B T \ll 1 \ . \tag{2.4}$$

The dimensionless energy density is thus measured by W.

It is important to recognize that the term "strong" does not here refer to wave energy densities $W \gtrsim 1$, although such regimes are certainly of interest in plasma physics. As is often the case in nonlinear wave theories, certain small parameters permit the reduction of impossibly nonlinear equations to more tractable, less highly nonlinear models. One such small parameter in the theories we shall describe is the ratio of the characteristic rate of nonlinear processes T^{-1}, to the plasma frequency ω_p:

$$\omega_p T \gg 1 \ . \tag{2.5}$$

The nonlinear phenomena we consider are time averaged over a plasma period $2\pi/\omega_p$.

III. NONLINEAR SCHRÖDINGER EQUATION

A. Derivation

One may understand many of the basic physical phenomena which underly strong Langmuir turbulence in terms of nonlinear index-of-refraction effects for a scalar field E in a dispersive, nonlinear medium. We assume the Fourier transform of E obeys a wave equation of the following form:

$$(\eta^2\omega^2 - v^2k^2)E_{\omega,k} = 0 \ . \tag{3.1}$$

Here, v is a characteristic linear velocity of the wave, such as the speed of light, and k is a mode wave number. The combination v^2k^2 represents the physical effect of linear dispersion. The quantity η is the refractive index of the medium, which contains both a linear part, $\eta^2_{\text{lin}}(\omega)$, and an additive nonlinear part, η^2_{NL}, which we here assume depends on the square of E [such a term usually arises from a perturbative expansion of the nonlinear current in the medium, in which case one makes use of a smallness parameter, such as that of Eq. (2.4)]:

$$\eta^2 = \eta^2_{\text{lin}} + \eta^2_{\text{NL}}(E^2) \ . \tag{3.2}$$

Suppose the linearized wave equation (with $\eta_{\text{NL}}=0$) possesses a solution for a nonzero frequency of a mode at zero wave number:

$$\omega = \omega_0(k=0) \ .$$

The Langmuir wave dispersion relation in Eq. (2.1) is of this type. (Other examples are radiation in a plasma or optical phonons in a solid.)

We now seek a solution to the *nonlinear* wave equation in which all frequencies are "close" to ω_0 (Hasegawa, 1975). That is, processes such as harmonic generation, which occurs at $2\omega_0$, will not be considered. This means that we expand all frequencies about ω_0 and keep only the lowest-order terms:

$$\omega^2\eta^2_{\text{lin}} \approx a(\omega-\omega_0) \ ,$$

$$\omega^2\eta^2_{\text{NL}} \approx \omega_0^2\eta^2_{\text{NL}} \ .$$

Note that η_{NL} will contain only frequencies that are low compared to ω_0. This is equivalent, in real time, to an average over the period ω_0^{-1} of terms such as $E(t)^2$ in the nonlinear index of refraction. The wave equation now has the following form:

$$[a(\omega-\omega_0) - v^2k^2 + \omega_0^2\eta_{\text{NL}}]E_{\omega,k} = 0 \ .$$

We introduce the slowly varying envelope $\mathscr{E}(r,t)$ of the carrier wave at ω_0:

$$E_k(t) = \mathscr{E}_k(t)e^{-i\omega_0 t} \ .$$

Transforming back into real time, we obtain a nonlinear Schrödinger equation, which, in dimensionless units, may be expressed as follows:

$$[i\partial_t + \nabla^2 + \eta^2_{\text{NL}}(|\mathscr{E}|^2)]\mathscr{E} = 0 \ . \tag{3.3}$$

Note that the nonlinear index of refraction depends only on the modulus of the envelope, as a result of the slow-time-variation requirement. In Eq. (3.3), the dimensionless units of time are a/ω_0^2, and of distance are v/ω_0.

The simplest and most instructive limit to examine is the one in which the nonlinear index of refraction is expanded to its lowest nontrivial order in $|\mathscr{E}|^2$:

$$\eta_{\text{NL}}(|\mathscr{E}|^2) = |\mathscr{E}|^2 \ . \tag{3.4}$$

We now obtain a cubically nonlinear Schrödinger equation:

$$(i\partial_t + \nabla^2 + |\mathscr{E}|^2)\mathscr{E} = 0 \ . \tag{3.5}$$

Here, the $|\mathscr{E}|^2$ term corresponds to nonlinear refraction, and the ∇^2 term to dispersion.

B. Stability

Let us first consider the stability in one dimension of a spatially uniform, time-independent envelope $\mathscr{E}_0$, which extends from $-\infty$ to $+\infty$. In this case it is desirable (and justifiable, in many physical models) to subtract from $|\mathscr{E}|^2$ in Eqs. (3.4) and (3.5) a constant equal to $|\mathscr{E}_0|^2$. This guarantees that $\mathscr{E}_0$ is an exact solution of the nonlinear equation (3.5) [and also that $\eta_{\text{NL}}(|\mathscr{E}_0|^2)=0$, in Eq. (3.4)].

A linear one-dimensional stability analysis of (3.5) may then be performed by assuming $\mathscr{E} = \mathscr{E}_0 + \mathscr{E}_1 e^{\gamma t}\cos kx$ (Nishikawa, 1968). This analysis reveals that the zero-order envelope $\mathscr{E}_0$ (which we shall refer to as the "pump" wave) is linearly unstable to "breakup" or "modulational instability" into a fastest-growing standing-wave envelope, with wave number $k \approx |\mathscr{E}_0|$ and growth rate $\gamma \approx |\mathscr{E}_0|^2$.

This result is, in fact, predictable from a dimensional analysis of Eq. (3.5): From a balance of the dispersion term (which contains a self-consistently determined scale length L^{-2}) against the nonlinear refraction term, we find that the natural breakup scale length is

$$L \approx |\mathscr{E}_0|^{-1} \ . \tag{3.6}$$

In the same manner, from a balance of the time derivative term (expressed as T^{-1}) against the nonlinear refraction term, we find that the growth rate is

$$T^{-1} \approx |\mathscr{E}_0|^2 \ . \tag{3.7}$$

The perturbation at wave number k grows because the peaks of its interference pattern with the pump correspond to high nonlinear index of refraction. Regions of high index of refraction are regions of total internal reflection for rays. The self-focusing of these rays increases the index of refraction still further, and so the process is unstable.

The long-time nonlinear one-dimensional (1D) evolution of an initially uniform envelope $\mathscr{E}_0$ has been studied numerically (Morales, Lee, and White, 1974). The growing sinusoidal standing-wave perturbation of wave number k steepens with time into a shape resembling an elliptic function, but this structure then gives way back to a state resembling the original uniform envelope. The system continues to show recurrent behavior (Thyagaraja, 1979,1981,1982; Yuen and Ferguson, 1978) and oscillates in time between the "brokenup" and uniform states. More general 1D solutions to Eq. (3.5) have been found recently (Tracy and Chen, 1983).

This result, that a *uniform* envelope breaks up into an envelope containing smaller "natural" lengths L, generalizes to broad initial envelopes which are only uniform over some large scale size L_0, but which ultimately go to zero as $x \to \pm\infty$. The condition for breakup of such finite-energy wave packets is simply that $L_0 \gg L$, or

$$|\mathscr{E}_0|^2 \gg L_0^{-2} . \tag{3.8}$$

The characteristic rate and wave number $k \approx L^{-1}$ for this breakup or "modulational instability" are then still given approximately by Eqs. (3.6) and (3.7).

C. One-dimensional solitons

Equation (3.4) is valid for finite-energy wave packets without any additive constants, and we obtain the classic form of the 1D scalar cubic nonlinear Schrödinger equation,

$$(i\partial_t + \partial_x^2 + |\mathscr{E}|^2)\mathscr{E} = 0 . \tag{3.9}$$

This equation possesses a very interesting particular solution, which we may find by looking for an envelope of form

$$\mathscr{E} = A(x)\exp(itL^{-2}/2),\quad L > 0 ,$$

where $A(x)$ is independent of t and goes uniformly to zero as $x \to \pm\infty$. With this substitution, the equation for A is second order in x, and formally identical to the equation for a particle in a nonlinear potential well, except that x plays the role of t. The solution that satisfies the boundary condition is the hyperbolic secant "soliton" solution,

$$A(x) = L^{-1}\mathrm{sech}(x/L) . \tag{3.10}$$

This solution is a standing-wave envelope, in the form of a localized disturbance of size L and maximum amplitude L^{-1}, which precisely retains its spatial shape as time evolves. [Linear wave packets, of course, do not have this property; they disperse in space as time goes on (Bohm, 1951).] The soliton envelope solution is very easily understood in terms of the nonlinear Schrödinger equation (3.9). The shape is preserved because the dispersion and nonlinear refraction exactly balance each other, a result which turns out to be unique to one-dimensional solutions. Hence the 1D soliton state is one in which

$$|\mathscr{E}|^2 \approx L^{-2} . \tag{3.11}$$

Clearly, if we take a soliton state as an initial condition, it will persist, but are there *different* initial states which will relax in time into a soliton state? The answer to this question requires study of a more general class of solutions to Eq. (3.9). As it turns out, Eq. (3.9) possesses an exact solution, found in 1971 by Zakharov and Shabat (1971), using the inverse scattering method, for the time evolution of an arbitrary initial envelope.

Inverse scattering solutions show that a broad initial envelope of length $L_0 \gg L \approx |\mathscr{E}_0|^{-1}$ can break up into an N-soliton state, containing roughly $N = L_0/L$ solitons, usually accompanied by a small amount of energy in "radiated" plane waves. This is the basis for the condition (3.8) for breakup. It is in this sense that soliton solutions similar to the type in (3.10) are "natural" nonlinear structures, or "attractors" for the solutions to the 1D cubic nonlinear Schrödinger equation. (The terminology "attractor" is more commonly used for dissipative systems which exhibit phase-space contraction.)

However, at least two qualifications are in order: First, the class of soliton solutions is more general than Eq. (3.10) and may involve propagating envelopes, rather than "standing-wave" structures. Thus solitons may be moving through each other.

Second, and more important, there are other kinds of solutions (Satsuma and Yajima, 1974). A "pulsating" soliton, or "breather" is obtained when the inequality (3.8) is only marginally filled (Goldman, Hafizi, and Rypdal, 1980; Satsuma and Yajima, 1974). For certain more energetic initial wave packets (real and not antisymmetric), it is common to find evolution into a "bound state" of several solitons, which manifests itself as a dynamically recurring multipeaked state. These evolutions are usually accompanied by propagating plane-wave envelopes, which are cast off as "radiation."

D. Higher-dimensional self-similar collapse

Our main concern, however, is with the behavior of solutions to the cubic nonlinear Schrödinger equation in higher dimensions. We retain the approximation of a scalar field. In this case, Eq. (3.9) reverts to (3.5):

$$(i\partial_t + \nabla^2 + |\mathscr{E}|^2)\mathscr{E} = 0 .$$

The solutions to this equation in two or three dimensions are significantly different from the solutions in one dimension. No exact solution such as the inverse scattering solution in one dimension is known to exist. Solitons like that in Eq. (3.10) are not stable initial conditions for 2D or 3D numerical solutions (Pereira *et al.*, 1977; Schmidt, 1975). They are no longer "natural" structures in these higher dimensions.

However, breakup still occurs when condition (3.8) is satisfied for an initial localized envelope of characteristic dimension L_0 (more precisely, $L_0^{-2} \approx L_x^{-2} + L_y^{-2}$, in two dimensions). The breakup or modulational instability size and growth rate are still given by Eqs. (3.6) and (3.7), in many cases involving localized envelopes which are not propagating (Vedenov and Rudakov, 1964). However, the situation is more complicated for an initial wave with finite wave vector (Bardwell and Goldman, 1976).

What happens in the nonlinear evolution of the "brokenup" initial envelope, now that we have a set of envelopes localized to the smaller size $L \approx \mathscr{E}^{-1}$, where $\mathscr{E}$ is the maximum amplitude of each newly localized field? The answer, provided by a number of different approaches, is that such a localized envelope in two or more spatial dimensions collapses or self-focuses to a point singularity in a finite time whenever its mean intensity exceeds its mean inverse square width. The condition for collapse is therefore

$$|\mathscr{E}|^2 \gtrsim L^{-2} , \tag{3.12}$$

which represents the dominance of ray self-focusing (into the region of high index of refraction $\eta_{\rm NL} \approx |\mathscr{E}|^2$) over the linear tendency of the rays to disperse or diffract.

One way to derive this result is through a "virial theorem" (Vlasov *et al.*, 1971; Zakharov, 1972; Glassey, 1977; Goldman and Nicholson, 1978) for the time evolution of the wave-packet-averaged width of an initial wave packet. This will be discussed briefly later.

A more useful insight for our purposes is provided by the existence of a *particular* class of solutions to the 3D nonlinear Schrödinger equation which, according to many numerical studies, act as "attractors" for initial localized wave packets that satisfy Eq. (3.12). These are "self-similar" solutions, which may be of the following form (Zakharov, 1972):

$$\mathscr{E} \approx \xi^{-1} f(r/\xi), \quad \xi = \xi(t) = (t_c - t)^{1/2} . \tag{3.13}$$

[It should be noted that *anisotropic* self-similar forms have also been proposed (Zakharov and Rubenchik, 1973; Zakharov, Mastryukov, and Synakh, 1974; Rudakov and Tsytovich, 1978; Sulem *et al.*, 1981).]

A solution of this *form* can easily be seen to satisfy Eq. (3.5). When it is inserted, the localized function $f(\mathbf{x})$ is seen to satisfy a time-independent equation related to (3.5). The function ξ goes from $\xi(0)=L$ to zero in the finite time t_c. [t_c is of the same order as T in (3.7).]

The form of solution in (3.13) is called "self-similar" because the initial spatial shape $f[\mathbf{r}/\xi(0)]$ is preserved at later times, although contracted, since $\xi(0)$ is replaced by the smaller scale sizes $\xi(t)$, which eventually go to zero. The solution $\mathscr{E}$ is therefore similar to itself at later times, except it becomes narrower, and also larger, because of the factor ξ^{-1} in front of f.

Numerical solutions of the evolution of a radially Gaussian envelope wave packet with assumed spherical symmetry show that it tends towards a self-similar shape as it collapses (Budneva *et al.*, 1975; Goldman *et al.*, 1980; Hafizi and Goldman, 1981). The self-similar region is in a small volume about the origin, which intensifies and narrows, eventually reaching a mathematical singularity at the finite time t_c.

Recent studies of the nonlinear Schrödinger equation in two dimensions also show self-similar collapse (Sulem *et al.*, 1983).

There are usually physical effects, missing from Eq. (3.12), which prevent the singular limit from being reached, although very high energy densities $|\mathscr{E}|^2$ may be achieved. It is also important to recognize that these high-energy densities are over a very small volume, so that the spatially integrated energy over the half-width forms only a small fraction of the total wave-packet energy (Goldman *et al.*, 1980).

E. Integral invariants and virial theorem

There are a small number of known integral invariants of the cubic Schrödinger equation (3.5) in higher dimensions (Vlasov *et al.*, 1971; Goldman *et al.*, 1980). We express the "number," momentum, and energy invariants, respectively, as follows:

$$N = \int d^D r \, |\mathscr{E}|^2 ,$$

$$P = \int d^D r \left[\frac{1}{2i} \right] (\mathscr{E}^* \nabla \mathscr{E} - \mathscr{E} \nabla \mathscr{E}^*) ,$$

$$\mathscr{H} = \int d^D r (\tfrac{1}{2}) (|\nabla \mathscr{E}|^2 - |\mathscr{E}|^4) .$$

These invariants are based on the symmetry of the cubic Schrödinger equation under gauge, space, and time transformations, and may be derived from a Lagrangian density for Eq. (3.5). [In addition, there is an angular momentum invariant which expresses rotational symmetry (Gibbons *et al.*, 1977).] The N invariant is actually an action integral and has the interpretation of total "boson" (sometimes called "plasmon") number for the Langmuir waves. The P invariant assumes the given form in the present case because current and momentum densities happen to be equivalent for Eq. (3.5).

The $\mathscr{H}$ invariant is actually a Hamiltonian for the cubic Schrödinger equation. The term $|\nabla \mathscr{E}|^2$ represents dispersion, and the term $|\mathscr{E}|^4$ represents the nonlinear refraction. It is important to note that $\mathscr{H}$ need not be positive definite, and is in fact negative when the nonlinear refraction term dominates the dispersion term. This is precisely the condition for collapse to occur, as we can demonstrate by means of a "virial theorem."

Consider a single localized wave packet which is not translating ($P=0$). Let us form measures of its spatial properties by taking spatial averages of r-dependent quantities, using the "probability" distribution $\rho = |\mathscr{E}|^2/N$ (i.e., as though we were doing quantum mechanics). Then it is quite easy to show that the mean spatial width of a wave packet, defined by $\langle (\Delta r)^2 \rangle = \langle (r - \langle r \rangle)^2 \rangle$, obeys a virial theorem (Goldman and Nicholson, 1978):

$$\partial_t^2 \langle (\Delta r)^2 \rangle = \mathscr{H}/N + (2-D) \langle |\mathscr{E}|^2 \rangle .$$

Hence if the dimension $D \geq 2$, and if $\mathscr{H} < 0$, then $\langle (\Delta r)^2 \rangle \to 0$ in a finite (collapse) time. This proves that collapse occurs in two or more dimensions when nonlinear refraction dominates dispersion. The condition for $\mathscr{H}$ to be less than zero is identical to the condition (3.12).

F. Graphic summary

The results of this section are summarized graphically in Fig. 1. Figure 1(a) illustrates modulational instability of a broad envelope of initial size L_0, which breaks up into smaller sizes $L \approx |\mathscr{E}|^{-1}$, provided $L_0 \gg L$ (i.e., for a sufficiently intense field $\mathscr{E}$). This phenomenon is illustrated in one dimension, but, in fact, occurs in higher dimensions under analogous conditions. Figure 1(b) shows a one-dimensional hyperbolic secant soliton solution to (3.9), and Fig. 1(c) illustrates its analog in higher dimensions, a self-similarly self-focusing wave packet, whose field strength $|\mathscr{E}|$ and initial inverse scale size L_0 are comparable.

IV. SELF-FOCUSING AND MODULATIONAL INSTABILITY IN OPTICS AND FLUIDS

We now describe a variety of interesting physical realizations of the phenomena of self-focusing, modulational instability, and soliton formation treated in Sec. III.

Figure 2 shows experimental observations of self-focusing and modulational instability in nonlinear optics. Figure 2(a) is a sketch of the configuration for observing spatial self-focusing of laser beams in nonlinear media,

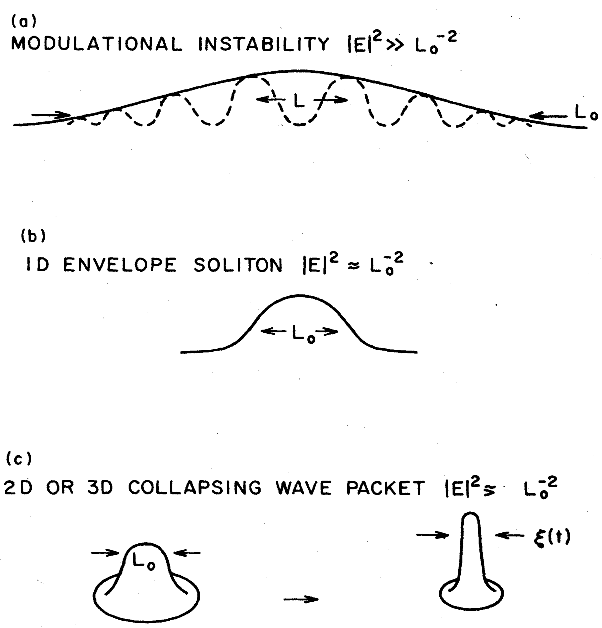

FIG. 1. Basic nonlinear wave envelope phenomena described by Eq. (3.5). (a) Modulational instability. Broad envelope of initial size L_0 breaks up into sizes $L \approx |\mathscr{E}|^{-1}$, provided $|\mathscr{E}|^2 \gg L_0^{-2}$. (b) One-dimensional soliton solution [Eq. (3.10)] when $|\mathscr{E}|^2 \approx L_0^{-2}$. (c) Two-dimensional or three-dimensional spatial collapse for $|\mathscr{E}|^2 \gtrsim L_0^{-2}$. A self-similar solution to Eq. (3.5) is illustrated, with scale size $\xi(t)$ given in Eq. (3.13).

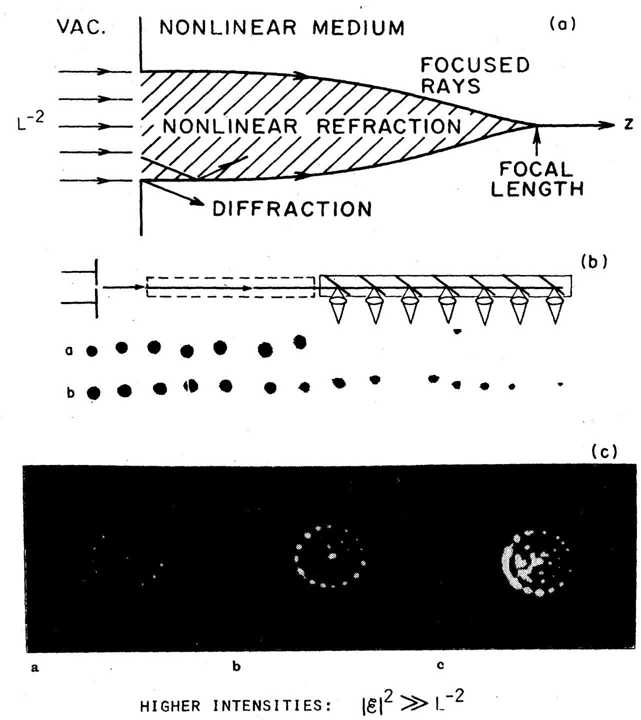

FIG. 2. Nonlinear optics realizations of self-focusing and modulational instability. (a) Sketch of configuration for spatial self-focusing of a laser beam discussed in text, when $|\mathscr{E}|^2 \gtrsim L^{-2}$ (L is the aperture width governing diffraction). (b) Apparatus and sequence of observed cross sections of laser beam. *a*, linear medium gives diffraction. *b*, nonlinear medium shows self-focusing (Garmire *et al.*, 1966). (c) Modulational instability of Fresnel diffraction pattern at higher intensities $|\mathscr{E}|^2 \gg L^{-2}$. Azimuthal breakup scale size decreases with intensity in accordance with Eq. (3.6) (Campillo *et al.*, 1973).

such as CS_2. The incident radiation is introduced through an aperture of radius L. When the incident intensity is sufficiently high, the nonlinear refraction $|\mathscr{E}|^2$ overcomes the linear diffraction L^{-2}, and self-focusing occurs. Here the (steady-state) self-focusing is along the z axis, and it is z which plays the role of time t in Eq. (3.5). The laser-beam self-focusing is in the radial direction, and ∇^2 is replaced by the radial part of the Laplacian.

Beneath the sketch is Fig. 2(b), which shows the actual experimental results from one of the earliest observations of self-focusing (Garmire, Chiao, and Townes, 1966). A sequence of cross sections of the laser beam inside the nonlinear medium is shown. In sequence *a*, the nonlinear medium is removed, and the successive beam cross sections show a slight radial increase due to diffraction. In sequence *b*, the nonlinear medium is present, and the successive beam cross sections show an unmistakable constriction, due to self-focusing.

For much higher incident intensities, the condition (3.8) is fulfilled, and modulational instability occurs. Figure 2(c) shows experimental observation of modulational instability when $|\mathscr{E}|^2 \gg L^{-2}$ (Campillo, Shapiro, and Suydam, 1973). Pictures *a*, *b*, and *c* represent cross sections at a fixed plane in the nonlinear medium of successively more intense incident radiation. The breakup is observed in the laser-beam Fresnel diffraction pattern. The dominant and striking feature of these photographs is the regularity of the spacing of the focal spots about the rings. This spacing can be predicted from the scaling arguments [Eq. (3.6)] of modulational instability applied to an azimuthal angular variable (Suydam, 1975). Note how the azimuthal breakup scale size decreases as the intensity of the laser beam is increased, in accordance with Eq. (3.6). The spots represent the spatially self-focused end result of spatial modulational instability which began closer to the entrance aperture.

Another application of these ideas comes from the theory of nonlinear deep-water waves. Engineers who test models of ships in long tanks generate long trains of water waves for this purpose. For decades they had been troubled by the generation of steep waves, which tend to break up into a series of wave groups, and discussed the malformation of those waves as being due to imperfections in their apparatus.

In 1967, Benjamin and Feir showed theoretically and experimentally that such deep-water waves can be described by a one-dimensional nonlinear Schrödinger equation, and are unstable to modulational instability (Feir, 1967). (Deep-water waves are not to be confused with nonlinear shallow-water waves, which are described by a Korteweg—de Vries equation.)

In the experiment shown in Fig. 3 (Yuen and Lake, 1975), waves are launched at one end of a water tank with a paddle, and absorbed by a "beach" at the other end. The paddle can be programmed to launch a particular waveform of interest. For both of the cases depicted, the

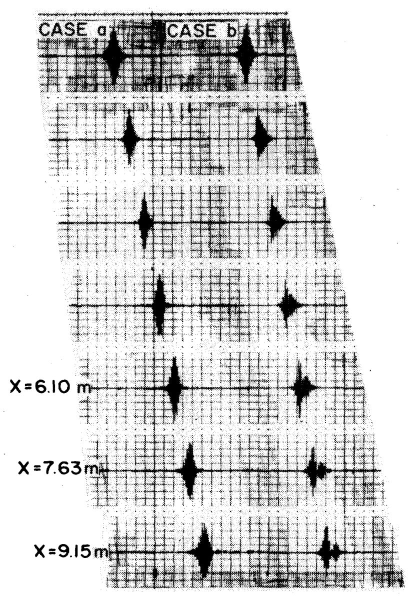

FIG. 3. Deep-water wave realizations of soliton and modulational instability. Spatial propagation of an initial wave packet is traced in the vertical sequence of profiles, starting at the top. Case a, initial pulse with soliton profile ($|\mathscr{E}|^2 \approx L^{-2}$) Eq. (3.10) retains shape. Case b, when $|\mathscr{E}|^2 = 2L^{-2}$, packet breaks up due to modulational instability (Yuen and Lake, 1975).

evolution of the wave can be traced in the vertical sequence of profiles, reading from the top down.

In case a, the initial pulse has a soliton profile, as in Eq. (3.10). (Note, this implies $|\mathscr{E}|^2 \approx L^{-2}$.) The subsequent propagation shows no significant change in shape. In case b, however, $|\mathscr{E}|^2 = 2L^{-2}$. According to Eq. (3.8), this is the condition for modulational instability (appropriately termed the Benjamin-Feir instability by hydrodynamicists). The evolution for case b indeed shows the breakup of the wave packet. (Note, the scale of intensity for the more intense wave packets is condensed in the figure.) The observed behavior is consistent with Zakharov and Shabat's inverse scattering theory.

Another example of modulational instability and self-focusing comes from astrophysics. The so-called Jeans instability is a modulational instability in which stars are formed out of gravitationally collapsing interstellar gas clouds (Spitzer, 1978). While this process is not described by precisely a nonlinear Schrödinger equation, there is a close relation, and common mathematical methods (such as generation of equivalent hydrodynamic equations in which $|\mathscr{E}|^2$ acts as a density and the phase of $\mathscr{E}$ is related to a flow velocity) lead to similar conditions on collapse or breakup.

V. DYNAMICS OF STRONG TURBULENCE IN PLASMAS

A. Modulational instability and Langmuir collapse: a brief history

We begin with a brief historical survey of the theoretical development of some of these ideas within the framework of plasma physics.

A key observation was provided by Vedenov and Rudakov (1964), who demonstrated that modulational instability can occur in ensemble or phase-averaged spectra of Langmuir turbulence. They considered a kinetic equation for the spectral function of the Langmuir wave field and found the condition for modulational instability to be

$$W \approx \langle |\mathscr{E}|^2 \rangle / 4\pi n k_B T \gg \Delta k^2 \lambda_D^2 , \tag{5.1}$$

where Δk is the effective half-width of the spectral function

$$W_k = \lim_{V \to \infty} \langle |\mathscr{E}_k|^2 \rangle / 4\pi n k_B T V \tag{5.2}$$

and W [also defined in Eq. (2.4)] is the integral over W_k,

$$W = \int (dk/2\pi) W_k . \tag{5.3}$$

These definitions are illustrated in Fig. 4.

The instability condition (5.1) is consistent with the condition (3.8), that we found from the dynamical (cubically nonlinear) Schrödinger equation for an unstable envelope field. Here the role of wave-packet scale size L_0 is played by the inverse spectral width Δk^{-1}.

Several years after the publication of this key result, plasma physicists began to consider modulational instabil-

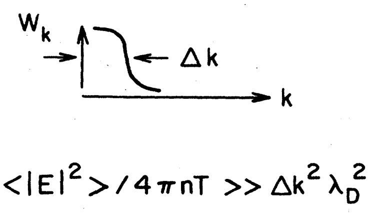

FIG. 4. Modulationally unstable Langmuir spectrum W_k, with total energy $W \gg \Delta k^2 \lambda_D^2$. The width of the spectrum is Δk. T is the temperature in energy units.

ity of coherent magnetohydrodynamics (MHD) plane waves. Taniuti and Washimi (1968) and also Karpman and Krushkal (1968) showed that the dynamics is governed by a cubic Schrödinger equation, and recognized the connection with optical self-focusing. Their treatment was one dimensional, however, and they were concerned with soliton solutions rather than with self-focusing behavior.

Nishikawa (1968) showed that modulational instability plays a prominent role in the parametric instability of a uniform Langmuir wave pump and has the same order-of-magnitude growth rate as the decay instability into another Langmuir wave and an ion-acoustic quasimode, in a plasma characterized by equal electron and ion temperatures. The modulational instability is sometimes called an "oscillating two-stream" instability, because the breakup occurs into a standing wave of wave number $k \approx W^{1/2}$, which decomposes into two oppositely propagating Langmuir waves ("streams").

In 1970, Hasegawa generalized the cubic Schrödinger equation model of modulational instability (Taniuti and Washima, 1968) to include a possible additional resonance with low-frequency waves. He showed how to generalize the Schrödinger equation so that the nonlinear "index of refraction" [e.g., $\eta_{\rm NL}$ in Eq. (3.3)] itself obeys a dynamical wave equation "driven" by an inhomogeneous source term proportional to $|\mathscr{E}|^2$. In the linear limit, this latter wave equation describes low-frequency waves. Although Hasegawa was concerned only with enhancement of modulational instability due to possible resonance with low-frequency (MHD or ion-cyclotron) waves, his equations had the same formal structure as the pair of equations that govern Langmuir collapse, and whose properties and major implications were elucidated by Zakharov.

In 1972, Zakharov published his seminal paper containing not only the more general set of dynamical equations which describe the evolution of nonlinear Langmuir waves, but also virial theorems and self-similar solutions which predict spatial self-focusing in more than one dimension. In these equations, which have come to be called the Zakharov equations, the nonlinearity comes from ponderomotive force, which is quadratically nonlinear in the Langmuir envelope field and can drive a low-frequency particle response in the electron density n_2. The density then couples back into the Langmuir wave equation through the nonlinear electric force $n_2 E$. In the adiabatic limit ($W \ll m/M$, where m/M is the electron-to-ion mass ratio), the Zakharov equations reduce to the cubic nonlinear Schrödinger equation. They contain considerably more physics, however, and include supersonic collapse as well as decay instabilities. We shall return to these equations later.

At about the same time, Zakharov and Shabat (1971) produced their inverse-scattering-method solution to the one-dimensional cubic Schrödinger equation in an infinite medium.

The first statistical model associated with the Zakharov equations came with the publication by Kingsep, Rudakov, and Sudan (1973) of a theory of "strong" Langmuir turbulence. They were motivated by numerical simulations of plasmas driven unstable by intense electron beams or powerful lasers. In their treatment, the strongly turbulent state consisted not of a Fourier superposition of plane waves, but of a collection of one-dimensional solitons, localized in space with random positions and an *ad hoc* distribution of amplitudes. They found a spectrum proportional to k^{-2} and noted that near $k \approx \lambda_D^{-1}$ there would occur strong absorption of wave energy by Landau damping, and that this would cause the spectrum to fall off faster in this (dissipative) region of $\mathbf{k}$ space. They also provided estimates of the statistical behavior of a set of "blobs" distributed in three dimensions, but did not attribute any internal dynamical structure such as self-focusing or damping to the blobs. While their theory provided a number of conceptual advances, the details have never been confirmed by experiment or by numerical simulation.

Galeev, Sagdeev, Sigov, Shapiro, and Shevchenko (1975) provided a more complete model of strong Langmuir turbulence, valid in three dimensions and based on the self-similarly collapsing solutions to the Zakharov equations. An "ideal gas" of such self-focusing wave packets was constructed, and an isotropic, stationary spectrum proportional to $k^{-5/2}$ was found at intermediate wave numbers. Here collapse provides the mechanism for energy transfer to high k. The spectrum at wave numbers near the Debye wave number is reduced to $W_k \propto k^{-9/2}$, due to energy transfer from (independent) wave packets to electrons. This wave-particle interaction is treated self-consistently by a modified quasilinear theory, in which the electrons undergo diffusion by repeated scattering off the packets as they damp, and the electron distribution function is thereby also modified. An isotropic velocity distribution going as $v^{-9/2}$ was found in the dissipative regime.

In a subsequent refinement of the Galeev *et al.* (1975) theory of strong Langmuir turbulence by Pelletier (1982), self-similar solutions are extended into the dissipative regime, and the Langmuir wave spectrum and electron distribution function are found to fall off less rapidly:

$$W_k \propto k^{-7/2}, \quad f_e(v) \propto v^{-7/2} .$$

Pelletier's theory will be dealt with later in this paper.

The idea of using nonlinear wave structures instead of plane waves as the building blocks of a statistical theory is stimulating and promising. Such an approach has also been employed in other branches of continuum physics, such as solid-state physics, where the partition function that describes the nonlinear statistical mechanics of structural phase transitions (Krumhansl and Schrieffer, 1976) had been constructed out of "kink" or topological solitons (representing long-range order in alternate equilibrium positions of ions) as well as out of the typical phonon plane-wave contribution. However, strong Langmuir turbulence (like certain models of fluid turbulence involving Burger's equations) is an application to a statistical system which is far from equilibrium.

We mention at this point that theories based on nonlinear "structures" as the building blocks of strong turbulence do not constitute the only possible approach to nonlinear systems. Recently a class of renormalized theories, which can be shown to be consistent perturbative schemes, have been applied to strong Langmuir turbulence. We cite the application of the direct-interaction approximation (DIA) as an example (Dubois, Rose, and Goldman, 1979; Dubois and Rose, 1981; Hanson and Nicholson, 1983).

It is perhaps worthwhile to digress a moment to compare the "independent structures" type of strong-turbulence theory with statistical closure theories such as the DIA. Independent structures models are closest in spirit to the well-known Kolmogorov (1941) scaling theory of incompressible fluid turbulence. These theories for Langmuir turbulence are based on the Zakharov equations (e.g., solitons, self-similar solutions). The assumption that these special local properties can be used globally in an independent soliton model has not been proven directly from the Zakharov equations. These models are quite tractable and have led to a set of detailed predictions. To date, however, no compelling long-time statistical data is available from numerical simulations in two or three dimensions or from experiment to compare with these predictions.

The statistical closure theories are closest in spirit to the approach of Kraichnan (1958,1959) in fluid turbulence, which has been successful in describing moderate Reynold's number turbulence. The closure methods, which are renormalized perturbation theories, are derived in a systematic but nonrigorous manner from the Zakharov equations. These theories suffer from the serious technical problem of being difficult to compute. Recent results of DuBois, Nicholson, and Rose (1984) and Rose (1984) for $d=1$ have shown qualitative agreement between the direct-interaction approximation predictions and mode simulation for the initial evolution of the modulational instability. DuBois, Rose, and Goldman (1979) (see also DuBois and Rose, 1981) introduced the DIA for Langmuir turbulence as a natural extension of weak-turbulence theory in which the modulational instability can be described in a statistical context; this theory provides a systematic description for this instability driven self-consistently by a spectrum of Langmuir modes and so provides a self-consistent generalization of the work of Vedenov and Rudakov (1964) and Bardwell and Goldman (1976). It is not expected that the DIA will successfully describe situations which are highly intermittent in both space and time, e.g., situations dominated by a few short-lived "solitons." In two or three dimensions, if there is sufficient symmetry (e.g., isotropic turbulence), the DIA equations may not be significantly harder to compute than for $d=1$.

B. Ponderomotive force and particle-in-cell simulation

The physics of strong Langmuir turbulence can be understood through the concept of ponderomotive force, which we next elucidate using a very simple, single-particle approach. Consider a single particle of mass m and charge q, moving in a standing Langmuir wave packet of form $E(x)\cos\omega t$, where ω is a frequency close to the plasma frequency, ω_p:

$$m\partial_t^2 x = qE(x)\cos\omega t \ .$$

When $E(x)$ is a slowly varying function (to be characterized more fully in a moment), the solution may be expanded in a "guiding-center" approximation:

$$x \approx x_0(t) - [qE(x_0)/m\omega^2]\cos\omega t \ .$$

The first term is the (drift) coordinate of the instantaneous guiding center of the motion, and the second is recognizable as the oscillating solution to the equation of motion in a *uniform* field equal in value to $E(x)$ at the drift coordinate $x=x_0$. The equation of the drift coordinate x_0 is easily seen to be

$$m\partial_t^2 x_0 = -(d/dx)[q^2E(x_0)/4m\omega^2]^2 \ . \qquad (5.4)$$

Note that the equation of motion is identical to that of a particle in a conservative force. That effective force, on the right side of (5.4), is the "ponderomotive force." It is proportional to E^2 and so to the energy density of the Langmuir field. Electrons will drift down a gradient in this energy density, whereas ions are virtually unaffected because of their high mass m.

The situation is illustrated in Fig. 5. A profile of $|E|$ which is localized in space is assumed in Fig. 5(a). This causes electrons to drift out of the localized region. The resulting separation of charge produces an ambipolar field which then drags the ions out of the localized region.

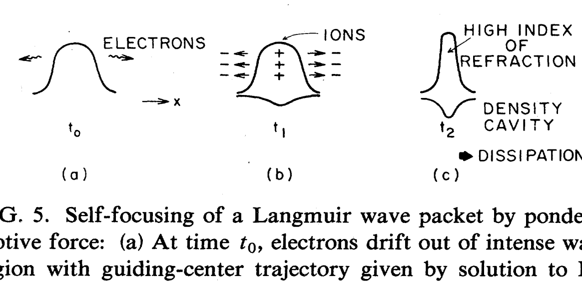

FIG. 5. Self-focusing of a Langmuir wave packet by ponderomotive force: (a) At time t_0, electrons drift out of intense wave region with guiding-center trajectory given by solution to Eq. (5.4). (b) Charge separation field pulls out ions at time t_1. (c) This results in a density cavity at time t_2, which self-focuses the wave packet to short scales, where dissipation to particles can occur.

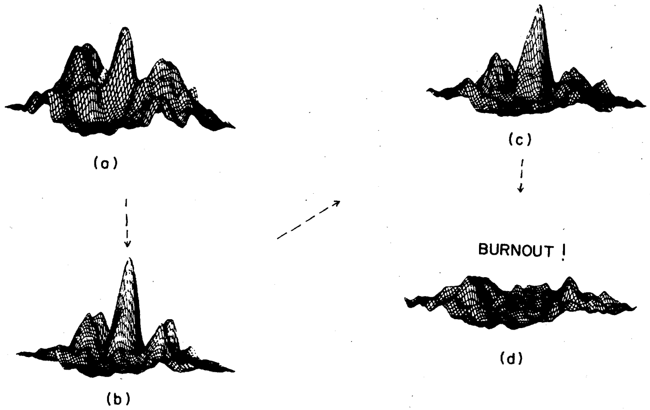

FIG. 6. Particle-in-cell simulation of two-dimensional Langmuir collapse and "burnout": (a) Initial wave packet, (b) and (c) self-focusing, (d) "burnout" due to surrender of short-scale wave energy to electrons (Anisimov *et al.*, 1982).

The result of evacuation of both species of charged particles is the creation of a density cavity, which can trap the Langmuir energy and, in more than one spatial dimension, produces self-focusing.

To understand these nonlinear index-of-refraction effects, we define an effective index of refraction for a Langmuir wave, in which the electron thermal velocity v_e replaces the speed of light c [see Eq. (2.1)]:

$$\eta=\sqrt{3}v_e k/\omega=(1-\omega_p^2/\omega^2)^{1/2} . \tag{5.5}$$

Since the index of refraction increases in spatial locations where the density is depressed (due, in this case to ponderomotive force), the localized field sees a higher index of refraction than the surrounding region. This is the origin of the "trapping" of the Langmuir energy density in such regions (think of a ray undergoing a total internal reflection). In two or more dimensions, this situation can be shown to be unstable: the Langmuir field becomes more localized, pushing out more particles, deepening the density cavity, increasing the index of refraction, and thereby localizing the field to ever smaller spatial dimensions as time passes. The resulting spatial self-focusing proceeds to a singularity in a finite time, according to the nonlinear Schrödinger or Zakharov equation descriptions, when dissipation is neglected.

However, dissipation due to Landau damping begins to cut in at short spatial scales—on the order of the Debye

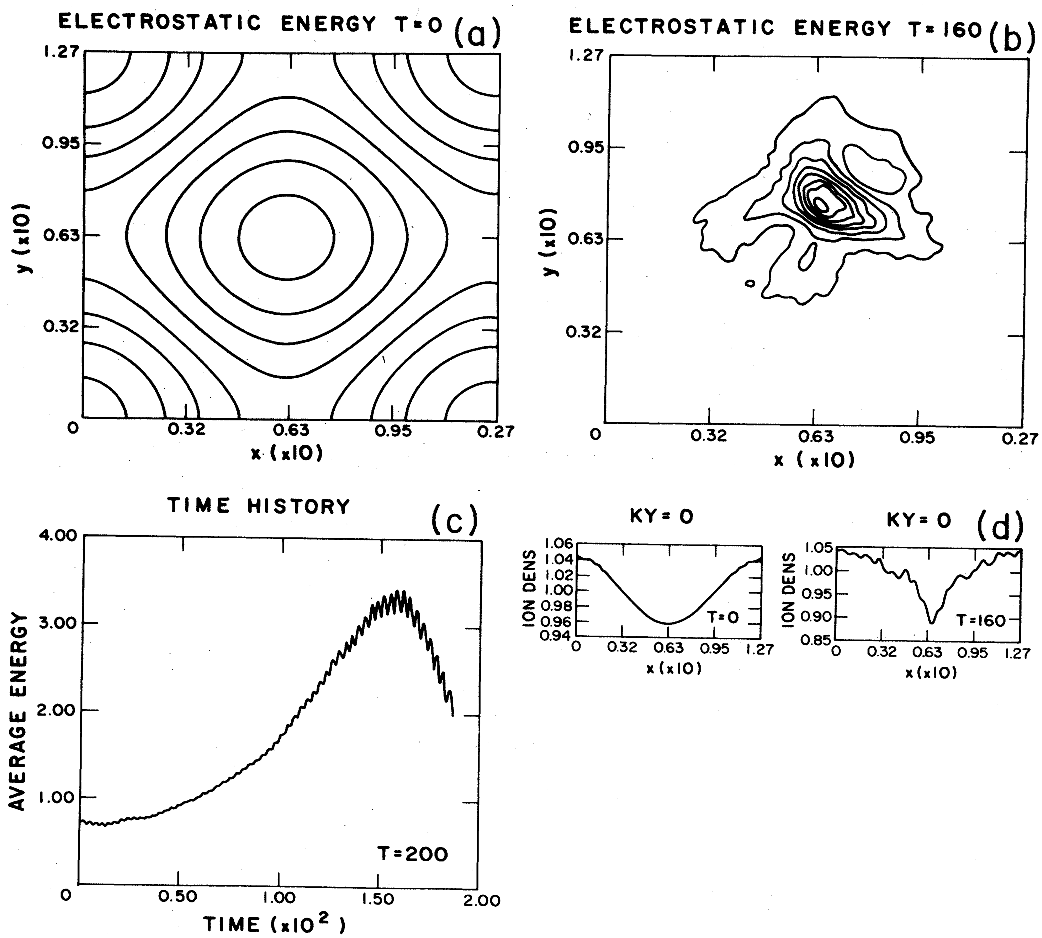

FIG. 7. Particle-in-cell simulation of two-dimensional Langmuir collapse. (a) Wave energy contour map at $t=0$. (b) Self-focused contours, a short time later. (c) Time history of average energy density. (d) x dependence of y-averaged density profile at initial and later times (Forslund, 1983).

length (large k in the Fourier spectrum of the field). This dissipation can cause the collapsing wave packet to "burn out" in a finite time at a finite energy density. (As we shall see later, dissipation may be put into the Schrödinger or Zakharov equations in a primitive linear fashion using straight-line particle trajectories.)

The collapse and burnout can be seen most dramatically in a recent series of particle-in-cell plasma simulations. These simulations are performed in two spatial dimensions for an electron-ion plasma. In Fig. 6, we see the result of a 2D simulation (Anisimov *et al.*, 1982) in which the initial state of the plasma consists of a localized wave packet (whose intensity exceeds the collapse threshold) and an associated density cavity. The dimensionless energy density $W \approx 0.1$, initially. This exceeds $(2\pi/L)^2$, where L is the initial localization size (the size of the cell). By Eq. (3.8), the collapse condition is fulfilled, and Figs. 6(a)—6(c) indeed show the collapse as the plasma and initial field are allowed to develop, with no impressed sources. At shorter scales, the burnout is evident in Fig. 6(d).

The results of Anisimov *et al.* have been confirmed in recent preliminary studies of Forslund (1983), using the Los Alamos particle-in-cell code, WAVE. In Fig. 7(a), we see the contours of electrostatic Langmuir wave energy of the initial state. In Fig. 7(b), a short time later, the contours clearly define a more localized packet. Figure 7(c) exhibits the time history of the electrostatic energy density (averaged over one plasma period). The energy increases during the self-focusing stage and begins to decrease during the burnout phase of the field evolution. Figure 7(d) shows the evolution of the x dependence of the (y-averaged) density profile, from an initial broad form to a more highly localized cavity.

Figure 8 is significant because it illustrates the evolution of the electron velocity distribution function: The initial Maxwellian distribution function is shown in a semi-log plot in Fig. 8(a) (v_x and v_y velocity-space profiles) and the time-evolved distribution is in Fig. 8(b). A tail has clearly developed on the distribution function, due to the transfer of wave energy to electrons during the burnout phase of wave-particle evolution.

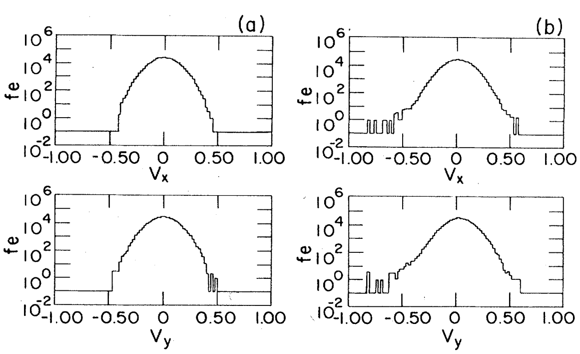

FIG. 8. Evolution of electron velocity distribution functions during particle-in-cell simulation of 2D Langmuir collapse portrayed in Fig. 7. (a) Initial Maxwellian distribution (semi-log plot). (b) Later distribution, showing formation of tail due to burnout (Forslund, 1983).

Such simulations are very encouraging because they permit interactive numerical studies at a fundamental level. Particles are pushed by Coulomb forces, so there is no question of the validity of partial differential equation (PDE) models or of the proper form of dissipation terms at short scales. In the next section we shall see that the PDE models do indeed reproduce much of this behavior.

C. The Zakharov equation: nonlinear Langmuir wave dynamics

We can easily provide a heuristic "derivation" of the Zakharov equations based on Eqs. (2.1)—(2.3) and the idea that density cavities can be produced by ponderomotive force [right side of Eq. (5.4)]. In dimensionless units, the generalization of the Langmuir wave dispersion relation (2.1) to include linear dissipation/growth [like γ_k in Eq. (2.3a)] and density cavity formation by ponderomotive force is

$$\omega = \omega_p(1+k^2+\delta n) - i\gamma_k \ . \tag{5.6}$$

The term k^2 comes from the linear dispersion. The term δn is the nonlinear density depression caused by ponderomotive force (proportional to $|\mathscr{E}|^2$). It arises formally because the plasma frequency ω_p, defined in Eq. (2.1), can be shown to depend on a sum of both the linear background density and the nonlinear density modification δn. The term δn can also be thought of as governing nonlinear refraction, since Eq. (5.5) has demonstrated that a change in the density (and hence in the plasma frequency ω_p) means a change in the index of refraction η.

The nonlinear dispersion relation (5.6) leads to a nonlinear wave equation in real space for the slowly varying Langmuir wave field envelope, $\mathscr{E}$, defined by

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}[\mathscr{E}(r,t)\exp(-i\omega_p t)] \ . \tag{5.7}$$

The resulting nonlinear wave equation for $\mathscr{E}$ is the first of the so-called Zakharov equations:

$$\nabla\cdot(i\partial_t + \nabla^2 - \delta n + i\hat{\gamma})\mathscr{E} = 0 \ . \tag{5.8}$$

Here, ∇^2 comes from the dispersive term in (5.6), δn from the nonlinear refraction, and γ from the damping. ($\hat{\gamma}\mathscr{E}$ is the Fourier inverse of $\gamma_k \mathscr{E}_k$ and hence a convolution. We should think of $\hat{\gamma}$ as a linear operator in real space.) The divergence is present because the Langmuir field is electrostatic ($\mathscr{E}_k = -\nabla\phi$). Generalization to include electromagnetic polarization is straightforward (Kuznetsov, 1974).

The second Zakharov equation is for the nonlinear density δn and may be derived from low-frequency ion and electron fluid equations, together with the quasineutral approximation. The adiabatic electron response is determined by the time average of the convective term, $\mathbf{v}\cdot\nabla\mathbf{v}$, in which $\mathbf{v}$ is the (hf) response to the Langmuir field. This term becomes the ponderomotive force (Nicholson, 1983). The result is

$$(\partial_t^2 - C_s^2\nabla^2)\delta n = C_s^2\nabla^2 |\mathscr{E}|^2 . \tag{5.9a}$$

In these equations, time is measured in units of ω_p^{-1}, distance in units of $(\frac{2}{3})^{1/2}\lambda_D$, field in units of $8(\pi n T)^{1/2}$, and density in units of $2n_0$, where n_0 is the background density. In simple units, then, sound speed is given by

$$C_s^2 = (\tfrac{3}{2})(m/M) . \tag{5.9b}$$

Density perturbations are driven by the (divergence of the) ponderomotive force, on the right side of Eq. (5.9a). These density responses in general are governed by the linear "ion-acoustic-wave" operator on the left side, which corresponds to the dispersion relation (2.2). The second derivative in time comes from ion inertia, and the Laplacian from particle pressure. Quasineutrality has been assumed, so δn can be thought of as either an electron or ion response (the ambipolar field has been eliminated).

The Zakharov equations possess solitary wave solutions in one dimension (Nishikawa *et al.*, 1974). There is no inverse scattering solution as for the cubic Schrödinger equation, but one-dimensional dynamics has been studied extensively (Gibbons *et al.*, 1977; Degtyarev *et al.*, 1980; Rowland *et al.*, 1981).

Higher-dimensional behavior has also been studied in detail (Rudakov and Tsytovich, 1978; Degtyarev and Zakharov, 1974; Degtyarev *et al.*, 1975,1976). Integral invariants exist, but there is no virial theorem proof of collapse for the Zakharov equations. As we shall see, there are a profusion of self-similar solutions in various limits, although none for the full Zakharov equations. Numerical studies do show, however, that collapse can occur under a variety of conditions.

D. Supersonic effects in the Zakharov equations

For those conditions under which ion inertia may be ignored, Eq. (5.9b) reduces to $\nabla^2\delta n = -\nabla^2|\mathscr{E}|^2$, or

$$\delta n = -|\mathscr{E}|^2 + \text{const} . \tag{5.10}$$

With this adiabatic result inserted in the first Zakharov equation (5.8), we obtain a cubic nonlinear Schrödinger equation generalized for a particular vector field.

Let us develop a criterion for when this adiabatic result is an appropriate limit. The issue is whether $|\partial_t^2\delta n| \ll |C_s^2\nabla^2\delta n|$ (adiabatic limit) or $|\partial_t^2\delta n| \gg |C_s^2\nabla^2\delta n|$ (supersonic limit). The question may be posed in terms of the characteristic velocity L/T, where T and L are characteristic time and space scales of δn and $\mathscr{E}$. From Eq. (5.8) we have

$$T^{-1} \approx L^{-2} \approx \delta n . \tag{5.11}$$

Hence the characteristic velocity for collapse (of a nontranslating Langmuir wave packet) is

$$L/T \approx |\delta n|^{1/2} . \tag{5.12}$$

The adiabatic or supersonic limit is obtained by comparing L/T in Eq. (5.12) with C_s in (5.9b):

$$L/T \ll C_s \text{ or } |\delta n| \ll m/M \text{ (adiabatic limit)} . \tag{5.13a}$$

Since $\partial_t^2\delta n \ll C_s^2\nabla^2\delta n$, Eq. (5.10) obtains, and (5.11) guarantees that the natural time and space scales of the cubic Schrödinger equation prevail. In this limit the virial theorem predicts that $|\mathscr{E}|^2$ (and hence δn) becomes singular in a finite time. However, according to (5.13a), when $|\delta n|$ or $|\mathscr{E}|^2$ begins to exceed the electron-to-ion mass ratio $C_s \propto m/M$, the inward velocity of spatial collapse, as measured by L/T in (5.12), becomes "supersonic," and $\partial_t^2\delta n$ can no longer be neglected. Unless dissipation has set in to stabilize or burn out the collapse, the Langmuir wave packet will always become supersonic when

$$L/T \gg C_s \text{ or } |\delta n| \gg m/M \text{ (supersonic limit)} . \tag{5.13b}$$

In the supersonic limit, Eq. (5.10) is no longer valid. The solution to Eq. (5.9) with $\nabla^2\delta n$ neglected is nonlocal in time, but we can see that δn scales as $(C_s T/L)^2|\mathscr{E}|^2$ or

$$\delta n \propto [(m/M)|\mathscr{E}|^2]^{1/2} . \tag{5.13c}$$

Hence the rate of collapse of (5.11) and (5.12) is slowed relative to the adiabatic case. Note that (5.13c) yields a useful alternate form of the condition (5.13b) for the supersonic limit, namely

$$|\mathscr{E}|^2 \gg m/M \text{ (supersonic limit)} . \tag{5.13d}$$

These arguments also apply when performing a linear stability analysis of the Zakharov equations for modulational instability of a "pump" wave (Nishikawa, 1968; Sudan, 1975; Bardwell and Goldman, 1976). The scaling is once more given by Eq. (5.11), so the modulational instability growth rate goes as

$$T^{-1} \propto [(m/M)|\mathscr{E}|^2]^{1/2} , \tag{5.14a}$$

and the fastest growing modulational unstable mode has wave number

$$k \propto [(m/M)|\mathscr{E}|^2]^{1/4} . \tag{5.14b}$$

E. Self-similar collapse and "burnout"

Numerical solutions of the Zakharov equations in two dimensions have demonstrated many of the effects observed in particle-in-cell simulations. Among the first to solve these equations were Pereira, Sudan, and Denavit (1977). Their two-dimensional solutions have shown modulational instability of an initial-value problem (with $W \approx 0.1$) in which the initial field contours were uniform in y and solitonlike in x. The subsequent evolution, after the linear instability stage, consisted of collapse, followed by burnout when γ_k was given by the linear Landau-damping term, Eq. (2.3b). In the absence of dissipation, the collapse was shown to be of an anisotropic self-similar form, given by

$$\mathscr{E} \approx (\xi\eta)^{-1} f(x/\xi, y/\eta) , \tag{5.15}$$

where

$$\eta(t)=\xi(t)^2 \to 0 \quad \text{as } t \to t_c \ .$$

These results are illustrated in Fig. 9. Note that $|\mathscr{E}|^2$ increases as the collapse to smaller scales proceeds in sequence (a), for $\gamma=0$. In sequence (b), Landau damping is present, and, after initial collapse, $|\mathscr{E}|^2$ begins to decrease due to burnout. At the same time, the density cavity which had trapped the Langmuir wave energy begins to break apart when there is no longer any ponderomotive force to carve it out. The broken up density cavity is expected to propagate away in the form of ion-acoustic waves [Eq. (2.2)], since the second Zakharov equation permits such propagation once the ponderomotive force has vanished.

There is not universal agreement concerning the "proper" form of self-similar solutions to the Zakharov equations. In the adiabatic regime, the reduction of the Zakharov equations to the cubic Schrödinger equation yields analytic self-similar solutions which are either isotropic (Zakharov, 1972) or anisotropic (Rudakov and Tsytovich, 1978), but which do not agree with Eq. (5.15). Other numerical work on the Zakharov equations in two dimensions yields dipole collapse which does not scale as Eq. (5.15) (Degtyarov and Zakharov, 1974,1975). Analytical

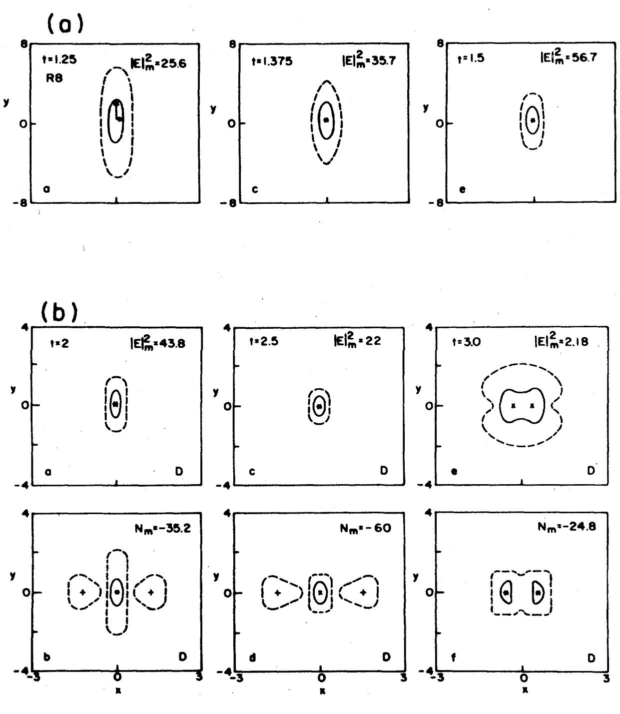

FIG. 9. Numerical solutions of Zakharov equations (5.8) and (5.9) in two dimensions, showing collapsing Langmuir wave packet at the late stages of evolution of an initially unstable (1D) soliton. (a) Self-similar collapse. (b) Top: Landau damping produces wave-packet burnout. Bottom: Density cavity begins to decompose into ion-acoustic waves as burnout occurs (Pereira *et al.*, 1977).

self-similar forms also exist in the supersonic regime, but are still different (Zakharov, 1972; Galeev *et al.*, 1975).

VI. PLASMA EXPERIMENTS AND OBSERVATIONS

A. Brief historical survey

Some of the effects of ponderomotive force are starting to be observed in experiments on real plasmas. After the pioneering work of Kim, Stenzel, and Wong (1974), a number of workers have observed laboratory density "cavitons" and associated Langmuir wave envelope localization, driven by electric fields or beams. We cite the recent observations of spiky turbulence and radiation driven by a cold electron beam (Cheung, Wong, Darrow, and Quian, 1982; Leung, Tran, and Wong, 1982; Michel, Paris, Schneider, and Tran, 1982; Wong *et al.*, 1983; and Wong and Cheung, 1984), and the two-dimensional collapse driven by an electron beam in a homogeneous plasma (Wong and Cheung, 1984) and driven by a capacitor electric field along a density gradient in an inhomogeneous plasma (Eggleston, Wong, and Darrow, 1982).

In Fig. 10, we see the results of one such experiment (Leung *et al.*, 1982, Wong, Cheung, and Tanikawa, 1983). Oppositely directed weak electron beams create a Langmuir envelope wave packet which is localized in space, and a slight density depression associated with it [Fig. 10(a)]. A coherent signal triggers a breakup or modulational instability, followed by spatial collapse, as shown in Figs. 10(b) and 10(c). Figure 11 shows the electron velocity distribution function before and after the collapse. Note the formation of a tail, reminiscent of the particle-in-cell simulation results, Fig. 8.

In an even more dramatic, very recent experiment, Wong and Cheung (1984) studied three-dimensional Langmuir collapse induced by a weak pulsed electron beam in a spatially homogeneous plasma. A single wave packet was observed to collapse in a highly reproducible manner from shot to shot, thus enabling an ensemble-averaged time history of collapse to be constructed. Since W was on the order of $0.3 \gg m/M$, the collapse would be expected to be supersonic. The early radial contraction of the packet was observed to be consistent with supersonic self-similar behavior [see Eq. (8.1)], although the axial contraction rate was different.

Another realm in which strong Langmuir turbulence is thought to be produced is the solar wind. There is direct evidence from spacecraft that weak electron beams in the solar wind can drive Langmuir turbulence at levels up to $W \approx 10^{-5}$. This can occur in conjunction with radio-wave emissions near the plasma wave frequency and its second harmonic, in the well-documented "Type-III" solar radio-wave emission (Gurnett and Frank, 1975; Gurnett and Anderson, 1977; Gurnett *et al.* 1980,1981; Lin *et al.*, 1981). The associated Langmuir turbulence has been studied using the Zakharov equations in one (Papadopoulos *et al.*, 1974; Rowland *et al.*, 1981) and two or more dimensions (Nicholson, Weatherall, Goldman, and Hoyng, 1978; Nicholson and Goldman, 1978; Hafizi

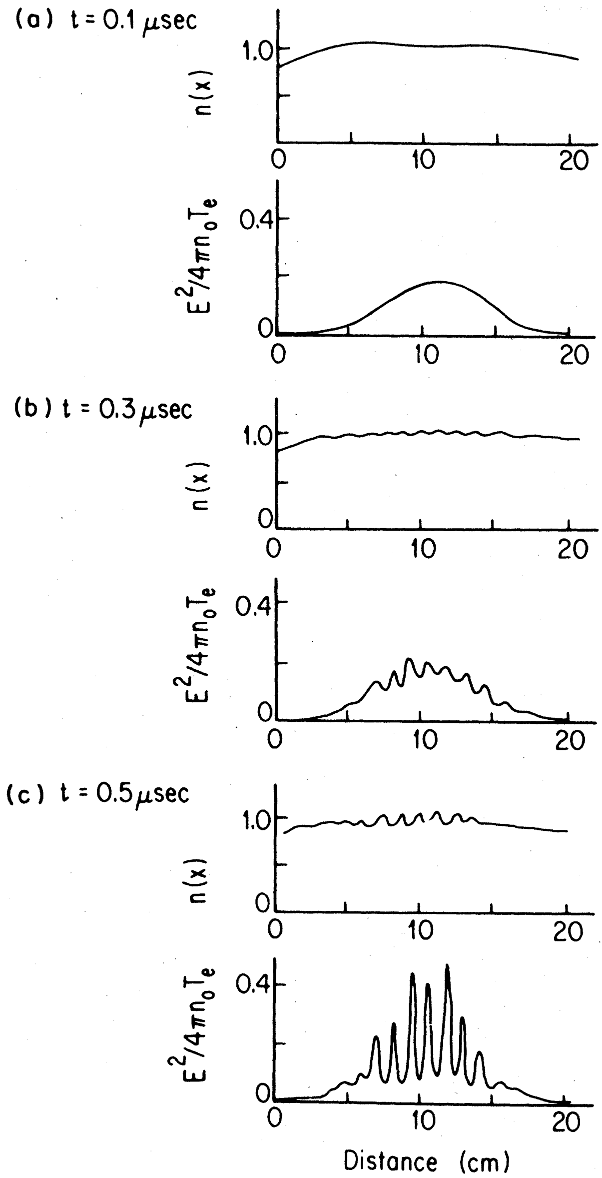

FIG. 10. Experimental study of modulational instability in a plasma. Spatial profiles of ion density and of Langmuir wave energy density. (a) Oppositely directed electron beams have driven a localized Langmuir envelope wave packet and slight density depression. (b) and (c) Coherent signal triggers modulational instability and spatial collapse (Leung *et al.*, 1982).

et al., 1982; Goldman, 1983; Weatherall *et al.*, 1983). We shall return to this subject momentarily.

A second example of Langmuir turbulence in the solar wind is offered by measurements near the planetary bow shocks of the Earth and Jupiter. In particular, near the Jovian bow shock very narrow spatial Langmuir envelopes (Fig. 12) and high-k Langmuir spectra have been detected (Gurnett *et al.*, 1981). It is still not clear whether these waveforms are collapsed wave packets, as alternate explanations for the production of short-scale low-

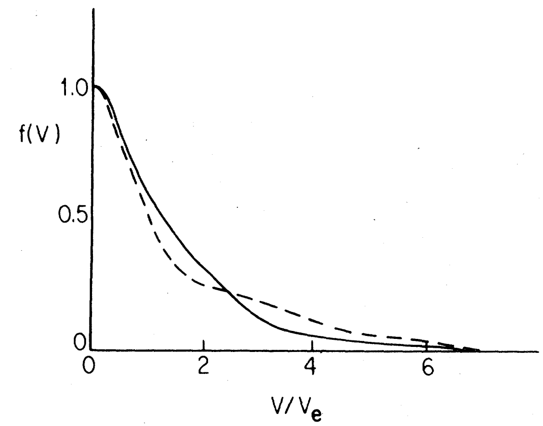

FIG. 11. Electron distribution function evolution for the experiment shown in Fig. 10. Note formation of high-energy tail when spatial wave collapse has proceeded to short scales (Leung *et al.*, 1982).

level turbulence have been offered (Russell and Goldman, 1983).

Sheerin *et al.* (1982) have suggested that Langmuir soliton collapse can occur in the *F* region of the Earth's ionosphere, when it is irradiated by high-power hf radar from a ground transmitter. Their numerical solutions of the Zakharov equations show that the radar electric field near the reflection point ($\approx$300 km) drives a modulational instability, and that the nonlinear stage of wave evolution involves self-focusing wave packets. This result is not inconsistent with radar backscatter observations.

Soviet scientists (Sigov and Khodyrev, 1976; Galeev *et al.*, 1977a,1977b,1977c; Al'terkop *et al.*, 1977) have consistently cited the importance of Langmuir wave self-focusing effects in laser-plasma interactions. Langmuir waves can be produced in laser target plasmas by resonant conversion of radiation at the critical density or, at lower densities, by stimulated Raman scattering, or the $2\omega_p$ instability. Hot electron formation due to burnout could have an adverse effect on the isentropic compression necessary for inertial fusion schemes. In addition, the observed harmonic generation (Rubenchik, 1981) in the radiation from laser-irradiated plasmas may be affected by strong Langmuir turbulence. In the laser-plasma context there are generally strong density gradients, and spatial inhomogeneity must be taken into account. Self-focusing of the electromagnetic radiation within laser-irradiated pellets has also been proposed (Valeo and Estabrook, 1975).

Finally, we cite several applications of these ideas to magnetic wave phenomena in plasma physics. Musher and Sturman (1975) have considered the collapse of lower hybrid waves. It is conceivable that this process is relevant to attempts to radiatively heat plasmas that are toroidally confined. In another magnetic application, upper-hybrid solitons and modulational instability of upper-hybrid waves have been studied (Kaufman and Stenflo, 1975; Porkolab and Goldman, 1976).

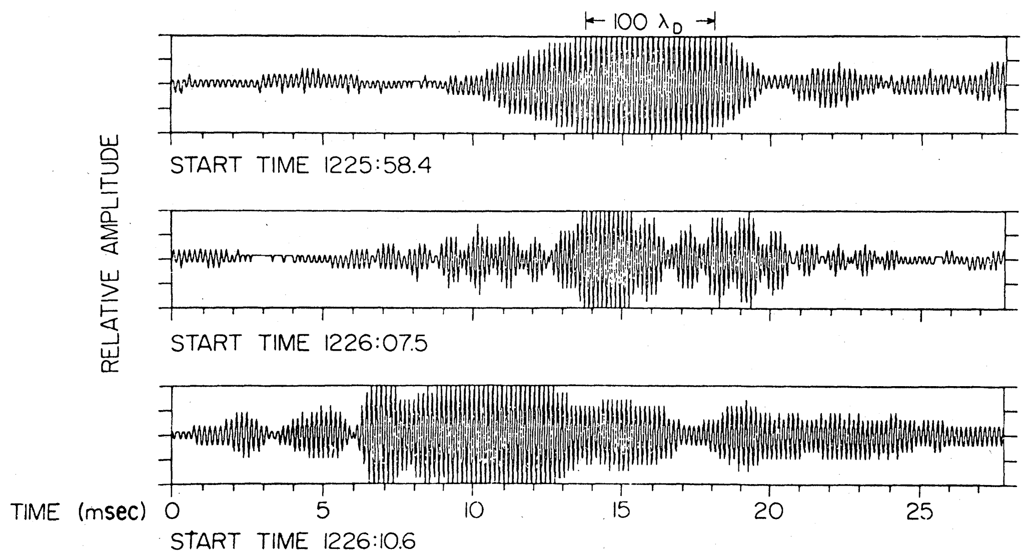

FIG. 12. Narrow Langmuir envelope wave packet excited by electron beam at Jupiter's bow shock. The electric field waveforms of three Langmuir wave bursts of unusually short duration are shown. These bursts consist of very intense wave packets, which have collapsed down to spatial scales as short as a few Debye lengths (Gurnett *et al.,* 1981).

B. Type-III solar radio-wave emission

We return now to a more detailed description of a possible example of strong Langmuir turbulence excited by weak electron beams in the solar wind, during so-called Type-III solar radio-wave emission. This example is interesting because it involves a class of three-wave interactions, also described by the Zakharov equations, and because it raises the issue of competition between a variety of possible saturation mechanisms for beam-excited Langmuir waves, including the traditional "weak" turbulence processes of (quasilinear) diffusion of particles by waves, and of transfer of wave energy out of resonance with the beam by scattering from ion-acoustic waves.

A complete account of the observations and plasma physics associated with Type-III bursts can be found in the review article by the present author (Goldman, 1983) and in the theoretical work of the following authors: Papadopoulos *et al.,* 1974; Bardwell and Goldman, 1976; Smith *et al.,* 1976,1979; Nicholson *et al.,* 1978; Papadopoulos and Freund, 1978; Goldstein *et al.,* 1979; Goldman *et al.,* 1980,1981; Kruchina *et al.,* 1980; Hafizi *et al.,* 1981,1983. For the present we touch on some of the relevant observations and one attempt at an explanation.

The scenario associated with a Type-III burst is illustrated in Fig. 13. An electron beam is excited on the surface of the sun, possibly as a result of the release of magnetic reconnection energy during a solar flare. The resulting electron beam propagates along open solar magnetic field lines to the Earth and beyond. In so doing, it passes through the denser plasma close to the sun and moves out in the solar wind, through plasma that is progressively more dilute. As the beam front progresses outwards, it excites electron plasma (Langmuir) waves.

The plasma waves are nonlinearly converted into radiation near the electron plasma frequency and near its second harmonic. Since the plasma frequency increases with density [see Eq. (2.1)], the frequency of radiation observed at a given point in interplanetary space decreases as a function of time, from high values ($\approx$200 MHz), generated when the beam is near the surface of the sun, to low values (tens of kHz), generated when the beam is closer to the Earth.

This Type-III emission was first observed from ground stations on the Earth, but since the Earth's ionosphere reflects radiation below around 5 MHz, only the emission from close to the sun could be observed. With the advent of satellite experiments, it became possible to observe the lower-frequency emission as well. Figure 13 shows the location of the solar orbiting satellites, Helios 1 and 2, and the Earth-orbiting spacecraft, ISEE-3 and IMP 6,8, all of

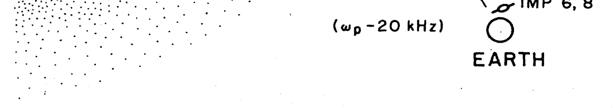

FIG. 13. Illustration of Type-III solar radio emission. Electron beam propagates out from the sun after a flare and excites Langmuir waves of progressively lower frequencies. These waves are nonlinearly converted into emission at ω_p and $2\omega_p$. Spacecraft that have detected the electron beam, Langmuir waves, and electromagnetic emission are shown (Goldman, 1983).

which have provided important data on the radiation spectrum, the Langmuir waves, and the electron beam.

In Fig. 14, we show the results of measurements of the beam and Langmuir waves from ISEE-3, some 200 Earth radii from the Earth (Lin *et al.*, 1981). An effective one-dimensional electron velocity distribution function is constructed and depicted in Fig. 14(a) at two different times. The dots represent the ambient local interplanetary plasma electron distribution function, before the arrival of the beam from the sun. The squares show the beam as a "bump" on the tail of the distribution, near $v=c/3$, some 35 minutes later.

In Fig. 14(b), the Langmuir turbulence measured by the spacecraft antenna at 31 kHz is given as a function of time. The first appearance of the turbulence is simultaneous with the first appearance of the "bump-on-tail" distribution function. This is consistent with Eq. (2.3), which says that negative damping, or Langmuir wave growth, is associated with a region of positive slope on the reduced one-dimensional electron distribution function. The instability is called the "bump-on-tail" instability in plasma physics, and the growth rate of Eq. (2.3) is called the "quasilinear" growth rate. The measured value of the integrated-wave energy density, $W=|E|^2/4\pi nT$, is on the order of 10^{-5}. This is an average that tends to emphasize wavelengths of order 8 km or longer (i.e., wavelengths longer than the beam-resonant Langmuir wavelength $\lambda=c/3f_p\approx 3$ km).

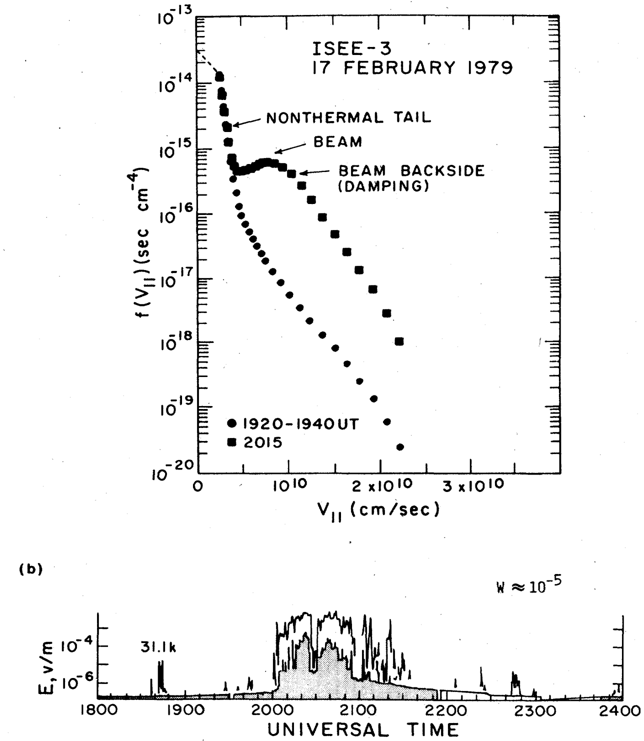

FIG. 14. Results of ISEE-3 spacecraft measurements of electron beam and Langmuir waves during a Type-III burst (not shown). (a) Reduced one-dimensional electron velocity distribution function before (●) and during (■) the passage of the beam. (b) Corresponding Langmuir turbulence, which onsets at the time of appearance of beam "bump." Logarithmic average weights wavelengths are greater than 8 km (Lin *et al.*, 1981).

In the next section, we show how certain aspects of the Langmuir wave dynamics underlying Type-III solar radio-wave emission can be modeled by the Zakharov equations.

VII. NONLINEAR EVOLUTION OF A "BUMP-ON-TAIL" INSTABILITY

A. "Driven" Zakharov equations

Let us return to the Zakharov equations, (5.8) and (5.9). In the numerical solutions of Pereira *et al.* (1977; Fig. 9), γ_k was taken to be purely dissipative and given by the Landau damping expression (2.3b). The wave evolution followed from an initial value for the field $\mathscr{E}$ and for the density cavity δn. This is an example of an "undriven" problem. However, we have seen from the discussion of Sec. VI that real plasmas are usually "driven" by external sources of free energy, such as an electron beam.

We may study the nonlinear dynamics of a plasma driven by a weak bump-on-tail electron beam, such as the one associated with solar Type-III radio-wave emissions, by using the Zakharov equations, but allowing γ_k to be negative in the regions of **k** space that correspond to wave growth, and positive in the regions that correspond to damping (Nicholson *et al.*, 1978; Hafizi *et al.*, 1983; Goldman, 1983). A bump-on-tail electron distribution function like that shown in Fig. 14 will give rise to both growth and damping of Langmuir waves at the appropriate wave vectors [see Eq. (2.3)]. The damping regions of **k** space arise from nonthermal Landau damping, which can be quite strong (damping rates on the order of the beam growth rate). Note that the dissipative region is close to the growth region for the electron distribution function of Fig. 14.

Turning now to the second Zakharov equation, (5.9), we shall allow for the ion-acoustic wave operator on the left to include possible damping of ion-acoustic waves through a linear operator $\hat{\nu}_i\partial_t$, for which the Fourier transform $\nu_i(k)$ is the appropriate damping rate. The Zakharov equations therefore assume the following form:

$$\nabla\cdot(i\partial_t+\nabla^2-\delta n+i\gamma_b)\mathscr{E}=0,\quad \gamma_{bk}\propto-\partial_v f_e|_{\omega_p/k} \tag{7.1a}$$

$$(\partial_t^2+\nu_i\partial_t-c_s^2\nabla^2)\delta n=c_s^2\nabla^2|\mathscr{E}|^2\,. \tag{7.1b}$$

B. Wave-wave interactions

The Zakharov equations contain wave-wave interactions of the conventional three-wave decay variety, as well as of the (four-wave) modulational and self-focusing type. Such three-wave dynamics are part of the basis of conventional "weak" Langmuir turbulence. To show how three-wave interactions arise in the Zakharov equations, suppose we consider the stability of an initial Langmuir wave field $\mathscr{E}_0$. Then, Eq. (7.1a) tells us that the beating of $\mathscr{E}_0$ against a density fluctuation δn (associated with an ion-acoustic wave) will produce a scattered Langmuir wave $\mathscr{E}_{\text{scatt}}$, and (7.1b) tells us that the scattered wave will beat against $\mathscr{E}_0$, to act as a source for δn:

$$\mathscr{E}_{\text{scatt}} \approx \mathscr{E}_0 \delta n \ ,$$

$$\delta n \approx \mathscr{E}_0 \mathscr{E}^*_{\text{scatt}} \ .$$

This feedback gives rise to a "wave-wave decay" or "stimulated scattering instability," with growth rate γ_{WW},

$$\gamma_{\text{WW}} \approx |\mathscr{E}_0|^2 \ . \tag{7.2a}$$

In this instability, the pump wave $\mathscr{E}_0$ gives energy to the scattered wave $\mathscr{E}_{\text{scatt}}$ and to the ion-acoustic wave, represented by δn. The linear stability analysis is, of course, actually performed on the Fourier transform of the Zakharov equations, so there is a wave vector attached to each of the three waves. The kinematics is illustrated in Fig. 15.

There is a special wave number in the kinematics, which is proportional, in physical units, to the square root of the electron-to-ion mass ratio m/M. We shall call this wave number k_*:

$$k_* \approx (m/M)^{1/2} k_D \ .$$

For a pump wave number much larger than k_*, the scattered Langmuir wave is always in the backward direction, relative to the pump wave vector [Fig. 15(a)], and reduced in magnitude by the small amount k_*. In this limit, the ion-acoustic frequency is negligible compared to the incident and scattered Langmuir wave frequencies, and the ion-acoustic wave only serves to take up momentum.

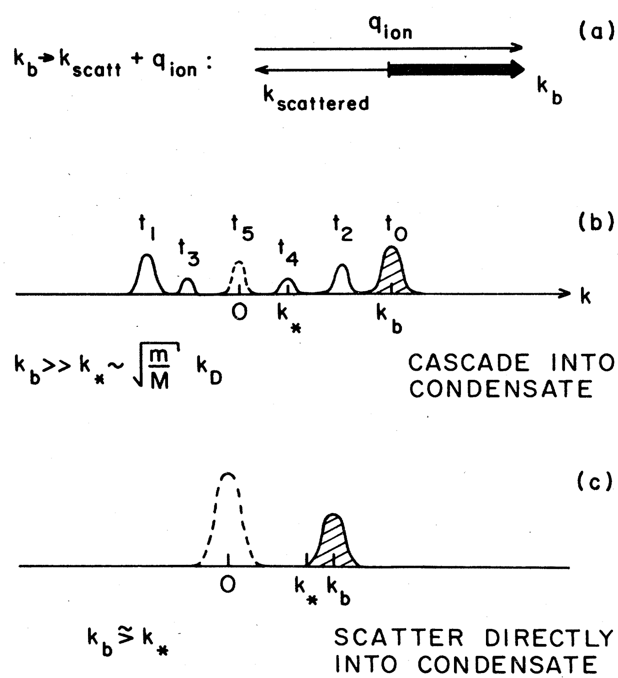

FIG. 15. Wave scattering and cascade described by Zakharov equations. (a) Wave-vector geometry for backscatter of a Langmuir wave $\mathbf{k}_b$ from an ion-acoustic wave $\mathbf{q}_{\text{ion}}$. (b) Cascade to low k when $k_b \gg k_*$. (c) Scatter directly into condensate when $k_b \gtrsim k_*$.

For somewhat smaller pump wave numbers, $k_0 \gtrsim k_*$, the scattered wave will have essentially zero wave number (large wavelength). This is because the ion-acoustic wave frequency becomes comparable to the dispersive (k^2) part of the pump wave frequency [Eq. (2.1)] at such small pump wave numbers and can no longer be neglected. For still smaller pump wave numbers, $k_0 < k_*$, the three-wave interaction is no longer possible kinematically (although modulational instability may occur, since it represents a "four-wave" interaction with nonresonant adiabatic ion response).

Let us now consider the pump wave to be one of the beam-driven waves. Its wave number is $k_b \approx \omega_p / v_b$, where v_b is the beam speed. For beams that are not very energetic ($v_b \ll \omega_p / k_*$), the beam-amplified pump wave backscatters. As the backscattered wave grows [at the rate given by Eq. (7.2)], it eventually becomes sufficiently energetic to act as a pump for a forward scatter. This process may continue for a number of back-and-forth scatters. At each scatter, the wave number is reduced by the amount k_*.

The result is a cascade of wave energy to lower wave numbers [see Fig. 15(b)]. Such cascades are well known in weak turbulence, and have been observed as numerical solutions to the Zakharov equations (Nicholson and Goldman, 1978). If there is sufficient dissipation at wave numbers lower than k_b, the beam instability can be saturated, and the resulting turbulence is termed "weak" (Kadomtsev, 1965; Kaplan and Tsytovich, 1967,1973). If there is insufficient dissipation, energy begins to build up at long wavelengths, in the so-called Langmuir wave condensate.

Since there is usually virtually no Landau damping at very long wavelengths, the condensate eventually becomes unstable to the modulational instability discussed by Vedenov and Rudakov (1964) (see Sec. V.A). The evolution of the modulational instability will involve the spatially collapsing wave packets discussed earlier. Modulational instability and collapse are both strong-turbulence effects, described by the Zakharov equations. The collapse causes an eventual transfer of energy up to large k, where thermal or nonthermal Landau damping (or, more appropriately, "transit-time" damping) provides the dissipation that saturates the instability. (Such damping may also lead to Langmuir wave-packet burnout in real space.)

For very energetic beams, we may have k_b on the order of k_*. In this case, the stimulated scattering will carry energy directly into the condensate [see, Fig. 15(c)], rather than indirectly (i.e., after a prior cascade). The Langmuir turbulence excited during Type-III solar emission behaves in this way, since $v_b \approx c/3$. For still more energetic beams (e.g., relativistic beams), the beam-driven waves may go unstable to modulational instability directly, rather than after scattering off ion-acoustic waves (Sudan, 1973,1975).

C. Wave-particle (quasilinear) saturation

The reader may wonder at this point what has happened to the traditional quasilinear processes (Sec. II.B) in the nonlinear evolution of Langmuir waves driven by a bump-on-tail electron beam. In quasilinear theory, the nonlinear interactions are between the beam-driven waves and the beam particles only. Wave-wave interactions such as stimulated scattering, modulational instability, and collapse are all omitted from the theory. Saturation starts to occur when the beam-driven modes become as energetic as the beam. At this time they begin to cause velocity-space diffusion of the beam particles, causing a plateau to appear where there was formerly a positive slope. This removal of free energy quenches the instability [Eq. (2.3)], as described in Sec. II.B.

In the Zakharov equations (7.1), the electron distribution function f_e is fixed, so there can be no reaction of the waves back on the beam particles. The Zakharov equation treatment (in this approximation) is complementary to the quasilinear description in this sense, so we must ask which is the proper description. The problem is illustrated in Fig. 16, which suggests that the beam-driven modes can either react back on the electron beam or transfer their energy to other waves, leading, in the strong-turbulence case, to the formation of a condensate which is unstable to collapse.

The answer is that the more rapid process dominates. If wave-wave interactions begin rapidly to transfer wave energy out of resonance with the beam at an early time (i.e., when the beam-excited modes contain only a small fraction of the beam energy), the beam modes will not be sufficiently energetic to react back on the beam and remove its free energy by quasilinear diffusion. In this case, the wave-wave interactions certainly cannot be ignored, and strong turbulence may prevail. Whether or not the waves will eventually alter the beam depends on the shape and intensity of the spectrum of fully developed wave turbulence. If the asymptotic spectrum contains appreciable wave energy in resonance with the beam, then quasilinear diffusion may once more become possible.

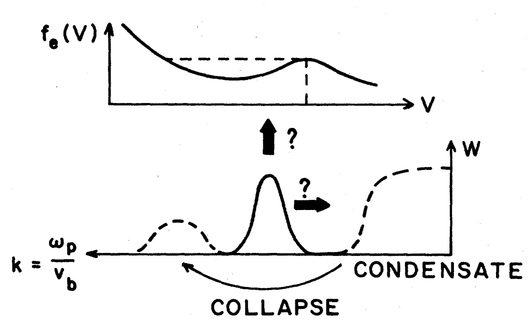

FIG. 16. Two possible saturation mechanisms for a "bump-on-tail" beam-excited Langmuir wave instability. The growing waves either react back on the resonant electrons, causing quasilinear plateau formation, or they scatter down to a nonresonant long-wavelength condensate, which may undergo modulational instability.

We may formulate a rather simple criterion for whether the evolution of Langmuir waves driven by a beam is initially dominated by wave-wave or by wave-particle interactions. The wave-wave scattering rate is given by Eq. (7.2a) or, in proper dimensionless units, by

$$\gamma_{\rm WW} = W/16 , \tag{7.2b}$$

where $W \equiv \langle |E|^2 \rangle / 4\pi n k_B T$, as in Eq. (2.4). Wave-wave scattering can only begin to depopulate the beam-resonant region of wave k space when $\gamma_{\rm WW}$ is comparable to or greater than the growth rate of the beam-resonant waves, γ_b. This occurs at a wave energy density

$$W = W_{cr} \approx 16\gamma_b/\omega_p . \tag{7.3}$$

If W_{cr} is significantly less than the beam energy density nmv_b^2 (in units of the background particle energy density nT), then wave-wave interactions will preempt the wave-particle interactions which cause quasilinear plateau formation. Only in this situation is it self-consistent to introduce an unchangeable beam distribution function into the Zakharov equations, as in (7.1a). We thus have as the criterion for wave-wave saturation of the instability

$$16\gamma_b/\omega_p \ll nm(\Delta v_b/v_b)v_b^2/nT . \tag{7.4a}$$

For a bump-on-tail distribution function, the maximum growth rate of the beam modes is given by $\gamma_b|_{\max} \approx (n_b/n_e)(v_b/\Delta v_b)^2$. Hence the criterion (7.4a) may be rewritten as

$$4v_e/v_b \ll (\Delta v_b/v_b)^2 . \tag{7.4b}$$

There are various limitations on the validity of this simple inequality. The (quasilinear) expression we have used for the maximum growth rate of the beam modes is only valid in the "warm-beam" limit $(n_b/n_e)^{1/3} \ll \Delta v_b/v_b$. In addition, for very high beam energy densities, the relevant wave-wave interaction for removal of beam-resonant wave energy density may be intense modulational instability (Sudan, 1973,1975; Galeev *et al.*, 1975,1976,1977a,1977b,1977c), rather than stimulated scattering. Also, for beams sufficiently strong that W_{cr} exceeds the mass ratio m/M, the rate of stimulated scattering is reduced from W to $(m/M)^{1/2}W^{1/2}$ [Eq. (5.13) and (5.14); Sudan, 1973,1975; Papadopoulos, 1975]. In any of these regimes, the condition (7.4b) is appropriately modified. However, in the example of strong turbulence we consider next, n_b/n_e is of order 10^{-6}, W_{cr} is of order 10^{-5}, and (7.4b) is the proper criterion, and is well satisfied.

Does the criterion (7.4a) also serve as a criterion for deciding if the Langmuir turbulence will be strong or weak? The answer is somewhat subtle. When we speak of weak and strong turbulence we are making a distinction that belongs in the domain of statistical theory. As mentioned earlier, three-wave interactions, together with quasilinear wave-particle interactions, constitute "weak" turbulence. Hence the dominance of one or the other is not by itself sufficient to distinguish when the "strong"-turbulence effects of modulational instability and collapse come into play. However, when the wave-wave scattering dominates, it is usually the case that a condensate forms and becomes modulationally unstable.

This has been shown for the parameters of the Type-III burst problem by Grognard (1982), who solved the complete one-dimensional weak-turbulence equations for a bump-on-tail electron distribution function and the self-consistent resonant and nonresonant (i.e., wave-scattered) portions of the Langmuir wave spectral function. He found that wave-wave interactions caused a long-wavelength condensate to build up to a high energy level, at which time the weak-turbulence equations broke down because the collapse threshold was exceeded. His equations also included advective effects of the beam propagation through space. The buildup of the condensate was found to be sufficiently rapid that a quasilinear plateau was never able to form on the advecting beam distribution function.

Of course, one can envisage other situations where weak turbulence may indeed be a sufficiently complete description. For example, a weak low-velocity beam could cause an extensive cascade to occur. If the plasma were sufficiently collisional, there might be enough collision damping (free-free absorption) to prevent the cascade from terminating in a long-wavelength condensate.

D. Numerical solutions to the Zakharov equations

We assume, based on the above discussion, that Eq. (7.1), the Zakharov equation description of the Langmuir turbulence excited by a Type-III burst-associated beam, is justified. There have been a number of numerical treatments of the two-dimensional dynamical wave evolution which exhibit the phenomena of modulational instability and collapse (Degtyarev and Zakharov, 1974; Pereira *et al.*, 1977; Nicholson *et al.*, 1978; Hafizi *et al.*, 1982; Nicholson *et al.*, 1983).

Here, we present an unpublished numerical treatment, due to Dr. J. Weatherall (1982), in which the measured velocity distribution function of Fig. 14 is used to calculate γ_{bk} in Eq. (7.1a). The beam-mode growth rate is very weak ($\gamma_b|_{\max} \approx 10^{-6}$), and nonthermal Landau damping occurs on either side of the bump. The condition $k_b \gtrsim k_*$ is satisfied, so scattering into a condensate is expected. The ion-acoustic wave that participate in the scattering process are heavily damped ($\nu_{ik} \approx c_s k$).

A grid of 128×128 is employed, and the initial conditions consist of random-phase low-level wave noise. The wave evolution is exhibited in Fig. 17, which shows the contours of constant $|E_k|^2$, $|E(x,y)|^2$, and $\delta n(x,y)$ (reading from left to right) at three different times (reading from top to bottom).

In Fig. 17(a), we see that the **k**-space energy has grown to a sufficiently high level ($W \approx 10^{-5}$) that the wave-wave interaction of stimulated scatter of beam-excited waves off ions is beginning to fill up a long-wavelength condensate faster than the beam can continue to drive up the level of resonant waves. It is at this point in time that saturation begins. In Fig. 17(b), we see that the energy density of waves in real space still appears to carry the random phasing of amplified initial noise. The pattern is one of valleys and hills produced by the interference of these plane waves. Figure 17(c) shows that ion-acoustic plane waves have built up, due to the stimulated scatter. Their momentum is essentially that of the beam-resonant Langmuir waves, since the scattered Langmuir waves reside in the (zero-momentum) condensate.

Figure 17(d) shows the energy density of Langmuir waves in real space at a later time. Now there is evidence of a more coherent, self-focused wave packet's having formed. Its origin is presumably the modulationally unstable **k**-space condensate.

In Fig. 17(e), we are at a much later time. The condensate is fully formed, and very little energy is in resonance with the beam; the total energy of the waves has remained uniform since the preceding figure because the rate of increase of total energy is proportional to the intensity of resonant energy. Figure 17(f) shows that the self-focused wave packet appears to be relatively stable and stationary. Presumably the small amount of incoming wave energy due to the beam instability is balanced by the small amount of nonthermal Landau damping of the spectrum at low and high k. In this steady state there does not appear to be any burnout of the collapsed wave packets, as in the particle-in-cell simulations (Figs. 6 and 7). Figure 17(g) shows the density cavity that has been dug out by the ponderomotive force of the collapsed wave packet. It too is stationary.

From these results we can draw several conclusions regarding the Langmuir turbulence that underlines Type-III solar radio emission. The long-wavelength condensate that forms has most of the spectral energy. There is a sizable energy component at scale sizes of 8 km and longer, so this part of the turbulence would indeed be observable and consistent with the limitations of the spacecraft measuring apparatus discussed above (short scales do not register, except in a logarithmic average). The mean energy density of the condensate agrees with observation ($W \approx 10^{-5}$) and also the wave-scattering criterion, Eq. (7.3). Preliminary estimates indicate that the condensate can give rise to the observed intensities of electromagnetic emission at the plasma frequency and its second harmonic (Newman, private communication). Collapse and cavitation occur, but although the collapsed wave packets have a high energy density, they do not have a high integrated energy. [However, the integral over $|\mathscr{E}|^4$ can be large, and is perhaps a significant measure (Kaufman, private communication).]

This computation took one hour of Cray-1 time. Although it appears that we have found an asymptotic steady state, it is hard to be certain that enough time has elapsed, or that we have sufficient resolution in **k** space to believe the absence of burnout, for example (Zakharov, Rubenchik, and Sagdeev, 1983).

It may be useful to try to classify the wave dynamics in the language of fluid turbulence: The so-called injection range of **k** space is where wave energy is introduced by

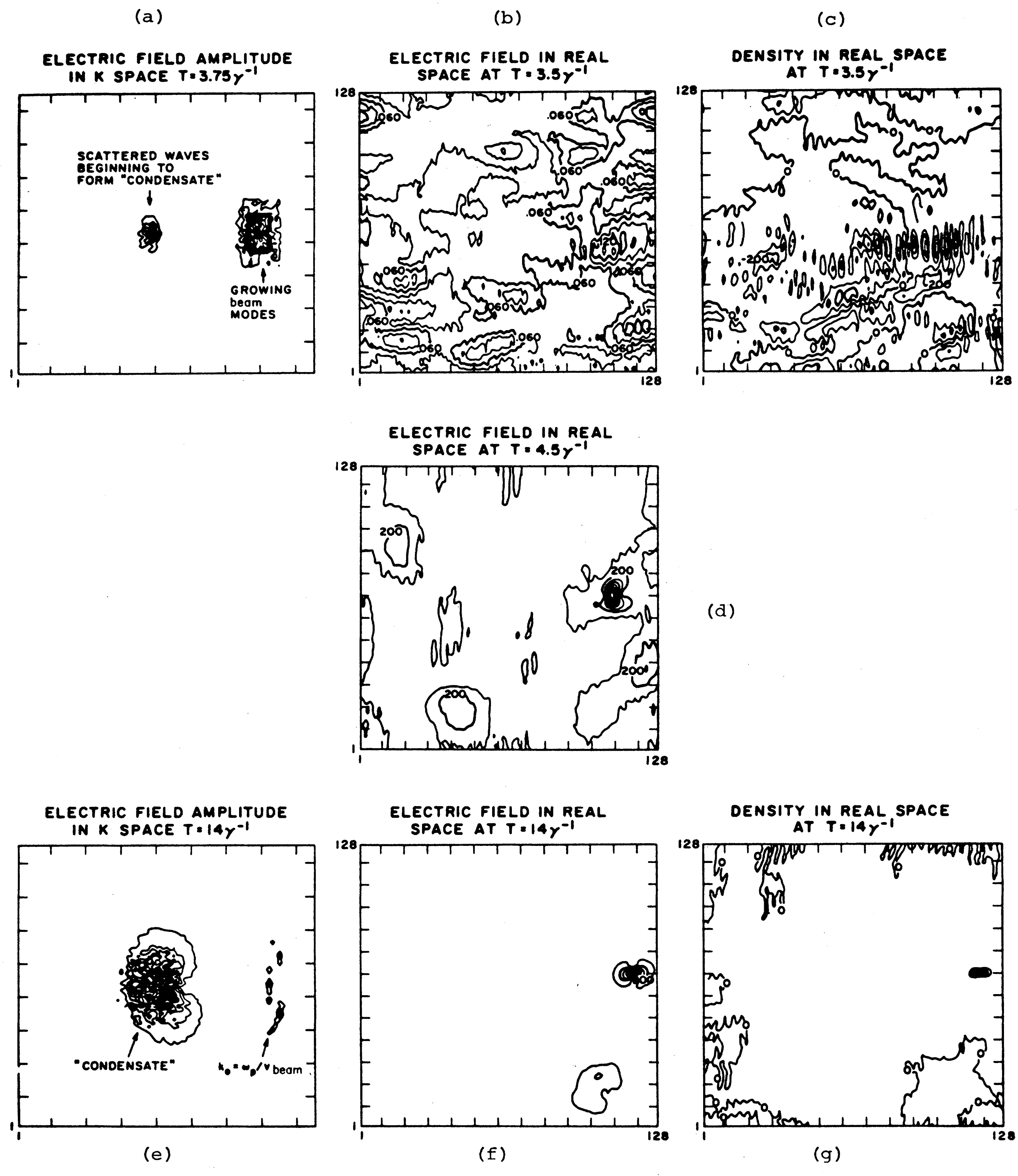

FIG. 17 Numerical solution of Zakharov equations for beam-excited strong Langmuir turbulence, corresponding to Type-III solar radio-wave burst parameters. (a) and (e) Wave energy density contours in **k** space at $t=3.75\gamma^{-1}$ and at $t=14\gamma^{-1}$, where γ is the beam-mode growth rate. Scatter into a condensate is evident. (b), (d), and (f) Wave energy density contours in real space, showing collapse of a Langmuir wave packet at $t=3.5\gamma^{-1}$, $4.5\gamma^{-1}$, and $14\gamma^{-1}$. (c) and (g) Contours of constant ion density in real space at $t=3.5\gamma^{-1}$ (showing ion-acoustic waves generated during scatter into the condensate) and at $t=14\gamma^{-1}$, showing density cavity supported by ponderomotive force of collapsed Langmuir wave packet. See text for further discussion. From J. Weatherall (1982).

the beam to the beam-resonant modes. The dissipation range is presumably at large k, but there is also some dissipation at the edges of the condensate; both are due to nonthermal Landau damping. The so-called inertial range is supposed to be a region of wave evolution that is free from sources or sinks. In the Type-III burst problem, it appears that the inertial range is absent or very narrow. Other recent studies have been made of narrow-inertial-range turbulence (Weatherall *et al.,* 1983). Such turbulence may be less amenable to a statistical theory than situations in which the inertial range is wider. We now turn to a discussion of a different parameter regime appropriate to one such statistical theory.

VIII. STATISTICAL THEORY OF STRONG LANGMUIR TURBULENCE

Our discussion so far has dealt with the physical phenomena and dynamical equations which underlie strong Langmuir turbulence. A statistical theory is often implied when we speak of "turbulence," although since the rise of modern dynamics this is no longer strictly necessary. A statistical theory is desirable because the resulting kinetic equations are usually (but not always) simpler than the original dynamical equations, and analytic solutions and scaling laws are often possible. Such a theory also has a distinctly different flavor, introducing concepts such as the rates of energy transfer between scales and various possible transport phenomena, such as particle or wave diffusion.

A brief history of statistical theories based on the Zakharov equations was presented in Sec. V.A. In the following discussion we limit ourselves to a description of the strong-Langmuir-turbulence theory developed by Galeev *et al.* (1975,1976,1977a,1977b,1977c), after the introduction of seminal ideas by Kingsep, Rudakov, and Sudan (1973). Our treatment is based on the corrected and refined version of this theory developed by Pelletier (1982).

The model is illustrated in Fig. 18. The basic structures are self-similarly collapsing Langmuir envelope wave packets, rather than plane waves. Thus a high degree of phase coherence is built in from the beginning, in contrast to the usual weak turbulence theories, which rely on the assumption of randomly phase plane waves. It is important to stress that it is largely an article of faith, based on some experience with numerical solutions of the Zakharov equations, that collapsing wave packets tend towards a self-similar form (i.e., that the self-similar solutions are "attractors").

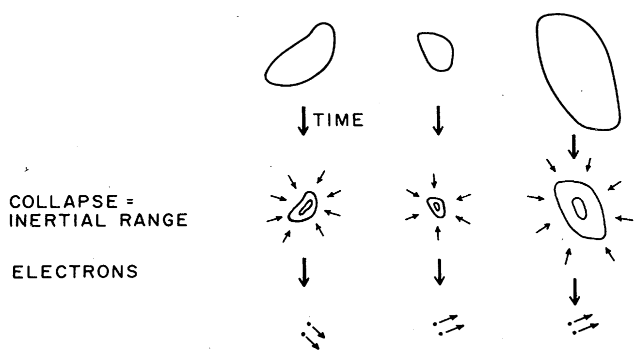

FIG. 18. Ideal gas of self-similarly collapsing Langmuir wave packets which surrender their energy to electrons at short scales. This is the model of strong Langmuir turbulence due to Galeev *et al.* (1975).

Additional assumptions are made in the original version of this theory: Weakly turbulent Langmuir plane waves are neglected altogether. The interaction between collapsing wave packets, which may proceed by the emission and absorption of ion-acoustic sound waves, is likewise neglected, as are possible effects of radiated ion-acoustic waves. Thus it is assumed we have an ideal gas of such self-similar wave packets. (Some of these restrictions are lifted in later work.)

Dissipation of wave energy is provided by quasilinear coupling to electrons at short scale sizes. That is, when a wave packet has collapsed to small scales (large k), it can surrender its energy to relatively low-velocity electrons ($v \approx \omega_p/k$). This transfer of energy is probably more correctly described as transit-time damping of the Langmuir wave packet by the local electron distribution function. However, in the statistical description considered here, we use Landau damping by the average distribution function. The trajectories of these electrons are permitted to diffuse in the collection of Langmuir wave packets, so the resulting self-consistent electron distribution function will be non-Maxwellian.

In the original work of Galeev *et al.* (1975), the turbulent "inertial" range arose from the dynamical process of free collapse. Energy was injected at large scales and extracted at short scales, but, for most of the range of collapse, sources and sinks of energy were assumed to be negligible. This is very much in the spirit of the concept of inertial range familiar from fluids: the dynamics in the inertial range is that of a conservative system (i.e., the Zakharov equations, with $\gamma_k = 0$ and with no inhomogeneous source term). Pelletier showed how to build the effects of dissipation into the self-similar solutions in a self-consistent manner, and this is the version we shall describe.

One virtue of the theory is that the underlying dynamical collapse can be taken to be three dimensional. In the resulting kinetic equations for the spectral function W_k of Langmuir waves and quasilinearly coupled electrons, a statistically isotropic and homogeneous steady state is sought and found. The spectrum is illustrated qualitatively in Fig. 19. An unspecified driver injects energy at large scales (small k). Modulationally unstable wave-packet collapse gives rise to an inertial range in which the spectrum falls as $k^{-5/2}$, and at large k, quasilinear Landau damping to electrons reduces the spectrum still further to a $k^{-7/2}$ shape, while the self-consistent electron distribution function is found to go as $v^{-7/2}$. These results are modified for realizations of this model (such as simulations) in spatial dimensions other than three.

Based on an isotropic supersonic self-similar solution to

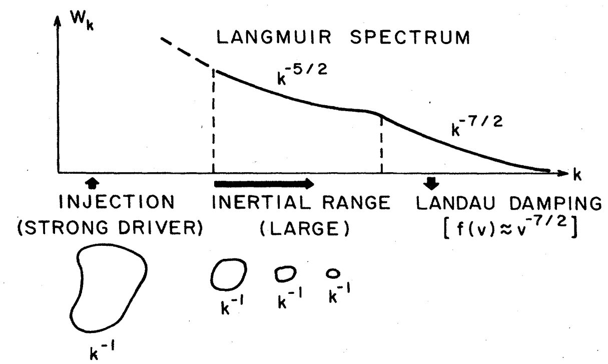

FIG. 19. Langmuir spectrum W_k, given by turbulence theory depicted in Fig. 18. Collapse to short scales corresponds to spectral transfer to large k. This spectrum is the solution to the kinetic equations (8.6) and (8.7).

the Zakharov equations, (5.8) and (5.9), the square modulus of the Langmuir wave envelope is given by

$$|\mathscr{E}|^2=\varepsilon\xi^{-3}F(r/\xi)\ , \tag{8.1}$$

where the time-dependent scale size ξ and the time-dependent wave-packet energy ϵ satisfy the following first-order (autonomous) ordinary differential equations in time:

$$\partial_t\xi \approx -\nu(\xi,\varepsilon)\xi\ , \tag{8.2}$$

$$\partial_t\varepsilon \approx -2\gamma(\xi^{-1},\varepsilon)\varepsilon\ . \tag{8.3a}$$

Here ν has the interpretation of the "collapse rate" and is given analytically (Pelletier, 1982) by

$$\nu=(\varepsilon/\xi^3)^{1/2}\ . \tag{8.3b}$$

The damping rate $\gamma(\xi^{-1},\varepsilon)$ is essentially the quasilinear Landau damping rate as a function of the scale size ξ (corresponding to wave number) and the wave energy ε (corresponding to a wave energy dependence in the quasilinear diffusion coefficient).

This form of self-similar solution can be understood as part of the solution of the Zakharov equations (5.8) and (5.9) in the following sense: The function F is not arbitrary, but must satisfy a time-independent equation based on (5.8), with the same boundary conditions as (5.8). The self-similar form of the density cavity, δn, has been suppressed, but it is a solution to (5.9) in the asymptotically supersonic limit, in which ∂_t^2 dominates ∇^2. A new set of dimensionless units has been adopted, in which C_s has been absorbed.

Up until this point we have merely characterized a particular (self-similar) dynamical solution to the Zakharov equation. A statistical hypothesis is introduced as follows: A noninteracting ensemble of self-similar solutions with a distribution of sizes and energies is assumed to exist and to have a spatial density $n_{\rm ss}$ set by the driven modulational instability. That is, we focus our attention on one self-similar solution, and allow ξ and ε to be random variables. Their distribution is governed by a probability density $\rho(\varepsilon,\xi)$, which obeys the following Liouville equation in the ξ-ε phase space:

$$\partial_t\rho=\partial_\xi(\rho\nu\xi)+\partial_\varepsilon(\rho\epsilon 2\gamma)\ . \tag{8.4}$$

This equation can easily be derived by letting $\rho=\delta[\varepsilon-\varepsilon(t)]\delta[\xi-\xi(t)]$, where $\varepsilon(t)$ and $\xi(t)$ satisfy Eqs. (8.3). The function ρ can now be used to perform averages over quantities which depend on the energy and the scale size. The quantity of most interest to us will be the *spectral* energy W_k. It is therefore desirable to introduce the spectral shape of a self-similar solution, which we define as $\xi g(\mathbf{k}\xi)$. The spectral energy W_k may then be defined by an appropriate average energy ε:

$$W_k \propto n_{\rm ss}\int d\varepsilon\, d\xi\, \varepsilon\rho(\varepsilon,\xi)[\xi g(\mathbf{k}\xi)]\ . \tag{8.5}$$

Likewise, we may define $\gamma_k(W_k)$ in terms of $\gamma(\xi^{-1},\varepsilon)$ and $\nu_k(W_k)$ in terms of $\nu(\xi,\varepsilon)$, by averaging over $\rho\xi g$. A kinetic equation is derived by taking the time derivative of Eq. (8.5) and using the Liouville equation (8.4). [Note, that ρ is the only time-dependent term on the right side of (8.5).] The resulting kinetic equation is

$$\partial_t W_k=-\partial_k[(ak^5W_k)W_k]-2\gamma_k W_k\ . \tag{8.6}$$

The first term on the right is the k-space divergence of the power flow which corresponds to collapse (see Fig. 19). The term in parentheses is the rate of change of k with time, corresponding to $\nu_k k$.

Let us first consider the inertial-range steady state, in which $\partial_t W_k$, the damping term, and any driver term [which may be added to the right side of Eq. (8.6)], are all set equal to zero. The vanishing of the divergence of the power flow then gives us

$$W_k=k^{-5/2}\ .$$

This is the k dependence of the spectrum in the inertial range. The argument here is very familiar from Navier-Stokes turbulence, where it is called the Kolmogorov hypothesis.

Next, let us consider the dissipative range, where the steady-state spectrum is determined from a balance of the two terms on the right side of Eq. (8.6), that is, from the balance of power flow with dissipation. How is γ_k determined? As in Eq. (2.3), it is a function of the electron velocity distribution $f(\mathbf{v})$. In the present theory, f is taken to be the average distribution function. An equivalent form is

$$\gamma_k \approx \int d\mathbf{v}\,\mathbf{k}\cdot\partial_{\mathbf{v}}f\delta(\omega_p-\mathbf{k}\cdot\mathbf{v})\ . \tag{8.7a}$$

We assume the distribution function $f(\mathbf{v})$ undergoes quasilinear velocity space diffusion governed by a diffusion coefficient $D_{\mathbf{v}}$, although this assumption may be difficult to support if the local electron distribution function is strongly modified in a single "transit-time" interaction with a collapsing wave packet:

$$\partial_t f=a_{\mathbf{v}}D_{\mathbf{v}}\partial_{\mathbf{v}}f\ , \tag{8.7b}$$

$$D \approx \int d\mathbf{k}\, W_k\delta(\omega_p-\mathbf{k}\cdot\mathbf{v})\hat{\mathbf{k}}\hat{\mathbf{k}}\ . \tag{8.7c}$$

These equations appear identical to the usual quasilinear equations (Sec. II.B). However, we should bear in mind that the spectral energy density W_k, which governs the electron velocity-space diffusion, must be a self-consistent solution to the kinetic equation (8.6). The diffusion is therefore physically off an ensemble of collapsing wave packets whose average plane-wave decomposition is given by W_k, rather than off randomly phased plane waves. A steady-state solution of Eqs. (8.6) and (8.7) can be found by assuming isotropy in **v** space and **k** space, and looking for power-law solutions. The result, generalized to arbitrary dimension D, is

$$W_k \approx k^{2+D/2}, \quad f(v) \approx v^{3D/2-1} .$$

It is easy to show further that the Landau damping rate γ_k is equal to the collapse rate ν_k, so that a collapsing wave packet vanishes due to dissipation in precisely the same time that it takes to collapse. The collapse time is found to be finite in three dimensions, but in two dimensions the scale size ξ decreases exponentially with time.

This model has a number of limitations besides the obvious one that the premise of an ideal gas of self-similarly collapsing wave packets is an unproven assumption. Pelletier points out that at large k the condition for strong turbulence [Eqs. (3.12) or (5.1)] is violated, so the dissipative range might include weak-turbulence effects.

A more serious shortcoming has to do with the effect of ion-acoustic waves on the large-scale Langmuir waves. Ion-acoustic waves can be generated in several ways. We have seen in Fig. 17(c) that ion-acoustic waves are produced in the early weak-turbulence evolution of beam-driven Langmuir waves. Another source may be the density cavities left after a collapsing Langmuir wave has burnt out due to Landau damping.

When the density cavity that had trapped the packet is no longer present to supply the ponderomotive force which had been self-consistently maintaining the cavity, the latter is free to radiate away all or part of its energy in the form of ion-acoustic waves (Sec. II). This is evident from the Zakharov equation (7.1b). When the right side is zero, the solution to the problem of the evolution of an initial cavity δn will always involve the generation of ion-acoustic waves with a short wavelength, commensurate with the small collapsed cavity size.

The effect of the ion-acoustic waves depends on several factors, including the number of spatial dimensions and the temperature of electrons relative to ions. If the electron and ion temperature in the plasma are equal, these waves will quickly dissipate away [i.e., ν_i in Eq. (7.1b) is large—see discussion of Sec. II]. When the electron temperature is high compared to the ion temperature, however, the short-scale ion-acoustic waves persist and may cause the anomalous absorption of long-scale Langmuir waves by a nonresonant conversion process. In this process, the long-wavelength Langmuir waves scatter off the short-scale ion-acoustic turbulence, and are converted into short-wavelength Langmuir waves which are strongly Landau damped by electrons (in the dissipative range).

One-dimensional numerical solutions of the Zakharov equations for the case $T_e \gg T_i$ have shown the generation of intense ion-acoustic turbulence, which has a profound effect on the evolution of the high-frequency Langmuir waves (Sagdeev and Khodryev, 1977; Galeev *et al.*, 1975,1976,1977a,1977b,1977c; Degtyarev *et al.*, 1980).

Another possible effect of ion-acoustic turbulence involves the formation of a "sea" of cavities capable of driving Langmuir wave packets localized to their own (short) scale length. Doolen *et al.* (1984) have recently claimed that long-time *one*-dimensional simulations of plasmas driven near $k=0$ do not agree with the predictions of the "independent-self-similar-collapse" statistical theory of Galeev *et al.* (1975) and Pelletier (1982). This is because trapped Langmuir wave fields renucleate in cavities whose (small) spatial scale is nearly the size at which strong dissipation occurs by Landau damping. Thus "solitons" are reborn near the dissipation subrange, and the well-developed inertial range required for self-similar collapse is not observed. (At long scale sizes a flat Langmuir spectrum is found numerically and thought to be associated with equipartion of mode energy.)

Finally we remark that the very recent pulsed-beam experiments on three-dimensional collapse by Wong and Cheung (1984) do not appear to show burnout or significant ion-acoustic turbulence. A single Langmuir wave packet forms and collapses supersonically in a manner consistent with supersonic self-similar collapse (in the radial direction).

It is quite likely that there are many different parameter regimes for strong Langmuir turbulence. The applicability of the statistical model described in this section remains to be demonstrated, either by comparison with higher-dimensional long-time numerical simulations or in terms of real experiments exhibiting many collapsing wave packets.

IX. CONCLUSIONS

In this review, we have attempted to show, on a heuristic level, some of the physics, mathematics, and philosophy that go into a particular set of nonlinear phenomena involving Langmuir waves in a nonmagnetic plasma. We have explored a number of concepts relevant to dynamical and statistical models of strong Langmuir turbulence. Included among the physical processes treated were wave modulational instability, self-focusing of wave packets and of self-consistent density cavities, wave cascade down to a low-k condensate, and driving and dissipation due to coupling of waves to particles.

Developments in this subject area appear to be accelerating, with a number of recent experiments and computer simulations demonstrating the existence of the basic physical phenomena of modulational instability, ponderomotive-force-induced cavities and localized wave packets, saturation of bump-on-tail beam instability by nonlinear wave interactions, and particle tail formation by collapsing Langmuir wave packets.

However, there are still a number of open questions which provide rich material for theoretical and experimental research.

There have been no simulations on numerical solutions of the Zakharov equation dynamical model for driven waves in three dimensions, although there are indications that there may be important differences between one, two, and three dimensions. Even in two dimensions the fine grid spacing and short time steps required for numerical analysis provide severe restrictions and lead to the consumption of hours of Cray-1 time. Two-dimensional particle-in-cell simulations which show strong-turbulence effects have not been performed in the presence of a driver, such as an electron beam. [However, very recent preliminary results by Russell (1984) appear to show collapse driven by a spatially uniform Langmuir field in a 2D particle-in-cell simulation.] Particle-in-cell simulations have also not been able to treat the important case in which a number of wave packets are simultaneously self-focusing, so that one could study their interaction, density, etc.

Statistical theories based on the Zakharov equations are beginning to emerge, but many assumptions have not been justified. In the model of Galeev *et al.* (1975), the assumed shape and independence of collapsing wave packets have not been fully justified, and the role of a driver, of ion-acoustic waves, and of cavities in the evolution of the Langmuir turbulence is poorly understood. Other statistical theories, such as the direct-interaction approximation (DuBois and Rose, 1981; DuBois, Rose, and Nicholson, 1984) have shown qualitative agreement with numerical studies of modulational instability in one dimension, but predictions for higher dimensions have not yet been extracted. Neither statistical theory has as yet been applied successfully to laboratory (or space) experiments. The large parameter space for strong Langmuir turbulence suggests that there may be room for more than one statistical model, depending on the strength, structure, and duration of the driver and on the background plasma conditions.

Finally, in the case of experiment, many of the effects need to be measured in more quantitative detail, to enable checks with theory. Further laboratory study of two- and three-dimensional wave evolution in a homogeneous, weakly driven plasma would be very desirable. The dynamical spatial evolution of nonlinear wave packets should be probed directly as a function of time, in a time-asymptotic regime. In particular, the existence of self-similar collapse should be examined, and the criteria for distinguishing between weak- and strong-turbulence saturation of beam instabilities should be established. In the case of spacecraft observations of Langmuir wave turbulence in the solar wind, the single most important development would be the ability to make absolute short-scale measurements of the Langmuir wave spectrum.

ACKNOWLEDGMENTS

The author wishes to thank D. F. DuBois, G. Pelletier, D. Forslund, J. Weatherall, D. Nicholson, B. Hafizi, L. Muschietti, D. Russell, H. T. Moon, R. Sudan, and A. Rubenchik for valuable discussions. This work was supported by the Air Force Office of Scientific Research, the National Science Foundation, the National Aeronautics and Space Administration, and the National Center for Atmospheric Research.

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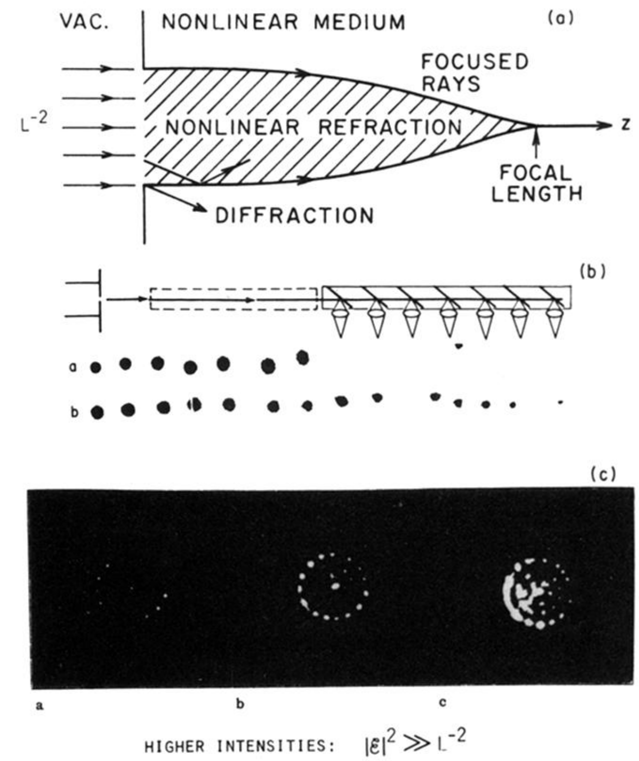

FIG. 2. Nonlinear optics realizations of self-focusing and modulational instability. (a) Sketch of configuration for spatial self-focusing of a laser beam discussed in text, when $|\mathscr{E}|^2 \gtrsim L^{-2}$ (L is the aperture width governing diffraction). (b) Apparatus and sequence of observed cross sections of laser beam. *a*, linear medium gives diffraction. *b*, nonlinear medium shows self-focusing (Garmire *et al.,* 1966). (c) Modulational instability of Fresnel diffraction pattern at higher intensities $|\mathscr{E}|^2 \gg L^{-2}$. Azimuthal breakup scale size decreases with intensity in accordance with Eq. (3.6) (Campillo *et al.,* 1973).

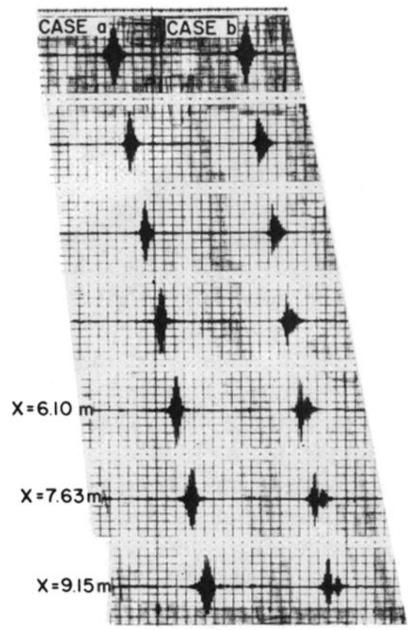

FIG. 3. Deep-water wave realizations of soliton and modulational instability. Spatial propagation of an initial wave packet is traced in the vertical sequence of profiles, starting at the top. Case *a*, initial pulse with soliton profile ($|\mathscr{E}|^2 \approx L^{-2}$) Eq. (3.10) retains shape. Case *b*, when $|\mathscr{E}|^2 = 2L^{-2}$, packet breaks up due to modulational instability (Yuen and Lake, 1975).