# On stars, their evolution and their stability

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## 1. INTRODUCTION

When we think of atoms, we have a clear picture in our minds: a central nucleus and a swarm of electrons surrounding it. We conceive them as small objects of sizes measured in angstroms ($\sim 10^{-8}$ cm); and we know that some hundred different species of them exist. This picture is, of course, quantified and made precise in modern quantum theory. And the success of the entire theory may be traced to two basic facts: *first,* the Bohr radius of the ground state of the hydrogen atom, namely,

$$\frac{h^2}{4\pi^2 me^2} \sim 0.5 \times 10^{-8} \text{ cm} , \tag{1}$$

where $h$ is Planck's constant, $m$ is the mass of the electron and $e$ is its charge, provides a correct measure of atomic dimensions; and *second,* the reciprocal of *Sommerfeld's fine-structure constant,*

$$\frac{hc}{2\pi e^2} \sim 137 , \tag{2}$$

gives the maximum positive charge of the central nucleus that will allow a stable electron-orbit around it. This maximum charge for the central nucleus arises from the effects of special relativity on the motions of the orbiting electrons.

We now ask: can we understand the basic facts concerning stars as simply as we understand atoms in terms of the two combinations of natural constants (1) and (2). In this lecture, I shall attempt to show that in a limited sense we can.

The most important fact concerning a star is its mass. It is measured in units of the mass of the sun, $\odot$, which is $2 \times 10^{33}$ g: stars with masses very much less than, or very much more than, the mass of the sun are relatively infrequent. The current theories of stellar structure and stellar evolution derive their successes largely from the fact that the following combination of the dimensions of a mass provides a correct measure of stellar masses:

$$\left[\frac{hc}{G}\right]^{3/2} \frac{1}{H^2} \simeq 29.2\odot , \tag{3}$$

where $G$ is the constant of gravitation and $H$ is the mass of the hydrogen atom. In the first half of the lecture, I shall essentially be concerned with the question: how does this come about?

## 2. THE ROLE OF RADIATION PRESSURE

A central fact concerning normal stars is the role which radiation pressure plays as a factor in their hydrostatic equilibrium. Precisely the equation governing the hydrostatic equilibrium of a star is

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho , \tag{4}$$

where $P$ denotes the total pressure, $\rho$ the density, and $M(r)$ is the mass interior to a sphere of radius $r$. There are two contributions to the total pressure $P$: that due to the material and that due to the radiation. On the assumption that the matter is in the state of a perfect gas in the classical Maxwellian sense, the material or the gas pressure is given by

$$p_{\text{gas}} = \frac{k}{\mu H}\rho T , \tag{5}$$

where $T$ is the absolute temperature, $k$ is the Boltzmann constant, and $\mu$ is the mean molecular weight (which under normal stellar conditions is $\sim 1.0$). The pressure due to radiation is given by

$$p_{\text{rad}} = \tfrac{1}{3}aT^4 , \tag{6}$$

where $a$ denotes Stefan's radiation-constant. Consequently, if radiation contributes a fraction $(1-\beta)$ to the total pressure, we may write

$$P = \frac{1}{1-\beta}\frac{1}{3}aT^4 = \frac{1}{\beta}\frac{k}{\mu H}\rho T . \tag{7}$$

To bring out explicitly the role of the radiation pressure in the equilibrium of a star, we may eliminate the temperature, $T$, from the foregoing equations and express $P$ in terms of $\rho$ and $\beta$ instead of in terms of $\rho$ and $T$. We find:

$$T = \left[\frac{k}{\mu H}\frac{3}{a}\frac{1-\beta}{\beta}\right]^{1/3}\rho^{1/3} \tag{8}$$

and

$$P = \left[\left[\frac{k}{\mu H}\right]^4 \frac{3}{a}\frac{1-\beta}{\beta^4}\right]^{1/3}\rho^{4/3} = C(\beta)\rho^{4/3} \tag{9}$$

(say).

The importance of this ratio, $(1-\beta)$, for the theory of stellar structure was first emphasized by Eddington. Indeed, he related it, in a famous passage in his book on *The Internal Constitution of the Stars,* to the "happening of the stars" (Eddington, 1926, p. 16). A more rational version of Eddington's argument which, at the same time, isolates the combination (3) of the natural constants is the following:

There is a general theorem (Chandrasekhar, 1936) which states that the pressure, $P_c$, at the centre of a star of a mass $M$ in hydrostatic equilibrium in which the density, $\rho(r)$, at a point at a radial distance, $r$, from the centre does not exceed the mean density, $\bar{\rho}(r)$, interior to the same point $r$, must satisfy the inequality,

$$\tfrac{1}{2}G(\tfrac{4}{3}\pi)^{1/3}\bar{\rho}^{4/3}M^{2/3} \le P_c \le \tfrac{1}{2}G(\tfrac{4}{3}\pi)^{1/3}\rho_c^{4/3}M^{2/3}\,, \tag{10}$$

where $\bar{\rho}$ denotes the mean density of the star and $\rho_c$ its density at the centre. The content of the theorem is no more than the assertion that the actual pressure at the centre of a star must be intermediate between those at the centres of the two configurations of uniform density, one at a density equal to the mean density of the star, and the other at a density equal to the density $\rho_c$ at the centre (see Fig. 1). If the inequality (10) should be violated then there must, in general, be some regions in which adverse density gradients must prevail; and this implies instability. In other words, we may consider conformity with the inequality (10) as equivalent to the condition for the stable existence of stars.

The right-hand side of the inequality (10) together with $P$ given by Eq. (9), yields, for the stable existence of stars, the condition,

$$\left[\left[\frac{k}{\mu H}\right]^4 \frac{3}{a}\,\frac{1-\beta_c}{\beta_c^4}\right]^{1/3} \le \left[\frac{\pi}{6}\right]^{1/3} G M^{2/3}\,, \tag{11}$$

or, equivalently,

$$M \ge \left[\frac{6}{\pi}\right]^{1/2}\left[\left[\frac{k}{\mu H}\right]^4 \frac{3}{a}\,\frac{1-\beta_c}{\beta_c^4}\right]^{1/2}\frac{1}{G^{3/2}}\,, \tag{12}$$

where in the foregoing inequalities, $\beta_c$ is a value of $\beta$ at the centre of the star. Now Stefan's constant, $a$, by virtue of Planck's law, has the value

$$a = \frac{8\pi^5 k^4}{15 h^3 c^3}\,. \tag{13}$$

Inserting this value $a$ in the equality (12) we obtain

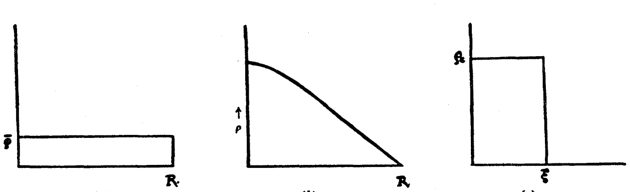


FIG. 1. A comparison of an inhomogeneous distribution of density in a star (b) with the two homogeneous configurations with the constant density equal to the mean density (a) and equal to the density at the centre (c).

$$\mu^2 M\left[\frac{\beta_c^4}{1-\beta_c}\right]^{1/2} \ge \frac{(135)^{1/2}}{2\pi^3}\left[\frac{hc}{G}\right]^{3/2}\frac{1}{H^2} = 0.1873\left[\frac{hc}{G}\right]^{3/2}\frac{1}{H^2}\,. \tag{14}$$

We observe that the inequality (14) has isolated the combination (3) of natural constants of the dimensions of a mass; by inserting its numerical value given in Eq. (3), we obtain the inequality,

$$\mu^2 M\left[\frac{\beta_c^4}{1-\beta_c}\right]^{1/2} \ge 5.48\odot\,. \tag{15}$$

This inequality provides an upper limit to $(1-\beta_c)$ for a star of a given mass. Thus,

$$1-\beta_c \le 1-\beta_*\,, \tag{16}$$

where $(1-\beta_*)$ is uniquely determined by the mass $M$ of the star and the mean molecular weight, $\mu$, by the quartic equation,

$$\mu^2 M = 5.48\left[\frac{1-\beta_*}{\beta_*^4}\right]^{1/2}\odot\,. \tag{17}$$

In Table I, we list the values of $1-\beta_*$ for several values of $\mu^2 M$. From this table it follows in particular, that for a star of solar mass with a mean molecular weight equal to 1, the radiation pressure at the centre cannot exceed 3% of the total pressure.

What do we conclude from the foregoing calculation? We conclude that to the extent Eq. (17) is at the base of the equilibrium of actual stars, to that extent the combination of natural constants (3), providing a mass of proper magnitude for the measurement of stellar masses, is at the base of a physical theory of stellar structure.

## 3. DO STARS HAVE ENOUGH ENERGY TO COOL?

The same combination of natural constants (3) emerged soon afterward in a much more fundamental context of resolving a paradox Eddington had formulated in the form of an aphorism: "a star will need energy to cool." The paradox arose while considering the ultimate fate of a gaseous star in the light of the then new knowledge that white-dwarf stars, such as the companion of Sirius, exist, which have mean densities in the range $10^5$–$10^7$ g cm$^{-3}$. As Eddington stated

TABLE I. The maximum radiation pressure, $(1-\beta_*)$, at the centre of a star of a given mass, $M$.

| $1-\beta_*$ | $M\mu^2/\odot$ | $1-\beta_*$ | $M\mu^2/\odot$ |
|---|---|---|---|
| 0.01 | 0.56 | 0.50 | 15.49 |
| 0.03 | 1.01 | 0.60 | 26.52 |
| 0.10 | 2.14 | 0.70 | 50.92 |
| 0.20 | 3.83 | 0.80 | 122.5 |
| 0.30 | 6.12 | 0.85 | 224.4 |
| 0.40 | 9.62 | 0.90 | 519.6 |

> I do not see how a star which has once got into this compressed state is ever going to get out of it.... It would seem that the star will be in an awkward predicament when its supply of subatomic energy fails (Eddington, 1926, p. 172).

The paradox posed by Eddington was reformulated in clearer physical terms by R. H. Fowler (1926). His formulation was the following:

> The stellar material, in the white-dwarf state, will have radiated so much energy that it has less energy than the same matter in normal atoms expanded at the absolute zero of temperature. If part of it were removed from the star and the pressure taken off, what could it do?

Quantitatively, Fowler's question arises in this way.

An estimate of the electrostatic energy, $E_V$, per unit volume of an assembly of atoms, of atomic number $Z$, ionized down to bare nuclei, is given by

$$E_V = 1.32\times 10^{11} Z^2 \rho^{4/3} , \tag{18}$$

while the kinetic energy of thermal motions, $E_{\rm kin}$, per unit volume of free particles in the form of a perfect gas of density, $\rho$, and temperature, $T$, is given by

$$E_{\rm kin} = \frac{3}{2}\frac{k}{\mu H}\rho T = \frac{1.24\times 10^8}{\mu}\rho T . \tag{19}$$

Now if such matter were released of the pressure to which it is subject, it can resume a state of ordinary normal atoms only if

$$E_{\rm kin} > E_V , \tag{20}$$

or, according to Eqs. (18) and (19), only if

$$\rho < \left[ 0.94\times 10^{-3} \frac{T}{\mu Z^2} \right]^3 . \tag{21}$$

This inequality will be clearly violated if the density is sufficiently high. This is the essence of Eddington's paradox as formulated by Fowler. And Fowler resolved this paradox in 1926 in a paper entitled "Dense Matter"—one of the great landmark papers in the realm of stellar structure: in it the notions of Fermi statistics and of electron degeneracy are introduced for the first time.

## 4. FOWLER'S RESOLUTION OF EDDINGTON'S PARADOX; THE DEGENERACY OF THE ELECTRONS IN WHITE-DWARF STARS

In a completely degenerate electron gas all the available parts of the phase space, with momenta less than a certain "threshold" value $p_0$—the Fermi threshold—are occupied consistently with the Pauli exclusion-principle, i.e., with two electrons per "cell" of volume $h^3$ of the six-dimensional phase space. Therefore, if $n(p)dp$ denotes the number of electrons, per unit volume, between $p$ and $p+dp$, then the assumption of complete degeneracy is equivalent to the assertion,

$$n(p) = \frac{8\pi}{h^3}p^2 \quad (p \le p_0) ,$$
$$= 0 \quad (p > p_0) . \tag{22}$$

The value of the threshold momentum, $p_0$, is determined by the normalization condition

$$n = \int_0^{p_0} n(p)dp = \frac{8\pi}{3h^3}p_0^3 , \tag{23}$$

where $n$ denotes the total number of electrons per unit volume.

For the distribution given by (22), the pressure $p$ and the kinetic energy $E_{\rm kin}$ of the electrons (per unit volume), are given by

$$P = \frac{8\pi}{3h^3}\int_0^{p_0} p^3 v_p dp \tag{24}$$

and

$$E_{\rm kin} = \frac{8\pi}{h^3}\int_0^{p_0} p^2 T_p dp , \tag{25}$$

where $v_p$ and $T_p$ are the velocity and the kinetic energy of an electron having a momentum $p$.

If we set

$$v_p = p/m \quad \text{and} \quad T_p = p^2/2m , \tag{26}$$

appropriate for non-relativistic mechanics, in Eqs. (24) and (25), we find

$$P = \frac{8\pi}{15h^3 m}p_0^5 = \frac{1}{20}\left[\frac{3}{\pi}\right]^{2/3}\frac{h^2}{m}n^{5/3} \tag{27}$$

and

$$E_{\rm kin} = \frac{8\pi}{10h^3 m}p_0^5 = \frac{3}{40}\left[\frac{3}{\pi}\right]^{2/3}\frac{h^2}{m}n^{5/3} . \tag{28}$$

Fowler's resolution of Eddington's paradox consists in this: at the temperatures and densities that may be expected to prevail in the interiors of the white-dwarf stars, the electrons will be highly degenerate and $E_{\rm kin}$ must be evaluated in accordance with Eq. (28) and *not* in accordance with Eq. (19); and Eq. (28) gives,

$$E_{\rm kin} = 1.39\times 10^{13}(\rho/\mu)^{5/3} . \tag{29}$$

Comparing now the two estimates (18) and (29), we see that, for matter of the density occurring in the white dwarfs, namely $\rho \sim 10^5$ g cm$^{-3}$, the total kinetic energy is about two to four times the negative potential-energy; and Eddington's paradox does not arise. Fowler concluded his paper with the following highly perceptive statement:

> The black-dwarf material is best likened to a single gigantic molecule in its lowest quantum state. On the Fermi-Dirac statistics, its high density can be achieved in one and only one way, in virtue of a correspondingly great energy content. But this energy can no more be expended in radiation than the energy of a normal atom or molecule. The only difference between black-dwarf matter and a normal molecule is that the molecule can

exist in a free state while the black-dwarf matter can only so exist under very high external pressure.

## 5. THE THEORY OF THE WHITE-DWARF STARS; THE LIMITING MASS

The internal energy ($=3P/2$) of a degenerate electron gas that is associated with a pressure $P$ is *zero-point energy;* and the essential content of Fowler's paper is that this zero-point energy is so great that we may expect a star to eventually settle down to a state in which all of its energy is of this kind. Fowler's argument can be more explicitly formulated in the following manner (Chandrasekhar, 1931a).

According to the expression for the pressure given by Eq. (27), we have the relation,

$$P=K_1\rho^{5/3} \quad \text{where} \quad K_1=\frac{1}{20}\left[\frac{3}{\pi}\right]^{2/3}\frac{h^2}{m(\mu_e H)^{5/3}}\,, \tag{30}$$

where $\mu_e$ is the mean molecular weight per electron. An equilibrium configuration in which the pressure, $P$, and the density, $\rho$, are related in the manner,

$$P=K\rho^{1+1/n}\,, \tag{31}$$

is an *Emden polytrope* of index $n$. The degenerate configurations built on the equation of state (30) are therefore polytropes of index 3/2; and the theory of polytropes immediately provides the relation,

$$K_1=0.4242(GM^{1/3}R) \tag{32}$$

or, numerically, for $K_1$ given by Eq. (30),

$$\log_{10}(R/R_\odot)=-\tfrac{1}{3}\log_{10}(M/\odot)-\tfrac{5}{3}\log_{10}\mu_e-1.397\,. \tag{33}$$

For a mass equal to the solar mass and $\mu_e=2$, the relation (33) predicts $R=1.26\times10^{-2}R_\odot$ and a mean density of $7.0\times10^5$ g cm$^3$. These values are precisely of the order of the radii and mean densities encountered in white-dwarf stars. Moreover, according to Eqs. (32) and (33), the radius of the white-dwarf configuration is inversely proportional to the cube-root of the mass. On this account, finite equilibrium configurations are predicted for all masses. And it came to be accepted that the white dwarfs represent the last stages in the evolution of all stars.

But it soon became clear that the foregoing simple theory based on Fowler's premises required modifications. For, the electrons at their threshold energies, at the centres of the degenerate stars, begin to have velocities comparable to that of light as the mass increases. Thus, already for a degenerate star of solar mass (with $\mu_e=2$) the central density (which is about six times the mean density) is $4.19\times10^6$ g cm$^3$; and this density corresponds to a threshold momentum $p_0=1.29mc$ and a velocity which is $0.63c$. Consequently, the equation of state must be modified to take into account the effects of special relativity. And this is easily done by inserting in Eqs. (24) and (25) the relations,

$$v_p=\frac{p}{m(1+p^2/m^2c^2)^{1/2}}$$

and (34)

$$T_p=mc^2[(1+p^2/m^2c^2)^{1/2}-1]\,,$$

in place of the non-relativistic relations (26). We find that the resulting equation of state can be expressed, parametrically, in the form

$$P=Af(x) \quad \text{and} \quad \rho=Bx^3\,, \tag{35}$$

where

$$A=\frac{\pi m^4c^5}{3h^3},\quad B=\frac{8\pi m^3c^3\mu_e H}{3h^3} \tag{36}$$

and

$$f(x)=x(x^2+1)^{1/2}(2x^2-3)+3\sinh^{-1}x\,. \tag{37}$$

And similarly

$$E_{\text{kin}}=Ag(x)\,, \tag{38}$$

where

$$g(x)=8x^3[(x^2+1)^{1/2}-1]-f(x)\,. \tag{39}$$

According to Eqs. (35) and (36), the pressure approximates the relation (30) for low enough electron concentrations ($x\ll1$); but for increasing electron concentrations ($x\gg1$), the pressure tends to (Chandrasekhar, 1931b)

$$P=\frac{1}{8}\left[\frac{3}{\pi}\right]^{1/3}hcn^{4/3}\,. \tag{40}$$

This limiting form of relation can be obtained very simply by setting $v_p=c$ in Eq. (24); then

$$P=\frac{8\pi c}{3h^3}\int_0^{p_0}p^3dp=\frac{2\pi c}{3h^3}p_0^4\,; \tag{41}$$

and the elimination of $p_0$ with the aid of Eq. (23) directly leads to Eq. (40).

While the modification of the equation of state required by the special theory of relativity appears harmless enough, it has, as we shall presently show, a dramatic effect on the predicted mass-radius relation for degenerate configurations.

The relation between $P$ and $\rho$ corresponding to the limiting form (41) is

$$P=K_2\rho^{4/3} \quad \text{where} \quad K_2=\frac{1}{8}\left[\frac{3}{\pi}\right]^{1/3}\frac{hc}{(\mu_e H)^{4/3}}\,. \tag{42}$$

In this limit, the configuration is an Emden polytrope of index 3. And it is well known that when the polytropic index is 3, the mass of the resulting equilibrium configuration is uniquely determined by the constant of proportionality, $K_2$, in the pressure-density relation. We have accordingly,

$$M_{\text{limit}}=4\pi\left[\frac{K_2}{\pi G}\right]^{3/2}(2.018)$$
$$=0.197\left[\frac{hc}{G}\right]^{3/2}\frac{1}{(\mu_e H)^2}=5.76\mu_e^{-2}\odot\,. \quad (43)$$

[In Eq. (43), 2.018 is a numerical constant derived from the explicit solution of the Lane-Emden equation for $n=3$.]

It is clear from general considerations (Chandrasekhar, 1931c) that *the exact mass-radius relation for the degenerate configurations must provide an upper limit to the mass of such configurations given by Eq. (43); and further, that the mean density of the configuration must tend to infinity,* while the radius tends to zero, and $M\rightarrow M_{\text{limit}}$. These conditions, straightforward as they are, can be established directly by considering the equilibrium of configurations built on the exact equation of state given by Eqs. (35)—(37). It is found that the equation governing the equilibrium of such configurations can be reduced to the form (Chandrasekhar, 1934b, 1935)

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left[\eta^2\frac{d\phi}{d\eta}\right]=-\left[\phi^2-\frac{1}{y_0^2}\right]^{3/2}, \quad (44)$$

where

$$y_0^2=x_0^2+1\,, \quad (45)$$

and $mcx_0$ denotes the threshold momentum of the electrons at the centre of the configuration and $\eta$ measures the radial distance in the unit

$$\left[\frac{2A}{\pi G}\right]^{1/2}\frac{1}{By_0}=l_1 y_0^{-1} \quad (46)$$

(say).

By integrating Eq. (44), with suitable boundary conditions and for various initially prescribed values of $y_0$, we can derive the exact mass-radius relation, as well as the other equilibrium properties, of the degenerate configurations. The principal results of such calculations are illustrated in Figs. 2 and 3.

The important conclusions which follow from the foregoing considerations are: *first,* there is an upper limit, $M_{\text{limit}}$, to the mass of stars which can become degenerate configurations, as the last stage in their evolution; and *second,* that stars with $M>M_{\text{limit}}$ must have end states which cannot be predicted from the considerations we have presented so far. And finally, we observe that the combination of the natural constant (3) now emerges in the fundamental context of $M_{\text{limit}}$ given by Eq. (43): its significance for the theory of stellar structure and stellar evolution can no longer be doubted.

## 6. UNDER WHAT CONDITIONS CAN NORMAL STARS DEVELOP DEGENERATE CORES?

Once the upper limit to the mass of completely degenerate configurations had been established, the question

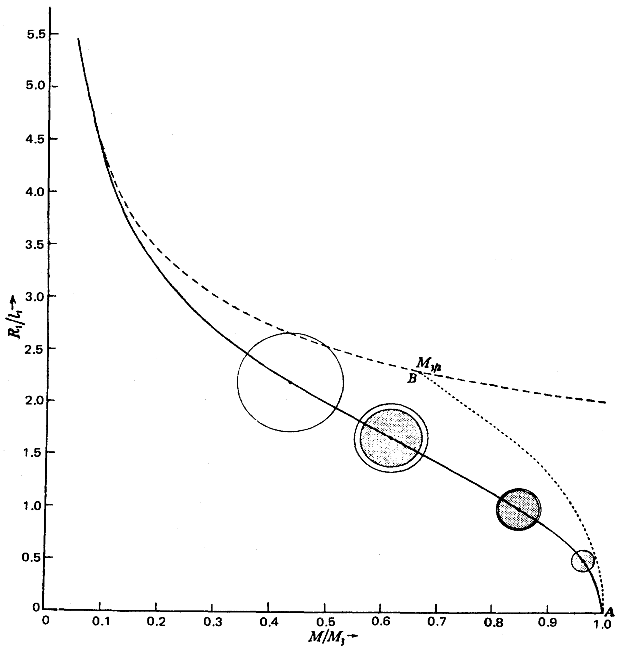


FIG. 2. The full-line curve represents the exact (mass-radius)-relation [$l_1$ is defined in Eq. (46) and $M_3$ denotes the limiting mass]. This curve tends asymptotically to the — — — curve appropriate to the low-mass degenerate configurations, approximated by polytropes of index 3/2. The regions of the configurations which may be considered as relativistic [$\rho>(K_1/K_2)^3$] are shown shaded. [From Chandrasekhar (1935).]

that required to be resolved was how to relate its existence to the evolution of stars from their gaseous state. If a star has a mass less than $M_{\text{limit}}$, the assumption that it will eventually evolve towards the completely degenerate state appears reasonable. But what if its mass is greater than $M_{\text{limit}}$? Clues as to what might ensue were sought in terms of the equations and inequalities of Secs. 2 and 3 (Chandrasekhar, 1932, 1934a).

The first question that had to be resolved concerns the circumstances under which a star, initially gaseous, will

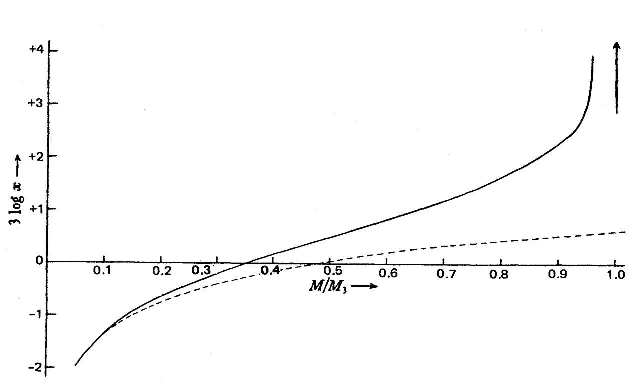


FIG. 3. The full-line curve represents the exact (mass-density)-relation for the highly collapsed configurations. This curve tends asymptotically to the dotted curve as $M\rightarrow 0$. [From Chandrasekhar (1935).]

develop degenerate cores. From the physical side, the question, when departures from the perfect-gas equation of state (5) will set in and the effects of electron degeneracy will be manifested, can be readily answered.

Suppose, for example, that we continually and steadily increase the density, at constant temperature, of an assembly of free electrons and atomic nuclei, in a highly ionized state and initially in the form of a perfect gas governed by the equation of state (5). At first the electron pressure will increase linearly with $\rho$; but soon departures will set in and eventually the density will increase in accordance with the equation of state that describes the fully degenerate electron-gas (see Fig. 4). The remarkable fact is that this limiting form of the equation of state is independent of temperature.

However, to examine the circumstances when, during the course of evolution, a star will develop degenerate cores, it is more convenient to express the electron pressure (as given by the classical perfect-gas equation of state) in terms of $\rho$ and $\beta_e$ defined in the manner [cf. Eq. (7)],

$$p_e=\frac{k}{\mu_e H}\rho T=\frac{\beta_e}{1-\beta_e}\frac{1}{3}aT^4 , \tag{47}$$

where $p_e$ now denotes the electron pressure. Then, analogous to Eq. (9), we can write

$$p_e=\left[\left[\frac{k}{\mu_e H}\right]^4\frac{3}{a}\frac{1-\beta_e}{\beta_e}\right]^{1/3}\rho^{4/3} . \tag{48}$$

Comparing this with Eq. (42), we conclude that if

$$\left[\left[\frac{k}{\mu_e H}\right]^4\frac{3}{a}\frac{1-\beta_e}{\beta_e}\right]^{1/3}>K_2=\frac{1}{8}\left[\frac{3}{\pi}\right]^{1/3}\frac{hc}{(\mu_e H)^{4/3}} , \tag{49}$$

the pressure $p_e$ given by the classical perfect-gas equation of state will be greater than that given by the equation if degeneracy were to prevail, not only for the prescribed $\rho$ and $T$, but for *all* $\rho$ and $T$ having the same $\beta_e$.

Inserting for $a$ its value given in Eq. (13), we find that the inequality (49) reduces to

$$\frac{960}{\pi^4}\frac{1-\beta_e}{\beta_e}>1 , \tag{50}$$

or, equivalently

$$1-\beta_e>0.0921=1-\beta_\omega \tag{51}$$

(say). (See Fig. 5.)

For our present purposes, the principal content of the inequality (51) is the criterion that for a star to develop degeneracy, it is necessary that the radiation pressure be less than 9.2% of $(p_e+p_{\rm rad})$. This last inference is so central to all current schemes of stellar evolution that the directness and the simplicity of the early arguments are worth repeating.

The two principal elements of the early arguments were these: *first,* that radiation pressure becomes increasingly dominant as the mass of the star increases; and *second,* that the degeneracy of electrons is possible only so long as the radiation pressure is not a significant fraction of the total pressure—indeed, as we have seen, it must not exceed 9.2% of $(p_e+p_{\rm rad})$. The second of these elements in the arguments is a direct and an elementary consequence of the physics of degeneracy; but the first requires some amplification.

That radiation pressure must play an increasingly dominant role as the mass of the star increases is one of the earliest results in the study of stellar structure that was es-

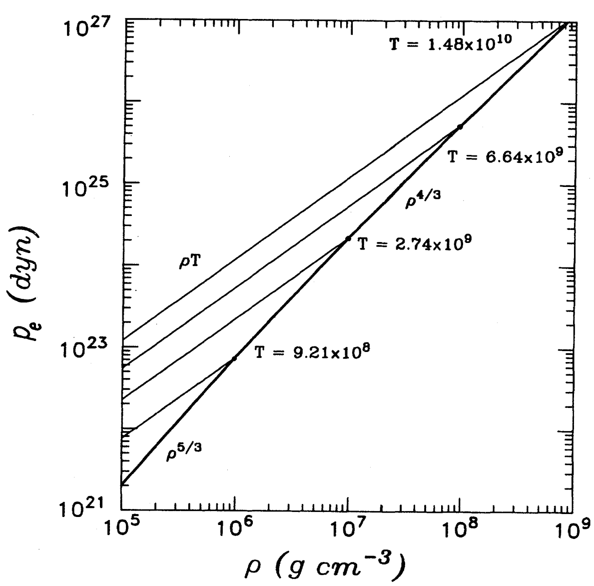


FIG. 4. Illustrating how by increasing the density at constant temperature degeneracy always sets in.

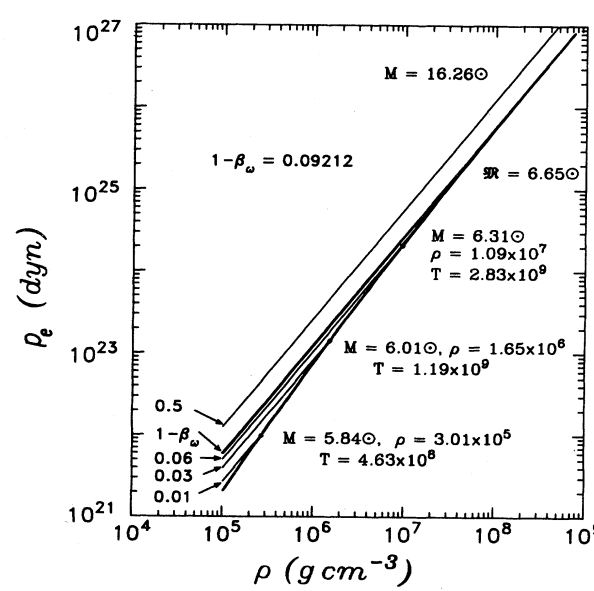


FIG. 5. Illustrating the onset of degeneracy for increasing density at constant $\beta$. Notice that there are no intersections for $1-\beta>0.092\,12$. In the figure, $1-\beta$ is converted into the mass of a star built on the standard model.

tablished by Eddington. A quantitative expression for this fact is given by Eddington's *standard model* which lay at the base of his early studies summarized in his *The Internal Constitution of the Stars.*

On the standard model, the fraction $\beta$ (=gas pressure/total pressure) is a constant through a star. On this assumption, the star is a polytrope of index 3 as is apparent from Eq. (9); and, in consequence, we have the relation [cf. Eq. (43)]

$$M=4\pi\left[\frac{C(\beta)}{\pi G}\right]^{3/2}(2.018) \tag{52}$$

where $C(\beta)$ is defined in Eq. (9). Equation (52) provides a quartic equation for $\beta$ analogous to Eq. (17) for $\beta_*$. Equation (52) for $\beta=\beta_\omega$ gives

$$M=0.197\beta_\omega^{-3/2}\left[\frac{hc}{G}\right]^{3/2}\frac{1}{(\mu H)^2}=6.65\mu^{-2}\odot=\mathfrak{M} \tag{53}$$

(say). On the standard model, then, stars with masses exceeding $\mathfrak{M}$ will have radiation pressures which exceed 9.2% of the total pressure. Consequently stars with $M>\mathfrak{M}$ cannot, at any stage during the course of their evolution, develop degeneracy in their interiors. Therefore, for such stars an eventual white-dwarf state is not possible unless they are able to eject a substantial fraction of their mass.

The standard model is, of course, only a model. Nevertheless, except under special circumstances, briefly noted below, experience has confirmed the essential qualitative correctness of the conclusions drawn from the standard model, namely that the evolution of stars of masses exceeding 7—8$\odot$ must proceed along lines very different from those of less massive stars. These conclusions, which were arrived at some fifty years ago, appeared then so convincing that assertions such as these were made with confidence:

> Given an enclosure containing electrons and atomic nuclei (total charge zero) what happens if we go on compressing the material indefinitely? (Chandrasekhar, 1932)

> The life history of a star of small mass must be essentially different from the life history of a star of large mass. For a star of small mass the natural white-dwarf stage is an initial step towards complete extinction. A star of large mass cannot pass into the white-dwarf stage and one is left speculating on other possibilities. (Chandrasekhar, 1934b)

And these statements have retained their validity.

While the evolution of the massive stars was thus left uncertain, there was no such uncertainty regarding the final states of stars of sufficiently low mass (Chandrasekhar, 1934a). The reason is that by virtue, again, of the inequality (10), the maximum central pressure attainable in a star must be less than that provided by the degenerate equation of state, so long as

$$\tfrac{1}{2}G(\tfrac{4}{3}\pi)^{1/3}M^{2/3}<K_2=\frac{1}{8}\left[\frac{3}{\pi}\right]^{1/3}\frac{hc}{(\mu_e H)^{4/3}} \tag{54}$$

or, equivalently

$$M<\frac{3}{16\pi}\left[\frac{hc}{G}\right]^{3/2}\frac{1}{(\mu_e H)^2}=1.74\mu_e^{-2}\odot\,. \tag{55}$$

We conclude that there can be no surprises in the evolution of stars of mass less than 0.43$\odot$ (if $\mu_e=2$). The end stage in the evolution of such stars can only be that of the white dwarfs. [Parenthetically, we may note here that the inequality (55) implies that the so-called "mini" black-holes of mass $\sim10^{15}$ g cannot naturally be formed in the present astronomical universe.]

## 7. SOME BRIEF REMARKS ON RECENT PROGRESS IN THE EVOLUTION OF MASSIVE STARS AND THE ONSET OF GRAVITATIONAL COLLAPSE

It became clear, already from the early considerations, that the inability of the massive stars to become white dwarfs must result in the development of much more extreme conditions in their interiors and, eventually, in the onset of gravitational collapse attended by the super-nova phenomenon. But the precise manner in which all this will happen has been difficult to ascertain in spite of great effort by several competent groups of investigators. The facts which must be taken into account appear to be the following.[1]

In the first instance, the density and the temperature will steadily increase without the inhibiting effect of degeneracy since for the massive stars considered $1-\beta_e>1-\beta_\omega$. On this account, "nuclear ignition" of carbon, say, will take place which will be attended by the emission of neutrinos. This emission of neutrinos will effect a cooling and a lowering of $(1-\beta_e)$; but it will still be in excess of $1-\beta_\omega$. The important point here is that the emission of neutrinos acts selectively in the central regions and is the cause of the lowering of $(1-\beta_e)$ in these regions. The density and the temperature will continue to increase till the next ignition of neon takes place followed by further emission of neutrinos and a further lowering of $(1-\beta_e)$. This succession of nuclear ignitions and lowering of $(1-\beta_e)$ will continue till $1-\beta_e<1-\beta_\omega$ and a relativistically degenerate core with a mass approximately that of the limiting mass (=1.4$\odot$ for $\mu_e=2$) forms at the centre. By this stage, or soon afterwards, instability of some sort is expected to set in (see following section, 8) followed by gravitational collapse and the phenomenon of the super-nova (of type II). In some instances, what was originally the highly relativistic degenerate core of approximately 1.4$\odot$, will be left behind as a neutron star. That this happens sometimes is confirmed by the fact

[1]I am grateful to Professor D. Arnett for guiding me through the recent literature and giving me advice in the writing of this section.

that in those cases for which reliable estimates of the masses of pulsars exist, they are consistently close to $1.4\odot$. However in other instances—perhaps, in the majority of the instances—what is left behind, after all "the dust has settled," will have masses in excess of that allowed for stable neutron stars; and in these instances black holes will form.

In the case of less massive stars ($M \sim 6$–$8\odot$) the degenerate cores, which are initially formed, are not highly relativistic. But the mass of the core increases with the further burning of the nuclear fuel at the interface of the core and the mantle; and when the core reaches the limiting mass, an explosion occurs following instability; and it is believed that this is the cause underlying super-nova phenomenon of type I.

From the foregoing brief description of what may happen during the late stages in the evolution of massive stars, it is clear that the problems one encounters are of exceptional complexity, in which a great variety of physical factors compete. This is clearly not the occasion for me to enter into a detailed discussion of these various questions.

## 8. INSTABILITIES OF RELATIVISTIC ORIGIN: (1) THE VIBRATIONAL INSTABILITY OF SPHERICAL STARS

I now turn to the consideration of certain types of stellar instabilities which are derived from the effects of general relativity and which have no counterparts in the Newtonian framework. It will appear that these new types of instabilities of relativistic origin may have essential roles to play in discussions pertaining to gravitational collapse and the late stages in the evolution of massive stars.

We shall consider first the stability of spherical stars for purely radial perturbations. The criterion for such stability follows directly from the linearized equations governing the spherically symmetric radial oscillations of stars. In the framework of the Newtonian theory of gravitation, the stability for radial perturbations depends only on an average value of the adiabatic exponent, $\Gamma_1$, which is the ratio of the fractional Lagrangian changes in the pressure and in the density experienced by a fluid element following the motion; thus,

$$\Delta P/P = \Gamma_1 \Delta\rho/\rho \ . \tag{56}$$

And the Newtonian criterion for stability is

$$\overline{\Gamma}_1 = \int_0^M \Gamma_1(r) P(r) dM(r) \div \int_0^M P(r) dM(r) > \tfrac{4}{3} \ . \tag{57}$$

If $\overline{\Gamma}_1 < 4/3$, *dynamical instability* of a global character will ensue with an $e$-folding time measured by the time taken by a sound wave to travel from the centre to the surface.

When one examines the same problem in the framework of the general theory of relativity, one finds (Chandrasekhar, 1964a; see also Chandrasekhar, 1964b, 1964c) that, again, the stability depends on an average value of $\Gamma_1$; but contrary to the Newtonian result, the stability now depends on the radius of the star as well. Thus, one finds that no matter how high $\overline{\Gamma}_1$ may be, instability will set in provided the radius is less than a certain determinate multiple of the *Schwarzschild radius,*

$$R_S = 2GM/c^2 \ . \tag{58}$$

Thus, if for the sake of simplicity, we assume that $\Gamma_1$ is a constant through the star and equal to 5/3, then the star will become dynamically unstable for radial perturbations, if $R_1 < 2.4R_S$. And further, if $\Gamma_1 \to \infty$, instability will set in for all $R < (9/8)R_S$. *The radius $(9/8)R_S$ defines, in fact, the minimum radius which any gravitating mass, in hydrostatic equilibrium, can have in the framework of general relativity.* This important result is implicit in a fundamental paper by Karl Schwarzschild published in 1916. [Schwarzschild actually proved that for a star in which the energy density is a uniform $R > (9/8)R_S$.]

In one sense, the most important consequence of this instability of relativistic origin is that if $\Gamma_1$ (again assumed to be a constant for the sake of simplicity) differs from and is greater than 4/3 only by a small positive constant, then the instability will set in for a radius $R$ which is a large multiple of $R_S$, and, therefore, under circumstances when the effects of general relativity, on the structure of the equilibrium configuration itself, are hardly relevant. Indeed, it follows (Chandrasekhar, 1965) from the equations governing radial oscillations of a star, in a first post-Newtonian approximation to the general theory of relativity, that instability for radial perturbations will set in for all

$$R < \frac{K}{\Gamma_1 - 4/3} \frac{2GM}{c^2} \ , \tag{59}$$

where $K$ is a constant which depends on the *entire*[2] march of density and pressure in the equilibrium configuration in the Newtonian framework. Thus, for a polytrope of index $n$, the value of the constant is given by

$$K = \frac{5-n}{18} \left[ \frac{2(11-n)}{(n+1)\xi_1^4 |\theta_1'|^3} \int_0^{\xi_1} \theta \left[ \frac{d\theta}{d\xi} \right]^2 \xi^2 d\xi + 1 \right] , \tag{60}$$

where $\theta$ is the Lane-Emden function in its standard normalization ($\theta = 1$ at $\xi = 0$), $\xi$ is the dimensionless radial coordinate, $\xi_1$ defines the boundary of the polytrope (where $\theta = 0$) and $\theta_1'$ is the derivative of $\theta$ at $\xi_1$.

[2]It is for this reason that we describe the instability as *global.*

TABLE II. Values of the constant $K$ in the inequality (59) for various polytropic indices, $n$.

| $n$ | $K$ | $n$ | $K$ |
|---|---|---|---|
| 0 | 0.452 381 | 3.25 | 1.285 03 |
| 1.0 | 0.565 382 | 3.5 | 1.499 53 |
| 1.5 | 0.645 063 | 4.0 | 2.253 38 |
| 2.0 | 0.751 296 | 4.5 | 4.530 3 |
| 2.5 | 0.900 302 | 4.9 | 22.906 |
| 3.0 | 1.124 47 | 4.95 | 45.94 |

In Table II, we list the values of $K$ for different polytropic indices. It should be particularly noted that $K$ increases without limit for $n \to 5$ and the configuration becomes increasingly centrally condensed.[3] Thus, already for $n=4.95$ (for which polytropic index $\rho_c = 8.09\times 10^6 \bar{\rho}$), $K \sim 46$. In other words, for the highly centrally condensed massive stars (for which $\Gamma_1$ may differ from 4/3 by as little as 0.01),[4] the instability of relativistic origin will set in, already, when its radius falls below $5\times 10^3 R_S$. Clearly this relativistic instability must be considered in the contexts of these problems.

A further application of the result described in the preceding paragraph is to degenerate configurations near the limiting mass (Chandrasekhar and Tooper, 1964). Since the electrons in these highly relativistic configurations have velocities close to the velocity of light, the effective value of $\Gamma_1$ will be very close to 4/3 and the post-Newtonian relativistic instability will set in for a mass slightly less than that of the limiting mass. On account of the instability for radial oscillations setting in for a mass less than $M_{\text{limit}}$, the period of oscillation, along the sequence of the degenerate configurations, must have a minimum. This minimum can be estimated to be about two seconds (see Fig. 6). Since pulsars, when they were discovered, were known to have periods much less than this minimum value, the possibility of their being degenerate configurations near the limiting mass was ruled out; and this was one of the deciding factors in favour of the pulsars being neutron stars. (But by a strange irony, for reasons we have briefly explained in Sec. 7, pulsars which have resulted from super-nova explosions have masses close to 1.4⊙!)

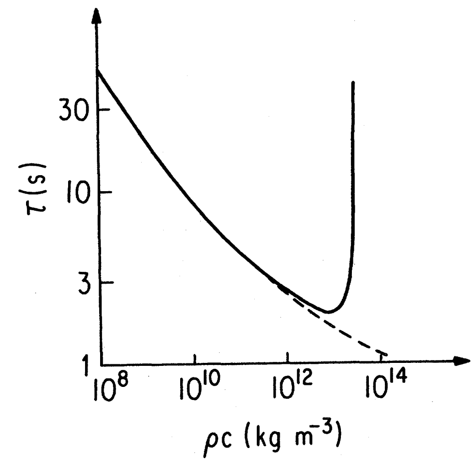


FIG. 6. The variation of the period of radial oscillation along the completely degenerate configurations. Notice that the period tends to infinity for a mass close to the limiting mass. There is consequently a minimum period of oscillation along these configurations; and the minimum period is approximately two seconds. [From Skilling (1968), p. 59.]

Finally, we may note that the radial instability of relativistic origin is the underlying cause for the *existence* of a maximum mass for stability: it is a direct consequence of the equations governing hydrostatic equilibrium in general relativity. [For a complete investigation on the periods of radial oscillation of neutron stars for various admissible equations of state, see Detweiler and Lindblom (1983).]

## 9. INSTABILITIES OF RELATIVISTIC ORIGIN: (2) THE SECULAR INSTABILITY OF ROTATING STARS DERIVED FROM THE EMISSION OF GRAVITATIONAL RADIATION BY NON-AXISYMMETRIC MODES OF OSCILLATION

I now turn to a different type of instability which the general theory of relativity predicts for rotating configurations. This new type of instability (Chandrasekhar, 1970a; see also Chandrasekhar, 1970b,1970c) has its origin in the fact that the general theory of relativity builds into rotating masses a dissipative mechanism derived from the possibility of the emission of gravitational radiation by nonaxisymmetric modes of oscillation. It appears that this instability limits the periods of rotation of pulsars. But first, I shall explain the nature and the origin of this type of instability.

It is well known that a possible sequence of equilibrium figures of rotating homogeneous masses is the Maclaurin sequence of oblate spheroids [for an account of these matters pertaining to the classical ellipsoids see Chandrasekhar (1968)]. When one examines the second harmonic oscillations of the Maclaurin spheroid, in a frame of reference rotating with its angular velocity, one finds

[3]Since this was written, it has been possible to show (Chandrasekhar and Lebovitz, 1984) that for $n \to 5$, the asymptotic behavior of $K$ is given by

$$K \to 2.3056/(5-n) ;$$

and, further, that along the polytropic sequence, the criterion for instability (59) can be expressed alternatively in the form

$$R < 0.2264 \left[\frac{\rho_c}{\bar{\rho}}\right]^{1/3} \frac{2GM}{c^2} \frac{1}{\Gamma_1 - 4/3} \quad (\rho_c/\bar{\rho} \gtrsim 10^6) .$$

[4]By reason of the dominance of the radiation pressure in these massive stars and of $\beta$ being very close to zero.

that for two of these modes, whose dependence on the azimuthal angle is given by $e^{2i\varphi}$, the characteristic frequencies of oscillation, $\sigma$, depend on the eccentricity $e$ in the manner illustrated in Fig. 7. It will be observed that one of these modes becomes neutral (i.e., $\sigma=0$) when $e=0.813$ and that the two modes coalesce when $e=0.953$ and become complex conjugates of one another beyond this point. Accordingly, the Maclaurin spheriod becomes *dynamically unstable* at the latter point (first isolated by Riemann). On the other hand, the origin of the neutral mode at $e=0.813$ is that at this point a new equilibrium sequence of triaxial ellipsoids—the ellipsoids of Jacobi—bifurcate. On this latter account, Lord Kelvin conjectured in 1883 that

> if there be any viscosity, however slight... the equilibrium beyond $e=0.81$ cannot be secularly stable.

Kelvin's reasoning was this: viscosity dissipates energy but not angular momentum. And since for equal angular momenta, the Jacobi ellipsoid has a lower energy content than the Maclaurin spheroid, one may expect that the action of viscosity will be to dissipate the excess energy of the Maclaurin spheroid and transform it into the Jacobi ellipsoid with the lower energy. A detailed calculation (Chandrasekhar, 1968) of the effect of viscous dissipation on the two modes of oscillation, illustrated in Fig. 7, does confirm Lord Kelvin's conjecture. It is found that viscous dissipation makes the mode, which becomes neutral at $e=0.813$, unstable beyond this point with an $e$-folding time which depends inversely on the magnitude of the kinematic viscosity and which further decreases monotonically to zero at the point, $e=0.953$ where the dynamical instability sets in.

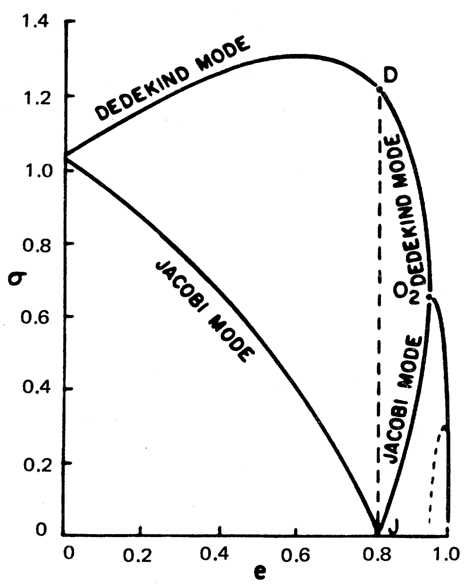


FIG. 7. The characteristic frequencies [in the unit $(\pi G\rho)^{1/2}$] of the two even modes of second-harmonic oscillation of the Maclaurin spheroid. The Jacobi sequence bifurcates from the Maclaurin sequence by the mode that is neutral ($\sigma=0$) at $e=0.813$; and the Dedekind sequence bifurcates by the alternative mode at $D$. At $O_2$ ($e=0.9529$) the Maclaurin spheroid becomes dynamically unstable. The real and the imaginary parts of the frequency, beyond $O_2$, are shown by the full line and the dashed curves, respectively. Viscous dissipation induces instability in the branch of the Jacobi mode; and radiation-reaction induces instability in the branch $DO_2$ of the Dedekind mode.

Since the emission of gravitational radiation dissipates *both* energy and angular momentum, it does *not* induce instability in the Jacobi mode; instead it induces instability in the *alternative* mode at the same eccentricity. In the first instance this may appear surprising; but the situation we encounter here clarifies some important issues.

If instead of analyzing the normal modes in the rotating frame, we had analyzed them in the inertial frame, we should have found that the mode which becomes unstable by radiation reaction at $e=0.813$, is in fact neutral at this point. And the neutrality of *this* mode in the inertial frame corresponds to the fact that the neutral deformation at this point is associated with the bifurcation (at this point) of a new triaxial sequence—the sequence of the Dedekind ellipsoids. These Dedekind ellipsoids, while they are congruent to the Jacobi ellipsoids, they differ from them in that they are at rest in the inertial frame and owe their triaxial figures to internal vortical motions. An important conclusion that would appear to follow from these facts is that in the framework of general relativity we can expect secular instability, derived from radiation-reaction to arise from a Dedekind mode of deformation (which is quasi-stationary in the inertial frame) rather than the Jacobi mode (which is quasi-stationary in the rotating frame).

A further fact concerning the secular instability induced by radiation-reaction, discovered subsequently by Friedman [(1978); see also Friedman and Schutz (1977)] and by Comins (1979a,1979b), is that the modes belonging to higher values of $m$ ($=3,4,\ldots$) become unstable at smaller eccentricities though the $e$-folding times for the instability become rapidly longer. Nevertheless it appears from some preliminary calculations of Friedman (1983) that it is the secular instability derived from modes belonging to $m=3$ (or 4) that limit the periods of rotation of the pulsars.

It is clear from the foregoing discussions that the two types of instabilities of relativistic origin we have considered are destined to play significant roles in the contexts we have considered.

## 10. THE MATHEMATICAL THEORY OF BLACK HOLES

So far, I have considered only the restrictions on the last stages of stellar evolution that follow from the existence of an upper limit to the mass of completely degenerate configurations and from the instabilities of relativistic origin. From these and related considerations, the conclusion is inescapable that black holes will form as one of the natural end products of stellar evolution of massive stars; and further that they must exist in large numbers in the present astronomical universe. In this last section I want to consider very briefly what the general theory of relativity has to say about them. But first, I must define precisely what a black hole is.

A black hole partitions the three-dimensional space into two regions: an inner region which is bounded by a smooth two-dimensional surface called the *event horizon;* and an outer region, external to the event horizon, which is asymptotically flat; and it is required (as a part of the definition) that no point in the inner region can communicate with any point of the outer region. This incommunicability is guaranteed by the impossibility of any light signal, originating in the inner region, crossing the event horizon. The requirement of asymptotic flatness of the outer region is equivalent to the requirement that the black hole is isolated in space and that far from the event horizon the space-time approaches the customary space-time of terrestrial physics.

In the general theory of relativity, we must seek solutions of Einstein's vacuum equations compatible with the two requirements I have stated. It is a startling fact that compatible with these very simple and necessary requirements, the general theory of relativity allows for stationary (i.e., time-independent) black-holes exactly a single, unique, two-parameter family of solutions. This is the Kerr family, in which the two parameters are the mass of the black hole and the angular momentum of the black hole. What is even more remarkable, the metric describing these solutions is simple and can be explicitly written down.

I do not know if the full import of what I have said is clear. Let me explain.

Black holes are macroscopic objects with masses varying from a few solar masses to millions of solar masses. To the extent they may be considered as stationary and isolated, to that extent, they are all, every single one of them, described *exactly* by the Kerr solution. This is the only instance we have of an exact description of a macroscopic object. Macroscopic objects, as we see them all around us, are governed by a variety of forces, derived from a variety of approximations to a variety of physical theories. In contrast, the only elements in the construction of black holes are our basic concepts of space and time. They are, thus, almost by definition, the most perfect macroscopic objects there are in the universe. And since the general theory of relativity provides a single unique two-parameter family of solutions for their descriptions, they are the simplest objects as well.

Turning to the physical properties of the black holes, we can study them best by examining their reaction to external perturbations such as the incidence of waves of different sorts. Such studies reveal an analytic richness of the Kerr space-time which one could hardly have expected. This is not the occasion to elaborate on these technical matters [the author's investigations on the mathematical theory of black holes, continued over the years 1974—1983, are summarized in his latest book, Chandrasekhar (1983b)]. Let it suffice to say that contrary to every prior expectation, all the standard equations of mathematical physics can be solved exactly in the Kerr space-time. And the solutions predict a variety and range of physical phenomena which black holes must exhibit in their interaction with the world outside.

The mathematical theory of black holes is a subject of immense complexity. But its study has convinced me of the basic truth of the ancient mottoes,

> The simple is the seal of the true

and

> Beauty is the splendour of truth.

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[The reader may wish to consult, additionally, Chandrasekhar (1980,1983a).]