Einstein gravity as a symmetry-breaking effect in quantum field theory*

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This article gives a pedagogical review of recent work in which the Einstein-Hilbert gravitational action is obtained as a symmetry-breaking effect in quantum field theory. Particular emphasis is placed on the case of renormalizable field theories with dynamical scale-invariance breaking, in which the induced gravitational effective action is finite and calculable. A functional integral formulation is used throughout, and a detailed analysis is given of the role of dimensional regularization in extracting finite answers from formally quadratically divergent integrals. Expressions are derived for the induced gravitational constant and for the induced cosmological constant in quantized matter theories on a background manifold, and a strategy is outlined for computing the induced constants in the case of an SU(n) gauge theory. By use of the background field method, the formalism is extended to the case in which the metric is also quantized, yielding a derivation of the semiclassical Einstein equations as an approximation to quantum gravity, as well as general formulas for the induced (or renormalized) gravitational and cosmological constants.

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I. INTRODUCTION

In the conventional formulation of general relativity, gravitation is described by rewriting the matter action in generally covariant form, and by adding to it the Einstein-Hilbert gravitational action

$$S_{\rm grav}=\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R-2\Lambda)\ , \qquad (1.1)$$

with G Newton's constant and R the curvature scalar, and with the cosmological constant Λ taken to be zero. The total action is then treated as a classical variational principle, to be extremized with respect to variations of the c-number metric $g_{\mu\nu}$. As discussed in the survey articles in Hawking and Israel (1979), the theory in this form accounts very well for all astronomical gravitational phenomena and has a structure which is understood in considerable theoretical detail. On the other hand, when treated as a fundamental quantum action, Eq. (1.1) leads to a nonrenormalizable quantum field theory. This problem has long been known, and has stimulated much theoretical effort aimed at achieving a satisfactory quantization of the Einstein-Hilbert action or its supergravity extensions. [For reviews of the current status of these approaches, see Hawking and Israel (1979) and Van Nieuwenhuizen (1981).]

An entirely different approach to quantum gravity derives from work by Zel'dovich (1967) and Sakharov (1967) on induced quantum effects. Zel'dovich studied the effect of vacuum quantum fluctuations on the

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cosmological constant; extending this idea, Sakharov proposed that Eq. (1.1) is not a fundamental microscopic action, but rather is an effective action induced by vacuum quantum structure (see also Klein, 1974). To quote the two key sentences from Sakharov's paper, "The presence of the action (1) [Eq. (1.1)] leads to a metrical elasticity of space, i.e., to generalized forces which oppose the curving of space. (¶) Here we consider the hypothesis which identifies the action (1) with the change in the action of quantum fluctuations of the vacuum if space is curved." Sakharov's proposal attracted attention from the outset (see Misner *et al.*, 1970), but further progress was hampered by the fact that in the free field models for which he made his estimates, the induced gravitational constant $G_{\rm ind}^{-1}$ is given by integrals which contain both quadratic and logarithmic divergences. It is only in the last few years that the technology of quantum field theory has advanced to the point where one can systematically study induced quantum effects in interacting field theories. These advances, and their application to induced Einstein gravity, are the subject matter of this review.

Since the topics discussed below span the areas of high-energy physics and relativity, in which different notational conventions are generally used, I have adopted the following compromise with respect to notation. I use microscopic units throughout,

$$\hbar=c=1\ , \tag{1.2}$$

so the only dimensional quantity is mass$=$(length)$^{-1}$. The coordinates x^μ are taken to have the dimension (length)^1, making the metric $g_{\mu\nu}$ dimensionless. In all flat space-time examples and discussions, I use the $+---$ signature convention of Bjorken and Drell (1965), while in all expressions which involve a curved manifold I follow the $-+++$ convention of Misner *et al.*, (1970). In the few places where it is necessary to change from one convention to the other, I will explicitly call attention to the transition.

II. FIELD THEORY PRELIMINARIES

A. Actions and canonical dimension accounting

The functional integral formulation of quantum field theory (see Abers and Lee, 1973) expresses transition amplitudes in the form

$$Z=\int d\{\phi\}e^{iS[\{\phi\}]}\ ,$$
$$S[\{\phi\}]=\int d^4x\,\mathscr{L}[\{\phi\}]\ , \tag{2.1}$$

with $\{\phi\}$ the set of fields present, S the action, and $\mathscr{L}$ the Lagrangian (or action) density. Since the argument of an exponential or a logarithm must be dimensionless, in the conventional accounting of canonical dimension in which

$$\dim[\text{length}]=-1,\ \ \dim[\text{mass}]=+1\ , \tag{2.2}$$

we have

$$\dim[S]=0,\ \ \dim[d^4x]=-4$$
$$\Longrightarrow \dim[\mathscr{L}]=4\ . \tag{2.3}$$

From Eq. (2.3) we can infer the canonical dimensionality of the fields and parameters from which elementary renormalizable matter theories are constructed. For a scalar φ^4 field theory we have

$$\mathscr{L}=\tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi-\tfrac{1}{2}m_0^2\varphi^2-\tfrac{1}{4}\lambda_0\varphi^4\ , \tag{2.4}$$

with m_0 the bare mass and λ_0 the bare coupling, and with

$$\dim[\partial_\mu\equiv\partial/\partial x^\mu]=1 \Longrightarrow \dim[\varphi]=1\ ,$$
$$\dim[m_0]=1\ ,$$
$$\dim[\lambda_0]=0\ . \tag{2.5}$$

For a spin-1 Abelian gauge field (the photon) we have

$$\mathscr{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\ ,$$
$$F_{\mu\nu}=\partial_\nu A_\mu-\partial_\mu A_\nu\ , \tag{2.6}$$

with

$$\dim[F_{\mu\nu}]=2\ ,$$
$$\dim[A_\mu]=1\ , \tag{2.7}$$

while for a spin-$\frac{1}{2}$ Dirac field we have

$$\mathscr{L}=\bar{\psi}(i\gamma^\mu\partial_\mu-m_0)\psi\ , \tag{2.8}$$

with

$$\dim[\psi]=\tfrac{3}{2}\ . \tag{2.9}$$

Minimal coupling of the photon to a Dirac field or a complex scalar field with bare charge e_0 yields the Lagrangian densities for quantum electrodynamics,

$$\mathscr{L}_{\rm QED1/2}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\psi}(i\gamma^\mu D_\mu-m_0)\psi\ ,$$
$$\mathscr{L}_{\rm QED0}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}|D_\mu\varphi|^2-\tfrac{1}{2}m_0^2|\varphi|^2-\tfrac{1}{4}\lambda_0|\varphi|^4\ ,$$
$$D_\mu=\partial_\mu+ie_0A_\mu\ ,$$
$$\Longrightarrow \dim[e_0]=0\ . \tag{2.10}$$

For a spin-1 non-Abelian gauge field (the massless gauge gluon) we have

$$\mathscr{L}=-\tfrac{1}{4}F^i_{\mu\nu}F^{i\mu\nu}\ ,$$
$$F^i_{\mu\nu}=\partial_\nu A^i_\mu-\partial_\mu A^i_\nu+g_0f^{ijk}A^j_\mu A^k_\nu\ , \tag{2.11}$$

with i the internal symmetry index, f^{ijk} the group structure constants, and g_0 the bare coupling constant. Minimal coupling of the gauge field to a Dirac field in the fundamental representation (with representation matrices $\frac{1}{2}\lambda^i$) gives the basic Lagrangian density for quantum chromodynamics,

$$\mathscr{L}_{\rm QCD} = -\tfrac{1}{4}F^i_{\mu\nu}F^{i\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m_0)\psi\,,$$
$$D_\mu = \partial_\mu + ig_0\tfrac{1}{2}\lambda^i A^i_\mu\,, \tag{2.12}$$

from which we infer the dimensional assignments

$$\dim[F^i_{\mu\nu}] = 2\,,$$
$$\dim[A^i_\mu] = 1,\ \dim[\psi] = \tfrac{3}{2}\,,$$
$$\dim[g_0] = 0\,. \tag{2.13}$$

All of the field theory models currently under study as candidates for unified matter theories [for reviews see Marciano and Pagels (1978); Fritzsch and Minkowski (1981)] are combinations of scalar, Dirac, and gauge field action building blocks of the basic types enumerated above. The characterizing feature of all such renormalizable actions is that their coupling constants (λ_0, e_0, g_0 above) have canonical dimension zero.

B. Effective actions

Consider now a renormalizable field theory with action

$$S[\{\phi^L\},\{\phi^H\}] = \int d^4x\, \mathscr{L}[\{\phi^L\},\{\phi^H\}]\,, \tag{2.14}$$

where $\{\phi^L\}$ are "light" field components whose dynamics we directly observe, while $\{\phi^H\}$ are "heavy" field components which influence the dynamics of the light components but are not directly observable. The $\{\phi^H\}$ can in general include fields with high physical masses and high-momentum[1] components of fields with light physical masses. Since the $\{\phi^H\}$ are hidden from view, it is convenient to rewrite the functional integral of Eq. (2.1) in the following form,

$$Z = \int d\{\phi^L\}d\{\phi^H\}e^{iS[\{\phi^L\},\{\phi^H\}]}$$
$$= \int d\{\phi^L\}e^{iS_{\rm eff}[\{\phi^L\}]}\,, \tag{2.15}$$

where the *effective action* $S_{\rm eff}[\{\phi^L\}]$ for the light fields is defined through

$$e^{iS_{\rm eff}[\{\phi^L\}]} = \int d\{\phi^H\}e^{iS[\{\phi^L\},\{\phi^H\}]}\,. \tag{2.16}$$

Clearly, the effective action, if exactly known, would give a complete description of the dynamics of the fields $\{\phi^L\}$. In practice, one usually works with only an approximation to $S_{\rm eff}$, obtained by keeping leading terms in an expansion in a small parameter. Some examples of commonly used effective actions are as follows.

1. The Heisenberg-Euler (1936) effective action in quantum electrodynamics (see also Schwinger, 1951).

Integrating out the electron fields in quantum electrodynamics gives an effective action describing the nonlinear interactions of photons,

$$e^{iS_{\rm eff}[F_{\mu\nu}]} = \int d\{\psi,\bar{\psi}\}e^{iS_{\rm QED1/2}[\{F_{\mu\nu},\psi,\bar{\psi}\}]}\,. \tag{2.17}$$

For field strengths which are slowly varying over an electron Compton wavelength, $S_{\rm eff}$ can be approximated by taking $F_{\mu\nu}$=constant, which gives a problem which can be solved in closed form. For weak, slowly varying fields (on a scale of an electron Compton wavelength), $S_{\rm eff}$ can be approximated by the first two terms in a series expansion

$$S_{\rm eff}[F_{\mu\nu}] = \int d^4x\, \mathscr{L}_{\rm eff}\,,$$
$$\mathscr{L}_{\rm eff} = \tfrac{1}{2}(E^2-H^2) + \frac{2\alpha^2}{45m^4}[(E^2-H^2)^2 + 7(E\cdot H)^2] + \cdots\,, \tag{2.18}$$

with E,H the electric and magnetic fields, α the fine-structure constant, and m the electron mass. If interpreted as a fundamental action and used (or, rather, misused) beyond the tree-approximation level, Eq. (2.18) would yield a nonrenormalizable perturbation expansion in powers of the dimensional effective coupling $2\alpha^2/45m^4$.

2. The four-fermion effective action approximation to the Weinberg (1967)-Salam (1968) weak interaction theory

At center-of-mass energies well below 100 GeV, the weak interactions are described by a current-current four-fermion effective action

$$S_{\rm eff}[\{\text{fermions}\}] = \int d^4x(\mathscr{L}^{\rm charged}_{\rm eff} + \mathscr{L}^{\rm neutral}_{\rm eff})\,,$$
$$\mathscr{L}^{\rm charged}_{\rm eff} = \frac{1}{\sqrt{2}}G_F(j^\lambda_{\rm ch} + J^\lambda_{\rm ch})(j^\dagger_{\rm ch\lambda} + J^\dagger_{\rm ch\lambda})\,,$$
$$\mathscr{L}^{\rm neutral}_{\rm eff} = \frac{1}{\sqrt{2}}G_F(j^\lambda_n + J^\lambda_n)(j_{n\lambda} + J_{n\lambda})\,,$$
$$j^\lambda_{\rm ch} = \bar{e}\gamma^\lambda(1-\gamma_5)\nu_e + \mu,\tau \text{ terms}\,,$$
$$J^\lambda_{\rm ch} = \bar{u}\gamma_\lambda(1-\gamma_5)(d\cos\theta_C + s\sin\theta_C) + \text{charm terms}\,,$$
$$j^\lambda_n = -\tfrac{1}{2}\bar{e}\gamma^\lambda(1-\gamma_5)e + \tfrac{1}{2}\bar{\nu}_e\gamma^\lambda(1-\gamma_5)\nu_e + 2\sin^2\theta_W\bar{e}\gamma^\lambda e + \mu,\tau \text{ terms}\,,$$
$$J^\lambda_n = \tfrac{1}{2}\bar{u}\gamma^\lambda(1-\gamma_5)u - \tfrac{1}{2}\bar{d}\gamma^\lambda(1-\gamma_5)d - 2\sin^2\theta_W(\tfrac{2}{3}\bar{u}\gamma^\lambda u - \tfrac{1}{3}\bar{d}\gamma^\lambda d) + \text{strange, charm terms}\,, \tag{2.19}$$

with e,ν_e,u,d,s the electron, electron neutrino, and up, down, and strange quark fields, respectively, and with θ_C and θ_W the Cabibbo and Weinberg angles.[2] Since the fermion fields have dimension $\frac{3}{2}$, the Fermi constant G_F has dimension -2, and has the empirical value

$$G_F \approx \frac{1.023\times 10^{-5}}{m^2_{\rm proton}}\,. \tag{2.20}$$

[1] I wish to thank B. Holdom for suggesting the inclusion of a momentum criterion in the definition, as a way of automatically including renormalization effects arising from overlapping divergences. For recent discussions of effective actions, see Weinberg (1980a) and Ovrut and Schnitzer (1980,1981).

[2] For a recent review of the phenomenology of the Weinberg-Salam model, see Kim *et al.* (1981).

As expected for a theory with a dimensional coupling constant, the use of Eq. (2.19) as a fundamental action leads to a nonrenormalizable perturbation expansion in G_F. This difficulty is resolved in the Weinberg-Salam gauge theory, in which in addition to the fermions, the fundamental action contains gauge and Higgs boson fields, and which has a renormalizable perturbation expansion in the gauge boson couplings g,g'. When the boson fields are integrated out, according to

$$e^{iS_{\text{eff}}[\{\text{fermions}\}]} = \int d\{\text{bosons}\}e^{iS_{\text{Weinberg-Salam}}[\{\text{fermions}\},\{\text{bosons}\}]}, \quad (2.21)$$

the effective action of Eq. (2.19) is obtained as a leading approximation, with the Fermi constant related to the electric charge e, the charged gauge boson mass M_W, and $\sin\theta_W$ by

$$\frac{1}{\sqrt{2}}G_F=\frac{g^2}{8M_W^2}, \qquad g=\frac{e}{\sin\theta_W}. \quad (2.22)$$

Let us now return to gravitation. The action of Eq. (1.1) contains dimensional couplings G^{-1} and Λ,

$$\begin{aligned} &\dim[G^{-1}]=\dim[\Lambda]=2, \\ &G^{-1/2}=m_{\text{Planck}}=1.22\times10^{19}\ \text{GeV}=l_{\text{Planck}}^{-1}, \\ &l_{\text{Planck}}=1.62\times10^{-33}\ \text{cm}, \\ &|\Lambda|\lesssim10^{-57}\ \text{cm}^{-2}, \end{aligned} \quad (2.23)$$

and, as expected for the case when the couplings are not dimensionless, leads to a nonrenormalizable quantum field theory. The viewpoint of this article will be that the gravitational action is not a fundamental microscopic action, but rather is a long-wavelength effective action similar to the ones discussed above. The fundamental action will be assumed to be renormalizable, and conditions on it will be formulated which guarantee that the effective gravitational action is calculable in terms of parameters of the microscopic theory.

C. Renormalizability and the dimensional algorithm

In quantum field theory, one in general encounters divergences when evaluating radiative corrections. In renormalizable field theories, all divergences can be eliminated by making divergent rescalings, or renormalizations, of a finite number of parameters of the theory, which cannot be calculated from first principles but are replaced by measured values at the end of the calculation.

For example, in spin-$\frac{1}{2}$ quantum electrodynamics, working for simplicity to one-loop order, one introduces renormalization constants $Z_{1,2,3,m,e}$, renormalized fields A_μ^r, $F_{\mu\nu}^r$, ψ^r, and renormalized charge and mass parameters e,m by writing

$$\begin{aligned} &A_\mu=Z_3^{1/2}A_\mu^r,\ \ F_{\mu\nu}=Z_3^{1/2}F_{\mu\nu}^r, \\ &\psi=Z_2^{1/2}\psi^r, \\ &e_0=Z_e^{1/2}e,\ \ m_0=Z_m m; \end{aligned} \quad (2.24a)$$

$$\begin{aligned} &F_{\mu\nu}F^{\mu\nu}=Z_3F_{\mu\nu}^rF^{r\mu\nu}, \\ &\bar\psi\gamma^\mu\partial_\mu\psi=Z_2\bar\psi^r\gamma^\mu\partial_\mu\psi^r, \\ &\bar\psi\gamma^\mu e_0A_\mu\psi=Z_1\bar\psi^r\gamma^\mu eA_\mu^r\psi^r=Z_2Z_e^{1/2}Z_3^{1/2}\bar\psi^r\gamma^\mu eA_\mu^r\psi^r, \\ &\bar\psi m_0\psi=Z_2Z_m\bar\psi^r m\psi^r. \end{aligned} \quad (2.24b)$$

From Eq. (2.24b) and the Ward identity (which is derived from current conservation) one learns that

$$Z_e^{1/2}Z_3^{1/2}=\frac{Z_1}{Z_2}=1, \quad (2.25)$$

leaving as the independent renormalization constants Z_e, Z_m, and Z_2. Thus the renormalization procedure calls for the bare e_0, m_0, and ψ to be adjusted to absorb all divergences, leaving finite e, m, and ψ^r to be identified with the measured values. To understand why e, for example, cannot be calculated, let us recall that in one-loop order, the divergence in Z_e has the form

$$Z_e=1+\frac{\alpha_0}{3\pi}\log M^2+O(\alpha_0^2),\ \ \alpha_0=\frac{e_0^2}{4\pi}, \quad (2.26)$$

with M a massive regulator. Under rescalings $M\to\xi M$ of the regulator mass, Z_e changes to

$$Z_e\to1+\frac{\alpha_0}{3\pi}\log M^2+\frac{\alpha_0}{3\pi}\log\xi^2+O(\alpha_0^2), \quad (2.27)$$

and hence the finite part of Z_e is regulator-scheme dependent. As a result, the finite quantity e extracted from the divergent bare charge e_0 remains a free parameter of the renormalized theory. In general in a renormalizable field theory, we expect to find one free renormalized coupling or mass parameter for each bare coupling or mass appearing in the unrenormalized Lagrangian density.

Continuing for the moment to work to one-loop order, the renormalization procedure given in Eq. (2.24b) for the various dimension-four terms in the action density can be rewritten in a compact matrix notation,

$$[\Psi]=[Z][\Psi^r],$$

$$[\Psi]=\begin{bmatrix}F_{\mu\nu}F^{\mu\nu}\\ \bar\psi\gamma^\mu\partial_\mu\psi\\ \bar\psi\gamma^\mu e_0A_\mu\psi\\ \bar\psi m_0\psi\end{bmatrix},\ \ [\Psi^r]=\begin{bmatrix}F_{\mu\nu}^rF^{r\mu\nu}\\ \bar\psi^r\gamma^\mu\partial_\mu\psi^r\\ \bar\psi^r\gamma^\mu eA_\mu^r\psi^r\\ \bar\psi^r m\psi^r\end{bmatrix}.$$

$$[Z]=\begin{bmatrix}Z_3&0&0&0\\0&Z_2&0&0\\0&0&Z_1&0\\0&0&0&Z_2Z_m\end{bmatrix}. \quad (2.28a)$$

Beyond one-loop order, the renormalization constants associated with the action density terms $F_{\mu\nu}F^{\mu\nu},\ldots$ are no longer simply products of the renormalization constants for the individual field, charge, and mass factors introduced in Eq. (2.24a), and the action density terms themselves will mix under renormalization. The appropriate generalization of Eq. (2.28a) then takes the form

$$[\Psi]=[Z][\Psi^r] ,$$

$$[\Psi^r]=\begin{bmatrix}(F_{\mu\nu}F^{\mu\nu})^r\\(\bar{\psi}\gamma^\mu\partial_\mu\psi)^r\\(\bar{\psi}\gamma^\mu e_0 A_\mu\psi)^r\\(\bar{\psi}m_0\psi)^r\end{bmatrix} , \tag{2.28b}$$

with $[\Psi]$ as in Eq. (2.28a), with $(F_{\mu\nu}F^{\mu\nu})^r, \ldots$, the renormalized action density terms, and with $[Z]$ a nondiagonal renormalization matrix.

As we have seen from the above example, in the generic case multiplicative renormalization takes the more general form of matrix multiplicative renormalization. The set of operators which can mix under this renormalization process is characterized by the following rule.

The dimensional algorithm [see Weinberg (1957), Zimmermann (1970), and Brown (1980)]. A composite operator in quantum field theory is defined (up to a constant factor) as the product of any number of fields or field derivatives at the same space-time point. The dimensional algorithm states: (i) The most general basis set of composite operators which can mix under renormalization are the polynomials of the same canonical dimension, and of the same symmetry type (spatial and internal) formed from the bare fields, the bare masses, and $\partial/\partial x^\mu$. (ii) The Lagrangian density for a renormalizable field theory must contain a complete basis set (apart from total derivatives) of Lorentz- and internal symmetry-invariant composite operators of canonical dimension four.

Let us illustrate the dimensional algorithm in the flat space-time cases of scalar φ^4 theory, QED$\frac{1}{2}$, and QCD, and then use it to deduce additional Lagrangian counterterms which must be added to assure renormalizability when these theories are embedded in a curved background manifold.

1. Scalar φ^4 theory in flat space-time

Excluding total derivatives, the only dimension-four composites even under $\varphi\rightarrow-\varphi$ (the internal symmetry of the model) are

$$\partial_\mu\varphi\partial^\mu\varphi,\ m_0^2\varphi^2,\ \varphi^4 , \tag{2.29a}$$

$$m_0^4 . \tag{2.29b}$$

The operators of Eq. (2.29a) are just the ones appearing in the Lagrangian density of Eq. (2.4), while in flat space-time Eq. (2.29b) is an irrelevant constant which can be dropped.

2. QED $\frac{1}{2}$ and QCD in flat space-time

For QED $\frac{1}{2}$, the only dimension-four composites (excluding total derivatives) are

$$F_{\mu\nu}F^{\mu\nu},\ \bar{\psi}\gamma^\mu D_\mu\psi,\ m_0\bar{\psi}\psi , \tag{2.30a}$$

$$m_0^4 , \tag{2.30b}$$

$$\bar{\psi}\gamma^\mu\partial_\mu\psi,\ m_0^2A_\mu A^\mu,\ A_\mu\partial^2A^\mu,\ (\partial_\mu A^\mu)^2 . \tag{2.30c}$$

The operators of Eq. (2.30a) are just the ones appearing in the Lagrangian density of Eq. (2.10), while in flat space-time Eq. (2.30b) is an irrelevant constant. The operators of Eq. (2.30c) are Lorentz scalars, but are not invariant under the internal symmetry (or gauge) transformation

$$A_\mu\rightarrow A_\mu+\partial_\mu\Phi ,$$

$$\psi\rightarrow e^{-ie_0\Phi}\psi , \tag{2.31}$$

and hence do not appear in the Lagrangian density. For QCD the classification of gauge-invariant Lorentz scalar operators constructed from the bare fields is analogous—one simply adds an internal symmetry index i, and changes the definition of the covariant derivative as in Eq. (2.12). A careful proof that $A_{\mu i}A^{\mu i}$ is not an internal symmetry invariant in the non-Abelian case, taking account of the complexities introduced by gauge-fixing and ghost terms, is given in Appendix A.

3. Additional Lagrangian density terms in a background curved space-time (Brown and Collins, 1980)

When spin-0, spin-$\frac{1}{2}$, or gauge spin-1 matter fields are quantized on a curved background manifold with metric $g_{\mu\nu}$, the action takes the form

$$S[\{\phi\},g_{\mu\nu}]=\int d^4x\sqrt{-g}\,\mathscr{L}[\{\phi\},g_{\mu\nu}] , \tag{2.32}$$

with $d^4x\sqrt{-g}$ the invariant volume element, and with $\mathscr{L}$ a scalar with respect to general-coordinate transformations. According to the dimensional algorithm, $\mathscr{L}$ must contain all scalar dimension-four polynomials which can be formed from the bare fields (including now $g_{\mu\nu}$), the bare masses, and $\partial/\partial x^\mu$, and which are invariant under the internal symmetries of the matter fields. The terms which can thus appear in $\mathscr{L}$ are easily enumerated, and may be conveniently grouped into the following four classes: (i) The generally covariant transcriptions of the Lagrangian densities of Eqs. (2.4)–(2.13), obtained in the usual manner by replacing ordinary derivatives ∂_μ by covariant derivatives ∇_μ with respect to the background metric. (ii) The bare mass terms m_0^4 of Eqs. (2.29b) and (2.30b), which contribute to the cosmological constant on a curved manifold,[3] as well

[3]The terms m_0^4 and m_0^2R, which appear in $\mathscr{L}$ multiplied by independent renormalization constants, may be considered, respectively, as the bare cosmological constant Λ_0/G_0 and the bare order-R Lagrangian density R/G_0. Prior to the discussion of the cosmological constant in Sec. VI.C, we shall not introduce bare parameters Λ_0, G_0 when not required to do so by the presence of dimensional parameters in the microscopic matter action.

as corresponding regulator mass terms M^4 if massive regulators are employed. (iii) Terms of first degree in the Riemann curvature tensor,[4]

$$\mathscr{O}_2 R \ , \tag{2.33}$$

with $\mathscr{O}_2$ a general-coordinate—scalar and internal symmetry-invariant operator of canonical dimension two. The allowed forms for $\mathscr{O}_2$ are[3]

$$m_0^2,\ M^2,\ \varphi^2 \ , \tag{2.34}$$

since as shown in Appendix A, $A_{\mu i}A^{\mu i}$ is excluded by gauge invariance. The differential operator $\mathscr{O}_2 = \nabla_\mu \nabla^\mu$ is omitted from the list because $\sqrt{-g}\,\nabla_\mu\nabla^\mu R$ is a total derivative, and $\mathscr{O}_2 = \nabla_\mu A^\mu$, with A^μ an Abelian gauge potential, is omitted because it is not gauge invariant. Moreover, as is also shown in Appendix A, the operator φ^2 is excluded by supersymmetry invariance when φ is a spin-0 partner of a massless supermultiplet.[5] (iv) Terms of second degree in the Riemann curvature tensor,

$$\begin{aligned} \mathscr{G} &= R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2 \ , \\ \mathscr{H} &= C_{\mu\nu\lambda\sigma}C^{\mu\nu\lambda\sigma}, \quad \mathscr{K} = R^2 \ , \end{aligned} \tag{2.35}$$

with $\mathscr{G}$ the Gauss-Bonnet density and $C_{\mu\nu\lambda\sigma}$ the Weyl conformal tensor, which in four dimensions has the form

$$\begin{aligned} C_{\mu\nu\lambda\sigma} = {} & R_{\mu\nu\lambda\sigma} - \tfrac{1}{2}(g_{\mu\lambda}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\lambda} - g_{\nu\lambda}R_{\mu\sigma} + g_{\nu\sigma}R_{\mu\lambda}) \\ & + \tfrac{1}{6}R(g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda}) \ . \end{aligned} \tag{2.36}$$

The results of this enumeration can be summarized in the following lemmas:

Lemma 1. For a general renormalizable matter field theory (spin-0+spin-$\frac{1}{2}$ + gauge spin-1 fields) in curved space-time, quantized in a manner which respects all gauge and supersymmetry internal symmetries, the Lagrangian density terms proportional to R are of the following types,

$$\begin{aligned} & m_0^2 R, \quad m_0 = \text{a bare mass} \ , \\ & M^2 R, \quad M = \text{a massive regulator} \ , \\ & \varphi^2 R, \quad \varphi = \text{a spin}-0 \text{ field not a member} \\ & \qquad \text{of a massless supermultiplet} \ . \end{aligned} \tag{2.37}$$

Lemma 2. If there are no bare masses or massive regulators and if all spin-0 fields belong to massless supermultiplets, then there are no terms in $\mathscr{L}$ proportional to R—that is, terms (iii) above are absent. Moreover, when these conditions are satisfied, terms (ii) above are also absent, and the structure of $\mathscr{L}$ reduces to

$$\begin{aligned} & \mathscr{L}[\{\phi\},g_{\mu\nu}] = \mathscr{L}_{\text{matter}}[\{\phi\},g_{\mu\nu}] + \mathscr{L}_{\text{grav}}[g_{\mu\nu}] \ , \\ & \mathscr{L}_{\text{grav}} = A_0\mathscr{G} + B_0\mathscr{H} + C_0\mathscr{K} \ , \end{aligned} \tag{2.38}$$

with $\mathscr{L}_{\text{matter}}[\{\phi\},g_{\mu\nu}]$ the generally covariant transcription of the flat space-time matter Lagrangian density $\mathscr{L}[\{\phi\}]$. The splitting of $\mathscr{L}$ into the "matter" and "gravitational" parts given in Eq. (2.38) is unique, since in the absence of dimensional constants $\mathscr{L}_{\text{matter}}$ and $\mathscr{L}_{\text{grav}}$ satisfy

$$\begin{aligned} & \mathscr{L}_{\text{matter}}[\{0\},g_{\mu\nu}] = 0 \ , \\ & \mathscr{L}_{\text{grav}}[\eta_{\mu\nu}] = 0 \ , \quad \mathscr{L}_{\text{matter}}[\{\phi\},\eta_{\mu\nu}] = \mathscr{L}[\{\phi\}] \ . \end{aligned} \tag{2.39}$$

D. Conditions for calculability of the gravitational effective action

We are now ready to return to a discussion of the gravitational effective action induced by quantized matter fields on a curved background. Following Eq. (2.16), we define the gravitational effective action by

$$e^{iS_{\text{eff}}[g_{\mu\nu}]} = \int d\{\phi\}\, e^{iS[\{\phi\},g_{\mu\nu}]} \ . \tag{2.40}$$

Since S_{eff} is a scalar under general-coordinate transformations, it may be represented as the integral over the manifold of a scalar density, which for slowly varying metrics can be formally developed in a series expansion in powers of $\partial_\lambda g_{\mu\nu}$,[6]

$$\begin{aligned} & S_{\text{eff}}[g_{\mu\nu}] = \int d^4x\, \sqrt{-g}\, \mathscr{L}_{\text{eff}}[g_{\mu\nu}] \ , \\ & \mathscr{L}_{\text{eff}}[g_{\mu\nu}] = \mathscr{L}_{\text{eff}}^{(0)}[g_{\mu\nu}] + \mathscr{L}_{\text{eff}}^{(2)}[g_{\mu\nu}] + O[(\partial_\lambda g_{\mu\nu})^4] \ , \\ & \mathscr{L}_{\text{eff}}^{(0)}[g_{\mu\nu}] = \frac{1}{16\pi G_{\text{ind}}}(-2\Lambda_{\text{ind}}), \quad \mathscr{L}_{\text{eff}}^{(2)}[g_{\mu\nu}] = \frac{1}{16\pi G_{\text{ind}}}R \ . \end{aligned} \tag{2.41}$$

What are the conditions for G_{ind}^{-1} and Λ_{ind} to be uniquely calculable in terms of the renormalized parameters of the flat space-time matter theory? Clearly, if the fundamental action $S[\{\phi\},g_{\mu\nu}]$ contains terms proportional to R, then the finite renormalization ambiguities arising from these terms will produce an undetermined finite contribution to G_{ind}^{-1}; in this case the induced gravitational con-

[4]No additional counterterms of first order in the curvature tensor can be formed by using the Ricci tensor $R_{\mu\nu}$, since these must have the form $\mathscr{O}_2^{\mu\nu}R_{\mu\nu}$, with $\mathscr{O}_2^{\mu\nu}$ a rank-two symmetric tensor of canonical dimension two. The only possibilities are $\mathscr{O}_2^{\mu\nu} = \nabla^\mu A^\nu + \nabla^\nu A^\mu$, which can be reduced to $\nabla_\mu A^\mu R$ by integration by parts and use of the Bianchi identity $\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R$, and $\mathscr{O}_2^{\mu\nu} = \mathscr{O}_2 g^{\mu\nu}$, which is equivalent to Eq. (2.33) of the text. Similarly, no additional counterterms can be formed by using the Weyl conformal tensor $C_{\mu\nu\lambda\sigma}$, and so the enumeration given in the text is complete.

[5]See Fayet and Ferrara (1977) for a discussion of supersymmetry field representations.

[6]In renormalizable theories, massless particle loops in general give rise to logarithms of $\partial_\lambda g_{\mu\nu}$ in the $(\partial_\lambda g_{\mu\nu})^4$ terms (that is, in the curvature-squared terms) of Eq. (2.41). For example, the existence of a conformal trace anomaly proportional to $\mathscr{H}$ indicates the presence of an effective action term proportional to $\int d^4x\, \sqrt{-g}\, \mathscr{H}\log\mathscr{H}$.

stant is renormalizable, but not calculable. On the other hand, if no terms proportional to R appear in $S[\{\phi\},g_{\mu\nu}]$, then $G_{\rm ind}^{-1}$ will be calculable, since there will now be no source of ambiguity proportional to R.[7] In this case the theory will yield a uniquely determined finite value for $G_{\rm ind}^{-1}$. So we have the following result:

Theorem [Adler (1980a)]. Under the conditions of lemma 2, a quantized matter theory in a curved background produces a calculable induced gravitational constant $G_{\rm ind}^{-1}$.

Consider next the induced cosmological constant $\Lambda_{\rm ind}$, which appears in the effective action in the dimension-four combination $\Lambda_{\rm ind}/G_{\rm ind}$. Ambiguities in $\Lambda_{\rm ind}$ can arise only from dimension-four terms in the flat space-time limit of $S[\{\phi\},g_{\mu\nu}]$ which are not determined by the renormalization conditions on the flat space-time matter theory. The decomposition of Eqs. (2.38)–(2.39) guarantees that no such additional dimension-four terms are present, and so we can conclude:

Theorem. Under the conditions of lemma 2, a quantized matter theory in a curved background produces a calculable induced cosmological constant $\Lambda_{\rm ind}$, and so the entire effective Einstein-Hilbert gravitational action is calculable.

The basic theorems just stated give *sufficient conditions* for the finiteness of the induced gravitational action. Of the three conditions in lemma 1, two—the absence of bare masses, and of scalar fields not in massless supermultiplets—are also necessary conditions. However, the exclusion of massive regulators is not necessary, and in Appendix A the analysis is generalized to the case where massive regulators are employed. As discussed in Sec. 4.4 of Fadde'ev and Slavnov (1980), massive regulators have useful formal properties, but they are awkward to use in explicit calculations. A superior method for diagram evaluations is the technique of dimensional regularization, which is discussed in Sec. III below. The subsequent sections of this review contain elaborations on the theorems of this section. For the theorems to have a nontrivial content, we must have a way of generating a nonzero scale for physical masses even when bare masses are zero (otherwise we get $G_{\rm ind}^{-1}=0$, which is calculable but trivial); this requires dynamical breaking of scale invariance, as discussed in detail in Sec. IV. In Sec. V, we derive explicit, formal expressions for $G_{\rm ind}^{-1}$ and $\Lambda_{\rm ind}$ in terms of expectations of operators in the flat space-time matter vacuum. Finally, in Sec. VI, I extend the discussion to include the effects of quantization of the metric.

III. DIMENSIONAL REGULARIZATION

A. Survey

The regularization of quantum field theory without introducing massive regulators can be accomplished by analytic regularization methods, in which divergent integrals are defined by analytic continuation in a dimensionless parameter (for a review, see Leibbrandt, 1975). It will suffice to limit the discussion to regularization methods for flat space-time, since we will see below (in Sec. V) that after doing the curvature arithmetic needed to extract expressions for $G_{\rm ind}^{-1}$ and $\Lambda_{\rm ind}$, we can explicitly take the flat space-time limit in the resulting formulas. The most widely used form of analytic regularization for flat space-time calculations is dimensional regularization, in which the dimension of the space-time manifold is continued from 4 to 2ω by the coordinate and momentum space replacements

$$\int d^4x \to \int d^{2\omega}x$$
$$\int d^4p \to \int d^{2\omega}p \ , \tag{3.1}$$

while keeping the formal structure of the action, in terms of fields and field derivatives, the same as in dimension four. After Wick rotation to 2ω-dimensional Euclidean space,[8] Feynman integrands in the continued theory are evaluated by using the following simple rules. The Kronecker delta δ^μ_ν obeys the usual composition law

$$\delta^\mu_\nu \delta^\nu_\sigma = \delta^\mu_\sigma \ , \tag{3.2a}$$

but its trace is modified to

$$\delta^\mu_\mu = 2\omega \ . \tag{3.2b}$$

From Eq. (3.2) the symmetric average of momentum factors can be uniquely deduced, giving, for example,

$$\langle p_\mu p_\nu \rangle_{\substack{\text{symmetric}\\ \text{average}}} = \frac{p^2}{2\omega}\delta_{\mu\nu} \ . \tag{3.3}$$

The Dirac γ matrices continue to obey a Clifford algebra

$$\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}1 \ , \tag{3.4a}$$

and are trace normalized so that

$$\mathrm{Tr}(\gamma_\mu\gamma_\nu) = 2^\omega \delta_{\mu\nu} \ , \quad \mathrm{Tr}(1) = 2^\omega \ , \tag{3.4b}$$

permitting one to deduce unique values for all spinor loops not containing an odd number of factors γ_5.[9] Using Eqs. (3.2)–(3.4) and rotational covariance, all perturbation theory calculations can be reduced to multiple integrals of scalar-valued integrands over the momentum space of dimension 2ω.

The basic momentum space integral which appears is

$$\int_E d^{2\omega}p\, f(p) \ , \tag{3.5}$$

and is uniquely specified, up to an overall normalization, by the following three conditions given by Wilson (1973),

[7]This type of argument was first used in connection with the calculability of mass relations by Weinberg (1972).

[8]I will use the notation $\int_E$ to denote a Euclidean integral, and will consistently use a $++++$ metric convention in Euclidean space formulas.

[9]For recent discussions of the dimensional regularization treatment of γ_5, see Gottlieb and Donohue (1979) and Ovrut (1981).

linearity:

$$\int_E d^{2\omega}p[af_1(p)+bf_2(p)]=a\int_E d^{2\omega}p\, f_1(p) + b\int_E d^{2\omega}p\, f_2(p)\ ,$$

translation invariance:

$$\int_E d^{2\omega}p\, f(p+q)=\int_E d^{2\omega}p\, f(p)\ ,$$

scaling law:

$$\int_E d^{2\omega}p\, f(sp)=s^{-2\omega}\int_E d^{2\omega}p\, f(p)\ . \tag{3.6}$$

The normalization which is conventionally used is

$$\int_E d^{2\omega}p\, e^{-p^2}=\pi^{\omega}\ , \tag{3.7}$$

but is fixed only at $\omega=2$; Collins (1975) shows that ambiguities in normalization away from $\omega=2$ can always be absorbed into the ambiguities of the renormalization constants discussed in Sec. II.C. Hence dimensional regularization gives a well-defined procedure for regularizing the ultraviolet divergences of quantum field theory.[10]

Using the rules of Eqs. (3.5)–(3.7), we find, for example, that

$$\int_E d^{2\omega}p(p^2+m^2)^{-\alpha}=\pi^{\omega}\frac{\Gamma(\alpha-\omega)}{\Gamma(\alpha)}(m^2)^{\omega-\alpha}\ . \tag{3.8}$$

For $\omega-\alpha<0$, the integral on the left is convergent in the ultraviolet and yields the expression on the right, which is meromorphic (analytic apart from isolated poles) in ω and α. The integral can then be defined by analytic continuation for $\omega-\alpha>0$, except at points where it develops poles. For example, when $\alpha=1$ we have

$$\int_E d^{2\omega}p\frac{1}{p^2+m^2}=\pi^{\omega}\Gamma(1-\omega)(m^2)^{\omega-1} = \begin{cases} \dfrac{\pi}{1-\omega}+\text{finite}\ , & \text{near } \omega=1 \\ \left[-\dfrac{\pi^2}{2-\omega}+\text{finite}\right]m^2, & \text{near } \omega=2\ , \end{cases} \tag{3.9}$$

showing that the pole at $\omega=1$ is associated with the two-dimensional logarithmically divergent integral

$$\int_E \frac{d^2p}{p^2}\sim\frac{\pi}{1-\omega},\quad \omega\to 1\ , \tag{3.10a}$$

while the pole at $\omega=2$ is associated with the four-dimensional logarithmically divergent integral

$$\int_E d^4p\frac{1}{(p^2)^2}\sim\frac{\pi^2}{2-\omega},\quad \omega\to 2\ . \tag{3.10b}$$

The integral of Eq. (3.10b) would be represented by $\pi^2 \log M^2$ using a conventional massive regulator, giving the useful correspondence

$$\log M^2 \leftrightarrow \frac{1}{2-\omega} \tag{3.11}$$

[10]A detailed axiomatization of the rules of dimensional regularization, along the lines sketched by Wilson (1973), has been given by Collins (unpublished).

between the representation of a four-dimensional logarithmic divergence in the massive regulator and the dimensional regularization schemes. In N-loop order in dimensional regularization, one in general encounters higher powers of logarithmic divergences $1/(2-\omega),\ldots,1/(2-\omega)^N$ near $\omega=2$; these divergences must be cancelled against corresponding poles in the renormalization constants Z in order to extract finite physical amplitudes at dimension four.

B. Vanishing of quadratic divergences

The formally quadratically divergent (and m^2-independent) integral

$$\int_E \frac{d^4p}{p^2} \tag{3.12}$$

is assigned the value 0 by dimensional regularization, since the right-hand side of Eq. (3.9) is proportional to m^2 at $\omega=2$. A more precise statement of this fact is given by the following:

Lemma. The only evaluation of the ultraviolet divergent, infrared convergent massless integral

$$I^{\omega,\alpha}=\int_E d^{2\omega}p(p^2)^{-\alpha},\quad \omega-\alpha>0\ , \tag{3.13}$$

which is meromorphic in ω and α and which agrees with the $m\to 0$ limit of Eq. (3.8), is $I^{\omega,\alpha}=0$. The proof follows immediately from the observations that: (i) when $\omega-\alpha>0$ is not a positive integer, the limit as $m\to 0$ of Eq. (3.8) exists, and is 0; and (ii) the only meromorphic extension (to $\omega-\alpha=$positive integer) of 0 is 0.[11] Working from $I^{\omega,\alpha}=0$, we can now prove the vanishing of

$$I^{\omega,\alpha,\beta}=\int_E d^{2\omega}p(p^2)^{-\alpha}(\log p^2)^{-\beta},\quad \omega-\alpha>0 \tag{3.14}$$

by repeated differentiation of $I^{\omega,\alpha}$ with respect to α, and by repeated application of the Weyl transform (Erdélyi, 1954)

$$W^{\beta}I^{\omega,\alpha,\beta'}\equiv\frac{1}{\Gamma(\beta)}\int_{\alpha}^{i\infty}d\gamma(\gamma-\alpha)^{\beta-1}I^{\omega,\gamma,\beta'} = \int_E d^{2\omega}p[W^{\beta}(p^2)^{-\alpha}](\log p^2)^{-\beta'}\ . \tag{3.15}$$

Since

$$W^{\beta}(p^2)^{-\alpha}=\frac{1}{\Gamma(\beta)}(p^2)^{-\alpha}\int_0^{i\infty}d\delta\,\delta^{\beta-1}(p^2)^{-\delta} =(p^2)^{-\alpha}(\log p^2)^{-\beta},\quad 0<\beta<1\ , \tag{3.16}$$

we have

[11]Any nonzero evaluation of $I^{\omega,\alpha}$ (such as the one given by Leibbrandt, 1975) is thus necessarily not a meromorphic function of ω. Such evaluations violate the basic philosophy of analytic regularization, which is essentially a calculus of meromorphic functions. The vanishing of $I^{\omega,\alpha}$ in dimensional regularization was first noted by 't Hooft and Veltman (1972), and is deduced as a theorem in the axiomatization of Collins (unpublished).

$$W^{\beta}I^{\omega,\alpha,\beta'}=I^{\omega,\alpha,\beta'+\beta},\quad 0<\beta<1\ ,$$
$$(-\partial/\partial\alpha)I^{\omega,\alpha,\beta'}=I^{\omega,\alpha,\beta'-1}\ , \tag{3.17}$$

and so by repeated operations any value of β can be reached, starting from $\beta=0$, where we have $I^{\omega,\alpha,0}=I^{\omega,\alpha}=0$. By continuing this procedure with respect to the index β we can generate powers of $\log\log p^2$, etc., giving finally:

Lemma. In dimensional regularization, for $\omega-\alpha>0$ we have

$$I^{\omega,\alpha,\beta,\gamma,\cdots}=\int_E d^{2\omega}p(p^2)^{-\alpha}(\log p^2)^{-\beta}(\log\log p^2)^{-\gamma}\cdots =0\ . \tag{3.18}$$

In particular, the generalized quadratically divergent integral vanishes,

$$I^{2,1,\beta,\gamma,\cdots}=\int_E \frac{d^4p}{p^2}(\log p^2)^{-\beta}(\log\log p^2)^{-\gamma}\cdots =0\ . \tag{3.19}$$

This result shows that computing radiative corrections to the basic quadratically divergent integral of Eq. (3.12) always gives 0, independently of whether one proceeds order by order in perturbation theory, which gives only positive powers of $\log p^2$ (corresponding to $I^{2,1,-n}$), or whether one uses the renormalization group to sum powers of $\log p^2$ into running coupling constant factors (see Sec. IV.C below), giving the more general integral of Eq. (3.18).

C. An application: the stress tensor trace anomaly in gauge theories

As an application of dimensional regularization, let us derive, to one-loop order, the flat space-time stress tensor trace anomaly in QED $\frac{1}{2}$. In spinor quantum electrodynamics, the symmetrized stress energy tensor is given by

$$T_{\mu\nu}=\tfrac{1}{4}\eta_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}-F_{\lambda\mu}F^{\lambda}{}_{\nu}+\frac{i}{4}[\bar\psi(\gamma_\nu D_\mu+\gamma_\mu D_\nu)\psi-\bar\psi(\overleftarrow{D}_\mu\gamma_\nu+\overleftarrow{D}_\nu\gamma_\mu)\psi]\ ,$$
$$D_\mu=\partial_\mu+ie_0A_\mu,\quad \overleftarrow{D}_\mu=\overleftarrow{\partial}_\mu-ie_0A_\mu\ . \tag{3.20}$$

Contracting with $\eta^{\mu\nu}$ and using Eq. (3.2b) and the spinor equation of motion

$$i\gamma^\mu D_\mu\psi=m_0\psi\ ,\qquad -i\bar\psi\overleftarrow{D}_\mu\gamma^\mu=m_0\bar\psi\ , \tag{3.21}$$

we get

$$T^\mu_\mu=-2(2-\omega)\tfrac{1}{4}F_{\lambda\sigma}F^{\lambda\sigma}+\bar\psi m_0\psi\ . \tag{3.22}$$

Although the first term on the right-hand side of Eq. (3.22) is proportional to $2-\omega$, it cannot be dropped as $\omega\to2$ because the factor $F_{\lambda\sigma}F^{\lambda\sigma}$ contains a pole series in $(2-\omega)^{-1}$. To exhibit these poles explicitly, Eqs. (2.24)−(2.26) and Eq. (3.11) are used to write

$$F_{\lambda\sigma}F^{\lambda\sigma}=Z_e^{-1}F^r_{\lambda\sigma}F^{r\lambda\sigma}\ ,$$
$$Z_e=1+\frac{\alpha_0}{3\pi}\log M^2+O(\alpha_0^2)\leftrightarrow 1+\frac{\alpha_0}{3\pi}\frac{1}{2-\omega}+O(\alpha_0^2)\ ,$$
$$Z_e^{-1}=1-\frac{\alpha_0}{3\pi}\frac{1}{2-\omega}+\left[\frac{\alpha_0}{3\pi}\right]^2\frac{1}{(2-\omega)^2}+\cdots\ , \tag{3.23}$$

where we have worked to one-loop order in the photon proper self-energy, and to iterated one-loop order in Z_e^{-1}. Substituting Eq. (3.23) into the first term of Eq. (3.22), we get (in the limit as $\omega\to2$)

$$-2(2-\omega)Z_e^{-1}=\frac{2\alpha_0}{3\pi}\left[1-\frac{\alpha_0}{3\pi}\frac{1}{2-\omega}+\left[\frac{\alpha_0}{3\pi}\right]^2\frac{1}{(2-\omega)^2}+\cdots\right]=\frac{2\alpha_0}{3\pi}Z_e^{-1}=\frac{2\alpha}{3\pi},\quad \alpha=\frac{e^2}{4\pi}\ . \tag{3.24}$$

Hence to one-loop order the stress energy tensor trace is

$$T^\mu_\mu=\frac{2\alpha}{3\pi}\tfrac{1}{4}F^r_{\lambda\sigma}F^{r\lambda\sigma}+\bar\psi m_0\psi\ . \tag{3.25}$$

The first term on the right-hand side, found by Coleman and Jackiw (1971), Crewther (1972), and Chanowitz and Ellis (1972, 1973), would be lost if one naively used the equations of motion without attention to regularization, and is called the trace anomaly. The derivation given above can be generalized to all orders in perturbation theory (Adler, Collins, and Duncan, 1977; Nielsen, 1977) and yields

$$T^\mu_\mu=\frac{2\beta(e)}{e}\tfrac{1}{4}(F_{\lambda\sigma}F^{\lambda\sigma})^r+[1+\delta(e)](\bar\psi m_0\psi)^r\ , \tag{3.26}$$

with β and δ finite functions of e which are defined through the renormalization group, and with the splitting of T^μ_μ into the two terms on the right-hand side made unique by the specification of certain zero momentum-transfer matrix elements of the composite operators $(F_{\lambda\sigma}F^{\lambda\sigma})^r$ and $(\bar\psi m_0\psi)^r$. The analogous formula for QCD (obtained by Collins, Duncan, and Joglekar, 1977 and Nielsen, 1977) reads[12]

$$T^\mu_\mu=\frac{2\beta(g)}{g}\tfrac{1}{4}(F^i_{\lambda\sigma}F^{i\lambda\sigma})^r+[1+\delta(g)](\bar\psi m_0\psi)^r\ . \tag{3.27}$$

For a pure SU(n) gauge theory with no quarks, the second term on the right-hand side is absent, and the trace anomaly formula simplifies to

[12]I have dropped equation of motion terms, which both vanish at nonzero momentum transfer and have vanishing zero-momentum−transfer vacuum expectation values.

$$T^{\mu}_{\mu}=\frac{2\beta(g)}{g}\tfrac{1}{4}(F^{i}_{\lambda\sigma}F^{i\lambda\sigma})^{r}\ . \tag{3.28}$$

Equation (3.28) will play an important role in the analysis, given in Sec. V.D below, of G_{ind}^{-1} and Λ_{ind} in an SU(n) gauge theory.

IV. SYMMETRY BREAKDOWN

A. Models with elementary scalars

Spontaneous symmetry breaking plays a crucial role in constructing gauge theory models, since it permits generation of the gauge boson masses needed to get realistic low-energy effective actions, while preserving the ultraviolet cancellations which guarantee renormalizability. The simplest model exhibiting spontaneous symmetry breaking is the scalar φ^4 field theory of Eq. (2.4),

$$\begin{aligned}\mathscr{L}&=T-V\ ,\\ T&=\tfrac{1}{2}(\partial_0\varphi)^2\ ,\\ V&=\tfrac{1}{2}(\partial_i\varphi)^2+\tfrac{1}{2}m_0^2\varphi^2+\tfrac{1}{4}\lambda_0\varphi^4\ .\end{aligned} \tag{4.1}$$

For constant φ, the potential V reduces to

$$V(\varphi)=\tfrac{1}{2}m_0^2\varphi^2+\tfrac{1}{4}\lambda_0\varphi^4\ , \tag{4.2}$$

and has the behavior sketched in Fig. 1. In the conventional case $m_0^2>0$, the potential has a single stable minimum at $\varphi=0$, as shown in Fig. 1(a). However, when the sign of m_0^2 is reversed to $m_0^2<0$, the extremum at $\varphi=0$ becomes unstable, and V develops a pair of stable minima at

$$\begin{aligned}\varphi&=\pm\overline{\varphi}\ ,\\ \overline{\varphi}&=(-m_0^2/\lambda_0)^{1/2}\ ,\end{aligned} \tag{4.3}$$

as shown in Fig. 1(b). Either the minimum at $\varphi=\overline{\varphi}$ or the minimum at $\varphi=-\overline{\varphi}$ can be used as the zeroth-order approximation in a perturbation expansion, by making a shift

$$\varphi=\pm\overline{\varphi}+\varphi' \tag{4.4}$$

and taking φ' as the new field variable. Mixing between the two configurations is not possible, because in the limit of an infinite space-time volume, they are separated by an infinite quantum-mechanical tunneling barrier. Thus the discrete $\varphi\leftrightarrow-\varphi$ symmetry of the Lagrangian is broken by the choice of one of the two classical minima as the quantum mechanical vacuum state.

The simplest field theory model in which a continuous symmetry is broken is obtained by making φ a complex scalar field

$$\varphi=\varphi_1+i\varphi_2\ , \tag{4.5}$$

with Lagrangian density

$$\mathscr{L}=\tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi^*-\tfrac{1}{2}m_0^2\varphi^*\varphi-\tfrac{1}{4}\lambda_0(\varphi^*\varphi)^2\ . \tag{4.6}$$

When $m_0^2<0$ the potential V has the behavior sketched in Fig. 2, and a suitable quantum-mechanical vacuum is obtained by making the shift

$$\varphi\rightarrow\overline{\varphi}+\varphi'\ , \tag{4.7a}$$

with $\overline{\varphi}$ any complex constant scalar satisfying

$$|\overline{\varphi}|^2=-m_0^2/\lambda_0\ . \tag{4.7b}$$

In this case a continuous symmetry is broken, and the excitation φ' which generates an infinitesimal rotation of $\overline{\varphi}$ is a zero-mass Goldstone mode. When the complex scalar field of Eqs. (4.5)–(4.6) is minimally coupled to a spin-1 gauge field, the zero-mass mode decouples from the physical degrees of freedom, and the spin-1 field be-

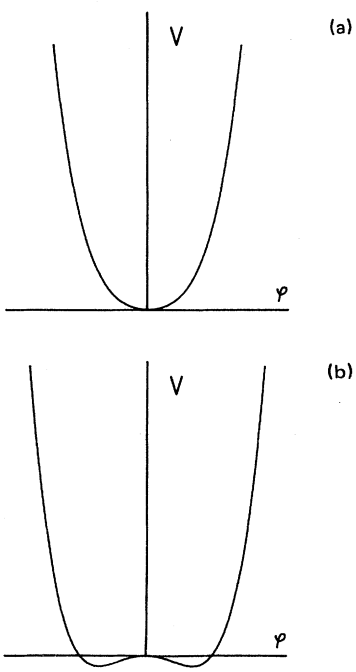

FIG. 1. (a) Potential V of Eq. (4.2), for $m_0^2>0$. (b) Potential V of Eq. (4.2), for $m_0^2<0$.

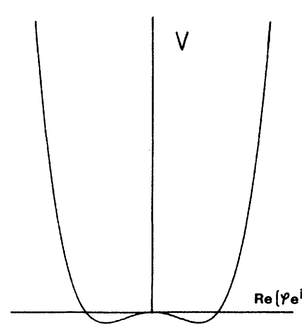

FIG. 2. Potential V in the complex scalar model, obtained by substituting $\varphi^2\rightarrow\varphi^*\varphi$ in Eq. (4.2). The curve shows an arbitrary section $\text{Im}(\varphi e^{i\theta})=0$ of the potential surface, $0\le\theta<2\pi$, plotted vs $\text{Re}(\varphi e^{i\theta})$.

comes massive. This is the so-called Higgs mechanism, which is used in the Weinberg-Salam model to generate intermediate vector boson masses. (For a detailed pedagogical review of these ideas and full references, see Bernstein, 1974).

The suggestion of linking spontaneous scale symmetry breaking with generation of the gravitational constant first appeared in the context of scalar meson models [see Fujii (1974), Englert, Truffin, and Gastmans (1976), Minkowski (1977), Chudnovsky (1978), Matsuki (1978), Smolin (1979), Zee (1979), Linde (1979, 1980) and Nieh (1982)]. The basic mechanism of the above-cited papers is to start from a Lagrangian density of the form

$$\mathscr{L}=\epsilon\varphi^2 R+T-V(\varphi^2)\ , \tag{4.8}$$

with V a symmetry-breaking potential as in Eq. (4.2). In the unstable symmetric phase $\overline{\varphi}=0$ there is no order-R term in $\mathscr{L}$, but in the stable broken-symmetry phase with $\overline{\varphi}^2=-m_0^2/\lambda_0$ an induced gravitational action is generated with

$$\frac{1}{16\pi G_{\text{ind}}}=\varepsilon\overline{\varphi}^2\ . \tag{4.9}$$

In such models, since both scalar fields and dimensional parameters ($m_0\neq 0$) appear, the induced gravitational constant is not calculable[13]; ε is an additional curved space-time parameter of the theory which is not determined by the flat space-time renormalized parameters (Brown and Collins, 1980).

B. Dynamical symmetry breaking: the renormalization group in asymptotically free gauge theories

In order to get a calculable and nonvanishing induced gravitational constant, we must turn our attention to field theory models with dynamical scale-invariance breaking. Such theories, by definition, are formally scale invariant at the classical Lagrangian or tree-approximation level, but exhibit spontaneous scale-invariance breaking as a result of quantum corrections in one- or higher-loop order. There are two reasonably well understood mechanisms by which dynamical scale-invariance breaking can occur. The first, which will be discussed in this section, is through the renormalization process itself, in infrared-singular theories such as unbroken non-Abelian gauge theories. The second, which will be described in Sec. IV.C below, is through the generation of a mass gap and a fermion pair condensate in relativistic versions of the Bardeen-Cooper-Schrieffer (BCS, 1957) theory of superconductivity. The two mechanisms are not really disjoint, and both are believed to be operative in non-Abelian gauge theories. This fact and some further gauge theory-superconductor analogies are discussed briefly in Sec. IV.D. The material which follows has been organized so that the reader who wishes to proceed most directly to the gravitational applications of Secs. V and VI can do so after reading Sec. IV.B alone.

The most important class of field theory models exhibiting dynamical spontaneous scale-invariance breaking are asymptotically free gauge theories [see 't Hooft (unpublished), Gross and Wilczek (1973), and Politzer (1973)]. Consider an SU(n) non-Abelian gauge field coupled to N_f massless fermions in the fundamental representation, as is described, for example, by $\mathscr{L}_{\text{QCD}}$ of Eq. (2.12) with $m_0=0$ and with ψ replicated N_f times. In tree approximation this theory contains no dimensional parameters, and so scale invariance is unbroken; moreover, since there are no scalar fields, all of the conditions of the theorems of Sec. II are satisfied. Let us now consider the effect of quantum corrections to the tree-approximation theory. When radiative corrections are included, the coupling constant g appears in calculations through the running coupling constant

$$g^2(-q^2)=\frac{g^2(\mu^2)}{1+\frac{1}{2}b_0 g^2(\mu^2)\log(-q^2/\mu^2)+\cdots}\ , \tag{4.10}$$

with q^2 the four-momentum squared, μ^2 an arbitrary subtraction point, and $g^2(\mu^2)$ the value of the coupling constant at $-q^2=\mu^2$. The appearance of the subtraction mass μ^2 is necessitated by the fact that radiative corrections to massless gauge theories are highly infrared divergent, making it impossible to introduce a renormalized coupling parameter by specifying the value of g^2 at $q^2=0$, as is done in the more familiar case of quantum electrodynamics. The parameter b_0 is determined by one-loop radiative corrections to be

$$b_0=\frac{1}{8\pi^2}\left[\frac{11}{3}n-\frac{2}{3}N_f\right]\ , \tag{4.11}$$

and is positive, provided that N_f is not too large. When b_0 is positive, Eq. (4.10) shows that the running coupling vanishes at large four-momentum squared, and the theory in this case is said to be asymptotically free.

Let us examine the structure of Eq. (4.10) in the approximation in which only one-loop radiative corrections are retained, while the higher-loop contributions to the running coupling constant, denoted by ..., are neglected. (As discussed in Appendix B.1, there is a well-defined sense in which a one-loop analysis is exact.) Evaluating Eq. (4.10) at $-q^2=\mu_1^2$, we get

$$\begin{aligned}\frac{1}{g^2(\mu_1^2)}-\frac{1}{g^2(\mu^2)}&=\tfrac{1}{2}b_0\log\left[\frac{\mu_1^2}{\mu^2}\right]\\ \Longrightarrow\frac{1}{g^2(\mu_1^2)}-\tfrac{1}{2}b_0\log\mu_1^2&=\frac{1}{g^2(\mu^2)}-\tfrac{1}{2}b_0\log\mu^2\ ,\end{aligned} \tag{4.12}$$

showing that the scale mass $\mathscr{M}(g(\mu),\mu)$ defined by

[13]Strictly speaking, to get a renormalizable model an additional term $\delta m_0^2 R$ must be included in $\mathscr{L}$; the spontaneous symmetry breaking then generates a change in the constant factor multiplying R from δm_0^2 to $\delta m_0^2+\epsilon\overline{\varphi}^2$.

$$\mathscr{M}(g(\mu),\mu)=\mu e^{-1/[b_0 g^2(\mu^2)]} \tag{4.13}$$

is subtraction-point independent. In technical terminology, the scale mass $\mathscr{M}(g(\mu),\mu)$ is said to be renormalization group[14] invariant to one-loop order, since it is left unchanged to this order by transformations of the renormalization point μ^2 and the renormalized coupling constant $g^2(\mu^2)$. When radiative corrections to all orders are kept, Eq. (4.13) generalizes to (Gross and Neveu, 1974; Lane, 1974a)

$$\mathscr{M}(g(\mu),\mu)=\mu e^{-\int^{g(\mu)} dg'/\beta(g')}, \tag{4.14}$$

with

$$\beta(g)=-\tfrac{1}{2}b_0 g^3+O(g^5) \tag{4.15}$$

the function appearing in the trace anomaly formula of Eq. (3.27), and again $\mathscr{M}(g,\mu)$ is said to be renormalization group invariant. An alternative, and frequently used, way of specifying that $\mathscr{M}$ has the functional form of Eq. (4.14) is obtained by requiring that $\mathscr{M}$ satisfy the Callan (1970)-Symanzik (1970) differential equation

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right]\mathscr{M}(g,\mu)=0\ . \tag{4.16}$$

Let us now apply the above analysis to determine the structure of physically observable parameters, such as effective action parameters. Since observables must be subtraction-point independent, they can depend on μ only through the scale mass $\mathscr{M}(g,\mu)$, and so we get the following important result:

Theorem [Gross and Neveu (1974)]. Any physical parameter $P(g,\mu)$ which has canonical dimension d_P in the accounting of Sec. II.A must be equal to $[\mathscr{M}(g,\mu)]^{d_P}$ up to a calculable number,

$$P(g,\mu)=\text{calculable number}\times[\mathscr{M}(g,\mu)]^{d_P}\ . \tag{4.17}$$

Equivalently, $P(g,\mu)$ must satisfy the homogeneous renormalization group equation

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\right]P(g,\mu)=0\ , \tag{4.18}$$

which for a quantity of canonical dimension d_P implies Eq. (4.17).

According to this theorem, it is the dimensional scale mass $\mathscr{M}$, rather than the dimensionless (but subtraction-point dependent) renormalized coupling $g^2(\mu^2)$, which in asymptotically free gauge theories plays a role analogous to that played by the renormalized fine-structure constant α in quantum electrodynamics. In other words, the renormalization process has replaced a one-parameter family of unrenormalized theories, characterized by their values of the dimensionless unrenormalized gauge coupling g_0, by a one-parameter family of renormalized theories, characterized by their values of the dimension-one scale mass $\mathscr{M}(g,\mu)$. This change in dimensionality of the effective parameter, when radiative corrections are included, clearly implies that there has been a dynamical breaking of scale invariance. The general phenomenon is called dimensional transmutation, after Coleman and Weinberg (1973), who discovered similar behavior in massless QED 0 (a theory which, like the massless non-Abelian gauge theory, is highly infrared divergent.)

C. Dynamical symmetry breaking: relativistic generalizations of the superconductor gap equation

Historically, the earliest suggestion that dynamical symmetry breaking plays an important role in particle physics was contained in the classic paper of Nambu and Jona-Lasinio (1961), who proposed a model for nucleon mass generation[15] based on an analogy with the BCS theory of superconductivity.[16] The Nambu—Jona-Lasinio model starts from a Lagrangian containing massless, interacting fermions, and then sets up a self-consistent equation for the dynamically generated fermion mass in analogy with the "gap equation" of superconductivity. In this section, I give a very schematic account of the basic approximation method used in the BCS and Nambu—Jona-Lasinio models, and show that it gives a dynamical version of the tree-approximation model for symmetry breaking described in Sec. IV.A.

Let us consider a fermion with bare propagator G_0^{-1}, proper self-energy part Σ, and full propagator G^{-1}, related to one another as usual by

$$G^{-1}=G_0^{-1}-\Sigma\ . \tag{4.19}$$

Assuming the fermions interact through a potential V, a simple self-consistent approximation for the proper self-energy is obtained by truncating the Dyson equation for Σ to include only the lowest-order skeleton diagram illustrated in Fig. 3. This gives

$$\begin{aligned}\Sigma&=\int VG\\ &=\int V(G_0^{-1}+\Sigma)[G_0^{-2}-\Sigma^2]^{-1}\ ,\end{aligned} \tag{4.20}$$

where $\int$ indicates symbolically a summation or integration over intermediate state (closed loop) variables. In models with dynamical symmetry breaking, the unbroken symmetry of the classical Lagrangian can be shown to

[14]For a pedagogical discussion of the renormalization group structure of non-Abelian gauge theories, see Stevenson (1981).

[15]A very important aspect of the Nambu—Jona-Lasinio model, which is not dealt with in this review, is the generation of the pion as a zero-mass bound state. There has been recent interest in analogs of this phenomenon in which the Higgs scalars or pseudoscalars in unified theories are dynamically generated composites of more fundamental fields; see Englert and Brout (1964); Jackiw and Johnson (1973); Cornwall and Norton (1973); Weinberg (1976); and Susskind (1979).

[16]For texts on the BCS theory, see Schrieffer (1964) and Fetter and Walecka (1971). The Ginzburg-Landau phenomenological theory is also described in these books.

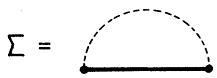

FIG. 3. Truncated Dyson equation for the self-energy part. The dashed line and dots denote the potential V in the BCS case, or the photon propagator and emission and absorption vertices in the JBW model case. The heavy line denotes a full electron propagator $G=(G_0^{-1}-\Sigma)^{-1}$, giving a nonlinear integral equation (the gap equation) for Σ.

imply that

$$\int VG_0^{-1}[G_0^{-2}-\Sigma^2]^{-1}=0 \ . \tag{4.21}$$

Substituting Eq. (4.21) into Eq. (4.20) then gives the general form of the "gap equation" for Σ,

$$\Sigma=\int V\Sigma[G_0^{-2}-\Sigma^2]^{-1} \ . \tag{4.22}$$

Equation (4.22) always has a trivial solution $\Sigma=0$, analogous to the trivial root $\varphi=0$ of the equation

$$0=V'(\varphi)=\varphi(m_0^2+\lambda_0\varphi^2) \ , \tag{4.23}$$

which governs the vacuum structure of the scalar meson model discussed in Sec. IV.A. However, when V has the (attractive) sign for which dynamical symmetry breaking occurs, there is also a nontrivial solution to Eq. (4.22), corresponding symbolically to the root of

$$1=\int V[G_0^{-2}-\Sigma^2]^{-1} \ , \tag{4.24}$$

and analogous to the symmetry-breaking roots $\varphi=\pm\bar{\varphi}$ of Eq. (4.23).

To solve Eq. (4.24) explicitly in the case of the BCS model, we make substitutions appropriate to the nonrelativistic kinematics of the superconductor problem [see Schrieffer (1964)],

$$\begin{aligned} &\int = i\int\frac{dk_0}{\pi}\int d^3k \ , \\ &G_0^{-2}=k_0^2-(k^2-k_F^2)^2+i\varepsilon \ , \\ &\Sigma^2=\Delta^2 \ , \end{aligned} \tag{4.25}$$

where k_F is the Fermi momentum, and we carry out the k_0 integration. Equation (4.24) then yields an algebraic equation for the energy gap characterizing the low-lying electronic excitations in a superconductor,

$$1=V\int_{|k^2-k_F^2|=0}^{|k^2-k_F^2|=\omega_D} d^3k\frac{1}{[(k^2-k_F^2)^2+\Delta^2]^{1/2}} \ , \tag{4.26}$$

with ω_D the Debye frequency, which serves as an effective ultraviolet cutoff in the BCS model. Because phase space in the neighborhood of the Fermi momentum is effectively one dimensional,

$$d^3k \approx 4\pi k_F^2 dk \ , \tag{4.27}$$

Eq. (4.26) is logarithmically divergent at the lower limit when $\Delta=0$, and for small Δ can be approximated by

$$1=NV\int_{c\Delta}^{\omega_D}\frac{d\omega}{\omega}=NV\log\left[\frac{\omega_D}{c\Delta}\right] \ , \tag{4.28}$$

with N the density of states at the Fermi surface and c a numerical factor of order unity. Solving Eq. (4.28) for Δ gives

$$\Delta=\frac{1}{c}\omega_D\exp\left[-\frac{1}{NV}\right] \ , \tag{4.29}$$

showing that the energy gap has a nonperturbative dependence on the interaction strength V, with an essential singularity at $V=0$. The detailed analysis of the BCS model shows that the energy gap Δ is proportional to the ground-state expectation value of a product of creation (or annihilation) operators for two electrons, with opposite momenta lying near the Fermi surface and opposite spins,

$$\Delta \propto \left\langle \psi_{\mathbf{k}\uparrow}^{\dagger}\psi_{-\mathbf{k}\downarrow}^{\dagger}\right\rangle_0 \ , \quad |\mathbf{k}| \sim k_F \ . \tag{4.30}$$

Thus, the presence of a nonvanishing energy gap in a superconductor implies the existence of a ground-state condensate of correlated electron pairs.

An analogous reduction of Eq. (4.22) (now using relativistic kinematics) can be carried out for the Nambu–Jona-Lasinio model and for its more recent gauge-theoretic extensions, in which the nonrenormalizable local four-fermion interaction used by Nambu and Jona-Lasinio is replaced by a renormalizable interaction mediated by vector meson exchange. [See Johnson, Baker, and Willey (1964), Jackiw and Johnson (1973), Cornwall and Norton (1973), and Lane (1974b).] For definiteness, let us consider the case of the Johnson-Baker-Willey (JBW, 1964) model for fermion mass generation in Abelian electrodynamics. These authors consider zero-bare mass spinor electrodynamics [that is, $\mathscr{L}_{\mathrm{QED}\,1/2}$ of Eq. (2.10), with $m_0=0$] in the approximation in which all photon self-energy parts are neglected. The dashed line in Fig. 3 then represents a bare photon propagator; thus to leading order of perturbation theory for the vertex parts, the analog of Eq. (4.22) is

$$\begin{aligned} \Sigma(p) \sim i\alpha_0\int d^4k\frac{1}{k^2}\Sigma(p-k) \\ \times[(p-k)^2-\Sigma(p-k)^2]^{-1} \ , \end{aligned} \tag{4.31}$$

where $\sim$ indicates that numerical constants of order unity have been omitted. In addition to the trivial solution $\Sigma=0$, Eq. (4.31) has a nonperturbative solution in which Σ has the asymptotic behavior

$$\Sigma(p) \sim m\left[\frac{m^2}{-p^2}\right]^{\delta} , \ \ \delta\sim\alpha_0 \ . \tag{4.32}$$

Equation (4.32) gives self-consistency because

$$\begin{aligned} i\int d^4k\frac{1}{k^2}\frac{m}{(p-k)^2}\left[\frac{m^2}{-(p-k)^2}\right]^{\delta} \\ \sim\frac{1}{\delta}m\left[\frac{m^2}{-p^2}\right]^{\delta} \sim\frac{1}{\alpha_0}\Sigma(p) \ , \end{aligned} \tag{4.33}$$

which follows from angular averaging and the elementary integral

$$\int_A^\infty \frac{dB}{B} B^{-\delta} = \frac{A^{-\delta}}{\delta} . \tag{4.34}$$

The parameter m in Eq. (4.32) is an arbitrary integration constant introduced by the boundary condition

$$\Sigma(p^2 = -m^2) = m , \tag{4.35}$$

and clearly corresponds to an electron physical mass. We see that as a result of dynamical symmetry breaking a mass scale has appeared in the solution to Eq. (4.31), even though no mass scale appears in the integral equation itself or in the fundamental Lagrangian from which it was derived. The vanishing of m_0 is mirrored in the fact that $\Sigma(p)$ has a softer ultraviolet behavior

$$\Sigma(p) \underset{p^2 \to \infty}{\longrightarrow} 0 \tag{4.36}$$

than would be found if a mass scale were introduced kinematically into the Lagrangian. Such ultraviolet softness (seen also in the discussion of asymptotically free gauge theories in Sec. IV.B above) is a very general feature of field theory models where the mass scale is introduced through dynamical scale-invariance breaking. The detailed analysis of the JBW and other Nambu—Jona-Lasinio type models shows that, associated with the generation of a nonvanishing fermion physical mass, the ground state contains a fermionic condensate, this time involving a nonvanishing fermion-antifermion expectation value of the form $\langle \bar{\psi}\psi \rangle_0$.

D. Gauge theory-superconductor analogies

Comparing Eq. (4.13) with Eq. (4.29), we see that there is a close similarity between the nonperturbative structure of the gauge theory one-loop scale mass $\mathscr{M}(g,\mu)$ and that of the superconductor energy gap Δ. As was noted in connection with Eqs. (4.26)—(4.28) above, the $e^{-1/NV}$ form in the superconductor case arises from the effectively one-dimensional phase space near the Fermi surface, which produces a logarithmically divergent one-loop perturbation theory contribution

$$\int \frac{d^3k}{k^2 - k_F^2} \sim \int_{k_F}^{k_{\max}} \frac{dk}{k - k_F} . \tag{4.37}$$

Similarly, the e^{-1/g^2} form in the gauge theory case arises from the logarithmic divergence of the one-loop contribution to $g^2(-q^2)^{-1}$ at $q^2=0$, which in turn comes from the nonvanishing and effectively one-dimensional phase space for a massless particle to decay into two massless particles, as expressed in the identity

$$|\mathbf{k}_1|\,|\mathbf{k}_2|\,\delta^3(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\delta(|\mathbf{k}|-|\mathbf{k}_1|-|\mathbf{k}_2|) = 2\pi \int_0^1 dx\, \delta^3(\mathbf{k}_2 - \mathbf{k}x)\delta^3[\mathbf{k}_1 - \mathbf{k}(1-x)] . \tag{4.38}$$

To see the effect of Eq. (4.38), let us recall that the S-wave phase space for a pair of particles of mass m, at center of mass energy $\sqrt{s}$, is

$$\rho(s) = \left[\frac{s - 4m^2}{s} \right]^{1/2} , \tag{4.39}$$

and vanishes at threshold for $m > 0$. However, when $m=0$, Eq. (4.39) reduces to $\rho(s)=1$, which is nonvanishing at threshold as required by Eq. (4.38). Consequently, the one-loop perturbation-theory integral

$$\int_0^{s_{\max}} \frac{ds'\rho(s')}{s' - q^2} \tag{4.40}$$

is logarithmically divergent at $q^2=0$.

As suggested by this phase-space analysis, and as discussed in more detail by Gross and Neveu (1974) and Lane (1974a, 1974b), the renormalization group mechanism for dynamical symmetry breaking on the one hand, and the superconductor gap equation mechanism on the other, are really two complementary aspects of the dynamical symmetry breaking which occurs in non-Abelian gauge theories. The gauge theory-superconductor analogy can be carried considerably further. Just as a superconductor contains an electron pair condensate proportional to the energy gap Δ, quantum chromodynamics contains a fermionic condensate $\langle \bar{\psi}\psi \rangle_0$ proportional to the third power $\mathscr{M}^3$ of the gauge theory scale mass $\mathscr{M}$, and very likely[17] contains a gluonic condensate $\langle F^i_{\lambda\sigma} F^{i\lambda\sigma} \rangle_0$ proportional to $\mathscr{M}^4$. When a superconductor and its energy gap are perturbed by a weakly varying electromagnetic field, the resulting dynamics is described by the induced effective action of the Ginzburg-Landau theory.[16] Correspondingly, when a non-Abelian gauge theory and its scale mass are perturbed by a weakly varying metric, the resulting dynamics, as we will see in detail below, is described by an induced effective action of the Einstein-Hilbert form.[18]

V. INDUCED GRAVITATIONAL AND COSMOLOGICAL CONSTANTS FOR MATTER THEORIES ON A BACKGROUND MANIFOLD

A. Path-integral derivation of formulas for G_{ind}^{-1} and Λ_{ind}

From the viewpoint of the theorem of Gross and Neveu discussed in Sec. IV.B, the induced gravitational

[17]For discussions of gluon pairing see Batalin, Matinyan, and Savvidi (1977); Savvidy (1977); Pagels and Tomboulis (1978); Vainstein, Zakharov, and Shifman (1978); Ambjörn and Olesen (1980); Fukuda and Kazama (1980); Kazama (1980); and Milton (1981). See also Sec. V.D below.

[18]The superconductor phase space analogy is discussed briefly in the "photon pairing" paper of Adler *et al.* (1976). One conclusion of their paper, that photon ladders cannot generate a graviton in flat space-time, is a special case of a recent general theorem of Witten and Weinberg (1980). The remainder of their paper and a subsequent paper of Adler (1976) attempted, unsuccessfully, to generate a gap equation as a curvature effect in a model which has no gap equation in the absence of curvature.

constant $G_{\rm ind}^{-1}$ and cosmological constant $\Lambda_{\rm ind}$ of a gauge field theory are simply physical parameters of canonical dimension two, defined through the response of the gauge field system to local perturbations in the space-time metric. This suggests that it should be possible to take formal derivatives with respect to deviations of the metric $g_{\mu\nu}$ from the Minkowski metric $\eta_{\mu\nu}$, thereby extracting expressions for $G_{\rm ind}^{-1}$ and $\Lambda_{\rm ind}$ in terms of flat space-time vacuum expectation values. Such an analysis will be carried out in this section, using the metric and curvature conventions of Misner *et al.* (1970).

The starting point of the derivation is the basic definition of the gravitational effective action given in Eq. (2.40) above,

$$e^{iS_{\rm eff}[g_{\mu\nu}]}=\int d\{\phi\}e^{iS[\{\phi\},g_{\mu\nu}]}\,, \tag{5.1}$$

with

$$S_{\rm eff}[g_{\mu\nu}]=\int d^4x\sqrt{-g}\left[\frac{1}{16\pi G_{\rm ind}}(R-2\Lambda_{\rm ind})+O[(\partial_\lambda g_{\mu\nu})^4]\right], \tag{5.2a}$$

$$S[\{\phi\},g_{\mu\nu}]=\int d^4x\,\overline{\mathscr{L}}[\{\phi\},g_{\mu\nu}]\,,$$
$$\overline{\mathscr{L}}[\{\phi\},g_{\mu\nu}]\equiv\sqrt{-g}\,\mathscr{L}[\{\phi\},g_{\mu\nu}]\,. \tag{5.2b}$$

I will assume that the microscopic action density $\overline{\mathscr{L}}$ is a function of the metric and its first and second derivatives,

$$\overline{\mathscr{L}}[\{\phi\},g_{\mu\nu}]=\mathscr{L}(\{\phi\},g_{\mu\nu},\partial_\lambda g_{\mu\nu},\partial_\lambda\partial_\sigma g_{\mu\nu})\,, \tag{5.3}$$

making the derivation general enough to encompass the case, discussed in Sec. VI below, where the metric itself (and not just the matter fields $\{\phi\}$) is path integral quantized. To proceed, let us calculate the conformal variation of Eq. (5.1) around a general background metric. This is done by acting on the left- and right-hand sides with the differential operator $2g_{\mu\nu}(y)\delta/\delta g_{\mu\nu}(y)$, where y is an arbitrary space-time point which will shortly be chosen as the origin, and then dividing by $i\exp(iS_{\rm eff})$. Inserting the expansion of Eq. (5.2a) in the left-hand side gives

$$2g_{\mu\nu}(y)\frac{\delta}{\delta g_{\mu\nu}(y)}\int d^4x\sqrt{-g}\left[\frac{1}{16\pi G_{\rm ind}}(R-2\Lambda_{\rm ind})+O[(\partial_\lambda g_{\mu\nu})^4]\right]=\frac{\int d\{\phi\}\,e^{iS[\{\phi\},g_{\mu\nu}]}2g_{\mu\nu}(y)\dfrac{\delta}{\delta g_{\mu\nu}(y)}\int d^4x\,\overline{\mathscr{L}}}{\int d\{\phi\}\,e^{iS[\{\phi\},g_{\mu\nu}]}}\,, \tag{5.4a}$$

where the quantities $g_{\mu\nu}$, $\overline{\mathscr{L}}$, R inside the x-integral are values at space-time point x, and where the functional integral $\int d\{\phi\}$ is still an integration over the values of the matter fields at all space-time points,

$$\int d\{\phi\}=\prod_z\int d\{\phi(z)\}\,. \tag{5.4b}$$

Equation (5.4a) can be evaluated using standard formulas for the first variations (with $T^{\mu\nu}$, as before, the renormalized matter stress-energy tensor),

$$\delta\sqrt{-g}=\tfrac{1}{2}\sqrt{-g}\,g^{\mu\nu}\delta g_{\mu\nu}\,,$$
$$\delta(\sqrt{-g}R)=-\sqrt{-g}\,(R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R)\delta g_{\mu\nu}+\text{total derivatives}\,, \tag{5.5}$$

$$\delta\overline{\mathscr{L}}=\tfrac{1}{2}\overline{T}^{\mu\nu}\delta g_{\mu\nu}\,,$$
$$\overline{T}^{\mu\nu}\equiv\sqrt{-g}\,T^{\mu\nu} \tag{5.6}$$
$$=2\left[\frac{\partial\overline{\mathscr{L}}}{\partial g_{\mu\nu}}-\partial_\lambda\frac{\partial\overline{\mathscr{L}}}{\partial(\partial_\lambda g_{\mu\nu})}+\partial_\lambda\partial_\sigma\frac{\partial\overline{\mathscr{L}}}{\partial(\partial_\lambda\partial_\sigma g_{\mu\nu})}\right].$$

Substituting these, and defining the point y to be the origin 0 in order to simplify the subsequent formulas, we get

$$\frac{1}{8\pi G_{\rm ind}}[R(0)-4\Lambda_{\rm ind}]+O[(\partial_\lambda g_{\mu\nu})^4]=\frac{\int d\{\phi\}e^{iS[\{\phi\},g_{\mu\nu}]}\overline{T}[g_{\mu\nu},0]}{\int d\{\phi\}e^{iS[\{\phi\},g_{\mu\nu}]}}\,, \tag{5.7}$$

with $\overline{T}[g_{\mu\nu},x]$ the stress-energy tensor trace functional defined by

$$\overline{T}[g_{\mu\nu},x]=\sqrt{-g}\,T^\mu_\mu$$
$$=2g_{\mu\nu}\left[\frac{\partial\overline{\mathscr{L}}}{\partial g_{\mu\nu}}-\partial_\lambda\frac{\partial\overline{\mathscr{L}}}{\partial(\partial_\lambda g_{\mu\nu})}+\partial_\lambda\partial_\sigma\frac{\partial\overline{\mathscr{L}}}{\partial(\partial_\lambda\partial_\sigma g_{\mu\nu})}\right]. \tag{5.8a}$$

Taking the flat space-time limit of Eq. (5.7) and introducing the abbreviated notation

$$T(x)\equiv\overline{T}[\eta_{\mu\nu},x]=T^\mu_\mu\,|_{g_{\mu\nu}=\eta_{\mu\nu}}\,, \tag{5.8b}$$

we obtain a formula for the induced cosmological term,

$$-\frac{1}{2\pi}\frac{\Lambda_{\rm ind}}{G_{\rm ind}}=\frac{\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}T(0)}{\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}}\,. \tag{5.9}$$

In order to extract a formula for the induced gravitational constant, we must take a further metric variation of Eq. (5.7). Since the left-hand side of Eq. (5.7) has no tensor structure, it suffices to specialize[19] to a metric which around $x=0$ has the conformally flat, constant-curvature form

[19]For a derivation which does not make this specialization, but instead proceeds from the general Riemann normal expansion $g_{\mu\nu}=\eta_{\mu\nu}-(\frac{1}{3})R_{\mu\alpha\nu\beta}x^\alpha x^\beta+\cdots$, see Adler (1980c). See also Brown and Zee (1982).

$$g_{\mu\nu}(x)=\eta_{\mu\nu}[1-\tfrac{1}{24}R(0)x^2+O(\nabla R,R^2)]$$
$$=\eta_{\mu\nu}+\delta g_{\mu\nu}\ ,$$
$$\delta g_{\mu\nu}(x)=-\eta_{\mu\nu}\tfrac{1}{24}R(0)x^2,\quad x^2=(x^i)^2-(x^0)^2\ . \quad (5.10)$$

Varying Eq. (5.7) around a Minkowski background, and dropping terms which are higher than second order in the expansion in powers of $\partial_\lambda g_{\mu\nu}$, we get

$$\delta\left[\frac{1}{8\pi G_{\text{ind}}}[R(0)-4\Lambda_{\text{ind}}]\right]=\frac{1}{8\pi G_{\text{ind}}}R(0)$$
$$=\frac{\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}\delta\overline{T}[g_{\mu\nu},0]}{\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}} \quad \text{(i)}$$
$$+\frac{\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}T(0)i\int d^4x\,\delta\overline{\mathscr{L}}}{\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}} \quad \text{(ii)}$$
$$-\frac{\left[\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}T(0)\right]\left[\int d\{\phi\}\,e^{iS[\{\phi\},\eta_{\mu\nu}]}i\int d^4x\,\delta\overline{\mathscr{L}}\right]}{\left[\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}\right]^2}\ . \quad \text{(iii)} \quad (5.11)$$

Terms (ii) and (iii) on the right-hand side can be evaluated by using Eqs. (5.6), (5.8), and (5.10), which give

$$i\int d^4x\,\delta\overline{\mathscr{L}}=-\frac{i}{48}R(0)\int d^4x\,x^2T(x)\ . \quad (5.12)$$

To evaluate term (i), we note that since $\delta g_{\mu\nu}$ vanishes as x^2 at $x=0$, the only terms which contribute to $\delta\overline{T}[g_{\mu\nu},0]$ are those in which $\delta g_{\mu\nu}$ is acted on by two derivatives. After a certain amount of algebra, we find

$$\delta\overline{T}[g_{\mu\nu},0]=2R(0)U(0)\ , \quad (5.13)$$

with $U(x)$ the functional defined by

$$U(x)=\tfrac{1}{12}g_{\mu\nu}g_{\alpha\beta}\left[g_{\lambda\theta}\frac{\partial^2\overline{\mathscr{L}}}{\partial(\partial_\lambda g_{\mu\nu})\partial(\partial_\theta g_{\alpha\beta})}-2g_{\theta\phi}\frac{\partial^2\overline{\mathscr{L}}}{\partial g_{\mu\nu}\partial(\partial_\theta\partial_\phi g_{\alpha\beta})}+g_{\theta\phi}\partial_\lambda\frac{\partial^2\overline{\mathscr{L}}}{\partial(\partial_\lambda g_{\mu\nu})\partial(\partial_\theta\partial_\phi g_{\alpha\beta})}\right.$$
$$\left.\left.-2g_{\lambda\theta}\partial_\sigma\frac{\partial^2\overline{\mathscr{L}}}{\partial(\partial_\lambda\partial_\sigma g_{\mu\nu})\partial(\partial_\theta g_{\alpha\beta})}-g_{\theta\phi}\partial_\lambda\partial_\sigma\frac{\partial^2\overline{\mathscr{L}}}{\partial(\partial_\lambda\partial_\sigma g_{\mu\nu})\partial(\partial_\theta\partial_\phi g_{\alpha\beta})}\right]\right|_{g_{\mu\nu}=\eta_{\mu\nu}}\ . \quad (5.14)$$

Inserting Eqs. (5.12)–(5.14) into Eq. (5.11) and dividing by $2R(0)$ gives the desired formula for G_{ind}^{-1},

$$\frac{1}{16\pi G_{\text{ind}}}=\frac{\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}U(0)}{\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}}$$
$$-\frac{i}{96}\int d^4x\,x^2\left[\frac{\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}T(0)T(x)}{\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}}-\frac{\left[\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}T(0)\right]\left[\int d\{\phi\}\,e^{iS[\{\phi\},\eta_{\mu\nu}]}T(x)\right]}{\left[\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}\right]^2}\right]\ . \quad (5.15)$$

If we define the subtracted functional $\widetilde{T}$ by

$$\widetilde{T}(x)=T(x)-\frac{\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}T(x)}{\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}}\ , \quad (5.16)$$

and note that the second term on the right-hand side is a constant, we can rewrite Eq. (5.15) as

$$\frac{1}{16\pi G_{\text{ind}}}=\frac{\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}U(0)}{\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}}$$
$$-\frac{i}{96}\int d^4x\,x^2\frac{\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}\widetilde{T}(0)\widetilde{T}(x)}{\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}}\ . \quad (5.17)$$

Finally, recalling the correspondence (Abers and Lee, 1973) between expectations of functionals and vacuum expectations of time-ordered products of the corresponding operators,

$$\langle A(0)\rangle_0=\frac{\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}A(0)}{\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}}\ , \quad (5.18)$$
$$\langle\mathscr{T}(A(x)B(0))\rangle_0=\frac{\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}A(x)B(0)}{\int d\{\phi\}e^{iS[\{\phi\},\eta_{\mu\nu}]}}\ ,$$

we can rewrite Eqs. (5.9) and (5.15)–(5.17) in the compact form

$$-\frac{1}{2\pi}\frac{\Lambda_{\text{ind}}}{G_{\text{ind}}}=\langle T(0)\rangle_0\ , \quad (5.19a)$$

$$\frac{1}{16\pi G_{\rm ind}}=\langle U(0)\rangle_0-\frac{i}{96}\int d^4x\,x^2[\,\langle \mathcal{T}(T(x)T(0))\rangle_0 -\langle T(0)\rangle_0^2]$$
$$=\langle U(0)\rangle_0-\frac{i}{96}\int d^4x\,x^2\langle \mathcal{T}(\tilde{T}(x)\tilde{T}(0))\rangle_0\,,$$
$$\tilde{T}(x)=T(x)-\langle T(x)\rangle_0\,. \tag{5.19b}$$

As noted above, we have so far carried along some extra generality, which will be needed to discuss the case when the metric is a quantum variable. When the metric is not quantized, the trace functional $\bar{T}[g_{\mu\nu},0]$ depends on derivatives of the metric only[20] through terms of order R^2, which come directly from the Lagrangian terms $\mathcal{G},\mathcal{H},\mathcal{K}$ of Eq. (2.35). The variations of these terms vanish in flat space-time, and so when the metric is not quantized, the functional U vanishes. Hence for matter theories on a background manifold, Eq. (5.19b) reduces to the form

$$\frac{1}{16\pi G_{\rm ind}}=\frac{-i}{96}\int d^4x\,x^2\langle \mathcal{T}(\tilde{T}(x)\tilde{T}(0))\rangle_0 \tag{5.20}$$

given by Adler (1980b) and Zee (1981).

B. Convergence and spectral analysis

From the explicit formula of Eq. (5.20), we can again analyze the conditions for $G_{\rm ind}^{-1}$ to be calculable. Since Eq. (5.20) is a flat space-time formula, it will be convenient at this point to switch to the Bjorken-Drell (1965) signature convention, in which Eq. (5.20) becomes

$$\frac{1}{16\pi G_{\rm ind}}=\frac{i}{96}\int d^4x\,x^2\langle \mathcal{T}(\tilde{T}(x)\tilde{T}(0))\rangle_0\,,$$
$$x^2=(x^0)^2-(x^i)^2\,. \tag{5.21}$$

As discussed in Sec. III above, we define the flat space-time matter theory by a renormalization procedure based on dimensional regularization, and so Eq. (5.21) is to be interpreted as a dimensional continuation limit

$$\frac{1}{16\pi G_{\rm ind}}=\frac{i}{96}\lim_{\omega\to 2}\int d^{2\omega}x\,x^2\langle \mathcal{T}(\tilde{T}(x)\tilde{T}(0))\rangle_0^\omega\,, \tag{5.22}$$

where $\langle\ \rangle_0^\omega$ denotes the vacuum expectation in the 2ω-dimensional theory. Equation (5.22) will give a calculable $G_{\rm ind}^{-1}$ if the integral on the right-hand side is regular at $\omega=2$, and as we have seen, the singularity structure in the ω plane is directly determined by the ultraviolet divergence structure of the dimension-four integral of Eq. (5.21). This can be studied by using the Wilson (1968) operator product expansion of the time-ordered product,[21]

$$\langle \mathcal{T}(\tilde{T}(x)\tilde{T}(0))\rangle_0=\frac{\langle \mathcal{O}_0\rangle_0}{(x^2)^4}\times\text{logs} +\frac{\langle \mathcal{O}_2\rangle_0}{(x^2)^3}\times\text{logs}+O\left[\frac{1}{(x^2)^2}\right]\,, \tag{5.23}$$

where "$\times$ logs" indicates the presence of power series in $\log x^2$, and where $\mathcal{O}_{0,2}$ are Lorentz-scalar, internal symmetry-invariant operators of canonical dimension 0 and 2, respectively [corresponding to the fact that $\tilde{T}$ has canonical dimension four, and hence the left-hand side of Eq. (5.23) has canonical dimension eight]. When Eq. (5.23) is inserted in Eq. (5.21) the order $(x^2)^{-4}$ terms give formally quadratically divergent integrals, which vanish by the lemma of Eq. (3.18) above, while the order $(x^2)^{-2}$ and higher terms are ultraviolet convergent. However, the order $(x^2)^{-3}$ terms give logarithmically divergent integrals and thus generate poles at $\omega=2$ in the dimensional continuation, unless no operators $\mathcal{O}_2$ are present in the theory, in which case $G_{\rm ind}^{-1}$ is calculable. We have therefore recovered the same calculability criterion as was obtained from the dimensional algorithm in Sec. II.D above.

Let us next attempt to put Eq. (5.21) into spectral form, which if possible, would yield information about the sign of $G_{\rm ind}$. From the standard[22] spectral analysis for a scalar operator φ, we have

$$-i\langle \mathcal{T}(\varphi(x)\varphi(0))\rangle_0=\int_0^\infty d\sigma^2\rho(\sigma^2)\Delta_F(x,\sigma)\,, \tag{5.24}$$

with ρ the spectral function defined by[23]

$$\rho(q^2)=(2\pi)^3\sum_n\delta^4(p_n-q)\,|\,\langle 0\,|\,\varphi(0)\,|\,n\,\rangle\,|^2\geq 0\,, \tag{5.25}$$

and with Δ_F the scalar Feynman propagator,

$$\Delta_F(x,\sigma)=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\frac{1}{k^2-\sigma^2+i\epsilon}\,. \tag{5.26}$$

Ignoring for the moment questions of convergence, let us set $\varphi=\tilde{T}$ in the above formulas and substitute into Eq. (5.21), giving

$$\frac{1}{16\pi G_{\rm ind}}=\frac{1}{96}\int_0^\infty d\sigma^2\rho(\sigma^2)\left[-\int d^4x\,x^2\Delta_F(x,\sigma)\right]\,. \tag{5.27}$$

A simple calculation then shows that

$$-\int d^4x\,x^2\Delta_F(x,\sigma)=\frac{\partial^2}{\partial k_\mu\partial k^\mu}\left.\frac{1}{(k^2-\sigma^2)}\right|_{k=0}=\frac{-8}{\sigma^4}\,, \tag{5.28}$$

and so Eq. (5.27) yields

$$\frac{1}{16\pi G_{\rm ind}}=-\frac{1}{12}\int_0^\infty d\sigma^2\frac{\rho(\sigma^2)}{\sigma^4}\,, \tag{5.29}$$

which if correct would imply that $G_{\rm ind}^{-1}$ has manifestly the wrong sign to give attractive gravitation. However, Eq. (5.29) is valid only if the integral on the right-hand side converges, which requires the vanishing of $\rho(\sigma^2)/\sigma^2$ as σ becomes infinite. But in gauge theories, we have seen in Sec. III.C above that $\tilde{T}$ contains a trace anomaly

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[20]The Lagrangian density $\bar{\mathcal{L}}$ also contains metric derivatives in the spin connections used in constructing the spinor kinetic terms, but these do not appear in the trace functional $\bar{T}$.

[21]For a proof of the operator product expansion in perturbation theory and a detailed discussion, see Zimmermann (1970).

[22]See Bjorken and Drell (1965), pp. 138–139 and pp. 387–390.

[23]Since ρ is gauge invariant, it can be evaluated in a canonical gauge to establish positivity.

term proportional to the hard operator $[(F^i_{\lambda\sigma})^2]^r$, as a result of which $\rho(\sigma^2)$ behaves asymptotically as $\sigma^4 \times$ logs, invalidating the spectral representation of Eq. (5.29). The failure of the spectral representation, as indicated by the quadratic divergence of Eq. (5.29), is just a reflection of the formal quadratic divergence of Eq. (5.22), arising from the leading $(x^2)^{-4}$ term in the operator product expansion of Eq. (5.23).

The breakdown of Eq. (5.29) can also be rephrased in the language of dispersion relations, by defining

$$\chi(k^2)=\int d^4x\, e^{ik\cdot x}(-i)\langle \mathscr{T}(\tilde{T}(x)\tilde{T}(0))\rangle_0 . \qquad (5.30)$$

If $\chi(k^2)-\chi(0)$ obeyed an unsubtracted dispersion relation, then Eq. (5.29) could be derived, but in fact one must make an additional subtraction, as in $\chi(k^2)-\chi(0)-k^2\chi'(0)$, before getting a quantity which obeys an unsubtracted dispersion relation. Substituting this dispersion relation into Eq. (5.21) then yields $G_{\rm ind}^{-1}\propto\chi'(0)$, which furnishes no *a priori* information about the sign of $G_{\rm ind}$. The calculations discussed in the next two sections suggest, in fact, that the sign of $G_{\rm ind}$ is sensitive to details of the infrared behavior of the matter theory.

C. Early model calculations of $G_{\rm ind}^{-1}$

According to Eq. (5.21), the leading perturbative contributions to $G_{\rm ind}^{-1}$ are those in which two insertions of the stress-energy tensor trace T are made in connected matter diagrams of low-loop order, as shown in Fig. 4. In theories with dynamical spontaneous symmetry breaking, such as SU(n) gauge theories, the diagrams of Fig. 4(a) and 4(b) are typically absent and the leading contributions to $G_{\rm ind}^{-1}$ begin at three-loop order. However, one way of simulating the ultraviolet softening produced by dynamical scale-invariance breaking is to consider a massive fermion or scalar meson theory, in which the leading contribution is the one-loop diagram of Fig. 4(a), and to include explicit, finite-mass Pauli-Villars regulators to control the ultraviolet divergences. This calculation has been performed by Sakharov (1975), Akama *et al.*[24] (1978), and Zee (1981), and Zee's results in particular were important in motivating the general derivation leading to Eq. (5.21). Zee considers a fermion loop of mass $m_0=m$, and by including two Pauli-Villars regulators with mass $m_{1,2}$, finds

$$\frac{1}{16\pi G_{\rm ind}}=\frac{2\pi^2}{3(2\pi)^4}I ,$$

$$I=\sum_{i=0}^{2} c_i m_i^2 \log m_i^2, \ \text{with} \sum_{i=0}^{2} c_i=0 , \ \sum_{i=0}^{2} c_i m_i^2=0 . \qquad (5.31)$$

By some simple algebra, Eq. (5.31) can be rewritten as

$$I=m_2^2\left[\frac{m_1^2-m^2}{m_1^2-m_2^2}\right]\log\frac{m_1^2}{m_2^2}-m^2\log\frac{m_1^2}{m_2^2} , \qquad (5.32)$$

an expression which is positive as long as $m^2<m_{1,2}^2$, but which can change sign when the regulator masses are smaller than m, illustrating the sensitivity of the sign of $G_{\rm ind}^{-1}$ to dynamical details. In order to give the observed magnitude of $G_{\rm ind}^{-1}$, Eq. (5.31) requires $m\sim m_{\rm Planck}=1.22\times10^{19}$ GeV, suggesting more generally that to get a realistic theory of Einstein gravitation as an induced quantum effect, the physics of dynamical scale-invariance breaking must take place at energies near the Planck mass.

According to the discussion of Sec. IV.B above, the simplest field theory model which has calculable induced gravitational and cosmological constants is a pure SU(2) gauge theory. A direct evaluation of Eq. (5.7) has been given in this case by Hasslacher and Mottola (1980), using the approximation of saturating the Euclidean continuation of the functional integral by a dilute gas of instantons.[25] Their result can be written as

$$\frac{1}{8\pi G_{\rm ind}}(R-4\Lambda_{\rm ind})+O(R^2)$$
$$=\int_0^{\rho_{\max}(R)}\frac{d\rho}{\rho^5}[C_1+C_2\rho^2R+\cdots]D(\mu\rho) , \qquad (5.33)$$

where the integral is over the instanton size parameter ρ, and where $\rho_{\max}(R)$ symbolically indicates a cutoff on this integration, of unknown form at present, produced by the infrared vacuum structure of the gauge theory. The instanton gas calculation gives a definite expression for the integrand of Eq. (5.33), written as a series expansion in R times the flat space-time instanton density[25] $D(\mu\rho)$,

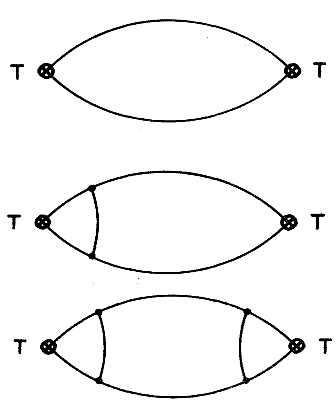

FIG. 4. Typical diagrams contributing to $G_{\rm ind}^{-1}$ in (a) one-, (b) two-, and (c) three-loop order, respectively, with the solid lines indicating matter field propagators. In an SU(n) gauge theory, the contributions of one- and two-loop order vanish, and the perturbation series for $G_{\rm ind}^{-1}$ begins at three-loop order, with a leading term proportional to g^4.

[24]See also Terazawa *et al.* (1977a,b) for related earlier work by this group.

[25]For a pedagogical review of instanton gas methods, see Coleman (1979). A simplified derivation of the instanton density $D(\mu\rho)$ (with μ the subtraction mass discussed in Sec. IV.C) is given by Bernard (1979).

$$C_1=\tfrac{22}{3}\ ,$$
$$C_2=-\tfrac{5}{3}(\alpha_z+\alpha_\rho+\alpha_g-\tfrac{1}{3}\beta)\ ,$$
$$\alpha_z=\tfrac{1}{6},\ \ \alpha_\rho=\tfrac{1}{8}\log\left[\frac{48}{\rho^2 R}\right]-\frac{7}{24}\ ,$$
$$\alpha_g=3\alpha_\rho,\ \ \beta=\tfrac{1}{6}\ . \tag{5.34}$$

In Eq. (5.34), C_1 gives the contribution to the cosmological constant arising from the instanton gas expectation of the trace anomaly of Eq. (3.28), while C_2 gives the corresponding contribution to the induced gravitational constant, obtained by summing contributions from the various small fluctuation modes around an instanton. Specifically, α_z, α_ρ, and α_g are, respectively, the contributions from the translational, dilatational and gauge zero modes, while β is the contribution from the nonzero modes. The $\log R$ terms in α_ρ and α_g arise because these zero modes make a contribution to Eq. (5.21) which is infrared divergent. Since an exact evaluation of the Euclidean continuation of the correlation function $\langle\mathscr{T}(\widetilde{T}(x)\widetilde{T}(0))\rangle_0$ is expected to show an exponential decay law for large separations x (see Sec. V.D below), Eq. (5.21) should in fact be strongly convergent in the infrared. Thus the divergence leading to the presence of $\log R$ in α_ρ and α_g appears to be an artifact of the dilute instanton gas approximation, and one expects the $R\log R$ terms in the integrand of Eq. (5.33) to be cancelled by corresponding terms in the integration cutoff $\rho_{\max}(R)$ and/or in corrections to the instanton picture, leaving a remainder of order R which is determined by the detailed dynamics of the infrared region. This means that the dilute instanton gas calculation, while demonstrating the existence and ultraviolet finiteness of the induced gravitational action in the gauge theory case, does not yield a quantitative calculation of $G_{\rm ind}^{-1}$.

D. A strategy for calculating $G_{\rm ind}^{-1}$ and $\Lambda_{\rm ind}$ in an SU(n) gauge theory

Because a pure Yang-Mills theory is the simplest field theory model with dynamical scale-invariance breaking, it would clearly be desirable to carry out quantitative calculations of $G_{\rm ind}^{-1}$ and $\Lambda_{\rm ind}$ in this case. I shall outline below a general strategy for doing this, assuming that one can, in principle, make arbitrarily good Monte Carlo[26] evaluations of the various gluon field vacuum expectations which are needed, together with calculations to any finite order of perturbation theory.

Let us begin with the induced cosmological term $\Lambda_{\rm ind}/G_{\rm ind}$. Substituting Eq. (5.8b) into Eq. (5.19a) and converting to the Bjorken-Drell metric convention (which was used in the derivation of Sec. III.C), we get

$$\frac{1}{2\pi}\frac{\Lambda_{\rm ind}}{G_{\rm ind}}=\langle T_\mu^\mu\rangle_0\ . \tag{5.35}$$

The vacuum expectation on the right can be expressed in terms of the gluon field strength by using the trace anomaly relation of Eq. (3.28), giving

$$\langle T_\mu^\mu\rangle_0=\left\langle\frac{\beta(g)}{2g}(F_{\lambda\sigma}^i F^{i\lambda\sigma})^r\right\rangle_0\ . \tag{5.36}$$

At this point it is convenient to choose a definition of the coupling constant (see Appendix B.1 for details) for which the one-loop renormalization group structure of Eqs. (4.11)–(4.13) is exact.[27] Combining Eq. (5.36) with Eqs. (4.11) and (4.15), we then find

$$\frac{1}{2\pi}\frac{\Lambda_{\rm ind}}{G_{\rm ind}}=\langle T_\mu^\mu\rangle_0$$
$$=-\frac{1}{8}\left[\frac{11}{3}n-\frac{2}{3}N_f\right]\left\langle\frac{\alpha_s}{\pi}(F_{\lambda\sigma}^i F^{i\lambda\sigma})^r\right\rangle_0\ ,$$
$$\alpha_s=\frac{g^2}{4\pi}\ , \tag{5.37}$$

where for a pure SU(n) Yang-Mills theory one would set $N_f=0$. Equation (5.37) expresses the induced cosmological term as a multiple of the extensively studied[17] gluon pairing amplitude $\langle(\alpha_s/\pi)((F_{\lambda\sigma}^i)^2)^r\rangle_0$. Since the gluon pairing amplitude has canonical dimension four, it is proportional to the fourth power of the renormalization-group-invariant scale mass $\mathscr{M}$ introduced in Sec. IV.B above,

$$\left\langle\frac{\alpha_s}{\pi}(F_{\lambda\sigma}^i F^{i\lambda\sigma})^r\right\rangle_0=c\mathscr{M}^4\ ,$$
$$\mathscr{M}=\mu e^{-1/(b_0 g^2)}\ , \tag{5.38}$$

with c a numerical constant of order unity. According to Eq. (5.38), the pairing amplitude has an essential singularity of the form $e^{-4/(b_0g^2)}$ at $g^2=0$, and vanishes identically in perturbation theory. This agrees with what would be found by making a Feynman diagram expansion of the left-hand side of Eq. (5.38) and evaluating the formally quartically divergent momentum space integrals by using the lemma of Eq. (3.18).

In order to express Eq. (5.38) directly in terms of an observable quantity, it is customary to introduce the string tension σ, defined as the coefficient of the asymptotic linear term in the heavy quark-antiquark static potential,

$$V_{\rm static}(R)\underset{R\to\infty}{=}\sigma R+O(1)\ . \tag{5.39}$$

Since the string tension has canonical dimension two, it is proportional to the square of $\mathscr{M}$,

$$\sigma=c'\mathscr{M}^2\ , \tag{5.40}$$

with c' a second numerical constant of order unity. Eliminating $\mathscr{M}$ from Eqs. (5.38) and (5.40), we get

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[26]For a review of statistical physics applications of Monte Carlo methods, see Binder (1976). Lattice gauge theories were introduced by Wilson (1974); see also Kogut and Susskind (1975) and the review by Creutz (1978). The application of Monte Carlo methods to lattice gauge theories was initiated by Creutz, Jacobs, and Rebbi (1979) and Creutz (1980).

[27]If the transformation of Appendix B.1 is not made, the general definition of the gluon pairing amplitude which corresponds to that of Eq. (5.38) is $(-2\beta/b_0g^3)\langle(\alpha_s/\pi)((F_{\lambda\sigma}^i)^2)^r\rangle_0$.

$$\left\langle \frac{\alpha_s}{\pi}(F^i_{\lambda\sigma}F^{i\lambda\sigma})^r \right\rangle_0 = c''\sigma^2 ,$$
$$c'' = c/(c')^2 , \tag{5.41}$$

which when substituted into Eq. (5.37) gives a relation between the induced cosmological term and the string tension,

$$\frac{\Lambda_{\rm ind}}{G_{\rm ind}} = -2\pi\tfrac{1}{8}(\tfrac{11}{3}n - \tfrac{2}{3}N_f)c''\sigma^2 . \tag{5.42}$$

Methods for making a Monte Carlo estimate of c'' in pure SU(2) and SU(3) gauge theories ($n=2,3$; $N_f=0$) have been discussed by Kripfganz (1981), by Banks *et al.* (1981), and by Di Giacomo and Paffuti (1982).

Let us consider next the expression for the induced gravitational constant $G^{-1}_{\rm ind}$ given in Eq. (5.21), which, we have seen, must be interpreted as a dimensional continuation limit. Again substituting the trace anomaly equation, and defining the coupling constant so that the one-loop renormalization group is exact, we get

$$\frac{1}{16\pi G_{\rm ind}} = \frac{i}{96}\int d^4x\, x^2\Psi(-x^2) ,$$
$$\Psi(-x^2) \equiv \langle \mathscr{T}(T(x)T(0))\rangle_0 - \langle T\rangle_0^2 , \tag{5.43}$$
$$T = -\tfrac{1}{4}b_0 g^2 (F^i_{\lambda\sigma}F^{i\lambda\sigma})^r .$$

To evaluate Eq. (5.43) it is convenient to make a Wick rotation to the Euclidean section, which is formally accomplished by making the substitutions $d^4x \to -i\,d^4x$, $x^2 \to -x^2$, giving

$$\frac{1}{16\pi G_{\rm ind}} = -\frac{1}{96}\int_E d^4x\, x^2\Psi(x^2) . \tag{5.44}$$

In order to devise a practical method for implementing the dimensional continuation limit implicit in Eq. (5.44),[28] we shall split the integration over the variable $x^2=t$ into an ultraviolet (UV) part $0\le t\le t_0$, and an infrared (IR) part $t_0 \le t < \infty$,

$$\frac{1}{16\pi G_{\rm ind}} = -\frac{\pi^2}{96}(I_{\rm UV}+I_{\rm IR}) ,$$
$$I_{\rm UV} = \int_0^{t_0} dt\, t^2\Psi(t) , \tag{5.45}$$
$$I_{\rm IR} = \int_{t_0}^{\infty} dt\, t^2\Psi(t) .$$

Let us suppose that the correlation function $\Psi(t)$ has been determined to high accuracy by Monte Carlo studies. In the infrared region, Ψ behaves for large t as

$$\Psi(t) \underset{t\to\infty}{\sim} e^{-m_g t^{1/2}} , \tag{5.46}$$

with m_g a parameter, called the glueball mass, which is related to the string tension by

$$m_g = c_g\sigma^{1/2} = c_g(c')^{1/2}\mathscr{M} , \tag{5.47}$$

with c_g a numerical constant. [Numerical Monte Carlo estimates of c_g for an SU(2) gauge theory, obtained by studying the plaquette-plaquette correlation function, have been given recently by Berg (1981) and by Bhanot and Rebbi (1981).] As a result of the good asymptotic behavior of Eq. (5.46), the infrared integral $I_{\rm IR}$ of Eq. (5.45) is convergent at $t=\infty$, and can be evaluated by numerical integration. Turning next to the ultraviolet integral $I_{\rm UV}$, let us write it in the form

$$I_{\rm UV} = I^c_{\rm UV} + \Delta I_{\rm UV} ,$$
$$I^c_{\rm UV} = \int_0^{t_0} dt\, t^2\Psi_c(t) ,$$
$$\Delta I_{\rm UV} = \int_0^{t_0} dt\, t^2[\Psi(t)-\Psi_c(t)] , \tag{5.48}$$

with $\Psi_c(t)$ a comparison function chosen so that: (i) the integral $\Delta I_{\rm UV}$ converges at $t=0$, and hence can be evaluated by numerical integration; and (ii) the dimensional continuation needed to evaluate $I^c_{\rm UV}$ can be carried out explicitly, leaving a convergent integral which can again be done numerically. The motivation behind the introduction of Ψ_c is the evident fact that, while discrete methods can be used to evaluate convergent integrals, they cannot be used to make analytic continuations.

The general form required for the comparison function $\Psi_c(t)$ can be inferred from the operator product expansion of Eq. (5.23). This expansion can be "improved" by using the renormalization group and asymptotic freedom, which permit a partial resummation of the power series of logarithms in the leading term of Eq. (5.23) into a joint power series in the running coupling constant $g^2(t)$ and (since we have made the transformation of Appendix B.1) in its logarithm $\log g^2(t)$. Defining the coordinate space running coupling by[29]

$$g^2(t) \equiv \frac{1}{-\frac{1}{2}b_0\log(\mathscr{M}^2t)} = \frac{g^2}{1-\frac{1}{2}b_0g^2\log(\mu^2t)} , \tag{5.49}$$

we have[30]

[28]Use of a coordinate space formalism is not necessary in order to implement the dimensional continuation limit. For example, one could equally well rewrite the spectral representation of Eq. (5.29) as

$$\frac{1}{16\pi G_{\rm ind}} = -\frac{1}{12}(J_{\rm UV}+J_{\rm IR}) ,$$
$$J_{\rm UV} = \int_{\sigma_0^2}^{\infty} d\sigma^2\frac{\rho(\sigma^2)}{\sigma^4},\quad J_{\rm IR} = \int_0^{\sigma_0^2} d\sigma^2\frac{\rho(\sigma^2)}{\sigma^4} ,$$

and evaluate $J_{\rm UV}$ by dimensional continuation. However, it is likely to be easier to extract the coordinate space function $\Psi(x^2)$ than the spectral function $\rho(\sigma^2)$ from Monte Carlo studies of the infrared region.

[29]The use of the same scale mass in Eq. (5.49) as in the one-loop version of Eq. (4.10) is a matter of convenience; redefining $\mathscr{M}$ by a constant factor simply redefines the expansion coefficients appearing in Eq. (5.50).

[30]In general, such renormalization-group-improved operator product expansions contain an additional fractional power $[\log(\mathscr{M}^2t)]^\delta$, with the exponent δ proportional to the difference in anomalous dimensions of the operators on the left- and right-hand sides. Since T^μ_μ and $\mathscr{O}_0 \propto 1$ both have zero anomalous dimension, this fractional power is absent from the leading term in the expansion. See Gross and Wilczek (1974), p. 982, for a detailed discussion of this point.

$$\Psi(t)=C_\Psi \frac{1}{t^4(-\log \mathcal{M}^2 t)^2}\left[1+\frac{1}{(-\log \mathcal{M}^2 t)}[a_{10}+a_{11}\log\log(\mathcal{M}^2 t)^{-1}]+\cdots\right]+O(t^{-2}) . \tag{5.50}$$

The leading term in Eq. (5.50) is proportional to

$$\frac{1}{(-\log \mathcal{M}^2 t)^2} \propto g^4(t) , \tag{5.51}$$

because, as seen from Eq. (5.43), the perturbation expansion for Ψ begins in order g^4; the constant C_Ψ is computed from lowest-order perturbation theory in Appendix B.2, with the result

$$C_\Psi=\frac{3\times 2^6}{(2\pi)^4}(n^2-1) . \tag{5.52}$$

[The two-loop contribution to the glueball propagator, which gives the coefficients a_{10}, a_{11} in the series of Eq. (5.50), has recently been calculated by Kataev *et al.* (1982).] No order t^{-3} term is present in the expansion of Eq. (5.50) because of the absence of dimension-two operators $\mathcal{O}_2$, while the order t^{-2} and higher terms make contributions to $I_{\rm UV}$ which are convergent at $t=0$. Hence it suffices to take as the comparison function Ψ_c the leading t^{-4} part of $\Psi(t)$,

$$\Psi_c(t)= C_\Psi \frac{1}{t^4(-\log \mathcal{M}^2 t)^2} \times \left[1+\sum_{n=1}^{\infty}\sum_{m=0}^{n} a_{nm}\frac{[\log\log(\mathcal{M}^2 t)^{-1}]^m}{(-\log \mathcal{M}^2 t)^n}\right], \tag{5.53}$$

and to restrict t_0 by the condition

$$\mathcal{M}^2 t_0 < 1 , \tag{5.54}$$

so that the logarithm $\log(\mathcal{M}^2 t)$ does not vanish in the integration range $0\le t\le t_0$ of $I^c_{\rm UV}$. Substituting Eq. (5.53) into $I^c_{\rm UV}$ and making the change of variable $u=\mathcal{M}^2 t$ gives

$$I^c_{\rm UV}=C_\Psi \mathcal{M}^2\int_0^{u_0}\frac{du}{u^2}\frac{\Theta(u)}{(\log u)^2},\quad u_0=\mathcal{M}^2 t_0 ,$$

$$\Theta(u)=1+\sum_{n=1}^{\infty}\sum_{m=0}^{n} a_{nm}\frac{(\log\log u^{-1})^m}{(\log u^{-1})^n} . \tag{5.55}$$

The evaluation of this integral by dimensional continuation is carried out in Appendix B.3, with the result

$$I^c_{\rm UV}=C_\Psi \mathcal{M}^2 \,\mathrm{Re}\left[\int_{\log(\mathcal{M}^2 t_0)^{-1}}^{i\infty} dv\frac{e^v}{v^2}\Theta(e^{-v})\right], \tag{5.56}$$

where Re indicates the real part, and where the integration contour is shown in Fig. 5. [As discussed in Appendix B.3, the need to take a real part in Eq. (5.56), reflecting the existence of a cut in the ω plane, arises from the fact that the running coupling constant variable $g^2(t)$ used in the "improved" expansion sums an infinite number of Feynman diagrams. The dimensional continuation of individual Feynman diagrams remains meromorphic in ω.] The integral of Eq. (5.56) can be done by numerical integration, and so the problem of evaluating Eq. (5.43) has been reduced to a sequence of steps which can each be implemented by discrete methods.

Up to this point in the discussion I have used the one-loop exact running coupling constant defined in Eq. (5.49), which transforms the renormalization group to its minimal, exponential form. However, in doing an actual calculation it is not advantageous to make the nonanalytic transformation of Appendix B.1; instead, it is better to work with a two-loop exact or more general definition of the running coupling constant $g^2(t)$, in terms of which $\Psi_c(t)$ takes the form of a simple power-series expansion

$$\Psi_c(t)=\tfrac{1}{4}b_0^2 C_\Psi \frac{[g^2(t)]^2}{t^4}\left[1+\sum_{n=1}^{\infty}c_n[g^2(t)]^n\right] . \tag{5.57}$$

Corresponding to this, Eqs. (5.55) and (5.56) take the form

$$\Theta(u)=1+\sum_{n=1}^{\infty}c_n[g^2(u/\mathcal{M}^2)]^n ,$$

$$I^c_{\rm UV}=\tfrac{1}{4}b_0^2 C_\Psi \mathcal{M}^2\,\mathrm{Re}\left\{\int_{\log(\mathcal{M}^2 t_0)^{-1}}^{i\infty} dv\, e^v \times[g^2(e^{-v}/\mathcal{M}^2)]^2\Theta(e^{-v})\right\}, \tag{5.58}$$

with the coefficient c_1 known from the above-cited work of Kataev *et al.*, and with the higher coefficients yet to be computed. In doing a calculation it is of course necessary to make an explicit choice both for the dividing point t_0, and for the accuracy to which the perturbation expansion Ψ_c is to be computed. A reasonable strategy for doing this, I believe, is as follows:

(i) Choose t_0 far enough into the ultraviolet so that perturbation theory is valid at t_0, and so that $|\Delta I_{\rm UV}/I_{\rm IR}|$ is small. Such a choice is always possible, since the fact that $\Delta I_{\rm UV}$ is a convergent integral implies that

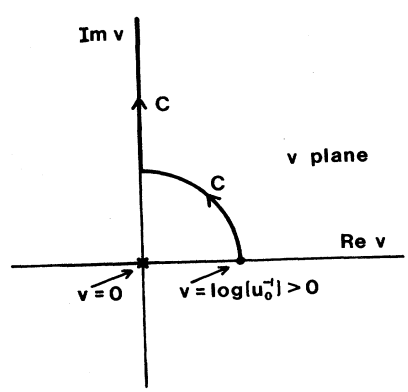

FIG. 5. Contour of integration C to be used in evaluating Eq. (5.56). The contour begins at $v=\log u_0^{-1}=\log(\mathcal{M}^2 t_0)^{-1}$ and must avoid the singularity at $v=0$.

$$\lim_{t_0\to 0} \Delta I_{\rm UV} = 0 \ . \tag{5.59}$$

(ii) Then, keeping t_0 fixed, compute a large enough number N of perturbation-theory coefficients c_n so that $I^c_{\rm UV}$ is well approximated by

$$\begin{aligned} I^{cN}_{\rm UV} &= \left[\int_0^{t_0} dt\, t^2 \Psi_c^N(t) \right]_{\text{dimensionally regularized}} \\ &= \tfrac{1}{4} b_0^2 C_\Psi \mathscr{M}^2 {\rm Re} \left\{ \int_{\log(\mathscr{M}^2 t_0)^{-1}}^{i\infty} dv\, e^v \right. \\ &\qquad \left. \times [g^2(e^{-v}/\mathscr{M}^2)]^2 \Theta^N(e^{-v}) \right\} , \end{aligned} \tag{5.60}$$

with $\Psi_c^N(t)$ and $\Theta^N(u)$, respectively, the truncations of the series of Eq. (5.57) and Eq. (5.58) to the first N terms. Such an approximation is possible because[31]

$$\lim_{N\to\infty} \Psi_c^N(t) = \Psi_c(t) \tag{5.61}$$

implies that

$$\lim_{N\to\infty} I^{cN}_{\rm UV} = I^c_{\rm UV} \ . \tag{5.62}$$

(iii) According to Eqs. (5.59) and (5.62), the total integral which we are calculating is given by the double limit

$$\begin{aligned} I &\equiv I_{\rm IR} + I^c_{\rm UV} + \Delta I_{\rm UV} \\ &= \lim_{t_0\to 0} \lim_{N\to\infty} (I_{\rm IR} + I^{cN}_{\rm UV}) , \end{aligned} \tag{5.63}$$

which with t_0 and N chosen according to (i) and (ii), is well approximated by

$$I \approx I_{\rm IR} + I^{cN}_{\rm UV} \ . \tag{5.64}$$

However, for fixed N we must be careful not to let t_0 become arbitrarily small in the approximated expression of Eq. (5.64), because as a result of the mismatch between $I_{\rm IR}$ and $I^{cN}_{\rm UV}$ and the quadratic divergence of the unregularized integral, we find

$$\lim_{t_0\to 0} (I_{\rm IR} + I^{cN}_{\rm UV}) = \infty \ . \tag{5.65}$$

In other words, the order of the limiting operations in Eq. (5.63) is significant, and is reflected in the procedure for choosing t_0 and N given in (i) and (ii) above.

In a recent paper, Zee (1982a) has given a model in which the infrared region is explicitly known, permitting the complete integral $I_{\rm UV} + I_{\rm IR}$ to be evaluated explicitly by dimensional regularization, and thus giving a simple illustration of the methods outlined above. Zee's model is a gauge theory in which the one-loop β-function coefficient b_0 is positive and small, while the two-loop β-function coefficient b_1 is negative [cf. Appendix B, Eq. (B1)], as happens, for instance, in QCD with 16 quark flavors. Such a theory is still asymptotically free, but has a nontrivial infrared stable fixed point at a small coupling constant $g_*^2 = -b_0/(2b_1)$. In the approximation of retaining only the leading term in an expansion in powers of g_*^2, one finds

$$\Psi(t) = \Psi_c(t) = \tfrac{1}{4} b_0^2 C_\Psi \frac{1}{t^4} \left[g^2(t) \left[1 - \frac{g^2(t)}{g_*^2} \right] \right]^2 , \tag{5.66}$$

with the two factors $g^2(t)[1-g^2(t)/g_*^2]$ arising directly from the two factors $\beta(g)/g$, which appear in Ψ when the trace anomaly formula of Eq. (3.28) is used. Hence, in this model the entire answer is given by the power-series expansion of Eq. (5.57), and the series terminates after only a finite number of terms. The explicit calculation shows that the sign of $G_{\rm ind}$ in this model depends strongly on the values of the β-function coefficients b_0 and b_1, and thus again is sensitive to infrared details. [For a further discussion, in the context of a survey of induced gravitation generally, see Zee (1982c).]

VI. EXTENSION TO A QUANTIZED METRIC

A. The general-coordinate invariant effective action, and derivation of the background metric Einstein equations

Up to this point the metric $g_{\mu\nu}$ has been treated as a purely classical variable, which determines the background geometry and thereby influences the dynamics of the quantized matter fields, but which is not itself quantized. While this classical metric formulation is useful as a model, there are a number of arguments indicating that it is not a satisfactory starting point for a fundamental theory. For example, Duff (1981) has pointed out that if the metric is not quantized, then the system of equations comprising the quantized matter fields and the classical Einstein equations for the metric is not invariant under metric-dependent redefinitions of the matter fields. Such redefinitions should be allowed in a completely consistent formulation, and Duff shows that they are in fact permitted if the metric is quantized. A second argument is simply that if the metric is treated as a classical variable, then the Einstein equations or the equivalent Einstein-Hilbert action principle must be postulated on an *ad hoc* basis. As we will see below, when the metric is quantized, the Einstein equations for the background metric emerge automatically as the leading long-wavelength approximation to the effective action formalism.

In discussing the dynamics of a quantized matter-metric system, it is necessary to give a procedure for identifying that part $\bar{g}_{\mu\nu}$ of the metric which we observe as the "classical" metric and a method for computing its effective action functional. I do this by using the background field method of DeWitt (1965), in which the total quantum metric $g_{\mu\nu}$ is split, in a self-consistent fashion, into the sum of a background metric $\bar{g}_{\mu\nu}$ and a quantum fluctuation $h_{\mu\nu}$,

[31]If the series for $\Psi_c(t)$ is only an asymptotic series, a summation procedure [such as Padé approximants or Borel summation; see Simon (1981)] is needed to extract, from the perturbation coefficients c_n, a sequence of approximants Ψ_c^N which satisfy Eq. (5.61).

$$g_{\mu\nu}=\bar{g}_{\mu\nu}+h_{\mu\nu} . \tag{6.1}$$

Elaborating on earlier work by 't Hooft (1975), recently Boulware (1981) and DeWitt (1981) have given an extension[32] of the background field method which preserves manifest general-coordinate covariance with respect to the background metric, and hence is an ideal vehicle for the discussion which follows.

To introduce the general-coordinate invariant effective action formalism, let us consider first the case in which no matter fields are present, so that the total microscopic action density consists solely of the term $\mathscr{L}_{\rm grav}[g_{\mu\nu}]$ introduced in Eq. (2.38). The partition function is then given formally by

$$Z=\int d[g_{\mu\nu}]e^{iS_{\rm grav}[g_{\mu\nu}]} ,$$
$$S_{\rm grav}[g_{\mu\nu}]=\int d^4x\,\sqrt{-g}\,\mathscr{L}_{\rm grav}[g_{\mu\nu}] , \tag{6.2}$$

but this expression is divergent because of the general-coordinate invariance of the action. To get a useful expression for Z, a gauge-fixing term and a compensating Fadde'ev-Popov (1967) determinant must be introduced into Eq. (6.2). Let us choose the gauge-fixing term in the action to have the form

$$S_{gf}[g^R_{\alpha\beta},g_{\mu\nu}]=\int d^4x\,\sqrt{-g^R}\,\mathscr{L}_{gf}[g^R_{\alpha\beta},g_{\mu\nu}] , \tag{6.3}$$

with $g^R_{\alpha\beta}$ an arbitrary fixed reference metric (which for the time being is distinct from $\bar{g}_{\mu\nu}$), and with $\mathscr{L}_{gf}$ constructed so as to transform formally as a general-coordinate scalar with respect to $g^R_{\alpha\beta}$, when the total quantum metric $g_{\mu\nu}$ is treated as a tensor with respect to $g^R_{\alpha\beta}$. A suitable gauge fixing for quantizing the curvature-squared action of Eq. (2.38) would be[33]

$$\mathscr{L}_{gf}[g^R_{\alpha\beta},g_{\mu\nu}]=\tfrac{1}{2}g^{R\lambda\sigma}g^{R\mu\nu}\nabla_{R\lambda}G_\mu\nabla_{R\sigma}G_\nu , \tag{6.4}$$

with ∇_R the covariant derivative with respect to $g^R_{\mu\nu}$, and with G_ν formally a covariant vector with respect to $g^R_{\mu\nu}$, given by

$$G_\nu=\nabla^\mu_R g_{\mu\nu}-\tfrac{1}{2}g^{R\mu\lambda}\nabla_{R\nu}g_{\mu\lambda} . \tag{6.5}$$

Equations (6.4) and (6.5) are a natural generalization of the usual harmonic coordinate condition; however, the precise form of $\mathscr{L}_{gf}$ (beyond the fact that it depends explicitly on the auxiliary metric $g^R_{\alpha\beta}$) will not play a role in the following discussion. The gauge fixing of Eqs. (6.4) and (6.5) completely breaks the invariance of the gravitational action under the group of general-coordinate transformations $g_{\mu\nu}\to g^\theta_{\mu\nu}$, which has the infinitesimal form[34]

$$\delta_\theta g_{\mu\nu}\equiv g_{\mu\lambda}\partial_\nu(\delta\theta^\lambda)+g_{\lambda\nu}\partial_\mu(\delta\theta^\lambda)+(\partial_\lambda g_{\mu\nu})\delta\theta^\lambda , \tag{6.6}$$

with $\delta\theta^\lambda$ an arbitrary infinitesimal contravariant vector. The Fadde'ev-Popov compensating determinant for the gauge-fixing action of Eq. (6.3) is defined by[35]

$$1=\int d[\theta]e^{iS_{gf}[g^R_{\alpha\beta},g^\theta_{\mu\nu}]}\Delta[g^R_{\alpha\beta},g_{\mu\nu}] , \tag{6.7}$$

with $d[\theta]$ the invariant measure on the manifold of the general-coordinate transformation group. Since the invariant measure satisfies

$$d[\theta\theta']=d[\theta'\theta]=d[\theta] \tag{6.8}$$

for any fixed general-coordinate transformation $g_{\mu\nu}\to g^{\theta'}_{\mu\nu}$, we learn from Eqs. (6.7) and (6.8) that the compensating determinant is invariant under general-coordinate transformations on $g_{\mu\nu}$,

$$\Delta[g^R_{\alpha\beta},g_{\mu\nu}]=\Delta[g^R_{\alpha\beta},g^{\theta'}_{\mu\nu}] . \tag{6.9}$$

According to the Fadde'ev-Popov ansatz,[35] a convergent path-integral representation for the partition function is then given by

$$Z=\int d[g_{\mu\nu}]\Delta[g^R_{\alpha\beta},g_{\mu\nu}]e^{iS_{\rm grav}[g_{\mu\nu}]+iS_{gf}[g^R_{\alpha\beta},g_{\mu\nu}]} . \tag{6.10}$$

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[32]See also Fradkin and Vilkovisky (1976), who use the gauge fixing

$$\mathscr{L}_{gf}=\tfrac{1}{2}g^{R\mu\nu}G_\mu G_\nu\,|_{g^R_{\mu\nu}=\bar{g}_{\mu\nu}}$$

to quantize the Einstein theory formally, and who suggest that it gives a generally covariant effective action for $\bar{g}_{\mu\nu}$. For the use of the gauge-invariant background field method to compute two-loop counter terms, see Abbott (1981) and Ichinose and Omote (1982).

[33]For a discussion of the complexities involved in representing higher-derivative gauge fixings in terms of a local "ghost" action density, see Kallosh (1978) and Nielsen (1978).

[34]As pointed out by Fradkin and Vilkovisky (1975) and reviewed by Batalin and Fradkin (1979), the presence of a term $\partial_\lambda g_{\mu\nu}$ in $\delta_\theta g_{\mu\nu}$ leads to a nonvanishing variation of the integration measure under general coordinate transformations,

$$\delta_\theta d[g_{\mu\nu}]\propto{\rm Tr}[\delta(\delta_\theta g_{\mu\nu}(x))/\delta g_{\lambda\sigma}(y)]$$
$$\propto\int d^4x\,\partial_\lambda\delta^4(0)\delta\theta^\lambda(x) .$$

The $\partial_\lambda\delta^4(0)$ term vanishes in covariant calculations using dimensional regularization, and is ignored in the discussion of the text, where $d[g_{\mu\nu}]$ is treated as being general-coordinate invariant. The variation of the integration measure cannot be ignored in setting up a canonical, Hamiltonian formalism using a massive regulator scheme; in this case it leads to an extra Jacobian factor in the path-integral formulas, which can be represented by a quartic local "ghost" action density. For a related analysis of the connection between Jacobian factors in the path-integral measure and chiral and conformal anomalies, see Fujikawa (1981).

[35]The discussion of Eqs. (6.7)−(6.14) is based on Sec. 3.3 of Fadde'ev and Slavnov (1980). Strictly speaking, the Lagrangian form of the path-integral formula given in Eq. (6.10) must be derived from the more fundamental Hamiltonian form, and the standard textbook discussions describe this step only for second-order actions. The derivation of Eq. (6.10) from the Hamiltonian formalism in the case of fourth-order, curvature-squared gravitational actions has been carried out by Boulware (1982).

To verify that Z is independent of the choice of the reference metric $g^R_{\alpha\beta}$, let us multiply the integrand of Eq. (6.10) by unity in the form

$$1=\int d[\theta]e^{iS_{gf}[g'^R_{\alpha\beta},g^\theta_{\mu\nu}]}\Delta[g'^R_{\alpha\beta},g^\theta_{\mu\nu}]\ , \tag{6.11}$$

giving

$$Z=\int d[\theta]d[g_{\mu\nu}]\Delta[g^R_{\alpha\beta},g_{\mu\nu}]\Delta[g'^R_{\alpha\beta},g^\theta_{\mu\nu}] \times e^{iS_{\rm grav}[g_{\mu\nu}]+iS_{gf}[g^R_{\alpha\beta},g_{\mu\nu}]+iS_{gf}[g'^R_{\alpha\beta},g^\theta_{\mu\nu}]}\ . \tag{6.12}$$

Making the substitution $g_{\mu\nu}\rightarrow g^{\theta^{-1}}_{\mu\nu}$, and using the fact that the action $S_{\rm grav}[g_{\mu\nu}]$, the compensating determinants Δ and the integration measure[34] $d[g_{\mu\nu}]$ are all general-coordinate invariant, and also using the invariance property $d[\theta]=d[\theta^{-1}]$, Eq. (6.12) becomes

$$Z=\int d[\theta^{-1}]d[g_{\mu\nu}]\Delta[g^R_{\alpha\beta},g^{\theta^{-1}}_{\mu\nu}]\Delta[g'^R_{\alpha\beta},g_{\mu\nu}] \times e^{iS_{\rm grav}[g_{\mu\nu}]+iS_{gf}[g^R_{\alpha\beta},g^{\theta^{-1}}_{\mu\nu}]+iS_{gf}[g'^R_{\alpha\beta},g_{\mu\nu}]}\ . \tag{6.13}$$

But now applying Eq. (6.7) once more (with θ replaced by θ^{-1}) we get

$$Z=\int d[g_{\mu\nu}]\Delta[g'^R_{\alpha\beta},g_{\mu\nu}]e^{iS_{\rm grav}[g_{\mu\nu}]+iS_{gf}[g'^R_{\alpha\beta},g_{\mu\nu}]}\ , \tag{6.14}$$

which differs from the original form in Eq. (6.10) by the replacement of $g^R_{\alpha\beta}$ by $g'^R_{\alpha\beta}$.

Let us now introduce an external source $J^{\lambda\sigma}$ coupled to the metric $g_{\lambda\sigma}$, so that the path-integral formula of Eq. (6.10) is modified to read

$$e^{iW[J^{\lambda\sigma},g^R_{\alpha\beta}]}\equiv Z[J^{\lambda\sigma},g^R_{\alpha\beta}] =\int d[g_{\mu\nu}]\Delta[g^R_{\alpha\beta},g_{\mu\nu}] \times e^{iS_{\rm grav}[g_{\mu\nu}]+iS_{gf}[g^R_{\alpha\beta},g_{\mu\nu}]-i\int d^4x\, g_{\lambda\sigma}J^{\lambda\sigma}}\ . \tag{6.15}$$

Both $J^{\lambda\sigma}$ and $g^R_{\alpha\beta}$ are indicated as arguments of Z in Eq. (6.15) because the source term breaks the general-coordinate invariance of the action. As a result, when $J^{\lambda\sigma}\neq 0$ the argument of Eqs. (6.11)–(6.14) cannot be applied, and hence the previously derived zero-source invariance,

$$0=\frac{\delta}{\delta g^R_{\alpha\beta}}Z[0,g^R_{\alpha\beta}]=\frac{\delta}{\delta g^R_{\alpha\beta}}W[0,g^R_{\alpha\beta}]\ , \tag{6.16}$$

cannot be extended to the case when a source is present. From the functional W, we can calculate the expectation value $\bar{g}_{\lambda\sigma}$ of the metric in the presence of the source $J^{\lambda\sigma}$ by using the formula

$$\bar{g}_{\lambda\sigma}[J^{\gamma\delta},g^R_{\alpha\beta}]=\langle g_{\lambda\sigma}\rangle_J=-\frac{\delta W}{\delta J^{\lambda\sigma}}\ , \tag{6.17}$$

which can be inverted to determine $J^{\lambda\sigma}$ implicitly as a functional of $\bar{g}_{\lambda\sigma}$ (and of $g^R_{\alpha\beta}$),

$$J^{\lambda\sigma}=J^{\lambda\sigma}[\bar{g}^{\gamma\delta},g^R_{\alpha\beta}]\ . \tag{6.18}$$

Let us now introduce the Legendre-transformed effective action functional Γ defined by

$$\Gamma=W+\int d^4x\, \bar{g}_{\lambda\sigma}J^{\lambda\sigma}\ . \tag{6.19}$$

Varying Eq. (6.19) (for fixed $g^R_{\alpha\beta}$) and using Eq. (6.17), we get

$$\delta\Gamma=\delta W+\int d^4x(\bar{g}_{\lambda\sigma}\delta J^{\lambda\sigma}+\delta\bar{g}_{\lambda\sigma}J^{\lambda\sigma}) =\int d^4x\, \delta\bar{g}_{\lambda\sigma}J^{\lambda\sigma}\ , \tag{6.20}$$

which shows that Γ is a functional only of $\bar{g}^{\lambda\sigma}$ and $g^R_{\alpha\beta}$,

$$\Gamma=\Gamma[\bar{g}^{\lambda\sigma},g^R_{\alpha\beta}]\ , \tag{6.21a}$$

and satisfies

$$\frac{\delta\Gamma}{\delta\bar{g}_{\lambda\sigma}}=J^{\lambda\sigma}\ . \tag{6.21b}$$

The partition function $Z[J^{\lambda\sigma},g^R_{\alpha\beta}]$ can be reexpressed in terms of the effective action Γ through the formula

$$e^{iW[J^{\lambda\sigma},g^R_{\alpha\beta}]}=\text{ext}_{g^{\lambda\sigma}}\left[e^{i\Gamma[g^{\lambda\sigma},g^R_{\alpha\beta}]-i\int d^4x\, g_{\lambda\sigma}J^{\lambda\sigma}}\right]\ , \tag{6.22}$$

where $\text{ext}_{g^{\lambda\sigma}}(\ )$ indicates that one is to take the extremum of the parenthesis over all values of $g^{\lambda\sigma}$. Equation (6.22) is verified by noting that the exponent on the right-hand side is extremized[36] at the metric $g^{\lambda\sigma}=\bar{g}^{\lambda\sigma}$ for which Eq. (6.21b) is satisfied, and that at the extremum it can be rewritten, by using Eq. (6.19), to give $W[J^{\lambda\sigma},g^R_{\alpha\beta}]$.

With these preliminaries completed, we are ready to introduce the general-coordinate invariant effective action functional $\Gamma_{\rm inv}[\bar{g}^{\lambda\sigma}]$, defined by identifying the reference metric $g^R_{\alpha\beta}$ with the expectation value $\bar{g}_{\alpha\beta}$ in the formulas given above,

$$\Gamma_{\rm inv}[\bar{g}^{\lambda\sigma}]\equiv\Gamma[\bar{g}^{\lambda\sigma},\bar{g}_{\alpha\beta}]\ . \tag{6.23}$$

To get an explicit formula for $\Gamma_{\rm inv}$, let us multiply Eq. (6.15) by $\exp(i\int d^4x\, \bar{g}_{\lambda\sigma}J^{\lambda\sigma})$ and change to $h_{\mu\nu}$, defined in Eq. (6.1), as the new functional integration variable. Making use of the identity

$$S_{gf}[\bar{g}_{\alpha\beta},\bar{g}_{\mu\nu}+h_{\mu\nu}]=S_{gf}[\bar{g}_{\alpha\beta},h_{\mu\nu}] =\tfrac{1}{2}\bar{g}^{\lambda\sigma}\bar{g}^{\mu\nu}\bar{\nabla}_\lambda G_\mu\bar{\nabla}_\sigma G_\nu\ ,$$
$$G_\nu=\bar{\nabla}^\mu h_{\mu\nu}-\tfrac{1}{2}\bar{g}^{\mu\lambda}\bar{\nabla}_\nu h_{\mu\lambda} \tag{6.24}$$

(which follows from the fact that $\bar{\nabla}_\lambda\bar{g}_{\mu\nu}=0$), we get the following functional integral representation for $\Gamma_{\rm inv}$,

$$e^{i\Gamma_{\rm inv}[\bar{g}^{\lambda\sigma}]} =\int d[h_{\mu\nu}]\Delta[\bar{g}_{\alpha\beta},\bar{g}_{\mu\nu}+h_{\mu\nu}] \times e^{iS_{\rm grav}[\bar{g}_{\mu\nu}+h_{\mu\nu}]+iS_{gf}[\bar{g}_{\alpha\beta},h_{\mu\nu}]-i\int d^4x\, h_{\lambda\sigma}J^{\lambda\sigma}[\bar{g}_{\alpha\beta}]}\ . \tag{6.25}$$

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[36] I will assume here, and later on, that the extremum problems which are encountered always have a unique solution.

The source current $J^{\lambda\sigma}$ in Eq. (6.25) is implicitly determined as a functional of $\bar{g}^{\lambda\sigma}$ by the requirement

$$0=\langle h_{\lambda\sigma}\rangle_J=-\frac{\delta\Gamma}{\delta J^{\lambda\sigma}}\ , \tag{6.26}$$

which is equivalent to

$$0=\int d[h_{\mu\nu}]\Delta[\bar{g}_{\alpha\beta},\bar{g}_{\mu\nu}+h_{\mu\nu}]h_{\lambda\sigma} \times e^{iS_{\rm grav}[\bar{g}_{\mu\nu}+h_{\mu\nu}]+iS_{gf}[\bar{g}_{\alpha\beta},h_{\mu\nu}]-i\int d^4x\, h_{\xi\eta}J^{\xi\eta}[\bar{g}_{\alpha\beta}]}\ . \tag{6.27}$$

To see that $\Gamma_{\rm inv}$ is a general-coordinate invariant functional of its argument, we note that we are free to take $h_{\mu\nu}$, which is a dummy integration variable, to transform as a tensor with respect to general-coordinate transformations of $\bar{g}_{\mu\nu}$. By construction, S_{gf} is then a scalar with respect to such transformations, and therefore from Eq. (6.7), the compensating determinant $\Delta[\bar{g}_{\alpha\beta},\bar{g}_{\mu\nu}+h_{\mu\nu}]$ is also a scalar. Equation (6.27) then determines $J^{\lambda\sigma}$ to transform as a tensor, and so the right-hand side of Eq. (6.25) is manifestly invariant under general-coordinate transformations of $\bar{g}_{\mu\nu}$.

Let us next show that the source-free partition function Z can be obtained by extremizing the gauge-invariant effective action functional. According to Eqs. (6.22) and (6.16), in the absence of an external source we have

$$Z={\rm ext}_{g^{\lambda\sigma}}(e^{i\Gamma[g^{\lambda\sigma},g^R_{\alpha\beta}]})\ , \tag{6.28a}$$

$$\frac{\delta}{\delta g^R_{\alpha\beta}}Z=0\ . \tag{6.28b}$$

The extremum in Eq. (6.28a) determines $g^{\lambda\sigma}$ to take a value $\bar{g}^{\lambda\sigma}\,[g^R_{\alpha\beta}]$ at which

$$\frac{\delta\Gamma}{\delta g^{\lambda\sigma}}[\bar{g}^{\lambda\sigma},g^R_{\alpha\beta}]=0\ , \tag{6.29}$$

and when expressed in terms of $\bar{g}^{\lambda\sigma}$, the reference-metric invariance of Eq. (6.28b) takes the form

$$\frac{\delta\Gamma}{\delta g^R_{\alpha\beta}}[\bar{g}^{\lambda\sigma},g^R_{\alpha\beta}]=0\ . \tag{6.30}$$

Since $\bar{g}^{\lambda\sigma}[g^R_{\alpha\beta}]$ is a continuous map from the manifold of reference metrics into itself, there is[37] a fixed point $g^R_{\alpha\beta}=g^*_{\alpha\beta}$ for which $\bar{g}^{\lambda\sigma}[g^*_{\alpha\beta}]=g^{*\lambda\sigma}$. At the fixed point, Eqs. (6.29) and (6.30) become

$$\frac{\delta}{\delta g^{\lambda\sigma}}\Gamma[g^{*\lambda\sigma},g^*_{\alpha\beta}]=\frac{\delta}{\delta g^R_{\alpha\beta}}\Gamma[g^{*\lambda\sigma},g^*_{\alpha\beta}]=0\ , \tag{6.31}$$

which together imply that

$$\frac{\delta}{\delta g^{\lambda\sigma}}\Gamma_{\rm inv}[g^{*\lambda\sigma}]=0\ , \tag{6.32}$$

and so[36] we have

$$Z={\rm ext}_{g^{\lambda\sigma}}(e^{i\Gamma_{\rm inv}[g^{\lambda\sigma}]})\ . \tag{6.33}$$

[37]I am assuming that $g^R_{\alpha\beta}$ and $\bar{g}^{\lambda\sigma}$ lie in a closed convex set, so that the conditions of the Schauder fixed point theorem are satisfied. I wish to thank J. and L. Chayes for a conversation about the conditions for the existence of a fixed point.

An alternative way of deriving Eq. (6.32) is to note that when variations of $g^R_{\alpha\beta}$ are included, Eq. (6.20) is modified to read

$$\delta\Gamma=\int d^4x\left[\left[\frac{\delta}{\delta g^R_{\alpha\beta}}W\right]\delta g^R_{\alpha\beta}+\left[\frac{\delta}{\delta J^{\lambda\sigma}}W+\bar{g}_{\lambda\sigma}\right]\delta J^{\lambda\sigma}+\delta\bar{g}_{\lambda\sigma}J^{\lambda\sigma}\right]\ , \tag{6.34}$$

which implies that

$$\frac{\delta}{\delta\bar{g}_{\lambda\sigma}}\Gamma^{\rm inv}[\bar{g}^{\lambda\sigma}]=\frac{\delta}{\delta g^R_{\lambda\sigma}}W\left[J^{\lambda\sigma}[\bar{g}^{\gamma\delta},\bar{g}_{\alpha\beta}],\bar{g}_{\alpha\beta}\right]+J^{\lambda\sigma}[\bar{g}^{\gamma\delta},\bar{g}_{\alpha\beta}]\ . \tag{6.35}$$

At the solution $\bar{g}_{\alpha\beta}=g^*_{\alpha\beta}$ of the equation

$$J^{\lambda\sigma}[\bar{g}^{\gamma\delta},\bar{g}_{\alpha\beta}]=0\ , \tag{6.36}$$

we learn from Eq. (6.16) that both terms on the right-hand side of Eq. (6.35) are zero, thus reproducing Eq. (6.32).

Having now established the procedure for identifying the background metric and calculating its dynamics, let us restore the matter fields to the analysis. Following the notation of Eqs. (2.1) and (2.38), this is done by making the substitutions

$$\begin{aligned} &d[g_{\mu\nu}]\to d[g_{\mu\nu}]d\{\phi\}\ ,\\ &S_{\rm grav}[g_{\mu\nu}]\to S_{\rm matter}[\{\phi\},g_{\mu\nu}]+S_{\rm grav}[g_{\mu\nu}]\ ,\\ &S_{\rm matter}[\{\phi\},g_{\mu\nu}]=\int d^4x\,\sqrt{-g}\,\mathscr{L}_{\rm matter}[\{\phi\},g_{\mu\nu}] \end{aligned} \tag{6.37}$$

in Eq. (6.10), giving[38]

$$Z=\int d[g_{\mu\nu}]d\{\phi\}\Delta[g^R_{\alpha\beta},g_{\mu\nu}] \times e^{iS_{\rm matter}[\{\phi\},g_{\mu\nu}]+iS_{\rm grav}[g_{\mu\nu}]+iS_{gf}[g^R_{\alpha\beta},g_{\mu\nu}]}\ . \tag{6.38}$$

Let us next divide the matter fields $\{\phi\}$ into "light" and "heavy" components[39] as in Sec. II.B, and find the effective action equations governing the dynamics of the light

[38]As in the earlier sections, I do not explicitly indicate the gauge-fixing procedure for the matter gauge fields.

[39]The heavy "matter" fields can include any fields which are not directly observable, including ones which are basically geometric or pregeometric in nature, and auxiliary fields. The only essential requirement for the discussion of Secs. VI.A and VI.B is that the partition function be representable in the form of Eq. (6.38) for some choice of heavy fields $\{\phi^H\}$. The discussion, as given, applies only to the case when the observed matter fields $\{\phi^L\}$ appear as elementary fields in the fundamental action. If, as has been much discussed recently, some of the light fields are effective fields for composites formed from the truly elementary fields, an extended effective action formalism is needed, along the lines discussed by Cornwall, Jackiw, and Tomboulis (1974). For a discussion of the effective action for composites in a nonrelativistic solid-state physics context, see Kleinert (1978).

fields. The most straightforward way of doing this is to introduce external sources $\{J^L\}$ and expectation values $\{\bar{\phi}^L\}$ for the light matter fields $\{\phi^L\}$, as well as an external source $J^{\lambda\sigma}$ and expectation value $\bar{g}^{\lambda\sigma}$ for the metric, and to construct the Legendre-transformed effective action functional $\Gamma[\{\bar{\phi}^L\}, \bar{g}^{\lambda\sigma}, g^R_{\alpha\beta}]$ in analogy with Eqs. (6.15)–(6.21) above. Following Eq. (6.28), the partition function Z of Eq. (6.38) can be expressed in the form

$$Z=\mathrm{ext}_{\bar{g}^{\lambda\sigma},\{\bar{\phi}^L\}}(e^{i\Gamma[\{\bar{\phi}^L\},\bar{g}^{\lambda\sigma},g^R_{\alpha\beta}]}) ,$$
$$\frac{\delta}{\delta g^R_{\alpha\beta}}Z=0 , \tag{6.39}$$

and the fixed point argument of Eqs. (6.28)–(6.33) can then be used to show that Eqs. (6.39) are equivalent to

$$Z=\mathrm{ext}_{\bar{g}^{\alpha\beta},\{\bar{\phi}^L\}}(e^{i\Gamma_{\mathrm{inv}}[\{\bar{\phi}^L\},\bar{g}^{\alpha\beta}]}) , \tag{6.40}$$

with Γ_{inv} the general-coordinate invariant effective action

$$\Gamma_{\mathrm{inv}}=\Gamma[\{\bar{\phi}^L\},\bar{g}^{\lambda\sigma},\bar{g}_{\alpha\beta}] . \tag{6.41}$$

Equation (6.40) gives an exact description of the dynamics of the light matter-metric system in terms of a classical variational principle

$$\frac{\delta}{\delta\bar{g}^{\alpha\beta}}\Gamma_{\mathrm{inv}}[\{\bar{\phi}^L\},\bar{g}^{\alpha\beta}]=\frac{\delta}{\delta\{\bar{\phi}^L\}}\Gamma_{\mathrm{inv}}[\{\bar{\phi}^L\},\bar{g}^{\alpha\beta}]=0 ; \tag{6.42}$$

that is, for an isolated system, the background metric and the light-field expectation values must evolve according to a principle of stationary effective action.

To put Eq. (6.42) in a more familiar form, let us assume the background metric to be slowly varying on the length scale of the heavy fields, so that the curvature dependence of Γ_{inv} can be approximated by writing

$$\Gamma_{\mathrm{inv}}[\{\bar{\phi}^L\},\bar{g}^{\alpha\beta}]=S_{\mathrm{eff,matter}}[\{\bar{\phi}^L\},\bar{g}^{\alpha\beta}]+S_{\mathrm{eff,grav}}[\bar{g}_{\alpha\beta}] + \text{small corrections} ,$$

$$S_{\mathrm{eff,matter}}[\{\bar{\phi}^L\},\bar{g}^{\alpha\beta}]=\text{minimal generally covariant extension of } \Gamma_{\mathrm{inv}}[\{\bar{\phi}^L\},\eta^{\alpha\beta}]-\Gamma_{\mathrm{inv}}[\{0\},\eta^{\alpha\beta}] ,$$

$$S_{\mathrm{eff,grav}}[\bar{g}_{\alpha\beta}]=\Gamma_{\mathrm{inv}}[\{0\},\bar{g}^{\alpha\beta}]+O[(\partial_\lambda\bar{g}_{\mu\nu})^4] = \int d^4x\sqrt{-\bar{g}}\frac{1}{16\pi G_{\mathrm{ind}}}(\bar{R}-2\Lambda_{\mathrm{ind}}) , \tag{6.43}$$

with $\bar{R}=R[\bar{g}_{\alpha\beta}]$ the curvature scalar constructed from $\bar{g}_{\alpha\beta}$. As defined in Eq. (6.43), $S_{\mathrm{eff,matter}}$ contains terms in Γ_{inv} which are $\bar{\phi}^L$ dependent and are of zeroth or first order in space-time derivatives of $\bar{g}_{\alpha\beta}$, while $S_{\mathrm{eff,grav}}$ contains terms independent of the matter fields $\bar{\phi}^L$, which are of zeroth through second order in space-time derivatives of $\bar{g}_{\alpha\beta}$. Substituting Eq. (6.43) into Eq. (6.42) gives the classical Einstein equations[40] and the effective classical equations for the matter fields,

$$\frac{1}{8\pi G^{\mathrm{ind}}}(\bar{G}^{\mu\nu}+\Lambda_{\mathrm{ind}}\bar{g}^{\mu\nu}) = T^{\mu\nu}_{\mathrm{matter}} = \frac{2}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}_{\mu\nu}}S_{\mathrm{eff,matter}}[\{\bar{\phi}^L\},\bar{g}^{\alpha\beta}] ,$$
$$\frac{\delta}{\delta\{\bar{\phi}^L\}}S_{\mathrm{eff,matter}}[\{\bar{\phi}^L\},\bar{g}^{\alpha\beta}]=0 . \tag{6.44}$$

Of course, making the approximations of Eq. (6.43) is only a matter of convenience in dealing with slowly varying background metrics, and the exact dynamics of $\bar{g}_{\alpha\beta}$ and $\{\bar{\phi}^L\}$, including the effect of higher derivative terms in $\Gamma_{\mathrm{inv}}[\{\bar{\phi}^L\},\bar{g}^{\alpha\beta}]$, is always governed by Eq. (6.42).

An alternative way of describing the dynamics of the light fields is to keep them as quantum variables and to introduce, inside the $\{\phi^L\}$ functional integration, an effective action which incorporates the quantum effects of the heavy fields [cf. Eq. (2.15) above]. To do this, we rewrite Eq. (6.38) in the form

$$Z=\int d\{\phi^L\}e^{iW[\{\phi^L\},g^R_{\alpha\beta}]} , \tag{6.45a}$$
$$e^{iW[\{\phi^L\},g^R_{\alpha\beta}]}=\int d[g_{\mu\nu}]d\{\phi^H\}\Delta[g^R_{\alpha\beta},g_{\mu\nu}] \times e^{iS_{\mathrm{matter}}[\{\phi\},g_{\mu\nu}]+iS_{\mathrm{grav}}[g_{\mu\nu}]+iS_{gf}[g^R_{\alpha\beta},g_{\mu\nu}]} . \tag{6.45b}$$

The dependence of W on $g^R_{\alpha\beta}$ results from the fact that the general covariance of Eq. (6.45b) is broken by the fixed (nonscalar) light fields $\{\phi^L\}$, which act in the same manner as does the source term in Eq. (6.15), and prevent the application of the argument of Eqs. (6.11)–(6.14). Let us now introduce an additional external source $J^{\lambda\sigma}$ for the metric and use it to construct a Legendre-transformed effective action functional for the metric, $\Gamma'[\{\phi^L\},\bar{g}^{\lambda\sigma},g^R_{\alpha\beta}]$, as in Eqs. (6.15)–(6.22). [The prime on Γ' is to distinguish it from the functional $\Gamma[\{\bar{\phi}^L\},\bar{g}^{\lambda\sigma},g^R_{\alpha\beta}]$ introduced following Eq. (6.38), which was constructed by Legendre transforming with respect to both the metric and the light fields.] This allows us to rewrite Eq. (6.45) in the form

$$Z=\int d\{\phi^L\}\mathrm{ext}_{\bar{g}^{\lambda\sigma}}(e^{i\Gamma'[\{\phi^L\},\bar{g}^{\lambda\sigma},g^R_{\alpha\beta}]}) , \tag{6.46}$$

[40]Derivations of the Einstein equations similar to that of Eqs. (6.38)–(6.44) have been given by Fradkin and Vilkovisky (1977a, 1977b), by DeWitt (1979), and by Horowitz (1981). Fradkin and Vilkovisky (1977a, 1977b) and DeWitt (1979, 1981) have emphasized that Eq. (6.42) contains corrections to the Einstein equations which are needed for rapidly varying metrics. For discussions of the "out-in" form of the semiclassical gravitational equations, see Kay (1981) and Horowitz (1981).

which gives an exact formulation of the quantum dynamics of the light fields and the background metric, expressed in terms of a general-coordinate noninvariant effective action Γ'. Because the integrand in Eq. (6.46) still depends on $g^R_{\alpha\beta}$, the fixed point argument of Eqs. (6.28)–(6.32) cannot be used to introduce a gauge-invariant effective action inside the light-field functional integration. An alternative way of seeing this is to note that the extremum in Eq. (6.46) makes $\bar{g}^{\lambda\sigma}$ a functional of the integration variables $\{\phi^L\}$, and so the fixed reference metric $g^R_{\alpha\beta}$ cannot be equated to $\bar{g}^{\lambda\sigma}$ inside the functional integration. To proceed further, let us consider the mean-field approximation to Eq. (6.46), obtained by pulling the extremum over $\bar{g}^{\lambda\sigma}$ to the outside of the functional integration (which should be a physically reasonable approximation for the slowly varying components of $\bar{g}^{\lambda\sigma}$),

$$Z_{mf} \approx \mathrm{ext}_{\bar{g}^{\lambda\sigma}} \int d\{\phi^L\} e^{i\Gamma'[\{\phi^L\},\bar{g}^{\lambda\sigma},g^R_{\alpha\beta}]} . \tag{6.47a}$$

Since Z is independent of $g^R_{\alpha\beta}$, Z_{mf} is independent of $g^R_{\alpha\beta}$ to within the accuracy of the mean-field approximation, and so we have

$$\frac{\delta}{\delta g^R_{\alpha\beta}} Z_{mf} \approx 0 . \tag{6.47b}$$

Equations (6.47a,b) have the same structure as Eqs. (6.28a,b) above, and thus within the mean-field approximation we can apply the fixed point argument of Eqs. (6.28)–(6.32), giving

$$Z_{mf} \approx \mathrm{ext}_{\bar{g}^{\alpha\beta}} \int d\{\phi^L\} e^{i\Gamma'_{\mathrm{inv}}[\{\phi^L\},\bar{g}^{\alpha\beta}]} , \tag{6.48}$$

with Γ'_{inv} the general-coordinate invariant effective action

$$\Gamma'_{\mathrm{inv}} = \Gamma'[\{\phi^L\},\bar{g}^{\lambda\sigma},\bar{g}_{\alpha\beta}] . \tag{6.49}$$

Assuming a slowly varying background metric and making an expansion of the primed effective action analogous to that of Eq. (6.43), we can approximate Eq. (6.48) by

$$Z'_{mf} \approx \mathrm{ext}_{\bar{g}^{\alpha\beta}} \int d\{\phi^L\} \times e^{iS'_{\mathrm{eff,matter}}[\{\phi^L\},\bar{g}^{\alpha\beta}] + S'_{\mathrm{eff,grav}}[\bar{g}_{\alpha\beta}]} . \tag{6.50}$$

This gives the field equations for the background metric in the form

$$\frac{1}{8\pi G'_{\mathrm{ind}}}(\bar{G}^{\mu\nu} + \Lambda'_{\mathrm{ind}}\bar{g}^{\mu\nu}) \approx \frac{\int d\{\phi^L\} e^{iS'_{\mathrm{eff,matter}}[\{\phi^L\},\bar{g}_{\alpha\beta}]} T'^{\mu\nu}_{\mathrm{matter}}}{\int d\{\phi^L\} e^{iS'_{\mathrm{eff,matter}}[\{\phi^L\},\bar{g}_{\alpha\beta}]}} = \langle 0^+ | T'^{\mu\nu}_{\mathrm{matter}} | 0^- \rangle ,$$

$$T'^{\mu\nu}_{\mathrm{matter}} = \frac{2}{\sqrt{-\bar{g}}} \frac{\delta}{\delta \bar{g}_{\mu\nu}} S'_{\mathrm{eff,matter}} , \tag{6.51a}$$

with $|0^+\rangle$ and $|0^-\rangle$ the "out" and "in" vacuum states for the observable matter fields. Thus, the background metric formalism, with the mean-field approximation of Eq. (6.47a) and an expansion for slowly varying metrics, gives the "out-in" form[40] of the semiclassical gravitational equations. The quantum field dynamics for the matter fields then follows in the usual fashion from the approximation to the partition function given in Eq. (6.50). Because the induced constants G'_{ind} and Λ'_{ind} do not include the quantum effects of the light matter fields, they are not identical to the constants G_{ind} and Λ_{ind} defined in Eq. (6.43), which do include such effects. However, since one expects

$$G'_{\mathrm{ind}}/G_{\mathrm{ind}} \approx 1 + O[(l_{\mathrm{Planck}}/l_{\mathrm{proton}})^2] , \tag{6.51b}$$

with l_{proton} the proton Compton wavelength, the difference between the primed and unprimed constants is numerically very small.

B. Formulas for G^{-1}_{ind} and Λ_{ind} with a quantized metric

To complete the analysis begun in Sec. VI.A, we must derive expressions for the induced gravitational and cosmological constants in terms of functional integrals over $h_{\mu\nu}$ and the matter fields,[41] and discuss the conditions under which these expressions yield finite answers. Since the gravitational effective action relevant to astronomy and astrophysics is insensitive to the state of motion of the long-wavelength components of the matter fields, it is most convenient to start the derivation of this section from the formula

$$e^{i\Gamma_{\mathrm{inv}}[\bar{g}_{\alpha\beta}]} = \mathrm{ext}_{\{\bar{\phi}^L\}} (e^{i\Gamma_{\mathrm{inv}}[\{\bar{\phi}^L\},\bar{g}_{\alpha\beta}]}) , \tag{6.52}$$

rather than from the functional $\Gamma_{\mathrm{inv}}[\{0\},\bar{g}_{\alpha\beta}]$ of Eq. (6.43). It is also convenient at this point to represent[33] the gravitational compensating determinant $\Delta[\bar{g}_{\alpha\beta}, g_{\mu\nu}]$ by an added action density $\sqrt{-g}\,\mathscr{L}_{\mathrm{ghost}}$, and to adopt the convention that a functional argument $h_{\mu\nu}$ implicitly indicates a dependence on the ghost fields and that the integration measure $d[h_{\mu\nu}]$ implicitly includes the ghost integration measure. By substituting the expansion of $\Gamma_{\mathrm{inv}} \approx S_{\mathrm{eff,grav}}$ from Eq. (6.43) into the left-hand side of Eq. (6.52), and noting that the right-hand side of Eq. (6.52) has a functional integral representation obtained by making the substitutions

$$d[h_{\mu\nu}] \to d[h_{\mu\nu}]d\{\phi\} , \quad S_{\mathrm{grav}} \to S_{\mathrm{matter}} + S_{\mathrm{grav}} \tag{6.53}$$

[41]In an older terminology, we must compute expressions for the renormalized gravitational and cosmological constants, including radiative corrections arising from virtual metric and matter fluctuations, in terms of the bare parameters appearing in the fundamental Lagrangian.

in Eq. (6.25), we can rewrite Eq. (6.52) in the form[42]

$$e^{i\int d^4x\{\sqrt{-\bar g}(1/16\pi G_{\rm ind})(\bar R-2\Lambda_{\rm ind})+O[(\partial_\lambda \bar g_{\mu\nu})^4]\}}$$
$$=\int d[h_{\mu\nu}]d\{\phi\}e^{i\int d^4x\,\tilde{\mathscr{L}}[\{\phi\},\bar g_{\alpha\beta},h_{\mu\nu}]}\,,$$
$$\tilde{\mathscr{L}}[\{\phi\},\bar g_{\alpha\beta},h_{\mu\nu}]$$
$$=\sqrt{-g}\,\{\mathscr{L}_{\rm matter}[\{\phi\},g_{\mu\nu}]+\mathscr{L}_{\rm grav}[g_{\mu\nu}]$$
$$+\mathscr{L}_{\rm ghost}[\bar g_{\alpha\beta},h_{\mu\nu}]\}$$
$$+\sqrt{-\bar g}\,\mathscr{L}_{gf}[\bar g_{\alpha\beta},h_{\mu\nu}]-h_{\lambda\sigma}J^{\lambda\sigma}[\bar g_{\alpha\beta}]\,,$$
$$g_{\mu\nu}=\bar g_{\mu\nu}+h_{\mu\nu}\,. \qquad (6.54)$$

Since we now wish to study the effective action at general values of $\bar g_{\alpha\beta}$, where it is not stationary, it is essential to retain the source term $J^{\lambda\sigma}[\bar g_{\alpha\beta}]$ in $\tilde{\mathscr{L}}$. The problem of extracting expressions for $G_{\rm ind}^{-1}$ and $\Lambda_{\rm ind}$ from Eq. (6.54) has the same formal structure as that set out in Eqs. (5.1) and (5.2) and solved in Sec. V.A. Hence the desired formulas are obtained by making the following substitutions in Eqs. (5.8), (5.14), (5.18), and (5.19),

$$\int d\{\phi\}\to\int d[\;]\equiv\int d[h_{\mu\nu}]d\{\phi\}\,,$$
$$g_{\mu\nu}\to\bar g_{\mu\nu}\,,$$
$$S[\{\phi\},\eta_{\mu\nu}]\to\tilde S\equiv\int d^4x\,\tilde{\mathscr{L}}[\{\phi\},\eta_{\alpha\beta},h_{\mu\nu}]\,,$$
$$\mathscr{L}\to\tilde{\mathscr{L}}[\{\phi\},\bar g_{\alpha\beta},h_{\mu\nu}]\,. \qquad (6.55)$$

In order to indicate explicitly the appearance of the source current in the following formulas, it is useful to introduce the notation

$$\tilde{\mathscr{L}}[\{\phi\},\bar g_{\alpha\beta},h_{\mu\nu}]=\hat{\mathscr{L}}[\{\phi\},\bar g_{\alpha\beta},h_{\mu\nu}]-h_{\lambda\sigma}J^{\lambda\sigma}\,,$$
$$\hat{\mathscr{L}}[\{\phi\},\bar g_{\alpha\beta},h_{\mu\nu}]=\sqrt{-g}\,[\mathscr{L}_{\rm matter}+\mathscr{L}_{\rm grav}+\mathscr{L}_{\rm ghost}]+\sqrt{-\bar g}\,\mathscr{L}_{gf}\,,$$
$$J^{\lambda\sigma}[\bar g_{\alpha\beta};y]\Big|_{\bar g_{\alpha\beta}=\eta_{\alpha\beta}}\equiv\mathscr{J}^{\lambda\sigma}\,,$$
$$\frac{\delta}{\delta\bar g_{\mu\nu}(x)}J^{\lambda\sigma}[\bar g_{\alpha\beta};y]\Big|_{\bar g_{\alpha\beta}=\eta_{\alpha\beta}}\equiv\mathscr{J}^{\lambda\sigma|\mu\nu}(y,x)\,, \qquad (6.56)$$

in terms of which

$$\tilde S=\int d^4x[\hat{\mathscr{L}}(\{\phi\},\eta_{\alpha\beta},h_{\mu\nu})-h_{\lambda\sigma}\mathscr{J}^{\lambda\sigma}]\,. \qquad (6.57)$$

[42]In Eq. (6.54) we have not required the total space-time volume to have a fixed value. Modifications required by a volume constraint and by the presence of boundaries are discussed by Hawking (1979). A volume constraint can be included by adding a Lagrange multiplier term $\kappa_0\sqrt{-g}$ to $\tilde{\mathscr{L}}$, which plays the role of a bare cosmological term and is discussed in more detail in Sec. VI.C below. Space-time boundaries require the addition to the Einstein-Hilbert action of a surface integral over the boundaries. Hasslacher and Mottola (1981) show that when the quantum fluctuations $h_{\mu\nu}$ in Eq. (6.54) are constrained to have zero normal derivative on a boundary, so that the boundary does not fluctuate, a surface term of the expected form automatically appears in the induced gravitational effective action.

The tensors $\mathscr{J}^{\lambda\sigma}$ and $\mathscr{J}^{\lambda\sigma|\mu\nu}$ are implicitly defined by the relations

$$0=\langle h_{\theta\tau}(0)\rangle\Big|_{\bar g_{\alpha\beta}=\eta_{\alpha\beta}} \qquad (6.58a)$$
$$\propto\int d[\;]e^{i\tilde S}h_{\theta\tau}(0)\,,$$
$$0=\frac{\delta}{\delta\bar g_{\mu\nu}(x)}\langle h_{\theta\tau}(0)\rangle\Big|_{\bar g_{\alpha\beta}=\eta_{\alpha\beta}}$$
$$\propto\int d[\;]e^{i\tilde S}V^{\mu\nu}(x)h_{\theta\tau}(0)\,, \qquad (6.58b)$$
$$V^{\mu\nu}(x)=2\frac{\delta}{\delta\bar g_{\mu\nu}(x)}\int d^4x\,\tilde{\mathscr{L}}[\{\phi\},\bar g_{\alpha\beta},h_{\mu\nu}]\Big|_{\bar g_{\alpha\beta}=\eta_{\alpha\beta}}$$
$$=V_1^{\mu\nu}(x)+V_2^{\mu\nu}(x)\,,$$
$$V_1^{\mu\nu}(x)=2\left[\left[\frac{\partial}{\partial\bar g_{\mu\nu}}-\frac{\partial}{\partial x^\lambda}\frac{\partial}{\partial(\partial_\lambda\bar g_{\mu\nu})}+\frac{\partial}{\partial x^\lambda}\frac{\partial}{\partial x^\sigma}\frac{\partial}{\partial(\partial_\lambda\partial_\sigma\bar g_{\mu\nu})}\right]\times\hat{\mathscr{L}}[\{\phi\},\bar g_{\alpha\beta},h_{\mu\nu};x]\right]\Big|_{\bar g_{\alpha\beta}=\eta_{\alpha\beta}}\,,$$
$$V_2^{\mu\nu}(x)=-2\int d^4z\,h_{\lambda\sigma}(z)\mathscr{J}^{\lambda\sigma|\mu\nu}(z,x)\,.$$

After simplifications using Eq. (6.58), the formulas for $\Lambda_{\rm ind}/G_{\rm ind}$ and $G_{\rm ind}^{-1}$ take the form

$$-\frac{1}{2\pi}\frac{\Lambda_{\rm ind}}{G_{\rm ind}}=\langle V_1(0)\rangle_0\,,$$
$$\frac{1}{16\pi G_{\rm ind}}=\langle U(0)\rangle_0-\frac{i}{96}\int d^4x\,x^2\left[\langle\mathscr{T}(\tilde V_1(x)\tilde V_1(0))\rangle_0-\langle\mathscr{T}(V_2(x)V_2(0))\rangle_0\right]\,,$$
$$\langle A(0)\rangle_0=\frac{\int d[\;]e^{i\tilde S}A(0)}{\int d[\;]e^{i\tilde S}}\,,$$
$$\langle\mathscr{T}(A(x)B(0))\rangle_0=\frac{\int d[\;]e^{i\tilde S}A(x)B(0)}{\int d[\;]e^{i\tilde S}}\,,$$
$$\tilde V_1(x)=V_1(x)-\langle V_1(x)\rangle_0\,,$$
$$V_1(x)=\eta_{\mu\nu}V_1^{\mu\nu}(x)\,,$$
$$V_2(x)=\eta_{\mu\nu}V_2^{\mu\nu}(x)\,,\quad\langle V_2(x)\rangle_0=0\,,$$
$$U(x)=\text{Eq. (5.14) with } g_{\mu\nu}\to\bar g_{\mu\nu},\ \mathscr{L}\to\hat{\mathscr{L}}\,. \qquad (6.59)$$

A second useful formula for $\Lambda_{\rm ind}/G_{\rm ind}$ can be obtained by using Eq. (6.35) to calculate the conformal variation of Γ, giving

$$-\frac{1}{2\pi}\frac{\Lambda_{\rm ind}}{G_{\rm ind}}=2\eta_{\lambda\sigma}\left[\frac{\delta}{\delta g^R_{\lambda\sigma}}W[\mathscr{J}^{\gamma\delta},\eta_{\alpha\beta}]+\mathscr{J}^{\lambda\sigma}\right]\,. \qquad (6.60)$$

Since

$$\frac{\delta}{\delta g^R_{\lambda\sigma}} W[\mathscr{J}^{\gamma\delta},\eta_{\alpha\beta}]=0 \text{ when } \mathscr{J}^{\lambda\sigma}=0 , \quad (6.61)$$

we learn from Eq. (6.60) that the condition for the cosmological constant $\Lambda_{\rm ind}$ to vanish is the vanishing of $\mathscr{J}^{\lambda\sigma}=J^{\lambda\sigma}[\eta_{\alpha\beta}]$. This is of course expected, since when $\Lambda_{\rm ind}$ vanishes, the induced gravitational action $\Gamma_{\rm inv}[\bar{g}_{\alpha\beta}]$ is stationary at a Minkowski background metric $\eta_{\alpha\beta}$. When $\eta_{\alpha\beta}$ is the stable ground state, the second-order fluctuation operator around $\eta_{\alpha\beta}$ has no negative eigenvalues, and the functional integral formula of Eq. (6.59) is then guaranteed to give a real value for $G_{\rm ind}^{-1}$.

Unlike the situation found in Sec. V.A, where $\langle U(0)\rangle_0$ vanished, the term $\langle U(0)\rangle_0$ in Eq. (6.59) contains nonvanishing contributions quadratic in the fluctuation metric, such as $\langle(\partial h_{\mu\nu}/\partial x^\lambda)^2\rangle_0$. Hence this term in the formula for $G_{\rm ind}^{-1}$ is qualitatively similar to the relation

$$G_{\rm ind}^{-1}\sim\langle R[h_{\mu\nu}]\rangle_0 \quad (6.62)$$

proposed by Mansouri (1979, 1981), in papers suggesting that Einstein gravitation is generated by dynamical scale-invariance breaking in conformally invariant, order-R^2 gravitational models. However, Eq. (6.62) [which omits the nonlocal V^2 terms of Eq. (6.59)] is not a quantitatively correct expression for $G_{\rm ind}^{-1}$.

C. Conditions for finiteness of $G_{\rm ind}^{-1}$ and $\Lambda_{\rm ind}$ and for the vanishing of $\Lambda_{\rm ind}$

Let us turn now to the issue of whether the formulas for $G_{\rm ind}^{-1}$ and $\Lambda_{\rm ind}$ given in Eq. (6.59) are finite. By construction, the fundamental Lagrangian density $\mathscr{L}_{\rm matter}+\mathscr{L}_{\rm grav}$ contains a complete basis of dimension-four operators formed from the fields which are present, together with a number (say, N) of dimensionless unrenormalized couplings. The dimensional algorithm of Sec. II.C then guarantees that $G_{\rm ind}^{-1}$ and $\Lambda_{\rm ind}$ will be calculable in terms of the corresponding N renormalized couplings.[43] If scale invariance remains unbroken, we get $G_{\rm ind}^{-1}=0=\Lambda_{\rm ind}$. If dynamical breaking of scale invariance occurs, we expect one of the N dimensionless couplings to be replaced by a scale mass $\mathscr{M}$, as discussed in Sec. IV.B, and the theory will then yield nonvanishing predictions for $G_{\rm ind}^{-1}$ and $\Lambda_{\rm ind}$ in terms of $\mathscr{M}$ and the remaining $N-1$ dimensionless couplings. The ideal case, of course, would be that in which the fundamental action contains only one dimensionless coupling, so that after dynamical symmetry breaking and dimensional transmutation, no free dimensionless coupling constants remain.

Let us consider next the conditions under which the induced cosmological constant $\Lambda_{\rm ind}$ vanishes, assuming initially that $\mathscr{L}_{\rm matter}+\mathscr{L}_{\rm grav}$ has a unifying symmetry which leaves only a single dimensionless coupling constant, and which requires the vanishing of the bare cosmological constant. Then after dimensional transmutation, $\Lambda_{\rm ind}$ will be calculable in terms of the scale mass $\mathscr{M}$ (which is expected[44] to be in the range $10^{14}-10^{19}$ GeV), but in general $\Lambda_{\rm ind}/\mathscr{M}^2$ will be a number of order unity, in violent contradiction to Eq. (2.23). The only way to save the situation is for the underlying theory to have a "hidden" symmetry which guarantees the vanishing of $\Lambda_{\rm ind}$, as discussed recently by Pagels (1982). The difficulty with implementing this mechanism is that in order for the hidden symmetry to restrict $\Lambda_{\rm ind}$ it must be an unbroken symmetry, and no natural candidate for such a symmetry is known.[45]

An interesting alternative possibility is suggested by recent work in which Ovrut and Wess (1982) use a cosmological constant as a mechanism for breaking supersymmetry. Suppose that the unifying symmetry allows only a single dimensionless coupling constant but does not restrict the value of the bare cosmological constant, so that we can freely add a term $\int d^4x\sqrt{-g}\,\kappa_0$ to the fundamental action. Because κ_0 has dimension four, any polynomial formed from κ_0 and the fields will have dimension greater than or equal to four, and so the added term does not require the introduction of any dimensional renormalization constants with dimension smaller than four. After dynamical symmetry breaking, the theory now has two dimensional parameters, κ_0 and $\mathscr{M}$, or equivalently, $\Lambda_{\rm ind}$ and $\mathscr{M}$. We can then impose as a renormalization condition the requirement that in the absence of real (as opposed to virtual) matter, the Minkowski metric $\eta_{\mu\nu}$ be the stable background metric, which will require[46]

$$\Lambda_{\rm ind}=0=\mathscr{J}^{\lambda\sigma} . \quad (6.63)$$

This leaves only one dimensional parameter $\mathscr{M}$, in terms of which all particle masses and Newton's constant are calculable.

In order to implement this alternative mechanism, we must have justifications both for assuming that the bare cosmological constant κ_0 is nonzero, and for imposing

[43]Note, however, that there *is* a dimension-two internal-symmetry scalar operator $\mathscr{O}_2=R[h_{\mu\nu}]$ which transforms as a Lorentz scalar with respect to $\eta_{\mu\nu}$, the limiting value of the background metric $\bar{g}_{\mu\nu}$ appearing in Eq. (6.59). As a consequence, the U and V^2 terms in Eq. (6.59) in general will each be divergent, with the infinities cancelling only in their sum.

[44]This range extends from the so-called "grand unification mass" of particle physics [see Weinberg (1980b) for a review] to the Planck mass.

[45]An unbroken "hidden" symmetry is also required if the unifying symmetry specifies a definite nonzero value for the bare cosmological constant. For a recent survey of quantum gravity with a cosmological constant, see Christensen and Duff (1980).

[46]The fact that stability of the Minkowski metric requires the vanishing of $\Lambda_{\rm ind}$ is noted and used as a renormalization condition in Brout *et al.* (1980).

the renormalization condition that the induced (or renormalized) cosmological constant Λ_{ind} vanish. A possible rationale for assuming that κ_0 is nonzero has been given by Hawking (1979), who points out that in order to construct a partition function Z for a fixed total space-time volume one must include a Lagrange multiplier for this volume, and this is formally equivalent to including a bare cosmological term in the fundamental action.[47] A possible rationale for the renormalization condition $\Lambda_{\text{ind}}=0$ could be provided by the observation that in a two-parameter theory, the ratio $\Lambda_{\text{ind}}/\mathscr{M}^2$ is not constant in nonequilibrium situations. If one could show that nonequilibrium processes in the early universe, such as back-reaction effects from particle production, resulted in the decay of Λ_{ind} towards an equilibrium value of zero,[48] then use of the renormalization condition $\Lambda_{\text{ind}}=0$ in the equilibrium analysis of Sec. VI.B would be justified.

D. Structure and properties of the fundamental gravitational action

In this final section I will comment very briefly on the structure and on some of the properties of the fundamental gravitational action. I have assumed in the preceding discussion a general order-R^2 gravitational action density of the form

$$\mathscr{L}_{\text{grav}}=A_0\mathscr{G}+B_0\mathscr{H}+C_0\mathscr{K}\ , \tag{6.64}$$

with $\mathscr{G},\mathscr{H},\mathscr{K}$ defined in Eq. (2.35). The study of gravitational actions of this type was initiated by Utiyama and DeWitt (1962), and a proof that they lead to a renormalizable perturbation theory has been given by Stelle (1977).[49] When dimensional regularization is employed, all three terms in Eq. (6.64) are in general needed, as shown in detail for the case of a scalar field by Brown (1977) and by Brown and Collins (1980). Even though the action formed from $\mathscr{G}$ is a topological invariant in four dimensions, it is not a topological invariant in 2ω dimensions, and so makes a nontrivial contribution when multiplied by the power series in $(\omega-2)^{-1}$ contained in the coefficient A_0. The only circumstance under which a term $\mathscr{T}(=\mathscr{G},\ \mathscr{H},\text{ or }\mathscr{K})$ can be omitted from Eq. (6.64) is when the theory with $\mathscr{T}$ deleted has special symmetries, which guarantee that no divergences with the structure of $\mathscr{T}$ are encountered. Thus, for example, a renormalizable theory of matter and gravitation could be formulated without including any order-R^2 terms in the fundamental action, only if $\mathscr{L}_{\text{matter}}$ itself had enough symmetry so that no divergences with the structure of $\mathscr{G}$, $\mathscr{H}$, or $\mathscr{K}$ were encountered. Whether such matter actions can be constructed is not presently known. A more realistic possibility for omitting terms from Eq. (6.64) is afforded by the case of classically conformally invariant theories, in which there are hints[50] that the induced $\mathscr{K}$ term may always have a finite coefficient, permitting one to take $C_0=0$ in Eq. (6.64). It is possible to

[47]The value of κ_0 would then presumably be a parameter characterizing the initial quantum fluctuation which led to the birth of the universe.

[48]For a review of gravitational particle production, see Parker (1977), while for an effective action formalism for particle production in the early universe, see Hartle (1977). In an earlier article, Parker (1969), p. 1066, postulated that "the reaction of the particle creation (or annihilation) back on the gravitational field will modify the expansion in such a way as to reduce the creation rate." Since $\Lambda_{\text{ind}}>0$ corresponds to positive vacuum energy, a naive extension of this postulate suggests that a state of the early universe with $\Lambda_{\text{ind}}>0$ will decay by gravitational particle production to an equilibrium with $\Lambda_{\text{ind}}=0$, at which point particle production ceases. Variants of this idea have appeared in models for the creation of the universe through a quantum tunneling event given by Brout *et al.* (1978), Brout *et al.* (1979), Guth (1981), Akatz and Pagels (1982), and Gott (1982). The models of Brout *et al.* and Gott postulate a transition from a particle producing de Sitter phase with $\Lambda_{\text{ind}}=0$, $T^{\mu\nu}_{\text{matter}}=-\kappa g^{\mu\nu}$, $\kappa\sim l^{-4}_{\text{Planck}}$ to a standard equation of state with $P=\frac{1}{3}\rho$ (P=pressure, ρ=density) as a result of back-reaction effects of particle production. When the term $-\kappa g^{\mu\nu}$ is transposed to the $G^{\mu\nu}+\Lambda_{\text{ind}}g^{\mu\nu}$ side of the Einstein equations, κ is equivalent to an initially nonvanishing $\Lambda_{\text{ind}}/G_{\text{ind}}$.

Attempts to find an instability associated with $\Lambda_{\text{ind}}\neq 0$, within the framework of the semiclassical approximation for the coupled matter-metric system [cf. Eq. (6.51) of the text] have not been successful. Abbott and Deser (1982) have shown that the de Sitter solutions obtained when the Einstein equations are solved with $\Lambda_{\text{ind}}\neq 0, T^{\mu\nu}=0$ are classically stable against small perturbations. Particle production calculations in de Sitter spaces using the semiclassical formalism have not yielded an unambiguous answer; see Parker (1977), p. 136, and Gibbons (1979), p. 666, for a discussion and references. Hence a dynamical argument to explain why $\Lambda_{\text{ind}}\approx 0$ would have to involve nonequilibrium phenomena and/or higher-loop quantum effects which are ignored in the semiclassical approximation.

[49]For further references, see the review of Weinberg (1979).

[50]There are two pieces of evidence that a bare $\mathscr{K}$ term may not be needed in conformally invariant theories, both coming from the study of conformally invariant matter theories on an unquantized background manifold. The first is that apart from a total divergence, the conformal trace anomaly has only $\mathscr{G}$ and $\mathscr{H}$ terms, which implies that the one-loop Lagrangian counterterm contains no divergences proportional to $\mathscr{K}$. [For a succinct discussion and references, see Tsao (1977).] The second is a general formula which Zee (1982b) has recently derived for the coefficient C_{ind} of the induced $\mathscr{K}$ term,

$$C_{\text{ind}}=-\frac{1}{13\,824}\int_E d^4x(x^2)^2\Psi(x^2)\ ,$$

in the notation of Eq. (5.44). Since in an asymptotically free gauge theory one has $\Psi\sim(x^2)^{-4}(\log x^2)^{-2}$ for large x^2 [cf. Eq. (5.50)], the integral for C_{ind} is just barely convergent. Zee's formula also shows that C_{ind} is negative definite, and so the theory is free of tachyons; in this connection see also Horowitz (1981) and Yamagishi (1982). Since the gravitational theory of Eq. (6.66) is asymptotically free, it seems a reasonable conjecture that Zee's results will generalize to the case in which the metric is also quantized.

construct renormalizable order-R^2 gravitational theories of greater complexity than Eq. (6.64) by adding new field degrees of freedom in a number of ways (for example, by including torsion[51] or superfields[52]). A prime consideration in searching for the correct gravitational action will almost certainly be that it should unify in a natural way with the fundamental matter action $\mathscr{L}_{\text{matter}}$, when that is finally known; this may involve the introduction of "pregeometric" fundamental variables[39,53] which are not directly classifiable as "matter" or "metric."

The momentum space graviton propagator calculated from the fundamental action density of Eq. (6.64) contains a term proportional to

$$\frac{1}{(k^2)^2}=\lim_{m^2\to 0}\frac{1}{m^2}\left[\frac{1}{k^2}-\frac{1}{k^2+m^2}\right]. \tag{6.65}$$

Since the second pole-term in Eq. (6.65) has an unphysical, negative residue, order-R^2 theories do not satisfy unitarity (with positive probabilities) at the tree level. However, unitarity is a statement about the asymptotic scattering states of a field theory and their S-matrix, and hence unlike renormalizability, is a dynamical, rather than a kinematic statement. Thus if radiative corrections play an important role in the dynamics (and they certainly do in theories with dynamical scale-invariance breaking), violations of tree-level unitarity do not necessarily imply violations of unitarity in the full theory. This point was first made a decade ago by Lee and Wick (1969, 1970), who showed that if fields which have negative-residue "ghost" propagators at the tree level become unstable as a result of radiative corrections, then the S matrix for the asymptotic scattering states can obey unitarity with positive probabilities. The relevance of the Lee-Wick mechanism for quantum gravity was first pointed out by Tomboulis (1977) and has since been discussed by a number of authors.[54] As a concrete example [see Hasslacher and Mottola (1981)], let us consider a conformally invariant order-R^2 theory with the fundamental action

$$\mathscr{L}_{\text{grav}}=A_0\mathscr{G}+B_0\mathscr{H},\quad B_0\equiv-\frac{1}{4\xi^2},$$
$$\mathscr{G}=\text{Gauss-Bonnet density [Eq. (2.35)]},$$
$$\mathscr{H}=C_{\mu\nu\lambda\sigma}C^{\mu\nu\lambda\sigma}, \tag{6.66}$$

with the sign of B_0 chosen to guarantee that the Euclidean continuation of the partition function is represented by a convergent functional integral. (The $\mathscr{G}$ term in the action plays no role in the following discussion and in general does not affect the field equations.) Taking into account the fact that radiative corrections induce an effective Newton's constant, and assuming that G_{ind} has the correct positive sign, a simple calculation shows that the spin-2 part of the full graviton propagator has the form

$$\frac{P^{(2)}_{\mu\nu\alpha\beta}}{k^2[\xi(k^2)^{-2}k^2+m^2(k^2)]}=\frac{P^{(2)}_{\mu\nu\alpha\beta}}{m^2(k^2)}\left[\frac{1}{k^2}-\frac{1}{k^2+\xi(k^2)^2m^2(k^2)}\right]. \tag{6.67}$$

Here $P^{(2)}_{\mu\nu\alpha\beta}$ is a spin-2 projection matrix, $m^2(k^2)$ is the amplitude [analogous to $(d/dk^2)\chi(k^2)$ of Eq. (5.30)] which gives G^{-1}_{ind} in the zero-momentum limit,

$$m^2(0)=\frac{1}{16\pi G_{\text{ind}}}, \tag{6.68}$$

and $\xi(k^2)$ is the (one-loop) running coupling constant for the action of Eq. (6.66),

$$\xi(k^2)^2=\frac{\xi(\mu^2)^2}{1+\frac{1}{2}b\xi(\mu^2)^2\log(k^2/\mu^2)}. \tag{6.69}$$

In the timelike region, where $k^2<0$, both $\xi(k^2)^2$ and $m(k^2)$ have imaginary parts, and consequently the propagator of Eq. (6.67) has two complex conjugate unstable ghost poles rather than a single stable ghost pole. Thus it appears that the Lee-Wick mechanism is applicable to order-R^2 gravitational theories; more detailed checks on this are now needed.

A further property of order-R^2 gravitational theories, which is illustrated by Eqs. (6.67) and (6.69), is that they are asymptotically free. This follows from work of Julve and Tonin (1978), as corrected and extended by Fradkin and Tseytlin (1981) [see also Tomboulis (1980) and Christensen (1982)], showing that $b>0$ in Eq. (6.69) and in the analogous equation for the running coupling constant associated with the $\mathscr{H}$ term in Eq. (6.64). The scale mass $\mathscr{M}$ which characterizes the strong coupling region for the fundamental theory is presumably the Planck mass m_{Planck}. At energies much higher than the Planck mass, the theory becomes weakly coupled, and so no singularities are expected.[55] At energies much lower than the Planck mass, the induced gravitational term dominates,

$$\xi(k^2)^{-2}k^2+m^2(k^2)\underset{k^2\to 0}{\to} m^2(0)=\frac{1}{16\pi G_{\text{ind}}}, \tag{6.70}$$

reflecting the presence of an extra power of k^2 multiplying $\xi(k^2)^{-2}$ in Eqs. (6.67) and (6.70), and giving gravitation the form seen in observational astronomy.

[51]Models with torsion have been discussed by Neville (1980) and by Sezgin and van Nieuwenhuizen (1980), who give further references.

[52]For a discussion of conformal supergravity see Kaku, Townsend, and van Nieuwenhuizen (1978).

[53]For attempts at pregeometric theories of gravitation, see Amati and Veneziano (1981), Terazawa and Akama (1980a, 1980b) and Terazawa (1981a, 1981b).

[54]See Adler (1980b), Hasslacher and Mottola (1981), Tomboulis (1980), and also Salam and Strathdee (1978).

[55]For discussions of singularity avoidance in order-R^2 theories, see Hu (1979), Tomboulis (1980), and Hasslacher and Mottola (1981).

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APPENDIX A: DETAILS FOR THE BASIC THEOREMS

1. Arguments excluding dimension-two Lorentz-scalar operators

a. Pure non-Abelian gauge theories in axial and covariant gauges

The necessity for gauge fixing and ghosts requires, in the case of nonAbelian gauge theories, that we give a somewhat more careful argument for the absence of dimension-two Lorentz-scalar and internal symmetry-invariant operators $\mathscr{O}_2$ than would be needed in the Abelian case. Let me give first the argument working in axial gauge

$$A_z^i=0 \ . \tag{A1}$$

Since axial gauge is a canonical gauge (Hanson *et al.*, 1976), no ghost fields are present. Hence invariance under the subgroup of the Lorentz group which leaves the z axis invariant and invariance under global internal symmetry transformations restrict a candidate for $\mathscr{O}_2$ to have the form

$$\mathscr{O}_2'=A_x^iA^{ix}+A_y^iA^{iy}+A_t^iA^{it} \ . \tag{A2}$$

Consider now the local gauge transformation

$$\delta A_\mu^i=\partial_\mu\Phi^i-g_0f^{ijk}A_\mu^j\Phi^k \ , \tag{A3}$$

with $\Phi^k=\Phi^k(x,y,t)$ independent of z, so that

$$\delta A_z^i=\partial_z\Phi^i-g_0f^{ijk}A_z^j\Phi^k=0 \ . \tag{A4}$$

Under the transformation of Eq. (A3) we have

$$\delta\mathscr{O}_2'=A_x^i\partial_x\Phi^i+A_y^i\partial_y\Phi^i+A_t^i\partial_t\Phi^i\neq 0 \ , \tag{A5}$$

and so $\mathscr{O}_2'$ is not invariant under the subclass of local gauge transformations which preserves the $A_z^i=0$ gauge condition. Thus Eq. (A2) is not a physically observable dimension-two Lorentz-scalar operator.

I give next a covariant gauge argument, following the notation of Kugo and Ojima (1979), which uses an inner product ($\cdot$) and an outer product ($\times$) to denote contraction of internal symmetry indices with δ_i^j and f^{ijk}, respectively. In covariant gauge, we have

$$\mathscr{L}=-\tfrac{1}{4}F_{\mu\nu}\cdot F^{\mu\nu}+\mathscr{L}_{\mathrm{GF}}+\mathscr{L}_{\mathrm{FP}} \ , \tag{A6}$$

with the gauge-fixing (GF) and Fadde'ev-Popov (FP) Lagrangian terms given by

$$\begin{aligned}\mathscr{L}_{\mathrm{GF}}&=-\partial^\mu B\cdot A_\mu+\frac{\alpha_0}{2}B\cdot B\\&=-\frac{1}{2\alpha_0}(\partial^\mu A_\mu)^2+\frac{\alpha_0}{2}\left[B+\frac{1}{\alpha_0}\partial^\mu A_\mu\right]^2-\partial_\mu(B\cdot A^\mu) \ ,\\ \mathscr{L}_{\mathrm{FP}}&=-i\partial^\mu\bar{c}\cdot D_\mu c \ ,\\ D_\mu&=\partial_\mu-g_0A_\mu\times, \quad c^\dagger=c, \quad \bar{c}^\dagger=\bar{c} \ .\end{aligned} \tag{A7}$$

In Eqs. (A6) and (A7), B is an auxilliary scalar field, α_0 is a gauge parameter, and c is the Fadde'ev-Popov ghost field. The Lagrangian density of Eq. (A6) is invariant under the Becchi-Rouet-Stora (BRS, 1976) transformation

$$\begin{aligned}\delta A_\mu&=\lambda D_\mu c \ ,\\ \delta c&=\lambda g_0(c\times c)/2 \ ,\\ \delta\bar{c}&=i\lambda B \ ,\\ \delta B&=0 \ ,\end{aligned} \tag{A8}$$

with λ an x-independent parameter which anticommutes with c and $\bar{c}$, and all physically observable operators must be similarly invariant. In covariant gauge, Lorentz invariance and invariance under global internal symmetry transformations restrict a candidate for $\mathscr{O}_2$ to have the form (for any constant β_0)

$$\mathscr{O}_2'=A_\mu\cdot A^\mu+\beta_0\bar{c}\cdot c \ . \tag{A9}$$

Under the transformation of Eq. (A8), the change in $\mathscr{O}_2'$ is

$$\delta\mathscr{O}_2'=2A^\mu\cdot\lambda\partial_\mu c+\beta_0[i\lambda Bc+\tfrac{1}{2}\lambda g_0\bar{c}\cdot(c\times c)]\neq 0 \ , \tag{A10}$$

and so $\mathscr{O}_2'$ is not a BRS invariant. Hence Eq. (A9) does not give a physically observable dimension-two Lorentz-scalar operator.

We have thus concluded, by working in either axial or covariant gauge, that in a pure non-Abelian gauge theory there are no Lorentz and internal symmetry-invariant operators $\mathscr{O}_2$, and hence no action density terms $\mathscr{O}_2R$ in curved space-time.

b. Massless supersymmetric theories with spin-0 fields

An extension of the above argument excludes Lorentz- and internal symmetry-invariant dimension-two operators $\mathscr{O}_2$ in massless supersymmetric theories with spin-0 fields. Let φ be a massless spin-0 field which has a Majorana spinor supersymmetry partner ψ. Under supersymmetry transformations, φ transforms[5] as

$$\delta\varphi = i(\bar{\psi}\alpha - \bar{\alpha}\psi) , \tag{A11}$$

with α an x-independent parameter which anticommutes with ψ. Lorentz invariance allows a candidate for $\mathscr{O}_2$ of the form

$$\mathscr{O}'_2 = \varphi^2 , \tag{A12}$$

but under supersymmetry transformations the change in $\mathscr{O}'_2$ is

$$\delta\mathscr{O}'_2 = 2\varphi\delta\varphi \neq 0 . \tag{A13}$$

Hence $\mathscr{O}'_2$ is not an internal symmetry invariant, and an action density term $\mathscr{O}'_2 R$ is excluded. (A dimension-two supersymmetry invariant is readily constructed by adding to $\mathscr{O}'_2$ a fermionic piece proportional to $\bar{\psi}\psi/m_0$, but this requires the introduction of a mass parameter m_0.)

2. Extension to massive regulator schemes

When massive regulators are employed, we learn from the enumeration of Sec. II.C that there are Lagrangian density terms of the form

$$T' = M^4, \quad U' = M^2 R , \tag{A14}$$

with M^4 and M^2 schematically indicating polynomials which are, respectively, quartic and quadratic in the regulator masses. The term T' contributes to the induced cosmological constant $\Lambda_{\rm ind}/G_{\rm ind}$ through the operator $T(0)$ of Eq. (5.19a), while the term U' contributes to the induced gravitational constant $G_{\rm ind}^{-1}$ through the operator $U(0)$ of Eq. (5.19b). The coefficients of T' and U' (which in general depend logarithmically on the regulator masses) are determined by the requirement that $\Lambda_{\rm ind}/G_{\rm ind}$ and $G_{\rm ind}^{-1}$ remain finite as the regulator masses tend to infinity. Consider now the differences

$$\delta(\Lambda_{\rm ind}/G_{\rm ind}) = (\Lambda_{\rm ind}/G_{\rm ind})_{\substack{\rm massive\\ \rm regulator}} - (\Lambda_{\rm ind}/G_{\rm ind})_{\substack{\rm dimensional\\ \rm regularization}} ,$$

$$\delta(G_{\rm ind}^{-1}) = (G_{\rm ind}^{-1})_{\substack{\rm massive\\ \rm regulator}} - (G_{\rm ind}^{-1})_{\substack{\rm dimensional\\ \rm regularization}} , \tag{A15}$$

between the finite induced constants calculated using massive regulators, and the finite values calculated using dimensional regularization. According to the dimensional algorithm, differences such as these between the finite values of connected, one-particle irreducible matrix elements evaluated in two different regularization schemes must be representable as the corresponding matrix elements of a Lagrangian density polynomial $\delta\mathscr{L}$ formed from the bare masses, the bare fields, and $\partial/\partial x^\mu$. The polynomial $\delta\mathscr{L}$ cannot contain the terms T' and U' of Eq. (A14), since any nonzero multiple of these bases is necessarily at least quadratically divergent as the regulator masses tend to infinity. The polynomial $\delta\mathscr{L}$ also cannot contain any field-dependent dimension-four Lagrangian terms which survive in the flat space-time limit, since these would give rise to differences in the flat space-time S matrices calculated in the two regularization schemes. When there are no bare masses and no scalar fields apart from members of massless supermultiplets, no other dimension-four operator is present in curved space-time, and $\delta\mathscr{L}$ then vanishes. We conclude that

$$\delta(\Lambda_{\rm ind}/G_{\rm ind}) = \delta(G_{\rm ind}^{-1}) = 0 ; \tag{A16}$$

that is, under the necessary conditions discussed in Sec. II.D, the renormalized induced gravitational action calculated using massive regulators is unique, and agrees with that calculated by using the method of dimensional regularization.

APPENDIX B: DETAILS FOR THE CALCULATION OF $G_{\rm ind}^{-1}$ IN SU(n) GAUGE THEORY

1. Transformation to one-loop exact renormalization group

In an SU(n) gauge theory, the behavior of physical parameters under changes in the renormalization subtraction point μ is governed [through Eq. (4.18)] by the function $\beta(g)$, which has the power-series expansion

$$\beta(g) = -(\tfrac{1}{2}b_0 g^3 + b_1 g^5 + b_2 g^7 + \cdots) . \tag{B1}$$

Only the first two coefficients $b_{0,1}$ are gauge invariant, and only these coefficients are invariant under coupling constant transformations $g \to g'$ of the form

$$g = g(g') = g' + \sum_{n=1}^{\infty} A_n (g')^{2n+1} , \tag{B2}$$

which are analytic in a neighborhood of $g=0$. 't Hooft (1979) pointed out that the noninvariance of $b_2, \ldots$ under the transformation of Eq. (B2) could be exploited to define a transformation which, in a formal perturbative sense, makes the transformed coefficients $b_2, \ldots$ vanish. Global conditions for the existence of a nonsingular 't Hooft transform were studied by Khuri and McBryan (1979); if singular transformations are not excluded [see Frishman, Horsely, and Wolff (1981) for arguments suggesting the physical relevance of singular coupling constant transformations], then a transformation to a two-loop exact renormalization group can always be made, giving

$$\beta(g) = -(\tfrac{1}{2}b_0 g^3 + b_1 g^5) . \tag{B3}$$

Following Adler (1981), let us now make a further, nonanalytic transformation to a new "reduced" running coupling constant g_R for which a one-loop renormalization group structure is exact. (In the applications of the one-loop exact running coupling constant in Sec. V.D of the text, the subscript R is omitted.) Writing $\bar{\alpha}_R \equiv g_R^2$, $\bar{\alpha} \equiv g^2$, the transformation is simply

$$\frac{1}{\bar{\alpha}_R} = -\frac{1}{2}b_0 \int_{\bar{\alpha}}^{\infty} \frac{d\bar{\alpha}'}{\bar{\beta}(\bar{\alpha}')} ,$$

$$\bar{\beta}(\bar{\alpha}) = g\beta = -(\tfrac{1}{2}b_0\bar{\alpha}^2 + b_1\bar{\alpha}^3) , \tag{B4}$$

which is easily seen to give a nonsingular mapping from the half-line $0<\overline{\alpha}<\infty$ to the half-line $0<\overline{\alpha}_R<\infty$. The renormalization group structure in the new variable $\overline{\alpha}_R$ is determined by $\overline{\beta}_R(\overline{\alpha}_R)$, given by

$$\overline{\beta}_R(\overline{\alpha}_R)=\overline{\beta}(\overline{\alpha})\frac{\partial\overline{\alpha}_R}{\partial\overline{\alpha}}=\overline{\beta}(\overline{\alpha})(-\overline{\alpha}_R^2)\frac{\partial(\overline{\alpha}_R^{-1})}{\partial\overline{\alpha}}=-\frac{1}{2}b_0\overline{\alpha}_R^2\ , \tag{B5}$$

and so has exactly a one-loop form.

Explicitly integrating Eq. (B4) gives for the transformation

$$\frac{1}{\overline{\alpha}_R}=\frac{1}{\overline{\alpha}}-a\left[\log\left[\frac{1}{a\overline{\alpha}}\right]+\log(1+a\overline{\alpha})\right],$$
$$a=\frac{2b_1}{b_0}\ , \tag{B6}$$

which for small $a\overline{\alpha}$ can be developed into a series expansion,

$$\frac{1}{\overline{\alpha}_R}=\frac{1}{\overline{\alpha}}-a\log\left[\frac{1}{a\overline{\alpha}}\right]+a\sum_{n=1}^{\infty}\frac{(-a\overline{\alpha})^n}{n}\ . \tag{B7}$$

Equation (B7) can be inverted to give an expansion for $\overline{\alpha}$ in terms of $\overline{\alpha}_R$ and $\log(a\overline{\alpha}_R)$,

$$\overline{\alpha}=\overline{\alpha}_R(1+\overline{\alpha}_R f)\ ,$$
$$f=\sum_{k=0}^{\infty}\overline{\alpha}_R^k f_k\ ,$$
$$f_0=a\log(a\overline{\alpha}_R)\ ,$$
$$f_1=f_0^2+af_0-a^2\ ,\ \ldots\ . \tag{B8}$$

Because f_0 contains a logarithm, the transformation is nonanalytic at $\overline{\alpha}_R=0$, which is why the coefficient b_1 can be transformed to zero. Substituting Eq. (B8) into a perturbation series which has been brought to 't Hooft's form yields a modified perturbation series in terms of the new running coupling constant g_R, for which the one-loop renormalization group is exact. The modified expansion has the form of a joint power series in $\overline{\alpha}_R$ and $\log(a\overline{\alpha}_R)$ in which, for a physical quantity with leading-order contribution at order $\overline{\alpha}_R^L$, the general term has the form $\overline{\alpha}_R^n[\log(a\overline{\alpha}_R)]^p$, with $n\geq L$ and with $p\leq n-\max(1,L)$.

2. Leading short-distance contribution to $\Psi(t)$

Asymptotic freedom implies that the leading short-distance contribution to $\Psi(t)$ is obtained by doing a lowest-order perturbation theory calculation, with the coupling constant g^2 replaced by the running coupling constant $g^2(t)$. Thus from Eqs. (5.43) and (5.49) we get

$$\Psi(t)=\frac{1}{4}\frac{1}{(-\log\mathscr{M}^2t)^2}[\langle\mathscr{T}(F^2(x)F^2(0))\rangle_{0E}-\langle F^2(x)\rangle_{0E}^2]\ , \tag{B9}$$

with F^2 a shorthand for $F^{ri}_{\lambda\sigma}F^{ri\lambda\sigma}$, and with the subscript $0E$ indicating the Euclidean vacuum expectation. In lowest-order perturbation theory, the square bracket in Eq. (B9) is given by

$$\begin{aligned}\langle\mathscr{T}(F^2(x)F^2(0))\rangle_{0E}-\langle F^2\rangle_{0E}^2&=2[\langle\mathscr{T}(F^i_{\lambda\sigma}(x)F^j_{\mu\nu}(0))\rangle_{0E}]^2\\&=2[\langle\mathscr{T}(\partial_{[\lambda,}A^i_{\sigma]}(x)\partial_{[\mu,}A^j_{\nu]}(0))\rangle_{0E}]^2\ ,\end{aligned} \tag{B10}$$

with [,] indicating antisymmetrization of indices. Substituting the Euclidean Feynman propagator

$$\langle\mathscr{T}(A^i_\sigma(x)A^j_\nu(y))\rangle_{0E}=\frac{\delta^{ij}\delta_{\sigma\nu}}{(2\pi)^2(x-y)^2}\ , \tag{B11}$$

and carrying out the differentiations and contractions, an elementary calculation gives

$$\begin{aligned}[\langle\mathscr{T}(\partial_{[\lambda,}A^i_{\sigma]}(x)\partial_{[\mu,}A^j_{\nu]}(0))\rangle_{0E}]^2&=\frac{3\times2^7}{(2\pi)^4}(n^2-1)\frac{1}{t^4}\\\Longrightarrow\Psi(t)&=\frac{3\times2^6}{(2\pi)^4}(n^2-1)\frac{1}{t^4(-\log\mathscr{M}^2t)^2}\ ,\end{aligned} \tag{B12}$$

yielding the value of C_Ψ given in Eq. (5.52) of the text.

3. Dimensional continuation evaluation of comparison integrals

I give here two evaluations of the integral of Eq. (5.55) by dimensional continuation. In the first calculation, only the power of u in the integrand is dimensionally continued, while the logarithms are kept in dimension four. In the second calculation [restricted for simplicity to the leading term in $\Theta(u)$] both the power of u and the logarithms are dimensionally continued, corresponding to use of the 2ω-dimensional vacuum expectation in Eq. (5.22). The two calculations give the same answer, as expected where a finite radiative correction is evaluated by different regularization methods. In the context of the second calculation, we can compare the analyticity properties in ω of the dimensional continuation of a finite sum of Feynman diagrams, with the analyticity properties of the infinite sum of Feynman diagrams contained in the running coupling constant factor $g^4(t)$.

In 2ω dimensions, the factor $d^{2\omega}x$ in Eq. (5.22) is proportional to $dt\,t^{\omega-1}$, and since $\widetilde{T}(x)$ has canonical dimension 2ω, the leading power behavior of the vacuum expectation $\langle\mathscr{T}(\widetilde{T}(x)\widetilde{T}(0))\rangle_0^\omega$ is $t^{-2\omega}$. Hence when the logarithmic sum $\Theta(u)/(\log u)^2$ is kept in four dimensions (and when a normalization factor of π^ω/π^2 is omitted), the continuation of the integral of Eq. (5.55) is

$$\int_0^{u_0}(du\,u^{\omega-1})u\left[u^{-2\omega}\frac{\Theta(u)}{(\log u)^2}\right]=\int_0^{u_0}du\,u^{-\omega}\frac{\Theta(u)}{(\log u)^2}\ , \tag{B13}$$

and is convergent at $u=0$ when $\mathrm{Re}\,\omega<1$. In order to put Eq. (B13) in a form where it can be analytically continued to $\omega=2$, let us first make the change of variable $u=e^{-v}$, giving

$$\int_{\log u_0^{-1}}^{\infty} dv \frac{e^{(\omega-1)v}}{v^2}\Theta(e^{-v}) , \tag{B14}$$

with the contour of integration running along the positive real axis. When $\mathrm{Re}\omega<1$ and $\mathrm{Im}\omega>0$, the integration contour can be deformed to the contour C of Fig. 5, while when $\mathrm{Re}\omega<1$ and $\mathrm{Im}\omega<0$, the contour can be deformed to a contour C^*, obtained by reflecting C in the real axis. Once the contour has been deformed to C or C^*, the integral of Eq. (B14) converges for any value of $\mathrm{Re}\omega$, and we can continue $\mathrm{Re}\omega$ to 2. Since Hermiticity of a quantum field theory requires that the regularization prescription be manifestly real (contour prescriptions can enter only through Feynman propagators), the limit as $\omega\to 2$ must be defined as the average of dimensional continuations to $\omega=2+i\epsilon$ and to $\omega=2-i\epsilon$. That is, we must average the evaluations of Eq. (B14) with $\omega=2$ on the contours C and C^*, or equivalently, take the real part of the evaluation on the contour C alone, yielding the formula given in Eq. (5.56) of the text. The inequivalence of the evaluations on C and C^* implies that the analytic continuation of Eq. (B14) to $\mathrm{Re}\omega>1$ has a branch cut running along the positive real axis from $\omega=1$ (space-time dimension two) to ∞.

To study the effect of dimensionally continuing the logarithmic terms in Eq. (5.55), we note that a momentum space factor $\log k^2$ continues into

$$\frac{(k^2)^{\omega-2}}{\omega-2}+\text{counterterm}=\frac{(k^2)^{\omega-2}-1}{\omega-2} , \tag{B15}$$

and corresponding to this, a coordinate space factor $\log(-x^2)=\log t$ continues into

$$\frac{(x^2)^{2-\omega}}{2-\omega}+\text{counterterm}=\frac{(x^2)^{2-\omega}-1}{2-\omega} . \tag{B16}$$

Hence let us consider the integral

$$I(\omega,\gamma)=\int_0^{u_0} du\, u^{-\omega}\frac{1}{\left[\dfrac{u^{\gamma}-1}{\gamma}\right]^2} , \tag{B17}$$

which when $\gamma=2-\omega$ describes the continuation of the leading logarithmic term in $\Theta(u)$, and when $\gamma=0$ reduces to the integral, studied above, in which the logarithmic factor is not continued,

$$I(\omega,0)=\int_0^{u_0} du\, u^{-\omega}\frac{1}{(\log u)^2} . \tag{B18}$$

To study the ω-plane analyticity of Eq. (B17), we expand the factor $(1-u^{\gamma})^{-2}$ into a power series in u^{γ} (since $u_0<1$, this is permitted for $\gamma>0$), and then do the u integrations assuming $\mathrm{Re}\omega<1$, giving

$$\begin{aligned} I(\omega,\gamma)&=\gamma^2\int_0^{u_0} du\, u^{-\omega}\sum_{n=0}^{\infty}(n+1)u^{n\gamma} \\ &=\gamma^2\sum_{n=0}^{\infty}(n+1)\frac{u_0^{n\gamma+1-\omega}}{n\gamma+1-\omega} . \end{aligned} \tag{B19}$$

When γ is regarded as a parameter independent of ω, Eq. (B19) shows that $I(\omega,\gamma)$ is a meromorphic function of ω, with poles at $\omega=1+n\gamma$, $n=0,1,\ldots$. In the limit as $\gamma\to 0$ for fixed ω, these poles coalesce into a branch cut running from $\omega=1$ to $\omega=+\infty$, which is just the analyticity structure of $I(\omega,0)$ which was inferred from the discussion following Eq. (B14) above. When the value $\gamma=2-\omega$, corresponding to continuation of the logarithm, is substituted into Eq. (B19), we get

$$\begin{aligned} I(\omega,2-\omega) &=(2-\omega)^2\sum_{n=0}^{\infty}(n+1)\frac{u_0^{2n+1-\omega(n+1)}}{2n+1-\omega(n+1)} \\ &=(2-\omega)u_0^{1-\omega}\,{}_2F_1\left[2,\frac{1-\omega}{2-\omega};\frac{1-\omega}{2-\omega}+1;u_0^{2-\omega}\right] , \end{aligned} \tag{B20}$$

with the hypergeometric function ${}_2F_1$ $(a=2,b;c=b+1;z)$ defined by

$${}_2F_1(2,b;b+1;z)=\sum_{n=0}^{\infty}\frac{(n+1)z^n}{b+n} . \tag{B21}$$

The singularities of Eq. (B21) are poles at $b_n=-n$, corresponding to poles in ω at ω_n given by

$$1\le\omega_n=\frac{2n+1}{n+1}<2 , \tag{B22}$$

and a cut along the real z axis from $z=1$ to $z=\infty$, corresponding to a cut along the real ω axis from $\omega=2$ to $\omega=\infty$. Hence there is an infinite accumulation of poles on the real axis to the left of $\omega=2$, and a branch cut on the real axis to the right of $\omega=2$. As a result, the limit $\omega\to 2$ cannot be taken along the real axis, and instead must be defined as the average of limits from above and below the real axis, giving the real part prescription of Eq. (5.56). The fact that $\omega=2$ is a branch point is a direct result of the fact that $I(\omega,2-\omega)$ is the sum of an infinite number of Feynman diagrams. If the sum in Eq. (B20) is truncated at $n=N$, corresponding to retaining only contributions to the running coupling constant through N-loop order, one gets a meromorphic function of ω which is regular at $\omega=2$. This is the result expected from the discussion of the dimensional continuation of individual Feynman diagrams given in Sec. III.

We must still show that, as $\omega\to 2$ in a real part or principal value sense, $I(\omega,2-\omega)$ approaches the leading term of Eq. (5.56) of the text. To do this, let us again make the change of variable $u=e^{-v}$, giving

$$I(\omega,2-\omega)=\int_{\log u_0^{-1}}^{\infty} dv\frac{e^{(\omega-1)v}}{v^2}\frac{1}{\left[\dfrac{1-e^{(\omega-2)v}}{(2-\omega)v}\right]^2} . \tag{B23}$$

For $\mathrm{Re}\omega<1$, $\mathrm{Im}\omega>0$, and v in the first quadrant, we have

$$\begin{aligned} &\mathrm{Re}[(\omega-1)v]=\mathrm{Re}(\omega-1)\mathrm{Re}v-\mathrm{Im}(\omega-1)\mathrm{Im}v<0 , \\ &\mathrm{Re}[(\omega-2)v]=\mathrm{Re}[(\omega-1)v]-\mathrm{Re}v<0 , \end{aligned} \tag{B24}$$

and so the contour of integration can be deformed to C without encountering poles coming from vanishing of the denominator in Eq. (B23). We can then set $\omega=2+i\epsilon$, giving

$$\mathrm{Re}[I(2+i\epsilon,-i\epsilon)]=\mathrm{Re}\left[\int_{\log u_0^{-1}}^{i\infty}\frac{dv}{v^2}e^{v}F(v,\epsilon)\right], \tag{B25}$$

with $F(v,\epsilon)$ given by

$$F(v,\epsilon)=\frac{e^{i\epsilon v}}{\left[\dfrac{1-e^{i\epsilon v}}{i\epsilon v}\right]^2},$$

$$F(v,\epsilon)=1+O(\epsilon^2|v|^2),\quad |v|\ll\epsilon^{-1},$$

$$F(v,\epsilon)=\frac{1}{\left[\dfrac{\sinh\frac{1}{2}|\epsilon v|}{\frac{1}{2}|\epsilon v|}\right]}\le 1,\quad v\ \text{imaginary}. \tag{B26}$$

Since the integral of Eq. (B25) is absolutely and uniformly convergent for all $\epsilon\ge 0$, we can take the limit as $\epsilon\to 0$ inside the integral, giving

$$I(2,0)=\lim_{\epsilon\to 0}\mathrm{Re}[I(2+i\epsilon,-i\epsilon)] =\mathrm{Re}\left[\int_{\log u_0^{-1}}^{i\infty}\frac{dv}{v^2}e^{v}\right]. \tag{B27}$$

The result of this rather tedious analysis thus reproduces the leading, $\Theta=1$, term of Eq. (5.56).

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