Large N limits as classical mechanics

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This paper discusses the sense in which the large N limits of various quantum theories are equivalent to classical limits. A general method for finding classical limits in arbitrary quantum theories is developed. The method is based on certain assumptions which isolate the minimal structure any quantum theory should possess if it is to have a classical limit. In any theory satisfying these assumptions, one can generate a natural set of generalized coherent states. These coherent states may then be used to construct a classical phase space, derive a classical Hamiltonian, and show that the resulting classical dynamics is equivalent to the limiting form of the original quantum dynamics. This formalism is shown to be applicable to the large N limits of vector models, matrix models, and gauge theories. In every case, one can explicitly derive a classical action which contains the complete physics of the $N = \infty$ theory. "Solving" the $N = \infty$ theory requires minimizing the classical Hamiltonian, and this has been possible only in simple theories. The relation between this approach and other methods which have been proposed for deriving large N limits is discussed in detail.

CONTENTS

I. INTRODUCTION

Many quantum theories possess natural generalizations in which the number of degrees of freedom is a free parameter. If N is some measure of the number of dynamical variables, then for a wide class of these theories the $N \to \infty$ limit is known to simplify the dynamics dramatically. This is true in theories ranging from the quantum mechanics of a point particle moving, in an N-dimensional central potential, to quantum spin models with spin N quantum spins, to quantum field theories containing SU(N) gauge fields. If the $N \to \infty$ limit of such a theory can be explicitly solved, then a systematic expansion in powers of $1/N$ can provide a very useful approach for studying the original finite N theory.[1]

Much of the recent interest in large N expansions is motivated by the desire to find reliable methods for analyzing the dynamics of quantum chromodynamics (QCD). One may introduce a parameter N into QCD by replacing the SU(3) gauge group with SU(N) ('t Hooft, 1974). $1/N$ then provides the only known expansion parameter which can be used in calculations of hadronic properties. Qualitative arguments suggest that the $N = \infty$ theory is surprisingly similar to the real world (Veneziano, 1976; Witten, 1979a). (For example, for large N one expects to see infinitely many narrow resonances that are purely composed of valence quarks. Exotics are absent, Zweig's rule is satisfied, and one-meson exchange dominates scattering amplitudes.) Therefore a $1/N$ expansion might be very reliable even at $N = 3$. Unfortunately, the $N = \infty$ theory has not yet been explicitly solved, and for this reason quantitative predictions are totally lacking.

Because brute force methods for solving the $N \to \infty$ limit (such as summing the appropriate class of Feynman diagrams) appear to be totally hopeless in theories like QCD, there has been considerable effort directed toward finding useful ways to reformulate the large N limit of various theories. By now, quite a few different methods have been proposed [under trade names such as "collective field methods" (Jevicki and Sakita, 1980a), "string equations" (Makeenko and Migdal, 1979), "master fields" (Witten, 1979), "constrained classical solutions" (Jevicki and Papanicolaou, 1980; Halpern, 1981a), etc.]. Each of these methods is known to work in at least some specific set of models. However, questions such as "Why does the method work?" or "For what class of theories does the method work?" have not been fully answered. This paper represents an attempt to answer some of these questions.

Essentially every method developed for studying large N limits has been based on the following fact. In every theory known to have a sensible large N limit, the vacuum expectation of any product of (reasonable) operators,[2] $\hat{A}\hat{B}$, satisfies the factorization relation,

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[1]For a review of some of the applications of $1/N$-expansions to field theories, see, for example, Coleman (1980).

[2]What constitutes a "reasonable" operator will be discussed in Sec. III.

$$\langle \hat{A}\hat{B} \rangle = \langle \hat{A} \rangle \langle \hat{B} \rangle + O(1/N) . \tag{1.1}$$

Therefore, the variance of any (reasonable) operator vanishes as $N \to \infty$,

$$\lim_{N \to \infty} (\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2) = 0 . \tag{1.2}$$

[One way to verify these statements, at least perturbatively, is to examine the set of Feynman diagrams which survive in the $N \to \infty$ limit. (1.1) is equivalent to the statement that disconnected graphs always dominate, and this may be verified on a graph-by-graph basis without having to sum the whole series.]

Equation (1.2) shows that fluctuations become irrelevant, at least for some set of operators, when N tends to infinity. Therefore in some sense quantum theories with large N behave like classical theories. It then seems natural to ask the question, "Is the $N \to \infty$ limit a classical limit?" By this we mean the following. Can one find a classical system (i.e., a classical phase space, Poisson bracket, and classical Hamiltonian) whose dynamics is equivalent to the $N \to \infty$ limit of a given quantum theory? In this paper the following strategy will be used to explore this question. First I present a general scheme for finding classical limits in arbitrary quantum theories. This formalism is based on a small set of assumptions which explicitly isolate the minimal structure any quantum theory should possess if it is to have a classical limit. Given any quantum theory satisfying these assumptions, an explicit algorithm may then be used to construct the classical phase space, define a consistent Poisson bracket, and find a classical Hamiltonian, such that the resulting classical dynamics agrees with the limiting form of the original quantum dynamics. I then examine various theories in order to see if their large N limits can be understood as examples of this general formalism. Specifically, I consider vector models, matrix models, and gauge theories, and in every case find that the large N limit is a classical limit in the sense described above. In fact, this method for deriving classical limits is applicable to every quantum theory known to me which, in some limit, satisfies factorization (Eq. 1.1). Besides trivial $\hbar \to 0$ limits, this includes all large N limits of the type discussed here, where the invariance group of the theory grows with N, as well as limits where the underlying symmetry group is fixed, but where quantum operators in larger and larger representations of the group appear. This latter type of limit describes, for example, the large spin limit of quantum spin models. In somewhat greater detail, the outline of this paper is as follows.

Section II contains a brief discussion of the $\hbar \to 0$ limit in quantum mechanics of point particles. I review a few of the standard properties of Gaussian coherent states, and show how they may be used to construct a very simple derivation of the $\hbar \to 0$ limit. The basic purpose of this section is to provide a concrete example which will be used to illustrate many of the features of the following rather abstract discussion.

Section III presents a general formalism for finding classical limits in arbitrary quantum theories. To apply this method one must (a) choose an appropriate group of unitary transformations, (b) generate a set of coherent states by applying elements of this group to a suitable initial state, and (c) show that in some limit of the theory different coherent states become orthogonal. This structure then automatically allows one to construct a unique classical phase space, define classical dynamics on this space, and show that the limit of the quantum dynamics is equivalent to this classical dynamics. In particular, one can form a classical action which contains all the physics of the original quantum theory that survives in the classical limit.

This formalism is applied to the large N limit of vector models in Sec. IV. This class of models includes theories ranging from quantum mechanics of point particles in N dimensions to N component ϕ^4 field theories. I show that all the assumptions of the general formalism are valid for these models, and explicitly derive the classical limit. All of the standard results concerning the large N limit of these models (such as the ground state energy, spectrum, etc.) may be obtained by minimizing the classical Hamiltonian and expanding the classical action about the minimum.

Section V discusses the large N limit of matrix models. Such theories are much more complicated than simple vector models. (For example, the large N limit of the perturbation series contains all planar diagrams.) Nevertheless, the general formalism is shown to be applicable to these models, and the classical limit is derived. However, only in the case of a single matrix has it been possible to explicitly minimize the resulting classical Hamiltonian and thereby "solve" the $N = \infty$ theory.

Lattice gauge theories are the subject of Sec. VI. The analysis is essentially identical to the preceding treatment of matrix models. The large N limit may be shown to be a classical limit, and the classical Hamiltonian may be derived. The resulting classical phase space is sufficiently complicated that only the one plaquette model has been explicitly solved.

Section VII discusses the relation between this approach for understanding the large N limit, and previously proposed methods. The advantages and disadvantages of each approach, as well as their interrelationships, are considered at some length. Various open problems are mentioned.

Finally, the appendix contains a brief discussion of several topics which are related to the general formalism of Sec. III.

A few "historical" remarks are appropriate to end this introduction. The method presented in Sec. III for deriving classical limits of general quantum theories was motivated by two recent papers, one by Berezin (1978) and the other by Simon (1980). Berezin considered the large N limit of vector models and showed that the limiting theory is a classical theory. In fact, he found that vector models with finite N may be regarded as the quantization of classical mechanics on Kähler manifolds. Berezin used coherent state methods which are very

similar to those employed in Sec. IV. The treatment here is somewhat simpler, but the results are equivalent. Further comments on Berezin's work will be found in Sec. VII. Simon discussed the classical limit of quantum spin systems, extending previous work by Lieb (1973). His work includes a derivation of the classical phase space for spin models whose spins represent the generators of any compact Lie group. This paper's treatment of generalized coherent states, as well as the identification of the classical phase space, is essentially patterned after Simon's discussion. I do not discuss quantum spin models in any detail in this paper simply because there already exists an extensive literature on the application of coherent state methods to spin models. [See Simon (1980), Lieb (1973), Fuller and Lenard (1979), Gilmore and Feng (1978), Gilmore (1979), Shankar (1980), and references therein.] However, it should be noted that the classical limits of quantum spin models provide beautiful examples of the general formalism described in Sec. III.

II. THE $\hbar \to 0$ LIMIT

Quantum mechanics is generally said to reduce to classical mechanics in the $\hbar \to 0$ limit. However, this statement really requires some qualification. The crucial fact is that one may form states whose uncertainty in both position and momentum vanishes as $\hbar$ tends to zero. If (and only if) the quantum system is prepared in such a state, then the quantum dynamics will reduce to classical dynamics when $\hbar \to 0$. A simple way formally to derive this result is as follows.

Consider a quantum theory describing n degrees of freedom, with basic position $\{\hat{x}_i\}$ and momentum $\{\hat{p}_i\}$ operators obeying the canonical commutation relations,

$$i[\hat{p}_i,\hat{x}_j]=\hbar\delta_{ij}\ ,\quad i,j=1,\ldots,n\ . \tag{2.1}$$

Introduce a set of Gaussian coherent states,[3] $\{|p,q\rangle\}$, with wave functions given by

$$\langle x\,|p,q\rangle=(\pi\hbar)^{-n/4}\exp\{(1/\hbar)[ip\cdot x-\tfrac{1}{2}(x-q)^2]\}\ . \tag{2.2}$$

Note that different coherent states are not orthogonal; their overlaps are given by

$$\langle p,q\,|p',q'\rangle=\exp\{-(1/4\hbar)[(p-p')^2+(q-q')^2+2i(p-p')\cdot(q+q')]\}\ . \tag{2.3}$$

This set of coherent states forms an overcomplete basis for the full Hilbert space. This is expressed by the completeness relation,

$$\hat{1}=\int (dp\,dq/2\pi\hbar)\,|p,q\rangle\langle p,q\,| \tag{2.4}$$

$[(dp\,dq/2\pi\hbar)\equiv\prod_{i=1}^{n}dp_i dq_i/2\pi\hbar)]$. Consequently, any quantum state $|\psi\rangle$ may be represented by its projections onto the different coherent states, $\psi(p,q)=\langle p,q\,|\psi\rangle$. This is convenient representation regardless of the choice of the Hamiltonian; the quantum dynamics is not required to preserve the form of the coherent states.

Similarly, any operator, $\hat{A}$, may be represented by its coherent state matrix elements, $\langle p,q|\hat{A}|p',q'\rangle$. However, to specify an operator uniquely it is not necessary to give all possible matrix elements. Due to the overcompleteness of the coherent state basis, an arbitrary operator $\hat{A}$ may be reconstructed from just the diagonal expectation values,

$$A(p,q)\equiv\langle p,q\,|\,\hat{A}\,|p,q\rangle\ . \tag{2.5}$$

[One way to see this is based on the observation that $\langle x|p,q\rangle e^{q^2/2\hbar}$ is an analytic function of $(p-iq)$. Therefore, for any (reasonable) operator $\hat{A}$, $\langle p,q|\hat{A}|p',q'\rangle/\langle p,q\,|p',q'\rangle$ is an analytic function of $(p+iq)$ and $(p'-iq')$. Consequently, one may recover arbitrary coherent state matrix elements of $\hat{A}$ by analytic continuation from the diagonal expectation values.]

Now consider the diagonal matrix elements of a product of two operators, $\hat{A}\hat{B}$. The completeness relation (2.4) may be used to write this as

$$(AB)(p,q)=\int (dp'dq'/2\pi\hbar)\,|\langle p,q\,|p',q'\rangle|^2\times\frac{\langle p,q\,|\,\hat{A}\,|p',q'\rangle}{\langle p,q\,|p',q'\rangle}\frac{\langle p',q'\,|\,\hat{B}\,|p,q\rangle}{\langle p',q'\,|p,q\rangle}\ . \tag{2.6}$$

To study the classical limit, one must compute the small $\hbar$ asymptotics of this integral. The first factor,

$$|\langle p,q\,|p',q'\rangle|^2=\exp\{-(1/2\hbar)[(p-p')^2+(q-q')^2]\}$$

becomes arbitrarily highly peaked about $p=p'$ and $q=q'$ as $\hbar\to 0$. However, the remaining factors have a smooth limit when $\hbar\to 0$.[4] Therefore one finds

$$\lim_{\hbar\to 0}(AB)(p,q)=a(p,q)b(p,q)\ , \tag{2.7}$$

where $a(p,q)\equiv\lim_{\hbar\to 0}A(p,q)$, etc. Similarly, one may expand the integral (2.6) about $p=p'$ and $q=q'$ and easily find that

$$\lim_{\hbar\to 0}\frac{i}{\hbar}[A,B](p,q)=\left[\frac{\partial a(p,q)}{\partial p}\frac{\partial b(p,q)}{\partial q}-\frac{\partial b(p,q)}{\partial p}\frac{\partial a(p,q)}{\partial q}\right]=\{a(p,q),b(p,q)\}_{PB}\ . \tag{2.8}$$

[3] The properties of Gaussian coherent states are discussed in many textbooks; Klauder and Sudarshan (1968), for example, contains a good discussion.

[4] $\hat{A}$ and $\hat{B}$ must be "classical operators" as defined in Sec. III. Such operators include arbitrary polynomials in $\hat{x}$ and $\hat{p}$ with no explicit $\hbar$ dependence.

These results show how the quantum theory reduces to classical mechanics when $\hbar \to 0$. Equation (2.7) implies that quantum operators become ordinary functions on the classical phase space, while Eq. (2.8) shows that quantum commutators become classical Poisson brackets. This implies that the quantum equations of motion, $\partial \hat{A}/\partial t = (i/\hbar)[\hat{H}, \hat{A}]$, reduce to the classical Hamiltonian equations, $\partial a(p,q)/\partial t = \{h(p,q), a(p,q)\}_{PB}$. If the quantum Hamiltonian is some given functions of $\hat{x}$ and $\hat{p}$, $\hat{H} = f(\hat{p}, \hat{x})$, then the classical Hamiltonian $h(p,q)$ equals $f(p,q)$ regardless of the ordering of the original quantum operators.[5]

The classical equations of motion follow from the classical action,

$$S_{cl}[p(t), q(t)] = \int dt\,[p(t)\dot{q}(t) - h(p(t), q(t))] \ . \quad (2.9)$$

One may regard the classical action as containing the complete physics of the theory in the $\hbar \to 0$ limit. For example, the limiting behavior as $\hbar \to 0$ of the ground-state energy, the spectrum, or any correlation function may be obtained from the classical action. This will be discussed in more detail at the end of the next section.

III. COHERENT STATES AND COADJOINT ORBITS

The key ingredient in the preceding discussion of the $\hbar \to 0$ limit was obviously the choice of coherent states. They provided

(i) a convenient partition of unity requiring only diagonal projections onto the coherent states,

(ii) a basis sufficiently overcomplete that any operator could be completely represented by its diagonal matrix elements alone,

(iii) a simple derivation of factorization based on the fact that different coherent states become orthogonal in the classical limit, and

(iv) an identification of the classical phase space as the manifold whose coordinates could be used to label different coherent states.

We will now see how each of these features may be naturally incorporated in a more general framework. The resulting abstract formalism will be applicable to every known theory possessing a factorizing limit.

Consider a family of quantum theories labeled by some parameter χ (such as $\hbar$ or $1/N$). χ is assumed to take values in some set of positive real numbers whose limit points include zero. We are intereseted in studying the limit of these theories as χ tends to zero. Each theory is defined on some Hilbert space $\mathbf{H}_\chi$ with some Hamiltonian $\hat{H}_\chi$, etc. [We speak of a one-parameter family of theories, as opposed to a single theory depending on the parameter χ, in order to emphasize that the basic structure of the theory (such as the Hilbert space, commutation relations, etc.) may vary with χ. For example, the number of basic degrees of freedom will differ in theories with different values of N.]

Let there be given some Lie group g which, within each theory, may be represented by a set of unitary operators. In other words, acting on each Hilbert space $\mathbf{H}_\chi$ is a group of unitary operators, $G_\chi \equiv \{\hat{U} | \hat{U} = D_\chi(u), u \in g\}$, which provides a representation of the abstract group g. We will refer to g as the *coherence group*. This coherence group has a Lie algebra $\mathbf{g}$ which may be represented within each theory by a set of anithermitian operators, $\mathbf{G}_\chi \equiv \{\hat{\Lambda} | \hat{\Lambda} = D_\chi(\lambda), \lambda \in \mathbf{g}\}$, which generate one-parameter subgroups of G_χ, $\exp t\hat{\Lambda} \in G_\chi$. Note that the abstract group g and its algebra $\mathbf{g}$ do not depend on the parameter χ.

Furthermore, within each theory let there be given some chosen normalized state, $|0\rangle_\chi \in \mathbf{H}_\chi$, which we will call the *base state*. Consider the states which are generated by applying elements of the coherence group to the base state,

$$|u\rangle_\chi \equiv \hat{U}\,|0\rangle_\chi \ , \quad \hat{U} \in G_\chi \ . \quad (3.1)$$

These are precisely the coherent states we will use.[6] Henceforth, we will occasionally drop the explicit subscript χ if we do not need to emphasize which particular theory we are working in.

We will use the previous $\hbar \to 0$ example to illustrate each feature of the general discussion. For convenience, I describe the case of just one degreee of freedom. In this example χ equals $\hbar$, and the coherence group g may be chosen to be the Heisenberg group. This may be regarded as a three-parameter group whose elements, $u(p,q,\alpha)$, obey the muliplication rule

$$u(p,q,\alpha)\,u(p',q',\alpha') = u(p+p', q+q', \alpha+\alpha'-qp') \ .$$

This group may be represented by the set of operators

$$G_\hbar = \{\hat{U}(p,q,\alpha) \equiv \exp(i\alpha/\hbar)\,\exp(ip\hat{x}/\hbar)\,\exp(-iq\hat{p}/\hbar)\} \ .$$

Note that $\hat{U}(p,q,\alpha)$ simply translates positions by q, and momenta by p. If we choose the base state to be a simple Gaussian,

$$\langle x\,|\,0\rangle_\hbar = (\pi\hbar)^{-1/4}\exp -(x^2/2\hbar) \ ,$$

then elements of the Heisenberg group precisely generate our previous coherent states up to an overall phase factor,

$$\hat{U}(p,q,\alpha)\,|\,0\rangle_\hbar = e^{i\alpha/\hbar}\,|\,p,q\,\rangle \ .$$

[5]Note that if the Hamiltonian $\hat{H}(\hat{p}, \hat{x})$ is replaced by $(1/g^2)\hat{H}(g\hat{p}, g\hat{x})$, and the rescaled operators $\hat{p}' \equiv g\hat{p}$ and $\hat{x}' \equiv g\hat{x}$ are used in place of $\hat{p}$ and $\hat{x}$, then every occurrence of $\hbar$ becomes $(g^2\hbar)$. Therefore the classical $(\hbar \to 0)$ limit is equivalent to the weak coupling $(g^2 \to 0)$ limit.

[6]This method for generating generalized coherent states based on arbitrary Lie groups was first discussed by Klauder (1963) and Perelomov (1972).

The choice of the coherence group is restricted by the following assumption. We require

Assumption 1. Each representation of the coherence group, G_χ, acts irreducibly on the corresponding Hilbert space $\mathbf{H}_\chi$.

In other words, there must be no nontrivial subspace of $\mathbf{H}_\chi$ which is left invariant under the action of all elements of the coherence group G_χ. To test for irreducibility one may use Schur's lemma, which states that a group acts irreducibly if and only if the only operators which commute with all elements of the group are proportional to the identity—i.e., G_χ acts irreducibly iff $\hat{U}\hat{A}\hat{U}^{-1}=\hat{A}$ for all $\hat{U}\in G_\chi$ implies $\hat{A}\propto\hat{1}$.

This assumption has the following consequences. Consider the operator $\hat{J}\equiv\int d\mu_L(u)|u\rangle\langle u|$, where $d\mu_L(u)$ is the (left) invariant measure on the coherence group—i.e., $d\mu_L(u'u)=d\mu_L(u)$ for any fixed $u'\in g$.[7] Note that $\hat{J}$ commutes with all elements of G_χ,

$$\hat{U}'\hat{J}\hat{U}'^{-1}=\int d\mu_L(u)\,|\,u'u\,\rangle\langle\,u'u\,|\,=\hat{J}$$

for all $\hat{U}'\in G_\chi$, so that by Schur's lemma $\hat{J}$ is proportional to $\hat{1}$. Therefore the irreducibility of G_χ automatically provides us with a natural completeness relation,

$$\hat{1}=c_\chi\int d\mu_L(u)\,|\,u\,\rangle\langle\,u\,|\ . \tag{3.2}$$

The constant c_χ depends on the normalization of the group measure and must be computed explicitly.

Assumption 1 has another important consequence. If a group acts irreducibly, then any operator may be expressed as a linear combination of elements of the group.[8] Therefore any operator $\hat{A}$ acting in $\mathbf{H}_\chi$ may be written in the form

$$\hat{A}=\int(d\lambda)f(\lambda)\exp(\chi\hat{\Lambda}) \tag{3.3}$$

for some weight function $f(\lambda)$. [Here $(d\lambda)$ indicates uniform measure on the Lie algebra $\mathbf{g}$. The factor of χ is inserted for later convenience.]

Assumption 1 may be easily verified in our $\hbar\to 0$ example. To do so, first consider the subgroup of the Heisenberg group which is generated by $(i\hat{x}/\hbar)$, $\{\hat{U}(p,0,0)=\exp(ip\hat{x}/\hbar)\}$. One may easily see that the only operators which commute with all elements of this subgroup are of the form $a(\hat{x})$, that is, which are solely constructed from $\hat{x}$. However, any operator of this form which also commutes with the subgroup generated by $(i\hat{p}/\hbar)$, $\{\hat{U}(0,q,0)=\exp(-iq\hat{p}/\hbar)\}$, must simply be a constant. Therefore the Heisenberg group acts irreducibly. Invariant measure on the Heisenberg group is given by $d\mu(u(p,q,\alpha))\equiv dp\,dq\,d\alpha$ ($d\mu$ is both left and right invariant). Therefore (3.2) agrees with our previous completeness relation (2.4). In this example, the operator representation (3.3) becomes

$$\begin{aligned}\hat{A}&=\int dp\,dq\,d\alpha\,f(p,q,\alpha)\exp i(p\hat{x}-q\hat{p}+\alpha)\\&=\int dp\,dq\,\bar{f}(p,q)\exp i(p\hat{x}-q\hat{p})\end{aligned}$$

[$\bar{f}(p,q)\equiv\int d\alpha f(p,q,\alpha)e^{i\alpha}$]. This is simply the well-known Weyl representation. Equation (3.3) is the natural generalization of this representation to any group which acts irreducibly.

Assumtpion 1 involves only the choice of the coherence group g. It places no restriction on the base state $|0\rangle_\chi$. In fact, (3.2) shows that a complete set of states may be generated by applying elements of the coherence group to any initial state in $\mathbf{H}_\chi$. However, the choice of base state will be restricted by our next assumption.

For any operator $\hat{A}$ acting in $\mathbf{H}_\chi$, let us define the *symbol* $A_\chi(u)$ as the set of coherent-state expectation values,[9]

$$A_\chi(u)=\langle\,u\,|\,\hat{A}\,|\,u\,\rangle_\chi\ ,\quad u\in g\ . \tag{3.4}$$

For each value of χ, we require

Assumption 2. Zero is the only operator whose symbol identically vanishes.

In other words, if $A_\chi(u)=0$ for all $u\in g$, then $\hat{A}$ must equal zero. This assumption implies that two different operators cannot have the same symbol. (Otherwise, the difference of the two operators would violate the assumption.) Therefore any operator may be uniquely recovered from its symbol. This means that it is sufficient to study the behavior of the symbols of various operators in order to characterize the theory completely.

Assumption 2 may be easily verified for the $\hbar\to 0$ example. One method, based on the analyticity of $\langle p,q|\hat{A}|p',q'\rangle/\langle p,q\,|\,p',q'\rangle$ in $(p+iq)$ and $(p'-iq')$, was discussed in the last section. The assumption may also be proven in a more direct fashion using an argument due to B. Simon. We will present this argument in some detail, since the method naturally generalizes to later examples.

Suppose the symbol of some operator, $A_\chi(u)$, vanishes for all $u\in g$. We may choose $u=e^{t_1\lambda_1}e^{t_2\lambda_2}\cdots e^{t_n\lambda_n}$ for

[7] All integrals over the coherence group need only be taken over the coset space $g_\chi\equiv g/h_\chi$, where h_χ is the isotropy subgroup of $|0\rangle\langle 0|$, $h_\chi\equiv\{\,u\in g\mid\,|\,u\,\rangle\langle u|=|0\rangle\langle 0|\,\}$. I will not bother to indicate this explicitly in the notation here. The integral $\int d\mu_L(u)\,|\langle u|0\rangle\,|^2$ is assumed to converge.

[8] This is a sloppy version of the Von Neumann density theorem, which implies that a group acts irreducibly if and only if the algebra of operators generated by the group is strongly dense in the set of all bounded operators. For a rigorous proof see, for example, Bratteli and Robinson (1979), Sec. 2.4.2.

[9] What I am calling the *symbol* of an operator is elsewhere referred to as the *lower symbol* (Simon, 1980) or the *covariant symbol* (Berezin, 1972). There is another natural association of operators with functions on the coherence group which may be used to define *upper* or *contravariant* symbols. These are described in the appendix, but are not used in the bulk of this paper.

arbitrary $\lambda_i \in \mathbf{g}$, differentiate with respect to each t_i, and find

$$0 = \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_n} A_\chi(u) \mid_{t_i=0}$$
$$= \langle 0 \mid [\cdots [[\hat{A}, \hat{\Lambda}_1], \hat{\Lambda}_2], \ldots, \hat{\Lambda}_n] \mid 0 \rangle_\chi . \quad (3.5)$$

Therefore expectations in the base state of multiple commutators of $\hat{A}$ with arbitrary generators of the coherence group vanish. Now for the Heisenberg group each $\hat{\Lambda}_i$ is some linear combination of $i\hat{p}/\hbar$, $i\hat{x}/\hbar$, and $i\hat{1}/\hbar$. Therefore we may choose each $\hat{\Lambda}_i$ to be either a creation, $\hat{a}^\dagger = (\hat{p} + i\hat{x})/\sqrt{2\hbar}$, or annihilation, $\hat{a} = (\hat{p} - i\hat{x})/\sqrt{2\hbar}$, operator. We will prove inductively that

$$\langle 0 \mid \hat{a} \cdots \hat{a} \hat{A} \hat{a}^\dagger \cdots \hat{a}^\dagger \mid 0 \rangle = 0 \quad (3.6)$$

for any number of creation or annihilation operators. Suppose (3.6) has been verified whenever the total number of creation plus annihilation operators is less than some number L. Consider a multiple commutator of the form (3.5) with L creation or annihilation operators. Expand the multiple commutator. One term has all annihilation operators to the left, and all creation operators to the right of $\hat{A}$. Every other term contains at least one annihilation operator which may be pushed right until it annihilates the base state, or one creation operator which may be pushed left. This process may produce commutator terms (since $[\hat{a}, \hat{a}^\dagger] = 1$); however, each commutator reduces the number of $\hat{a}$'s plus $\hat{a}^\dagger$'s by two. Therefore the expectation of the multiple commutator contains one new term of the form in (3.6) with L creation and annihilation operators, plus lower-order terms which have already been shown to vanish. Therefore (3.6) holds for L creation and annihilation operators, and by induction holds for any number. This shows that matrix elements of $\hat{A}$ between any two states formed by applying polynomials of creation operators to the base state vanish. But such states are known to be dense, thereby implying that $\hat{A} = 0$. This proves the assumption.

Henceforth, whenever I speak of an "operator," I will actually mean some given family of operators consisting of one operator acting in each Hilbert space $\mathbf{H}_\chi$. χ obviously determines which operator in a given family is appropriate; normally I will not bother to add it as an explicit label.

Clearly, an arbitrary operator need not have a sensible limit as $\chi \to 0$. In order to have some control over this limit, we will introduce a restricted class of operators, $\mathbf{K}$, consisting of operators $\hat{A}$ whose coherent state matrix elements, $\langle u \mid \hat{A} \mid u' \rangle_\chi / \langle u \mid u' \rangle_\chi$, have finite limits as $\chi \to 0$ for all $u, u' \in g$. We will refer to such operators as *classical operators.*

Since classical operators form only a subset of all operators, it is possible that measurements with any classical operator will fail to distinguish between different coherent states. Therefore we will call two coherent states, $|u\rangle$ and $|u'\rangle$, *classically equivalent* if

$$\lim_{\chi \to 0} A_\chi(u) = \lim_{\chi \to 0} A_\chi(u') \quad (3.7)$$

for all $\hat{A} \in \mathbf{K}$. We will write $u \sim u'$ if $|u\rangle$ and $|u'\rangle$ are classically equivalent.

We may now formulate our third assumption, which states that classically inequivalent coherent states become orthogonal as $\chi \to 0$. Specifically, we require

Assumption 3. $\phi(u,u') \equiv -\lim_{\chi \to 0} \chi \ln \langle u \mid u' \rangle_\chi$ exists for all $u, u' \in g$, and satisfies

(i) if $u \neq u'$, $\mathrm{Re}\,\phi(u,u') > 0$, and

(ii) if $u \sim u'$, $\mathrm{Re}\,\phi(u,u') = 0$, and

$$\frac{\partial}{\partial t} [\phi(u, e^{t\lambda} u) - \phi(u, e^{t\lambda} u')] \mid_{t=0} = 0 \quad \text{for all } \lambda \in \mathbf{g} .$$

This shows that if $|u\rangle$ and $|u'\rangle$ are classically inequivalent, then their overlap $\langle u \mid u' \rangle$ decreases exponentially as $\chi \to 0$. Therefore for any classical operator $\hat{A} \in \mathbf{K}$, $\langle u | \hat{A} | u' \rangle_\chi$ must become highly peaked about $u \sim u'$ as $\chi \to 0$ (otherwise, $\langle u \mid \hat{A} \mid u' \rangle_\chi / \langle u \mid u' \rangle_\chi$ will have no limit). In other words, classical operators cannot "move" the coherent states. This excludes any fixed element $\hat{U} \in G_\chi$ of the coherence group (except $\hat{1}$). However, Assumption 3 shows that $(\chi \hat{\Lambda})$ and $\exp(\chi \hat{\Lambda})$ are acceptable classical operators, for any $\hat{\Lambda} \in \mathbf{G}_\chi$, since

$$\lim_{\chi \to 0} \frac{\langle u \mid \chi \hat{\Lambda} \mid u' \rangle_\chi}{\langle u \mid u' \rangle_\chi} = \frac{\partial}{\partial t} \phi(u, e^{t\lambda} u') \mid_{t=0} \quad (3.8)$$

and

$$\lim_{\chi \to 0} \frac{\langle u \mid e^{\chi \hat{\Lambda}} \mid u' \rangle_\chi}{\langle u \mid u' \rangle_\chi} = \exp \left[-\frac{\partial}{\partial t} \phi(u, e^{t\lambda} u') \mid_{t=0} \right] . \quad (3.9)$$

Consequently, a general classical operator, $\hat{A} \in \mathbf{K}$, may be represented in the form (3.3) for some weight function $f(\lambda)$ with compact support [and for which $\lim_{\chi \to 0} f(\lambda)$ exists as a distribution]. Note that

$$\lim_{\chi \to 0} A_\chi(u) = \int (d\lambda) \lim_{\chi \to 0} f(\lambda) \exp \left[-\frac{\partial}{\partial t} \phi(u, e^{t\lambda} u) \mid_{t=0} \right] .$$

This shows that if $\lim_{\chi \to 0} \chi \Lambda \chi(u) = \lim_{\chi \to 0} \chi \Lambda_\chi(u')$ for all $\hat{\Lambda} \in \mathbf{G}_\chi$, then $|u\rangle$ and $|u'\rangle$ are classically equivalent. In other words, expectation values of the set of operators $\{\chi \hat{\Lambda} | \hat{\Lambda} \in \mathbf{G}_\chi\}$ are sufficient to distinguish classically inequivalent coherent states. Finally, note that (3.9) plus the last part of Assumption 3 implies that if $|u\rangle$ and $|u'\rangle$ are classically equivalent, then

$$\lim_{\chi \to 0} \frac{\langle u \mid \hat{A} \mid u' \rangle_\chi}{\langle u \mid u' \rangle_\chi} = \lim_{\chi \to 0} A_\chi(u) \quad \text{for all } \hat{A} \in \mathbf{K} . \quad (3.10)$$

Assumption 3 allows us to prove factorization for any pair of operators $\hat{A}, \hat{B} \in \mathbf{K}$. We may use the completeness relation (3.2) to write the symbol of the product $\hat{A}\hat{B}$ as

$$(AB)_\chi(u)=c_\chi \int d\mu_L(u')\,|\langle u\,|\,u'\rangle_\chi|^2 \times \frac{\langle u\,|\,\hat{A}\,|\,u'\rangle_\chi}{\langle u\,|\,u'\rangle_\chi}\frac{\langle u'\,|\,\hat{B}\,|\,u\rangle_\chi}{\langle u'\,|\,u\rangle_\chi}\,. \quad (3.11)$$

As $\chi\to 0$ the integral becomes highly peaked about points which are equivalent to u. If we define the region $R_\chi\equiv[u'\in g|\mathrm{Re}\,\phi(u,u')\le\sqrt{\chi}]$ consisting of a small neighborhood around each point $u'\sim u$, then Assumption 3 implies that the contribution from the region outside R_χ is exponentially small. However, inside R_χ, (3.10) implies that

$$\langle u\,|\,\hat{A}\,|\,u'\rangle_\chi/\langle u\,|\,u'\rangle_\chi=A_\chi(u)+o(1)\,.$$

Therefore

$$\begin{aligned}(AB)_\chi(u)&=c_\chi\int_{R_\chi}d\mu_L(u')\,|\langle u\,|\,u'\rangle_\chi|^2 \times\frac{\langle u\,|\,\hat{A}\,|\,u'\rangle_\chi}{\langle u\,|\,u'\rangle_\chi}\frac{\langle u'\,|\,\hat{B}\,|\,u\rangle_\chi}{\langle u'\,|\,u\rangle_\chi}+o(1)\\ &=A_\chi(u)B_\chi(u)\cdot c_\chi\int_{R_\chi}d\mu_L(u')\,|\langle u\,|\,u'\rangle_\chi|^2+o(1)\\ &=A_\chi(u)B_\chi(u)+o(1)\,,\end{aligned}$$

since $c_\chi\int d\mu_L(u')|\langle u\,|\,u'\rangle_\chi|^2=1$. Consequently, the factorization

$$\lim_{\chi\to 0}[(AB)_\chi(u)-A_\chi(u)B_\chi(u)]=0 \quad (3.12)$$

holds for any pair of classical operators.

This whole discussion is a simple generalization of the $\hbar\to 0$ example. In that case classical operators obviously include arbitrary $\hbar$ independent polynomials in $\hat{x}$ and $\hat{p}$. Since measurements with classical operators can determine the mean position and momentum of a state, classically equivalent coherent states can differ only in their overall phase. To verify Assumption 3, one must simply compute the overlap $\langle u\,|\,u'\rangle_\hbar$, and check that

$$\begin{aligned}&\phi(u(p,q,\alpha),u'(p',q',\alpha'))\\ &=\tfrac{1}{4}[(p-p')^2+(q-q')^2+2i(p-p')(q+q')+4i(\alpha-\alpha')]\end{aligned}$$

satisfies the stated conditions. The operators $\{\hat{U}(p,q,\alpha)=e^{i\alpha/\hbar}e^{ip\hat{x}/\hbar}e^{-iq\hat{p}/\hbar}\}$ are not classical operators and do not obey factorization. However, the rescaled operators $\{\exp(\hbar\hat{\Lambda})=\exp i(\alpha+p\hat{x}-q\hat{p})\}$ translate positions and momenta only by $O(\hbar)$ and are prefectly acceptable classical operators.

The previous assumptions have given us some control over the structure of the theory as $\chi\to 0$, but have left the quantum dynamics completely unrestricted. To control the $\chi\to 0$ limit completely we need to place one condition on the quantum Hamiltonian $\hat{H}_\chi$. We require[10]

Assumption 4. $(\chi\hat{H}_\chi)$ is a classical operator.

This condition will ensure that the coupling constants in the Hamiltonian are scaled in a manner that maintains sensible dynamics as $\chi\to 0$.

These assumptions will suffice to show that the complete quantum theory reduces to classical mechanics as $\chi\to 0$. However, before this can be demonstrated the appropriate classical phase space must first be described. To facilitate this, we first review certain aspects of the structure of the coherence group.[11]

Consider the Lie algebra $\mathbf{g}$. It is a linear space, and therefore has a *dual space* $\mathbf{g}^*$ consisting of linear functionals acting on $\mathbf{g}$. If we introduce a basis in $\mathbf{g}$, $\{e_i\}$, and the corresponding biorthogonal basis in $\mathbf{g}^*$, $\{e^{*i}\}$, then the application of any element of the dual space, $\zeta=\zeta_i e^{*i}\in\mathbf{g}^*$, on any element of the Lie algebra, $\lambda=\lambda^i e_i\in\mathbf{g}$, is given by

$$\langle\zeta,\lambda\rangle=\zeta_i\lambda^i\,. \quad (3.13)$$

The communicator of any two elements, $\lambda,\mu\in\mathbf{g}$, of the Lie algebra may be represented as

$$[\lambda,\mu]=\lambda^i\mu^j c_{ij}^k e_k\,,$$

where $\{c_{ij}^k\}$ are the structure constants of $\mathbf{g}$. There is a natural action of elements of the coherence group on elements of the algebra $\mathbf{g}$, given by

$$Ad[u](\lambda)\equiv u\lambda u^{-1} \quad (3.14)$$

for any $u\in g,\lambda\in\mathbf{g}$. This is simply the *adjoint representation* of g. There is a corresponding action of the group on elements of the dual space given by

$$Ad^*[u]\equiv Ad[u^{-1}]^*\,. \quad (3.15)$$

This is the *coadjoint representation*.[12] It is defined so that (3.13) is invariant,

$$\begin{aligned}\langle Ad^*[u](\zeta),Ad[u](\lambda)\rangle&=\langle\zeta,Ad[u^{-1}]Ad[u](\lambda)\rangle\\ &=\langle\zeta,\lambda\rangle\,.\end{aligned}$$

One may now consider the set of points generated by the action of $Ad^*[u]$ applied to any given element $\zeta_o\in\mathbf{g}^*$. This is a *coadjoint orbit,*

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[10]The quantum equations of motion are taken to be $\partial\hat{A}/\partial t=i[\hat{H},\hat{A}]$, with no explicit factors of χ appearing.

[11]For more information on the following material see, for example, Kirillov (1976).

[12]In any semisimple Lie group the Cartan-Killing form, $(\lambda,\mu)\equiv\mathrm{tr}([\lambda,e_i][e_i,\mu])$, provides a nondegenerate invariant scalar product for the Lie algebra. This then generates a natural isomorphism between the algebra $\mathbf{g}$ and its dual space $\mathbf{g}^*$. Given any element of the dual space, $\zeta\in\mathbf{g}^*$, the corresponding element of the Lie algebra, $\bar{\zeta}\in\mathbf{g}$, is uniquely defined by the requirement that $\langle\zeta,\lambda\rangle=(\bar{\zeta},\lambda)$ for all $\lambda\in\mathbf{g}$. Consequently for semisimple groups the adjoint and coadjoint representations are equivalent. Unfortunately, none of the coherence groups considered in this paper is semisimple, and we are therefore forced to distinguish between working on the Lie algebra $\mathbf{g}$ and on its dual space $\mathbf{g}^*$.

$$\Gamma=\{\, Ad^*[u](\zeta_o) \mid u\in g \,\} \,. \tag{3.16}$$

We will later see that each set of classically equivalent coherent states may be naturally associated with some point on a single coadjoint orbit. In other words, the set of these equivalence classes, $[u]\equiv\{\, u'\in g | u\sim u'\}$, is isomorphic to a particular coadjoint orbit $\Gamma\in\mathbf{g}^*$. This is important for the following reason. To define a classical phase space one must not only specify the classical manifold, but must also give a consistent definition of a Poisson bracket. This requires the existence of an invariant symplectic structure on the classical manifold. Coadjoint orbits provide a particularly natural setting for classical mechanics, because they automatically possess the required symplectic structure.[13] This may be described as follows.

Consider an arbitrary point on some coadjoint orbit, $\zeta\in\Gamma$. The tangent space to Γ at ζ, $T_\zeta(\Gamma)$, is a subspace of $\mathbf{g}^*$ given by

$$T_\zeta(\Gamma)=\{\, \xi\in\mathbf{g}^* \mid \langle\xi,\cdot\rangle=\langle\zeta,[\lambda,\cdot]\rangle \text{ for some } \lambda\in\mathbf{g} \,\} \,. \tag{3.17}$$

These are simply the tangent vectors to the curves $Ad^*[e^{t\lambda}](\zeta)$ on Γ passing through ζ. We may regard elements of the Lie algebra $\mathbf{g}$ as linear functionals acting on the tangent space $T_\zeta(\Gamma)$. However, we should then identify different elements of $\mathbf{g}$ if they yield the same value when applied to any vector in $T_\zeta(\Gamma)$. Therefore elements of the cotangent space $T^*_\zeta(\Gamma)$ ("one-forms") may be regarded as equivalence classes of $\mathbf{g}$. For each $\lambda\in\mathbf{g}$, the equivalence class $[\lambda]$ is given by

$$[\lambda]=\{\, \lambda'\in\mathbf{g} \mid \langle\zeta,[\lambda',\mu]\rangle=\langle\zeta,[\lambda,\mu]\rangle \text{ for all } \mu\in\mathbf{g} \,\} \,.$$

Explicitly, the cotangent space $T^*_\zeta(\Gamma)$ is given by the quotient space

$$T^*_\zeta(\Gamma)=\mathbf{g}/\mathbf{h}_\zeta \,, \tag{3.18}$$

where $\mathbf{h}_\zeta\equiv\{\, \lambda\in\mathbf{g} \mid \langle\zeta,[\lambda,\mu]\rangle=0 \text{ for all } \mu\in\mathbf{g} \,\}$ is the Lie algebra of the isotropy subgroup of ζ,

$$h_\zeta\equiv\{\, u\in g \mid Ad^*[u](\zeta)=\zeta \,\} \,.$$

Note that the coadjoint orbit Γ is equivalent to the coset space g/h_ζ. We may now define a linear functional ω_ζ acting on $T^*_\zeta(\Gamma)\otimes T^*_\zeta(\Gamma)$ by

$$\omega_\zeta([\lambda],[\mu])\equiv\langle\zeta,[\lambda,\mu]\rangle \,, \tag{3.19}$$

or in components, $\omega_\zeta([\lambda],[\mu])=\lambda^i\mu^j c_{ij}^k\zeta_k$. The value of $\omega_\zeta([\lambda],[\mu])$ is clearly independent of which representative λ of the equivalence class $[\lambda]$ is used. ω_ζ is obviously antisymmetric and so may be regarded as a bivector on Γ. Furthermore, ω_ζ is nondegenerate, since if $\omega_\zeta([\lambda],[\mu])=0$ for all $[\mu]$, then $[\lambda]=0$ by our definition of equivalence.[14] Therefore ω_ζ provides an invertible mapping of the cotangent space $T^*_\zeta(\Gamma)$ onto the tangent space $T_\zeta(\Gamma)$. The inverse mapping, $\tilde{\omega}_\zeta$, may be regarded as a two-form on Γ. $\tilde{\omega}_\zeta$ is automatically closed, $d\omega_\zeta=0$, simply as a consequence of the definition (3.19). Consequently $\tilde{\omega}_\zeta$ is the exterior derivative of some one-form on Γ, $\tilde{\omega}_\zeta=d\vartheta_\zeta$, $\vartheta_\zeta\in T^*_\zeta(\Gamma)$.[15] (This construction will be described in more explicit terms later).

The bivector ω_ζ may be used to define a Poisson bracket as follows. Given any function $f(\zeta)$ on the coadjoint orbit, its gradient, $df(\zeta)\in T^*_\zeta(\Gamma)$, is one-form on Γ. We define the Poisson bracket of two such functions $f(\zeta)$ and $g(\zeta)$ by

$$\{\, f(\zeta), g(\zeta) \,\}_{PB}\equiv\omega_\zeta(df(\zeta),dg(\zeta)) \,. \tag{3.20}$$

If $f(\zeta)$ and $g(\zeta)$ are defined in a neighborhood of Γ, then one may use the component form

$$\{\, f(\zeta), g(\zeta) \,\}_{PB}=\frac{\partial f(\zeta)}{\partial\zeta_i}\frac{\partial g(\zeta)}{\partial\zeta_j}c_{ij}^k\zeta_k \,. \tag{3.21}$$

One may easily verify that this definition satisfies the Jacobi identity and therefore yields a consistent definition of the Poisson bracket.

We may now define classical dynamics on the coadjoint orbit Γ. The classical Hamiltonian, $h_{cl}(\zeta)$, is some given function on Γ. The classical equations of motion are simply

$$\frac{d}{dt}f(\zeta)=\{\, h_{cl}(\zeta), f(\zeta) \,\}_{PB} \tag{3.22}$$

for any function $f(\zeta)$ defined on Γ. These equations of motion may be derived from the classical action,

$$S_{cl}[\zeta(t)]=\int dt\{\, -\langle\dot{\zeta}(t),\vartheta_\zeta\rangle-h_{cl}(\zeta(t)) \,\} \,. \tag{3.23}$$

So far, we have not bothered to introduce independent coordinates on Γ. In practice it will be useful to do so. Let $\{z_\alpha\}$ be an arbitrary set of coordinates on Γ, defined as explicit functions of the natural coordinates in $\mathbf{g}^*$, $z_\alpha\equiv z_\alpha(\zeta_i)$. The components of an arbitrary vector $\xi\in T_\zeta(\Gamma)$, in the coordinate basis defined by $\{z_\alpha\}$, are $\xi_\alpha\equiv(\partial z_\alpha/\partial\zeta_i)\zeta_i$. Similarly, an arbitrary one-form, $\sigma\in T^*_\zeta(\Gamma)$, may be expressed as $\sigma=\sigma^\alpha dz_\alpha$. The components of the bivector ω_ζ are

$$\omega_{\alpha\beta}=\frac{\partial z_\alpha}{\partial\zeta_i}\frac{\partial z_\beta}{\partial\zeta_j}c_{ij}^k\zeta_k(z) \,,$$

and the Poisson bracket becomes

$$\{\, f(z), g(z) \,\}_{PB}=\frac{\partial f}{\partial z_\alpha}\frac{\partial g}{\partial z_\beta}\omega^{\alpha\beta} \,. \tag{3.24}$$

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[13] This fact has been extensively used in the theory of group representations (Kirillov, 1962, 1976; Auslander and Kostant, 1967). It also underlies the method of "geometric quantization" developed by Kostant (1970) and Souriau (1970).

[14] This shows that coadjoint orbits are always even dimensional.

[15] ϑ_ζ need not be defined globally. This does not affect our applications. Note that ϑ_ζ is defined only up to the addition of an arbitrary gradient $df(\zeta)$.

The two-form $\tilde{\omega}_\zeta$ equals $\frac{1}{2}\tilde{\omega}^{\alpha\beta}(z)dz_\alpha \wedge dz_\beta$, where $(\tilde{\omega}^{\alpha\beta}) = (\omega_{\alpha\beta})^{-1}$. (Hence, $\tilde{\omega}^{\alpha\beta}\omega_{\beta\gamma} = \delta^\alpha_\gamma$.) The closure of $\tilde{\omega}_\zeta$ reads

$$\tilde{\omega}^{[\alpha\beta,\gamma]} \equiv \frac{1}{3}\left\{\frac{\partial\tilde{\omega}^{\alpha\beta}}{\partial z_\gamma} + \frac{\partial\tilde{\omega}^{\beta\gamma}}{\partial z_\alpha} + \frac{\partial\tilde{\omega}^{\gamma\alpha}}{\partial z_\beta}\right\} = 0 .$$

Consequently,

$$\tilde{\omega}^{\alpha\beta} = \frac{\partial\vartheta^\beta}{\partial z_\alpha} - \frac{\partial\vartheta^\alpha}{\partial z_\beta}$$

for some one-form $\vartheta(z) = \vartheta^\alpha(z)\,dz_\alpha$. The classical action becomes

$$S_{cl}[z(t)] = \int dt\{ -z_\alpha(t)\,\vartheta^\alpha(z(t)) - h_{cl}(z(t))\} . \tag{3.25}$$

This construction of classical mechanics on coadjoint orbits may be illustrated by considering the Heisenberg group. The Lie algebra $\mathbf{g}$ is three dimensional, and we may choose a basis $\{e_i\}$ where the only nonzero structure constants are $c^3_{12} = -c^3_{21} = 1$ (e_1, e_2, and e_3 may be represented by $i\hat{p}/\hbar$, $i\hat{x}/\hbar$, and $i\hat{1}/\hbar$, respectively). The action of the adjoint representation is given by

$$Ad[u(p,q,\alpha)](\lambda^i e_i) = \bar{\lambda}^i e_i ,$$

where $\bar{\lambda}^1 = \lambda^1$, $\bar{\lambda}^2 = \lambda^2$, and $\bar{\lambda}^3 = \lambda^3 - p\lambda^1 - q\lambda^2$. Similarly, the action of the coadjoint representation is

$$Ad^*[u(p,q,\alpha)](\zeta_i e'^i) = \bar{\zeta}_i e'^i ,$$

where $\bar{\zeta}_i = \langle \zeta, Ad[u^{-1}](e_i)\rangle$, or, explicitly, $\bar{\zeta}_1 = \zeta_1 + p\zeta_3$, $\bar{\zeta}_2 = \zeta_2 + q\zeta_3$, and $\bar{\zeta}_3 = \zeta_3$. Therefore coadjoint orbits are simply two-dimensional planes specified by $\zeta_3 =$ constant (provided ζ_3 is nonzero; the $\zeta_3 = 0$ orbits are single points—we ignore this dull possibility in the following discussion.) Since the orbits are flat, all tangent spaces equal to the $\zeta_3 = 0$ subspace, $T_\zeta(\Gamma) = \{\zeta_1 e'^1 + \zeta_2 e'^2 \mid \zeta_1, \zeta_2 \in \mathscr{R}\}$. The cotangent spaces are given by $T^*_\zeta(\Gamma) = \mathbf{g}/e_3$, since $\langle \xi, e_3\rangle = 0$ for all $\zeta \in T_\zeta(\Gamma)$. Let us relabel ζ_1 as p and ζ_2 as q; (p,q) obviously provide natural coordinates on a given orbit Γ. The bivector ω_ζ is given by $\omega_\zeta([\zeta],[\mu]) = (\lambda^1\mu^2 - \lambda^2\mu^1)\zeta_3$, its inverse is $\tilde{\omega}_\zeta = \zeta_3^{-1}dq \wedge dp$, and $\vartheta = -\zeta_3^{-1}p\,dq$ satisfies $d\vartheta = \tilde{\omega}_\zeta$. Therefore (3.21) yields the standard Poisson bracket,

$$\{f(p,q), g(p,q)\}_{PB} = \zeta_3\left[\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}\right] , \tag{3.26}$$

up to a constant overall factor of ζ_3. Finally, the classical action (3.23), is simply

$$S_{cl}[p(t),q(t)]$$

$$= \int dt\{\zeta_3^{-1}p(t)\dot{q}(t) - h_{cl}(p(t),q(t))\} . \tag{3.27}$$

This shows that classical mechanics on the coadjoint orbits of the Heisenberg group essentially reproduces the standard classical dynamics of a point particle.

We must now show what this discussion of coadjoint orbits has to do with the original quantum theory we wished to study. The basic connection is provided by the following observation. For any $\lambda \in \mathbf{g}$, consider $\lim_{\chi\to 0}(-i)\langle 0|\chi\hat{\Lambda}|0\rangle_\chi$. (Since $\chi\hat{\Lambda}$ is a classical operator, the limit exists.) This expectation is a linear functional of λ and therefore equals $\langle \zeta_o, \lambda\rangle$ for some element $\zeta_o \in \mathbf{g}^*$ of the dual space. Arbitrary coherent state expectations are given by

$$\lim_{\chi\to 0}(\chi/i)\Lambda_\chi(u) = \lim_{\chi\to 0}(\chi/i)\langle 0 \mid \hat{U}^{-1}\hat{\Lambda}\hat{U} \mid 0\rangle$$

$$= \langle \zeta_o, Ad[u^{-1}](\lambda)\rangle$$

$$= \langle Ad^*[u](\zeta_o), \lambda\rangle . \tag{3.28}$$

Therefore each coherent state, $|u\rangle$, may be associated with some point, $Ad^*[u](\zeta_o)$, on a single coadjoint orbit $\Gamma \in \mathbf{g}^*$. Since expectation values of $\{\chi\hat{\Lambda}\}$ distinguish classically inequivalent states, the coherent states which are mapped onto any given point $\zeta \in \Gamma$ are all classically equivalent. In other words, each point on the coadjoint orbit may be regarded as uniquely labeling a set of classically equivalent coherent states.

We will now show that for any classical operators, $\hat{A}, \hat{B} \in \mathbf{K}$, the following statements hold for all $u \in g$:

$$\lim_{\chi\to 0} A_\chi(u) = a(\zeta) , \tag{3.29}$$

$$\lim_{\chi\to 0}(AB)_\chi(u) = a(\zeta)b(\zeta) , \tag{3.30}$$

$$\lim_{\chi\to 0}\frac{i}{\chi}[A,B]_\chi(u) = \{a(\zeta), b(\zeta)\}_{PB} . \tag{3.31}$$

Here $\zeta = Ad^*[u](\zeta_o) \in \Gamma$.

The first relation (3.29) simply expresses the fact that in the $\chi \to 0$ limit, the symbol of a classical operator may be regarded as a function on the coadjoint orbit Γ. This is just a restatement of the fact that points on Γ label equivalence classes of coherent states.

The second relation (3.30) is a restatement of factorization (3.12) and has already been established.

The last relation (3.31) could be verified by computing the subleading terms in (3.11). Fortunately, we may avoid this tedious computation by arguing as follows. First, let $\hat{A} = (\chi/i)\hat{\Lambda}$ and $\hat{B} = (\chi/i)\hat{\Lambda}'$ be arbitrary elements of the Lie algebra (times χ/i). Then $a(\zeta) = \langle \zeta, \lambda\rangle$, $b(\zeta) = \langle \zeta, \lambda'\rangle$ and, from (3.21), $\{a(\zeta), b(\zeta)\}_{PB} = \langle \zeta, [\lambda, \lambda']\rangle$. To evaluate the left-hand side of (3.31), note that $(i/\chi)[\hat{A}, \hat{B}] = (\chi/i)[\hat{\Lambda}, \hat{\Lambda}']$ is again an element of $\mathbf{G}_\chi$ times (χ/i). Therefore $\lim_{\chi\to 0}(i/\chi)[A,B]_\chi(u)$ also equals $\langle \zeta, [\lambda, \lambda']\rangle$. Hence the set of operators $\{(\chi/i)\hat{\Lambda}\}$ satisfies (3.31). Next, let $\hat{A} = \exp(\chi\hat{\Lambda})$ and $\hat{B} = \exp(\chi\hat{\Lambda}')$. Then from equation (3.9) we find $a(\zeta) = \exp i\langle \zeta, \lambda\rangle$, $b(\zeta) = \exp i\langle \zeta, \lambda'\rangle$, and

$$\{a(\zeta), b(\zeta)\}_{PB} = \langle \zeta, [\lambda', \lambda]\rangle \exp i\langle \zeta, \lambda + \lambda'\rangle .$$

We may now evaluate $(i/\chi)[\hat{A}, \hat{B}]$, dropping all terms whose expectation values vanish as $\chi \to 0$. Using factorization (3.30), one finds

$$\frac{i}{\chi}[A,B]_\chi(u)=-i\chi([\Lambda,\Lambda']\exp\chi(\Lambda+\Lambda'))_\chi(u)+o(1)$$

$$=\langle\zeta,[\lambda',\lambda]\rangle\exp i\langle\zeta,\lambda+\lambda'\rangle+o(1)$$

in agreement with (3.31). The representation (3.3) may then be used to extend this result to all classical operators. This proves the stated relation.

Finally, note that Assumption 4 plus equation (3.31) implies that the quantum equations of motion, $\partial\hat{A}/\partial t=i[\hat{H},\hat{A}]$, reduce to the classical Hamilton equations, $a(\zeta)=\{h_{cl}(\zeta),a(\zeta)\}_{PB}$, where the classical Hamiltonian is given by

$$h_{cl}(\zeta)=\lim_{\chi\to 0}\chi H_\chi(u)\ . \tag{3.32}$$

These results show that in the $\chi\to 0$ limit, the complete quantum theory reduces to classical mechanics on the coadjoint orbit Γ. We emphasize that all information about the $\chi\to 0$ theory is contained in the classical action $S_{cl}[\zeta(t)]$, (3.23). For example, the limiting behavior of the ground state energy is given by $E_0\sim(1/\chi)\varepsilon_0$, where ε_0 is the minimum of the classical Hamiltonian $h_{cl}(\zeta)$ over the coadjoint orbit Γ. Expanding the classical action about this minimum and diagonalizing the quadratic terms will yield the set of small oscillation frequencies, $\{\omega_i\}$. These frequencies give the $\chi\to 0$ limit of the spectrum, in the sense that the excitation energy to a particular excited state is given by $\Delta E\sim\sum_i n_i\omega_i$, for some set of non-negative integers $\{n_i\}$. Furthermore, the limiting behavior of the connected part of any correlation function of time-ordered products of classical operators may be computed from tree diagrams generated by $(1/\chi)S_{cl}[\zeta(t)]$. [This implies that the connected part of the vacuum expectation of any product of n classical operators vanishes as $(\chi)^{n-1}$.] Finally, the behavior of $S_{cl}[\zeta(t)]$ away from the minimum determines the dynamics of highly excited [$\Delta E\sim O(1/\chi)$] collective excitations. These statements may be easily derived using functional integrals based on the coherent state completeness relation (3.2). This is discussed in further detail in the appendix.

Naturally, the above results reproduce the expected $\hbar\to 0$ behavior. The particular coadjoint orbit which emerges in that case is the one with

$$\zeta_3=\frac{\hbar}{i}\langle 0|\frac{i}{\hbar}\hat{1}|0\rangle=1\ .$$

On this orbit (3.27) exactly reproduces the standard classical dynamics. The ground state corresponds to the minimum of the Hamiltonian; expanding the classical action about this point yields free propagators and bare vertices, from which one can construct tree diagrams giving the leading $\hbar\to 0$ behavior of any observable.

In the next few sections this formalism will be used to study the $N\to\infty$ limit of various theories. In every case the procedure will be the same. One must

(i) Choose the coherence group g and the base state $|0\rangle_\chi$, and construct the coherent states $\{|u\rangle\}$.

(ii) Verify Assumptions 1—4 thereby deriving the classical phase space.

(iii) Compute the classical Hamiltonian $h_{cl}(\zeta)$ and minimize it, thereby "solving" the $N=\infty$ theory.

IV. VECTOR MODELS

In this section we shall examine the large N limit of O(N) vector models.[16] These are O(N) invariant theories whose fundamental degrees of freedom form O(N) vectors. For convenience, we will look at only linear bosonic models [i.e., models where the O(N) symmetry is realized linearly]; nonlinear models may always be reached as limits of linear models. Identical methods may also be applied to O(N) fermionic theories; see Berezin (1978) and Papanicolaou (1981) for explicit discussions using similar methods.

The basic operators in this set of theories are the position, $\{\hat{x}_i(\alpha)\}$, and conjugate momentum, $\{\hat{p}_i(\alpha)\}$, operators normalized so that

$$i[\hat{p}_i(\alpha),\hat{x}_j(\beta)]=\frac{1}{N}\delta_{ij}\delta_{\alpha\beta},\ i,j=1,\ldots,N$$
$$\alpha,\beta=1,\ldots,n\ . \tag{4.1}$$

Here i and j are O(N) vector indices, α and β label the different O(N) vectors, and n is the total number of vectors. [I have included an unconventional factor of $1/\sqrt{N}$ in the definition of $\hat{x}_i(\alpha)$ and $\hat{p}_i(\alpha)$. This choice will allow us to avoid explicit rescalings of the coupling constants in the Hamiltonian as $N\to\infty$.]

The Hamiltonian is assumed to be O(N) invariant. Consequently we may completely restrict our attention to the O(N) invariant sector of the theory. The Hilbert space $\mathbf{H}_N$ is the space of O(N) invariant wave functions, and all physical operators may be constructed from the following basic invariants:

$$\hat{A}(\alpha,\beta)\equiv\tfrac{1}{2}\sum_{i=1}^{N}[\hat{x}_i(\alpha)\hat{x}_i(\beta)]\ , \tag{4.2a}$$

$$\hat{B}(\alpha,\beta)\equiv\tfrac{1}{2}\sum_{i=1}^{N}[\hat{x}_i(\alpha)\hat{p}_i(\beta)+\hat{p}_i(\beta)\hat{x}_i(\alpha)]\ , \tag{4.2b}$$

$$\hat{C}(\alpha,\beta)\equiv\tfrac{1}{2}\sum_{i=1}^{N}[\hat{p}_i(\alpha)\hat{p}_i(\beta)]\ . \tag{4.2c}$$

The Hamiltonian $\hat{H}_N$ will be taken to be N times an arbitrary polynomial in $\hat{A}(\alpha,\beta)$, $\hat{B}(\alpha,\beta)$, and $\hat{C}(\alpha,\beta)$ with no explicit N dependence,

$$\hat{H}_N\equiv N\,h[\hat{A}(\alpha,\beta),\hat{B}(\alpha,\beta),\hat{C}(\alpha,\beta)]\ . \tag{4.3}$$

To apply the previous formalism we choose $\chi=1/N$. The coherence group G_N is defined as the group generated by the operators $\hat{A}(\alpha,\beta)$ and $\hat{B}(\alpha,\beta)$. In other words, the Lie algebra $\mathbf{G}_N$ is given by

[16] All results may be easily extended to U(N) or Sp(N) vector models.

$$\mathbf{G}_N=\{\,\hat{\Lambda}(a,b)\equiv iN\sum_{\alpha,\beta}[a(\alpha,\beta)\hat{A}(\beta,\alpha)+b(\alpha,\beta)\hat{B}(\beta,\alpha)]\,\}\ . \tag{4.4}$$

$a=||a(\alpha,\beta)||$ is an arbitrary n-dimensional real symmetric matrix, and $b=||b(\alpha,\beta)||$ is an arbitrary $n\times n$ real matrix. These operators satisfy the commutation relations

$$[\hat{\Lambda}(a_1,b_1),\hat{\Lambda}(a_2,b_2)]=\hat{\Lambda}(a_{12},b_{12})\ , \tag{4.5}$$

where $a_{12}=(a_2b_1-a_1b_2)+(b_1^t a_2-b_2^t a_1)$ and $b_{12}=(b_2b_1-b_1b_2)$. (b^t is the transpose of b.) $\mathbf{G}_N$ may be regarded as a representation of the $2n$-dimensional matrix algebra,

$$\mathbf{g}=\{\lambda(a,b)\equiv\begin{bmatrix}-b & 0\\ a & b^t\end{bmatrix}\,|\,a=a^t\}\ , \tag{4.6}$$

since $\lambda(a,b)\in\mathbf{g}$ and $\hat{\Lambda}(a,b)\in\mathbf{G}_N$ obey identical commutation relations. Note that

$$\hat{\Lambda}(a,b)=iN\tfrac{1}{2}(-\hat{p},\hat{x})\,\lambda(a,b)\begin{bmatrix}\hat{x}\\ \hat{p}\end{bmatrix}\ .$$

The algebra $\mathbf{g}$ generates the group g given by

$$g=\{\,u(\psi,\phi)\equiv\begin{bmatrix}\phi^{-1} & 0\\ \psi\phi^{-1} & \phi^t\end{bmatrix}\,|\,\psi=\psi^t\,\}\ . \tag{4.7}$$

Note that $u(\psi,\phi)^{-1}=u(-(\phi^t)^{-1}\psi\phi^{-1},\phi^{-1})$. If $u(\psi,\phi)=\exp\lambda(a,b)$, then $\phi=\exp b$, and $\psi=\int_0^1 d\tau\times e^{\tau b^t}ae^{\tau b}$. The algebra G_N generates a group of unitary transformations, $G_N=\{\hat{U}(\psi,\phi)\equiv\exp\hat{\Lambda}(a,b)\}$, which provides a faithful representation of g.

Elements of G_N act on $\hat{x}$ and $\hat{p}$ as

$$[\hat{\Lambda}(a,b),\hat{x}_i(\alpha)]=\sum_\beta b(\alpha,\beta)\hat{x}_i(\beta)\ , \tag{4.8a}$$

and

$$[\hat{\Lambda}(a,b),\hat{p}_i(\alpha)]=-\sum_\beta[b(\beta,\alpha)\hat{p}_i(\beta)+a(\alpha,\beta)\hat{x}_i(\beta)]\ , \tag{4.8b}$$

or, equivalently,

$$[\hat{\Lambda},\begin{bmatrix}\hat{x}\\ \hat{p}\end{bmatrix}]=-\lambda\begin{bmatrix}\hat{x}\\ \hat{p}\end{bmatrix}\ .$$

Finite transformations, $\hat{U}\in G_N$, act as follows:

$$\hat{U}\begin{bmatrix}\hat{x}\\ \hat{p}\end{bmatrix}\hat{U}^{-1}=u^{-1}\begin{bmatrix}\hat{x}\\ \hat{p}\end{bmatrix}\ ,$$

or

$$\hat{U}(-\hat{p},\hat{x})\hat{U}^{-1}=(-\hat{p},\hat{x})\,u\ . \tag{4.9}$$

This shows that the momenta transform into linear combinations of momentum and position vectors, but position vectors only mix among themselves. The motivation behind this choice for the coherence group will be discussed in Sec. VII.

We will choose the base state, $|0\rangle_N$, to be the standard Gaussian, given by

$$\Psi_0(x)\equiv\langle x\,|\,0\rangle=C\exp-\tfrac{1}{2}N\sum_{\alpha=1}^{n}x_i(\alpha)x_i(\alpha) \tag{4.10}$$

[$C\equiv(\pi/N)^{-nN/4}$]. Note that $(\hat{p}-i\hat{x})|0\rangle=0$. The coherent states $|u\rangle\equiv\hat{U}\,|\,0\rangle$ satisfy $\hat{U}(\hat{p}-i\hat{x})\hat{U}^{-1}|u\rangle=\{(\phi^t)^{-1}(\hat{p}-\psi\hat{x})-i\phi\hat{x}\}|u\rangle=0$. Therefore the coherent state wave functions are given by

$$\Psi_u(x)\equiv\langle x\,|\,u\rangle=C(z)\exp-\tfrac{1}{2}N\sum_{\alpha,\beta}x_i(\alpha)z(\alpha,\beta)x_i(\beta)\ , \tag{4.11}$$

where $C(z)=\det[N(z+\bar{z})/2\pi]^{N/4}$ and

$$z=||z(\alpha,\beta)||\equiv\phi^t\phi-i\psi\ . \tag{4.12}$$

z is a complex symmetric matrix which may be used uniquely to label the coherent states. Under the action of the group, $|z\rangle\to\hat{U}|z\rangle=|ad^*[u](z)\rangle$, where

$$ad^*[u(\phi,\psi)](z)\equiv\phi^t z\phi-i\psi\ . \tag{4.13}$$

For future convenience, we define $w\equiv(z+\bar{z})^{-1}$, and $v\equiv i(z-\bar{z})/2$; v is an arbitrary real symmetric matrix, while w must also be positive definite.

Since the coherent states are all Gaussian, computing their overlaps is trivial. One finds

$$\langle u_1\,|\,u_2\rangle=\exp\tfrac{1}{4}N\,\mathrm{tr}\,[\ln(\bar{z}_1+z_1)+\ln(\bar{z}_2+z_2)-2\ln(\bar{z}_1+z_2)]\ . \tag{4.14}$$

Finally, note that $\hat{A}$, $\hat{B}$, and $\hat{C}$ are all classical operators. Let us compute their symbols. Consider the $2n\times 2n$ matrix,

$$\hat{J}=\sum_{i=1}^{N}\begin{bmatrix}\hat{x}_i(\alpha)\\ \hat{p}_i(\alpha)\end{bmatrix}\otimes[-\hat{p}_i(\beta),\hat{x}_i(\beta)]+\frac{i}{2}\begin{bmatrix}\delta_{\alpha\beta} & 0\\ 0 & \delta_{\alpha\beta}\end{bmatrix}\cdot$$
$$=\begin{bmatrix}-\hat{B} & 2\hat{A}\\ -2\hat{C} & \hat{B}^t\end{bmatrix}\cdot$$

The symbol of $\hat{J}$ is given by

$$J(u)=\langle 0\,|\,\hat{U}^{-1}\begin{bmatrix}\hat{x}\\ \hat{p}\end{bmatrix}\otimes(-\hat{p},\hat{x})U\,|\,0\rangle+\frac{i}{2}\mathbb{1}$$
$$=\langle 0\,|\,u\begin{bmatrix}\hat{x}\\ \hat{p}\end{bmatrix}\otimes(-\hat{p},\hat{x})u^{-1}\,|\,0\rangle+\frac{i}{2}\mathbb{1}$$
$$=\tfrac{1}{2}uIu^{-1}\ ,$$

where

$$I=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\cdot$$

Using the definitions of $u(\phi,\psi)$, $w=\frac{1}{2}(\phi^t\phi)^{-1}$, and $v=\psi$, we find

$$A(u)=\tfrac{1}{2}w\ , \tag{4.15a}$$

$$B(u)=v\,w\ , \tag{4.15b}$$

and

$$C(u)=\tfrac{1}{2}v\,w\,v+\tfrac{1}{8}w^{-1}\ . \tag{4.15c}$$

These results show that one may reconstruct $z=\frac{1}{2}w^{-1}-iv$ from the expectations of $\hat{A}$ and $\hat{B}$ in a given coherent state. Since z uniquely labels the coherent states, this shows that classically equivalent states must in fact be equal.

The preliminaries are all we will need to derive the $N\to\infty$ limit. To verify Assumption 1 (irreducibility of G_N), we first note that the only O(N) invariant operators which commute with the set of transformations $\{\hat{U}=\exp iN\,\mathrm{tr}(a\hat{A})\in G_N\}$ are those solely constructed from the basic operators $\{\hat{A}(\alpha,\beta)\}$. However, the set of transformations, $\{\hat{U}=\exp iN\,\mathrm{tr}(b\hat{B})\in G_N\}$ dilate and rotate the set of vectors $\{\hat{x}(\alpha)\}$ by arbitrary amounts. Therefore only constant operators commute with the full group G_N. This proves Assumption 1.

Assumption 2 ($Z(u)=0$ implies $\hat{Z}=0$) may be verified either by using analyticity in $z=\phi^t\phi-i\psi$ or by a direct argument analogous to the discussion following (3.5). Suppose there exists some operator $\hat{Z}$ whose symbol $Z(u)$ is identically zero. Then expectations in the base state of arbitrary multiple commutators of generators of G_N with $\hat{Z}$ vanish, that is, Eq. (3.5) holds. By taking linear combinations we may choose each generator to be (any component of)

$$\hat{L}\equiv\hat{B}-2i\hat{A}=\hat{x}\otimes(\hat{p}-i\hat{x})-i/2\ ,$$

or $\hat{L}^\dagger$. Using the commutation relations plus the fact that $\hat{L}\,|0\rangle=(1/2i)\mathbb{1}|0\rangle$, one may show by a simple induction that

$$\langle 0|\hat{L}\cdots\hat{L}\hat{Z}\hat{L}^\dagger\cdots\hat{L}^\dagger|0\rangle=0$$

for any number of $\hat{L}$'s or $\hat{L}^\dagger$'s. However, polynomials in $\hat{L}^\dagger$ applied to the base state $|0\rangle$ clearly form a dense set of O(N) invariant states. Therefore $\hat{Z}$ equals zero.

Assumption 3 (coherent states orthogonal in the $\chi\to 0$ limit) follows from examining Eq. (4.14). $\phi(u_1,u_2)\equiv-\lim_{N\to\infty}(1/N)\ln\langle u_1\,|\,u_2\rangle$ is given by

$$\phi(u_1,u_2)=-\tfrac{1}{4}\mathrm{tr}\,[\ln(\bar{z}_1+z_1)+\ln(\bar{z}_2+z_2)-2\ln(\bar{z}_1+z_2)]$$

and satisfies $\mathrm{Re}\,\phi(u_1,u_2)>0$ if and only if $z_1\neq z_2$.

Last, any N-independent polynomial in $\hat{A}$, $\hat{B}$, and $\hat{C}$ is clearly a classical operator. Therefore any Hamiltonian of the form (4.3) satisfies assumption 4 ($\chi\hat{H}\chi$ classical).

Thus, all the assumptions of the general formalism are satisfied, and consequently in this set of models the $N\to\infty$ limit is a classical limit.

The classical phase space is a coadjoint orbit of the coherence group g. The orbit, Γ, is specified by (3.28), which yields

$$\langle\zeta(z),\lambda\rangle\equiv\lim_{N\to\infty}\frac{i}{N}\Lambda_N(u)$$

$$=\tfrac{1}{2}\mathrm{tr}\,(w\,a+2w\,v\,b)\ , \tag{4.16}$$

where $\hat{\Lambda}=\hat{\Lambda}(a,b)\in G_N$, $u=u(\psi,\phi)\in g$, and $z=\frac{1}{2}w^{-1}-iv=\phi^t\phi-i\psi$. The matrices v and w provide convenient independent coordinates on Γ. Equation (4.13) shows how these coordinates transform under the action of the coadjoint representation. The gradient of any function $f(v,w)$ on Γ may be given by[17]

$$\boldsymbol{d}f=\lambda(f_{,w}-\tfrac{1}{2}(vf_{,v}w^{-1}+w^{-1}f_{,v}v),\ \tfrac{1}{2}f_{,v}w^{-1})\ . \tag{4.17}$$

[We define the derivative with respect to any symmetric matrix s by

$$(f_{,s})_{\alpha\beta}=[\partial f/\partial s(\alpha,\beta)\text{ if }\alpha\le\beta]+[\partial f/\partial s(\beta,\alpha)\text{ if }\beta\le\alpha]$$

Therefore

$$\tfrac{1}{2}\mathrm{tr}(\delta s\,f_{,s})=\sum_{\alpha\le\beta}\delta s(\alpha,\beta)[\partial f/\partial s(\alpha,\beta)].$$

If $z\to z+\delta z$, then $f(z)\to f(z)+\delta f(z)$, where

$$\delta f(z)=\tfrac{1}{2}\mathrm{tr}\,(\delta v f_{,v}+\delta w f_{,w})=\langle\delta\zeta(z),\boldsymbol{d}f\rangle\ .$$

The Poisson bracket may now be computed from the definition (3.20). One finds that v and w are naturally canonically conjugate,

$$\{f(z),g(z)\}_{PB}=\tfrac{1}{2}\mathrm{tr}\,(f_{,v}g_{,w}-f_{,w}g_{,v})\ . \tag{4.18}$$

The classical Hamiltonian follows from Eqs. (3.32) and (4.3). It is given by

$$h_{cl}(v,w)=h(\tfrac{1}{2}v,vw,\tfrac{1}{2}vwv+\tfrac{1}{8}w^{-1})\ . \tag{4.19}$$

Finally, the classical action (3.23) becomes

$$S_{cl}[v(t),w(t)]=\int dt\{\tfrac{1}{2}\mathrm{tr}[v(t)w(t)-h_{cl}(v(t),w(t))]\}\ . \tag{4.20}$$

This contains the complete physics of the theory in the $N\to\infty$ limit.

Let us apply these formulas to two simple examples. First, consider a theory describing a single-point particle moving in an N-dimensional spherically symmetric potential. The quantum Hamiltonian is $\hat{H}=N[\frac{1}{2}\hat{p}^2+V(\hat{x}^2)]$. [Remember that $\hat{p}$ and $\hat{x}$ have been scaled by $1/\sqrt{N}$. In terms of unscaled operators, $\hat{H}=\frac{1}{2}\hat{p}^2+NV(\hat{x}^2/N)$. This shows that the coefficient of a term $(\hat{x}^2)^n$ in the potential must be scaled by $N^{-(n-1)}$ in order to obtain a smooth large N limit.] The classical phase space has just two coordinates, (v,w) and the classical Hamiltonian describing the $N\to\infty$ limit is simply

[17] $\boldsymbol{d}f(z)$ is an element of the cotangent space $T_z^*(\Gamma)$, which consists of equivalence classes of elements of the Lie algebra $\mathbf{g}$. Equation (4.17) gives one representative of the appropriate equivalence class.

$$h_{cl}(v,w)=\tfrac{1}{2}v^2w+\tfrac{1}{8}w^{-1}+V(w)\ .$$

If we relabel w as r^2 and v as p/r, then $\{p,r\}_{PB}=1$, and the Hamiltonian becomes

$$h_{cl}(p,r)=\tfrac{1}{2}p^2+\tfrac{1}{8}r^{-2}+V(r^2)\ . \tag{4.21}$$

This is just the classical Hamiltonian of a particle with angular momentum $L^2=\frac{1}{4}$. The minimum of the Hamiltonian is at (p_0,r_0), where $p_0=0$ and $8r_0^4V'(r_0^2)=1$. The ground-state energy is $E_0=N\varepsilon_0+O(1)$, where $\varepsilon_0\equiv h_{cl}(p_0,r_0)=r_0^2V'(r_0^2)+V(r_0^2)$. Expanding the classical action (4.20) around the point (p_0,r_0) yields

$$S_{cl}=\int dt[-\varepsilon_0+p\delta-\tfrac{1}{2}(p^2+\omega_s^2\delta^2)+O(\delta^3,p^2\delta)]\ ,$$

where $\delta\equiv r-r_0$ and $\omega_s^2=8V'(r_0^2)+4r_0^2V''(r_0^2)$. ω_s is the energy gap to the lowest O(N)-invariant excited state. This is reflected, for example, in the ground-state expectation

$$\langle[\hat{x}(t)^2,\hat{x}(0)^2]\rangle=-\frac{4i}{N}\frac{r_0^2}{\omega_s}\sin\omega_s t+O(1/N^2)\ ,$$

which reveals the presence of an intermediate state with energy ω_s above the ground state. One may also probe the O(N) noninvariant spectrum, even though we have restricted the Hilbert space to be O(N) invariant. For example, consider the vacuum expectation of the operator $\hat{x}(t)\cdot\hat{x}(0)$,

$$G(t)\equiv\lim_{N\to\infty}\langle\hat{x}(t)\cdot\hat{x}(0)\rangle\ .$$

Using the quantum equation of motion, $\ddot{\hat{x}}+2V'(\hat{x}(t)^2)\hat{x}(t)=0$, plus factorization, one sees that $\ddot{G}(t)+\omega_v^2G(t)=0$, where $\omega_v^2=2V'(r_v^2)$. Since $G(0)=1/(2\omega_v)$, and $\dot{G}(0)=-i/2$, one finds $G(t)=(1/2\omega_v)\times\exp-i\omega_v t$. This reveals the presence of an O(N)-vector multiplet of excited states with energy ω_v above the ground state. The large N limit of any other observable may be computed in an analogous manner.

Finally, consider an N-component ϕ^4 field theory in d dimensions. The quantum Hamiltonian is

$$\hat{H}=N\int d^dx\{\tfrac{1}{2}\hat{\pi}(x)^2+\tfrac{1}{2}[\nabla\hat{\phi}(x)]^2+\tfrac{1}{2}\mu^2\hat{\phi}(x)^2+\tfrac{1}{4}\lambda(\hat{\phi}(x)^2)^2\}\ ,$$

and

$$i[\hat{\pi}_i(x),\hat{\phi}_j(x')]=\frac{1}{N}\delta_{ij}\delta^d(x-x')\ .$$

One may immediately apply the previous formula to find the $N\to\infty$ limit of the theory. The "matrices" $v(x,x')$ and $w(x,x')$ provide natural coordinates on the coadjoint orbit Γ. Note that $w(x,x')$ is simply the equal time expectation $\langle z|\hat{\phi}(x)\cdot\phi(x')|z\rangle$. The classical Hamiltonian which generates the $N=\infty$ dynamics is

$$h_{cl}=\int d^dx\,d^dx'd^dx''[\tfrac{1}{2}v(x,x')w(x',x'')v(x'',x)]+\int d^dx[\tfrac{1}{8}w^{-1}(x,x)+\tfrac{1}{2}(-\nabla_x^2+\mu^2)w(x,x')+\tfrac{1}{4}\lambda(w(x,x))^2]\,|_{x'=x}\ . \tag{4.22}$$

The Hamiltonian is minimized when $v=v_0$ and $w=w_0$, where

$$v_0(x,x')=0\ , \tag{4.23}$$

$$w_0(x,x')=\tfrac{1}{2}\int\frac{d^dk}{(2\pi)^d}(k^2+\mu^2+\lambda\sigma)^{-1/2}\exp ik\cdot(x-x')\ ,$$

and $\sigma\equiv w_0(x,x)$ satisfies the standard gap equation,

$$2\sigma=\int\frac{d^dk}{(2\pi)^d}(k^2+\mu^2+\lambda\sigma)^{-1/2}\ . \tag{4.24}$$

The vacuum expectation

$$\langle\hat{\phi}(x)\cdot\hat{\phi}(x')\rangle=w_0(x,x')\sim\exp-m\,|x-x'|$$

as $|x-x'|\to\infty$, where $m^2=\mu^2+\lambda\sigma$. This shows that the theory contains an O(N)-vector multiplet of physical particles with mass m. Expanding the classical action about (v_0,w_0) and inverting the quadratic terms yields a Green's function which describes the propagation of a pair of physical particles, including their mutual interactions. Higher terms in the expansion describe further multibody interactions. In general, the connected part of the scattering amplitude of $2n$ particles is of order $N^{-(n-1)}$.[18]

V. MATRIX MODELS

The next set of models we consider is matrix models. These are theories where the number of degrees of freedom grows like N^2 as N tends to infinity. Specifically, I shall discuss U(N)-invariant models of Hermitian matrices. All results may be easily extended to, for example, O(N)-, Sp(N)-, or U(N)$\times$U(N)-invariant models which describe real symmetric, Hermitian self-dual quaternionic, or arbitrary complex matrices, respectively.

Consider a theory where the basic degrees of freedom are described by the set of operators, $\{\hat{M}_{ij}^{\alpha}\}$, and their conjugates, $\{\hat{E}_{ij}^{\alpha}\}$, normalized so that

[18] All these results for simple vector models have been previously derived using many different methods. See, for example, Coleman, Jackiw, and Politzer (1974), Cornwall, Jackiw, and Tomboulis (1974), Halpern (1980a), Jevicki and Papanicolaou (1980), and Mlodinow and Papanicolaou (1980).

$$i[\hat{E}^\alpha_{ij},\hat{M}^\beta_{kl}]=\frac{1}{N}\delta_{il}\delta_{jk}\delta^{\alpha\beta}\ ,\quad i,j,k,l=1,\ldots,N \qquad \alpha,\beta=1,\ldots n\ . \tag{5.1}$$

(ij) and (kl) are to be regarded as matrix indices, while α and β simply label the different matrices. We assume that $(\hat{M}^\alpha_{ij})^\dagger=\hat{M}^\alpha_{ji}$ and $(\hat{E}^\alpha_{ij})^\dagger=\hat{E}^\alpha_{ji}$; in other words, $\hat{M}^\alpha=||\hat{M}^\alpha_{ij}||$ and $\hat{E}^\alpha\equiv||\hat{E}^\alpha_{ij}||$ form N-dimensional Hermitian matrices.

The Hamiltonian is assumed to be invariant under the transformation $\hat{M}^\alpha\to V\hat{M}^\alpha V^\dagger$, $\hat{E}^\alpha\to V\hat{E}^\alpha V^\dagger$, for any $V\in$U(N). Hence, we may restrict our attention to the U(N)-invariant sector of the theory. Physical operators must have all the U(N) indices contracted. Typical examples are $\tilde{\mathrm{tr}}(\hat{E}^\alpha)^2$ and $\tilde{\mathrm{tr}}(\hat{M}^{\alpha_1}\hat{M}^{\alpha_2}\cdots\hat{M}^{\alpha_k})$. $\tilde{\mathrm{tr}}$ stands for a normalized trace over U(N)-indices, $\tilde{\mathrm{tr}}\,\hat{Q}\equiv(1/N)\sum_{i=1}^N Q_{ii}$. For any finite sequence of integers between one and n, $\Gamma\equiv\{\alpha_1,\alpha_2,\ldots,\alpha_k\}$, let us define $\hat{M}^\Gamma\equiv\hat{M}^{\alpha_1}\hat{M}^{\alpha_2}\cdots\hat{M}^{\alpha_k}$. Note that $(\hat{M}^\Gamma)^\dagger=\hat{M}^{\bar{\Gamma}}$ where $\bar{\Gamma}\equiv\{\alpha_k,\ldots,\alpha_2,\alpha_1\}$ is the reversed sequence.

The Hamiltonian will be taken to be N^2 times an arbitrary U(N)-invariant polynomial in $\hat{E}^\alpha$ and $\hat{M}^\alpha$ with no explicit N depenence. For example,

$$\hat{H}=N^2\sum_{\alpha=1}^n \tfrac{1}{2}\tilde{\mathrm{tr}}\,[(\hat{E}^\alpha)^2+(\hat{M}^\alpha-\hat{M}^{\alpha-1})^2+\omega^2(\hat{M}^\alpha)^2+g(\hat{M}^\alpha)^4] \tag{5.2}$$

is an acceptable choice which we will use as an explicit example in the following dicussion. ($\tilde{\mathrm{tr}}$ is considered to have no N dependence).

To apply the previous formalism we must find a group of unitary transformations which acts irreducibly. This necessarily requires an infinite dimensional group. We shall choose a group which involves arbitrary products of the matrices $\{\hat{M}^\alpha\}$. Specifically, the Lie algebra $\mathbf{G}_N$ is given by[19]

$$\mathbf{G}_N=\{\,\hat{\Lambda}(a,b)\equiv iN^2\tilde{\mathrm{tr}}(a[\hat{M}]+\tfrac{1}{2}(\hat{E}^\alpha b^\alpha[\hat{M}]+b^\alpha[\hat{M}]\hat{E}^\alpha))\,\}\ , \tag{5.3}$$

where $a[\hat{M}]=\sum_\Gamma a^\Gamma\hat{M}^\Gamma$, $b^\alpha[\hat{M}]=\sum_\Gamma b^{\alpha,\Gamma}\hat{M}^\Gamma$, and $\sum_\Gamma$ indicates for a sum over all sequences, $\sum_\Gamma\equiv\sum_{k=0}^\infty\left[\prod_{i=1}^k\sum_{\alpha_i=1}^n\right]\delta_\Gamma^{\alpha_1\cdots\alpha_k}$. The coefficients $\{a^\Gamma\}$ and $\{b^{\alpha,\Gamma}\}$ must satisfy $a^\Gamma=(a^{\bar{\Gamma}})^*$ and $b^{\alpha,\Gamma}=(b^{\alpha,\bar{\Gamma}})^*$ so that $\alpha[\hat{M}]$ and $b^\alpha[\hat{M}]$ are Hermitian. Furthermore, $\{a^\Gamma\}$ may be chosen to be cyclically symmetric, so that $a^{\Gamma_1\Gamma_2}=a^{\Gamma_2\Gamma_1}$ for all sequences Γ_1,Γ_2. Henceforth, to avoid a proliferation of $\sum_\Gamma$ signs, we will treat Γ like any other index and automatically sum over all sequences whenever repeated Γ's appear. For example, $(\partial a[\hat{M}]_{ij}/\partial\hat{M}^\alpha_{kl})=a^{\Gamma_1\alpha\Gamma_2}(\hat{M}^{\Gamma_1})_{ik}(\hat{M}^{\Gamma_2})_{lj}$. The commutation relations are

$$[\hat{\Lambda}(a_1,b_1),\hat{\Lambda}(a_2,b_2)]=\hat{\Lambda}(a_{12},b_{12})\ , \tag{5.4}$$

where

$$a_{12}[\hat{M}]_{ij}=b_1^\alpha[\hat{M}]_{kl}\frac{\partial a_2[\hat{M}]_{ij}}{\partial\hat{M}^\alpha_{kl}}-b_2^\alpha[\hat{M}]_{kl}\frac{\partial a_1[\hat{M}]_{ij}}{\partial\hat{M}^\alpha_{kl}}\ ,$$

and

$$b_{12}^\alpha[\hat{M}]_{ij}=b_1^\beta[\hat{M}]_{kl}\frac{\partial b_2^\alpha[\hat{M}]_{ij}}{\partial\hat{M}^\beta_{kl}}-b_2^\beta[\hat{M}]_{kl}\frac{\partial b_1^\alpha[\hat{M}]_{ij}}{\partial\hat{M}^\beta_{kl}}\ .$$

Equivalently,

$$a_{12}^\Gamma=(b_1^{\alpha,\Gamma'}a_2^{\Gamma_1\alpha\Gamma_2}-b_2^{\alpha,\Gamma'}a_1^{\Gamma_1\alpha\Gamma_2})\,\delta_{\Gamma'}^{\Gamma_1\Gamma'\Gamma_2}$$

and

$$b_{12}^{\alpha,\Gamma}=(b_1^{\beta,\Gamma'}b_2^{\alpha,\Gamma_1\beta\Gamma_2}-b_2^{\beta,\Gamma'}b_1^{\alpha,\Gamma_1\beta\Gamma_2})\,\delta_{\Gamma'}^{\Gamma_1\Gamma'\Gamma_2}\ .$$

Elements of $\mathbf{G}_N$ act on $\hat{E}^\alpha$ and $\hat{M}^\alpha$ as follows:

$$[\hat{\Lambda}(a,b),\hat{M}^\alpha_{ij}]=b^\alpha[\hat{M}]_{ij} \tag{5.5}$$

$$[\hat{\Lambda}g(a,b),\hat{E}^\alpha_{ij}]=-\tfrac{1}{2}\left[\hat{E}^\beta_{lk}\frac{\partial b^\beta[\hat{M}]_{kl}}{\partial\hat{M}^\alpha_{ji}}+\frac{\partial b^\beta[\hat{M}]_{kl}}{\partial\hat{M}^\alpha_{ji}}\hat{E}^\beta_{lk}\right]-\frac{\partial\,\mathrm{tr}\,(a[\hat{M}])}{\partial\hat{M}^\alpha_{ji}}\ . \tag{5.6}$$

Finite transformations, $\hat{U}(\psi,\phi)\in G_N$, may be labeled by the functionals $\psi[\hat{M}]\equiv\psi^\Gamma\hat{M}^\Gamma$ and $\phi^\alpha[\hat{M}]\equiv\phi^{\alpha,\Gamma}\hat{M}^\Gamma$. If $\hat{U}(\psi,\phi)$ equals $\exp\hat{\Lambda}(a,0)\exp\hat{\Lambda}(0,b)$, then $\psi[M]=a[M]$, and

$$\phi^\alpha[M]=M^\alpha+b^\alpha[M]+\tfrac{1}{2}b^\beta[M]_{ij}(\partial b^\alpha[M]/\partial M^\beta_{ij})+\cdots=\exp\{\,\mathrm{tr}\,(b^\beta[M]\partial/\partial M^\beta)\,\}M^\alpha\ .$$

$\phi^\alpha[M]$ has an inverse,

$$\phi^\alpha_{-1}[M]\equiv\exp\{\,-\mathrm{tr}\,(b^\beta[M]\partial/\partial M^\beta)\,\}M^\alpha\ ,$$

such that $\phi^\alpha_{-1}[\phi[M]]=M^\alpha$. Elements of G_N obey the multiplication rule

$$\hat{U}(\psi_1,\phi_1)\hat{U}(\psi_2,\phi_2)=\hat{U}(\psi_{12},\phi_{12})\ , \tag{5.7}$$

where $\phi^\alpha_{12}[M]=\phi^\alpha_2[\phi_1[M]]$ and $\psi_{12}[M]=\psi_2[\phi_1[M]]+\psi_1[M]$. Consequently,

$$\hat{U}(\psi,\phi)^{-1}=\hat{U}(-\psi[\phi_{-1}],\phi_{-1})\ . \tag{5.8}$$

These transformations act on $\hat{E}^\alpha$ and $\hat{M}^\alpha$ as follows:

$$\hat{U}\hat{M}^\alpha_{ij}\hat{U}^{-1}=\phi^\alpha[\hat{M}]_{ij}\ , \tag{5.9}$$

$$\hat{U}\hat{E}^\alpha_{ij}\hat{U}^{-1}=\left[\frac{\partial\hat{M}^\beta_{kl}}{\partial\phi^\alpha[\hat{M}]_{ji}}\right]\left[\hat{E}^\beta_{lk}-\frac{\partial\,\mathrm{tr}\,\psi[\hat{M}]}{\partial\hat{M}^\beta_{kl}}\right]-\frac{i}{2N}\frac{\partial}{\partial\hat{M}^\beta_{kl}}\left[\frac{\partial\hat{M}^\beta_{kl}}{\partial\phi^\alpha[\hat{M}]_{ji}}\right]$$

$$=\left[\frac{\partial\hat{M}^\beta_{kl}}{\partial\phi^\alpha[\hat{M}]_{ji}}\right]\hat{E}^\beta_{lk}+\frac{\partial}{\partial\phi^\alpha[\hat{M}]_{ji}}(-\mathrm{tr}\,\psi[\hat{M}]+\frac{i}{2N}\ln J[\hat{M}])\ . \tag{5.10}$$

[19] A precise definition of this infinite-dimensional Lie algebra requires more mathematical sophistication than I possess. Following established tradition, we will dispense with excessive mathematical rigor and blindly proceed until confronted by obvious problems. Questions concerning, for example, appropriate growth conditions for the coefficients $\{a^\Gamma\}$ and $\{b^{\alpha,\Gamma}\}$ will simply be ignored. If only a single matrix is present, then a more precise definition may be given. See footnote 22.

Here, $J[M] \equiv \det[\partial\phi^\alpha[M]_{ij}/\partial M^\beta_{kl}]$ is the Jacobian for the change of variables from $\{M^\alpha_{ij}\}$ to $\{\phi^\alpha[M]_{ij}\}$.

Once again, the most convenient choice for the base state, $|0\rangle_N$, is given by a simple Gaussian,

$$\Psi_0[M] \equiv \langle M \mid 0\rangle = C \exp -\{\tfrac{1}{2}N^2 \sum_\alpha \tilde{\mathrm{tr}}\,(M^\alpha)^2\} \quad (5.11)$$

$[C \equiv (\pi/N)^{nN^2/4}]$. The coherent states $|u\rangle \equiv \hat{U}|0\rangle$ satisfy $\hat{U}(\hat{E}^\alpha - i\hat{M}^\alpha)\hat{U}^{-1}|u\rangle = 0$. Using (5.8) and (5.9), one finds that the coherent state wave functions are given by

$$\Psi_u[M] \equiv \langle M \mid u\rangle = C(J[M])^{1/2} \exp -\{N^2 \tilde{\mathrm{tr}}\,(\tfrac{1}{2}\phi^\alpha[M]^2 - i\psi[M])\}\,. \quad (5.12)$$

We shall need to compute the symbols, $\Lambda(a,b)(u)$, of elements of the Lie algebra. Using (5.7) and (5.8), one may evaluate $\hat{U}^\dagger \hat{\Lambda}(a,b)\hat{U}$, and find

$$\langle u \mid \hat{\Lambda}(a,b) \mid u\rangle = \langle 0 \mid \hat{\Lambda}\Big(a[\phi_{-1}] + \mathrm{tr}\left[b^\alpha[\phi_{-1}]\frac{\partial}{\partial\phi^\alpha_{-1}}\right]\psi[\phi_{-1}], \mathrm{tr}\left[b^\alpha[\phi_{-1}]\frac{\partial}{\partial\phi^\alpha_{-1}}\right]\phi^\beta[\phi_{-1}]\Big) \mid 0\rangle$$

$$= iN^2\langle 0 \mid \tilde{\mathrm{tr}}\, a[\phi_{-1}[\hat{M}]] + b^\alpha[\phi_{-1}[\hat{M}]]_{ij}\frac{\partial\,\tilde{\mathrm{tr}}\,\psi[\phi_{-1}[\hat{M}]]}{\partial\phi^\alpha_{-1}[\hat{M}]_j} \mid 0\rangle\,. \quad (5.13)$$

This expresses the symbol $\Lambda(a,b)(u)$ as a Gaussian average of a functional of $\{M^\alpha\}$. Except in the special case of a single matrix (discussed below) it does not appear possible to express the expectation in any more explicit form. Fortunately, the representation above will be sufficient for our purposes. Note that (5.13) implies that classically equivalent states must give identical expectations (as $N\to\infty$) to the operators $\{\tilde{\mathrm{tr}}\,\hat{M}^\Gamma\}$, for all sequences Γ. Since Gaussian expectations factorize,

$$\lim_{N\to\infty}\langle 0 \mid (\tilde{\mathrm{tr}}\,\hat{M}^{\Gamma_1})(\tilde{\mathrm{tr}}\,\hat{M}^{\Gamma_2}) \mid 0\rangle = \lim_{N\to\infty}\langle 0 \mid \tilde{\mathrm{tr}}\,\hat{M}^{\Gamma_1} \mid 0\rangle\langle 0 \mid \tilde{\mathrm{tr}}\,\hat{M}^{\Gamma_2} \mid 0\rangle\,,$$

this implies that expectations of any product of such operators, $(\tilde{\mathrm{tr}}\,\hat{M}^{\Gamma_1})\cdots(\tilde{\mathrm{tr}}\,\hat{M}^{\Gamma_2})$, must also be equal. Furthermore, $\psi[M]$ can at most differ by a constant between any two classically equivalent states.

We may now examine the various assumptions made in Sec. III in order to see if the general formalism described there is applicable to the $N\to\infty$ limit of these models. We set $\chi = 1/N^2$.

Assumption 1 (irreducibility of G_N) follows from an argument analogous to those used previously. Consider the subgroup of G_N which is generated by $\{\hat{\Lambda}(a,0)\in \mathbf{G}_N\}$. Any U($N$)-invariant operator which commutes with this subgroup must be solely constructed from the matrices $\{\hat{M}^\alpha\}$, i.e., must be a multiplication operator in the representation where all $\{\hat{M}^\alpha_{ij}\}$ are diagonal. However, the subgroup of G_N generated by $\{\hat{\Lambda}(0,b)\in \mathbf{G}_N\}$ contains transformations which independently translate each eigenvalue of each matrix M^α, $M^\alpha \to f_\alpha(M^\alpha)$, where each $f_\alpha(z)$ is an arbitrary monotonically increasing function of a single variable. Consequently, operators commuting with G_N cannot depend on the eigenvalues of the matrices $\{M^\alpha\}$. Furthermore, G_N also contains transformations which mix the different matrices, such as $M^\alpha \to c^{\alpha\beta}M^\beta$, where $||c^{\alpha\beta}||$ is an arbitrary invertible matrix. Such transformations completely change the inner products between eigenvectors of the different matrices $\{M^\alpha\}$. Therefore operators commuting with G_N can depend on neither the eigenvalues nor the eigenvectors of the matrices $\{M^\alpha\}$, and so must be constants. This proves Assumption 1.

Unlike our previous examples, simple analyticity arguments are not sufficient to verify Assumption 2 (operators uniquely determined by their symbols). However, an inductive proof similar to those used previously may be constructed. Let

$$\hat{L}^{\alpha,\Gamma}_\pm = N\,\mathrm{tr}\,[\hat{M}^{\alpha\Gamma} \pm \frac{i}{2}(\hat{E}^\alpha\hat{M}^\Gamma + \hat{M}^\Gamma\hat{E}^\alpha)]\,.$$

The following relations may be easily verified:

$$[\hat{L}^{\alpha,\Gamma}_+, \hat{L}^{\beta,\Gamma'}_+] = \delta^{\Gamma_1\alpha\Gamma_2}_{\Gamma'}\hat{L}^{\beta,\Gamma_1\Gamma\Gamma_2}_+ - \delta^{\Gamma_1\beta\Gamma_2}_{\Gamma}\hat{L}^{\alpha,\Gamma_1\Gamma'\Gamma_2}_+\,, \quad (5.14a)$$

$$[\hat{L}^{\alpha,\Gamma}_-, \hat{L}^{\beta,\Gamma'}_-] = -\delta^{\Gamma_1\alpha\Gamma_2}_{\Gamma'}\hat{L}^{\beta,\Gamma_1\Gamma\Gamma_2}_- + \delta^{\Gamma_1\beta\Gamma_2}_{\Gamma}\hat{L}^{\alpha,\Gamma_1\Gamma'\Gamma_2}_-\,, \quad (5.14b)$$

$$[\hat{L}^{\alpha,\Gamma}_+, \hat{L}^{\beta,\Gamma'}_-] = \delta^{\Gamma_1\alpha\Gamma_2}_{\Gamma'}\hat{L}^{\beta,\Gamma_1\Gamma\Gamma_2}_- + \delta^{\Gamma_1\beta\Gamma_2}_{\Gamma}\hat{L}^{\alpha,\Gamma_1\Gamma'\Gamma_2}_+\,, \quad (5.14c)$$

and

$$\hat{L}^{\alpha,\Gamma}_+ \mid 0\rangle = \frac{1}{8N^2}\delta^{\gamma_1\Gamma_1\alpha\gamma_2\Gamma_2}_{\Gamma}(\hat{L}^{\gamma_1,\Gamma_1}_+ + \hat{L}^{\gamma_1,\Gamma_1}_-)(\hat{L}^{\gamma_2,\Gamma_2}_+ + \hat{L}^{\gamma_2,\Gamma_2}_-) \mid 0\rangle\,. \quad (5.15)$$

Now suppose that $\hat{Z}$ is an operator whose symbol $Z(u)$ vanishes for all $u\in g$. Then (3.5) holds, and this implies that expectations in the base state of multiple commutators of any number of $\hat{L}^{\alpha,\Gamma}_\pm$'s vanish,

$$\langle 0 \mid [\cdots[[\hat{Z}, \hat{L}^{\alpha_1,\Gamma_1}_\pm], \hat{L}^{\alpha_2,\Gamma_2}_\pm], \cdots, \hat{L}^{\alpha_k,\Gamma_k}_\pm] \mid 0\rangle = 0\,. \quad (5.16)$$

Consider any particular multiple commutator of this form, and let S be the length of the sequence $\alpha_1\Gamma_1\alpha_2\Gamma_2\cdots\alpha_k\Gamma_k$ (in other words, S equals the total number of matrices $\{M^\alpha\}$ or $\{E^\alpha\}$ in any term of the multiple commutator). Apply the obvious procedure; expand the multiple commutator, push all $\hat{L}_+$'s to the

right of $\hat{Z}$ right, and all $\hat{L}_-$'s to the left of $\hat{Z}$ left. Equation (5.14) shows that any commutator terms which arise reduce S by two. Furthermore, (5.15) shows that any $\hat{L}_+$ which is applied to $|0\rangle$ (or $\hat{L}_-$ applied to $\langle 0|$) also drops S by two. This implies (by a trivial induction) that

$$\langle 0|\hat{L}_+^{\alpha_1,\Gamma_1}\cdots\hat{L}_+^{\alpha_k,\Gamma_k}\hat{Z}\hat{L}_-^{\beta_1,\Gamma_1'}\cdots\hat{L}_-^{\beta_l,\Gamma_l'}|0\rangle=0 \tag{5.17}$$

for all sets of operators $\{\hat{L}_+^{\alpha_i\Gamma_i}\}$ and $\{\hat{L}_-^{\beta_j\Gamma_j'}\}$. However, arbitrary products of $\hat{L}_-$'s applied to $|0\rangle$ clearly form a dense set of U(N)-invariant states. Consequently (5.17) implies that $\hat{Z}=0$, and this proves Assumption 2.

Assumption 3 requires that classically inequivalent coherent states become orthogonal as $N\to\infty$. Classically inequivalent states must be sufficiently different so as to give different expectation values for operators such as $\tilde{\mathrm{tr}}\hat{M}^\Gamma$ or $\tilde{\mathrm{tr}}\hat{E}^\alpha\hat{M}^\Gamma$. That such states become orthogonal as $N\to\infty$ seems almost obvious due to the rapid growth of the number of degrees of freedom. Unfortunately, I have been unable to express the overlap, $\langle 0|u\rangle=\langle 0|\hat{U}|0\rangle$, in any form which is sufficiently explicit to allow a completely rigorous proof of Assumption 3.[20] This appears to be a purely technical problem which we will simply ignore.

Finally, matrix elements of operators such as $\tilde{\mathrm{tr}}(\hat{M}^\Gamma)$ or $\tilde{\mathrm{tr}}(\hat{E}^\alpha)^2$ are easily seen to have finite limits as $N\to\infty$. Therefore, Hamiltonians such as (5.2) satisfy Assumption 4 [$(1/N^2)\hat{H}_N$ classical] and generate sensible dynamics as $N\to\infty$.

We may now use the general formalism of Sec. III to find the classical dynamics which describes the $N\to\infty$ limit of these models. The particular coadjoint orbit of g which will provide the classical phase space is defined by

$$\langle\zeta,\lambda(a,b)\rangle=\lim_{n\to\infty}(1/iN^2)\Lambda(a,b)(u)\ ,$$

where $\zeta\equiv Ad^*[u](\zeta_o)$. Equation (5.13) provides the "explicit" expression. We need to find a reasonably convenient set of coordinates for the coadjoint orbit; however, expressing $\langle\zeta,\lambda(a,b)\rangle$ in terms of the original labels (that is, $\{\phi^{\alpha\Gamma}\}$ and $\{\psi^\Gamma\}$) is very difficult. We may avoid this problem by simply choosing the set of expectations

$$W^\Gamma=\lim_{N\to\infty}\langle u|\tilde{\mathrm{tr}}\hat{M}^\Gamma|u\rangle \tag{5.18}$$

plus $\{\psi^\Gamma\}$ as coordinates on the coadjoint orbit. Note that $\{\psi^\Gamma\}$ and $\{W^\Gamma\}$ are both cyclically symmetric (so that $\psi^{\Gamma_1\Gamma_2}=\psi^{\Gamma_2\Gamma_1}$ and $W^{\Gamma_1\Gamma_2}=W^{\Gamma_2\Gamma_1}$ for all sequences Γ_1,Γ_2). Therefore the classical phase space has one pair of coordinates for each cyclically identified sequence. We will refer to these equivalence classes of sequences as "loops" and will indicate the identification by enclosing the label for the sequence in parentheses, as in $\psi^{(\Gamma)}$. Let

$$\Omega^{(\Gamma)(\Gamma')}\equiv\lim_{N\to\infty}\frac{1}{N}\langle u|\frac{\partial\,\mathrm{tr}\hat{M}^{\bar{\Gamma}}}{\partial\hat{M}_{ij}^\alpha}\frac{\partial\,\mathrm{tr}\hat{M}^{\Gamma'}}{\partial\hat{M}_{ji}^\alpha}|u\rangle$$
$$=\delta_\Gamma^{\Gamma_1\alpha\Gamma_2}W^{(\bar{\Gamma}_1\bar{\Gamma}_2\Gamma_2'\Gamma_1')}\delta_{\Gamma'}^{\Gamma_1'\alpha\Gamma_2'}\ . \tag{5.19}$$

Ω should be regarded as a matrix in the space of all loops. Ω is Hermitian and positive definite and so has an inverse, Ω_{-1}, such that

$$\sum_{(\Gamma')}\Omega_{-1}^{(\Gamma)(\Gamma')}\Omega^{(\Gamma')(\Gamma'')}=\delta^{(\Gamma)(\Gamma'')}\ . \tag{5.20}$$

The gradient of any function on the coadjoint orbit, $f(\{\psi^{(\Gamma)}\},\{W^{(\Gamma)}\})$, may be represented by

$$df(\{\psi^{(\Gamma)}\},\{W^{(\Gamma)}\})=\lambda(a,b)\ , \tag{5.21}$$

where

$$b^{\alpha,\Gamma}=\delta_\Gamma^{\Gamma_2\Gamma_1}\Omega_{-1}^{(\Gamma')(\Gamma_1\alpha\Gamma_2)}(\partial f/\partial\psi^{(\Gamma')})$$

and

$$a^{(\Gamma)}=(\partial f/\partial W^{(\Gamma)})-b^{\alpha,\Gamma'}\psi^{\Gamma_1\alpha\Gamma_2}\delta_\Gamma^{\Gamma_1\Gamma'\Gamma_2}\ .$$

Equivalently,

$$b^\alpha[M]=(\partial f/\partial\psi^{(\Gamma')})\Omega_{-1}^{(\Gamma')(\Gamma)}(\partial\,\mathrm{tr}M^{\bar{\Gamma}}/\partial M^\alpha)\ ,$$

and

$$a[M]=(\partial f/\partial W^{(\Gamma)})M^\Gamma-b^\alpha[M]_{ij}\psi^\Gamma(\partial M^\Gamma/\partial M_{ij}^\alpha)\ .$$

Inserting these expressions into the definition of the Poisson bracket, (3.20), and using the commutation relations, (5.4), we find

$$\{f,g\}_{PB}=\left\{\frac{\partial f}{\partial\psi^{(\Gamma)}}\frac{\partial g}{\partial W^{(\Gamma')}}-\frac{\partial g}{\partial\psi^{(\Gamma)}}\frac{\partial f}{\partial\omega^{(\Gamma')}}\right\}\cdot$$
$$\times\lim_{N\to\infty}\frac{1}{N}\langle u|\frac{\partial\,\mathrm{tr}\hat{M}^{\bar{\Gamma}}}{\partial\hat{M}_{ij}^\alpha}\frac{\partial\,\mathrm{tr}\hat{M}^{\Gamma''}}{\partial\hat{M}_{ji}^\alpha}\Omega_{-1}^{(\Gamma'')(\Gamma')}|u\rangle$$
$$=\sum_{(\Gamma)}\left\{\frac{\partial f}{\partial\psi^{(\Gamma)}}\frac{\partial g}{\partial W^{(\Gamma)}}-\frac{\partial g}{\partial\psi^{(\Gamma)}}\frac{\partial f}{\partial W^{(\Gamma)}}\right\}\cdot \tag{5.22}$$

Thus $\{\psi^{(\Gamma)}\}$ and $\{W^{(\Gamma)}\}$ are naturally canonically conjugate.

Following the general formalism, the classical Hamiltonian, $h_{cl}(\{\psi^{(\Gamma)}\},\{W^{(\Gamma)}\})$, is given by the $N\to\infty$ limit of the coherent state expectation of the quantum Hamiltonian. For the choice of Hamiltonian in (5.2) this is given by

$$h_{cl}=\lim_{N\to\infty}\frac{1}{N}\sum_{\alpha=1}^{n}\frac{1}{2}\left\{\langle u|\left|\frac{\partial}{\partial\hat{M}_{ij}^\alpha}\left\{\mathrm{tr}\,\psi[\hat{M}]+\frac{i}{2}\mathrm{tr}(\phi^\beta[\hat{M}])^2-\frac{i}{2N}\ln J[M]\right\}\right|^2|u\rangle\right.$$
$$\left.+\langle u|\,\mathrm{tr}\{(\hat{M}^\alpha-\hat{M}^{\alpha-1})^2+\omega^2(\hat{M}^\alpha)^2+g(\hat{M}^\alpha)^4\}|u\rangle\right\}\cdot \tag{5.23}$$

[20] One may verify that $\phi(0,u)\equiv-\lim_{N\to\infty}(1/N^2)\ln\langle 0|u\rangle$ has strictly positive curvature about the point $u=0$ in all directions other than the one corresponding to constant phase rotations.

We must express this expectation in terms of the classical coordinates $\{\psi^{(\Gamma)}\}$ and $\{W^{(\Gamma)}\}$. The following observation will enable us to accomplish this. If $f[\tilde{\mathrm{tr}}(\hat{M}^\Gamma)]$ is any functional of $\{\tilde{\mathrm{tr}}(\hat{M}^\Gamma)\}$, then

$$\lim_{N\to\infty} N\langle u \mid \left|\frac{\partial}{\partial M^\alpha_{ij}} f[\tilde{\mathrm{tr}}(\hat{M}^\Gamma)]\right|^2 \mid u\rangle = \langle u \mid \frac{\partial \bar{f}}{\partial \hat{M}^\alpha_{ji}} \frac{\partial\, \mathrm{tr}\hat{M}^{\Gamma'}}{\partial \hat{M}^\alpha_{ij}} \mid u\rangle \Omega^{(\Gamma')(\Gamma)}_{-1} \langle u \mid \frac{\partial\, \mathrm{tr}\hat{M}^\Gamma}{\partial \hat{M}^\beta_{kl}} \frac{\partial f}{\partial \hat{M}^\beta_{kl}} \mid u\rangle \ . \tag{5.24}$$

This follows from factorization plus the definition of $\Omega^{(\Gamma)(\Gamma')}$ in Eq. (5.19). Using this relation, one finds

$$h_{cl}(\{\psi^{(\Gamma)}\},\{W^{(\Gamma)}\}) = \tfrac{1}{2}\psi^{(\bar{\Gamma})}\Omega^{(\Gamma)(\Gamma')}\psi^{(\Gamma')} + \tfrac{1}{2}\omega^{(\bar{\Gamma})}\Omega^{(\Gamma)(\Gamma')}_{-1}\omega^{(\Gamma')} + V[W^{(\Gamma)}] \ , \tag{5.25}$$

where

$$\begin{aligned}\omega^{(\Gamma)} &\equiv \lim_{N\to\infty} \langle u \mid \frac{\partial\, \tilde{\mathrm{tr}}\hat{M}^\Gamma}{\partial \hat{M}^\alpha_{ij}} \frac{\partial}{\partial \hat{M}^\alpha_{ji}} \left\{ \frac{1}{2}\mathrm{tr}\,(\phi^\beta[\hat{M}])^2 - \frac{1}{2N}\ln J[\hat{M}] \right\} \mid u\rangle \\ &= \lim_{N\to\infty} \frac{1}{2}\langle u \mid \left[\frac{\partial}{\partial \hat{M}^\alpha_{ij}} \frac{\partial}{\partial \hat{M}^\alpha_{ji}} \tilde{\mathrm{tr}}(\hat{M}^\Gamma) \right] \mid u\rangle \\ &= W^{(\Gamma_2)} W^{(\Gamma_3\Gamma_1)} \delta^{\Gamma_1\alpha\Gamma_2\alpha\Gamma_3}_{\bar{\Gamma}} \ ,\end{aligned} \tag{5.26}$$

and

$$\begin{aligned}V[W^{(\Gamma)}] &= \lim_{N\to\infty} \tfrac{1}{2} \sum_{\alpha=1}^{n} \langle u \mid \tilde{\mathrm{tr}}\{ (\hat{M}^\alpha - \hat{M}^{\alpha-1})^2 + \omega^2(\hat{M}^\alpha)^2 + g(\hat{M}^\alpha)^4 \} \mid u\rangle \\ &= \sum_{\alpha=1}^{n} \{ (1+\tfrac{1}{2}\omega^2) W^{(\alpha\alpha)} - W^{(\alpha(\alpha-1))} + \tfrac{1}{2} g W^{(\alpha\alpha\alpha\alpha)} \} \ .\end{aligned} \tag{5.27}$$

This is about as far as we can go in the discussion of general matrix models. Equation (5.25) gives the explicit form of the classical Hamiltonian. To "solve" the $N=\infty$ theory completely, one must find the minimum of the $h_{cl}(\{\psi^{(\Gamma)}\},\{W^{(\Gamma)}\})$. This appears to be extremely difficult.

In the special case of a single matrix, one may completely solve the $N=\infty$ theory.[21] Let us briefly see how the preceding discussion simplifies in this case.

If only a single matrix is present, then the set of sequential products of matrices, $\{\hat{M}^\Gamma\}$, reduces to a single set of integer powers, $\{\hat{M}^k\}$, $k=0,1,2,\ldots,\infty$. Therefore elements of the Lie algebra, $\mathbf{G}_N=\{\hat{\Lambda}(a,b)\}$, may be labeled by two "arbitrary" real functions of a single variable, $a(z)\equiv\sum_{k=0}^{\infty} a_k z^k$ and $b(z)\equiv\sum_{k=0}^{\infty} b_k z^k$.[22] This algebra generates a group of finite transformations $G_N=\{\hat{U}(\psi,\phi)\}$, where $\psi(z)$ is an arbitrary function, and $\phi(z)$ is necessarily monotonically increasing. If $\hat{U}(\psi,\phi)=\exp\hat{\Lambda}(a,0)\exp\hat{\Lambda}(0,b)$, then the formal expression $\phi(z)=\exp[b(z)\partial/\partial z]z$ may be solved by the implicit definition,

$$\int_z^{\phi(z)} dz'/b(z') = 1 \ .$$

The action of the coherence group is given by (5.9) and (5.10). When only a single matrix is present, the Jacobian $J(M)\equiv\det[\partial\phi(M)_{ij}/\partial M_{kl}]$ may be directly expressed in terms of the eigenvalues $\{\mu_i\}$ of the matrix M. One finds (Mehta, 1967),

$$J(M) = \prod_{i=1}^{N} \phi'(\mu_i) \prod_{i\neq j} \left\{ \frac{[\phi(\mu_i)-\phi(\mu_j)]}{(\mu_i-\mu_j)} \right\} . \tag{5.28}$$

All expectations in the coherent states may now be reduced to averages over a single Gaussian random matrix. Such averages may be computed explicitly for large N by writing the integration measure in terms of the eigenvalues $\{\mu_i\}$ and by using saddlepoint methods to evaluate the resulting integrals. One finds that the Gaussian measure $dM\exp -N\,\mathrm{tr}\,M^2$ leads to a density of eigenvalues given by Wigner's semicircle distribution (Wigner, 1959),

$$d\rho(\mu) = \frac{1}{\pi}(2-\mu^2)^{1/2}\Theta(2-\mu^2)\,d\mu \tag{5.29}$$

(Θ is the step function; $\Theta(x)=1$ if $x\geq 0$, 0 otherwise). This allows us to express the symbols of elements of the Lie algebra, (5.13), in the form

$$\Lambda(a,b)(u) = iN^2 \int d\rho(\phi(\mu))\{ a(\mu) + b(\mu)\psi'(\mu) \} + O(1) \ . \tag{5.30}$$

[21] In fact, the theory is exactly soluble for arbitrary N. See Brezin *et al.* (1978) and Marchesini and Onofri (1980).

[22] Requiring $a(z)$ and $b(z)$ to be entire functions which are bounded on the real axis, and whose derivatives (to all orders) are bounded on the real axis, appears to be a sensible definition. Such functions include, for example, functions with smooth Fourier transforms which decrease faster than any exponential (on the real axis). This set of functions is closed under the commutation relations, generates a well-defined set of one-parameter subgroups, $\exp t\lambda$, $\lambda\in\mathbf{g}$, and leads to a coherence group which acts irreducibly.

As usual, these expectations determine the appropriate coadjoint orbit for the classical phase space. As shown previously,

$$W^k \equiv \lim_{N\to\infty} \langle u \,|\, \widetilde{\mathrm{tr}} M^k \,|\, u \rangle = \int d\rho(\phi(\mu))\mu^k$$

and

$$\psi^k \equiv \frac{1}{k!}\frac{\partial^k \psi(z)}{(\partial z)^k}\bigg|_{z=0} \quad (\text{for } k=1,2,\ldots)$$

provide canonically conjugate coordinates on the classical phase space. However, these are not the most convenient set of coordinates. If we define $w(\mu)\equiv d\rho(\phi(\mu))/d\mu$ and $v(\mu)\equiv\psi(\mu)$, then one can show that $v(\mu)$ and $w(\mu)$ are canonically conjugate, $\{v(\mu),w(\mu')\}_{PB}=\delta(\mu-\mu')$.[23] [Alternatively, $\mu(x)$, defined on the interval $0<x<1$ by $\int_{-\infty}^{\mu(x)} d\rho(\phi(\mu))=x$, and $v(x)\equiv\psi'(\mu(x))$, are also conjugate, $\{v(x),\mu(x')\}_{PB}=\delta(x-x')$.]

Finally, the classical Hamiltonian may be expressed in a very simple form. Using (5.28), the expectation of $\widetilde{\mathrm{tr}}\hat{E}^2$ may be computed as follows:

$$\begin{aligned}
\langle u \,|\, \widetilde{\mathrm{tr}}\hat{E}^2 \,|\, u \rangle &= \frac{1}{N}\langle u \,|\, \left| \frac{\partial}{\partial M_{ij}} \left\{ \mathrm{tr}\,\psi(M) + \frac{i}{2}\mathrm{tr}\,\phi(M)^2 - \frac{i}{2N}\ln J(M) \right\} \right|^2 | \, u \rangle \\
&= \int d\rho(\phi(\mu)) \left| \frac{\partial}{\partial\mu} \left\{ \psi(\mu) + \frac{i}{2}\phi(\mu)^2 - i\int d\rho(\phi(\mu'))\ln\frac{\phi(\mu)-\phi(\mu')}{\mu-\mu'} \right\} \right|^2 + O(1/N^2) \\
&= \int d\rho(\phi(\mu))\{\psi'(\mu)^2 + [\fint d\rho(\phi(\mu'))/\mu-\mu']^2\} + O(1/N^2) \\
&= \int d\mu\, w(\mu)\{v'(\mu)^2 + \frac{\pi^2}{3}w(\mu)^2\} + O(1/N^2) \,.
\end{aligned}$$

Here we have used the defining relation for the Wigner distribution, $\mu = \fint d\rho(\mu')/\mu-\mu'$, and the fact that

$$\int d\mu\, w(\mu)(\fint d\mu' w(\mu')/\mu-\mu')^2 = \frac{\pi^2}{3}w(\mu)^3 \,.$$

This latter relation may be easily derived by writing the principal-value integrals as averages of contour integrals passing above and below the pole, and then symmetrizing over the ordering of the contours [see Mondello and Onofri (1981) or Shapiro (1981) for more detailed discussion]. Therefore the final classical Hamiltonian is given by

$$h_{cl}[v(\mu),w(\mu)] = \tfrac{1}{2}\int d\mu\, w(\mu)\{v'(\mu)^2 + \frac{\pi^2}{3}w(\mu)^2 + \omega^2\mu^2 + g\mu^4\} \,, \tag{5.32}$$

Minimizing this Hamiltonian, subject to the constraint $\int d\mu\, w(\mu)=1$, yields

$$v_0(\mu)=0$$

and

$$w_0(\mu) = \frac{1}{\pi}(2e-\omega^2\mu^2-g\mu^4)^{1/2}\Theta(2e-\omega^2\mu^2-g\mu^4) \,, \tag{5.33}$$

where the Lagrange multiplier, e, is determined by the condition $1=\int d\mu\, w(\mu)$. The $N\to\infty$ limit of the ground-state energy is given by

$$\begin{aligned}
\lim_{N\to\infty} E_0/N^2 &= h_{cl}[v_0(\mu),w_0(\mu)] \\
&= \tfrac{1}{3}\int d\mu\, w(\mu)(\omega^2\mu^2 + g\mu^4 + e) \,.
\end{aligned} \tag{5.34}$$

The classical action is given by

$$S_{cl}[v,w] = \int d\mu\{v(\mu)\dot{w}(\mu) - \tfrac{1}{2}w(\mu)[v'(\mu)^2 + \frac{\pi^2}{3}w(\mu)^2 + \omega^2\mu^2 + g\mu^4]\} \,. \tag{5.35}$$

One may expand the classical action about the minimum to find the small oscillation frequencies. One finds

$$\omega_j = j\omega_s \quad \text{for } j=1,2,3,\ldots, \tag{5.36}$$

[23] $v(\mu)$ and $w(\mu)$ are not strictly independent coordinates. $w(\mu)$ must satisfy the constraint $\int d\mu\, w(\mu)=1$, and adding a constant to $v(\mu)$ does not affect the dynamics.

where

$$\omega_s^{-1} = \int d\mu\, w(\mu)/(2e - \omega^2\mu^2 - g\mu^4) .$$

This is known to reproduce correctly the $N\to\infty$ limit of the U(N)-invariant spectrum of the theory (Shapiro, 1981; Mondello and Onofri, 1981).

VI. GAUGE THEORIES

As a final example, we will examine the $N\to\infty$ limit of U(N)-lattice gauge theories. [I choose to work on a lattice in order to make the theory well defined. The choice of a U(N) instead of SU(N)-gauge theory is made for convenience; the difference between the groups is irrelevant for large N. Furthermore, one may show that U(N), O(N), and Sp(N) lattice gauge theories all have equivalent large N limits (Lovelace, 1982). I will not include fermions in the following discussion, although they may be easily inserted (Yaffe, 1982).] The analysis is essentially identical to that presented in the preceeding section on matrix models. Consequently, this treatment will be as brief as possible.

Consider a U(N)-Hamiltonian lattice gauge theory. The basic operators are the link variables, $\{\hat{V}^\alpha_{ij}\}$, and their conjugate momenta, $\{\hat{E}^\alpha_{ij}\}$. α labels the links of lattice. For each link α, $\hat{V}^\alpha \equiv ||\hat{V}^\alpha_{ij}||$ is an N-dimensional unitary matrix, $(\hat{V}^\alpha)^\dagger = (\hat{V}^\alpha)^{-1}$, and $\hat{E}^\alpha \equiv ||\hat{E}^\alpha_{ij}||$ is Hermitian, $(\hat{E}^\alpha)^\dagger = \hat{E}^\alpha$. Each link α is assumed to have a standard orientation. Links with orientation opposite to the standard will be denoted by $\bar{\alpha}$, and $\hat{V}^{\bar{\alpha}} \equiv (\hat{V}^\alpha)^\dagger$. The commutation relations are

$$[\hat{E}^\alpha_{ij}, V^\beta_{kl}] = \frac{1}{2N}\delta^{\alpha\beta}\delta_{kj}\hat{V}^\alpha_{il} , \tag{6.1a}$$

$$[\hat{E}^\alpha_{ij}, \hat{V}^{\bar{\beta}}_{kl}] = \frac{-1}{2N}\delta^{\alpha\beta}\delta_{il}\hat{V}^{\bar{\alpha}}_{kj} , \tag{6.1b}$$

and

$$[\hat{E}^\alpha_{ij}, \hat{E}^\beta_{kl}] = \frac{1}{2N}\delta^{\alpha\beta}(\delta_{kj}\hat{E}^\alpha_{il} - \delta_{il}\hat{E}^\alpha_{kj}) . \tag{6.1c}$$

$\hat{E}^\alpha_{ij}$ may be represented by $(1/2N)\hat{V}^\alpha_{ik}(\partial/\partial\hat{V}^\alpha_{jk})$. [Our conjugate momenta, $\{\hat{E}_{ij}\}$, are related to the more conventional generators $\{\hat{E}^a\}$ (which satisfy $[\hat{E}^a, \hat{E}^b] = -if^{abc}\hat{E}^c$) by $\hat{E}_{ij} \equiv (1/N)t^a_{ij}\hat{E}^a$. Here $\{f^{abc}\}$ are the structure constants of U(N), and the matrices $\{t^a\}$ represent the generators of U(N). They satisfy $[t^a, t^b] = if^{abc}t^c$, $\mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$, and $\sum_{a=0}^{N^2-1} t^a_{ij}\, t^a_{kl} = \frac{1}{2}\delta_{il}\delta_{kj}$.]

Gauge transformations are specified by giving an arbitrary element of U(N) for every site of the lattice, $\Omega^s \in \mathrm{U}(N)$. If α denotes the link running from site s to site s', then under a gauge transformation $\hat{V}^\alpha \to (\Omega^s)\hat{V}^\alpha(\Omega^{s'})^\dagger$ and $\hat{E}^\alpha \to (\Omega^s)\hat{E}^\alpha(\Omega^s)^\dagger$. For any ordered set of links, $\Gamma = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, which forms a single closed curve beginning and ending at some site s, let us define $\hat{V}^\Gamma \equiv \hat{V}^{\alpha_1}\hat{V}^{\alpha_2}\cdots\hat{V}^{\alpha_k}$. Note that $\hat{V}^\Gamma$ transforms covariantly under a gauge transformation, $\hat{V}^\Gamma \to (\Omega^s)\hat{V}^\Gamma(\Omega^s)^\dagger$. Henceforth, if α labels a particular link of the lattice, then we will use Γ^α to denote an arbitrary closed curve which begins with the link α. (Γ) will denote closed curves irrespective of their starting point; in other words, (Γ) labels loops.

We will assume that the lattice is cubic and choose the standard Kogut-Susskind Hamiltonian (Kogut and Susskind, 1975),

$$\hat{H} \equiv N^2\,\tilde{\mathrm{tr}}\{\lambda\sum_\alpha(\hat{E}^\alpha)^2 - \lambda^{-1}\sum_p(\hat{V}^{\partial p} + \hat{V}^{\partial\bar{p}})\} . \tag{6.2}$$

Here $\lambda \equiv g^2N$ and $p(\partial p)$ indicates (the boundary of) an arbitrary plaquette. $\hat{H}$ is, of course, gauge invariant.

We will choose the Lie algebra of the coherence group to be given by

$$\mathbf{G}_N = \{\,\hat{\Lambda}(a,b) \equiv iN^2\,\tilde{\mathrm{tr}}\,(a[\hat{V}] + E^\alpha b^\alpha[\hat{V}] + b^\alpha[\hat{V}]E^\alpha)\,\} , \tag{6.3}$$

where $a[\hat{V}] \equiv \sum_{(\Gamma)} a^{(\Gamma)}\hat{V}^\Gamma$ and $b^\alpha[\hat{V}] \equiv \sum_{\Gamma^\alpha} b^{\alpha,\Gamma^\alpha}\hat{V}^{\Gamma^\alpha}$. $a[\hat{V}]$ and $b^\alpha[\hat{V}]$ must be Hermitian. The commutation relations are

$$[\hat{\Lambda}(a_1,b_1), \hat{\Lambda}(a_2,b_2)] = \hat{\Lambda}(a_{12}, b_{12}) , \tag{6.4}$$

where

$$a_{12}[\hat{V}]_{ij} = i(b_1^\alpha[\hat{V}]\hat{V}^\alpha)_{kl}\frac{\partial a_2[\hat{V}]_{ij}}{\partial\hat{V}^\alpha_{kl}} - i(b_2^\alpha[\hat{V}]\hat{V}^\alpha)_{kl}\frac{\partial a_1[\hat{V}]_{ij}}{\partial\hat{V}^\alpha_{kl}} ,$$

and

$$b^\alpha_{12}[\hat{V}]_{ij} = i(b_1^\beta[\hat{V}]\hat{V}^\beta)_{kl}\frac{\partial b_2^\alpha[\hat{V}]_{ij}}{\partial\hat{V}^\beta_{kl}} - i(b_2^\beta[\hat{V}]\hat{V}^\beta)_{kl}\frac{\partial b_1^\alpha[\hat{V}]_{ij}}{\partial\hat{V}^\beta_{kl}} + i\delta^{\alpha\beta}[b_2^\alpha[\hat{V}], b_1^\alpha[\hat{V}]]_{ij} .$$

This algebra generates a coherence group which is very similar to the group used for treating matrix models. Elements of the group, $\hat{U}[\psi,\phi] \in G_N$, are labeled by the functionals $\psi[\hat{V}] \equiv \psi^{(\Gamma)}\hat{V}^\Gamma$ and $\phi^\alpha[\hat{V}] \equiv \phi^{\alpha,\Gamma^\alpha}\hat{V}^{\Gamma^\alpha}$. $\psi[\hat{V}]$ must be Hermitian and $\phi^\alpha[\hat{V}]$ unitary. The action of the coherence group is given by

$$\hat{U}\hat{V}^\alpha_{ij}\hat{U}^{-1} = (\phi^\alpha[\hat{V}]\hat{V}^\alpha)_{ij} \tag{6.5}$$

and

$$\hat{U}\hat{E}^{\alpha}_{ij}\hat{U}^{-1}=(\phi^{\alpha}[\hat{V}]\hat{V}^{\alpha})_{ik}\left[\frac{\partial\hat{V}^{\beta}_{mn}}{\partial(\phi^{\alpha}[\hat{V}]\hat{V}^{\alpha})_{jk}}\right]\left[(\hat{V}^{\bar{\beta}}\hat{E}^{\beta})_{nm}-\frac{i}{2}\frac{\partial\,\mathrm{tr}\,\psi[\hat{V}]}{\partial\hat{V}^{\beta}_{mn}}\right]$$
$$+\frac{1}{4N}(\phi^{\alpha}[\hat{V}]\hat{V}^{\alpha})_{ik}\frac{\partial}{\partial\hat{V}^{\beta}_{mn}}\left[\frac{\partial\hat{V}_{mn}}{\partial(\phi^{\alpha}[\hat{V}]\hat{V}^{\alpha})_{jk}}\right]$$
$$=(\phi^{\alpha}[\hat{V}]\hat{V}^{\alpha})_{ik}\left[\left[\frac{\partial\hat{V}^{\beta}_{mn}}{\partial(\phi^{\alpha}[\hat{V}]\hat{V})_{jk}}\right](\hat{V}^{\bar{\beta}}\hat{E}^{\beta})_{nm}-\frac{i}{2}\frac{\partial}{\partial(\phi^{\alpha}[\hat{V}]\hat{V})_{jk}}\left[\mathrm{tr}\,\psi[\hat{V}]-\frac{i}{2N}\ln J[\hat{V}]\right]\right]. \tag{6.6}$$

$J[V]\equiv\det[\partial(\phi^{\alpha}[V]V^{\alpha})_{ij}/\partial V^{\beta}_{kl}]$ is the Jacobian for the change of variables from $\{V^{\alpha}_{ij}\}$ to $\{(\phi^{\alpha}[V]V^{\alpha})_{ij}\}$. Note that

$$J[V]^{*}=J[V]\prod_{\alpha}\{\det(V^{\alpha})/\det(\phi^{\alpha}[V]V^{\alpha})\}^{2}.$$

The base state, $|0\rangle$, will be chosen to be the state which is annihilated by all conjugate momenta, $\hat{E}^{\alpha}|0\rangle=0$, for all links α. Its wave function is simply a constant,

$$\Psi_0[V]\equiv\langle V\,|\,0\rangle=1.$$

This base state generates a set of coherent states whose wave functions are given by

$$\Psi_u[V]\equiv\langle V\,|\,u\rangle=(J[V])^{1/2}\exp iN^2\,\tilde{\mathrm{tr}}(\psi[V]). \tag{6.7}$$

We may now apply the general formalism developed earlier. The arguments needed to verify the assumptions made in Sec. III are essentially identical to those presented in the last section.[24] I will not bother to repeat that discussion here. Note that the coupling constant λ in the Hamiltonian (6.2) must remain fixed in order for the dynamics to have a sensible limit as $N\to\infty$.

The coadjoint orbit which provides the appropriate classical phase space is specified by

$$\langle\xi,\,\lambda(a,b)\rangle=\lim_{N\to\infty}(1/iN^2)\langle u\,|\,\hat{\Lambda}(a,b)\,|\,u\rangle$$
$$=\lim_{N\to\infty}\langle u\,|\left[\tilde{\mathrm{tr}}\,a[\hat{V}]+i(b^{\alpha}[\hat{V}]\hat{V}^{\alpha})_{ij}\frac{\partial\,\tilde{\mathrm{tr}}\,\psi[\hat{V}]}{\partial\hat{V}^{\alpha}_{ij}}\right]|\,u\rangle. \tag{6.8}$$

Reasonably convenient coordinates on this coadjoint orbit are provided by the coefficients $\{\psi^{(\Gamma)}\}$ plus the expectation values of Wilson loops,

$$W^{(\Gamma)}\equiv\lim_{N\to\infty}\langle u\,|\,\tilde{\mathrm{tr}}V^{(\Gamma)}\,|\,u\rangle. \tag{6.9}$$

By applying exactly the same procedure that was used in the last section, one may derive the Poisson bracket for the classical phase space and compute the classical Hamiltonian. One finds that the coordinates $\{\psi^{(\Gamma)}\}$ and $\{W^{(\Gamma)}\}$ are naturally canonically conjugate,

$$\{f,\,g\}_{PB}=\sum_{(\Gamma)}\left[\frac{\partial f}{\partial\psi^{(\Gamma)}}\frac{\partial g}{\partial W^{(\Gamma)}}-\frac{\partial f}{\partial W^{(\Gamma)}}\frac{\partial g}{\partial\psi^{(\Gamma)}}\right]. \tag{6.10}$$

The classical Hamiltonian, $h_{cl}(\{\psi^{(\Gamma)}\},\{W^{(\Gamma)}\})$, is given by the expectation of (6.2),

$$h_{cl}=\lim_{N\to\infty}(1/N^2)\langle u\,|\,\hat{H}_N\,|\,u\rangle$$
$$=\lim_{N\to\infty}\frac{1}{N}\left\{\lambda\sum_{\alpha}\frac{1}{4}\langle u\,|\left|\hat{V}^{\alpha}_{ij}\frac{\partial}{\partial\hat{V}^{\alpha}_{kj}}\left\{\mathrm{tr}\,\psi[\hat{V}]-\frac{i}{2N}\ln J[\hat{V}]\right\}\right|^{2}|\,u\rangle-\frac{1}{\lambda}\sum_{p}\langle u\,|\,\mathrm{tr}\{\hat{V}^{\partial p}+\hat{V}^{\partial\bar{p}}\}\,|\,u\rangle\right\}.$$

Let us define the following Hermitian, positive definite matrix in "loop space,"

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[24]The easiest way to verify Assumption 2 is to regard the gauge theory as the $\varepsilon\to0$ limit of a theory in which the matrices $\{V^{\alpha}\}$ are arbitrary complex matrices and a term $(1/2\varepsilon)\,\mathrm{tr}(V^{\alpha}V^{\bar{\alpha}}-1)$ is added to the Hamiltonian. (This is completely analogous to the construction of nonlinear sigma models from limits of linear sigma models.) One may then use exactly the same procedure developed earlier for matrix models.

$$\Omega^{(\Gamma)(\Gamma')} \equiv \lim_{N\to\infty} \frac{1}{N} \langle u \mid \frac{\partial \operatorname{tr} \hat{V}^{(\bar{\Gamma})}}{\partial \hat{V}^{\bar{\alpha}}_{ji}} \frac{\partial \operatorname{tr} \hat{V}^{(\Gamma')}}{\partial \hat{V}^{\alpha}_{ij}} \mid u \rangle$$

$$= \Delta_{\Gamma}^{\Gamma_1 \alpha \Gamma_2} W^{(\bar{\Gamma}_1 \bar{\Gamma}_2 \Gamma'_2 \Gamma'_1)} \Delta_{\Gamma'}^{\Gamma'_1 \alpha \Gamma'_2} . \tag{6.11}$$

$\Delta_{\Gamma}^{\Gamma_1 \alpha \Gamma_2}$ is a "signed" delta function, defined to equal $\pm \delta_{\Gamma}^{\Gamma_1 \alpha \Gamma_2}$, depending on the orientation of Γ ($+$ if Γ traverses the link α in the standard orientation, $-$ otherwise). The classical Hamiltonian may then be expressed in the form,

$$h_{cl}(\{\psi^{(\Gamma)}\}, \{W^{(\Gamma)}\}) = \tfrac{1}{4} \lambda \psi^{(\bar{\Gamma})} \Omega^{(\Gamma)(\Gamma')} \psi^{(\Gamma')} + \lambda \omega^{(\bar{\Gamma})} \Omega_{-1}^{(\Gamma)(\Gamma')} \omega^{(\Gamma')} - \frac{1}{\lambda} \sum_{p} (W^{(\partial p)} + W^{(\partial \bar{p})}) , \tag{6.12}$$

where $\Omega_{-1}^{(\Gamma)(\Gamma')}\, \Omega^{(\Gamma')(\Gamma'')} = \delta^{(\Gamma)(\Gamma'')}$ and

$$\omega^{(\Gamma)} \equiv \lim_{N\to\infty} \langle u \mid \frac{\partial \operatorname{tr} \hat{V}^{(\bar{\Gamma})}}{\partial \hat{V}^{\bar{\alpha}}_{ji}} \frac{\partial}{\partial \hat{V}^{\alpha}_{ij}} \left[\frac{1}{4N^2} \ln J[\hat{V}] \right] \mid u \rangle$$

$$= \lim_{N\to\infty} \langle u \mid [\hat{E}^{\alpha}_{ij}, [\hat{E}^{\alpha}_{ji}, \operatorname{tr} \hat{V}^{(\bar{\Gamma})}]] \mid u \rangle$$

$$= \lim_{N\to\infty} \frac{1}{4N} \langle u \mid \left[\hat{V}^{\alpha}_{ik} \frac{\partial}{\partial \hat{V}^{\alpha}_{jk}} \hat{V}^{\alpha}_{jl} \frac{\partial}{\partial \hat{V}^{\alpha}_{il}} \tilde{\operatorname{tr}}(\hat{V}^{(\bar{\Gamma})}) \right] \mid u \rangle$$

$$= \tfrac{1}{4} l(\Gamma) W^{(\bar{\Gamma})} + \tfrac{1}{2} W^{(\bar{\Gamma}_3 \bar{\alpha} \bar{\Gamma}_1)} W^{(\bar{\Gamma}_2 \bar{\alpha})} \Delta_{\Gamma}^{\Gamma_1 \alpha \Gamma'} \Delta_{\Gamma'}^{\Gamma_2 \alpha \Gamma_3} . \tag{6.13}$$

Here $l(\Gamma) \equiv V^{\alpha}_{ij} (\partial \operatorname{tr} V^{(\Gamma)} / \partial V^{\alpha}_{ij}) / (\operatorname{tr} V^{(\Gamma)})$ is the signed "length" of the loop (Γ). [In deriving (6.13) we have used the fact that $(1/N^2)(\ln J[V] - \ln J[V]^{\dagger}) = O(1/N)$.]

Equation (6.12) gives the explicit form of the classical Hamiltonian.[25] Solving the $N = \infty$ theory requires finding the minimum of $h_{cl}(\{\psi^{(\Gamma)}\}, \{W^{(\Gamma)}\})$. Regrettably, if the lattice contains more than one plaquette, this minimization appears to be very difficult. The one plaquette theory, however, may be completely solved. I will briefly summarize the explicit results which may be obtained in that case.

If only a single plaquette is present, then the theory may be completely expressed in terms of a single unitary matrix, given by the product of the link variables around the plaquette, $\hat{V} \equiv \hat{V}^1 \hat{V}^2 \hat{V}^3 \hat{V}^4$, and its conjugate momentum $\hat{E} \equiv \hat{E}^1$. Consequently, elements of the coherence group may be labeled by two functions of a single variable, $\psi(z)$ and $\phi(z)$, defined on the unit circle [$\psi(z)$ must be real and $\phi(z)$ must provide an invertible mapping of the unit circle onto itself]. The Jacobian $J[V] = \det[\partial(\phi[V]V)_{ij} / \partial V_{kl}]$ may then be explicitly computed in terms of the eigenvalues $\{v_i\}$ of the matrix V. One finds (Mehta, 1967)

$$J[V] = \prod_i (\psi(v_i) + v_i \phi'(v_i)) \prod_{i \neq j} |[v_i \phi(v_i) - v_j \phi(v_j)] / (v_i - v_j)| . \tag{6.14}$$

Expectations such as (6.8) may be expressed in the form

$$\langle u \mid \hat{\Lambda}(a,b) \mid u \rangle = iN^2 \oint d\rho [v\phi(v)] \{ a(v) + iv b(v) \psi'(v) \} , \tag{6.15}$$

where $d\rho(z) \equiv dz/(2\pi i z)$ is the density of eigenvalues for the base state and the integration is over the unit circle. The set of Wilson loops reduces to simple powers of the matrix V, $W^k \equiv \lim_{N\to\infty} \langle u | \tilde{\operatorname{tr}}(\hat{V})^k \mid u \rangle$, $k = \pm 1, \pm 2, \ldots$. Instead of the coordinates $\{\psi^k\}$ and $\{W^k\}$ we may use the density of the eigenvalues, $w(\vartheta) \equiv \{ d\rho(e^{i\vartheta}) \phi(e^{i\vartheta})) / d\vartheta \}$, and $v(\vartheta) \equiv \psi(e^{i\vartheta})$.[26] One may easily show that $v(\vartheta)$ and $w(\vartheta)$ are canonically conjugate, $\{ v(\vartheta), w(\vartheta') \} = \delta(\vartheta - \vartheta')$. Finally, the classical Hamiltonian may be expressed in terms of these variables. One finds

$$h_{cl} = \lambda \oint d\rho(v\phi(v)) \left| \frac{\partial}{\partial v} \left\{ \psi(v) - i \oint d\rho(v' \phi(v')) \ln \left| \frac{v\phi(v) - v'\phi(v')}{v - v'} \right| \right\} \right|^2 - \lambda^{-1} \oint d\rho(v\phi(v))(v + v^{-1})$$

$$= \oint d\rho(v\phi(v)) \left\{ \lambda \mid \psi'(v) \mid^2 + \tfrac{1}{4} \lambda \left| \oint d\rho(v' \phi(v'))(v + v')/v - v' \right|^2 - \lambda^{-1}(v + v^{-1}) \right\}$$

$$= \int_{-\pi}^{\pi} d\vartheta\, w(\vartheta) \left\{ \lambda v'(\vartheta)^2 + \frac{\pi^2}{3} \lambda w(\vartheta)^2 - 2\lambda^{-1} \cos\vartheta - \frac{\lambda}{12} \right\} . \tag{6.16}$$

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[25] This form of the classical Hamiltonian has been previously derived by Sakita (1980).

[26] $v(\vartheta)$ is real and $w(\vartheta)$ is real and positive. $v(\vartheta)$ and $w(\vartheta)$ do not quite represent independent dynamical variables; $w(\vartheta)$ must satisfy $\int d\vartheta\, w(\vartheta) = 1$, and adding a constant $v(\vartheta)$ does not affect the dynamics.

(Exactly the same contour integral tricks that were used in the last section have been used here.) Minimizing (6.16) subject to the constraint $\int_{-\pi}^{\pi} d\vartheta\, w(\vartheta)=1$ yields[27]

$$v_0(\vartheta)=0$$

and (6.17)

$$w_0(\vartheta)=\frac{1}{\pi}(e+2\lambda^{-2}\cos\vartheta)^{1/2}\Theta(e-2\lambda^{-2}\cos\vartheta)\ ,$$

where the constant e is determined by the condition $1=\int_{-\pi}^{\pi} d\vartheta\, w_0(\vartheta)$. Finally, the ground-state energy is given by

$$\lim_{N\to\infty}(E_0/N^2)=h_{cl}[v_0,w_0]$$
$$=\lambda\left[e-\tfrac{2}{3}\pi^2\int_{-\pi}^{\pi} d\vartheta\, w_0(\vartheta)^3\right]\ . \quad (6.18)$$

Note that the structure of the original lattice appears in the classical Hamiltonian (6.12) only indirectly through the set of possible loops $\{\Gamma\}$. Suppose the original lattice, Λ, is invariant under some translation T. If one identifies all loops in Λ which are equivalent under T, then the resulting set of equivalence classes of loops is isomorphic to the set of all topologically trivial loops on a smaller periodic lattice $\tilde{\Lambda}$, formed by identifying all sites of Λ equivalent under T. Comparing the classical Hamiltonians for the lattices Λ and $\tilde{\Lambda}$, one finds that the two will be identical if (a) the expectations (and conjugate momenta) of all Wilson loops in Λ which are equivalent under T are equal, and (b) the expectations (and conjugate momenta) of all topologically nontrivial loops in $\tilde{\Lambda}$ vanish. The minimum of the classical Hamiltonian will automatically satisfy these conditions if appropriate global symmetries remain unbroken. [Condition (a) requires unbroken translation invariance under T, while (b) requires unbroken invariance under gauge transformations which are only periodic up to elements of the center of the U(N) gauge group.] Repeating this argument allows one to show that the large N limits of gauge theoreis on all periodic sublattice of Λ are equivalent, provided that all global symmetries remain unbroken. In particular, the large N limit of a theory on a d-dimensional cubic lattice should be equivalent to the limit of a theory on a periodic lattice containing just one site (i.e., a matrix model of d matrices). [Eguchi and Kawai (1982) have recently discussed an analogous result for Euclidean lattice gauge theories.]

VII. DISCUSSION

We have presented a general formalism for finding classical limits in arbitrary quantum theories, based on certain assumptions shown to be sufficient to construct a classical phase space and derive the appropriate classical dynamics. These assumptions appear to isolate cleanly the minimal structure required for any classical limit; however, proving any form of necessity appears to be very difficult. Using this formalism, it has been shown that for a large class of theories the $N\to\infty$ limit is a classical limit. This class of theories (plus their obvious generalizations) includes essentially all known theories with sensible large N limits. In every case considered, exactly the same procedure has worked. The only input required is a suitable choice for the coherence group and an appropriate base state.

In any theory where the fundamental quantum operators can be divided into "coordinates," $\{\hat{x}(\alpha)\}$, and conjugate "momenta," $\{\hat{p}(\alpha)\}$, there is a natural choice for the coherence group. If $\{f_i(\hat{x}(\alpha))\}$ is a minimal set of physical operators such that every physical operator constructed from the "coordinates" $\{\hat{x}(\alpha)\}$ can be expressed as a function of the f's and if $\{g_j(\hat{x}(\alpha),p(\beta))\}$ is the set of operators obtained from the set $\{f_i\}$ by replacing any single coordinate, $\hat{x}(\alpha)$, by the corresponding momentum, $\hat{p}(\alpha)$, then the group generated by the operators $\{f_i(\hat{x}(\alpha))\}$ and $\{g_j(\hat{x}(\alpha),\hat{p}(\beta))\}$ will act irreducibly on the physical Hilbert space. Every one of the coherence groups we have considered earlier may be regarded as an example of this prescription. Note that the generators of the coherence groups have always been at most linear in the conjugate momenta. This feature ensures that one can always exponentiate the generators, and thereby construct the group of finite transformations, in a reasonably explicit manner.

We have always used the simplest possible choice for the base state. It turns out that only for these simple choices is it easy to prove Assumption 2 (operators uniquely specified by their symbols). It is not known whether, in an arbitrary quantum theory, there necessarily exists any choice for the base state which will satisfy this assumption.[28] Similarly, questions of uniqueness (such as whether or not using a different coherence group for gauge theories might allow one to avoid loop spaces) have not been adequately answered. Different choices for the coherence group or base state must give equivalent results, and it seems very doubtful that there exist any choices satisfying the required assumptions which are more convenient than the choices we have made.

Next, we will discuss how the approach we have used to derive large N limits relates to previously proposed methods. We will begin by considering various methods which have had limited applicability.

Large N limits were first studied in vector models, originally in the context of statistical mechanics (Stanley,

[27]The following results have been previously derived using different methods. See Jevicki and Sakita (1980b) and Wadia (1980a).

[28]Note that Assumption 2 was used only to justify restricting attention to the symbols of physical operators. It is conceivable that there exist theories for which a weaker form of the assumption may be appropriate. See the appendix for further consequences of this assumption.

1968), and later from the viewpoint of particle physics (Wilson, 1973). Certain specific models, such as ϕ^4 field theories, are sufficiently tractable that one may simply sum all Feynman diagrams that survive in the $N\to\infty$ limit (Dolan and Jackiw, 1974; Schnitzer, 1974; Coleman, Jackiw, and Politzer, 1974; Gross and Neveu, 1974). Equivalently, functional integral methods may be used to compute directly the $N=\infty$ limit of the effective action (Halpern, 1980). One finds an effective action which is nonlocal in both space and time and not obviously equivalent to any classical action which is local in time. However, minimizing the effective action in, for examples, ϕ^4 theories, leads to exactly the same gap equation as (4.24), and one may easily see that all other results also agree. The classical action, (4.20), is in fact closely related to the effective action one obtains from the second Legendre transform of the generating functional (Cornwall, Jackiw, and Tomboulis, 1974).

More recently, Berezin (1978) studied vector models with the specific intention of understanding the classical nature of the $N\to\infty$ limit. His paper is somewhat obscure and relies heavily on his earlier work on quantization on Kähler manifolds (Berezin, 1974, 1975). Presumably for this reason, it has received less attention than it deserved. Berezin used coherent state methods similar to those employed in Sec. IV. The major difference is that he chose to include the operators $\hat{C}(\alpha,\beta)\equiv\frac{1}{2}\hat{p}_i(\alpha)\hat{p}_i(\beta)$ among the generators of the coherence group. This enlarged group actually leads to exactly the same set of coherent states. However, including all bilinear operators in the Lie algebra **G** conveniently allows one to express the action of the coherence group in Fock space.[29] This makes it very easy to carry out the discussion for both Bose and Fermi theories in parallel. Berezin expresses the resulting classical mechanics in a form which appears quite different from (4.19) and (4.20); however, this is simply a consequence of his choice of coordinates on the classical phase space. [He uses complex coordinates which reflect the curvature of the Kähler manifold and are not naturally canonically conjugate. A stereographic projection linearizes the phase space (Jevicki and Papanicolaou, 1980) and relates his coordinates to those used in Sec. IV.]

Mlodinow and Papanicolaou (1980, 1981) have also studied certain vector models using related techniques termed "pseudospin" methods. Instead of employing coherent states, they choose to work directly at the operator level and study the algebra of the O(N)-invariant bilinear operators, $\hat{A}(\alpha,\beta)$, $\hat{B}(\alpha,\beta)$, and $\hat{C}(\alpha,\beta)$. This algebra reflects the structure of the canonical commutation relations and for Bose theories is equivalent to Sp($2n,\mathscr{R}$) [$n=1$ or 2 is the number of O(N)-vectors in their work]. They rewrite the operators $\hat{A}$, $\hat{B}$, and $\hat{C}$ in terms of a new set of elementary Bose creation and annihilation operators (a generalized Holstein-Primakoff representation) and show that in the large N limit all of these new operators may be treated classically. This then leads to the same classical Hamiltonian as in (4.19). In my opinion, this operator level approach is less convenient than the coherent state approach for deriving the $N\to\infty$ limit. However, it appears to be more convenient for deriving systematic corrections in $1/N$, since one need deal with only a finite number of O(N)-invariant basis states, instead of with an overcomplete set of coherent states. Mlodinow and Papanicolaou have computed the first three terms in the $1/N$ expansion for the ground-state energy of systems such as helium, and hydrogen in a magnetic field, and obtained surprisingly good results (N equals the dimension of space here). Recently, Papanicolaou (1981) has also used this "pseudospin" method to discuss the large N limit of the two-dimensional $(\bar{\psi}\psi)^2$ model.

Berezin, Mlodinow, and Papanicolaou have all relied heavily on the fact that for any vector model one can immediately rewrite any physical operator, such as the Hamiltonian, in terms of the bilocal "pseudospin" operators $\hat{A}$, $\hat{B}$, and $\hat{C}$. However, in order to understand the generalization of these methods to more complicated theories, it is important to realize that one need not include all three sets of operators in the Lie algebra of the coherence group (≡"pseudospin" algebra). As shown in Sec. IV, including only $\hat{A}$ and $\hat{B}$ (i.e., operators at most linear in momenta) still produces a coherence group which acts irreducibly. Furthermore, any physical operator can be expressed as a linear combination of elements of the coherence group, or equivalently, as a function of just $\hat{A}(\alpha,\beta)$ and $\hat{B}(\alpha,\beta)$. This was stated in equation (3.3), which follows from the Von Neumann density theorem (see footnote 8). However, finding this representation can be very difficult. This problem is neatly avoided in our discussion of the $N\to\infty$ limit. All we ever require is the symbol of an operator, and computing this is a deductive operation.

Next, we turn to a discussion of methods which are, or claim to be, applicable to a wider class of theories than just vector models. We begin with the method which is, in some ways, closest to the approach used in this paper. This is the "collective field method" of Jevicki and Sakita (1980a).

The collective field method is based on the idea of directly rewriting a quantum theory in terms of an overcomplete set of commuting physical operators. Specifically, all wave functions are taken to be functionals of an overcomplete set of physical variables, such as the set of bilinears $\{x(\alpha)\cdot x(\beta)\}$ for vector models or the set of Wilson loops $\{\tilde{\mathrm{tr}}V^\Gamma\}$ for gauge theories; this set of variables is the "collective field." Using the chain rule, one then expresses the kinetic terms in the Schrödinger equation in terms of derivatives with respect to the collective field variables. Massaging the resulting expression by means of a similarity transformation then produces the "collective field Hamiltonian." Finally, it is argued that

[29]This also produces a coherence group which is semisimple, unlike all of the groups used in this paper. Consequently for this group one need not distinguish between the Lie algebra and its dual space. (See footnote 12 for further discussion.)

in the large N limit one may treat the collective field variables and their conjugates as independent classical variables. Therefore one finds a classical Hamiltonian appropriate for studying the large N limit. (Infinite terms formally suppressed by powers of $1/N$ are typcially discarded at this stage.) The ground-state energy, for example, may then be computed by minimizing this Hamiltonian.

In every example we have considered it turns out that the coherent-state method used in this paper produces exactly the same classical Hamiltonian as does the collective field method. This was basically inevitable, since both methods work entirely within the physical Hilbert space and produce classical Hamiltonians which are expressed in terms of physical variables. Despite this equivalence of the final results, the two methods of derivation differ considerably. We would like to discuss this difference in somewhat greater detail because it will bring up an important point.

Consider, for example, a one-matrix model such as the one described in Sec. V. The U(N)-invariant spectrum consists of a set of modes whose frequencies increase linearly, $\omega_j = j\omega_s$. For finite N, there are N such modes. Therefore the zero-point energy of these modes is of order N^2 and contributes to the leading large N behavior of the ground-state energy. Consequently, any method for deriving the large N limit of the model must correctly account for this zero-point energy. If one simply rewrites the Schrödinger equation for this model in terms of the eigenvalues of the original matrix and tries to neglect the gradient terms, then this zero-point energy will be missed and one will obtain the wrong answer.[30] The collective field method manages to avoid this problem. Instead of writing the theory in terms of the complete set of N eigenvalues, $\{\lambda_i\}$, it uses a continuous function with a smooth large N limit, the density of eigenvalues $\rho(\lambda)$. One might think that this would only make the problem worse, since the collective field Hamiltonian has no cutoff on the number of modes and the zero-point energy is now $\infty \times N$ instead of $O(N^2)$. However, it turns out that the correct answer is obtained by simply dropping all such terms which are (formally) suppressed by powers of $1/N$. Understanding why this is true, within the collective field approach, is not easy.

The coherent-state formalism provides a much cleaner method for deriving the large N classical Hamiltonian. Instead of requiring operator level manipulations, one simply computes the expectation of the quantum Hamiltonian in a specified set of (normalizable) coherent states. The zero-point energy is automatically included correctly.

Along this same line, I should mention the work of Lovelace (1981). He considered the general problem of changing variables to an independent set of gauge-invariant coordinates and claimed that adding a simple term based on the volume of a gauge orbit to the potential energy would reproduce the results of the collective field method. This prescription amounts to simply neglecting certain portions of the kinetic energy and does not appear correctly to include zero-point energy contributions.

The large N limit of many simple models can be solved by formulating Schwinger-Dyson equations for correlation functions of time-ordered products of physical operators. Normally, such equations generate an infinite hierarchy of relations involving arbitrarily complicated correlation functions. However, in the large N limit factorization [Eq. (1.1)] can be used to simplify the equations, and one can derive a closed set of nonlinear equations which specify the behavior of a minimal set of physical observables. In a variety of simple models, one can explicitly solve these equations, and thereby derive the leading $N \to \infty$ behavior of physical correlation functions (Paffuta and Rossi, 1980; Friedan, 1981; Brower, Rossi, and Tan, 1981a, 1981b; Kazakov and Kostov, 1980; Wadia, 1981).

Many people have attempted to study gauge theories by formulating Dyson equations for the vacuum expectations of Wilson loops (Gervais and Neveu, 1979; Nambu, 1979; Polyakov, 1979; Foerster, 1979; Eguchi, 1979; Migdal, 1980). Makeenko and Migdal (1979, 1980) were the first to emphasize that in the large N limit one can derive a closed set of equations involving the expectations of single Wilson loops. Inevitably, these equations are extremely difficult to solve.

A Hamiltonian approach, such as we have used leads to an explicit minimization problem, and varying the classical Hamiltonian (Eq. 6.12) generates a closed set of equations for the expectations of equal-time Wilson loops. In contrast, the Migdal-Makeenko equations require expectations of arbitrary space-time Wilson loops and do not follow from any explicit minimization problem.[31] It is not clear if the equations have a unique solution. Furthermore, these equations actually contain much less information than the large N classical Hamiltonian (6.12). An explict solution would at most determine the large N limit of the expectation of any single Wilson loop. However, this is not sufficient information to determine the large N behavior of all physical observables. In order to compute, for example, the gauge-invariant spectrum (i.e., glueball states), one must be able to determine the connected part of the expectation of a product of two Wilson loops. This information is not contained in the Migdal-Makeenko equations.

When applied to gauge theories, any method which

[30]This has been discussed in detail in the context of a self-dual quaternionic matrix model by Aragao de Carvalho and Fateev (1981).

[31]Jevicki and Sakita (1981) have shown that the Migdal-Makeenko equations follow from an effective action for loops derived from the Euclidean functional integral. However, evaluating the action requires computing a Jacobian which is only implicitly defined through a functional differential equation.

works entirely within the physical (i.e., gauge-invariant) Hilbert space seems inevitably to lead to some sort of loop space. Because analysis in such a space is intractable, a number of people have recently investigated alternative approaches which might avoid this problem.

Witten (1979) [see also Coleman (1979)] argued on the basis of factorization (Eq. 1.1) that the support of the Euclidean functional integral must reduce to a single gauge orbit when $N\to\infty$. This would imply that the large N limit of the expectation of any time-ordered product of physical operators could be computed by simply replacing all quantum gauge fields by a single classical field (unique up to gauge transformations). Such a field configuration has been termed a "master field". The major problem with this approach is that no constructive method for finding a suitable master field is known. The basic idea of a "unique" gauge orbit at $N=\infty$ is somewhat ill-defined, and it is not always clear how to define the infinite-dimensional space in which the master field is supposed to live. In some sense, one can always package an arbitrarily large amount of information into the "sum" over the group indices of the master field. Consider, for example, a ϕ^4 field theory. The original configuration space consists of real fields, $\phi_i(x)$, where the index i runs from 1 to N. The large N limit of the two-point function is given by

$$\lim_{N\to\infty}\langle T\{\hat{\phi}_i(x)\hat{\phi}_i(x')\}\rangle=\int\frac{d^{d+1}k}{(2\pi)^{d+1}}\frac{e^{i(k\cdot(x-x'))}}{(k^2+m^2)}\ , \tag{7.1}$$

where $m^2=\mu^2+\lambda\sigma$ and σ satisfies the gap equation (4.24). Construction of a master field for this theory is straightforward. One simply (a) allows the master field to be complex instead of real, (b) interprets the index i as the label for an arbitrary $(d+1)$-dimensional momentum vector, and (c) chooses the master field $\Phi_k(x)$ to be given by (Levine, 1980; Halpern, 1981b)

$$\Phi_k(x)=\frac{\exp\{-i(k\cdot x)\}}{(2\pi)^{(d+1)/2}(k^2+m^2)^{1/2}}\ . \tag{7.2}$$

Replacing every occurrence of $\hat{\phi}(x)\cdot\hat{\phi}(x')$ in any time-ordered correlation function with $\int d^{d+1}k\Phi_k^*(x)\Phi_k(x')$ will then yield the correct large N limit. Obviously, this construction simply reflects the fact that the large N limit of the spectrum consists of a single particle of mass m. Note that the master field does not allow one to compute the leading $N\to\infty$ behavior of one-particle irreducible correlation functions, and that therefore one cannot extract, for example, the large N behavior of physical scattering amplitudes from the master field. Because its derivation was solely based on factorization, the master field actually contains much less information than the classical Hamiltonian (4.22)

Since there is no constructive approach for finding such "master fields" without having first solved the theory, it is obviously desirable to find an alternative approach which will have more predictive power. Such an approach is potentially provided by the recently developed method of "constrained classical solutions." (Unlike the usage in the rest of this paper, in the following discussion "classical solutions," "classical hamiltonians," etc., will be understood to refer specifically to the $\hbar\to 0$ limit.) The first step in the development of this method was provided by the observation by Jevicki and Papanicolaou (1980) that the effective Hamiltonian describing the $N=\infty$ dynamics of a point particle, (4.21), is identical to the classical Hamiltonian of a point particle whose angular momentum (squared) is equal to 1/4. (If we had not scaled out factors of $\hbar$ and N, L^2 would equal $\hbar^2N^2/4$). Minimizing the original classical Hamiltonian subject to this constraint does in fact lead to the correct large N limit of the ground-state energy. A similar result was shown to hold in linear σ models. Subsequent work by Jevicki and Levine (1980, 1981), and Kessler and Levine (1981) has extended this idea to general vector models and to the single Hermitian matrix model. In every case, the equations of motion which follow from the large N effective Hamiltonian may be shown to be equivalent to the original classical equations of motion subject to a suitable constraint. In the one-matrix model, for example, the constraint may be written as $J_{ab}\equiv i[M,\dot{M}]_{ab}=\hbar(1-\delta_{ab})$. Bardakci (1981a) has also discussed one-vector and one-matrix models from a somewhat different viewpoint and obtained equivalent results. The clearest explanation about what's going on, and why this approach works, has been given by Halpern (1981a). A brief sketch of his argument follows.

Consider a theory which is known to satisfy factorization in the large N limit, such as a one-vector model. Imagine computing the vacuum expectation of any "index-ordered" product of field operators. ["Index-ordered" means that the quantum operators are ordered in a way that allows a natural contraction of the group indices. In vector models this means that neighboring pairs of vectors are contracted, as in $\langle 0|\hat{x}(t)\cdot\hat{x}(t')\,\hat{x}(t)\cdot\hat{x}(t'')|0\rangle$.] Consider inserting a complete set of quantum eigenstates after each field operator. This obviously requires working in the full Hilbert space, which contains states transforming under all possible irreducible representations of the symmetry group. However, the restriction to index-ordered products of field operators significantly restricts the types of intermediate states which can contribute. In vector models each intermediate state must transform either as an O(N) vector or O(N) singlet. Furthermore, factorization implies that the only O(N)-invariant state which can contribute to the leading $N\to\infty$ behavior is the ground state. Therefore in order to compute any ordered expectation in, for example, the one-vector model, one requires only the following matrix elements,

$$\langle n,i\,|\,\hat{x}_j\,|\,0\rangle=\delta_{ij}\,q_n/\sqrt{N}\ . \tag{7.3}$$

Here, n labels the (unknown) number of O(N)-vector eigenstates, i is an O(N) vector index labeling the states within a multiplet, and q_n is a reduced matrix element. Because we have taken matrix elements between quantum eigenstates, the reduced matrix elements have a sim-

ple time dependence,

$$q_n(t)=e^{i\omega_n t}q_n(0)\ , \tag{7.4}$$

where $\omega_n \equiv E_n - E_0$ is the (unknown) excitation energy of the nth eigenstate. The quantum equation of motion is simply $\ddot{\hat{x}}_i + 2V'(\hat{x}^2)\hat{x}_i = 0$ [for the usual Hamiltonian $\hat{H}=N\{\frac{1}{2}\hat{p}^2 + V(\hat{x}^2)\}$]. Taking matrix elements and using factorization yields

$$\ddot{q}_n + 2V'(q\cdot q^*)\,q_n = 0\ . \tag{7.5}$$

Therefore q_n may be regarded as a complex vector satisfying the original classical equations of motion. However, the "index" n has nothing to do with the original O(N)-vector index; rather, it labels the set of vector eigenstates which can couple to the ground state. The relevant constraint arises from the vacuum expectation of the commutation relations and reads

$$\sum_n (q_n^*\dot{q}_n - \dot{q}_n^* q_n) = i\ . \tag{7.6}$$

Assuming that the frequencies $\{\omega_n\}$ are not degenerate (7.5)—(7.6) have a unique solution (up to an overall phase). This solution allows one to calculate the large N limit of any ordered correlation functions in the one-vector model. (It should be emphasized that the leading behavior always comes from the maximally disconnected part of the correlation function; the constrained classical solution does not retain enough information to compute the connected part).

Halpern has shown that the same approach works in any vector model and in the one-matrix model. Extending the analysis to gauge theories appears to be straightforward (Bardakci, 1981b). In each case one isolates the relevant set of transition matrix elements which can contribute to the large N-limit of any ordered correlation function. Next, one defines reduced matrix elements in such a way that the quantum equations of motion are equivalent to classical equations of motion for the reduced matrix elements. Matrix elements of the canonical commutation relations then generate a set of constraints which the reduced matrix elements must satisfy.

This approach of solving for constrained classical solutions is complementary to the coherent-state or collective field methods in the following sense. The latter methods work entirely within the invariant sector of the Hilbert space and generate an effective Hamiltonian describing the $N=\infty$ dynamics. Examining small oscillations about the minimum of the effective Hamiltonian allows one to compute the $N\to\infty$ limit of the spectrum of gauge- [or O(N)-] invariant states. The constrained classical solution does not contain this information. Rather, the time dependence of the constrained solution contains information about the spectrum of noninvariant states. (In gauge theories these would be static quark antiquark states.) In general, it is not at all clear that solving for the appropriate constrained classical solution is any easier than minimizing the large N effective Hamiltonian. [In the one-matrix model, solving for the constrained solution has only recently been accomplished (Halpern and Schwartz, 1981).]

Note that methods which use factorization from the outset, such as the Migdal-Makeenko derivation of loop equations or the preceding constrained classical solution approach, at most allow one to compute the vacuum expectation of any single physical operator. These methods do not, in general, produce enough information to determine the large N limit of the invariant spectrum, scattering amplitudes, or other quantities which require knowledge of the connected part of correlation functions of products of physical operators. Only methods leading to the complete large N effective action retain enough information to allow one to compute such quantities.

The above arguments summarize why, at least in my opinion, methods which derive the effective $N=\infty$ classical Hamiltonian appear to provide the most useful approach known for studying large N limits. It seems doubtful that any significantly more convenient approach can be developed. The problem of solving the large N limit of gauge theories, for example, is reduced to a minimization problem of an explicitly known functional (6.12). The fact that the relevant variables are defined on loops appears to be unavoidable. Although we lack an analytic solution to this problem, minimizing (6.12) numerically should be perfectly feasible. This appears to be well worth the effort, since an explicit solution would allow one directly to compute the large N spectrum of QCD.

Finally, we should say a few words about $1/N$ corrections. In general, it is not sufficient simply to turn around and try to quantize the large N classical Hamiltonian (Mondello and Onofri, 1981). Besides obvious factor ordering problems, there can be explicit corrections of order $1/N$ in the Hamiltonian. Furthermore, there can also be corrections coming from a nontrivial measure for the classical variables. One can study these problems by deriving functional integral representations based on the overcomplete set of coherent states. This is briefly discussed in the appendix. In principle, this representation can be used to derive systematic corrections in powers of $1/N$; however, only in simple theories has it been possible to carry out explicit calculations.

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APPENDIX

This appendix discusses some further applications of coherent states, such as the representation of operators in terms of diagonal projections onto coherent states and

functional integral representations based on coherent states.

In Sec. III we defined the symbol of any operator $\hat{A}$ as the set of coherent state expectation values,

$$A(u)=\langle u\,|\,\hat{A}\,|\,u\rangle\ ,\quad u\in g\ . \tag{A1}$$

Henceforth we will call $A(u)$ the "lower symbol" (or "covariant" symbol) of $\hat{A}$. Lower symbols provide a natural mapping of quantum operators into functions on the coherence group. One may also define a natural mapping of functions into operators. If $\mathring{A}(u)$ is some function on the coherence group such that

$$\int d\mu_L(u)\mathring{A}(u)\,|\,u\rangle\langle u\,|=\hat{A}\ , \tag{A2}$$

then $\mathring{A}(u)$ will be called an "upper symbol" (or "contravariant" symbol) of $\hat{A}$. Upper symbols of a given operator need not be unique, and in general may not even exist. Only operators which can be expressed as weighted sums of diagonal projections onto the coherent states will have upper symbols.

The irreducibility of the coherence group led to the completeness relation (3.2), and this implies that

$$\int d\mu_L(u)A(u)=\mathrm{Tr}(\hat{A})=\int d\mu_L(u)\mathring{A}(u)\ . \tag{A3}$$

[We have absorbed the constant c_χ in the completeness relation into the measure $d\mu$. The second part of (A3) obviously requires that $\hat{A}$ possess an upper symbol.] One may easily show that the norm of an operator is bounded below by the maximum of its lower symbol and bounded above by the maximum of any upper symbol (Simon, 1980; Berezin, 1972). Furthermore, if $\Phi(x)$ is any convex function of a real variable [such as $\exp(x)$] and if $\hat{A}$ is a self-adjoint operator, then (Berezin, 1972; Lieb, 1973),

$$\begin{aligned}\int d\mu_L(u)\ \Phi(A(u))&\le\mathrm{Tr}[\Phi(\hat{A})]\\&\le\int d\mu_L(u)\ \Phi(\mathring{A}(u))\ .\end{aligned} \tag{A4}$$

[These bounds follow from a simple argument based on Jensen's inequality. See Simon (1980).] These relations motivate the names "upper" and "lower" symbols.

A natural question to ask in a particular theory is whether all operators have upper symbols. If they do, then the set of coherent projections, $\{\,|\,u\rangle\langle u\,|\ \ |\,u\in g\,\}$, will be said to be *complete*. (Completeness of the coherent projections in the space of all operators should not be confused with completeness of the set of coherent states in the Hilbert space. Completeness of the coherent states is an immediate consequence of Assumption 1.) It turns out that Assumption 2 [$Z(u)=0$ implies $\hat{Z}=0$] is equivalent to the requirement that the coherent projections be complete (Simon, 1980). The basic argument is surprisingly simple.[32] Suppose that Assumption 2 were false. Then there would exist some nonzero operator $\hat{Z}$ whose lower symbol $Z(u)$ was identically zero. This would imply that

$$\int d\mu_L(u)\mathring{A}(u)Z(u)=0 \tag{A5}$$

for any function $\mathring{A}(u)$. Since $Z(u)=\langle u\,|\,\hat{Z}\,|\,u\rangle$, (A5) may be rewritten as

$$\mathrm{Tr}(\hat{A}\hat{Z})=0\ , \tag{A6}$$

where $\hat{A}\equiv\int d\mu_L(u)\mathring{A}(u)\,|\,u\rangle\langle u\,|$. Hence, the operator $\hat{Z}$ would be orthogonal to all operators which possess upper symbols. In other words, the set of operators with upper symbols would have a nontrivial orthogonal complement, thereby implying that the set of coherent projections was not complete. Therefore completeness of the coherent projections implies the validity of Assumption 2. To prove the converse, one simply inverts the argument above.

This shows that in any theory satisfying Assumption 2, every operator has an upper symbol. However, the preceding argument is about as nonconstructive as one can get. Regrettably, no general constructive procedure for finding the upper symbol of an arbitrary operator is known. In specific cases, if an explicit upper symbol for the Hamiltonian can be found, then the inequalities (A4) can provide matched upper and lower bounds on the partition function. This can yield detailed information on the rate at which the classical limit is approached [in contrast to Eq. (3.29)–(3.31), which simply show that the limit exists.] This procedure has been successfully applied in a variety of quantum spin systems (Lieb, 1973; Fuller and Lenard, 1979; Gilmore and Feng, 1978; Gilmore, 1979; Shankar, 1980) and even in atomic physics (Thirring, 1981). Unfortunately, constructing upper symbols in more complicated theories appears to be very difficult.

Next, we discuss functional integrals based on coherent states. The coherent-state representation will provide a simple and direct method for showing that the tree diagrams generated from the large N classical action (3.23) correctly reproduce the large N limit of any connected correlation function of physical operators.

Consider, for example, computing the partition function $Z=\mathrm{Tr}(\exp-\beta\hat{H})$. The basic ingredient needed to derive any functional integral is a convenient completeness relation. We will use the coherent-state completeness relation, (3.2). Repeatedly inserting this into the trace leads to

[32]The following sketch is quite sloppy. In particular, the phrase *all operators* actually means a dense set of bounded operators in a particular topology. See Simon (1980) for details.

$$Z=\lim_{k\to\infty}\int\prod_{i=1}^{k}d\mu_L(u_i)\langle u_k\,|\,(1-\varepsilon\hat{H})\,|\,u_{k-1}\rangle\cdots\langle u_2\,|\,(1-\varepsilon\hat{H})\,|\,u_1\rangle\langle u_1\,|\,(1-\varepsilon\hat{H})\,|\,u_k\rangle \tag{A7}$$

($\varepsilon\equiv\beta/k$). Assumption 3 [$\langle u\,|\,u'\rangle\sim\exp-(1/\chi)\phi(u,u')$] and Assumption 4 [$(1/\chi)\hat{H}$ classical] imply that this integral is highly peaked about $u_i\sim u_{i-1}$. Therefore [using (3.10)] we may write (A7) as

$$Z\sim\lim_{k\to\infty}\int\prod_{i=1}^{k}d\mu_L(u_i)\exp-\frac{1}{\chi}\sum_{i=1}^{k}\{\,\phi(u_i,u_{i-1})+\varepsilon\chi H_\chi(u_i)\,\}\ . \tag{A8}$$

Next, we may split each integral over the coherence group g into an integral over the coset space g/h times an integral over the subgroup h, where h generates the set of coherent states which are classically equivalent to the base state. (In other words, $h=\{\,u\in g\,|\,u\sim 1\,\}$). Recall that the coset space g/h is equivalent to the coadjoint orbit Γ which provides the classical phase space. If we write $u_i=e^{\lambda_i}u_{i-1}$, then it may be shown that

$$\sum_{i=1}^{k}\phi(u_i,u_{i-1})=\sum_{i=1}^{k}i\langle\delta\zeta_i,\vartheta_{\zeta_i}\rangle+O(\delta\zeta)^2\ ,$$

where $\zeta_i=Ad^*[u_i](\zeta_0)$ and $\delta\zeta_i=(Ad^*[e^{\lambda_i}]-1)(\zeta_i)$. Finally, using the definition of the classical Hamiltonian, $\lim_{\chi\to\infty}\chi H_\chi(u)=h_{cl}(\zeta)$, we find

$$Z\sim\lim_{k\to\infty}\int\prod_{i=1}^{k}d^\Gamma\mu(\zeta_i)\exp-\frac{1}{\chi}\sum_{i=1}^{k}\{\,i\langle\delta\zeta_i,\vartheta_{\zeta_i}\rangle+\varepsilon\,h_{cl}(\zeta)\,\}\equiv\int D\mu[\zeta(t)]\exp-\frac{1}{\chi}S_{cl}^E[\zeta(t)]\ , \tag{A9}$$

where

$$S_{cl}^E[\zeta(t)]\equiv\int_0^\beta dt\{\,i\langle\dot{\zeta}(t),\vartheta_\zeta\rangle+h_{cl}(\zeta(t))\,\}$$

is the Euclidean classical action, and

$$d^\Gamma\mu(\zeta)\equiv\int d\mu_L(u)\,\delta(\zeta-Ad^*[u](\zeta_0))$$

is the invariant measure on the coadjoint orbit Γ. In a similar fashion, one may consider the expectation of any time-ordered product of classical operators, and find

$$\begin{aligned}\langle A(t_1)\cdots B(t_2)\rangle&\equiv Z^{-1}\mathrm{Tr}\,[T\{\,\hat{A}(t_1)\cdots\hat{B}(t_2)\exp-\beta\hat{H}\,\}]\\&\sim Z^{-1}\int D\mu[\zeta(t)]\,a(\zeta(t_1))\cdots b(\zeta(t_2))\exp-\frac{1}{\chi}S_{cl}^E[\zeta(t)]\ .\end{aligned} \tag{A10}$$

Equations (A9) and (A10) have the standard form of any semiclassical functional integral; all dependence on χ is isolated in a single factor of $(1/\chi)$ multiplying the classical action. We emphasize that these expressions are valid only as $\chi\to 0$. Small χ is the only justification for expanding $\phi(u_i,u_{i-1})$ and $\langle u_i\,|\,\hat{H}\,|\,u_{i-1}\rangle/\langle u_i\,|\,u_{i-1}\rangle$ about $u_i\sim u_{i-1}$. Nevertheless, one may see that the terms we have neglected to not contribute to the leading $\chi\to 0$ behavior of the connected part of correlation functions such as (A10). This limiting behavior may be computed simply by expanding the classical action $S_{cl}^E[\zeta(t)]$ about the minimum of the classical Hamiltonian and evaluating the lowest-order connected tree diagrams. Corrections to this leading behavior can come from several sources; explicit higher-order loop diagrams generated from the classical action, the nontrivial measure $D\mu[\zeta(t)]$, plus all the terms we have neglected in deriving equations (A9) and (A10).[33]

[33]See Klauder (1979) plus references therein for careful discussion of some of the subtleties involved in deriving exact functional integrals based on ordinary Gaussian coherent states.

REFERENCES

Aragao de Carvalho, C., and V. A. Fateev, 1981, J. Phys. A **14**, 1925.

Auslander, L., and B. Kostant, 1967, Bull. Am. Math. Soc. **73**, 692.

Bardakci, K., 1981a, Nucl. Phys. B **178**, 263.

Bardakci, K., 1981b, "Classical Equations for Non-Abelian Gauge Theories in the Large N Limit," Princeton, Institute for Advanced Study preprint 81-0136.

Berezin, F. A., 1972, Izv. Akad. Nauk SSR Ser. Mat. **36**, 1134 [Math. USSR Izv. **6**, 1117, (1972)].

Berezin, F. A., 1974, Izv. Akad. Nauk SSR Ser. Mat. **38**, 1116 [Math. USSR Izv. **8**, 1109, (1974)].

Berezin, F. A., 1975, Izv. Akad. Nauk SSR Ser. Mat. **39**, 363 [Math. USSR Izv. **9**, 341 (1975)].

Berezin, F. A., 1978, Commun. Math. Phys. **63** 131.

Bratteli, O., and D. W. Robinson, 1979, *Operator Algebras and Quantum Statistical Mechanics 1,* (Springer, New York).

Berezin, E., C. Itzykson, G. Parisi, and J. B. Zuber, 1978, Commun. Math. Phys. **59**, 35.

Brower, R. C., P. Rossi, and C.-I. Tan, 1981a, Phys. Rev. D **23**, 942.

Brower, R. C., P. Rossi, and C.-I. Tan, 1981b, Phys. Rev. D **23**, 953.

Coleman, S., 1980, "1/N", SLAC preprint Pub-2484 (Erice lectures, 1979).
Coleman, S., R. Jackiw, and H. D. Politzer, 1974, Phys. Rev. D **10**, 2491.
Cornwall, J. M., R. Jackiw, and E. Tomboulis, 1974, Phys. Rev. D **10**, 2428.
Dolan, L., and R. Jackiw, 1974, Phys. Rev. D **9**, 2904.
Eguchi, T., 1979, Phys. Lett. B **87**, 91.
Eguchi, T. and H. Kawai, 1982, "Reduction for Dynamical Degrees of Freedom in the Large N Gauge Theory," Tokyo Univ. preprint UT-378.
Foerster, D., 1979, Phys. Lett. B **87**, 87.
Friedan, D., 1981, Commun. Math. Phys. **78**, 353.
Fuller, W., and A. Lenard, 1979, Commun. Math. Phys. **67**, 69; **69**, 99.
Gervais, J. L., and A. Neveu, 1979, Phys. Lett. B **80**, 255.
Gilmore, R., 1979, J. Math. Phys. **20**, 891.
Gilmore, R., and D. H. Feng, 1978, Phys. Lett. B **76**, 26.
Gross, D. J., and A. Neveu, 1974, Phys. Rev. D **10**, 3235.
Halpern, M. B., 1980, Nucl. Phys. B **173**, 504.
Halpern, M. B., 1981a, Nucl. Phys. B **188**, 61.
Halpern, M. B., 1981b, Nuovo Cimento A **61**, 207.
Halpern, M. B., and C. Schwartz, 1981, Phys. Rev. D **24**, 2146.
Hooft, G. 't., 1974, Nucl. Phys. B **72**, 461.
Jevicki, A., and H. Levine, 1980, Phys. Rev. Lett. **44**, 1443.
Jevicki, A., and H. Levine, 1981, Ann. Phys. **136**, 113.
Jevicki, A., and N. Papanicolaou, 1980, Nucl. Phys. B **171**, 362.
Jevicki, A., and B. Sakita, 1980a, Nucl. Phys. B **165**, 511.
Jevicki, A., and B. Sakita, 1980b, Phys. Rev. D **22**, 467.
Jevicki, A., and B. Sakita, 1981, Nucl. Phys. B **185**, 89.
Kazakov, V. A., and I. Kostov, 1980, Nucl. Phys. B **176**, 199.
Kessler, D. A., and H. Levine, 1981, Ann. Phys. **133**, 13.
Kirillov, A. A., 1962, Usp. Mat. Nauk **17**, (4), 57 [1962, Russian Math. Surveys **17**, (4), 53 (1962)].
Kirillov, A. A., 1976, *Elements of the Theory of Representations,* (Springer, Berlin).
Klauder, J. R., 1963, J. Math. Phys. **4**, 1055; 1058.
Klauder, J. R., 1979, Phys. Rev. D **19**, 2349.
Klauder, J. R., and E. C. G. Sudarshan, 1968, *Fundamentals of Quantum Optics* (Benjamin, New York).
Kogut, J., and L. Susskind, 1975, Phys. Rev. D **11**, 395.
Kostant, B., 1970, in "Quantization and Unitary Representations," *Lectures in Modern Analysis and Applications III,* edited by C. T. Taam, Lecture Notes in Mathematics, (Springer, Berlin), Vol. 170, p. 87.
Levine, H., 1980, "Comments of the Large N Master Field Approach," Harvard University preprint 80/A076.
Lieb, E., 1973, Commun. Math. Phys. **31**, 327.
Lovelace, C., 1981, Nucl. Phys. B **190**, 45.
Lovelace, C. 1982, "Universality at Large N", Rutgers Univ. preprint 82-01.
Makeenko, Yu. M., and A. A. Migdal, 1979, Phys. Lett. B **88**, 135; 1980, **89**, 437(E).
Makeenko, Yu. M., and A. A. Migdal, 1980, Phys. Lett. B **97**, 253.
Marchesini, G., and E. Onofri, 1980, J. Math. Phys. **21**, 1103.
Mehta, M. L., 1967, *Random Matrices,* (Academic, New York).
Migdal, A. A., 1980, Ann. Phys. (N.Y.) **126**, 279.
Mlodinow, L. D., and N. Papanicolaou, 1980, Ann. Phys. (N.Y.) **128**, 314.
Mlodinow, L. D., and N. Papanicolaou, 1981, Ann. Phys. **131**, 1.
Mondello, M., and E. Onofri, 1981, Phys. Lett. B **98**, 277.
Nambu, Y., 1979, Phys. Lett. B **80**, 372.
Paffuta, G., and P. Rossi, 1980, Phys. Lett. B **92**, 321.
Papanicolaou, N., 1981, Ann. Phys. **136**, 210.
Perelomov, A. M., 1972, Commun. Math. Phys. **26**, 222.
Polyakov, A. M., 1979, Phys. Lett. B **82**, 247.
Sakita, B., 1980, Phys. Rev. D **21**, 1067.
Schnitzer, H., 1974, Phys. Rev. D **10**, 1800.
Shankar, R., 1980, Phys. Rev. Lett. **45**, 1088.
Shapiro, J., 1981, Nucl. Phys. B **184**, 218.
Simon, B., 1980, Commun. Math. Phys. **71**, 247.
Souriau, J. M., 1970, *Structures des systèmes dynamiques,* (Dunod, Paris).
Stanley, H. E., 1968, Phys. Rev. **176**, 718.
Thirring, W., 1981, Commun. Math. Phys. **79**, 1.
Veneziano, G., 1976, Nucl. Phys. B **117**, 519.
Wadia, S. R., 1980a, Phys. Lett. B **93**, 403.
Wadia, S. R., 1981, Phys. Rev. D **24**, 970.
Wigner, E. P. 1959, *Proceedings of the 4th Canadian Mathematical Congress, Banff 1957,* (University of Toronto, Toronto), p. 174.
Wilson, K., 1973, Phys. Rev. D **7**, 2911.
Witten, E., 1979a, Nucl. Phys. B **160**, 57.
Witten, E., 1979b, "The 1/N Expansion in Atomic and Particle Physics," Harvard University preprint 79/A078; Cargese Summer Institute 1979 (QCD 161:S77:1979), p. 403.
Yaffe, L. G. 1982 (unpublished).