Electron scattering by ionized impurities in semiconductors

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Theories of electron scattering by ionized impurities in semiconductors are reviewed. The early foundations based on the Born approximation and their subsequent refinements are discussed thoroughly. The phase-shift method which is not restricted to the Born approximation is also presented. The situation in heavily doped semiconductors is described. The theories are then compared critically with experiments. Finally, conclusions are drawn and some plausible lines of future work are outlined.

CONTENTS

LIST OF ACRONYMS

BH	Brooks-Herring
CA	Csavinszky-Adawi
CW	Conwell-Weisskopf
FC	Falicov-Cuevas

GLOSSARY OF SYMBOLS

a	Lattice parameter
a_B	Effective Bohr radius
a_c	Temperature-independent inverse correlation length
a_r	Range of potential
C_e	Factor reducing ionized impurity scattering mobility due to electron-electron scattering
$D(q;k)$	Lorentzian function
d	Average distance between impurities
E	Energy of an electron
E_F	Fermi energy
$\mathcal{E}$	Electric field
e	Electron charge
$\mathcal{F}_j$	Fermi-Dirac integral of order j
f	Distribution function of the carriers
f_0	Equilibrium distribution function
$H(k)$	Screening function
$\hbar$	Planck's constant divided by 2π
$j_l(kr)$	Spherical Bessel function
$\mathbf{k}$	Wave vector
k_B	Boltzmann's constant
k_0	$(2m^*k_BT/\hbar^2)^{1/2}$
L	Screening length
L_p	Correlation length of potential
m^*	Band-edge effective mass
m_0	Mass of an electron in vacuum
N	Ionized impurity concentration
N^+	Ionized donor concentration
N^-	Ionized acceptor concentration
N_D	Total donor concentration
N_A	Total acceptor concentration
ΔN	Additional impurities (after irradiation)
n	Uniform electron concentration
$n(r)$	Electron concentration at a distance r from an impurity ion
$P(\mathbf{k},\mathbf{k}')$	Probability of transition from wave vector $\mathbf{k}$ to wave vector $\mathbf{k}'$ per unit time
$\mathbf{q}$	Change in wave vector upon scattering
$\mathbf{r}$	Position coordinate
T	Absolute temperature
$\langle U^2\rangle_{ms}$	Mean-square value of potential fluctuations
$V(r)$	Impurity potential
v	Electron velocity
W	Pair correlation function
Z	Charge on the impurity in units of e
z	$\cos\beta$
β	Angle of scattering
β_s	Inverse screening length
Γ	Relaxation rate
$\nabla_{\mathbf{k}}$	Gradient operator in $\mathbf{k}$ space
$\delta(x)$	Dirac delta function
δ_B	Correction due to departure from Born approximation
δ_D	Impurity dressing correction
δ_M	Multiple-scattering correction
$\delta_l(E)$	Phase shift of lth partial wave
δ_l^B	Phase shift in the Born approximation
ε	Static permittivity of the semiconductor

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ε_0	Free-space permittivity
η	$E_F(k_BT)^{-1}$
η_0	$E(k_BT)^{-1}$
θ	Angle between k and $\mathcal{E}$
λ	Electron wavelength
μ	Drift mobility
μ_{BH}	Drift mobility in the Brooks-Herring approximation
μ_{CW}	Drift mobility in the Conwell-Weisskopf approximation
μ_H	Hall mobility
$\rho(E)$	Density of states
$\sigma(\beta)$	Differential scattering cross section
σ_c	Scattering cross section
τ	Relaxation time
τ_{BH}	Relaxation time in the Brooks-Herring approximation
τ_{CW}	Relaxation time in the Conwell-Weisskopf approximation
τ_D	Electron deflection time
Ω	Crystal volume

I. INTRODUCTION

A. Significance of ionized impurity scattering

Impurities influence the electronic behavior of semiconductors in two ways. The bound states of a carrier localized at an impurity and the associated energy to create a delocalized state determine the *type* and *number* of the charge carriers and their *lifetime* through recombination involving such bound states. The impurities also scatter the carriers, leading to changes in their motion under the influence of an electric field; therefore the *mobility* is the affected observable.

Scattering of carriers by ionized impurities is most important at low temperatures, where phonon effects are small. Sufficiently large doping levels may extend this regime towards room temperature. Figure 1 shows an example of carrier mobility as a function of temperature for two levels of doping with impurities. The mobility limited by ionized impurity scattering decreases with decreasing temperature, since the low-temperature carriers have correspondingly smaller thermal velocities, which lead to a larger change in momentum through the Coulomb interaction between the impurity ion and the charged carriers. The low-temperature mobility is thus a measure of the impurity or defect content of a semiconductor specimen. The maximal mobility and the temperature of this maximum are in fact often quoted by materials suppliers and materials users as a measure of the purity and perfection of a crystal; higher mobility and lower temperature of maximal mobility indicate better quality. A mobility analysis within the ionized impurity scattering regime provides information about chemical impurities which is in most cases more sensitive than conventional chemical analysis. Advances in our understanding of scattering by ionized impurities are intimately linked to advances in the materials sciences.

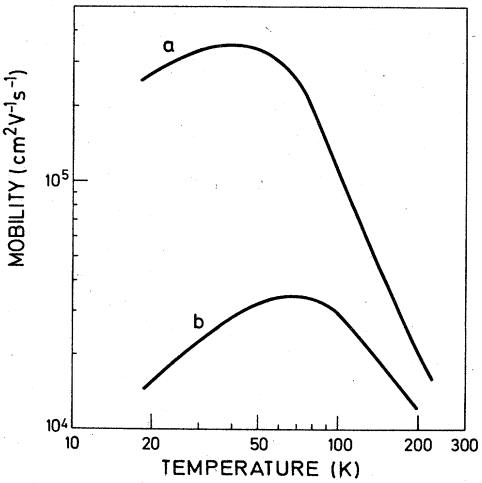

FIG. 1. Electron mobility versus temperature for n-type GaAs. Curves a and b are for room-temperature carrier concentrations of 3×10^{13} and 1.5×10^{15} cm^{-3}, respectively.

Bound states at shallow and deep impurities are currently a lively topic of experimental and theoretical investigations. Shallow donors and acceptors may be described by a hydrogenic "effective-mass theory" while the theory of deep impurity levels is much more difficult (Pantelides, 1978). The present great efforts devoted to the bound states will undoubtedly stimulate new interest in the parallel problems concerning ionized impurity scattering. We therefore consider it timely and useful to present this review on the current state of theory and experiment for this specific process affecting mobility.

B. Outline of this review

We start in Sec. II with the Boltzmann equation formalism which is most frequently used in describing the carrier transport properties in semiconductors. The early theories of ionized impurity scattering which look upon the scattering as a small perturbation on the carrier motion, or in other words, using the Born approximation, are outlined in Sec. III in order to facilitate further discussion. The problems with these theories and their subsequent refinements are then presented in Sec. IV. Corrections to the Born approximation are considered in Sec. V. Section VI contains a discussion of the comparative importance of the various refinements of the Brooks-Herring theory. Section VII is devoted to a description of phase-shift analysis which is not restricted to the Born approximation. The problems which are of current interest in heavily doped semiconductors are covered in Sec. VIII. We compare theory with experiment in Sec. IX, for elemental as well as for compound semiconductors, and discuss the cases of agreement and disagreement. Finally, in Secs. X and XI we offer some conclusions and discuss the prospects for future work.

II. SOLUTION OF THE BOLTZMANN TRANSPORT EQUATION

The Boltzmann equation has had remarkable success in describing the transport properties of semiconductors (see, for example, Ziman, 1960; Conwell, 1967; Nag, 1980). The calculation, with the help of the Boltzmann equation, of carrier mobility as limited by

ionized impurity scattering is described below.[1]

We concentrate on n-type materials, i.e., take the current carriers to be electrons. Furthermore, we consider an isotropic conduction-band structure. This assumption holds well for most direct-gap semiconductors. For indirect-gap semiconductors the conduction band valleys are anisotropic, and here further elaborations are necessary in describing electron transport (Blatt, 1957a; Paige, 1964). The case of anisotropic band structures will be discussed in Sec. IV.C.

Charge-carrier transport problems are concerned with the determination of the *distribution function* $f(\mathbf{k}, \mathbf{r}, t)$ in the presence of applied fields. This function gives the probability that the state corresponding to the Bloch wave vector $\mathbf{k}$ at a point in the crystal given by the position vector $\mathbf{r}$ is occupied by a current carrier at time t. If we restrict our attention to conditions of homogeneity and to steady-state problems under the influence of a time-independent electric field $\mathcal{E}$, f becomes a function of $\mathbf{k}$ only. The Boltzmann equation giving the function f then reads

$$(e\mathcal{E}/\hbar)\cdot\nabla_{\mathbf{k}} f = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} , \tag{1}$$

where e is the magnitude of the electron charge, $\hbar$ is Planck's constant divided by 2π, $\nabla_{\mathbf{k}}$ is the gradient operator in the $\mathbf{k}$ space, and $(\partial f/\partial t)_{\text{coll}}$ indicates the rate of change of f due to collisions.

We introduce a spherical coordinate system for $\mathbf{k}$ with the polar axis along $\mathcal{E}$. Because of symmetry, $f(\mathbf{k})$ can be a function of k and the polar angle θ only, and can be expanded in Legendre polynomials. At low fields the drift velocity is much smaller than the thermal velocity, and here the Legendre polynomial expansion functions fall off rapidly. It is therefore adequate to truncate the series after the second term. We shall hence write

$$f(\mathbf{k}) = f_0(E) - \frac{e\hbar\mathcal{E}}{m^*} k\Phi(E)\frac{\partial f_0}{\partial E}\cos\theta , \tag{2}$$

where E is the energy for wave vector $\mathbf{k}$ and m^* is the band-edge effective mass. The function $f_0(E)$ is the equilibrium Fermi-Dirac distribution function; in the special case of nondegeneracy, $f_0(E)$ reduces to the Maxwell-Boltzmann form. The equation for the function $\Phi(E)$ is obtained by substituting Eq. (2) into Eq. (1). For this purpose we need the expression for $(\partial f/\partial t)_{\text{coll}}$, which we consider below.

If $P(\mathbf{k}, \mathbf{k}')$ denotes the *a priori* probability of transition of an electron from the state $\mathbf{k}$ to the state $\mathbf{k}'$ per unit time, we may write (Blatt, 1968)

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = \frac{\Omega}{8\pi^3}\int \{P(\mathbf{k}', \mathbf{k}) f(\mathbf{k}')[1 - f(\mathbf{k})] - P(\mathbf{k}, \mathbf{k}') f(\mathbf{k})[1 - f(\mathbf{k}')]\} d\mathbf{k}' , \tag{3}$$

where Ω is the crystal volume. For a nondegenerate electron distribution, $f \ll 1$, so that the quantities $[1 - f(\mathbf{k})]$ and $[1 - f(\mathbf{k}')]$ in Eq. (3) may be replaced by unity. We shall, however, retain these terms to see the effects of degeneracy. Detailed balance under thermal equilibrium gives

$$P(\mathbf{k}', \mathbf{k}) f_0(E')[1 - f_0(E)] = P(\mathbf{k}, \mathbf{k}') f_0(E)[1 - f_0(E')] , \tag{4}$$

where E' is the energy for wave vector $\mathbf{k}'$.

Substituting Eq. (2) in Eq. (3), using Eq. (4), and retaining the terms up to the first order in $\mathcal{E}$, we obtain

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = \frac{\Omega}{8\pi^3}\frac{e\hbar\mathcal{E}}{m^* k_B T}\int P(\mathbf{k}, \mathbf{k}') f_0(E)[1 - f_0(E')] \times [\Phi(E')k'\cos\theta' - \Phi(E)k\cos\theta] d\mathbf{k}' , \tag{5}$$

where we have used the identity

$$f_0(E)[1 - f_0(E)] = -k_B T\frac{\partial f_0}{\partial E} , \tag{6}$$

k_B and T being the Boltzmann constant and the lattice temperature, respectively.

We shall now introduce a spherical coordinate system with the $\mathbf{k}$ direction as the polar axis. In this system let (k', β, ϕ) be the spherical coordinates of $\mathbf{k}'$, the azimuthal angle ϕ being measured from the $\mathbf{k} - \mathcal{E}$ plane (Fig. 2). One may then write

$$k'\cos\theta' = k'\cos\beta\cos\theta + k'\sin\beta\cos\phi\sin\theta \tag{7}$$

and

$$d\mathbf{k}' = k'^2 dk'\sin\beta\, d\beta\, d\phi . \tag{8}$$

We shall substitute Eq. (7) and Eq. (8) into Eq. (5) and use Eq. (6). Noting further that $P(\mathbf{k}, \mathbf{k}')$ does not depend on ϕ due to symmetry, we obtain

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = \frac{\Omega}{4\pi^2}\frac{e\hbar\mathcal{E}}{m^*} k\cos\theta\int_{k'}\int_{z=-1}^{+1} P(\mathbf{k}, \mathbf{k}')\frac{1 - f_0(E')}{1 - f_0(E)}\left(\Phi(E) - \frac{k'}{k} z\Phi(E')\right)\frac{\partial f_0}{\partial E} k'^2 dk'\, dz , \tag{9}$$

where $z = \cos\beta$, and the range of integration over k' must be determined from energy conservation.

We are here concerned with scattering by ionized impurity centers. These centers are massive and are considered fixed in a scattering event. Furthermore the scattering is considered perfectly elastic, so that $k' = k$. Equation (4) then gives $P(\mathbf{k}, \mathbf{k}') = P(\mathbf{k}', \mathbf{k})$; also, Eq. (9) can be written as

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = -\frac{f(\mathbf{k}) - f_0(E)}{\tau} , \tag{10}$$

where

$$\frac{1}{\tau} = \frac{\Omega}{4\pi^2}\int_{k'}\int_{z=-1}^{+1}(1 - z)P(\mathbf{k}, \mathbf{k}')k'^2 dk'\, dz . \tag{11}$$

Equation (10) indicates that when the external forces setting up the distribution function $f(\mathbf{k})$ are suddenly removed, the perturbation in the distribution function decays exponentially with the time constant τ. Hence τ

[1]The equation was originally given by Boltzmann (1872) for a dilute gas. A good discussion of this equation may be found in the books by Ferziger and Kaper (1972) and Ziman (1960).

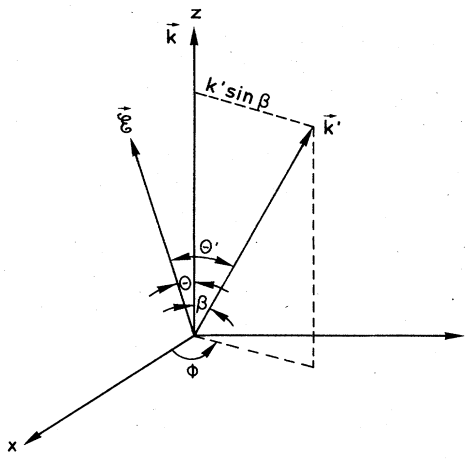

FIG. 2. Spherical coordinate system for k′.

is known as the *relaxation time* for the scattering. It is noteworthy that the dependence of τ on the degeneracy of the electron distribution, if any, enters through the function $P(\mathbf{k},\mathbf{k}')$. For ionized impurity scattering, we shall see that $P(\mathbf{k},\mathbf{k}')$ is influenced by degeneracy through the screening effect of the free carriers. The relaxation time for ionized impurity scattering is therefore also affected by degeneracy if free-carrier screening is included. It must be stressed, however, that the mobility which involves an average of τ over the carrier distribution is affected by degeneracy even in the absence of screening [see Eqs. (16) and (18) below].

The probability that an electron in the state k is scattered in a time interval dt is dt/τ. We may also introduce here the expression for the *scattering cross section* σ_c (Schiff, 1968; Seeger, 1973). If N is the number density of the scattering centers, each with cross section σ_c, and v is the velocity of the electron, the probability that the electron is scattered in a time interval dt is $N\sigma_c v\,dt$. Since $v=(1/\hbar)(\partial E/\partial k)$, we have

$$\sigma_c=\frac{\hbar}{\tau N(\partial E/\partial k)} =\frac{\hbar}{N(\partial E/\partial k)}\,\frac{\Omega}{4\pi^2}\int_{k'}\int_{z=-1}^{+1}(1-z)P(\mathbf{k},\mathbf{k}')k'^2dk'\,dz\,. \tag{12}$$

σ_c is related to the *differential scattering cross section* $\sigma(\beta)$ by

$$\sigma_c=2\pi\int_{-1}^{+1}\sigma(\beta)(1-z)dz\,. \tag{13}$$

Comparing Eq. (13) with Eq. (12), we have

$$\sigma(\beta)=\frac{\hbar}{N(\partial E/\partial k)}\,\frac{\Omega}{8\pi^3}\int_{k'}P(\mathbf{k},\mathbf{k}')k'^2dk'\,. \tag{14}$$

Substituting Eq. (10) in the Boltzmann equation (1), using Eq. (2), and retaining only the terms linear in $\mathcal{E}$, we obtain

$$\Phi(E)=\frac{m^*\tau}{\hbar^2k}\left(\frac{\partial E}{\partial k}\right). \tag{15}$$

The *drift mobility* μ is given by

$$\mu=\frac{\int(1/\hbar)\nabla_k Ef(k)dk}{\mathcal{E}\int f(\mathbf{k})d\mathbf{k}} =\frac{e}{3m^*}\int_0^\infty\Phi\left(-\frac{\partial f_0}{\partial k}\right)k^3dk\Big/\int_0^\infty f_0k^2dk=\frac{e\langle\tau\rangle}{m^*}\,, \tag{16}$$

where the angular brackets represent an average given by

$$\langle x\rangle=\frac{m^*}{3\hbar^2}\int_0^\infty x\frac{\partial E}{\partial k}\left(-\frac{\partial f_0}{\partial k}\right)k^2dk\Big/\int_0^\infty f_0k^2dk\,. \tag{17}$$

The *Hall mobility* μ_H is given by

$$\mu_H=\frac{e}{m^*}\,\frac{\langle\tau^2\rangle}{\langle\tau\rangle}\,. \tag{18}$$

It is clear that μ and μ_H involve f_0 and are hence affected by the degeneracy of the electron distribution.

There are methods which obtain mobilities without going through the Boltzmann equation treatment. Luong and Shaw (1971) use the Kubo formula. Another method has been proposed by Gerlach (1974). He first calculates the energy loss which arises when a scattering center moves through the electrons at rest, and then obtains the energy loss by Joule heating. A (static or frequency-dependent) resistivity results. The dielectric function of the solid is needed; it can be used so as to include screening and plasmon effects automatically. The results agree with those of the Boltzmann formalism; see Farvaque and Gerlach (1976). Scattering on pairs and clusters of impurities has also been treated by means of this technique (Gerlach and Harbecke, 1980), as has the anisotropic mobility due to dislocations in A^{III}-B^{V} compounds (Booyens, Vermaak, and Proto, 1978).

III. THE BORN APPROXIMATION APPROACH

The determination of mobility, as seen above, reduces to the problem of evaluating the transition probability $P(\mathbf{k},\mathbf{k}')$ for ionized impurity scattering. The task is relatively simple if $P(\mathbf{k},\mathbf{k}')$ is evaluated in the Born approximation, i.e., to the first order in the interaction potential between the particle and the scattering center. The Born approximation is good if the scattering potential falls off rapidly at large distances, and if the carrier energy or the temperature is high (Born, 1926; see also the textbook by Messiah, 1970). In this approximation, time-dependent perturbation theory gives for a parabolic band (Schiff, 1968; Rode, 1975)

$$P(\mathbf{k},\mathbf{k}')=\frac{2\pi}{\hbar}\,\frac{Ne^2}{\Omega}\,|V(q)|^2\delta(E'-E)\,, \tag{19}$$

where the Dirac delta function indicates the elastic nature of the scattering event. The randomly located N centers per unit volume are assumed to scatter independently, $\mathbf{q}=\mathbf{k}'-\mathbf{k}$ is taken to be much smaller than a reciprocal lattice vector, and $V(q)$ is related to the Fourier transform of an impurity potential $V(r)$:

$$V(q)=\int V(r)\exp(-i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}\,. \tag{20}$$

The relaxation time or the scattering cross section for ionized impurity scattering has been obtained by various workers by making different approximations to the form of the potential $V(r)$. These will be discussed now. We

start with the early foundations, as these are frequently used in analyzing experimental results.

A. Conwell-Weisskopf approximation

Here the potential $V(r)$ due to an impurity atom with charge Ze is taken as a Coulomb potential (Conwell and Weisskopf, 1950)

$$V(r)=Ze(4\pi\varepsilon r)^{-1}, \tag{21}$$

where ε is the static permittivity of the material, and the SI system of units is used. The potential in question is the difference between the potential in the actual crystal and that in an ideal crystal and is represented by Eq. (21) except for values of r smaller than the lattice parameter a. Here we treat the electron as a classical particle with a well-defined position or trajectory and take the use of Eq. (21) as a good approximation. This assumption will hold at low impurity densities for the following reasons. To the extent that the electron has a well-defined **k** value, it is delocalized in real-space ($\Delta x\Delta k \gtrsim 1$). The classical or semiclassical picture can be justified in terms of wave packets of Bloch functions, and the spatial extent of the wave packet may be much less than the interimpurity distances if the impurity concentration is not high. Also, in such cases, the distances of closest approach to the impurity in Coulomb scattering are ordinarily larger than the lattice parameter a.

Substituting Eq. (21) in Eq. (20) we have

$$|V(q)|=\frac{Ze}{4\pi\varepsilon}\left|\int_0^{2\pi}d\phi\int_0^{\infty}r\,dr\int_0^{\pi}\exp(-iqr\cos\alpha)\sin\alpha\,d\alpha\right| = Ze(q^2\varepsilon)^{-1}. \tag{22}$$

If we employ this expression in Eqs. (19) and (11), we obtain

$$\frac{1}{\tau}=\frac{NZ^2e^4E^{-3/2}}{16\sqrt{2}\,\pi\varepsilon^2m^{*1/2}}\int_{-1}^{+1}\frac{dz}{1-z}, \tag{23}$$

where we have used the relation

$$E=\hbar^2k^2/(2m^*) \tag{24}$$

for a parabolic band structure.

The integral in Eq. (23) diverges for $z=1$, i.e., for $\beta=0$ (zero-angle scattering). This corresponds to electrons passing very far from the scattering center. At such distances the electron is more likely to be scattered by other centers than by the particular center in question. Conwell and Weisskopf (1950) assumed that the electron is scattered at any time only by the center to which it is closest. Thus the electron is deflected by a particular center only when it comes within a distance $b=d/2$ of that center, where $d=N^{-1/3}$, the average distance between the impurities. This implies that the *impact parameter* (which is the perpendicular distance between the scattering center and the projection of the initial line of approach of the electron) is cut off at $d/2$. The divergence of the integral in Eq. (23) is thus resolved by setting the upper limit of integration equal to $\cos\beta_0$, where (Conwell and Weisskopf, 1950)

$$\tan\left(\frac{\beta_0}{2}\right)=\frac{Ze^2N^{1/3}}{4\pi\varepsilon E}. \tag{25}$$

One then obtains from Eq. (23) the relaxation time τ

$$\frac{1}{\tau_{CW}}=\frac{NZ^2e^4E^{-3/2}}{16\sqrt{2}\,\pi\varepsilon^2m^{*1/2}}\ln\left(\operatorname{cosec}^2\frac{\beta_0}{2}\right) =\frac{NZ^2e^4}{16\sqrt{2}\,\pi\varepsilon^2m^{*1/2}}E^{-3/2}\ln\left(1+\frac{16\pi^2\varepsilon^2E^2}{Z^2e^4N^{2/3}}\right), \tag{26}$$

where the subscript CW indicates that this formula is due to Conwell and Weisskopf. Equation (16) then yields for a nondegenerate electron gas[2]

$$\mu_{CW}=\frac{64\sqrt{2}\,\pi^{1/2}\varepsilon^2}{3m^{*1/2}(k_BT)^{5/2}NZ^2e^3}\int_0^{\infty}\frac{E^3\exp(-E/k_BT)dE}{\ln(1+16\pi^2\varepsilon^2E^2/Z^2e^4N^{2/3})}. \tag{27}$$

The integral in Eq. (27) cannot be evaluated analytically. However, since the logarithm is a slowly varying function, a good approximation to the value of the integral is obtained by assuming that the logarithm has a constant value which is equal to the value it attains at the maximum of the function in the numerator. This maximum occurs at $E=3k_BT$. Evaluating the simplified integral, we obtain

$$\mu_{CW}=\frac{128\sqrt{2}\,\pi^{1/2}\varepsilon^2(k_BT)^{3/2}}{m^{*1/2}NZ^2e^3}\left[\ln\left(1+\frac{144\pi^2\varepsilon^2k_B^2T^2}{Z^2e^4N^{2/3}}\right)\right]^{-1}. \tag{28}$$

Neglecting the variation of the logarithm, we observe that the mobility due to ionized impurity scattering is approximately proportional to $T^{3/2}$ and to $m^{*-1/2}$. Deviations from the $T^{3/2}$ dependence on temperature may, however, occur in practical situations because of the log term (see, for example, Shockley, 1950). If the relaxation time were proportional to $E^{3/2}$, the Hall-to-drift mobility ratio for a nondegenerate distribution would be $\mu_H/\mu=315\pi/512=1.93$.

The CW treatment has been criticized for the artificiality of the way in which the small-angle scattering was cut off (Sclar, 1956). Also, the effect on the potential of the distribution of space charges around the impurity was not taken into account. The space-charge distribution around the impurity produces screening so as to weaken the Coulomb potential at large distances. The problem has been considered by Dingle (1955), Mansfield (1956), and Brooks and Herring (Brooks, 1951, 1955), and is discussed in the following subsection.

B. Dingle and Brooks-Herring approximations

For simplicity, we assume for the present that only the electrons contribute to screening. The modifications due to screening by other charges will be considered later in this section.

The electron concentration $n(r)$ at a distance r from an ionized impurity is given by (Dingle, 1955)

$$n(r)=\frac{1}{\sqrt{2}\,\hbar^3}\left(\frac{m^*k_BT}{\pi}\right)^{3/2}\mathcal{F}_{1/2}\left(\eta+\frac{eV(r)}{k_BT}\right), \tag{29}$$

[2]The case of a degenerate electron gas is considered by Johnson and Lark-Horovitz (1947). We also refer the reader to Shockley (1950) for a discussion of degenerate semiconductors.

where $\eta = E_F (k_B T)^{-1}$, E_F being the Fermi energy, and $\mathcal{F}_j$ is the Fermi-Dirac integral of order j (Blakemore, 1962):

$$\mathcal{F}_j(y_0) = \frac{1}{\Gamma(j+1)} \int_0^\infty \frac{y^j \, dy}{1 + \exp(y - y_0)} . \tag{29'}$$

$V(r)$ is the impurity potential to be determined from Poisson's equation

$$\nabla^2 V(r) = e[n(r) - n]/\varepsilon , \tag{30}$$

where n is the uniform electron concentration and is derived from Eq. (29) by putting $V(r) = 0$.

Equation (29) is now used in Eq. (30), and $eV(r)/k_B T$ is assumed to be small compared with η. One may then linearize Eq. (30) by expanding the function $\mathcal{F}_{1/2}$ in a Taylor series and retaining only the terms up to the first order in $eV(r)/k_B T$. Using the boundary conditions $V(r \to 0) = Ze(4\pi\varepsilon r)^{-1}$ and $V(r \to \infty) = 0$, one obtains the "screened Coulomb potential"

$$V(r) = \frac{Ze}{4\pi\varepsilon r} \exp(-\beta_s r) , \tag{31}$$

where β_s is the *inverse screening length* or the *inverse screening radius:*

$$\beta_s^2 = \frac{ne^2}{\varepsilon k_B T} \frac{\mathcal{F}_{-1/2}(\eta)}{\mathcal{F}_{1/2}(\eta)} . \tag{32}$$

The form of the potential given by Eq. (31) occurs in many branches of physics. In nuclear physics it is known as the "Yukawa potential"; in physics of electrolytes it is familiar as the "Debye-Hückel potential." The screening length is generally termed the "Thomas-Fermi" or "Debye-Hückel" screening length, depending on whether the carriers obey the Fermi-Dirac or the classical distribution.

Equation (31) is used in the theory of ionized impurity scattering by assuming that the screening length is much larger than the de Broglie wavelength, or more adequately that $4k^2/\beta_s^2 \gg 1$. This condition is satisfied for low carrier concentrations and high temperatures and is incorporated in the Born approximation which we conveniently use in describing the scattering (Rode, 1975).

Substituting Eq. (31) in Eq. (20), we have

$$|V(q)| = \frac{Ze}{4\pi\varepsilon} \left| \int_0^{2\pi} d\phi \int_0^\infty r \, dr \int_0^\pi \exp[-r(\beta_s + iq\cos\alpha)] \sin\alpha \, d\alpha \right| = \frac{Ze}{\varepsilon q} \int_0^\infty \exp(-r\beta_s) \sin qr \, dr = \frac{Ze}{\varepsilon (q^2 + \beta_s^2)} . \tag{33}$$

Using this in Eqs. (19) and (11), we obtain the Brooks-Herring formula:

$$\frac{1}{\tau_{BH}} = \frac{NZ^2 e^4}{16\sqrt{2}\,\pi\varepsilon^2 m^{*1/2}} E^{-3/2} \left(\ln(1+b) - \frac{b}{1+b} \right) , \tag{34}$$

where

$$b = \frac{4k^2}{\beta_s^2} = \frac{8m^* E}{\hbar^2 \beta_s^2} . \tag{35}$$

The Born approximation presupposes[3] that $b \gg 1$, so that the quantity within the bracket in Eq. (34) may be replaced by $\ln b$. This condition is usually satisfied for nondegenerate materials: For example, for GaAs at room temperature $b \simeq 3.3 \times 10^{18}/n[\mathrm{cm}^{-3}]$ and is thus larger than unity for reasonably small concentrations, n.

The mobility is then calculated for a nondegenerate electron gas as in the CW theory: The logarithm is taken outside the integral, replacing E in it by $3k_B T$ at which the rest of the integrand is a maximum. The result is

$$\mu_{BH} = \frac{128\sqrt{2}\,\pi^{1/2}\varepsilon^2 (k_B T)^{3/2}}{m^{*1/2} N Z^2 e^3} \left(\ln \frac{24 m^* \varepsilon (k_B T)^2}{ne^2 \hbar^2} \right)^{-1} . \tag{36}$$

The Brooks-Herring (BH) mobility [Eq. (36)] has the same form as the CW result [Eq. (28)] except for the differing logarithmic factor. In the BH analysis the low-angle scattering is automatically limited by screening considerations.[4]

Neglecting the energy dependence of the terms within the large parenthesis in Eq. (34), one finds, as in the CW model, that the Hall-to-drift mobility ratio is a constant, namely, 1.93. More exact treatments show, however, that this ratio is less than 1.93 and depends on temperature and carrier concentration (Blatt, 1957c; Chattopadhyay, 1981a). It is further seen that if the temperature dependence of the screening radius is considered, the temperature dependence of mobility will be different from $T^{3/2}$. Let us first consider the case in which the carrier concentration does not change with temperature. In this case, the screening radius clearly increases with temperature. As a result, the average relaxation time $\langle \tau \rangle$ increases less rapidly with tem-

[3]The same condition for the validity of the BH treatment also follows from a solution of the perturbation problem in a uniform electron gas (March and Murray, 1962).

[4]A formal bridge between the CW and the BH formula has recently been provided by Ridley (1977). He observes that both theories consider a characteristic length defining the range of the scattering potential—screening length in the case of BH, half the average distance between centers in the case of CW. Also, both theories assume that within the characteristic length only one center is present, only that center scatters, and all other centers are ineffective. Ridley (1977) quantifies the assumption that only one center is operative by introducing the probability that another center does not interfere. This is done by weighing the differential scattering cross section $\sigma(\beta)$ by the probability $P(b)$ that there is no scattering center with impact parameter less than b, where b is the impact parameter for the angle β. This procedure ensures a rapidly decreasing scattering probability at large impact parameters and removes the divergence problem of the CW treatment. Furthermore, a smooth transition from the screened limit (BH) to the unscreened limit (CW) is obtained.

perature than it would if the temperature variation of the screening radius were ignored (Blatt, 1957a). This would make the mobility increase with temperature less rapidly than $T^{3/2}$. An additional temperature dependence of the screening radius will have to be incorporated if the carrier concentration also varies with temperature. This may occur in the temperature range of partial ionization, or in the intrinsic range which may be important in narrow-gap semiconductors at modest temperatures, or for medium-gap semiconductors at high temperatures.

Equation (36) is appropriate when only one kind of impurity (donor) is present and the impurity concentration N equals the carrier concentration n. In a *compensated* semiconductor both donors and acceptors are present, and the neighboring ionized impurities also contribute to screening (Brooks, 1955; Falicov and Cuevas, 1967). This contribution has been accounted for by Brooks and Herring on the basis of a linearized Poisson's equation: Only the terms up to the first order in $eV(r)/k_BT$ and in the departure of the local impurity densities from the average values are considered, and a random distribution of impurities is assumed (Falicov and Cuevas, 1967). One then finds that the relaxation time is given by Eq. (34) with N equal to $(N^+ + N^-)$, and with β_s^2 given by Eq. (32) with the term $(e^2/\varepsilon k_BT)(N_D - N^- - n)(n + N^-)/N_D$ added to the right-hand side. Here N^+ and N^- are the ionized donor and acceptor concentrations, respectively, and N_D is the total donor concentration. This extra term in β_s^2 is, however, insignificant at sufficiently high temperatures when all the donors are ionized. It is important at low temperatures when carrier freeze-out occurs, and the ionized impurity concentration N largely exceeds the electron concentration n.

IV. PROBLEMS AND REFINEMENTS WITHIN THE BORN APPROXIMATION

The BH theory has on the whole been preferred over the CW theory. Several attempts have been made to improve the BH treatment by removing one or more of its simplifying assumptions or by allowing for effects neglected by it. Table I gives a summary of these phenomena with a schematic explanation of the effects, their treatments, and their ranges of significance. We shall now discuss these refinements in detail.

A. Wave-vector-dependent dielectric function

1. Polarization due to the colliding electron

In the BH analysis, $V(r)$ is treated as the potential for the colliding electron, and no deformation of the screening cloud in the field of the colliding electron is taken into account. Actually, the colliding electron will be screened by the other electrons with a hole scooped out by itself, i.e., the screening cloud will be polarized by the colliding electron. The effect of the screening will thus be somewhat less than that predicted by Brooks and Herring. This problem is dealt with by Takimoto (1959), who shows that the transformed potential incorporating the polarization effect is given by

$$|V(q)| = \frac{Ze}{\varepsilon[q^2 + H(k)]}, \tag{37}$$

where

$$H(k) = -\frac{e^2}{\varepsilon}\sum_{\mathbf{k}} \frac{f(E_{\mathbf{k}-\mathbf{q}}) - f(E_{\mathbf{k}})}{E_{\mathbf{k}-\mathbf{q}} - E_{\mathbf{k}}}, \tag{38}$$

$f(E_{\mathbf{k}})$ being the distribution function of electrons with energy $E_{\mathbf{k}}$. Writing Eq. (37) as

$$|V(q)| = Ze(q^2\varepsilon_q)^{-1}, \tag{39}$$

we have for the q-dependent dielectric function

$$\varepsilon_q = \varepsilon + \varepsilon H(k) q^{-2}. \tag{40}$$

A comparison of Eq. (37) with Eq. (33) shows that the screening parameter is modified to $H(k)$. When $q \to 0$, $H(k)$ reduces to β_s^2, and the polarization effect is negligible. For larger q, however, the screening parameter $H(k)$ deviates from β_s^2. For a nondegenerate electron gas, one has (Takimoto, 1959; Hall, 1962)

$$H(k) = \beta_s^2 F(\xi), \tag{41}$$

where

$$F(\xi) = \frac{1}{\sqrt{\pi}\,\xi}\int_0^\infty x \exp(-x^2) \ln\left|\frac{x+\xi}{x-\xi}\right| dx \tag{42}$$

and

$$\xi^2 = \hbar^2 q^2 (8m^* k_B T)^{-1}. \tag{43}$$

The corresponding mobility is given by (Takimoto, 1959)

$$\mu_T = \frac{64\sqrt{2\pi}\,\varepsilon^2 (k_BT)^{3/2}}{3m^{*1/2} N e^3 Z^2} \int_0^\infty \frac{\eta_0^3 \exp(-\eta_0) d\eta_0}{J_\gamma(\sqrt{\eta_0})}, \tag{44}$$

where

$$J_\gamma(\sqrt{\eta_0}) = 2\int_0^{\sqrt{\eta_0}} \xi^3 [\xi^2 + \gamma^2 F(\xi)]^{-2} d\xi, \tag{45}$$

$$\gamma^2 = ne^2\hbar^2 (8\varepsilon m^*)^{-1} (k_BT)^{-2},$$

and

$$\eta_0 = E(k_BT)^{-1}. \tag{46}$$

Takimoto (1959) simplified the integral in Eq. (44) by replacing $J_\gamma(\sqrt{\eta_0})$ by $J_\gamma(\sqrt{3})$, since the rest of the integral peaks at $\eta_0 = 3$. This approach is consistent with the CW or BH approximations. Hall (1962) gave suitable forms of the function $F(\xi)$ for an accurate evaluation of the integral in Eq. (44).

It is interesting at this point to compare the CW, BH, and Takimoto results. In the CW approximation $F(\xi) = 0$, and in the BH theory $F(\xi) = 1$. In the Takimoto model, $F(\xi)$ is unity at $\xi = 0$ and monotonically decreases to zero with increasing ξ (Takimoto, 1959). This is the expected result, since the polarization of the screening electron is less for small q and also for large T when the thermal agitation causes less accumulation of electrons around the impurity. At low concentrations, the CW, BH, and Takimoto values of electron mobility are nearly the same. As the carrier concentration increases, however, the three results are different: The CW value of mobility is smaller than the BH value, and the Takimoto value is intermediate between them. Takimoto has shown that for the simple case $N = n$, the

TABLE I. Survey of refinements to the Brooks-Herring theory and special effects.

Phenomenon considered	Sec. or Eq. in this paper	Range of applicability: Doping	Range of applicability: Temperature	Type of semiconductor	Mathematical method	Typical effect on mobility (with maximal order of magnitude)	Refs. for theory	Experimental evidence and Refs.
Polarization by colliding electron	Eqs. (37), (44)	high	low	particularly anisotropic bands	incorporation of the effect in impurity potential	mobility decreases by about a factor of 5	a,b	Ge, Si Refs. b,c
Valence dielectric screening	Eqs. (47), (48)	high	high	all	incorporation of the effect in impurity potential	mobility decreases by 10%	d,e	Si Ref. e
Nonparabolicity	IV.C	high	high	narrow-gap materials	inclusion of nonparabolicity in Boltzmann Eq.	mobility decreases by a factor of about 2	f,g	InSb Ref. h
Subsidiary minimum	IV.C	high	low	materials with higher-lying subsidiary minima	incorporation of dielectric screening	mobility abruptly increases by a factor of about 2 at a critical concentration	i	GaSb Ref. i
Conduction band anisotropy	IV.C	all	all	materials with off-center minima (Ge,Si)	inclusion of tensorial effective mass	decisive	j,k	Ge Ref. k
Valence band details	IV.C	all	all	p type	inclusion of the effect in Boltzmann equation	decisive	l	
Carrier-carrier scattering	IV.D	low	all	nondegenerate materials	use of variational principle or Kubo formula	mobility decreases by 40%	m,n	InSb Ref. h

TABLE I. *(Continued).*

Phenomenon considered	Sec. or Eq. in this paper	Range of applicability: Doping	Temp.	Type of semiconductor	Mathematical method	Typical effect on mobility (with maximal order of magnitude)	Refs. for theory	Experimental evidence and Refs.
Nonlinear screening	IV.E	medium	low	all	solution of nonlinear Poisson's Equation	mobility increases by about factor of 2	o	
Multiple scattering	Eqs. (58), (59)	medium	high	all	perturbation approach	mobility decreases by about 35%	p	GaAs Ref. p
Impurity dressing	Eqs. (58), (60)	medium	high	all	perturbation approach	mobility decreases by 20%	p	GaAs Ref. p
Failure of Born approximation	Eqs. (61), (62), (65)	high	low	all donors differ from acceptors!	perturbation approach or phase-shift analysis	depends on conditions	p,q	GaAs Ref. p
Impurity central-cell correction	VII.C	high	moderate, high	all, particularly high-permittivity materials, depends on dopant	phase-shift analysis	mobility decreases by about 40%	r	Si Ref. r; PbTe Ref. s
Spatial distribution of impurities	VIII, X, XI	high	moderate, low	all, depends on sample uniformity	detailed account of space charges, dipoles, correlation functions	mobility may change by large amounts		see text

[a]Takimoto (1959).
[b]Krieger and Meeks (1973).
[c]Krieger *et al.* (1974).
[d]Resta and Resca (1979).
[e]Scarfone and Richardson (1980).
[f]Rode (1975).
[g]Nag (1980).
[h]Bate *et al.* (1965).
[i]Robinson and Rodriguez (1964, 1965).
[j]Baranski *et al.* (1977).
[k]Krieger *et al.* (1971, 1972).
[l]Ralph (1977).
[m]Appel (1961).
[n]Luong and Shaw (1971).
[o]Chattopadhyay (1981a)
[p]Moore (1967).
[q]Boardman and Henry (1973).
[r]Ralph *et al.* (1975).
[s]Morita (1963).

differences between the three results become significant for concentrations above 10^{16} cm^{-3} for the following set of parameters: $T=80$ K, $\varepsilon/\varepsilon_0=16.1$, and $m^*=0.25m_0$ (see Fig. 3). The result is explained as follows. At low carrier concentrations, screening is less, and the potential is nearly Coulombic. Forward scattering predominates here, and so the polarization effect is small. At large concentrations the screening is important, and large-angle scattering makes the polarization effect significant.

2. Valence dielectric screening

The dielectric response of the valence electrons to the field of the ionized impurity is taken to be described by the static dielectric constant of the material in both the CW and BH theories. In recent years considerable attention has been paid to more realistic valence dielectric screening (Penn, 1962; Srinivasan, 1969; Grimes and Cowley, 1975; Resta, 1977; Cornolti and Resta, 1978). Attempts were made by Csavinszky (1976a, 1978) to incorporate in the impurity ion potential the dispersive screening of the dielectric medium beyond that carried by the static dielectric constant. The impurity ion potentials derived by him were also used by others to calculate the ionized impurity scattering mobilities (Paesler, 1978; Richardson and Scarfone, 1978, 1979; Theodorou and Queisser, 1979). Unfortunately these impurity ion potentials were shown to be incorrect and corrected subsequently (Csavinszky and Morrow, 1979; Meyer 1979).

Following an approach different from Csavinszky's, Resta (1979) developed a screening theory in a doped semiconductor at nonzero temperature. The theory was later applied by Resta and Resca (1979) to calculate the ionized impurity scattering mobility. Essentially, the theory gives a Fourier transform of the screened impurity ion potential in the form

$$|V(q)| = (Ze/q^2)[\varepsilon(q)+q^{-2}\varepsilon\beta_s^2]^{-1}, \tag{47}$$

where $\varepsilon(q)$ is the wave-vector-dependent dielectric function (Resta, 1979). The effective dielectric function of the doped semiconductor is thus

$$\varepsilon_{eff}(q)=\varepsilon(q)+\varepsilon q^{-2}\beta_s^2 . \tag{48}$$

This result actually means a summation of the susceptibilities due to the valence electrons and due to the free carriers. As $q\to 0$, $\varepsilon(q)\to\varepsilon$, the static permittivity, and $\varepsilon(q)$ decreases at larger q (Resta, 1977; Cornolti and Resta, 1978). The inverse Fourier transform of Eq. (47) shows that the corresponding potential is the simple screened Coulomb potential [Eq. (31)] for distances r larger than R, where R is very close to the nearest-neighbor distance in the crystal (Resta, 1979). For $r<R$, the potential is larger than the standard screened Coulomb potential. Since R is much less than any realistic value of β_s^{-1}, carrier screening clearly acts over a much larger distance scale than the dispersive valence screening. As the scattering potential is enhanced by the dispersive valence screening, one expects a reduction in mobility from the BH value. The effect is expected to assume importance at high temperatures, for then the carriers, due to their enhanced kinetic energy, are able to come closer to the impurity ion and feel the influence of dispersive valence screening. Calculations of Resta and Resca (1979), who also give an analytic expression for mobility, corroborate this expectation. They give numerical results for silicon over the concentration range 10^{15} to 10^{21} cm^{-3}. The departure from the Brooks-Herring result tends to zero at both high and low concentration limits. At intermediate concentrations ($\sim 10^{18}$–10^{19} cm^{-3}) the departure is a maximum but is not more than 5% at 300 K. It should be mentioned that Resta and Resca (1979) use classical statistics in their analysis. Using Fermi-Dirac statistics, Scarfone and Richardson (1980) get slightly different results. They find that the dispersive valence screening effects at 300 K are negligible for concentrations below 10^{18} cm^{-3} for Si and Ge, and for concentrations below 10^{19} cm^{-3} for GaAs. With increasing doping above these values, the mobility decreases monotonically from the BH value. However, the effect is not large, being less than 10% for concentrations as high as 10^{21} cm^{-3}. A recent discussion by Csavinszky and Morrow (1981) and Scarfone and Richardson (1981) concerns the problem of boundary conditions for the potential.

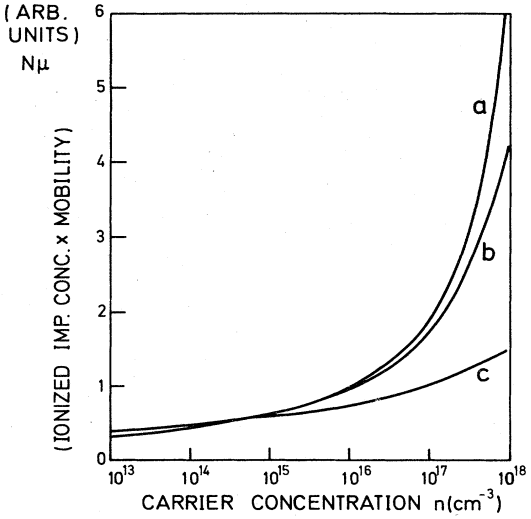

FIG. 3. Comparison of Brooks-Herring (curve a), Takimoto (curve b), and Conwell-Weisskopf (curve c) values of electron mobility, multiplied by impurity density N, for the following parameters: $m^*=0.25m_0$, $T=80$ K, $\varepsilon/\varepsilon_0=16.1$ (after Takimoto, 1959).

B. Considerations in the low-temperature limit

The conventional Debye-Hückel screening length vanishes at a temperature of absolute zero; as a result, the BH form of the potential breaks down at very low temperatures. Alfred (1977) showed that this difficulty can be overcome by considering the dielectric response formulation of localized point-charge screening in a noninteracting electron gas. For a nondegenerate electron distribution he obtains the following analytic expression for the screened impurity potential:

$$V(r)=\frac{Ze}{4\pi\varepsilon r}\left(\exp(-k_0^2r^2)-\sqrt{\pi}\,k_0r\,\mathrm{erfc}(k_0r)+\frac{\sqrt{\pi}\,k_0}{\beta_s}\exp(\omega^2)\,\mathrm{erf}(\omega)\exp(-\beta_sr)+\frac{\sqrt{\pi}\,k_0}{2\beta_s}\exp(\omega^2)\zeta(k_0,\beta_s,r)\right), \tag{49}$$

where $k_0^2=2m^*k_BT/\hbar^2$, $\omega=\beta_s/2k_0$, $\zeta(k_0,\beta_s,r)=\exp(\beta_sr)\times\mathrm{erfc}(k_0r+\omega)-\exp(-\beta_sr)\,\mathrm{erfc}(k_0r-\omega)$; erf and erfc are error functions. At ordinary temperatures when $\beta_s/2k_0\ll1$, there is no significant deviation from the Brooks-Herring potential at large distances. As $T\to0$, and $\beta_s\to\infty$, $V(r)$ approaches the form $Ze(4\pi\varepsilon r)^{-1}\times\exp(-k_0^2r^2)$ and problems with the vanishing screening length do not arise.

Stern (1974) has argued that the breakdown of the screening length at absolute zero occurs from the assumption of *discrete* impurity levels. He considers the *potential fluctuations* due to the Coulomb potentials associated with the ionized impurities. The mean-square value of such potential fluctuations for randomly distributed impurities is

$$\langle U^2\rangle_{\mathrm{ms}}=(e^2/\varepsilon)^2[(N^++N^-)/8\pi]\,\beta_s^{-1}. \tag{50}$$

The donor level is consequently broadened into a band with a density of states given by

$$\rho(E)=[N_D/(2\pi\langle U^2\rangle_{\mathrm{ms}})^{1/2}]\exp[-(E-E_D)^2/2\langle U^2\rangle_{\mathrm{ms}}]\,, \tag{51}$$

where E_D is the center of the broadened donor level. When $\langle U^2\rangle_{\mathrm{ms}}^{1/2}$ is greater than k_BT, the level broadening becomes important, and the inverse screening length β_s is given by

$$\beta_s^2=\frac{e^2}{\varepsilon}\int_{-\infty}^{\infty}\rho(E)\frac{df}{dE}\,dE\,, \tag{52}$$

where f is the distribution function. It is assumed that the level broadening is small compared with the energy gap between the donor level and the conduction band edge. Equations (50) through (52) must be solved self-consistently for β_s and $\langle U^2\rangle_{\mathrm{ms}}$, taking account of the condition

$$N_D-N_A=\int_{-\infty}^{\infty}\rho(E)f(E)dE\,, \tag{53}$$

which determines the Fermi level. Here N_D and N_A are the donor and acceptor concentrations. A finite value of β_s at absolute zero can be obtained in this way. A transition to the conventional result is expected when k_BT is of the order of $\langle U^2\rangle_{\mathrm{ms}}^{1/2}$. Thus again the problems as $T\to0$ are avoided.

The low-temperature screening was earlier considered by Morgan (1965). He used a self-consistent theory but introduced an additional length parameter $a_0=[3/4\pi(N_D+N_A)]^{1/3}$. We observe that although low-temperature screening has been considered by several authors, calculations of mobility with the associated potentials have not attracted attention.

Larsen (1975) has shown that at low temperatures when most of the carriers are not free but are localized on impurities, the screened Coulomb potential represents an average potential and not an actual physical potential around an impurity. The use of a screened Coulomb potential in ionized impurity scattering calculations therefore amounts to averaging the potential first and then computing the scattering. The correct procedure would be to average the scattering from the different charge configurations which give the average potential. Further work is needed to investigate whether the two procedures are equivalent in any physical situation.

Fujita *et al.* (1976) considered that the lifetime of an electron in a quantum state is finite due to the presence of other impurities. This results in a replacement of the δ function in Eq. (19) by $\pi^{-1}D(q;k)$ where $D(q;k)$ is the Lorentzian function:

$$D(q,k)=\hbar\Gamma/[(E_{k+q}-E_k)^2+\hbar^2\Gamma^2]\,,$$

Γ being the relaxation rate. The constraint of energy conservation is thus relaxed, and this causes an enhancement of the relaxation time. In the limit of small Γ, however, the Lorentzian function reduces to π times the δ function. For a very pure semiconductor with a low carrier concentration, an unscreened Coulomb potential can be used. A finite relaxation rate is found in this case by Fujita *et al.* (1976) without introducing an artificial cutoff as in the CW theory. Zubarev *et al.* (1977) performed a numerical calculation and showed that for Ge below liquid helium temperature with lowest concentrations ($n<10^{11}$ cm^{-3}) the mobility behaves like

$$c_1\varepsilon^2(k_BT)^{3/2}e^{-3}N^{-1}m^{*-1/2}\ln[c_2\varepsilon^2m^{*1/2}(k_BT)^{5/2}(e^4\hbar N)^{-1}]\,.$$

For concentrations between 10^{12} and 10^{14} cm^{-3}, the mobility is $\mu=c_3\hbar^{1/2}\varepsilon(k_BT)^{1/4}e^{-1}N^{-1/2}m^{*-3/4}$; here μ is proportional to $T^{1/4}N^{-1/2}m^{*-3/4}$ in contrast with the CW result. For higher concentrations (10^{16} cm^{-3} and above), one must use a screened Coulomb potential. The replacement of $D(q;k)$ by $\pi\delta(E_{k+q}-E_k)$ is justified in this case, yielding the Brooks-Herring formula (Fujita *et al.*, 1976; Zubarev *et al.*, 1977). Further discussion of this work is relegated to Sec. IX, where theory is compared with experiment (see also Capek, 1980).

C. Consideration of band-structure details

The mobility theory given above is developed for a simple isotropic parabolic band structure. In the case of narrow-gap semiconductors (e.g., InSb), conduction-band *nonparabolicity* necessitates modifications of the mobility expressions. Such modifications of the BH mobility introduced by nonparabolicity have been considered by Barrie (1956), and more recently by Askerov (1970), Rode and Knight (1971), Rode (1975), Nag (1980), and Neumann, Tsipivka, and Unger (1981). In a nonparabolic band the density of states is larger than that in a parabolic band, which alters the screening length given by Eq. (32). This enhancement of the density of states decreases the mobility, but the effect is slightly offset by the admixture of p-type valence-band wave functions. Also, recent calculations of mobility employ numerical evaluation of the integral in Eq. (16) without making approximations (Rode, 1975; Nag, 1980).

The screening problem in a band structure consisting

of a central minimum at the Γ point and a number of subsidiary higher-lying minima at other points in the Brillouin zone was tackled by Robinson and Rodriguez (1964, 1965). The particular case of GaSb was considered, and the subsidiary minima were characterized by an isotropic spherical effective mass. Since the effective mass in the subsidiary minima is much larger than that in the central minimum, the transport properties are largely those of the central-valley electrons. At liquid nitrogen temperatures for carrier concentrations of 10^{18}–10^{19} cm^{-3}, the central-valley electrons were taken to be completely degenerate, but the subsidiary-valley electrons were taken to be incompletely so. Dielectric screening was incorporated. The key feature is that since the subsidiary minima have a much larger density of states than the central minimum, when the subsidiary valleys are occupied the carriers in them dominate the screening of the Coulomb potential of the charged impurities.

The influence of the spheroidal energy surfaces, as in germanium and silicon, on ionized impurity scattering has been discussed by Brooks (1955). He observes that small-angle scattering contributes predominantly to the relaxation time. Assuming that a relaxation time exists for an anisotropic configuration, one would expect from this observation that the relaxation times for impurity scattering along the two principal axes of the energy spheroids of a single minimum will be different because of the difference in effective masses. Ham (1955) calculated the relaxation times along the principal spheroid directions assuming small-angle scattering. For Ge, the relaxation time along the major axis was found to be about 12 times that along an axis perpendicular to it. For Si, the same quantity is about four. Ito (1963) has noted that electron screening reduces the importance of small-angle scattering, demanding a more careful analysis. He used the Herring-Vogt transformation (Herring and Vogt, 1956, 1957) and terminated a spherical harmonic expansion of the electron distribution function after the $l=1$ term to obtain the relaxation time tensor. The procedure is justified for small anisotropy in the relaxation times. Ito, however, finds the opposite results for Ge at low temperatures and so his results are questionable. Similar results have been reported by Samoilovich *et al.* (1961a), working with terminated expansions.

Brooks (1955) points out that even when the relaxation times along the principal spheroid directions are known, it is not certain how they should be averaged over all directions to obtain a mobility. Some plausible averagings indicate that an isotropic effective mass can be used with some success in mobility calculations. Such average effective masses were used by other workers in treating ionized impurity scattering in anisotropic band structures (Long and Myers, 1959; Paige, 1964).

The problem of ionized impurity scattering in anisotropic band structures also received the attention of Boiko (1959), Samoilovich *et al.* (1961b), Andrianov *et al.* (1964), Dakhovski (1963), and Eagles and Edwards (1965). For a review of work on anisotropic band structures we refer the reader of Baranski *et al.* (1977).

Elaborate calculations of ionized impurity scattering in degenerate materials with anisotropic band structures were performed by Krieger and his co-workers (Krieger *et al.*, 1971, 1972, 1974; Krieger and Meeks, 1973). For saturation-stressed, degenerately doped *n*-type Ge they obtained upper and lower bounds to the transverse resistivity predicted by BH scattering by exactly solving the Boltzmann equation including the mass anisotropy for two different scattering rates which either overestimate or underestimate the BH scattering rate. The variational principle was also employed to obtain an upper bound for the transverse and longitudinal resistivity for BH scattering (Krieger *et al.*, 1971, 1972). In later publications Krieger *et al.* observed that the use of dielectric screening [Eq. (37)] instead of that due to Thomas-Fermi [Eq. (33)] causes a significant decrease in the intervalley contribution to the screening in unstressed many-valley semiconductors with strongly anisotropic band structures (Krieger and Meeks, 1973; Krieger *et al.*, 1974).

Scattering of holes from impurity potentials is recently dealt with by Ralph (1977). In his treatment mass anisotropy is neglected, but the effects of degeneracy and spin-orbit splitting are included.

D. Electron-electron scattering

Electron-electron scattering tends to equalize the energy among the electrons and thus modifies the mobility (Spitzer and Harm, 1953). For nondegenerate materials with electron concentration n equal to the ionized impurity concentration N, Appel (1961) finds on using a variational principle that electron-electron scattering lowers the mobility below the Brooks-Herring value by a factor of 0.573, independent of temperature. Working with the Kubo formula, Luong and Shaw (1971) have revised this factor to 0.632, also temperature-independent. For extremely degenerate materials, however, electron-electron scattering does not contribute because conservation of momentum and energy near the Fermi surface implies conservation of total electron velocity. Thus electron-electron scattering decreases the mobility by a factor which rises from about 0.6 at low electron concentrations to unity at very high electron concentrations. This factor C_e for arbitrary degeneracy has been approximately calculated by Bate *et al.* (1965) (see Fig. 4).

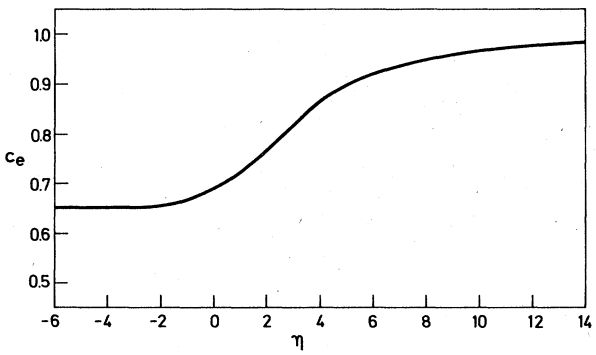

FIG. 4. Plot of the quantity C_e, by which the electron-electron scattering reduces the ionized impurity scattering mobility, as a function of the reduced Fermi energy η (after Bate *et al.*, 1965).

The problem of electron-electron scattering in many-anisotropic-valley semiconductors has been discussed by Morgan (1963, 1965), Price and Krieger (1966), and Meeks and Krieger (1969). In this case, electron-electron collisions affect the mobility even if the material is degenerate because electrons in different valleys can scatter, remaining in their respective valleys, and influence the resistivity due to the anisotropy.

E. Nonlinear screening

The linearization of Poisson's equation [Eq. (30)] by retaining only terms up to the first order in $eV(r)/k_BT$ is unsatisfactory at low temperatures. Falicov and Cuevas (1967) tried to get around the problem by solving Poisson's equation without resorting to an expansion in a series of $eV(r)/k_BT$. They considered a *compensated* semiconductor with the carrier and the impurity ion distributions "frozen" in the state of minimum potential energy. The static screening of the individual charges was accounted for by considering the mutual statistical correlation between them. The exact forms of the correlation functions are not known, and there is evidence that they are complicated (Falicov and Cuevas, 1967). Falicov and Cuevas made some simplifications by assuming that charge distributions of the same species are uncorrelated, and the number of electrons are small, and by using a temperature-independent exponential correlation function between acceptors and donors. They then obtained a relaxation time τ which differs from Eq. (34) only in that the sign of the last term within large parentheses is positive and that the temperature-dependent inverse screening length β_s is replaced by the static temperature-independent *inverse correlation length* a_c:

$$a_c^3 = 8\pi (N_D - N_A) , \tag{54}$$

where N_D and N_A are the donor and the acceptor concentrations.

Attempts have also been made to improve on the simple screened Coulomb potential by retaining higher-order terms in $eV(r)/k_BT$ in the Taylor series expansion of $\mathcal{F}_{1/2}$ in Eq. (29). A method is suggested in which the nonlinear Poisson's equation [Eq. (30)] is solved as a linear combination of screened Coulomb potentials with coefficients to be determined by the variational principle (Csavinszky, 1976b). A variational solution of the nonlinear Poisson's equation including valence dielectric screening has been proposed recently (Brownstein, 1980). A numerical solution to the nonlinear Poisson's equation has also been performed (Meyer, 1979). An analytic expression for the impurity potential keeping terms up to the second order in $eV(r)/k_BT$ in Eq. (29) has been obtained by Csavinszky (1964) and Adawi (1966). This expression reads

$$V_{CA}(r) = V(r)\{1 + \alpha_s[\exp(2r\beta_s)\,\mathrm{Ei}(-3r\beta_s) - \mathrm{Ei}(-r\beta_s) - \ln 3]\} , \tag{55}$$

where $V(r)$ is the standard screened Coulomb potential given by Eq. (31),

$$\alpha_s = \frac{e^2\beta_s}{16\pi\varepsilon k_BT}\,\frac{\mathcal{F}_{-3/2}(\eta)}{\mathcal{F}_{-1/2}(\eta)} , \tag{56}$$

and $\mathrm{Ei}(-\psi)$ is the exponential integral

$$-\mathrm{Ei}(-\psi) = \int_{\psi}^{\infty} x^{-1}\exp(-x)dx . \tag{57}$$

Equation (55) is a satisfactory solution to the nonlinear Poisson's equation when $\alpha_s \ll 1$ (Adawi, 1966). The approximate calculations of Adawi (1966) as well as the exact numerical solutions of Meyer (1979) show that in the nonlinear theory the screening charge is increased in the vicinity of the impurity ion but is decreased at large distances. The scattering potential is therefore weaker than the standard screened Coulomb potential, resulting in a corresponding enhancement of mobility over the BH value. This result is confirmed by detailed calculations of mobility in GaAs (Chattopadhyay, 1981a) (see Fig. 5). The departure from the BH mobility is found to be most important at intermediate carrier concentrations, since at such concentrations the departure from the conventional screened Coulomb potential is most significant (Csavinszky, 1964).

F. Multiple scattering and impurity dressing

The BH mobility is based on the assumption that the carrier is scattered separately and incoherently by the impurity ions. When the electron interacts with several ionized impurities simultaneously, multiple scattering takes place. The perturbing or "dressing" effects of the impurities on the electron energy levels and wave functions are also neglected in the BH theory.

Rode (1975) has suggested that multiple scattering occurs if the electron deflection time τ_D in the impurity potential is greater than the mean time between collisions, τ. Taking the deflection time τ_D to be approximately that required by the electron to travel $\sqrt{2}\,x$ (screening length), this implies that $\mu^2 < \varepsilon/m^*n$ for a nondegenerate material. Since the mobility μ for ionized impurity scattering decreases at low temperatures, multiple-scattering corrections due to this criterion need attention at low temperatures.

Multiple-scattering as well as dressing corrections

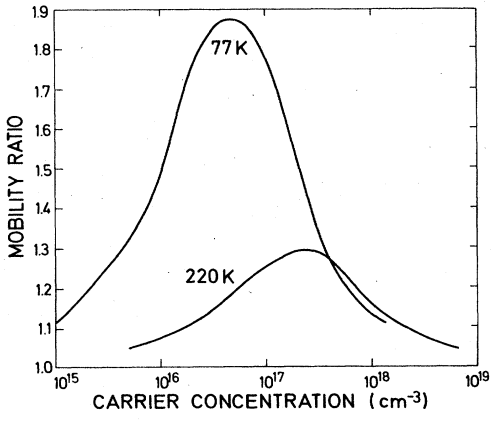

FIG. 5. Ratio of the electron drift mobility in the nonlinear screening model to that in the linear screening (Brooks-Herring) model as a function of carrier concentration at two different temperatures for n-type GaAs (after Chattopadhyay, 1981a).

have been calculated by Moore and Ehrenreich (1966) and by Moore (1967). They denoted these corrections by $\delta_M(\mathbf{k})$ and $\delta_D(\mathbf{k})$, respectively. For isotropic semiconductors the δ's are functions of k, i.e., of the energy E only. When the δ's are less than unity, $1+\delta_M(k)+\delta_D(k)$ will be a slowly varying function of energy. The corrected mobility can then be expressed as

$$\mu = \mu_{BH}[1+\delta_M(k_p)+\delta_D(k_p)]^{-1}, \tag{58}$$

when μ_{BH} is the Brooks-Herring mobility and k_p is the value of k at which $E^3 f_0'(E)$ is a maximum. For singly ionized impurities, Moore and Ehrenreich (1966) showed that

$$\delta_M(k_p) = \frac{\lambda_p^2 a_r^3}{4\pi a_B^2 d^3} Q_M\left(\frac{4\pi^2 a_r^2}{\lambda_p^2}\right), \tag{59}$$

and

$$\delta_D(k_p) = \left[1-(\lambda_p^2 a_r^3/16\pi a_B^2 d^3)\Big/\left(1+\frac{\lambda_p^2}{4\pi^2 a_r^2}\right)\right]^{-2} - 1, \tag{60}$$

where λ_p is the electron wavelength ($=2\pi/k_p$), a_r is the potential range ($a_r/2$ = screening length), a_B is the effective Bohr radius ($a_B = 4\pi\hbar^2\varepsilon m^{*-1}e^{-2}$), and d is the mean distance between impurities ($d = N^{-1/3}$). For $4\pi^2 a_r^2/\lambda_p^2 > 10$, where deviations from the Born approximation are not large, Q_M is slowly varying and ranges in value between 1.6 to 1. Under this condition the multiple-scattering correction would be small when λ_p or a_r is small compared with d. Moore and Ehrenreich (1966) and Moore (1967) gave numerical value of δ_M for uncompensated n-type GaAs. At higher temperatures when the Boltzmann statistics apply, $a_r \propto T^{1/2}n^{-1/2}$ and $\lambda_p \propto T^{-1/2}$. Therefore $\delta_M \propto T^{1/2}n^{-1/2}$. Multiple scattering is thus more important at higher temperatures and at low impurity concentrations. The underlying reason for this apparently unexpected result is that at high temperatures and low impurity concentrations the screening length increases, and the electron may feel the influence of more than one impurity at a time. This result conflicts with the result of the Rode criterion mentioned before. Moore and Ehrenreich (1966) and Moore (1967) have further shown that the dressing correction δ_D is typically 30–50 % of δ_M in the region where $4\pi^2 a_r^2/\lambda_p^2 > 10$.

The results of Moore and Ehrenreich were applied to the case of anisotropic band structures under conditions of degeneracy by Krieger *et al.* (1971, 1972). Their findings will be discussed in Sec. IX, where we compare theory with experiments.

V. CORRECTIONS DUE TO DEPARTURE FROM THE BORN APPROXIMATION

The Born approximation used in the BH derivation is valid when $4k^2/\beta_s^2 \gg 1$, i.e., at high temperatures. The validity of results based on the Born approximation is therefore questionable when the temperature is lowered. Moore and Ehrenreich (1966) and Moore (1967) have obtained corrections to the Born approximation by considering higher-order terms in the scattering rate. For an isotropic semiconductor this correction factor δ_B is a function of the electron energy only. Assuming that $\delta_B \ll 1$, the mobility incorporating corrections due to higher-order Born approximations, multiple scattering, and dressing effects is expressed as (Moore, 1967)

$$\mu = \mu_{BH}[1+\delta_B(k_p)+\delta_M(k_p)+\delta_D(k_p)]^{-1}, \tag{61}$$

where k_p has the same significance as in Eq. (58). For singly ionized impurities,

$$\delta_B(k_p) = \frac{\lambda_p^2}{\pi^2 a_r a_B} Q_B\left(\frac{4\pi^2 a_r^2}{\lambda_p^2}\right), \tag{62}$$

where the symbols are explained following Eqs. (59) and (60). Moore (1967) considered the case of uncompensated n-type GaAs and showed that Q_B is a slowly varying function between values of 0.6 and 1.0 when $4\pi^2 a_r^2/\lambda_p^2 > 10$. In this situation the main temperature and concentration dependence of δ_B arises from the factor $\lambda_p^2(\pi^2 a_r a_B)^{-1}$. If the temperature is high enough for Boltzmann statistics to apply, $a_r \propto T^{1/2}n^{-1/2}$ and $\lambda_p \propto T^{-1/2}$ and so $\delta_B(k_p) \propto T^{-3/2}n^{1/2}$. Thus the correction to the Born approximation assumes importance as the temperature decreases. Also with decreasing temperature the electron gas becomes degenerate, and so δ_B becomes less dependent on temperature at very low temperatures. The concentration dependence of the factor $(\lambda_p^2/\pi^2 a_r a_B)$ also changes to $n^{-1/2}$ in the degenerate region (Moore, 1967). The results obtained by Moore (1967) on δ_B were applied to the anisotropic-mass materials under degenerate statistics by Krieger *et al.* (1971, 1972).

A different approach avoiding the Born approximation has also been pursued by several workers. This is the phase-shift analysis, discussed in Sec. VII. Before introducing the phase-shift method, however, it is worthwhile to examine the relative importance of the various refinements of the BH treatment discussed so far. This is done in the following section.

VI. RELATIVE IMPORTANCE OF THE REFINEMENTS OF THE BROOKS-HERRING THEORY

Incorporation of the various refinements of BH analysis in estimating mobilities depends on the material of interest, the temperature and the doping level involved, and the desired accuracy of the result. Nevertheless, it is useful to frame some general guidelines as to the relative importance of the refinements in a typical situation. (See the survey in Table I.)

The effects due to band nonparabolicity and wave-function admixture are important for a narrow-gap semiconductor like InSb; they are less significant for a large-gap material like GaAs (Rode, 1975; Nag, 1980).

Electron-electron scattering is important for an isotropic material at low concentrations where the electron gas is nondegenerate; it reduces the BH mobility by about 40% at all temperatures (Bate *et al.*, 1965). At high concentrations where the electron gas is degenerate, the influence of electron-electron scattering is negligible.

Takimoto screening is important at low temperatures and high concentrations; for the parameter values of Fig. 3 at 80 K and a concentration of 10^{18} cm^{-3} it reduces the BH mobility by a factor of about 1.8. At room temperature, however, Takimoto screening is insignificant even at 10^{18} cm^{-3} for the same choice of parameters (Takimoto, 1959). In degenerately doped many-

valley semiconductors with highly anisotropic band structure, Takimoto screening is significantly important because of a strong reduction in the intervalley contribution to screening (Krieger and Meeks, 1973). Valence dielectric screening, however, appears not to affect the mobility significantly and may generally be neglected in practical cases of interest.

Multiple-scattering corrections are found to dominate over Born and dressing corrections for n-type GaAs above concentrations of about 10^{18} cm^{-3}, the correction factor δ_M in Eq. (61) being typically 0.35 while the correction factors δ_B and δ_D lie in the range 0.1–0.2 (Moore, 1967). For lower concentrations, Moore (1967) also gives the values of the correction factors. However, as Moore himself admits, these results are questionable due to the failure of the perturbation approach at lower concentrations. Reliable estimates of the relative importance of the corrections for such concentration regimes are lacking in the literature.

VII. PHASE-SHIFT ANALYSIS

A. General

For carrier scattering by spherically symmetric potentials falling off rapidly at large distances, the scattering cross section can be expressed in terms of the phase shifts of the partial waves of the free-electron system (Schiff, 1968). Friedel (1954, 1958) showed that the phase shifts also give the number of electrons attracted to, or repelled from, the vicinity of the ionized center. The fact that the center is effectively screened so as to appear neutral at large distances imposes a condition on the phase shifts. This condition is known as the *Friedel sum rule*, which for conduction electrons in a semiconductor with spherical energy surfaces can be written as (Stern, 1967; Boardman and Henry, 1973)

$$\frac{2}{\pi}\sum_{l=0}^{\infty}(2l+1)\int_0^{\infty}\frac{d\delta_l(E)}{dE}f(E)dE=Z\,, \tag{63}$$

where E is the energy of the scattered electron, $f(E)$ is the Fermi-Dirac distribution function, $\delta_l(E)$ is the phase shift introduced by the ionized impurity into the lth partial wave, and Z is the valence of an impurity atom minus the valence of a host atom in the crystal. The left-hand side of Eq. (63) represents the total number of electrons displaced from the vicinity of the impurity center, which must balance the excess valency Z of the scatterer.

To obtain the phase shift of the lth partial wave it is necessary to solve the radial portion of the reduced Schrödinger wave equation

$$\frac{d^2\chi_l}{dr^2}+\left(k^2-\frac{2m^*}{\hbar^2}eV(r)-\frac{l(l+1)}{r^2}\right)\chi_l=0\,, \tag{64}$$

where $V(r)$ is the potential characteristic of the scattering. The relaxation time τ without making use of the Born approximation is given by (Sclar, 1956)

$$\frac{1}{\tau}=\frac{N}{\hbar}\left(\frac{\partial E}{\partial k}\right)\sigma_c=\frac{4\pi\hbar}{m^*k}N\sum_{l=0}^{\infty}(l+1)\sin^2(\delta_l-\delta_{l+1})\,. \tag{65}$$

The drift and the Hall mobilities are then computed using Eqs. (16) and (18).

B. Phase shifts and the Born approximation

In the Born approximation the phase shifts δ_l are much less than unity and are given by (Schiff, 1968; Messiah, 1970)

$$\delta_l^B=\frac{2m^*ke}{\hbar^2}\int_0^{\infty}j_l^2(kr)V(r)r^2dr\,, \tag{66}$$

where the superscript B indicates that the Born approximation is used, and $j_l(kr)$ is a spherical Bessel function. The magnitudes of the phase shifts decrease with increasing l (Csavinszky, 1963). Using Eq. (66) in Eq. (63) and noting that $\sum_l(2l+1)j_l^2(x)=1$, we find that the sum rule imposes the following condition on the potential:

$$\frac{e}{\sqrt{\pi}}\left(\frac{2m^*}{\hbar^2}\right)^{3/2}(k_BT)^{1/2}\mathfrak{F}_{-1/2}(\eta)\int_0^{\infty}V(r)r^2dr=Z\,, \tag{67}$$

If $V(r)$ is a screened Coulomb potential, i.e.,

$$V(r)=\frac{Ze}{4\pi\varepsilon r}\exp(-r/L)\,, \tag{68}$$

then Eq. (67) gives

$$L^{-2}=L_B^{-2}=e^2(4\pi^{3/2}\varepsilon)^{-1}(2m^*/\hbar^2)^{3/2}(k_BT)^{1/2}\mathfrak{F}_{-1/2}(\eta)\,, \tag{69}$$

which agrees with the solution of the Poisson equation for linearized screening [Eq. (32)]. Thus if the Born approximation is valid, then the phase shifts for the conventional screened Coulomb potential obey the Friedel sum rule. This result was pointed out by Stern (1967) and also by Krieger and Strauss (1968).

If the scattering potential differs from the conventional screened Coulomb one, then it must be adjusted to satisfy the Friedel sum rule in the Born approximation, i.e., to satisfy Eq. (67). This approach was used by Chattopadhyay (1981b), who took the form of Eq. (55) as a phenomenological model potential and adjusted α_s and β_s to satisfy Eq. (67). The adjusted potential is closer to the conventional screened Coulomb potential than the unadjusted potential is. The result is a lowering of the mobility ratios depicted in Fig. 5; for example, the maximum of the 77 K curve is now reduced to 1.47. This modified value is, however, still quite sizable and raises the question of using the simple screened Coulomb potential at intermediate carrier concentrations.

Equation (67) has another important implication. It shows that if the Born approximation does not hold, and the scattering potential is of the form shown in Eq. (68), then the screening length L as derived from the Poisson's equation is incorrect. This point was examined by Krieger and Strauss (1968) and by Boardman and Henry (1973), who used Eq. (68) as a model potential and adjusted L to satisfy the Friedel sum rule. Their numerical results indicate that the exact results are quite close to the Born results for large L and k, which just reflects the criterion $4k^2L^2\gg1$ for the validity of the Born approximation. At low temperatures the self-consistent screening radii for donors ($Z=+1$) and for acceptors ($Z=-1$), satisfying the Friedel sum rule, differ. The quantity for donors is smaller, and that for acceptors is larger, than the conventional screening

radius. Typically, this occurs below about 40 K for germanium. Thus the phase-shift analysis distinguishes between the two types of scattering centers; the Born approximation does not. Boardman and Henry (1973) have further shown that when the self-consistent screening length L satisfying the Friedel sum rule is used, the cross section of electron-donor scattering is less than that when the conventional length L_B is used (see Fig. 6).

Assuming that only the zeroth-order partial-wave phase shift is important, Kuchar *et al.* (1976) obtained an expression relating the actual scattering cross section σ_c to the Born approximation cross section σ_c^B. This expression is

$$\sigma_c = \sigma_c^B \left[\left(1 \pm \frac{m^*}{\hbar^2} \frac{e^2}{4\pi\varepsilon\beta_k} \right)^2 + \frac{\sigma_c^B k^2}{4\pi} \right]^{-1}, \tag{70}$$

where the positive sign is for a repulsive potential and the negative sign is for an attractive potential. The quantity β_k equals β_s and $1.19k$, respectively, for $k/\beta_s < 2\pi$ and $k/\beta_s > 2\pi$. Interestingly, if $m^* e^2 (4\pi\varepsilon\beta_k\hbar^2)^{-1} = 1$, a scattering *resonance* occurs. Resonance scattering has also been discussed by Sclar (1956) and Blatt (1957b).

Meyer and Bartoli (1981) have recently provided a treatment with results in approximate analytic form for electrons and holes in Ge, Si, and GaAs in comparison with the Born approximation.

C. Impurity core potential considerations

Using the phase shifts of the partial waves, Ralph *et al.* (1975) examined the central-cell correction to the impurity potential, which had been ignored in the BH treatment. They employed a potential determined empirically from the bound-state energies of the donor, and found a reduction in mobility at high impurity densities. An enhancement of the scattering cross section due to impurity central-cell effects was also reported by Grinberg (1978). El-Ghanem and Ridley (1980) considered the impurity core potential, assuming a square-well potential followed by a Coulomb tail. They considered the screening to be weak, and invoked the Conwell-Weisskopf cutoff. Only the zeroth-order phase shift was retained, and the $l=0$ component of the internal wave function was fitted to the $l=0$ component of the external wave function. The neutral part of the scattering was found to be weak for a deep-level impurity and strong for a shallow level. Also, the scattering cross section was found to be larger for a positively charged than for a negatively charged center, a result in agreement with Blatt (1957b). For positively charged centers the electron is drawn closer to the center and hence scattered more. The opposite occurs for negatively charged centers. El-Ghanem and Ridley (1980) show that resonance scattering, if the resonance is sharp, has only a weak influence on mobility, but a significantly large effect on the ratio of Hall and drift mobilities. The latter may reach a value of 3.7 for sharp resonance, and hence may be a sensitive detector of resonance scattering.

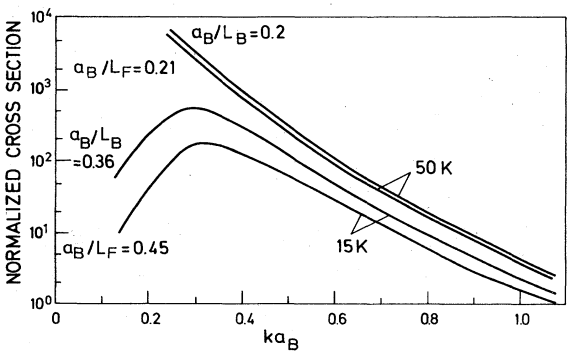

FIG. 6. Variation of the cross section (normalized by the square of the effective Bohr radius) of electron-donor scattering with the wave vector (normalized by the effective Bohr radius a_B). The quantity L_B is the conventional screening length given by Eq. (69); L_F is the adjusted screening length satisfying the Friedel sum rule (after Boardman and Henry, 1973).

VIII. THE HEAVY-DOPING REGIME

The particular effects occurring in heavily doped semiconductors have been reviewed by Bonch-Bruevich (1966), Fistul (1969), and more recently by Abram *et al.* (1978). This regime is of practical interest in view of the current use of high impurity concentrations in semiconductor devices. If the doping is very high, the simple picture of individual impurity states no longer holds. The effects of the fluctuating potential arising from randomly spaced impurities assume importance. Wave functions on impurities overlap and generate an impurity band which has a low-energy tail of states bound to deep potential fluctuations from close impurity clusters. At higher impurity concentrations the impurity band merges with the conduction band and we have a tail of localized states attached to the conduction band of extended states. The doping concentrations over which these regimes take place are determined by the extent to which the impurities are on the average separated relative to the effective Bohr radius.

In the region of heavy doping, the extra carriers cause a change of the band structure also. The band gap is reduced due to the exchange energy with the extra carriers and the modified screening of the valence electron exchange energy. The band-edge effective mass is also changed slightly (Abram *et al.*, 1978), and heavy doping may produce vacancies or vacancy complexes and structural disorder in the material.

Transport calculations in heavily doped semiconductors are beset with difficulties. Localized states occur at the bottom of the band, and perturbation calculations fail. Contributions to conductivity from localized states come through phonon excitation to extended states or through phonon excitation to another localized state, i.e., through *hopping*. We shall not consider these processes any further, but refer the reader to Mott and Davis (1971).

When the relevant carriers are above the band tail, the mobility theory described in the previous sections works satisfactorily. In a recent paper Yanchev *et al.* (1979) considered some modifications peculiar to the heavily doped region in the Boltzmann equation for-

malism. We shall describe this work here in view of the good agreement they obtained with experiments in liquid-phase-epitaxy (LPE)-grown n-type GaAs at 77 K with concentrations in the range 5×10^{17}–10^{19} cm^{-3}.

Yanchev *et al.* (1979) calculated the mobility due to electron scattering in a smooth potential caused by fluctuations in the impurity concentration rather than in individual impurity potentials. This potential energy $V(r)$ in the heavily doped case obeys Gaussian statistics and is characterized by its pair correlation function $W(r-r')=\langle V(r)V(r')\rangle$. Here the brackets denote averaging over positions of impurities. The relaxation time τ is

$$\tau^{-1}=\alpha_y\gamma^2(\hbar k_F L_p E_F)^{-1}, \tag{71}$$

where $k_F=(3\pi^2n)^{1/3}$ is the Fermi wave number, E_F is the Fermi energy, L_p is the correlation length of the potential, and γ its rms value such that $V(r)$ changes by an amount γ over a distance L_p. The factor α_y can be expressed in terms of the pair correlation function W. Using the pair correlation function for a random impurity distribution, Yanchev *et al.* (1979) got results close to those predicted by the BH theory but differing significantly from experimental results. They then used the correlation in the impurity distribution due to Coulomb repulsion between ionized donors in the melt prior to the solidification of the material. They obtained a weak dependence of mobility on electron concentration agreeing with experiment. The correlated impurity distribution produces an ionic screening given by the Debye length $L_i=(\varepsilon k_B T_0/Ne^2)^{1/2}$. This, together with the electron screening described by the length $L=\beta_s^{-1}$, where β_s is given by Eq. (32), characterizes a plasma consisting of the ionized impurities and free carriers which existed at the freezing temperature T_0. The total correlation length L_p therefore satisfies the relation $L_p^{-2}=L^{-2}+L_i^{-2}$.

The quantity γ and the numerical factor α_y were obtained from the Fourier transform of the pair correlation function. There was a problem of divergence of the integral involved in the determination of γ at small wave numbers. Yanchev *et al.* (1979) avoided the divergence by cutting off the integral at L^{-1}, the argument behind this step being that the long-range fluctuations causing the divergence should be screened at distances of the order of L. The electron mobility was finally determined from $\mu=e\tau/m^*$.

In their calculation Yanchev *et al.* (1979) corrected the free-electron Fermi energy $E_F=\hbar^2(3\pi^2n)^{2/3}/2m^*$, taking account of finite temperature, electron-electron exchange, and band tailing. The change in effective mass due to many-body interactions was also incorporated. Their calculated results in relation to experiments will be discussed in the following section.

IX. COMPARISON OF THEORY WITH EXPERIMENT

A direct analysis of the experimental data on electron mobilities in the light of theories of ionized impurity scattering is often complicated due to the presence of other scattering mechanisms. Addition of reciprocal mobility contributions is a poor approximation, and a more careful treatment is necessary (Debye and Conwell, 1954; Rode, 1975; Nag, 1980; Debney and Jay, 1980). Uncertainties in the contributions of the other scattering mechanisms also make it difficult to assess exactly the contribution of the ionized impurity scattering. At low temperatures the mobility is mostly limited by ionized impurity scattering. But at such temperatures freeze out of carriers often generates a strong temperature dependence of the concentrations of ionized impurities and carriers, thereby hampering analysis of the data. Attention to some of these difficulties has, however, been paid by some investigators, and estimates of the ionized impurity scattering contribution are obtained in several cases. We summarize below the information gathered in the cases of elemental and compound semiconductors.

A. The elemental semiconductors: Ge and Si

Debye and Conwell (1954) obtained experimental data on ionized impurity scattering limited mobility for electrons in Ge. They found that the mobility increases with temperature as T^s where s lies between 1.0 and 1.5. An approximate $T^{3/2}$ law for the mobility is predicted both by the CW and by the BH formulas. Blatt (1957a) pointed out that the temperature dependence of the screening radius could predict a less rapid increase of mobility with temperature, in agreement with the data of Debye and Conwell. Brooks (1955) suggested that the departure from the $T^{3/2}$ law may also be caused by the impurity cell effects. The impurity cell would tend to increase the magnitude of the scattering and make it less strongly dependent on energy. Csavinsky (1963) remarked that an additional temperature dependence is introduced by dipole scattering which is associated with the ionized impurity scattering. The thermal motion of the impurity creates a dipole, since the screening cloud around the ion does not rigidly follow the motion of the impurity atom. This should introduce a temperature dependence, since the rms displacement of the ion is a function of temperature.

Long and Myers (1959) have measured electron mobility in n-type Si samples of varying impurity content in the range 3.5×10^{13}–7.8×10^{15} cm^{-3} between 30 and 100 K. They admit that there is an appreciable error in the measurement of donor and acceptor concentrations, but they also show that the BH formula gives higher values of mobility than those measured. Brown and Bray (1962) find that for holes in p-type Ge, the BH formula describes the ionized impurity scattering satisfactorily from 30 to 300 K for impurity concentrations less than 10^{15} cm^{-3}. The formula, however, overestimates the mobility for higher impurity concentrations or lower temperatures. Li (1977) summarizes the results on Si and notes the overestimation of mobility by the BH model. He suggests carrier-carrier scattering and scattering anisotropy to account for the lower experimental mobilities. Norton and Levinstein (1972) have made a mobility analysis of Ge and found good agreement with the BH theory. They, however, fit the data with several adjustable parameters.

Including corrections in the BH formula originally treated by Moore and Ehrenreich (1966) and by Moore (1967), Krieger *et al.* (1971, 1972) have found good agreement with the experiments for both the transverse and longitudinal resistivity of saturation-stressed de-

generately doped Ge at very low temperatures. For unstressed Ge, however, the experimental isotropic resistivity was found to be 5 to 7 times larger than the theoretical estimate (Krieger and Meeks, 1973). The Moore-Ehrenreich corrections to the BH theory were not sufficient to explain the discrepancy between theory and experiment in this case. However, use of dielectric screening led to a significant increase in the predicted resistivity of many-valley materials with anisotropic energy bands, and gave good agreement with experiment both for Ge (Krieger and Meeks, 1973) and for Si (Krieger *et al.*, 1974) (see Figs. 7 and 8).

The experimental result that the mobilities of P-doped Si are smaller than those of As- and Sb-doped Si has been attributed to central-cell effects by Ralph *et al.* (1975). The effects are large at high impurity densities, being about 22% in Si with 10^{18} cm^{-3} doping with P at 300 K. Inclusion of central-cell effects brought the results into closer agreement with experiment (see Fig. 9). The central-cell correction in Si is found to be smaller for Sb donors than for As, P, and Bi donors. Similar results are found for Ge also, in agreement with the observations of Furukawa (1960) and Cuttriss (1961).

Experimental results on compensated materials are not convincingly explained. Attempts have been made by Falicov and Cuevas (1967) to explain the observations of Cuevas (1967) on the basis of their correlation theory. In a highly compensated material, the number of free carriers is negligible, and screening is primarily due to impurities. The Falicov-Cuevas (FC) calculation predicts a correlation between charges which gives a static screening. At higher concentrations of the major impurities, an increase of their number improves the screening but alters only slightly the number

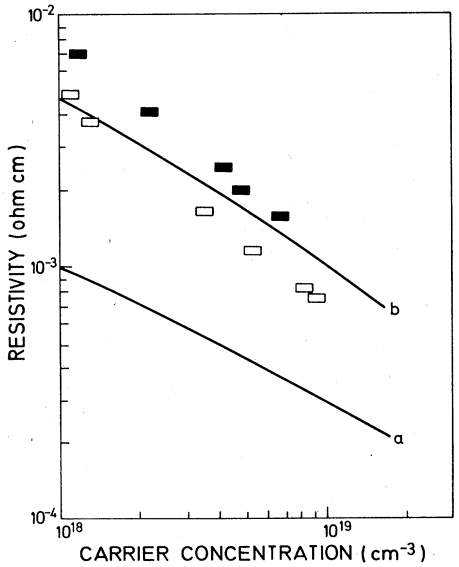

FIG. 7. Resistivity of unstressed degenerately doped n-type Ge as a function of carrier concentration. Curve a gives results predicted by Brooks-Herring scattering. Curve b gives results including dielectric screening. Bars are experimental data at 4 K (after Krieger and Meeks, 1973).

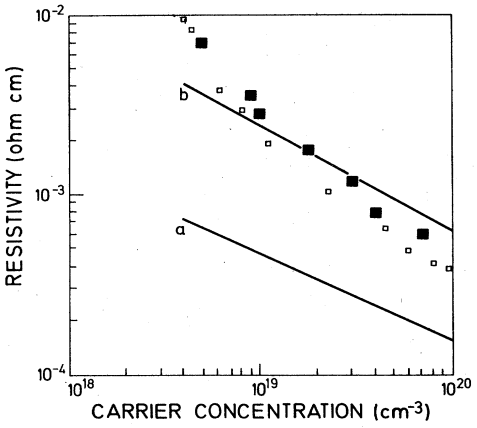

FIG. 8. Resistivity of unstressed degenerately doped n-type Si as a function of carrier concentration. Curve a represents results predicted by Brooks-Herring scattering. Curve b gives results including dielectric screening. Points are experimental results at 4.2 K (after Krieger *et al.*, 1974).

of ionized scattering centers. This is consistent with the experimental results of Cuevas (1967) on n-type Ge. He found that at higher concentrations of majority impurities it is possible to get larger mobilities. The theory did not represent exactly the experimental mobility (see Fig. 10), which might indicate that the pair correlation function was much more complicated than conjectured. Stern (1974), however, shows that the experimental results of Cuevas (1967) at very low temperatures may possibly be explained from level-broadening considerations without explicitly introducing the exponential correlation theory.

B. Compound semiconductors

Experiments in the ionized impurity scattering conduction regime have also been performed for compound

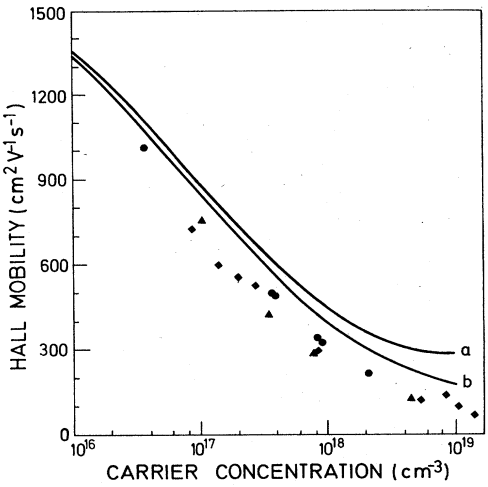

FIG. 9. Hall mobility at 300 K as a function of electron concentration for uncompensated n-type Si. Curve a represents the combined effects of phonon and Coulomb scattering in anisotropic bands. Curve b includes the central-cell scattering calculated for phosphorus. Points are experimental data (after Ralph *et al.*, 1975).

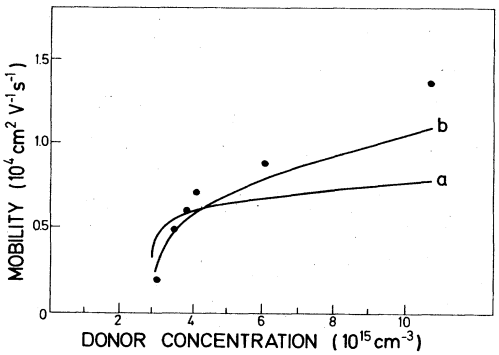

FIG. 10. Mobility versus donor concentration of compensated n-type Ge at 20 K. Curve a represents the theoretical values for a simple exponential correlation function; curve b represents those for a different correlation function. Points are experimental results (after Falicov and Cuevas, 1967).

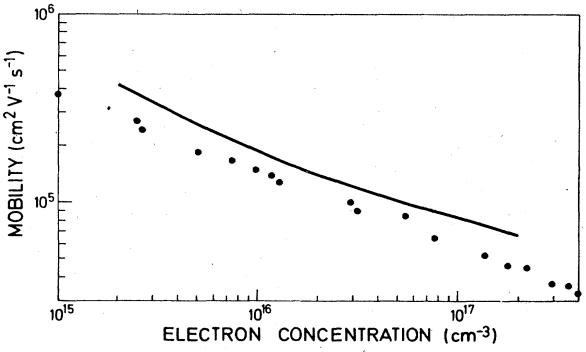

FIG. 11. Electron mobility versus concentration for presumably uncompensated n-type InSb at 80 K. The curve represents the calculated combined mobility due to polar optic and ionized impurity scattering using the Brooks-Herring theory with the inclusion of band nonparabolicity, Takimoto screening, and electron-electron scattering. Points are experimental data (after Bate *et al.*, 1965).

semiconductors. Emel'yanenko *et al.* (1976) carried out measurements of electrical conductivity and the Hall effect in n-type GaAs and found the screening of the impurities to be of Debye nature from an analysis of the data. Evidence of the screening of ionized impurities due to free carriers was obtained in optical experiments on GaAs (Bludau *et al.*, 1976; Paesler and Queisser, 1977, 1978; Queisser and Theodorou, 1979). Illumination of the semiconductor sample created photocarriers which enhanced the screening of the impurity ions and hence the mobility. These experiments were conducted at low temperatures where ionized impurity scattering predominates.

The Hall mobility and the Shubnikov–de Haas effect in degenerate samples of GaSb at low temperatures have been experimentally investigated by Sagar (1960), Strauss (1961), Becker *et al.* (1961), and Becker and Fan (1963). These observations are explained by Robinson and Rodriguez (1964, 1965) on the basis of their two-band screening model. In the experiments, as the concentration of the conduction electrons is increased beyond a critical value n_c, the Hall mobility and the amplitude of the Shubnikov–de Haas oscillations exhibit abrupt increases. This behavior is explained when the band structure of GaSb is considered in treating the screening. As the carrier concentration is increased through a small range about n_c, the subsidiary valleys are populated. The screening length decreases and, despite the onset of intervalley scattering, causes an increase in the lifetime and mobility of the light electrons.

Bate *et al.* (1965) analyzed electron mobility data in InSb at 80 K over a wide range of ionized donor concentrations (1×10^{15}–3×10^{17} cm^{-3}). They accounted for the ionized impurity scattering using the BH theory modified to include band nonparabolicity, the Takimoto screening term, and electron-electron scattering. The contribution of polar optic phonon scattering was also incorporated: Combination of the two scattering processes gave mobility values slightly higher than the measured ones (see Fig. 11). Mathur *et al.* (1980) also found that the simple BH formula overestimates the mobility in InSb. Wolfe *et al.* (1970) found good agreement between theory and experiment in high-purity GaAs using the BH formula. Rode observed that the BH theory overestimates slightly the mobility in GaAs (Rode, 1975), and also probably in GaP (Rode, 1972). Similar conclusions were drawn by Neumann and Nam (1978) in connection with holes in GaAs.

Nuclear reactions following irradiation by thermal neutrons, have been used by several workers to dope semiconductor samples, both elements and compounds. Starting with a relatively pure material one can calculate the doping and compensation introduced by neutron transmutation (Davis *et al.*, 1948). Experiments in this direction for Ge have been done by Fritzsche and Cuevas (1960), and by Thomas and Covington (1975). Similar experiments for InSb have been reported by Mirianashvili *et al.* (1965). More recently, Kuchar *et al.* (1974, 1976) have used neutron transmutation doping to study ionized impurity scattering in InSb. The nuclear reactions produce mainly Sn and a smaller amount of Te impurities, both of which act as donors in InSb. The impurities so created are randomly distributed because of the random distribution of the neutrons in the thermal pile of the nuclear reactor. The number of additional impurities ΔN produced by the nuclear reactions is determined from the total flux of thermal neutrons. The Hall effect is used to give the number of additional carriers, i.e., the number of additional donors ΔN produced by thermal neutrons. Also, the BH formula is used and N is adjusted to fit the measured mobility in the region where ionized impurity scattering predominates. This gives a third method of determining ΔN. All three values of ΔN are found to be in good agreement.

Moore and Ehrenreich (1966) and Moore (1967) have obtained good agreement with the experimental mobilities in n-type GaAs over impurity concentrations in the range 10^{17}–10^{19} cm^{-3} at 77 K and also at 300 K by incorporating corrections due to higher Born approximations for incoherent scattering, and those due to multiple scattering and impurity dressing in the Brooks-Herring formula. The effect of polar optic phonons is also considered in their analysis. The nature of the agreement obtained by Moore with the experimental data

at 77 K is depicted in Fig. 12. Interestingly, over a similar range of concentrations Yanchev *et al.* (1979) find that correlated impurity distribution is important in describing the experimental data of Arnaudov *et al.* (1977), which are weakly dependent on concentration. There is a small discrepancy at the highest impurity concentrations (see Fig. 13); Yanchev *et al.* attribute this discrepancy to the band nonparabolicity, which they do not include in their analysis.

The Falicov-Cuevas (FC) formula has been found by some investigators to describe the behavior in compensated semiconductors. For example, Kranzer and Gornik (1971) obtain good agreement with experiments for strongly compensated n-type InSb using the correlation theory of Falicov and Cuevas (1967). Voronova (1978) also shows that the nature of electron mobility variation in compensated GaAs in the temperature range 20–100 K generally agrees with the FC formula.

Electrical evidence for the pairing of impurities in CdS and CdTe was deduced from anomalously high mobilities by Woodbury (1974). Similar phenomena were suspected for GaN by Vesely *et al.* (1974). Such dipole scattering has been treated by Stratton (1962) and by Fukuda and Fukai (1967) as well as by Boardman (1965).

An interesting effect of the short-range central-cell part of the impurity potential is found in degenerate n-type PbTe at low temperatures (Morita, 1963; Shimizu, 1963). Since the static dielectric constant in PbTe is very large ($\varepsilon/\varepsilon_0 = 400$), the Coulombic potential $e^2/4\pi\varepsilon r$ is very small when r exceeds the nearest-neighbor ionic distance a_i. If the carrier concentration is large ($\sim 10^{18}$ cm^{-3}), the Fermi energy of the electrons is much larger than $e^2/4\pi\varepsilon a_i$. Hence the very weak Coulomb tail may be neglected and the impurity scattering arises from the short-range central-cell part alone. This leads to a mobility dependent on the carrier concentration n as $n^{-4/3}$, which has been experimentally observed (Kanai *et al.*, 1961). When $n < 10^{18}$ cm^{-3}, the Coulomb tail cannot be neglected, and must be considered to explain the experimental data of Kanai *et al.* (1961).

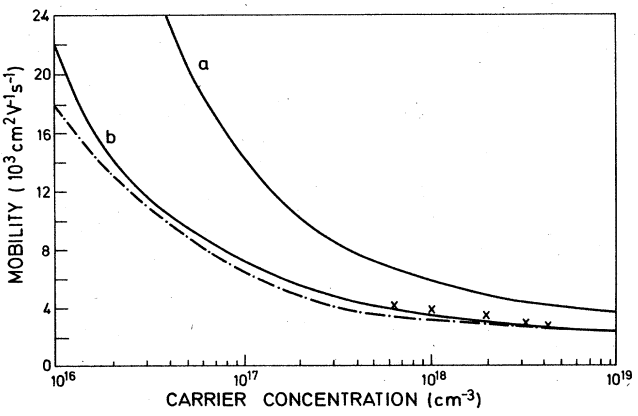

FIG. 12. Electron mobility of n-type GaAs versus concentration at 77 K. Curve a represents the results of the Brooks-Herring theory; curve b gives the theoretical values including corrections due to higher Born approximations, multiple scattering, and impurity dressing. The dash-dotted curve and the points represent experimental results (after Moore, 1967).

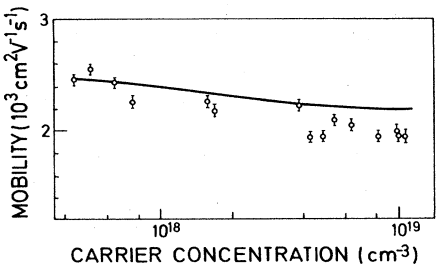

FIG. 13. Electron mobility versus carrier concentration in heavily doped GaAs at 77 K. The solid line represents the calculated results for correlated impurity distributions. Points are experimental results for liquid-phase epitaxial layers (after Yanchev *et al.*, 1979).

The lifetime broadening effect considered by Fujita *et al.* (1976) predicted the mobility to be inversely proportional to the square root of the impurity concentration for very pure semiconductors. Although data for a quantitative check with their theory are not available at present, the lifetime broadening effect is manifested in the cyclotron resonance width; here experiment and theory support $\sqrt{N}$ dependence for very pure Ge and InSb ($N \sim 10^{12}$ cm^{-3}) (Fujita *et al.*, 1976; Zubarev *et al.*, 1977).

X. CONCLUSIONS

We have outlined the regions where various corrections to the BH theory are important. When the Born approximation fails or when the scattering potential deviates from the conventional screened Coulomb form, it is desirable to perform phase-shift calculations to analyze the experimental data. As already mentioned, the former situation occurs at low temperatures and high concentrations and the latter when impurity cell effects are included. At present, the computational labor often precludes the performance of a full phase-shift analysis. An area which future investigations may probe is the modification of the phase-shift treatments to include wave-vector-dependent dielectric functions, influence of electron-electron scattering, and multiple-scattering effects. The problem of nonlinear screening at low temperatures and intermediate carrier concentrations also needs attention.

There are also problems at present with compensated semiconductors. The exponential correlation theory of Falicov and Cuevas (1967) is not always satisfactory for a quantitative agreement with experiments. Usually, the departure of the experimental mobility from the phonon-scattering limited value is attributed to ionized impurity scattering which is described by the BH formula (Rode, 1975). Such an analysis in essence lumps together all mobility-depressing mechanisms, thus attributing the total to the compensating acceptors, whose number is therefore overestimated. Very large compensation ratios are thus often quoted for semiconductors.

As a check on this problem, compensation ratios have been measured independently by temperature-dependent photoluminescence involving donor-acceptor pairs in GaAs (Wagner *et al.*, 1977; Kamiya and Wagner, 1977). Values of compensation ratio obtained for five liquid-phase epitaxial samples were markedly lower than those derived from mobility analysis. A substantial portion of this disparity may be ascribed to nonuniformity of the layers but there still remains a factor of between 1.5 and 3.5 which is most likely due to inadequacy of the mobility analysis. Chemical analysis of Sn-doped GaAs by secondary ion mass spectroscopy (Maier *et al.*, 1980) also shows that the amphoteric Sn cannot be made responsible for the very high compensation which would result from mobility data.

In compensated materials there may be additional scattering mechanisms like space-charge scattering (Weisberg, 1962). In an n-type material, inhomogeneous distribution of donors or acceptors would produce small p-type regions, each surrounded by a space-charge region. The mobility expression due to scattering at such space-charge regions has been worked out (Conwell and Vassell, 1968). In some cases there may also be scattering at local potentials introduced by certain impurities; such scattering may even be more important than space-charge scattering (Chattopadhyay *et al.*, 1980). Inclusion of these additional processes in mobility analysis causes a significant reduction in the theoretically estimated compensation ratio. Proper consideration of the additional scattering processes, coupled with refined treatment of impurity scattering and developments of alternative methods for the determination of compensation ratios, would thus lead to improved material characterization and a deeper understanding of carrier mobilities in compensated materials in the future. A promising but not-much-explored way to study ionized impurity scattering seems to be neutron transmutation doping (Fritzsche and Cuevas, 1960; Kuchar *et al.*, 1976), provided one can solve annealing and radiation damage problems.

Transport in heavily doped semiconductors is also not well understood at present. The Brooks-Herring theory gives room-temperature mobilities significantly higher than the measured values in Si with dopant concentration in the range 10^{19}–10^{20} cm^{-3} (Finetti *et al.*, 1979; Thurber *et al.*, 1980). Slight improvement can be obtained by considering enhanced donor and electron interactions (Finetti *et al.*, 1979). In a recent work, Saso and Kasuya (1980a) have considered the self-consistency between the screening and the scattering of electrons in degenerate semiconductors. They use the Friedel sum rule to determine the screening parameter and phase-shift analysis to include higher-order Born scattering terms. In a subsequent publication, Saso and Kasuya (1980b) have calculated the resistivity of unstressed and uniaxially saturation-stressed degenerately doped Ge, taking account of the anisotropic effects in the solution of the Boltzmann equation. Many-body and multiple-scattering effects have been incorporated as enhancement factors. Agreement with experiment is obtained for concentrations higher than 10^{18} cm^{-3}. The authors attribute the disagreement with experiment for lower concentrations to scattering from strongly correlated impurity clusters, but no detailed treatment is given. Some experimental data on n-type GaAs at 77 K over the range 5×10^{17} to 10^{19} cm^{-3} were explained by Moore (1967) by introducing Born, multiple-scattering, and dressing corrections in the BH formula. Other data over the same range showing less dependence on concentrations were, on the contrary, explained by Yanchev *et al.* (1979) on the basis of electron scattering in a Gaussian potential due to a correlated impurity distribution. Problems also arise when carriers are located in the band tail (Voronova, 1978). Information on the merging of the impurity band with the host band can be obtained from the Hall coefficient (Mott and Davis, 1971; Mott, 1974). Evidence of band tailing in heavily doped germanium has been found from the change of resistivity with uniaxial compression along [111] (Fritzsche and Cuevas, 1962; Cuevas and Fritzsche, 1965; Katz, 1965). Band-tail states may also be studied by moving the Fermi level into the tail by compensation. This approach is used by Redfield (1975), who has investigated electron conductivity in heavily doped closely compensated GaAs. Redfield's results qualitatively agree with the concept of conduction in a tail of localized states. At low fields and low temperatures, however, the conductivity behaves in a manner not expected from the variable-range hopping dependence on temperature. This is attributed by Mott (1976) to possible correlation effects. A better understanding of behavior in the heavily doped regime remains a goal for future work.

XI. OUTLOOK

This review has shown that our understanding of the seemingly simple process of impurity scattering is still unsatisfactory. Although the basic physics is clear, there are theoretical details which are difficult to calculate and problematic to test against experimental data. The main reasons for the shortcomings of mobility analysis lie in the superposition of several scattering mechanisms and in the multiple averaging of atomistic events.

The trend towards improved materials control and refined structuring techniques of semiconductors will demand and facilitate developments in this field of impurity scattering. What would be desirable is the development of observational methods towards a selective study of scattering from known centers instead of the averaged mobility measurements. Such techniques would have to provide sharply defined energies and momenta of the probing carriers and would have to have high resolution to detect the resultant response.

A more precise method of determining impurity scattering would be beneficial in several ways. First, experimental checks could distinguish between the various theories discussed in this review, for example, concerning dielectric screening, multiple scattering, validity of the Born approximation, and other questions of basic importance. Second, the crystal grower and the materials user would obtain much needed information on the vital but as yet unresolved problem of compensation and impurity distribution. Third, a whole set of new phenomena might be tackled which remain buried today under the unspecificity of the data.

Among these new phenomena to be studied by scattering of carriers are features of the spatial distribution of ionized impurities. There are also the problems of dipole scattering from closely spaced centers of unlike charge (Stratton, 1962; Fukuda and Fukai, 1967; Dimitrov, 1976, 1977) and scattering by charges on defects such as dislocation lines (Read, 1955; Mantovani *et al.*, 1980). Scattering by neutral impurities is another subject calling for attention (Blakemore, 1980). In spite of many attempts to unravel these effects from other mobility-determining phenomena, little basic information is currently available. The spatial separation of carrier-donating dopant ions and the carriers themselves has recently attracted much attention; this *modulation doping* (Dingle *et al.*, 1978) is potentially useful for achieving enhanced mobilities in ion-free conductance channels and accordingly higher device speeds.

Improvements in experimental techniques would be desirable for gathering information on the scattering centers. One obvious case is the distinction between impurities of different chemical species through the differences in the central-cell potential, for which indications have so far been seen only at high doping levels (Ralph *et al.*, 1975). Completely ignored thus far, the magnetic moments of the scattering centers ought to provide effects similar to the anomalous Hall effect of metallic specimens with magnetically active impurities (for a review, see Bergmann, 1979). Another fascinating prospect has been offered recently by the suggestion that we search for fractional elementary charge in semiconductors (Chaudhuri *et al.*, 1980). A different suggestion results from abandoning our assumption of a fixed lattice position of the impurity; inelastic phenomena arise, such as the resonant excitation of localized vibrational modes, which have thus far been seen only optically (Spitzer, 1971).

These few remarks should indicate that the topic of electron scattering by ionized impurities in semiconductors holds new challenges for investigators, as well as opportunities in theory and experiment.

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