QCD and instantons at finite temperature

David J. Gross

Department of Physics, Princeton University, Princeton, New Jersey 08544

Robert D. Pisarski

J. W. Gibbs Laboratories, Yale University, New Haven, Connecticut 06520

Laurence G. Yaffe*

Department of Physics, Princeton University, Princeton, New Jersey 08544

The current understanding of the behavior of quantum chromodynamics at finite temperature is presented. Perturbative methods are used to explore the high-temperature dynamics. At sufficiently high temperatures the plasma of thermal excitations screens all color electric fields and quarks are unconfined. It is believed that the high-temperature theory develops a dynamical mass gap. However in perturbation theory the infrared behavior of magnetic fluctuations is so singular that beyond some order the perturbative expansion breaks down. The topological classification of finite-energy, periodic fields is presented and the classical solutions which minimize the action in each topological sector are examined. These include periodic instantons and magnetic monopoles. At sufficiently high temperature only fields with integral topological charge can contribute to the functional integral. Electric screening completely suppresses the contribution of fields with nonintegral topological charge. Consequently the θ dependence of the free energy at high temperature is dominated by the contribution of instantons. The complete temperature dependence of the instanton density is explicitly computed and large-scale instantons are found to be suppressed. Therefore the effects of instantons may be reliably calculated at sufficiently high temperature. The behavior of the theory in the vicinity of the transition from the high-temperature quark phase to the low-temperature hadronic phase cannot be accurately computed. However, at least in the absence of light quarks, semiclassical techniques and lattice methods may be combined to yield a simple picture of the dynamics valid for both high and low temperature, and to estimate the transition temperature.

CONTENTS

I. INTRODUCTION

It is widely believed that the strong interactions are generated by a non-Abelian [SU(3)] gauge theory of colored quarks and gluons which are permanently confined in color singlet hadronic bound states (Gross and Wilczek, 1973b; S. Weinberg, 1973). This theory is called quantum chromodynamics (QCD).[1] It is described by the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \sum_{i=1}^{N_f} \bar{\psi}_i (i\not{D} - m_i)\psi_i ,$$

where $F^a_{\mu\nu}$ is the SU(3) field strength and ψ_i are quark fields of various types (flavors). The theory is parametrized by one coupling constant and the quark mass parameters m_i. In terms of these it purports to explain all the properties of hadrons.

Much of the support for QCD derives from its ability (unique among four-dimensional field theories) to produce the almost noninteracting behavior of quarks at short distances (Gross and Wilczek, 1973b; Politzer, 1973; Coleman and Gross, 1973). This feature of the theory, known as asymptotic freedom, explains the approximate scaling observed in the deep inelastic scattering of leptons off hadrons and leads to many quantitative predictions of scaling deviations at high energy (Gross and Wilczek, 1973a, 1974; Georgi and Politzer, 1974). The success of these predictions, as well as many other confirmations of the predictions of pertur-

*Current address: 452-48, Department of Physics, California Institute of Technology, Pasadena, California 91125.

[1]For an overall review of QCD see, for example, Marciano and Pagels (1978).

bative QCD at short distances[2] (e.g., narrow charm–anticharm bound states, quark and gluon jets, etc.), has greatly increased our confidence in the theory.

QCD also appears to be consistent with much of the successful phenomenology of the strong interactions—the observed symmetry patterns of hadrons, the notion of confinement of color, approximate chiral symmetry, and the bag and string models of hadrons. Recently, much progress has been made toward an understanding of the dynamical basis for color confinement using either semiclassical approximations (Callan, Dashen, and Gross, 1979b, 1980) or lattice approximations (Creutz, 1980) to the theory. The semiclassical treatment has also given indications of how dynamical chiral symmetry breaking might occur and how the bag model might emerge (Callan, Dashen, and Gross, 1978, 1979a).

Now that we possess a theory of the strong interactions it is natural to explore the properties of hadronic matter in unusual environments, in particular at high temperature or high baryon density. There are many reasons for such an investigation. First one might hope to find or to produce in nature such extreme conditions and thereby test the theory in a new domain. There are three places where one might look for the effects of high temperature and/or large baryon density on the structure of hadronic matter. One is in the interior of neutron stars, where the density is significantly greater than nuclear density. Another is during the collision of heavy ions at very high energy per nucleon, in which states of high density and temperature might be produced. Finally the standard cosmological models allow one to extrapolate back to about 10^{-5} sec after the big bang when the universe as a whole was at temperatures comparable to nucleon rest energies. In all of these cases an understanding of the physics requires knowledge of the equation of state and the nature of the phases of hadronic matter. If, as we shall argue is likely, phase transitions occur when the temperature and/or the density are increased, then one might hope to observe qualitatively striking phenomena.

There are, in addition, purely theoretical reasons for exploring the thermodynamics of hadrons. Presented with a new theory involving novel and unfamiliar physical mechanisms, it is of great value to explore its properties in as wide a set of circumstances as possible. It is particularly important to try to extend the theory to explain phenomena which are far removed from the observations that originally motivated the theory. This effort can test the consistency and reasonableness of the theory, increase confidence in its predictive power, and deepen one's understanding of its structure.

Thus, shortly after the unified gauge theory of weak and electromagnetic interactions was proposed (S. Weinberg, 1967; Salam, 1968), investigations of spontaneous symmetry breaking at finite temperature were carried out (Kirzhnits, 1972; Kirzhnits and Linde, 1972, 1975). These studies used the analogy between superconductivity and the spontaneous symmetry breaking (or Higgs mechanism) in gauge theories to argue that at high temperature a transition occurs to a phase in which the condensate of Higgs particles (or Cooper pairs) disappears and the original symmetry is restored. The perturbative methods developed by Bernard (1974), Weinberg (1974), and Dolan and Jackiw (1974) to study gauge theories at finite temperature allow one to compute the critical temperature and other thermal properties of any phase of a gauge theory which does not contain unbroken, non-Abelian subgroups.[3]

In the case of QCD it is of even greater importance to extend the theory to unusual environments, since the physics of confinement is much less understood. Furthermore, one might hope that QCD would be easier to solve at high temperature or density. We shall see below to what extent this is true.

The questions that can be asked, and partially answered, about QCD at, for example, finite temperature are many. First of all one is interested in qualitative issues: Does confinement persist at high temperatures, or is there a phase transition to a nonconfined phase? If there is a phase transition, what is the nature of the high-temperature phase? What collective excitations exist in this phase? Is there a mass gap? Quantitatively one would like to calculate the equation of state of hadronic matter at finite temperature, evaluate the temperature dependence of the quark interactions, deduce the nature of possible phase transitions, and calculate the value of the critical temperature. Before turning to these issues let us briefly review the existing literature in this area.

One of the first attempts to explore the properties of hadrons at high temperatures was carried out by Hagedorn (1965; Hagedorn and Ranft 1968). On the basis of a statistical bootstrap hypothesis he found that the density of hadronic states increases exponentially with energy, and argued that this implies the existence of a limiting temperature, above which hadronic matter cannot exist. The Veneziano model of hadronic scattering amplitudes, and later the dual string model, bolstered this notion since they produced an exponentially increasing number of (narrow) hadronic resonances (Fubini and Veneziano 1969; Huang and S. Weinberg, 1970). However, the arguments for a limiting temperature are suspect once the temperature is greater than the width of the resonances. If indeed hadrons are not "elementary" particles, but rather bound states of constituents, then the exponential increase of the density of resonances might equally well indicate a phase transition to a state composed of free constituents. Cabbibo and Parisi (1975) argued, in the framework of quark models of hadrons where quarks are permanently confined in hadrons, that the exponentially increasing density of states simply means that above some critical temperature quarks are liberated. We shall see in this paper that QCD supports this notion.

Another argument for a quark-liberating phase transition appeared shortly after the discovery of asymptotic freedom and the focus on QCD as the theory of the

[2] For a review of the applications of perturbative QCD see Frazer and Henyey, eds. (1979) and Mahanthappa and Randa, eds. (1979).

[3] For a comprehensive review of these results see Linde (1979).

strong interactions. Collins and Perry (1975) showed, using renormalization group arguments, that as the baryon density increases perturbation theory becomes more reliable. In the absence of infrared singularities, physical observables could be expanded, at sufficiently high density or temperature, in powers of an effective coupling which becomes arbitrarily small. Collins and Perry argued, furthermore, that at finite density the plasmon effect, i.e., the screening by the medium of colored electric fields, eliminates infrared divergences and that the Fermi momentum plays the role of an infrared cutoff. Thus they claimed that the equation of state at large density could be perturbatively calculated and that it approaches that of an ideal relativistic gas of quarks and gluons. Similar arguments may be applied when the temperature (or both temperature and density) is large. This suggests the possibility that any physical observable could be calculated using an asymptotic expansion in an effective coupling which decreases as the temperature or density is raised.

Further work on the properties of QCD at finite temperature and density was carried out by Kisslinger and Morley (1976a, 1976b), who stressed the role of screening at finite temperature. Freedman and McLerran (1977b), and Baluni (1978) have studied perturbative QCD at finite density and have computed the thermodynamic potential to $O(g^4)$. Kapusta (1979a) has evaluated the thermodynamic potential at finite temperature and density to $O(g^3)$. Although these explicit low-order computations yielded sensible, infrared finite results, screening has not been shown to remove infrared divergences in higher orders. This has been discussed by Linde (1979), who pointed out that finite temperature may not provide a genuine infrared cutoff. We shall examine this issue in much greater detail below.

All of these authors have stressed the fact that zero-temperature renormalization prescriptions suffice to eliminate all ultraviolet divergences, i.e., no new temperature or density dependent infinities appear. This has been shown explicitly in one-loop (S. Weinberg, 1974) and two-loop (Kisslinger and Morley, 1976a, 1976b; Morley and Kisslinger, 1979) orders; for a more general treatment see Taylor (1980).

Several of the above authors[4] have attempted to use these perturbative results in astrophysical applications, such as neutron star calculations. In addition there have been other attempts[5] to use perturbation theory at finite temperature or density to actually calculate the value of the phase temperature (or density). These calculations typically employ perturbation theory in a region where the coupling is strong, and they are therefore unreliable. In fact, unless the temperature is unreasonably low, the higher-order corrections to the equation of state (at least through the low orders which have been calculated) are small, and there are no substantial deviations from ideal gas behavior. In order to establish the existence of a transition from a perturbative phase to a confining phase one clearly requires a nonperturbative treatment which is capable of producing confinement.

Lattice gauge theories (Wilson, 1974; Kogut and Susskind, 1975) provide a model of QCD which in the unphysical limit of strong coupling can be easily solved, and which exhibits in this limit linear confinement. Polyakov (1978) and Susskind (1979) have studied the temperature dependence of the strong coupling lattice gauge theory. They have given convincing arguments that, as the temperature is increased, these theories undergo a phase transition to an unconfined phase. This important result illustrates how a confining theory can lose confinement. For strong coupling, the energy eigenstates are closed strings of electric flux whose energy is proportional to their length. Free quarks cannot exist at low temperature since the infinitely long strings (required by flux conservation) which are attached to them have infinite energy. However, as the temperature is raised the probability of finding closed loops of flux increases. Since the number of such closed loops increases exponentially with their length (*à la* Hagedorn), above some critical temperature entropy overwhelms energy and a condensate of strings is formed. One can then have free colored sources, since the addition of one more flux string does not substantially change the free energy. [This heuristic description of the work of Polyakov and Susskind is essentially the same as that given by Banks and Rabinovici (1979).] Polyakov and Susskind argue that this mechanism persists as one approaches the continuum limit by letting the coupling vanish; however, they are unable to deal, in a qualitative fashion, with this limit.

In this paper we shall discuss in detail the properties of QCD at finite temperature from the point of view of perturbation theory, semiclassical methods (instantons), and effective lattice gauge theories. Most of the results presented below are new, especially those relating to the effects of instantons.[6] The resulting physical picture, however, substantially agrees with previous discussions. The basic scenario which emerges is as follows.

At sufficiently high temperature QCD definitely loses confinement. Thermal excitations produce a plasma of quarks and gluons which screens all (color) electric flux. This is reflected in the behavior of the correlation function of the timelike component of the gauge

---

[4]See Collins and Perry (1975); Morley and Kisslinger (1979); and Freedman and McLerran (1978). See also Baym and Chin (1976) and Keister and Kisslinger (1976).

[5]See, for example, Morley and Kisslinger (1979), Kalashnikov and Klimov (1979), and Shuryak (1979, 1980). Several authors (e.g., Kalashnikov and Klimov, 1979, and Kapusta, 1979a) have also claimed that a phase transition will be signaled if the pressure passes through zero. However, even if perturbative calculations were reliable, this is not a valid criterion for a phase transition. The pressure is normally defined by subtracting the perturbative vacuum energy; nonperturbative contributions will then make the zero-temperature, zero-density pressure positive. Consequently, the pressure should never vanish, even at a phase transition.

[6]Many authors (Harrington and Shepard, 1978; Shuryak, 1978, 1980; Kapusta, 1979b; Källman, 1979; Bilic and Miller, 1979, 1980) have attempted to estimate the contribution of instantons at finite temperature or density. None of these papers actually calculates the temperature or chemical potential dependence of the instanton density, and in fact previous treatments have seriously overestimated the finite-temperature instanton density. (See Sec. VI.)

field,

$$\langle A_0(x)A_0(y)\rangle \sim \exp - m_{el}\,|\mathbf{x}-\mathbf{y}|\,.$$

The electric screening length m_{el}^{-1} is perturbatively calculable and is of order $(gT)^{-1}$, where g is the running coupling ($g^2 \sim 1/\ln T$). The heavy quark potential contains only a screened, short-range interaction, $V(R) \sim (\exp - m_{el}R)/R$. However, this electric screening does not remove all long-range correlations. In fact, the infrared behavior of the theory is controlled by the dynamics of the spatial gauge field A_i. For high temperature, this infrared behavior is equivalent to that of a three-dimensional pure gauge theory with coupling g^2T. Three-dimensional gauge theories are believed to develop dynamically a mass gap proportional to the (dimensional) coupling. However, this cannot be computed perturbatively. Beyond a certain order, high-temperature perturbation theory actually breaks down due to the singular perturbative infrared behavior of the spatial gauge fields. Thus, for sufficiently high temperature, QCD yields an unconfined phase with a computable electric screening length of order $(gT)^{-1}$ and a (perturbatively) uncomputable mass gap of order (g^2T).

Topological charge is not automatically quantized at finite temperature; finite action configurations exist with any value of topological charge. However, due to the dynamical effects of electric screening, only configurations with integral topological charge actually contribute to the functional integral. Consequently, the θ dependence of the theory may be reliably computed by expanding about instantons (which exist at any temperature). Large-scale instantons are suppressed, essentially due to the electric screening, so that one obtains a well defined, unambiguous contribution. The instanton contribution to the free energy is negligible compared to the perturbative corrections at any temperature where both calculations are reliable.

As the temperature is lowered, there must clearly be a phase transition to a confined phase (assuming that zero-temperature QCD confines). Such a transition, from a low-temperature confining phase to a high-temperature color screening phase, is consistent with the picture, gleaned from strong coupling lattice gauge theories, of a condensation of electric flux tubes as the temperature increases. The transition temperature cannot be reliably computed using either perturbative or strong coupling lattice methods alone, and it is not even clear whether the transition is of first or second order. In quarkless QCD one can estimate the transition temperature by combining semiclassical and lattice techniques and constructing an effective lattice gauge theory which summarizes the large-distance dynamics. This yields a crude, but consistent, picture of the behavior of QCD at any temperature.

The remainder of this paper is largely devoted to developing and supporting this scenario in detail. The outline of the discussion is as follows.

In Sec. II we discuss how to express the partition function as a functional integral over periodic fields. Particular attention is paid to the relation between the spatial boundary conditions and the definition of a physical state. The physical observables we shall be considering are then introduced and discussed.

In Sec. III we give a topological classification of the smooth, finite-energy gauge fields which may contribute to the functional integral. A complete classification of periodic gauge fields, satisfying a weak asymptotic condition that ensures finite energy, is given in terms of three sets of invariants. Two of these are the familiar Pontryagin index ν and the values of the quantized magnetic charges q_α. The third is related to the asymptotic spatial behavior of the observable

$$\Omega(\mathbf{x}) = P\exp\int_0^\beta dt\, A_0(t,\mathbf{x}), \quad \beta = 1/kT$$

which may be thought of as a closed, periodic timelike Wilson loop. Its eigenvalues are gauge invariant and at spatial infinity approach constant values, λ_α^∞. Any finite-energy gauge field is classified by λ_α^∞, q_α, and ν, in terms of which the topological charge

$$Q = \frac{1}{32\pi^2}\int_0^\beta dt\int d^3x\, \hat{\mathrm{t}}\mathrm{r} F_{\mu\nu}\tilde{F}_{\mu\nu}$$

is given by

$$Q = \nu + \sum_\alpha q_\alpha (\ln\lambda_\alpha^\infty)/2\pi i\,.$$

This is a generalization of a previously derived result of Christ and Jackiw (1980). Many of the details of our derivation are presented in Appendix B.

We expect that there exists a solution of the classical Yang–Mills equations corresponding to the minimal action field for each of these parameters. These classical fields are of interest in semiclassical approximations to the functional integral, and consist of vacuum fields, periodic instantons, and magnetic monopoles. In the remainder of this section we discuss the properties of the known solutions.

Section IV is devoted to a study of perturbation theory at high temperature. We first review the calculation of the free energy to $O(g^3)$. (Odd powers of g arise due to the presence of electric screening, which forces one to resum perturbation theory so as to include the electric mass m_{el} in the gluon propagator.) The one-loop gluon propagator is discussed in detail here and in Appendix C. We carefully continue the periodic Euclidean propagator back to Minkowski space and show that even though electric fields are screened by the plasma of excitations, static magnetic fields are unscreened to this order. Higher-order contributions are then analyzed using simple power counting arguments. Perturbation theory is found to break down, and in fact only the first five terms of the free energy [to $O(g^5)$] are perturbatively calculable. A "magnetic" mass of order (g^2T) should be generated; however, a reliable calculation requires a complete solution of the three-dimensional pure gauge theory. These effects may be heuristically understood as a consequence of the presence of topologically unstable magnetic monopoles.

In Sec. V we discuss the θ dependence at finite temperature, which arises due to the possibility of adding the surface integral $i\theta Q \sim i\theta\int F\tilde{F}$ to the Yang–Mills action. At finite temperature it is not immediately clear that the θ dependence should be periodic, since one can construct finite action fields with nonintegral topological charge Q. Such a field must have $\Omega(\mathbf{x}) \neq 1$ as $\mathbf{x} \to \infty$. We show that the contribution of such "infinite-range" fields vanishes in the infinite volume limit. An explicit illustration of this phenomenon is exhibited by calculating the contribution of inequivalent vacuum fields with A_0 = constant. (Details are presented in Appendix D.) The basic reason for this dynamical quantization of topological charge is once again the presence of electric screening. We compute the leading high-temperature behavior of the effective action and show that the contribution of any gauge field is suppressed by a factor of

$$\exp -\left(\frac{\beta}{2} \int d^3x \,\hat{\mathrm{tr}} m_{el}^2 T^2 (\Omega - 1)^2\right).$$

Therefore, only fields for which $\Omega(\mathbf{x}) \to 1$ at infinity, and hence which have Q = integer, contribute. Consequently, the functional integral may be expanded about the minimal action configuration in each topological charge sector. These fields are precisely the periodic instantons. Large scale instantons are also suppressed by the electric screening so that the semiclassical approximation is increasingly reliable as the temperature rises.

The precise contribution of instantons to the partition function is evaluated in Sec. VI. Here we generalize 't Hooft's calculation of the fluctuations about an instanton field to finite temperature. The instanton determinants are computed for an SU(N) gauge theory (details are given in Appendix E), yielding the complete, temperature-dependent, instanton density.

In Sec. VII the actual phase transition is discussed. We first examine the behavior of the free energy. We find that there are no substantial deviations from ideal gas behavior, even when instanton effects are included, until the temperature is so low that these calculations become untrustworthy. We then construct, for the quarkless theory, a crude effective lattice Lagrangian, using semiclassical methods to evaluate the lattice coupling as a function of lattice spacing and temperature. At zero temperature this yields an abrupt transition from weak to strong coupling behavior, and an estimate of the string tension in terms of the renormalization scale parameter in good agreement with numerical lattice calculations. We find that as the temperature increases the same picture persists for a while, although the value of the string tension decreases slowly. However, a substantial change is found when $T \approx \frac{1}{2}\sqrt{\sigma}$, where σ is the value of the zero-temperature string tension. Beyond this point instantons are greatly suppressed, and one is in a simple perturbative phase. Thus we have a global, albeit crude, picture of quarkless QCD, valid for all distances and temperatures, which undergoes a phase transition at $T_c \approx \frac{1}{2}\sqrt{\sigma}$. If this were the real world T_c would then equal ≈ 200 MeV. We then discuss the problems of extending this discussion to include dynamical, light quarks.

Section VIII contains some remarks concerning applications of this work, a list of open problems, and suggestions for future research. Our notation is summarized in Appendix A.

Each section has been written in as self-contained a fashion as possible. Therefore, readers who are primarily interested in a single topic, such as perturbation theory, are encouraged to turn directly to the relevant section.

II. FORMAL PROPERTIES OF FINITE-TEMPERATURE FUNCTIONAL INTEGRALS

The finite-temperature behavior of any theory is specified by the partition function

$$Z = \mathrm{Tr}(e^{-\beta H}) \tag{2.1}$$

and the thermal expectations of physical observables,

$$\langle \mathcal{O} \rangle = \frac{1}{Z}\,\mathrm{Tr}(e^{-\beta H}\mathcal{O}). \tag{2.2}$$

$\beta = T^{-1}$ is the inverse temperature ($k_B \equiv 1$).

In the standard fashion one may derive functional integral representations for these quantities (Feynman and Hibbs, 1965). For gauge theories, one finds (Abers and Lee, 1973; Fadeev, 1976)

$$Z = \int \mathfrak{D}A_\mu \mathfrak{D}\bar{\psi}\mathfrak{D}\psi \, \exp\left(-\frac{1}{g^2}\, S(A, \bar{\psi}, \psi)\right), \tag{2.3}$$

where

$$S = \int_0^\beta dt\,(\mathcal{E} + \mathfrak{L}_{\text{matter}}),$$

$$\mathcal{E} = \tfrac{1}{4} \int d^3x \,\hat{\mathrm{tr}} F_{\mu\nu} F_{\mu\nu},$$

and

$$\mathfrak{L}_{\text{matter}} = \sum_{i=1}^{N_f} \int d^3x \, \bar{\psi}_i (\not{D} + m_i)\psi_i .$$

(See Appendix A for a review of our notation.) Owing to the (Hilbert space) trace in the definition of Z, the functional integral is restricted to fields satisfying the periodicity conditions,[7]

$$\begin{aligned} &A_\mu(\beta, \mathbf{x}) = A_\mu(0, \mathbf{x}), \\ &\psi(\beta, \mathbf{x}) = -\psi(0, \mathbf{x}), \quad \bar{\psi}(\beta, \mathbf{x}) = -\bar{\psi}(0, \mathbf{x}). \end{aligned} \tag{2.4}$$

We should like to sketch the derivation of this result in order to exhibit the relation between the choice of boundary conditions and the physical definition of the partition function. It will be convenient to work in $A_0 = 0$ gauge. The matter fields will be ignored for simplicity; they cause no change in the following procedure.

One begins with the quantum Hamiltonian,

$$H = \int d^3x \, \frac{1}{2}\left(g^2 (\mathbf{E}^a)^2 + \frac{1}{g^2}(\mathbf{B}^a)^2\right),$$

and a Hilbert space of states spanned by $\{|\mathbf{A}(x)\rangle\}$. $\mathbf{E}^a(x)$ and $\mathbf{A}^a(\mathbf{x})$ are canonically conjugate. One may write $e^{-\beta H}$ as $\lim_{N\to\infty}(e^{-\varepsilon H})^N$, $\varepsilon \equiv \beta/N$, and repeatedly use the completeness relations to find

$$\langle \mathbf{A}'(\mathbf{x}) | e^{-\beta H} | \mathbf{A}''(\mathbf{x}) \rangle = \int \mathfrak{D}\mathbf{A}(t, \mathbf{x}) \exp\left(-1/g^2 \int_0^\beta dt \int d^3x \, \tfrac{1}{2} \hat{\mathrm{tr}}(\dot{\mathbf{A}}^2 + \mathbf{B}^2)\right),$$

where $\mathbf{A}(\beta, \mathbf{x}) = \mathbf{A}'(\mathbf{x})$ and $\mathbf{A}(0, \mathbf{x}) = \mathbf{A}''(\mathbf{x})$ are fixed. The

[7]The antiperiodicity condition satisfied by ψ and $\bar{\psi}$ is a simple consequence of the anticommuting coherent state representation of an operator trace.

trace of $e^{-\beta H}$ in the full Hilbert space,

$$\int \mathfrak{D}\mathbf{A}(\mathbf{x})\,\langle \mathbf{A}(\mathbf{x})|e^{-\beta H}|\mathbf{A}(\mathbf{x})\rangle\,,$$

could now be represented as a functional integral over periodic gauge fields, $\mathbf{A}(\beta,\mathbf{x})=\mathbf{A}(0,\mathbf{x})$. This answer, however, would not be the physical partition function. The difficulty is simply that Gauss's law has not been imposed and, consequently, the full Hilbert space contains many unphysical states. Physical states must satisfy

$$\mathbf{D}\cdot\mathbf{E}(\mathbf{x})|\psi_{\text{phys}}\rangle=0 \quad \text{for all } \mathbf{x}\,, \tag{2.5}$$

which expresses the conservation of electric flux. (This constraint commutes with the Hamiltonian.) In order to satisfy Eq. (2.5) it is sufficient to require

$$\exp\left(-i\int d^3x\,\hat{\text{tr}}[\mathbf{D}\Lambda(\mathbf{x})\cdot\mathbf{E}(\mathbf{x})]\right)|\psi_{\text{phys}}\rangle=|\psi_{\text{phys}}\rangle \tag{2.6}$$

for all $\Lambda(\mathbf{x})=\Lambda^a(\mathbf{x})T^a$ with compact support. Since $\Omega(U)\equiv\exp-i\int(D\Lambda\cdot E)$ is the unitary operator which implements the (time-independent) gauge transformation $U=e^{\Lambda}$, Eq. (2.6) shows that the imposition of Gauss's law is equivalent to the requirement that physical states remain invariant under all gauge transformations whose generators $\Lambda(\mathbf{x})$ vanish at infinity. Just such states may be selected by inserting the projection operator

$$P=\int_{\Lambda(\infty)=0}\mathfrak{D}\Lambda\,\Omega(e^{\Lambda})$$

$$=\int_{\Lambda(\infty)=0}\mathfrak{D}\Lambda\exp\left(-i\int d^3x\,\hat{\text{tr}}[\mathbf{D}\Lambda(\mathbf{x})\cdot\mathbf{E}(\mathbf{x})]\right) \tag{2.7}$$

into the functional integral. Since $\langle\mathbf{A}|\Omega(U)=\langle\mathbf{A}^U|$ (where $\mathbf{A}^U=U(\boldsymbol{\partial}+\mathbf{A})U^\dagger$), one finds the correct representation,

$$Z=\text{Tr}(Pe^{-\beta H})=\int_{\Lambda(\infty)=0}\mathfrak{D}\Lambda\mathfrak{D}\mathbf{A}\langle\mathbf{A}^U|e^{-\beta H}|\mathbf{A}\rangle$$

$$=\int_{\Lambda(\infty)=0}\mathfrak{D}\Lambda(\mathbf{x})\int_{\mathbf{A}(\beta,\mathbf{x})=\mathbf{A}^U(0,\mathbf{x})}\mathfrak{D}\mathbf{A}(t,\mathbf{x})\exp\left(-\frac{1}{g^2}\int_0^\beta dt\int d^3x\,\tfrac{1}{2}\hat{\text{tr}}(\dot{\mathbf{A}}^2+\mathbf{B}^2)\right). \tag{2.8}$$

This exhibits Z as a functional integral over fields which are periodic up to a "twist." To derive the strictly periodic form [Eq. (2.3)] one may redundantly insert the projection operator P more than once and define

$$Z=\lim_{N\to\infty}\text{Tr}(Pe^{-\epsilon H})^N=\int\mathfrak{D}\mathbf{A}(t,\mathbf{x})\mathfrak{D}\Lambda(t,\mathbf{x})\exp\left(-\frac{1}{g^2}\int_0^\beta dt\int d^3x\,\tfrac{1}{2}\hat{\text{tr}}[(\dot{\mathbf{A}}-\mathbf{D}\Lambda)^2+\mathbf{B}^2]\right)$$

$$=\int_{A_\mu(\beta,\mathbf{x})=A_\mu(0,\mathbf{x})}\mathfrak{D}A_\mu(t,\mathbf{x})\exp\left(-\frac{1}{4g^2}\int_0^\beta dt\int d^3x\,\hat{\text{tr}}F_{\mu\nu}^2\right). \tag{2.9}$$

Here $\Lambda(t,\mathbf{x})$ has been renamed $A_0(t,\mathbf{x})$ and $A_\mu(t,\mathbf{x})$ is now strictly periodic. $A_0(t,\mathbf{x})$ must vanish at spatial infinity.

This form for the partition function contains an infinite factor of the volume of the local gauge group. This may be removed by applying the standard Faddeev–Popov gauge-fixing procedure (Abers and Lee 1973; Faddeev, 1976; Faddeev and Popov, 1967). It is important to note, however, that the local gauge group is now composed of gauge transformations which are periodic in time, $U(\beta,\mathbf{x})=U(0,\mathbf{x})$. Consequently the Faddeev–Popov determinant must be defined on the space of periodic functions. Equivalently, the ghost fields which are used to represent this determinant must be anticommuting but periodic (Bernard, 1974). (There has been some confusion in the literature about this point.[8])

This derivation shows that if all states which satisfy Gauss's law are to contribute to the partition function, then $A_0(t,\mathbf{x})$ must vanish at spatial infinity. Relaxing this boundary condition is equivalent to redefining the projection operator P so as to further restrict the definition of a physical state. For example, allowing fields where $A_0\to$ constant as $\mathbf{x}\to\infty$ to contribute to the functional integral corresponds to including a projection onto global color charge zero states,

$$\int d\Lambda^a\exp(i\Lambda^aQ^a)\,,\quad Q^a=\int_{|\mathbf{x}|\to\infty}d^2\mathbf{S}\cdot\mathbf{E}^a.$$

Such a projection would prevent charged states from contributing to the partition function. Since high temperature may cause nonconfinement, physical charged states may be present and presumably should not be excluded from the theory. This will be discussed further in Sec. V. Similarly, different θ sectors may be separated by allowing the appropriate behavior for A_0. See Sec. V for details.

We would now like to review the various observables which are of interest in studying finite-temperature QCD. All thermodynamic quantities follow from the free energy density,

$$\mathfrak{F}(T)=-(\ln Z)/\beta V\,. \tag{2.10}$$

(V is the spatial volume.) The pressure $P(T)$ is simply equal to minus the free energy density. The entropy density $s=\partial P/\partial T$ and the specific heat $c_v=T(\partial s/\partial T)$. Note that the entropy and the specific heat must be positive; therefore, the pressure must be concave upward. The pressure must be continuous across any phase transition.

When using perturbation theory it is of course natural

[8]See, for example, Baluni (1978).

to study the behavior of the gluon and quark propagators. Although these are not gauge invariant and hence not physical observables, the propagators contain a wealth of information about correlations in the theory. In order to study correlations in the gauge field in a gauge-invariant fashion, one may use the Wilson loop (Wilson, 1974),

$$W[C]=\mathrm{tr}P\exp\left(\int_C dx^\mu A_\mu\right), \qquad (2.11)$$

where C is any closed contour and P denotes path ordering.

The Wilson loop is also the standard confinement criterion used in zero-temperature, quarkless QCD. It may be interpreted as the amplitude for an infinitely heavy quark–antiquark pair to propagate around the loop. If one considers a rectangular loop C of width R and length t, then the expectation of the Wilson loop yields the static quark–antiquark potential,

$$V(R)=\lim_{t\to\infty} -(\ln\langle W[C]\rangle)/t.$$

$V(R)$ is the minimal energy of a state containing a static quark–antiquark pair separated by a distance R.

One is not interested in the minimal energy of the quark pair at finite temperature, but rather in the average over the thermal ensemble of the energy of the quark and antiquark. In other words, one wishes to compute the trace of $e^{-\beta H}$ over all states containing an external source and sink of color electric flux, separated by the distance R. This is given by the expectation of "Wilson strings,"

$$\Omega(\mathbf{x})\equiv P\exp\left(\int_0^\beta dt\, A_0(t,\mathbf{x})\right). \qquad (2.12)$$

Owing to the periodic boundary conditions, $\Omega(\mathbf{x})$ may be considered as a closed, timelike Wilson loop. One easily finds that

$$\langle \mathrm{tr}[\Omega(\mathbf{x})]\,\mathrm{tr}[\Omega^\dagger(0)]\rangle=\exp[-\beta V(|\mathbf{x}|,\beta)] \qquad (2.13)$$

where $V(R,\beta)$ is the finite-temperature static quark potential.

Note that spacelike Wilson loops do not function as confinement criteria at finite temperature. They are not related to the static quark potential (2.13), and should simply be thought of as measuring correlations in the spatial gauge field $\mathbf{A}(\mathbf{x})$.

III. PERIODIC FIELDS AND CLASSICAL SOLUTIONS

A. Classification

We should now like to discuss the different types of gauge fields which may contribute to the functional integral (2.3). In order to do so, we must assume that the functional measure $[\mathfrak{D}Ae^{-S}]$ is concentrated on fields which are small fluctuations about smooth, finite-energy (that is $\mathcal{E}<\infty$) configurations.[9] Therefore we shall examine the topological classification of smooth, finite-energy, periodic gauge fields. Only the results of the analysis are presented here; for details see Appendix B.

In order to ensure finite energy,

$$\mathcal{E}=\tfrac{1}{2}\int \hat{\mathrm{t}}\mathrm{r}(\mathbf{E}^2+\mathbf{B}^2)<\infty,$$

we assume that $\hat{\mathrm{t}}\mathrm{r}(\mathbf{E}^2+\mathbf{B}^2)=O(1/r^{3+\varepsilon})$ as $r=|\mathbf{x}|\to\infty$, or

$$\begin{aligned}\mathbf{E}&=D_0\mathbf{A}-\partial A_0=o^{3/2}\\ \mathbf{B}&=(\partial+\mathbf{A})\times\mathbf{A}=o^{3/2}\end{aligned} \qquad (3.1)$$

$[o^{3/2}\equiv O(1/r^{3/2+\varepsilon})$, etc.]. This is a sufficient, but not necessary, condition for finite energy. It is not known if finite energy alone is sufficient for the following classification. No other spatial boundary conditions need be used.

The essential ingredient in our classification is an examination of the behavior of the matrix

$$\Omega(\mathbf{x})=P\exp\left(\int_0^\beta dt\,A_0(t,\mathbf{x})\right).$$

Under any proper (i.e., periodic) gauge transformation $U(t,\mathbf{x})$,

$$\Omega(\mathbf{x})\to U(0,\mathbf{x})\Omega(\mathbf{x})U^{-1}(0,\mathbf{x}).$$

Thus the eigenvalues of Ω are gauge invariant and hence are physical observables at finite temperature.

Using Eqs. (3.1) and (2.4) one easily finds

$$\mathbf{D}(\mathbf{A}(0,\mathbf{x}))\Omega(\mathbf{x})=\partial\Omega(\mathbf{x})+[\mathbf{A}(0,\mathbf{x}),\Omega(\mathbf{x})]=o^{3/2}. \qquad (3.2)$$

Therefore, for any integer n,

$$n\,\mathrm{tr}(\Omega^{n-1}\mathbf{D}\Omega)=\partial\,\mathrm{tr}(\Omega^n)=o^{3/2}. \qquad (3.3)$$

This shows that the eigenvalues $\{\lambda\}$ of $\Omega(\mathbf{x})$ approach a limit $\{\lambda^\infty\}$, independent of direction, as $|\mathbf{x}|\to\infty$. Thus $\Omega(\mathbf{x})$ may be considered as providing a mapping from the sphere at spatial infinity into the equivalence class of λ^∞. Consequently, the topology of $\Omega(\mathbf{x})$ for $|\mathbf{x}|\to\infty$ is classified by the winding of this mapping within the equivalence class of λ^∞. This winding is in fact characterized by the quantized magnetic charges,

$$q_\alpha=\lim_{R\to\infty}\frac{1}{4\pi i}\int_{\{|\mathbf{x}|=R\}} d^2\mathbf{S}\cdot\hat{\mathrm{t}}\mathrm{r}(P_\alpha\mathbf{B}). \qquad (3.4)$$

[$P_\alpha(\mathbf{x})$ is a projection onto an eigenspace of $\Omega(\mathbf{x})$. The magnetic charges $\{q_\alpha\}$ arise as winding numbers of the mapping of S_2 onto the coset space G/H, where H is the isotropy subgroup of λ^∞.]

One further integer is required to characterize the remaining topology of Ω. This is the well known Pontryagin index ν. [The index ν is the winding number for mappings of S_3 onto the full group G. After any "twist" of Ω at infinity associated with the magnetic charges $\{q_\alpha\}$ is removed by a (singular) gauge transformation, the resulting field may be regarded as a mapping of compactified three space (or S_3) onto the group G, leading to the familiar Pontryagin index.]

Thus all periodic gauge fields may be classified by the asymptotic eigenvalues $\{\lambda^\infty\}$ of Ω, the magnetic charges $\{q_\alpha\}$, and the Pontryagin index ν. We show in Appendix B that the topological charge,

$$Q=\frac{1}{32\pi^2}\int_0^\beta dt\int d^3x\,\hat{\mathrm{t}}\mathrm{r}F_{\mu\nu}\tilde{F}_{\mu\nu},$$

[9]This assumption is certainly required for any type of semiclassical approximation. Whether or not it remains true in the full theory is an open question. It could be that infinite energy, or at zero temperature infinite action, configurations contribute in a theory like QCD where the infrared coupling can be arbitrarily large. There is a well known example of such a phenomenon in the two-dimensional x-y model, where infinite energy vortices can exist at sufficiently high temperature (which is analogous to our strong coupling).

is given by[10]

$$Q=\nu+\sum_{\alpha}\frac{\ln\lambda_{\alpha}^{\infty}}{2\pi i}\,q_{\alpha}=\nu+\frac{1}{8\pi^2}\int d^2\mathbf{S}\cdot\hat{\mathrm{t}}\mathrm{r}[(\ln\Omega)\mathbf{B}]. \quad (3.5)$$

Note that these topological quantities ($\{\lambda^{\infty}\}$, q_{α}, ν) remain invariant under any continuous local deformation of the gauge field. Consequently, one might hope to find the classical solution corresponding to the minimal action field for any given values of these parameters. Since the gauge field action is bounded by the topological charge,

$$S=\tfrac{1}{4}\int F\cdot F\geq\left|\tfrac{1}{4}\int F\cdot\tilde{F}\right|=8\pi^2|Q|\,, \quad (3.6)$$

one might expect the classical solutions to saturate this inequality. In other words, one expects an exact self-dual ($F_{\mu\nu}=\tilde{F}_{\mu\nu}$) or anti-self-dual ($F_{\mu\nu}=-\tilde{F}_{\mu\nu}$) classical solution for every possible value of the classification parameters $\{\lambda^{\infty}\}$, q_{α}, and ν. In the remaining parts of this section we shall discuss the properties of these classical solutions.

B. Vacuum fields

The simplest classical solution is of course pure vacuum, $F_{\mu\nu}=0$. However, at finite temperature all such solutions are not gauge equivalent (under proper, periodic gauge transformations) (Batakis and Lazarides, 1978). Instead, the eigenvalues $\{\lambda^{\infty}\}$ of Ω distinguish inequivalent vacuum fields. Representatives of these fields are clearly given by

$$\mathbf{A}(t,\mathbf{x})=0\,,\quad \beta A_0(t,\mathbf{x})=\sigma\,, \quad (3.7)$$

where σ may be taken to be constant, diagonal, and traceless. Obviously $\nu=q_{\alpha}=0$ and $\lambda^{\infty}=\exp\sigma$.

C. Periodic instantons

Next we consider solutions with zero magnetic charge and nonzero Pontryagin index. Such solutions describe periodic instantons. We shall see how such periodic solutions may be constructed from the zero-temperature multiple-instanton solutions.

The general SU(N) multi-instanton solution with Pontryagin index K contains $4NK$ parameters (Atiyah *et al.*, 1978; Christ *et al.*, 1978; Corrigan *et al.*, 1978). This solution is believed to describe K instantons with independent positions, sizes, and group orientations. Assuming that this is the case, then a periodic instanton may be constructed from the multi-instanton solution which describes an infinite string of instantons located at $\mathbf{x}=0$ and $x_0=n\beta$, $n\in\mathbb{Z}$, with identical sizes and with gauge orientations given by $(\omega)^{-n}$, $\omega\in G$. (That is, the gauge orientation rotates by ω between any two nearest-neighbor instantons.) This self-dual solution has one unit of topological charge in the physical strip, $0\leq x_0\leq\beta$, and is periodic up to a gauge transformation

$$A(t+\beta,\mathbf{x})=\omega^{-1}A(t,\mathbf{x})\omega\,.$$

Furthermore, $\Omega(\mathbf{x})\to 1$ as $|\mathbf{x}|\to\infty$ since the instanton fields are well localized. Consequently, if one applies the (improper) gauge transformation $U(t,\mathbf{x})=\omega^{t/\beta}$, then a strictly periodic field with $\Omega(\mathbf{x})\to\omega$ as $|\mathbf{x}|\to\infty$ will be obtained. (ω^{α} for $\alpha\in\mathbf{Z}$ is defined by choosing any one parameter subgroup which interpolates between ω and 1. ω may be taken diagonal without loss of generality.) This field will describe a single periodic instanton with $Q=+1$, $S=8\pi^2$, and $\lambda^{\infty}=\omega$.

Periodic multiple-instanton solutions may be similarly constructed from zero-temperature solutions with several different strings of instantons. Anti-instanton solutions may obviously be constructed in an identical manner. These solutions will describe any number of periodic instantons (or anti-instantons), each with an independent position, size, and orientation. It should be stressed that raising the temperature does not increase the classical action of an instanton. Hence classically the temperature does not provide a cutoff on the scale size of an instanton. This is unlike the behavior in scale-noninvariant theories, where the action of a periodic instanton increases with increasing temperature.

We should now like to examine the behavior of the fields of the periodic instantons. Unfortunately, explicit parametrization of the zero-temperature multi-instanton solutions are available only when all instantons have identical gauge orientations. As a result, we shall only be able to present explicit expressions for solutions with $\lambda^{\infty}=1$. Fortunately, we shall be able to argue later (Sec. V) that only fields with $\lambda^{\infty}=1$ actually contribute to the functional integral.

We may use the convenient 't Hooft solution (1976; see Jackiw, Nohl, and Rebbi, 1977) to describe aligned instantons,

$$\begin{aligned}&\Pi^{-1}\partial^2\Pi=0\,,\\&A_{\mu}=\Pi\bar{\eta}^{a}_{\mu\nu}(\tau^a/2i)\partial_{\nu}\Pi^{-1}\,,\\&F_{\mu\nu}=\tfrac{1}{2}\Pi\tau\cdot\partial\bar{\eta}^{a}_{\mu\nu}(\tau^a/2i)\tau^{\dagger}\cdot\partial\Pi^{-1}\,.\end{aligned} \quad (3.8)$$

Explicitly,

$$\Pi(x)=1+\sum_{n=1}^{K}\rho_n^2/(x-z_n)^2$$

describes K instantons with positions $\{z_n\}$ and sizes $\{\rho_n\}$. Taking $\rho_n=\rho$ and $z_n=n\beta\hat{e}_0$, $n\in\mathbf{Z}$, one finds the periodic single instanton (Harrington and Shepard, 1978a)

$$\Pi(t,\mathbf{x})=1+\frac{\pi\rho^2}{\beta\mathrm{r}}\sinh\frac{2\pi}{\beta}\mathrm{r}\Big/\left(\cosh\frac{2\pi}{\beta}\mathrm{r}-\cos\frac{2\pi}{\beta}t\right). \quad (3.9)$$

($\mathrm{r}\equiv|\mathbf{x}|$.) Note that this exhibits the periodic instanton in a "singular" gauge where A_{μ} has a pure gauge singularity at $x=t=0$. This gauge singularity may be removed by a periodic gauge transformation. (For example, transforming to axial gauge results in an everywhere regular, periodic solution.) Note that (in any periodic gauge) $\Omega(\mathbf{x})=P\exp(\int_0^{\beta}A_0)$ equals -1 at $\mathbf{x}=0$ and approaches $+1$ as $|\mathbf{x}|\to\infty$.

For distances $|x|\ll\beta$

$$\Pi(x)=(1+\tfrac{1}{3}\lambda^2)+\rho^2/x^2+\lambda^2 O(x^2/\beta^2)\,, \quad (3.10)$$

where $\lambda\equiv\pi\rho/\beta$. If we let $\rho'^2=\rho^2/(1+\frac{1}{3}\lambda^2)$, then

[10] A simplified version of this result has been derived by Christ and Jackiw (1980). These authors considered only static fields with integer topological charge and imposed several unnecessarily strict boundary conditions.

$$A^a_\mu = \frac{2\rho'^2}{x^2}\frac{\bar\eta^a_{\mu\nu}x^\nu}{(x^2+\rho'^2)}[1+O(x^2/\beta^2)^2]\,,$$

$$F^a_{\mu\nu} = -4\rho'^2\frac{\bar\eta^a_{\alpha\beta}}{(x^2+\rho'^2)^2}I_{\alpha\mu}I_{\beta\nu}+O(x^2/\beta^4)\,,$$

where $I_{\alpha\mu}\equiv\delta_{\alpha\mu}-2x_\alpha x_\mu/x^2$. Thus, viewed on scales much less than β, the finite-temperature instanton is identical to a zero-temperature instanton with a renormalized size $\rho'^2=\rho^2/(1+\frac{1}{3}\lambda^2)$. If $\rho'\ll|x|\ll\beta$ (which requires $\rho\ll\beta$), then the instanton may be characterized as a four-dimensional self-dual dipole.

For distances $|\mathbf{x}|\equiv r\gg\beta$,

$$\Pi(x)=1+\lambda\rho/r+O(e^{-r/\beta})\,,$$

$$A^a_0\sim\frac{-x^a}{r^2(1+r/\lambda\rho)}\,,\quad A^a_i\sim\frac{\varepsilon^{aij}x^j}{r^2(1+r/\lambda\rho)}\,, \tag{3.11}$$

$$E^a_i=B^a_i\sim-[x^ax^i-(r/\lambda\rho)(\delta^{ai}r^2-3x^ax^i)]/r^4(1+r/\lambda\rho)^2\,.$$

If $x\gg\lambda\rho$ then

$$E^a_i=B^a_i\sim\lambda\rho(\delta^{ai}-3x^ax^i/r^2)/r^3$$

and the solution may be characterized as a three-dimensional dipole field. If $\beta\ll r\ll\lambda\rho$ (which requires $\rho\gg\beta$), then

$$E^a_i=B^a_i\sim-\hat x^a\hat x^i/r^2$$

and the fields describe a dyon with unit electric and magnetic charges. See Fig. 1 for a schematic picture of these regions.

The periodic instanton obviously has several different length scales associated with it, including ρ', β, and $\lambda\rho$. In discussing the behavior of instantons at finite temperature it will obviously be essential to identify the relevant scale for each physical effect. For example, most of the action density,

$$\tfrac{1}{4}\hat{\mathrm{t}}\mathrm{r}\,F_{\mu\nu}F_{\mu\nu}=\tfrac{1}{2}\partial^2(\partial_\mu\ln\Pi)^2\,,$$

is concentrated in a region of size ρ' about the center of the instanton. Note that $0\leq\rho'^2\leq3(\beta/\pi)^2$ and that

$$\tfrac{1}{4}\hat{\mathrm{t}}\mathrm{r}\,F^2|_{x=0}\geq\tfrac{1}{3}(2\pi/\beta)^4\,.$$

Thus at nonzero temperature the field strengths do not spread over an increasingly large region as $\rho\to\infty$.

Aligned multiple periodic instantons may obviously be constructed similarly. Specifically,

$$\Pi(t,\mathbf{x})=1+\sum_{k=1}^{K}\frac{\pi\rho_k^2}{\beta r_k}\sinh\frac{2\pi}{\beta}r_k\Big/\left(\cosh\frac{2\pi}{\beta}r_k-\cos\frac{2\pi}{\beta}t_k\right) \tag{3.12}$$

(where $r_k\equiv|\mathbf{x}-\mathbf{z}_k|$ and $t_k=t-\tau_k$) describes K periodic instantons located at $(\tau_k,\mathbf{z}_k)$. Further examination of the fields proceeds exactly as above.

Discussion of the classical interaction of instantons with anti-instantons, or with general background fields, will be deferred until Sec. VII.

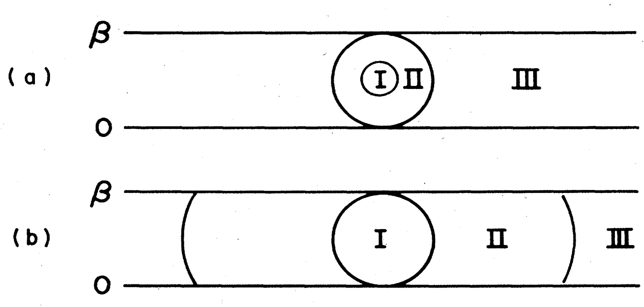

FIG. 1. (a) Small instanton $\rho\ll\beta$. I: core region $x\sim\rho$, II: four-dimensional dipole region $\rho\ll x\ll\beta$, III: three-dimensional dipole region $\beta\ll x$. (b) Large instanton $\beta\ll\rho$. I: core region $x\sim\beta$, II: dyon region $\beta\ll x\ll\lambda\rho$, III: three-dimensional dipole region $\lambda\rho\ll x$.

D. Magnetic monopoles

We now turn to solutions with nonzero magnetic charge. Naturally, these will be referred to as magnetic monopole solutions. It should be emphasized that these are gauge field configurations with A_0 taking the place of the conventional (adjoint) Higgs field.

The simplest solution is the original Prasad–Sommerfield (PS) magnetic monopole (Prasad and Sommerfield, 1975). This static, self-dual solution has the form

$$A^a_0(\mathbf{x})=(\mu r\coth\mu r-1)\hat x^a/r\,,$$
$$A^a_i(\mathbf{x})=(\mu r\,\mathrm{csch}\,\mu r-1)\varepsilon_{iab}\hat x^b/r\,. \tag{3.13}$$

The solution has an energy $\mathcal{E}=8\pi^2(\mu/2\pi)$. Asymptotically,

$$A^a_0\sim\left(\mu-\frac{1}{r}\right)\hat x^a\,,\quad A^a_i\sim\varepsilon_{aij}\hat x^j/r\,,$$

$$E^a_i=B^a_i\sim-\hat x^a\hat x^i/r^2\,.$$

Note that $A_\nu(\mathbf{x})$ is everywhere regular and vanishes at the origin. Unlike the case of the instanton, the energy depends on the scale of the solution, μ. However, $\lim_{\mathbf{x}\to\infty}A_0=\mu\hat x^a\neq0$ (as required of any regular field with nonzero topological charge[11]), and thus the value of μ may be considered as a boundary condition for the solution. The energy is a minimum for any local deformation of the fields. To obtain an anti-self-dual solution one simply changes the sign of A_0.

The existence of this solution is at first sight rather surprising, since all axially symmetric self-dual solutions were explicitly constructed by Witten (1977) and were found to describe multiple-instanton configurations. The resolution of this apparent paradox is that the Prasad–Sommerfield monopole is simply a gauge transform of the $\rho\to\infty$ limit of the periodic instanton. Rossi (1979) found that

$$A^{(\mathrm{PS})}_\nu=U(\partial_\nu+A^{(I)}_\nu)U^{-1}\,,$$

where

$$U(\mathbf{x},t)=\exp[-(\tau\cdot\hat{\mathbf{x}}/2i)\theta(r,t)]\,,$$
$$\theta(r,t)=\tan^{-1}\left[\frac{\sinh\mu r\,\sin\mu t}{\cosh\mu r\,\cos\mu t-1}\right]\,, \tag{3.14}$$

and

$$A^{(I)}_\nu=-\bar\eta^a_{\nu\sigma}(\tau^a/2i)\partial_\sigma\ln[(\sinh\mu r/\mu r)/(\cosh\mu r-\cos\mu t)]\,.$$

[The $\rho\to\infty$ limit simply serves to eliminate the 1 in Eq. (3.9), resulting in the conformal invariant superpoten-

[11]This is shown in Appendix B.

tial (Jackiw, Nohl, and Rebbi, 1977). It is known that the conformal solution possesses a residual gauge freedom which, if the instanton positions lie on a circle (or line), corresponds to rotating the instanton positions around the circle. This is why the $\rho\to\infty$ limit of the periodic instanton is actually static up to a gauge transformation.[12]]

The static (PS) monopole obviously may be considered as a finite-temperature field for any period β. Note that

$$\Omega(\mathbf{x})\sim\exp[\beta\mu(\tau\cdot\hat{\mathbf{x}}/2i)]$$

as $|\mathbf{x}|\to\infty$ and $\Omega(0)=1$. Thus the (PS) monopole provides finite-temperature solutions with

$$\{\lambda^\infty\}=\{e^{i\beta\mu/2},e^{-i\beta\mu/2}\},$$

$q^\alpha=\{+1,-1\}$, $\nu=[\beta\mu/2\pi]$, and $Q=\beta\mu/2\pi$. If we choose $\beta=2\pi/\mu$ so that $Q=1$, then the gauge transformation [Eq. (3.14)] which takes the $\rho\to\infty$ periodic instanton into the static monopole is not periodic, but rather antiperiodic.

We have found (or at least described) exact solutions with any values for $\{\lambda^\infty\}$ and ν, and with zero or one unit of magnetic charge. The existence of multiply charged solutions remains an open problem. Static multimonopole solutions are believed to exist, have a known number of parameters (E. Weinberg, 1979a), and may be approximately constructed for large separation (Manton, 1977). However, attempts to find an explicit construction have so far proved unsuccessful (Adler, 1979; E. Weinberg, 1979b). Unfortunately, such multimonopole solutions cannot be constructed as a limit of the explicit multi-instanton solutions [Eq. (3.12)]. Taking the limit $\rho_n\to\infty$ (or dropping the 1) simply yields a periodic solution with $E_i^a=B_i^a\sim-x^ax^i/r^2$ and $\Omega\sim1$ as $\mathbf{x}\to\infty$. These solutions may not be gauge transformed into static fields. Perhaps multimonopole solutions may be obtained from appropriate limits of the general periodic instanton with $\lambda^\infty\neq1$. However, explicit parametrizations of the general multi-instanton solution appear necessary to verify this conjecture.

IV. PERTURBATION THEORY IN HIGH-TEMPERATURE QCD

In this section we shall study QCD at high temperatures by examining the perturbative behavior of the theory. Since the running coupling $g(T)$ vanishes as $T\to\infty$, one might hope that perturbation theory would be reliable at sufficiently high temperature. We shall see below to what extent this expectation is true.

We shall apply standard perturbative expansions to the functional integral (2.3). However, one is immediately confronted with the problem of deciding which of the inequivalent classical vacuua (3.7) to expand about. We shall temporarily assume that the traditional choice, $A_\mu=0$, is correct and shall carefully justify this choice in Sec. V.

We choose to work in Feynman ($\alpha=1$) gauge and hence write

$$Z=\int\mathfrak{D}A_\mu\mathfrak{D}\bar\psi\mathfrak{D}\psi\mathfrak{D}\bar c\mathfrak{D}c\exp\Big(-\int d^4x\big[\tfrac12A_\mu^a(-\partial^2\delta^{ab}\delta_{\mu\nu})A_\nu^b+\bar\psi_i\not\partial\psi_i+\bar c^a(-\partial^2\delta^{ab})c^b+\mathcal{L}_{\rm int}(A_\mu)\big]\Big),$$
$$\mathcal{L}_{\rm int}=gf^{abc}(\partial_\mu A_\nu^a)A_\mu^bA_\nu^c+\frac{g^2}{4}(f^{abc}A_\mu^bA_\nu^c)^2+g\bar\psi_i\not A^a(\lambda^a/2i)\psi_i+gf^{abc}(\partial_\mu\bar c^a)A_\mu^bc^c\,. \tag{4.1}$$

For convenience, we have specialized to the chirally symmetric limit, $m_i=0$. A_μ has been rescaled to gA_μ. The coupling g should be understood to be the renormalized coupling defined at a scale which we may choose to be T. [Counterterms are not explicitly indicated in (4.1).]

A. Free energy

The lowest-order contribution to Z is obtained by simply dropping $\mathcal{L}_{\rm int}$ and performing the resulting Gaussian integrals. Thus,

$$\begin{aligned}Z&=\det{}_+^{-1/2}(-\partial^2\delta_{\mu\nu}\delta^{ab})\det{}_+(-\partial^2\delta^{ab})\det{}_-^{N_f}(\not\partial)\\&=\det{}_+^{-1}(-\partial^2\delta^{ab})\det{}_-^{N_f}(\not\partial)\,.\end{aligned}\tag{4.2}$$

The subscripts + or − indicate that the determinant is to be evaluated on the space of periodic or antiperiodic functions, respectively. These determinants may be easily calculated (see Appendix D). One finds

$$\ln\det{}_\pm(-\partial^2)=-\frac{\pi^2V}{\beta^3}\left(\frac{1}{45}-\frac{(1\mp1)}{48}\right).$$

Thus the leading contribution to the free energy density, $\mathfrak{F}=-\ln Z/\beta V$, is given by

$$\mathfrak{F}_0=-\frac{\pi^2T^4}{45}(N^2-1+\tfrac74NN_f)\,. \tag{4.3}$$

This is, of course, simply the free energy of a gas of noninteracting, massless particles.

The perturbative corrections to this result may be evaluated by expanding Eq. (4.1) in powers of $\mathcal{L}_{\rm int}$ and computing the resulting Feynman diagrams. Zero-temperature renormalization prescriptions eliminate all ultraviolet divergences; no temperature-dependent infinities remain (S. Weinberg, 1974; Morley and Kisslinger, 1979). Kapusta (1979a) has evaluated the first two corrections to Eq. (4.3). He finds

$$\begin{aligned}\mathfrak{F}=\mathfrak{F}_0&+\frac{g^2T^4}{16}\frac19(N^2-1)(N+\tfrac54N_f)\\&-\frac{g^3T^4}{12\pi}(N^2-1)[\tfrac13(N+N_f/2)]^{3/2}+O(g^4)\,.\end{aligned}\tag{4.4}$$

The first $O(g^2)$ correction comes from the two-loop graphs shown in Fig. 2. The $O(g^3)$ term is the leading contribution from the sum of ring diagrams shown in Fig. 3. These diagrams are increasingly infrared divergent and must be resummed to form $\frac12\,\mathrm{Tr}\ln(\mathfrak{D}_{\mu\nu}/\mathfrak{D}^0_{\mu\nu})$. $\mathfrak{D}_{\mu\nu}$ is the full gluon propagator. [This is the first term

[12]R. Jackiw is thanked for reminding the authors of this point.

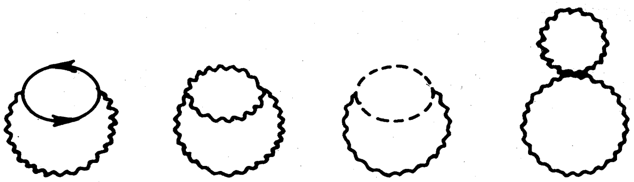

FIG. 2. $O(g^2)$ contributions to the free energy.

in the skeleton expansion of the free energy in terms of full propagators and proper vertices (Freedman and McLerran, 1977a) and will be discussed further below.] The presence of this nonperturbative (i.e., nonanalytic in g^2) term in the free energy is a consequence of color screening. To understand this phenomenon it is much more instructive to consider directly correlation functions such as the gluon propagator. We consider this topic next.

B. Gluon self-energy

The full gluon propagator $\mathfrak{D}^{ab}_{\mu\nu}(x)=\langle A^a_\mu(x)A^b_\nu(0)\rangle$ may be expressed in terms of the one-particle irreducible self-energy, $\Pi^{ab}_{\mu\nu}(\omega,\mathbf{k})$,

$$\mathfrak{D}^{ab}_{\mu\nu}(\omega_n,\mathbf{k})=[(\omega_n^2+\mathbf{k}^2)\delta^{ab}\delta_{\mu\nu}+\Pi^{ab}_{\mu\nu}(\omega_n,\mathbf{k})]^{-1}\,.$$

[Remember that $\mathfrak{D}_{\mu\nu}(t,\mathbf{x})$ is periodic in time; consequently its Fourier transform involves a sum over the discrete frequencies $k_0=\omega_n\equiv(2\pi/\beta)n$.] The timelike direction $\hat{e}_0$ is physically distinguished at finite temperature due to the finite period β. As a result one may construct four independent, symmetric, $O(3)$ covariant tensors depending on a single vector k. For example,

$$A_{\mu\nu}=\delta_{\mu i}(\delta_{ij}-k_ik_j/\mathbf{k}^2)\delta_{j\nu}\,,$$

$$B_{\mu\nu}=(\delta_{\mu0}-k_\mu k_0/k^2)k^2/\mathbf{k}^2(\delta_{\nu0}-k_\nu k_0/k^2)\,,$$

$$C_{\mu\nu}=\frac{1}{\sqrt{2}}(\delta_{\mu0}-k_\mu k_0/k^2)k_\nu/|\mathbf{k}|+\frac{1}{\sqrt{2}}k_\mu/|\mathbf{k}|(\delta_{\nu0}-k_\nu k_0/k^2)\,,$$

$$D_{\mu\nu}=k_\mu k_\nu/k^2\,.$$

Note that

$$A_{\mu\nu}+B_{\mu\nu}=\delta_{\mu\nu}-k_\mu k_\nu/k^2\,.$$

Thus the self-energy (which is always diagonal in color) may be decomposed as

$$\Pi^{ab}_{\mu\nu}=\delta^{ab}\Pi_{\mu\nu}=(\alpha A_{\mu\nu}+\beta B_{\mu\nu}+\gamma C_{\mu\nu}+\delta D_{\mu\nu})\delta^{ab}\,. \tag{4.5}$$

$\mathfrak{D}_{\mu\nu}$ satisfies a Ward identity, which implies that $k_\mu k_\nu\mathfrak{D}_{\mu\nu}=\alpha\equiv 1$. At zero temperature this condition plus Euclidean invariance implies that $k_\mu\Pi_{\mu\nu}=0$, or

$$\Pi_{\mu\nu}=\tfrac{1}{3}\Pi_{\alpha\alpha}(\delta_{\mu\nu}-k_\mu k_\nu/k^2)\,.$$

However, at finite temperature this merely provides one relation among the above coefficients, namely,

$$\delta=\tfrac{1}{2}\gamma^2/(k^2+\beta)\,. \tag{4.6}$$

Note that at zero frequency $\gamma(\omega=0,\mathbf{k})$ must vanish due to (Euclidean) time reversal invariance. Consequently, the static self-energy $\Pi_{\mu\nu}(\omega=0,\mathbf{k})$ is always transverse.

To $O(g^2)$, the self-energy is given by the one-loop diagrams in Fig. 3(b). Explicitly, these yield

$$\Pi^{ab}_{\mu\nu}(k)=\frac{g^2}{2}\delta^{ab}N_f\int_-\frac{d^4q}{(2\pi)^4}\,\mathrm{tr}[\gamma_\mu(\not k+\not q)\gamma_\nu\not q]/q^2(k+q)^2$$
$$+\frac{g^2}{2}\delta^{ab}N\int_+\frac{d^4q}{(2\pi)^4}\{2k_\mu k_\nu-4(k+q)_\mu q_\nu-4q_\mu(k+q)_\nu+2\delta_{\mu\nu}[(k+q)^2+q^2-2k^2]\}/q^2(k+q)^2\,. \tag{4.7}$$

Note that to one-loop order, $\Pi_{\mu\nu}$ is transverse for any frequency.

To study the screening caused by thermal fluctuations we should like to examine $\Pi_{\mu\nu}(\omega,\mathbf{k})$ for low spatial momentum, $\mathbf{k}\sim 0$. We show in Appendix C that as $\mathbf{k}\to 0$,

$$\Pi_{\mu\nu}(\omega_n=0,\mathbf{k})\sim\tfrac{1}{3}g^2T^2(N+N_f/2)\delta_{\mu0}\delta_{\nu0} \tag{4.8}$$

and

$$\Pi_{\mu\nu}(\omega_n\neq 0,\mathbf{k})\sim[\tfrac{1}{3}g^2T^2(N+N_f/2)+f(\omega_n)]\tfrac{1}{3}\delta_{\mu i}\delta_{\nu i}\,. \tag{4.9}$$

[See Appendix C for the explicit form of $f(\omega_n)$. It will be unnecessary for our discussion.]

The result (4.8) shows that A_0 develops a one-loop mass due to the thermal fluctuations. This mass, which we shall call the "electric" mass m_{el}, is given by

$$m_{\mathrm{el}}^2=\Pi_{00}(\omega=0,\mathbf{k}=0)=\tfrac{1}{3}g^2T^2(N+N_f/2)\,. \tag{4.10}$$

Note that both quark and gluon fluctuations contribute to the mass. The possibility of this electric mass is a direct consequence of the fact that at finite temperature the only way to approach zero (four) momentum, $k=0$, is to first set $k_0=\omega_n=0$ and then let $\mathbf{k}\to 0$. Hence $\Pi_{00}(\omega=0,\mathbf{k})$ is unconstrained by the transversality of the self-energy and so need not vanish at $\mathbf{k}=0$.

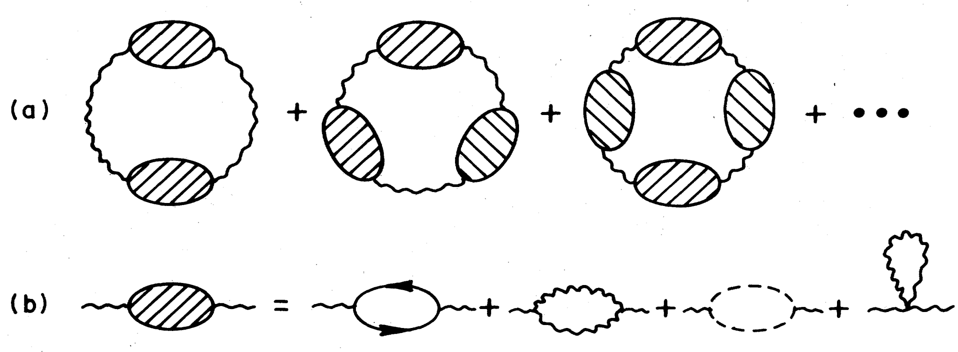

FIG. 3. (a) $O(g^3)$ contribution to the free energy. (b) $O(g^2)$ gluon self-energy.

However, the static spatial self-energy $\Pi_{ij}(\omega=0,\mathbf{k})$ must be transverse; $k_i\Pi_{ij}(\omega=0)=0$. Consequently,

$$\Pi_{ij}(\omega=0,\mathbf{k})=\tfrac{1}{2}(\delta_{ij}-k_ik_j/\mathbf{k}^2)\Pi_{kk}(\omega=0,\mathbf{k})\,. \qquad (4.11)$$

If $\Pi_{ii}(k=0)$ is nonzero, then the two transverse components of **A** will have developed a "magnetic" mass, $m^2_{\text{mag}}=\frac{1}{2}\Pi_{ii}(\omega=0,\mathbf{k}=0)$. The one-loop self-energy (4.7) is insufficiently divergent as $\mathbf{k}\to 0$ to develop the directional singularity (4.11) required for a magnetic mass (see Appendix C). Consequently, to one-loop order the spatial components of the gauge field **A** remain massless.

The electric mass (4.8) implies that

$$\langle A_0(x)A_0(y)\rangle \sim e^{-m_{\text{el}}|\mathbf{x}-\mathbf{y}|}$$

as $|\mathbf{x}-\mathbf{y}|\to\infty$. (The nonzero frequency correlations fall off much more rapidly, as $e^{-\omega_n|\mathbf{x}-\mathbf{y}|}$.) Thus A_0 acquires a finite correlation length of $(m_{\text{el}})^{-1}$. (See below, however.) Provided that higher-order corrections remain unimportant, this result implies that

$$\langle \text{tr}\Omega(\mathbf{x})\,\text{tr}\Omega(0)^\dagger\rangle \sim 1+g^4O(e^{-2m_{\text{el}}|\mathbf{x}|})\,. \qquad (4.12)$$

[The one gluon exchange term, of order $g^2\exp(-m_{\text{el}}|\mathbf{x}|)$, vanishes due to the separate traces in (4.12).] Consequently, the heavy quark potential behaves as

$$V(R)\sim g^4O(e^{-2m_{\text{el}}R}) \quad \text{as } R\to\infty\,,$$

showing that heavy quarks are *unconfined* at high temperatures. This lack of confinement is caused by the screening of the (color) charge of the heavy quarks due to the thermal fluctuations. (We shall discuss the region of validity of this result below.)

So far we have discussed the behavior of the gluon propagator in Euclidean space. This is the relevant domain for considering the perturbative behavior of the theory (see Sec. IV.C). However, one may choose to examine the Minkowski space behavior of the propagator; that is,

$$D_{\mu\nu}(x)\equiv \text{Tr}\{e^{-\beta H}T[A_\mu(x)A_\nu(0)]\}/Z\,,$$

where $A_\mu(t,\mathbf{x})=e^{iHt}A_\mu(0,\mathbf{x})e^{-iHt}$ is a genuine Heisenberg field operator. This is the appropriate correlation function for use in examining the real-time (linear) response of the system to perturbations which displace it from thermal equilibrium.

The Minkowski space propagator is simply the analytic continuation of the Euclidean propagator, $D_{\mu\nu}(t,\mathbf{x})=\mathfrak{D}_{\mu\nu}(it,\mathbf{x})$. However, the Fourier transform of the Minkowski propagator, $D_{\mu\nu}(k_0,\mathbf{k})$, is not just the continuation of $\mathfrak{D}_{\mu\nu}(\omega_n,\mathbf{k})$ (Kadanoff and Baym, 1962; Dolan and Jackiw, 1974). Rather, one must first continue $\mathfrak{D}_{\mu\nu}(\omega_n,\mathbf{k})$ to arbitrary (Euclidean) energy ω. This continuation $\mathfrak{D}_{\mu\nu}(\omega,\mathbf{k})$ is uniquely defined by the requirement that it not have an essential singularity at $\omega=\infty$ (Baym and Mermin, 1961). The resulting $\mathfrak{D}_{\mu\nu}(\omega,\mathbf{k})$ is analytic in the right and left ω half-planes. Across the imaginary axis it will have some discontinuity

$$\rho_{\mu\nu}(k_0,\mathbf{k})=\mathfrak{D}_{\mu\nu}(ik_0-\varepsilon,\mathbf{k})-\mathfrak{D}_{\mu\nu}(ik_0+\varepsilon,\mathbf{k})\,.$$

$\rho_{\mu\nu}(k_0,\mathbf{k})$ is the spectral density; it defines the possible energies for an excitation of momentum **k**. [In fact,

$$\rho_{\mu\nu}(k)=\int d^4x\, e^{-ikx}\langle[A_\mu(x),A_\nu(0)]\rangle\,.]$$

The Minkowski space propagator $D_{\mu\nu}(k)$ may now be reconstructed as (Dolan and Jackiw, 1974)

$$D_{\mu\nu}(k)=i\int_{-\infty}^{\infty}\frac{dk_0'}{2\pi}\rho_{\mu\nu}(k_0',\mathbf{k})\left(\frac{1+f(k_0')}{k_0-k_0'+i\varepsilon}-\frac{f(k_0')}{k_0-k_0'-i\varepsilon}\right)$$
$$=\mathfrak{D}_{\mu\nu}(i(k_0+i\varepsilon),\mathbf{k})+f(k_0)\rho_{\mu\nu}(k)\,.$$

Here $f(k_0)=1/(e^{\beta k_0}-1)$.

Applying this procedure to the one-loop results (4.8) and (4.9), we find (in Feynman gauge) that

$$\mathfrak{D}_{\mu\nu}(k)\sim\frac{\delta_{\mu 0}\delta_{\nu 0}}{k^2+m^2_{\text{el}}(\mathbf{k}^2/k^2)}+\frac{\delta_{\mu i}\delta_{\nu i}}{k^2+\frac{1}{3}[m^2_{\text{el}}+f(k_0)](k_0^2/k^2)}$$

as $\mathbf{k}\to 0$. If we take $\mathbf{k}\to 0$ for $k_0\neq 0$, then

$$\mathfrak{D}_{\mu\nu}(k_0\neq 0,\mathbf{k}=0)=\frac{\delta_{\mu i}\delta_{\nu i}}{k_0^2+\frac{1}{3}[m^2_{\text{el}}+f(k_0)]}+\frac{\delta_{\mu 0}\delta_{\nu 0}}{k_0^2}\,. \qquad (4.13)$$

We show in Appendix C that $f(k_0)$ vanishes as $k_0\to 0$. Consequently, the continued propagator (4.13) has a pole at $k_0^2=-\frac{1}{3}m^2_{\text{el}}+O(g^3)$. Thus the spectral density equals

$$\rho_{\mu\nu}(k_0,\mathbf{k}=0)=2\pi\varepsilon(k_0)[\delta_{\mu i}\delta_{\nu i}\delta(k_0^2-\tfrac{1}{3}m^2_{\text{el}})+\delta_{\mu 0}\delta_{\nu 0}\delta(k_0^2)]\,. \qquad (4.14)$$

This shows that the transverse, zero-momentum excitations have an energy of $m_{\text{el}}/\sqrt{3}$. This is the analog of the usual plasmon (Pines, 1964). One might be tempted to conclude from this that all transverse gluons have acquired a mass $m_{\text{el}}/\sqrt{3}$, that all color fluctuations are screened, and that no long-range forces exist at finite temperature (Kisslinger and Morley, 1976a, 1976b). However, this is wrong. The problem is that $\mathfrak{D}_{\mu\nu}(k)$ is not analytic about $\vec{k}=0$. So, for example, the limits $k_0\to 0$ and $\mathbf{k}\to 0$ do not commute. In fact, if we first set $k_0=0$ and then let $\mathbf{k}\sim 0$, we find

$$\mathfrak{D}_{\mu\nu}(k_0=0,\mathbf{k}\sim 0)=\frac{\delta_{\mu 0}\delta_{\nu 0}}{(\mathbf{k}^2+m^2_{\text{el}})}+\frac{\delta_{\mu i}\delta_{\nu i}}{\mathbf{k}^2} \qquad (4.15)$$

and $\rho_{\mu\nu}(k_0=0,\mathbf{k}\sim 0)=0$.

Since, for a static electric field

$$\langle \mathbf{E}(\mathbf{x})\mathbf{E}(\mathbf{x}')\rangle=\langle\partial A_0(\mathbf{x})\partial A_0(\mathbf{x}')\rangle+(\text{higher orders})\,,$$

Eq. (4.15) may be interpreted as showing that a static external electric field is screened by the plasma of thermal excitations. $(m_{\text{el}})^{-1}$ is the electric screening length. However, for a static magnetic field

$$\langle \mathbf{B}(\mathbf{x})\mathbf{B}(\mathbf{x}')\rangle=\langle\partial\times\mathbf{A}(\mathbf{x})\partial\times\mathbf{A}(\mathbf{x}')\rangle+(\text{h.o.})\,.$$

Consequently, Eq. (4.15) shows that a static external magnetic field is unscreened and so penetrates the plasma. Thus (to one-loop order) the plasma of thermal excitations acts like a conductor but not a superconductor. [This same behavior is found in a finite-density, zero-temperature quark gas (Freedman and McLerran, 1977b).][13]

This discussion has been slightly oversimplified at

[13]Naturally, one may also consider the one-loop quark propagator and its continuation back to Minkowski space. One finds (for massless quarks) that $S(p)\sim \not{p}/(p^2+m_q^2)$ for $p\sim 0$ with $m_q^2=\pi^2(N^2-1/2N)g^2T^2/8$. This chirally invariant "mass" physically reflects the presence of a coherent polarization cloud which surrounds the quark.

two points which will now be clarified. First, although $\Pi_{00}(\omega=0,\mathbf{k}=0)$ is nonvanishing, the explicit one-loop expression (4.7) is not analytic in $\mathbf{k}$ about the origin. Thus the Fourier transform

$$\int d^3k\, e^{i\mathbf{k}\cdot\mathbf{x}}/[\mathbf{k}^2+\Pi_{00}(\mathbf{k})]\,,$$

which occurs in $\mathfrak{D}_{00}(t,\mathbf{x})$ does not in fact decay exponentially with a correlation length of $(m_{\rm el})^{-1}$. Second, the pole in

$$\mathfrak{D}_{ij}(ik_0,\mathbf{k}=0)\sim(-k_0^2\delta_{ij}+\Pi_{ij}(\omega,\mathbf{k}\doteq 0))^{-1}$$

(which determines the plasmon mass) is not precisely at $k_0=\pm m_{\rm el}/\sqrt{3}$ but rather is shifted by an imaginary amount of order g^3T (into the physical sheet). See Appendix C for the explicit expression. Fortunately, these difficulties with the analytic structure of $\Pi_{\mu\nu}$ are simply artifacts of our failure to do a fully self-consistent calculation. Specifically, the internal gluon propagators in Eq. (4.7) represent the bare massless gluons. Higher-order corrections will include terms which replace these bare propagators with full propagators. Resumming these terms will yield a self-consistent integral equation for the self-energy which is an approximation to the full Schwinger–Dyson equations. (This will be discussed further in the next subsection.) Since $\Pi_{\mu\nu}$ is $O(g^2)$ the only relevant effect of this procedure will be to correct the analyticity properties of the gluon propagator. This improved propagator will then yield a finite static correlation length of $(m_{\rm el})^{-1}$ and a real plasmon mass equal to $m_{\rm el}/\sqrt{3}$ (up to higher-order corrections).

C. Higher orders

At sufficiently high temperature one might expect the lowest-order results to yield an increasingly accurate representation of the free energy. Standard renormalization group arguments yield

$$\mathfrak{F}(T,g,\Lambda)=(T/T_0)^4\mathfrak{F}(T_0,\bar{g}(T),\Lambda)\,, \qquad (4.16)$$

where Λ is the renormalization point and $\bar{g}^2(T)$ is the effective coupling which vanishes (as $1/\ln T$) for large T. Thus *if* the free energy has a simple power series asymptotic expansion for small g^2, then $\mathfrak{F}(T)$ could be calculated to arbitrary precision for large enough temperature. Unfortunately this is not the case. The singular infrared behavior of the Green's functions vitiates a naive expansion in powers of g^2.

One source of infrared divergences arises from multiple insertions of the one-loop gluon self-energy. For example, consider the contribution to the free energy of the graphs in Fig. 3(a), in which a timelike gluon circulates around the loop. These terms behave as

$$\frac{1}{\beta}\int d^3k\frac{1}{n}[\Pi_{00}(\omega=0,\mathbf{k})/\mathbf{k}^2]^n \quad (n\geq 2)\,.$$

They are infrared divergent due to the generation of an electric mass $[\Pi_{00}(0,\mathbf{k}=0)\neq 0]$. Clearly these divergences reflect the need to reexpress the perturbative expansion in terms of the full propagators instead of bare massless propagators. This may be achieved quite simply by adding the full self-energy, $\frac{1}{2}A^\mu\Pi_{\mu\nu}A^\nu$, to the free Lagrangian and subtracting it from the interaction terms.[14] This yields an improved expansion using full propagators, which eliminates the above infrared divergences. The leading term in the free energy will now contain $-\frac{1}{2}\ln\det(\mathfrak{D}_{\mu\nu})$ in place of the free gluon determinant in Eq. (4.2). This includes the zero-frequency contribution

$$\frac{1}{2\beta}\int\frac{d^3k}{(2\pi)^3}\,\mathrm{tr}\,\{\ln[1+\Pi^{ab}_{\mu\nu}(0,\mathbf{k})/\mathbf{k}^2]-\Pi^{ab}_{\mu\nu}(0,\mathbf{k})/\mathbf{k}^2\}$$
$$=-\frac{(N^2-1)}{12\pi}T\,m_{\rm el}^3+O(g^4)\,,$$

which [using Eq. (4.10)] immediately yields the $O(g^3)$ term in (4.4). Similarly, order g^4 corrections to $m_{\rm el}^2$ will induce $O(g^5)$ corrections in $\mathfrak{F}$, etc. A systematic expansion of this sort requires that we determine the self-energy from the Schwinger–Dyson equations and insert it into a skeleton expansion of the free energy. (For details of the skeleton expansion for $\mathfrak{F}$ see Freedman and McLerran, 1977a.)

We shall now discuss the behavior of this improved expansion. If we were dealing with QED, instead of QCD, the above procedure would eliminate all infrared divergences, and the free energy could be expanded to arbitrary order in powers of e. The timelike photons acquire in lowest order a static electric mass of order eT, and thus their propagators are finite at zero momenta. This is not the case for spacelike photons, which remain massless. However, since their only static interaction is with electrons, whose energy can never vanish due to the antiperiodic boundary conditions [i.e., $\omega_n^-=(2n+1)\pi/\beta$], all diagrams containing electron loops are infrared finite and can be expanded in the external momenta. Consequently, each term in the improved expansion is infrared finite. [In fact, as we shall discuss below, the behavior of QED for low momenta, $\mathbf{q}\lesssim eT$, is equivalent to a three (space-time)-dimensional theory of free photons (i.e., $\mathbf{A}$) and a neutral scalar field (i.e., A_0) of mass $\sim eT$, with a quartic coupling of order e^4T.] The only surprising feature is that the perturbative expansion is a series in e instead of e^2.

Unfortunately, the infrared behavior of QCD is much more singular due to the self-interactions of the gluons. Consider the contribution to $\mathfrak{F}$ of an arbitrary n-loop diagram consisting of spatial gluons with zero energy. The contribution that arises from the region where all spatial momenta are of order q will behave as

$$[g^2T(q^3)/(q^4)]^{n-1}\approx(g^2T/q)^{n-1}$$

for small q. (Note that each new loop contributes a factor of g^2, a factor of Tq^3 from phase space, and a factor of $1/q^4$ from the new vertices and propagators.) Thus due to the self-coupling of the massless spacelike gluons, higher-order corrections to $\mathfrak{F}$ are increasingly divergent.

Fortunately, the remedy for this disastrous situation may be found in these very singularities! The singularities arise because the spatial gluons were assumed to be massless. This is true to lowest order, since the one-loop self-energy (4.7) was not sufficiently infrared singular to produce the directional singularity (4.11) re-

[14]The ghost propagator should be similarly resummed.

quired to generate a magnetic mass. However, higher orders, starting with the two-loop self-energy, are singular enough to potentially generate a magnetic mass m_{mag} of order g^2T. After resuming the perturbation expansion so as to replace bare with full propagators, all infrared divergences will be removed by such a mass, since the momenta flowing through spacelike gluon lines will be cut off at $q \approx m_{mag}$. The contribution of the low-momentum region, $q \lesssim m_{mag}$, for an n-loop graph will then be of order $(g^2T/m_{mag})^{n-1}$, which is finite.

If the infrared divergences are cured in the above fashion, does this mean that one can calculate the expansion of $\mathfrak{F}$ in powers of g? The answer, unfortunately, is no (Linde, 1979). Since the magnetic mass is at most of order g^2T, beyond some point increasingly complex graphs will be of the same order in g^2.

Thus at some point in the expansion of $\mathfrak{F}$, or in the expansion of Green's functions for external moment $q \lesssim m_{mag}$, we lose perturbative calculability. For the free energy one finds that this occurs first at order g^6, and therefore while the first five terms of $\mathfrak{F}$ are perturbatively calculable, beyond this point an infinite number of diagrams contribute to order g^6. The value of m^2_{mag} itself is incalculable, since it receives contributions from n-loop graphs of order $(g^2T)^n/(g^2T)^{n-2} \approx g^4T^2$. (For that reason we have not attempted to calculate the two-loop contribution to m_{mag}.)

Clearly, perturbation theory breaks down. At best we may assume that $\mathfrak{F}$ is expandable to order g^6 and that the coefficient of g^6 is finite, although incalculable. If m_{mag} vanishes, then the expansion of $\mathfrak{F}$ would actually diverge. Although this result is a straightforward consequence of the infrared power counting arguments, one may feel that a simple physical picture is missing. In other words, why must the magnetic mass be $O(g^2T)$ instead of, for example, $O(g^3T)$? Why is the free energy incalculable at $O(g^6)$ instead of $O(g^4)$? [By contrast, the fact that the one-loop electric mass is $O(gT)$ is easy to understand. It follows from noting that external electric fields are screened by the charged particles in the thermally excited plasma. The average separation between particles is $\sim 1/T$ (since the density of gluons or quark–antiquark pairs is $\sim T^3$) and their coupling to the electric field is $\sim g$. Consequently, the screening length is $\sim (gT)^{-1}$.]

We should like to argue that the above results should have been expected. First, since the infrared divergences treated above arise from regions where all internal energies vanish, the singularities are the same as would arise in a three-dimensional gauge theory. In general the infrared behavior at high temperature of a d-dimensional theory is given by an equivalent $(d-1)$-dimensional theory. Here the equivalent theory is a three-dimensional gauge theory, whose coupling is g^2T. The static component of A_0 behaves like a scalar (Higgs) field in the adjoint representation, whose mass is $m_{el} \sim gT$. The quarks and the nonstatic components of the gauge field all behave as massive particles with mass $\sim T$. Since these are much larger than the fundamental scale of the theory, i.e., g^2T, one expects A_0, the quarks, and all nonstatic fields to decouple as $T \to \infty$. The decoupling theorem (Applequist and Carrazone, 1975) assures us that this is true up to corrections of order $g^2T/m_{el} \sim g$ or $g^2T/T \sim g^2$, and up to a renormalization of the coupling (g^2T) due to the heavy particles. Since the three-dimensional pure gauge theory is not merely renormalizable, but actually superrenormalizable, these renormalization effects should be fully computable. In fact, one may explicitly examine all superficially ultraviolet divergent graphs and see that the heavy particles induce no effects which are not suppressed by powers of g. (The electric mass for A_0 and the first five terms of the free energy could be considered as such renormalization effects since they are sensitive to momentum $q \gg g^2T$. These explicitly calculable terms have already been taken into account and, consequently, may be disregarded now.)

So, we learn that the leading infrared behavior of high-temperature QCD is the same as a three-dimensional pure gauge theory at a coupling g^2T.[15] The three-dimensional theory is completely finite. The only mass scale which appears is the coupling g^2T. Consequently, the mass gap (i.e., m_{mag}) can only be a pure number (possibly zero) times g^2T. Similarly, the three-dimensional free energy is a constant times $(g^2T)^3$. Perturbation theory is obviously useless for computing the infrared properties of the three-dimensional theory; there is no small, dimensionless parameter. Thus the previous breakdown of perturbation theory is simply reflecting the (perturbative) incomputability of three-dimensional non-Abelian gauge theories.[16]

However, one does believe that non-Abelian gauge theories are confining and possess a mass gap in any dimension less than or equal to four. Consequently, one expects a nonzero magnetic mass of order g^2T. (Similarly, one expects spacelike Wilson loops to exhibit area law behavior. Remember that this three-dimensional "confinement" has nothing to do with the behavior of real heavy quarks.)

This shows why the magnetic mass was $O(g^2T)$ and why the $O(g^6)$ terms in the free energy were incalculable. We also learn slightly more from this approach. In our previous power counting arguments we did not bother to worry about possible factors of $\ln g^2$. However, the fact that the heavy particles induce no renormalization effects which are not suppressed by powers of g shows that, for example, no such logs appear in the $O(g^2T)$ magnetic mass. (Factors of $\ln g$ could appear in the terms which are down by powers of g. In principle such logs could sum up to form g^γ. If $\gamma < -1$ we would have a nonperturbative breakdown of the decoupling theorem; we can only assume that this does not occur.)

[15]This shows that the high-temperature limit of the quantum partition function reduces to the classical partition function. For a recent discussion of this point see Dolan and Kiskis (1979).

[16]One may have entertained the notion that all the infrared divergences, indicated by the power counting analysis, might miraculously cancel among themselves, so that the theory would remain perturbatively calculable. However, this possibility is equivalent to the assumption that the sum of all n-loop diagrams $(n \geq 2)$ of the three-dimensional pure gauge theory identically vanish. This would imply that the three-dimensional theory is trivial, which is absurd. Hence perturbation theory *must* break down.

Finally, we should like to present an extremely intuitive picture of the mechanism producing this magnetic mass. We shall argue that it is due to magnetic screening by topologically unstable magnetic monopoles.

Among the many fluctuations contributing to the original functional integral, let us consider the effects of spatial magnetic monopoles. By this we mean fields with long-range magnetic fields, $\mathbf{B}\sim 1/r^2$, and with $\Omega(\mathbf{x})\sim 1$ for $\mathbf{x}\to\infty$. For example, we may consider a static field with $A_0=0$ and where $\mathbf{A}$ describes a Wu-Yang monopole (Wu and Yang, 1969). From Sec. III we know that such fields will not possess topologically conserved magnetic charge and hence will be unstable. Such a field will be characterized by an arbitrary spatial scale size R. As $R\to\infty$, the energy of the field may be made arbitrarily small. However, the field may be made a constrained solution subject to a single constraint which fixes the scale size R. Owing to the $1/r^2$ long-range behavior of the magnetic field, the minimal energy will be $\mathcal{E}=c/g^2R$ for some constant c. (The $1/g^2$ appears because $\mathbf{A}\sim 1/g$.) Let us imagine expanding the functional integral about these unstable monopoles. We shall have to treat the position and size of the monopole as collective coordinates (Polyakov, 1977). Consequently, their contribution to the functional integral will (suppressing irrelevant factors) behave like

$$\delta Z \sim \int\frac{dR}{R}\int\frac{d^3x}{R^3}\exp(-c\beta/g^2R)$$

$$\sim \int d^3x\int\frac{dR}{R}\exp[-(c\beta/g^2R+3\ln R)]. \qquad (4.17)$$

The $3\ln R$ clearly represents the entropy associated with the three translational degrees of freedom. Since the energy favors increasing the monopole size, while the entropy opposes this, there will be an optimal size which maximizes the contribution. This occurs at $R=c/3g^2T$. Factors of g^2 may be simply scaled out of the integral [Eq. (4.17)] and we learn that these monopoles give a contribution of order (g^6T^4) to the free energy. This does not have the typical e^{-1/g^2} form characteristic of a topologically stable configuration. Rather, we see that these unstable monopoles are indistinguishable from perturbative fluctuations about the vacuum. Since $\mathbf{A}(\mathbf{x})\sim O(1/g|\mathbf{x}|)$ we may estimate the monopole contribution to the gluon propagator $\langle\mathbf{A}\,\mathbf{A}\rangle$ by averaging $\mathbf{A}(\mathbf{x})$ $\mathbf{A}(\mathbf{y})$ over all monopole positions and multiplying by the density [Eq. (4.17)]. In momentum space, we find (as $\mathbf{k}\to 0$)

$$\delta\langle\mathbf{A}\,\mathbf{A}\rangle\sim g^4T^2/\mathbf{k}^4 .$$

This looks just like a mass insertion of an $O(g^2T)$ magnetic mass.

Thus we see how a magnetic mass of order g^2T can arise from magnetic screening due to a finite density of topologically unstable magnetic monopoles. Note that we are not claiming to be able to do a reliable semiclassical calculation by expanding around these monopoles. This is impossible precisely because the unstable monopoles are in fact indistinguishable from perturbative fluctuations. However, they provide a very intuitive picture of how the magnetic mass can arise.

This description is very similar to Polyakov's treatment of three-dimensional compact QED. The only difference is that the Abelian monopoles in compact QED are topologically stable and therefore have a minimal action of $O(1/g^2)$. Consequently, the monopoles are very dilute and only generate a mass gap of order e^{-1/g^2} (Polyakov, 1977).

The overall picture of the high-temperature phase of QCD is that of an electrically screening phase with a correlation length $(m_{\rm el})^{-1}\sim 1/gT$ and a magnetic mass gap $m_{\rm mag}\sim g^2T$. Owing to Debye screening heavy quarks are not confined. Although the electric mass may be reliably calculated in lowest-order perturbation theory, a genuine calculation of the magnetic mass appears to require the complete solution of the three-dimensional pure gauge theory. In Sec. VII we shall discuss the expected behavior of QCD as the temperature is lowered.

V. θ DEPENDENCE AT HIGH TEMPERATURE

Owing to dimensional transmutation, it would appear that quarkless QCD has no free parameters. However, one may add to the action [Eq. (2.3)] the surface integral

$$(i\theta/32\pi^2)\int d^4x\,\hat{\rm tr}F_{\mu\nu}\tilde{F}_{\mu\nu}=i\theta Q ,$$

thereby apparently introducing an adjustable parameter θ.

On the other hand, as noted in Sec. II, $A_0(t,\mathbf{x})$ must vanish at spatial infinity if all states satisfying Gauss's law are to contribute to the partition function. This implies that the topological charge Q is always zero (see Appendix B). Consequently, no θ dependence can arise, since nontrivial dependence requires that field configurations with $Q\neq 0$ contribute to the functional integral.

Should one allow fields with nonzero Q to contribute? Consider the contribution of fields satisfying $\Omega(\mathbf{x})\to 1$ as $\mathbf{x}\to\infty$, for which Q is always an integer (see Sec. II). The effect of the $e^{i\theta Q}$ term in the functional integral is to project out states—θ states—which transform as $|\psi\rangle_\theta \to e^{in\theta}|\psi\rangle_\theta$ under any gauge transformation which is constant at infinity and has winding number n (Jackiw and Rebbi, 1976). Thus by relaxing the boundary conditions on A_0 at spatial infinity one can restrict the theory to a smaller subspace of physical states. [Note that the above projection is consistent with Gauss's law (2.6), since a regular gauge transformation whose generator vanishes at infinity has winding number zero.] Conversely, imposing the boundary condition $A_0(t,\infty)=0$ means that one is summing over *all* physical states, including all θ states, in the partition function.

What is one to do? Since one can show that no physical operator can connect states with different values of θ, it follows that θ labels completely disjoint sectors of the theory. In particular, a complete physical theory may be built on each θ vacuum (Callan, Dashen, and Gross, 1976; Jackiw and Rebbi, 1976). It will be shown below that different θ worlds have different physical properties. Therefore, in order to calculate the expectation values of observables in a pure state, not in a mixture, one must project onto a single θ sector.

One might now wonder whether additional physically distinct sectors will be revealed if fields with $\Omega(\infty)\neq 1$ are included. However, this does not appear to be the case. For example, including fields where A_0 is constant as $\mathbf{x}\to\infty$ is equivalent to inserting a projection onto global color charge zero states. Thus one is separating color charge superselection sectors. However, different charge sectors should not yield physically inequivalent theories. Thus there appears to be no formal reason for including "infinite range" fields with $\Omega(\infty)\neq 1$.

As further confirmation of this idea, we shall next argue that for sufficiently high temperature, even if such infinite-range fields are included in the functional integral, their relative contribution to the partition function vanishes in the thermodynamic ($V\to\infty$) limit. (This behavior is analogous to that of a Coulomb gas. One need only include configurations which satisfy charge neutrality in the partition function. If this constraint is relaxed then one finds that typical fluctuations in the average charge density are of order $V^{-1/2}$. Hence the contribution of configurations with an imbalanced charge density vanishes in the thermodynamic limit.)

A. Infinite-range fields

We shall argue in several steps that infinite-range fields do not contribute to the functional integral for large T. First let us consider the contribution to the partition function that arises from expanding about an infinite-range exact, or constrained, classical solution. The simplest of these is a classical vacuum, $A_0=$ constant. Let $A_\mu=a_\mu+g\overline{A}_\mu$, where $a_\mu\equiv\sigma\delta_{\mu 0}$ and for convenience choose background gauge,

$$D_\mu(a)\overline{A}_\mu=[\partial_\mu+a_\mu,\overline{A}_\mu]=0\,.$$

Expanding the functional integral in powers of $\overline{A}$ yields the contribution

$$\begin{aligned}I(\sigma)&=\det{}_+^{-1/2}[-D(a)^2\delta_{\mu\nu}]\det{}_+[-D(a)^2]\\&\quad\times\det{}_-^{N_f}[\not{D}(a)][1+O(g^2)]\\&=\det{}_+^{-1}(D^2_{\mathrm{adj}})\det{}_-^{N_f}(\not{D}_{\mathrm{fund}})[1+O(g^2)]\,.\end{aligned}$$

These determinants are computed in Appendix D. Using Eqs. (D4) and (D5) we find

$$\begin{aligned}I(\sigma)=\exp\Bigg[&\pi^2\frac{V}{\beta^3}\frac{1}{45}(N^2-1)-2NN_f)+\frac{N_f}{12}\mathrm{tr}[1-(\ln\Omega^{\mathrm{fund}}/\pi i)^2]^2\\&-\frac{1}{6}\mathrm{tr}[(\ln\Omega^{\mathrm{adj}}/\pi i)(\ln\Omega^{\mathrm{adj}}/2\pi i-1)]^2+O(g^2)\Bigg].\qquad(5.1)\end{aligned}$$

(Here $\Omega=e^{\beta\sigma}$ is represented in either the fundamental or adjoint representations.)

Assuming that the temperature is sufficiently large so that the $O(g^2)$ corrections are small, we see that this contribution is maximized when $\Omega(\mathbf{x})=1$ or $A_\mu=0$. The contribution from any sector with $\Omega(\mathbf{x})\neq 1$ is suppressed by a factor of $\exp[-(cV/\beta^2)]$, for some $c>0$, relative to the leading ($A_\mu=0$) contribution. This becomes complete suppression in the thermodynamic ($V\to\infty$, β fixed) limit. Thus, for sufficiently high temperature, the free energy density

$$\mathfrak{F}=\lim_{V\to\infty}(-\ln Z)/\beta V$$

receives no contribution from small fluctuations about $\Omega=$ constant $\neq 1$ fields.

This same conclusion will be valid if we expand about any other exact or constrained solution with $\Omega(\infty)\neq 1$. As shown in Appendix B, any finite-energy field may be transformed to a gauge where $\Omega(\mathbf{x})$ goes to a constant as $\mathbf{x}\to\infty$. Consequently, the determinants in the background field will yield the same large-distance behavior as Eq. (5.1) multiplied by finite volume independent factors. Hence such fields will make no contribution to the free energy density.

Finally, to verify that this conclusion is not an artifact of expanding about classical solutions, we shall determine the leading high-temperature behavior of the quantum effective action and show that the same result emerges.

The effective action $\mathcal{S}[A_\mu]$ is a functional of a periodic gauge field, $A_\mu(t,\mathbf{x})$. To define the high-temperature limit we take

$$A_\mu(t,\mathbf{x})=a_\mu(2\pi t/\beta,\mathbf{x})\,,\qquad(5.2)$$

where $a_\mu(\tau,\mathbf{x})$ is periodic in τ with period 2π. The analysis of $\mathcal{S}[a_\mu]$ essentially follows from Weinberg's classic analysis of symmetry restoration at high temperature (S. Weinberg, 1974). The effective action is the generating functional of all proper vertices; it may be expanded in a sum over all 1PI graphs with a_μ inserted on the external legs. Consider an arbitrary graph contributing to $\mathcal{S}[a_\mu]$ with superficial ultraviolet degree of divergence D. It yields a contribution of the form

$$\sum_{n_1,\ldots,n_{n-1}}\int d^3k_1,\ldots,d^3k_n a_\mu(n_1,\mathbf{k}_1),\ldots,a_\nu(n_n,\mathbf{k}_n)\times\delta^4\Big(\sum_i k_i\Big)I_{\mu,\ldots,\nu}(\omega_{n_1},\mathbf{k}_1;\ldots;\omega_{n_n},\mathbf{k}_n)\,.$$

If we rescale all internal momenta by T we find

$$I(\omega_i,\mathbf{k}_i)=T^D I(2\pi n_i,\mathbf{k}_i/T)\,.$$

Consequently, the contribution will be of order (T^{D-1}) provided $I(\omega_i,\mathbf{k}_i)$ has a finite limit as all external spatial momenta vanish. However, due to the increasingly singular infrared behavior, $I(\omega_i,\mathbf{k}_i)$ does not in general have a smooth zero-momentum limit. Fortunately, we were able to argue in the last section that in the high-temperature phase A_0 acquires an "electric" mass of order gT and $\mathbf{A}$ acquires "magnetic" mass of order g^2T. Consequently, if the expansion for $\mathcal{S}[a_\mu]$ is resummed to produce full propagators, then no such infrared divergence will occur and we will be able to scale T out of the integrand.

This shows that the leading high-temperature behavior of the effective action is controlled by the terms with the largest superficial degree of divergence. These are simply the gluon self-energy graphs. Therefore we find[17]

$$\mathcal{S}[A_\mu]=\tfrac{1}{2}\int d^4x\,\hat{\mathrm{tr}}(A_\mu\mathcal{D}^{-1}_{\mu\nu}A_\nu)+O((T)^0)\,.\qquad(5.3)$$

Using the previous results for $\Pi_{\mu\nu}(\omega_n,\mathbf{p})$ (Sec. IV.B) this becomes

[17]The free energy $\beta V\mathfrak{F}$, which is the leading term in $\mathcal{S}[A_\mu]$, has been omitted since it is independent of A_μ.

$$\mathcal{S}[A_\mu]\sim\frac{\beta}{2}\int d^3x\,\hat{\mathrm{tr}}\left[\left(T\int_0^\beta dt\,\mathbf{E}^2\right)+m_{\mathrm{el}}^2\left(T\int_0^\beta dt\,A_0\right)^2\right]$$

up to terms suppressed by powers of g^2 or β. Recall that the electric mass

$$m_{\mathrm{el}}^2=\Pi_{00}(\omega=0,\mathbf{p}=0)=\tfrac{1}{3}g^2T^2(N+N_f/2).$$

This result may be written in the manifestly gauge-invariant form

$$\mathcal{S}[A_\mu]\sim\frac{1}{g^2}\int d^4x\,\hat{\mathrm{tr}}[\tfrac{1}{4}F_{\mu\nu}^2+\tfrac{1}{2}m_{\mathrm{el}}^2T^2(\Omega-1)^2]\,. \qquad (5.4)$$

(Here g has been scaled back out of A_μ.) The important feature here is the appearance of the term

$$(\Omega-1)^2\sim\left(\int_0^\beta dt\,A_0(t,\mathbf{x})\right)^2.$$

This provides a mass for the static component of A_0 and clearly will cause the effective action of any field with $\Omega(\infty)\neq1$ to diverge. Naturally, if A_0 is taken to be a constant, then this result agrees with the $O(T)$ term in Eq. (5.1).

Therefore, we conclude that for sufficiently high temperature only fields with $\Omega(\infty)=1$ contribute to the functional integral. The topological charge Q is "dynamically" quantized due to the screening behavior of the thermal fluctuations. Consequently, for sufficiently high temperature, the theory will be periodic in θ with period 2π.

B. Instantons

Nontrivial θ dependence can only arise due to the contribution of fields with topological charge $Q=n\neq0$. To compute the θ dependence we may try to expand the functional integral in the sector with $Q=n$ about the minimal action field with topological charge n. These fields are precisely the periodic instantons discussed in Sec. III.

At zero temperature one cannot reliably compute the θ dependence semiclassically. In the expansion about instantons the difficulty appears as the absence of any large-distance cutoff on the instanton scale size ρ (Callan, Dashen, and Gross, 1978, 1979a). However, at finite temperature one expects the temperature T to provide a physical scale which may serve as a cutoff. Classically, this does not happen; the classical action of an instanton is $8\pi^2/g^2$ independent of its scale size. Fortunately, quantum effects can produce this cutoff. Any field with nonzero topological charge must have a background electric field. But thermal (=quantum) fluctuations will screen this electric field (see Sec. IV. B) and consequently will suppress the contribution of such fields.

To see this suppression explicitly we may simply insert the instanton field A_μ^I [Eq. (3.8)] into the quantum effective action [Eq. (5.4)].[18] This will yield the high-temperature behavior of the instanton density. One finds

$$n(\rho)\sim\exp(-\mathcal{S}[A^I])$$
$$\sim\exp\{-[8\pi^2/g^2+\tfrac{1}{3}(2N+N_f)(\pi\rho T)^2]\}\,. \qquad (5.5)$$

This shows that large-scale instantons, $\rho\gg\beta$, will be exponentially suppressed. (The complete temperature dependence of the instanton density will be computed in the next section.)

Consequently, one may reliably calculate the one-loop contribution of instantons to the functional integral. The θ dependence of the free energy will have the characteristic instanton form

$$\frac{\partial\mathcal{F}}{\partial\theta}=(\sin\theta)\int_0^\infty d\rho\,n(\rho)\sim(\sin\theta)T^4\exp\{-[8\pi^2/g^2(T)]\}\,. \qquad (5.6)$$

This will be computed in detail in the following section.

This semiclassical expansion about an instanton will have precisely the same reliability for high temperature as the perturbative expansion about $A_\mu=0$. Specifically, the first few corrections will be directly calculable; however, due to the necessity of a self-consistent treatment of magnetic screening, beyond a certain point all succeeding corrections will be perturbatively incalculable (see Sec. IV.C).

The overall picture presented here is completely analogous to the behavior found by Affleck in the two-dimensional CP^N model. In both QCD and the CP^N model the quantization of topological charge is a dynamical consequence of electric screening. θ dependence may be reliably calculated at high temperature and has the characteristic exponentially small form (5.6) indicative of instantons. [Of course, in two dimensions the absence of transverse degrees of freedom eliminates all difficulties with magnetic fluctuations. Furthermore, one may use a $1/N$ expansion to independently confirm the instanton results (Affleck, 1980a, b).]

By now it should be clear that θ is a genuine, physically relevant, periodic parameter of high-temperature QCD.[19] Surely θ remains a physically relevant parameter of QCD even at $T=0$, as naive semiclassical arguments would indicate. Whether θ remains periodic, with period 2π, and whether further nonperturbative parameters appear at low temperature is an open dynamical question depending on the contribution of field configurations with $\Omega(\infty)\neq1$.

VI. THE INSTANTON DENSITY

The one-loop contribution to the partition function from fluctuations about single instantons is given (for $\theta=0$) by

$$Z_I=Z_0\int d^4x\,d\rho\,n(\rho),$$

where Z_0 is the perturbative result, and the instanton

[18] One may question the applicability of Eq. (5.4) to an instanton, since the chosen high-temperature limit (5.2) does not preserve the precise form of the instanton. However, one may easily show that only the large-distance, static part of the instanton field contributes to the leading behavior [Eq. (5.5)].

[19] Faced with the problem of assuring the naturalness of the choice $\theta=0$ for the strong interactions, in the presence of CP violating weak interactions, some authors (e.g., Linde, 1980) have argued that θ is not a real, adjustable parameter in QCD. Our results confirm the reality of θ.

density $n(\rho)$ is equal to ('t Hooft, 1976)

$$n(\rho)=\nu[\det J]^{1/2}[\det{}'(-D^2\delta_{\mu\nu}-2F_{\mu\nu})_{\rm adj}]^{-1/2}$$

$$\times[\det(-D^2)_{\rm adj}]\prod_{i=1}^{N_f}[\det(\not{D}+m_i)_{\rm fund}]\exp(-8\pi^2/g_0^2)\,. \tag{6.1}$$

Here $D_\mu=\partial_\mu+A_\mu$, where $A_\mu(t,\mathbf{x})$ is the classical field describing an instanton of the given size. All quantum fluctuation determinants are understood to be normalized by the corresponding vacuum determinants. det′ indicates that zero modes are to be omitted in the determinant; they are removed by the collective coordinate procedure. The instanton group volume ν equals the volume of SU(N) divided by the volume of the little group of the instanton. J is the collective coordinate Jacobian, given by

$$J_{\alpha\beta}=\frac{1}{g^2}\left(\frac{\overline{\mu}^2}{2\pi}\right)\int d^4x\,\hat{\rm tr}(\phi_\mu^{(\alpha)}\phi_\mu^{(\beta)})\,. \tag{6.2}$$

Let $\{z_\alpha\}$ denote the various collective coordinates of the instanton (the position, scale size, and group orientation). $\phi_\mu^{(\alpha)}$ is the deformation of the instanton field, $(\partial/\partial z_\alpha)A_\mu$, placed in background gauge, $D_\mu\phi_\mu^{(\alpha)}=0$. [Hence $\phi_\mu^{(\alpha)}$ is a zero mode of the gauge field determinant.] $\overline{\mu}$ is the Pauli–Villars regulator mass. (The factors of $\overline{\mu}/\sqrt{2\pi}$ come from the omission of the zero modes in the regulator determinant.[20])

We shall restrict our discussion to the case of vanishingly small fermion masses, $m_i\approx 0$. In this limit,

$$\det(\not{D}+m_i)\sim(m_i/\overline{\mu})\det{}'(\not{D})$$

due to the presence of a zero mode of $\not{D}$ (see below).

For any self-dual field,[21]

$$\det{}'(-D^2\delta_{\mu\nu}-2F_{\mu\nu})_{\rm adj}=[\det(-D^2)_{\rm adj}]^4$$

and

$$\det{}'(\not{D})_{\rm fund}=[\det(-D^2)_{\rm fund}]^2\,.$$

Furthermore, since the instanton is contained in an SU(2) subgroup,

$$\det(-D^2)_{\rm adj}=\det(-D^2)_1[\det(-D^2)_{1/2}]^{2(N-2)}$$

and

$$\det(-D^2)_{\rm fund}=\det(-D^2)_{1/2}\,.$$

[Here the subscripts 1/2 and 1 indicate the isospin of the SU(2) subgroup.] Thus

$$n(\rho)=\nu(\det J)^{1/2}[\det(-D^2)_1]^{-1}[\det(-D^2)_{1/2}]^{-2(N-2)}$$

$$\times[\det(-D^2)_{1/2}]^{2N_f}\left(\prod_i(m_i/\overline{\mu})\right)e^{-8\pi^2/g_0^2}\,. \tag{6.3}$$

Evaluating these determinants at zero temperature, one finds the zero-temperature instanton density ('t Hooft, 1976a; Yaffe, 1978; Bernard, 1979),

$$n(\rho,T=0)=\frac{C_N}{\rho^5}(4\pi^2/g^2)^{2N}\left(\prod_i\xi\rho m_i\right)\exp(-8\pi^2/g^2)\,, \tag{6.4}$$

where g^2 is the one-loop renormalized coupling defined at the scale ρ,

$$8\pi^2/g^2=8\pi^2/g_0^2-\tfrac{1}{3}(11N-2N_f)\ln\rho\overline{\mu}\,,$$

$\xi=1.338\,76$, and $C_N=(0.260\,156)\xi^{-(N-2)}/(N-1)!\,(N-2)!\,.$

To find the temperature dependence of the instanton density, we must reevaluate the determinants of Eq. (6.3) for the periodic instanton field (3.9). Due to the basic periodicity conditions (2.4) the gluon and ghost determinants must be evaluated on the space of periodic fluctuations, and the fermion determinant over antiperiodic fluctuations.

A. Zero modes

Gauge field zero modes, $\phi_\mu^{(\alpha)}$, must be periodic; must lead to self-dual perturbations of the field strength, $\overline{\eta}^a_{\mu\nu}D_\mu\phi_\nu^{(\alpha)}=0$; and must be in background gauge, $D_\mu\phi_\mu^{(\alpha)}=0$ (Brown, Carlitz, and Lee, 1977). Explicit expressions for the $4N$ instanton zero modes at any temperature are as follows.

Dilatation zero mode,

$$\phi_\mu^{(0)}=\frac{\partial}{\partial\rho}A_\mu=\frac{2}{\rho}\overline{\eta}^a_{\mu\nu}\partial_\nu\Pi^{-1}(\tau^a/2i)\,.$$

Translation zero modes,

$$\phi_\mu^{(\nu)}=F_{\mu\nu}=-\partial_\nu A_\mu+D_\mu A_\nu$$

$$=\tfrac{1}{2}\Pi\tau\cdot\partial\overline{\eta}^a_{\mu\nu}(\tau^a/2i)\tau^\dagger\cdot\partial\Pi^{-1}\,.$$

Isospin-1 global gauge zero modes,

$$\phi_\mu^{(a)}=(D_\mu)_1\Pi^{-1}(\tau^a/2i)$$

$$=-(\partial_\mu\Pi\delta^{ab}+\varepsilon^{abc}\overline{\eta}^c_{\mu\nu}\partial_\nu\Pi)\Pi^{-2}(\tau^b/2i)\,.$$

Isospin-$\frac{1}{2}$ global gauge modes,

$$\phi_\mu^{(j,q)}=(D_\mu)_{1/2}\Pi^{-1/2}(u_q^j/2i)-\text{h.c.}$$

$$=\tau\cdot\partial\Pi^{-1/2}\tau_\mu^\dagger u_q^j/2i-\text{h.c.}$$

Here $q=3,\ldots,N$; $j=\pm1$, ±2, and the matrix u_q^j is given by $(u_q^j)_{kl}=\delta_{jk}\delta_{ql}$ for $j>0$, and $u_q^{-j}\equiv iu_q^j$.

The fact that $\Pi^{-1}\partial^2\Pi=0$ (for any 't Hooft solution) allows one to reduce the normalization integrals of all zero modes to surface integrals at spatial infinity. These surface integrals may be immediately evaluated using the asymptotic form of Π [Eq. (3.11)] and remarkably one finds that the values are completely temperature independent. For example, the norm of the dilatation mode is

$$\int d^4x\,\hat{\rm tr}\,\phi_\mu^{(0)}\phi_\mu^{(0)}=\frac{2}{\rho^2}\int d^4x\,\partial^2\Pi^{-2}=16\pi^2\,.$$

Similarly, $\int d^4x\,\hat{\rm tr}\,\phi_\mu^{(\alpha)}\phi_\mu^{(\alpha)}$ equals $8\pi^2$ for the translation modes, $4\pi^2\rho^2$ for the isospin-1 modes, and $2\pi^2\rho^2$ for the isospin-$\frac{1}{2}$ modes. All off-diagonal overlaps vanish.

Thus the collective coordinate Jacobian $[\det J]^{1/2}$ is identically equal to its zero-temperature value, $2^7\rho^{-5}(\sqrt{\pi}\rho\overline{\mu}/g)^{4N}$.

The normalized fermion zero mode $\hat{\psi}_0$ is given by (Grossman, 1977)

---

[20] For a unified treatment of gauge fixing and collective coordinates, see Yaffe (1979).

[21] See 't Hooft (1976); Bashilov and Pokrovsky (1978); Brown *et al.* (1978).

$$\hat{\psi}_0(x)\hat{\psi}_0^\dagger(y)=\frac{1}{32\pi^2\rho^2}\Pi^{1/2}(x)[\tau\cdot\partial\phi(x)/\Pi(x)]\not{f}^\dagger\frac{(1+\gamma_5)}{2}\times\not{f}[\tau^\dagger\cdot\partial\phi(y)/\Pi(y)]\Pi^{1/2}(y)\,,$$

where

$$\phi(x)=\sum_{n=-\infty}^{\infty}(-1)^n\rho^2/(x-n\beta\hat{e}_0)^2=(\Pi-1)\frac{\cos\pi t/\beta}{\cosh\pi r/\beta}\,.$$

$\hat{\psi}_0(t,\mathbf{x})$ is antiperiodic in time and decreases as $O(e^{-\pi r/\beta}/r)$ as $r\to\infty$.

B. Determinants

We must now evaluate the temperature dependence of the determinant of $(-D^2/-\partial^2)$ (for both isospin $\frac{1}{2}$ and 1). To do so, we write

$$\ln\det(-D^2/-\partial^2)|_T=\ln\det(-D^2/-\partial^2)|_{T=0}+\delta\,, \qquad (6.5)$$

where

$$\delta=\int_0^T dT'\frac{\partial}{\partial T'}\mathrm{Tr}\ln(-D^2/-\partial^2)|_{T'}\,.$$

The first term is the known zero-temperature determinant. The temperature-dependent correction δ must now be computed.

At this point it is important to recognize that

$$T\frac{\partial}{\partial T}\ln\det(-D^2/-\partial^2)$$

is a dimensionless function of the single variable $\lambda=\pi\rho T$. Thus

$$\delta=\int_0^\lambda d\lambda'\frac{\partial}{\partial\lambda'}\mathrm{Tr}\ln(-D^2/-\partial^2)$$
$$=\int_0^\lambda d\lambda'\mathrm{Tr}\left(\frac{\partial}{\partial\lambda'}-D^2\right)\Delta$$
$$=\int_0^\lambda d\lambda'\mathrm{Tr}\left[\left(\frac{\partial A_\mu}{\partial\lambda'}\right)(-D_\mu\Delta-\Delta D_\mu)\right]. \qquad (6.6)$$

Here Δ is the scalar propagator $(-D^2)^{-1}$.

This form is helpful due to the fact that explicit expressions for the propagators in the field of the general 't Hooft solution are known (Brown *et al.*, 1978). However these propagators are appropriate for multiple instantons in infinite Euclidean space and do not satisfy the required periodicity conditions. Fortunately, the correct propagators may be easily constructed by noting that the solution to $-D^2\Delta(x,y)=\delta^4(x-y)$ with periodic or antiperiodic boundary conditions in time is equivalent to the solution of

$$-D^2\Delta(x,y)=\sum_{n=-\infty}^{\infty}(\pm1)^n\delta^4(x-y+n\beta\hat{t})\,,$$

with boundary conditions of regularity at infinity. Consequently, the correct finite-temperature propagators are given by

$$\Delta^\pm(x,y)=\sum_{n=-\infty}^{\infty}(\pm1)^n\tilde{\Delta}(x,y+n\beta\hat{t})\,, \qquad (6.7)$$

where $\tilde{\Delta}(x,y)$ is the aperiodic scalar propagator of Brown *et al.* (1978). Using this result, we have

$$\delta=-\int_0^\lambda d\lambda'\sum_{n=-\infty}^{\infty}(\pm1)^n\int d^4x\times\mathrm{tr}\{[(\partial_{\lambda'}A_\mu)D_\mu+D_\mu(\partial_{\lambda'}A_\mu)]\tilde{\Delta}(x,y)|_{y=x+n\beta\hat{t}}\}\,. \qquad (6.8)$$

Note that only the $n=0$ term of this sum involves the $(x-y)^{-2}$ singularity of the scalar propagator; all $n\neq0$ terms are manifestly finite.

Let us first apply this for isospin $\frac{1}{2}$. In this case,

$$\tilde{\Delta}(x,y)=F(x,y)/[4\pi^2(x-y)^2\Pi(x)^{1/2}\Pi(y)]^{1/2}\,, \qquad (6.9)$$

where

$$F(x,y)=1+\sum_n\rho_n^2\,\tau\cdot(x-w_n)\tau^\dagger\cdot(y-w_n)/(x-w_n)^2(y-w_n)^2$$

The $n=0$ term of Eq. (6.8) has been computed previously by Brown and Creamer (1978). For the periodic instanton, their result becomes the contribution

$$A(\lambda)=\frac{1}{12}\left[\int_{\mathfrak{M}}\frac{d^4x}{(4\pi)^2}\left(\frac{\partial\Pi}{\Pi}\right)^4-\int_{\mathbb{R}^4}\frac{d^4x}{(4\pi)^2}\left(\frac{\partial\Pi_0}{\Pi_0}\right)^4\right]. \qquad (6.10)$$

Here $\Pi_0(x)=1+\rho^2/x^2$ describes a single zero-temperature instanton. The first integral ranges over a physical strip, $-\beta/2\leq t\leq\beta/2$, while the second integral covers all Euclidean space. (Note that the singularities in the two integrands cancel.) A remarkably simple alternative derivation of this result is sketched in Appendix E.

The remaining $n\neq0$ terms of (6.9) may be directly computed using the explicit expression (6.9). Surprisingly, the only nonzero contribution is the surface term

$$-\int_0^\lambda d\lambda'\sum_n{}'(\pm1)^n\int d^4x\tfrac{1}{2}\mathrm{tr}\tau\cdot\partial\left[\tau_\mu^\dagger\left(\frac{\partial}{\partial\lambda'}\partial_\mu\ln\Pi\right)\tilde{\Delta}(x,x+n\beta\hat{t})\right]=\tfrac{1}{3}\eta\lambda^2 \qquad (6.11)$$

where $\eta=+1$ for periodic, and $-\frac{1}{2}$ for antiperiodic boundary conditions. The derivation of this result is contained in Appendix E.

So, for isospin $\frac{1}{2}$ we find

$$\delta_{1/2}=\tfrac{1}{3}\eta\lambda^2+A(\lambda)\,. \qquad (6.12)$$

The analogous calculation for isospin 1 is considerably more involved. We defer the lengthy details to Appendix E and merely quote the simple result (for periodic boundary conditions),

$$\delta_1=\tfrac{4}{3}\lambda^2+16A(\lambda)\,. \qquad (6.13)$$

Now, $A(\lambda)$ behaves as $-\frac{1}{6}\ln\lambda$ as $\lambda\to\infty$, and as $-\lambda^2/36$ as $\lambda\to0$. Unfortunately, we have been unable to compute the complete integral analytically. It may, however, be reduced to a two-dimensional integral which we have carefully evaluated numerically. We find that $A(\lambda)$ may be fitted extremely well by the expression

$$A(\lambda)\simeq-\tfrac{1}{12}\ln(1+\lambda^2/3)+\alpha(1+\gamma\lambda^{-3/2})^{-8}\,, \qquad (6.14)$$

where $\alpha=0.012\,897\,64$ and $\gamma=0.158\,58$. This expression has a maximum absolute error of 5×10^{-5}. [Note that an absolute error in $\ln\det(-D^2)$ becomes a relative error in the instanton density.]

C. Results

Combining the above, we find the complete instanton density,

$$n(\rho,T)=\frac{C_N}{\rho^5}(4\pi^2/g^2)^{2N}\left(\prod_{i=1}^{N_f}\xi\rho m_i\right)\exp(-\{8\pi^2/g^2+\tfrac{1}{3}\lambda^2(2N+N_f)+12A(\lambda)[1+\tfrac{1}{6}(N-N_f)]\})$$
$$=n(\rho,0)\exp(-\{\tfrac{1}{3}\lambda^2(2N+N_f)+12A(\lambda)[1+\tfrac{1}{6}(N-N_f)]\}). \tag{6.15}$$

This result may be independently checked in both asymptotic limits, $\lambda\to 0$ and $\lambda\to\infty$. If $\lambda\to 0$, then the separation between the individual "bare" instantons which form the periodic instanton becomes arbitrarily large. Consequently, in this limit, we need not bother with the details of the exact 't Hooft solution; we could instead use a pure superposition of zero-temperature instantons (in singular gauge) to form an approximate periodic solution. Properties of such an approximate solution may be computed in a power series in the inverse separation between instantons. In particular, it is known that if $N_f=0$ then all corrections to the instanton density are of order (inverse separation)^4, or in our case $O(\lambda^4)$ (Levine and Yaffe, 1979). This is easily seen to agree with our result (6.15) by noting that

$$d\rho\, n(\rho,T)=d\rho' n(\rho',0)[1+O(\lambda^4)], \tag{6.16}$$

and recalling that $\rho'=\rho/(1+\frac{1}{3}\lambda^2)^{1/2}$ is the size of the zero-temperature instanton whose superposition most nearly agrees with the exact periodic instanton of scale ρ. [This result (6.16) is also valid for the instanton–anti-instanton density in the presence of massless quarks.]

If $T\to\infty$ then, as explained in Sec. V, perturbation theory may be used to calculate the leading high-temperature behavior. The high-temperature limit of (6.15) agrees with the expected $\exp(-\rho^2T^2)$ cutoff in Eq. (5.5).

Finally, we may use the renormalization group to justify replacing the Pauli-Villars coupling g^2 with the renormalization group improved running coupling (Gross and Wilczek, 1973b; Politzer, 1973; Caswell, 1974; Jones, 1974).

$$\frac{4\pi^2}{g^2(\rho)}=\frac{1}{6}(11N-2N_f)\ln 1/\rho\Lambda+\frac{1}{2}\frac{[17N^2-N_f(13N^2-3)/2N]}{(11N-2N_f)}\ln\ln 1/\rho\Lambda+O(1/\ln\rho\Lambda). \tag{6.17}$$

Here, the Pauli-Villars renormalization scale Λ is defined precisely so as to absorb all $O(1)$ terms in (6.17). If a different renormalization scale, Λ', is used, then the coefficient of the instanton density (6.15) must be redefined as

$$C_N(\Lambda')=C_N\cdot(\Lambda/\Lambda')^{(11N-2N_f)/3}. \tag{6.18}$$

The resulting instanton density is plotted in Fig. 4 for several temperatures. Shown is the dimensionless density, $d(\rho)\equiv\rho^5 n(\rho)$, for $N=2$ and $N_f=0$. Note how the density as a function of ρ decreases and broadens as the temperature rises. The instanton contribution to the free energy density may be immediately computed. It will be discussed further in the next section.

VII. THE PHASE TRANSITION

In this section we shall explore the nature of the phase transition from a confined to an unconfined phase, using all the tools available to us—perturbation theory, semiclassical methods, and strong coupling lattice gauge theory expansions. For the most part we shall discuss quarkless QCD. We will return to a discussion of the real world at the end of the section.

A. Perturbation theory

In the previous sections we have argued that, at high temperature, QCD can be treated using ordinary perturbation theory. The high-temperature phase is that of a nonconfining plasma in which heavy quarks experience a short-range, screened Coulomb interaction, timelike gluons acquire an electric mass of order gT, and spacelike gluons a magnetic mass of order g^2T. All observables can be expanded in an asymptotic expansion in powers of the effective coupling $g^2(T)$, whose coefficients are calculable up to some finite order. Thus the free energy per unit volume $\mathcal{F}$, which is equal to minus the pressure P, is calculable up to order g^6 and given by (Kapusta, 1979a)

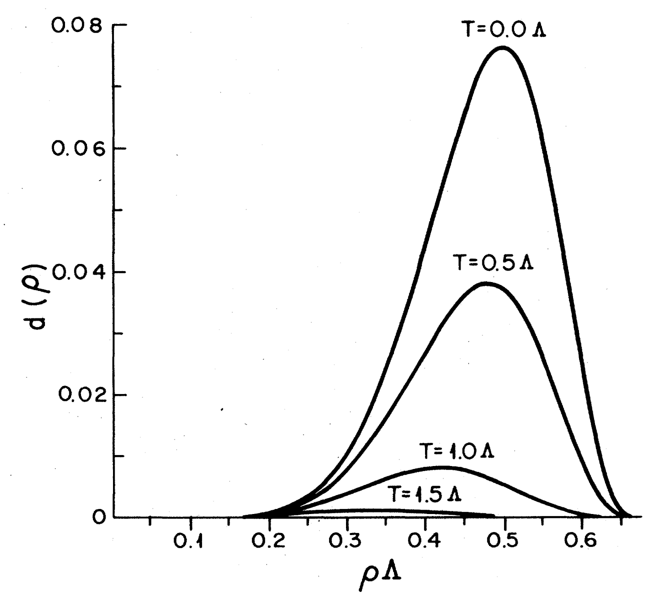

FIG. 4. Instanton density $d(\rho)\equiv\rho^5 n(\rho)$ for $N=2$ and $N_f=0$.

$$P(T)=-\mathfrak{F}(T)$$
$$=\frac{\pi^2(N^2-1)}{45}T^4$$
$$\times\left[1-\frac{5}{4}\left(\frac{g^2(T)N}{4\pi^2}\right)+\frac{10}{\sqrt{3}}\left(\frac{g^2(T)N}{4\pi^2}\right)^{3/2}+O(g^4)\right], \quad (7.1)$$

where

$$\frac{g^2(T)N}{4\pi^2}=\frac{1}{(11/6)\ln T/\Lambda}-\frac{(17/22)\ln\ln T/\Lambda}{[(11/6)\ln T/\Lambda]^2}+O\left(\frac{1}{\ln T/\Lambda}\right)^2. \quad (7.2)$$

To the calculated order in g^2 the above result is independent of our renormalization procedure. In other words, a redefinition of g^2 according to $g'^2=g^2+cg^4$ is equivalent, for small $g^2(T/\Lambda)$, to a rescaling of Λ, i.e.,

$$\frac{1}{g'^2(T/\Lambda)}=\frac{1}{g^2(T/\Lambda)}-c\equiv\frac{1}{g^2(T/\Lambda')},$$

where

$$\ln\frac{\Lambda'}{\Lambda}=\frac{24\pi^2}{11N}c.$$

But this only affects the yet uncalculated terms of order g^4 in $\mathfrak{F}(T)$. These do indeed depend on Λ or, equivalently, on the way in which the coupling constant is defined.

The resulting value of the pressure is plotted in Fig. 5. What do we learn from this perturbative calculation? First, note the large contribution of the g^3 term, which overwhelms the g^2 term for $g^2(T)N/4\pi^2 \gtrsim 1/20$ or $T/\Lambda \leqslant 4\times10^4$. This, however, is probably not an indication of the precocious breakdown of perturbation theory, since the g^3 term should be regarded as a correction to the ideal gas pressure due to the nonvanishing electric mass, and need not be indicative of the magnitude of higher-order terms. A more reasonable estimate of the point at which perturbation theory breaks down is when the corrections to the ideal gas behavior are of order 1, namely when $g^2N/4\pi^2\approx\frac{1}{5}-\frac{1}{3}$, or $T/\Lambda=10-4$ and the corrections are 25–70%, respectively. Thus while it is of interest that the perturbative corrections, once $g^2N/4\pi^2 \geqslant \frac{1}{20}$, tend to increase the pressure or decrease the free energy, one certainly cannot trust these results for $T/\Lambda\approx1.8$ where the entropy density $S=\partial P/\partial T$ vanishes. Therefore the fact that the positivity of the entropy is apparently violated below this point can hardly be used to estimate a transition temperature. Similarly, if one neglects the $O(g^3)$ term in the pressure, then the fact that the resulting curve crosses zero has no significance. Not only is the perturbative calculation completely unreliable, the zero of the pressure is also irrelevant. We have arbitrarily defined the free energy so that all perturbative contributions vanish at zero temperature; however, nonperturbative contributions will cause the zero-temperature pressure to be nonvanishing.

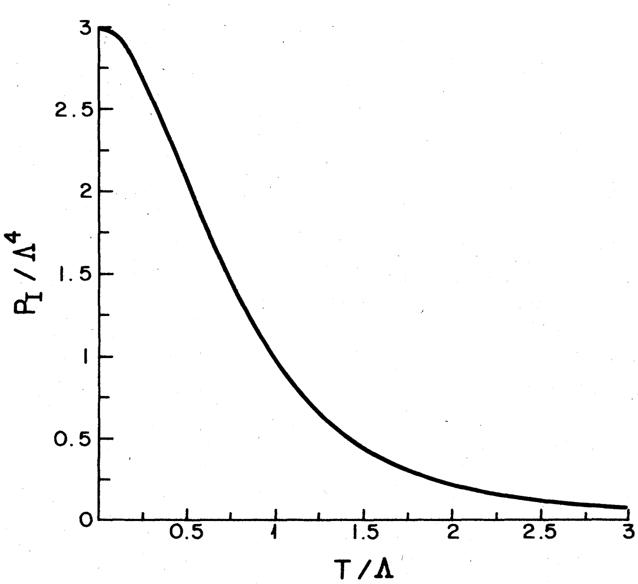

FIG. 5. Pressure versus temperature for SU(2). P_0= ideal gas contribution, $P_2=O(g^2)$ contribution, $P_3=O(g^3)$ contribution.

The contribution of instantons to the pressure is easily calculable in the dilute gas approximation, using the density of instantons derived above. For $\theta=0$, the instanton plus anti-instanton contribution is

$$P_{\rm inst}(T)=2\int d\rho\, n(\rho,T). \quad (7.3)$$

For $T=0$ such a calculation requires a knowledge of the value of the cutoff on the integration over the instanton scale size ρ. However, for large T the integration is exponentially cut off for $\rho\approx1/\pi T$. Thus as long as T is sufficiently large the calculation is reliable, and yields for large T

$$P_{\rm inst}(T)\approx C_N(\Lambda)T^4\left(\frac{4\pi^2}{g^2(T)}\right)^{2N}e^{-8\pi^2/g^2(T)}\left[1+O\left(\frac{1}{\ln T/\Lambda}\right)\right]\cdots. \quad (7.4)$$

This result is again independent of our definition of g^2 or the value of Λ, as long as T/Λ is sufficiently large, since

$$C_N(\Lambda')=\left(\frac{\Lambda}{\Lambda'}\right)^{11N/3}C_N(\Lambda).$$

The actual numerical value of $P_{\rm inst}(T)$ is plotted [for SU(2)] in Fig. 6, from which we see that it yields an exceedingly small correction to $P_{\rm pert}(T)$.

A more important correction due to instantons is the coupling constant renormalization that they induce. This effect will be discussed below. For the present discussion this simply means that the perturbative analysis breaks down slightly sooner than we would otherwise expect.

It is hardly surprising that we cannot explore the transition, as the temperature is lowered, from the unconfined to the confined phase using solely weak coupling techniques. After all, once the thermal cutoff $1/T$ is larger than the characteristic confinement length scale, these techniques are unable to describe the strong coupling infrared behavior of the theory. The above calculations simply support the claim that the high-temperature phase is nonconfining, and yield a crude estimate of the region in which perturbation theory is reliable.

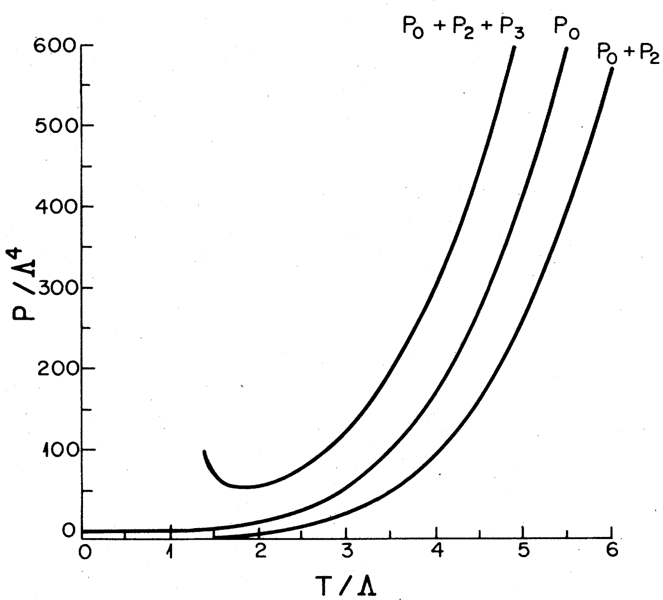

FIG. 6. Instanton pressure $P_I = 2\int d\rho\ n(\rho)$ for SU(2).

It is much more informative to approach the phase transition using an approximate description of the theory which should be valid for large-distance phenomena and which can describe color confinement. Such a description is provided by the strong coupling lattice gauge theory.

B. The strong coupling lattice model

One of the most useful approaches to the study of the large-scale structure of non-Abelian gauge theories is via the lattice formulation introduced by Wilson (1974), or its Hamiltonian form as introduced by Kogut and Susskind (1975). In the former approach the dynamical variables are special unitary matrices, U_l, associated with links on a four-dimensional lattice. The action is defined to be

$$S_E(U) = -\frac{1}{g_E^2}\sum_p \mathrm{tr}\left(\prod_{\partial p} U_l + \mathrm{h.c.}\right), \tag{7.5}$$

where the sum runs over all plaquettes on the lattice. In the latter approach the theory is described by a Hamiltonian (in $A_0 = 0$ gauge),

$$H = \frac{g_H^2}{2a}\sum_l E_l^2 - \frac{2}{g_H^2 a}\sum_p \mathrm{tr}\left(\prod_{\partial p} U_l + \mathrm{h.c.}\right), \tag{7.6}$$

where the dynamical variables are special unitary matrices U_l on the links of a spatial lattice (of size a), and E_l (the electric flux operators) are the conjugate momenta to U_l. These two versions of lattice gauge theory are closely related, and for small g^2 (i.e., in the continuum limit) one can derive Eq. (7.6) from Eq. (7.5) (Creutz, 1977).

The latticized gauge theory is simplest in the limit of strong coupling, where, for $T=0$, it exhibits linear confinement. This can be seen in the Lagrangian form by expanding the expectation value of a Wilson loop (with rectangular dimensions $R\times t$) in powers of $1/g_E^2$:

$$\begin{aligned}\langle W_L\rangle &= \int dU_l\, e^{-S(U_l)}\prod_{\mathrm{Loop}} U_l \Big/ \int dU_l\, e^{-S(U_l)} \\ &= \left(\frac{1}{Ng_E^2}\right)^{Rt/a^2}\left[1 + O\left(\frac{1}{g_E^2}\right)\right].\end{aligned} \tag{7.7}$$

[For SU(2) $1/Ng_E^2$ should be replaced with $1/g_E^2$.] This yields a linear heavy quark-antiquark potential, with a slope

$$\sigma \equiv \frac{1}{a^2}\ln Ng_E^2 .$$

In the Hamiltonian version one neglects the "magnetic" term in H, for large g_H^2, and then finds that the lowest state with a quark and antiquark source separated by a distant R has energy

$$E = \frac{g_H^2}{2a}\left(\frac{N^2-1}{2N}\right)\frac{R}{a} \equiv \sigma R .$$

In both cases one sees that in the strong coupling limit external colored sources are connected by strings (flux tubes), whose energy per unit length is nonzero.

Polyakov (1978) and Susskind (1979) have given convincing arguments that, as the temperature is increased, these theories undergo a phase transition to an unconfined phase. A simple, intuitive, and crude version of their argument can be given (Banks and Rabinovici, 1979), if we ignore the fact that in a non-Abelian theory strings can split. In that case the spectrum of H, for large coupling, is that of closed non-backtracking strings of length L and energy σL. The number of such strings grows, for large L, as $5^{L/a}$ (5 is simply the number of directions in which a nonbacktracking string can grow on a three-dimensional spatial lattice). Thus the partition function, $Z \sim \sum_L 5^{L/a} e^{-\beta\sigma L}$, has a singularity at $\beta = \ln 5/\sigma a$ indicating that the critical temperature T_c is less than

$$T_c \lesssim \frac{\sigma a}{\ln 5} = \frac{N^2-1}{2N}\frac{g_H^2}{(\ln 5)\, a}. \tag{7.8}$$

(Above T_c neglected interaction effects stabilize the free energy.) One can argue that the inclusion of the magnetic terms in H should lower this transition temperature, since they tend to deconfine color. For Abelian theories these arguments can be made precise; for non-Abelian theories they are reasonable, although the estimate given by Eq. (7.8) should not be taken too seriously.

This is an important result, since it shows that a confining theory can lose confinement at high temperature. Furthermore, it yields a pretty picture of the nature of the phase transition as a condensation of strings. Thus at low temperature free quarks cannot exist, since the infinitely long strings attracted to them would cost infinite energy, but at high temperature the addition of one more flux tube does not substantially change the free energy of the condensate of fluctuating flux tubes.

The above picture is totally consistent with the phase transition one expects from perturbative arguments at large temperature; however, it too cannot really be used to estimate T_c or to study the nature of the phase transition. In this approach it is even difficult to see electric screening in the high-temperature phase. In fact, in order to recover continuum QCD from the lattice theory, one must let g^2 approach zero [as $1/\ln(1/a\Lambda)$] as a vanishes. One would then have to include corrections (to all orders) in the strong coupling expansion, and argue that in this limit [where $\sigma(T=0)$ is kept fixed as g and a approach zero] T_c has a finite value. Such a calculation would be even more difficult than establishing confinement at $T=0$, i.e., showing that confinement persists in the continuum limit.

Lattice gauge theories can also be used in a different manner as a way to construct an approximate description of the dynamics of QCD valid for large-distance phenomenon. Here one imagines constructing a lattice action which will correctly describe the physics of QCD for distances greater than a chosen lattice spacing a, by integrating out all degrees of freedom involving distances less than a. Recently Callan, Dashen, and Gross (1979b, 1980) have argued that a crude effective lattice action of this sort can be constructed using semiclassical techniques, and used to calculate the string tension σ in terms of the continuum renormalization scale parameter Λ. We shall extend their methods to finite temperature, construct for $T\neq 0$ such an effective action, and argue that for $T\geq T_c$ it undergoes a Polyakov–Susskind phase transition. This will allow us to estimate T_c in terms of Λ.

C. The semiclassical effective Lagrangian

If one were to derive an exact effective lattice action for QCD, it would necessarily be much more complicated than the simple Wilson action [Eq. (7.5)]. However, it is not unreasonable that such a term alone could provide an adequate description of the behavior of certain observables, in particular planar Wilson loops. Making this assumption, one may attempt to calculate g_E^2 as a function of the lattice spacing a and the renormalization scale parameter Λ using semiclassical techniques. One may then evaluate the string tension by applying strong coupling techniques to the resulting Wilson action, *if* the two methods have overlapping domains of validity. This is the approach recently explored by Callan, Dashen, and Gross (1979b, 1980).

To evaluate $g_E^2(a\Lambda)$ they note that the Wilson term in the effective action dominates for weak, slowly varying fields ($U_l\approx 1$), and that the coupling renormalization due to fluctuations on scales less than a can be determined by semiclassical methods as long as $g_E^2(a\Lambda)/8\pi^2$ is sufficiently small. This coupling renormalization receives contributions from ordinary perturbative (Gaussian) fluctuations and from instantons of size $\rho\lesssim a$. The contribution of instantons can be thought of as producing a vacuum permeability μ, as if the instantons were a gas of paramagnetic four-dimensional magnetic dipoles with density $n(\rho)$ and dipole moment proportional to ρ^2 (Callan, Dashen, and Gross, 1978, 1979a). The resulting lattice coupling is given by

$$g_E^2(a\Lambda_L)=g_{AF}^2(a\Lambda_L)\mu(a)\,, \tag{7.9}$$

where

$$\mu(a)=\eta(a)+[1+\eta(a)^2]^{1/2}$$

and

$$\eta(a)=\left(\frac{8\pi^2}{N^2-1}\right)\int_0^{a_c(a)} d\rho\, n(\rho)\, \rho^4\left(\frac{4\pi^2}{g^2(\rho)}\right)$$
$$=\left(\frac{8\pi^2}{N^2-1}\right)C_N(\Lambda_L)\int_0^{a_c(a)}\frac{d\rho}{\rho}\left(\frac{4\pi^2}{g_{AF}^2(\rho)}\right)^{2N+1} e^{-8\pi^2/g_{AF}^2(\rho)}\,.$$

Here $g_{AF}^2(\rho\Lambda_L)$ is the effective coupling as determined by ordinary perturbation theory, i.e., Eq. (7.2), and $a_c(a)$ is a cutoff on the instanton scale size representing the fact that only fluctuations on scales less than the lattice spacing have been integrated out. The constant $C_N(\Lambda_L)$ depends on the precise value of the lattice renormalization point as $(\Lambda_L)^{-11N/3}$ [see Eq. (6.18)]. For SU(2) $C_2(\Lambda_L)=1.58\times 10^9$, since,

$$\frac{\Lambda_{PV}}{\Lambda_L}=(2.86)\exp\left(\frac{3\pi^2}{11}\,\frac{(N^2-1)}{N^2}\right),$$

(Hasenfratz and Hasenfratz, 1980) and the best estimate for a_c is $2a/3$.

Given the above one may evaluate the quantity $\bar{\sigma}=\ln g_E^2(a\Lambda_L)/a^2$ for a range of lattice spacing a. *If* the effective lattice theory is in the strong coupling domain then this quantity will be independent of a and in fact will equal the physical string tension σ. Thus the behavior of $\bar{\sigma}(a)$ may be used (1) to check the consistency of the basic assumption that semiclassical fluctuations alone are sufficient to drive one to the region where strong coupling expansions may be applied; (2) to calculate the string tension σ in terms of Λ_L; and (3) to compare with numerical lattice gauge theory results.

The comparison with Creutz's recent Monte Carlo evaluation of $g_E^2(a)$ is quite impressive (Creutz, 1980; Kogut, Pearson, and Shigemitsu, 1979). Both treatments yield a sharp transition from weak to strong coupling behavior at $g^2\approx 2$ and predict [for SU(2)] $\sqrt{\sigma}\approx(70-100)\Lambda_L$.[22] Note that Λ_L is much smaller than the Λ's commonly employed in conventional renormalization schemes such as the Pauli–Villars scheme used above, or the dimensional regularization ($\overline{MS}$) scheme; for SU(2) $\Lambda_L=\Lambda_{PV}/21.55=\Lambda_{\overline{MS}}/20.74$. Hence, in terms of, for example, the previous Pauli–Villars scheme, $\sqrt{\sigma}\sim(3.5-5)\Lambda_{PV}$.

We shall now extend these methods to construct an effective lattice gauge theory for finite temperature. It is quite straightforward to extend the calculation of the coupling renormalization to finite T. First, one must use the density of instantons as calculated for finite temperature; i.e., $n(\rho,T)$ as given by Eq. (6.15). Second, one must ask whether periodic instantons continue to renormalize the coupling as if they were four-dimensional magnetic dipoles.

To answer this question we must consider the response of a dilute gas of instantons, for finite T, to a slowly varying background field. This is most simply done by evaluating the gauge field propagator in the dilute gas approximation. Following the discussion of Callan, Dashen, and Gross (1978, 1979a), one finds the contribution to the propagator from instantons of size ρ,

---

[22] Callan, Dashen, and Gross (1979b, 1980) and Creutz (1980) both neglected to include the effects of the second-order term in the β function. This produces roughly a factor of two change in $\sqrt{\sigma}/\Lambda$.

$$\int d^4x\, e^{-ip\cdot x}\langle A_\mu(x)A_\nu(0)\rangle \sim \frac{(\delta_{\mu\nu}-p_\mu p_\nu/p^2)}{p^2}|F(p,T)|^2\,,$$

where

$$F(p,T)=\int d^4x\, e^{ip\cdot x}\,\partial^2 \ln \Pi(x)\,,$$

and $\Pi(x)$ is the 't Hooft potential for the instanton field [Eq. (3.9)]. However, for small momenta $F(p,T)$ reduces to a surface integral independent of T,

$$F(0,T)=\beta\int d^3\Sigma\cdot(\partial\Pi/\Pi)=\int d^4x\,\partial^2\Pi=-4\pi^2\rho^2\,.$$

Therefore, the coupling constant renormalization, for the small momenta which are relevant for slowly varying background fields, is given by the same formulas as for $T=0$, except that $n(\rho)$ is replaced by $n(\rho,T)$. (Naturally, one may reach this same conclusion by a direct calculation of the interaction of an instanton with a slowly varying background field.)

We now proceed to construct an effective lattice action for a lattice of size a at temperature T. As long as $aT \ll 1$ one can ignore the fact that the lattice is periodic in the time direction. Once again we calculate $g_E^2(a\Lambda_L)$ as a function of the lattice spacing a, and examine the behavior of the would-be string tension, $\tilde\sigma(a)$. The temperature dependence is illustrated in Fig. 7, from which we learn the following [for SU(2)]:

(1) At zero temperature there is an abrupt transition from weak to strong coupling behavior. Beyond $g^2\sim 2$, $\tilde\sigma(a)$ remains constant to $\sim\pm 5\%$. The region $2<g^2<25$ corresponds roughly to one doubling in the lattice spacing. This transition from weak to strong coupling behavior occurs at a very weak coupling, $g^2/8\pi^2\sim 1/40$, well within the region where semiclassical methods should be valid.

(2) As T increases, the value of the string tension decreases.

$$\sqrt{\sigma(0)}\approx 70\Lambda_L\,,\quad \sqrt{\sigma(10\Lambda_L)}\approx 67\Lambda_L\,,\quad \sqrt{\sigma(20\Lambda_L)}\approx 62\Lambda_L\,,\quad \text{etc.}$$

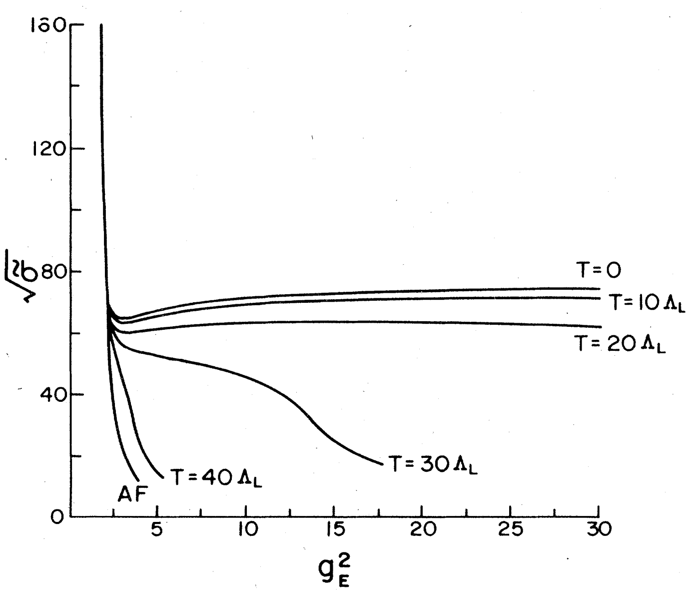

FIG. 7. $\tilde\sigma=\ln[g_E(a\Lambda_L)/a^2]$ vs $g_E^2(a\Lambda_L)$ for SU(2).

TABLE I. Critical temperature estimated from effective lattice theory at temperature T. $\bar a(T)$ equals the distance at which the transition from weak to strong coupling behavior occurs.

T/Λ_L	$[\tilde\sigma(T)]^{1/2}/\Lambda_L$	$\bar a(T)\Lambda_L$	$T_c^{\min}/\Lambda_L$
0	70	0.0116	39
10	67	0.0118	38
20	62	0.0123	37
30	~50	~0.013	36

(3) When $T\approx(30-40)\Lambda_L$ one no longer finds an abrupt transition from weak to strong coupling behavior, and $\ln g^2(a)/a^2$ does not approach a constant value for large a. The instantons, which gave rise to the sharp transition for $T=0$, are almost totally suppressed for $T \geq 40\Lambda_L$.

This suggests perhaps that a phase transition is taking place at about $T_c\approx(30-40)\,\Lambda_L\approx\frac{1}{2}\sqrt{\sigma}$. It is not surprising that for $T\geq 30\Lambda_L$ the calculated $\tilde\sigma$ tends to fall with increasing lattice spacing, since one would expect for this temperature that the corrections to the strong coupling relation between σ and g^2 would be important. After all, it is these temperature-dependent corrections which lead to the phase transition of the strong coupling lattice theory.

Another estimate of T_c can be obtained directly from the strong coupling lattice theory. As explained above, one finds a simple estimate, or upper bound on the transition temperature, $T_c\lesssim\sigma a/\ln 5$. Although this estimate was based on a Hamiltonian calculation, it is equally valid for the Euclidean theory since (by definition) both methods yield the same value of σ. We may therefore calculate T_c as a function of the lattice spacing for a given temperature, and take as our estimate the minimum value of $\tilde\sigma(a)a/\ln 5$. These results are exhibited in Table I. The best estimate of T_c occurs for a lattice spacing about equal to the value $\bar a$ where the transition from weak to strong coupling occurs. We find $T_c^{\min}\approx(40-35)\Lambda_L$. This is consistent with our previous estimates. If we translate this estimate of T_c into units more suitable for discussing ordinary continuum perturbation theory (say, minimal subtraction), it corresponds to $T_c\sim(1-1.5)\Lambda_{\overline{MS}}$, which indeed is in the region where, in the calculation of $P(T)$, we found large deviations from ideal gas behavior.

In summary we have constructed a very crude effective lattice action for quarkless QCD whose coupling $g_E^2(a,T)$ is a function of a and T. For small temperature, $T\lesssim 30\Lambda_L$, we see an abrupt transition from weak to strong coupling behavior at a distance $\bar a(T)$. This action describes a confining phase with flux tubes of radius $\approx\bar a(T)$ connecting external colored sources. As T increases, $g^2(a,T)$ decreases, leading to a smaller value of the string tension $\sigma(T)$ and a larger flux tube radius. For $T\gtrsim\Lambda_L$ the instantons responsible for the abrupt phase transition are absent, and the action describes an unconfined phase. The value of the critical temperature $T_c\sim(30-40)\Lambda_L$ agrees with the estimate of the critical temperature for flux tube condensation.[23]

[23]The quoted numbers are all for the SU(2) theory. For SU(3) $\sqrt{\sigma}\sim 110\Lambda_L$ and $T_c\sim 50\Lambda_L$.

What is unclear from the above simple treatment is the exact nature of the phase transition. We are unable to establish whether it is first or second order, or equivalently whether the string tension $\sigma(T)$ remains finite or vanishes at T_c.

D. The real world

We should now like to examine how the above scenario is modified by the presence of dynamical quarks. Clearly, one can still imagine the possibility of two different phases—a confined phase where physical states are colorless bound states of quarks, and an unconfined phase with free quarks and color screening. However, one no longer has a simple order parameter which distinguishes these phases. For example, a Wilson loop, or string [i.e., $\langle\Omega(0)\Omega^\dagger(R)\rangle$] is always screened, by a single quark–antiquark pair in the confined phase, or by a polarization cloud of gluons and quarks in the unconfined phase. Similarly, both phases have a mass gap (provided the quarks are not massless). Of course, the two phases are distinguished by completely different structures of the physical Hilbert space. This is reflected in, for example, whether or not multiparticle cuts exist in correlation functions of local gauge-invariant operators, which cannot be accounted for by the single particles produced by any local, gauge-invariant operator.

If the quarks are massless then one must also consider chiral properties. The low-temperature phase is believed to be chirally asymmetric with $\bar{\psi}\psi$ acquiring a nonzero expectation value. However, the high-temperature phase is surely chirally symmetric. Dynamical quarks do not affect the reliability of high-temperature perturbation theory (provided there are not so many that they destroy asymptotic freedom), and such a perturbative phase is manifestly chirally symmetric. In fact, all correlations between quarks decay as $\exp(-\pi T|\mathbf{x}|)$ at large distances, clearly preventing any dynamical symmetry breaking.

Thus there must be a chiral phase transition at some temperature. It may either be separate from the confining phase transition, or there may be a single combined transition. Unfortunately, we have even less information about the transition region than previously. As before, perturbation theory is unable to describe both phases, and certainly cannot be used to find a phase transition. Furthermore, in the presence of dynamical quarks, one is unable to construct even a simple lattice model of the relevant physics. Merely inserting quarks in a lattice theory without destroying the chiral symmetry is a major problem. One can still study the effects of instantons in the presence of dynamical quarks; however, it is considerably more involved. Massless quarks cause instantons and anti-instantons to become bound into pairs. Each pair now acts as a four-dimensional dipole, just like a single instanton, and additionally different pairs can exchange massless quarks. One can construct a model of chiral symmetry breaking at zero temperature which yields dynamical quark mass generation due to instanton mediated interactions (Callan, Dashen, and Gross, 1978, 1979a; Caldi, 1977). As the temperature increases, the instanton density decreases, which weakens the effective interactions between quarks. At some point the pion will become unbound and one will lose the chiral symmetry breaking. However, using instantons alone, one cannot adequately treat the long-distance color dynamics needed to understand the interplay between the confinement and chiral aspects of the phase transition.

Giving up on an honest treatment of the phase transition, we would like to mention how the overall picture of a transition to an unconfined phase may be seen in a simple phenomenological model, the bag model (Chodos *et al.*, 1974). This provides a nice description of low-lying hadrons as little pockets of "perturbative" vacuum immersed in the "true" vacuum. The true vacuum has an energy density which is less than the perturbative vacuum by an amount $\mathcal{B}$, the bag constant, and it expels all color electric flux. Consequently the flux generated by the quarks is confined within the bag. The inward pressure due to the bag constant must be balanced by the outward pressure due to the electric flux, as well as the quark kinetic pressure; this determines the size of the bag. An important ingredient of the bag model for light quarks is the chiral properties of the "true" vacuum. Outside the bag quarks behave as if they have a large, dynamically generated mass (which is actually infinite in the simple MIT bag model). This also serves to confine light quarks inside the bag.

If we now examine this model at finite temperature, three different effects can appear. First, the bag constant $\mathcal{B}$ may depend on temperature. After all, in QCD some of the nonperturbative fluctuations which lower the energy of the vacuum (such as instantons) will be suppressed at high temperature. Hence, if the bag model is intended to mimic QCD, then $\mathcal{B}$ should decrease with increasing temperature. Second, the chiral properties of the "true" vacuum might be temperature dependent. Third, thermal fluctuations will create a gas of bags whose density increases with increasing temperature. All of these effects lead to nonconfinement at high temperature. Since the expected radius of a bag behaves as $\mathcal{B}^{-1/4}$, as $\mathcal{B}(T)$ decreases the bags will grow. Clearly, if $\mathcal{B}\to 0$ then we shall have lost confinement. However, even before that point dramatic effects will occur. Specifically, as the density of thermally excited bags grows, the bags will begin to overlap and join. At some point, quarks will be able to percolate through a network of bags extending throughout space. In other words, the bags will have condensed, producing nonconfinement (Cabbibo and Parisi, 1975). Also, if the chiral phase transition occurs at low temperature, then the nature of the hadronic bags might change radically even before confinement breaks down.

VIII. CONCLUSIONS

Since the introduction contained a summary of the overall picture of high-temperature QCD, we shall confine our concluding discussion to a few remarks about the many outstanding problems. Clearly much work remains to be done. Perhaps the most straightforward problem is that of extending the perturbative analysis of the free energy to $O(g^4)$ and $O(g^5)$. This requires calculating all three-loop vacuum graphs. At this order the results depend on the specific scheme used to define the coupling. To go beyond this order, as discussed in Sec. IV, one must be able to completely solve the three-dimensional pure gauge theory. This is likely to be an open problem for quite some time; as always, proving the existence of a mass gap in gauge theories appears to be the most fundamental problem.

In the lattice approach, one would like to be able to calculate the transition temperature of the pure gauge theory, T_c, as a function of the coupling g^2. Series extrapolation techniques do not seem very promising here since, even for arbitrarily strong coupling, the transition is associated with a breakdown of the strong coupling expansion. However, the Monte Carlo methods which have recently been applied to gauge theories appear to be well suited for this application. It should be possible to numerically compute expectations for a sufficiently large lattice so that clear indications of a phase transition could be seen. The heavy quark potential, or $\langle \Omega(R)\Omega^{\dagger}(0)\rangle$, is a natural order parameter for the transition; one must be able to distinguish between the exponential falloff $\exp -\sigma R/T$ of the confining phase and the screened form $1+O(\exp(-m_{el}R))$ of the unconfined phase. If possible, one would like to be able to extract the electric mass m_{el} and compare it with the perturbative prediction. No doubt results of this nature will be available soon.[24]

Clearly, improvements can be made in the semiclassical derivation of effective lattice theories. A more careful treatment of how the constraints which are inserted cut off instantons would be particularly welcome. However, it must be remembered that all of the previous methods were developed for a fake theory—quarkless QCD. Therefore, the most pressing problem must surely be the development of nonperturbative methods which can adequately treat dynamical quarks. In particular, genuine calculational methods for quarks are totally lacking.

Finally, after spending all this time on the theory itself, we should like to add a few words concerning the possible applications of high-temperature QCD. Since interesting effects require a temperature of order some typical strong interaction scale, say 100 MeV or 10^{12} K, the most immediate application of high-temperature QCD is to cosmology. Within the standard big bang model, the universe expands and the temperature steadily drops and, at a time around 10^{-6} sec, hadronic temperatures of order 1 GeV are reached. In particular, we have seen that QCD clearly predicts a phase transition from an unconfined quark phase to a confined hadronic phase. This is, in fact, the last phase transition to occur, and therefore presumably the most accessible experimentally. (Earlier phase transitions may be associated with weak interactions at $\sim 10^5$ GeV, technicolor interactions at 10–100 TeV, or grand unified interactions around 10^{15} GeV.) Possible effects associated with this transition which one might contemplate include (1) creation of the fluctuations in the baryon density which are needed to form galaxies and clusters of galaxies, (2) effects of a strongly first-order transition, such as the generation of additional entropy and increased expansion in the supercooled phase due to the latent heat, and (3) perturbations in the photon spectrum due to recombination radiation emitted when hadrons are first formed. Unfortunately, observing any such effect appears to be hopeless. Although we are unable to compute realiably the transition temperature, latent heat, etc., it seems inevitable that in QCD (a theory with no small parameter) any such dimensional quantity will be of typical hadronic size. Thus, for example, if the transition is first order, then surely the nucleation rate is closer to hadronic rates of 10^{22} sec^{-1} than to the global expansion rate at that time of $\sim 10^5$ sec^{-1}.[25] But this makes it impossible for any dramatic supercooling effects to occur. Similarly, recombination energies cannot be many orders greater than the prevailing thermal energies, and any increase in entropy cannot be by a factor much greater than order one. Finally, any discussion of fluctuations immediately confronts the fact that at $T \sim 10^{12}$ K the baryon number inside the particle horizon (i.e., within the causally connected portion of the universe) is much less than a solar mass, far too small to nucleate galaxies.

Other possible applications of high-temperature QCD typically also involve large baryon density.[26] These include topics such as quark stars (Collins and Perry, 1975; Freedman and McLerran, 1978; Baym and Chin, 1976; Keister and Kisslinger, 1976) and statistical treatments of heavy-ion collisions (Schroeder, 1980). Consequently, it is of interest to include a chemical potential for quarks in QCD and to study the combined (T, μ)-phase diagram. Perturbative calculations may be easily extended to include a chemical potential and have been performed through $O(g^3)$ (Kapusta, 1979a) for $T \neq 0$ and $O(g^4)$ at $T=0$ (Freedman and McLerran, 1979b; Baluni, 1978). However, at higher orders one should encounter exactly the same breakdown of perturbation theory due to gluonic infrared divergences discussed previously (Sec. IV). Even the $(\mu \neq 0, T=0)$ calculations appear suspect (Linde, 1979), since a careful treatment inevitably seems to require a limit from nonzero temperature (Baluni, 1978). Therefore much further work will surely be required before this area of QCD is fully understood.

ACKNOWLEDGMENTS

We wish to thank John Collins for helpful discussions during part of this work, Ian Affleck for interesting conversations about the relation to his work, and Jim Peebles for stimulating discussions about possible cosmological applications. The research of D. Gross and L. Yaffe was partially supported by NSF grant No. PHY78-01221. The research of R. Pisarski was partially supported by DOE contract EY-76-C-02-3075.

[24] Recent Monte Carlo studies of SU(2) Yang-Mills theory at finite temperature (McLerran and Svetitsky, 1980; Kuti, Polonyi, and Szlachanyi, 1980) yield good evidence for a phase transition at temperature $T_c \approx 1/2\sqrt{\sigma}$.

[25] Recently, G. Lasher (1979) has discussed the cosmological implications of a strongly first-order phase transition from quark to hadronic matter. He argues that this transition might have a big effect on the history of the universe and might lead to galaxy formation. However this is based on an assumed nucleation rate which we feel is too small by many orders of magnitude.

[26] For a recent review of possible applications of QCD in this area, see Shuryak (1980).

APPENDIX A: NOTATION

We work in Euclidean space throughout this paper, with metric $g_{\mu\nu}=\delta_{\mu\nu}$, $\varepsilon_{0123}\equiv+1$, and Hermitian gamma matrices satisfying $\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}$.

The color gauge group G will be taken to be SU(N). The gauge field $A_\mu(x)$ is an element of the Lie algebra of G, and may be represented as $A_\mu=A_\mu^a T^a$, where $\{T^a\}$ are representations of the generators of the Lie algebra; $[T^a,T^b]=f^{abc}T^c$. Unless otherwise specified we shall take A_μ to be in the fundamental representation. Thus $A_\mu=A_\mu^a(\lambda^a/2i)$ is an arbitrary traceless, N-dimensional, anti-Hermitean matrix. The fundamental representation generators $\{\lambda^a/2i\}$ are normalized so that

$$\hat{\mathrm{tr}}[(\lambda^a/2i)(\lambda^b/2i)]\equiv-2\,\mathrm{tr}[(\lambda^a/2i)(\lambda^b/2i)]=\delta^{ab}.$$

The completeness of the λ's is expressed by

$$\tfrac{1}{2}\lambda_{ij}^a\lambda_{mn}^a=\delta_{in}\delta_{jm}-\frac{1}{N}\delta_{ij}\delta_{mn}.$$

The adjoint representation generators are given by $(T_{\mathrm{adj}}^a)_{mn}=f_{man}$.

The covariant derivative $D_\mu\equiv\partial_\mu+A_\mu$ and the field strength

$$F_{\mu\nu}\equiv[D_\mu,D_\nu]=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu].$$

A_μ is taken in whichever representation the covariant derivative is to act upon. We shall frequently split space and time components and hence define the electric field $E_i=F_{0i}$ and the magnetic field $B_i=\frac{1}{2}\varepsilon_{ijk}F_{jk}$. The dual field strength $\tilde{F}_{\mu\nu}\equiv\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$. This interchanges electric and magnetic fields.

For dealing with classical solutions, it is very convenient to use the matrices $\tau_\mu\equiv(-i,\boldsymbol{\tau})$ and $\tau_\mu^\dagger\equiv(i,\boldsymbol{\tau})$. [$\{\tau_a\}$ are the ordinary Pauli matrices. $\{\tau_a/2i\}$ may be considered as generators of any fixed SU(2) subgroup of G.] These matrices satisfy the relations

$$\tau_\mu^\dagger\tau_\nu=\delta_{\mu\nu}+i\tau^a\eta_{\mu\nu}^a,$$
$$\tau_\mu\tau_\nu^\dagger=\delta_{\mu\nu}+i\tau^a\bar{\eta}_{\mu\nu}^a,$$

where $\eta_{\mu\nu}^a$ and $\bar{\eta}_{\mu\nu}^a$ are 't Hooft's eta symbols. $\eta(\bar{\eta})$ projects out self-dual (anti-self-dual) tensors and satisfies various identities listed in 't Hooft, 1976.

All of our fields are defined on a slice of Euclidean space, $x\in\mathfrak{M}\equiv\{x\,|\,0\le x_0<\beta\}$. Thus we write

$$\int d^4x=\int_{\mathfrak{M}}d^4x\equiv\int_0^\beta dt\int d^3x.$$

Owing to the periodicity conditions (2.4), the time dependence of all fields may be represented by a fourier sum over discrete frequencies,

$$A_\mu(t,\mathbf{x})=\frac{1}{\beta}\sum_n A_\mu(\omega_n,\mathbf{x})e^{i\omega_n^+t},\quad \omega_n^+\equiv2n\pi/\beta$$

$$\psi(t,\mathbf{x})=\frac{1}{\beta}\sum_n\psi(\omega_n,\mathbf{x})e^{i\omega_n^-t},\quad \omega_n^-=(2n+1)\pi/\beta.$$

In general, we write

$$f(x)=\int_\pm\frac{d^4p}{(2\pi)^4}f(p)e^{ipx},$$

where

$$\int_\pm\frac{d^4p}{(2\pi)^4}\equiv\int\frac{d^3p}{(2\pi)^3}\frac{1}{\beta}\sum_n$$

and p_0 only takes on the discrete values $\omega_n^\pm$ depending on the periodicity of $f(x)$. It is frequently convenient to reexpress these frequency sums in terms of contour integrals, as follows:

$$\begin{aligned}\frac{1}{\beta}\sum_n f(\omega_n^\pm)&=\frac{1}{4\pi}\int_C dz\,f(z)(e^{i\beta z/2}\pm e^{-i\beta z/2})/(e^{i\beta z/2}\mp e^{-i\beta z/2})\\&=\int_{-\infty}^{\infty}\frac{dz}{2\pi}f(z)\mp\int_{-\infty+i\epsilon}^{\infty+i\epsilon}\frac{dz}{2\pi}[f(z)+f(-z)]/(e^{-i\beta z}\mp1).\end{aligned}\tag{A1}$$

(The contour C encloses the real axis counterclockwise.) This formula is valid for any function $f(z)$ which is analytic in a neighborhood of the real axis.

APPENDIX B: TOPOLOGICAL CLASSIFICATION

Let $A_\mu(t,\mathbf{x})$ be a smooth gauge field satisfying the conditions

$$\mathbf{E},\mathbf{B}=O(1/r^{3/2+\epsilon})\equiv o^{3/2}\quad\text{as }|\mathbf{x}|\to\infty\tag{B1}$$

and

$$A_\mu(\beta,\mathbf{x})=A_\mu(0,\mathbf{x}).\tag{B2}$$

$\mathbf{B}=o^{3/2}$ implies that $\mathbf{A}=g\boldsymbol{\partial}g^{-1}+o^{1/2}$ for some $g(\mathbf{x})\in G$. We may gauge transform by g^{-1} and consequently assume without loss of generality that

$$\mathbf{A}=o^{1/2}.\tag{B3}$$

Now

$$\Omega(\mathbf{x})=P\exp\left(\int_0^\beta dt\,A_0(t,\mathbf{x})\right)$$

Consequently,

$$\begin{aligned}\boldsymbol{\partial}\Omega(\mathbf{x})&=\int_0^\beta dt'\left[P\exp\left(\int_0^{t'}A_0\right)\right]\boldsymbol{\partial}A_0(t',\mathbf{x})\left[P\exp\left(\int_{t'}^\beta A_0\right)\right]\\&=\int_0^\beta dt'\frac{\partial}{\partial t'}\left\{\left[P\exp\left(\int_0^{t'}A_0\right)\right]\right.\\&\qquad\left.\times\mathbf{A}(t',\mathbf{x})\left[P\exp\left(\int_{t'}^\beta A_0\right)\right]\right\}+o^{3/2}\\&=[\Omega,\mathbf{A}(0,x)]+o^{3/2}.\end{aligned}$$

Here we used $\boldsymbol{\partial}A_0=\mathbf{D}_0\mathbf{A}+o^{3/2}$ and $\mathbf{A}(\beta,\mathbf{x})=\mathbf{A}(0,\mathbf{x})$. This establishes Eq. (3.2). $\mathbf{D}\Omega=o^{3/2}$ implies

$$\boldsymbol{\partial}\,\mathrm{tr}(\Omega^n)=o^{3/2}\quad\text{for any integer }n.\tag{B4}$$

Let $\{\lambda\}$ be the set of eigenvalues of $\Omega(\mathbf{x})$. Equation (B4) shows that this set of eigenvalues approaches a limit $\{\lambda^\infty\}$, independent of direction, as $|\mathbf{x}|\to\infty$. Let μ_α, $\alpha=0,\ldots,\kappa$ be the eigenvalues of $\{\lambda^\infty\}$ with multiplicities m_α. We may label the eigenvalues $\{\lambda\}$ of $\Omega(\mathbf{x})$ as $\{\lambda_{\alpha i}(\mathbf{x}),\ i=1,\ldots,m_\alpha;\ \alpha=0,\ldots,\kappa\}$ so that $\lambda_{\alpha i}(\mathbf{x})$ is everywhere continuous and

$$\lambda_{\alpha i}(\mathbf{x})=\mu_\alpha+o^{1/2},\quad\boldsymbol{\partial}\lambda_{\alpha i}(\mathbf{x})=o^{3/2}\quad\text{as }|\mathbf{x}|\to\infty.\tag{B5}$$

Thus

$$\Omega(\mathbf{x})=V(\mathbf{x})\lambda^\infty V^{-1}(\mathbf{x})+o^{1/2}.$$

(λ^∞ is considered as a diagonal matrix here.) Note that $V(\mathbf{x})$ is only defined up to right multiplication by any element of G which commutes with λ^∞. That is, $V\in G/H$

where $H=G_{\lambda^\infty}$ is the isotropy group of λ^∞. ($G_{\lambda^\infty}=\{g\in G|g\lambda^\infty g^{-1}=\lambda^\infty\}$.) So $\Omega(\mathbf{x})$ for $|\mathbf{x}|=R\to\infty$ may be regarded as a mapping of S_2 into the equivalence class of λ^∞. Equivalently, Ω provides a mapping of $S_2\to G/H$. These mappings fall into topologically distinct homotopy classes labeled by $\pi_2(G/H)$. Now $\pi_2(G/H)=\pi_1(H)$ since $\pi_1(G)$ and $\pi_2(G)$ are trivial. Clearly

$$H=[\mathrm{U}(1)]^\kappa[\bigotimes_\alpha \mathrm{SU}(m_\alpha)]$$

and $\pi_1(H)=(\mathbb{Z})^\kappa$ since $\pi_1[\mathrm{U}(1)]=\mathbb{Z}$. ($\kappa+1$ is the number of unequal eigenvalues of λ^∞.) These integers are in fact just the quantized magnetic flux,

$$q_\alpha=\lim_{R\to\infty}\frac{1}{4\pi i}\int_{S=\{\mathbf{x}||\mathbf{x}|=R\}} d^2\mathbf{S}\cdot\hat{\mathrm{tr}}(P_\alpha\mathbf{B})\,. \tag{B6}$$

Here $P_\alpha(\mathbf{x})$ is a projection operator onto the subspace spanned by eigenvectors of $\Omega(\mathbf{x})$ with eigenvalues $\{\lambda_{\alpha i}(\mathbf{x}), i=1,\ldots,m_\alpha\}$. [$P_\alpha(\mathbf{x})$ is continuously defined in the region

$$\mathcal{R}_\alpha=\{\mathbf{x}\in\mathbb{R}^3|\lambda_{\alpha i}(\mathbf{x})\neq\lambda_{\beta j}(\mathbf{x})\ \forall\ \beta\neq\alpha,\ \forall\ i,j\}\,.$$

Equation (B5) implies that $\mathcal{R}_\alpha$ contains the region $|\mathbf{x}|>d$ for some radius $d<\infty$.] Note that $\sum_0^\kappa q_\alpha=0$ since $\mathrm{tr}(\mathbf{B})=0$.

To see that q_α defines a topological integer, we diagonalize $\Omega(\mathbf{x})$ on the two-sphere S. Thus

$$\Omega(\mathbf{x})=V(\mathbf{x})\lambda(\mathbf{x})V^{-1}(\mathbf{x})\,.$$

Columns of V are the eigenvectors $\psi_{\alpha i}(\mathbf{x})$ of Ω.

$$P_\alpha(\mathbf{x})=\sum_i\psi_{\alpha i}(\mathbf{x})\psi^\dagger_{\alpha i}(\mathbf{x}).$$

Now, even though $\Omega(\mathbf{x})$ and $P_\alpha(\mathbf{x})$ are continuous everywhere on S, it may be impossible to continuously define $V\in G$ everywhere on S. [$V(\mathbf{x})$ is uniquely defined only as an element of the coset space $G/G_{\lambda(\mathbf{x})}$.] We may, however, remove a point P from the surface S and define $V(\mathbf{x})$ continuously on $S\backslash P$. As $\mathbf{x}\to\mathbf{x}_0\equiv P$ we must have $V(\mathbf{x})=v(\mathbf{x}_0)h(\mathbf{x})$, $h(\mathbf{x})\in G_{\lambda(\mathbf{x}_0)}$. $[h,\lambda(\mathbf{x}_0)]=0$ implies that h is block diagonal, i.e., $h_{(\alpha i)(\beta j)}=\delta_{\alpha\beta}M^\alpha_{ij}$. Let

$$\mathbf{A}=V(\boldsymbol{\partial}+\mathbf{A}')V^{-1}\quad\text{for }\mathbf{x}\in S\backslash P\,.$$

Therefore,

$$\mathbf{D}\Omega=V\{\boldsymbol{\partial}\lambda+[\mathbf{A}',\lambda]\}V^{-1}=o^{3/2}$$

so that

$$[\mathbf{A}',\lambda]=o^{3/2}\quad\text{or }\mathbf{A}'=\mathbf{a}+o^{3/2}\,, \tag{B7}$$

where $\mathbf{a}\in G_\lambda$. The projection P_α becomes $P_\alpha=Vp_\alpha V^{-1}$, where

$$p_\alpha=\sum_j\hat{e}_{(\alpha j)}\otimes\hat{e}^\dagger_{(\alpha j)}$$

and

$$[p_\alpha,h]=0\ \ \forall\ h\in G_\lambda\,.$$

We now compute the magnetic charge (B6):

$$\begin{aligned}q_\alpha&=\frac{-1}{2\pi i}\int_{S\backslash P}d^2\mathbf{S}\cdot\mathrm{tr}[p_\alpha(\boldsymbol{\partial}\times\mathbf{A}'+\mathbf{A}'\times\mathbf{A}')]\\&=\frac{-1}{2\pi i}\int_{S\backslash P}d^2\mathbf{S}\cdot\boldsymbol{\partial}\times\mathrm{tr}(p_\alpha\mathbf{A}')=\frac{1}{2\pi i}\oint_P d\mathbf{l}\cdot\mathrm{tr}(p_\alpha V^{-1}\boldsymbol{\partial}V)\\&=\oint_P\frac{d\mathbf{l}}{2\pi i}\cdot\mathrm{tr}(M^{\alpha^{-1}}\boldsymbol{\partial}M^\alpha)=\sum_j\frac{1}{2\pi}[f^\alpha_j(2\pi)-f^\alpha_j(0)]\in\mathbb{Z}\,.\end{aligned} \tag{B8}$$

In the first step we noted that the only part of $\mathbf{A}'$ which contributes in the $\mathbf{A}'\times\mathbf{A}'$ is the part which does not commute with λ. However, this part falls off as $o^{3/2}$, too rapidly to contribute to the surface integral. Next, we used the fact that $\mathrm{tr}(p_\alpha\mathbf{A})$ is continuous across P so that only the derivative term in $\mathbf{A}'$ can contribute to the infinitesimal line integral around P. In the last step we diagonalized $M^\alpha=um^\alpha u^{-1}$, $m^\alpha_{ij}=\delta_{ij}e^{if^\alpha_j(\phi)}$. [$\phi$ is the angle of polar coordinates (on S) centered at P.] The last line is clearly the integer winding number which measures the twist of m^α on an infinitesimal loop around P. This shows that the quantized magnetic charges $\{q^\alpha\}$ label the different homotopy classes of $\pi_1(H)=\pi_2(G/H)$ and thus characterize the winding of $\Omega(\mathbf{x})$ at infinity.

Next, we should like to relate the "topological charge" Q to the values of magnetic charge. We have

$$\begin{aligned}Q&=\frac{1}{32\pi^2}\int d^4x\,\hat{\mathrm{tr}}\tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\\&=\frac{1}{32\pi^2}\int d^4x\,\partial_\mu\,\hat{\mathrm{tr}}\varepsilon_{\mu\nu\alpha\beta}(A_\nu F_{\alpha\beta}-\tfrac{2}{3}A_\nu A_\alpha A_\beta)\\&=\frac{1}{16\pi^2}\int d^4x\{\partial_t\hat{\mathrm{tr}}(\mathbf{A}\cdot\mathbf{B}-\tfrac{1}{3}\mathbf{A}\cdot\mathbf{A}\times\mathbf{A})\\&\qquad+\boldsymbol{\partial}\cdot\hat{\mathrm{tr}}[\mathbf{A}\times\mathbf{E}-A_0(\boldsymbol{\partial}\times\mathbf{A})]\}\,.\end{aligned} \tag{B9}$$

For a periodic gauge field satisfying conditions (B1) and (B3), this becomes

$$Q=\frac{1}{16\pi^2}\int_0^\beta dt\,d^2\mathbf{S}\cdot\hat{\mathrm{tr}}(\boldsymbol{\partial}A_0\times\mathbf{A}).$$

This shows that any periodic field where A_0 vanishes rapidly at infinity will have zero topological charge. However, this expression is not the most useful for relating Q to the topology of Ω. For that, it is convenient to transform temporarily to $A_0=0$ gauge. Thus $A'_\mu=\mathrm{U}(\partial_\mu+A_\mu)\mathrm{U}^{-1}$ where

$$\mathrm{U}(t,\mathbf{x})=P\exp\left(\int_0^t dt'A_0(t',\mathbf{x})\right)\,.$$

In this gauge $A'_0=0$ and

$$\begin{aligned}\mathbf{A}'(\beta,\mathbf{x})&=\Omega(\mathbf{x})[\boldsymbol{\partial}+\mathbf{A}'(0,\mathbf{x})]\Omega^{-1}(\mathbf{x}),\\\mathbf{E}'(\beta,\mathbf{x})&=\Omega(\mathbf{x})\mathbf{E}'(0,\mathbf{x})\Omega^{-1}(\mathbf{x})\,.\end{aligned}$$

Now, from (B9),

$$\begin{aligned}Q&=\frac{1}{16\pi^2}\int d^3x\ \hat{\mathrm{tr}}[\mathbf{A}'\cdot(\mathbf{B}'-\tfrac{1}{3}\mathbf{A}'\times\mathbf{A}')]|_0^\beta\\&=\nu(\Omega^{-1}\boldsymbol{\partial}\Omega)+\frac{1}{16\pi^2}\int d^3x\,\boldsymbol{\partial}\cdot\hat{\mathrm{tr}}[\Omega^{-1}\boldsymbol{\partial}\Omega\times\mathbf{A}(\mathbf{x},0)]\,,\end{aligned} \tag{B10}$$

where

$$\nu(\mathbf{J})\equiv\frac{1}{16\pi^2}\int d^3x\ \tfrac{1}{3}\hat{\mathrm{tr}}(\mathbf{J}\cdot\mathbf{J}\times\mathbf{J})\,. \tag{B11}$$

The surface term in Eq. (B10) would vanish if $\boldsymbol{\partial}\Omega$ was $o^{3/2}$. In this case, Ω would approach a constant at infinity and hence could be regarded as a mapping of $S_3\to G$. Such mappings are classified by the homotopy group $\pi_3(G)=\mathbb{Z}$. $\nu(\Omega^{-1}\boldsymbol{\partial}\Omega)$ is precisely this integer-valued winding number.

In general, if $\boldsymbol{\partial}\Omega\neq o^{3/2}$ then we need to separate in

(B10) the effect of the winding of Ω at infinity. To do so, we extend the definition of $V(\mathbf{x})$ introduced previously so that

$$\Omega(\mathbf{x}) = V(\mathbf{x})\omega(\mathbf{x})V(\mathbf{x})^{-1}. \tag{B12}$$

We require that $\omega(\mathbf{x})$ be continuous throughout $\mathbb{R}^3$, and that $V(x)$ be continuously defined on $\mathbb{R}^3\backslash\{L\}$ for some set of strings $\{L\}$. We may choose $V(\mathbf{x})$ so that

$$\partial\omega(\mathbf{x}) = o^{3/2} \text{ and } \omega(\mathbf{x}) \to \lambda^\infty + o^{1/2}. \tag{B13}$$

Let

$$\mathbf{A}' = V^{-1}(\partial + \mathbf{A})V = V^{-1}\partial V + o^{1/2}.$$

Equations (B13) and (3.2) imply that $[\mathbf{A}', \omega] = o^{3/2}$ or $\mathbf{A}' = \mathbf{a} + o^{3/2}$ where $[\mathbf{a}, \omega] = 0$.

As $\mathbf{x}$ approaches some $\mathbf{x}_0 \in \{L\}$, we must have $V(\mathbf{x}) \to v(\mathbf{x}_0)h(\mathbf{x})$ where $h(\mathbf{x}) \in G_{\omega(\mathbf{x}_0)}$. [Otherwise $\omega(\mathbf{x})$ could not be continuous at $\mathbf{x}_0$.] $\mathbf{A}' - V^{-1}\partial V$ and $[V^{-1}\partial V, \omega]$ are bounded as $\mathbf{x} \to \mathbf{x}_0 \in \{L\}$. We show how to construct such a $V(\mathbf{x})$ below.

Inserting this decomposition into (B10), one finds

$$\begin{aligned} Q &= \nu(\omega^{-1}\partial\omega) + \frac{1}{16\pi^2}\int_{\mathbb{R}^3\backslash\{L\}} d^3x\, \partial \cdot \hat{\mathrm{tr}}[\omega^{-1}\partial\omega \times \mathbf{A}' + (\mathbf{D}'\omega)\omega^{-1} \times V^{-1}\partial V] \\ &= \nu(\omega^{-1}\partial\omega) + \frac{1}{16\pi^2}\int_{\partial(\mathbb{R}^3\backslash\{L\})} d^2\Sigma \cdot \hat{\mathrm{tr}}\{(\omega^{-1}\partial\omega + \partial\omega\omega^{-1} + [A', \omega]\omega^{-1}) \times \mathbf{A}'\} \\ &= \nu(\omega^{-1}\partial\omega) + \frac{1}{8\pi^2}\int_{\partial(\mathbb{R}^3\backslash\{L\})} d^2\Sigma \cdot \hat{\mathrm{tr}}(\omega^{-1}\partial\omega \times \mathbf{A}'). \end{aligned} \tag{B14}$$

In these steps we used the above conditions to eliminate terms which fall off too rapidly at infinity to give a surface term, or are bounded as one crosses a string in $\{L\}$. On $\partial(\mathbb{R}^3\backslash\{L\})$ we may represent ω as $\omega(\mathbf{x}) = \exp[-i\theta(\mathbf{x})]$ with $\theta(\mathbf{x})$ continuous and $[\theta(\mathbf{x}), \mathbf{A}']$ bounded and $o^{3/2}$. Consequently

$$\begin{aligned} Q &= \nu(\omega^{-1}\partial\omega) + \frac{1}{2\pi i}\int_{\partial(R^3\backslash\{L\})} d^2\Sigma \cdot \mathrm{tr}\left(\frac{\theta}{2\pi}\mathrm{B}'\right) \\ &= \nu(\omega^{-1}\partial\omega) + \sum_\alpha \frac{\ln\mu_\alpha}{2\pi i} \cdot q^\alpha. \end{aligned} \tag{B15}$$

Since $\partial\omega = o^{3/2}$, $\omega(x)$ may be regarded as a mapping of $S_3 \to G$, and $\nu(\omega^{-1}\partial\omega)$ is just the integer-valued Pontryagin index. [The branch of $\ln\mu_\alpha$ must be appropriately chosen so that ν contributes the correct integer part to Q. This branch is determined by the requirements above which imply that when two eigenvalues of $\lambda(\mathbf{x})$ become degenerate on $\{L\}$ so must the eigenvalues of $\ln\lambda(\mathbf{x})$.] This establishes the quoted relation (3.5).

To show that it is always possible to transform the fields into the "asymptotically Abelian" form used above, let us first consider the basic case of a field where λ^∞ has two different eigenvalues, μ_0 and μ_1. Let $\mathcal{R}$ be the region where no two eigenvalues of $\lambda(\mathbf{x})$ in different blocks cross. If $q^0 = -q^1 \neq 0$ then the complement of $\mathcal{R}$ must be nonvanishing [otherwise $\Omega(\mathbf{x})$ could not be continuous]. Let y be some point on the boundary of $\mathcal{R}$ where $\lambda_{0i} = \lambda_{1j}$ for some i and j. Choose a string $L \in \mathcal{R}$ running from y to ∞. We may choose a gauge transformation $V(x)$ continuous on $\mathbb{R}^3\backslash L$, which obeys

$$\begin{aligned} \lim_{\varepsilon\to 0} V(\mathbf{x}_0 + \varepsilon\mathbf{x}) = V(\mathbf{x}_0 + \varepsilon\mathbf{x}')[1 &+ \psi_{0i}\psi_{0i}^\dagger(e^{iq^0\theta} - 1) \\ &+ \psi_{1j}\psi_{1j}^\dagger(e^{iq^1\theta} - 1)] \end{aligned}$$

for $\mathbf{x}_0 \in L$, $\mathbf{x}$ and $\mathbf{x}'$ orthogonal to L, and $\mathbf{x}\cdot\mathbf{x}' = \cos\theta$. V may be chosen, for example, to block diagonalize Ω in the region $\mathcal{R}$ and otherwise may be filled in an arbitrary, continuous manner. This leaves $\omega = V^{-1}\Omega V$ continuous across L, and topologically trivial at infinity. All boundedness and asymptotic conditions above are satisfied. Finally, $\theta = i\ln\omega$ must be chosen so that $(\psi_{0i}, \theta\psi_{0i}) = (\psi_{1j}, \theta\psi_{1j})$ at y. This implies $[\theta(\mathbf{x}), G_{\omega(\mathbf{x})}] = 0$ for $\mathbf{x} \in \partial(\mathbb{R}^3\backslash L)$.

Generalizing this procedure to the case of an arbitrary set of magnetic charges is straightforward. One simply successively considers points where eigenvalues in different blocks cross, introduces strings running from these points to infinity, and iteratively constructs a gauge transformation V which removes all winding at infinity.

APPENDIX C: GLUON SELF-ENERGY

The one-loop gluon self-energy, in Feynman gauge, is given by

$$\begin{aligned} \Pi_{\mu\nu}(p) &= g^2 N_f \int_- \frac{d^4q}{(2\pi)^4}\frac{I_{\mu\nu}(p,q)}{q^2(p+q)^2} \\ &\quad - g^2 N \int_+ \frac{d^4q}{(2\pi)^4}\frac{I_{\mu\nu}(p,q) + J_{\mu\nu}(p)}{q^2(p+q)^2}, \end{aligned} \tag{C1}$$

where

$$I_{\mu\nu}(p,q) = 2(p+q)_\mu q_\nu + 2q_\mu(p+q)_\nu - \delta_{\mu\nu}((p+q)^2 + q^2 - p^2)$$

and

$$J_{\mu\nu}(p) = \delta_{\mu\nu}p^2 - p_\mu p_\nu.$$

At zero temperature one may evaluate $\Pi_{\mu\nu}$ using standard techniques (e.g., dimensional regularization). One finds

$$\Pi_{\mu\nu}(p)\big|_{T=0} = \frac{1}{3}\frac{g^2}{(4\pi)^2}(5N - 2N_f)(\delta_{\mu\nu}p^2 - p_\mu p_\nu)\ln p^2/\mu^2, \tag{C2}$$

where μ is the renormalization point.

At finite temperature, $\Pi_{\mu\nu}(p)$ has in general three independent components (see Sec. IV.B). However, since the one-loop expression (C1) is transverse for any energy, this reduces the number of components to two (for example, Π_{00} and Π_{ii}). Furthermore, we shall restrict our evaluation of $\Pi_{\mu\nu}(p_0, \mathbf{p})$ to the limit of low spatial momentum, $\mathbf{p} \sim 0$. In this limit the one-loop self-energy has only one independent component.

To see this, consider first the case of nonzero external energy. When $p_0 \neq 0$, the denominator of (C1) may be expanded in powers of $\mathbf{p}$ without causing any infrared divergence. Consequently, $\Pi_{\mu\nu}(p_0 \neq 0, \mathbf{p})$ has a finite asymptotic expansion in powers of $\mathbf{p}$. The transversality of $\Pi_{\mu\nu}$ then implies

$$\Pi_{\mu\nu}(p_0 \neq 0, \mathbf{p}) \sim \tfrac{1}{3}[\Pi_{\alpha\alpha}(p_0, \mathbf{p}=0)]\delta_{\mu i}\,\delta_{\nu i} + O(\mathbf{p})\,. \tag{C3}$$

On the other hand, at zero external energy, $\Pi_{\mu\nu}$ may be decomposed as

$$\Pi_{\mu\nu}(p_0=0,\mathbf{p}) = [\Pi_{00}(p_0=0,\mathbf{p})]\delta_{\mu 0}\delta_{\nu 0} + \tfrac{1}{2}[\Pi_{ii}(p_0=0,\mathbf{p})]\delta_{\mu i}(\delta_{ij} - p_i p_j/\mathbf{p}^2)\delta_{j\nu}\,.$$

$\Pi_{ii}(p_0, \mathbf{p}=0)$ can be nonvanishing only if $\Pi_{ij}(p_0=0,\ \mathbf{p})$ has a directional singularity at $\mathbf{p}=0$. However, the only part of (C1) which cannot be expanded in powers of $\mathbf{p}$ when $p_0=0$ is the contribution from the region where $q_0=0$ and $\mathbf{q}\sim 0(\mathbf{p})$. This contribution is of order $|\mathbf{p}|^3\mathbf{p}^2/\mathbf{p}^4 \sim O(\mathbf{p})$ and hence vanishes at zero momentum. Consequently, $\Pi_{ii}(p=0)=0$ so that

$$\Pi_{\mu\nu}(p_0=0,\mathbf{p}) \sim [\Pi_{\alpha\alpha}(p_0=0,\mathbf{p}=0)]\delta_{\mu 0}\delta_{\nu 0} + O(p)\,. \tag{C4}$$

Thus we need only compute the trace of the one-loop self-energy at zero spatial momentum. Equation (C1) yields

$$\Pi_{\mu\mu}(p) = -2g^2 N_f \int_- \frac{d^4q}{(2\pi)^4}\frac{[(p+q)^2 + q^2 - p^2]}{q^2(p+q)^2} + 2g^2 N \int_+ \frac{d^4q}{(2\pi)^4}\frac{[(p+q)^2 + q^2 - (5/2)p^2]}{q^2(p+q)^2}\,.$$

Equation (A1) may now be used to extract the temperature-dependent part,

$$\delta\Pi_{\mu\mu}(p) \equiv \Pi_{\mu\mu}(p) - \Pi_{\mu\mu}(p)\big|_{T=0}\,.$$

We find

$$\begin{aligned}
\delta\Pi_{\mu\mu}(p_0,\ \mathbf{p}=0) &= 2g^2 \int \frac{d^4q}{(2\pi)^4}\frac{4}{q^2}[N/(e^{-i\beta q_0+\varepsilon}-1) + N_f/(e^{-i\beta q_0+\varepsilon}+1)] \\
&\quad - g^2 \int \frac{d^4q}{(2\pi)^4}\frac{p^2}{q^2}\left(\frac{1}{(p+q)^2} + \frac{1}{(p-q)^2}\right)[5N/(e^{i\beta q_0+\varepsilon}-1) + 2N_f/(e^{-i\beta q_0+\varepsilon}+1)] \\
&= 4g^2 \int \frac{d^3q}{(2\pi)^3}\frac{1}{|\mathbf{q}|}[N/(e^{\beta|\mathbf{q}|}-1) + N_f/(e^{\beta|\mathbf{q}|}+1)] \\
&\quad -2g^2\,\mathrm{Re}\int \frac{d^3q}{(2\pi)^3}\frac{p_0}{|\mathbf{q}|(p_0+2i|\mathbf{q}|)}[5N/(e^{\beta|\mathbf{q}|}-1) + 2N_f/(e^{\beta|\mathbf{q}|}+1)] \\
&= \frac{2g^2}{\pi^2}\int_0^\infty dq\, q[N/(e^{\beta q}-1) + N_f/(e^{\beta q}+1)] - \frac{g^2p_0^2}{4\pi^2}\int_0^\infty \frac{dqq}{(q^2+p_0^2/4)}[5N/(e^{\beta q}-1) + 2N_f/(e^{\beta q}+1)] \\
&= \tfrac{1}{3}g^2T^2(N+N_f/2) - \frac{g^2p_0^2}{8\pi^2}(5N+2N_f)\left[\ln\beta p_0/4\pi - \psi\left(\frac{\beta p_0}{4\pi}\right) - \frac{2\pi}{\beta p_0}\right] \\
&\quad + \frac{g^2p_0^2}{4\pi^2}(2N_f)\left[\ln\beta p_0/2\pi - \psi\left(\frac{\beta p_0}{2\pi}\right) - \frac{\pi}{\beta p_0}\right] \\
&= \tfrac{1}{3}g^2T^2(N+N_f/2) - \frac{g^2p_0^2}{(4\pi)^2}(5N-2N_f)\ln(\beta p_0/4\pi)^2 \\
&\quad + \frac{g^2p_0^2}{8\pi^2}\left\{5N\left[\frac{1}{n} + \psi\left(\frac{n}{2}\right)\right] - 2N_f\left[2\psi(n) - \psi\left(\frac{n}{2}\right) - \ln 4\right]\right\}.
\end{aligned} \tag{C5}$$

Here $p_0 = 2\pi n/\beta$ and $\psi(z) = \Gamma'(z)/\Gamma(z)$.

Note that the $p_0^2 \ln p_0^2$ term in (C5) precisely cancels the $p^2\ln p^2$ zero-temperature term (C2). As $p_0 \to \infty$,

$$\delta\Pi_{\mu\mu}(p_0,\mathbf{p}=0) \sim -\tfrac{1}{2}g^2T^2N\,.$$

As $p_0 \to 0$,

$$\delta\Pi_{\mu\mu}(p_0,\mathbf{p}=0) \sim \tfrac{1}{3}g^2T^2(N+N_f/2).$$

These results may be inserted into (C3) and (C4) to find the quoted behavior for $\Pi_{00}(p_0=0,\mathbf{p}=0)$ and $\Pi_{ii}(p_0\neq 0,\mathbf{p}=0)$.

APPENDIX D: CONSTANT FIELD DETERMINANTS

Let

$$A_\mu = \frac{2\pi}{\beta}\left(\frac{q}{2i}\right)\delta_{\mu 0}\,,$$

where q is diagonal, real, and traceless,

$$(q)_{jk} \equiv q^j\delta_{jk}\,.$$

The fundamental representation fermion determinant may obviously be expressed as

$$\begin{aligned}
\ln\det{}_-(\not{D}_{\text{fund}}) &= \sum_j \ln\det{}_-(\not{\partial} + q^j\not{\xi}) \\
&= 2\sum_j \ln\det{}_-[-(\partial_\mu + q^j\xi_\mu)^2]\,,
\end{aligned} \tag{D1}$$

where $\xi \equiv (\pi/\beta i)\hat{e}_0$. Similarly, the adjoint representation gauge field determinant becomes

$$\ln\det{}_+(-D^2_{\text{adj}}) = \sum_{j,k}\ln\det{}_+\{-[\partial_\mu + (q^j - q^k)\xi_\mu]^2\} - \ln\det{}_+(-\partial^2)\,. \tag{D2}$$

This follows from noting that

$$(D_\mu^{\text{adj}}[v])_{jk} = (\partial_\mu v + [A_\mu, v])_{jk} = (\partial_\mu + (q^j - q^k)\xi_\mu)v_{jk}$$

and that each component of $v = v^a\lambda^a/2i$ except its trace may be regarded as independent. [Here we are choosing to regard an arbitrary adjoint representation field as a traceless, N-dimensional, anti Hermitian matrix, v, in place of the equivalent (N^2-1)-dimensional vector, $v^a = \hat{\mathrm{tr}}(v\lambda^a/2i)$.]

Thus we must evaluate $\ln\det[-(\partial_\mu + q\xi_\mu)^2]$ for either periodic or antiperiodic boundary conditions. Using (A1) we find

$$\ln\det_\pm[-(\partial_\mu + q\xi_\mu)^2] = \beta V \int_\pm \frac{d^4k}{(2\pi)^4} \ln\left[\left(\omega_n^\pm - \frac{\pi}{\beta q}\right)^2 + \mathbf{k}^2\right]$$

$$= \ln\det[-(\partial_\mu + q\xi_\mu)^2]\big|_{T=0} \pm \beta V \int \frac{d^4k}{(2\pi)^4} \frac{\ln\left[\left(k_0 - \frac{\pi}{\beta q}\right)^2 + \mathbf{k}^2\right] + \ln\left[\left(k_0 + \frac{\pi}{\beta q}\right)^2 + \mathbf{k}^2\right]}{(e^{-i\beta k_0} \mp 1)}.$$

The regulated, zero-temperature determinant is independent of q and defined to equal one. Hence

$$\ln\det_\pm[-(\partial_\mu + q\xi_\mu)^2] = \pm 2\beta V \,\mathrm{Re} \int \frac{d^4k}{(2\pi)^4} \ln k^2/(e^{-i\beta k_0 - i\pi q} \mp 1)$$

$$= 2V\,\mathrm{Re}\int \frac{d^3k}{(2\pi)^3} \ln(1 \mp e^{-\beta|\mathbf{k}| + i\pi q})$$

$$= -\frac{V}{\pi^2}\,\mathrm{Re}\int_0^\infty dk\,k^2 \sum_{n=1}^\infty (\pm e^{-\beta k + i\pi q})^n/n$$

$$= -\frac{2}{\pi^2}\frac{V}{\beta^3}\sum_{n=1}^\infty (\pm 1)^n \cos n\pi q/n^4$$

$$= -\pi^2 \frac{V}{\beta^3}\left(\frac{1}{45} - \frac{1}{24}(1 - [q]_\pm^2)^2\right), \qquad \text{(D3)}$$

where $[q]_+ = [(q)\bmod 2] - 1$, and $[q]_- = [(q+1)\bmod 2] - 1$. Hence $-1 \leq [q]_\pm < 1$.

Thus we find

$$\ln\det_-(\not{D}_{\mathrm{fund}}) = -2\pi^2 \frac{V}{\beta^3}\sum_{j=1}^N \left(\frac{1}{45} - \frac{1}{24}(1 - [q^j]_-^2)^2\right)$$

$$= -2\pi^2\frac{V}{\beta^3}\left(\frac{N}{45} - \frac{1}{24}\mathrm{tr}[1 - (\ln\Omega^{\mathrm{fund}}/\pi i)^2]^2\right) \qquad \text{(D4)}$$

and

$$\ln\det_+(-D^2_{\mathrm{adj}}) = -\pi^2\frac{V}{\beta^3}\left[\sum_{j,k=1}^N \left(\frac{1}{45} - \frac{1}{24}(1 - [q^j - q^k]_+^2)^2\right) - \frac{1}{45}\right]$$

$$= -\pi^2\frac{V}{\beta^3}\left(\frac{N^2-1}{45} - \frac{1}{6}\mathrm{tr}[(\ln\Omega^{adj}/\pi i) \times (1 - \ln\Omega^{\mathrm{adj}}/2\pi i)]^2\right). \qquad \text{(D5)}$$

($\ln\Omega^{\mathrm{fund}}$ and $\ln\Omega^{\mathrm{adj}}$ refer to the logarithm of the matrix representing Ω in the given representation. In the fundamental representation the eigenvalues of $\ln\Omega$ must lie within $[-\pi i, \pi i]$, while in the adjoint representation the eigenvalues are to be chosen in the internal $[0, 2\pi i]$.)

APPENDIX E: INSTANTON DETERMINANTS

1. Isospin 1/2

The temperature-dependent part of the isospin-$\frac{1}{2}$ determinant may be expressed in terms of the scalar propagator,

$$\delta\ln\det_\pm(-D^2/-\partial)_{1/2} = \int_0^\lambda d\lambda\,\mathrm{Tr}\left[\left(\frac{\partial}{\partial\lambda}(-D^2)\right)\Delta_\pm\right],$$

where

$$\Delta_\pm(x,y) = \sum_{n=-\infty}^{\infty} (\pm 1)^n \tilde{\Delta}(x, y + n\beta\hat{t}),$$

$$\tilde{\Delta}(x,y) = F(x,y)/4\pi^2(x-y)^2\Pi(x)^{1/2}\Pi(y)^{1/2}$$

and

$$F(x,y) = 1 + \sum_m \rho^2 \tau\cdot x_m \tau^\dagger \cdot y_m / x_m^2 y_m^2 .$$

($x_m \equiv x - m\beta\hat{t}$, etc.) Since

$$A_\mu = \tfrac{1}{4}(\tau_\mu\tau_\nu^\dagger - \tau_\nu\tau_\mu^\dagger) f_\nu \quad (f_\nu \equiv \partial_\nu \ln\Pi),$$

we have

$$\frac{\partial}{\partial\lambda}(-D^2) = \tfrac{1}{4}(\partial_\lambda f^2) - \tfrac{1}{2}\tau\cdot\partial\tau^\dagger\cdot\partial_\lambda f + \tfrac{1}{2}\tau\cdot\partial_\lambda f\tau^\dagger\cdot\partial .$$

Thus

$$\delta\ln\det_\pm(-D^2/-\partial^2) = A(\lambda) + \int_0^\lambda d\lambda \sum_n{}' (\pm 1)^n \Big(\int d^4x\,\mathrm{tr}[\tfrac{1}{4}(\partial_\lambda f^2)\tilde{\Delta} + \tfrac{1}{2}\tau\cdot\partial_\lambda f\tau^\dagger\cdot\partial\tilde{\Delta} - \tfrac{1}{2}\tilde{\Delta}\tau\cdot\overleftarrow{\partial}\tau^\dagger\cdot\partial_\lambda f]\big|_{y=x+n\beta\hat{t}}$$

$$- \tfrac{1}{2}\mathrm{tr}\,\tau\cdot\partial[\tau^\dagger\cdot\partial_\lambda f\tilde{\Delta}(x, x+n\beta\hat{t})]\Big), \qquad \text{(E1)}$$

where

$$A(\lambda) = \int_0^\lambda d\lambda\,\mathrm{Tr}\left[\left(\frac{\partial}{\partial\lambda}(-D^2)\right)\tilde{\Delta}\right].$$

This term has been previously examined by Brown and Creamer (1978). Their result is given in Eq. (6.10). The remaining terms above are easily evaluated using the relations

$$\tau^\dagger\cdot\vec{\partial}F(x,y)/(x-y)^2=-2\tau^\dagger\cdot(x-y)\Pi(x)/(x-y)^4$$

and

$$F(x,y)/(x-y)^2\tau\cdot\overleftarrow{\partial}=-2\Pi(y)\tau\cdot(x-y)/(x-y)^4 .$$

So

$$\begin{aligned}\delta\ln\det{}_\pm(-D^2/-\partial^2)_{1/2}=A(\lambda)+\int_0^\lambda d\lambda\sum_n{}'(\pm1)^n\Bigg[\int d^4x\,\mathrm{tr}\Big(&\tfrac14(\partial_\lambda f^2)\tilde\Delta-\tfrac14\tau\cdot\partial_\lambda f\tau^\dagger\cdot f\tilde\Delta-\tfrac14\tilde\Delta\tau\cdot f\tau^\dagger\cdot\partial_\lambda f\\&-\tau\cdot\partial_\lambda f\frac{\tau^\dagger\cdot(x-y)}{(x-y)^4}+\frac{\tau\cdot(x-y)}{(x-y)^4}\tau^\dagger\cdot\partial_\lambda f\Big)\Big|_{y=x+n\beta\hat t}\\&-\int d^3\Sigma\cdot\partial_\lambda\mathbf{f}/[4\pi^2(n\beta)^2]\Bigg]\\=A(\lambda)-\int_0^\lambda d\lambda\,\partial_\lambda\Big(\int d^3\Sigma\cdot\mathbf{f}&\sum{}'(\pm1)^n/4\pi^2n^2\beta^2\Big)\\=A(\lambda)-\frac{\int d^3\Sigma\cdot\mathbf{f}}{12\beta^2}\eta_\pm&=A(\lambda)+\tfrac13\lambda^2\eta_\pm ,\end{aligned}\tag{E2}$$

where

$$\eta_\pm=3\sum_n{}'(\pm1)^n/\pi^2n^2=(1\text{ for }+;\ -\tfrac12\text{ for }-).$$

This verifies Eq. (6.12).

We should also like to sketch a rather elegant alternative derivation of this result which requires no knowledge of the exact instanton propagators. This derivation is based on the approach of Callias and Taubes (1979).

The isospin-$\frac12$ operator $-D(A)^2$, for any 't Hooft solution, may be expressed in the factorized form

$$-D^2=-\Pi^{1/2}\tau\cdot\partial\Pi^{-1}\tau^\dagger\cdot\partial\Pi^{1/2} .\tag{E3}$$

Following Callias we generalize this factorization, defining

$$L_\nu=\Pi^{\nu/2}i\tau\cdot\partial\Pi^{-\nu/2},\quad L_\nu^\dagger=\Pi^{-\nu/2}i\tau^\dagger\cdot\partial\Pi^{\nu/2} .\tag{E4}$$

Therefore, $-D^2=L_1L_1^\dagger$. Now let

$$H=\begin{pmatrix}0 & L_\nu\\ L_\nu^\dagger & 0\end{pmatrix},\quad H^2=\begin{pmatrix}L_\nu L_\nu^\dagger & 0\\ 0 & L_\nu^\dagger L_\nu\end{pmatrix}$$

and observe that

$$\frac{\partial}{\partial\nu}H_\nu=\frac{\gamma}{2}[\ln\Pi,H_\nu]\quad\text{or}\quad\frac{\partial}{\partial\nu}H_\nu^2=\frac{\gamma}{2}[[\ln\Pi,H_\nu],H_\nu],$$

where

$$\gamma=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

and $\{\gamma,H_\nu\}=0$.

Now, since

$$\ln a/b=\int_0^\infty\frac{ds}{s}(e^{-sb}-e^{-sa})$$

we may define the finite Pauli–Villars regulated determinant to be

$$\ln\det(-D^2/-\partial^2)=\mathrm{Tr}\frac{(1+\gamma)}{2}\int_0^\infty\frac{ds}{s}\sum_{i=0}^R e_i(e^{-s(H_0^2+m_i^2)}-e^{-s(H_1^2+m_i^2)}) .$$

Here $e_i=(-1)^i$, $m_0=0$, and $m_1,\ldots,m_R\to\infty$. The regulator masses are chosen so that

$$\sum_{i=1}^R e_i=0,\quad\sum_i e_im_i^2=0,\quad\text{and}\ \sum_i e_im_i^2\ln m_i^2=0 .$$

Thus

$$\begin{aligned}\ln\det(-D^2/-\partial^2)&=\int_0^1 d\nu\,\mathrm{Tr}\frac{(1+\gamma)}{2}\int_0^\infty\frac{ds}{s}\sum_i e_i\frac{\partial}{\partial\nu}\{-e^{-s(H_\nu^2+m_i^2)}\}\\&=\int_0^1 d\nu\,\mathrm{Tr}\frac{(1+\gamma)}{2}\int_0^\infty ds\sum_i e_i\frac{\gamma}{2}[[\ln\Pi,H_\nu],H_\nu]e^{-s(H_\nu^2+m_i^2)}\\&=\int_0^1 d\nu\,\mathrm{Tr}\,\gamma\ln\Pi\sum_i e_iH_\nu^2/(H_\nu^2+m_i^2)=-\int_0^1 d\nu\,\mathrm{Tr}\,\gamma\ln\Pi\sum_i e_im_i^2/(H_\nu^2+m_i^2)\\&=\int_0^1 d\nu\,\mathrm{Tr}\ln\Pi\sum_i e_im_i^2[(L_\nu^\dagger L_\nu+m_i^2)^{-1}-(L_\nu L_\nu^\dagger+m_i^2)^{-1}] .\end{aligned}\tag{E5}$$

This remarkable result (due to Callias and Taubes) expresses the regulated determinant solely in terms of the regulator propagators. All dependence on the original massless field has been eliminated. [At this point, Callias and Taubes rewrite the propagators in exponential form and attempt to calculate the small t expansion of $\exp[-t(L_\nu L_\nu^\dagger)]$. We shall deal instead with the propagators directly.]

Now

$$(L_\nu^\dagger L_\nu+m^2)^{-1}=[(-\partial^2+m^2)-V_+]^{-1}$$

and

$$(L_\nu L_\nu^\dagger+m^2)^{-1}=[(-\partial^2+m^2)-V_-]^{-1} ,$$

where

$$V_\pm = \mp i\nu\tau_a \eta^{\pm a}_{\alpha\beta} f_\alpha \partial_\beta - \frac{\nu}{2}\left(\frac{\nu}{2} \mp 1\right) f^2 .$$

For arbitrarily large mass we expect to be able to expand $(LL^\dagger + m^2)^{-1}$ in powers of V. Inserting this expansion in (E5) yields a series of one-loop diagrams with free massive propagators, vertices $V_\pm$, and one insertion of $\ln\Pi$. All but the first few terms of this expansion should vanish as powers of $(1/m)$ as $m\to\infty$. In fact, only the diagrams shown in Fig. 8 need be computed. Expanding in external momenta and dropping all terms of $O(1/m_i^2)$, one easily finds that graphs (a)–(d) yield, respectively,

$$\tfrac{1}{6}\int \frac{d^4x}{(4\pi)^2}\ln\Pi(-\partial^2 f^2)\,,\quad \tfrac{1}{8}\int \frac{d^4x}{(4\pi)^2}\ln\Pi\, f^4$$

$$\tfrac{1}{6}\int \frac{d^4x}{(4\pi)^2}\ln\Pi\left(\frac{\partial_\alpha\partial_\beta\Pi}{\Pi}\right) f_\alpha f_\beta,\ \text{and}\ -\tfrac{3}{8}\int d^4x \ln\Pi\, f^4 .$$

Integrating by parts a few times, the sum of these terms becomes

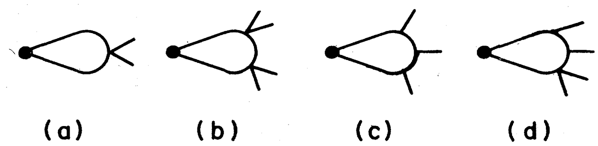

FIG. 8. Nonvanishing contributions for the isospin-$\frac{1}{2}$ determinant.

$$\tfrac{1}{12}\int \frac{d^4x}{(4\pi)^2}\left(\frac{\partial\Pi}{\Pi}\right)^4 .$$

This expression is logarithmically singular at the position of each instanton. This is due to the fact that our expansion of the propagators was only valid in the region where $\ln\Pi$ varied slowly on the scale of m_i^{-1}. This is true everywhere except in a tiny region about the center of each instanton. (Within this region all terms in the expansion contribute.) However, if we merely calculate the difference between the multi-instanton determinant and the corresponding sum of single instanton determinants, then the contributions of the near regions cancel, and we find

$$\ln\det(-D^2/-\partial^2) - \sum_i \ln\det(-D_i^2/-\partial^2) = \tfrac{1}{12}\left[\int_{\mathfrak{M}} \frac{d^4x}{(4\pi)^2}\left(\frac{\partial\Pi}{\Pi}\right)^4 - \int_{\mathbf{R}^4} \frac{d^4x}{(4\pi)^2}\sum_i\left(\frac{\partial\Pi_i}{\Pi_i}\right)^4\right].$$

For the finite-temperature instanton, this is precisely the term $A(\lambda)$. However, we have lost the λ^2 surface term in (E2). Fortunately, it may be easily found. In the manipulations leading to (E5) we repeatedly used the cyclic property of the trace. This actually involves integration by parts. For the finite-temperature instanton, one of these integrations yields a surface term. This happens in the step between the second and third equalities of (E5). The surface term is easily found to be

$$-\int d^3\Sigma_\mu \tfrac{1}{2}\mathrm{tr}[\tau_\mu\tau^\dagger\cdot f \sum_i e_i(-\partial^2+m_i^2)^{-1}_\pm(x,x)] .$$

At zero temperature, the regulated propagator, i.e.,

$$\sum_i e_i(-\partial^2+m_i^2)^{-1}(x,y)$$

simply vanishes as $x\to y$ since

$$\sum_i e_i = \sum_i e_i m_i^2 = \sum_i e_i m_i^2 \ln m_i^2 \equiv 0 .$$

However, at finite temperature there is an extra finite contribution from the terms which make $(-\partial^2+m_i^2)_\pm$ periodic (or antiperiodic). Thus

$$\sum_i e_i(-\partial^2+m_i^2)^{-1}_\pm(x,x) = \sum_n{}'(\pm 1)^n \sum_i e_i(-\partial^2+m_i^2)^{-1}(x,x+n\beta t) = \sum_n{}'(\pm1)^n/4\pi^2(n\beta)^2 + O(e^{-\beta m_i}).$$

Therefore, the surface term becomes

$$-\int d^3\mathbf{\Sigma}\cdot \mathbf{f}\eta_\pm/12\beta^2 ,$$

which agrees with (E2).

Unfortunately, this direct approach to calculating the instanton determinant does not seem to generalize to the isospin-1 case. The isospin-1 scalar operator, $-D_{ab}^2$, may be expressed as $\frac{1}{2}\mathrm{tr}(\tau^a L_2 L_2^\dagger \tau^b)$ with L_2 given by (E4). If we consider $L_2L_2^\dagger$ as acting on the space of 2×2 matrices, then this shows that $-D_{ab}^2$ is equivalent to $L_2L_2^\dagger$ restricted to the subspace of traceless matrices. This extra projection makes it difficult to generalize the previous approach. For this reason, we are forced to resort to the tedious analysis of the exact periodic propagator presented next.

2. Isospin 1

Following Brown *et al.* (1978) and Eq. (5.3), the periodic isospin-1 scalar propagator is given by

$$\Delta^{ab}(x,y) = \Delta_0^{ab} + \Delta_1^{ab} + \Delta_2^{ab} ,$$

where

$$\Delta_0^{ab}(x,y)=\tfrac{1}{2}\,\mathrm{tr}[\tau^a F(x,y)\tau^b F(y,x)]/4\pi^2(x-y)^2\Pi(x)\Pi(y)\,,$$

$$\Delta_1^{ab}(x,y)=\sum_m{}'\tfrac{1}{2}\,\mathrm{tr}[\tau^a F(x,y_m)\tau^b F(y_m,x)]/4\pi^2(x-y_m)^2\Pi(x)\Pi(y)\,,$$

and

$$\Delta_2^{ab}(x,y)=\sum_m C^{ab}(x,y_m)/4\pi^2\Pi(x)\Pi(y)\,.$$

The function $C^{ab}(x,y_m)$ will be exhibited shortly.
We must compute

$$\delta\ln\det(-D^2/-\partial^2)_1=\int_0^\lambda d\lambda\,\mathrm{Tr}\{[\partial_\lambda(-D^2)](\Delta_0+\Delta_1+\Delta_2)\}$$

$$\equiv P_0+P_1+P_2\,. \tag{E6}$$

Note that

$$\partial_\lambda(-D^2)^{ab}=2(\partial_\lambda f^2)\delta^{ab}-2\varepsilon^{abc}\bar\eta^c_{\mu\nu}(\partial_\lambda f_\nu)\partial_\mu\,.$$

The contribution of Δ_0 has been previously computed by Brown and Creamer (1978). They found that this term is simply four times the corresponding isospin-$\frac{1}{2}$ result,

$$P_0=4A(\lambda)\,, \tag{E7}$$

where $A(\lambda)$ is given in Eq. (6.10).

The analysis of P_1 is much more involved. Let

$$F(x,y)=F_0(x,y)+i\tau^a F_a(x,y)\,,$$

where

$$F_0(x,y)=1+\rho^2\sum_l x_l\cdot y_l/x_l^2 y_l^2$$

and

$$F_a(x,y)=\rho^2\bar\eta^a_{\mu\nu}\sum_l x_l^\mu y_l^\nu/x_l^2 y_l^2\,.$$

Then

$$P_1=P_{1a}+P_{1b}+P_{1c}\,,$$

where

$$P_{1a}=\int_0^\lambda d\lambda\int\frac{d^4x}{4\pi^2\beta^2}\frac{2\partial_\lambda f^2}{\Pi^2}\sum_m{}'\,[3F_0^2(x,x_m)-F_a^2(x,x_m)]/m^2\,,$$

$$P_{1b}=-\int_0^\lambda d\lambda\int\frac{d^4x}{\pi^2\beta^3}\frac{4\partial_\lambda f_a}{\Pi^2}\sum_m{}'\,F_0(x,x_m)F_a(x,x_m)/m^3\,,$$

and

$$P_{1c}=-\int_0^\lambda d\lambda\int\frac{d^4x}{\pi^2\beta^2}\frac{2\bar\eta_{\mu\nu}}{\Pi^2}\partial_\lambda f_\mu\sum_m{}'\,\partial_\nu[F_0(x,y_m)F_a(x,y_m)]_{y=x}/m^2\,.$$

Now

$$F_0(x,x_m)=1+2r^2\rho^2 I_0(m),\quad F_a(x,x_m)=x^a m\beta\rho^2 I_0(m)\,,$$

$$\partial_\nu F_0(x,y)\big|_{y=x_m}=\rho^2[(m\beta^\nu-x^\nu)I_0(m)+\beta^\nu I_1(m)+2m(x^\nu\beta t-x^2\beta^\nu)J_0(m)+2m\beta(\beta^\nu t-\beta x^\nu)J_1(m)]$$

and

$$\partial_\nu F_a(x,y)\big|_{y=x_m}=\rho^2[\bar\eta^a_{\nu\sigma}(x-m\beta)^\sigma I_0(m)-\bar\eta^a_{\nu\sigma}\beta^\sigma I_1(m)-2x^a x^\nu m\beta J_0(m)+2x^a\beta^\nu m\beta J_1(m)]\,.$$

Here,

$$\beta^\nu\equiv\beta\delta^{\nu0},\quad x\equiv(t,\mathbf{r})\equiv\frac{\beta}{2\pi}(\tau,\mathbf{z})$$

and we have defined

$$I_0(m)=\sum_l 1/x_l^2x_{l+m}^2=2(\Pi-1)/\rho^2(m^2\beta^2+4r^2)$$

$$I_1(m)=\sum_l l/x_l^2x_{l+m}^2=(\Pi-1)(2t-m\beta)/\beta\rho^2(m^2\beta^2+4r^2)$$

$$J_0(m)=\sum_l 1/x_l^4x_{l+m}^2=(\Pi-1)(m^2\beta^2+12r^2)/2r^2\rho^2(m^2\beta^2+4r^2)+\frac{\pi^2[2r\sin\tau\sinh z-m\beta(1-\cosh z\cos\tau)]}{\beta^3 m r^2(m^2\beta^2+4r^2)(\cosh z-\cos\tau)^2}$$

$$J_1(m)=\sum_l l/x_l^4x_{l+m}^2=(\Pi-1)[t(m^2\beta^2+12r^2)-4m\beta r^2)]/2\beta\rho^2r^2(m^2\beta^2+4r^2)^2$$
$$-\pi^2\frac{[(tm\beta+2r^2)(1-\cosh z\cos\tau)+r(m\beta-2t)\sin\tau\sinh z]}{m\beta^4r^2(m^2\beta^2+4r^2)(\cosh z-\cos\tau)^2}\,.$$

These and all following sums may be easily evaluated using Eq. (A1). For later convenience, define

$$\mathfrak{M}_{i,j}=\sum_m{}'1/[m^{2i}(m^2\beta^2+4r^2)^j]\,,$$

and let $f_z=\partial_z\ln\Pi$, $f_\tau=\partial_\tau\ln\Pi$, and $f_{\mu'}^2=f_z^2+f_\tau^2$. Plugging in, we find

$$P_{1a}=\frac{1}{8\pi^4}\int d\lambda\; d^3z\; d\tau\frac{\partial_\lambda f^2_{\mu'}}{\Pi^2}\left[3\mathfrak{M}_{1,0}+\frac{6\beta^2z^2}{\pi^2}(\Pi-1)\mathfrak{M}_{1,1}+\frac{3\beta^4z^4}{\pi^4}(\Pi-1)^2\mathfrak{M}_{1,2}-\frac{\beta^4z^2}{\pi^2}(\Pi-1)^2\mathfrak{M}_{0,2}\right], \tag{E8}$$

$$P_{1b}=\frac{-\beta^2}{2\pi^6}\int d\lambda\; d^3z\; d\tau\frac{\partial_\lambda zf_z}{\Pi^2}(\Pi-1)\left[\mathfrak{M}_{1,1}+\frac{\beta^2z^2}{\pi^2}(\Pi-1)\mathfrak{M}_{1,2}\right], \tag{E9}$$

and $P_{1c}=P_{1c\alpha}+P_{1c\beta}+P_{1c\gamma}+P_{1c\delta}$, where

$$\begin{aligned}P_{1c\alpha}&=\frac{6}{\pi^2\beta^2}\int d\lambda\int d^4x\frac{\partial_\lambda f_\mu}{\Pi^2}\sum_m{}'\frac{\rho^2}{m^2}[(x^\mu-m\beta^\mu)I_0(m)-\beta^\mu I_1(m)]F_0(x,x_m)\\&=-\tfrac{3}{2}P_{1b},\end{aligned} \tag{E10}$$

$$\begin{aligned}P_{1c\beta}&=-\frac{4}{\pi^2\beta}\int d\lambda\int d^4x\frac{\partial_\lambda f_a}{\Pi^2}\sum_m{}'\frac{\rho^2}{m}x^a[tJ_0(m)-\beta J_1(m)]F_0(x,x_m)\\&=-\frac{\beta^2}{2\pi^6}\int d\lambda\; d^3z\; d\tau\frac{\partial_\lambda zf_z}{\Pi^2}\left[\beta^2(\Pi-1)\left(\mathfrak{M}_{0,2}+\frac{\beta^2z^2}{\pi^2}(\Pi-1)\mathfrak{M}_{0,3}\right)\right.\\&\qquad\left.+\lambda^2\frac{(1-\cosh z\cos\tau)}{(\cosh z-\cos\tau)^2}\left(\mathfrak{M}_{1,1}+\frac{\beta^2z^2}{\pi^2}(\Pi-1)\mathfrak{M}_{1,2}\right)\right],\end{aligned} \tag{E11}$$

$$\begin{aligned}P_{1c\gamma}&=\frac{2}{\pi^2\beta}\int d\lambda\int d^4x\frac{\partial_\lambda f_a}{\Pi^2}\sum_m{}'\frac{\rho^2}{m^2}[mI_0(m)+I_1(m)-2mr^2J_0(m)]F_a(x,x_m)\\&=-\frac{\beta^2}{2\pi^6}\int d\lambda\; d^3z\; d\tau\frac{\partial_\lambda zf_z}{\Pi^2}\left(\frac{\beta^2z^2}{\pi^2}(\Pi-1)^2\mathfrak{M}_{0,3}-\lambda^2\beta^2\frac{(1-\cosh z\cos\tau)}{(\cosh z-\cos\tau)}(\Pi-1)\mathfrak{M}_{0,2}\right),\end{aligned} \tag{E12}$$

$$\begin{aligned}P_{1c\delta}&=\frac{4}{\pi^2\beta}\int d\lambda\int d^4x\frac{\partial_\lambda f_0}{\Pi^2}\sum_m{}'\frac{\rho^2}{m}\{r^2J_0(m)F_0(x,x_m)-[tJ_0(m)-\beta J_1(m)]r^aF_a(x,x_m)\}\\&=\frac{\beta^2}{2\pi^6}\int d\lambda\; d^3z\; d\tau\frac{\partial_\lambda zf_\tau}{\Pi^2}\frac{\lambda^2\sin\tau\sinh z}{(\cosh z-\cos\tau)^2}\left[\mathfrak{M}_{1,1}+(\Pi-1)\left(\frac{\beta^2z^2}{\pi^2}\mathfrak{M}_{1,2}-\beta^2\mathfrak{M}_{0,2}\right)\right].\end{aligned} \tag{E13}$$

Before evaluating these sums, we turn to the contribution of

$$\Delta_2^{ab}(x,y)=\sum_m C^{ab}(x,y_m)/4\pi^2\Pi(x)\Pi(y).$$

$C^{ab}(x,y)$ is given by

$$C^{ab}(x,y)=\sum_{r\neq s}\frac{2\Phi^a_{rs}(x)\Phi^b_{rs}(y)}{[\beta(r-s)]^2}-\sum_{r\neq s}\sum_{t\neq u}\frac{\rho^2\Phi^a_{rs}(x)}{[\beta(r-s)]^2}\frac{\Phi^b_{tu}(y)}{[\beta(t-u)]^2}h_{rs,tu},$$

where

$$\Phi^a_{rs}(x)=\rho^2\bar\eta^a_{\mu\nu}x^\mu_r x^\nu_s/x_r^2x_s^2=\rho^2\beta(r-s)x^a/x_r^2x_s^2$$

since $\bar\eta^a_{j0}=\delta_{aj}$ and

$$h_{rs,tu}=f_{su}-f_{st}+f_{rt}-f_{ru},$$

with f_{rs} a particular constant matrix. Owing to the periodicity of the finite-temperature instanton, f_{rs} is translationally invariant, $f_{rs}=f_{(r-s),0}$. Consequently, when we compute $\sum_m C^{ab}(x,y_m)$, the second term involving $h_{rs,tu}$ drops out. Therefore we find

$$\begin{aligned}P_2&=\int_0^\lambda d\lambda\int d^4x[\partial_\lambda(-D^2)^{ab}]\sum_m\frac{C^{ba}(x,y_m)|_{y=x}}{4\pi^2\Pi(x)\Pi(y)}\\&=\frac{\rho^4}{\pi^2}\int d\lambda\int d^4x[(\partial_\lambda f^2)\mathbf{x}^2+2(\partial_\lambda\mathbf{x}\cdot\mathbf{f})]\sum_m{}'[I_0(m)/\Pi]^2\\&=\frac{\beta^4}{4\pi^6}\int d\lambda\, d^3z\, d\tau(2\partial_\lambda zf_z+z^2\partial_\lambda f^2_{\mu'})\left(\frac{\Pi-1}{\Pi}\right)^2\mathfrak{M}_{0,2}.\end{aligned}$$

Now we must evaluate the sums

$$\mathfrak{M}_{i,j}=\sum_m{}' m^{-2i}(m^2\beta^2+4r^2)^{-j}.$$

One finds

$$\mathfrak{M}_{0,1}=\frac{\pi^2}{\beta^2}\left(\frac{\coth z}{z}-\frac{1}{z^2}\right),$$

$$\mathfrak{M}_{0,2}=\frac{\pi^4}{2\beta^4z^2}\left(\frac{\coth z}{z}+\frac{1}{\sinh^2 z}-\frac{2}{z^2}\right),$$

$$\mathfrak{M}_{0,3}=\frac{\pi^6}{8\beta^6 z^4}\left(\frac{3\coth z}{z}+\frac{3}{\sinh^2 z}+\frac{2z\cosh z}{\sinh^3 z}-\frac{8}{z^2}\right),$$

$$\mathfrak{M}_{1,0}=\frac{\pi^2}{3},$$

$$\mathfrak{M}_{1,1}=\frac{\pi^4}{\beta^2 z^2}\left(\frac{1}{3}+\frac{1}{z^2}-\frac{\coth z}{z}\right),$$

$$\mathfrak{M}_{1,2}=\frac{\pi^6}{2\beta^4 z^4}\left(\frac{2}{3}+\frac{4}{z^2}-\frac{3\coth z}{z}-\frac{1}{\sinh^2 z}\right).$$

Finally, adding up the contributions (E7)–(E14), we find

$$\delta\ln\det(-D^2/-\partial^2)_1=4A(\lambda)+\frac{1}{8\pi^2}\int_0^\lambda d\lambda\int d^3z\,d\tau\,\Pi^{-2}\Bigg((\partial_\lambda f^2_{\mu\nu})\tilde{B}(\lambda,z,\tau)+2(\partial_\lambda z f_z)\tilde{C}(\lambda,z,\tau)$$
$$+4\lambda^2\frac{[-\partial_\lambda f_z(1-\cosh z\cos\tau)+\partial_\lambda f_\tau\sin\tau\sinh z]}{z(\cosh z-\cos\tau)^2}\tilde{D}(\lambda,z,\tau)\Bigg), \tag{E15}$$

where

$$\tilde{B}(\lambda,z,\tau)=\frac{3}{\pi^2}\mathfrak{M}_{1,0}+6(\Pi-1)\frac{\beta^2z^2}{\pi^4}\mathfrak{M}_{1,1}+(\Pi-1)^2\frac{\beta^2z^2}{\pi^4}\left(\mathfrak{M}_{0,2}+\frac{3z^2}{\pi^2}\mathfrak{M}_{1,2}\right)$$
$$=1+6(\Pi-1)\left(\frac{1}{3}+\frac{1}{z^2}-\frac{\coth z}{z}\right)+(\Pi-1)^2\left(1+\frac{5}{z^2}-\frac{4\coth z}{z}-\frac{1}{\sinh^2 z}\right),$$

$$\tilde{C}(\lambda,z,\tau)=(\Pi-1)\frac{\beta^2}{\pi^4}(\mathfrak{M}_{1,1}-2\beta^2\mathfrak{M}_{0,2})+(\Pi-1)^2\frac{\beta^4}{\pi^4}\left(2\mathfrak{M}_{0,2}+\frac{z^2}{\pi^2}\mathfrak{M}_{1,2}-\frac{4\beta^2z^2}{\pi^2}\mathfrak{M}_{0,3}\right)$$
$$=\frac{(\Pi-1)}{z^2}\left[\Pi\left(\frac{1}{3}-\frac{2\coth z}{z}+\frac{3}{z^2}-\frac{1}{\sinh^2 z}\right)+(\Pi-1)\left(\frac{1}{z^2}-\frac{z\cosh z}{\sinh^3 z}\right)\right],$$

and

$$\tilde{D}(\lambda,z,\tau)=\frac{z^2\beta^2}{\pi^4}\left[\mathfrak{M}_{1,1}+(\Pi-1)\beta^2\left(\frac{z^2}{\pi^2}\mathfrak{M}_{1,2}-\mathfrak{M}_{0,2}\right)\right]$$
$$=\left(\frac{1}{3}+\frac{1}{z^2}-\frac{\coth z}{z}\right)+(\Pi-1)\left(\frac{1}{3}-\frac{2\coth z}{z}\,\frac{1}{\sinh^2 z}+\frac{3}{z^2}\right).$$

Lastly, we may explicitly evaluate the integral over λ by recalling that $\Pi(\lambda,z,\tau)=1+\lambda^2h(z,\tau)$. The following integrals are helpful:

$$\int_0^\lambda d\lambda\,\partial_\lambda f^2_{\mu\nu}[\alpha+\beta(\Pi-1)+\gamma(\Pi-1)^2]/\Pi^2=\frac{f^2_{\mu\nu}}{6\Pi^2}[\alpha(3+2\Pi+\Pi^2)+\beta(3+\Pi)(\Pi-1)+3\gamma(\Pi-1)^2]$$

and

$$\int_0^\lambda d\lambda\,\partial_\lambda\hat{v}\cdot f(\Pi-1)[\alpha+\beta(\Pi-1)]/\Pi^2=\frac{\hat{v}\cdot f}{\Pi^2}\,\frac{(\Pi-1)}{3}[\alpha(1+\tfrac{1}{2}\Pi)+\beta(\Pi-1)].$$

Hence, we find the final result,

$$\ln\det(-D^2/-\partial^2)_1=4A(\lambda)+B(\lambda)+C(\lambda)+D(\lambda), \tag{E16}$$

where

$$B(\lambda)=\frac{1}{8\pi^2}\int d^3z\,d\tau\left(\frac{\partial_\mu\Pi}{\Pi}\right)^2\left[\Pi^2+4(\Pi-1)\left(\frac{1}{z^2}-\frac{\coth z}{z}\right)-(\Pi-1)^2\left(\frac{3\coth z}{z}-\frac{7}{2z^2}+\frac{1}{2\sinh^2 z}\right)\right],$$

$$C(\lambda)=\frac{1}{12\pi^2}\int d^3z\,d\tau\left(\frac{z\partial_z\Pi}{\Pi^3}\right)\left(\frac{\Pi-1}{z^2}\right)\left[\Pi\left(\frac{1}{2}-\frac{3\coth z}{z}+\frac{9}{2z^2}-\frac{3}{2\sinh^2 z}\right)+(\Pi-1)\left(\frac{1}{z^2}-\frac{z\cosh z}{\sinh^3 z}\right)\right],$$

$$D(\lambda)=\frac{\lambda^2}{12\pi^2}\int d^3z\,d\tau\frac{[(-\partial_z\Pi)(1-\cosh z\cos\tau)+(\partial_\tau\Pi)\sin\tau\sinh z]}{z\Pi^3(\cosh z-\cos\tau)^2}\left[\Pi+3\left(\frac{1}{z^2}-\frac{\coth z}{z}\right)-(\Pi-1)\left(\frac{5\coth z}{z}-\frac{7}{z^2}+\frac{2}{\sinh^2 z}\right)\right],$$

and $A(\lambda)$ is given in Eq. (6.10).

3. Computations

We have evaluated the above integrals numerically. One must be particularly careful to treat correctly the contributions from both large and small distances. For $z\gg 1$, one may use the asymptotic form

$$\Pi=1+2\lambda^2/z+\lambda^2O(e^{-z})$$

and evaluate the resulting integrals analytically. For very short distances, $z^2+\tau^2 \ll 1$, one must expand the integrand in a Taylor series, since, although the complete integrand is finite at the origin, many cancellations between singular terms are occurring.

$A(\lambda)$ was found to agree with the simple expression (6.14) for all values of λ to within the quoted error. $B(\lambda)+C(\lambda)+D(\lambda)$ was found to equal $12A(\lambda)+\frac{4}{3}\lambda^2$ to within our numerical accuracy of $\sim 10^{-7}$. We are embarrassed to say that we have not been able to establish this analytically.

REFERENCES

Abers, E., and B. Lee, 1973, Phys. Rep. **9C** (1).
Adler, S., 1979, private communication.
Affleck, I., 1980a, Nucl. Phys. B **162**, 461.
Affleck, I., 1980b, Nucl. Phys. B **171**, 420.
Applequist, T., and J. Carrazone, 1975, Phys. Rev. D **11**, 2856.
Atiyah, M., W. Drinfeld, N. Hitchin, and Yu. Manin, 1978, Phys. Lett. A **65**, 185.
Baluni, V., 1978, Phys. Rev. D **17**, 2092.
Banks, T., and E. Rabinovici, 1979, Nucl. Phys. B **160**, 349.
Bashilov, Yu., and S. Pokrovski, 1978, Nucl. Phys. B **143**, 431.
Batakis, N., and G. Lazarides, 1978, Phys. Rev. D **18**, 4710.
Baym, G., and S. Chin, 1976, Phys. Lett. B **64**, 117.
Baym, G., and N. Mermin, 1961, J. Math. Phys. **2**, 232.
Bernard, C., 1974, Phys. Rev. D **9**, 3312.
Bernard, C., 1979, Phys. Rev. D **19**, 313.
Bilic, N., and D. Miller, 1979, Phys. Lett. B **87**, 239.
Bilic, N., and D. Miller, 1980, Phys. Rev. D **22**, 472.
Brown, L., R. Carlitz, D. Creamer, and C. Lee, 1978, Phys. Rev. D **17**, 1583.
Brown, L., R. Carlitz, and C. Lee, 1977, Phys. Rev. D **16**, 417.
Brown, L., and D. Creamer, 1978, Phys. Rev. D **18**, 3695.
Cabbibo, N., and G. Parisi, 1975, Phys. Lett. B **59**, 67.
Caldi, D., 1977, Phys. Rev. Lett. **39**, 121.
Callan, C., R. Dashen, and D. Gross, 1976, Phys. Lett. B **63**, 334.
Callan, C., R. Dashen, and D. Gross, 1978, Phys. Rev. D **17**, 2717.
Callan, C., R. Dashen, and D. Gross, 1979a, Phys. Rev. D **19**, 1826.
Callan, C., R. Dashen, and D. Gross, 1979b, Phys. Rev. D **20**, 3279.
Callan, C., R. Dashen, and D. Gross, 1980, Phys. Rev. Lett. **44**, 4351.
Callias, C., and C. Taubes, 1979, "Functional Determinants in Euclidean Yang-Mills Theory," MIT preprint CTP #783.
Caswell, W., 1974, Phys. Rev. Lett. **33**, 244.
Chodos, A., R. Jaffe, K. Johnson, C. Thorn, and V. Weisskopf, 1974, Phys. Rev. D **9**, 3471.
Christ, N., and R. Jackiw, 1980, Phys. Lett. B **91**, 228.
Christ, N., E. Weinberg, and N. Stanton, 1978, Phys. Rev. D **18**, 2013.
Coleman, S., 1977, "The Uses of Instantons" (1977 Erice lectures), Harvard University preprint HUTP-78/A004.
Coleman, S., and D. Gross, 1973, Phys. Rev. Lett. **31**, 851.
Collins, J., and M. Perry, 1975, Phys. Rev. Lett. **34**, 135.
Corrigan, E., D. Fairlie, S. Templeton, and P. Goddard, 1978, Nucl. Phys. B **140**, 31.
Creutz, M., 1977, Phys. Rev. D **15**, 1128.
Creutz, M., 1980, Phys. Rev. D **21**, 2308.
Dashen, R. and D. Gross, 1980, Phys. Rev. D, in press.
Dolan, L., and R. Jackiw, 1974, Phys. Rev. D **9**, 3320.
Dolan, L., and J. Kiskis, 1979, Phys. Rev. D **20**, 505.
Faddeev, L. D., 1976, in *Methods in Field Theory*, edited by R. Balian and J. Zinn-Justin (North-Holland, Amsterdam), p. 1.
Faddeev, L. D., and V. Popov, 1967, Phys. Lett. B **25**, 29.
Feynman, R., and A. Hibbs, 1965, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York).
Frazer, W., and F. Henyey, Eds., 1979, *Quantum Chromodynamics (La Jolla Institute, 1978)*, Proceedings of the Summer Workshop on Quantum Chromodynamics (AIP, New York).
Freedman, B., and L. McLerran, 1977a, Phys. Rev. D **16**, 1130.
Freedman, B., and L. McLerran, 1977b, Phys. Rev. D **16**, 1169.
Freedman, B., and L. McLerran, 1978, Phys. Rev. D **17**, 1109.
Fubini, S., and G. Veneziano, 1969, Nuovo Cimento A **64**, 8811.
Georgi, H., and H. Politzer, 1974, Phys. Rev. D **9**, 416.
Gross, D., and F. Wilczek, 1973a, Phys. Rev. D **8**, 3633.
Gross, D., and F. Wilczek, 1973b, Phys. Rev. Lett. **30**, 1343.
Gross, D., and F. Wilczek, 1974, Phys. Rev. D **9**, 980.
Grossman, B., 1977, Phys. Lett. A **61**, 86.
Hagedorn, R., 1965, Nuovo Cimento Suppl. **3**, 147.
Hagedorn, R., and J. Ranft, 1968, Nuovo Cimento Suppl. **6**, 169.
Harrington, B., and H. Shepard, 1978a, Phys. Rev. D **17**, 2122.
Harrington, B., and H. Shepard, 1978b, Phys. Rev. D **18**, 2990.
Hasenfratz, A., and P. Hasenfratz, 1980, Phys. Lett. **93B**, 165.
Huang, K., and S. Weinberg, 1970, Phys. Rev. Lett. **25**, 895.
Jackiw, R., C. Nohl, and C. Rebbi, 1977, Phys. Rev. D **15**, 1642.
Jackiw, R., and C. Rebbi, 1976, Phys. Rev. Lett. **37**, 172.
Jones, D., 1974, Nucl. Phys. B **75**, 531.
Kadanoff, L., and G. Baym, 1962, *Quantum Statistical Mechanics* (Benjamin, New York).
Kalashnikov, O., and V. Klimov, 1979, Phys. Lett. B **88**, 328.
Källman, C., 1979, Phys. Lett. B **85**, 392.
Kapusta, J., 1979a, Nucl. Phys. B **148**, 461.
Kapusta, J., 1979b, Phys. Rev. D **20**, 989.
Keister, B., and C. Kisslinger, 1976, Phys. Lett. B **64**, 117.
Kirzhnits, D., 1972, Zh. Eksp. Teor. Fiz. Pis'ma Red. **15**, 745 [1972, JETP Lett. **15**, 529].
Kirzhnits, D., and A. Linde, 1972, Phys. Lett. B **42**, 471.
Kirzhnits, D., and A. Linde, 1974, Zh. Eksp. Teor. Fiz. **67**, 1263 [1975, Sov. Phys.—JETP **40**, 628].
Kisslinger, M., and P. Morley, 1976a, Phys. Rev. D **13**, 2765.
Kisslinger, M., and P. Morley, 1976b, Phys. Rev. D **13**, 2771.
Kogut, J., R. Pearson, and J. Shigemitsu, 1979, Phys. Rev. Lett. **43**, 484.
Kogut, J., and L. Susskind, 1975, Phys. Rev. D **11**, 395.
Kuti, J., J. Polonyi, and K. Szlachanyi, 1980, "Monte Carlo Study of SU(2) Gauge Theory at Finite Temperature," KFKI-1980-67.
Lasher, G., 1979, Phys. Rev. Lett. **42**, 1646.
Levine, H., and L. Yaffe, 1979, Phys. Rev. D **19**, 1225.
Linde, A., 1979, Rep. Prog. Phys. **42**, 389.
Linde, A., 1980, Phys. Lett. **93B**, 327.
Mahanthappa, K., and J. Randa, Eds., 1979, *Quantum Flavordynamics, Quantum Chromodynamics, and Unified Theories* (Plenum, New York).
Manton, N., 1977, Nucl. Phys. B **126**, 525.
Marciano, W., and H. Pagels, 1978, Phys. Reports **36C** (3).
McLerran, L. D., and B. Svetitsky, 1980, "A Monte Carlo Study of SU(2) Yang-Mills Theory at Finite Temperature," SLAC preprint Pub-2572.
Morley, P., and M. Kisslinger, 1979, Phys. Reports **51C** (2).
Pines, D., 1964, *Elementary Excitations in Solids* (Benjamin, New York).
Politzer, H., 1973, Phys. Rev. Lett. **30**, 1346.
Polyakov, A., 1977, Nucl. Phys. B **120**, 429.

Polyakov, A., 1978, Phys. Lett. B **72**, 477.
Prasad, M., and C. Sommerfield, 1975, Phys. Rev. Lett. **35**, 760.
Rossi, P., 1979, Nucl. Phys. B **149**, 170.
Salam, A., 1968, in *Elementary Particle Physics*, edited by Nils Svartholm (Almqvist and Wiksell, Stockholm), 367.
Schroeder, L., Ed., 1979, *1st Workshop on Ultra-Relativistic Nuclear Collisions*, LBL pub. 8957.
Shuryak, E., 1978, Phys. Lett. B **79**, 135.
Shuryak, E., 1979, Phys. Lett. B **81**, 65.
Shuryak, E., 1980, Phys. Reports **61C** (2).
Susskind, L., 1979, Phys. Rev. D **20**, 2610.
Taylor, J. G., 1980, J. Math. Phys., in press.
't Hooft, G., 1976a, Phys. Rev. D **14**, 3432; D **18**, 2199 (E).
't Hooft, G., 1976b, unpublished. See Jackiw, Nohl, and Rebbi, 1977.
Weinberg, E., 1979a, Phys. Rev. D **20**, 936.
Weinberg, E., 1979b, private communication.
Weinberg, S., 1967, Phys. Rev. Lett. **19**, 1264.
Weinberg, S., 1973, Phys. Rev. Lett. **31**, 494.
Weinberg, S., 1974, Phys. Rev. D **9**, 3357.
Wilson, K., 1974, Phys. Rev. D **10**, 2445.
Witten, E., 1977, Phys. Rev. Lett. **38**, 121.
Wu, T. T., and C. N. Yang, 1969, in *Properties of Matter under Unusual Conditions*, edited by H. Mark and S. Fernbach (Interscience, New York), p. 349.
Yaffe, L., 1979, Nucl. Phys. B **151**, 247.
Yaffe, L., 1978, unpublished.