Random-matrix physics: spectrum and strength fluctuations

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It now appears that the general nature of the deviations from uniformity in the spectrum of a complicated nucleus is essentially the same in all regions of the spectrum and over the entire Periodic Table. This behavior, moreover, is describable in terms of standard Hamiltonian ensembles which could be generated on the basis of simple information-theory concepts, and which give also a good account of fluctuation phenomena of other kinds and, apparently, in other many-body systems besides nuclei. The main departures from simple behavior are ascribable to the moderation of the level repulsion by effects due to symmetries and collectivities, for the description of which more complicated ensembles are called for. One purpose of this review is to give a self-contained account of the theory, using methods—sometimes approximate—which are consonant with the usual theory of stochastic processes. Another purpose is to give a proper foundation for the use of ensemble theory, to make clear the origin of the simplicities in the observable fluctuations, and to derive other general fluctuation results. In comparing theory and experiment, the authors give an analysis of much of the nuclear-energy-level data, as well as an extended discussion of observable effects in nuclear transitions and reactions and in the low-temperature thermodynamics of aggregates of small metallic particles.

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I. GENERAL CONSIDERATIONS

A. Statistical properties

We consider in this review the properties of random-matrix ensembles and the application of such ensembles to physics, particulary to a study of the spectrum and the transition strengths in complicated systems. It is often taken for granted that one resorts to statistical studies of such systems only because detailed properties—of the spectrum, for example—are not really open to calculation. This is indeed one of the reasons but not really the main one. Such studies, whose nature is essentially different from the study of individual levels, are of interest because they reveal new features of the system. The situation is analogous to that in statistical mechanics, where properties such as temperature and entropy are exhibited best by systems of many particles. Even in systems where the individual levels and their quantum descriptions are better known than the corresponding phase-space or related descriptions, statistical methods are often essential for a more complete understanding.

The study of many-particle spectra, at least at low excitation energy, has been mainly concerned with properties of individual levels or short sequences of related levels—rotational bands, for example—and with individual transitions and sets of similarly related ones. In contrast to that, our present purpose is to deal, at low energies as well as high, with energy-level fluctuations (i.e., departures from uniformity in the spectrum) and fluctuations in transition strengths. This represents only one part of the old but rapidly growing field of *statistical spectroscopy*, but it is nonetheless a subject of considerable interest in itself and one with a long history of distinguished experimental and theoretical work. The early history of the relevant nuclear experiments and the general ideas about nuclear structure to which they gave rise are well reviewed by Lynn (1968), while random-matrix methods for the detailed theoretical analysis were introduced and considerably developed by Wigner (1951a, 1955, 1967). For earlier reviews, summaries and general discussion see Wigner (1967), Porter (1965a), Mehta (1967), Brody *et al.* (1972), Pastur (1973), Carmeli (1974), and, for recent conference proceedings, Garg (1972) and Sheldon (1976). The subject is significant, moreover, for other complex systems besides nuclei—atoms, for example, and metallic powders, whose thermodynamic properties are linked with spectral fluctuations, as we shall discuss in Sec. IX.

This review, though concerned mainly with nuclei (as most analyses have been in the past), can perhaps be regarded as an introduction to a larger subject, that of the general properties of spectral fluctuations in complicated systems, of the way in which they are influenced by the symmetries, collective modes, and other general features, and conversely of the information which they carry about the system. Because the concerns of this review are different than those of earlier random-matrix reviews, we do things differently: We are not so much concerned with exact as with simple calculations of fluctuation measures (though we do give some new exact results as well); we pay more attention than do

earlier reviews to the stationarity properties of the measures (the way in which they vary over the spectrum); we consider the effects of collectivities and symmetries on the strength distributions. We consider, moreover, the ergodic properties of the ensembles which we are mainly concerned with, drawing on quite recent work of Pandey (1979), which goes far toward giving a theoretical justification for the use of random-matrix ensembles.

We consider the energy-level fluctuations first, leaving until later the strength fluctuations, the understanding of which makes use of the same concepts. The study of both kinds of fluctuations has developed because of the remarkable series of resonances found in the scattering of slow neutrons on heavy nuclei. The resonances are narrowed because of the strong surface reflection of long-wavelength neutrons, and as a consequence one can often observe up to a few hundred resonances, essentially all (if the target state is even-even) with the same "exact" quantum numbers (J, π; isospin is also good, but is usually irrelevant). These levels occur typically at 6 to 8 MeV excitation and are separated from the ground-state domain by a "no-man's land" containing perhaps a million levels whose properties are not easily accessible to study. It is not then surprising that there has been little attempt to reconcile the concepts used in the two energy regions, the consequence being an (unnecessary) splitting of the subject into two noncommunicating domains. In order to span the no-man's land, and for other purposes as well, we shall make use not only of experimental data in both domains, but of shell-model calculations as well. We shall see, as we proceed, that the fluctuation properties extend over the entire spectrum (in the sense that a run of levels taken at any excitation displays almost the same properties), and indeed over the entire Periodic Table, a result which we regard as of real significance. Insofar as *averaged* properties (as opposed to *fluctuations*) are concerned, the barrier between high and low excitations has been breaking down during recent years; it is a good thing now that the same thing is happening for the fluctuations.

B. Level repulsion in spectra

Consider the set of spectra shown in Figs. 1(a)–1(c), where we have brought together runs of 50 levels taken from three very different sources: the slow-neutron resonance region of ^{167}Er (Liou *et al.*, 1972), the neighborhood of an isobaric-analogue state in ^{49}V (Prochnow *et al.*, 1972), and a section of a large shell-model calculation with a realistic interaction (Soyeur and Zuker, 1972). The levels in each spectrum have the same (J, π), and the scales have been chosen so that the average spacing D is the same for each.

The spectra are similar in their general nature; for example, the number of spacings much smaller than the average spacing is statistically the same for all three, as we have made evident by marking those which are smaller than $D/4$. The similarity of the spectra is made more obvious in Figs. 2(a)–2(c), which give histograms of the nearest-neighbor spacings for each spectrum. They resemble each other closely enough that we may regard the three spectra of Figs. 1(a)–1(c)

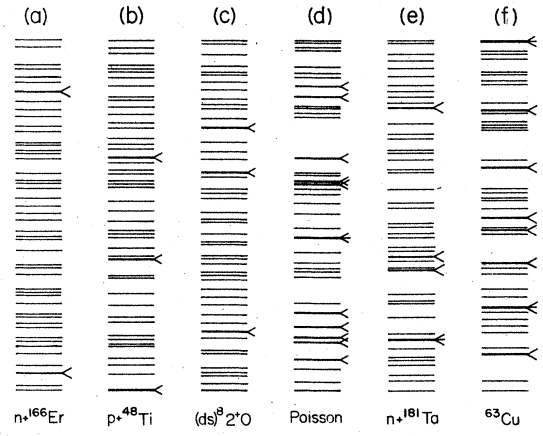


FIG. 1. Segments of complex spectra, each containing 50 levels and rescaled to the same spectrum span. The first two show experimental results for neutron and proton resonances, while Fig. 1(c) shows the central region of a 1206-dimensional, $J^\pi = 2^+$, $T = 0$, shell-model spectrum; in these three cases all the states have the same exact symmetries. Figure 1(d) shows a Poisson sequence, while Figs. 1(e) and 1(f) show spectra with mixed exact symmetries, the first an experimental spectrum with $J = 3^+, 4^+$ and the second a shell model spectrum with $J = \frac{1}{2}^-, \frac{3}{2}^-, \ldots, \frac{19}{2}^-$. The "arrowheads" mark the occurrence of pairs of levels with spacings smaller than one quarter of the average.

as having the same nearest-neighbor spacing distributions, thus exhibiting a regularity of a statistical nature, even though the spectra themselves differ in the three examples. The continuous curve shown in the figures (Wigner's distribution, whose theoretical basis we discuss below) fits them quite well.

Small spacings in our examples have a small proba-

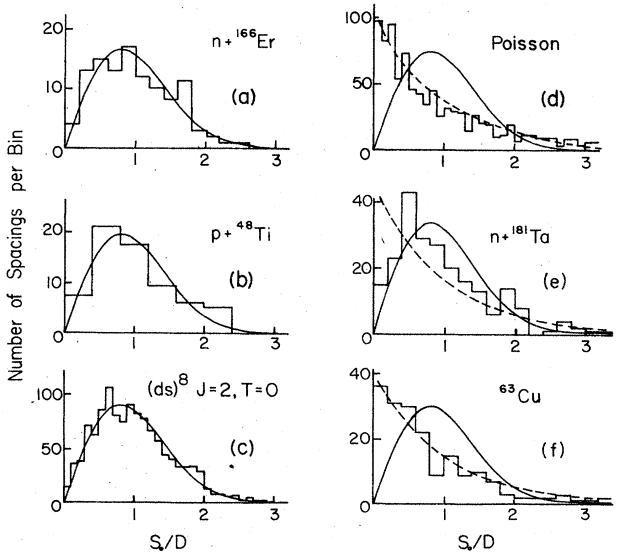


FIG. 2. Nearest-neighbor spacing histograms for the six cases of Fig. 1, constructed by considering all the available levels instead of the 50 used in Fig. 1. Spacings S_0 are expressed in terms of the local spacing unit D, derived via the unfolding operation of Sec. III.A, where necessary. The Wigner distribution, Eq. (1.5), is shown for all cases, and the Poisson distribution (1.4) also for the last three.

bility of occurrence, and the case is similar for large spacings. (The relevant limiting case, in which there are no small or large spacings, is the uniform spectrum, which we may describe as *rigid*. We shall see that our spectra have a high degree of rigidity.) Taking for granted that the individual states have complicated structures, we might have guessed that the levels should form a completely random sequence, being distributed in energy as the pulses from a radioactive target are distributed in time. In that case, as we show in Eq. (1.4) below, the spacing distribution would have been of Poisson type, in which small spacings predominate. This is shown in the random-sequence spectrum and corresponding histogram [Figs. 1(d) and 2(d)], which are seen to be very different from the examples of Figs. 1(a)–1(c) and 2(a)–2(c), which, by comparison with the random case, may be regarded as displaying a *level repulsion*.

As we proceed, we shall see that this repulsion, which was first discussed by von Neumann and Wigner (1929), and whose relevance to our present subject was first stressed by Landau and Smorodinski (1959) and by Wigner (1957a, 1957b, 1959; see also Gurevich and Pevsner, 1957), dominates all the spectral fluctuations. To get a first idea about its origin, let us consider the Hamiltonian as defined, with respect to some fixed basis, by its matrix elements which we may regard as the "coordinates" of H in the "matrix-element space," H then being represented as a vector in this space. The repulsion may be regarded as arising from the fact that the subspace for which the corresponding spectrum has a degeneracy is of a dimensionality less by two than that of the general matrix-element space, so that in some sense a degeneracy is "unlikely." Alternatively, if we think of the matrix elements as functions of a parameter λ, we cannot, in general, force a crossing by varying λ but must instead take the matrix elements as functions of at least two parameters which are independently varied. In the one-parameter case one will find that, if two levels approach each other as λ is varied, then instead of crossing they will turn away as if repelled; see, for example Landau and Lifshitz (1965). A similar repulsion for resonant states, defined by S-matrix poles, will be described in Sec. VIII; see also McVoy *et al.* (1967).

To get started, a simple heuristic calculation, due to Wigner (1967), is worthwhile. For a random sequence the probability that a level will be in the small interval $(E+S, E+S+dS)$, proportional, of course, to dS, will be independent of whether or not there is a level at E. This will be modified if we introduce level repulsion. Given a level at E, let the probability that the *next* level $(S \geq 0)$ be in $(E+S, E+S+dS)$ be $P(S)dS$. Then for $P(S)$, the nearest-neighbor spacing distribution, we have

$$P(S)dS = P(1 \in dS \,|\, 0 \in S) P(0 \in S), \tag{1.1}$$

where $P(n \in S)$ is the probability that the interval of length S contains n levels and $P(n \in dS \,|\, m \in S)$ is the conditional probability that the interval of length dS contains n levels, when that of length S contains m levels. The second factor in (1.1) is $\int_S^\infty P(x)dx$, the probability that the spacing is larger than S, while the first one will be dS times a function of S, $r_{10}(S)$, depending explicitly on the choices, 1 and 0, of the discrete variables n, m. Then

$$P(S) = r_{10}(S) \int_S^\infty P(x)dx, \tag{1.2}$$

which we can solve easily to find

$$P(S) = C r_{10}(S) \exp\left(-\int^S r_{10}(x)dx\right). \tag{1.3}$$

The Poisson law follows if we take $r_{10}(S) = 1/D$, where D is the mean local spacing so that $1/D$ is the density of levels. Wigner's law follows from the assumption of a *linear* repulsion, defined by $r_{10}(S) = \alpha S$. The arbitrary constants are determined by $\int P(x)dx = 1$, $\int xP(x)dx = D$. We find for the Poisson and Wigner cases, respectively,

$$P(S) = (1/D)\exp(-S/D), \quad S \geq 0 \tag{1.4}$$

$$P(S) = (\pi S/2D^2)\exp(-\pi S^2/4D^2), \quad S \geq 0 \tag{1.5}$$

the second of which displays the repulsion explicitly, since $P(0) = 0$, in contrast to the Poisson form, which has a maximum at $S = 0$. Like the Poisson, the Wigner distribution (1.5) is a standard one in statistics (Cramér, 1946, p. 236), being that for the square root of the sum of the squares of two independent Gaussian random variables of type[1] $G(0, D\sqrt{2/\pi})$, and often called a "Rayleigh distribution."

There are two major difficulties with the derivation of (1.5). In the first place, the (unbounded) linear form for $r_{10}(S)$ must be wrong for large S. Secondly, even for small S, why should we assume a repulsion, $r_{10}(S) = \alpha S$? Although we shall see in Sec. II that there are some simple plausibility arguments for this form, the result cannot be correct for every system, since we can obviously construct an H to have any given spectrum and hence a spacing law very different from either of the forms (1.4) and (1.5). But evidently certain Hamiltonians are in some sense more "likely" to occur than others; in Wigner's derivation there are thus tacit assumptions about the relative probability of different Hamiltonians. In Sec. II we formalize these considerations by introducing an ensemble of Hamiltonians. A fundamental difficulty now is that there appears to be no natural and physically significant weighting function for such an ensemble, there being no equivalent to the Liouville theorem which supplies such a function in classical statistical mechanics. Many different forms are possible, depending on the quantum numbers which we take to be exactly conserved and on the importance which, apart from that, we attach to various features of the Hamiltonian, such as its two-body nature.

It will follow that a probability argument which pays no attention to the specific features of the Hamiltonian cannot explain the nature, the origin, or the consequences of level repulsion.

Finally, we remark that our above discussion about level repulsion applies only for a pure sequence, i.e., one whose levels all have the same values of the exact quantum numbers. Cases of mixed sequences are given in Figs. 1(e)–1(f) which show a segment of the ^{182}Ta spectrum, derived from slow neutrons on ^{181}Ta (Hacken

[1] $G(a,b)$ has centroid a and variance b^2. Thus the corresponding probability density is $\rho(x) = (2\pi b^2)^{-1/2}\exp[-(x-a)^2/2b^2]$.

et al., 1978) for which $J=7/2^+$, and a low-lying region deriving from a ^{63}Cu shell-model calculation (Wong, 1970). In the first spectrum there are interwoven runs of levels with $J=(3^+,4^+)$ and, in the second, with $J=(1/2^- - 19/2^-)$. In these cases the level repulsion is moderated by the vanishing of Hamiltonian matrix elements connecting two different J values, the spectra have moved toward random, and the spacing distributions of Figs. 2(e)–2(f) toward Poisson.

If we regard the different J sequences as essentially independent, and each described by (1.5), it is possible (Gurevich and Pevsner, 1957; Lane, 1957; Mehta, 1967) to calculate the resulting distribution. We could therefore take for granted that the fundamental problem is that of understanding the pure sequence, since all others may be obtained from it by superposition. An unsolved subsidiary problem is to understand why different sequences should be independent, since they derive, after all, from the same Hamiltonian.

C. Fluctuations and level repulsion in the ground-state domain

The spectra of Fig. 1 refer to high excitation energy. To relate them to low-energy spectra we span the gap, as described above, by making use of large shell-model calculations. Fig. 3(a) shows the spacing histogram for levels 10–110 of the same (1206-dimensional) shell-model spectrum from which levels 576–625 were used in Fig. 1(c). It displays a level repulsion similar to that shown in Fig. 2(c) for the entire spectrum, indicating that the same spacing distribution applies over a wide energy range. The same is true, in fact, for other fluctuation properties (though certain small differences will appear on detailed examination). The results have been verified more explicitly by comparing separate segments of the spectrum.

We cannot conclude from this that the fluctuation pattern extends all the way to the ground state, because of the rapid secular variation of the density which sets in at the extremes of the spectrum (shell-model examples of this are to be seen in the first two parts of Fig. 4,

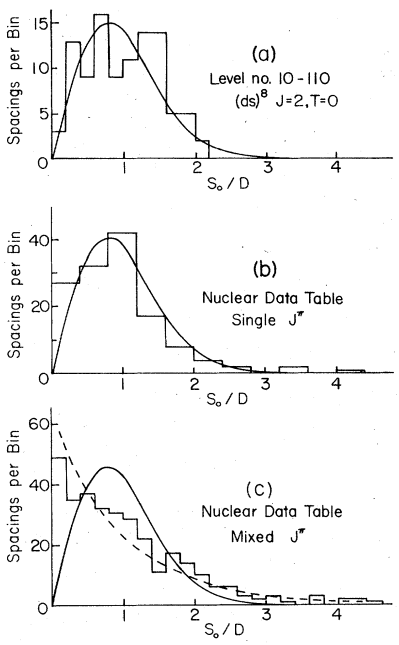


FIG. 3. Nearest-neighbor spacing histograms: (a) for a low-lying segment of the 1206-dimensional shell-model spectrum used in Figs. 1(c) and 2(c); (b) for the nuclear data table, considering spacings between the ground state and the lowest excited state of the same exact symmetries; (c) for the nuclear table considering the lowest spacing irrespective of the exact symmetries. Spacings have been renormalized by unfolding in the first case and by assuming that $D\propto A^{-1}$ in the second and third. In the latter cases the "collective" nuclei have been excluded.

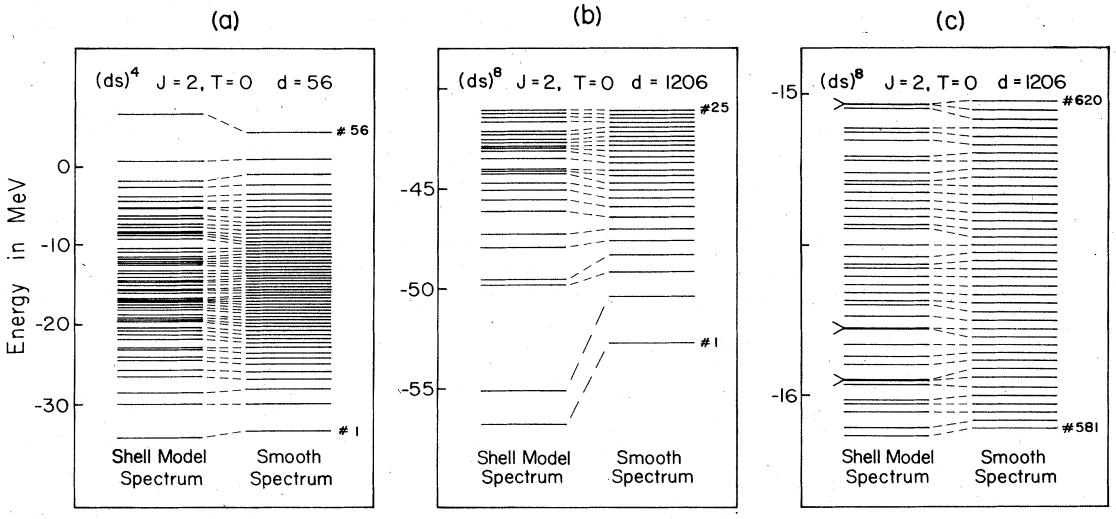


FIG. 4. Examples of shell model spectra and the smoothed spectra derived from them by using (1.8), as illustrated in Fig. 5. The smoothed distribution function, $\hat{F}(E)$, is calculated in terms of a few low-order spectral moments, by using a truncated Gram-Charlier expansion (Cramér, 1946) for the eigenvalue density. The first example shows a complete 56-dimensional spectrum; the other two show small segments of the 1206-dimensional spectrum. In all cases the level deviations are of the order of a single spacing unit.

other aspects of which we shall consider below). But we can consider the ground-state fluctuations in a different manner (Flores and Mello, 1973; Brody *et al.*, 1976) by plotting the spacings between the two lowest levels of the same J and π for different nuclei as a function of the mass number A. Except for the very light nuclei, a spacing, locally averaged in A, can now be determined quite consistently; in fact, it is found to be well represented by a $1/A$ law. The nuclei can be separated into two categories: There is a first group consisting of the even-even, rotational, magic, and doubly magic nuclei (which we might roughly describe as "collective"), for which the spacings differ only marginally from the average; and there is a main group composed of the other nuclei. After normalizing the data so that the average spacing is constant, this second group yields a spacing histogram [Fig. 3(b)] roughly of Wigner type [compare it, for example, with Fig. 2(a)], displaying the level repulsion seen in the slow-neutron-resonance region. If, on the other hand, the normalized spacings between ground and first-excited states are taken irrespective of the good quantum numbers involved, the spacing distribution [Fig. 3(c)] is comparable with Figs. 2(d)–2(f) and therefore of Poisson type.

These are surprising results. They involve a statistical property which covers the entire range of energies of the discrete nuclear spectra, but studied now, not by examining many spacings taken from a single spectrum, but rather by combining spacings taken one each from the series of different nuclei. What we have done here is analogous to the standard procedure in (classical) statistical mechanics: Whenever the time average of a function for an individual system has not been calculable, we have constructed a large set of (theoretical) replicas of the system and have taken the average over this set or *ensemble*.

For *theoretical* studies on complex systems the introduction of ensembles is essential, since it is out of the question to integrate the equations of motion or, equivalently, in the present circumstances, to construct and diagonalize a sufficiently extended shell-model matrix. In each case computer calculations are possible and important, as we have stressed for the level fluctuations, in supplementing theoretical studies; but they do not replace them, if only because, in the latter case, one can never be certain that the results from relatively small matrices are properly "asymptotic" or applicable to very complex nuclei. The equivalence of phase and time averaging in statistical mechanics defines the essence of the "ergodic" and related problems which of course have their counterpart in the present case. We shall have to consider these problems later (Sec. X), but in the meantime we stress that nearest-neighbor spacings, at least, have the same distribution for the main group of nuclei when taken *along an individual spectrum* a million levels above the ground state as when taken *across an ensemble* at the ground state itself; and this may be taken as evidence for ergodic behavior (or, more strictly, for some combination of ergodicity and stationarity). On the other hand, the spectra of the excluded nuclei are dominated by collectivities or symmetries (or both) and, since these are not effective at high excitation, the equivalence of the spacing laws found above for the main group is not to be expected for these nuclei. In more general treatments of fluctuations we should be able to take these nuclei into account, as well.

We must mention that the type of ensemble needed is different from those of conventional statistical mechanics (Wigner, 1955). Its members are different Hamiltonians rather than different states of the same Hamiltonian. A second major difference is that the behavior is studied along the energy axis rather than the time axis. The analog of time is therefore energy, but the analogy is not perfect, because, while thermodynamic systems evolve along the time axis, we have at present no way of describing mathematically the transition from one level to the next. Moreover, it is mainly the region of discrete energy levels (or narrow resonances) which is of interest to us so that we can limit ourselves to matrix representations of the Hamiltonians (exceptions to this will come in Sec. VIII, where we deal with more general features of nuclear reactions). Then, as described in Appendix A, only a small low-lying part of the spectrum can have any claim to a *direct* one-to-one correspondence (as opposed to a statistical one) between the model states and those of the physical system.

D. Spectral rigidity and the separation of fluctuations and secular behavior

Extrapolating from the results of the preceding section, we suspect now that fluctuation properties, always measured in locally defined units, are the same at high and low energies, even though the spacing unit varies by a very large factor (by $\sim 10^4$ for a heavy nucleus, for which $D \sim 10$ eV in the slow neutron region and ~ 100 keV at the ground state). In order to cross more confidently the no-man's land separating the regions we should understand more about the way in which the spectrum expands as we come down in energy; in particular, we should ask if there are random aspects to this behavior, or if it is understandable in terms of a small number of physically significant parameters. A tentative answer to this comes from shell-model calculations, examples of which, due to Chang (1970) and Soyeur and Zuker (1972), have been given in Fig. 4. In each case a shell-model spectrum is compared with a fluctuation-free spectrum which is derived from it and represents *by a small number of parameters* the secular behavior of the original eigenvalue density; a measure of the long-range rigidity is given then by the rms deviation, measured in local spacing units, between the spectrum and its fluctuation-free form. In the 1206-dimensional spectrum shown [^{24}Mg with $J=2$, $T=0$, described via $(ds)^8$] the secular variation, or, equivalently, the smoothed spectrum itself, is described well by four parameters; the remarkable fact is that the smoothed spectrum deviates, on the rms average, by only a single spacing unit from its parent, this behavior holding over the entire spectrum and being found also in many other examples of varying dimensionalities. In the central region, then, where the secular variation is negligible, the spectrum deviates surprisingly little from the extreme of a rigid spectrum, the uniformly spaced or "picket-fence" spectrum. Fig. 4(c) gives an example. We have no reason to doubt that this rigidity applies also to physical spec-

tra; indeed, the formal analysis of Sec. IV, in which we shall rederive and extend results of Dyson and Mehta (1963), who first discovered the spectral rigidity, will strongly suggest that this is true.

In giving these examples we have not indicated where the "smoothed" or "locally uniform" spectrum comes from, yet, because fluctuations are defined in terms of departure from local uniformity, an understanding of this is quite essential; Sec. III, in fact, is devoted to it, but in the meantime some simple considerations will make things clearer. Consider the density $\rho(x)$ for a discrete spectrum (a sum of delta functions) and its distribution function $F(x)$ (a staircase function, as shown in Fig. 5):

$$\rho(x)=d^{-1}\sum_r \delta(x-E_r), \quad F(x)=\int_{-\infty}^{x}\rho(z)dz\,. \tag{1.6}$$

The dimensionality d of the space (the total number of levels considered) is inserted here so that $\int\rho(x)dx=1$, $\rho(x)$ then being a probability density; if we wished to consider an unbounded spectrum we would omit the d factor and deal thereby with the actual density of levels. $F(x)$ has discontinuities at the eigenvalues:

$$\begin{aligned} &F(E_{r-})=(r-1)/d, \quad r=1,2,\ldots,d \\ &F(E_{r+})=r/d\,. \end{aligned} \tag{1.7}$$

If we have a smoothed (i.e., continuous) approximation to $F(x)$—say, $\hat{F}(x)$—we would *define* (Ratcliff, 1971) the corresponding spectrum as the set of values $\hat{E}_r$ which satisfy

$$\hat{F}(\hat{E}_r)=(r-\tfrac{1}{2})/d\,, \tag{1.8}$$

and then $(E_r-\hat{E}_r)/D$, with D the local average spacing, will be the level deviation.

The problem now is how to carry out a proper smoothing, whereby $F(x)\to\hat{F}(x)$, of a given spectrum. Empirical methods for doing this, via a running average, for example, are, for the purposes of studying the fluctuations, without a physically or mathematically significant basis. Broadly speaking, the situation is as follows: There must be some natural limiting distribution (the same for a wide class of systems) and a corresponding expansion of the distribution function in terms of components of varying "wavelengths" (the longest of the order of the spectrum span and the shortest of the order of the mean spacing) built upon the limiting distribution. If we order the components, which we can usefully call "excitations," according to their "wave number" by a parameter ζ $(=1,2,3,\ldots)$, the secular behavior will then be given by truncating the expansion at some value $\hat{\zeta}$, this operation introducing a local smoothing. In doing this we are, of course, taking for granted that there is a real separation of the secular behavior and the fluctuations, and in actual cases this should be verified.

We shall see in Sec. IV how this is to be implemented, but in the meantime an extremely instructive example is given in Fig. 6, which deals once again with the 1206-

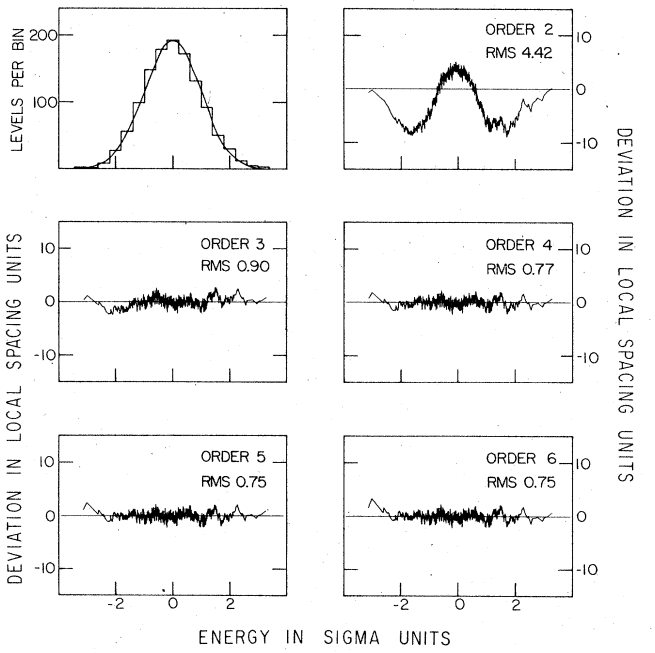


FIG. 6. Partial normal-mode decomposition of a shell-model spectrum. The spectrum, due to Soyeur and Zuker (1972, and private communication), is for the 1206-dimensional $(ds)^8$ with $J=2$, $T=0$. Energy is measured with respect to the spectral centroid, in units of the spectral width ("sigma units"). The spectral histogram in the upper-left figure shows that the distribution is essentially Gaussian. The level-to-level deviations between the true spectrum and its Gaussian version are displayed in the second figure ("order 2") and are large (rms average = 4.42 spacings). Incorporating the "skewness" correction into the density ("order 3") reduces this to 0.90, and the fourth order "excess" correction to 0.77. Corrections of orders 5, 6 have a negligible effect on the rms value and, moreover, on the detailed spectrum of the deviations. Since these deviations show ~200 sign changes over the spectrum, we can expect no further modification of the deviations until we reach very high ζ values. The residual rms value for $\zeta\sim4$ is then a measure of the very short-range fluctuations, and there are no fluctuations of intermediate wavelength. Numerical analyses of this kind were first given by Chang and Zuker (1972), whose straightforward Gram-Charlier expansion, however, does not always display convergence in the sense of $\langle\delta x^2/D^2\rangle_{av}$ nondecreasing as the order increases. This arises from the fact that the Gram-Charlier expansion is in terms of polynomials orthogonal with a continuous (Gaussian) weight function, while the δx_i spectrum is discrete. Instead of using polynomial orthogonality, which is then approximate, the coefficients in the normal-mode decomposition can be fixed by minimizing $\langle\delta x^2/D^2\rangle_{av}$ for each order; this much more complicated procedure is used here. The present calculations are due to Wong (unpublished) and the theory to Mon and French (1975), as discussed in IV.F ahead.

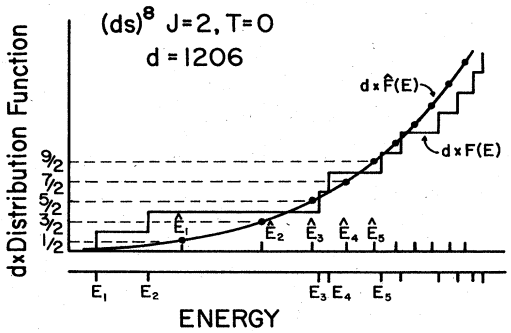


FIG. 5. An exact distribution function $F(E)$, its smoothed approximation $\hat{F}(E)$, and the spectra E_i and $\hat{E}_i$ which they represent. The example shown is that of Fig. 4(b).

dimensional shell-model spectrum, the smoothed spectra derived from it, and the residual level-to-level fluctuations. The first inset figure gives a simple histogram of the spectrum and the corresponding Gaussian density. In the other figures, which show the level-to-level deviations, the cut-off parameter $\hat{\xi}$ is varied between two (which defines a Gaussian distribution) and six; we see a very rapid convergence, the "final" rms value for the deviation (~ 0.75 local spacings) being attained with $\hat{\xi}=4$. It is clear, moreover, from the figures that to reduce the deviation significantly would require extremely high-order excitations of the Gaussian, excitations of order comparable with the dimensionality or, better stated, of wavelength comparable with the spacing.

The fluctuations in the spectrum dealt with in Fig. 6 thus appear to have an existence which is independent of the secular behavior. This is completely in line with the result deduced above from the slow-neutron and nuclear-table data that at least the nearest-neighbor spacing distributions survive unaltered, even when the spacing parameter varies over a 10^4 range. The conclusion is supported by the results for other large spectra which show the same rapid convergence of the density *to within fluctuations*; for example, we find for three 839-dimensional $(ds)^{12}$ spectra (^{28}Si, $J=T=0$, with three different effective interactions) that, with $\hat{\xi}=6$, the rms deviations are 0.77, 0.79, and 0.82 mean local spacings. Since the values are similar for the four cases, the possibility is seen that the fluctuations for pure sequences always display the same pattern (Dyson, 1972b). We shall be able in Sec. IV to give an analytic treatment of this phenomenon.

E. Strength fluctuations and collective behavior

Besides the energy-level fluctuations we must consider fluctuations in the strengths (or "widths") for the decay modes of the excited nuclei. The basic assumption used here is that of Porter and Thomas (1956; see also Scott, 1954) that the transition matrix elements (whose squares are proportional to the strengths) connecting states in one narrow energy band with those in another may be treated as similar independent zero-centered Gaussian random variables; one assumes further that the same is true for the states in a wider energy band as long as the strengths are expressed in terms of their local average. It is clear now that if we are able to take into account the secular variations of the strengths, we have the necessary ingredients for a theory of the strength fluctuations.

A general discussion of this can wait until Sec. VII. We content ourselves now with very briefly describing evidence for a surprising connection between statistical and collective behavior (Draayer *et al.*, 1977). This starts with the proposition that the Porter-Thomas description of the transition amplitudes is more or less valid even down to the ground-state domain (in correspondence therefore with the behavior of the energy-level fluctuations). This derives from inspection of a limited number of shell-model calculations and is supported by an analysis of some of the data; it would be predicted on the basis of ensemble calculations given in Sec. VII (which, however, pay no attention to the presence of strong collectivities).

Consider now, not the individual strengths, but the sum of the strengths which originate with a given starting state, and ask how this sum varies as we vary the starting state over an energy band. If the strength from every starting state splits into ν equal parts, we would get a χ^2_ν distribution whose centroid $\mathcal{S}_s$ and variance σ_s^2 are related by $\sigma_s^2=2\mathcal{S}_s^2/\nu$.

It is elementary to extend this result (as done in Sec. VII) to take into account the secular variation of the strength sum with starting-state energy, and the fact that the strength from a given state does not split evenly among the final states. We then find the *effective* number of accessible states (an analog of the effective number of open channels in reaction theory) to be

$$\hat{\nu}=\frac{2\hat{\mathcal{S}}_s^2}{\hat{\sigma}_s^2}, \tag{1.9}$$

where $\hat{\mathcal{S}}_s$ and $\hat{\sigma}_s^2$ are certain local averages of the strength centroids and variances.

A small value of $\hat{\nu}$, say $\hat{\nu}\sim 1$, defines a situation in which the strength originating in the starting region is very little fragmented, most of it going to a single final state. There are two different cases here; the states in the starting region may have comparable total strengths (but all unfragmented), or there may be a dominant starting state, from which originates most of the strength of the region considered. In the latter case, which is more dramatic, there is a single very strong transition, while in the former, there may be many fairly strong ones. But each case we would recognize as displaying a strong collectivity.

Given the parameters of the model spaces and Hamiltonians, the trick now is in the actual evaluation of $\hat{\mathcal{S}}_s$ and $\hat{\sigma}_s^2$, which depend, of course, on the smoothed strength and the smoothed square of the strength. Good methods have recently been developed (Draayer *et al.*, 1977), and tested for calculating these quantities in terms of certain traces (or integrals) which can be evaluated directly from the parameters of the model space and the Hamiltonian. In conjunction with (1.9) they then yield an explicit statistical method for predicting certain types of collective behavior. Figure 11 in Sec. VII shows two shell-model examples, for $E2$ and $M1$ transitions in the ground-state region, in which the test is quite successful, strong collectivity being predicted by (1.9) and found via the explicit shell-model calculation. It is successful also for $E4$ transitions; (1.9) predicts that there should be no significant collectivity, and the calculations show none.

It is too early yet to say whether (1.9) will be generally useful for the prediction of collectivity. It derives, after all, from the Porter-Thomas law, which we might expect to be disturbed by the existence of strong collectivities; it is not, then, *a priori* obvious that its use in such cases is really self-consistent. On the other hand, the only Porter-Thomas feature which is made use of is the relationship (1.9) between the centroid and the variance of the strength distribution; and besides that we have excellent agreement in the shell-model cases. So there are good grounds for optimism.

In any case, it does seem clear that, not only do fluc-

tuations extend to the ground state, but they carry information about phenomena which are usually regarded as highly "nonstatistical." Somewhat similar relationships are found between statistical behavior and symmetries (going beyond the elementary results displayed in Figs. 1 and 2). A general lesson is that we must not think of separating phenomena in complicated systems into a nonstatistical part connected with "real" physics, and another part connected with statistical behavior. In some cases this separation might be appropriate, but, in general, things are both more complicated and a great deal more interesting than that, and we stand to learn a lot from the two kinds of physical behavior and from the interplay between them.

F. Preview

In the next section, following Wigner, we introduce ensembles of Hamiltonians in order to deal with fluctuations; we discuss the extent to which they satisfy certain reasonable criteria for physical significance, and come face to face with certain difficult and not fully solved problems which arise from the fact that mathematically tractable ensembles do not *a priori* appear to be physically most reasonable.

Since we shall often describe fluctuations in terms of random excitations of the smoothed eigenvalue density we give in Sec. III a brief account of level densities. We shall have to consider carefully the meaning of the *characteristic density* for an ensemble; our study of this will give us a feeling for the kinds of interactions which give departures from standard results and eventually lead us to the important notion of ergodicity. The Stieltjes-transform method which is used a little in this section is effective for a wide class of ensemble densities (and for fluctuation properties, as well).

In Sec. IV we derive the two-point correlation function for the Gaussian orthogonal ensemble (GOE) and go on then to derive some basic results about the motions of levels away from their ensemble-averaged positions. In Sec. V we use the results of Sec. IV to derive the standard GOE measures[2] for fluctuations, most of which have been given originally for the central region of the spectrum by Dyson and Mehta (1963); we find new relationships between them and answer some long-standing general questions, for example, about the independence of the measures; these results, along with the ergodicity and stationarity described in Sec. X, go a considerable distance toward explaining why statistical behavior is observable in the ground-state domain. In Secs. IV and V we also deal with Gaussian unitary and symplectic ensembles.

In Sec. VI we discuss the analysis of the modern slow-neutron data and of some of the proton-capture data. We describe briefly the difficulties which arise because of missed or wrongly identified levels and because of the methods used to circumvent them. Values of the standard measures are given for each spectrum. We treat spectra deriving from various large shell-model and Monte Carlo matrix calculations in the same way and discuss the significance of the comparisons between these results and the data. In Sec. VII we do somewhat the same for the strengths, but we give also a more extended treatment than given above of the origins and justification of the Porter-Thomas distribution and of the relationships between strength fluctuations, collective behavior, and symmetries. Data analysis, on the other hand, is considered only briefly.

In Sec. VIII we make somewhat of a diversion from the main lines of the review, giving a compact discussion of the major techniques used, and some of the results derived, for fluctuations in reaction cross sections. We do this in small part because most of the data described earlier have been derived from reactions, so that we should really understand how such continuum phenomena can be described by (discrete) matrix techniques; in fact, this can only be done to a limited extent and for sharp resonances. It is much more important to understand that the same general *ideas* (as opposed to methods) which we have used in the earlier sections have their counterparts here also, in a domain which, for example because of interest in heavy-ion reactions, is growing rapidly in importance. We cannot claim complete success in integrating the two subjects of cross-section fluctuations on the one hand and energy-level and strength fluctuations on the other, but we do emphasize the many ways in which they are in correspondence.

Section IX gives an up-to-date account of the statistical mechanics of metallic powders, a subject which is quite active both theoretically and experimentally, and which has already challenged some of the natural assumptions which one would make in studying such a subject. It seems fairly clear that the low-lying excitation spectrum for the particles involved is a discrete one, so that the essential physics at low temperatures is determined by the first few spacings. Are the levels randomly spaced, do they exhibit a Wigner repulsion, or is the spectrum of a different nature than either of these possibilities? It is easy enough to calculate the thermodynamic properties for a collection of particles with any given type of spectrum; but there has not developed any real agreement concerning the class of spectra which should obtain; the intuitions and calculations of different authors have given rise to quite different results. It seems, indeed, that one must not even take for granted that there is any really appropriate random-matrix ensemble, a good thing to keep in mind when dealing with other problems.

Section X is devoted to a simple discussion of ergodicity. In order to justify the use of ensemble averages for dealing with the properties of single systems it must be shown that, with respect to the quantities being compared (strength distributions, fluctuation measures, and so forth), the ensemble has an appropriate ergodic behavior. Only very limited attention has been heretofore paid to this important question; however, as the result of recent work (Pandey, 1979, French and Pandey, to be published); the essential results for the conventional ensembles are now available. They show that a strong "locally generated" ergodicity obtains and that the fluc-

[2]By "fluctuation measure" we mean "statistic," a quantity which can be calculated from empirical data and used for testing the theory. The reader should be warned not to confuse this with "measure" as used in integration, for which we use "weight."

tuation measures in particular are also stationary; the corresponding spectra are then closely analogous to the output of an ergodic stationary random process. Some attention is also given in Sec. X to ergodicity in reactions.

In the final section we review the ways in which random ensembles enter into physics, comment on the general concepts and results which seem most significant, and mention some outstanding problems and methods which should be useful for dealing with them.

II. RANDOM-MATRIX ENSEMBLES

A. Introduction

In this section we introduce ensembles of Hamiltonians, which, in contrast to the nuclear-table ensemble, label their members by continuous parameters; for these ensembles, moreover, physical requirements do not specify any obvious unique weighting.

If it is found that a certain measure of the fluctuations varies across the ensemble by only a small amount, then that measure *may* be useful in studying experimental fluctuations; it certainly *will* be useful, provided also that (1) the ensemble-averaged value for the measure is calculable analytically (and its variance, as well, for otherwise we would not in general know that the distribution of the measure is peaked sharply enough), and (2) the ensemble contains, with a "reasonable" weighting, model Hamiltonians which we know from experience to give a good description of general features of the system.

With regard to the first condition, numerical matrix diagonalizations are not adequate for a general understanding. As for the second, it is not easy to define a "reasonable" weighting of satisfactory Hamiltonians, but even if there is no such reasonable weighting in the given ensemble, its fluctuation results might still describe the experimental situation; if this is so for various measures and for a number of cases which we study, we might well be prepared to use the ensemble for other measures and in other cases. But there would then be a gap in our understanding, a problem to be solved, namely, why unreasonable Hamiltonians give proper results. We stress that this is a problem but not necessarily a mystery, for it is common in statistical physics that specified quantities are independent of many of the parameters of the system. This consideration is relevant, for we shall see that the Gaussian orthogonal ensemble (GOE), which has generated most of our understanding of energy-level fluctuations, is made up of quite unreasonable (multibody-interaction) Hamiltonians, the probability of coming upon a reasonable one being negligible and vanishing in the asymptotic limit usually considered, that of large model-space dimensionality.

Our ensembles will all be of matrices; except for Sec. VIII we pay no attention to the continuous spectrum, even though, strictly speaking, most of the experimental data come from resonance states in the continuum. Certain conditions on the matrices will be imposed by the invariance properties of the Hamiltonian; in particular, for systems which are time-reversal invariant and, besides that, are either rotation invariant or of integral angular momentum, the matrices may be taken to be real symmetric (this in fact is the most important case). Other requirements, which for the most part have been considered only during the past few years (French and Wong, 1970, 1971; Bohigas and Flores, 1971a, 1971b; Dyson, 1972b; Wong and French, 1972; Brody *et al.*, 1972), and parallel with an increase in the quality of the slow-neutron data (Camarda *et al.*, 1972, and later papers of the Columbia group), are much more involved with the dynamics of the system and with quite general conditions which should be satisfied if the statistical description is to be appropriate. Adapted in part from Dyson (1972b), the other requirements might be taken as follows (though they are obviously not independent of each other).

1. *Connection with dynamics.* The definition of the ensemble should be based on the dynamics of the system (or at least on the general principles of many-body theory). We may ask, moreover, that, as in conventional statistical mechanics, the ensemble deal in the least biased fashion with the available relevant information.

2. *Significance of nonstatistical parameters.* Any parameters in the definition of the ensemble should be linked to nonstatistical properties of the system to be described; this then allows adapting the general model, for example, to a specific nucleus. We have seen in Sec. I that the secular behavior depends only on a small number of low-order moments. In some cases these may then be taken as the nonstatistical parameters; they are, in fact, the concern of much recent work on spectral distributions.

3. *Ergodicity.* The statistical properties of individual members of the ensemble should almost always coincide (to within suitably narrow error bounds) with the ensemble average. Such behavior ensures that ensemble predictions can be used for individual systems.

4. *Physical relevance.* The ensemble averages should reproduce the statistical properties as observed experimentally.

5. *Mathematical tractability.* It would be good if the ensemble were mathematically tractable, so that analytical results might be obtained. There is in fact a conflict between reasonableness of the Hamiltonians and mathematical tractability of the ensembles, one which has usually been resolved in favor of the latter; thus, as indicated above, the fundamental (and tractable) ensemble, the GOE, contains almost exclusively quite "unreasonable" Hamiltonians.

Points (1), (2), and (3) are necessary for a physically reasonable ensemble; point (4) means that it will also be successful. Point (5) is not an absolute necessity, since one can always resort to Monte Carlo calculations, but there are obvious shortcomings in that. As we proceed it will be seen that considerable progress has been made in satisfying these requirements.

We now have basically two ways of starting. In the first we examine the statistical behavior of a complicated system as *generated* by its microscopic (many-particle) structure; in the second, which is the usual one, we ignore that to begin with, essentially *postulate* a statistical behavior by adopting a particular ensemble, and ask later about the connection with the microscopic

structure. We start with the first, though we shall have to proceed quickly to the second.

B. Embedded ensembles; the statistical extension of the shell model

Let us consider m particles in a set of orbits which correspond to N single-particle states; then a two-body interaction is completely defined once we have the values of the antisymmetrized two-particle matrix elements $H_{\Lambda\Lambda'}$, connecting two-particle states Λ, Λ', with the same angular momentum and parity (and, e.g., isospin where it is relevant). Given the $H_{\Lambda\Lambda'}$, the matrix elements for the m-particle system can be calculated by standard techniques as

$$H_{ij}(m)=\langle m;i\,|H|\,m;j\rangle=\sum_{\Lambda,\Lambda'} C_{\Lambda\Lambda'}^{mij} H_{\Lambda\Lambda'}\,, \tag{2.1}$$

where the C's are "geometrical" coefficients, characteristic of the system, but independent of the interaction. The two-particle matrix which defines the Hamiltonian may then be described as "embedded" in the m-particle matrix, in the sense that the matrix elements of the larger system are linear combinations of those of the smaller. The m-particle matrix breaks up into diagonal blocks characterized by the exact symmetries; we concentrate on a typical block. As we saw in Sec. I, if we should take more than one such block, the corresponding spectra—which we take to be statistically independent—may be superimposed; if there are enough blocks then, with none predominating, a central-limit theorem comes into operation, which for the spacing distribution, to take the best-studied example, yields a Poisson distribution, whatever the single blocks show (Leff, 1964a; some special cases are studied by Gurevich and Pevsner, 1957; Lane, 1957; and Mehta, 1967, Appendix A22).

The two-body random ensemble (TBRE; French and Wong, 1970; Bohigas and Flores, 1971a) follows when we take the $H_{\Lambda\Lambda'}$ as real random variables, independently distributed according to the same zero-centered law. The exact form of this law becomes unimportant when the particle number is large, but, as we shall see below, a Gaussian form is to be favored. Statistical independence of the $H_{\Lambda\Lambda'}$, on the other hand, is of consequence; with a natural weight in the matrix-element space it makes equally probable all two-body interactions which preserve the general symmetries we have accepted. Note very carefully, however, that the m-particle matrix elements ($m>2$) are by no means independent; for instance, the ($J=3$, $T=1$) matrix in the nuclear ds shell with 12 particles has a dimension of 6706, so that there are 22 488 571 different matrix elements, of which only 63 are independent.

No significant progress has been made in dealing analytically with the TBRE. Large-scale Monte Carlo calculations have, however, been carried out, the matrices being constructed, element by element, in accordance with the distribution law, and then diagonalized to produce the complete spectra, from which the fluctuation measures are also calculated. One finds for these ensembles, in agreement with a great variety of nuclear shell-model calculations for "realistic" interactions (i.e., those whose dominant features are compatible with nucleon-nucleon scattering), that the spectra are close to Gaussian. Results for the fluctuation measures will be discussed, along with other "data," in Sec. VI. In general terms, the TBRE comes close to satisfying the criteria for a physically satisfactory ensemble, failing, however, in the last, that of mathematical tractability.

The most severe mathematical difficulties with TBRE are due to the angular momentum constraints. These difficulties are present also in an ensemble, TBRE(J_z), studied by Yépez (1975), in which J_z instead of J^2 is conserved; but the numerical calculations are easier. Its fluctuation properties seem to be identical to those of the TBRE, a result no longer surprising. Another type of ensemble (French, 1973), much closer to being mathematically tractable, abandons the J restrictions entirely. This eases the mathematical difficulties sufficiently for a useful generalization to be possible: Instead of supposing simply two-body forces, as for the TBRE, k-body forces can be considered, with $k\geq 1$. Equation (2.1) will still apply, but now with $H_{\Lambda\Lambda'}$ the k-body matrix elements in any k-particle basis. If we take them to be real symmetric and choose them from a zero-centered Gaussian[3] distribution with diagonal variances twice the nondiagonal ones, we have, in the k-particle space, what we shall call a Gaussian orthogonal ensemble (GOE) and, for m-particles, an example of an embedded GOE, or EGOE for short. For $m=k$, of course, the two ensembles coincide.

Physically, $k=2$ is the case of greatest interest (or perhaps $k=1+2$), though, in fact, it seems likely that the fluctuation measures are essentially independent of k for $k>1$ (interacting particles). Short-range fluctuation measures have not yet been analytically derived for this EGOE, but its secular properties have been (for $m\gg k$ the characteristic spectrum is Gaussian), as well as its long-wavelength fluctuation properties and its ergodic behavior for the eigenvalue density. The details will be discussed in later sections.

C. The Gaussian orthogonal ensembles (GOE)

The one significant feature of the Hamiltonian which is preserved in the EGOE but not in the GOE itself is the fact that the H's define k-body interactions. If we ignore this feature (because of the mathematical difficulties which it generates and because it seems to have little effect on the fluctuations, anyway—though a profound one on the density) we come to the GOE itself as a fluctuation model.

There is a GOE for each dimensionality d. We start with $d=2$. Then any ensemble of real symmetric H's is specified by giving the joint probability distribution for the three independent H_{ij}. The eigenvalue spacing S is given by

$$S^2=(H_{11}-H_{22})^2+(2H_{12})^2\,, \tag{2.2}$$

whose vanishing specifies two conditions. From the

[3]The asymptotic results depend only very weakly, or not at all, on the shape of the distribution. Wigner's semicircular density [(3.17) below] was first derived for the "random-sign ensemble," which has elements all of unit magnitude but of randomly chosen sign.

remarks following (1.5) we see that if we form an ensemble by taking $(H_{11}-H_{22})/2\equiv x_1$ and $H_{12}\equiv x_2$ to be independent random variables, with distribution identical to that of $G(0,v)$, we shall find the Wigner result (1.5) for the *ensemble* distribution of the spacings; the average spacing D and the variances v^2 of x_1, x_2 are then related by $D=\sqrt{2\pi v}$. Though "derived" in Sec. I by an argument involving a single matrix of (large) dimensionality d, it is reproduced here as the distribution of the spacings which are found, one from each matrix, in the ensemble of two-dimensional matrices. Thus by comparison we see that this ensemble may be regarded as implying a linear repulsion. See Appendix B for a more general result.

If we complete the specification of the matrices by taking the eigenvalue centroid $x_3=(H_{11}+H_{22})/2$ also to be independent, the ensemble is isotropic, i.e., invariant under orthogonal transformations of the x_i basis (since the joint density function and the volume element $dx_1dx_2dx_3=dH_{11}dH_{22}dH_{12}$ are both invariant). This ensemble is the two-dimensional Gaussian orthogonal ensemble (GOE), which we can equally well describe by stipulating that the H_{ij} are independently chosen, the diagonal elements as $G(0,\sqrt{2v})$ and the off diagonal as $G(0,v)$; in verifying this, recall that variances add for independent variables. The linear-repulsion factor may be described in geometrical terms as arising from projecting this distribution onto the (x_1,x_2) plane which carries the information about the spacings; specifically, we see from (2.2) that $S/2$ is the radial coordinate in the plane, so that the factor giving the linear repulsion derives from the two-dimensional volume element $2\pi SdS/4$. Observe also that we may take the x_3 distribution to be anything we wish without changing the spacing distribution. Thus we have a (trivial) class of ensembles which give the same spacings.

These considerations do not at all *prove* that there should be level repulsion. A different probability weighting for the matrices may give different results. For example, every two-dimensional Hermitian matrix can be written in terms of the unit matrix **1**, the Pauli matrices $\boldsymbol{\sigma}$, and the real unit vector **n**; and its eigenvalues will be independent of **n**. Thus

$$H=\alpha\mathbf{1}+\eta\boldsymbol{\sigma}\cdot\mathbf{n}\underset{n_z=1}{\longrightarrow}\begin{bmatrix}\alpha+\eta & 0\\ 0 & \alpha-\eta\end{bmatrix}, \tag{2.3}$$

so that $S=2|\eta|$ is distributed as the positive half of the marginal distribution of η. With Gaussian, for example, small spacings predominate and we have no level repulsion. This ensemble and the GOE contain the same matrices but with different weights and it is this which gives rise to the differing results.

To define the d-dimensional GOE we again take the matrices to be real symmetric, with distinct matrix elements independently distributed according to

$$\begin{aligned}&H_{ij}\sim G(0,v),\quad i\neq j\\ &H_{ii}\sim G(0,\sqrt{2}v).\end{aligned} \tag{2.4}$$

For the GOE normalization we take $v^2d=1$. It is convenient, however, to introduce the parameter $\beta=1,2,4$ for orthogonal, unitary, and symplectic ensembles (Sec. II.D below) and then (with $v_1^2=v^2$ for GOE)

$$\beta v_\beta^2 d=1. \tag{2.5}$$

Then, for GOE we have the result (which would be $d-1+2/\beta$ in the general case)

$$\begin{aligned}\overline{\mathrm{Tr}(H^2)}\equiv[\mathrm{Tr}(H^2)]_e&=\sum_{i\neq j}H_{ij}^2+\sum_i H_{ii}^2\\ &=d(d-1)v^2+2v^2d=d+1\end{aligned} \tag{2.6}$$

so that the ensemble- and spectral-averaged H^2 matrix element (i.e., the mean-squared energy) will be $(1+1/d)\to 1$ for large dimensionality.* It will turn out that for large d, the (smoothed) spectrum is semicircular (see Sec. III); and, since for a semicircle of radius R the variance is $\sigma^2=R^2/4$, it follows that the radius is 2.

The probability density for a matrix, in the space of the independent matrix elements (volume element $=\prod_{i\leq j}dH_{ij}$), is now, with $\beta=1$,

$$P_\beta(H)=2^{-d/2}(2\pi v_\beta^2)^{-1/2[d+\beta d(d-1)/2]}\exp[-\mathrm{Tr}(H^2)/4v_\beta^2], \tag{2.7}$$

and, being dependent only on a trace, it is invariant under orthogonal transformations. We remark also that the statistical independence of the matrix elements and orthogonal invariance of the ensemble are sufficient to define the GOE (Friedrichs and Shapiro, 1957; Porter and Rosenzweig, 1960).

Occasionally it seems to be taken for granted that the necessary invariance of physical results under orthogonal (more generally, unitary) basis transformations implies a corresponding necessary invariance for ensembles, but this is not at all the case (see Appendix C). Orthogonal invariance describes a "physical uniformity," which implies that with respect to the ensemble of H's, all basis states, and therefore all states, behave in the same way. From a physical standpoint, statistical independence and the uniform treatment of the matrix elements imply the avoidance of matrix-element correlations which would generate special features of the Hamiltonian. From the statistical standpoint we would be led to introduce this kind of uniformity if we were ignorant of all features of the system except those defined by the underlying symmetries implied by GOE. Other ensembles defined with more restrictions would be appropriate if further information were available.

Starting from this point of view, Balian, whose work involves an extension and more systematic treatment of notions first used in this domain by Bronk (1965) and Porter (1965b), has given a prescription for deriving ensembles in terms of the standard concepts of information theory (Balian, 1968).[4] The basic quantity in Balian's formulation is the information functional

*Here and throughout, the overbar will show the ensemble average. Owing to typesetting limitations, we use the notation $[(\mathrm{Tr}H^2)]_e$ interchangeably.

[4]See also the conference report (Balian, 1969) and the especially interesting discussion which follows it.

$$I\{P(H)\} \equiv \int d(H)P(H)\ln P(H) = [\ln P(H)]_e , \tag{2.8}$$

where, as discussed by Porter (1965b), $d(H)$, the volume element in the matrix space (= $\prod_{i<j} dH_{ij}$ for real symmetric matrices, as given above), is defined in terms of a metric $ds^2 = \mathrm{Tr}(\delta H \delta H^*)$. Then the probability distribution $P(H)$ for a random matrix constrained to satisfy some given conditions is determined by a principle of maximal randomness, namely, that the information $I\{P(H)\}$ is minimal, corresponding to the least biased distribution. The constraints are generally expressed by specifying the ensemble averages θ_i of some quantities $f_i(H)$ depending on the random matrices H:

$$\bar{f}_i \equiv \int d(H)P(H)f_i(H) = \theta_i . \tag{2.9}$$

In this case minimization yields

$$P(H) = \exp\left(\sum_i \lambda_i f_i(H)\right), \tag{2.10}$$

where the Lagrange multipliers λ_i are determined by adjusting (2.10) to (2.9). The normalization of $P(H)$ is, of course, always one of the constraints; if, in the space of real symmetric matrices, the only other constraint is the "strength" of the interaction, as defined by $[\mathrm{Tr}(H^2)]_e$, we obtain the GOE, in which the orthogonal invariance is then seen to follow as a consequence of the postulates. All the other ensembles that we consider could be derived in this formalism.

Every real symmetric d-dimensional matrix is found in the GOE, and thus every matrix is accompanied by all of its equivalent matrices, OHO^{-1} with orthogonal O; these enter with the same weight and of course have the same spectrum. If now Ψ_r is the state associated with the eigenvalue E_r of H, then $O\Psi_r$ will be the eigenstate for E_r of OHO^{-1}, and the complete set $\{O\Psi_r\}$, seen as vectors in a d-dimensional space, will uniformly cover the unit sphere. We may call Ψ_r a "uniform" vector; every GOE eigenvector is uniform. The uniformity is maintained if, for example, we select from the ensemble only those matrices for which some or all of the eigenvalues are specified, or more generally if we select a subset by any procedure which is independent of the Hamiltonians themselves.

The space of the matrix elements is $\binom{d}{2}+1$ dimensional, while that of the eigenvalues E_i is only d dimensional. Let us then transform from the matrix-element representation to a "diagonal" representation, defined by the eigenvalues $(E_1, \ldots, E_d)$ and the parameters $[p_1, \ldots, p_l$ with $l = \binom{d}{2}]$, which generate the orthogonal transformations. Since

$$P(H) = C\exp\left(-\sum_i E_i^2/4v^2\right),$$

the weight of the matrices in the new basis depends on the p_s only via the Jacobian of the transformation $\{H_{ij}\} \to \{E_r, p_s\}$. Then we can integrate over the p_s and find the joint probability distribution in the eigenvalue subspace. The result of this, as carried out very simply by Wigner (1967), is, again with $\beta = 1$ for GOE, though, just as with (2.7), the result applies for the other β values,

$$P_\beta(E_1, \ldots, E_d) = C_\beta \prod_{r<s} |E_r - E_s|^\beta \exp\left(-\frac{1}{4v_\beta^2}\sum_i E_i^2\right). \tag{2.11}$$

In one sense this solves the entire GOE problem, since, for example, by integrating over all the variables but one, we find the density, over all but two, the two-point correlation function, and so forth. But here very difficult technical problems are encountered. These have been solved to a considerable extent by Wigner, Mehta, Dyson, and many others [see the papers collected by Porter (1965a), as well as Wigner (1967), Mehta (1967, 1971), and Dyson (1970)] by mathematically elegant methods which depend rather critically on the invariance properties of the ensemble. In Secs. IV and V we shall use a simpler method, not so dependent on these properties, to calculate approximately the fluctuation measures which we shall need.

We know now that the last criterion of Sec. II.A, that of mathematical tractability, is pretty well satisfied. This, in fact, is the only one of these criteria which has been considered in constructing the GOE. We shall see in Sec. X that the ergodicity requirement is also satisfied. The other requirements have been ignored entirely; the GOE has only two free parameters, a trivial scale factor, and the dimensionality which is hard to connect with experiment. Nevertheless, as will appear in Secs. V to VII, the fluctuation behavior of the GOE agrees well with experimental observations.

The GOE, when realized in a many-particle system, describes a situation in which all the particles interact simultaneously,[5] the other interactions forming a subset of negligible weight (zero in the limit) for large matrices. Besides that, the uniformity which we have imposed is incompatible with special features which characterize real systems, such as, for nuclei, the importance of a quadrupole-quadrupole interaction, or the occurrence of giant-resonance states whose excitation modes are very different from those of their neighbors. Since these features of the GOE are *a priori* unreasonable, it is important that we understand the significance of a comparison of GOE results with experiment. If they are *not* in agreement, the natural step is to construct ensembles which better satisfy the criteria listed above. If, on the other hand, they *are* in agreement, this implying that the fluctuations are independent of particle rank and other properties of H, the significant problem is to find the reason for success and the range over which we may expect it. Stated differently, the problem is to find what systems give rise to GOE fluctuations and to find when deviations may be expected; these deviations, in fact, would have to be large in order to overcome the (theoretical) variations represented by the variances of the fluctuation measures and those arising from small sample size—small both in the

[5]This is because the GOE ignores the constraints that H be $(k<m)$-body. If the m particles are distributed over N single-particle states, the dimensionality is $d(m) = \binom{N}{m}$, and the number of independent matrix elements is $[\frac{1}{2}d(m)] \sim \frac{1}{2}d^2(m)$; but, since a k-body interaction is realizable in a k-particle space, the number of *its* independent matrix elements is $d^2(k)/2$, which in practical cases will be very much smaller unless $k = m$ (see also Appendix D).

number of levels accessible in a given nucleus and in the number of nuclei for which data are available. In the case of GOE success, we would find ourselves studying, not properties of specific nuclei, but instead more general laws which might well apply to very different systems.

In either case the appropriate procedure appears to be the same, namely, to study other ensembles and to seek simpler descriptions of the fluctuations for these ensembles, as well as for the GOE.

D. Gaussian ensembles and underlying symmetries; the GUE and the GSE

We have commented on the underlying space-time symmetries which, when preserved, permit us to take the Hamiltonian matrices to be real symmetric. The orthogonal invariance of the ensemble then follows from the fact that the real-symmetric property is preserved under orthogonal transformations, but under no larger subgroup of the unitary transformations, and from the desire to have a uniform measure for the ensemble members. The latter is analogous to the notion of *equal a priori probability* in statistical mechanics. Different ensembles result if the conditions which give rise to the orthogonal ensemble are not satisfied. In particular, if the Hamiltonian is not time-reversal invariant, then, irrespective of its behavior under rotations, the Hamiltonian matrices are complex and the canonical group which generates the uniform measure for the ensemble is the unitary group. If we have time-reversal invariance, but not rotation invariance, then, for systems with half-integral angular momentum, the H matrices are "quaternion real" and the ensemble is generated by symplectic transformations.

These general results are due originally to Dyson (1962a); a more detailed derivation is given by Porter (1965b). In an elegant but difficult later paper Dyson (1962e) establishes connections between the mathematics of the three ensembles (which, in a sense which he defines, are the fundamental "irreducible" ones; hence his "threefold way") and other mathematical structures, and states the theorem which gives rise to them.

Hence we have Gaussian orthogonal, unitary, and symplectic ensembles,[6] GOE, GUE, and GSE, characterized, respectively, by $\beta=1,2,4$. For the GUE, $H=H_0+iH_1$, with $H_0=H_0^*=H_0^T$ and $H_1=H_1^*=-H_1^T$, the two matrices being independently constructed and their off-diagonal matrix elements having variances v_2^2. For GSE we have an ensemble of "quaternion real" matrices

$$H=H_0 1-i\sum_{k=1}^{3} H_k\sigma_k,$$

with $H_0=H_0^*=H_0^T$, $H_k=H_k^*=-H_k^T$ for $k=1,2,3$; the four matrices are independent and their off-diagonal matrix elements have variances v_4^2. Equation (2.7), with the varying values of β and the volume element

$$\left(\prod_{i=1}^{d} dH_{ii}\right)\left(\prod_{i<j=1}^{d}\prod_{k=0}^{\beta-1} dH_{k,ij}\right),$$

gives the joint probability density in all cases. Similarly, the eigenvalue distribution is given by (2.11).

The asymptotic eigenvalue density and the correlation functions of general order have been derived in all three cases [Dyson (1970); Mehta (1971); see also Porter (1965a) and Mehta (1967)]. We shall give, in Secs. IV and V, good approximations for the two-point functions and fluctuation measures.

A sum of a GOE and an independent, similarly defined, ensemble of Hermitian *antisymmetric* matrices has been introduced to study the breaking of time-reversal invariance (Wigner, 1967; Favro and McDonald, 1967; Rosenzweig *et al.*, 1968); we come to that in Sec. VII.F. Such an ensemble describes a GOE → GUE transition, induced by moderating the GOE constraint on the matrix elements by increasing the magnitude of the antisymmetric part. The transition will be seen to be strikingly unsymmetrical due to the fact that there is no constraint at the GUE end. A similar ensemble would generate a GSE → GUE transition which would also be unsymmetrical (unlike GOE → GSE, for which there is a constraint at both ends). All three of these are special cases of the deformed ensembles of Sec. II.G.

E. Gaussian ensembles with limited traces

In the GOE there is an obvious factoring of the "angular" and "radial" structures, as in (2.11). The GOE can thus be thought of in terms of shells defined by $\lambda_0 \leq \lambda \leq \lambda_0+\Delta\lambda$, where $\lambda^2=\mathrm{Tr}(H^2)$, so that λ defines the "magnitude" of H; similarly for the GUE and GSE. The fluctuations, being expressed, as always, in terms of local spacing units, are then independent of λ, so that each shell has the same renormalized spectra and the same fluctuations. The fixed-trace ensemble of Rosenzweig (1963) deals with a single GOE shell, i.e., with a single shell in the space of Hamiltonian matrix elements where the GOE is defined, and is thus analogous to a microcanonical rather than a canonical ensemble. It is therefore equivalent to the GOE for fluctuations calculated by spectral averaging (i.e., averaging along the spectra of individual H's, also called "energy averaging"). This does not imply exact equality for the theoretical fluctuations obtained by means of *ensemble* averaging, but any departure from equality would then be due to a failure of ergodicity in one or both of the ensembles (in fact, in the fixed-trace ensemble). Mehta (1967) has shown, however, that for large dimensionality the two ensembles give the same ensemble-averaged results; since GOE ergodicity obtains, anyway, only for large d, we can therefore say that, insofar as physical applications are concerned, the fixed-trace ensemble is identical with the GOE. The same is true for the bounded-trace ensemble of Bronk (1964).

F. The circular ensembles

The treatment of the Hamiltonian magnitudes has been handled in a different and very elegant way by Dyson (1962a), who indirectly introduces a weight in which all H's (not just those with the same renormalized spectra) are treated identically. The starting point of this is a mapping of the Hamiltonian onto a unitary matrix S, for example, by $S=e^{i\alpha H}$. Then, as long as we consider

[6] We shall often refer to them as the "standard" or "canonical" ensembles.

domains in which the mapping has no zeros or singularities (being therefore invertible), we can, by Taylor expansion, relate the original and mapped spectra; and these, to lowest order in the expansion, are identical to within a scale factor. The fluctuations can then be studied in terms of the eigenvalues of unitary matrices which, with appropriate weighting and with symmetry restrictions, constitute the *circular* ensembles (circular, since the eigenvalues all lie on the unit circle).

An important general result (Dyson, 1970; Mehta, 1971) is that, in the large-d limit, the Gaussian and circular ensembles give the same fluctuations; this should not be very surprising in view of the general comments above. The fluctuation properties of the circular ensembles are stationary by construction. Moreover, we shall show in Sec. X that the Gaussian ensembles have an ergodicity which is "locally generated," so that, for large enough d, we can ensure agreement as close as we wish between spectral and ensemble averaging, while still averaging over an energy domain as small as we wish. Then, because of the local mapping relationship between the ensembles, the ergodic behavior of one class will imply that of the other.

G. Deformed ensembles

By construction the set $\{H\}$ of operators which forms a GOE (or one of the other canonical ensembles) is isotropic with respect to the zero operator as center. If we know, however, that the true H has in it a large component of a particular operator K, we might question the adequacy of $\{H\}_{\rm GOE}$ as a model. We could then introduce a new "constrained" ensemble, in which H varies not over the entire GOE but over the surface of a cone within the GOE; the axis of the cone would lie along K, and the angle of the cone would then determine the magnitude of the K component in H relative to that of H itself. But the ensemble $H_\alpha = H + \alpha K$, where α is a real constant and H the GOE, achieves the same result much more simply, since the relative magnitude of any given component of the Hamiltonians is negligibly small in the GOE. We shall speak of the H_α ensemble as a "deformed" GOE (though "off-centered" or "centroid-modified" might be better). See Secs. III.E and IV.G for formal results and specific applications.

K might correspond to a weak deformation of a canonical ensemble, and, if then it does not preserve the ensemble symmetry specified by $\beta = 1, 4$, we have the transition between canonical fluctuations, as discussed above. If αK is "large" compared with H, we might be taking account of the damping of collectivities, the αK operator representing perhaps a "schematic" Hamiltonian ("pairing + quadrupole," to take a simple example) and the GOE those parts of the Hamiltonian which, for the purposes at hand, need not be specified in detail. The deformation of a partitioned ensemble, which describes the combination of independent subsystems, or the Hamiltonian, which preserves a given model symmetry defined by the partitioning structure, is of obvious interest in describing the subsystem interactions or the breaking of the model symmetry; and there are, of course, other kinds of applications.

With an *infinitesimal* deformation we find a simple equation which implicitly forms the basis for our treatment of fluctuations in Sec. IV. Let $M_p(\alpha)$ be the pth moment of $\rho_\alpha(x)$, the eigenvalue density for $H + \alpha K$. Then, with $d \times \langle G \rangle$ denoting the trace of G, a convention we shall always use,

$$\begin{aligned} M_p(\alpha) &= \langle (H + \alpha K)^p \rangle = \sum_t \alpha^t \langle \{H^{p-t} K^t\} \rangle \\ &= \langle H^p \rangle + \alpha p \langle H^{p-1} K \rangle + \alpha^2 \langle \{H^{p-2} K^2\} \rangle + \cdots \\ &= \sum_s \alpha^s M_p^{(s)} , \end{aligned} \tag{2.12}$$

where $\{\ \}$ denotes the sum over all possible orderings, $\binom{p}{t}$ of them, of H and K in the product. Recalling that $\langle Q \rangle = \int \rho(x) Q(x) dx$, we find by partial integration that $\langle H^{p-r} G \rangle$ is the pth moment of

$$\frac{(-1)^r p!}{(p-r)!} \left(\frac{d}{dx} \right)^r [G(x) \rho(x)] ,$$

where $\rho(x)$ [$\equiv \rho_{\alpha=0}(x)$ in the notation above] is defined by the moments $M_p \equiv M_p(0) = \langle H^p \rangle$, and $G(x) = \langle x | G | x \rangle$, the expectation value of G in the state $|x\rangle$. Using this we have

$$\begin{aligned} &\rho_\alpha(x) = \rho(x) - \alpha \frac{d}{dx} [K(x) \rho(x)] + \cdots , \\ &\left(\frac{\partial F_\alpha(x)}{\partial \alpha} \right)_{\alpha = 0} = - K(x) \rho(x) . \end{aligned} \tag{2.13}$$

The latter equation has been used, *inter alia*, for calculating expectation values, spin-cut-off factors in level densities, and so forth (Chang and French, 1973). Since it is valid for an individual H, and thus for the separate members of an ensemble, it may be combined with itself to give a two-point equation [(K3) of Appendix K] which leads to the fluctuations.

H. Other ensembles

In another extension of the concepts underlying the GOE, Dyson (1962d; see also Wigner, 1959) writes (2.11), with a different normalization and extended to the three basic Gaussian ensembles, as

$$P_\beta(E_1, \ldots, E_d) = C_\beta \exp(-\beta W) , \tag{2.14}$$

where

$$W = \tfrac{1}{2} \sum_{i=1}^{d} E_i^2 - \sum_{i<j} \ln |E_i - E_j| \tag{2.15}$$

can be interpreted as the energy of d equal line charges at positions E_i along an infinite line, the first term being a harmonic oscillator that serves to confine them, and the second their interaction potential. If now we place the line charges at random positions along the line and let the system evolve under the joint action of (2.15) together with a fluctuating and a dissipative force which give rise to a Brownian motion, then, after equilibrium has been reached, the positions at any instant will be described by the Gaussian ensemble appropriate to the value of β chosen. In the second paper Dyson (1972b) carries the idea further and supposes instead of (2.15) a potential of the form

$$W = \sum_{i=1}^{d} \int r(\lambda) \ln |\lambda - E_i| d\lambda - \sum_{i<j} \ln |E_i - E_j| , \tag{2.16}$$

where $r(\lambda)$ is an otherwise arbitrary non-negative smooth function. This leads to the same local fluctuations but with $r(E)$ as the prescribed density. Ensembles with specified density have been considered also by Balian (1968). The essential decoupling of the secular behavior and the fluctuations of the spectrum make it far from surprising that such ensembles can be constructed.

The fact that the GOE spectrum has (formally) no lower bound might be regarded as a defect; to remedy this, Wigner (1972) proposed an ensemble of matrices $H=\mathrm{Re}M^{*}M$, where M is an asymmetric complex matrix whose real and imaginary parts sample a Gaussian distribution of unit variance. Dyson (1971; see also Bohigas *et al.*, 1971), observing that if $M=A+iB$, then $H=A^{T}A+B^{T}B$, generalized this to $H=\sum_{i=1}^{k}A_i^{T}A_i$, where T denotes the transpose, and evaluated the density, which is found to have effectively both an upper and lower bound. We shall give in Sec. III.E a different derivation of this density; as will be discussed in Sec. IV.G, the fluctuations are identical with GOE.

The classical mathematical treatment of the Gaussian ensembles—in particular, the GOE (Mehta, 1967)—depends heavily on the properties of the Hermite polynomials. The idea that other sets of orthogonal polynomials might correspond to useful ensembles has been studied by Leff (1964b) and by Fox and Kahn (1964). Of the cases examined, which include the Jacobi and the Laguerre polynomials also, the Legendre polynomials are of particular interest in that the asymptotic level density, instead of having the highly nonphysical semicircular form for the GOE (which we discuss in the next section), is concave upwards and increases rapidly.

III. THE LEVEL DENSITY

The level fluctuations may be regarded as microscopic fluctuations of the level density, or equivalently as being generated by excitations of the ensemble-averaged density. It is then clear that to understand fluctuations we should first learn something about the averaged density itself.

A. Experimental densities

The experimental density has of course a "secular" variation (i.e., a long-range variation with energy) which is a major subject of interest in the usual level-density studies. The earliest calculation of this was given by Bethe (1936; see also Van Lier and Uhlenbeck, 1937). In these calculations, which have established the general procedures used for almost all of the subsequent work on level densities, the specific residual interactions are ignored, so that the formal problem is essentially combinatorial. As one would expect, then, and as is characteristic of complicated fermion systems with single-particle spectra unbounded from above, the level density is found to increase with energy in an exponential fashion, $\sim\exp(aE^{1/2})$, which is in good accord with experiment. In Bethe's paper a simple but elegant use of the central-limit theorem gave also a (Maxwellian) decomposition of the level density according to angular momentum. A great deal of work has been devoted to various extensions of Bethe's theory, to take account of pairing effects by quasiparticle methods, to introduce schematic interactions as Casimir operators of reasonable groups and so forth, and to evaluate the parameters of the theory. For recent reviews see Lynn (1968) and Huizenga and Moretto (1972).

We have already encountered two general questions connected with the secular variation.

1. Why is there apparently little or no secular variation of fluctuation patterns, i.e., why are fluctuation measures, expressed in terms of the local spacing unit, apparently independent of energy? And, related to this, what gives rise to the sharp separation between the fluctuations and the secular variation which we have seen in Fig. 6?

2. In particular, why do the patterns extend to the very end of the spectrum in the ground-state domain? An intuitive picture of a level position as being determined by a balancing of the level-repulsion "pressure" from above and below would suggest that there could be sizeable "end" effects.

We shall be able to give reasonable, if partial, answers to these questions.

Level-density theory may, of course, be regarded as simply an aspect of the study of spectra. But the concepts and methods are so different from those made use of in studying nuclear properties in the ground-state domain that contacts between the two activities have been disappointingly few; this applies in particular to the neglect or very rough treatment of residual interactions in the level-density theory, as contrasted with the dominant role assigned to the residual interaction in the usual shell-model and related theories.

However, as we have mentioned in Sec. II, there exist by now methods, somewhat akin to Khinchin's (1949) procedures for statistical mechanics, by which the level densities can be calculated for *interacting* particles described by indefinitely large model spaces. These methods are now beginning to be applied for level-density calculations (Chang *et al.*, 1971; French and Chang, 1972; Ayik and Ginocchio, 1974; Wong and Lougheed, 1978; Dalton *et al.*, 1980); however, they give us no access to fluctuations, as they apply only to low-order moments. We shall not, therefore, go further into this matter, since our present interest is not so much with the experimental densities as with the densities appropriate to the models which we set up for fluctuations. This is justified by the sharp separation implied in the first question above.

These model densities will also have a secular variation, but unlike the "true" densities, they will for the most part describe a finite number of levels, therefore spanning a finite energy range. They will not be monotonically increasing with energy, usually having instead a single maximum at or near the centroid, in the neighborhood of which they may be regarded as constant. Theoretical calculations of fluctuation measures are usually made in the central region in order to take advantage of mathematical simplifications arising from this constancy. We shall, however, be able to extend our approximate GOE calculations over the entire spectrum, and, beyond that, will be able to see (Pandey, 1979) that the significant fluctuation measures are in

fact stationary (not varying over the spectrum) except perhaps when the level repulsion is moderated at low energies because of the existence of a conserved or almost conserved symmetry.

A good way to emphasize the separation between secular variations and spectral fluctuations is to introduce a new energy variable which maps to a constant the secular ("fluctuation-free") density. We can take $E \to W(E) = d\int_{-\infty}^{E} \bar{\rho}(x)dx$, where $\bar{\rho}(x)$ is the smoothed density; there are other procedures more or less equivalent to this (and of course introducing the circular ensembles, as in Sec. II.F, does the same thing in a different way). Such a mapping is known as an *unfolding* of the spectrum; it results in a fluctuation pattern which, to the extent that the secular-fluctuation separation is valid, is invariant, or close to it, over the whole W spectrum. Such an invariance indicates that, in the sense of stochastic processes, the spectra are stationary or quasi-stationary. For the GOE we shall understand later (Sec. X) the origins of this behavior.

B. Densities, convolutions, and binary correlations for noninteracting particles

Consider a system generated by distributing m noninteracting particles over N single-particle states. The blocking effects due to the Pauli principle can be ignored if the system is sufficiently "dilute", i.e., if $m \ll N$, and we shall take this to be the case. Then we have, for the Hamiltonian in diagonal representation, and the eigenvalue density,

$$H = \sum \varepsilon_i n_i, \quad \rho_1(x) = N^{-1} \sum \delta(x - \varepsilon_i), \tag{3.1}$$

where n_i is the number operator for the ith single-particle state. Because particles do not interact, the m-particle density is a convolution of the $(m-1)$-particle and the single-particle densities,

$$\rho_m(x) = \int \rho_{m-1}(y)\rho_1(x-y)dy,$$

and thus, in an abbreviated notation,

$$\rho_m = \rho_{m-1} \otimes \rho_1 = \rho_1 \otimes \rho_1 \otimes \cdots \otimes \rho_1. \tag{3.2}$$

But by the central-limit theorem, and with very weak restrictions[7] on $\rho_1(x)$, the m-fold convolution of ρ_1 with itself becomes Gaussian, with centroid and variances easily seen to be

$$\mathcal{E}(m) = m\mathcal{E}(1), \quad \sigma^2(m) = m\sigma^2(1) \tag{3.3}$$

$$\mathcal{E}(1) = N^{-1}\sum \varepsilon_i, \quad \sigma^2(1) = N^{-1}\sum [\varepsilon_i - \mathcal{E}(1)]^2. \tag{3.4}$$

Note carefully that this is *not* an ensemble result but is valid for every individual one-body H. Thus the characteristic many-particle density is Gaussian when the particles are noninteracting.

The Gaussian density has arisen as a result of multiple convolution. But since the convolution result does not apply for interacting particles (whose energies are not additive), we rederive the density from another point of view, as arising from the dominance of binary Hamiltonian associations; at the same time we shall see how fast the spectrum tends to Gaussian as we add particles.

Consider the moments of the spectrum of the one-body H given by (3.1), fixing the energy zero so that $\sum_i \varepsilon_i = 0$ which assures us that the moments are in fact *central* moments. For $M_p(m)$, the pth moment of the m-particle system, we have

$$M_p(m) = \frac{1}{d(m)} \sum_i [E_i(m)]^p = \frac{1}{d(m)} \operatorname{Tr}^{(m)}(H^p) \equiv \langle H^p \rangle^m, \tag{3.5}$$

where

$$d(m) = \binom{N}{m}$$

is the model-space dimensionality, $E_i(m)$ are the eigenvalues of H [note that $E_i(1) = \varepsilon_i$] and, for any operator G, $\operatorname{Tr}^{(m)}G$ is its trace over the m-particle space. As indicated in Sec. II, we write, where the last form is used if m is not specified,

$$d(m)^{-1}\operatorname{Tr}^{(m)}G \equiv d(m)^{-1}\langle\langle G \rangle\rangle^m = \langle G \rangle^m \to \langle G \rangle, \tag{3.6}$$

so that $\langle G \rangle^m$ is the *average* expectation value or eigenvalue of G. To evaluate $M_p(m)$ we recall that $n_i^2 = n_i$; furthermore, $\langle n_i n_j n_k \ldots \rangle^m$ is easily seen to have the value

$$\binom{m}{l}\binom{N}{l}^{-1},$$

where l is the number of different n's in the product, and therefore grows faster with particle number the higher the value of l. Let us now decompose the moment $M_p(m) = \langle H^p \rangle^m$ into terms according to the partitions of p; that is to say, multiply H^p out, collect the terms with the same $n_i n_j n_k \ldots$ structure, and use the result just given; there are, for example, five terms for $p = 4$ and eleven for $p = 6$.

The general result is easily derived (Mon and French, 1975), but we content ourselves with examples. The term associated with the partition [22] of $p = 4$ is

$$\begin{aligned} 3\sum_{i\neq j} \varepsilon_i^2 \varepsilon_j^2 \langle n_i^2 n_j^2 \rangle^m &= 3\binom{m}{2}\binom{N}{2}^{-1}\left[\left(\sum \varepsilon_i^2\right)^2 - \sum \varepsilon_i^4\right] \\ &= 3\binom{m}{2}\binom{N}{2}^{-1}[N^2(\langle \varepsilon^2 \rangle)^2 - N\langle \varepsilon^4 \rangle] \\ &\simeq 6\binom{m}{2}(\langle \varepsilon^2 \rangle)^2, \end{aligned} \tag{3.7}$$

where $\langle \varepsilon^p \rangle = N^{-1}\sum(\varepsilon_i)^p = M_p(1)$. In the last step we have dropped a term of relative order N^{-1}, including the $\langle \varepsilon^4 \rangle$ term, which, deriving as it does from the $i \neq j$ restriction on the left-hand side, is a "blocking" correction. The [4] partition gives $m\langle \varepsilon^4 \rangle$ when N is large, and is

[7] For finite N (and, of course, $|\varepsilon_i| < \infty$), which is the case of interest to us, there are essentially no restrictions as long as we can ignore the Pauli effects, so that we get close to a Gaussian distribution but still have $m \ll N$. Otherwise, we could in principle be foiled if the single-particle spectrum were "singular" (Sec. VII.D); but such cases are of little interest. For a compact review of the restrictions in more general cases see Fortet (1958).

thus much smaller for large m than the [22] contribution. More generally, a partition, $[\lambda_1\lambda_2\cdots\lambda_s]$, involving s parts, gives a term proportional to

$$\binom{m}{s}\{\langle\varepsilon^{\lambda_1}\rangle\langle\varepsilon^{\lambda_2}\rangle\ldots\langle\varepsilon^{\lambda_s}\rangle\}\,.$$

However, all partitions containing 1's (e.g., the other $p=4$ partitions, [31], [211], and [1111]) vanish, since $\langle\varepsilon\rangle=0$; this being the case, the dominant term will be the partition into the largest number of 2's, that is to say, $[2^{p/2}]$ or $[32^{(p-3)/2}]$, according as p is even or odd. Thus, when the particle number is large, the binary correlation terms, corresponding to the counting of Hamiltonian pairs, are dominant.

As a further example, we find that, for $p=6$, seven of the eleven terms vanish because of the vanishing of the centroid, and then

$$M_6(m)\to 90\binom{m}{3}\langle\varepsilon^2\rangle^3+20\binom{m}{2}\langle\varepsilon^3\rangle^2 +30\binom{m}{2}\langle\varepsilon^4\rangle\langle\varepsilon^2\rangle+\binom{m}{1}\langle\varepsilon^6\rangle\,, \tag{3.8}$$

the terms here arising respectively from the partitions [222], [33], [42], and [6].

When there are many particles, the dominant contributions are then $m\langle\varepsilon^2\rangle$, $6\binom{m}{2}\langle\varepsilon^2\rangle^2$, and $90\binom{m}{3}\langle\varepsilon^2\rangle^3\to m\langle\varepsilon^2\rangle$, $3(m\langle\varepsilon^2\rangle)^2$ and $15(m\langle\varepsilon^2\rangle)^3$ for $p=2,4,6$, respectively. The coefficients are the number of ways of making binary associations among p objects; for arbitrary even $p=2\nu$ this number is $(2\nu-1)!!$, so that as $m,N\to\infty$ with $m/N\to 0$,

$$M_{2\nu}(m)\to(2\nu-1)!!\,[M_2(m)]^\nu\,, \tag{3.9}$$

which shows the asymptotic distribution to be Gaussian (see Appendix E).

C. Interacting particles, ensembles, and binary correlations

For interacting particles, not every H generates a Gaussian or a close-to-Gaussian spectrum in some natural limit; nor does every H generate the same asymptotic spectrum. For example, if $H=h^2$, where h is a traceless one-body operator, we easily find that the many-particle density, appropriately scaled, is $\rho(x)\sim x^{-1/2}\exp(-x/2)$, a χ^2 distribution in a single variable [h^2 is, of course, not a pure two-body operator; but the deviation from Gaussian is not due to a mixing of particle ranks (see Appendix F)].

Obviously, there is then no unique characteristic spectrum as there is for noninteracting particles. Yet it is a fact, borne out by very many shell-model calculations done with a wide variety of "realistic" interactions and yielding excellent agreement with experiment (in the ground-state region where the results should be of direct physical significance), that the spectra are very close to Gaussian.

To understand this, consider first a set of exceptional cases, those with $H=\sum_{\alpha=1}^{l}h_\alpha^2$, where $l\ll N$ and the h_α are traceless nonsingular one-body operators, which have the same variance and which are orthogonal in the one-particle space, $\langle h_\alpha h_\beta\rangle^1=c\delta_{\alpha\beta}$. The asymptotic eigenvalue distribution is of χ_l^2 form, which is obvious for $l=1$, since the central limit theorem (CLT) generates a Gaussian distribution for h_α, but may not be obvious for $l>1$, since the h_α do not in general commute [see French and Draayer (1979) for this case]. The given structure for H is of formal interest as the bilinear Casimir operator of the l-dimensional Lie algebras, and of physical interest because, for $l=5$, it represents the quadrupole interaction, $Q\cdot Q$, an important component of reasonable Hamiltonians. Since χ_l^2 becomes Gaussian for large l, and since realistic H's are not dominated[8] by a few h_α^2 terms, it is not then surprising that the resultant spectra will be close to Gaussian. Once again this behavior can be traced to the dominance of binary associations in the moments.

An alternative procedure is to calculate an ensemble-averaged density (along with an appropriate measure for the deviations from average, which we come to in Sec. IV.F. We start with a GOE of k-body Hamiltonians. Let $\alpha\equiv\alpha(k)$ be the (second-quantized) state operator which, acting on the vacuum, generates the k-body state $\psi_\alpha(k)$; if $W_{\alpha\beta}$ are the matrix elements, then the members of this GOE will be given by

$$H=\sum_{\alpha,\beta}W_{\alpha\beta}\alpha\beta^+. \tag{3.10}$$

We have, of course (with the bar denoting the ensemble average), that $W_{\alpha\beta}=W_{\beta\alpha}$, $\overline{W}_{\alpha\beta}=0$, $[W_{\alpha\beta}W_{\gamma\delta}]_e=0$ if the matrix elements are distinct, and $[(W_{\alpha\beta})^2]_e=(1+\delta_{\alpha\beta})v^2$. As in Eq. (2.5), let us normalize the GOE so that the variance v^2 of the off-diagonal elements is $[d(k)]^{-1}=\binom{N}{k}^{-1}$; the average k-particle expectation value of H^2 is then $1+\binom{N}{k}^{-1}$ which goes to 1 in the large-N limit.

When the matrices of this ensemble act in the k-particle space, we get back the GOE; if they act in an m-particle space with $m>k$, we obtain what we have called the "embedded" GOE or EGOE. Observe that in either case the odd moments vanish because of the symmetry about zero of the $W_{\alpha\beta}$ distributions. In complete analogy with the noninteracting particle case let us now define an *associated Hamiltonian pair* and its ensemble average:

$$\underline{HH}\equiv\sum_{\alpha\beta}W_{\alpha\beta}W_{\beta\alpha}\alpha\beta^+\beta\alpha^+, \tag{3.11}$$

$$\overline{\underline{HH}}=\binom{N}{k}^{-1}\sum_{\alpha\beta}(1+\delta_{\alpha\beta})\alpha\beta^+\beta\alpha^+ =\left[\binom{N-n+k}{k}+1\right]\binom{N}{k}^{-1}\binom{n}{k}\xrightarrow[d\to\infty]{}\binom{n}{k}\,, \tag{3.12}$$

where, to produce the second-last form, we have used the elementary results (Mon and French, 1975) that $\sum\alpha\alpha^+=\binom{n}{k}$, $\sum\beta^+\beta=\binom{N-n}{k}$, $\alpha\alpha^+\alpha\alpha^+=\alpha\alpha^+$.

The dominant role of the binary associations now arises from the distribution of the $W_{\alpha\beta}$. Writing $H_{\alpha\beta}$

[8] As suggested above, $Q\cdot Q$ is the largest "single" component, its squared norm being characteristically one quarter of the total. But it has a negative coefficient so that the part of the spectrum which deviates most from Gaussian comes harmlessly at high excitation.

$= W_{\alpha\beta\alpha\beta^+}$, we have as an example with $p=4$, that $\langle H_{\alpha\beta}H_{\beta\alpha}H_{\alpha\gamma}H_{\gamma\alpha}\rangle^m$, which on averaging gives $\langle\overline{(\underline{HH})^2}\rangle^m \to \binom{m}{k}^2$. The fourth-order-association sequences, as, $\langle H_{\alpha\beta}H_{\beta\alpha}H_{\alpha\beta}H_{\beta\alpha}\rangle^m$, involve restricted summations (with only two free summation indices) and may be expected therefore to give a vanishing relative contribution in the limit $d\to\infty$, a result borne out in the calculations of the next section. Similarly, for higher even p values, the binary associations give the dominant term just as in the noninteracting particle case without ensemble averaging. We have now established a correspondence between the two cases: in the next two sections we make clearer what this implies for the ensemble-averaged eigenvalue densities and for the densities for individual systems.

D. Inhibited and uninhibited associations. The semicircular-to-Gaussian transition

For noninteracting particles the dominance of binary associations leads to a Gaussian density when there are many particles. The same is true with interacting particles, but strictly only for the averaged density. Suppose that $m \gg k$ and consider the moment of fixed order $p=2\nu$. When $m \gg pk$ it is clear that in $\langle H^{2\nu}\rangle^m$ the H's, except for the requirement of pairwise association, act for the most part on different particles. Then an $\alpha\beta^+$ term of H in (3.10) effectively commutes with all H's, so that "free" pairwise association is allowed, and

$$H^{2\nu} \to (\underline{HH})^\nu \times \{\text{number of binary associations of } 2\nu \text{ objects}\} = (2\nu-1)!!(\underline{HH})^\nu , \tag{3.13}$$

just as with noninteracting particles. Then (Mon and French, 1975; see also Gervois, 1972)

$$\overline{M}_{2\nu}(m) \to (2\nu-1)!!\,[\overline{M}_2(m)]^\nu , \quad \overline{M}_{2\nu+1}(m)\to 0 , \tag{3.14}$$

so that the ensemble-averaged EGOE density becomes Gaussian: $\overline{\rho} \to G(0,\binom{m}{k}^{1/2})$ for many particles. In our first treatment of noninteracting particles the central-limit theorem made its appearance via the process of multiple convolution, in the second via binary associations. In the present case the convolution is not so obvious but we can still characterize the transition to Gaussian as being effected by a generalization of the CLT already used for noninteracting particles.

If, instead of $m \gg k$, we take the other extreme, $m=k$, binary associations are still dominant, but now only a certain subset of the fully correlated terms are contributing, since for "almost all" traceless operators in the k-particle space (including H itself) the operator $\underline{HOH}$ is smaller than $\underline{HHO}$; indeed, the ratio goes to zero as $d(k)\to\infty$, so that terms containing $\underline{HOH}$ in an operator sequence may be ignored. This derives from the fact that, in $\underline{HOH}$ acting in the k-particle space, only the diagonal matrix elements, which are down in number by a factor $\sim d^{-1}(k)$ from the totality of matrix elements, can survive the contraction (see Appendix G), whereas no such restriction is found with $\underline{HHO}$. We may thus describe the GOE moments $\langle H^p\rangle^k$ as generated by a counting of the *unlinked* binary association ($\overline{\underline{HH}}\,\overline{\underline{HH}}$ and $\overline{\underline{HHHH}}$ contributing for $p=4$, but not $\overline{\underline{HHHH}}$), so that we are dealing with "inhibited" correlations, those which are nested in the same way as correctly paired brackets, in contrast with EGOE for which all binary associations contribute equally.

Let us now evaluate the ensemble-averaged GOE density. Since in $\langle H^p\rangle^k$ the first H (on the extreme right) can be correlated with each of the others, we have when $m=k$

$$\begin{aligned}\overline{M}_{2\nu}(k) = \langle\overline{H^{2\nu}}\rangle^k &= \sum_{\tau=0}^{2\nu-2}\langle\overline{H^{2\nu-2-\tau}\underline{HH^\tau H}}\rangle^k \\ &= \sum_{\tau=0}^{2\nu-2}\langle\overline{H^{2\nu-2-\tau}}\rangle^k\langle\overline{H^\tau}\rangle^k\langle\overline{H^2}\rangle^k \\ &= \sum_{\tau=0}^{2\nu-2}\overline{M}_{2\nu-2-\tau}(k)\overline{M}_\tau(k) ,\end{aligned} \tag{3.15}$$

since, in the limit, $M_0(k)=1=\overline{M}_2(k)$.

We have used here the fact that the H^τ factor can have only internal associations, for otherwise the pair which encloses it would generate an operator of rank higher than k which would vanish in the k-particle space. The recursion relationship can be solved easily (Wigner, 1955) to give

$$\overline{M}_{2\nu} = \frac{1}{\nu+1}\binom{2\nu}{\nu} , \tag{3.16}$$

which are known as the "Catalan numbers" (Riordan, 1968; Gardner, 1976). Since the odd moments vanish, we know all the moments of the (ensemble-averaged) density and hence the density, which turns out to be[9]

$$\overline{\rho}(x) = \frac{1}{2\pi}(4-x^2)^{1/2} , \tag{3.17}$$

the famous "semicircle" of Wigner (1955). Here the characteristic form emerges only through ensemble averaging in contrast to the case of noninteracting particles. The semicircle shows up very well in Fig. 7, which gives a histogram of the spectra for 50 members of a ($d=294$) ensemble.

As we add particles to the system, starting with the GOE ($m=k$), we see that the density expands in scale $[\sigma^2=\binom{m}{k}]$; it also changes in shape from semicircular to Gaussian, the transition being due to the gradual lessening of the inhibitions on binary associations; a histogram for an individual spectrum is to be seen in Fig. 6. This spectrum is not chosen from an EGOE but represents instead a calculation with a specific "realistic" Hamiltonian; binary correlations are still dominant, however, and since the particle number is considerably larger than the particle rank of the interaction, a good Gaussian emerges.

This is a good place to stress that, in the simplest representation (a direct-product one in which particles are assigned to definite single-particle states), there are obvious selection rules associated with a k-body interaction; only k particles can be transferred by H to different single-particle states. Thus if $m>k$, the H matrix has a number of zeros. However, an H with the appropriate zeros is *not* necessarily k body, for, be-

[9] For the exact density (i.e., for finite d) see Mehta and Gaudin (1960).

sides the zeros, there are also necessary linear constraints on the nonvanishing matrix elements. It is easy to see then that an ensemble which satisfies the selection rules but not the constraints is still m-body and gives a semicircular rather than Gaussian spectrum. This has been demonstrated by Monte Carlo calculations.

The rate at which the semicircle – Gaussian transition occurs may be measured by the statistician's "excess" or "kurtosis," γ_2. It is an elementary exercise to evaluate this,[10] and one finds (Mon and French, 1975)

$$\gamma_2 \equiv \frac{\bar{M}_4(m) - 3\bar{M}_2^2(m)}{\bar{M}_2^2(m)} = \binom{m-k}{k}\binom{m}{k}^{-1} - 1 \underset{m\gg k}{\longrightarrow} -\frac{k^2}{m}. \tag{3.18}$$

A departure from Gaussian characterized by $|\gamma_2| > 0.3$, say, would be recognized by the eye. This value is reached for $k = 1$ to 5 with $m = 4, 12, 26, 47, 74$, respectively, so that, for the conventional $(1+2)$-body Hamiltonian, 6 to 10 particles should give an excellent Gaussian. In practice one finds good Gaussians with fewer particles than that. The semicircle-to-Gaussian transition can, of course, be generated by varying the particle rank of the interaction rather than the particle number. The transition is shown very nicely in this way by Fig. 8, taken from French and Wong (1971), in which the spectra of k-body interactions with $k = 2$ to 7 are shown and compared with the corresponding semicircle and Gaussian. We are seeing here quite clearly the action of a CLT *for interacting particles* which forces the density towards Gaussian as we add particles. The origin of this CLT is to be found in the dominance of binary associations for most Hamiltonians in the ensemble. Finally, we remark that Monte Carlo calculations (French and Wong, 1970; Bohigas and Flores, 1971a; Yépez, 1975) give an essentially Gaussian density for the TBRE and TBRE-J_z, not surprising in view of their close relationship to the EGOE. We have already commented in Sec. II.H on the density for the "Legendre" ensemble (Leff, 1964b). The techniques of Balian (1968) and Dyson (1972b), or a straightforward Hamiltonian mapping (Appendix H), enable one to generate ensembles with any predetermined density.

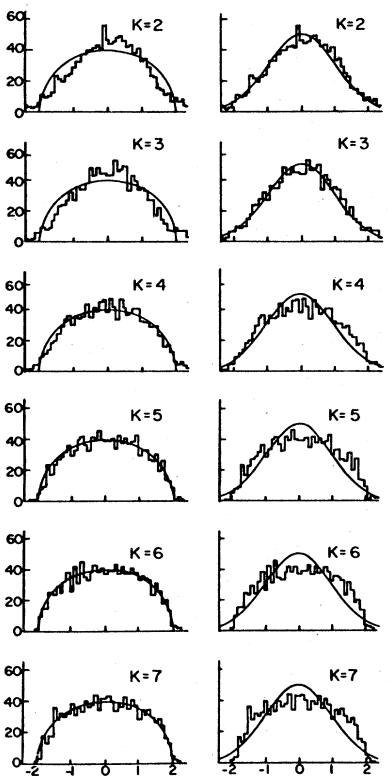


FIG. 8. Spectral histograms for ensembles, 25 members each, of k-body J-conserving interactions with $k = 2$–7, which act in the 50-dimensional space f^7 with $J = \frac{7}{2}$ and identical particles ($T = \frac{7}{2}$). Energies are measured in units of the spectral width. Each spectrum is given twice, compared on the left with a semicircular, and on the right with a Gaussian, spectrum. We see the transition from semicircle to Gaussian as k decreases. For details see French and Wong (1971).

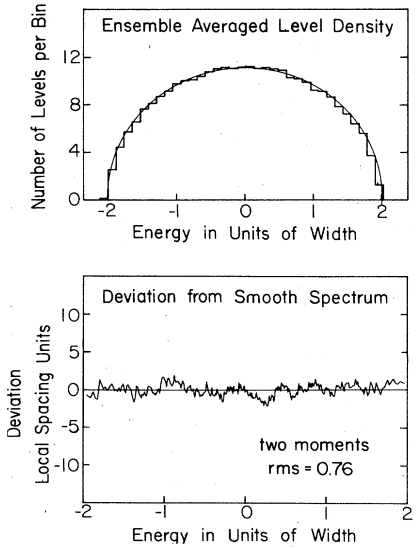


FIG. 7. Spectral histogram for a 50-member GOE with $d = 294$ is given at the top. The bottom figure, corresponding to Fig. 6, shows the second-order spectral deviations. The analytic treatment is given in IV.F.

[10]For $k = 1$ the result follows from $M_4(m)$, as given in Sec. III.B and the semicircular moment of (3.16).

E. Eigenvalue densities for deformed and other ensembles

We have mentioned in Sec. II.G the way in which deformed ensembles may be used to study fluctuations in systems with almost conserved symmetries. Another use which we envision is for eigenvalue densities and transition strengths; here, for example, if our interest is in quadrupole collectivity, we would represent the system Hamiltonian $\hat{H}$ as $\lambda Q \cdot Q + H_\perp$, where λ is so chosen that the decomposition is orthogonal, $\langle Q \cdot Q(\hat{H} - \lambda Q \cdot Q)\rangle^m = 0$. Then, instead of dealing with the specified $H_\perp$, we could represent it by an ensemble of the appropriate norm, this being a sensible procedure when the $Q \cdot Q$ component is large in $\hat{H}$ (i.e., when the correlation coefficient between $\hat{H}$ and $Q \cdot Q$ is large).

In this section therefore we begin by considering the

ensemble average of ρ_α, the density for $H_\alpha = H + \alpha K$, where α is real, K is an arbitrary fixed real symmetric matrix, and H is a GOE (we comment later on the EGOE case). The representation for K need not be specified, since H is orthogonally invariant. The general result, given by Pastur (1972; see also Pastur, 1973), is

$$\bar{\rho}_\alpha(x) = -\frac{1}{\pi}\,\mathrm{Im}\bar{g}_\alpha(x)\,, \tag{3.19}$$

$$\bar{g}_\alpha(x) = \left\langle \frac{1}{x - \alpha K - \bar{g}_\alpha(x)} \right\rangle . \tag{3.20}$$

Here $\bar{g}_\alpha(x)$, which is determined by the second equation, is the ensemble average of the Stieltjes transform of the density:

$$g_\alpha(x) = \left\langle \frac{1}{x - H_\alpha} \right\rangle = \int_{-\infty}^{\infty} \frac{\rho_\alpha(y)}{x - y}\, dy\,. \tag{3.21}$$

It is to be understood that x is endowed with a small positive imaginary part which is taken to be zero in the final results. It is worthwhile remarking that the reciprocal (resolvent) function encountered here arises from a standard representation of the delta function; thus

$$\begin{aligned}\rho_\alpha(x) &= \frac{1}{d}\sum_{j=1}^{d}\delta(x - E_j) \\ &= -\frac{1}{\pi d}\,\mathrm{Im}\sum_{j=1}^{d}(x + i\varepsilon - E_j)^{-1} \\ &= -\frac{1}{\pi}\,\mathrm{Im}g_\alpha(x)\,,\end{aligned} \tag{3.22}$$

where ε is positive infinitesimal.

As an example, consider the GOE itself. We have $\alpha = 0$ and then $\bar{g}_0^2 - x\bar{g}_0 + 1 = 0$, which gives

$$\bar{g}_0(x) = \tfrac{1}{2}[x \pm (x^2 - 4)^{1/2}]\,, \tag{3.23}$$

the negative sign being called for since the positive sign would yield a negative density [similarly in (3.24) below]. From this equation, along with (3.19), we arrive at the semicircle (3.17). By the same procedure we can demonstrate a result, whose genesis should be clear, namely, that if K is another (independent) GOE, the final density is also semicircular.

As a nontrivial example, which is of interest in certain physical models, take K to be a matrix with a spectrum $(1, 0^{d-1})$, i.e., a single unit eigenvalue and $(d-1)$ zero eigenvalues; this ("pairing") matrix will be described in Sec. VII.D as the basic example of a "singular" matrix. The density in this case has been recently derived in several different ways by Edwards and Jones (1976), Jones *et al.* (1978), and Pandey and French (1979). However, the result follows immediately as a special case of (3.19) and (3.20). Thus

$$\begin{aligned}\bar{g}_\alpha(x) &= \frac{1}{d}\frac{1}{x - \alpha - \bar{g}_\alpha(x)} + \frac{d-1}{d}\frac{1}{x - \bar{g}_\alpha(x)} \\ &\simeq \frac{x - (x^2-4)^{1/2}}{2} + \frac{1}{d}\frac{2\alpha}{[x + (x^2-4)^{1/2} - 2\alpha](x^2-4)^{1/2}}\,,\end{aligned} \tag{3.24}$$

where the last form follows by solving the first equality in the approximation $\bar{g}_\alpha \simeq \theta + d^{-1}\phi$. Equation (3.19) now yields

$$\begin{aligned}\bar{\rho}_\alpha(x) &= \bar{\rho}_0(x) + \frac{1}{2\pi d}\frac{x - 2\alpha}{(\alpha + \alpha^{-1} - x)(4 - x^2)^{1/2}} \\ &\quad + \frac{1}{d}\delta(x - \alpha - \alpha^{-1}) \\ &\simeq \bar{\rho}_0(x) + d^{-1}\delta(x - \alpha - \alpha^{-1}), \quad |\alpha| > 1 \\ \bar{\rho}_\alpha(x) &= \bar{\rho}_0(x) + \frac{1}{2\pi d}\frac{x - 2\alpha}{(\alpha + \alpha^{-1} - x)(4 - x^2)^{1/2}} \\ &\simeq \bar{\rho}_0(x), \quad |\alpha| < 1\end{aligned} \tag{3.25}$$

$\bar{\rho}_0(x)$ being, of course, the semicircle (3.17); the d^{-1}-order correction to the semicircle serves to yield the correct normalization of $\bar{\rho}_\alpha(x)$. The special feature of the spectrum is that as the strength of the random interaction is increased (i.e., as $|\alpha|$ increases), the isolated special state moves toward the semicircle, disappearing into it when $|\alpha| = 1$.

To derive the Pastur result, and large extensions of it for other ensembles and for functions other than the density, we follow recent work of Pandey (to be published), who uses the method of binary correlations described above.[11] We first derive the ensemble average of

$$f_L \equiv f(L, G) \equiv \left\langle L\frac{1}{1 - HG} \right\rangle = \sum_{p=0}^{\infty}\langle L(HG)^p\rangle\,, \tag{3.26}$$

which, among its several applications, will lead us to Pastur's result. Here L and G are arbitrary matrices of dimensionality d. For simplicity of notation we write f_L instead of $f(L, G)$; note then that $f_G = f(G, G)$. The odd-p terms in (3.26) vanish, of course, on ensemble averaging, while for even $p \geq 2$ we have

$$\begin{aligned}\overline{\langle L(HG)^p\rangle} &= \sum_{r=0}^{p-2}\overline{\langle L\underbrace{HG(HG)^r}HG(HG)^{p-r-2}\rangle} \\ &= \sum_{r=0}^{p-2}\overline{\langle G(HG)^r\rangle}\,\overline{\langle LG(HG)^{p-r-2}\rangle}\,,\end{aligned} \tag{3.27}$$

a generalization of (3.15). Substituting in the expansion of $\bar{f}_L$, we find, after an interchange of summations, that

$$\bar{f}_L = \langle L\rangle + \bar{f}_G\bar{f}_{LG} = \left\langle L\frac{1}{1 - \bar{f}_G G}\right\rangle, \tag{3.28}$$

where the last form follows by using the first equality in (3.28) applied successively for the matrices L, LG, $LG^2, \ldots$, and then summing the resultant geometric series.

As a first application we take $L = G = (x - \alpha K)^{-1}$; then $f_L = g_\alpha(x)$, and (3.28) will yield the Pastur result (3.20). With an imaginary part in x, the convergence in (3.26)

[11] Instead of using Eqs. (3.19) and (3.20), we could in principle construct $\bar{\rho}_\alpha$ from its moments; and this is one of the methods used for the special case (3.25). As described in Appendix I, the general moment evaluation has in fact been done by using a counting theorem (4.17) given in Sec. IV and used there for fluctuations. However, except for some simple cases, the moment results are difficult to invert, they do not display the simplicity of Pastur's result, and they are not easily adaptable to more general ensembles. We remark, too, that the original Pastur result has been very recently rederived also by Edwards and Warner (1980).

is already assured. It will be clear, of course, from (3.28) that taking $L=G$ will be the first step in dealing with the more general quantity $\bar{f}_L$. It will be helpful to note that $L(1-\bar{f}_G G)^{-1}=LG^{-1}(G^{-1}-\bar{f}_G)^{-1}$ as long as G is invertible, as it will be in our applications. In fact, it will usually be so that $G=(x-\alpha K)^{-1}$ where K is the arbitrary operator introduced at the beginning.

One of the methods used in dealing with the effect of random (GOE) interactions on the simple operator with spectrum $(1,0^{d-1})$ is the Brillouin-Wigner expansion. The application was to calculate, for $|\alpha|>1$, the average position of the (isolated) level outside the semicircle, and its variance. We can use the theorem of (3.28) to deal with the behavior of isolated levels of a general operator K, isolation here implying that the level does not cross any of its neighbors as a result of the random interaction. The results which emerge are that

$$\bar{E}=\alpha k+\alpha^{-1}\langle Q(k-K)^{-1}Q\rangle\,,$$
$$\mathrm{var}E=\frac{2}{\alpha d}\left(\frac{d\bar{E}}{dk}\right)=\frac{2}{d}[1-\alpha^{-2}\langle Q(k-K)^{-2}Q\rangle]\,, \tag{3.29}$$

where E and k are the perturbed and unperturbed positions of the isolated level, and where Q is the projection operator for the $(d-1)$-dimensional subspace orthogonal to the eigenstate $|k\rangle$ of K. A proof may be given by following the steps used for the simple operator (Pandey and French, 1979) and recognizing that when one uses (3.28), the Brillouin-Wigner sums may in fact be carried out. The expression for $\bar{E}$, but not the variance, also follows directly from the Pastur result.

We remark parenthetically that a third application will be to transition strengths, which are discussed more conventionally in Sec. VII. For systems described by the deformed ensemble $(H+\alpha K)$ we take $L=P_i G$, with $G=(x-\alpha K)^{-1}$ as for the Pastur result, where $P_i=|i\rangle\langle i|$ is the projection operator for a given fixed state $|i\rangle$; we see from (3.26) that the poles and the residues of f_L are simply the eigenvalues E and the squared amplitudes $|\langle i|E\rangle|^2$ of the matrix H_α. Thus the imaginary part of $\bar{f}_{P_iG}$ will be $-\pi[|\langle i|E\rangle|^2\rho_\alpha(E)]_e$, an extension of the Pastur result. The GOE result that $[|\langle i|E\rangle|^2]_e=d^{-1}$ follows trivially, but the general result and its extensions to the other ensembles discussed below would seem to be useful in describing the secular variation of the locally averaged strength.

The ensembles described above represent a deformation of the corresponding GOE by varying the matrix-element centroids. For more general modifications of H we should allow also for nonuniformity of the matrix-element variances. We consider the ensemble $V=\sum_{ij}v_{ij}P_iHP_j$ and the corresponding deformed ensemble $V_\alpha=V+\alpha K$, where, of course, H is a GOE. Here the space is divided into disjoint subspaces generated by the projection operators P_i (which could be one-dimensional, as in the third example above), and the $v_{ij}=v_{ji}$ define the variances for the matrix elements of V. Redefining f_L of (3.26) by $f_L=\langle L(1-VG)^{-1}\rangle$, and following the procedure leading to (3.27) and (3.28), we obtain in this case

$$\bar{f}_L=\langle L\rangle+\sum_{ij}v_{ij}^2\bar{f}_{P_jG}\bar{f}_{LP_iG}=\left\langle L\frac{1}{1-\sum_{ij}v_{ij}^2\bar{f}_{P_jG}P_iG}\right\rangle\,, \tag{3.30}$$

where once again $G=(x-\alpha K)^{-1}$. Note that we may well be interested in the V ensemble itself, for which $\alpha=0$; while G in this case is trivial, this may by no means be true of Eq. (3.30). Recall also that (3.30) follows from the dominance of binary correlations and may not be valid in certain singular cases, for example, when V has only a few matrix elements which are not identically zero.

The basic equation (3.30) is very complicated in general. It simplifies in special cases, for example, when v_{ij} is separable $(=v_iv_j)$. Another special case would be a class of V ensembles which yield semicircular densities, in particular, those which display a uniformity with respect to the given basis states, a uniformity described by $\bar{f}_{P_iG}=d^{-1}\bar{f}_G$. A third such case is the ensemble of matrices with binary partitioning which represent a random interaction between two nonrandom subsystems; in this case (3.30) is solved via two coupled functional equations for $\bar{f}_{P_1G}$ and $\bar{f}_{P_2G}$.

In Sec. II.H we mentioned Dyson's (1971) ensemble, $\tilde{H}=\sum_{i=1}^{k}A_i^TA_i$, where the A_i are independent real asymmetric Gaussian random matrices whose matrix elements have zero centroids and variances d^{-1}. We show how the present methods give the density for this ensemble and a functional equation which applies to a deformation of it, $\tilde{H}\to\tilde{H}_\alpha=\tilde{H}+\alpha K$. Redefining the function f_L of (3.26) by $f_L=\langle L(1-\tilde{H}G)^{-1}\rangle$ with $G=(x-\alpha K)^{-1}$, we have

$$\begin{aligned}\overline{\langle L(\tilde{H}G)^p\rangle}&=\sum_{i=1}^{k}\overline{\langle LA_i^TA_iG(\tilde{H}G)^{p-1}\rangle}\\&\quad+\sum_{r=0}^{p-2}\sum_{i=1}^{k}\overline{\langle LA_i^TA_iG(\tilde{H}G)^rA_i^TA_iG(\tilde{H}G)^{p-r-2}\rangle}\\&=k\overline{\langle LG(\tilde{H}G)^{p-1}\rangle}+\sum_{r=0}^{p-2}\overline{\langle G(\tilde{H}G)^r\rangle\langle L(\tilde{H}G)^{p-r-1}\rangle}\,,\end{aligned} \tag{3.31}$$

so that, after summation, we have

$$\begin{aligned}\bar{f}_L&=\langle L\rangle(1-\bar{f}_G)+\bar{f}_G\bar{f}_L+k\bar{f}_{LG}\\&=\langle L\rangle+\frac{k\bar{f}_{LG}}{1-\bar{f}_G}\\&=\left\langle L\frac{1-\bar{f}_G}{1-\bar{f}_G-kG}\right\rangle\,,\end{aligned} \tag{3.32}$$

in which the second step follows by solving the first equality for $\bar{f}_L$ and the last form follows just as in the corresponding Eq. (3.28).

For $\alpha=0$, with $L=G=x^{-1}$, we then obtain

$$\bar{f}_G=\frac{(x-k+1)-[(x-k+1)^2-4x]^{1/2}}{2x}\,, \tag{3.33}$$

which, via (3.19), with $\bar{g}_\alpha$ replaced by $\bar{f}_G$, gives the Dyson result for the nondeformed ensemble $\tilde{H}$:

$$\bar{\rho}_0(x)=\frac{[2(k+1)x-(k-1)^2-x^2]^{1/2}}{2\pi x}\,,$$
$$(\sqrt{k}-1)^2\le x\le(\sqrt{k}+1)^2\,. \tag{3.34}$$

We turn briefly to the unitary (GUE) and symplectic (GSE) cases as defined in Sec. II.D. Our binary-correlation method is seen to be valid in these cases also, and thus the moments (3.15) and (3.16) and the semicircular

density (3.17) for all β. The semicircular density obtains also if we choose different normalizations for the H_0 and H_k matrices, as we do, for example, for the ensemble introduced in Sec. II.D for studying time-reversal invariance. In similar fashion, we obtain the Pastur result for the deformed versions of all three ensembles.

For an embedded version of the first problem ($H_\alpha = H + \alpha K$) the average eigenvalue density follows easily in the limit of large particle number. Since different operators in a trace act on different particles, and therefore effectively commute in this limit, we have for the averaged moments (Pandey and French, 1979)

$$\overline{M_p(\alpha)} \equiv \overline{\langle (H+\alpha K)^p \rangle^m} \simeq \sum_{r=0}^{p} \binom{p}{r} \overline{\langle (\alpha K)^r H^{p-r} \rangle^m}$$

$$= \sum_{r=0}^{p} \binom{p}{r} \overline{\langle (\alpha K)^r \rangle^m \langle H^{p-r} \rangle^m}, \tag{3.35}$$

which are the moments of a convolution of the K spectrum and the continuous Gaussian EGOE density ρ_G of zero centroid and variance $\binom{m}{k}$. For large $|\alpha|$ the K spectrum is preserved, but the levels have a Gaussian spread whose magnitude is characteristic of the EGOE density. For small $|\alpha|$ the K spectrum is lost in the background.

IV. CORRELATIONS AND FLUCTUATIONS

A. Introduction: two-point and k-point functions

The secular behavior of a spectrum, which is describable in terms of long-wavelength excitations of a characteristic density, and the fluctuations, which correspond to short-wavelength excitations, are in a real sense opposite in nature. They may in fact (as is shown in Fig. 6, and in the experimental data, as well) be said to have an independent existence.

On the other hand, we are not so much interested in the fluctuations themselves as in some significant measures for them; and when we take that into account, we may ask whether there is a unified way of studying the two behaviors and if so whether it leads to any real simplication in understanding the fluctuations. The answer to both questions is in the affirmative.

All of the statistical properties can be discussed in terms of the k-point functions S_k, with $k = 1, 2, \ldots$,

$$S_k = \overline{s}_k, \quad s_k = \prod_{i=1}^{k} \rho(x_i), \tag{4.1}$$

or, equivalently, in terms of the k-point cluster functions (Dyson, 1962c), expressible as polynomials in the functions S_k which are homogeneous in the ρ's. Here $\rho(x_i)$ is the eigenvalue density evaluated at any point x_i along the energy axis. There are, of course, corresponding functions with ensemble averaging replaced by spectral averaging.

As we shall see in the next chapter, the fluctuation measures of interest (that is, those which can be calculated without large statistical uncertainty from the data) are related to the density $\overline{\rho}(x)$ itself and to the two-point (or covariance) function,

$$S^\rho(x,y) = \overline{\rho(x)\rho(y)} - \overline{\rho}(x)\overline{\rho}(y). \tag{4.2}$$

In other words, we have essentially only "two-point" fluctuations, and these are relatively simple in nature. When used with a discrete spectrum, as given by (3.1), $\overline{\rho(x)\rho(y)}$ has both a two-level part and a singular one-level self-correlation part which we may subtract from it to produce a "true" two-level function. Taking the density to be $\rho(x) = d^{-1}\sum\delta(x - E_i)$, we have

$$\overline{\rho(x)\rho(y)} = \{\overline{\rho(x)\rho(y)}\}_2 + \{\overline{\rho(x)\rho(y)}\}_1$$

$$= \{\overline{\rho(x)\rho(y)}\}_2 + d^{-1}\,\delta(x-y)\overline{\rho}(x) \tag{4.3}$$

and then, for Dyson's two-level cluster function Y_2,

$$\{\overline{\rho}(x)\overline{\rho}(y)\}^{-1} S^\rho(x,y) = [\overline{\rho}(x)d]^{-1}\delta(x-y) - Y_2(r,X)$$

$$= \delta(r) - Y_2(r;X) \to \delta(r) - Y_2(r), \tag{4.4}$$

in which $X = (x+y)/2$ defines the center of the interval, while $r = (y-x)/D(x)$ measures the length of the interval in average local spacing units.[12] We shall see that for "small" intervals (i.e., those over which the ensemble-averaged density may be taken as constant; with large enough d, of course, such an interval may contain as many levels as we wish) all three of the standard Gaussian ensembles give $Y_2(r,X)$ to be independent of X; we accordingly write it as $Y_2(r)$, as we have done in the last form of (4.4). This independence defines stationarity; it tells us that all two-point fluctuations, measured in terms of local spacing units, are the same in all parts of the spectrum. By construction, of course, the same result holds for the circular ensembles. It is adequate for us to take small intervals only; the covariance function will be seen to decrease as an inverse power of r and becomes therefore negligible when r is an appreciable fraction of d.

Just as with the two-point function S^ρ we encounter a term with a single δ-function singularity, so with the k-point function we find terms with up to $(k-1)$-fold δ-function products. We shall have little or no need of these higher-order functions until we discuss ergodicity and stationarity in Sec. X.

In what follows we shall calculate (4.2) or, equivalently,

$$S^F(x,y) = \int_{-\infty}^{x} dx' \int_{-\infty}^{y} dy'\, S^\rho(x',y')$$

$$= \overline{F(x)F(y)} - \overline{F(x)}\,\overline{F(y)} \tag{4.5}$$

for the GOE in the asymptotic $d \to \infty$ limit, by calculating the moments of $S^\rho(x,y)$ under the assumption that the binary associations dominate when traces of operators are to be computed. The resulting expressions for $S^F(x,y)$, Eqs. (4.18)–(4.20), show the essential simplicity of the two-point function. We proceed differently from the conventional way which starts with the joint distribution for the eigenvalues (2.11) and then by multiple integrations [$(d-1)$ of them for the density, $(d-2)$ for the two-point function, and so forth] produces the desired functions. The k-point functions for the three standard ensembles have in fact been evaluated by Dyson (1970) and Mehta (1971); the methods used are elegant, but the results are far from transparent;

[12] The renormalization here is essentially equivalent to the unfolding in Sec. III.A.

moreover, because, in deriving the starting equation (2.11) for the GOE, heavy use has been made of the invariance properties of the ensemble, it is not immediately obvious to what extent the final results depend on this property and how much they would be modified if the exact invariance requirement were dropped, while, of course, maintaining the symmetry properties of the individual matrices [see Wigner (1958) for a relevant discussion). It is better for us to settle for an approximate (though surprisingly accurate) calculation, for which we use a quite different starting point (French *et al.*, 1978a). For good measure we shall extend the results to the unitary and symplectic ensembles and will in fact show that they are exact for the unitary case (Pandey 1978).

B. Binary-association expansions

We begin by considering moments of the two-point function in both its variables,

$$\Sigma^2_{p,q} \equiv \int S^p(x,y)x^p y^q dx\,dy = \overline{\langle H^p\rangle\langle H^q\rangle} - \overline{\langle H^p\rangle}\,\overline{\langle H^q\rangle}\,. \tag{4.6}$$

In Sec. III.D we saw that for the GOE one-point function, the density, only unlinked binary associations need be considered; our GOE normalization ($v^2 d = 1$) leads to $[\langle H^2\rangle]_e = \langle \underbrace{\overline{HH}}\rangle = 1$, so that the density moments simply count the number of such associations in the corresponding power of H. For the reasons discussed there we need consider only binary associations for $\Sigma^2_{p,q}$; but now the two members of an associated pair may belong to the same trace or may be found one in each; calling the number of pairs in the latter case ζ, the number of cross-associated pairs, we can group the terms in (4.6) according to ζ (Mon and French, 1975); thus

$$\Sigma^2_{p,q} = \sum_{\zeta=1}^{m} \langle \underbrace{\overline{H^p\rangle\langle H^q}}_{\zeta}\rangle\,, \quad m = (p,q)_<\,. \tag{4.7}$$

Here $\underbrace{\quad}_{\zeta}$ indicates that there are ζ such pairs, formed with one H in $\langle H^p\rangle$ and the other in $\langle H^q\rangle$. The term with $\zeta = 0$ has been cancelled by $\langle \overline{H^p}\rangle\langle \overline{H^q}\rangle$. Inspection of (4.7) suggests that $S^p(x,y)$ can be expanded as $\sum_\zeta f_\zeta(x) f_\zeta(y)$; but to make the sum in (4.7) break off at m, it is necessary that the moments $\int f_\zeta(x)x^p dx$ should vanish when $\zeta > p$, and similarly for y. Thus the f_ζ must oscillate, and ζ should be the number of zeros in the interval $[-2,2]$, the domain of the variables. We shall see, in fact, that, with the natural choice, f_ζ represents a polynomial excitation of the semicircle, the polynomials being related to those orthonormal with the semicircular density as weight function. ζ becomes then the polynomial order, thus defining a characteristic wavelength for the corresponding excitation, and the vanishing of the moment follows from the fact that x^p is a linear superposition of polynomials of orders p and lower.

In the factor H^p of the ζ term there are then $(p-\zeta)$ H's occurring as $(p-\zeta)/2$ "allowed" binary associations, and similarly for the $\langle H^q\rangle$ factor; "allowed" here means that these pairs are unlinked among themselves and do not contract around any of the remaining H's which are cross-linked between the two traces. Since each allowed pair gives a factor of unity ($\langle \underbrace{\overline{HH}}\rangle = 1$), we get then, as a factor, μ^p_ζ, the number of "allowed" ways of inserting $(p-\zeta)$ points into a lattice containing ζ points. The added H's must not contract about any of the lattice points; on the other hand, because of the cyclic invariance of traces, they *can* contract around the ends of the lattice, which is therefore better thought of as ζ points arranged on a circle. For $\zeta = 2$, $p = 4$, for example, we have, letting X label the lattice points, the allowed arrangements XH^2X, H^2XX, $HXXH$, XXH^2, so that $\mu^4_2 = 4$. Then extending the orthogonal ensemble results ($\beta = 1$) to the unitary ($\beta = 2$) and simplectic ($\beta = 4$) cases, as well, we have

$$\Sigma^2_{p,q} = \sum_{\zeta=1}^{m} \mu^p_\zeta \mu^q_\zeta \langle \underbrace{\overline{H^\zeta\rangle\langle H^\zeta}}_{\zeta}\rangle = \frac{2}{\beta d^2}\sum_{\zeta=1}^{m} \zeta \mu^p_\zeta \mu^q_\zeta\,, \tag{4.8}$$

where we have used the first of two essential correlation results, valid for fixed ζ when d is large, which are derived, along with a third, in Appendix J

$$\langle \underbrace{\overline{H^\zeta\rangle\langle H^\zeta}}_{\zeta}\rangle = \frac{2\zeta}{\beta d^2}\,, \tag{4.9}$$

$$\langle \underbrace{\overline{H^\zeta H^\zeta}}_{\zeta}\rangle = 1\,. \tag{4.10}$$

It should be observed that an uncorrelated H in any density moment averages to zero, because of the symmetry about zero of the matrix-element distributions; thus the odd moments vanish. A similar result holds for higher-order quantities, as, for example, the covariance of an odd and even moment. Thus μ^p_ζ and $\Sigma^2_{p,q}$, and the similar quantities λ^p_ζ, $\Sigma^2_{p,q}(K)$ encountered below, vanish unless p, q, ζ have the same parity. We remark also that with the normalizations of the matrix-element distributions given in Eq. (2.5) the ensemble-averaged density is the same (semicircular) for all three cases and so therefore will be the orthonormal polynomials introduced below.

To evaluate the μ^p_ζ we first express them in terms of closely related quantities λ^p_ζ which count the number of ways of inserting $(p-\zeta)$ points into the same linear lattice of ζ points as before,[13] but now without allowing contractions around the ends. Of the four arrangements above for the case $\zeta = 2$, $p = 4$, $HXXH$ is then forbidden, so that $\lambda^4_2 = 3$. It is clear that the two kinds of counting are closely related. Indeed, we can construct the μ^p_ζ by first inserting $(p-\zeta-t)$ points, pairwise, into the $\zeta - 1$ internal slots of the lattice, which can be done in $\lambda^{p-\zeta-t}_{\zeta-2}$ ways; the remaining t points are added at the ends, which can be done in $(t+1)\lambda^t_0$ ways, the factor $(t+1)$ accounting for the number of ways of dividing the t points between the two ends. Summing over the possible values of t, we have

$$\mu^p_\zeta = \sum_t (t+1)\lambda^{p-\zeta-t}_{\zeta-2}\lambda^t_0\,. \tag{4.11}$$

Now the λ^p_ζ can be obtained from the moments of the

[13] See Appendix K and French *et al.* (1978) for a derivation of the two-point function in terms of a modified two-point function (a parametric derivation of the function which we are seeking). The λ^p_ζ play the same role with respect to the modified function as do the μ^p_ζ for the true one.

density itself; from their definition (3.5) we have

$$\bar{M}_{p+q} \equiv \langle \overline{H^{p+q}} \rangle = \sum_{\zeta=0}^{m} \langle \underbrace{\overline{H^p H^q}}_{\zeta} \rangle = \sum_{\zeta=0}^{m} \lambda_\zeta^p \lambda_\zeta^q . \tag{4.12}$$

Here the λ's are relevant, rather than the μ's, because the trace, while invariant under overall cyclic permutations, is not so under cyclic permutations in one of the factors; in other words, when inserting H's to form H^p, the presence of H^q forbids contractions around the ends, and analogously for H^q. From the integral definition of the moments we have an alternative expansion,

$$\begin{aligned}\bar{M}_{p+q} &= \int \bar{\rho}(x) x^{p+q} dx \\ &= \int \bar{\rho}(x) x^p y^q \delta(x-y) dx\, dy \\ &= \sum_{\zeta \geq 0} \int \bar{\rho}(x) x^p v_\zeta(x) dx \int \bar{\rho}(y) y^q v_\zeta(y) dy \\ &= \sum_{\zeta \geq 0} I_\zeta^p I_\zeta^q \end{aligned} \tag{4.13}$$

in the third form of which we have expanded the δ function in orthogonal polynomials. Since both λ_ζ^p and I_ζ^p vanish when $\zeta > p$, it is easy to see that to within a sign, which we now fix in accord with the standard convention for $v_\zeta(x)$, we must have $I_\zeta^p = \lambda_\zeta^p$, which identifies λ_ζ^p as the pth moment of a v_ζ excitation of the semicircle,

$$\lambda_\zeta^p = \int \bar{\rho}(x) x^p v_\zeta(x) dx . \tag{4.14}$$

The density $\bar{\rho}(x)$ now determines, by (4.14), the λ_ζ^p, and these in turn, via (4.11), give the μ_ζ^p which generate the moments $\Sigma^2_{p,q}$ (4.6), whose inversion gives the two-point function $S^\rho(x,y)$ which we are seeking. In the next section we carry out these steps explicitly.

C. The evaluation of the two-point function

For GOE the density $\bar{\rho}(x)$, given by Wigner's semicircle (3.17), defines the orthonormal polynomials as

$$\begin{aligned} v_\zeta(x) = U_\zeta(x/2) &= (-1)^\zeta \frac{\sin[(\zeta+1)\psi(x)]}{\sin\psi(x)} \\ &= \sum_m (-1)^m \binom{\zeta - m}{m} x^{\zeta - 2m}, \end{aligned} \tag{4.15}$$

where $U_\zeta(x)$ is the standard Chebyshev polynomial of the second kind defined for the range $(-1,1)$, and $\psi(x)$ is the angle between the negative x axis and the radius vector. The inverse of (4.14) gives [Abramowitz and Stegun (1964); note the correction given as a footnote by French *et al.*, 1978]

$$x^p = \sum_{\zeta=0}^{p} \lambda_\zeta^p v_\zeta(x) , \quad \lambda_\zeta^p = \frac{\zeta+1}{p+1} \binom{p+1}{\frac{p-\zeta}{2}} \tag{4.16}$$

if $(p-\zeta)$ = even, from which by evaluating the sum in (4.11) we find (Riordan, 1968, p. 169)

$$\begin{aligned} \mu_\zeta^p = \frac{p}{\zeta} \lambda_{\zeta-1}^{p-1} &= \binom{p}{\frac{p-\zeta}{2}} = \frac{p}{\zeta} \int \bar{\rho}(x) x^{p-1} v_{\zeta-1}(x) dx \\ &= -\frac{1}{\zeta} \int x^p \frac{d}{dx} \{\bar{\rho}(x) v_{\zeta-1}(x)\} dx , \end{aligned} \tag{4.17}$$

where in the third step we have used (4.14) and in the fourth an integration by parts. But now from (4.2), (4.5), and (4.8) we see that

$$\begin{aligned} S^\rho(x,y) &\equiv \frac{\partial}{\partial x} \frac{\partial}{\partial y} S^F(x,y) \\ &= \frac{2}{\beta d^2} \frac{\partial}{\partial x} \frac{\partial}{\partial y} \left(\bar{\rho}(x) \bar{\rho}(y) \sum_{\zeta \geq 1} \zeta^{-1} v_{\zeta-1}(x) v_{\zeta-1}(y) \right) . \end{aligned} \tag{4.18}$$

Since our treatment of the highest-order excitations is not completely trustworthy, we have an unwanted freedom in the explicit evaluation of the ζ sum in (4.18). We can, for example, use a sharp cutoff, taking $\zeta_{\max} \sim d$, or an exponential cutoff, where we evaluate by inserting an $e^{-\alpha\zeta}$ factor (with $\alpha \sim d^{-1}$), replacing the sum by an integration. Using the first procedure, along with (4.15) and (4.18), we find (French *et al.*, 1978) that, as long as (x,y) are separated by at least a few levels,

$$\begin{aligned} S^F(x,y) &= \frac{1}{\beta\pi^2 d^2} \ln \frac{2(x-y)}{4 - xy - [(4-x^2)(4-y^2)]^{1/2}} \\ &\xrightarrow[\text{small } |x-y|]{} \frac{1}{\beta\pi^2 d^2} \ln \frac{4\sin^2\psi}{|x-y|} \\ &= \frac{1}{\beta\pi^2 d^2} \ln \frac{4d\sin^3\psi}{\pi|r|} , \end{aligned} \tag{4.19}$$

where r, whose absolute value $|r|$ gives the number of levels in the (x,y) interval, is given by $r = 2d\pi^{-1}[\psi(y) - \psi(x)]\sin^2\psi[(x+y)/2] \simeq 2d\pi^{-1}\delta\psi \sin^2\psi$. For the variance $(x=y)$, instead of the covariance, we find, by inserting a preliminary cutoff in the ζ expansion, the result that

$$S^F(x,x) = \frac{1}{\beta\pi^2 d^2} \ln(2d\sin\psi) = \frac{1}{\beta\pi^2 d^2} \ln[2\pi d\bar{\rho}(x)] . \tag{4.20}$$

We find similarly for the density correlation (in which, as indicated, we renormalize to constant local spacing),

$$\frac{S^\rho(x-y)}{\bar{\rho}(x)\bar{\rho}(y)} = -\frac{4}{\beta d^2} \frac{4 - xy}{(4-x^2)(4-y^2)(x-y)^2} \xrightarrow[r \gtrsim 1]{} -\frac{1}{\beta\pi^2 r^2} . \tag{4.21}$$

We see from (4.21) that the asymptotic (large-r) form, first given, for $\beta = 1$, by Dyson (1962c) for the central region of the spectrum, applies, in fact, over the whole spectrum. It will, however, turn out that the two-point function given by (4.19) and (4.20) gives a slow variation of the spectral measures over the spectrum (French *et al.*, 1978), which would imply that the spectrum (or, more strictly, its map to a spectrum with constant density $\bar{\rho}$) is the analog of a *quasi-stationary* process. This raises the question as to whether the (mapped) spectrum is in fact truly *stationary*; Sec. X will give an affirmative answer, derived from the exact two-point function. We ask therefore whether a different cutoff in the ζ expansion will give stationary results as derived from the *approximate* two-point

function. In fact, the cutoff which does so is more natural *a priori* than the ones we have mentioned.

As a reasonable criterion we argue that the cutoff method should give a stationary self-correlation (one-level) contribution to the two-point function. Also, in line with the usual notions of a normal-mode expansion, we should, in order to produce the Y_2 function (4.4), subtract out a self-correlation term separately for each normal mode (Pandey, 1978). The procedure implied by this will give strictly stationary results which will turn out to be exact for the unitary ensemble, $\beta=2$ (though not for $\beta=1,4$). We would be inclined to argue that for all three cases it gives the "true" binary-association contribution to the two-point function.

We use a sharp cutoff with $\zeta_{\max}=2\beta' d\sin^2\psi$ in which the ψ dependence of β' will be fixed below. Then, writing $\psi(x)=\psi_1$, $\psi(y)=\psi_2$, we have for the self-correlation term

$$\frac{\delta(x-y)}{d\bar{\rho}(x)}\to d^{-1}\sum_{\zeta=0}^{\zeta_{\max}}\frac{\sin(\zeta+1)\psi_1\sin(\zeta+1)\psi_2}{\sin\psi_1\sin\psi_2}$$
$$=\frac{1}{2d\sin\psi_1\sin\psi_2}\times\sum_{\zeta=0}^{\zeta_{\max}}[\cos(\zeta+1)(\psi_1-\psi_2)-\cos(\zeta+1)(\psi_1+\psi_2)]$$
$$\xrightarrow[d\to\infty]{}\beta'\left(\frac{\sin\beta'\pi r}{\beta'\pi r}\right) \tag{4.22}$$

and for $S^{\rho}(x,y)$

$$\frac{S^{\rho}(x,y)}{\bar{\rho}(x)\bar{\rho}(y)}\to\frac{1}{2\beta d^2\sin^2\psi_1\sin^2\psi_2}\sum_{\zeta=1}^{\zeta_{\max}}\zeta\cos\zeta\psi_1\cos\zeta\psi_2$$
$$=\frac{1}{4\beta d^2\sin^2\psi_1\sin^2\psi_2}\times\sum_{\zeta=1}^{\zeta_{\max}}\zeta[\cos\zeta(\psi_1-\psi_2)+\cos\zeta(\psi_1+\psi_2)]$$
$$\xrightarrow[d\to\infty]{}-\frac{(\beta')^2}{\beta}\left(\frac{1-\cos\beta'\pi r}{(\beta'\pi r)^2}-\frac{\sin\beta'\pi r}{\beta'\pi r}\right). \tag{4.23}$$

The first forms in (4.22) and (4.23) follow by using (4.15), respectively, in the self-correlation term of S^{ρ} (4.4) and in S^{ρ} itself (4.18). The final forms follow by taking the limit $\zeta_{\max}\to\infty$; the sums involving $(\psi_1+\psi_2)$ vanish in the limit.

Now, using (4.4) to combine (4.22) and (4.23), we see that, for the coefficient of the self-correlation term, $\sin(\beta'\pi r)/(\beta'\pi r)$, to vanish, we must have $\beta'=\beta$. We thus have a natural cutoff defined by $\zeta_{\max}=2\beta d\sin^2\psi$. Our final result is then

$$Y_{2,\beta}(r)=\frac{1-\cos\beta\pi r}{\beta(\pi r)^2}=\frac{2\sin^2\left(\frac{\beta\pi r}{2}\right)}{\beta(\pi r)^2}, \tag{4.24}$$

which gives the binary-correlation part of the exact function. Combining this with (4.4) we have the forms for $S^{\rho}(x,y)$. The results are exact for the unitary ensemble ($\beta=2$) but not for $\beta=1,4$; for these the exact forms, given in Appendix L, give rise to the $1/r$ expansions

$$Y_{2,1}=\frac{1}{\pi^2r^2}+\frac{1-\cos^2\pi r}{\pi^4r^4}+\cdots$$
$$Y_{2,4}=-\frac{\cos2\pi r}{4r}+\frac{1+2\pi^{-1}\sin2\pi r}{4\pi^2r^2}-\frac{1+\cos^2 2\pi r}{16\pi^4r^4}+\cdots. \tag{4.25}$$

These are different in their fluctuating components from the results of (4.24); but the locally smoothed versions, when the interval contains at least a few levels, are essentially identical, so that the fluctuation measures, which depend on integrated versions of $Y_2(r)$, will be almost identical. This is, however, not the case for very small r, say, $r\ll1$; for example, the exact forms give unity for $r=0$, while the approximate (4.24) gives $\beta/2$. Thus, except for $\beta=2$, excitations of wavelength $\lesssim D$ are inadequately treated by the polynomial expansion, and similarly therefore for measures of the very short-range fluctuations, such as Q of Sec. V.F.

It is well known that the unitary ensemble is structurally simpler than the other two (Wigner, 1962) and is indeed "almost trivial from a mathematical point of view" (Mehta and Dyson, 1963, p. 713). It is perhaps not then surprising that our simple procedure produces the exact result. On the other hand, we give in Sec. V a general result connecting the widths of spacing distributions with the variance of the number of levels in a fixed interval; if we combine that with two general theorems (Dyson, 1962c; Mehta and Dyson, 1963) which relate the statistical behavior of the three ensembles, we can, as shown in Sec. V, derive the orthogonal and symplectic two-point functions from the unitary one, ending up with results of good accuracy for all three cases.

D. Level motions

An essential feature of the spectrum, for which we have seen evidence in Sec. I, and which dominates the behavior of the fluctuations in a fixed region of the spectrum, becomes apparent when we consider the motions of the individual levels and the correlation between the motions of two levels.

Let us agree that, when we speak of a level motion, we have in mind the variation in the energy of a given numbered level, the levels being ordered by energy and the motions measured in locally-averaged spacing units, as H moves through the ensemble; during this motion the state vector uniformly covers the unit sphere in the d-dimensional space. (Of course, our real interest is in the way things behave when we move along the spectrum of a given H which adequately describes the system; the relationship between the two ways of looking at things is one aspect of ergodicity.)

The essential result, which we have seen signs of in Sec. I, is that the individual levels move by only a small amount [$\sim\pi^{-1}(\ln d)^{1/2}$ in the spectrum center for large d], while the relative motion of two levels, which are on the average rD apart, is even smaller [$\sim\pi^{-1}(\ln r)^{1/2}$, independent of d for large d], so that the motions are very highly correlated. Let us see where these results, first derived quite differently by Dyson

(1962c) and Dyson and Mehta (1963), come from; for more details and a justification of the last step in (4.28) below, see French *et al.*, (1978). If x_t, with $1 \leq t \leq d$, is the tth eigenvalue (the ordering being by energy) and $\bar{x}_t$ its ensemble-averaged value, then with $x_t = \bar{x}_t + \delta x_t$,

$$\rho(x) = d^{-1} \sum_t \delta(x - \bar{x}_t - \delta x_t) \to \int_{-\infty}^{\infty} \delta(x - x' - \delta x') \bar{\rho}(x') dx'$$

$$\simeq \bar{\rho}(x) - \frac{d}{dx} [\bar{\rho}(x) \delta x] , \tag{4.26}$$

which defines $\delta x (\equiv \delta x_t)$ as a random variable. Integrating this gives, with $D(x) = [d\bar{\rho}(x)]^{-1}$ = the average level spacing,

$$\frac{\delta x}{D(x)} \simeq - [F(x) - \bar{F}(x)] d \tag{4.27}$$

and then, for the mean squared level motion expressed in local spacing units and evaluated at the spectrum center in accordance with the discussion above,

$$\frac{\overline{(\delta x)^2}}{[D(x)]^2} \simeq d^2 S^F(x,x) = \frac{1}{\beta \pi^2} \ln[2d \sin\psi(x)]$$

$$= \frac{1}{\beta \pi^2} \ln[2\pi d \bar{\rho}(x)] \underset{x=0}{\longrightarrow} \frac{1}{\beta \pi^2} \ln 2d . \tag{4.28}$$

In deriving (4.28) we have for convenience used (4.20) in place of the marginally better result which would follow from the procedure leading to (4.24). For $d = 10^6$ the rms motion is 1.2 level spacings and, for $d = 10^{12}$, only 1.7 spacings.

For the covariance of two level motions, and their correlation coefficient C, we find similarly

$$\frac{\overline{\delta x \delta y}}{D^2} \simeq d^2 S^F(x,y) \to \frac{1}{\beta \pi^2} \ln\left(\frac{4d}{\pi r}\right) , \tag{4.29}$$

$$C \simeq \frac{\overline{\delta x \delta y}}{(\overline{\delta x^2}\, \overline{\delta y^2})^{1/2}} \to 1 - \ln(\pi r/2)/\ln(2d) . \tag{4.30}$$

Thus, as we increase the dimensionality, the level motions grow logarithmically, but, at the same time, become more highly correlated, the quantity $(1 - C)$ going to zero as the inverse of a logarithm. However, for two random variables with the same variance σ^2 and with correlation coefficient C, it is a simple standard result that the difference of the variables has variance $2\sigma^2(1 - C)$ and thus, for the variance of an interval whose end-points are levels numbered t, $t + r$, we have[14]

$$\sigma^2(r-1) \simeq d^2[S^F(x,x) + S^F(y,y) - 2S^F(x,y)]$$

$$\to 2d^2[S^F(x,x) - S^F(x,y)] \to \frac{2}{\beta \pi^2} \ln(\pi r/2) \tag{4.31}$$

[in the second step, we have, as usual, taken for granted that x, y are close enough that the curvature of the density in the interval (x,y) is negligible]. Thus, as we have indicated above, while a given (numbered) level moves, for large d, $\sim \pi^{-1}(\ln d)^{1/2}$, the fluctuation in the span of the levels is smaller and independent of d. This comes about, of course, because the individual-level motions are dominated by the long-wavelength excitations [small ζ, as in (4.37) ahead] and these cannot distinguish two close levels.

We have now the very important result, first given by Dyson and Mehta (1963) for the center of the spectrum, but seen now to apply over the entire spectrum, that the spectrum forms an almost rigid ("semicrystalline") system, a behavior which is unaffected by the secular deformation of the "crystal" corresponding to the rapid decrease in the level spacing (by a factor $\sim 10^4$) as we go from the ground-state domain to the upper limit of the discrete spectrum (the slow-neutron region). This phenomenon leads us to the view, *inter alia*, that the amount of information which can be extracted from spectra, and also from strength distributions (which derive by parametric derivatives on spectra) is quite limited. We return to this later.

E. The "ensemble density" for GOE ($\beta = 1$)

Any given spectrum can be described, as has been done in Fig. 6, in terms of a normal-mode expansion, the modes being labeled by the parameter ζ. If we regard the R_ζ, the coefficients in the expansion of the distribution function $F(x)$, as random variables whose statistical properties are appropriately determined, the expansion defines not just a single spectrum but a whole ensemble of them, giving then the "ensemble density" (*a random function*[15]) $\rho_e(x)$ and distribution function $F_e(x)$. Thus by inspection of (4.18) we see that if we take

$$\bar{R}_\zeta = 0 , \quad \overline{R_\zeta R_{\zeta'}} = \frac{2\zeta}{d^2} \delta_{\zeta\zeta'} \tag{4.32}$$

(the higher-order statistical properties being, of course, determined only by higher-order correlation functions which we have not evaluated), then the density and two-point function will be reproduced with

$$F_e(x) = \bar{F}_e(x) - \bar{\rho}(x) \sum_{\zeta \geq 1} \zeta^{-1} R_\zeta v_{\zeta-1}(x) , \tag{4.33}$$

where $2\pi \bar{F}(x) = 2\psi(x) - \sin 2\psi(x)$. The ensemble density follows by differentiation, with $\bar{F}'(x) = \bar{\rho}(x) = \pi^{-1} \sin\psi$ and

$$[\bar{\rho}(x) v_{\zeta-1}(x)]' = (-1)^{\zeta+1} [2\pi \sin\psi(x)]^{-1} \cos\zeta\psi(x) ,$$

which is essentially a Chebyshev polynomial of the first kind.

If we make random choices on the R_ζ, subject, of course, to (4.32), we can, by evaluating the ζ sums, construct an *ensemble of spectra* which will have essentially the same density and two-point correlations as the GOE but with different higher-order fluctuations. We see that (4.33), along with its higher-order extensions, solves the GOE problem in a very different way than does the standard procedure of generating the joint probability distribution (2.11). "Two-point" spectra have been numerically constructed using (4.33) in order to see whether, by visual comparison with GOE

[14] To avoid later confusion we mention that, by a very silly convention which it is now too late to change, the variance of the nearest-neighbor spacing distribution is $\sigma^2(0)$, the second-nearest $\sigma^2(1)$, etc.

[15] From one standpoint this is simply another representation of the density (3.1), but it is expressed now not in terms of the eigenvalues, whose statistical properties are complicated, but in terms of quantities R_ζ, whose low-order statistical properties are both simple and adequate for our purpose.

spectra, one can detect the effects of higher-order correlations. One finds, in fact, that the two-point spectra are much "milder," i.e., have far fewer small and large spacings; but the test so far has not been conclusive and we do not show the spectra.

F. EGOE fluctuations

As we have said, the embedded Gaussian orthogonal ensemble of k-body interactions represents *a priori* a better model than the GOE, which necessarily describes simultaneous interactions between all the particles. The EGOE eigenvalue density is well understood (Sec. III). There is as yet no theory for the short-range fluctuations,[16] but we can make some progress toward understanding, in general terms, why they apparently coincide with GOE fluctuations. Besides that, we shall be able to understand the very sharp separation of secular behavior and fluctuations, an example of which we have seen in Fig. 6.

Let us proceed by evaluating the moment covariances just as we have done with GOE. The differences are threefold (Mon and French, 1975).

(1) Instead of a unit value, as with GOE, the basic pair-correlated average $\overline{HH}$, Eq. (3.12), gives $\binom{m}{k}$, where m is the particle number and k is the particle rank.

(2) Instead of the value $2\zeta/d^2$, Eq. (4.9), the basic cross-correlated average becomes $2\zeta\binom{m}{k}^2\binom{N}{k}^{-2}$, where N is the number of single-particle states so that $\binom{N}{k}$ is the GOE dimensionality [see Mon and French, (1975), Eq. (20)]. The fact that the normalization is $\binom{m}{k}^2$, rather than the $\binom{m}{k}^\zeta$ which one might have guessed, gives rise to a very strong attenuation of the mode amplitudes with increasing ζ.

(3) The counting associated with the correlated H's in the first factor of the covariance is no longer μ_ζ^p, as in Eqs. (4.8) and (4.17), but rather

$$\binom{p}{\zeta}(p-\zeta-1)!!\binom{m}{k}^{(p-\zeta)/2},$$

and similarly with $p\to q$ for the second factor. Here $\binom{p}{\zeta}$ counts the number of ways of choosing ζH's to be cross-correlated between the factors, while $(p-\zeta-1)!!\binom{m}{k}^{(p-\zeta)/2}$, the Gaussian moment of order $(p-\zeta)$, counts the number of pairwise associations of the remaining H's in the first factor, as in Eq. (3.14). We are assuming here that $m\gg k$, the only case we consider.

As in the analog of Eq. (4.8), we have then

$$\begin{aligned}\Sigma^2_{p,q}(m)&=\langle\overline{H^p}^m\langle H^q\rangle^m\rangle-\langle\overline{H^p}^m\rangle\langle\overline{H^q}^m\rangle\\&=2\binom{N}{k}^{-2}\sum_{\zeta\geq1}\zeta\binom{p}{\zeta}\binom{q}{\zeta}(p-\zeta-1)!!\\&\quad\times(q-\zeta-1)!!\binom{m}{k}^{(p+q)/2+2-\zeta}.\end{aligned}\tag{4.34}$$

Now, just as with GOE in the preceding section, we express the ensemble density and distribution function in terms of random variables S_ζ, whose linear and quadratic properties we know. We have

$$\begin{aligned}&\rho_e(x)=\rho_G(x)\Big(1+\sum_{\zeta\geq3}\zeta!^{-1}S_\zeta He_\zeta[(x-\mathcal{E})/\sigma]\Big),\\&F_e(x)=F_G(x)-\sigma\rho_G(x)\sum_{\zeta\geq3}\zeta!^{-1}S_\zeta He_{\zeta-1}[(x-\mathcal{E})/\sigma],\\&\overline{S_\zeta}=0\,;\ \overline{S_\zeta S_{\zeta'}}=2\zeta\delta_{\zeta\zeta'}\binom{m}{k}^{2-\zeta}\binom{N}{k}^{-2},\end{aligned}\tag{4.35}$$

where in $\rho_G\equiv G(\mathcal{E},\sigma)$ the centroid and variance fluctuations are already included[17] (hence the explicit expansion begins with $\zeta=3$, rather than $\zeta=1$). We have used here the fact that $\sigma^p(\zeta^p)(p-\zeta-1)!!$ is the pth moment of $\rho_G(x)He_\zeta[(x-\mathcal{E})/\sigma]$.

The significant aspect of (4.35) is the damping of the higher-ζ (shorter wavelength) excitations, the origin of which we have traced above to the behavior of the basic cross-correlated trace. If the damping should persist to the highest ζ, we would see no spectral fluctuations at all, but instead would find a uniform spectrum in the limit of large particle number. It is easy enough to see, however, that when $p\gtrsim m/k$, different H's in a moment trace can no longer act on different particles and our procedure will then no longer be adequate. Thus the results (4.35) are guaranteed valid only for $\zeta\ll m/k$. Let us keep this in mind as we use (4.35) to treat the normal-mode decomposition of the level motion.

We have via (4.35) the assurance that the variance of the level motion, expressed in terms of the square of the local spacing, is (for $m\gg k$)

$$\begin{aligned}\frac{\overline{(\delta x)^2}}{D(x)^2}&=2d^2\sigma_0^2[\rho_G(x)]^2\binom{N}{k}^{-2}\\&\quad\times\sum_{\zeta\geq1}\zeta(\zeta!)^{-2}\binom{m}{k}^{2-\zeta}[He_{\zeta-1}(x/\sigma_0)]^2\\&\xrightarrow[x=0]{}\frac{1}{\pi}\binom{N}{m}^2\binom{N}{k}^{-2}\sum_{\substack{\zeta\geq1\\(\zeta\,\text{odd})}}2^{\zeta-1}\zeta\Big(\frac{\zeta-1}{2}!\Big)^2\binom{m}{k}^{2-\zeta}\\&=\frac{1}{\pi}\binom{N}{m}^2\binom{N}{k}^{-2}\binom{m}{k}\\&\quad\times\left[1+\frac{1}{12}\binom{m}{k}^{-2}+\frac{1}{320}\binom{m}{k}^{-4}+\cdots\right]\end{aligned}\tag{4.36}$$

(and similarly of course for energies away from the spectrum center, for which there will also be even-ζ contributions). The long-range contributions to $[(\delta x)^2]_e$ are very large (really because the number of statistically independent matrix elements is far smaller than the number of levels[18]), but they die out rapidly as

[16] Except for $k=1$, which gives Poisson for $m\geq2$. See Appendix M.

[17] This is formally a little better than the choice we have made in (4.33). It is easy to see that when we consider the ζ decomposition of $(\delta x)^2$, as in Eq. (4.36) ahead, the ζ dependence given for $\zeta\geq3$ properly extends to $\zeta=1,2$, and similarly for ζ decompositions of other quantities.

[18] Or, looked at from the standpoint of the GOE, because the m-particle level spacing, which supplies the unit, is unnaturally small for the GOE dimensionality and normalization.

the order increases and would make no contribution to the fluctuations measured over a range containing a small number of levels (~100 in the slow-neutron experiments). The question now is whether as we increase ζ, these fluctuations disappear well before we reach the $\zeta \sim m/k$ limit of validity. Observing that

$$\pi^{-1}\binom{N}{m}^{2}\binom{N}{k}^{-2}\binom{m}{k}^{2-\zeta} \sim 1$$

for

$$\zeta \sim 2(m/k)\ln(N/m)/\ln(m/k)\,,$$

and taking account of the ζ-dependent coefficient in the summation (4.36), we have an affirmative answer to the question and a clear prediction then that the mode "intensity" in the $[(\delta x)^2]_e$ decomposition (and hence in the decomposition of other fluctuation measures) goes rapidly to zero with increasing ζ and stays there at least until the limit $\zeta \sim m/k$ is reached; this is exactly as we have found and would predict for the case of Fig. 6. Since, however, this case is not exactly EGOE, but rather an ensemble of mixed $k=1$ and $k=2$ interactions, and with specified J, T, a strict numerical comparison is not appropriate. We could regard the matrix dimensionality (1206) and particle number (8) as defining, via $\binom{N}{8}=1206$, an effective $N \sim 13$; then for $k=1$ and $k=2$ we would find $(\delta x)_{\rm rms}/D \sim 0.3$, 4.5, respectively. The observed value is in between.

We should remark that the intensity spectrum is very different for $[(\delta x)^2]_e$ in the GOE. We have from (4.20) and (4.27) the result that the contribution to $[(\delta x)^2]_e$ for modes with $\zeta \leq t$ is

$$\frac{\overline{(\delta x)^2}}{D(x)^2} \simeq \frac{2}{\pi^2}\sum_{\substack{\zeta=1\\(\zeta=\text{odd})}}^{t}\zeta^{-1}\sin^2\zeta\psi(x) \xrightarrow[x=0]{} \frac{2}{\pi^2}\sum_{\substack{\zeta=1\\(\zeta=\text{odd})}}^{t}\zeta^{-1}$$

$$=\frac{1}{\pi^2}\left(\gamma\ln 2s+\frac{1}{6s^2}-\frac{7}{60s^4}+\cdots\right), \qquad (4.37)$$

where $s=t$ if t is even and $(t+1)$ if t is odd. The summed result here is compatible with (4.28), derived for $t=d$ via an integration. The GOE intensity then falls off slowly (as ζ^{-1}) but regularly, there being no sharp separation into secular and short-range fluctuation parts. One is inclined to believe, because of the apparent identity of the GOE and EGOE fluctuations, that a secular map (semicircle → Gaussian) of the GOE spectrum to EGOE would lead to the same sharp separation. But this has not been verified.

Returning to the EGOE, we remark that the rapid attentuation of long-range fluctuations as ζ increases is due to the operation of a central limit theorem. The short-range fluctuation modes, however escape the action of the *CLT*, for, as we increase particle number, obtaining thereby a spectrum which is smoother on the scale supplied by the spectrum variance, the spacing unit which supplies the scale for the fluctuations rapidly decreases, so that the fluctuations are not smoothed away.

As we shall see in Sec. VI, these fluctuations seem more or less independent of dimensionality (over the narrow range which can be studied) and very close to GOE fluctuations and to those observed in nuclei. One outcome of this is that fluctuations do not give a method for detecting multiparticle interactions, and the *a priori* unreasonable GOE does represent the observations. But there is no real theory yet for EGOE fluctuations, that gap being one of the most significant ones in the entire subject. We can now only speculate as to why GOE and EGOE short-range fluctuations are apparently very similar and perhaps asymptotically identical, despite their very different normal-mode decompositions of $[(\delta x)^2]_e$. For one thing, in evaluating very high-order moments $\langle H^p\rangle^m$, we are tempted to introduce $\mathcal{H}=H^s$ as a new Hamiltonian with s large enough that H^s is dominated by its high-particle-rank components. Then $\langle H^p\rangle^m \to \langle \mathcal{H}^{p/s}\rangle^m$, which behaves as a matrix trace, related therefore to GOE. As another clue we can speculate that it is not statistical *independence* of the matrix elements which matters in the GOE but instead only the fact that they are *uncorrelated* in low orders. The EGOE matrix elements are, of course, not independent, but we would expect a small correlation for most pairs; even in the $(ds)^{12}$ example above, the significant thing might then not be the $(2\times10^7-63)$ reduction in degrees of freedom, but, rather, the fact that $63 \gg 1$.

G. Recent developments

We briefly describe some very new methods (Pandey, to be published) which have already been applied to some problems, and which promise to yield more general fluctuation results. They represent an extension to fluctuations of the Stieltjes-transform technique introduced by Pastur (1972) for dealing with the density and described in Sec. III.E.

For a given ensemble our interest now is in the two-point function $S^\rho(x,y)$ which is to be evaluated via its double Stieltjes transform $S^g(x,y)$:

$$S^\rho(x,y)=-\frac{1}{4\pi^2}\left[S^g(x,y)+S^g(x^*,y^*)-S^g(x^*,y)-S^g(x,y^*)\right], \qquad (4.38)$$

$$S^g(x,y)=\overline{g(x)g(y)}-\bar g(x)\bar g(y)$$

$$=\iint_{-\infty}^{\infty}\frac{S^\rho(x',y')}{(x-x')(y-y')}\,dx'dy'\,.$$

Recall that, when used as arguments of Stieltjes transforms, x,y have small positive imaginary parts and that, from (3.22), $2\pi i\rho(x)=g^*(x)-g(x)$. It is important also to recall that, as in Sec. III.E, Stieltjes transforms are generating functions for moments; thus the success of the transform method is closely related to the existence of simple moment recursion relations. In particular, for the two-point moments of the canonical Gaussian ensembles, we have, in binary-correlation approximation,

$$\Sigma^2_{p,q}=2\sum_{r=0}^{p-2}(\bar M_r\Sigma^2_{p-r,q})+\frac{2q}{\beta d^2}\bar M_{p+q-2}\,, \qquad (4.39)$$

which, by the procedure of Sec. III.E, yields

$$S^g(x,y)=-\frac{4-xy+[(x^2-4)(y^2-4)]^{1/2}}{\beta d^2(x-y)^2[(x^2-4)(y^2-4)]^{1/2}}\,. \qquad (4.40)$$

Now, without any explicit evaluation of the moments, inversion of (4.40) leads immediately to the two-point function (4.21).

Consider next special cases of the deformed ensembles which for the present purpose it is convenient to write as $H^\gamma = K + \gamma H$, where H is one of the three canonical ensembles, and the otherwise arbitrary operator K preserves the basic underlying symmetry of H, being, for example, representable as a real symmetric matrix when $\beta = 1$; it follows that H_α is characterized by a specific value of β. We then find as a generalization of (4.40), with $g^\gamma(x) = \langle (x - H^\gamma)^{-1} \rangle$, that

$$S^\epsilon_\gamma(x,y) \equiv \overline{g^\gamma(x) g^\gamma(y)} - \overline{g^\gamma}(x)\overline{g^\gamma}(y) = \frac{2}{\beta d^2} \frac{\partial^2}{\partial x \partial y} \ln\left[1 - \gamma^2 \left(\frac{\overline{g^\gamma}(x) - \overline{g^\gamma}(y)}{x - y}\right)\right], \quad (4.41)$$

which represents in compact form the transform of the two-point function for the deformed ensemble in terms of the one-point transforms. The two-point function and fluctuations follow from its inverse. If K itself is defined statistically, an extra term is encountered in (4.41). As a first application of (4.41) we could rederive the variance (3.29) of the position of an isolated level. Otherwise, we could treat the two-point fluctuations more generally, excluding, however, spectral regions in which the ensemble-averaged density is zero or singular or otherwise not well behaved. These things have been done.

Consider the effects on the fluctuations as we increase the random part, γH, of H^γ. Let us agree that K and H have comparable norms. As γ increases from zero, the fluctuations must go over from those of K to the fluctuations of H. The remarkable result which emerges from (4.41) is that the transition is extremely rapid, occurring for $\gamma \sim d^{-1/2}$ and being therefore discontinuous in the large-d limit. In other words, the effects of K on the fluctuations disappear rapidly as γ increases and as a consequence a very wide class of deformed ensembles gives the same canonical fluctuations.

As a special case of the deformed ensembles we have the partitioned ones for which the rapid transition was already predicted, via a Brownian-motion argument, in Dyson's (1962d) paper. Besides the deformed ensembles, there has also been a recent study of the general V_α ensembles of Sec. III.E, in which the matrix-element variances are taken differently for different elements, and which provide a better treatment for the transition which occurs with the partitioned ensembles.

As we have mentioned in Sec. II, symmetry effects on the fluctuations can be studied in terms of deformed and V_α ensembles. Once again the principal result is that the presence of even a small symmetry-breaking part in the interaction matrix would wash out entirely the effect of the symmetry on the fluctuation properties. A special case, that of the breaking of time-reversal invariance, will be dealt with explicitly in Sec. VII.F.

We remark finally that similar Stieltjes-transform calculations give the two-point fluctuations of Dyson's ensemble $\tilde{H}$ (Sec. II.H) as being identical to those of GOE. Adding to the list of these results the Monte Carlo results for the EGOE, we begin to see the emergence of a simple picture for the fluctuations, viz., that the standard ensembles yield the basic fluctuation patterns.

V. FLUCTUATION MEASURES

A. Introduction

We have, in Sec. I, given the Wigner surmise (1.5) for the distribution of the nearest-neighbor spacings, and several histograms showing a comparison of Wigner's equation with experiment and with shell-model results. Usually, however, it is preferable to compare a few characteristic parameters of the distributions ("statistics" or "fluctuation measures" as we have agreed to call them), rather than the distributions themselves. For example, it is to be expected (see Appendix N and Fig. 9) that as $d \to \infty$ the rth nearest-neighbor spacing distributions become Gaussian for large r. Since the centroid is necessarily $(r+1)D$, the only significant measure connected with the spacing distribution for large r would then be the width $\sigma(r)$.

It will turn out that "number variances" (Dyson and Mehta, 1963), the variances $\Sigma^2(r)$ of the number n of levels in a (small) interval of length $r \times D$ located at x, determine everything about the local two-point fluctuations; thus every measure of these fluctuations is expressible in terms of the $\Sigma^2(r)$. We have, with $y - x = rD$, where then $r = \overline{n}$ (if we agree that $y > x$), that

$$n \equiv n(x,y) = d \int_x^y \rho(x')dx' = d[F(y) - F(x)], \quad (5.1)$$

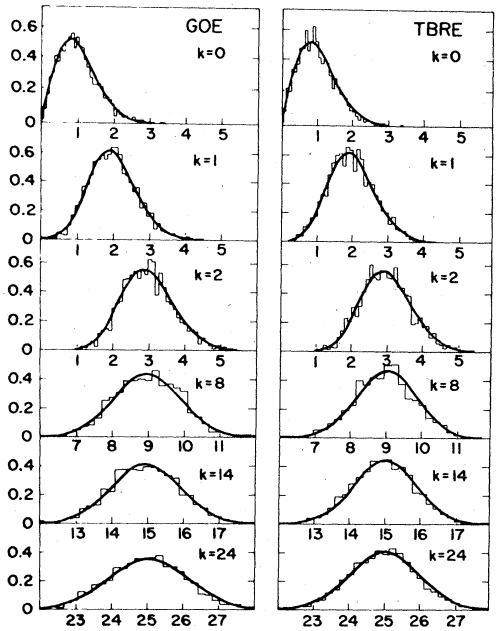


FIG. 9. Spacing histograms of various orders k for 294-dimensional GOE and TBRE of 50 members each (Wong and French, 1972). The distributions are compared for $k = 0$–2 with theoretical GOE results (Mehta, 1960; Gaudin, 1961; Kahn, 1963; Mehta and des Cloizeaux, 1972) and, for higher k values, with Gaussian forms. Note that the GOE and TBRE distributions are essentially identical. Similar analyses have been made, by O. Bohigas and R. U. Haq, of the slow-neutron data for nuclei with $A > 100$, and close agreements with GOE predictions have been found (private communication from R. U. Haq).

so that

$$\Sigma^2(r) = d^2 \int\int_x^y s^\rho(x', y')dx'dy'$$
$$= d^2[S^F(x,x) + S^F(y,y) - 2S^F(x,y)]$$
$$= 2\int_0^r (r-r')[\delta(r') - Y_2(r')]dr' , \qquad (5.2)$$

where in the last step we have used local stationarity. We see that $\Sigma^2(r)$ is itself stationary. The integrated quantities $S^F(x,x)$ and $S^F(x,y)$ are, of course, not separately stationary, so that only the difference, as in (5.2), enters into the evaluation of the local measures. Moreover, as we shall see ahead, the standard measures are dominated by the two-point function and, to the extent that that is so, are, via (5.2), functionals of the $\Sigma^2(r)$; thus the question of the independence of the measures is settled. We could introduce measures for correlations of higher than second order, though little has been done along that line (see, however, Sec. V.C). It is not really known whether higher-order measures, which should require longer runs of levels, can be determined with adequate accuracy from presently available data; the answer is probably in the negative. The variances of the measures are also important, for without them we should not know how to assess the significance of the comparison with experiment. For some measures analytic forms have been given for ensemble variances (Dyson and Mehta, 1963); in other cases both ensemble and spectral variances can be determined adequately by Monte Carlo calculations. Spectral variances are measures themselves and many of the standard measures are in fact variances. In principle, of course, the variance of a k-point measure involves correlation functions of orders up to $2k$, though in at least some cases the contributions from the higher-order correlations appear to be very small.

There is a close relationship between the spacing variances $\sigma^2(r)$, which, despite (5.3) below, are not precisely two-point measures, and the number variances $\Sigma^2(r+1)$, which, by (5.1), are. The first of these, for which r is necessarily integral, measures the mean-square fluctuation in the length of an interval containing a fixed number of levels; the second (for which r is not restricted to integral values) measures the fluctuation in the number of levels contained in an interval of fixed length. At least for large r the two should then be essentially the same (if we measure the first in spacing units, as we shall). However, if the relationship between $\Sigma^2(r)$ and $\sigma^2(r-1)$ is considered more closely (French *et al.*, 1978), allowing for the fact that the conditions at the ends of the interval are different for the two measures, it is found that to good precision

$$\Sigma^2(r) = \sigma^2(r-1) + \tfrac{1}{6}, \qquad (5.3)$$

this result being valid for many ensembles including the three standard ones (though not for Poisson for which the term $\frac{1}{6}$ does not appear) and for noncentral spectral regions, as well (an aspect of stationarity). There are certain mysteries connected with (5.3), which, as we shall see ahead, is of more consequence than might appear; it would be good to have a better derivation and understanding of it.

Analytic treatments of the measures are based on ensemble averaging. For experimental data we would, of course, use spectral measures; in the case of $\Sigma^2(r)$ the fixed-length interval would be moved along the spectrum, making any necessary correction for a secular variation in the eigenvalue density. It is obviously important that the measures should be shown to be ergodic. Analytic proofs of ergodicity and stationarity in the GOE and in the other two standard ensembles, as well, have been given by Pandey [(1979); see Sec. X] for all of the k-point functions and hence for all of the measures expressible as "simple" functions of them; for the repulsion statistic ω, which may or may not fit into this class, ergodic behavior has been made plausible by Monte Carlo calculations.

Finally, we stress that the discussions of this chapter are based primarily on the GOE; this raises the question of how far the results can be used for other ensembles. Two points can be made. First, there is considerable evidence for certain other ensembles that their *fluctuation* behavior—which the measures described here are intended to determine—is that of the GOE or at least close to it; for a number of ensembles (e.g., the circular orthogonal ensemble and Rosenzweig's microcanonical ensemble) this can be proved analytically; for the EGOE, as discussed briefly in Sec. IV, the major effect of embedding seems representable as a secular map of the spectrum, which would leave all fluctuation results essentially unaltered; for the closely related TBRE, and for certain other ensembles, Monte Carlo calculations make the conclusion plausible. Note, however, that this fluctuation behavior is not found for *all* ensembles, and in particular, as we shall see ahead, not for the unitary and simplectic ensembles, whose underlying algebraic structures are essentially different from that of the orthogonal ensembles. For many of the results, in fact, we give also the extensions to these ensembles ($\beta=2$ for unitary and $\beta=4$ for symplectic). Secondly, it is possible to some extent to use certain of the fluctuation measures to decide whether extraneous factors such as missed or spurious levels affect a given experimental spectrum and then use others to confirm that the fluctuation properties of a sufficiently "pure" spectrum are in accord with expectation. But such procedures have yielded little data to document a significant deviation from GOE behavior.

B. Correlation coefficients

Correlation coefficients between spacings yield important information about a spectrum, for example, about the range over which a spectrum anomaly (represented, for example, by a spacing very much smaller or larger than its local average) propagates. We first consider correlations between spacings of various orders separated by various distances (usually a small multiple of D), or even partially overlapping, the essential results for which, though derived somewhat differently, are due to Garrison (1964; see also Bohigas and Giannoni, 1975). We consider also the closely related correlations between number statistics (5.1).

Given four random variables $Z_1 \cdots Z_4$ we immediately have

$$\operatorname{covar}(Z_1 - Z_2, Z_3 - Z_4) = \tfrac{1}{2}[\operatorname{var}(Z_1 - Z_4) + \operatorname{var}(Z_2 - Z_3) - \operatorname{var}(Z_1 - Z_3) - \operatorname{var}(Z_2 - Z_4)] . \tag{5.4}$$

For the deviation of the tth order spacings from their mean values we write, with $t \geq 0$,

$$S_i^t = D^{-1}(E_{i+t+1} - E_i) - (t+1) , \tag{5.5}$$

whose variance (which by stationarity we take to be independent[19] of i) is $\sigma^2(t)$; we define $\sigma^2(s) = 0$ for $s < 0$. Now, on dividing through by $\sigma(t)\sigma(t')$, we have, for fixed t, t', r, the general correlation coefficient $C(r:t,t')$ between two spacings of orders t, t' whose first levels are separated by $(r+1)$ spacings, with $r \geq -1$ ($r = 1$ then implies adjacent spacings)

$$2\sigma(t)\sigma(t')C(r;t,t') = \sigma^2(|r-t|-1) + \sigma^2(r+t'+1) - \sigma^2(|r+t'-t+1|-1) - \sigma^2(r) . \tag{5.6}$$

This result is valid for spectral averaging in which i ranges over the levels of a single spectrum (and then, for example, $\sigma^2(r) = p^{-1}\sum_{i=1}^{p}(S_i^r)^2$); it is also valid for ensemble averaging, in which for fixed i the energy varies over the ensemble.

As a special case for separated nearest-neighbor spacings we have (Garrison, 1964), with $C(r;0,0) \equiv C(r)$ and taking $r \geq 0$,

$$2\sigma^2(0)C(r) = \sigma^2(r+1) - 2\sigma^2(r) + \sigma^2(r-1) . \tag{5.7}$$

Combining (5.6) for $t = t'$ with the (approximate) result (4.31) for the spacing variances, we find for $(r+1)^2 \gg (t+1)^2$ the result (Pandey, 1979)

$$C(r;t,t) \equiv C(r;t) \simeq -\frac{(t+1)^2}{\beta\pi^2\sigma^2(t)(r+1)^2} , \tag{5.8}$$

where the inclusion of β^{-1} validates the result for the standard ensembles. We see that a spacing which spans a given energy range is essentially uncorrelated with a similar one separated from it by a considerably larger interval. This, of course, is an obvious requirement for ergodic behavior. For $t = 0$ the approximation (5.8) is good for $r \geq 1$, giving, for GOE, -0.089 for $r = 1$ instead of the -0.087 which follows from Mehta's numerical evaluations of the $\sigma^2(r)$ (Mehta, 1975) or from the analytic evaluations of French *et al.* (1978), corrected as discussed in the next section. For adjacent nearest-neighbor spacings ($r = 0$) the correlation coefficient is -0.271.

We find similarly for *adjacent* tth-order spacings an exact relationship

$$C(t;t,t) \equiv \hat{C}(t) = \frac{\sigma^2(2t+1)}{2\sigma^2(t)} - 1 , \tag{5.9}$$

whose value, -0.271 [$= C(0)$] for $t = 1$ and -0.38 for $t = 3$, approaches $-\frac{1}{2}$ for large t. Observe that adjacent high-order spacings are strongly anticorrelated.

Using the limits on the correlation coefficients, $|C| \leq 1$, it is easy to find restrictions on the growth of the spacing variances; in particular,

$$\sigma^2(r) \leq (r+1)^2\sigma^2(0) , \qquad 0 \leq \sigma^2(2r+1) \leq 4\sigma^2(r) . \tag{5.10}$$

Other restrictions of higher order may be derived similarly. Brody (unpublished) has considered the relevance of the restrictions to the "rigidity" of spectra.

Turning now to the number statistics, as in (5.1) but writing n_{ij} for the number of levels in (x_i, x_j), we again make use of (5.4). The basic covariance is then

$$\operatorname{covar}(n_{12}, n_{34}) = \tfrac{1}{2}[\Sigma^2(1,4) + \Sigma^2(2,3) - \Sigma^2(1,3) - \Sigma^2(2,4)] , \tag{5.11}$$

where of course $\Sigma^2(i,j)$ is the number variance (ensemble or spectral as the case may be) for the interval indicated. Once again, and by the same procedure, we find an inverse-square fall-off of the correlation between the numbers in two separated intervals.

C. The number and spacing variances

Analytic forms for the number variances $\Sigma^2(r)$ follow from the two-point function via (5.2). For the two-point function of (4.24), we find

$$\begin{aligned}\Sigma^2(r) &= \frac{2}{\beta\pi^2}[\ln\beta\pi r + \gamma + 1 - \cos\beta\pi r - \mathrm{Ci}(\beta\pi r)] + r\left(1 - \frac{2}{\pi}\mathrm{Si}(\beta\pi r)\right) \\ &= \frac{2}{\beta\pi^2}[\ln\beta\pi r + \gamma + 1] + O\left(\frac{1}{\pi^2 r}\right) ,\end{aligned} \tag{5.12}$$

which is exact only for $\beta = 2$. For the other cases the exact results (Dyson and Mehta, 1963; Pandey, 1979), derived from the two-point function of Dyson and Mehta (Appendix L), are

$$\begin{aligned}\Sigma_1^2(r) &= 2\Sigma_2^2(r) + \frac{1}{\pi^2}[\mathrm{Si}(\pi r)]^2 - \frac{1}{\pi}\mathrm{Si}(\pi r) \\ &= \frac{2}{\pi^2}\left(\ln(2\pi r) + \gamma + 1 - \frac{\pi^2}{8}\right) + O\left(\frac{1}{\pi^2 r}\right) , \\ \Sigma_4^2(r) &= \tfrac{1}{2}\Sigma_2^2(r) + \frac{1}{4\pi^2}[\mathrm{Si}(2\pi r)]^2 \\ &= \frac{1}{2\pi^2}\left(\ln(4\pi r) + \gamma + 1 + \frac{\pi^2}{8}\right) + O\left(\frac{1}{\pi^2 r}\right) .\end{aligned} \tag{5.13}$$

In fact, the terms $O(1/\pi^2 r)$ are quite small. For $r = 1$ we find 0.0043, -0.0018, -0.0125 for $\beta = 1, 2, 4$ respectively, while for $r = 5$ the corrections[20] are 0.0002, -0.0001, -0.0025. Further discussion is given in Appendix L.

The test of Eq. (5.3) also works out satisfactorily. Using Mehta's (1975) evaluations of the $\sigma^2(r)$ we find, for $r = 1$–5, respectively, that $[\Sigma^2(r) - \sigma^2(r-1)]$ is $0.161, 0.167, 0.168, 0.168, 0.168$ instead of the expected $\frac{1}{6}$.

The argument that only binary associations are relevant thus yields a good understanding of the closely related $\sigma^2(r)$ and $\Sigma^2(r+1)$ and hence of the two-point fluctuations in general. Our understanding is not quite

[19] We have ignored small corrections in (5.6) and elsewhere which arise from the corresponding fluctuations in spectral averaging.

[20] We note also that the asymptotic form of (5.12) is larger than that of (5.13) by an (r-independent) constant, 0.110 for $\beta = 1$, and smaller by 0.063 for $\beta = 4$.

complete, because, while our $\Sigma^2(r)$ values will be exact for the unitary ensemble, they differ by a small constant[21] (independent of r) for GOE; and similarly then for the $\sigma^2(r)$. This error can be eliminated (French *et al.*, 1978) by using either Mehta's value for $\sigma^2(0)$ (=0.286) or the "Wigner surmise" value $[(4-\pi)/\pi = 0.273]$ as a boundary condition for the $\sigma^2(r)$ and hence for the $\Sigma^2(r)$, as well.

For the nearest-neighbor spacing the GOE distribution is not Gaussian,[22] and its shape is therefore of interest. In the examples given in Sec. I we have seen that Wigner's distribution, which may be derived from the two-dimensional GOE, fits the data quite satisfactorily. The results for the GOE of asymptotically large dimensionality, which have been given by Mehta (1960) and Gaudin (1961), are only very slightly different [a graphical comparison is given in Mehta's book (1967, Fig. 1.3)], but have in particular different slopes at the origin. It is of some interest that this slope (which is hardly measurable with presently available data) involves the three-point cluster function[23] Y_3, though only its value at the origin; the specific form is

$$[-1+3Y_2(0)-Y_3(0)-Y_2'(r)_{r=0_+}],$$

which is valid for all ensembles (Pandey, 1978). For all three of the standard ensembles (but not, for example, for Poisson) the first three terms cancel when evaluated exactly, since $Y_k(0)=(k-1)!$, and then the slope is simply $-Y_2'(0)$. For GOE the value is $\pi^2/6$ [Wigner, as reported in Mehta (1967, p. 128)] whereas the Wigner value from Eq. (1.5) is $\pi/2$. For $\beta=2,4$ it has been given as zero by Kahn (1963); a zero value also follows for $\beta=2$ from our Eq. (4.24).

It is worth remarking that the results for the number variance can be extended (Dyson and Mehta, 1963) to the case where s different independent sequences of levels are mixed, the fraction of levels belonging to sequence i being f_i (whence $\Sigma f_i=1$). The result for an arbitrary quasistationary ensemble is that

$$\Sigma^2_{\rm mix}(r)=\sum_i \Sigma^2(f_i r), \tag{5.14}$$

a general k-point extension of which (Pandey, 1979) is given in Sec. X.C.

Finally, we mention two remarkable theorems which relate the fluctuations of the three standard ensembles. The simpler theorem (Mehta and Dyson, 1963) asserts that the spectra of the symplectic ensemble may be realized by choosing alternate eigenvalues from the orthogonal ensemble. The content of the other theorem (Dyson, 1962c; Gunson, 1962) is that the spectra of the unitary ensemble may be realized by choosing alternate eigenvalues from a random superposition (mixing) of two independent equivalent orthogonal ensembles. As an immediate result of these theorems we have

$$\sigma_2^2(r)=\tfrac{1}{4}\sigma^2_{\rm mix}(2r+1),$$
$$\sigma_4^2(r)=\tfrac{1}{4}\sigma_1^2(2r+1). \tag{5.15}$$

Using the σ^2 vs Σ^2 relationship of (5.3) we have then (Pandey, 1978)

$$\Sigma_2^2(r)=\tfrac{1}{2}[\Sigma_1^2(r)+\tfrac{1}{4}],$$
$$\Sigma_4^2(r)=\tfrac{1}{4}[\Sigma_1^2(2r)+\tfrac{1}{2}]. \tag{5.16}$$

The Eqs. (5.16) are easily seen, via (5.12) and (5.13), to be asymptotically exact, and hence very close to exact for all values of r; the very small deviations encountered with the exact forms must be ascribed to small errors in (5.3). In going from (5.15) to (5.16) we have used the result that $\Sigma^2_{\rm mix}(r)=2\Sigma_1^2(r/2)$, which follows from (5.14), and have assumed also that (5.3) is valid for binary mixtures. It is curious that this latter assumption, which is validated by the results, could not, however, be valid for a many-component mixture, since that leads to a random (Poisson) spectrum, for which the $\frac{1}{6}$ term in (5.3) disappears. Note also that the two ensemble theorems combine with the relatively simple unitary-ensemble results to produce asymptotically exact two-point fluctuation measures for all three ensembles. The real origin of the theorems is, however, still obscure (Handelman, 1978).

Concerning the direct experimental realization of the unitary and symplectic ensembles we see that every GOE spectrum generates two symplectic spectra. More interesting is that, with an odd target of angular momentum J_0, the two slow-neutron-resonance sequences $(J_0\pm\frac{1}{2})$, which are conventionally regarded as independent, would, if their densities were equal, give directly two realizations of the unitary spectrum. See Lynn (1968) for good examples of each of these. Note that the relative $(J_0\pm\frac{1}{2})$ densities are determined by the more or less calculable "spin cutoff" factor, so that one may verify whether the condition for the unitary case is in fact satisfied. For an unequal mixture of two GOE spectra the corresponding calculation has not been done.

D. The Δ statistics

These measures are concerned with the departure from uniformity (even spacing) of an observed run of levels. Depending on whether we describe things in terms of the distribution function or the density function, we have two natural definitions. The first (Dyson and Mehta, 1963) is[24]

$$\Delta_3(r)=\frac{d}{2L}\min_{A,B}\int_{x-L}^{x+L}[F(x')-Ax'-B]^2dx', \tag{5.17}$$

where $2L=rD$ is the interval length, $d\times F$ is the staircase function with unit steps, and A,B are chosen to minimize the integral. Alternatively, we may define (French *et al.*, 1978)

$$\Delta^*=\frac{1}{nD^2}\min_{A,B}\sum_{s=1}^{n}(x_s-As-B)^2, \tag{5.18}$$

in which we are minimizing the (spectral-averaged)

[21] The numerical value depends on the method of cutoff used in the ζ sum of Eq. (4.18). With a sharp ψ-independent cutoff the approximate values are smaller by 0.07.

[22] For the unitary ensemble the distribution, which has been given by Kahn (1963), is closer to Gaussian.

[23] Y_k is as defined by Dyson (1962c); for $Y_k(0)$ all the relative coordinates are put to zero. See Sec. X.C.

[24] The statistics Δ_1, Δ_2, also introduced by Dyson and Mehta, are not usually employed, since their variances are significantly larger than for Δ_3.

mean-squared deviation of the observed levels from the positions they would have if the spectrum were uniform. Roughly speaking, Δ_3 assigns weights proportional to the spacings, while Δ^* assigns equal weight to each level.

The obvious questions are whether these measures carry different information—unlikely on the face of it—and whether they are largely independent of the spacing variances $\sigma^2(k)$ (which has been tacitly assumed to be the case). These questions are answered in the negative by French *et al.*, (1978); see also Bohigas and Giannoni, 1975). Instead of following their derivation, which is valid to order r^{-1}, where r is the average number of levels in the interval, we follow Pandey (1979, App. II) who finds, on minimization of (5.17), that

$$\Delta_3=\frac{d^2}{8L^2}\int\int_{x-L}^{x+L}\left(1+\frac{3(x'-x)(x''-x)}{L^2}\right)\times[F(x')-F(x'')]^2dx'dx'' . \tag{5.19}$$

Ensemble averaging, with the use of stationarity, enables us to carry out one integration, giving then the result, valid for all ensembles including Poisson,

$$\overline{\Delta_3}(r)=\frac{2}{r^4}\int_0^r(r^3-2r^2s+s^3)\Sigma^2(s)ds . \tag{5.20}$$

Observe now that $\overline{\Delta_3}$ is precisely a two-point measure. Inserting the (asymptotic) Σ_β^2 values of Eqs. (5.12) and (5.13), we find

$$\overline{\Delta_{3,\beta}}(r)=\tfrac{1}{2}\Sigma_\beta^2(r)-\frac{9}{4\beta\pi^2}, \tag{5.21}$$

which, combined with Eq. (5.3), expresses the essential equivalence of three measures. Equation (5.21) is satisfied by the asymptotic forms of $\overline{\Delta_3}$ and Σ^2 (Dyson and Mehta, 1963) and also by the binary-correlation approximations for these quantities, the latter agreement confirming again the validity of the binary-correlation calculations. The Poisson value, which derives from (5.20) with $\Sigma^2(s)=s$, is $r/15$, as given already by Dyson and Mehta.

On carrying out the minimization for Δ^*, we find (French *et al.*, 1978)

$$\Delta^*(r)=\frac{1}{2r^2}\sum_{i,j=1}^{r}\left[1+\frac{12}{r^2-1}\left(i-\frac{r+1}{2}\right)\left(j-\frac{r+1}{2}\right)\right]\times(\delta_i-\delta_j)^2 \underset{r\gg1}{\longrightarrow}\Delta_3(r)-\tfrac{1}{2}, \tag{5.22}$$

the last step being valid for the Gaussian ensembles, but not for Poisson. We see that Δ^*, somewhat easier to evaluate from a given spectrum, carries the same information as Δ_3. If we take the $(\delta_i-\delta_j)$ variables to be multivariate Gaussian, the variances of the Δ measures follow easily from (5.19) and (5.22). For Δ_3 we find a constant value, about twice as large as that given by Dyson and Mehta (0.012). The approach to the asymptotic value is, however, slow and for $r=10$ the value found is less than the limiting value by about one third. For mixed sequences $\overline{\Delta_3}$ follows immediately from (5.14) and (5.20).

Two general remarks about these Δ statistics may be helpful. The first concerns the relationship between Δ^* and the rms deviations which we have encountered in Sec. I in discussing the separation of the secular and fluctuation parts of a spectrum. A minor difference is that the sum in Sec. I was extended over the entire spectrum, the linear reference spectrum (Ar + B) being replaced by the spectrum derived via the low-moment Gram-Charlier expansion. The more essential difference is that the procedure there involved no minimization; instead, the deviations were taken between the actual position of a given numbered level and its position in the smoothed spectrum. We can take for granted that the measure used in Sec. I when applied to a subspectrum, will for "small" d (say, $d<10^{12}$), give essentially the Δ^* value; in the large-d limit, however, the measure diverges logarithmically, as we see from (4.28), while Δ^* becomes independent of d. This difference in behavior arises because with Δ^* we allow the spectrum to "slip," no longer measuring deviations with respect to the fixed centroid positions, as we did earlier. From this example we see that caution is in order when we attempt to infer from Monte Carlo results (always with really small d) what the asymptotic behavior might be.

E. The *F* statistic

The F statistic was introduced by Dyson (1972a) in order to search for missing and spurious levels in an otherwise pure sequence. It is defined by

$$F_i=\sum_{j\neq i}f(y_{ij}), \tag{5.23}$$

where

$$y_{ij}=\frac{x_i-x_j}{L},\qquad f(y)=\begin{cases}\frac{1}{2}\ln\dfrac{1+(1-y^2)^{1/2}}{1-(1-y^2)^{1/2}}, & |y|<1\\ 0, & |y|\geq 1,\end{cases} \tag{5.24}$$

and the sum extends over all x_j within $x_i\pm L$. Writing $L=nD$, expanding $f(y)$ to quadratic terms, and dropping terms of order $n^{-1}\ln n$, we find (French *et al.*, 1978) that

$$\overline{F}\to\sum_{j=1}^{n-1}\ln\frac{1+(1-j^2/n^2)^{1/2}}{1-(1-j^2/n^2)^{1/2}}+\sum_{j=1}^{n-1}\frac{1-2j^2/n^2}{j^2(1-j^2/n^2)^{3/2}}\sigma^2_{j-1,\beta}=n\pi-\ln 2n+\tfrac{5}{3}\sigma_\beta^2(0)-\tfrac{11}{6}\left(1-\frac{1}{\beta\pi^2}\right)+O\left(\frac{1}{n}\right) \underset{\beta=1}{\longrightarrow} n\pi-\ln n\pi-0.72 , \tag{5.25}$$

in the second step of which we have made an Euler-Maclaurin summation. The final value agrees well with that of Dyson (1972a), $n\pi-\ln n\pi-0.66$, who gives also the variance of F as $\sigma^2(F)=\ln\pi n$. An exact integral form for $\overline{F}$, which shows its two-point nature, is

$$\overline{F}=\int_0^n[1-Y_2(r)]\ln\left(\frac{n+(n^2-r^2)^{1/2}}{n-(n^2-r^2)^{1/2}}\right)dr =n\pi-2\int_0^n dq\int_0^q\frac{Y_2(r)}{(q^2-r^2)^{1/2}}dr . \tag{5.26}$$

If, within a pure level sequence, a level is missing or a spurious one has been introduced, locally the calculated F values should differ markedly from those given by (5.25) with a reasonable value for n, say, 10 to 14. As (5.24) makes clear, a missing level will cause a drop in F, a spurious one a rise. A deviation greater than $2\sigma(F)$ has been used by the Columbia group (Liou *et al.*, 1972) as an impurity indicator.

A further test may be made on the basis that the experimental set of F values can be expected to have a Gaussian distribution for a pure sequence (Liou *et al.*, 1972), while a not too large number of missing or spurious levels will cause departures from a Gaussian form. But this is by no means a satisfactory test. First, in the presence of sufficient impurities of either type, the Gaussian distribution of F can be regained whenever the impurity levels are uncorrelated, as is indeed reasonable to suppose. Secondly, the Monte Carlo calculations mentioned below show that the only statistically significant effect on the distribution is to increase its width beyond the theoretical value given by (5.26); we may expect therefore that the ratio of the experimental and theoretical values of the variance of F is a better indicator of the quality of the data than is the shape of the distribution. We may also conclude that, in using F for local testing of a spectrum, as described above, the theoretical $\sigma(F)$ should be used, rather than the experimental one, as is sometimes done.

F. The energy statistic Q

This statistic, also introduced by Dyson and Mehta (1963; see also Dyson, 1962c), derives its name from an analogy between it and the energy of the two-dimensional Coulomb gas. It is concerned with short-range correlations in the spectrum. Basically,

$$Q=-\sum_{i<j} f(x_i,x_j)\ln 2\pi|x_i-x_j|\,, \tag{5.27}$$

but with (relatively complicated) counter terms added in order to moderate a strong variation of Q when individual levels move as we go across the ensemble. The summations extend over the levels in an interval which would contain on the average n levels (its length therefore being nD), and $f(x_i,x_j)$ is a weight function, which in practice is taken to be unity when $|x_i-x_j|\leq rD$ and zero otherwise; one takes $r=2$ to 4 so that the short-range correlations are emphasized. The ensemble average and variance of Q are found to be

$$\begin{aligned}\overline{Q}_\beta &= n[U_\beta-(\beta\pi^2 r)^{-1}]\,,\\ \sigma^2(Q_\beta) &= n[C_\beta+\theta(\beta\pi^2 r)^{-1}]\,,\end{aligned} \tag{5.28}$$

where $\theta\simeq 1$. Dyson (1962c) gives

$$U_\beta=\tfrac{1}{2}\int Y_2(r)\ln 2\pi r\,dr=1-\tfrac{1}{2}\gamma-\tfrac{1}{2}\ln 2=0.365$$

for $\beta=1$ and $=\frac{1}{2}(1-\gamma)=0.211$ for $\beta=2$. $C_\beta=\frac{3}{2}-\pi^2/8=0.266$ and $2-\pi^2/6=0.355$ for $\beta=1,2$, respectively. Our binary correlation results give $U_\beta=\frac{1}{2}(1-\gamma-\ln\beta/2)$, correct, of course, for $\beta=2$, but too large by 0.193 for $\beta=1$. The latter difference, which is significant, reminds us that the shortest-range correlations (which are emphasized in Q) are not treated well in our approximation. We note finally that $\overline{Q}$ is precisely a two-point measure, being expressible as an integral over Y_2.

G. The Λ statistic

A fluctuation measure sometimes used to detect impurities, in the sense of missing or spurious levels, in an experimental level sequence is the Λ statistic developed by Monahan and Rosenzweig (1970, 1972). It compares the nearest-neighbor spacing distribution with a theoretical distribution that ideally should be the ensemble average. Let $F^*(S)$ be this theoretical distribution function, and $F(S)$ the experimentally found one, which will, of course, be a staircase function; then Λ is defined as

$$\Lambda(n)=\frac{n}{D}\int_0^\infty [F(S)-F^*(S)]^2 dS\,, \tag{5.29}$$

where D is the average of the n spacings S. $F^*(S)$ here is usually taken to be the integral of the Wigner distribution (1.5), namely, $1-\exp(-\pi S^2/4D^2)$, which we know to be a close approximation to the correct distribution function for the GOE. There is practically no theoretical background for the use of this measure. If the spacings had the Wigner distribution but no correlation, linear or otherwise (an unrealistic situation), the expected value of Λ would be 0.293. The existence of long-range order in the level sequence depresses the actual value of $\Lambda(n)$ to well below this and makes it a function of n; Monahan and Rosenzweig derived its properties from Monte Carlo calculations based on GOE matrices of a single fixed dimensionality, so that it is not known to what extent their results are asymptotic to the large-d limit.

H. The repulsion parameter

Last we discuss a parameter which measures the degree of level repulsion. It is based on a generalization of Wigner's distribution (1.5) for the nearest-neighbor spacings (Brody, 1973) and may be written

$$P_\omega(z)=Az^\omega\exp(-\alpha z^{1+\omega}),\quad z=\frac{S}{D}\,, \tag{5.30}$$

where S is the nearest-neighbor spacing. The two parameters A and α in (5.30) are found from the condition that both the average of z and the area under the curve must be equal to unity; they turn out to be

$$A=(1+\omega)\alpha\,,\quad \alpha=\left[\Gamma\left(\frac{2+\omega}{1+\omega}\right)\right]^{1+\omega}. \tag{5.31}$$

The value of ω is found by fitting (5.30) to the data (or, for that matter, to a theoretical distribution) by least squares. The problem is nonlinear, but convergence is usually rapid. A simple method for carrying out the fit is, starting with an estimate ω_0, to compute the values of $\alpha_0\equiv\alpha(\omega_0)$ and

$$\phi_i=1-\exp(-\alpha_0 z_i^{1+\omega_0})\,, \tag{5.32}$$

which give the probabilities of finding spacings smaller than z_i, for all z_j in the sample; plotted in ascending order as a function of i, the ϕ_i will be well approxi-

TABLE I. The effects on various measures of randomly deleting a number of levels from a given spectrum, adding a number to it, or doing both. The spectra are either GOE, in which case ensemble averages are given, or are taken from the (^{238}U + n) experiment (Rahn *et al.*, 1972a). The results in the first two rows are mostly analytical, the Poisson values being given for comparison.

Ensemble	$\sigma(F)$	Δ_3	Q	ω	$\Sigma(10)$	$C(0)$	$\sigma(0)$	$\sigma(3)$	$\sigma(6)$	$\sigma(10)$
GOE	1.86	0.46	34.0	0.95	0.95	−0.27	0.53	0.75	0.82	
Poisson	7.6 ± 1.8	6.67	153	0	3.16	0	1.00	2.00	2.65	3.32
GOE − 1	2.1 ± 0.3	0.54	34.7		1.00	−0.26	0.54	0.76	0.86	0.93
GOE − 3	2.4 ± 0.4	0.65	36.6		1.16	−0.25	0.55	0.79	0.91	1.01
GOE − 5	2.4 ± 0.4	0.75	37.6	0.92 ± 0.06	1.12	−0.24	0.56	0.83	0.95	1.04
GOE − 10	2.7 ± 0.5	0.77	42.1	0.89 ± 0.08	1.11	−0.19	0.58	0.91	1.09	1.23
GOE + 1	2.1 ± 0.4	0.54	36.3		1.15	−0.26	0.55	0.77	0.87	0.93
GOE + 3	2.6 ± 0.5	0.78	39.3		1.25	−0.24	0.56	0.81	0.94	1.03
GOE + 5	2.8 ± 0.5	0.84	43.0	0.91 ± 0.07	1.18	−0.21	0.57	0.86	1.00	1.11
GOE + 10	3.4 ± 0.9	1.17	49.7	0.84 ± 0.09	1.39	−0.19	0.59	0.95	1.16	1.30
GOE^{+5}_{-5}		1.23		0.89 ± 0.08		−0.18	0.58	0.96	1.14	1.31
^{238}U	1.69	0.49	30.3	1.21 ± 0.08		−0.27	0.51	0.58	1.64	1.81
^{238}U + 10		1.17 ± 0.66	36.1	1.18 ± 0.12	1.04	−0.20	0.54	0.79	0.98	1.22
^{238}U − 10		0.87 ± 0.35	42.0	1.22 ± 0.14	1.11	−0.21	0.55	0.76	0.89	1.10
$^{238}\mathrm{U}^{+5}_{-5}$				1.19 ± 0.14						

mated by a straight line if $\omega_0 = \omega$; otherwise,

$$\delta\omega = (1+\omega_0)(1-r\delta_m) \tag{5.33}$$

gives an estimate of the change $\delta\omega$ required to obtain a better approximation. Here r is the number of spacings x_i and δ_m is the extremum of a quadratic fitted to the first differences $d_i = \phi_i - \phi_{i-1}$.

The value of ω is zero for the exponential distribution of spacings which results when the levels are statistically unrelated; for Wigner's distribution (1.5) it is unity; for the GOE its value is 0.953, which can be experimentally distinguished from unity when there are more than about 150 levels. The method we have outlined for its determination can also yield an estimate of the error limits on it, which are of importance in the application of ω to experimental data.

There is no adequate theory underlying the repulsion parameter. But it is applicable to a rather wide variety of situations, its numerical estimation yields error limits which can to some extent make up for our ignorance of its ensemble variance, and practical experience in its use has shown that it is rather insensitive to spectrum impurities. Moreover, its ergodicity has been made plausible numerically (Mello *et al.*, 1976). In the GOE the conditions for ergodic behavior are satisfied after unfolding over the whole spectrum. In the TBRE, however, ω is not stationary; while in the central region of the TBRE spectrum it has the same value 0.95 as for the GOE, it decreases towards the ground-state region, where it is found to be about 0.82.

I. Effect of impurities on the fluctuation measures

Even in a set of experimental data of very high quality, it is not possible to eliminate completely the errors represented by missing or spurious levels. We must therefore learn about the effects of such errors on the fluctuation measures.

Consider first what happens as we add δp levels to a GOE sequence containing p levels; let $f = \delta p/(p+\delta p)$ be the mixing ratio. We can agree that the spurious levels are independent of the pure sequence. Then, using (5.14) and (5.20), as well, for $\bar{\Delta}_3$, we have for the number variance and $\bar{\Delta}_3$

$$\begin{aligned} \Sigma^2_{\mathrm{mix}}(r) - \Sigma^2_{\mathrm{GOE}}((1-f)r) &= \Sigma^2_{sp}(fr) \xrightarrow[P]{} fr\,, \\ \bar{\Delta}_{3\ \mathrm{mix}}(r) - \bar{\Delta}_{3,\mathrm{GOE}}((1-f)r) &= \bar{\Delta}_{3,sp}(fr) \xrightarrow[P]{} \frac{fr}{15}\,. \end{aligned} \tag{5.34}$$

In the final forms here we have assumed (P) that the spurious levels are themselves part of a Poisson sequence, as they are in the GOE tests in Table I below, and as we would expect also in experimental cases. Things are more complicated if we have δp missing levels but the same results apply for small enough f,[25] say, $fr \lesssim 1$. Note that in each case the measures of (5.34) increase, corresponding to a decrease in the level repulsion. Similar results can be derived for a combination of spurious and missing levels, as well as for other measures, but for the most part these have not been explicitly worked out. Instead we give in Table I some of our Monte Carlo simulations. But we refer first to a recent study by Coceva and Stefanon (1979), done as part of a thorough analysis of their own experimental results, to which we return in Sec. VI; see in particular their Fig. 3 for the variation of Δ_3, Q, and σ_{10}.

The first two lines in Table I give analytical results for the GOE and analytical and numerical results for a Poisson spectrum. In the rest of the table we show the results for randomly deleting (−) or adding (+) the indicated number of levels. The original spectrum is taken either from experiment or from the center of an unfolded 294-dimensional GOE; the final subspectrum in each case contains 100 levels.

To begin with, we remark that there is satisfactory agreement with (5.34). It is found also that the distribution of the F values does not deviate significantly from a Gaussian shape in the cases tested. Beyond that it will be seen that the various measures have by no means the same sensitivity to level impurities. Thus

[25] Observe that if we take r to span the entire sample (as experimentalists essentially always do for Δ_3, and as is done in Table I), then fr is the number of spurious or missing levels.

a 10% impurity causes only an 8% change in ω, but a 20% one in σ_F; however, when measured in terms of their sampling standard deviation, Δ_3 is the most sensitive impurity test, showing a variation of about 5 times $\sigma_{\Delta 3}$, while again ω is very insensitive. Unfortunately, the sampling variances for the $\sigma(r)$ and $C(0)$ are known only from Monte Carlo results.

These differences in the behavior of the various statistical measures have led to the preference of some of them as tests for impurities in an experimental level sequence, while others have been regarded as better suited for verifying the validity of the underlying theory. In particular, Δ_3, F, and $\Sigma(r)$ have often been used to test for missing or spurious levels. But, as we have seen, all these parameters, except probably ω, are derived from the two-point function and are related to each other; thus such a discrimination in their uses is of dubious justification.

VI. ANALYSIS OF THE DATA ON ENERGY-LEVEL FLUCTUATIONS IN NUCLEI

A. Introduction

When considering the data, we shall take into account not only the experimental results, but also those derived from shell-model calculations and Monte Carlo simulations, to which techniques developed for the experimental results may be applied and from which we can draw conclusions relevant to experiment. In this section we discuss the level-spacing fluctuations; Sec. VII will deal with the strength distributions and related matters.

As we saw in Sec. I, most useful for the study of energy-level fluctuations is a long sequence of states, all with the same conserved quantum numbers (J and π, and, for nuclei, the isospin T) and spanning a narrow energy range. With a long sequence we have good statistics; in a narrow band of energy the underlying global properties (average level density and strength) will not change appreciably; if the states have the same exact symmetries, we do not encounter the (inessential) complications associated with impure sequences. These requirements, which are often too stringent, can sometimes be relaxed. If there is a secular variation of the density over the region spanned by the sequence of levels, we may be able to "unfold" the data, as described in Sec. III; this is particularly useful with Monte Carlo results, with experimental results for proton resonances, and with the nuclear-data ensemble. It is clear that in order for this mapping to produce the required results it must be one-to-one and sufficiently smooth. If necessary, we can also extend the analysis to the case of mixed exact-symmetry subsequences. This has its limits, because sequences of too many admixed subsequences yield Poisson distributions which carry no information of interest; as a practical matter one would deal with mixtures of no more than two pure subsequences.

Before proceeding we remark that there are also early analyses of nearest-neighbor spacing distributions in atomic spectra [see, for example, Porter and Rosenzweig (1960); Rosenzweig and Porter (1960); Trees (1961)], and there are indications of renewed interest in atomic energy-level fluctuations more generally (J. C. Parikh, private communication; V.K.B. Kota and V. Potbhare, unpublished).

B. Slow-neutron resonances

Most of the data which we consider below are due to the Columbia group (Hacken *et al.*, 1978, and earlier papers referred to therein); for other work see Karzhavina *et al.*, (1968), Kirouac and Eiland (1975), and Coceva and Stefanon (1979).

Slow-neutron resonances supply the most complete data for testing the fluctuation theory, the time-of-flight technique providing the necessary resolution for the closely spaced resonances (for heavy nuclei D is often as small as a few electron volts). In the resonance region we can expect compound-nuclear states of both parities and with comparable densities; but in general the p-wave resonances (those of higher orbital angular momentum need not be considered) will be very much sharper, their strengths much reduced by a barrier penetration effect. So we have basically a sequence of s-wave resonances with some intrinsically strong p-wave interlopers at the higher energies which may be mistaken for weak s-wave resonances. This problem is least with targets in the rare-earth region for which the ratio of p-wave and s-wave strengths is at a minimum. Rare-earth targets with even N and Z should then come closest to giving a pure sequence, in this case of $\frac{1}{2}^+$ levels. The other problem, of course, is that some weak s-wave resonances will have widths below the detection threshold.

The observed resonance width Γ_n at an energy E is related to the reduced width γ_l^2 by the penetration factor of Sec. VII.C below; γ_l is the matrix element of the transition operator. We shall see in Sec. VII that the reduced neutron widths follow the Porter-Thomas distribution which we have already encountered in Sec. I, and with the use of this we can often make plausible corrections for both the "weak s-wave" and "strong p-wave" problems. For widths of the same parity, we have

$$P(t_l)dt_l = (2\pi t_l)^{-1/2}\exp(-\tfrac{1}{2}t_l)dt_l, \quad t_l = \gamma_l^2/\langle\gamma_l^2\rangle . \quad (6.1)$$

If the number of missed levels is small, an estimate of it can be made by using this equation, since the experimental mean reduced strength $\langle\gamma_l^2\rangle$, the only parameter required in order to use it, is not adversely affected by a small number of missing weak resonances. Knowledge of the number of missed levels, as given by (6.1), is already of value; corrections to the set of levels can often be made by using measures such as the F test of Dyson (1972a), described in Sec. V.

The question of spurious states is slightly more complicated. Because of the different energy dependence in the barrier penetration factor [$\gamma_s^2 = \Gamma_n E^{-1/2}$ for s wave, and $\gamma_p^2 = \Gamma_n(\hbar^2/2MR^2)E^{-3/2}$ for p wave], one expects only s-wave resonances to be observable at extremely low energies. At higher energies, however, a strong p-wave resonance may produce an observable width. If the averaged width for each partial wave is known, a Bayesian test, based on the fact that the distribution for t_l is Porter-Thomas, can be applied to

make the distinction. Bollinger and Thomas (1968) give the probability for an observed resonance to be p wave as

$$P(p|\Gamma_n) = [1 + cE^{-1/2}\exp(t_p - t_s)]^{-1}, \quad (6.2)$$

where c is a constant depending on the ratio $\langle\gamma_p^2\rangle/\langle\gamma_s^2\rangle$ but independent of Γ_n and E. The E dependence comes from the difference in barrier penetration factors; the power $-\frac{1}{2}$, together with the exponential factor, follows from Eq. (6.1). For most resonances, the Bayesian test is able to produce a probability close to either 0 or 1, giving an indication of the most probable l value of the observed low-energy resonance. But if the probability is intermediate, it is better not to correct.

Recently, a much more careful analysis of the neutron-resonance data aimed specifically at fluctuation studies has been proposed by Coceva and Stefanon (1979) and applied to their data on ($n + {}^{156}$Gd). Since the purity of the sequence may greatly affect the statistics, it may not be adequate to use the simple Bayesian test; they argue that more reliable information can be extracted from the data by a maximum-likelihood test involving the energy-dependent experimental resolving power, as well as the number of degrees of freedom, ν, of the chi-square distribution for the neutron strength, the average level spacing, D, and the average s- and p-wave neutron widths, $\langle\Gamma_n^0\rangle$ and $\langle g\Gamma_n^1\rangle$, where g is the degeneracy. Once the sequence is determined, the experimental resolving power should be taken into account in calculating the measures and comparing with expected values.

C. Data from fissile nuclei

The fission width of a resonance depends on the height of the fission barrier, which is believed to be J dependent. The Saclay group (Derrien *et al.*, 1967; Michaudon, 1973; Jain and Blons, 1975) makes use of this dependence to separate the neutron resonances observed in $n + {}^{239}$Pu, for which the target has angular momentum $\frac{1}{2}$, into two categories according to their fission widths, assigning $J = 0$ and 1 to the separate groups. For a long mixed sequence, this represents the only presently available technique for assigning angular momenta to all the neutron resonances.

The energy-level fluctuations (Jain and Blons, 1975) for the $J = 1$ sequence (the $J = 0$ sequence is too short for a good analysis) and also for neutron resonances on the neighboring fissile nucleus ^{240}Pu (Böckhoff *et al.*, 1967) differ considerably from the fluctuations derived from experiments with other targets (see Table II). However, careful analysis (Michaudon, 1973) shows that there are pronounced modulations of the fission cross sections which may be taken as an indication of the existence of intermediate structure in the region. If this is indeed the case, it may be inappropriate to consider the Pu data on the same footing as those derived from pure sequences [for a different possibility, however, see the last section of Coceva and Stefanon (1979)]. Data on other fissile targets are available, but none for which a claim of adequate purity has been made. Thus the data with ^{239}Pu and ^{240}Pu targets form at present a class by themselves. For further discussion see Garrison (1972).

D. Proton-resonance data

The proton experiments have the advantage that angular distributions can be measured and therefore J values determined. On the other hand, they do not have the energy resolution of neutron experiments. As a result, it is only in recent years that long sequences of energy levels have become available for light nuclei ($A \lesssim 65$) at low excitation energies (Prochnow *et al.*, 1972; Wilson *et al.*, 1975; Bilpuch *et al.*, 1976; Mitchell, 1980). The larger mean spacing value, however, implies an energy span for a long sequence which is too large for the assumption of a constant average level spacing, and therefore unfolding becomes necessary; for this purpose the authors referred to make use of an exponential form for the density, $P(E) \sim \exp(E/T)$. There are uncertainties associated with the existence of intermediate structure in the region, so that some of the standard methods for testing the purity of a sequence cannot be used here. But of course the intermediate structures give rise as well to features of special interest; these we shall comment upon in Sec. VII.

E. The nuclear-table ensemble

The first few levels in the spectra, when considered together for a wide class of nuclei, may be regarded as giving a realization of an ensemble (Flores and Mello, 1973; Cota *et al.*, 1974; Brody *et al.*, 1976). As discussed in Sec. I.C, it is necessary to renormalize the level spacings in accordance with the A dependence of the level density, $D \sim A^{-1}$; it is also appropriate to remove from consideration the class of "collective" nuclei, especially since specific mechanisms are known which give rise to their anomalous spacing behavior. The data give rise to good nearest-neighbor spacing distributions and values for the repulsion parameter ω, but most of the fluctuation measures discussed in Sec. V cannot be applied, since the major part of the data consists of a single spacing from each spectrum. It has been possible to obtain data about spacings of orders 2, 3, 4, but from smaller and smaller numbers of nuclei; moreover, there are ambiguities in "unfolding" the data with respect to the energy dependence.

F. Shell-model and Monte Carlo data

As we have shown in Sec. I, numerical random-matrix and shell-model spectra have been of considerable use in giving a general view of the behavior of complicated systems; they have been useful, also, in providing values for some of the fluctuation measures which have resisted analytical treatment, in determining their variances, and in studying their ergodic properties. These spectra are, of course, exempt from problems of experimental technique, of level identification, and of missed and spurious levels; on the other hand, they display the secular variation of density, which, as discussed in Sec. III, must be removed without altering the fluctuation properties before comparisons with other data can be made. In ad-

TABLE II. Statistics of pure sequence experimental data. All data are from neutron resonance experiments except the Pu isotopes (fission) and ^{44}Ca and ^{48}Ti (proton resonances). For references: Ca73, Camarda *et al.* (1973); Ha74, Hacken *et al.* (1974); Ha78, Hacken *et al.* (1978); Ja75, Jain and Blons (1975); Ka68, Karzhavina *et al.* (1968); Ki75, Kirouac and Eiland (1975); Liou *et al.* (1972); Li72a, Liou *et al.* (1972a); Li73, Liou *et al.* (1973); Li74, Liou *et al.* (1974); Li75, Liou *et al.* (1975); Li75a, Liou *et al.* (1975a); Li75b, Liou *et al.* (1975b); Li75c, Liou *et al.* (1975c); Ra72, Rahn *et al.* (1972); Ra72a, Rahn *et al.* (1972a); Ra74, Rahn *et al.* (1974); Wi75, Wilson *et al.* (1975). Asterisks denote experimental measures which differ by more than a standard deviation from the average, as determined either analytically in some cases or by Monte-Carlo calculations in others. Comparison values are not known for $\Sigma(10)$ and ω. Experimental results for ^{156}Gd+n, and a detailed analysis of them, are given by Coceva and Stefanon (1979).

Target	Ref.	E_{max} (eV)	Sequence n	add	delete	D (eV)	$S_0\times10^{-4}$	$S_1\times10^{-4}$	$C(0)$	Δ_3 Obs.	Δ_3 Theor.	Q Obs.	Q Theor.	Spacing width σ_0	σ_3	σ_6	σ_{10}	$\sigma(F)$	$\Sigma(10)$	ω
^{166}Er	Li72	4200	109	1	24p	38.4	1.7	≤0.7	−0.22	0.46	0.47	30	36	0.53	0.76	0.81	0.93	1.8	0.96	0.85
^{168}Er		4700	50	2	24p	95.3	1.5	0.7	−0.29	0.29	0.39	15	16	0.50	0.73	0.65	0.61	1.9	0.78	1.03
^{170}Er		4800	31	0	30p	155	1.5	0.8	−0.09*	0.36	0.34	10	9.5	0.52	0.94*	0.82	1.00	1.7	0.85	
^{152}Sm	Ra72	3665	70	0	0	51.8	2.2		−0.26	0.40	0.42	25	23	0.56	0.77	0.81	0.78	1.9	0.91	0.96
^{154}Sm		3046	27	0	0	115	1.8		−0.32	0.38	0.32	2.3*	8.1	0.66*	0.79	0.92	0.83	2.3	1.08	
^{232}Th	Ra72a	3000	178	0	62p	16.8	0.84	~0.9	−0.19	0.39*	0.51	69*	60	0.54	0.83	0.86	1.03	2.2	1.05	0.83
^{238}U		3015	146	0	74p	20.8	1.1	~1.4	−0.24	0.42	0.49	32*	49	0.50	0.60	0.62	0.79	1.7	0.80	1.21
^{172}Yb	Li73	3900	55	0	0	70.3	1.68		−0.24	0.41	0.40	18	18	0.55	0.88*	1.00*	0.89	1.7	1.04	0.79
^{174}Yb		3350	19	1	1p	162	1.62		−0.36*	0.31	0.29	2.0*	5.4	0.67*	0.74	0.71	1.08	3.4*	0.70	
^{176}Yb		3850	21	1	3p	185	2.29		−0.20	0.34	0.30	1.1*	6.1	0.55	0.77	0.78	0.49*	2.2	0.83	
^{182}W	Ca73	2650	41	0	0	66.3	2.40		−0.37*	0.26	0.37	8.4*	13	0.47	0.61	0.68	0.73	1.6	0.73	
^{184}W		2600	30	0	1p	81.3	2.35	(1.0)	−0.28	0.45	0.34	8.0	9.5	0.64	0.88*	1.09*	1.05	2.0	1.26	
^{186}W		1230	14	0	0	90	2.23		−0.30	0.17	0.26	4.9	6.8	0.63	1.21*	1.68*	1.75*	5.4*	1.96	
^{110}Cd	Li74	5000	29	3	27p	174	0.50	2.8	−0.18	0.41	0.33	11	8.8	0.50	0.84	1.01*	0.94	2.2	0.85	
^{112}Cd		7000	52	9	26p	137	0.53	2.5	−0.32	0.30	0.39	3.9*	17	0.38*	0.65	0.69	0.53*	2.7*	0.67	0.62
^{114}Cd		3400	19	2	16p	183	0.70	3.2	−0.28	0.22	0.29	5.2	5.4	0.48	0.64	0.28	0.64*	1.1	0.82	
^{154}Gd	Ra74	269	19			14.5	2.0		−0.56*	0.22	0.28									
^{158}Gd		4000	47		2p	86	1.5		−0.14*	0.29	0.38									
^{160}Gd		4000	20	1	13p	202	1.8	1.7	−0.33	0.32	0.30									
^{160}Dy	Li75	900	33	2	0	27.3	2.0	3.0	−0.25	0.39	0.35	12	10	0.45	0.89	0.86	0.78	3.3	1.11	0.98
^{162}Dy		3000	48	2	17p	64.6	1.9	1.1	−0.29	0.34	0.39	9.2*	16	0.39*	0.53*	0.65	0.78	2.0	0.83	
^{164}Dy		5000	34	2	27p	147	1.7	1.3	−0.33	0.41	0.35	13	11	0.58	0.79	0.79	0.73	1.8	0.87	
^{150}Sm	Ki75	556	11			56.5	3.3		−0.14	0.15	0.24	3.8	3.4	0.36*	0.61	0.71				
^{152}Sm		3365	70			53.8	2.2			0.40	0.42									
^{154}Sm		3047	27			112.6	1.7			0.37	0.32									
^{239}Pu(1$^+$)	Ja75	300	91			3.2	1.13	2.33	−0.23	2.25*	0.45	47*	29	0.59	0.96*	1.25*	1.56*	4.9*	1.60	0.7
^{240}Pu		680	43			15.8	1.13	2.33	−0.23	0.65*	0.37	18*	14	0.56	1.09*	1.17*	1.00	3.1*	1.27	0.7
^{146}Nd	Ka68	6723	18			345	4.5		−0.23	0.61	0.29	2.2*	5.4	0.78*	1.23*	1.24*	1.34*	5.0*	1.17	
^{148}Nd		8781	29			242	3.6		0.65*	3.26*	0.33	10	8.8	0.65*	2.49*	2.43*	3.15*	11.0*	3.61	
^{150}Nd		3843	16			236	2.0		−0.20	0.32	0.27	1.8	4.8	0.62	0.81	0.60		2.5*	3.73	
^{44}Ca($\frac{1}{2}^-$)	Wi75	2.5 MeV (1.99 MeV)	52			a			−0.32	0.39	0.39	20	17	0.63	0.92	0.78	1.01	2.1	0.86	
^{44}Ca($\frac{1}{2}^+$)		2.4 MeV (1.88 MeV)	39			a			−0.35	0.34	0.36	4.3	12	0.52	0.68	0.87	0.77	3.2*	1.03	0.96
^{48}Ti($\frac{1}{2}^+$)		3.1 MeV (2.15 MeV)	66			a			−0.21	0.51	0.42	26	22	0.57	0.87	1.03*	0.98	3.4*	1.20	0.98

[a] Unfolded by constant temperature law. The energy given in brackets is the minimum energy.

dition, there are certain special problems which require some comments.

In the first place, the local density varies across the ensemble. It therefore makes a difference whether one expresses level spacings for an individual spectrum in terms of a unit spacing D defined by averaging over that spectrum, or in terms of an ensemble-averaged unit which is therefore the same for all spectra in the ensemble. This may be regarded as an aspect of nonergodic behavior which obtains in particular when the matrices are finite dimensional, as with Monte Carlo calculations. With $\sigma_s^2(r)$ defined by the first procedure of *spectral* averaging and $\sigma_e^2(r)$ by the second one of *ensemble* averaging, it is intuitively clear that $\sigma_s^2(r) < \sigma_e^2(r)$, the difference between them increasing with the spacing order r. It has, in fact, been shown by French (1973) that

$$\sigma_e^2(r) = \sigma_s^2(r) + [\sigma_s^2(r) + (r+1)^2] \frac{\Sigma^2}{D^2} , \tag{6.3}$$

where Σ^2 is the ensemble variance of the spectrally averaged D.

In the GOE, the correction implied by (6.3) is negligible for the spacing orders and dimensionalities of interest, since here $\Sigma = d^{-1/2}$ and goes to zero in the limit of infinite dimensionality, a consequence of the GOE's strong ergodic properties. Thus the dimensionality d plays here the role of the number of degrees of freedom. But for the EGOE and the TBRE this number, the "effective dimensionality," is given by the (much smaller) number of *independent* matrix elements, and then the correction can be significant. For EGOE its value has been given analytically (French, 1973), while for TBRE a value can be obtained from Monte Carlo data (Wong and French, 1972). We give this correction because at one point serious confusion arose as to whether or not multibody interactions might be detectable via fluctuation measures, which would be feasible if Monte Carlo calculations were to give different results for GOE and TBRE. In fact, they appeared to do so (Bohigas and Flores, 1972); the fact that experimental results seemed to favor the GOE was then used to argue that ensembles of the TBRE type were invalid [see discussion at the 1971 Albany Conference (Garg, 1972)]. The deviations, however, were spurious, because the GOE-TBRE comparison was made for $\sigma_e^2(r)$ instead of $\sigma_s^2(r)$, which is obviously the appropriate measure. In several of the experimental papers, however, arguments based on the spurious GOE-TBRE deviations are repeated, as pointed out by Bohigas *et al.* (1974).

G. Energy-level data and their fluctuation measures

In Tables II through IV we present the values for ten fluctuation measures calculated for a selection of spectra from all the sources we have discussed. In interpreting the data in the tables, it must be remembered that the measures for which we give values are closely linked through their association with the two-point correlation function (see Sec. V). These relations, however, are between the ensemble-averaged values, so that the individual deviations (and the effects of spec-

TABLE III. Shell-model and ensemble data. The ^{24}Mg and ^{28}Si shell-model spectra are due to Soyeur and Zuker (1972). The three ^{28}Si cases involve differently renormalized effective interactions. The numbers in brackets in the Δ_3, Q columns are theoretical values. For asterisks see Table II.

Case	Matrix dimension	n	$C(0)$	Δ_3	Q	Spacing width σ_0	σ_3	σ_6	σ_{10}	$\sigma(F)$	$\Sigma(10)$	ω
Ensemble												
GOE	∞	100 (Analytic)	−0.27	0.46 ± 0.11	34.0 ± 5.2	0.53	0.75	0.82		1.9		0.95
TBRE	294	100 (Unfolded)	−0.26 ± 0.8	0.46	34.5	0.52	0.73	0.80	0.85	2.0 ± 0.4	0.96	0.95
GOE	294	100	−0.27 ± 0.8	0.48	34.2	0.54	.0.75	0.84	0.89	1.9 ± 0.3	0.88	0.94
Nuclear table		135										0.69
Shell model												
^{24}Mg 2^+ $(T=0)$	1206	120 (Unfolded)	−0.30	0.33 (0.48)	29.7 (41 ± 6)	0.50	0.67	0.68	0.73	1.3*	0.81	0.91
^{28}Si 0^+ $(T=0)$a	839	84 (Unfolded)	−0.13*	0.56 (0.44)	31.7 (29 ± 5)	0.51	0.87	1.00	0.86	1.9	1.01	
^{28}Si 0^+ $(T=0)$b	839	84 (Unfolded)	−0.24	0.51	34.7	0.52	0.87	0.97	0.87	1.9	0.95	
^{28}Si 0^+ $(T=0)$c	839	84 (Unfolded)	−0.32	0.42	26.7	0.48	0.60	0.72	0.89	2.7*	0.95	
^{22}Na 2^+ $(T=1)$	525	54 (Unfolded)		0.42 (0.40)	14.9 (18 ± 4)	0.56	0.83	0.82	0.88	1.7	0.87	

TABLE IV. Statistics of two mixed sequence experimental data. For references see Table II.

Target (J)	Ref.	E_{max} (eV)	n	Sequence add	Sequence delete	D (eV)	$S_0\times10^{-4}$	$S_1\times10^{-4}$	$C(0)$	Δ_3 Obs.	Δ_3 Theo.	Q Obs.	Q Theo.	σ_0	σ_8	σ_6	σ_{10}	$\Sigma(10)$
^{167}Er($\frac{7}{2}$)	Li72	118	30	0	0	4.06	1.9		−0.31	0.29	0.53	10	21 ± 7	0.63	0.58	0.73	0.79	0.80
^{183}W($\frac{1}{2}$)	Ca73	701	52	2	0	13.2	1.65		−0.09	0.59	0.62	19	32 ± 7	0.64	0.94	0.73	0.97	1.07
^{111}Cd($\frac{1}{2}$)		1800	74	11	20*p*	24	0.38	3.5	−0.14	0.63	0.69	20	46 ± 8	0.43	0.72	0.90	1.11	1.09
^{115}In($\frac{9}{2}$)	Ha74	500	53	4	27*p*	9.4	0.26	2.7	−0.05	0.57	0.65	21	36 ± 7	0.48	0.78	0.88	0.88	1.07
^{161}Dy($\frac{5}{2}$)	Li75	140	53	5	0	2.67	1.7	<3.0	−0.17	0.62	0.65	19	36 ± 7	0.45	0.71	0.92	1.08	1.31
^{163}Dy($\frac{5}{2}$)		220	32	2	0	6.85	2.0	<3.0	−0.21	0.34	0.55	11	22 ± 6	0.53	0.88	0.79	0.67	0.73
^{175}Lu($\frac{7}{2}$)	Li75a	200	57	5	0	3.65	1.8		−0.24	0.32	0.65	20	39 ± 7	0.49	0.69	0.69	0.81	0.77
^{177}Hf($\frac{7}{2}$)	Li75b	100	45	3	0	2.22	2.7		−0.36	0.40	0.62	17	31 ± 7	0.61	0.73	0.83	0.89	1.03
^{203}Tl($\frac{1}{2}$)	Li75c	6600	14	0	8*p*	490	1.3	0.25	+0.25	0.65	0.35	14[a]	9 ± 3	0.68				
^{147}Sm($\frac{7}{2}$)	Ki75	291.4	43			7.00	4.3		−0.11	0.70	0.61	26	29 ± 6	0.65	1.08	1.28	1.22	1.35
^{149}Sm($\frac{7}{2}$)		127.1	53			2.38	5.0			0.51	0.65							
^{151}Sm($\frac{5}{2}$)		105.2	64			1.72	3.3		−0.14	0.54	0.69	34	43 ± 8	0.66	1.00	1.13	0.99	0.92
^{181}Ta($\frac{7}{2}$)	Ha78	1000	224	0	0	4.2	1.7		−0.33	2.59		90	73 ± 15	0.92	1.29	1.57	1.87	

tral impurities) may be rather different for the different measures; it is for this reason that we have grouped together those measures, like Δ_3 and Q, which are commonly thought of as useful for verifying the purity of the spectral data. As a guide to the quality of the experimental data, we include in the tables the number of levels added by the experimentalists on the basis of the statistical tests described above, and the number of levels deleted from the raw data as being, with high probability, p-wave resonances. Except for the Δ_3 and Q measures the theoretical variances are unknown, but reasonable values have been determined numerically for most of them; we indicate with asterisks the experimental measures which differ by more than a standard deviation from the averages.

Examining the single-sequence data in Table II, in comparison with the ensemble data of Table III, we will observe, in the first place, that there are a number of spectra, such as those of ^{167}Er or the $\frac{1}{2}^-$ series for ^{45}Sc, where the agreement between the experimental measures and the ensemble predictions for the GOE are excellent for all ten measures; there are also cases, such as the ^{113}Cd spectrum, where the agreement is poor for almost all the measures. But as we noted above, for the first two cases we have reason to believe that the experimental conditions were very favorable, while in the case of ^{113}Cd the number of levels that the experimental group felt it necessary to add and to delete in itself indicates the difficulties with the data. There are also spectra, such as ^{239}U, and ^{240}Pu and ^{241}Pu as analyzed by the Saclay group, where the quality of the data appears to be reasonable, but in which nonetheless there are important deviations from the ensemble predictions. Where, however, the sequence is short or where the data are extensively "treated" by adding or deleting many levels, we find that more of the ten fluctuation measures tend to deviate significantly from the ensemble values.

Several conclusions may be drawn.

(a) While the quality of the data is variable, the better spectra clearly confirm that the random-matrix theory adequately describes the fluctuation properties, at least as far as they are captured by the ten measures for which we give data.

(b) On the basis of these data, it is not possible to make a clear-cut choice of one ensemble as appropriate. Some results slightly favor the TBRE (or a similar type of ensemble, for example, the EGOE), such as the tendency for the repulsion parameter ω to be somewhat lower than the GOE value in the ground-state domain, and perhaps similarly for the spacing variances: As has been indicated, the TBRE has much the same statistical behavior in the central region as the GOE, but towards the ground-state region the repulsion which it shows decreases.

(c) The hope that the presence of three-body interactions could be detected is not borne out by the ensemble results of Table III, since no significant differences are found between the Monte Carlo TBRE and GOE measures. The effects of imposing a fixed particle rank (two-body) on the interaction are therefore small, so that the extra information specified for these ensembles is largely irrelevant for the fluctuations.

(d) The possibility that the behavior of spectrum fluctuations in the fissile nuclei is somewhat different from that for the stable nuclei cannot be excluded. In its favor there are the deviant values observed for the higher-order spacing variances, the correlation coefficient and the repulsion parameter for the plutonium isotopes, and of the repulsion parameter for ^{239}U; against it is the "well-behaved" nature of the ^{233}Th spectrum, and the fact that, while for plutonium the repulsion is usually low, it is unexpectedly high for uranium; indeed, none of the physically reasonable ensembles discussed in earlier chapters would predict such a high value. It is a pity that other spectral sequences, with quality similar to that quoted for plutonium, have not been measured by the fission process used for that case; any definite conclusion must await further data.

(e) The disagreements between experimental and theoretical values observed for some spectra tend to be correlated for the various measures. This could, however, be due to the interrelationships between the measures or to the fact that they all show the effect of spectrum impurities.

These conclusions are based on conventional analyses of a broad range of data. On the other hand, Coceva and Stefanon (1979), who ask the same questions as we do, come to the same general answers on the basis of a more detailed study of their own data for a single case (^{156}Gd $+n$). Their more sophisticated analysis, if applied to further high quality data, might well result in a better appreciation of the agreement between theory and experiment. It may be possible also to construct and make use of sharper (spectrally averaged) measures, as mentioned in Sec. X.C below.

For mixed-sequence spectra the measures should be considerably different than for pure sequences [see, for example, (5.14)]; the differences are to be ascribed to the moderation of the level repulsion represented by the vanishing of interactions between the subspaces. Table IV gives values for fluctuation measures for a selected set of spectra which contain, in roughly equal proportions, two intermixed subsequences; as mentioned earlier, there is little point in studying spectra involving a mixture of many subsequences. Agreements with theory are in general adequate; whether the discrepancies are to be attributed to the poorer quality of the data (shorter sequences and larger corrections), or to uncertainties in the mixing ratio, or even to a failure of statistical independence of the two sequences is unknown.

The fact that ensemble evaluations of the theoretical fluctuation measures agree with spectral evaluations of the experimental ones will be seen in Sec. X to be well understood in terms of a strong ergodic behavior. Closely related is the fact that the fluctuations extend, essentially unchanged, into the ground-state domain. This, of course, is an aspect of stationarity; it is of major importance in giving a permit for statistical studies in the ground-state domain.

The GOE has often been regarded in the past as the "basic" ensemble, and this seems appropriate. Its special *a priori* significance arises from the fact that it specifies minimal information about the system. Its *a posteriori* significance comes not only from the good description of the data which it yields but from the fact, which is now emerging (see Sec. IV.G), that a wide class of ensembles gives essentially the same two-point function.

The good agreement between theory and the better quality data is significant in providing experimental confirmation for the various general concepts introduced in the preceding sections, in particular, the statistical nature of level repulsion, the existence of long-range correlations that manifest themselves as spectral rigidity, the ergodic behavior of the properties studied, and the dominance of binary associations as seen in the two-point functions and the measures derived from it. These concepts acquire importance because of the great generality and wide range of applicability of the theory.

VIII. TRANSITION STRENGTHS

A. Introduction

In discussing the analysis of the energy-level fluctuations, we have referred often to the particle widths and their distributions, quite naturally so, since it is via these widths that the (unbound) resonance levels are detected and identified. The width distributions have, of course, a significance of their own, deriving directly from the statistical properties of the states; and it is this which we treat in this section, not only for particle widths, but for widths corresponding to other kinds of excitations or transitions, as well.

A width measures the rate at which the system makes a transition from one state to another under the influence of some disturbance. It is expressible in terms of the square of the matrix element of the transition operator between the two states and it shows up in some cases as an actual width of a resonance. Transitions, of course, may take place between states in the discrete or continuous spectrum or between one of each. In the first of these three cases the use of matrix methods is clearly appropriate and needs no specific justification, but this is by no means true for the others, which we come to in Sec. VIII. In this Section we discuss everything via matrix methods.

We are interested then in the distribution of, and correlations between, the transition amplitudes $\gamma_{ij} = \langle \Psi_i | T | \Psi_j \rangle \equiv T_{ij}$,[26] where T is a transition operator and i,j label H eigenvectors. The basic result is the Porter-Thomas distribution (Porter and Thomas, 1956; see also Brink, 1955; Wigner, 1967), which gives the transition amplitudes, normalized by the square root of the locally averaged strength, as Gaussian random variables, the correlations between different strengths

[26]The γ_{ij} notation is common in nuclear reaction theory, while T_{ij} is more appropriate in matrix theory. We shall use the latter. We shall below use both $\langle i|\lambda\rangle$ and $x_{i\lambda}$ for the overlap between the H-eigenvector Ψ_i and the fixed basis state $\hat{\psi}_\lambda$. Note that, with the assumption that the system is time-reversal invariant, the $x_{i\lambda}$ and the matrix elements of transition operators may be taken as real, and we do so. We write $\rho_{A,B,\ldots}(x_1, x_2, \ldots)$ for the joint probability density of the variables $A, B, \ldots$.

being, except in one important case, quite weak and vanishing asymptotically (as $d \to \infty$).

The Porter-Thomas distribution, as usually derived, would seem to be appropriate if the states involved have very different structures; for example, with a scattering resonance, one state might be describable in terms of closed shells with a few valence particles, the other in terms of many hole-particle excitations of this simple structure. When this is the case, it is reasonable to treat the excited states Ψ_i statistically but to regard the other member of the transition pair (which, instead of Ψ_j, we might then label as ψ_c, c defining a reaction channel) as being fixed, not varying as H moves across the ensemble. It is this procedure which gives rise very simply to the Porter-Thomas distribution, as we shall see below. Of course, if the states have very different structures, the strength, for transition operators of interest, will be fragmented into many small components (small compared with their theoretical maxima) and it is reasonable then to expect general statistical laws to apply. We shall sometimes refer to this as the "external" case, one of the states in the transition pair being outside the vector space of the statistically treated states.

However, as we have mentioned in Sec. I, and will describe in some detail below, there is some evidence that, like other statistical laws, something like the Porter-Thomas distribution is valid much more generally than has often been assumed. It appears to hold, for example, with γ transitions between low-lying states, when one cannot reasonably assume that the states involved have very different structures, or, equivalently, that the strength is greatly fragmented. In this circumstance we should regard both states as belonging to the "statistical space," both therefore varying with H. We shall consider this "internal" case, also, and derive forms for the strength distribution, making clear the circumstances under which they reduce to the Porter-Thomas form.

Although from a purely technical standpoint the random-matrix theory for transition strengths is very much simpler than for eigenvalues, the strengths carry much more information about the system (see, for example, Appendix C) and therefore make far more demands on the models used. The most interesting questions concern, not simply how well the GOE or a similar theory fits the data, but rather what extensions of the simplest statistical model are called for, either as modifications of the ensembles or, more explicitly, in terms of direct-reaction effects, single-particle excitations, and so forth (the two are, of course, related). Moreover, the relevant experiments, needed, in particular to study correlations between processes, are hard to do, and the runs of data are much shorter than in the energy-level case, so that the statistical errors are large. The whole subject then is huge and complicated; and though we shall comment briefly on some of the strength data, we cannot attempt any general review of it [see, for example, Lynn (1968); Lane (1976); Bilpuch *et al.*, (1976); MacDonald (1979)]. We shall, however, study closely some formal aspects which have been ignored and consider also expectation values and some effects of symmetries and collectivities.

B. The distribution of eigenvector components in the GOE

We derive in this section some basic results, which we shall need ahead, for the distribution of a component of a given H eigenvector, Ψ_i, and more generally the joint distribution of several components and the correlations between them. Many of the results derive directly from simple theorems of d-dimensional geometry; others are well known, also, having been derived, in some cases by different methods, largely by Ullah and Porter (1963a) and Ullah (1964, 1967).

Because the GOE is an orthogonally invariant ensemble, each H eigenvector uniformly covers the d-dimensional unit sphere as H moves through the ensemble. In the two-dimensional case each vector—Ψ_i, say—moves uniformly on a circle, so that, for the component along a fixed axis (say, $\hat{\psi}_\lambda$), we have the probability density $\rho_\theta(x) = (2\pi)^{-1}$ and therefore

$$\rho_{\langle i|\lambda\rangle}(x) = \rho_{\sin\theta}(x) = \frac{1}{\pi|\cos\theta|} . \tag{7.1}$$

In three dimensions the corresponding probability density is found to be constant in the $(-1, 1)$ interval,

$$\rho_{\langle i|\lambda\rangle}(x) = \tfrac{1}{2}, \quad |x| \leq 1 . \tag{7.2}$$

The corresponding result in d-dimensions is a standard one of d-dimensional geometry, easily derived by introducing polar coordinates. But it will be more efficient for us to start with the joint probability distribution

$$\rho_{\langle i|\lambda_1\rangle,\langle i|\lambda_2\rangle,\ldots,\langle i|\lambda_d\rangle}(x_1, x_2, \ldots, x_d) = \pi^{-d/2}\Gamma(d/2)\delta\left(\sum_{i=1}^{d} x_i^2 - 1\right) \tag{7.3}$$

for all the components of Ψ_i, and derive the required results by integrating over the unwanted components. The form (7.3) follows from the facts that (1) the distribution can depend, because of orthogonal invariance, only on the norm, $r = (\sum_{i=1}^{d} x_i^2)^{1/2}$, of the vector and must be proportional to the indicated δ function because of the normalization (so that any multiplying function of r would reduce to a constant); and (2) for integrating a spherically symmetric function $g(r)$ the volume element $d\mathcal{V}$ is given by

$$\Gamma(d/2)d\mathcal{V} = 2\pi^{d/2}r^{d-1}dr ,$$

so that

$$\Gamma(d/2)\int d\mathcal{V}\,\delta(r^2 - 1) = \pi^{d/2} .$$

For the joint probability distribution for l components—say, $\langle i|\lambda_1\rangle, \ldots, \langle i|\lambda_l\rangle$—we simply carry out a $(d-l)$ fold integration on (7.3) using Cartesian coordinates, encountering thereby only elementary integrals. The result is

$$\rho_{\langle i|\lambda_1\rangle,\ldots,\langle i|\lambda_l\rangle}(x_1, \ldots, x_l) = \pi^{-l/2}\Gamma(d/2)\,\Gamma\left(\frac{d-l}{2}\right)^{-1}\left(1 - \sum_{i=1}^{l} x_i^2\right)^{(d-l-2)/2} . \tag{7.4}$$

The density vanishes, of course, as in other equations ahead, if the normalization restriction $\sum_{i=1}^{l} x_i^2 \leq 1$ is not satisfied. It should be clear that the same form (7.4) gives the density for the same component (fixed $\hat{\psi}_l$) of l different H eigenvectors, and similarly for other equations which we give below. This result (7.4) is given without derivation in Lynn (1968), being ascribed there to Ullah (1967). The form is indeterminate for $l=d$, but that result, of course, is given by (7.3).

For $l=1$ we find the standard result that

$$\rho_{\langle i|\lambda\rangle}(x) = \pi^{-1/2}\Gamma(d/2)\Gamma\left(\frac{d-1}{2}\right)^{-1}(1-x^2)^{(d-3)/2}, \tag{7.5}$$

of which (7.1) and (7.2) are special cases. The odd moments of this distribution vanish, while for the even moments we find

$$M_{2\nu} = \frac{(2\nu-1)!!}{2^\nu}\frac{\Gamma(d/2)}{\Gamma(d/2+\nu)} = \frac{(2\nu-1)!!(d-2)!!}{(d+2\nu-2)!!} \sim \frac{(2\nu-1)!!}{d^\nu}\left\{1 - \frac{2}{d}\binom{\nu}{2} + \frac{4}{d^2}\left[3\binom{\nu}{4} + 4\binom{\nu}{3} + \binom{\nu}{2}\right] + \cdots\right\}. \tag{7.6}$$

The last form derives from expanding out the factorials and is an expansion in inverse powers of d; the first term of this shows that the "standardized" variable $d^{1/2}\langle i|\lambda\rangle$ becomes asymptotically a Gaussian random variable of zero centroid and unit variance[27]:

$$\rho_{d^{1/2}\langle i|\lambda\rangle}(x) \xrightarrow[d\to\infty]{} G(0,1). \tag{7.7}$$

Going beyond that, we have, with $\rho_G \equiv G(0,\sigma)$ and He_ν the Hermite polynomial with weight function $\exp(-x^2/2\sigma^2)$, that

$$\int dx\, \rho_G(x) x^{2\nu} \mathrm{He}_\zeta(x/\sigma) = (2\nu-1)!!2^{(\zeta/2)}\frac{\nu!}{[\nu-(\zeta/2)]!}\sigma^{2\nu} \tag{7.8}$$

and then, from (7.6)

$$\rho_{\langle i|\lambda\rangle}(x) \sim \left[1 - \frac{1}{4d}\mathrm{He}_4(d^{1/2}x) + \frac{1}{d^2}\left(\frac{1}{32}\mathrm{He}_8(d^{1/2}x) + \frac{1}{3}\mathrm{He}_6(d^{1/2}x) + \frac{1}{2}\mathrm{He}_4(d^{1/2}x)\right) + \cdots\right]\left(\frac{d}{2\pi}\right)^{1/2}\exp(-x^2 d/2), \tag{7.9}$$

which gives the lower-order corrections to (7.7). The asymptotic Gaussian form is one of the two basic results, the other, which we encounter below, being concerned with correlations between amplitudes. For the square of the amplitude, using $\rho_{Z^2}(x) = x^{-1/2}\rho_Z(\sqrt{x}\,)$, which is valid for a distribution symmetrical about zero, we have

$$\rho_{\langle i|\lambda\rangle^2}(x) = \frac{\Gamma(d/2)}{\sqrt{\pi}\,\Gamma\left(\frac{d-1}{2}\right)}x^{-1/2}(1-x)^{(d-3)/2} \xrightarrow[d\to\infty]{} \left(\frac{d}{2\pi}\right)^{1/2}x^{-1/2}\exp(-xd/2), \tag{7.10}$$

the asymptotic form of which is a χ_1^2 distribution.

Different components of the same H-eigenvector (and therefore the same component of different H eigenvectors) cannot be independent, since, if for no other reason, a quadratic correlation between $\langle i|\lambda\rangle$ and $\langle i|\mu\rangle$ (and between $\langle i|\lambda\rangle$ and $\langle j|\lambda\rangle$) is imposed by normalization.[28] On the other hand, from the asymptotic form of (7.4),

$$\rho_{\langle i|\lambda_1\rangle,\ldots,\langle i|\lambda_l\rangle}(x_1,\ldots,x_l) \xrightarrow[d\to\infty]{} \prod_{i=1}^{l}\left(\frac{d}{2\pi}\right)^{1/2}\exp(-dx_i^2/2), \tag{7.11}$$

which is valid when $(d-l)\gg 1$, we see that the components are asymptotically independent, rather than merely uncorrelated. Thus for large dimensionality the correlations may be characterized as "weak." We shall see that they may often be ignored (but not always, as, for example, in Sec. VII.H below).

The correlations, of course, follow from (7.4), but it is instructive to derive them rather differently. Observe first that the average of any product of amplitudes will vanish unless every symbol $i,j,\ldots,\lambda,\mu$ occurs an even number of times (since, for example, $\langle i|\lambda\rangle$ takes on plus and minus values with equal probability as Ψ_i moves over the unit sphere, and similarly for products not satisfying the stated rule). Moreover, such averages involving $\langle i|\lambda\rangle$ and $\langle i|\mu\rangle$ will be independent of i,λ,μ and will have the same values as those involving $\langle i|\lambda\rangle$ and $\langle j|\lambda\rangle$; the case is similar for other averages. These results follow immediately from the orthogonal invariance. Now, since $a|\lambda\rangle + b|\mu\rangle$ is itself a basis state, with norm $(a^2+b^2)^{1/2}$, we have for the 2νth moment of the component of Ψ_i along that state, with $M_{2\nu}$ given by (7.6),

$$\overline{(a\langle i|\lambda\rangle + b\langle i|\mu\rangle)^{2\nu}} = (a^2+b^2)^\nu\overline{\langle i|\tau\rangle^{2\nu}} = (a^2+b^2)^\nu M_{2\nu}. \tag{7.12}$$

Expanding both sides and equating terms in $a^r b^s$, we have for the mixed moments the exact results

$$M_{2\nu-2\alpha,2\alpha} \equiv \overline{\langle i|\lambda\rangle^{2\nu-2\alpha}\langle i|\mu\rangle^{2\alpha}} = \binom{\nu}{\alpha}\binom{2\nu}{2\alpha}^{-1}\overline{\langle i|\lambda\rangle^{2\nu}} = (2\nu-2\alpha-1)!!(2\alpha-1)!!\frac{(d-2)!!}{(d+2\nu-2)!!}. \tag{7.13}$$

[27] We shall often write this as $\rho_{\langle i|\lambda\rangle}(x) \xrightarrow[d\to\infty]{} G(0,d^{-1/2})$, though, of course, the limiting distribution for $\langle i|\lambda\rangle$ is simply $\delta(x)$—similarly, in other equations ahead, for example, Eq. (7.10). When confusion might arise, or when we wish to distinguish between the large-d and the ($d=\infty$) cases, we shall use also the notation $\xrightarrow[\text{large } d]{}$.

[28] We shall see, in fact, that there is a weaker correlation between $\langle i|\lambda\rangle^2$ and $\langle j|\mu\rangle^2$, with all symbols different. We use the term "quadratic correlation" for a linear correlation between squares.

For the correlation between strengths (squares of amplitudes) and the corresponding correlation coefficient, defined for two random variables in the standard way and taking values in the interval $(-1,1)$, we have then

$$\begin{aligned}&\overline{\langle i|\lambda\rangle^2\langle i|\mu\rangle^2}=\overline{\langle i|\lambda\rangle^2\langle j|\lambda\rangle^2}=\frac{1}{d(d+2)}\ ,\\ &C(\langle i|\lambda\rangle^2,\langle i|\mu\rangle^2)=C(\langle i|\lambda\rangle^2,\langle j|\lambda\rangle^2)\\ &\qquad=-\frac{1}{(d-1)}\ .\end{aligned}\tag{7.14}$$

It is to be understood that, in these equations, and similarly below, $i\neq j, \lambda\neq\mu$. Thus, for example, $[\langle i|\lambda\rangle^4]_e$, which by (7.6) is $3/d(d+2)$, would not be given correctly by (7.13) or (7.14).

The procedure used here for two components works for an arbitrary number l of them ($1\leq l\leq d$). We use multinomial expansions and find them

$$\begin{aligned}&\overline{\langle i|\lambda_1\rangle^{2\nu_1}\langle i|\lambda_2\rangle^{2\nu_2}\cdots\langle i|\lambda_l\rangle^{2\nu_l}}\\ &\qquad=\frac{(\nu:\nu_1,\nu_2,\ldots,\nu_l)}{(2\nu:2\nu_1,2\nu_2,\ldots,2\nu_l)}\overline{\langle i|\mu\rangle^{2\nu}}\ ,\end{aligned}\tag{7.15}$$

in which $\nu=\Sigma\nu_i$, $(\nu;\nu_1,\ldots,\nu_l)=\nu!/\Pi_i\nu_i!$, while $[\langle i|\mu\rangle^{2\nu}]_e$ is given by (7.6) and contains the entire dependence on the dimensionality. The corresponding case of several (orthonormal) eigenvectors and a single basis vector is given by the same equation.

Besides these "one-vector" results we give also "two-vector" results, for different components of different eigenvectors. These results, due originally to Ullah and Porter (1963a), follow from the vector orthonormality, by ensemble averaging $(\sum_\lambda x_{i\lambda}^2)(\sum_\mu x_{j\mu}^2)=1$ and $(\sum_\lambda x_{i\lambda}x_{j\lambda})^2=0$. We find

$$\begin{aligned}&\overline{\langle i|\lambda\rangle^2\langle j|\mu\rangle^2}=\frac{d+1}{(d-1)d(d+2)}\ ,\\ &\overline{\langle i|\lambda\rangle\langle j|\lambda\rangle\langle i|\mu\rangle\langle j|\mu\rangle}=-\frac{1}{(d-1)d(d+2)}\\ &C(\langle i|\lambda\rangle^2,\langle j|\mu\rangle^2)=\frac{1}{(d-1)^2}\ .\end{aligned}\tag{7.16}$$

We remark finally that we have established in (7.15) an implicit connection between the joint distribution function for the components of the same H eigenvector (determined by the moments, including the mixed ones) and the marginal distribution for a single component. Let us make this explicit in terms of the multidimensional characteristic function, which is

$$\begin{aligned}\phi(t_1,\ldots,t_l)&=\int\cdots\int e^{-it_1x_1\cdots-it_lx_l}\rho(x_1,\ldots,x_l)dx_1\cdots dx_l\\ &=\sum_{\nu_1,\ldots,\nu_l=0}^{\infty}\frac{(-it_1)^{2\nu_1}\cdots(-it_l)^{2\nu_l}}{(2\nu_1)!\cdots(2\nu_l)!}\frac{(\nu:\nu_1,\ldots,\nu_l)}{(2\nu:2\nu_1\cdots 2\nu_l)}M_{2\nu}\\ &=\sum_{\nu_1,\ldots,\nu_l=0}^{\infty}(\nu:\nu_1\cdots\nu_l)[(-it_1)^{2\nu_1}\cdots(-it_l)^{2\nu_l}](2\nu)!^{-1}M_{2\nu}\\ &=\sum_\nu(-1)^\nu t^{2\nu}(2\nu)!^{-1}M_{2\nu}=n!\left(\frac{2}{t}\right)^n J_n(t),\quad t^2=\sum_{i=1}^{l}t_i^2,\quad n=(d-2)/2\ .\end{aligned}\tag{7.17}$$

This form is independent of l and, of course, identical with the characteristic function for the single-component distribution, but with the one-dimensional t replaced by the l-dimensional radial t. We may invert the Fourier transform (7.17) either one step at a time, in Cartesian coordinates or more simply in l-dimensional polar coordinates, and thereby, of course, regain the density (7.4).

C. Strength distributions and correlations: the external case

We are interested in the distribution of the strength $|T_{ij}|^2$, where T is the transition operator, and the matrix element connects Hamiltonian eigenstates. We rely, of course, on ergodicity, which we discuss in Sec. X, to replace the spectral distribution (for a given H) by the distribution over the ensemble, for which at the beginning we assume a GOE.

As we have indicated above there are two separate cases, in accordance with our treatment statistically of one or both of the H eigenstates. In this section we consider the first case. Let us agree that we have a d'-dimensional model space S containing a d-dimensional subspace $\mathfrak{s}(\mathfrak{s}\subset S)$, which we propose to treat statistically. Let us choose for S a basis $|i\rangle$ with $i=1\ldots d'$ such that $|i\rangle$ with $i=1\ldots d$ is a basis for $\mathfrak{s}$; then the states $|i\rangle$ with $i=d+1,\ldots d'$ are a basis for the "nonstatistical" subspace of S. It will be appropriate to take the states $|i\rangle$ as the Hamiltonian eigenstates.

The ordinary Porter-Thomas distribution results when we assume that one state of the transition pair—say, $|c\rangle$—is fixed and outside of $\mathfrak{s}$ (i.e., is to be treated nonstatistically). Then the transition amplitude T_{ic} is simply the overlap between the statistical state $|i\rangle$ and the "giant resonance" state $\mathcal{P}T|c\rangle$, where

$$\mathcal{P}=\sum_{j=1}^{d}|j\rangle\langle j|\ ,\tag{7.18}$$

defined in S, is the projection operator into the statistical subspace. Except for the fact that $\mathcal{P}T|c\rangle$ is not normalized to unity, its squared norm being

$$\sigma_c^2=\sum_{i=1}^{d}|T_{ic}|^2=\langle c|T^+\mathcal{P}T|c\rangle\ ,\tag{7.19}$$

the strength distribution is given by the results of the last section; we note that the properties of the transition operator T enter into the strength distribution only via the normalization (7.19). Using the elementary result that, for a random variable z and positive constant σ, $\rho_{\sigma z}(x)=\sigma^{-1}\rho_z(x/\sigma)$, we have from (7.5) that

$$\rho_{T_{ic}}(x)=\frac{\Gamma(d/2)}{\sqrt{\pi}\,\Gamma\frac{d-1}{2}}\sigma_c^{-1}\left(1-\frac{x^2}{\sigma_c^2}\right)^{(d-3)/2}$$

$$\xrightarrow[d\to\infty]{}\left(\frac{d}{2\pi\sigma_c^2}\right)^{1/2}\exp(-x^2d/2\sigma_c^2), \tag{7.20}$$

both forms of which have a variance $d^{-1}\sigma_c^2$. For the strength itself,

$$\rho_{|T_{ic}|^2}(x)\xrightarrow[d\to\infty]{}\left(\frac{d}{2\pi\sigma_c^2 x}\right)^{1/2}\exp(-xd/2\sigma_c^2)$$

$$=(2\pi\overline{|T_{ic}|^2}x)^{-1/2}\exp(-x/2\overline{|T_{ic}|^2}), \tag{7.21}$$

the usual Porter-Thomas (χ_1^2) distribution.[29] For the case of isolated resonance, the "observed" width Γ_{ic} incorporates a channel penetrability factor, as discussed in Sec. VIII, which usually varies slowly with the energy

$$\Gamma_{ic}=2P_c\gamma_{ic}^2=2P_c|T_{ic}|^2 \tag{7.22}$$

and then

$$\rho_{\Gamma_{ic}}(x)=\frac{1}{(2\pi\overline{\Gamma}_{ic})^{1/2}}x^{-1/2}\exp(-x/2\overline{\Gamma}_{ic}). \tag{7.23}$$

The finite-d forms of (7.21) and (7.23) also follow directly from (7.20).

It must be borne in mind that these equations give (for GOE) the probability of finding any given value of the strength; they have nothing to do with how the strength varies along the spectrum of the initial or final states. Moreover, because of the GOE uniformity discussed in Sec. II, the distributions do not depend at all on the energy E_i. When we apply the distribution to experimental data, we must, however, take account of any *secular* variation, in the strength, along the initial spectrum, arising, for example, from the slow energy variation in the orbital structure of the initial states. A basic assumption here (common to all extensions of GOE results to more complicated situations) is that a simple mapping of local GOE results is permissible. In the present case we accomplish this by renormalizing the strength according to its *local* (slowly energy-varying) average, i.e., we take $[|T_{ic}|^2]_e$ in (7.21) or $\overline{\Gamma}_{ic}$ in (7.23) to be "local constants" rather than global ones, allowing thereby for their secular variation with respect to the initial and final energies.

It should be clear that, with one exception, transition amplitudes are uncorrelated; the exceptional case is for amplitudes connecting the same statistical state $|i\rangle$ with two different (and therefore orthogonal) nonstatistical ones $|c\rangle$, $|c'\rangle$. The correlation in this case arises from the fact that the projections into $\mathcal{S}$ of the giant-resonant states, $T|c\rangle$ and $T|c'\rangle$, are not in general orthogonal, so that one projection "contains" a multiple (given by $\cos\theta_{cc'}$, the cosine of the angle between $\mathcal{P}T|c\rangle$ and $\mathcal{P}T|c'\rangle$) of the other. We have then, valid for all d,

$$C(T_{ic},T_{ic'})=\cos\theta_{cc'}=(\sigma_c\sigma_c)^{-1}\langle c'|T^+\mathcal{P}T|c\rangle$$

$$=(\sigma_c\sigma_c)^{-1}\sum_{j=1}^{d}T_{c'j}T_{jc}, \tag{7.24}$$

with the obvious extension if the two transitions are generated by different operators, in which case we can consider $c=c'$ in the same way.

It follows, moreover, from the *asymptotic independence* of the amplitudes that with the same exception, which then gives a *non-negative* correlation

$$C(|T_{ic}|^2,|T_{ic'}|^2)\xrightarrow[d\to\infty]{}\cos^2\theta_{cc'}, \tag{7.25}$$

the strengths are asymptotically uncorrelated. It is elementary, but worthwhile, to derive the finite-d results which follow from (7.15) or (7.24). We find

$$C(|T_{ic}|^2,|T_{jc}|^2)=-\frac{1}{(d-1)},\quad i\neq j$$

$$C(|T_{ic}|^2,|T_{jc'}|^2)=-\frac{d}{(d-1)^2}\left(\cos^2\theta_{cc'}-\frac{1}{d}\right),\quad i\neq j,\ c\neq c'$$

$$C(|T_{ic}|^2,|T_{ic'}|^2)=\frac{d}{(d-1)}\left(\cos^2\theta_{cc'}-\frac{1}{d}\right),\quad c\neq c'. \tag{7.26}$$

It is reasonable to regard these correlation results as a part of the Porter-Thomas law, the content of which then is that the transition amplitudes in Hamiltonian representation are Gaussian random variables which, except in the case of strengths connecting the same eigenvector to two different channel states, may be taken as independent in the usual case of large dimensionality; even in the exceptional case the joint probability distribution is multivariate Gaussian. The statistical operation involved here is that of ensemble averaging but, relying on ergodicity (Sec. X), we would replace that by averaging over the spectrum. Moreover, while the asymptotic independence of T_{ic} and T_{jc} is of major consequence, leading immediately to ergodicity, it is the correlations between T_{ic} and $T_{ic'}$ which are of most interest in the data analysis.

The sum of statistically independent transition widths

$$\Gamma_i=\sum_{c=1}^{l}\Gamma_{ic}, \tag{7.27}$$

each of which is distributed as χ_1^2, is by definition distributed as χ_l^2 if the χ_1^2 parameters $\overline{\Gamma}_{ic}$ are all equal. We can then use the relation between the χ_l^2 centroid and variance ($\sigma^2/\mathcal{E}^2=2/l$) to define an effective $l=\hat{l}$ for the case where the variables are not independent or the parameters not equal (Porter and Thomas, 1956; Draayer *et al.*, 1977). Thus

$$\frac{2}{\hat{l}}=\frac{\mathrm{var}(\Gamma_i)}{(\overline{\Gamma}_i)^2}=\frac{2\sum_{cc'}\overline{\Gamma}_{ic}\overline{\Gamma}_{ic'}\cos^2\theta_{cc'}}{\sum_c\overline{\Gamma}_{ic})^2}$$

$$\xrightarrow[l=2]{}2\left\{\frac{\alpha+\alpha^{-1}+2\gamma^2}{\alpha+\alpha^{-1}+2}\right\}. \tag{7.28}$$

We have already, in (1.9), used the first form of $2/\hat{l}$ and will return to it below. The second form derives from (7.25); it is easy to see that $1\leq\hat{l}\leq l$, the lower bound being attained when the Γ_{ic} are completely correlated and the upper bound when they are uncorrelated and the parameters equal. In the last form, α is the ratio of the two average widths and γ^2 the correlation

[29] For the unitary instead of the orthogonal ensemble the distribution is χ_2^2 (exponential) instead of χ_1^2. This, in fact, was the first general form proposed, by Scott (1954), for the strength distribution.

coefficient, written as positive because of (7.25). This form is used in the analysis of mixed transitions (two different multipolarities in γ transitions, for example); it shows that, according to the simple statistical model, such admixtures lead in general to an increase ($\bar{l} > 1$) in the effective number of "degrees of freedom."

D. Strength distributions and correlations: the internal case

Things are considerably different if we treat statistically both members of the transition pair, which, as we have said above, should be appropriate in certain important cases. In particular, for a Hermitian transition operator the results depend on its spectrum (since we consider an orthogonally invariant ensemble, all transition operators with the same spectrum give rise to the same distributions). The interesting questions then are what kinds of distributions and correlations follow for various general classes of operators and under what circumstances the results coincide with Porter-Thomas. We follow French and Pandey (unpublished).

We can distinguish three general cases. In the first the entire space is treated statistically, the transition operator being Hermitian, $T = T^+$; similarly for the second, but with a non-Hermitian operator, $T \neq T^+$. For these we have effectively $d' = d$ and $S = \mathfrak{s}$. The third case is for a non-Hermitian operator which divides the space into two statistical Hamiltonian eigenspaces (i.e., subspaces invariant for the ensemble of H's), $S = \mathfrak{s}_1 + \mathfrak{s}_2$. It is natural to regard statistical variations in $\mathfrak{s}_1$ and $\mathfrak{s}_2$ as independent, the ensemble then being taken as two independent (GOE) subensembles; we have here a natural extension of the "external" case of Sec. VII.C. As an example, consider single-particle addition, as with neutron capture, for which T is the projection into S of a linear combination of creation operators. Then $T \neq T^+$, $T^2 = 0$ so that T is not diagonalizable, and T effectively partitions S into m-particle and $(m+1)$-particle subspaces and acts in one direction between them.

To begin with, we consider the first two cases, in which the space is not restricted by any symmetry of H. A single GOE is then adequate. T is diagonalizable in S if it is Hermitian. In a general basis $|\lambda\rangle, |\mu\rangle, \ldots$, we have for the (real) matrix elements

$$T_{ij} = \sum_{\lambda,\mu} T_{\lambda\mu} x_{i\lambda} x_{j\mu} , \tag{7.29}$$

in which the random vectors $|i\rangle$, $|j\rangle$ are constrained by orthogonality and thus correlated (weakly so for large d). We now carry out the d-dimensional orthogonal transformations in two steps. In the first we keep the H eigenstate $|j\rangle$ fixed so that $|i\rangle$ moves, under $(d-1)$-dimensional orthogonal transformations, in a $(d-1)$-dimensional subspace of $\mathfrak{s}$, orthogonal to $|j\rangle$; during this stage the giant-resonance state $T|j\rangle$ and its projection, $\mathcal{P}_{(d-1)}T|j\rangle$, onto the hyperplane are of course fixed vectors, the latter with squared norm $\sigma_j^2 = \langle j|T^+\mathcal{P}_{(d-1)}T|j\rangle$. The distributions which result from these restricted transformations, which we write as ρ^*, are precisely those of the external case above, but with $d \to (d-1)$. Thus

$$\rho^*_{T_{ij}}(x:\sigma_j) = \frac{\Gamma\left(\frac{d-1}{2}\right)}{\pi^{1/2}\Gamma\left(\frac{d-2}{2}\right)} \sigma_j^{-1}\left(1 - \frac{x^2}{\sigma_j^2}\right)^{(d-4)/2} \xrightarrow[d\to\infty]{} \left(\frac{d}{2\pi\sigma_j^2}\right)^{1/2} \exp(-x^2 d/2\sigma_j^2) . \tag{7.30}$$

In the second stage we allow $|j\rangle$ to vary uniformly over the unit hypersphere, during which, of course, the norm of the projected giant-resonance state also varies. Let us write the distribution of σ_j as $\rho_{\sigma_j}(x)$. Then combining the two distributions gives for the transition-amplitude distribution,

$$\rho_{T_{ij}}(x) = \int \rho^*_{T_{ij}}(x:z)\rho_{\sigma_j}(z)dz , \tag{7.31}$$

whose moments, by (7.6), are then

$$M_{2\nu}(T_{ij}) \equiv \overline{T_{ij}^{2\nu}} = \frac{(2\nu-1)!!}{2^\nu} \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d-1}{2}+\nu\right)} \xrightarrow[d\to\infty]{} \frac{(2\nu-1)!!}{d^\nu} \overline{\sigma_j^{2\nu}} ,$$

$$M_{2\nu+1}(T_{ij}) \equiv \overline{T_{ij}^{2\nu+1}} = 0 . \tag{7.32}$$

Just as in (7.19), we have

$$\sigma_j^2(T) \equiv (T^+T)_{jj} - (T_{jj})^2 = (\tilde{T}^+\tilde{T})_{jj} - (\tilde{T}_{jj})^2 = \{(\tilde{T}^+ - \tilde{T}_{jj})(\tilde{T} - \tilde{T}_{jj})\}_{jj} \xrightarrow[d\to\infty]{} (\tilde{T}^+\tilde{T})_{jj} = (G_T)_{jj} ,$$

$$\tilde{T} \equiv T - \langle T\rangle , \quad G_T \equiv \tilde{T}^+\tilde{T} , \tag{7.33}$$

where $\tilde{T}$ is then the traceless operator corresponding to T and G_T is its Hermitian square. The asymptotic form in (7.33) follows from the fact that $\tilde{T}_{jj}$ goes to zero in "almost all" cases[30]; this is so because its ensemble average is zero, while its variance, when scaled by the spectral variance of T, is of order d^{-1}, as follows from (7.29) with $i = j$, along with the basic "one-vector" result (7.15); or see (7.60) below.

Now by taking powers of σ_j^2 and using (7.15), we can, for low powers, evaluate $[\sigma_j^{2\nu}]_e$ and hence, by (7.32), the moments of the T_{ij} distribution. But more simply, for asymptotic d, we can use the last form of (7.33), whose moments, being those of an expectation value, derive from (7.60) below. Parallel with the ensemble moments of a matrix element (7.32) we introduce the spectral moments of a Hermitian operator. If the eigenvalues of Q are q_λ, we have

$$M_p(Q) \equiv d^{-1}\sum_\lambda (q_\lambda)^p = \langle Q^p\rangle . \tag{7.34}$$

We shall similarly write $\mathfrak{M}_p$ for the central moments of both distributions.

Note then that $\mathfrak{M}_p(Q) = M_p(\tilde{Q})$. We find[31]

[30] This can be regarded as an aspect of ergodicity. We shall use this concept several times in this section [Eq. (7.37), for example].

[31] For Hermitian T we have $T_{ij} = \sum_\lambda t_\lambda x_{i\lambda} x_{j\lambda}$ where t_λ are its eigenvalues. Then $\sigma_j^2 = \sum_\lambda t_\lambda^2 x_{j\lambda}^2 - (\sum_\lambda t_\lambda x_{j\lambda}^2)^2$, and

$$(d-1)(d+1)(d+2)(d+4)(d+6)M_4(T_{ij}) = 3d[2(d+2)\mathfrak{M}_4(T) + (d^2+6d+12)\mathfrak{M}_2^2(T)].$$

Note that $M_\nu(G_T) = \mathfrak{M}_{2\nu}(T)$ for Hermitian T.

$$M_2(T_{ij})=\frac{\overline{\sigma_j^2}}{(d-1)}=\frac{d+1}{(d-1)(d+2)}M_1(G_T)-\frac{1}{(d-1)(d+2)}\mathfrak{M}_2(T)\xrightarrow[d\to\infty]{}\frac{1}{d}M_1(G_T),$$

$$M_4(T_{ij})=\frac{\overline{3\sigma_j^4}}{(d-1)(d+1)}\xrightarrow[d\to\infty]{}\frac{3}{d^2}\left[M_1^2(G_T)+2d^{-1}M_2(G_T)\right], \tag{7.35}$$

$$M_6(T_{ij})=\frac{\overline{15\sigma_j^6}}{(d-1)(d+1)(d+3)}\xrightarrow[d\to\infty]{}\frac{15}{d^3}\left[M_1^3(G_T)+6d^{-1}M_2(G_T)M_1(G_T)+8d^{-2}M_3(G_T)\right].$$

The skewness, of course, vanishes, while, for the excess which measures the departure from Gaussian,

$$\gamma_2(T_{ij})\xrightarrow[d\to\infty]{}\frac{6M_2(G_T)}{dM_1^2(G_T)}\xrightarrow[T=T^+]{}\frac{6}{d}\left[\gamma_2(T)+3\right]. \tag{7.36}$$

As examples for Hermitian T, $\gamma_2(T)=0,-1,+12$, respectively, for Gaussian, semicircular, and χ_1^2 distributions; then $\gamma_2(T_{ij})\xrightarrow[d\to\infty]{}0$. These are examples of what we shall call "nonsingular" distributions. With operators which lead to such distributions we can expect to find Porter-Thomas for the internal transitions just as for the external.

In contrast, the pairing operator K, whose spectrum is $(0^{d-1},1)$, is an example of a singular operator. For large d its spectral excess is $\gamma_2(T)\to d$, and then $\gamma_2(T_{ij})\to 6$, so that the T_{ij} distribution is far from Gaussian; as we shall see below, T_{ij} has, in fact, a K_0 distribution (7.39).

A general criterion for a singular operator T is easily stated in terms of the properties of its Hermitian square G_T. When rescaled to unit average, $G_T\to G_T/M_1(G_T)$, its ($\nu\geq 2$) spectral moments diverge with d as $d^{\nu-1}$, in contrast to a nonsingular operator for which there is no such divergence. Singular spectra most important for us are bimodal, with one mode carrying all but a few of the levels and with the centroid separation $\sim d$ times the width of the dominant mode. See Appendix O for further discussion. Observe that the GOE, deformed by a pairing interaction (Sec. III.E), gives for large α an excellent example of a singular operator.

A measure of the fractional variation in σ_j^2 as H moves over the ensemble is supplied by

$$\frac{\mathrm{var}(\sigma_j^2)}{(\overline{\sigma_j^2})^2}=\frac{\overline{\sigma_j^4}}{(\overline{\sigma_j^2})^2}-1\xrightarrow[d\to\infty]{}\frac{2}{d}\frac{M_2(G_T)}{M_1^2(G_T)}$$

$$\xrightarrow[\text{nonsingular T}]{}0. \tag{7.37}$$

We see that, for a nonsingular T, σ_T^2 for large d is very sharply peaked at its average value $M_1(G_T)$; for essentially all H's in the GOE we would find closely the same value of σ_j^2. It follows then that, for nonsingular operators in the large-d limit, $\rho_{\sigma_j}(x)\to\delta(x-M_1^{1/2}(G_T))$, and since

$$\rho^*_{T_{ij}}(x)\xrightarrow[d\to\infty]{}G(0,d^{-1/2}\sigma_j),$$

we find that

$$\rho_{T_{ij}}(x)\xrightarrow[\text{nonsingular T}]{d\to\infty}G(0;d^{-1/2}M_1^{1/2}(G_T)), \tag{7.38}$$

the Porter-Thomas result. We see, on the other hand, that a singular T does not give rise to Porter-Thomas, since $\mathrm{var}(\sigma_j^2)/[(\sigma_j^2)]_e^2\sim 1$.

In geometrical terms it is easy to see the distinction between the two cases. If, for example, G_T has a singular spectrum $(1,0^{d-1})$, then only one component of ψ_j contributes, via the action of T, to the giant resonance, and this component varies strongly as H moves over the ensemble; this behavior is not to be expected when G_T has a "rich" spectrum, for then, as one component decreases, others will take its place in generating the giant resonance.

In the usual P-T distribution the amplitudes are Gaussian random variables. By (7.38) the same is true for the internal case when the transition operator is nonsingular. However, for the basic singular operator T [for which G_T has a spectrum $(1,0^{d-1})$], $T_{ij}=x_{i\lambda}x_{j\mu}$, being then the product of two asymptotically independent, identically distributed Gaussian random variables $G(0,d^{-1/2})$. For a pair of operators ξ,η, whose joint probability density is $\rho_{\xi,\eta}(x,y)$, we have

$$\begin{aligned}\rho_{\xi\times\eta}(x)&=\int_{-\infty}^{\infty}\frac{ds}{|s|}\rho_{\xi,\eta}\left(s,\frac{x}{s}\right)\\&\xrightarrow[\text{Ind}]{}\int_{-\infty}^{\infty}\frac{ds}{|s|}\rho_\xi(s)\rho_\eta(x/s)\\&\xrightarrow[\text{Gaussian}]{}\frac{d}{2\pi}\int_{-\infty}^{\infty}\frac{ds}{|s|}\exp\left[-\frac{d}{2}\left(s^2+\frac{x^2}{s^2}\right)\right]\\&=\frac{d}{2\pi}\int_0^\infty\frac{dz}{z}\exp\left[-\frac{xd}{2}\left(\frac{z}{x}+\frac{x}{z}\right)\right]=\frac{d}{\pi}K_0(d|x|),\end{aligned} \tag{7.39}$$

where the second form follows if the variables are independent (Ind), and the third if, besides that, they are Gaussian with centroid zero and variance d^{-1}, as we are assuming.[32] In the last form K_0 is the usual modified Bessel function of the second kind; it has a logarithmic singularity at $x=0$ and falls off exponentially for $|x|d\gg 1$. The moments of a K_0 distribution generated by variables ξ,η are, because of their independence, simply the products of the moments of the separate ξ,η distributions; thus in the present case

$$M_{2\nu}\to\frac{(2\nu-1)!!^2}{d^{2\nu}},\quad M_{2\nu+1}=0, \tag{7.40}$$

which could have been used also to derive the K_0 distribution. Note that both the Gaussian and the K_0 forms are revealed by the asymptotic T_{ij} moments (7.35).

Since by (7.29) T_{ij} is given as a sum of such variables, we see that, in strong contrast to the "external" case, in which they are Gaussian, here the basic random variables are of K_0 type. Only if enough of them, with coefficients $T_{\lambda\mu}$, which are comparable in magnitude and more or less random in phase, combine to give the transition amplitude, will that amplitude (by the opera-

[32] K_0 variables have also been encountered by Whitehead *et al.* (1978) in a somewhat related problem.

tion of the CLT) be close to Gaussian and give rise to a Porter-Thomas strength distribution. It is, in fact, easy enough to see how fast[33] the approach to Gaussian is for T_{ij}. If we combine s K_0 variables with centroid zero and variance d^{-2} (as above) we have for the cumulants K_2 (=variance) and K_4 ($=\gamma_2\times(K_2)^2$), where γ_2 is the "excess" (do not confuse K_ν with the Bessel function),

$$K_2=s/d^2,\quad K_4=6s/d^4,\quad \gamma_2=6/s, \tag{7.41}$$

the K_ν results following from (7.40) (for $s=1$) and from the fact that the cumulants are additive under convolution of the densities (which gives the density function for the summed variables). We have remarked above that a departure from Gaussian corresponding to an excess $|\gamma_2|\lesssim 0.25$ is hardly detectible by eye; that would require about 25 contributing amplitudes. However, accurate measurements of the distributions are not usually feasible so that a satisfactory Gaussian might be achieved with fewer amplitudes than that, perhaps as few as a dozen.

The need for a randomness in phase, which is sometimes forgotten, becomes apparent if we take $T_{\mu\lambda}=T_\lambda T_\mu$. If the T_λ are themselves comparable, then in (7.29) we are combining a very large number of quadratic amplitudes comparable in magnitude. But the result obviously is $(\sum_\lambda x_{i\lambda}T_\lambda)(\sum_\mu x_{j\mu}T_\mu)$, which, in a T-diagonal representation, reduces to a single term and gives us back our K_0 distribution.

For completeness we work out the asymptotic ($d\to\infty$) strength correlations, which, as we have stressed, describe an essential aspect of the strength distributions. Correlations between T_{ij} and T_{ji}, which are, of course, identical for Hermitian T, seem to be of little interest for non-Hermitian T; however, the linear and quadratic correlations are given in Appendix P. There being no linear correlations between any other pairs of transition amplitudes, we are then left with two cases of correlations, between disjoint strengths and between strengths with a single state in common.

Since the random variables $x_{k\lambda}$ in the expansions (7.29) of two disjoint amplitudes, T_{ij} and T_{rs}, are asymptotically independent, we have the result that *disjoint strengths are asymptotically uncorrelated for all transition operators T.*

The other significant correlations can be derived also via (7.29). For a nicer way we consider the ensemble average of

$$\sigma_j^4=\Big(\sum_{i\neq j}|T_{ij}|^2\Big)^2,$$

where σ_j is the norm of the projected giant-resonance state, as in (7.33). We find that

$$\operatorname{var}(\sigma_j^2)=(d-1)\operatorname{var}(T_{ij}^2)+(d-1)(d-2)\operatorname{covar}(T_{ij}^2,T_{kj}^2), \tag{7.42}$$

which leads to the desired correlation coefficient

$$C(T_{ij}^2,T_{kj}^2)=\frac{(d-1)\overline{\sigma_j^4}-(d+1)(\overline{\sigma_j^2})^2}{3(d-1)\overline{\sigma_j^4}-(d+1)(\overline{\sigma_j^2})^2}=\frac{1}{3}\frac{\overline{T_{ij}^4}-3(\overline{T_{ij}^2})^2}{\overline{T_{ij}^4}-(\overline{T_{ij}^2})^2}\xrightarrow[d\to\infty]{}\frac{M_2(G_T)}{3M_2(G_T)+dM_1^2(G_T)}, \tag{7.43}$$

in which we have used (7.32) and (7.35).

We find, similarly, for the related strength sequence ($i\to j\to k$) that $C(T_{ij}^2,T_{jk}^2)$ is asymptotically given also by (7.43). See Appendix P for nonasymptotic results.

For nonsingular operators these correlations also vanish just as in the simple Porter-Thomas case. For singular operators, on the other hand, the correlations are finite, taking the value $C=\frac{1}{4}$ when G_T is the simple pairing operator. For this latter case the result is easily verified *ab initio*.

Finally we turn to the two-GOE case. We have T_{ij} with eigenstates $|i\rangle$, $|j\rangle$ varying over subspaces $\mathcal{S}_1,\mathcal{S}_2$ of dimensionalities d_1,d_2 and projection operators $\mathcal{P}_1,\mathcal{P}_2$, respectively. This can be regarded as an extension of the external case, but now with the channel states also belonging to an independent orthogonal ensemble. As before, we can carry out the ensemble averaging in two independent steps. We have

$$M_{2\nu}(T_{ij})=\frac{(2\nu-1)!!}{2^\nu}\frac{\Gamma(d_1/2)}{\Gamma(d_1/2+\nu)}\overline{\sigma_j^{2\nu}}=\frac{(2\nu-1)!!}{2^\nu}\frac{\Gamma(d_1/2)}{\Gamma(d_2/2+\nu)}\overline{\sigma_i^{2\nu}}, \tag{7.44}$$

where $\sigma_j^2\equiv\langle j|T^+\mathcal{P}_1T|j\rangle$ and $\sigma_i^2\equiv\langle i|T\mathcal{P}_2T^+|i\rangle$ are expectation values in $|j\rangle$ and $|i\rangle$ of the operators $\mathcal{P}_2T^+\mathcal{P}_1T\mathcal{P}_2$ and $\mathcal{P}_1T\mathcal{P}_2T^+\mathcal{P}_1$, respectively. Note that the operators are Hermitian and therefore the expectation-value moments, as required in (7.44), derive from (7.60) below. For the first few moments we have, with $\langle\langle G\rangle\rangle$ the trace as in Eq. (3.6),

$$M_2(T_{ij})=\frac{1}{d_1d_2}\langle\langle G\rangle\rangle,$$

$$M_4(T_{ij})=\frac{3}{d_1(d_1+2)d_2(d_2+2)}\left(2\langle\langle G^2\rangle\rangle+\langle\langle G\rangle\rangle^2\right), \tag{7.45}$$

where G stands for either of the two operators, their traces being equal because every strength which starts in one subspace has its counterpart which starts in the other. We have again a K_0 distribution for T_{ij} if the G's are pairinglike, whereas most operators do lead to the usual Porter-Thomas distribution (for asymptotic d_1 and d_2, of course). The correlation results are also similarly derived. In their essential forms, then, the results are the same as in the single-GOE case.

E. Porter-Thomas distribution and experimental data

The main question is whether locally renormalized transition amplitudes behave as independent Gaussian random variables. For transitions between the states of a given system, defined by a specific H, we would then rely on the ergodic behavior in making a comparison between theoretical ensemble results and experimental spectral ones; there is, of course, also the possibility of combining results for several systems which may ex-

[33] This, of course, follows also from (7.36), which is easily seen to be in agreement with Eq. (7.41) below.

tend, for example, across the nuclear table. In dealing with this question we cannot attempt any kind of detailed review of the data; we will examine instead only a few of the especially relevant experiments and computer calculations.

One part of the question is whether the locally renormalized strengths have a χ_1^2 (Porter-Thomas) distribution. The experience here is that the width distribution—for example, that of Γ_n in (n,γ) reactions—is indeed Porter-Thomas, at least as long as no doorway states or other intermediate-state structures are present (that case we consider in Sec. VII.G). In fact, as we have seen in Sec. VI, many statistical tests of the experimental data, for missing levels and for partial identification of spin, are based on the assumption that this *is* the distribution. As an example with very high-quality data, that of the neutron widths for ^{166}Er $+n$, 174 widths[34] with neutron energy ≤ 9.5 keV (Liou *et al.*, 1972), we show in Fig. 10 a histogram of the data drawn to logarithmic scale, and the corresponding P-T distribution. The fit is quite good.

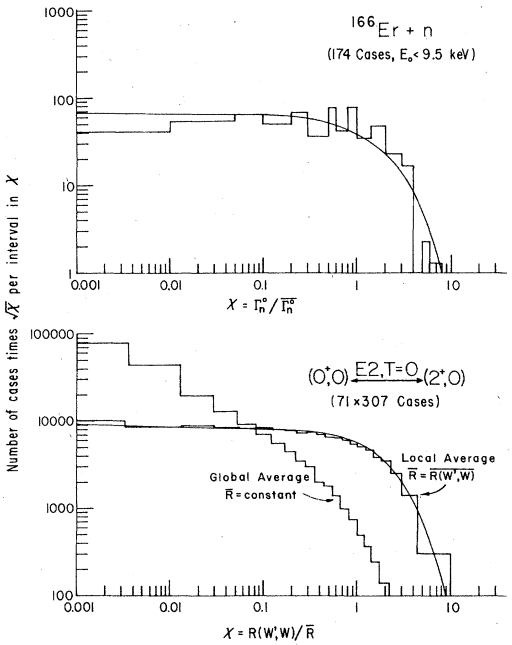


FIG. 10. Histogram of the reduced neutron widths for ^{166}Er $+n$ is compared with Porter-Thomas; good agreement is seen. A similar comparison is made for $E2$ strengths for transitions between two shell-model spaces $(ds)^6$, $T=1$, $J=0,2$. When the strengths are locally renormalized, as called for in the theory, the agreement is again good; but when this renormalization is not carried out, the distribution has a radically different shape. The experimental widths are from Liou *et al.* (1972), and the shell-model widths from Draayer *et al.* (1977).

[34]We are free to use a longer run of levels for testing the strength distribution than for the energy-level fluctuations (109 levels in this case), since one may choose a method for analyzing the experimental strength distribution which discounts to some extent the small widths, which are more liable to be missed or wrongly interpreted. There is no simple way to do the corresponding thing with the spectrum so that the run of levels must be more restricted; see, however, Coceva and Stefanon (1979).

A plot of this nature, due originally to Garrison (1964) and reproduced in Lynn (1968), shows the same comparison, with about the same quality of agreement, for the best neutron width data (144 widths) available at that time for even targets. Once again it is clear that "the Porter-Thomas distribution is a good representation of the data" (Lynn, 1968, p. 223).

One could test this further by calculating a χ^2 confidence limit, or by using a maximum-likelihood procedure based on the number of degrees of freedom (Porter and Thomas, 1956; Lynn, 1968). In the latter procedure, more commonly used, one calculates ν, the number of degrees of freedom, assuming that the distribution itself is χ_ν^2 with ν not necessarily an integer. A statistically significant departure from $\nu=1$ is then an indication of a failure of the Porter-Thomas assumption. However, there is a tendency for the data to have systematic errors which almost always result in larger calculated ν values. For example, $\nu=1.20^{+0.02}_{-0.15}$ was found (Wasson *et al.*, 1971) for 12 final states and 23 s-wave resonances with $E_n<600$ keV in ^{238}U$(n,\gamma)^{239}$U; and the systematic error, due, for example, to unresolved components in the γ-ray peaks, was estimated to increase ν by as much as 20%. It is probable that many other known $\nu>1$ (n,γ) results may be due to the same cause (Bollinger, 1970). With some simple statistical assumptions on the errors Coceva *et al.* (1971) have given unbiased estimates of ν and find good agreement with the Porter-Thomas distribution for $M1$ transition strengths in the ^{105}Pd(n,γ) reaction; see Corvi and Stefanon (1974) and Stefanon and Corvi (1977) for other examples, ^{115}In(n,γ) and ^{177}Hf(n,γ). For $E1$ transitions the same authors report ν values slightly larger than unity, which may again be due to an inadequate accounting of the errors.

Among other examples, γ rays from nine 3^- and seven 4^- resonant states to a total of 41 final $2^+, 3^+, 4^+$ states below 2600 keV in the ^{149}Sm(n,γ) reaction have been analyzed by Becvar *et al.* (1974). It is found that for transitions to the eight 2^+ final states there is no significant departure from the P-T distribution. However, whereas the maximum-likelihood tests for the thirty-three 3^+ and 4^+ final states give results consistent with P-T, the same tests for the lowest thirteen 3^+ and 4^+ states (below 2196 keV) give a small departure. The authors have made a detailed analysis of the probable errors; they believe that the deviation from Porter-Thomas is a real one.

The effects of the secular variation in the strength are best studied by means of shell-model calculations, for which they are large. In Fig. 10 we also give the $E2$ strength distribution derived from the ~20 000 transitions between the $J=2$, $T=0$ and $J=0$, $T=0$ $(ds)^6$ shell-model states constructed for a "realistic" Hamiltonian by Draayer *et al.* (1977) in connection with their spectral-distribution studies. The strength distribution is seen to be drastically different from Porter-Thomas. On the other hand, when the strengths are locally renormalized, to eliminate the secular variation, the distribution, given also in Fig. 10, agrees beautifully with P-T—similarly for $M1$ and $E4$ transitions in $(ds)^6$, and for distributions in which the strengths are taken from restricted energy domains (which would, when suffi-

ciently restricted, yield P-T without renormalization).

These shell-model calculations give an excellent further demonstration that the fluctuations, when properly measured, are invariant under mappings and embeddings of the Hamiltonian. In the same connection it has been demonstrated, by Monte Carlo calculations, that in the restricted two-body interaction ensemble, TBRE(J_g) of Sec. II, $\rho_{\langle i|\lambda\rangle}(x)$ is well represented as a Gaussian and the pair of random variables $\langle i|\lambda\rangle, \langle i|\mu\rangle$, as well as the pair $\langle i|\lambda\rangle, \langle j|\mu\rangle$, is essentially independent.

We turn to the question of strength correlations in circumstances where the ordinary P-T distribution should apply; we consider, in particular, the case of "resonance" correlations in which a "statistical" state, a member of a sequence, makes transitions to two channel states. According to Eq. (7.25) the correlation coefficient in this case is not necessarily zero, but it is *non-negative* for the ordinary statistical model.

Early experiments seemed often to show significant correlations which in some cases disappeared with better γ-ray or neutron energy resolution. Careful studies of the reaction $^{238}U(n,\gamma)$ referred to above (Wasson *et al.*, 1971) have shown, except in one case, no significant correlations. These authors examined γ transitions to 15 final states originating from each of the 23 s-wave resonances with $E_n < 600$ keV. For each of the 105 γ-ray final-state pairs one can then determine a correlation coefficient from the series of resonances. A question which arises in the analysis is whether the observed correlations may arise from finite-sample effects which are large because the resonance series is short. This question is studied by comparing the histograms of the correlations with that arising from a large number of similar examples constructed via Monte Carlo calculations which make use of a set of *uncorrelated* χ_1^2 strength distributions. The conclusion is that the one very large correlation coefficient (+0.81 between γ rays with energies 3991 and 3982 keV) which is observed is not guaranteed to be a real effect; the case is similar for correlations between the neutron and γ-ray widths.

On the other hand, Beer *et al.* (1968), in an earlier experiment, find in the $^{169}Tm(n,\gamma)$ reaction a large correlation (+0.274) between the neutron and γ-ray widths and a smaller average γ-γ correlation (+0.088), which they regard as significant, perhaps as indicating that the excited states have a larger-than-statistical component of a single-particle excitation of the ground state (Bockelman, 1959; Lane and Lynn, 1960).

In both of these experiments the correlations are "initial state" or "resonance" correlations, involving a sequence of such states, in contrast to "final-state" correlations, which result from averaging over the final states for any two given initial states. Many examples of large correlations of both types are listed by Lane (1976, p. 533), and in fact "there is a tendency for a given target and neutron partial wave to show both or neither."

Transition-amplitude correlations have also been determined for proton resonances—in particular, on ^{48}Ti (Mitchell, 1980, and private communication). The relation between the strength and amplitude correlations as given by (7.24) and (7.25) is not satisfied for much of the data. No explanation has yet been given for this anomaly.

Finally, we mention some work in progress (Auger *et al.*, 1980) concerning $E1$ electromagnetic transitions in the ground-state region of light nuclei. With the somewhat old data of Perdrisat (1966) for nuclei with $A \leq 40$ the renormalized widths for all nuclei belonging to the main group of the nuclear-table ensemble fit the Porter-Thomas distribution reasonably well. This complements the picture given in Secs. I and VI for eigenvalue fluctuations in the nuclear-table ensemble.

F. Time-reversal invariance

The principal question (Wigner, 1967) is whether one can detect, in the strength and energy-level fluctuations, the influence of that part of the Hamiltonian which is not invariant under time reversal. The possibility of doing this comes from the fact that the fluctuations are quite different for the GOE, which prescribes invariance, and the GUE, which ignores it. We therefore introduce the ensemble (Rosenzweig *et al.*, 1968; Mehta and Rosenzweig, 1968; Favro and McDonald, 1967; Camarda, 1976; McDonald, 1980)

$$\begin{aligned} &H_\alpha = (1+\alpha^2)^{-1/2}(H + i\alpha T), \quad \alpha = \alpha^* \geq 0 \\ &\langle \overline{H_\alpha^2} \rangle = 1 + d^{-1}\eta \xrightarrow[d\to\infty]{} 1, \\ &\eta \equiv (1-\alpha^2)/(1+\alpha^2), \end{aligned} \tag{7.46}$$

where H is a GOE with the standard normalization (2.5) and T is an independent ensemble of real *antisymmetric* matrices with matrix elements distributed similarly to those of H. For $\alpha = 0, 1$ we have the standard GOE and GUE; for $\alpha = \infty$, which is physically uninteresting, at least in the present context, the corresponding analytical results have been given by Mehta and Rosenzweig (1968). For other α values, and, in particular, therefore for $0 < \alpha < 1$, which gives the physically interesting orthogonal → unitary transition, only Monte Carlo results have been previously given. We consider these results before proceeding to an analytic treatment.

Among the quantities considered are the nearest-neighbor spacing distribution for small spacings, the variance-to-centroid ratio for the strength distribution, and the value of Δ_3 [which for the standard ensembles are respectively proportional to S^β, equal to $2/\beta$, and given by (5.21)]. For small spacings Rosenzweig *et al.* (1968), who use the parameter $\epsilon^2 = \alpha^2/(1+\alpha^2)$, find a very rapid approach to the unitary case as α increases; in fact, for $\alpha^2 \simeq 0.01$, with $d = 20$, the result is already very close to unitary. For the strength distribution they find an even more rapid transition which is strongly d dependent; "in particular, a discontinuous jump... as ϵ^2 varies from 0 to a small finite value is not excluded (p. 446)." If such should be the case (and our analysis will show that it is), the value of the "effective" dimensionality would be required. This point is explicitly made by Rosenzweig *et al.*, who analyze some of the strength data and give the results in terms of upper limits inferred from the results with $d = 100$ matrices. Monte Carlo calculations have also been more recently made by Camarda (1976) who studies strength distributions and Δ_3 values for $d = 40, 80$. He stresses that the strengths supply a better test than does Δ_3. His corresponding analysis of the data suggests that $\alpha < 0.05$.

Turning now to an analytical treatment we observe first that, with the normalization (7.46), the recursion relation (3.15) for the ensemble-averaged density moments applies for all α and, hence, so does the semicircular density (3.17). For the two-point function we follow Sec. IV, the only difference being in the value of the completely cross-associated trace product which supplies the amplitude in the modal expansion. Following the procedure of Appendix J we find easily that, for large d

$$\overline{\underbrace{\langle H^{\zeta}_{\alpha}\rangle\langle H^{\zeta}_{\alpha}\rangle}_{\zeta}} = \frac{\zeta}{d^2}\left[1+\left(\frac{1-\alpha^2}{1+\alpha^2}\right)^{\zeta}\right] = \frac{\zeta}{d^2}\,(1+\eta^{\zeta})\,, \qquad (7.47)$$

which, with $\alpha=0,1$, gives back the $\beta=1,2$ results of (4.9). Equation (7.47) carries the entire α dependence of our two-point function, which can now be constructed by replacing $2/\beta$ by $(1+\eta^{\zeta})$ inside the summation in (4.18) and then evaluating the elementary sum anew.

With $g_{\zeta}(x)$ defined by inspection of (4.18), we have the two-point function

$$S^{o}_{\alpha}(x,y) = \sum_{\zeta=1}^{\sim d}(1+\eta^{\zeta})g_{\zeta}(x)g_{\zeta}(y)$$

$$= S^{o}_{\mathrm{GUE}}(x,y) + \sum_{\zeta=1}^{\sim d}\eta^{\zeta}g_{\zeta}(x)g_{\zeta}(y)\,. \qquad (7.48)$$

Let us partition the range of α^2 by the values $\alpha^2 = 0$(GOE), d^{-1}, 1(GUE), $\infty(MR)$, where $(0, \sim d^{-1})$ will be seen to define the transition region, and, as indicated, $\alpha^2=\infty$ gives the ensemble of Mehta and Rosenzweig (1968) [see also Dyson (1962e) and Cristofori *et al.* (1966)]. We ask now for the relative contribution of the second sum. For $\alpha^2 \sim d^{-1}$ we see that η^{ζ} has a nonzero value even for $\zeta \sim d$, its maximum value; therefore in the first region, $0 \leq \alpha^2 \lesssim d^{-1}$, we should carry out the explicit summation. In the second region, as $\alpha^2 d$ becomes $\gg 1$, the damping becomes very large in the second sum, which therefore does not contribute; S^{o}_{α} thus becomes the unitary two-point function. We now see a transition from GOE to GUE which sets in at $\alpha \sim d^{-1/2}$, well before the value $\alpha=1$, which defines the standard unitary ensemble, is reached. Moreover, as d increases, the transition region shrinks, so that the transition becomes very fast, discontinuous in the limit. The range of α values which define the transition region is in general agreement with the Monte Carlo calculations. We observe that the "cluster-type" calculation of Favro and McDonald (1967), which displays the changing slope of the nearest-neighbor spacing distribution, also applies to this region. For $\alpha=\infty$, (7.48) gives the GOE expansion, but with odd-ζ terms missing, so that we get half the GOE sum, and thereby the GUE result. For asymptotic d this is in agreement with the results of Mehta and Rosenzweig [(1968; their Eqs. (22, 24)], which are, of course, exact.[35]

A main conclusion from these results is that for almost the entire range of strengths of the symmetry-breaking Hamiltonian, i.e., for $\alpha \gtrsim d^{-1/2}$, the energy-level fluctuations are independent of the strength and are described by the unitary ensemble. The transition which occurs is due to the release of a symmetry constraint near the GOE which has no counterpart near the GUE. The first question which now arises, in determining from the data a value for the magnitude of the symmetry-breaking interaction, concerns the adequacy of the Hamiltonian (7.46) with its multibody interactions. The second is how to calculate the effective dimensionality. Both of these have been solved (French, Kota, and Pandey, to be published) but the analysis is not yet completed.

[35] For finite but large d they display some special features, in a small central region of the spectrum, which are not to be found in the true unitary ensemble (or in our results).

G. Isobaric analog resonances

In high-resolution proton-resonance experiments an isobaric analog ($T_>$) state, which resides in a sea of ($T_<$) background states, is excited, it being usually identified by its very large strength—for example, in the (p,p) reaction— resulting from direct coupling to the proton channel. The widths of the background states are enhanced by a spreading of the analog due to charge-dependent residual forces. In heavy nuclei, and also at very high excitation energies, the background states are very close and overlapping, and consequently the analog state is seen as a smooth envelope which has a Breit-Wigner shape. However, in less heavy nuclei, especially below their neutron thresholds, the states are well resolved; they are characterized by the normal strength fluctuations which are modified, in the vicinity of the analog, by a strong level-to-level secular variation due to the symmetry mixing. The only questions which we shall consider are whether the secular variation can in a satisfactory way be eliminated and whether the fluctuations are indeed of Porter-Thomas type. We shall by no means enter into the complexities of the data analysis which have been much discussed in the literature, but shall merely indicate the way in which random matrices enter and say a little about the answers to our questions. General analyses, which have been mainly directed toward establishing the secular trend and interpreting the parameters in terms of "microscopic" quantities, have been made by Bilpuch *et al.* (1976), following Lane (1969) and Lane *et al.* (1974), and by MacDonald (unpublished; see also MacDonald, 1979), based on the original theoretical considerations of MacDonald and Mekjian (1967) and Kerman and de Toledo Piza (1968). Excellent accounts in a more general context are given by Mahaux (1973) and Mahaux and Weidenmüller (1979).

If the strength extends far from the analog, the usual Porter-Thomas distribution should apply; this has been verified by Bilpuch *et al.* (1971). But, as we shall see, Porter-Thomas does not apply in the vicinity of the analog state, at least in its usual form.

We consider a doorway state ϕ_0 at energy ϵ_0 and the background states ϕ_m at energies ϵ_m $(m=1,2,\ldots,d)$; these are solutions of a model Hamiltonian H_0. The background states have the same set of exact quantum numbers, while the doorway state has one or more quantum numbers different from these; for example, an analog-state doorway differs from its background with respect to the isospin symmetry. We have also channel states $|c\rangle$. The symmetry-breaking interaction V is described in the ϕ space by a real symmetric matrix

whose only nonvanishing elements are $V_{m0}=V_{0m}=\langle\phi_m|V|\phi_0\rangle$ with $m\neq 0$. V has also matrix elements V_{mc} connecting the ϕ states (including $m=0$) with the channel states. For $m\neq 0$, V_{mc}^2 is the "internal width." The total Hamiltonian $H=H_0+V$ has eigenvalues E_λ ($\lambda=0,1,\ldots,d$) given by the solutions of the secular equation

$$Z-\varepsilon_0-\sum_{m=1}^{d}\frac{V_{m0}^2}{Z-\varepsilon_m}=0 \tag{7.49}$$

and the eigenvector components $X_{\lambda m}$ in the ϕ basis by

$$\begin{aligned} &X_{\lambda m}=X_{\lambda 0}V_{m0}/(E_\lambda-\varepsilon_m),\quad m\neq 0\\ &X_{\lambda 0}^2=(\partial E_\lambda/\partial\varepsilon_0)=\left(1+\sum_{m=1}^{d}V_{m0}^2/(E_\lambda-\varepsilon_m)^2\right)^{-1}. \end{aligned} \tag{7.50}$$

The states decay via the residual interaction V. Then the reduced width $\gamma_{\lambda c}^2$ in the channel c is given by

$$\gamma_{\lambda c}^2\simeq V_{\lambda c}^2=X_{\lambda 0}^2\left(V_{c0}+\sum_{m=1}^{d}V_{cm}V_{m0}/(E_\lambda-\varepsilon_m)\right)^2. \tag{7.51}$$

We now take the ϕ_m with $m\neq 0$ as forming a statistical space, assuming in particular that they are chosen as a representative set of GOE eigenstates. The V_{mc} and V_{m0} have then a multivariate Gaussian distribution with linear and quadratic averages given by

$$\begin{aligned} &\overline{V}_{mc}=\overline{V}_{m0}=0,\\ &\overline{V_{mc}V_{m'c}}=\omega_c^2\delta_{mm'},\\ &\overline{V_{m0}V_{m'0}}=\omega_0^2\delta_{mm'},\\ &\overline{V_{mc}V_{m'0}}=r\omega_c\omega_0\delta_{mm'}, \end{aligned} \tag{7.52}$$

valid for $m,m'\neq 0$. Note that r is the correlation coefficient between the two classes of matrix elements.

From (7.51) we see that when the internal widths are zero, the decay widths for different channels are completely correlated. Large correlations are in fact observed (Bilpuch *et al.*, 1976). However, even if we ignore the energy-level fluctuations (which seems appropriate), to extract the $\gamma_{\lambda c}^2$ distribution from (7.51) appears to be a very difficult problem.

We can simplify (7.51) in the case of very weak mixing by retaining only quadratic terms in V_{m0}; if we do the same for V_{mc}, which is usually satisfactory, we find, as in Lane (1969), that

$$\begin{aligned} &E_\lambda=\varepsilon_\lambda+\frac{V_{\lambda 0}^2}{\varepsilon_\lambda-\varepsilon_0},\quad \lambda\neq 0\\ &E_0=\varepsilon_0+\sum_{m=1}^{d}\frac{V_{m0}^2}{\varepsilon_0-\varepsilon_m},\\ &\gamma_{\lambda c}^2=\left(V_{\lambda c}+V_{c0}\frac{V_{0\lambda}}{\varepsilon_\lambda-\varepsilon_0}\right)^2,\quad \lambda\neq 0\\ &\gamma_{0c}^2=V_{c0}^2-\sum_{m=1}^{d}(\gamma_{mc}^2-V_{mc}^2). \end{aligned} \tag{7.53}$$

From the third of these equations we see that, since $\gamma_{\lambda c}$ is given as a sum of two Gaussian random variables, the *ensemble* distribution of the strength is Porter-Thomas. However, the *spectral* distribution of the strength is not Porter-Thomas, especially in the vicinity of the doorway, because of the strong secular dependence. Carrying out the ensemble averaging we find, with $\lambda\neq 0$, that $\gamma_{\lambda c}^2$ has a χ_1^2 distribution with

$$\overline{\gamma_{\lambda c}^2}=\omega_c^2+\frac{2r\omega_0\omega_c V_{c0}}{(\varepsilon_\lambda-\varepsilon_0)}+\frac{\omega_0^2V_{0c}^2}{(\varepsilon_\lambda-\varepsilon_0)^2} \tag{7.54}$$

and that different strengths involving the same channel are independent. γ_{0c}^2 behaves differently; its mean follows from (7.54), together with the last equation of (7.53); the quadratic and higher averages are rather complicated in form and the distribution itself is not simple. Because of the approximations made (7.53) and (7.54) are of limited applicability, but, as confirmed by inspection of Monte Carlo calculations (Brody *et al.*, 1973) with the (ε_λ) taken as an unfolded GOE spectrum, they do give a reasonable view of the overall picture. For example, even when the spectrum (ε_λ) is symmetrically distributed around ε_0, they yield $\gamma_{\lambda c}^2$, which is asymmetric (Robson, 1965), the asymmetry, as we see in (7.54), being induced by the correlation coefficient r.

It seems quite probable that the methods which we have used in Sec. III.E will give a better solution to the statistical problem which originates with (7.51). In fact, these methods, in which the ensemble averaging is done at the outset, bear a certain relationship to those of MacDonald and Mekjian (1967) and Kerman and de Toledo Piza (1968) which lead to the following identity for the nonaveraged Lorentz-weighted strength function:

$$\begin{aligned} S(E,I)&\equiv\frac{I}{\pi}\sum_{\lambda=0}^{d}\frac{\gamma_{\lambda c}^2}{(E-E_\lambda)^2+I^2}\\ &=\frac{1}{2\pi}\frac{V_{0c}^2(\Gamma_I+2I)}{(E-E_0-\Delta_I)^2+\frac{1}{4}(\Gamma_I+2I)^2},\\ \Gamma_I(E)&=2I\sum_{m=1}^{d}\frac{V_{m0}^2}{(E-\varepsilon_n)^2+I^2},\\ \Delta_I(E)&=\sum_{m=1}^{d}\frac{(E-\varepsilon_m)V_{m0}^2}{(E-\varepsilon_m)^2+I^2}. \end{aligned} \tag{7.55}$$

The last form for $S(E,I)$ is valid for zero intrinsic widths of the background; for $V_{mc}\neq 0$ it should contain additional terms with some new parameters [see the above references or Lane (1969)]. For I large enough, the parameters Γ_I and Δ_I (and other parameters, in the general case) become energy-independent; Γ_I simply becomes an estimator of $2\pi\omega_0^2/D^2$.

The data have been successfully analyzed by Bilpuch *et al.* (1976), with the proposition (Lane *et al.*, 1974) that the continuous smooth function $S(E,I)$ can be approximated by a locally-averaged histogram strength function. It has been pointed out recently by MacDonald (1978, 1979) that this approximation should be good only for the strong-coupling case, whereas most of the data yield weak-to-intermediate coupling results. He has carried out an analysis (MacDonald, private communication) of the data, by using the Lorentz-weighted strength function, having shown via corresponding Monte Carlo calculations that by choosing I properly, the investigator can considerably reduce the statistical error. For an exchange concerning this question see Lane *et al.* (1979) and MacDonald (1980).

H. Expectation values, sum rules, and giant resonances

Expectation values in nuclear states are sometimes measurable, for example, quadrupole moments and single-particle occupancies, the latter by use of single-

nucleon-transfer sum rules. Indeed, such sum rules, which, like others, determine the expectation values in an initial state by summing strengths to final states (with perhaps an energy weighting or a J or T selection), fix the "basic" multipole moments of the initial state (French, 1964, 1966; Moinester, 1967; Hansen, 1967; de López *et al.*, 1967; Moinester *et al.*, 1969; Moinester and Alford, 1970; Clement, 1973; Clement and Perez, 1977, 1978), and so our two examples are related. The more complicated $E2$ sum rule fixes the expectation value of a $Q\cdot Q$ operator which is in some cases closely related to an SU(3) Casimir operator.

These remarks only remind us of the obvious, namely, that expectation values are of great interest. They do not make clear why we should be concerned with them here, since averaging an expectation value over a GOE, for example, will necessarily give the same result for all states and, because of the summation over final states, very small fluctuations (vanishing as $d\to\infty$) in most cases. Thus to make contact with physics we must, when dealing with the expectation value itself, not average over an "isotropic" ensemble. Moreover, while ensemble averaging, with an appropriate renormalization, might somehow be adequate for dealing with local behavior, we still must ask how expectation value fluctuations are to be observed or otherwise made use of. It will turn out that such fluctuations will depend very much on the microscopic structures of the states and the excitation operator.

Let us first solve the simplest formal problem, that of the GOE distribution of the expectation values of an operator G in the Hamiltonian eigenstates. There are two main cases of interest: (1) $G=T=T^+$, in which the off-diagonal elements of G might be transition amplitudes, as, for example, with G the quadrupole moment operator, Q_0; (2) $G=T^+T$, in which G ($=G_T$ of Sec. VII.D) might be a sum-rule operator. Note from the initial remarks that there will not always be a clear distinction between the two cases. But in both, G is Hermitian, and, without loss of generality, we can subtract out its trace; thus

$$G=G^+,\quad \langle G\rangle=0\,. \tag{7.56}$$

Our interest is in the ensemble distribution of $G_{ii}=\langle i|G|i\rangle$; we consider this in terms of its moments $[G_{ii}^{\nu}]_e=\mathfrak{M}_\nu(G_{ii})$, which, just as with T_{ij}, must be homogeneous polynomials of the spectral moments $\mathfrak{M}_p(G)$. It follows then that $\mathfrak{M}_1(G_{ii})=\overline{G_{ii}}=0$. It should be clear also that $\mathfrak{M}_2(G_{ii})\sim d^{-1}\mathfrak{M}_2(G)$, this because, in the representations defined by "most" H's in the ensemble, G is highly nondiagonal, so that the span of its centroid spectrum is small compared with its eigenvalue spectrum. These results are easily verified by writing G_{ii} in the representation in which G (with eigenvalues g_λ) is diagonal,

$$G_{ii}=\sum_\lambda g_\lambda (x_{i\lambda})^2,\quad \sum_\lambda g_\lambda=0\,. \tag{7.57}$$

It is straightforward, in fact, and instructive, to derive the general moment of G_{ii}. Using (7.57), along with the standard result for the power of a polynomial (Abramowitz and Stegun, 1964; Riordan, 1968), we have

$$\overline{(G_{ii})^\nu}=\sum_{\substack{\pi(\nu)\\ \lambda_1,\lambda_2,\ldots,\lambda_l\\ (\lambda_1\neq\lambda_2\cdots\neq\lambda_l)}}\delta_\pi\,\overline{\left(x_{i\lambda_1}^{2\nu_1}x_{i\lambda_2}^{2\nu_2}\cdots x_{i\lambda_l}^{2\nu_l}\right)}\,g_{\lambda_1}^{\nu_1}\cdots g_{\lambda_l}^{\nu_l} \tag{7.58}$$

$$\delta_\pi=\frac{(\nu:\nu_1\cdots\nu_l)}{\Pi(\omega_t!)}\,,$$

$\omega_t=$ number of t's in the partition π. The $\pi(\nu)$ are the (unordered) partitions of $\nu\to(\nu_1,\nu_2,\ldots,\nu_l)$, and $(\nu:\nu_1\cdots\nu_l)$ is the multinomial coefficient of (7.15). Using (7.15) to evaluate the ensemble average, we have

$$\overline{(G_{ii})^\nu}=\frac{(2\nu-1)!!(d-2)!!}{(d+2\nu-2)!!}\sum_{\pi(\nu)}\frac{(\nu:\nu_1\cdots\nu_l)^2}{(2\nu;2\nu_1\cdots2\nu_l)}\frac{1}{\Pi(\omega_t!)}\sum_{\substack{\lambda_1,\lambda_2,\ldots,\lambda_l\\ (\lambda_1\neq\lambda_2\cdots\neq\lambda_l)}}g_{\lambda_1}^{\nu_1}\cdots g_{\lambda_l}^{\nu_l}\,, \tag{7.59}$$

in which, as in (7.6), the first factor is simply the 2νth moment of $x_{i\lambda}$. Special cases of (7.59) have been given by Rosenzweig and Porter (1961); see also Ullah and Porter (1963b).

The restricted eigenvalue summation encountered here defines a νth-order symmetric function of the eigenvalues of G, which by a standard theorem on symmetric functions may be expanded as a sum of homogeneous products of the spectral moments $\mathfrak{M}_\mu$ of G. In the notation of Kendall and Stuart (1969) the restricted sums are the "augmented symmetric functions," while the spectral moments, when multiplied by d, are the "power sums," $s_\mu=d\times\mathfrak{M}_\mu(G)$. For very low orders—say, $\nu=2,3$—the expansion is quite simple. For general order a useful explicit form is not available, but there are tables (David *et al.*, 1966; Kendall and Stuart, 1969), the results given in the latter reference being more than adequate for us. Using these, we have

$$\begin{aligned}
&\mathfrak{M}_1(G_{ii})=0\,,\\
&\mathfrak{M}_2(G_{ii})=2(d+2)^{-1}\mathfrak{M}_2(G)\,,\\
&\mathfrak{M}_3(G_{ii})=8[(d+2)(d+4)]^{-1}\mathfrak{M}_3(G)\,,\\
&\mathfrak{M}_4(G_{ii})=12[(d+2)(d+4)(d+6)]^{-1}\\
&\qquad\times[4\mathfrak{M}_4(G)+d(\mathfrak{M}_2(G))^2]\,,\\
&\mathfrak{M}_5(G_{ii})=[(d+2)(d+4)(d+6)(d+8)]^{-1}\\
&\qquad\times[384\mathfrak{M}_5(G)+160d\mathfrak{M}_3(G)\mathfrak{M}_2(G)]\,,
\end{aligned} \tag{7.60}$$

and then

$$\begin{aligned}
\gamma_1(G_{ii})&\xrightarrow[d\to\infty]{}2\left(\frac{2}{d}\right)^{1/2}\frac{\mathfrak{M}_3(G)}{[\mathfrak{M}_2(G)]^{3/2}}=2\left(\frac{2}{d}\right)^{1/2}\gamma_1(G),\\
\gamma_2(G_{ii})&\xrightarrow[d\to\infty]{}\frac{12}{d}\frac{\mathfrak{M}_4(G)}{[\mathfrak{M}_2(G)]^2}=\frac{12}{d}[\gamma_2(G)+3]\,.
\end{aligned} \tag{7.61}$$

These results indicate that, *for nonsingular operators* G,

the $[\mathfrak{M}_2(G)]^\alpha$ term is dominant in $\mathfrak{M}_{2\alpha}$, so that

$$\mathfrak{M}_{2\alpha}(G_{ii}) \to (2\nu-1)!![2\mathfrak{M}_2(G)/d]^\alpha ,$$

whereas the odd moments are down by order $d^{-1/2}$. Thus the expectation value distribution should become asymptotically Gaussian[36]

$$\rho_{G_{ii}}(x) \underset{d\to\infty}{\longrightarrow} G\left(0:\left(\frac{2}{d}\right)^{1/2}\sigma(G)\right) \quad \text{(for nonsingular } G\text{)}, \tag{7.62}$$

while a simple calculation using (7.57) and (7.15) tells us that expectation values in different eigenstates are only weakly correlated,

$$C(G_{ii}, G_{jj}) = -(d-1)^{-1}. \tag{7.63}$$

Now take $G=T$ and observe that, by Eqs. (7.38) and (7.62), the matrix elements of a nonsingular Hermitian operator G, in Hamiltonian representation, are distributed as zero-centered Gaussians, the G_{ii} variance being twice that of the G_{ij}, just as with the matrix elements of H in a fixed basis. Thus these matrices generate an asymptotic GOE, modified, for finite d, by the weak correlations (of order d^{-1} and d^{-2}) between the matrix elements, whose spectrum can be given, by a choice of G, any preassigned form consistent with uniformity.

That the expectation value should be Gaussian was mentioned long ago by Porter (1965b), the proof being ascribed to H. Robbins. The result, however, is not valid for a singular G, as we see from (7.61), using the facts that $\gamma_1(G)\sim d^{1/2}$, $\gamma_2(G)\sim d$. For the prototypical case, the pairing operator, we know from Eqs. (7.56) and (7.57) that $G_{ii}=x_{i1}^2-d^{-1}$, whose asymptotic distribution is (off-centered) χ_1^2. For this we have $\gamma_1(G_{ii})=2\sqrt{2}$, $\gamma_2(G_{ii})=12$, which may be checked by using

$$\mathfrak{M}_p(G)=(d-1)d^{-(p+1)}[(d-1)^{p-1}+(-1)^p]$$

in (7.61).

Two-point measures for GOE expectation value fluctuations are easily calculated in terms of the two-point function $S_G^e(x,y)$ given in Appendix K, but they are not very interesting. For nonsingular sum-rule operators, $G=T^\dagger T$ (and these are the ones of most interest to us; the results, however, are easily extended to more general G), the expectation value fluctuations are small, the distribution of G_{ii} being narrow when measured in terms of the spectral width $\sigma(G)$; this comes about because, for almost all H's in the GOE, many essentially independent components contribute to the strength expansion of G. But in practice one finds "collectivities," even for many nonsingular operators, which result in some states having very much larger strength sums than any in their neighborhood. Another (related) point of disagreement between GOE and observation is that the GOE measures are necessarily independent of the energy, whereas, for example, one manifestation of strong quadrupole collectivity in nuclei is that the $Q\cdot Q$ expectation value is very large near the ground state (giving rise to large $E2$ sums) and falls off at higher energies. This behavior arises from the large negative (spectral) correlation between $Q\cdot Q$ and H, which, of course, is not to be found in the GOE. Since secular and fluctuation behavior might well be decoupled, this would not, however, rule out a GOE description for the fluctuations.

A Monte Carlo study of these phenomena for one-particle one-hole excitations of a closed-shell nucleus has been made by Touchard *et al.* (1977). They deal specifically with Gamow-Teller transitions between ^{208}Bi and ^{208}Pb, representing the model space in terms of 16 hole-particle excitations. In a basis in which the transition amplitudes connecting these states to the closed shell are all positive, the Hamiltonian is H_0, defined by a set of single-particle energies taken from experiment, supplemented by a GOE-type interaction ensemble in which, however, the matrix-element distributions have the same nonzero centroids. This ensemble is a GOE deformed by an operator K, the sum of the one-body H_0 and the pairing Hamiltonian of Sec. III.E. It represents a generalization of the schematic (separable) Hamiltonian which, when added to H_0, generates, for a strong enough interaction, one collective state well separated in energy from the others and carrying a large fraction of the total strength. In the specific basis the separable interaction has matrix elements all of the same sign. For the ensemble, on the other hand, there is a probability p, determined by the centroid and variance of the distribution, that a minor fraction of the matrix elements will have a sign different from that of the majority. Touchard *et al.* determine the values of p for a number of the standard interactions, finding for all of them except the separable "surface-delta" interaction a value quite close to $p=0.2$. By studying the properties of the strength distribution as a function of p, considering, in particular, the location of the collective state and the fraction of the strength which it carries, they show, *inter alia*, that the existence of a giant resonance is governed by the value of the parameter p and that for the interactions considered a giant resonance is indeed to be expected.

The Stieltjes-transform method of Sec. III.E gives a *formal* solution to the problem of Touchard *et al.* but we are not yet in a position to extract numerical results from the equations given there [in particular (3.28)]. We turn instead to another aspect of collectivity (Draayer *et al.*, 1977) mentioned in Sec. I.E and used in Sec. VII.C, in which we combine the secular variation of the strength sum with an *assumed* Porter-Thomas description of the local strength fluctuations.

As above, we take $G=T^\dagger T$, the Hermitian square of the excitation operator T, and we consider the ensemble variance of the summed strength,

$$S_i=\sum_j |T_{ij}|^2=\sum_j R_{ij} ,$$

which originates with the H eigenvector Ψ_i. Taking for granted that strengths are uncorrelated, we have

$$\begin{aligned} &\mathcal{E}(S_i)\equiv\bar{S}_i=\sum_j \bar{R}_{ij} ,\\ &\sigma^2(S_i)\equiv\overline{S_i^2}-(\bar{S}_i)^2=\sum_j [\overline{R_{ij}^2}-(\bar{R}_{ij})^2] . \end{aligned} \tag{7.64}$$

Now, labeling the states by their energies $i\to|W\rangle$,

[36] That this is indeed the case is easily verified from Eq. (7.58); it will be seen also that, as always, Gaussian behavior arises from the dominance of binary associations (i.e., from the partition $2\nu\to[2^\nu]$) in the moment traces.

$j \to |z\rangle$, introducing the final-space eigenvalue density, $d' \times \rho'(z)$ (where then d' is the final-space dimensionality), and writing $R_{ij} \to R(z, W)$, we find

$$\hat{\nu} \equiv \frac{2\mathcal{S}^2(S_i)}{\sigma^2(S_i)} = \frac{2d'[\int \rho'(z)\overline{R(z,W)}dz]^2}{\int \rho'(z)\{\overline{R^2(z,W)} - [\overline{R(z,W)}]^2\}dz} = \frac{d'[\int \rho'(z)\overline{R(z,W)}dz]^2}{\int \rho'(z)\overline{R(z,W)^2}dz}, \tag{7.65}$$

where the last step follows from the χ_1^2 form of the R distribution, and as discussed above [(1.9) and (7.28)], $\hat{\nu}$ may be interpreted as the effective number of accessible states. A small value of $\hat{\nu}$ will indicate a high degree of collectivity in the region with energy $\sim W$. The special feature of (7.65) is that we can plausibly assume an ergodic behavior and replace the ensemble average by a spectral one. This, moreover, may be calculated by truncating the formally exact expansion,

$$R(z, W) = [\rho'(z)\rho(W)d']^{-1}\langle T^+\delta(H-z)T\delta(H-W)\rangle^m = (dd')^{-1}\sum_{\mu,\nu=0}^{\infty}\langle T^+P'_\mu(H)TP_\nu(H)\rangle^m P'_\mu(z)P_\nu(W), \tag{7.66}$$

which in a many-particle system is strongly convergent. Here $\langle\rangle^m$ denotes an average over the W space, while $P'_\mu(z), P_\nu(W)$ are sets of orthonormal polynomials with $\rho'(z)$ and $\rho(W)$ as weight functions. Truncation of the expansion defines the smoothing (a "global" one) which is called for. By spectral-distribution methods it is possible to construct the smoothed $R(z, w)$ expansion and hence to evaluate $\hat{\nu}$. Two shell-model examples are given in Fig. 11. The procedure used here is interesting, in general terms, because of the connection established between statistical and collective behavior. One "practical" use would appear to be in the prediction of such collectivities, given the model space and the operators H, T.

Finally, we have a comment about a symmetry aspect of expectation value fluctuations. The level-to-level fluctuations in the expectation value of G will be small if G is closely correlated (or anticorrelated) with H, for they must vanish in the extreme case of complete correlation in which G is effectively a multiple of H. Since $Q\cdot Q$ is strongly correlated with H, the summed $E2$ strength for $0^+ \to 2^+$ transitions in a many-particle nucleus should then be weakly fluctuating, whereas that for the inverse transitions, $2^+ \to 0^+$, should be strongly so, since for these the sum-rule operator is not $Q\cdot Q$, but $Q\cdot \mathcal{P}(0^+)Q$, with $\mathcal{P}$ an angular-momentum projection operator. See Draayer *et al.* (1977) for shell-model examples and further discussion.

VIII. THE STATISTICAL THEORY OF NUCLEAR REACTIONS

A. Introduction

In the previous sections we have discussed ensembles of Hamiltonians constructed in a discrete (matrix) representation, even though the results are for the most part compared with experimental data on nuclear resonances in the continuum. Is this a consistent procedure? As far as the levels are concerned, there is no

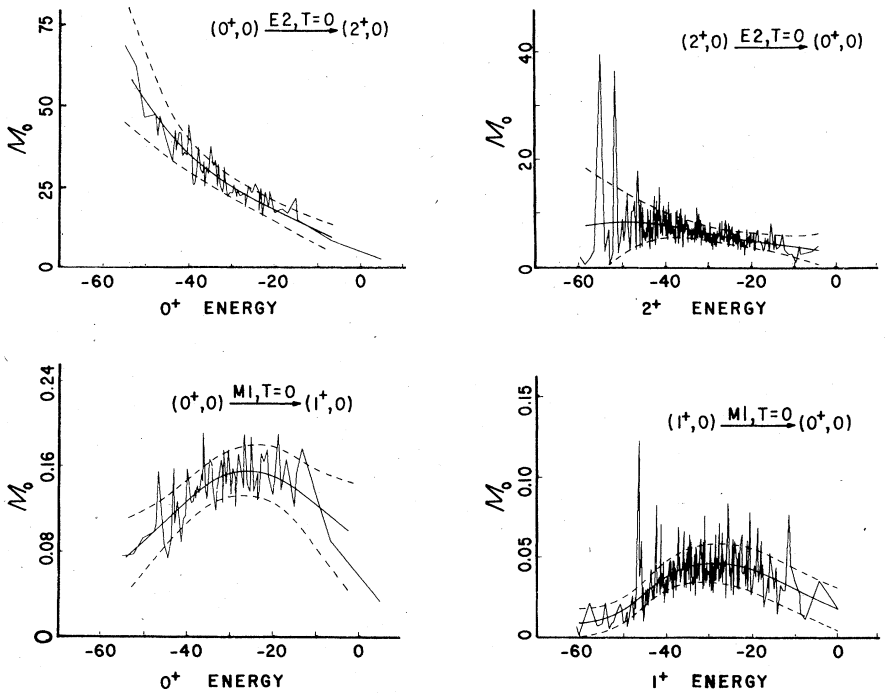


FIG. 11. Level-to-level fluctuations in the total strength $\mathfrak{M}_0(W)$ originating with a given level. The transitions are isoscalar $M1$ ($J=1 \longleftrightarrow J=0$) and isoscalar $E2$ ($J=2 \longleftrightarrow J=0$) between $T=0$ states in $(ds)^6$. Shown are the shell model $\mathfrak{M}_0(W)$, the smoothed values $\overline{\mathfrak{M}}_0(W)$ (solid lines), and Porter-Thomas values for $\overline{\mathfrak{M}}_0(W) \pm \hat{\Sigma}_0(W)$ (dashed lines). $\hat{\Sigma}_0(W)$ is the rms deviation predicted by the polynomial theory. Note that, for strong collectivity which results when there is only a single accessible state, we have $\hat{\Sigma}_0 = 2^{1/2}\overline{\mathfrak{M}}_0$, as is seen near the ground state in $(2^+,0) \to (0^+,0)$. See Draayer *et al.* (1977) for further discussion.

real problem in justifying it, for we could imagine that a very slight increase in the average potential-well depth would bind the observed states (which at least in the slow-neutron case span a very narrow energy range), and we can take for granted that this would involve no significant change in the spacings. However, we shall apply matrix results also to the widths themselves, and here some further justification is necessary. The question is: How do the widths arise from a description in which the continuum never appears? To answer this we can think, for instance, that the problem is posed in R-matrix language (Lane and Thomas, 1958). Here one considers the Schrödinger equation in the "internal" region subject to boundary conditions on its surface. This problem gives rise to discrete eigenvalues E_i only; these are the poles of the R matrix. On the other hand, the residues of the $R_{cc'}$ matrix element are connected with the eigenvector associated with level i and are given by $\gamma_{ic}\gamma_{ic'}$, where γ_{ic}, the reduced amplitude of level i into channel c, is given by an integral over the (multidimensional) surface S_c corresponding to an appropriate channel radius a_c

$$\gamma_{ic} = \left(\frac{\hbar^2}{2M_c a_c}\right)^{1/2} \int \Psi_i \psi_c^* dS_c . \tag{8.1}$$

Here ψ_c is the channel wave function which we can regard as given, Ψ_i is the Hamiltonian (matrix) eigenfunction, and M_c is the reduced mass for channel c.

When one connects the "internal" to the "external" region and calculates the S matrix (needed to compute cross sections) in terms of the R matrix, the poles of the former will be complex, the real part being the position of the resonances and the imaginary part the half-width. A case of particular importance is that of levels very narrow compared with their separation, because one can extract cleanly the properties of these levels from experiment (which is not feasible for overlapping resonances). In this case one can relate in a simple fashion the parameters of the S matrix to those of the R matrix. For a suitable choice of the boundary conditions, the real parts of the poles of S coincide with the poles of R and their partial widths Γ_{ic} (whose sum over the various channels is the total width, or twice the imaginary part of the pole) are related to the R-matrix reduced width by

$$\Gamma_{ic} = 2P_c\gamma_{ic}^2 , \tag{8.2}$$

a result already used in Sec. VII. Here P_c, the penetrability in channel c, is usually a slowly varying function of the energy.

Alternatively, one could adopt the point of view of Feshbach's (1958, 1962; see also Löwdin, 1966) formalism of nuclear reactions, in which one divides Hilbert space into two orthogonal parts with the projection operators P and Q, P corresponding to the incident particle plus the target in the ground state, or any other state giving rise to an open channel, and Q corresponding to the rest. The operator QHQ will then have discrete eigenvalues only; here the states acquire a width through the coupling QHP of the Q space to the continuum; the square of the matrix element associated with this coupling is proportional to the width of the level. In this case we can think that the Hamiltonian matrix we have been talking about in the previous sections is the matrix representation of QHQ.

In this chapter we shall go much beyond the case of isolated resonances by using random-matrix theory to describe the discrete aspects of the problem, i.e., the eigenvalues and eigenvectors of H in the internal region if we use the R-matrix formalism, or those of QHQ in Feshbach's formalism. The S matrix associated with the problem is constructed therefrom for isolated as well as for overlapping resonances.

Historically, it was indeed in connection with the quantum theory of collisions that the earliest references to the statistical theory of spectra are found (Wigner, 1951a, 1951b). The concept of a statistical R function with a definite distribution of poles and residues is introduced there for the first time, and classes of R functions are studied, whose members have essentially the same statistical properties; thus also the ergodic problem makes its appearance.

There is, however, in these investigations no clue concerning the form of the distribution functions for poles and residues, which were studied only later as the distributions of eigenvalues and eigenvectors of an ensemble of random matrices. This study started properly with a paper by Wigner (1955) and was then followed by a host of papers quoted throughout this review article.

The relevance of these distribution laws in the theory of average neutron reaction cross sections in the resonance region and, later on, in the continuum region of overlapping resonances, was soon recognized. The notion of a statistical S matrix, an extension of that of Wigner's statistical R function, enabled Moldauer (1964a) to find a generalization of the Hauser-Feshbach (1952) model for the compound-nucleus cross section. The problem of deriving the Hauser-Feshbach formula, both with and without the presence of direct reactions, has since then been the subject of many investigations.

The idea of constructing a statistical theory of nuclear reactions, using as basic ingredients random interaction-matrix elements which preserve the two-body selection rules, was taken up by Agassi *et al.* (1975), who developed a formalism in which the existence of a hierarchy of states (Griffin, 1966) with increasing complexity (the two-body interaction connecting only closer members of the hierarchy) can be taken into account explicitly. In this theory particle emission may occur before the more complex states of the hierarchy have been reached. This provides a theory of preequilibrium emission, a special case of which leads to the usual Hauser-Feshbach model; an alternative treatment of this problem has been given by Feshbach *et al.* (1980). The idea is to give a theoretical basis to the mostly intuitive formulations of precompound processes that have normally been used to describe the experimental data.

The discovery that the randomness of the S-matrix residues would lead, in the continuum region of overlapping resonances, to a cross section which, instead of being smooth, would show random fluctuations led Ericson (1963) to study moments of the cross section higher than the first one (the average cross section) and eventually the whole distribution. The study of the

autocorrelation function of the cross section at two different energies has proved to be a very powerful tool for extracting the widths of the levels in the region of overlapping resonances. In what follows, we shall give a more detailed description of the topics we have just mentioned.

B. The statistical R function

In the scattering of particles with an energy low enough that only elastic scattering occurs, one can include all the information concerning the short-range nuclear interaction in the inverse of the logarithmic derivative of the radial wave function evaluated at the nuclear radius, $r=a$. This is known as the Wigner-Eisenbud R function (Wigner and Eisenbud, 1947), and can be expanded as

$$R(E)\equiv\left[\frac{r\phi_l}{d(r\phi_l)/dr}\right]_{r=a}=\sum_\lambda\frac{\gamma_\lambda^2}{E_\lambda-E},\quad E_\lambda,\gamma_\lambda=\text{real}. \tag{8.3}$$

In order to obtain the scattering at angular momentum l, one first forms the expression

$$Q(E)=\frac{F'R-F}{-G'R+G}=Q_0+\sum_\lambda\frac{w_\lambda}{e_\lambda-E}\ , \tag{8.4}$$

where F and G are the regular and irregular solutions for angular momentum l outside the nuclear radius $r=a$. $Q(E)$ is thus influenced by the long-range interaction. The cross section σ_l then becomes

$$\sigma_l=\frac{4\pi}{k^2}(2l+1)\frac{Q^2}{1+Q^2}\ . \tag{8.5}$$

In an energy region where the density of resonances is very large, one can consider an interval wide enough to contain many resonances but narrow enough that the average properties do not change appreciably throughout it; F and G can thus be considered practically constant. Equation (8.4) then relates R and Q through a bilinear (or, as Wigner calls it a "fractional") transformation. The influence of the long-range interaction on the statistical properties of the resonances can thus be considered by asking how a fractional transformation affects the distribution of poles and residues.

The very useful concept of a "uniform statistical R function" is introduced (Wigner, 1951a, 1951b) with the properties that the densities of residues and poles are constant and that the distribution of residues and distances between nearest-neighbor poles does not depend on energy, so that it can be extended to all energies, $-\infty<E<\infty$. In a case of physical interest, this cannot be accomplished without changing R quite drastically everywhere, except in the energy interval I under study. This is done as follows: Given R, we retain those resonances that are contained in I; for energies well within I, the outside resonances contribute to R practically as a constant R^0. We substitute for them a new infinite set of outside resonances having the same statistical properties as those inside; adding these new resonances and subtracting the almost constant value R^1 which they contribute to the inside of I, we are led to a new R function

$$R_s(E)=(R^0-R^1)+\sum_{E_\lambda\in I}\frac{\gamma_\lambda^2}{E_\lambda-E}+\sum_{E_\lambda\notin I}\frac{(\gamma_\lambda^{\text{new}})^2}{E_\lambda^{\text{new}}-E}, \tag{8.6}$$

called a "uniform statistical R function", which coincides with the true R well inside I, but is now stationary, having the same statistical properties everywhere. In a different language we might call this an R function "tangent" to the true one. It plays a role similar to that of the "unfolded" spectra considered above.

The surmise, which is not proved but made very plausible, is that the distribution of level spacings $e_{\lambda+1}-e_\lambda$, Eq. (8.4), is the same as the distribution of the $(E_{\lambda+1}-E_\lambda)$, Eq. (8.3). In other words, the statistical properties of the level spacings are independent of the long-range interaction. A similar statement applies to the level widths; the distribution of the w_λ, if measured in terms of their average, is the same as the distribution of the γ_λ^2 if these are measured in units of *their* average. These relations are expected to be valid for "most statistical distributions of poles and spacings, just as the ergodic hypothesis in statistical mechanics is valid for most Hamiltonians" (Wigner, 1951b, p. 797).

An important fractional transformation is the orthogonal one,

$$\tilde{Q}=\frac{aR+b}{-bR+a},\quad\text{with } a^2+b^2=1 \tag{8.7}$$

which can also be written as

$$\tilde{Q}=\frac{R\cos\phi+\sin\phi}{-R\sin\phi+\cos\phi}=\frac{R+\tan\phi}{1-R\tan\phi}\ . \tag{8.8}$$

If we write $R=\tan f$, we see that $\tilde{Q}=\tan(f+\phi)$. Then the transformation (8.8) replaces f by $f+\phi$ and leaves $df/dE\equiv f'$ invariant. Hence $f'=R'(1+R^2)^{-1}$ will be called the *invariant derivative*. It follows that R and its orthogonal transforms have the same form

$$R=\tan\int_\epsilon^E f'(E)dE\ , \tag{8.9}$$

except that, for different transforms, the lower limit ϵ is different. The poles of R are so placed that the area below f' which is between two successive poles is π; see Fig. 12. Thus, starting from one pole E_1, one can locate the next pole E_2 by determining the abscissa E_2 so that the area under f' between E_1 and E_2 is just π; the corresponding γ^2 is equal to $1/f'(E_2)$. The levels and widths of the orthogonal transforms of R can be obtained by a similar construction, starting, however, at another abscissa and thus altering ϵ. It seems reasonable that, unless $f'(E)$ has a very special structure, the chance of hitting on a given value of $f'(E)$ will be in the long run the same, no matter at which abscissa one started originally, thus giving the same statistical properties to R and its orthogonal transforms. This reminds us of the arguments invoked in classical statistical mechanics, where one expects the chance of occupying a certain volume in phase space to be independent, in the long run, of the initial conditions. This probability is proportional to the volume in question; even this point has an analog in our present problem, for if we define $g_\lambda=1/\gamma_\lambda^2$, then (Wigner, 1952) the probability $G(g)\delta g$ that g be between g and $g+\delta g$ is proportional to the total length of the domains in which $f'(E)$

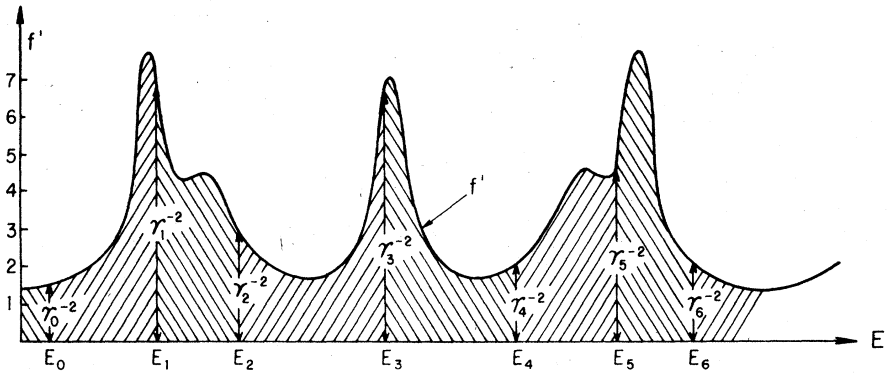


FIG. 12. Construction to obtain the position of the poles $E_1, E_2, E_3, \ldots$ and the reciprocals of the corresponding residues $\gamma_1^2, \gamma_2^2, \gamma_3^2, \ldots$ of an R function from its invariant derivative $f' = R'/(1+R^2)$. The area of the shaded regions is π, the ordinates which border them give the positions of the poles and their length is equal to the reciprocal of the corresponding residues (with opposite sign). The f' curve is common to all R functions of a family (8.8). If the construction is started from a different E_0, the poles and residues of another member of the family will be obtained. [After Wigner (1951a).]

is between g and $g+\delta g$, multiplied by the ordinate $f' = g$ in these domains. A statement similar to this can be made concerning the spacing distribution.

In the case of a random distribution for the poles of the invariant derivative $f'(E)$, it is possible (Wigner, 1952) to reduce to quadratures both the pole and the spacing distribution for R, but it is not known what the result of the integration is. It might be interesting to evaluate the integrals numerically to find explicitly these two distributions. How much further the analogy with statistical mechanics can be pursued is not known at present but might well be worth exploring.

C. Average cross sections

Since the notion of a compound nucleus was proposed (Bohr, 1936) the calculation of the cross section for a reaction proceeding through a compound system has been the subject of many investigations. The equation proposed by Bethe (1937) for the average cross section,

$$\langle\sigma_{cc'}\rangle = \frac{2\pi}{D}\frac{\langle\Gamma_{\lambda c}\rangle\langle\Gamma_{\lambda c'}\rangle}{\sum_{c''}\langle\Gamma_{\lambda c''}\rangle}, \tag{8.10}$$

incorporates the idea of independence of formation and decay of the compound nucleus and is valid in the limit of small width-to-spacing ratio ($\Gamma/D \ll 1$). Here $\langle\sigma\rangle$ means an energy average and $\langle\Gamma_{\lambda c}\rangle$ is an average over the levels λ. What is known as the Hauser-Feshbach (HF) formula (Wolfenstein, 1951; Hauser and Feshbach, 1952),

$$\langle\sigma_{cc'}\rangle \sim \frac{T_c T_{c'}}{\sum T_c''} \tag{8.11}$$

is expressed in terms of "transmission coefficients" (Blatt and Weisskopf, 1952) which have to be calculated in some model, like the continuum model or the optical model (Feshbach *et al.*, 1954). The HF formula for the average compound-nucleus cross section is computationally quite simple. With the incorporation of the appropriate vector-addition coefficients to take into account the angular momenta involved in the problem, it has become a powerful tool for extracting the quantum numbers of the excited levels. When there is the possibility of a direct reaction, the HF formula (8.11) is assumed to reproduce the "fluctuation" or compound-nucleus cross section in the separation

$$\langle\sigma_{cc'}\rangle = \sigma_{cc'}^{\rm dir} + \sigma_{cc'}^{fl}. \tag{8.12}$$

This is associated with a separation of the S matrix into an energy-averaged part $\langle S_{cc'}\rangle$, the optical S matrix, slowly varying with energy, and a fluctuating part $S_{cc'}^{fl}$

$$S_{cc'} = \langle S_{cc'}\rangle + S_{cc'}^{fl}, \tag{8.13}$$

so that, in units of $(2l+1)\pi\lambda^2$,

$$\sigma_{cc'}^{\rm dir} = |\delta_{cc'} - \langle S_{cc'}\rangle|^2, \quad \sigma_{cc'}^{fl} = \langle |S_{cc'}^{fl}|\rangle^2. \tag{8.14}$$

Much effort has been devoted to providing a proof and extensions of the HF formula. We shall discuss here the following main lines of approach to the problem, namely those of (1) Moldauer (1964a), (2) Hofmann *et al.* (1975), (3) Agassi *et al.* (1975), (4) Kawai *et al.* (1973), and (5) Mello (1979).

D. Moldauer's method

Using energy averages and the expansion of the S matrix (Lane and Thomas, 1958)

$$S_{cc'} = S_{cc'}^0 + e^{-i(\phi_c+\phi_{c'})}(\delta_{cc'} + 2\mathcal{S}_{cc'}),$$

$$\mathcal{S}_{cc'} = \sum_\lambda A_\lambda^{cc'} + \sum_{\lambda\neq\mu}\sum_{c''} A_\lambda^{cc''}A_\mu^{c''c'} + \cdots, \tag{8.15}$$

$$A_\lambda^{cc'} = \frac{\frac{i}{2}\gamma_{\lambda c}\gamma_{\lambda c'}}{E_\lambda - E - i\Gamma_\lambda/2}.$$

Moldauer (1961) has been able to give, in the small Γ_c/D limit, an expansion of the energy-averaged cross section

$$\langle\sigma_{cc'}\rangle = |S_{cc'}^0|^2 + \left\langle\frac{T_{\lambda c}T_{\lambda c'}}{\sum_a T_{\lambda a}}\right\rangle\left(1 - \tfrac{1}{2}\langle T_{\lambda c} + T_{\lambda c'}\rangle\Phi + \cdots\right), \tag{8.16}$$

which takes into account the statistical distribution of levels and widths.

Here

$$\tau_{\lambda c}=\frac{2\pi\gamma_{\lambda c}^2}{D} \tag{8.17}$$

and

$$\Phi\left(\frac{\Gamma}{D}\right)=i\frac{D}{\pi}\int_{-\infty}^{\infty}\frac{W(\varepsilon)}{\varepsilon+i\Gamma}\,d\varepsilon, \tag{8.18}$$

where $W(\varepsilon)$ is the density of resonances at an energy ε measured from some fixed reference level. The leading resonant term in (8.16) differs from what one would find in a HF formula by the so called "width fluctuation" factor found earlier (Lane and Lynn, 1957)

$$W_{cc'}=\left\langle\frac{\tau_{\lambda c}\tau_{\lambda c'}}{\tau_\lambda}\right\rangle\left(\frac{\langle\tau_{\lambda c}\rangle\langle\tau_{\lambda c'}\rangle}{\langle\tau_\lambda\rangle}\right)^{-1},\quad \tau_\lambda=\sum_c\tau_{\lambda c} \tag{8.19}$$

which gives, in a simple two-channel case, $\frac{1}{2}$ for $c\neq c'$ and $\frac{3}{2}$ for $c=c'$, assuming independent and identical Porter-Thomas distributions for $\tau_{\lambda c}$ and $\tau_{\lambda c'}$ (see Sec. VII.C).

It is found that the expansion converges rapidly only when total and channel widths are not too large. The method can apparently be used fairly safely for $\langle\tau_\lambda\rangle<1.5$, about half the value for which the expansion converges. The correction terms have magnitudes which, in the range of rapid convergence, go up to 50% and can therefore be comparable in importance to the fluctuation correction. The results were applied successfully to describe the inelastic scattering of neutrons from ^{92}Zr (Moldauer, 1961).

In the absence of direct reactions one can obtain a relation (Moldauer, 1963) between the optical S matrix $\langle S_{cc'}\rangle$ and the parameters of the R matrix, such as $\langle\tau_{\lambda c}\rangle$. This is valid for arbitrary values of $\langle\tau_\lambda\rangle$, but its relation to the fluctuation cross section $\langle|S_{cc'}^{fl}|^2\rangle$ is not specified there; it is discussed later in this chapter.

Deviations from the HF formula (8.11) can be expected also in the continuum region (Satchler, 1963), where the average total width Γ is much larger than the average spacing D and where the compound-nucleus levels strongly overlap. Using a model in which the residues $g_{\lambda c}g_{\lambda c'}$ of $S_{cc'}$ are random numbers uncorrelated with the poles, and which have no level-level or channel-channel correlations, one arrives at the result

$$\overline{S_{ab}^{fl}(S_{cd}^{fl})^*}=\delta_{ac}\delta_{bd}\frac{P_{aa}P_{bb}}{\sum_d P_{dd}}W_{ab}, \tag{8.20}$$

Here P is Satchler's penetration matrix

$$\mathbf{P}=\mathbf{1}-\bar{\mathbf{S}}\bar{\mathbf{S}}^+=\overline{\mathbf{S}^{fl}(\mathbf{S}^{fl})^+}, \tag{8.21}$$

which is diagonal in this case and given by

$$P_{ab}=\frac{2\pi}{D}\overline{(G_aG/\Gamma)}\delta_{ab}. \tag{8.22}$$

W_{ab} is again a fluctuation factor,

$$W_{ab}=\frac{\overline{(G_aG_b/\Gamma)}\,\overline{(G^2/\Gamma)}}{\overline{(G_aG/\Gamma)}\,\overline{(G_bG/\Gamma)}}, \tag{8.23}$$

where

$$G_a=|g_a|^2,\quad G=\sum_a G_a. \tag{8.24}$$

In the special case $G_a=G$ and $G=\Gamma$, this becomes the same correction required when averaging over discrete resonances, Eq. (8.19). In the limit of very many open channels, $W_{ab}\simeq\overline{G_aG_b}/(\overline{G_a}\,\overline{G_b})\simeq 1$ if $b\neq a$ and $\simeq 2$ if $b=a$, where we have assumed that the $g_{\lambda a}$ have a Gaussian distribution with the same dispersion for the real and imaginary parts.

If there are direct reactions, these results are valid in an "eigenchannel" representation. The transformation that takes us back to the physical channels will then, in general, produce a result which is not factorizable in the simple way described by HF.

Moldauer (1964a) found it very advantageous to avoid the series expansion of Eq. (8.16) by expressing his results in terms of averages over the resulting S-matrix parameters. A considerable improvement in the treatment of the problem is achieved via the statistical collision matrix. In terms of the statistical properties of $S(E)$ in the neighborhood of any E_0 we define a stationary random matrix function (Yaglom, 1962) of E. If an appropriate ergodicity obtains, we may then (as in Sec. X) replace energy averages by ensemble averages. One can derive this statistical collision matrix from a resonance formalism by expanding the wave function in terms of the eigenfunctions of a boundary-value problem with complex boundary conditions. If one chooses the logarithmic derivative of the wave functions to coincide with that of the outgoing wave for that channel at the energy $E=E_0$, one obtains the Kapur-Peierls (1938) collision matrix

$$\mathbf{S}=\boldsymbol{\Omega}[\mathbf{1}+2i\mathbf{P}^{1/2}\boldsymbol{\mathfrak{R}}\mathbf{P}^{1/2}]\boldsymbol{\Omega}, \tag{8.25}$$

where $\boldsymbol{\Omega}$ and P are the usual diagonal matrices (Lane and Thomas, 1958) containing the Coulomb and hard-sphere phase-shifts and the penetrability factors, respectively, and the complex $\boldsymbol{\mathfrak{R}}$ matrix can be expanded as

$$\mathfrak{R}_{cc'}=\sum_\lambda\frac{\theta_{\lambda c}\theta_{\lambda c'}}{E_\lambda-E-i\Gamma_\lambda/2}. \tag{8.26}$$

For energies different from E_0, the boundary conditions change, and by expanding S in the vicinity of E_0 one can estimate the size of the interval Δ within which (8.25) is a good approximation to the collision matrix. It is next assumed that for all resonances in the interval, the values of $\theta_{\lambda c}$, Γ_λ and E_λ constitute a proper sample from the appropriate stationary ensemble of resonance parameters. With the same device used in connection with Eq. (8.6) to define the uniform statistical R function, one now obtains the statistical collision matrix

$$S_{cc'}^s(E,E_0)=S_{cc'}^0(E_0)-i\sum_\lambda\frac{g_{\lambda c}g_{\lambda c'}}{E-E_\lambda+i\Gamma_\lambda/2}, \tag{8.27}$$

where all Ω_c and P_c are evaluated at E_0; the $g_{\lambda c}$ are connected to the $\theta_{\lambda c}$ of (8.26) by the penetrability factors of (8.25). S^s is a good approximation to S in Δ, but has stationary statistical properties for all energies ($-\infty<E<\infty$).

There is, however, a great drawback in this analysis, since the statistical properties of states defined with complex boundary conditions have not yet been studied. One can instead propose a definition in terms of the eigenvalues and eigenfunctions of the Wigner-Eisenbud R-matrix theory (Wigner and Eisenbud, 1947; Lane and

Thomas, 1958). The eigenvalues E_λ^0 and the reduced amplitudes $\gamma_{\lambda c}$ are now real, so that the real **R** matrix becomes

$$R_{cc'}=\sum_\lambda \frac{\gamma_{\lambda c}\gamma_{\lambda c'}}{E_\lambda^0-E}. \tag{8.28}$$

Separating out the part $R_{cc'}^0$ containing those resonances that lie outside Δ and calling $S_{cc'}^0$ its contribution to the **S** matrix, we have

$$S_{cc'}(E)=S_{cc'}^0(E)+2i\Omega_c P_c^{1/2}\sum_{\lambda,\,\mu\in\Delta}\alpha_{\lambda c}\alpha_{\mu c'}A_{\lambda\mu}P_{c'}^{1/2}\Omega_{c'}, \tag{8.29}$$

where

$$\alpha_{\lambda c}=\sum_{c'}[(1-\mathbf{R}^0\mathbf{L}^0)^{-1}]_{cc'}\gamma_{\lambda c'} \tag{8.30}$$

L_{cc}^0 is the difference between the logarithmic derivative of the outgoing wave and that of the eigenfunctions in the internal region, at the surface associated with channel c. The "level matrix" **A** is defined as

$$\mathbf{A}=(\mathbf{e}-\mathbf{E}-\xi)^{-1}, \tag{8.31}$$

where

$$\begin{aligned} &e_{\lambda\mu}=E_\lambda^0\delta_{\lambda\mu},\quad E_{\lambda\mu}=E\delta_{\lambda\mu},\\ &\xi_{\lambda\mu}=\sum_{cc'}\gamma_{\lambda c}[\mathbf{L}^0(1-\mathbf{R}^0\mathbf{L}^0)^{-1}]_{cc'}\gamma_{\mu c'}. \end{aligned} \tag{8.32}$$

Supposing that at the energy E_0 we diagonalize the complex symmetric matrix $\mathbf{e}-\xi$ (assuming it has distinct roots) by means of the complex orthogonal transformation **D** (Lane and Thomas, 1958)

$$[\mathbf{D}(\mathbf{e}-\xi)\mathbf{D}^{-1}]_{\lambda\mu}=(E_\lambda-i\Gamma_\lambda/2)\delta_{\lambda\mu}, \tag{8.33}$$

we can write

$$\sum_{\lambda\mu}\alpha_{\lambda c}\alpha_{\mu c'}A_{\lambda\mu}=\sum_\lambda\frac{\theta_{\lambda c}\theta_{\lambda c'}}{E_\lambda-E-i\Gamma_\lambda/2}, \tag{8.34}$$

where the complex amplitudes $\theta_{\lambda c}$ are

$$\theta_{\lambda c}=\sum_\mu D_{\lambda\mu}\alpha_{\mu c}. \tag{8.35}$$

Introducing (8.34) into (8.29) we obtain again a Kapur-Peierls type expansion. Adding as before the appropriate resonances outside the interval Δ in order to realize stationarity, we obtain the statistical **S** matrix of Eq. (8.27), with the following important relations between the residues and the imaginary part of the poles:

$$\begin{aligned} &\Gamma_\mu=\sum_c\Gamma_{\mu c},\quad \Gamma_{\mu c}=\frac{|g_{\mu c}|^2}{N_\mu}=\frac{2P_c|\theta_{\mu c}|^2}{N_\mu},\\ &N_\mu=\sum_\nu|D_{\mu\nu}|^2\geq 1. \end{aligned} \tag{8.36}$$

We shall find it convenient to define the quantities

$$\Theta_\mu\equiv\sum_c\Theta_{\mu c},\quad \Theta_{\mu c}\equiv\frac{2\pi|g_{\mu c}|^2}{D}N_\mu=\frac{2\pi\Gamma_{\mu c}}{D}N_\mu^2. \tag{8.37}$$

Moldauer (1964a) estimates that, when the ratio of the average total width to the average resonance spacing is small compared to $\sqrt{n/2}$, where n is the number of important competing open channels, one can diagonalize **A** of (8.31) by perturbation theory, so that to lowest order $\theta_{\mu c}\simeq\alpha_{\mu c}$ and $N_\mu\simeq 1$. In this case, the statistical distributions of the E_μ and $\theta_{\mu c}$ are approximately the same as those of the E_μ^0 and $\alpha_{\mu c}$, which in turn follow from the considerations of earlier chapters. For overlapping resonances, however, even if R^0 were zero, so that $\alpha_{\lambda c}=\gamma_{\lambda c}$ would be a real Gaussian variable, the fact that the matrix $D_{\lambda\mu}$ of (8.35) is orthogonal and complex (no longer approximately equal to the unit matrix as for isolated resonances) introduces an extra degree of freedom in the $\theta_{\lambda c}$ that was not present before, i.e., an imaginary part. For overlapping resonances there is no analytical treatment available at present for the distribution of the **S**-matrix parameters, but one can resort to Monte Carlo calculations. Moldauer (1964b) generated the **R** matrix by producing random uncorrelated and normally distributed real amplitudes $\gamma_{\mu c}$ with zero mean and unit standard deviation; for the E_μ^0 a Wigner "anticorrelated" distribution was chosen, centered about $E=0$ and having unit average spacing; it was generated by drawing at random alternately from the upper and lower halves of a Wigner distribution of spacings, to simulate the effect of the negative correlation of spacings actually present in a random-matrix model. The case of 100 competing channels, each with transmission coefficient equal to unity and boundary conditions adjusted so that all energy shifts (Lane and Thomas, 1958) vanished, was studied most thoroughly. Diagonalization of the level matrix $A_{\lambda\mu}$ was performed for dimensionalities up to 50.

Fig. 13 shows the distribution of poles $E_\mu-i\Gamma_\mu/2$ for one such collision matrix. Notice that $\langle\Gamma\rangle/D=6.4$ and $\sqrt{n/2}=5$, so that the perturbation expansion mentioned above would not be applicable.

The nearest-neighbor spacing distribution of the E_μ is compared in Fig. 14 with Wigner's distribution, Eq. (1.5), which characterizes the spacings of the E_μ^0 and with Poisson's distribution (1.4) which one would have for random uncorrelated levels. It is clear that the level repulsion present in the E_μ^0 is markedly reduced in the E_μ. However, small spacings are less common than in the Poisson case.

In Fig. 15 the distribution of the total widths is shown to be slightly broader than the χ^2 distribution with 100 degrees of freedom, which would characterize the distribution of the quantities $2\sum_c P_c\gamma_{\mu c}^2$; this is the case even though the average values of the two quantities do not differ appreciably:

$$\langle\Gamma_\mu\rangle_\mu=6.40,\quad \sum_c 2P_c\langle\gamma_{\mu c}^2\rangle_\mu=6.37 \tag{8.38}$$

The channel parameters $|g_{\mu c}|^2$, $\Gamma_{\mu c}$, $\Theta_{\mu c}$ of Eqs. (8.36) and (8.37) were found to have quite similar distributions, resembling the Porter-Thomas law of the $\gamma_{\mu c}^2$ but somewhat deficient in small values. This is shown in Figs. 16 and 17, where the distributions of the $\Theta_{\mu c}$ and the $\Gamma_{\mu c}$ for 25 channels in one of the cases are compared with the χ^2 distributions with one and two degrees of freedom, corresponding to the Porter-Thomas and exponential distribution laws, respectively.

Going back to the expansion (8.27) of the **S** matrix, one can prove (Moldauer, 1964a) that its ensemble average is given by

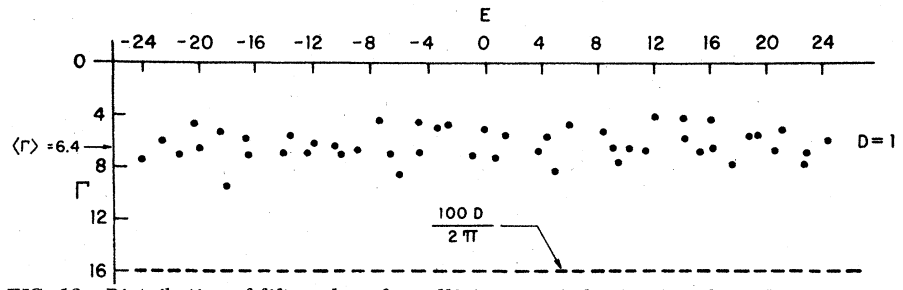


FIG. 13. Distribution of fifty poles of a collision matrix having 100 channels with random uncorrelated R-matrix amplitudes and R-matrix pole spacings selected from the Wigner distribution with anticorrelated neighboring spacings. The transmission coefficients of all channels are unity. [After Moldauer (1964b).]

$$\bar{S}_{cc'} = S^0_{cc'} - \frac{\pi}{D}\overline{g_{\mu c} g_{\mu c'}} \,. \tag{8.39}$$

$\bar{S}_{cc'}$ describes the direct contribution to the process $c \to c'$; the second term of (8.39), which is independent of μ because the ensemble is stationary, is the contribution to that direct process from those resonances near the energy in question. It is indeed the covariance of the corresponding R-matrix quantity $\overline{\gamma_{\mu c}\gamma_{\mu c'}}$ that was studied in the previous section.

The fluctuation cross section, defined as

$$\sigma^{fl}_{cc'} = \overline{|S_{cc'}|^2} - |\bar{S}_{cc'}|^2 \tag{8.40}$$

in units of $(2l+1)\pi\lambda^2$, can be written in terms of the $\Theta_{\mu c}$ of (8.37) as

$$\sigma^{fl}_{cc'} = \left(\frac{\overline{\Theta_{\mu c}\Theta_{\mu c'}}}{\Theta_\mu}\right) - M_{cc'} \,, \tag{8.41}$$

where

$$M_{cc'} = \frac{2\pi^2}{D^2}\left\{|\overline{g_{\mu c} g_{\mu c'}}|^2 - \overline{\left[g_{\mu c} g_{\mu c'} g^*_{\nu c} g^*_{\nu c'} \Phi\left(\frac{\Gamma_\mu + \Gamma_\nu}{2D}\right)\right]_{\mu\neq\nu}}\right\}. \tag{8.42}$$

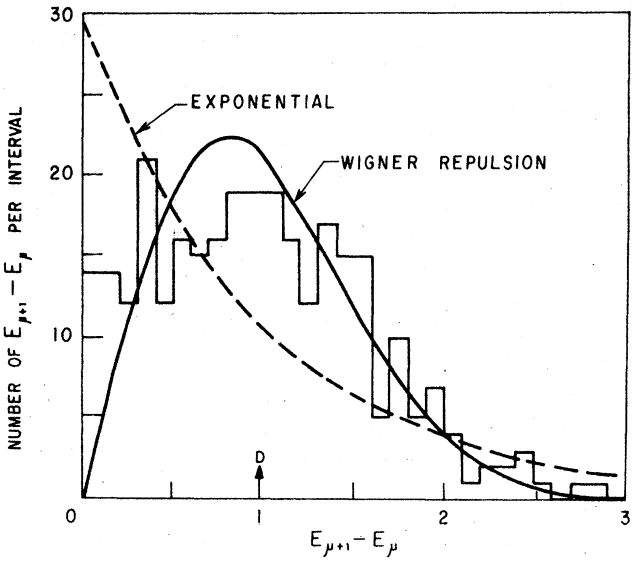


FIG. 14. Distribution of the energy spacings compared with the Wigner and Poisson (exponential) distributions. [After Moldauer (1964b).]

Here again, $M_{cc'}$ is independent of μ and ν because of stationarity. The function Φ was defined in (8.18) in terms of the resonance spacing distribution; it goes to unity for large values of the argument and it is identically unity for correlated levels.

The transmission coefficient can be split into a direct and a compound contribution as

$$T_c = 1 - |\bar{S}_{cc}|^2 = T^{\rm dir}_c + P_{cc} \,, \tag{8.43}$$

where the Satchler penetration matrix element P_{cc} of Eq. (8.21) is now given by

$$P_{cc} = \bar{\Theta}_{\mu c} - \sum_{c'} M_{cc'} \equiv \bar{\Theta}_{\mu c} - X_c \,. \tag{8.44}$$

A few years ago an important simplification was discovered (Moldauer, 1975a, 1975b, 1976, 1978). In the exact expression for the fluctuation cross section, besides a Hauser-Feshbach term there are terms which depend on resonance-resonance correlations of various kinds, terms which depend on channel-channel correlations, and non-HF terms that remain important in the absence of all correlations. Moldauer finds numerical evidence that all these different kinds of terms cancel to leave a remainder that is generally quite close to the

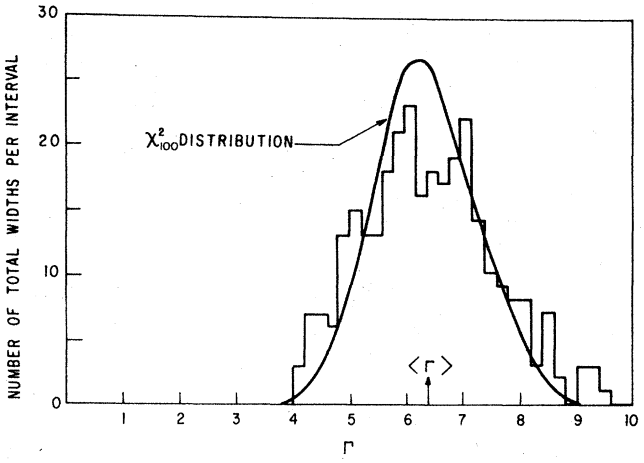


FIG. 15. Distribution of the total widths (twice the imaginary parts) of the poles of Fig. 13 compared with the chi-squared distribution with 100 degrees of freedom. [After Moldauer (1964b).]

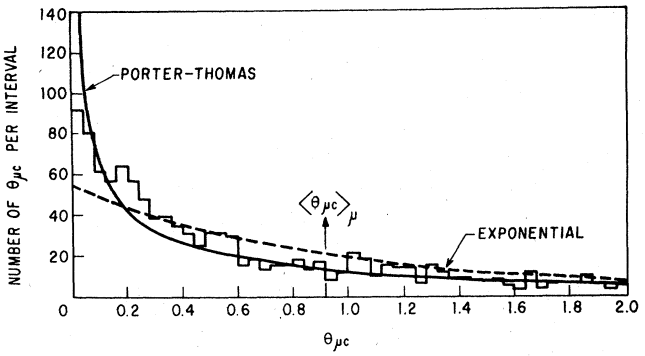


FIG. 16. Combined distributions of the $\Theta_{\mu c}$ of 25 of the channels of one of the collision matrices of Fig. 13 compared with the Porter-Thomas (χ_1^2) and exponential (χ_2^2) distributions. [After Moldauer (1964b).]

Hauser-Feshbach prediction. We write (8.41) as

$$\sigma_{cc'}^{fl}=\frac{\bar{\Theta}_{\mu c}\bar{\Theta}_{\mu c'}}{\bar{\Theta}_\mu}W_{cc'}-M_{cc'}, \tag{8.45}$$

where the width fluctuation factor $W_{cc'}$ is given by

$$W_{cc'}=\left(\frac{\overline{\Theta_{\mu c}\Theta_{\mu c'}}}{\Theta_\mu}\right)\left(\frac{\bar{\Theta}_{\mu c}\bar{\Theta}_{\mu c'}}{\bar{\Theta}_\mu}\right)^{-1}. \tag{8.46}$$

Consider now the case of n equivalent open channels. By this we mean that all $P_{cc}, \bar{\Theta}_{\mu c}, X_c$ appearing in (8.44) are independent of c. We can then write (8.45) as

$$\sigma_{cc'}^{fl}=\frac{P}{n}W_{cc'}+\left(\frac{X}{n}W_{cc'}-M_{cc'}\right). \tag{8.47}$$

The specific cancellation we referred to above is that the term in brackets in (8.47) vanishes; this is a nontrivial consequence of unitarity which has been called M cancellation; as a result of it we are left with a Hauser-Feshbach formula with a width fluctuation factor.

Finally, we refer to remarks in the literature (Blatt and Weisskopf, 1952; Feshbach, 1958) that the ratio $\bar{\Gamma}_c/D$, appearing in statistical theories of nuclear reactions, satisfies

$$\frac{\bar{\Gamma}_c}{D}\lesssim\frac{2}{\pi},\quad \bar{\Gamma}_c\equiv\langle\Gamma_{\mu c}\rangle_\mu. \tag{8.48}$$

Difficulties encountered in trying to prove this inequality are discussed by Moldauer (1969) and Feshbach and Mello (1972).

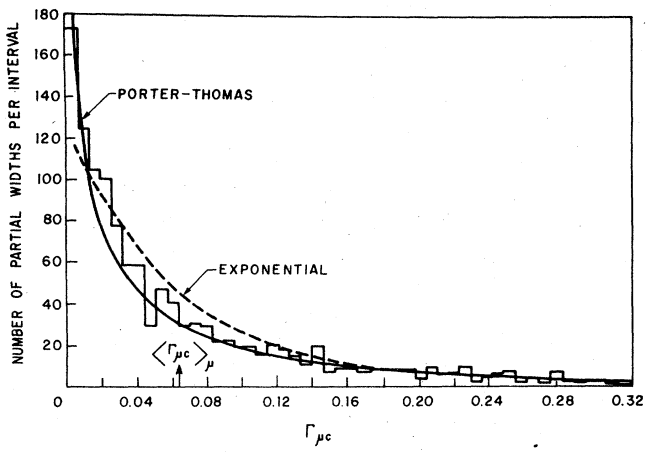


FIG. 17. Combined distribution of the partial widths of 25 of the channels of one of the collision matrices of Fig. 13 compared with the Porter-Thomas (χ_1^2) and exponential (χ_2^2) distributions. [After Moldauer (1964b).]

E. The Hofmann, Richert, Tepel, and Weidenmüller (HRTW) method

The basic element of this approach is the Engelbrecht-Weidenmüller (1973) transformation, which reduces the general problem to one without direct reactions.

By its very definition, the penetration matrix **P** of Eq. (8.21) is Hermitian and can be diagonalized by a unitary transformation **U**, so that[37]

$$(\mathbf{UPU}^+)_{ab}=\delta_{ab}p_a, \tag{8.49}$$

with

$$0\leq p_a\leq 1. \tag{8.50}$$

The important observation of Engelbrecht and Weidenmüller is that, with **U** given by (8.49) and T denoting transposition,

$$\mathbf{U\bar{S}U}^T=\text{diagonal}. \tag{8.51}$$

This suggests the introduction of the new scattering matrix

$$\tilde{\mathbf{S}}\equiv\mathbf{USU}^T, \tag{8.52}$$

which is again symmetric and unitary. Its average is diagonal, so that direct reactions vanish in this new representation.

The analysis is carried out (Hofmann, Richert, Tepel, and Weidenmüller, 1975) by writing the S matrix

$$\mathbf{S}=(1+i\mathbf{K})(1-i\mathbf{K})^{-1} \tag{8.53}$$

in terms of a K matrix, which in turn is written as the sum of a background $\mathbf{K}^{(0)}$ and a resonant part $\mathbf{K}^{res}$ containing N resonances

$$\mathbf{K}=\mathbf{K}^{(0)}+\mathbf{K}^{res}, \tag{8.54}$$

$$K_{ab}^{res}=\sum_{\mu=1}^{N}\frac{\gamma_{\mu a}\gamma_{\mu b}}{E_\mu-E}. \tag{8.55}$$

The dimensionality of the matrices is n, the number of open channels.

The E_μ and $\gamma_{\mu a}$ are taken from a Gaussian orthogonal ensemble, so that they are uncorrelated with each other and the $\gamma_{\mu a}$ are normally distributed with zero mean (see Sec. VII). For $\mu\neq\nu$, the $\gamma_{\mu a}$ and $\gamma_{\nu b}$ are uncorrelated, and the distribution of the $\gamma_{\mu a},\gamma_{\mu b},\ldots$ is independent of μ and is therefore stationary. The statistical behavior of the $\gamma_{\mu a}$ is thus determined by the matrix

$$C_{ab}=\overline{\gamma_{\mu a}\gamma_{\mu b}}, \tag{8.56}$$

where the bar indicates again an ensemble average.

The matrix so constructed is another example of a

[37] Here and below $a, b, c, \ldots$ denote channel indices.

statistical K matrix, and the corresponding S matrix (8.53) another example of a statistical S matrix. They will depend on the two real symmetric matrices $K^{(0)}$ and C and therefore upon $n(n+1)$ input parameters.

In the absence of direct reactions, both $K^{(0)}$ and C must be diagonal, since $\bar{S}$ is. The corresponding statistical S matrix depends only upon $2n$ parameters. If the $\tilde{S}$ defined by Eq. (8.52) is, indeed, free of direct reactions, then the transformation U converts a statistical S matrix with nondiagonal $K^{(0)}$ and nondiagonal C into a statistical S matrix, $\tilde{S}$, for which the associated matrices $\tilde{K}^{(0)}$ and $\tilde{C}$ are both diagonal. To be more precise, let us write $\tilde{S}$ in the form

$$\tilde{S}=\frac{1+i\tilde{K}}{1-i\tilde{K}}\ , \tag{8.57}$$

where $\tilde{S}$ and S are related as in Eq. (8.52), U being for the moment an *arbitrary* unitary matrix. $\tilde{K}$ and K can be shown to be related by a linear fractional transformation, as in the relation (8.4) between Q and R, and $\tilde{K}$ can be shown to be of the form

$$\tilde{K}_{ab}=\tilde{K}_{ab}^{(0)}+\sum_{\mu=1}^{N}\frac{\tilde{\gamma}_{\mu a}\tilde{\gamma}_{\mu b}}{\tilde{E}_{\mu}-E}, \tag{8.58}$$

where the $\tilde{E}_{\mu}$ and the $\tilde{\gamma}_{\mu a}$ are real and independent of E. Moreover, it can also be shown (Hofmann, Richert, Tepel, and Weidermüller, 1975) that if the original E_{μ} and $\gamma_{\mu a}$ arise from a GOE, the new $\tilde{E}_{\mu}$ and $\tilde{\gamma}_{\mu a}$ also arise from a GOE in the limit $N\gg n$. As we saw before, Wigner (1951a) constructed a special class of functions whose pole and spacing distributions were shown to remain invariant under an orthogonal fractional transformation. Here we have the corresponding theorem for more than one open channel and for an ensemble which is of more physical interest.

Since the average of $\tilde{S}$ is diagonal, $\tilde{K}_{ab}^{(0)}$ and $\tilde{C}_{ab}=[\tilde{\gamma}_{\mu a}\tilde{\gamma}_{\mu b}]_e$ are both diagonal; hence $\tilde{S}$ has all the properties of a statistical S matrix without direct reactions. This result was first found in Monte Carlo calculations.

It is possible to show that $[\tilde{S}_{ab}^{fl}(\tilde{S}_{cd}^{fl})^*]_e$ vanishes unless $a=c$, $b=d$, or $a=b$, $c=d$; this can be proved (Hofmann, Richert, Tepel, and Weidenmüller, 1975) without resorting to a pole expansion of the S matrix, the result being then true for arbitrary values of the transmission coefficients. Therefore in the evaluation of

$$\overline{S_{ab}^{fl}(S_{cd}^{fl})^*}=\sum_{efgh}U_{ea}^{*}U_{fb}^{*}\overline{\tilde{S}_{ef}^{fl}(\tilde{S}_{gh}^{fl})^*}U_{gc}U_{hd}\ , \tag{8.59}$$

only quantities like $[|\tilde{S}_{ab}^{fl}|^2]_e$ and $[\tilde{S}_{aa}^{fl}(\tilde{S}_{cc}^{fl})^*]_e$ will enter. These, in turn, can be shown to depend, except for a trivial phase factor, only upon the transmission coefficient $p_c=1-|\bar{S}_{cc}|^2$. This result is nontrivial, since the $\tilde{S}$ matrix contains the diagonal background matrix $\tilde{K}_{ab}^{(0)}=\delta_{ab}k_a^{(0)}$ and the diagonal covariance matrix $\tilde{C}_{ab}=\delta_{ab}c_a$, thus depending upon *two* parameters, $k_a^{(0)}$ and c_a, per channel. These are related to the transmission coefficient p_a by the formula

$$p_a=\frac{4\pi c_a/D}{(1+\pi c_a/D)^2+(k_a^{(0)})^2}\ . \tag{8.60}$$

The curve $p_a=$ constant is an ellipse in the $(c_a, k_a^{(0)})$ plane; it is certainly not obvious *a priori* that all points on this ellipse should, aside from a phase factor, yield the same values for $[\tilde{S}_{ab}^{fl}(\tilde{S}_{cd}^{fl})^*]_e$.

As for the specific dependence upon the p_a's, we know that, for weak absorption in all channels (Engelbrecht and Weidenmüller, 1973; Weidenmüller, 1974), the inelastic cross sections in the absence of direct reactions factorize, giving independence of formation and decay of the compound nucleus. Outside this regime the factorization hypothesis was tested numerically by writing

$$\overline{|\tilde{S}_{ab}^{fl}|^2}=\frac{V_aV_b}{\sum_c V_c},\quad a\neq b\,;\quad \overline{|\tilde{S}_{aa}^{fl}|^2}=\frac{V_a^2}{\sum_c V_c}W_a^{\mathrm{HRTW}}, \tag{8.61}$$

$$\overline{\tilde{S}_{aa}^{fl}(\tilde{S}_{bb}^{fl})^*e^{-2i\phi_a+2i\phi_b}}=\chi_a\chi_b,\quad a\neq b \tag{8.62}$$

W_a^{HRTW} being a factor that enhances the elastic cross section. One has the following unitarity requirement

$$p_a=V_a+\frac{V_a^2}{\sum_c V_c}(W_a^{\mathrm{HRTW}}-1). \tag{8.63}$$

As shown by Tepel *et al.* (1974), this equation determines uniquely the quantities V_a in terms of W_a^{HRTW} and the set of p_c. There remains the problem of finding the dependence of W_a^{HRTW} and χ_a on the transmission coefficients p_a. Monte Carlo calculations can be performed and curves fitted to these results. Numerically it is found that $W_a^{\mathrm{HRTW}}\approx 2$ for $p_a=1$; for $p_a=0$ the curves were fitted to a value of 3, although it has to be noted that the error bars of the Monte Carlo calculation are quite large in this low-absorption limit. It is found that $\chi_a\approx 0$ for $p_a=0$ and $p_a=1$, showing a maximum for some intermediate p.

There is unfortunately no analytical treatment for the dependence of the averages $[|\tilde{S}_{ab}^{fl}|^2]_e$ and $[\tilde{S}_{ac}^{fl}(\tilde{S}_{bb}^{fl})^*]_e$ on the transmission factors: What is found by Tepel *et al.* (1974), Hofmann, Richert, Tepel, and Weidenmüller (1975), and Hofmann, Richert, and Tepel (1975) must be viewed as simple parametrizations which reproduce the numerically generated cross sections in a statistically satisfactory fashion. We shall see, however, in the next section, that the method of Agassi *et al.* (1975) provides a framework for the calculation of these and other quantities, in the limit $\Gamma\gg D$.

F. The Agassi, Weidenmüller, and Mantzouranis (AWM) method

This method is appropriate in the case of overlapping resonances ($\Gamma\gg D$), where it provides a framework for dealing analytically with all concepts of physical interest, Hauser-Feshbach theory, Ericson fluctuations, distribution of S matrix elements, influence of direct reactions, correlations between amplitudes, presence of doorway states, etc. The theory also provides a unified microscopic statistical description of preequilibrium and equilibrium processes, appropriate for mass numbers $A\geq 40$ and incident projectiles $A\leq 4$.

In the analysis of the previous sections, it was taken for granted that the compound system reaches internal statistical equilibrium before it decays. It turns out that the deviations from this model, observed even at excitation energies several tens of MeV above neutron threshold, are amenable to a statistical interpretation.

At such energies the decay time t_{dec} of the compound nucleus becomes comparable to the time t_{eq} it takes the compound system to reach statistical equilibrium, and a fraction of the particles emitted thus correspond to "preequilibrium" of "precompound" emission (Griffin, 1966; Blann, 1975). The description of Agassi *et al.* (1975) is supposed to apply if the condition

$$t_{resp} \ll (t_{eq}, t_{dec}) \ll t_{per} \tag{8.64}$$

is met. Here t_{resp} is the duration of a nucleon-nucleon collision and $t_{per} = \hbar/D$ is an estimate of a natural "recurrence" time. In terms of the associated energies

$$\Delta E = \hbar/t_{resp}, \quad \Gamma^{\downarrow} = \hbar/t_{eq}, \quad \Gamma^{\uparrow} = \hbar/t_{dec}, \quad D = \hbar/t_{per} \tag{8.65}$$

the condition is

$$\Delta E \gg (\Gamma^{\downarrow}, \Gamma^{\uparrow}) \gg D. \tag{8.66}$$

The largest of the ratios $D/\Gamma^{\uparrow}, D/\Gamma^{\downarrow}, \Gamma^{\uparrow}/\Delta E, \Gamma^{\downarrow}/\Delta E$ defines a small dimensionless parameter y, which can be considered as the expansion parameter of the theory.

The description of preequilibrium processes is carried out in terms of the shell model and, accordingly, classes of shell-model states with m "excitons" (m being the number of particles plus the number of holes) are introduced (Griffin, 1966). It is assumed that the interaction among the states of a given class is strong enough to produce a rapid equilibration within each class, much faster than the decay from one class to a more complicated one. The bound states in the continuum are thus grouped into classes of m excitons; it is assumed that the Hamiltonian H has been diagonalized within the states of a given class m, giving eigenfunctions $\phi_{m\mu}$ and eigenvalues $E_{m\mu}$, where μ is a running index in class m. The specific distribution law for the resulting eigenvalues $E_{m\mu}$ is actually not needed in the analysis. In this new basis, the matrix elements of the Hamiltonian H are given by

$$\begin{aligned}
&\langle \phi_{m\mu} | H | \phi_{m\nu} \rangle = \delta_{mn}\delta_{\mu\nu} E_{m\mu} + (1 - \delta_{mn}) \langle \phi_{m\mu} | V | \phi_{m\nu} \rangle , \\
&\sqrt{\pi}\, \langle \phi_{m\mu} | H | \chi_E^c \rangle \equiv \gamma_{m\mu}^c , \\
&\langle \chi_E^c | H | \chi_{E'}^{c'} \rangle = \delta_{cc'}\delta(E - E') E + \langle \chi_E^c | V | \chi_{E'}^{c'} \rangle ,
\end{aligned} \tag{8.67}$$

where χ_E^c is the wave function associated with channel c, and V denotes the residual interaction. The quantities $\gamma_{m\mu}^c$ and $\langle \phi_{m\mu} | V | \phi_{n\nu} \rangle$, $m \neq n$, are taken as zero-centered Gaussian random variables; their distributions are thus completely specified in terms of the second moments, which are assumed to be

$$\begin{aligned}
&\overline{\gamma_{m\mu}^c \gamma_{m'\mu'}^{c'}} = \delta_{cc'}\delta_{mm'}\delta_{\mu\mu'} \overline{(\gamma_m^c)^2} , \\
&\overline{\langle \phi_{m\mu} | V | \phi_{n\nu} \rangle \gamma_{m'\mu'}^{c'}} = 0 , \\
&\overline{\langle \phi_{m\mu} | V | \phi_{n\nu} \rangle \langle \phi_{n'\nu'} | V | \phi_{m'\mu'} \rangle} = (\delta_{mm'}\delta_{\mu\mu'}\delta_{nn'}\delta_{\nu\nu'} \\
&\qquad + \delta_{mn'}\delta_{\mu\nu'}\delta_{nm}\delta_{\nu\mu'}) \overline{V_{mn'}^2} .
\end{aligned} \tag{8.68}$$

The two-body character of the interaction V implies that the matrix elements $\langle \phi_{m\mu} | V | \phi_{n\nu} \rangle$ vanish[38] unless $|m - n| = 2$; however, as discussed in Sec. III.D, simply satisfying these selection rules does not at all ensure that we have a two-body Hamiltonian; in fact, just as with the GOE itself, we shall have simultaneous interactions of all the particles. For a true two-body interaction a large number of linear constraints on the matrix elements must also be satisfied, but these are ignored by Agassi *et al.* However, satisfying the selection rules alone is sufficient to produce significant deviations (supported by the experimental evidence) from the standard Hauser-Feshbach result, which one would recover by assuming a single class of states only or, presumably, several classes, all of them connected through an interaction of GOE type. It is not known, however, what modifications would result from the use of the full TBRE.

The central result for the avarage cross section, evaluated to leading order in the small parameter y mentioned above, has the form (direct reactions not occurring at all or having been eliminated by the Engelbrecht-Weidenmüller transformation)

$$\overline{|S_{ab}^{fl}|^2} = (1 + \delta_{ab}) \sum_{mn} T_m^a \Pi_{mn} T_n^b . \tag{8.69}$$

Here, T_m^a is the transmission coefficient for the population of states with m excitons from the channel a and Π_{mn} can be interpreted as the relative probability of reaching class n from class m. The symmetric probability matrix Π is found to obey the probability balance-equation (see also Nörenberg and Weidenmüller, 1976)

$$\Pi_{mn} \frac{2\pi\Gamma_n}{D_n} - \sum_k \Pi_{mk} T_{kn} = \delta_{mn} . \tag{8.70}$$

The flux of probability between classes is generated by the coefficient $T_{mn} = T_{nm}$, defined in terms of the appropriate transition matrix elements. The loss of probability in one class m due to transitions into other classes and into the channels is described by the width Γ_m, while D_m is the average spacing of levels in class m.

As a special case, suppose that only a single class of levels contributes to the reaction. We may then omit the label m and the result is

$$\overline{|S_{ab}^{fl}|^2} = (1 + \delta_{ab}) \frac{T^a T^b}{\sum_c T^c} , \tag{8.71}$$

which is the Hauser-Feshbach formula, including an elastic enhancement factor of 2, as expected for overlapping resonances.

In the presence of direct reactions and only one class of states, the result

$$\overline{S_{ab}^{fl} (S_{cd}^{fl})^*} = \frac{P_{ac}P_{bd} + P_{ad}P_{bc}}{\mathrm{Tr}\,\mathbf{P}} \tag{8.72}$$

is obtained, where S_{ab}^{fl} and $(S_{cd}^{fl})^*$ are evaluated at the same energy. For the autocovariance function of S_{ab}^{fl} one obtains

$$\overline{S_{ab}^{fl}(E) S_{ab}^{fl*}(E + \varepsilon)} = \frac{-i\Gamma^{corr}}{\varepsilon - i\Gamma^{corr}} C(0) , \tag{8.73}$$

where $\Gamma^{corr} = \mathrm{Tr}\,\mathbf{P}/(2\pi\rho)$ is the correlation length, $\mathbf{P}$ is the penetration matrix (8.21), and ρ is the level density; $C(0)$ is given by

$$C(0) = |S_{ab}^{fl}|^2 = (P_{aa}P_{bb} + P_{ab}P_{ba}) / \mathrm{Tr}\,\mathbf{P} .$$

[38] $|m - n| = 4$, although possible, is energetically not favored. This case is therefore neglected.

Equations (8.71) and (8.72) give analytic expressions, valid for $\Gamma \gg D$, for the quantities calculated only numerically in the HRTW method of the preceding section. Terms of higher order in y, correcting the Hauser-Feshbach formula, can be calculated in the absence of direct reactions and with only a single class of states. The result is in agreement with the formulas suggested on the basis of numerical studies (Tepel *et al.*, 1974; Hofmann, Richert, Tepel, and Weidenmüller, 1975), where, by letting $W_a^{\mathrm{HRTW}} = 2$, for all a, Eq. (8.63), one finds V_a in terms of the transmission factor T_a.

G. The Kawai, Kerman, and McVoy (KKM) method

The usual representation of the $S_{cc'}$ matrix element

$$S_{cc'}(E) = S_{cc'}^{(0)}(E) - i\sum_{\lambda} \frac{\hat{g}_{\lambda c}\hat{g}_{\lambda c'}}{E-\varepsilon_{\lambda}} \tag{8.74}$$

is replaced in this treatment (Kawai *et al.*, 1973) by an alternative but equivalent expression

$$S_{cc'}(E) = \langle S_{cc'}(E)\rangle - i\sum_{q} \frac{g_{qc}(E)\,g_{qc'}(E)}{E-\varepsilon_q}, \tag{8.75}$$

where the background is chosen to be the *optical S* matrix. The average of the second term is then zero. It is proved that the phases of the "partial-width amplitudes" $g_{qc}(E)$ exhibit a slow q dependence across the energy-averaging interval I, which causes the resonance sum in (8.75) to average to zero. The use of this "optical-background representation" for $S_{cc'}(E)$ considerably simplifies the statistical arguments employed in the discussion of overlapping resonances and fluctuations.

In Feshbach's (1958, 1962) reaction formalism, using the projection operators P and $Q = 1 - P$, where P projects onto all open channels, $\hat{g}_{\lambda c}$ can be written as

$$\hat{g}_{\lambda c} = \sqrt{2\pi}\,\langle \chi_c | H_{PQ} | \lambda\rangle, \quad H_{PQ} \equiv PHQ \tag{8.76}$$

and the g_{qc} of (8.75) as

$$g_{qc} = \sqrt{2\pi}\,\langle \psi_c^{\mathrm{opt}} | V_{PQ}(E) | q\rangle, \tag{8.77}$$

where ψ_c^{opt} is now an optical-model wave function and

$$V_{PQ}(E) = H_{PQ}\left(\frac{i(I/2)}{E - H_{QQ} + i(I/2)}\right)^{1/2}. \tag{8.78}$$

The states $|q\rangle$ result from the diagonalization of the matrix

$$E_Q\delta_{QQ'} + \langle Q | V_{QP}\frac{1}{E - \mathcal{H}_{PP}^{\mathrm{opt}}} V_{PQ} | Q'\rangle, \tag{8.79}$$

where $\mathcal{H}_{PP}^{\mathrm{opt}}$ is the many-channel optical Hamiltonian, which gives rise to the optical S matrix $\langle S_{cc'}\rangle$. In (8.78), $|Q\rangle$ represents an eigenstate of H_{QQ} with eigenvalue E_Q. Let us consider the limit $\Gamma \gg D$. It is argued that the potential term in (8.79) is a complex matrix with phases which vary randomly from one element to the next because of the complexity of the multiparticle bound states $|Q\rangle$. It is then assumed that the resulting g_{qc} have random phases and are uncorrelated for different q's.

In the presence of direct reactions one finds in this limit the result

$$\sigma_{ab}^{fl} = \frac{2\pi}{\Gamma D}\left(X_{aa}X_{bb} + X_{ab}X_{ba}\right). \tag{8.80}$$

Here X_{ab} is a Hermitian matrix in channel space defined as

$$X_{ab} = \langle g_{qa}g_{qb}^{*}\rangle_q = X_{ba}^{*} \tag{8.81}$$

and related to the penetration matrix P of Eq. (8.21) by

$$P_{ab} = \frac{2\pi}{\Gamma D}\left[X_{ab}\,\mathrm{Tr}\mathbf{X} + (\mathbf{X}^2)_{ab}\right] \tag{8.82}$$

and hence can be obtained from a knowledge of the complete open-channel optical S matrix. For example, if the number n of directly coupled open channels is large, Eq. (8.80) can be approximated up to order $1/n$ by

$$\sigma_{ab}^{fl} = \frac{1}{\sum_c P_c}\left(P_{aa}P_{bb} + P_{ab}P_{ba} + \frac{\mathrm{Tr}(\mathbf{P}^2)}{(\mathrm{Tr}\mathbf{P})^2}P_{aa}P_{bb} - \frac{P_{aa}(\mathbf{P}^2)_{bb} + P_{bb}(\mathbf{P}^2)_{aa}}{\mathrm{Tr}\mathbf{P}}\right). \tag{8.83}$$

For $a \neq b$, the leading term in this expression is the Hauser-Feshbach one, and it does dominate in the large-n limit. For compound elastic scattering ($a = b$), however, the first and second terms are equal and dominate as $n \to \infty$, giving a result which is twice the Hauser-Feshbach prediction.

In principle one could calculate, within this formalism, other quantities of physical interest, such as Ericson fluctuations, the distribution of the elements of the S matrix, etc. It would be interesting to compare the results with the predictions of the formalism of Agassi *et al.* (1975).

H. Ericson fluctuations

At energies where the total width Γ of the levels is larger than the mean spacing D, the cross section is simultaneously dominated by a large number of resonances, the amplitudes of which interfere strongly. The cross section will then show fluctuations (Ericson, 1963) on an energy scale $\sim\Gamma$.

As a simple guess, it was assumed by Ericson and Mayer-Kuckuk (1966), and only later proved for $\Gamma \gg D$ by Agassi *et al.* (1975), that the real and imaginary parts of the complex amplitude $f = \xi + i\eta$ are distributed as independent Gaussians with zero mean and the same width; the cross section, which is proportional to $|f|^2$, will then be distributed according to a χ^2 distribution with two degrees of freedom, which is simply an exponential

$$\rho_{\Lambda}(x) = e^{-x}, \quad \Lambda = \sigma(\theta)/\langle\sigma(\theta)\rangle. \tag{8.84}$$

Several experiments on statistical cross sections represented by one single amplitude have cross-section probability distributions well described by Eq. (8.84). A typical example with unusually good statistics is shown in Fig. 18, taken from Halbert *et al.* (1965).

Ericson's analysis (1963) is equivalent to taking, as a simple model for the reaction amplitude, the expression

$$S_{ab} = S_{ab}^{B} - i\sum_{\mu}\frac{G_{\mu,ab}}{E - E_{\mu} + i\Gamma/2} \tag{8.85}$$

with the assumption that the total width Γ_{μ}, which is composed of many partial widths, does not fluctuate

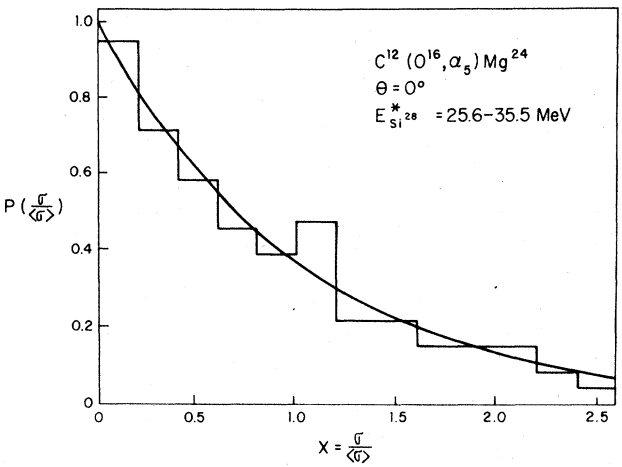


FIG. 18. Probability distribution of the differential cross section for $^{12}C(^{16}O,\alpha_5)^{24}Mg$ for the α_5 group leading to the 6.00 MeV excited state in ^{24}Mg. The excitation energy in the compound nucleus ^{28}Si is about 30 MeV. [After Halbert *et al.* (1965), reprinted by Ericson and Mayer-Kuckuk (1966)]. Reproduced with permission from Annual Reviews of Nuclear Science (Annual Reviews, Inc.), Vol. 16, 1966.

very much around its average and can thus be replaced by a constant Γ. Assume also that the $G_{\mu,ab}$ and the E_μ are not correlated. Writing

$$G_{\mu,ab} = \bar{G}_{\mu,ab} + G^{fl}_{\mu,ab}, \tag{8.86}$$

we can express S_{ab} as

$$S_{ab}(E) = \bar{S}_{ab} - i\sum_\mu \frac{G^{fl}_{\mu,ab}}{E - E_\mu + i\Gamma/2}. \tag{8.87}$$

Assuming that the $G^{fl}_{\mu,ab}$ for different levels are uncorrelated,

$$\overline{G^{fl}_{\mu,ab}(G^{fl}_{\nu,ab})^*} = \delta_{\mu\nu}\overline{|G^{fl}_{\mu,ab}|^2}, \tag{8.88}$$

one easily finds that the autocovariance of S_{ab} is given by

$$\overline{S^{fl}_{ab}(E)S^{fl*}_{ab}(E+\varepsilon)} = \frac{-i\Gamma}{\varepsilon - i\Gamma}\overline{|S^{fl}_{ab}|^2}, \tag{8.89}$$

which agrees with the correct result (8.73) only if $\Gamma/(nD) \ll 1$, where n is the number of open channels (Moldauer, 1975a; Bauer *et al.*, 1979). For the covariance of the cross section

$$C_{ab}(\varepsilon) \equiv \overline{[\sigma_{ab}(E+\varepsilon) - \bar{\sigma}_{ab}][\sigma_{ab}(E) - \bar{\sigma}_{ab}]}, \tag{8.90}$$

one finds

$$C_{ab}(\varepsilon) = \frac{\Gamma^2}{\varepsilon^2 + \Gamma^2}\sigma^{fl}_{ab}(2\sigma^{dir}_{ab} + \sigma^{fl}_{ab}). \tag{8.91}$$

The Lorentzian factor is due to the fact that the intermediate compound system decays exponentially at a rate h/Γ. This leads to a normalized variance

$$v \equiv \frac{C_{ab}(0)}{(\bar{\sigma}_{ab})^2} = 1 - y^2_{ab}, \qquad y_{ab} = \sigma^{dir}_{ab}/\bar{\sigma}_{ab}. \tag{8.92}$$

In most nuclear reactions, several amplitudes contribute simultaneously, owing to the additional freedom associated with the spins of the particles involved. In this case the magnitude of the fluctuation effects is reduced, since the observed cross section is the sum of several partial cross sections. One then defines the effective number n of channels (Brink and Stephen, 1963), so that

$$v = \frac{1}{n}(1 - y^2), \tag{8.93}$$

provided the relative amount of direct interactions is the same in each channel.

Equation (8.91) for the autocovariance function has been very useful in extracting Γ [which we shall now call Γ^{corr}, in view of the exact result (8.73)] and, through a relation of the type

$$T_c = 2\pi\overline{\Gamma_{\mu c}/D}, \tag{8.94}$$

in replacing the sum of transmission coefficients $\sum_c T_c$ which occurs in the denominator of the HF formula by Eberhard *et al.* (1969)

$$\sum_c T_c \simeq \frac{2\pi\Gamma^{corr}}{D}. \tag{8.95}$$

We refer the reader to Agassi *et al.* (1975) for a complete analytical treatment of the autocorrelation function in the limiting case of overlapping resonances and even in the presence of direct reactions but with only one class of levels (i.e., without precompound decay). A similar treatment could also be done with the framework of KKM.

It is known (Moldauer, 1969; Feshbach and Mello, 1972) that (8.94) is not appropriate when $\Gamma \gg D$. A more adequate expression would be (Moldauer, 1975a)

$$\sum_c \ln\frac{1}{1 - T_c} = \frac{2\pi\bar{\Gamma}}{D}. \tag{8.96}$$

Nevertheless, the use of Eq. (8.95) in conjunction with the HF formula has generally led to consistent results. In order to understand this, Moldauer (1975a) has done some illustrative Monte Carlo calculations. Figure 19 shows a typical width distribution histogram for the case of 20 open channels with all $T_c = 0.91$. The hypothesis that all the widths are equal is seen to be wrong: The distribution is very broad and very skewed; there are no very small widths at all. A peak in the distribution occurs near the minimum width ($\sim\bar{\Gamma}/3$) and a long tail extends to several times the average width $\bar{\Gamma}$. The average width $\bar{\Gamma}$ agrees reasonably well with the prediction of (8.96), whereas the correlation width Γ^{corr} predicted by (8.95) lies lower, at the peak of the distribution. On the other hand, fitting the numerically computed autocovariance function with the Lorentzian

$$C(\varepsilon) = \frac{(\Gamma^{corr})^2}{\varepsilon^2 + (\Gamma^{corr})^2}C(0) \tag{8.97}$$

produces a Γ^{corr} which is in good agreement with that predicted by (8.95).

In conclusion, Γ^{corr} is not related to the average width $\bar{\Gamma}$, but rather to the minimum width. Moreover, whereas $\bar{\Gamma}$ satisfies (8.96), Γ^{corr} satisfies (8.95), and it is this last combination that has been used to fit the experimental data in quite a satisfactory way.

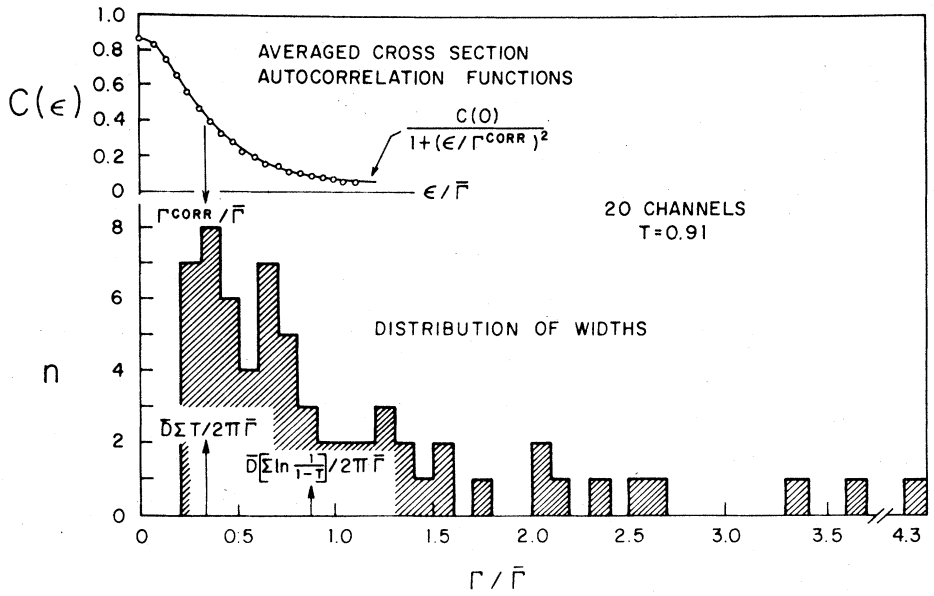


FIG. 19. Histogram of the number n of statistically computed widths falling within each width interval for the case of 20 statistically equivalent channels, each having $T_c = 0.91$. The points show the average of statistically computed cross-section autocorrelation functions for the same case. [After Moldauer (1975a).]

I. Comparison of the different methods described above

The HRTW and the KKM approaches are compared by Hofmann, Richert, Tepel, and Weidenmüller (1975). It is found[39] that the KKM result is recovered when we put the parameter W_a^{HRTW} of (8.61) equal to 2 and the χ_a equal to zero for all a. This case is realized only in the limit of strong absorption in all channels. The condition of validity of the KKM results is thus not merely $\Gamma \gg D$, but the more restrictive condition $p_a \simeq 1$ for all a, where the p_a (8.49) are the eigenvalues of the transmission matrix **P**.

The properties that the $S_{cc'}$ matrix element should have in order to reach this strong-absorption limit have been recently investigated (McVoy and Mello, 1979) in terms of the poles and zeros of $S_{cc'}$. In a statistical model, with assumptions similar to those used in KKM, they find that the density of the upper-plane zeros plays an important role in certain aspects of the scattering, particularly in determining the ratio $\sigma_{ab}^{\mathrm{dir}}/\sigma_{ab}^{fl}$ and the time delay of the wave packets. When the imaginary parts $-\Gamma_\lambda/2$ of the poles are chosen to be all equal to $-\Gamma/2$, it is found that the ensemble-averaged density of zeros is symmetric about the line of poles and has a width of about $\sigma_{ab}^{fl}/\sigma_{ab}^{\mathrm{dir}}$ (see Fig. 20). Incidentally, one can also show (Bauer *et al.*, 1979) that for a packet consisting of an incoherent superposition of plane waves, the centroid in time, $\bar{\tau}_{ab}$, that is to say, the time delay, of the scattered packet, satisfies $\bar{\tau}_{ab} = 2\pi/D_+$, where D_+ is the mean horizontal separation of the upper-half plane zeros. Increasing $\sigma_{ab}^{fl}/\sigma_{ab}^{\mathrm{dir}}$ decreases D_+ and so increases the time delay; the exact relation for this model is found to be

$$\bar{\tau}_{ab} = \frac{1}{\Gamma}\exp(-\sigma_{ab}^{\mathrm{dir}}/\sigma_{ab}^{fl}). \tag{8.98}$$

Moldauer (1975b) has investigated the effects upon compound cross sections of direct amplitudes that couple each channel with at most one other channel. Three theoretical formulas for the inelastic fluctuation cross-section enhancements were compared with the results of computer experiments. Both the M-cancellation formula (Moldauer, 1975a, 1976) and our Eq. (8.61), taken from Hofmann, Richert, Tepel, and Weidenmüller (1975), give a good account of the computer results. The M-cancellation formula is somewhat simpler to apply and gives slightly better agreement. The KKM formula (Kawai *et al.*, 1973) yields correct results in its domain of validity, namely, the strong-absorption limit.

An attempt has been made (Kerman and Sevgen, 1976) to present a general formalism in which the various theories appear as particular cases. In a background-

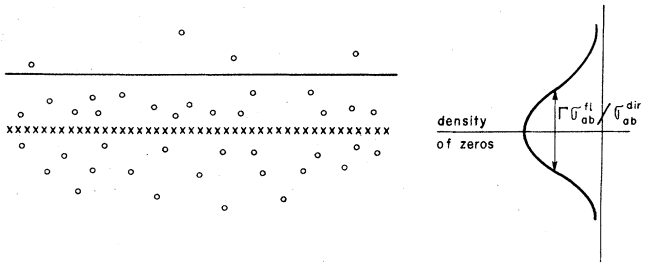


FIG. 20. Poles and zeros of an $S_{cc'}$ matrix element with random residues statistically independent of the poles, the latter having all the same imaginary part. The ensemble-averaged density of zeros is symmetric about the line of poles and has a width $\sim\Gamma\sigma_{ab}^{fl}/\sigma_{ab}^{\mathrm{dir}}$. See McVoy and Mello (1979).

[39] Notice again that the formulas that were tested only numerically in HRTW can be calculated analytically in the limit of $\Gamma \gg D$ (Agassi *et al.*, 1975).

plus-pole expansion of the S matrix

$$S_{ab} = S^B_{ab} - i\sum_\mu \frac{g_{\mu a} g_{\mu b}}{E - \varepsilon_\mu} \tag{8.99}$$

the approximation is made of neglecting the energy variation of the various quantities involved; moreover, the dimension of the S matrix is taken as fixed, independent of energy, whereas the realistic situation is that a number of channels may be opening, over the averaging interval I used to define the average $\langle \mathbf{S} \rangle$. Since the form itself is approximate for the above reasons, Kerman and Sevgen (1976) argue that one need not demand exact analytic unitarity. They therefore propose an S matrix with the familiar expansion, which should be nearly unitary on the real-energy axis and which should be unitary on the average, i.e.,

$$\begin{aligned} &\mathbf{S}(E)\mathbf{S}^*(E) = \mathbf{1} - \mathbf{B}(E)\,, \\ &|B_{cc'}(E)| \ll 1\,, \\ &\langle \mathbf{B}(E) \rangle = 0\,. \end{aligned} \tag{8.100}$$

The freedom obtained by relaxing analytic unitarity allows a representation with no level-level correlations.

The matrix $\langle \mathbf{S} \rangle$ has three components

$$\begin{aligned} &\langle \mathbf{S} \rangle = \mathbf{S}^B + \mathbf{Z} - \mathbf{Y}\,, \\ &Z_{ab} = i\sum_\mu g_{\mu a} g_{\mu b} \frac{E - \varepsilon_\mu}{(E - \varepsilon_\mu)^2 + I^2}\,, \\ &Y_{ab} = I\sum_\mu \frac{g_{\mu a} g_{\mu b}}{(E - \varepsilon_\mu)^2 + I^2} = \frac{\pi}{D} \langle g_{\mu a} g_{\mu b} \rangle\,, \end{aligned} \tag{8.101}$$

and can also be written as

$$\langle \mathbf{S} \rangle = (\mathbf{1} - \mathbf{P})^{1/2} \mathbf{M}\,, \tag{8.102}$$

where P is the transmission matrix and **M** is unitary.

One can recover Moldauer's (1964a) results by putting

$$\mathbf{B}(E + iI) = 0, \quad \mathbf{Z} = 0 \tag{8.103}$$

and those of Englebrecht and Weidenmüller (1973) by

$$\mathbf{B}(E + iI) = 0, \quad \mathbf{S}^B (\mathbf{S}^B)^+ = \mathbf{1}\,. \tag{8.104}$$

On the other hand, the KKM formalism corresponds to neglecting level-level correlations and putting

$$\mathbf{Z} = \mathbf{Y} = 0 \tag{8.105}$$

so that

$$\begin{aligned} &\mathbf{B}(E + iI) = \mathbf{P}\,, \\ &\mathbf{S}^B = \langle \mathbf{S} \rangle = (\mathbf{1} - \mathbf{P})^{1/2} \mathbf{M}\,. \end{aligned} \tag{8.106}$$

It is argued that a decrease in the correlations of the parameters is achieved in this way.

It remains, of course, to be seen what the physical significance is of this violation of unitarity, which is the new ingredient of the theory. One might think of the analogy with energy being conserved only on the average in the canonical ensemble or with particle number in the grand canonical ensemble. However, in these cases this is not just a computational device but represents the interaction with the surrounding medium. In the scattering problem, one has indeed the interesting case of covering many thresholds with the incident beam, but when this is not the case it is not clear if any physical interpretation can be given to this flux nonconservation, however useful it is as a mathematical device.

In the statistical theories of nuclear reactions reviewed so far in this section, the S matrix is written in terms of *microscopic* quantities (like the poles and residues of the S or of the K matrix, or the matrix elements of the underlying Hamiltonian), for which a statistical law is *assumed* and used to calculate the fluctuation cross section $\sigma^{fl}_{ab} \sim [\,|S^{fl}_{ab}|^2]_e$. We have seen that, if the compound system has reached equilibrium, σ^{fl}_{ab} can be expressed entirely in terms of the *macroscopic* quantities $\bar{S}_{ab}$, i.e., the optical S matrix elements, as in the familiar Hauser-Feshbach theory, in which σ^{fl}_{ab} is calculated in terms of the transmission factors, computed from the optical model for each channel.

Thus, at least in the case of σ^{fl}_{ab}, the microscopic quantities play the role of a scaffolding, which is eliminated at the end in favor of the macroscopic quantities $\bar{S}_{ab}$. It is natural to ask whether the scaffolding can be eliminated from the very beginning, by proposing a trial statistical law directly for the S matrix elements, the input to the problem being the exact expectation values $\bar{S}_{ab}$ [see Mello (1979)]. This question will be considered in the next section.

J. A statistical theory of nuclear reactions based on a variational principle

In this section we shall construct an ensemble of S matrices such that the quantities that are physically relevant in the case of equilibrium, i.e., the optical S matrix elements, are introduced from the very beginning (Mello, 1979). The ensemble averages will then be expressed in terms of $\bar{\mathbf{S}}$.

A convenient measure (Krieger, 1967; Mello, 1979) for unitary symmetric matrices S, as given by Dyson (1962a), is $\mu(d\mathbf{S})$, defined uniquely by the property of remaining invariant under the transformation $\mathbf{S} \to \mathbf{U}\mathbf{S}\mathbf{U}^T$ (U being any unitary matrix), which in turn preserves the property of unitarity and symmetry. In a simple one-channel problem $[S = \exp(i\theta)]$, $\mu(dS) = d\theta$.

In the case of complete absorption, $\bar{\mathbf{S}} = 0$, we shall make the assumption that the frequency of occurrence of S in a subspace of the space of unitary symmetric matrices is proportional to Dyson's measure for that subspace. The results obtained in this limit for the average and variance (Mello, 1979; Mello and Seligman, 1980) of the elastic and inelastic cross sections agree with those obtained by HRTW, KKM, and Moldauer (1975a).

For the general case $\bar{\mathbf{S}} \neq 0$, one can propose a trial probability density for S, containing a suitably chosen number, k, of functions $f_i(\mathbf{S})$, and k parameters; the latter can be varied so as to give the "best" trial probability density, in the sense of a variational principle (Levine and Bernstein, 1976; Alhassid and Levine, 1978); essentially, one finds the best upper bound to the statistical entropy of the exact distribution, and when this is reached, the expectation values of the $f_i(\mathbf{S})$ associated with the exact and trial probability densities coincide. If we reach the *exact* entropy, then the trial and exact *distributions* would coincide. In our case the functions $f_i(\mathbf{S})$ are chosen as the real and imaginary parts of the various matrix elements S_{ab};

then we have $k=\frac{1}{2}n(n+1)$, and we fix the expectation value $\bar{\mathbf{S}}$, i.e., the optical S matrix. The trial distribution is then (Mello, 1979)

$$p(d\mathbf{S})=\frac{\exp(-\mathrm{Re\,Tr}\beta\mathbf{S})\,\mu(d\mathbf{S})}{\int \exp(-\mathrm{Re\,Tr}\beta\mathbf{S})\,\mu(d\mathbf{S})}\ , \tag{8.107}$$

where the matrix β has to be fixed so that $\bar{\mathbf{S}}$ has the required value.

When $\bar{\mathbf{S}}\neq 0$ but is small, one can expand the exponential in (8.107) in a power series and keep the lowest-order terms. We shall restrict the discussion to the case of $\bar{\mathbf{S}}$ diagonal and real, the most general S being obtainable by an Engelbrecht-Weidenmüller (1973) transformation. In this strong-absorption regime, $[|S_{ab}^{fl}|^2]_e$, for $a\neq b$, factorizes as $\xi_a\xi_b$; for $a=b$ we can define an elastic enhancement factor W_a as in Eq. (8.61). Then the ξ_a are determined entirely (Hofmann, Richert, Tepel, and Weidenmüller, 1975) by unitarity and the values of the W's. The latter can be written (Mello and Seligman, 1980) as $W_a=2+K|\bar{S}_{aa}|^2+\cdots$. The factors K from the present formalism and from HRTW are compared in Table V, in which n indicates the number of channels. The agreement is seen to be very reasonable.

The cross average $[S_{11}^{fl}S_{22}^{fl*}]_e$ is, in this regime, identically zero for KKM, whereas the present method and that of HRTW give a result proportional to $\bar{S}_{11}\bar{S}_{22}^{*}$; the factor of proportionality (Mello and Seligman, 1980) is compared in Table VI, which shows a very good agreement.

Away from the strong-absorption regime, a simple two-equivalent-channel case has been considered (Mello, 1979). The elastic enhancement starts with two for complete absorption, it increases as in HRTW, but then it goes back to two for weak absorption, in disagreement with the results of Moldauer (1975a) and Hofmann, Richert, Tepel, and Weidenmüller (HRTW, 1975), in which, despite the error bars in the Monte Carlo calculations, an increase in W from about two to about three can be seen, as we go from strong to weak absorption. There is, however, a restriction which we have so far not taken into account. From the analytic structure of S, taken to be ergodic (French *et al.*, 1978b), one can prove (Mello and Seligman, 1980) that the average of products of S matrix elements (involving only S but not S*) must coincide with the product of the averages of the various factors. One can prove that this condition is exactly fulfilled for Dyson's measure $\mu(d\mathbf{S})$, but is gradually spoiled as we go away from this extreme. However, it can be incorporated as a set of new restrictions by means of Lagrange multipliers. This has been done (de los Reyes *et al.*, 1980) for quadratic combinations of S matrix elements in the two-channel problem mentioned above, and the results indeed go in the right direction.

TABLE V. The factor K described in the text for the present approach (Mello and Seligman, 1980) and that of Hofmann, Richert, Tepel, and Weidenmüller (HRTW, 1975).

n	Variational method	HRTW
2	0.60	0.53
3	0.44	0.40
5	0.30	0.30
10	0.16	0.23

TABLE VI. The ratio $[S_{11}^{fl}S_{22}^{fl*}]_e/\bar{S}_{11}\bar{S}_{22}^{*}$ for the present approach (Mello and Seligman, 1980) and that of Hofmann, Richert, Tepel, and Weidenmüller (HRTW, 1975).

n	Variational method	HRTW
2	0.200	0.197
5	0.050	0.055
10	0.015	0.015

One can also calculate the variance of the cross section (Mello and Seligman, 1980) for any number n of channels in the region of strong absorption. If $n\gg 1$, $\mathrm{var}\sigma_{ab}\simeq(\bar{\sigma}_{ab})^2$ as in Ericson's theory (Ericson, 1963), meaning essentially that $\mathrm{Re}S_{ab}$ and $\mathrm{Im}S_{ab}$ become Gaussian for $n\gg 1$. For arbitrary n, corrections to this result are predicted.

We may summarize by saying that, starting from the statistical ansatz for $\mu(d\mathbf{S})$, and including successively known physical constraints, it is possible to obtain a satisfactory theory for the domain of strong absorption; the two-channel example gives indications of how this theory can be completed for the general case.

IX. SPECTRUM FLUCTUATIONS AND THE THERMODYNAMIC PROPERTIES OF SMALL METALLIC PARTICLES

A. Introduction

In our discussion of spectrum fluctuations so far, we have mostly considered applications to nuclei; but level fluctuations are also relevant when studying other systems which show a discrete spectrum. In this section we shall be interested in the thermodynamic properties of small particles for which the electronic level spectrum is discrete, with a mean level spacing D. In what follows we shall always assume that a free quasi-electron picture is as good an approximation for describing valence electrons in a small metallic particle as it is for describing valence electrons in the bulk metal. In other words, all the results quoted below should have as a natural limit the ones obtained for the free-electron picture of metals in bulk, when D tends to zero and the spectrum becomes quasicontinuous.

When D is of the order of other relevant energy parameters, such as the thermal energy kT or the quantum energy $\hbar\omega$ associated with an oscillatory mode of frequency ω, it is clear that small particles will exhibit an anomalous thermodynamic behavior; there are three different reasons for this (Kubo, 1962, 1969, 1977). Firstly, in such cases only a few electrons will be excited above the Fermi level, and, as a consequence, D will have a direct influence in determining the partition function Z and all properties derived from it. This is referred to as the "quantum size effect".

Secondly, for particles of characteristic radius a, when

$$D\simeq kT \tag{9.1}$$

and the temperature is low enough, there might not be enough energy to ionize the particle, so that the number of electrons remains fixed in each particle and the Fermi-Dirac distribution does not apply. In fact, the ionization energy is of the order of $e^2/2a \sim 10^{-1}$ eV for a ~60 Å; on the other hand, $D \sim 10^{-3}$ eV for particles of this size, so that charge fluctuations are strongly suppressed. This implies that we must deal with the canonical rather than with the grand canonical ensemble of statistical mechanics.

Finally, since the shape and size of the small particles cannot be controlled beyond certain limits, the discrete spectrum will show fluctuations, i.e., deviations from uniformity (equal spacings). This will also have an influence on the values of the partition function and therefore on the thermodynamic properties of the powder of small metallic particles. It is precisely at this point that contact is made with the discussion of previous sections.

The history of the quantum size effect dates back to four decades ago when Fröhlich (1937) calculated the specific heat at constant volume, C_v, assuming that the electronic spectrum in the small particles is an equally spaced one. He naturally obtained an exponential dependence of C_v on $1/T$ at low temperature.

Later on, Kubo (1962) argued that, due to the irregularities of the particle surface, any systematic property of the electron spectrum would have disappeared; as a consequence, the spectrum would be a completely random one with a Poisson spacing distribution. In such a case, a linear T dependence of C_v obtains in the limit of very low temperature.

A few years later, Gor'kov and Eliashberg (1965) pointed out that, since the electrons in the small particle have a wavelength of the order of atomic dimensions, surface irregularities of atomic size induce random interactions of the electrons with the walls which determine the quasiparticle levels. According to these authors the situation is then analogous to that encountered in determining the distribution of nuclear levels at high excitation, so that random-matrix theory should be applicable.

One might make this assumption reasonable in the following way (Mehta, 1967). The electronic energies are the eigenvalues of a fixed Hamiltonian, but with random boundary conditions which may be incorporated into a random matrix through the use of fictitious potentials. The appropriate ensemble of random matrices to be used depends on the particular conditions of the powder: (a) if the number of electrons is even[40] and there is no magnetic field, the GOE is applicable, since the system is time-reversal invariant; (b) when the number of electrons is odd and there is no magnetic field, the symplectic ensemble applies; and (c) when there is a magnetic field, the Hamiltonian is no longer invariant under time reversal and a suitable ensemble of random matrices should be of unitary type.

In any case, and independently of which of these random-matrix ensembles is used, there will be level repulsion. This affects the T dependence of C_v at low temperatures. For GOE C_v varies as T^2, for the symplectic ensemble a T^5 variation is obtained, and for the unitary case C_v is proportional to T^4. However, the important thing to notice is that the arguments of Kubo and of Gor'kov and Eliashberg are qualitative, no real justification for the relevance of any of the spacing distributions being given. We shall further discuss this matter in Sec. IX.D.

In what follows we shall indicate how the thermodynamic and electromagnetic properties of small metallic particles can be calculated; we then compare the theoretical predictions with experimental results, classifying the latter in such a way that they inform us about the existence of the quantum size effect, the charge neutrality and the appropriate spacing distribution to be used. We finally comment on some recent theoretical work concerning the appropriateness of random-matrix theory for the problem of small metallic particles.

B. Calculation of the properties of small metallic particles

The numerical calculation of both thermodynamic and electromagnetic properties of a powder of small metallic particles can be easily performed for low temperatures within the free-electron picture, once the discrete electron spectrum is known. The calculation is simple, since, due to the presence of the Boltzmann factor, only a few states around the Fermi level determine the partition function Z when (9.1) holds. From Z the thermodynamic properties follow; for example, for a powder containing N small particles the heat capacity C_v is given by

$$\frac{C_v}{k} = \frac{1}{(kT)^2} \sum_{\alpha=1}^{N} \frac{S_0^{(\alpha)} S_2^{(\alpha)} - (S_1^{(\alpha)})^2}{(S_0^{(\alpha)})^2} , \tag{9.2}$$

where

$$S_p^{(\alpha)} = \sum_{\lambda} g_\lambda^{(\alpha)} (E_\lambda^{(\alpha)})^p e^{-E_\lambda^{(\alpha)}/kT} . \tag{9.3}$$

Here $E_\lambda^{(\alpha)}$ and $g_\lambda^{(\alpha)}$ are the energy and the (spin) degeneracy of the level λ of the αth particle which contains $n^{(\alpha)}$ independent fermions; the canonical ensemble has been used.

In Fig. 21 we present the values of C_v as a function of $\theta = kT/D$ for three different cases (Blaisten *et al.*, 1977): when the electron spectrum is a completely random sequence of levels, when it is a GOE spectrum, and when it is equally-spaced; all three spectra have the same level density. The calculation is restricted to values of $\theta \leq 0.2$, since for larger values the detailed properties of the spectrum are smeared out; for $\theta \gtrsim 1$ the values of C_v no longer depend on the spectrum fluctuations, and for $\theta \gg 1$ the spectra appear to be quasi-continuous and the bulk values follow.

One can see from Fig. 21 that C_v is largest for the Poisson case and smallest for the picket-fence spectrum. This follows immediately from the action of the Boltzmann factor at low temperatures, since small spacings are more likely to appear for a completely random spectrum than for the GOE case, while there are no small spacings for the uniform system.

[40] The GOE does not apply when the number of electrons is odd, since the Hamiltonian is not invariant under rotations due to the surface irregularities (see Sec. II).

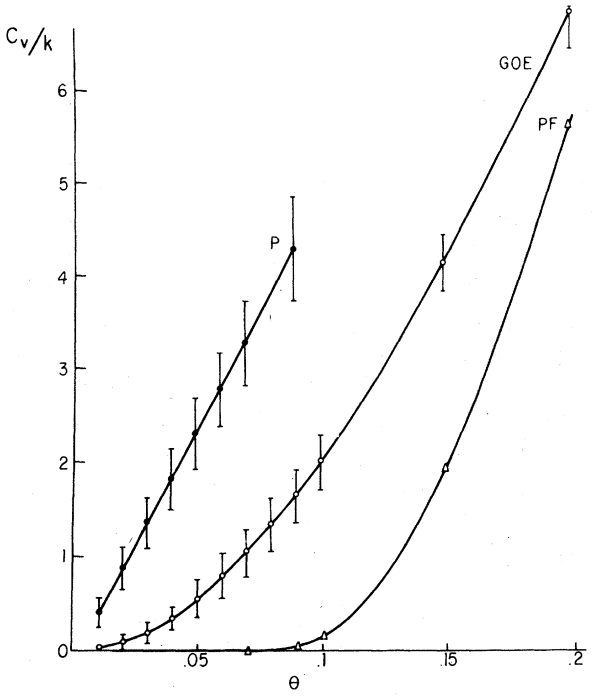


FIG. 21. Values of C_v/Nk versus $\theta = kT/D$ for a Poisson (P), a GOE, and a uniform (PF) spectrum. The values correspond to an average over an odd number and an even number of electrons, and the error bars are due to the finite sampling of 2000 spectra. Taken from Blaisten *et al.* (1977).

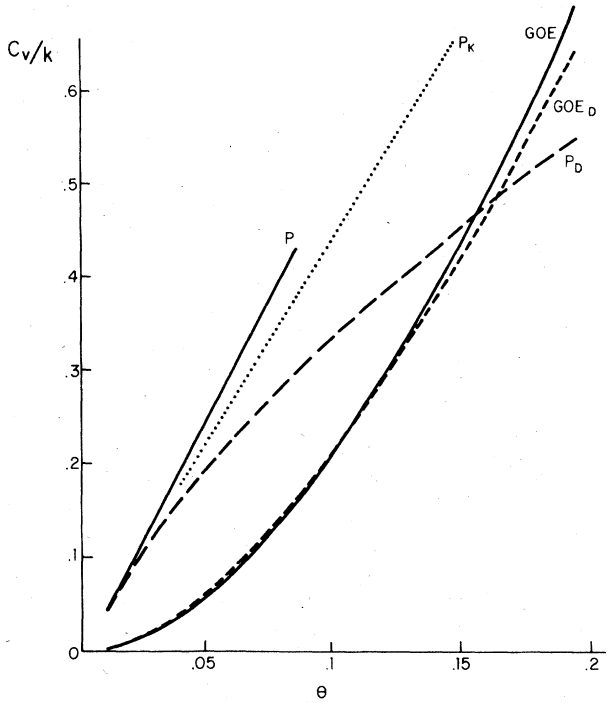


FIG. 22. Comparison of the C_v/Nk values of Fig. 21 with those calculated by Denton *et al.* (1971, 1973) indicated by broken lines (P_D, GOE_D) for two different types of spectra: Poisson and GOE. The result obtained by Kubo (1962) for the Poisson spectra (P_K) is also shown. Taken from Blaisten *et al.* (1977).

A widely used approximation to a calculation of the above type was first proposed by Denton *et al.* (1971, 1973). The authors first obtain an analytic expression for the partition function corresponding to an equally spaced spectrum. This represents a zero-order approximation. The influence of the level fluctuations is then taken into account by modifying the spectrum so that the first few electronic levels above the Fermi state belong to the spectrum of the appropriate ensemble.

It is clear therefore that the approximation of Denton *et al.* will be better the more uniform the spectrum is; it should provide better values for the GOE than for the Poisson case. This turns out to be true, as can be seen in Fig. 22, where the specific heat values obtained by Denton *et al.* are compared with the "exact" ones of Fig. 21.

The magnetic susceptibility χ_M was also obtained by Denton *et al.* within the same approximation. The values of χ_M for both the GOE and the Poisson spectra are given as functions of θ in Fig. 23. In the odd-electron case (for which, strictly speaking, GOE is not applicable) both types of spectra show the same Curie-law behavior[41] at low values of θ. For the even-electron case we find very different behaviors. When there is level repulsion, as with GOE, χ_M is proportional to θ for small θ, while with a random distribution which emphasizes small spacings, χ_M does not vanish, but instead increases as $\theta \to 0$.

With respect to the electric properties of small metallic particles, we focus our attention on the far-infrared absorption, for which both measurements and

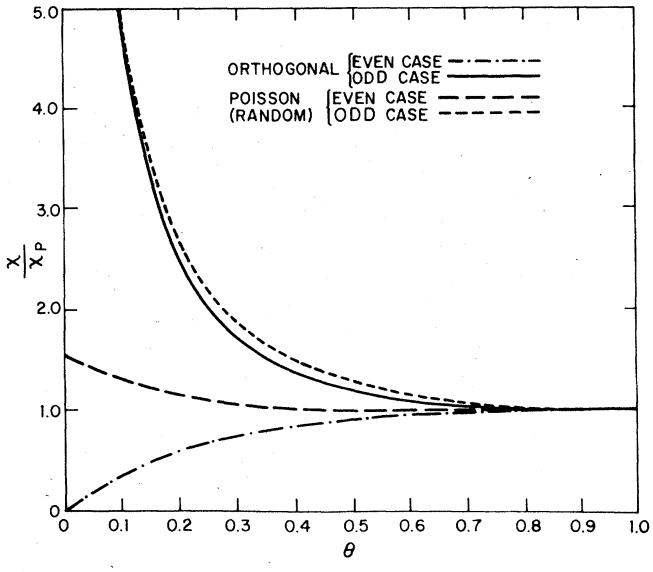


FIG. 23. Magnetic susceptibility χ, in units of the Pauli value χ_P, as a function of $\theta = kT/D$ for GOE and Poisson spectra. Taken from Denton *et al.* (1973).

[41] This comes about because of the unpaired electron and contrasts with what happens with the metal in bulk, where the T^{-1} behavior of χ_M is suppressed because only a fraction T/T_F of the electrons can be excited, this leading to Pauli susceptibility; here T_F is the Fermi temperature. See, for example, Kittel (1976).

interesting theoretical calculations have been performed. We comment on the theoretical calculation of Tanner *et al.* (1975), which revises the original one performed by Gor'kov and Eliashberg (1965), in which a large electric polarizability (as compared with the bulk value) was predicted for small metallic particles.

According to Tanner *et al.* (1975), the problem of finding the absorption coefficient for small particles divides into three parts, the first two being of essentially electrostatic nature: (1) to calculate the effective dielectric constant; (2) to find the relation between the internal and applied fields, including depolarization effects, originally not taken into account properly by Gor'kov and Eliashberg (see Strassler *et al.*, 1972); and (3) to calculate the linear response to the electric field of a system with discrete energy levels which obey a given spacing distribution. The resulting expression for the dielectric susceptibility χ_E is

$$\chi_E = \frac{1}{40\pi}\frac{e^2}{Da} + \frac{139}{1200\pi^2}\frac{e^2}{\hbar v_F}A(\omega)\,, \tag{9.4}$$

where the first term is the static one and the second depends on the frequency in a way determined through $A(\omega)$ by the spacing distribution assumed. In (9.4) a is the particle radius and v_F is the Fermi velocity. The resulting χ_E as a function of ω shows large variations when the simplectic ensemble is used and a much smoother variation when the GOE is assumed; no explicit value for $A(\omega)$ in the case of a Poisson spectrum is given.

C. Experimental evidence

It was not until five years after the appearance of Kubo's original paper that Taupin (1967) published the results of the first NMR and EPR measurements on small particles of lithium. Since then a whole set of these experiments measuring χ_M in small particles have been performed on aluminum (Kobayashi *et al.*, 1970, 1971, 1974; Fujita *et al.*, 1970; Granquist *et al.*, 1976), gold (Dupree *et al.*, 1967; Monot *et al.*, 1971), lithium (Charvolin *et al.*, 1966, 1967; Fujita *et al.*, 1977; Borel and Millet, 1977), silver (Monot *et al.*, 1974), tin (Kobayashi *et al.*, 1974), platinum (Marzke *et al.*, 1976; Gordon *et al.*, 1977), and copper (Kobayashi *et al.*, 1972; Yee and Knight, 1975; Kobayashi, 1977). Typical data are shown in Fig. 24 where χ_M is given for silver (Monot *et al.*, 1974). The appearance of the Curie-type law $\chi_M \sim T^{-1}$ for the temperature dependence of χ_M has been interpreted by most of the above-mentioned authors as conclusive evidence that the electronic spectrum is discrete in small metallic particles, i.e., that the quantum size effect exists.

There are some experimental results in apparent disagreement with the above conclusion. Thus in measuring the nuclear spin-lattice relaxation time in small superconductors in a magnetic field, Kobayashi *et al.* (1975) found the experimental results to be in agreement with the theory of Sone (1976) in which a continuous spectrum is assumed. A plausible explanation for this has been given by Simanek (1977); he shows that the electron energy levels are broadened because the small particles in actual samples tend to form clusters, so

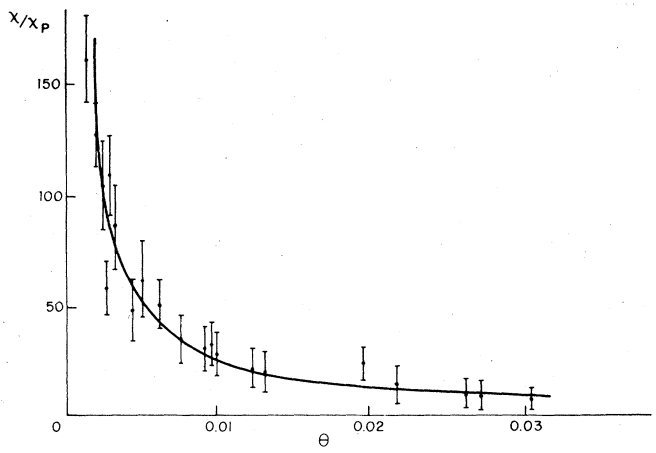


FIG. 24. Magnetic susceptibility χ, in units of the Pauli value χ_P, as a function of $\theta = kT/D$ for small particles of silver. Taken from Monot and Millet (1976).

that there is intergrain electron tunneling in small superconductors.

Another type of measurement not in agreement with the quantum size effect is that of the optical constants and their temperature dependence (Kreibig, 1974). Kawabata and Kubo (1966) discussed the plasma-resonance light absorption, which takes place via the plasma-oscillation modes of conduction electrons in the small particles which are coupled to the transverse electromagnetic field through the existence of the particle surface. In a quantum-mechanical treatment the surface produces the discreteness of the electron energy levels through the appropriate boundary conditions. In a classical description, on the other hand, the particle surface is regarded as scattering the conduction electrons, and so affecting their mean free path. The two ways of dealing with the problem lead to different size dependences of the optical plasma-resonance absorption and of its temperature variation. For silver particles (Kreibig, 1974) the classical theory seems to agree better, the predictions of Kawabata and Kubo (1966) being off by a factor of two. However, more recent measurements in gold particles by the same author (Kreibig, 1977) point in the direction of the quantum size effect. In summary, the weight of all the experimental evidence together supports the existence of this effect.

Consider next the postulate that we have a fixed number of electrons. The evidence comes mainly from comparing the magnetic susceptibility calculated using the canonical ensemble with that using the grand canonical ensemble. At very low values of θ, the values of χ_M obtained with the canonical ensemble are larger than those obtained assuming a variable number of electrons in each small particle. For an equally spaced spectrum the results are shown in Fig. 25, where they are compared with experimental values for platinum (Marzke *et al.*, 1976). It is clear that the canonical ensemble is favored for this spectrum (and similarly for other spectra).

Finally, the appropriate spacing distribution to be used is more difficult to establish. If one considers the magnetic susceptibility χ_M as a function of θ, he

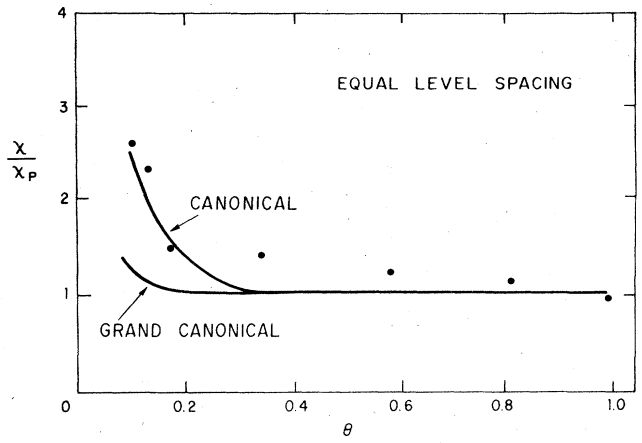


FIG. 25. Difference between the magnetic susceptibility calculated using a canonical and a grand canonical ensemble. In both cases an equally spaced spectrum was used. $\theta = kT/D$. The dots correspond to experimental values for small particles of platinum. Reprinted with permission from Marzke, *et al.* (1976), Solid State Commun. **18**, 1025 (Pergamon, New York).

sees that the various level distributions give approximately the same order of agreement; for example, for platinum, the Poisson distribution fits best at large values of θ and the random-matrix ensembles give a somewhat better agreement at low values of θ, as can be seen from Fig. 26. However, when the differing susceptibilities of particles having an even or odd number of atoms have been deduced, as for copper by Yee and Knight (1975), it has been observed that particles with an even number of electrons "tend" to have a zero spin susceptibility; one might conclude then that the symplectic or the orthogonal ensembles are more realistic than the Poisson distribution. However, the precision of the current measurements and the existence of a particle-size distribution in the sample does not at present really permit a distinction to be made between the different possibilities for the spacing distributions.

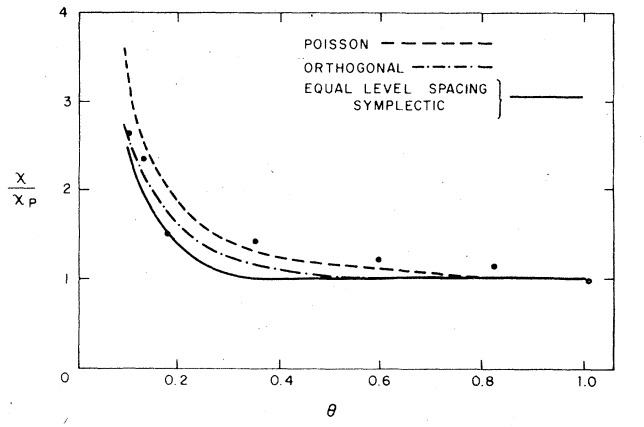
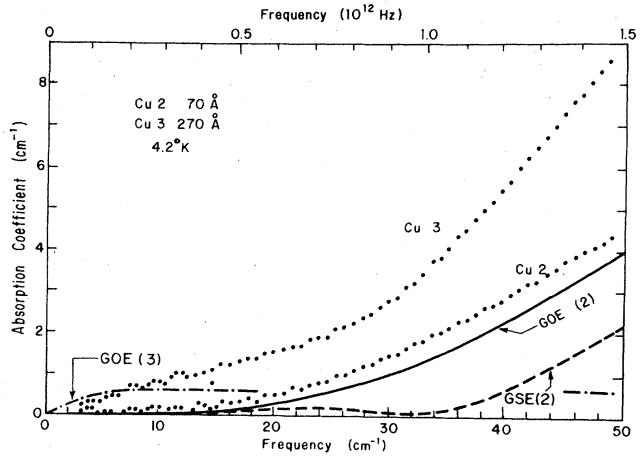


FIG. 26. Magnetic susceptibility as a function of $\theta = kT/D$ for Poisson, GOE, and equally spaced spectra. The experimental points, indicated by dots, correspond to small particles of platinum. Reprinted with permission from Marzke *et al.* (1976), Solid State Commun. **18**, 1025 (Pergamon, New York).

Besides the difficulties appearing in the experiments with χ_M, there are further ones when dealing with the electric polarizability. According to the theory of Gor'kov and Eliashberg (1965), the electric polarizability should be greatly enhanced over the bulk value for very small metallic particles obeying one of the spacing distributions predicted by random-matrix theory. This enhancement has not been found for silver (Dupree and Smithard, 1972), for gold (Meier and Wyder, 1972), or for aluminum (Granqvist *et al.*, 1976, 1977). Tanner *et al.* (1975), who performed the first measurements for the far-infrared absorption in small metallic particles (that is, absorption at frequencies comparable with the mean energy spacing), suggest that the modified theory of Gor'kov and Eliashberg leading to (9.4) gives a good description of the data. However, this is too optimistic, as can be seen from Fig. 27 which gives results for Cu and Al. Neither the GOE nor the symplectic ensemble provides a satisfactory explanation of the data. Unfortunately, the corresponding absorption coefficient versus frequency curves are not given for the Poisson case.

An alternative explanation for these results comes



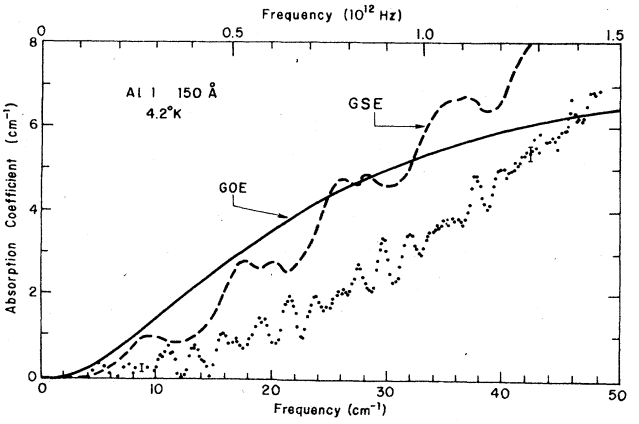


FIG. 27. Far-infrared absorption coefficient for Cu and Al as a function of frequency. Two different samples, of diameters ~70 Å and 270 Å, are used for Cu. Predicted values for GOE and for the symplectic ensemble GSE are also shown. Taken from Tanner *et al.* (1975).

from including the electron-electron interaction. This was done for the static case by Lushnikov and Simonov (1973), and for $\omega \neq 0$ by Monsivais and Flores (unpublished). The enhancement of the electric polarizability predicted by Gor'kov and Eliashberg (1965) no longer appears, and a better agreement with the experiments of Tanner *et al.* (1975) is obtained.

As a résumé of the actual experimental situation one could say the following: The effects of discrete spectra seem to be well established; it also seems that the description with a canonical ensemble, appropriate to systems with a fixed number of electrons, is the more realistic; but distinguishing among the different spacing distributions is extremely difficult, so none is definitely established. These conclusions are well justified when no superconductive transition is involved. Dealing with superconductive small particles seems to be rather complicated, no good theory existing up to now. This is unfortunate, because for superconducting vanadium particles the specific heat has been measured (Comsa *et al.*, 1976), as well as the magnetic susceptibility for several materials (Novotny and Meincke, 1973; Matsuo *et al.*, 1974). From these measurements the transition temperature T_c, from the superconductive to the normal state, can be obtained for particles of different sizes. On the other hand, by explicitly considering the discrete nature of the electron spectrum in the equations appropriate for strong-coupling superconductors (Eliashberg, 1960), an expression can be obtained for T_c as a function of D and for spectra with different fluctuation properties (Barojas *et al.*, 1980). Comparison of these results with the experimental ones for small particles of lead show that a completely random spectrum is appropriate, while an equally spaced one is not. For weak-coupling superconductors, such as Al, BCS theory should be applicable for not too small particles. Then it is simple to obtain T_c for discrete spectra with different fluctuation properties. Again, a Poisson spectrum fits the Al data better than a GOE or equally spaced spectrum (Barojas *et al.*, 1979).

D. Recent theoretical considerations

Up to now we have briefly described the early theoretical considerations, and have given a short account of the relevant experimental results, which, however, do not make clear the nature of the single-electron spectra. We now turn to a more detailed consideration of that.

In what follows we shall first give evidence that the problem cannot be formulated *in a straightforward way* in terms of matrix ensembles. We present first a qualitative argument and then reinforce it by analyzing a one-dimensional example (Barojas *et al.*, 1977a). These arguments show that the problem of perturbing the boundary, when Dirichlet conditions are imposed, leads to infinite matrix elements. We then turn to the recent formulation of Tavel *et al.* (1979; see also Tavel, 1978). At the end of this section we shall present numerical results which are in disagreement with each other, showing that a clearcut solution to this problem has not yet been obtained.

Let us assume a free-electron model. Then the problem of perturbing the boundary shape can be formulated in the following way. Take a region R' in which the solution to Schrödinger's equation

$$(\nabla^2 + k_n^2)\phi_n = 0 \tag{9.5}$$

is known, with ϕ_n obeying given boundary conditions on S', the boundary of R'. We then perturb S' to obtain a new region R with boundary S. The problem is to determine new eigenvalues κ_n^2 and new eigenfunctions ψ_m which satisfy the same boundary conditions as ϕ_n, but now on the surface S.

For small metallic particles, as we already mentioned, the number of electrons remains fixed; that is, the small particle behaves as if it had impenetrable walls at low temperatures; therefore Dirichlet conditions should be appropriate and we can see qualitatively that difficulties arise in casting the problem of boundary perturbations in matrix form using the unperturbed wave functions ϕ_n as a basis. They are similar to the problems encountered in many-body theory when dealing with a hard-core potential using an uncorrelated basis.

Indeed, consider the equation

$$(\nabla^2 - V + \kappa_m^2)\psi_m = 0\,, \tag{9.6}$$

with

$$V(\mathbf{r}) = V_0, \quad \mathbf{r} \in R', \ \mathbf{r} \notin R$$
$$= 0, \quad \mathbf{r} \in R\,, \tag{9.7}$$

and such that

$$\psi_m(\mathbf{r}) = 0, \quad \mathbf{r} \in S\,. \tag{9.8}$$

It is clear that when V_0 increases, ψ_m will decrease exponentially in the region between S and S' and that ψ_m will become zero on S only in the limit $V_0 \to \infty$. Since ϕ_n is not necessarily zero on S, the matrix elements of V with respect to the unperturbed basis diverge.

This argument does not apply for Neumann boundary conditions. Our problem can indeed be formulated in matrix language in this case, using ϕ_n as a basis. This, together with the analysis of convergence, is done, for example, in the book by Morse and Feshbach (1953).

In the one-dimensional case, when R' is the interval $\{0,1\}$ and R is the interval $\{0, 1-\delta\}$, we obtain, for Dirichlet boundary conditions, the following expression for the Hamiltonian matrix elements,

$$\langle \phi_r | H | \phi_s \rangle = 4(1-\delta)^{-1} \sin[\pi r(1-\delta)] \sin[\pi s(1-\delta)] \sum_{m=1}^{\infty} m^4 [m^2 - (1-\delta)^2 r^2]^{-1} [m^2 - (1-\delta)^2 s^2]^{-1}\,, \tag{9.9}$$

and, for the case of Neumann boundary conditions,

$$\langle \phi_r | H | \phi_s \rangle = 4(1-\delta) rs \sin[\pi r(1-\delta)] \sin[\pi s(1-\delta)] \sum_{m=1}^{\infty} m^2 [m^2 - (1-\delta)^2 r^2]^{-1} [m^2 - (1-\delta)^2 s^2]^{-1}\,, \tag{9.10}$$

where, for convenience, 1 and $1-\delta$ are assumed to be incommensurable. In Eq. (9.9) each term in the series approaches a constant and the series does not converge, whereas in Eq. (9.10) no problems arise, since each term of the series goes as $1/m^2$ in the large-m limit.

We see then that the problem of boundary perturbations with Dirichlet conditions cannot be formulated as a straightforward matrix problem, leading us to suspect that a random-matrix description of the small metallic particle might be questionable. In the method of Tavel *et al.* (1979) an equivalence is established between the eigenvalue problem for a simpler operator (i.e., the kinetic energy) in a complicated domain (i.e., the perturbed region R bounded by S) and that of a complicated operator in the simple domain R'. The complicated operator can be expressed in terms of the simple one plus a sum of terms of first and higher orders in the surface deformation parameter.

These authors assume the existence of a mapping $\mathbf{r} \to \mathbf{r}' = \mathbf{r}'(\mathbf{r})$ which is one-to-one, has an inverse, and maps R into R' in a continuous fashion. The Jacobian

$$J = \left| \frac{\partial \mathbf{r}'_i}{\partial \mathbf{r}_j} \right|$$

characterizes the mapping. The (Hermitian) transformed operator is then

$$\hat{H}(\mathbf{r}') = J^{-1/2}(\mathbf{r}')H[\mathbf{r}(\mathbf{r}')]J^{1/2}(\mathbf{r}'), \tag{9.11}$$

which has the same eigenvalues κ_m^2 in the simple domain R' as the original (simple) Hamiltonian H has in the complicated domain R. If one now expands $\mathbf{r}'(\mathbf{r})$ as a power series in the "deformation parameter" λ, (9.11) can be used to obtain a perturbation series in λ. For example, the first order correction to the eigenvalues is

$$(k_n^{(1)})^2 = \int_{R'} \phi_n^* \Delta V \phi_n d\tau, \tag{9.12}$$

where

$$\Delta V = \left(\frac{\partial H[\mathbf{r}'(\mathbf{r})]}{\partial \lambda} \right)_{\lambda=0} + \tfrac{1}{2}[H(\mathbf{r}), \Delta J], \tag{9.13}$$

ΔJ being equal to $(\partial J/\partial\lambda)_{\lambda=0}$. All this can be worked out explicitly once the function $\mathbf{r}'(\mathbf{r})$ is known, as we shall indicate towards the end of this section.

We turn now to the results of several numerical calculations, in which the nearest-neighbor spacing distribution of the perturbed spectrum is obtained, Consider first an assembly of small platelets (i.e., two-dimensional systems) in which platelet δ is a rectangle of sides $(1-\delta)$ and 1, δ being a random variable that generates the assembly (Barojas *et al.*, 1977b). The exact perturbed eigenvalues are then

$$E_{n_1,n_2} = \frac{\hbar^2\pi^2}{2m_e}\left(\frac{n_1^2}{(1-\delta)^2} + n_2^2 \right), \tag{9.14}$$

where m_e is the electron mass and n_1, n_2 are non-negative integers. Let us now consider a single platelet, that is, a fixed δ. The resulting spectral spacing distribution approaches closely a Poisson distribution, the value of the repulsion parameter being $\omega = 0.023 \pm 0.035$. We then vary δ at random, and consider the spacing between the Fermi level corresponding to $n^{(6)}$ electrons, and the next level[42]; we again obtain a Poisson distribution, the value of ω now being 0.003 ± 0.05. These results indicate that the spacing distributions for this case are ergodic, but different from those predicted by random-matrix theory; in particular, perturbing the boundary does not induce level repulsion in the spectrum.

A completely different result is obtained by Tavel *et al.* (1979), who use Eq. (9.11) and assume a slightly deformed spherical region, so that

$$r = r_0\left(1 + \lambda \sum_{lm} [a_{lm}(Y_{lm} + Y_{lm}^*) + ib_{lm}(Y_{lm} - Y_{lm}^*)]\right). \tag{9.15}$$

Here r_0 is the average radius and λ is the parameter that adjusts the height of the deformation; the real coefficients a_{lm} and b_{lm} are varied at random to generate the assembly of deformed surfaces. When the calculation is restricted to a single degenerate L state,[43] and when all a_{lm} and b_{lm} have the same random distribution, the perturbed eigenvalues (up to first order in λ) are such that a nearest-neighbor distribution of the Wigner type is obtained. This corresponds to what the authors call the surface of maximum roughness. When coefficients with larger values of l are less and less important (i.e., the surface is smoother) the distribution is no longer of the Wigner form, nor is it Poisson; but in this case the distributions are no longer ergodic.

Finally, we compute the spacing distributions from "realistic" calculations of electronic energy levels in small clusters of atoms. The electronic spectrum has been obtained in two cases for the purpose of studying the catalytic properties of small clusters. The first example refers to the electronic structure of a face-centered truncated octahedron, which has six square faces and eight hexagonal ones, all having equal edges. The eigenergies of d electrons, in the framework of the tight-binding approximation, have been obtained by Cyrot-Lackmann (1977). Using these eigenvalues, one can again obtain the spacing distribution, and a Poisson law follows. The second example emerges from the study of the complex $TiCl_4C_2H_7$ (catalyst + C_2H_4) in the context of a Zieglar-Natta–type reaction (Novaro *et al.*, 1978). The *ab initio* all-electron calculation was done within a linear-combination-of-atomic-orbitals–self-consistent-field (LCAO-SCF) approximation with a minimal basis set. Again the spacing distribution is of the Poisson type.

It is clear from all of these calculations that the appropriate theoretical spacing distribution has not been established. It might even be that the nature of the statistical behavior in complicated single-particle systems is so different from that generated in many-particles systems that the usual random-matrix considerations are not appropriate. In that case it could turn out, for example, that the recent studies [Balian and

[42] In the numerical calculation the Fermi level of the particle containing $n^{(6)}$ electrons was determined, assuming a constant electron density.

[43] Assuming rather small particles ($r_0 \sim 16$ Å), the authors estimate that the splitting of the $(2L+1)$-degenerate levels is smaller than the separation between the L levels in the perfectly spherical surface. This might not be the case when r_0 is larger, as frequently occurs in actual experiments.

Bloch (1974) and earlier papers referred to therein] of the Helmholtz equation in a cavity ("Weyl's problem") might lead to theoretical results which can more confidently be compared with those of the beautiful experiments which we have referred to and others which are now promised or in progress.

X. ERGODIC PROPERTIES OF RANDOM-MATRIX ENSEMBLES

A. Introduction

In the preceding sections we have repeatedly mentioned that an ensemble should have certain ergodic properties. We shall here examine what these properties are and how far they apply to the random-matrix ensembles we have discussed. Ergodic problems in conventional statistical mechanics have been extensively studied [see, for example, Khinchin (1949); Münster (1959)]. A clear exposition of the basic concepts will be found in Balescu (1975), and Lebowitz and Penrose (1973). For ergodicity in random processes, which is much more relevant for us, see, for example, Yaglom (1962) or Cramér and Leadbetter (1967).

Consider a quantity $f(E,\xi)$, defined for H^ξ, a member[44] of the ensemble, and dependent on the details of (i.e., a functional of) the spectrum in the neighborhood of E, say, within the *measuring interval* $(E\pm\frac{1}{2}\delta E)$; for example, with $\Delta_3(\bar{n})$ we have that $\delta E=\bar{n}D(E)$. f has a spectral average over an *averaging interval* ΔE at E

$$\langle f(E,\xi)\rangle_p=\frac{1}{\Delta E}\int_{E-\frac{1}{2}(\Delta E)}^{E+\frac{1}{2}(\Delta E)} f(E',\xi)dE'$$
$$=\frac{1}{p}\int_{-p/2}^{p/2} f(E+rD,\xi)dr\,, \tag{10.1}$$

$$\langle f(E,\xi)\rangle_p=\frac{1}{p}\sum_{E_i\in\Delta E} f(E_i,\xi)\,, \tag{10.2}$$

where the integral applies to a function defined for all points on the energy axis, the sum for one which exists only at the levels E_i of the H^ξ matrix. The subscript p gives essentially the number of levels in ΔE; the precise definition will depend on the kind of averaging used and the nature of the measure. In the last form of (10.1) we assume where necessary that the spectrum has been unfolded.[45] The function f also possesses an ensemble average given by

$$\bar{f}(E)=\int f(E,\xi)d\Phi(\xi)=\int f(E,\xi)P(H^\xi)\prod_{i\le j}dH_{ij}^\xi\,, \tag{10.3}$$

where $\Phi(\xi)$ is the distribution function for the ensemble; the corresponding density is $P(H)$, given for GOE by (2.7).

We could take the averaging interval ΔE to span the entire spectrum; in that case we would have $p=d$. Ergodicity could then be defined as the equality of the two averages

$$\langle f(\xi)\rangle=\bar{f}(E) \tag{10.4}$$

in the $d\to\infty$ limit (which enters here just as does the infinite-time limit in statistical mechanics). Now (10.4) could not hold unless each side is equal to a constant, f_0, say, since they have different and independent arguments. Hence (10.4) separates into two conditions:

$$\langle f(\xi)\rangle=f_0=\bar{f}(E)\,, \tag{10.5}$$

the second of which defines stationarity, while the first implies that the chosen H^ξ is a "characteristic" member of the ensemble. Note that $f_0=\langle\bar{f}\rangle$. It should be clear that Eqs. (10.4) and (10.5) could hold *almost always* at best, since there will be exceptional members in the ensemble and exceptional regions in the spectra.

However, averaging over the complete spectrum is not appropriate for our purposes, nor is it necessary. Only small segments of the spectrum are available experimentally so that we must really study the results of local instead of global spectral averaging. The definition of ergodicity, Eq. (10.4), is now given in the limit $p\to\infty$, following $d\to\infty$ (so that $p/d\to 0$). In practice we would have $d\gg p\gg\delta E/D$. In most cases of interest, though not with proton resonances, the *averaging interval* ΔE will be small enough that, even without unfolding, the density can be taken as constant.

Our main ergodicity requirement is the first equality of (10.5), which we restate as follows:

$$\mathrm{var}_{(\xi)}\langle f(E,\xi)\rangle_p\equiv\overline{\langle f(E,\xi)\rangle_p^2}-[\overline{\langle f(E,\xi)\rangle_p}]^2\xrightarrow[p\to\infty]{}0\,, \tag{10.6}$$

where $\mathrm{var}_{(\xi)}$ denotes the ensemble variance.[46] We may speak of the ergodicity obtained via (10.6) as "locally generated;" this only stresses the fact that the averaging interval can be a negligible fraction of the spectrum span. On the other hand, the experimental data are available in the form of finite samples, and it becomes then imperative to know not only that the variance vanishes in the limit but also its magnitude as a function of p.

As for the second equality of (10.5), viz., stationarity over the averaging interval, we stress that this is included in the definition of ergodicity merely to facilitate an easy interpretation of the physically important quantity f_0, to which the spectral average converges in the limit. In more general terms we may omit it and proceed with (10.6) to verify ergodicity. Except for "p-de-

[44] We have explicitly used the label ξ to emphasize that the quantities are defined for a particular member of the ensemble. We shall often drop the label for brevity as we have done in all the previous sections. Triangular brackets will denote spectral averaging. $D(E)$ will denote the ensemble-averaged local spacing at E, which, as discussed in the next section, we can replace by a spectral-averaged local spacing. A spectral average is implied in the definition of some measures, as in the spacing variances. This is not so, for example, with $\Delta_3(\bar{n})$ when $\bar{n}$ covers the total observed spectrum; on the other hand, for much smaller $\bar{n}$ an experimental evaluation of the spectral average (10.1) may be carried out. This freedom, in fact, may be used to define sharper measures, as we shall see in Sec. X.C.

[45] Alternatively, one could, for local averaging, introduce the notation of a "tangent ensemble."

[46] In the integral of Eq. (10.1) it is to be understood that the measuring interval δE moves continuously over the averaging interval. It is sometimes convenient to approximate the integral as a sum over a number, $\Delta E/\delta E$, of contiguous nonoverlapping domains. The conditions given below for ergodic behavior apply equally well in both cases. The approximate procedure (which really involves a new definition of spectral averaging) gives a variance (10.6) larger than the integral method, but the difference is negligible when the number of intervals is large. The case is similar for Eq. (10.2).

pendent" quantities, an example of which is given toward the end of the next section, we can assume what we may refer to as "local" stationarity; this is trivially achieved by local unfolding. Note carefully that the local stationarity does not at all require or imply the global stationarity discussed in Secs. I.C and IV.A.

We introduce the autocovariance function[47] for f

$$S^f(E_1,E_2)=\overline{f(E_1)f(E_2)}-\bar{f}(E_1)\bar{f}(E_2)\,. \tag{10.7}$$

As we have discussed above, we can usually take S^f to be locally stationary, and therefore write it as a function of the relative coordinate r only. In the first case (10.1) we have $r=(E_2-E_1)/D$, and in the second case (10.2) r is the number of levels in (E_1,E_2) with one of the ends included. $S^f(r)$ is an even function of r and its absolute value is bounded by $S^f(0)$. We have now, for (10.1) and (10.2), respectively,

$$\begin{aligned}\mathrm{var}_{(e)}\langle f(E)\rangle_p&=\frac{1}{p^2}\int\int_{-p/2}^{p/2}S^f(E+r_1D,E+r_2D)dr_1dr_2\\&=\frac{2}{p^2}\int_0^p(p-r)S^f(r)dr\,,\end{aligned} \tag{10.8}$$

$$\begin{aligned}\mathrm{var}_{(e)}\langle f(E)\rangle_p&=\frac{1}{p^2}\sum_{E_i,E_j\in\Delta E}S^f(E_i,E_j)\\&=\frac{2}{p^2}\sum_{r=1}^{p}(p-r)S^f(r)+\frac{S^f(0)}{p}\,,\end{aligned} \tag{10.9}$$

where we have integrated (or summed) over the center-of-mass variable. Since S^f determines the binary fluctuations of f (i.e., fixes all its two-point measures), we see that the fluctuations of f and its ergodic behavior are intimately related.

We can now use (10.8) and (10.9), either for a direct evaluation of the variance or for inferring the asymptotic form of the variance from that of S^f. Since we already know S^f for the one-point functions, we shall, in Sec. X.B, study their ergodicity by the first procedure. In other cases the explicit evaluation of the variance could be a formidable problem, and for these we would use the second procedure.

A widely used result, giving a sufficient condition for the vanishing of the variance, is Slutsky's theorem (Slutsky, 1938; see also Yaglom, 1962),

$$\frac{1}{p}\int_0^p S^f(r)dr\xrightarrow[p\to\infty]{}0\,, \tag{10.10}$$

$$\frac{1}{p}\sum_{r=1}^{p-1}S^f(r)\xrightarrow[p\to\infty]{}0\,. \tag{10.11}$$

A variant of the theorem, which is adequate for our purpose, is that the variance vanishes if $S^f(r)$ vanishes in the limit of large $|r|$, or is asymptotically a sinusoidal function of r. The theorems are valid for finite $S^f(0)$. It will turn out in some cases that $S^f(r)$ has delta-function singularities at $r=0$; in these cases we shall evaluate the integral over the singular part separately.

We may also encounter functions which are anomalous in that the variance given by (10.6) vanishes for all p. This is really an indication of an improper normalization of f. For example, if $f(E)$ has a proper ergodicity (i.e., an ergodicity generated by taking p large enough), then $d^{-1}f(E)$ is such an anomalous function and its ergodicity is of no interest.

When the autocovariance function vanishes for all $r\neq 0$ (as it does for transition strengths, discussed in Sec. X.D below), the spectral average becomes a Gaussian random variable for large enough p, this by an application of the simple CLT. We have already referred, in Appendix N to a theorem (Diananda, 1953) which leads to the same result when the autocovariance function vanishes for all $r>r_0$, a fixed number; we have relied on a plausible extension of this to argue that the kth-order spacing becomes Gaussian for large enough k. The same argument leads us to expect that most of the ergodic measures should become Gaussian under spectral averaging for large enough p. Then, for large but finite p, the variance properly measures the statistical error.[48] Unfortunately, the variance has not been calculated for most of the spectral-averaged two-point fluctuation measures so that the final comparisons in Sec. VI between theory and experiment are rather vague.

In the next section we demonstrate ergodicity for one-point measures, and in Sec. X.C for the two-point and higher-order correlation functions which leads to ergodic behavior for more complicated measures. It will turn out that these functions, when calculated for the unfolded spectra, are globally stationary, in the sense of Eq. (10.5). As we have seen, this very strong property is not necessary for ergodic behavior, but it goes a long way toward explaining the experimentally observed stationarity (Sec. I.C). In Sec. X.D we consider ergodicity and stationarity for transition strengths, and in Sec. X.E for an S-matrix ensemble.

B. Level density and other one-point measures

Ergodic behavior of the level density has been discussed by several authors. Grenander (1963) proved that, for ensembles whose matrix elements have distributions symmetric about zero, with identical variances and uniformly bounded moments of all finite order, the moments of the level density for almost all ensemble members go over, in the large-d limit, to those of Wigner's semicircle (3.17), in the sense that their variances about these values tend to zero. Grenander's results, extended to the other canonical ensembles, follow from our Eq. (4.8) above. Somewhat stronger results, with fewer restrictions on the distributions of the matrix elements, were later obtained by Arnold (1967) and Olson and Uppuluri (1972). By means of binary-association expansions, described in Sec. III and IV, Mon and French (1975) were able to derive ergodicity also for the EGOE. No explicit spectral averaging appears in these calculations, but there

[47]The term autocorrelation function, commonly used, is best restricted to the autocovariance function of the standardized function, $f/(\mathrm{var}_e f)^{1/2}$, as Bartlett (1966) suggests, since this is dimensionless and within the range -1 to 1.

[48]When the distribution of $\langle f\rangle_p$ is known, we can calculate confidence limits—say, the probability that an observed value lies in the range $\bar{f}\pm\alpha(\mathrm{var}\langle f\rangle_p)^{1/2}$. It is in this sense that we refer to $(\mathrm{var}\langle f\rangle_p)^{1/2}$ as the statistical error of the estimation procedure. For nonzero $\bar{f}$ we shall also refer to the "relative error" $(\mathrm{var}\langle f\rangle_p)^{1/2}/\bar{f}$, whose square is the "figure of merit" as used by Dyson and Mehta (1963).

is an implicit one in that results valid for moments of order $\ll d$ are extended to all orders, this generating a spectral smoothing which plays the same role.

Let us now return to the methods of the first section. We are interested in Eq. (10.1) with $f=\rho$. Since

$$d\times\int_{E-(1/2)\Delta E}^{E+(1/2)\Delta E}\rho(E')dE'$$

counts the number of levels in ΔE, its variance is $\Sigma^2(p)$, already given formally by Eq. (5.2) and explicitly for the standard ensembles by Eqs. (5.12) and (5.13). Then (Pandey, 1979)

$$\mathrm{var}_{(e)}\langle\rho(E)\rangle_p=\frac{1}{d^2(\Delta E)^2}\Sigma^2(p)=[\bar{\rho}(E)]^2\Sigma^2(p)/p^2\,, \quad (10.12)$$

which vanishes as $p^{-2}\ln p$ for the Gaussian ensembles, and as p^{-1} for the Poisson case [for which $\Sigma^2(p)=p$], yielding an ergodic behavior of the level density in all cases. Observe that, because of the level repulsion, the convergence is faster for the Gaussian cases. In deriving ergodicity we have used here the direct evaluation of the relevant variance instead of the general argument of Sec. X.A. See Appendix Q for the alternative derivation.

Ergodicity can similarly be shown for the number statistic, $n(E)$, the number of levels in an interval of fixed length $\bar{n}D$ at E. It is found (Pandey, 1979) that

$$\mathrm{var}_{(e)}\langle n(E)\rangle_p=\bar{n}^2\Sigma^2(p)/p^2 \quad (10.13)$$

as long as spectral averaging is done with $p/\bar{n}$ contiguous nonoverlapping domains. Note that the figure of merit for $\langle n(E)\rangle$ is independent of $\bar{n}$, and therefore identical with that for $\langle\rho(E)\rangle$ itself, as we see from (10.12) and (10.13), or could have predicted from the fact that n is an additive statistic. By the same procedure (Pandey, 1979; Bohigas and Giannoni, unpublished) the figure of merit for the spectral-averaged kth-order spacing[49] ($k<p$) is $\sigma^2(p-1)/p^2$, which via Eq. (5.3) differs only trivially from that for $\langle\rho\rangle$ and $\langle n\rangle$ for the Gaussian ensembles (and not at all for Poisson); the resultant ergodicity would follow also from Eq. (5.8).

The practical use of the one-point measures lies in estimating D. For example, D can be estimated in the obvious way (from the spectral-averaged spacings) with a fractional error $p^{-1}\sigma(p-1)$, which, for $p=100$, is 10% for Poisson, but only 1.2%, 0.9%, and 0.7% for the $\beta=1,2,4$ ensembles. D can be estimated in other ways, also—for example, as the spacing parameter for the "best" uniform spectrum, whose derivation is the essential feature in evaluating Δ_3 or Δ^*. Thus from (5.18) the estimator for D turns out to be

$$A_{\min}(E;p)=\frac{p\sum_{i=1}^{p}iE_i-\sum_{i=1}^{p}i\sum_{i=1}^{p}E_i}{p\sum_{i=1}^{p}i^2-(\sum_{i=1}^{p}i)^2}$$
$$=\frac{12}{p(p^2-1)}\sum_{i=1}^{p}\left(i-\frac{p+1}{2}\right)E_i\,, \quad (10.14)$$

where the E_i are the energy levels (p of them) in ΔE. We have easily that $\bar{A}_{\min}=D$ and

$$\mathrm{var}_{(e)}A_{\min}=\frac{72D^2}{p^2(p^2-1)^2}\sum_{i,j}\left(i-\frac{p+1}{2}\right)\left(\frac{p+1}{2}-j\right)\sigma^2(|i-j|-1)$$
$$=\frac{12D^2}{p^2(p^2-1)^2}\sum_{k=1}^{p-1}(-p^3+3p^2k-2k^3+p-k)\sigma^2(k-1)\xrightarrow[\text{large }p]{}\frac{9D^2}{\beta\pi^2p^2}\,, \quad (10.15)$$

the last form being valid for the Gaussian ensembles. Its variance being $\sim p^{-2}$ instead of $p^{-2}\ln p$, this estimator is asymptotically better than the simpler ones above. Its variance indeed is quite close to that ($8D^2/\pi^2p^2$) of the optimal estimator for GOE given by Dyson and Mehta (1963), involving a spectral averaging with a "semicircular" weighting which also pays little attention to the ends of the spectrum. For $p=100$, however, the improvement over the simpler estimator is negligible, 1.0% instead of 1.2% for GOE.

Finally, we stress that not all one-point measures are ergodic. For example, the levels in the fixed interval ΔE define moments

$$M_\nu(E)=\frac{1}{p}\sum_{E_i\in\Delta E}E_i^\nu=\frac{d}{p}\int_{E-(1/2)\Delta E}^{E+(1/2)\Delta E}E'^\nu\rho(E')dE'=\frac{1}{p}\int_{-p/2}^{p/2}(E+rD)^\nu\frac{\rho(E+rD)}{\bar{\rho}(E+rD)}dr \quad (10.16)$$

whose variances, for $\nu\geq 1$, diverge for asymptotic p. Thus for $\nu=1$, $\bar{M}_1(E)=E$ and from Eqs. (4.4), (5.2), and (10.16)

$$\mathrm{var}_{(e)}M_1(E)=\frac{1}{p^2}\int\int_{-p/2}^{p/2}(E+r_1D)(E+r_2D)[\delta(r_1-r_2)-Y_2(r_1-r_2)]dr_1dr_2$$
$$=\frac{2}{p^2}\int_0^p(p-r)\left(E^2+\frac{p^2D^2}{12}-\frac{prD^2}{6}-\frac{r^2D^2}{6}\right)[\delta(r)-Y_2(r)]dr$$
$$=D^2\left(\frac{1}{p^2}\int_0^p r\Sigma^2(r)dr-\frac{\Sigma^2(p)}{4}\right)+E^2\frac{\Sigma^2(p)}{p^2}\,, \quad (10.17)$$

where the second step follows after integrating out the center-of-mass variable, and the third step after two partial integrations; the first term in the last form is of order $\ln p$ for the Gaussian ensembles and is of order p for the Poisson case—similarly for the higher moments. These nonergodic results are not surprising, however, since the levels remain highly correlated, Eq. (4.30), even for large separations.

C. Two-point and higher-order functions

We know that a k-point fluctuation measure can be expressed in terms of the k-point and lower-order correlation functions defined in (4.1). As can be seen from (10.7), the corresponding autocovariance function in-

[49] We have $(p+1)$ levels in this case and the spectral averaging is over $p/(k+1)$ contiguous nonoverlapping kth spacings.

volves the $2k$-point function. The one-point measures could be dealt with as in the preceding section because the properties of the two-point function are well understood; not so for the higher-order functions. It is only recently (Pandey, 1979) that the ergodicity of a general k-point function has been established; from this follows the ergodicity of the fluctuation measures of Sec. V.

Following (4.1) we define

$$T_k \equiv \left(\prod_{i=1}^{k} \bar{\rho}(x_i)\right)^{-1} S_k = \overline{\prod_{i=1}^{k} [\rho(x_i)/\bar{\rho}(x_i)]}, \tag{10.18}$$

which are the k-point functions for unfolded spectra. Just as in (4.4), T_k has delta-function singularities due to self-correlation of the levels, having a product of at most $(k-1)$ delta functions in any term; T_1 is, of course, unity. By ignoring all the self-correlation terms, we get *k-level* correlation functions[50] R_k, which are simply the joint-probability densities for k levels. Dyson's (1962c) k-level cluster functions Y_k are obtained by subtracting out lower-order correlation effects from R_k. Formally,

$$Y_k(x_1,x_2,\ldots,x_k) = \sum_G (-1)^{k-m}(m-1)! \times \prod_{j=1}^{m} R_{G_j}(x_t \text{ with } t \text{ in } G_j), \tag{10.19}$$

where G stands for any division of the indices $(1,2,\ldots,k)$ into unordered subsets $(G_1,G_2,\ldots,G_m)$. Thus $Y_1 = 1$ and Y_2 is given by (4.4). The inverse of (10.19) is

$$R_k(x_1,x_2,\ldots,x_k) = \sum_G (-1)^{k-m} \times \prod_{j=1}^{m} Y_{G_j}(x_t \text{ with } t \text{ in } G_j). \tag{10.20}$$

Now, let us take the k points (x_i) to be defined in a small segment of the spectrum. We write $x_i = x + r_i D(x)$ where x is the centroid, so that $\sum_{i=1}^{k} r_i = 0$. The functions Y_k are well defined and finite everywhere, being functions of the r_i and x. Moreover, the Y_k (and hence the T_k and the R_k) are globally stationary if they are independent of x, being a function then of the relative coordinates $r_{ij} \equiv (r_i - r_j)$ only. The Poisson ensemble, which does not display any level repulsion, is by construction stationary for its cluster functions; in fact, $Y_k = 0$ for all $k > 1$. Dyson (1970) has evaluated the Y_k for the circular ensembles, and Mehta (1971) for the Gaussian ones, in each case of arbitrary dimensionality; in the large-d limit the Gaussian ensembles give Y_k which are identical at the center of the semicircle with those for the circular ensembles.

The circular ensembles are, of course, stationary by construction. They are inadequate, however, for dealing with the physical problem of global stationarity discussed, for example in Sec. I.C, for which the significant question is whether such stationarity obtains even when there is a large secular variation in the density. It becomes then of consequence to know whether the Gaussian ensembles[51] give stationary results. This question has been dealt with by Pandey [(1979); see also Mehta, as reported by Dyson (1972b)], who has shown that, in the large-d limit, global stationarity does, in fact, obtain. See Appendix R for the Y_k forms, and Sec. IV for Y_2. From the stationarity of the Y_k follows that of all fluctuation measures, except those, such as the moment variances in (10.17), in which a secular energy dependence is built into the definition.

Let us turn now to ergodicity. Consider the spectral average (10.1) with $f = s_k$, as defined in (4.1); the dummy variable is the center-of-mass coordinate, x, in s_k. For this we have the autocovariance function

$$S_k^o(r) \equiv \overline{s_k(x; r_1,\ldots,r_k) s_k(x+rD; r_1,\ldots,r_k)} - \bar{s}_k(x; r_1,\ldots,r_k)\bar{s}_k(x+rD; r_1,\ldots,r_k), \tag{10.21}$$

in which the singularities, due to self-correlation of the levels, result from a product of at most $(2k-1)$ delta functions. However, as we shall see below in an example, the "observable" quantities are not s_k, but at least a $(k-1)$-fold integral over s_k. This leaves us with only one delta function involving r in S_k^o, which contributes a p^{-1} order term to the variance in (10.8). On the other hand, since the $(k>1)$ cluster functions vanish whenever one of the relative coordinates r_{ij} is increased indefinitely, the continuous part of the S_k^o, which is bounded, vanishes for all k in the limit of large r. In fact (Pandey, 1979),

$$S_k^o(r) \xrightarrow[\text{larger}]{} O(r^{-\alpha}), \tag{10.22}$$

where $\alpha = 2$ for the orthogonal and unitary ensembles and $\alpha = 1$ for the symplectic one. These considerations, as worked out more explicitly in Appendix R for $k = 1$, yield the ergodic behavior of any k-point function.

Let us now briefly consider bounds for the errors involved in replacing one kind of average by the other in the standard (ergodic) fluctuation measures. Suppose that for a quantity f our sample gives $\hat{p}$ contiguous nonoverlapping measuring intervals, e.g., $\hat{p} = p/\bar{n}$ for $\Sigma^2(\bar{n})$. Then it is plausible that

$$\text{var}_{(e)}\langle f\rangle_{\hat{p}} \leq \frac{\text{var}_{(e)} f}{\hat{p}}, \tag{10.23}$$

where the spectral averaging is done over the $\hat{p}$ nonoverlapping intervals.[52] The upper bound follows by ignoring the correlations (assumed to be negative) between the values of f in distinct measuring intervals; since (10.23) is an equality for the Poisson case, we shall refer to the bound as the Poisson estimate. The inequality can be made plausible (though a rigorous proof is lacking except for the one-point measures) either by considering the asymptotic behavior of S_k^o, Eq. (10.22), as done by Pandey (1979), or by using the

[50] For a spectral realization of these functions see below.

[51] Ideally, of course, we should make use of an embedded ensemble, but, since that is not tractable, we must continue to rely on the assumption that embedding is equivalent to a mapping which does not affect the fluctuation behavior. In view of the apparent nonstationary behavior of the ω-statistic, discussed in Sec. V.H, there is room for caution here.

[52] We have already agreed that the averaging over overlapping domains results in a smaller $\text{var}_{(e)}\langle f\rangle_{\hat{p}}$.

general notion that the spectral rigidity follows from negative correlations. Thus for the number statistic the last form of (10.13) is smaller than the Poisson estimate $\bar{n}\Sigma^2(\bar{n})/p$, by a factor $\bar{n}/p$ which in practical cases can be quite small.

The number variance Σ^2 is calculated from a given spectrum as $\langle(n-\langle n\rangle)^2\rangle$. The Poisson estimate for the mean-square error in this case is $(\bar{n}/p)\,\mathrm{var}_{(e)}[(n-\langle n\rangle)^2]$, which, with a Gaussian assumption for n, gives the figure of merit $(2\bar{n}/p)^{1/2}$. Similarly for $\sigma^2(k)$ we find a Poisson figure of merit $[2(k+1)/p]^{1/2}$.

The ensemble variances for the Δ measures, and for Q, F, have been given by Dyson and Mehta, as in Sec. V. To take account of a further averaging over a set of measuring intervals, the same Poisson formula (10.23) can be applied. For Δ_3, in particular, it is worth remarking that, whereas the standard experimental analysis makes use of the entire available spectrum as a measuring interval [thereby giving $\Delta_3(p)$], an alternative procedure would use a smaller measuring interval followed by an average over the intervals. The Poisson estimate for $\langle\Delta_3(\bar{n})\rangle$ gives then a figure of merit which can be minimized with respect to $\bar{n}$, for example, in GOE by solving $\Delta_3(\bar{n})=2/\pi^2$, giving $\bar{n}\sim 8$, independent of p. The general use of this kind of thing would be in seeking the best possible measures for comparing experiment with theory. Whether Poisson estimation is sufficiently accurate and whether the optimal Δ_3 is better than other measures are presently unknown.

No Poisson or other estimate is known for the ω and Λ statistics, but a numerical study (Mello *et al.*, 1976) for ω favors ergodic behavior, and indicates that ω has a Gaussian ensemble distribution with variance close to p^{-1}.

Since most of the standard fluctuation measures follow from the two-point function, one might also ask if the function itself can be estimated from the data and, if so, with what accuracy. In fact, this has been done by Ideno and Ohkubo (1971, p. 620; see also Ideno, 1974) in order to search for "non-random distributions of neutron-resonance levels." While we do not yet understand the full implications of their results,[53] and therefore do not discuss them here, we do consider the essential parts of their procedure, extending it to the general k-point case.

We start with a level at E_i and consider small intervals of length εD around $(k-1)$ points at distances $r_j D$, $j=1,2,\ldots,k-1$, from E_i. The quantity of interest, say, $\hat{R}_k(E_i; r_1, r_2, \ldots, r_{k-1})$, is the product of the number of levels in each interval. We calculate $\hat{R}_k$ for each level in the sample and define the spectral average[54]

$$\begin{aligned}\langle\hat{R}_k(E; r_1, r_2, \ldots, r_{k-1})\rangle_p &= \frac{1}{p}\sum_{E_i \text{ in } \Delta E}\hat{R}_k(E_i; r_1, r_2, \ldots, r_{k-1}) \\ &= \frac{d^{k-1}}{p}\sum_{E_i \text{ in } \Delta E}\prod_{j=1}^{k-1}\int_{(r_j-\varepsilon/2)D}^{(r_j+\varepsilon/2)D}\rho(E_i+y_j)\,dy_j \\ &= \frac{1}{p[\bar{\rho}(E)]^k}\int_{-p/2}^{p/2}\rho(E+rD)\,dr\prod_{j=1}^{k-1}\int_{r_j-\varepsilon/2}^{r_j+\varepsilon/2}\rho(E+rD+r'_jD)\,dr'_j\,, \end{aligned} \tag{10.24}$$

where we shall take $|r_j|>\varepsilon$ and $|r_j-r_i|>\varepsilon$ for all i,j. We shall also take ε to be small enough so that the last form in (10.25) below is a good approximation, but at the same time large enough so that the figure of merit, resulting from (10.26) below, is within a reasonable limit. Typically, $\varepsilon=0.1$. We have then

$$\begin{aligned}\overline{\langle\hat{R}_k(E; r_1, r_2, \ldots, r_{k-1})\rangle_p} &= \int_{r_1-\varepsilon/2}^{r_1+\varepsilon/2}\cdots\int_{r_{k-1}-\varepsilon/2}^{r_{k-1}+\varepsilon/2}R_k(E; r'_1, r'_2, \ldots, r'_{k-1})\,dr'_1\cdots dr'_{k-1} \\ &\xrightarrow[\text{small }\varepsilon]{} \varepsilon^{k-1}R_k(E; r_1, r_2, \ldots, r_{k-1})\,, \end{aligned} \tag{10.25}$$

so that $\varepsilon^{-k+1}\langle\hat{R}_k\rangle$ gives a spectral realization of R_k. It is easy to see that, for evaluating $\mathrm{var}_{(e)}\langle\hat{R}_k\rangle$ to lowest order in ε, we must use the term with the maximum number of delta functions (k of them in this case, with the above restrictions on the r_i) in the autocovariance function of $\hat{R}_k$. We find then that

$$\mathrm{var}_{(e)}\langle\hat{R}_k(E)\rangle_p \to p^{-1}\varepsilon^{k-1}R_k(E; r_1, r_2, \ldots, r_{k-1})\,, \tag{10.26}$$

with the obvious ergodic property. The relative error in the estimation procedure is $(p\varepsilon^{k-1}R_k)^{-1/2}$, which, for a given p, increases as k increases (because of its ε dependence), making plausible the proposition of Sec. V that a higher-than-two-point measure is not observable with any reasonable accuracy from the data.

The ergodic properties derived here for the three Gaussian ensembles are for energy levels with the same set of exact quantum numbers. Following Pandey (1979) we show below that the ergodic properties extend to the mixed spectra also. For a superposition of l independent ensembles, the cluster functions are

$$Y_k(E; r_1, r_2, \ldots, r_k) = \sum_{i=1}^{l}(f_i)^k Y_{k,i}(E; f_i r_1, f_i r_2, \ldots, f_i r_k)\,, \tag{10.27}$$

where the $Y_{k,i}$ are the functions for the ith-component ensemble, contributing on the average a fraction f_i of the energy levels to the mixed spectrum ($\sum_{i=1}^{l} f_i = 1$). Thus (5.14) follows from (5.2) and (10.27) with $k=2$. The nonsingular part of the autocovariance function, Eq. (10.21), is a linear combination of the corresponding functions of the component ensembles. Thus, if all the component ensembles are ergodic for the k-

[53] They find stronger long-range correlations, resulting from periodic behavior in the spectrum. Their papers were brought to the authors' attention very recently by M. L. Mehta.

[54] Note that, because of trivial end effects, we cannot make use of the complete averaging interval [so that we should really write Eq. (10.24) for $\hat{p}<p$ levels], nor can we permit any of the r_i to increase indefinitely.

point functions, the superposed ensemble is also ergodic.

It is clear that the convergence (as $p \to \infty$) to ergodic results will in general be slower than the individual convergences. In fact, with $f_i \sim l^{-1}$ we have from (10.27) for $k > 1$ that $Y_k \to 0$ as $l \to \infty$, establishing thereby the Poisson ensemble as a limiting ensemble. Our Eq. (10.23) can be regarded as yet another demonstration of this well-known result.

D. Ergodic behavior for the GOE strength distribution

The Porter-Thomas distribution (7.21) in its finite (or asymptotic) form is an exact (or asymptotically exact) ensemble-averaged result. The question is whether or not the ensemble average equals or approaches the often measurable spectral average.

Ergodicity for eigenvector components was first considered by Brody and Mello (1971; see also Mello and Brody, 1972), who exploited a mathematical analogy between this problem and that of the ideal gas whose ergodic properties, connected with fluctuations about the Maxwell-Boltzmann law, had been already derived by Khinchin (1949). We shall proceed here quite differently, in order to be able to deal with more general aspects of strength ergodicity (French and Pandey, to be published).

Suppose, for example,[55] that the quantity of interest reduces to $x_{i\lambda}^2$ whose *ensemble* average, by (7.6), is d^{-1}. But since $\sum_i x_{i\lambda}^2 = 1$, we see that its *spectral* average is also d^{-1}. We have, moreover, from (7.19) above, that, in the GOE treatment of the (external) Porter-Thomas distribution, a change of basis reduces the strength to the $x_{i\lambda}^2$ form; thus for that function we have a strict ergodicity, the two averages being exactly equal for every value of d. From the standpoint of the strength distributions this merely tells us that the centroid can be calculated either as an ensemble or spectral average.

This strict equality does not, of course, hold for higher moments. Consider, for example, $x_{i\lambda}^4$. Its ensemble average by (7.6) is $3/[d(d+2)]$. But if $\hat{\psi}_\lambda$ is an H eigenfunction (being then identical with some Ψ_j), the spectral average is d^{-1} [one state carrying unit "strength" and $(d-1)$ states carrying zero], which is the *largest* possible value of $x_{i\lambda}^4$. If instead the $\hat{\psi}_\lambda$ in question were a uniform sum of all the H eigenvectors, we would find that $x_{i\lambda}^2$ would be d^{-1} for every i, so that the spectral average of $x_{i\lambda}^4$ would be d^{-2}, which is its *smallest* possible value; the ensemble average must, of course, lie between the two limits. Each of these states is quite "special," but the second, in which the strength is uniformly distributed, should be much less so than the first, in which it is completely concentrated. This is borne out by the fact that, for large d, the spectral value in the first case "misses" (is larger than) the ensemble value, $3/d^2$, by a factor $d/3$, while in the second it misses (is smaller than) the ensemble value only by a factor of 3.

We consider, as a characteristic case, adequate for discussing the Porter-Thomas distribution, that the quantity of interest is a function, say, $\Omega(x_{i\lambda})$, of only a single component. We do not restrict ourselves to averaging over the complete spectrum (d levels), but over a subset p of them,[56] as we have done for other quantities. Let us label the subset by $i = 1, 2, \ldots, p$, not necessarily implying any energy ordering by this labeling. The spectral average is now

$$\langle \Omega^\xi \rangle_p = p^{-1} \sum_{i=1}^{p} \Omega^\xi(x_{i\lambda}) . \tag{10.28}$$

The superscript ξ, which we usually do not write, reminds us that the value specifically depends on the H^ξ matrix. The question now is whether $\langle \Omega \rangle_p$ for most members of the ensemble approaches the ensemble-averaged value $\overline{\Omega}$.

We know that the components are stationary, i.e., the ensemble distribution of $x_{i\lambda}$, and therefore $\overline{\Omega}(x_{i\lambda})$, do not depend on i (or λ). Thus $\langle \overline{\Omega} \rangle_p = \overline{\Omega}$ and the only condition that remains to be satisfied for ergodicity is the equivalent of (10.6), namely, that

$$\begin{aligned} \mathrm{var}_{(e)} \langle \Omega \rangle_p &= p^{-1} \mathrm{var}_{(e)} \Omega(x_{i\lambda}) \\ &\quad + (1 - p^{-1}) \mathrm{covar}_{(e)} [\Omega(x_{i\lambda}), \Omega(x_{j\lambda})] \\ &\xrightarrow[p \to \infty]{} 0 , \end{aligned} \tag{10.29}$$

where covar denotes the covariance of the two random variables in the argument; here the first step follows from (10.9) and the fact that the variance is the same for all i, as is the covariance for all pairs $i \neq j$.

Clearly (10.29) is satisfied if and only if the covariance appearing in it vanishes. If $\mathrm{var}_{(e)}\Omega$ also vanishes the (p-independent) ergodicity thus generated would be trivial, as discussed in Sec. X.A. Let us limit ourselves to the case where Ω is a polynomial function. We know from Eq. (7.13) that $\mathrm{var}_{(e)}(x_{i\lambda}^\mu) \sim d^{-\mu}$, while $\mathrm{covar}_{(e)}(x_{i\lambda}^\mu, x_{j\lambda}^\nu)$ vanishes identically unless μ, ν are both even, in which case it is of order $d^{(-1/2)(\mu+\nu)-1}$. To demonstrate nontrivial ergodicity we now renormalize the variables by taking

$$y_{i\lambda} = d^{1/2} x_{i\lambda} \tag{10.30}$$

and deal with $\Omega(y_{i\lambda})$, where $\Omega(y)$ is independent of d. Specifically, we find, with $\varepsilon_\mu = 0, 1$, according as μ is odd or even, that[57]

$$\mathrm{covar}_{(e)}(\langle y_{i\lambda}^\mu \rangle, \langle y_{i\lambda}^\nu \rangle) = \frac{\varepsilon_{\mu+\nu}}{p} [(\mu+\nu-1)!!] - \frac{\varepsilon_\mu \varepsilon_\nu}{2d} \mu\nu(\mu-1)!!(\nu-1)!! + O\left(\frac{1}{pd}, \frac{1}{d^2}\right), \tag{10.31}$$

which, term by term, corresponds to (10.29). Since the second term vanishes as $d \to \infty$, we find that $\mathrm{var}_{(e)} \langle \Omega(y_{i\lambda}) \rangle_p \sim p^{-1}$, so that we have an ergodic behavior. This will follow similarly for polynomials in several components of

[55] The notation is the same as in Sec. VII, i indicating the eigenvector, while λ indicates a basis vector in the statistical space; both i and λ run from 1 to d. In the absence of any analytic results for the EGOE we continue to deal with the GOE. We suspect, however, that once again GOE results are relevant for EGOE.

[56] Alternatively, we can fix i and average over λ. The results would be the same.

[57] For $\mu = \nu$ a slightly simpler form (asymptotically p^{-1}) emerges if we renormalize each moment by its standard derivation.

the same or different vectors.

In Sec. X.B we remarked that ergodicity of the spectral moments leads to that of a spectrally smoothed density function, the need for smoothing arising from the discrete nature of the Hamiltonian spectrum. The same behavior is found with the $y_{i\lambda}$ distribution. For a specified H we have the spectral density function for the renormalized amplitude $y_{i\lambda}$

$$g(y)=p^{-1}\sum_{i=1}^{p}\delta(y-y_{i\lambda})\,. \tag{10.32}$$

Since the ensemble average for $\delta(y-y_{i\lambda})$ is $\rho_G(y)$, a Gaussian density of zero centroid and unit variance, so also $\bar{g}(y)=\rho_G(y)$. For the two-point function it follows then—for example, from (10.31)—that

$$\mathrm{covar}_{(e)}[g(y_1),g(y_2)]=p^{-1}[\delta(y_1-y_2)\rho_G(y_1)-\rho_G(y_1)\rho_G(y_2)]+O(d^{-1})\,, \tag{10.33}$$

which, for integrated versions of $g(y)$—say, a histogram—yields the expected p^{-1}-order variance. The case is similar for strengths, i.e., the $y^2_{i\lambda}$ distribution.

In the data analysis one usually resorts to the maximum-likelihood procedure and, in order to test the Porter-Thomas distribution, estimates the number of degrees of freedom, ν_p, from a sample of size p. It is a property of such estimators as ν_p that, for large p, they become Gaussian, with a variance which vanishes in the limit; for the general result see Kendall and Stuart (1967) and for the specific ν_p result Porter and Thomas (1956). We could have predicted this from (10.33), also.

Ergodicity extends also to the internal case discussed in Sec. VII. We have $\Omega(T_{ij})$, a polynomial function of the transition amplitude T_{ij}. We consider the two-GOE case, for which the states $|i\rangle$ and $|j\rangle$ belong to two disjoint statistical spaces; the single GOE case is equally easy to deal with and gives essentially the same results. Let us agree that the T_{ij} have been renormalized so that $\mathrm{var}_{(e)}\Omega$ is a nonzero finite number in the limit of large dimensionalities. Our sample has p_1 and p_2 states in the two spaces; this implies a double spectral averaging

$$\langle\Omega(T_{ij})\rangle_{p_1,p_2}=(p_1p_2)^{-1}\sum_{i=1}^{p_1}\sum_{j=1}^{p_2}\Omega(T_{ij})\,, \tag{10.34}$$

whose ensemble average is simply $\bar{\Omega}(T_{ij})$ and whose variance is given by

$$\begin{aligned}\mathrm{var}_{(e)}\langle\Omega(T_{ij})\rangle_{p_1,p_2}=&\frac{\mathrm{var}_{(e)}\Omega(T_{ij})}{p_1p_2}+\frac{(p_1-1)}{p_1p_2}\mathrm{covar}_{(e)}[\Omega(T_{ij}),\Omega(T_{ij'})]\\&+\frac{(p_2-1)}{p_1p_2}\mathrm{covar}_{(e)}[\Omega(T_{ij}),\Omega(T_{i'j})]+\left(1-\frac{p_1+p_2-1}{p_1p_2}\right)\mathrm{covar}_{(e)}[\Omega(T_{ij}),\Omega(T_{i'j'})]\,,\end{aligned} \tag{10.35}$$

in which $i\neq i'$ and $j\neq j'$. It follows from Sec. VII.D that in the limit of large dimensionalities the fourth term on the right-hand side of (10.35) vanishes. We have then the ergodicity of Ω in the double limit $(p_1,p_2)\to\infty$. As in (10.33) the ergodicity of the T_{ij} distribution follows in turn from this.

E. Ergodicity of the *S*-matrix ensembles

As we saw in Sec. VIII, the statistical theory of nuclear reactions commonly makes use of an ensemble of S matrices; this implies the appearance of an ergodic problem, which has so far been treated along two different lines. Richert and Weidenmüller (1977) succeeded in reducing the ergodicity of quantities such as $S_{ab}(E)S^*_{cd}(E)$, in the notation of Sec. VIII, to that of the widths and energy levels in the underlying ensemble of Hamiltonians; the ergodicity of these quantities they took for granted. We report here the work of French *et al.* (1978b).

We do not propose to examine exhaustively the ergodic properties of the quantities involved in each of the models described in Sec. VIII; indeed, in such cases as the uncorrelated model used for Eq. (8.20) this would be pointless. We shall concentrate on the formalism of Agassi *et al.* [(1975); see section VIII.F], which is valid in the limit $\Gamma\gg D$.

The energy levels and widths are obtained by these authors from the eigenvalues and eigenvectors of matrices belonging to the GOE; for these, as we have seen, we essentially have stationarity. The K matrices of (8.55) and the corresponding S matrices (8.53) are then "statistical" matrices in the sense discussed in connection with Eqs. (8.6) and (8.27); they constitute ensembles which have locally the same behavior as the "real" S matrix or K matrix over the interval $\Delta E=pD$.

We can now apply the results of Sec. X.A to the elements of the S matrix considered as functions of the energy. The appropriate autocovariance functions,[58] *viz.*, $[S^{fl}_{ab}(E)S^{fl*}_{ab}(E+\varepsilon)]_e$, is already evaluated in Eqs. (8.72) and (8.73); the function is finite for $\varepsilon=0$ and vanishes for asymptotic ε, establishing thereby the ergodic behavior of $S_{ab}(E)$. Similar conclusions could be drawn already for the simple model of (8.85) under the assumptions stated after that equation, as is shown in (8.89); but it is not evident that the conclusions remain valid when the simplifying assumptions are abandoned, so that the explicit derivation leading to Eqs. (8.72) and (8.73) is essential. The method of Sec. X.A also allows us to obtain the ensemble variance from (10.8);

[58] We use the complex conjugate of $S^{fl}_{ab}(E+\varepsilon)$ as the natural extension of the definition (10.7) to complex quantities [see, for example, Prohorov and Rozanov (1969), p. 119]; note that using S^{fl} rather than S itself avoids the need to subtract the second term in (10.7). Note also that the autocovariance functions defined here, as in Sec. VIII, and in Eq. (10.38) below, are complex conjugates of the $C(\varepsilon)$ functions in the paper referred to.

we find that

$$\mathrm{var}_{(e)}\langle S_{ab}\rangle_{\Delta E}=\frac{\pi\Gamma^{\mathrm{corr}}}{\Delta E}|S_{ab}^{fl}|^2 . \tag{10.36}$$

This result can be extended to products of matrix elements. For instance, let

$$G=S_{ab}(E)S_{cd}^{*}(E) . \tag{10.37}$$

Then an argument along similar lines yields for the autocovariance function of G,

$$S^{G}(\varepsilon)=\frac{i\Gamma^{\mathrm{corr}}}{\varepsilon+i\Gamma^{\mathrm{corr}}}|\bar{S}_{ab}|^2\frac{P_{cc}P_{dd}+P_{cd}P_{dc}}{\mathrm{Tr}P}-\frac{i\Gamma^{\mathrm{corr}}}{\varepsilon-i\Gamma^{\mathrm{corr}}}|\bar{S}_{cd}|^2\frac{P_{aa}P_{bb}+P_{ab}P_{ba}}{\mathrm{Tr}P}+\frac{(\Gamma^{\mathrm{corr}})^2}{\varepsilon^2+(\Gamma^{\mathrm{corr}})^2}\frac{(P_{aa}P_{bb}+P_{ab}P_{ba})(P_{cc}P_{dd}+P_{cd}P_{dc})}{(\mathrm{Tr}P)^2} . \tag{10.38}$$

Again, this is finite for $\varepsilon=0$ and goes to zero when $\varepsilon\to\infty$, satisfying the criterion for ergodicity. As above, we derive the variance of G and find

$$\mathrm{var}_{(e)}\langle G\rangle_{\Delta E}=\frac{\pi\Gamma^{\mathrm{corr}}}{\Delta E}\overline{|G-\bar{G}|^2} , \tag{10.39}$$

in complete analogy to (10.36). This particular result has a further interest in that G becomes the cross section σ_{ab} for the process $a\to b$, when $a=c, b=d$; we have then

$$\mathrm{var}_{(e)}\langle\sigma_{ab}\rangle_{\Delta E}=\frac{\pi\Gamma^{\mathrm{corr}}}{\Delta E}\mathrm{var}_{(e)}\sigma_{ab} , \tag{10.40}$$

so that for such cross sections ergodicity holds; the variance of σ_{ab} itself in the ensemble is, of course, finite in view of the physical meaning of this quantity. Ergodicity of higher powers of S_{ab} follows easily from the fact (Agassi *et al.*, 1975) that, in the limit of sufficiently high channel number, S is a normally distributed matrix variate.

XI. CONCLUSION

Since we have given a preview in the first section, we need not attempt a summary but can instead conclude with some general remarks.

A random-matrix ensemble supplies a model for the behavior of complicated systems. Averaging over the ensemble is used, together with an assumption of an appropriate ergodic behavior, to take account of part or all of the interaction, producing thereby the statistically-averaged eigenvalue density as well as measures for the fluctuations and correlations in eigenvalues, strengths and other related quantities. Among the kinds of problems to which in turn these results may be applied we mention the study of the directly measurable fluctuations, the effects on observable quantities (e.g., cross sections) of fluctuations which are not directly measurable, symmetry effects such as those generated by the breaking of an underlying fundamental symmetry (time-reversal invariance, in particular) or connected with a model symmetry, the broadening of collective excitations, and other such effects produced by those parts of the interaction not included in the collective Hamiltonian.

Among the ensembles we mention first the three standard ones, orthogonal, unitary, and symplectic, which, while taking account of the basic rotational and time-reversal symmetries, give a realization of the doctrine of equal *a priori* probabilities, important in statistical mechanics more generally; when we calculate a particular quantity in such an ensemble, we are taking for granted that its value is almost completely determined by "statistical" behavior. As further ensembles we may consider a function of the matrices defined by a standard ensemble; this ensemble has asymptotically ($d\to\infty$) the same local fluctuations as long as the function is invertible and defined independently of d; a sum of independent ensembles, which is trivial if they are identical but not in general (such a sum has been used in Sec. VII.F for studying time-reversal invariance); a sum of functions of random matrices, as in the ensemble $\tilde{H}=\sum A_i^T A_i$ of Sec. II.H; various kinds of partitioned ensembles, which may be used for describing random interactions between subsystems. Any of these ensembles may be supplemented by the addition of an operator K whose role might be (1) to take account of a component of the Hamiltonian, which we believe to be especially significant for the purpose at hand; if this is the case, the quantity being calculated will of course, carry "non-statistical" information; (2) to test the response of the system as the Hamiltonian varies along the line $(H+\alpha K)$, with varying α, in the operator space, this giving one way of treating fluctuations; or (3) to break a symmetry preserved by the original ensemble. Adding K to an ensemble amounts to changing the centroids of the matrix-element distributions. A modification of the variances leads also to ensembles of interest, in particular, to the partitioned ensembles.

All of the preceding ensembles have matrices with elements chosen independently. The major examples of ensembles which do not have this property are generated by embedding any of the above, i.e., by constructing an "independent" ensemble in a k-particle space and allowing it to operate in an $(m>k)$-particle one. These ensembles carry one piece of important information, namely, that the interaction is of k-body type, in contrast to ensembles such as the GOE which describe simultaneous interactions between all particles. Since one takes for granted that most Hamiltonians in real life are (1+2)-body, there is an *a priori* favoring of the embedded ensembles. Though embedding of a standard ensemble has, via the central limit theorem, a marked effect on the eigenvalue density, there is good evidence, both from Monte Carlo calculations and from the fact that GOE results agree well with experiment, that it has little or no effect on the local fluctuations. In view of the decoupling between the secular variation of the EGOE density and fluctuations (Secs. I.E and IV.F) this is no longer surprising; alternatively, one might be

tempted to argue that fluctuation measures described by low-order correlation functions should not depend on high-order correlations between the defining matrix elements. Finally, we remark that, with some obvious exceptions (mapping by a noninvertible map, such as $A = H^2$, for example, in which two essentially independent parts of the H are superimposed, or embedding the sum of two ensembles of different particle rank), the stability of the local fluctuation measures seems to extend also to other ensembles and other mappings.

The most straightforward use of random ensembles is as a model for the directly measurable fluctuations, especially as they appear in the energy level (or resonance) spectra of complex nuclei. Eigenvalue fluctuations are in principle complicated; but, although runs containing up to a few hundred levels are available, it appears that only the relatively simple two-point measures can be determined with reasonable accuracy. These can, of course, be written as integrals over the two-point function, or, equivalently, over $\Sigma^2(s)$, which measures the ensemble variance of the number of levels in an interval which contains s levels on the average; examples of these forms are given by Eqs. (5.20) and (5.26). When one writes things in this way, it becomes clear that the measures which have been introduced during the development of the subject are not all "independent" and *a fortiori* do not form an optimal set.

Experimentalists have made quite sophisticated analyses of the data and comparisons with theory; the agreement is in general quite good. It could be, as mentioned in Sec. X.C, that sharper measures can be constructed; if this is the case, then, once their averages and variances are known, they could be used to say more about the quality of the agreement.

There is considerable evidence that the pattern of the nuclear energy-level fluctuations is the same over the entire spectrum and over the whole range of nuclei, as long as energy intervals are measured in local spacing units and proper allowance is made for symmetry effects and collective effects more generally. In cases where symmetry effects can be ignored, it appears that spectra supply realizations of a quasistationary discrete stochastic process, and that spectral fluctuations carry very little information; this is compatible with the fact that experimental fluctuation measures agree well with those given by the GOE, which, as discussed in Sec. II.C, is that ensemble which carries minimal information beyond the specification of the fundamental underlying symmetries. The paucity of the information is connected with the remarkable Dyson-Mehta spectral rigidity (Sec. IV.D) by which a spectrum appears as a highly correlated structure as far as the levels are concerned (though weakly correlated for the spacings). It should not be taken for granted that a minimal-information law is devoid of interest; on the contrary, it is of fundamental interest, precisely *because* it is essentially parameter free. An analog in another domain would be the exponential law of radioactive decay which derives from the fundamental principles of quantum mechanics; the present results, of course, derive from simpler statistical laws.

Since these statistical laws allow for exceptional cases, we are, of course, left with the question whether ensemble predictions, which we have proved to be valid for "most" members of the ensemble, actually apply to a given physical system. The question here has been recently stated by Mahaux and Weidenmüller (1979), p. 28 as follows: *Perhaps the most fundamental problem concerns the foundations of both CN* [compound nucleus] *theory and the statistical theory of nuclear spectra: Which properties of the nucleon-nucleon interaction justify the use of random-matrix models for the nuclear Hamiltonian*? For the smoothed features of spectra and related distributions we are pretty well able to answer this question. No answer which is at all complete is available for the fluctuations, but from experience, we believe[59] that the requirements on H are mild ones. There must be interactions, for, as we have seen, the Hamiltonian for noninteracting particles in a given many-particle model space leads only to the trivial Poisson fluctuations (this is not meant, however, to rule out the H's used for small-particle thermodynamics, for there a new random element enters via the boundary conditions). The Hamiltonian should not have a singular spectrum, as it would, for example, if the interaction were essentially a separable one. If otherwise it should result in degeneracies or in preserving symmetries, this should be taken account of in the ensemble. Apart from these restrictions "most," or perhaps all, Hamiltonians in many-particle spaces might well give rise to canonical fluctuations, this applying in particular to the standard Hamiltonians of nuclear physics. Parenthetically, it should be clear that, because of the limited amount of information contained in the fluctuations, highly detailed level-to-level calculations, as often carried out especially for nuclei, should in many cases not be worth the great labor involved in making them.

Some of these conclusions will be modified by symmetry effects. We have distinguished two kinds of symmetries, the basic underlying symmetries, which are represented by the three standard ensembles, and the less fundamental model symmetries. Different basic symmetries give rise to quite different values of the fluctuation measures (represented by the three values of β in many of the equations of Sec. IV). The striking effects of *good* model symmetries are visible when we have an *interweaving* of symmetries, giving rise to a moderation of the level repulsion and a corresponding change, toward the Poisson values, in the fluctuation measures. These effects do, of course, carry information. The effects of a weak symmetry *breaking*, for either class of symmetries, appear to be quite dramatic, since it seems that small admixtures are adequate to destroy the moderating effects of the symmetry; for large matrices this gives rise, as the admixing parameter is "turned on," to an almost discontinuous change in the measures, as we have discussed in Sec. IV.G and demonstrated explicitly for time-reversal invariance in Sec. VII.F. Thus a measurement (as opposed to setting an upper limit) of a weak symmetry breaking would be

[59] With a "good" symmetry each H eigenfunction belongs to a single irreducible representation; not so for a "broken" symmetry. We have symmetry "interweaving" when the symmetry is good, but close-lying states belong to different representations.

possible only when the admixing parameter is confined to a narrow region of small values. With the possible exceptions of isospin and parity (which in a different cataloging are not model symmetries at all), any model symmetries are either good or are broken to such an extent that they are irrelevant for fluctuations.

The discussion of "small-particle thermodynamics" in Sec. IX centers not on directly measurable fluctuations but on indirect effects generated by fluctuations in the ground-state region. Special interest centers on this subject because of the possibility of realizing all three standard ensembles, orthogonal, unitary, and symplectic, in experiments on small metallic particles. On the other hand, it is not clear that these ensembles, or any close relatives of them, supply adequate models for eigenvalue fluctuations in the single-particle systems which describe the states of low excitation in small particles. There is need for further theoretical study of this question.

The review in Sec. VIII of ensembles in nuclear-reaction theory also concerns the "indirect" effects to which they give rise, e.g., in generating departures from the Hauser-Feshbach formula. Here the effects of a standard ensemble are superimposed on those which would be generated by a simpler reaction theory. The problems encountered here are difficult and different ways of dealing with them lead to competing reaction theories, including one (Sec. VIII.J), not yet fully developed, which is especially compatible with the notion of defining ensembles by their information content.[60] A second domain in which indirect effects are prominent is in generating widths for giant resonances, and other deviations from the results of simple "schematic" theories. For example, the Hamiltonian of the old "pairing+quadrupole" nuclear model, which shows that the competition between these two effects is responsible for some striking results in heavier nuclei, may be supplemented by a random Hamiltonian ensemble and would then go very much further. The parameters in the extended model can be calculated by spectral-averaging methods from a microscopic Hamiltonian, while the solution to the equations of motion for the resulting "deformed ensemble" can be sought by the methods of Sec. III.E.

To the extent that transitions can be regarded as occurring between a fixed giant-resonance state and a set of eigenfunctions of H, a member of a GOE (which form a "statistical space"), the essential result for transitions is that the transition amplitude is a Gaussian random variable, no matter what the nature of the transition operator. The same embedding which generates the more realistic Gaussian eigenvalue density from the GOE semicircle (and which is well understood) gives the same basic (Porter-Thomas) result for locally renormalized transition amplitudes; the formal process which does this, and the fluctuations for embedded ensembles in general, are not yet completely understood, though the results are no longer surprising. The statistical model used in deriving the Porter-Thomas result is not obviously adequate for transitions between low-lying states. With a more general model the basic amplitude is not a Gaussian, but rather a K_0 random variable; and the resulting transition amplitude, given as a sum of such variables, depends on the nature of the transition operator (specifically, on the spectrum of its Hermitian square), becoming Gaussian again when the spectrum has a nonsingular character. It is unclear whether the indicated departures from Porter-Thomas could be detected.

With regard to the methods of calculation, we have not made use of the classical methods which rely on the invariance properties of the basic ensembles to produce the joint eigenvalue distribution. Instead we have called upon the simple idea used by Wigner in his original derivation of the semicircle (and extended to fluctuations by Mon and one of the present authors), namely, that the eigenvalue densities (and the two-point functions) for a wide class of ensembles are calculable in terms of their moments by considering binary Hamiltonian correlations only. Moment methods, applied in the most straightforward way, are, however, often complicated, and it is good that Stieltjes transforms (Secs. III.E and IV.G) are easier to deal with and result in forms from which more general features can be discerned. These have led already to new results, for example, exhibiting wide classes of ensembles which give the canonical fluctuations. This appears to bring us closer to confirming that the three standard ensembles really define the essential patterns of the fluctuations to be expected for complicated systems, and to a more general understanding about the classes of operators which give rise to particular kinds of spectral fluctuations. Besides that, these methods should also make possible new ways of studying the properties of complicated systems.

Note added in proof. In very recent work, Bohigas, Haq, and Pandey (to be published) determine distribution and correlation function for the significant measure, and used them in a much more thorough analysis of the combined energy-level data than is given in Sec. VI. An important outcome of this work is a demonstration that, when sample size and the related effects are properly taken account of, a remarkably close agreement is found between experiment and theory.

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[60] See also two papers by Bloch (1968) with the illuminating title "Statistical theory of nuclear reactions as a communication problem."

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APPENDICES

We identify the section in which each appendix is referred to for the first time.

APPENDIX A: MODEL SPECTRUM VERSUS TRUE SPECTRUM (SEC. I.C)

The eigenvalue spectrum of the matrix has a high-energy cutoff, while that for most systems of interest is monotonically increasing. If we realize the matrices by distributing m particles over a set of N single-particle states, the cutoff will be imposedby restricting the single-particle spectrum. Since (for $m \ll N/2$) the spectrum span for such systems increases rapidly with particle number, it follows that the higher states, which would have contributions from single-particle states above the cutoff, are improperly described.

APPENDIX B: LEVEL REPULSION IN TWO DIMENSIONS (SEC. II.C)

For two-dimensional real symmetric matrices with $2x_1 = (H_{11} - H_{22})$ and $x_2 = H_{12}$, as in Sec. II.C, the general spacing distribution is

$$P(S) = \tfrac{1}{2}S \int_0^1 x^{-1/2}(1-x)^{-1/2} \rho(\tfrac{1}{2}\sqrt{x}\, S, \tfrac{1}{2}\sqrt{1-x}\, S)\, dx \quad \text{(B1)}$$

where $\rho(y,z)$ is the (marginal) joint probability density function for the variables x_1, x_2.

APPENDIX C: ORTHOGONAL INVARIANCE (SEC. II.C)

Given an ensemble (H), we can convert it into an ensemble which is orthogonally invariant in a trivial sense and which has exactly the same spectra (and therefore spectral fluctuations) as (H) itself. We need only, for every H in (H), add in all of its orthogonal transforms $O^{-1}HO$, with the proper weighting as defined by the d-dimensional rotation group, and then make a proper H- or spectrum-dependent adjustment of the relative weights for inequivalent H's; if (H) is already orthogonally invariant, it is unaffected by this operation. We do see from this that orthogonal invariance cannot be detected from the spectra alone. However, the operation is founded on transforms generated by basis changes, as a consequence of which different ensemble members are specified with respect to different bases. The "physical" orthogonal invariance which is of concern to us implies that all the matrix transforms of any member, H, are found in the ensemble, properly weighted and constructed *all in the same basis*; that is a property which we cannot affect by purely formal transformations; it has major consequences for the strength distributions.

APPENDIX D: UNITARY DECOMPOSITION OF NORMS (SEC. II.C)

Formally, when the GOE is realized in an m-particle space, its operators decompose into sets of operators labeled by a parameter ν $(=0, 1, \ldots, m)$; the trace of H^2, whose square root (the *Euclidean norm*) gives a proper measure of the size of H, decomposes (French, 1973) according to

$$d^{-1}(m)(\mathrm{Tr}\, H^2) = \langle H^2 \rangle^m \binom{N}{m}^{-2} \sum_{\nu=0}^{m} \frac{N-2\nu+1}{N+1} \binom{N+1}{\nu}^2$$

$$\xrightarrow[\text{large } N]{} \langle H^2 \rangle^m \sum_{\nu} (m!/\nu!)^2 N^{2(\nu-m)} . \quad \text{(D1)}$$

The decomposition here is according to the representations of the unitary group $U(N)$, whose (column) structure is $[N-\nu, \nu]$. The parameter ν is the "unitary rank," which for large N is essentially identical with the particle rank (a ν-body interaction has particle rank ν). We see then explicitly the rapid increase with ν of the norm of the ν-body part of H.

APPENDIX E: GAUSSIAN BEHAVIOR FOR NONINTERACTING PARTICLES (SEC. III.B)

The first corrections to the even moment of order 2ν are seen to arise from partitions $(3^2, 2^{\nu-3})$ and $(4, 2^{\nu-2})$, both being of relative order m^{-1}. For the Gaussian limit, of course, the odd moments must also be "small;" specifically, the odd-order *cumulants*, scaled by the appropriate power of $\sigma(m)$, must vanish asymptotically (i.e., for large particle number m). We find that

$$M_{2\nu+1}(m) \xrightarrow[\text{large } m]{} \tfrac{1}{3}\nu(2\nu+1)!!\, \sigma^{2\nu-2}(m) M_3(m) ,$$

so that the *skewness* γ_1 (the lowest-order reduced cumulant)

$$\sim m^{-1/2}\langle \varepsilon^3 \rangle / (\langle \varepsilon^2 \rangle)^{3/2} = m^{-1/2}\gamma_1(1) .$$

This is, in fact, an exact result in the $d \to \infty$ limit which we have assumed. We find a corresponding decrease to zero with increasing m for the higher-order rescaled odd cumulants.

APPENDIX F: h^2 AND ITS EQUIVALENT TWO-BODY OPERATOR (SEC. III.C)

h^2 is a $(1+2)$-body operator. Separate it into its two parts and, before recombining, multiply the one-body part by $(n-1)/(m-1)$, where n is the operator for the total particle number $(=\sum_{n_i})$. The result is a two-body operator which for any fixed m has the same matrix elements as h^2.

APPENDIX G: FORBIDDEN CONTRACTIONS (SEC. III.D)

A better argument uses the fact that a natural measure for an operator G in a k-particle space is $\langle\langle G^+G \rangle\rangle^k$, so that we should compare $\langle \underbrace{HO}\underbrace{H}\underbrace{O^+H} \rangle^k$ with $\langle \underbrace{H}\underbrace{H}O^+O\underbrace{H}\underbrace{H} \rangle^k$. More formally (Mon and French, 1975), the criterion involves the way in which O transforms under the group of unitary transformations in the N-dimensional single-particle space; only the unitary-scalar part of O survives the correlation operation $\underbrace{HOH}$ in the k-particle space. Since the scalar representation is one-dimensional, while the total operator space is $\binom{N}{k}^2$ dimensional, the result follows. Moreover, $\underbrace{H}\underbrace{H}$ is itself unitarily scalar; thus contractions about a correlated pair are fully contributing and the binary correla-

tions which contribute to the GOE are all those in which H's correlate only across pairs already correlated.

APPENDIX H: MAPPING (SEC. III.D)

See also Sec. II.F. For every member of the given ensemble take a new Hamiltonian $f(H)$; the density, of course, is fixed by f. Allow the dimensionality to increase while continually rescaling the spectrum to a fixed variance. Then, in closed regions of the energy in which f is single-valued and without zeros or singularities, the f fluctuations, measured in units defined by the local spacing and spanning a fixed number of levels (and hence a smaller and smaller energy domain), will, for large enough d, be as close as we wish to those of H. Alternatively, we remark that the cluster functions of Secs. IV and X, which describe the fluctuations, are invariant under such maps.

APPENDIX I: MOMENTS FOR DEFORMED ENSEMBLES (SEC. III.E)

To evaluate the ensemble average of the moments $M_p(\alpha)=\langle (H+\alpha K)^p\rangle$, we make use of two counting theorems, the first of which we come to below, while the second gives the polynomial moments μ_ζ^p, Eq. (4.17), as the number of allowed ways of inserting $(p-\zeta)/2$ correlated pairs of H's into a lattice containing ζ barriers over which contractions are forbidden. Since $\overline{HHK}=(1+d^{-1})K\to K$ while $\overline{HKH}=d^{-1}(K+\mathrm{Tr}\,K)\to\langle K\rangle$, we see that the singular operator discussed in the text, for which $\langle K\rangle=d^{-1}$, is such a barrier, and, since $K^\zeta=K$ for $\zeta=1,2,\ldots$, so also are its powers. The moments in this case are (Pandey and French, 1979)

$$\overline{M}_p(\alpha)=\sum_{\zeta=0}^{p}\mu_\zeta^p\langle(\alpha K)^\zeta\rangle$$

$$=\overline{M}_p(0)+d^{-1}\sum_{\zeta=1}^{p}\mu_\zeta^p\alpha^\zeta\,, \tag{I1}$$

which can be inverted via the polynomials in (4.15) and (4.17) and yield the results (3.25) in the text.

However, contractions around an arbitrary operator K and its powers do not in general give values which vanish in the large-d limit and must therefore be specifically evaluated—for example $[\langle K^{\zeta_1}HK^{\zeta_2}H\rangle]_e\to\langle K^{\zeta_1}\rangle\langle K^{\zeta_2}\rangle$, this generating a term $\mu^p_{\zeta_1+\zeta_2+2}\langle K^{\zeta_1}\rangle\langle K^{\zeta_2}\rangle$ for the pth moment. As above, we have terms with ζK's and $(p-\zeta)H$'s, where then $\zeta=0,1,\ldots,p$. Consider a "core" involving ζK's and r-correlated pairs of H's such that (i) each correlation line encloses at least one K (but not all of them, since, because of circular symmetry, that would be equivalent to none); (ii) the lines do not cross each other; and (iii) they partition the H's into exactly $l\le\zeta$ parts. Note carefully that, for $l=0,1$, we have necessarily $r=0$ but, for $l>1$, we have $r\ge l-1$.

The theorem which does this very complicated r-counting (the first one referred to above) has not been rigorously derived, but has been made plausible by consideing various l terms. Next, the remaining $[(p-\zeta)/2-r]$-correlated pairs of H's are inserted into this lattice of $(\zeta+2r)$ barriers, the number of ways of doing this yielding the factor $\mu^p_{\zeta+2r}$ by the second counting theorem. With a final r summation we arrive at the result (Pandey, unpublished) that

$$\overline{M}_p(\alpha)=\sum_{\zeta\ge0}\sum_{l\ge0}\alpha^\zeta\tilde{K}_{\zeta l}\binom{\frac{p-\zeta}{2}+1}{l}\frac{1}{\frac{p-\zeta}{2}+1}\mu_\zeta^p\,, \tag{I2}$$

where $\tilde{K}_{\zeta l}$ is the coefficient of η^ζ in $(\sum_{r\ge1}\eta^r\langle K^r\rangle)^l$, or, equivalently,

$$\sum_{l\ge0}\tilde{K}_{\zeta l}\binom{\frac{p-\zeta}{2}+1}{l}$$

is the coefficient of η^ζ in

$$\Big(\sum_{r\ge0}\eta^r\langle K^r\rangle\Big)^{(p-\zeta)/2+1}.$$

APPENDIX J: EVALUATION OF BASIC CROSS-CORRELATED TRACES (SEC. IV.B)

The third equation referred to in the text is

$$\underbrace{\overline{\langle KH^\zeta\rangle\langle KH^\zeta\rangle}}_{\zeta}=\frac{2\langle K^2\rangle}{\beta d^2} \tag{J1}$$

for a *traceless* K.

Consider first $\beta=1$. For (4.9) we write (Mon and French, 1975) $\langle H^\zeta\rangle=d^{-1}\Sigma H_{ij}H_{jk}\cdots H_{ti}$; ensemble averaging over the product of the two correlated traces gives $d^{-\zeta}$ (since $\overline{H_{ij}H_{ji}}=d^{-1}$ and since there are ζ Hamiltonian pairs), while summing over the ζ free indices gives d^ζ; a factor 2 comes because H_{ij} correlates with either H_{ij} or H_{ji} (so that, since the index sequence is determined in one "ring," we can choose that order or its inverse for the second); the factor ζ comes from the cyclic invariance whereby the second ring may be rotated with respect to the first. The other two equations (4.10) and (J1) follow in the same way. The significant difference between (4.9) and (J1) is that a cyclic rotation is permitted in the second ring in $[\underbrace{\langle H^\zeta\rangle\langle H^\zeta\rangle}]_e$, but not in $[\underbrace{\langle KH^\zeta\rangle\langle KH^\zeta\rangle}]_e$.

For $\beta=2$ we note that, while $\overline{H_{ij}H_{ji}}=d^{-1}$, $\overline{H^2_{ij}}_e=0$ (for $i\ne j$). Therefore the ring in one trace can correlate only with its inverse in the other. Hence there is no factor 2 in (4.9) and (J1).

For $\beta=4$ (see Sec. III.E), while the ring in one trace can correlate with either kind in the other, a different kind of inhibition comes into play, reducing the number by a factor $\frac{1}{4}$; this new restriction is that the product of the Pauli matrices in the ring must also have a nonvanishing trace. For correlations in the same trace, as in (4.10),

$$\overline{H_{ij}H_{ji}}=(4d)^{-1}[1+(-i\sigma)\cdot(-i\sigma)^*]=d^{-1}1\,.$$

For cross correlations we note that all even powers of $\boldsymbol{\sigma}_i$ and the triple product $\sigma_1\sigma_2\sigma_3$ can give a nonvanishing trace; the total number of such terms (1 or -1) in $(1-i\Sigma\sigma_j)^\zeta$ is $4^{\zeta-1}$. Corresponding to each term we have either the same product or its Hermitian adjoint in the other trace, and therefore the final result contains all positive terms. Since the variance of $H_{k,ij}$ is $(4d)^{-1}$, we have, for (4.9),

$$2\times4^{\zeta-1}\times\zeta\times(4d)^{-\zeta}\times d^\zeta\times d^{-2}=\zeta/2d^2\,,$$

similarly for the second and third. We remark finally

that, while the above derivations of (4.9) are valid for $\zeta > 2$, the result is valid for $\zeta = 1, 2$, also, as can be verified by a direct evaluation of the variances of $\langle H \rangle$ and $\langle H^2 \rangle$.

APPENDIX K: THE TWO-POINT FUNCTION VIA PARAMETRIC DIFFERENTIATION (SEC. IV.B)

For a derivation of $S^o(x,y)$ via parametric differentiation we introduce

$$\begin{aligned} &\rho_K(x) = \langle x|K|x\rangle \rho(x) \equiv K(x)\rho(x)\,, \\ &S^o_K(x,y) = \overline{\rho_K(x)\rho_K(y)} - \overline{\rho_K(x)}\,\overline{\rho_K(y)}\,, \end{aligned} \tag{K1}$$

where K is an arbitrary fixed traceless operator; note that $\rho_K(x)$, the density weighted by the K expectation value, is not a proper density function (since it is not non-negative definite), but that need not concern us. Writing F_α as the distribution function for $H + \alpha K$, we have

$$S^F_{\alpha\beta}(x,y) \equiv \overline{F_\alpha(x) F_\beta(y)} - \overline{F_\alpha(x)}\;\overline{F_\beta(y)} \xrightarrow[\alpha,\beta\to 0]{} S^F(x,y)\,, \tag{K2}$$

$$\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta} S^F_{\alpha\beta}(x,y)\Big|_{\alpha=\beta=0} = S^\rho_K(x,y)\,. \tag{K3}$$

We see that $S^\rho_K(x,y)$ defines the function which, with respect to α,β variations, is tangent to $S^F_{\alpha\beta}(x,y)$ at the (α,β) origin. $S^\rho_K(x,y)$ is easily calculated and thus the question is whether the tangent function adequately determines S^F itself. The answer is that it does, to the extent that binary associations are dominant.

To evaluate S^ρ_K we borrow from (7.60) and (7.63) the result that

$$\overline{K_{ii}K_{jj}} - \overline{K_{ii}}\,\overline{K_{jj}} = \frac{2\langle K^2\rangle}{(d-1)(d+2)}(d\delta_{ij} - 1)\,. \tag{K4}$$

Since the K_{ii} and the eigenvalues E_i are independently distributed, we have for the (p,q) moment of S^ρ_K

$$\begin{aligned} \Sigma^2_{pq}(K) &\equiv \overline{M_p(K)M_q(K)} - \overline{M}_p(K)\overline{M}_q(K) \\ &= \frac{1}{d^2}\sum_{ij}\overline{K_{ii}K_{jj}}\;\overline{E_i^p E_j^q} \\ &= \frac{2\langle K^2\rangle}{(d-1)(d+2)}(\overline{M}_{p+q} - \overline{M}_p\overline{M}_q) \\ &\xrightarrow[d\to\infty]{} \frac{2\langle K^2\rangle}{d^2}(\overline{M}_{p+q} - \overline{M}_p\overline{M}_q) = \frac{2\langle K^2\rangle}{d^2}\sum_{\zeta\ge 1}\lambda_\zeta^p\lambda_\zeta^q\,. \end{aligned} \tag{K5}$$

Here the second form follows by writing $M_p(K) = d^{-1}\Sigma K_{ii}E_i^p$ ($\Rightarrow \overline{M}_p(K) = \langle K\rangle \overline{M}_p = 0$), the third form by using (K4). The final form, which makes use of (4.12), gives the moments of S^o_K in a form analogous to that given by (4.8) for S^o itself. By exploiting the counting theorem (4.11) we can, in fact, show that $\mu^p_\zeta = (p/\zeta)\lambda^{p-1}_{\zeta-1}$; moreover, the amplitudes in the ζ expansions (4.8) and (K5) can be related by (J1). Thus we are able to determine S^o in terms of S^o_K, thereby "solving" Eq. (K3). Incidentally, the moments (K5) fix the auxiliary two-point function as

$$\begin{aligned} S^o_K(x,y) &= 2\langle K^2\rangle(d-1)^{-1}(d+2)^{-1}[\delta(x-y)\overline{\rho}(x) - \overline{\rho(x)\rho(y)}] \\ &\xrightarrow[d\to\infty]{} 2\langle K^2\rangle d^{-2}[\delta(x-y)\overline{\rho}(x) - \overline{\rho}(x)\overline{\rho}(y)] \\ &= 2d^{-2}\langle K^2\rangle\overline{\rho}(x)\overline{\rho}(y)\sum_{\zeta\ge 1} v_\zeta(x)v_\zeta(y)\,, \end{aligned} \tag{K6}$$

where, in proceeding to the second form, we have dropped a d^{-2} fluctuation term and, in the last, the v_ζ are given by (4.15). The results, of course, are written for GOE, but the forms extend immediately to $\beta = 2, 4$.

APPENDIX L: EXACT FORMS FOR $Y_{2,\beta}$ (SEC. IV.C)

The exact functions, as given by Dyson (1962c, 1970) and Mehta (1967, 1971), are

$$\begin{aligned} Y_{2,1}(r) &= [s(r)]^2 - Js(r)\,Ds(r)\,, \\ Y_{2,2}(r) &= [s(r)]^2\,, \\ Y_{2,4}(r) &= [s(2r)]^2 - Is(2r)\,Ds(2r)\,, \end{aligned} \tag{L1}$$

where

$$\begin{aligned} s(r) &= \frac{\sin(\pi r)}{\pi r}\,, \\ Ds(r) &= \frac{ds(r)}{dr}\,, \\ Is(r) &= \int_0^r s(r')\,dr'\,, \\ Js(r) &= Is(r) - \varepsilon(r)\,, \\ \varepsilon(r) &= \tfrac{1}{2}, \quad r > 0 \\ &= 0, \quad r = 0 \\ &= -\tfrac{1}{2}, \quad r < 0\,. \end{aligned} \tag{L2}$$

Dyson and Mehta deal mainly with the *two-level form factor*, which is the Fourier transform of the Y_2 function,

$$b(k) \equiv \int_{-\infty}^{\infty} Y_2(r)e^{2\pi i k r}\,dr\,. \tag{L3}$$

For example, the number variance Σ^2 (5.2), is given (Dyson and Mehta, 1963) as

$$\Sigma^2(r) = \int_{-\infty}^{\infty}\frac{\sin^2\pi r t}{(\pi t)^2}[1 - b(t)]\,dt\,. \tag{L4}$$

For our Y_2 (4.24), we have

$$b_\beta(k) = \begin{cases} 1 - \dfrac{2|k|}{\beta}\,, & |k| \le \beta/2 \\ 0\,, & |k| \ge \beta/2\,, \end{cases} \tag{L5}$$

whereas, for the exact functions, we have

$$\begin{aligned} b_1(k) &= \begin{cases} 1 - 2|k| + |k|\ln(1+2|k|)\,, & |k| \le 1 \\ -1 + |k|\ln\left(\dfrac{2|k|+1}{2|k|-1}\right), & |k| \ge 1\,, \end{cases} \\ b_4(k) &= \begin{cases} 1 - \tfrac{1}{2}|k| + \tfrac{1}{4}|k|\ln|(|k|-1)|\,, & |k| \le 2 \\ 0\,, & |k| \ge 2\,. \end{cases} \end{aligned} \tag{L6}$$

APPENDIX M: FLUCTUATIONS FOR NONINTERACTING PARTICLES (SEC. IV.F)

For noninteracting particles the $(m \geq 2)$-particle EGOE density is essentially an m-fold convolution of the GOE density. Thus every m-particle fluctuation domain receives contributions from very different regions of the single-particle spectrum, which are asymptotically independent and give rise therefore to Poisson fluctuations; see, for example, Bloch (1969). Alternatively, without interactions there are many preserved unitary symmetries, these being defined by an additive decomposition of the U(N) algebra of transformations in the single-particle space. This argument, in fact, yields Poisson fluctuations also for some interacting-particle cases.

APPENDIX N: THE DISTRIBUTION OF HIGH-ORDER SPACINGS (SEC. V.A)

The question is whether the correlations between nearest-neighbor spacings S_i [the S_i^0 of (5.5)] modify the asymptotic Gaussian distribution which, in their absence, would result for large k (via the central limit theorem) from the fact that a kth order spacing is a sum of $(k+1)$ identically distributed S_i. The correlation coefficients between the S_i (5.8) are small, except for adjacent spacings, and since, for separated spacings, $C(S_i, S_j) \sim |i-j|^{-2}$, we may regard S_i, S_j as uncorrelated when $|i-j| \geq m$ (where, for the accuracy of interest to us, a reasonable value would be $m \sim 5$). We may describe the S_i as forming an *"m-correlated"* stationary sequence. The Diananda theorem (1953; see also Hannan, 1960) would assert, *inter alia*, that a kth-order spacing generated from a corresponding *m-dependent* sequence has a Gaussian distribution for large k. In light of results about higher-order correlation functions which are discussed in Sec. X, we have little trouble in believing that well-separated nearest-neighbor spacings are not only uncorrelated but also independent, and then the Gaussian form for the spacing distribution follows.

APPENDIX O: SINGULAR AND NONSINGULAR SPECTRA (SEC. VII.D)

Consider the reduced central spectral moments, $\mu_p = \mathfrak{M}_p / \mathfrak{M}_2^{p/2}$, of a Hermitian operator G whose eigenvalues are x_i. We standardize the spectrum by $y_i = (x_i - \langle G \rangle)/\mathfrak{M}_2^{1/2}$, so that the μ_p are the moments of the y_i distribution. We seek the $\hat{\mu}_p$, the extrema of μ_p with respect to variations of the x_i (i.e., of G). For $i = 1, 2, \ldots, d$ we have the derivatives

$$\frac{\partial \mu_p}{\partial x_i} = (\mathfrak{M}_2 d)^{-1} p (y_i^{p-1} - \mu_{p-1} - y_i \mu_p) , \tag{O1}$$

which are identically zero for $p = 0, 1, 2$. For extrema $\partial \mu_p / \partial x_i = 0$, with $p \geq 3$. It is easy to see that these define degenerate spectra for which the y_i can attain only two distinct values, $\hat{y}_1$ and $\hat{y}_2$, say. There are $(d-1)$ solutions, with standardized spectra $(\hat{y}_1^{d_1}, \hat{y}_2^{d-d_1})$, where $d_1 = 1, 2, \ldots, d-1$, $\hat{y}_1 = [(d-d_1)/d]^{1/2} = -\hat{y}_2^{-1}$. Note that, since the results are independent of p, we have simultaneous extrema for all p.

The absolute maxima are given by $(d_1, d-d_1)_< \equiv d_< = 1$, which defines the pairing spectrum, for which $\hat{\mu}_p \xrightarrow[d\to\infty]{} d^{(p-2)/2}$. Solutions of (O1) with $d_< \ll d$ give $\hat{\mu}_p$ values with the same d dependence, these forming the singular spectra, of which the pairing spectrum is the basic example. At the other extreme, for solutions with $d_< \sim d/2$, the $\hat{\mu}_p$ are d independent for large d, as they are also for spectra such as semicircular, Gaussian, χ_ν^2. We can thus classify spectra (and thereby operators), including nonextremal ones, according to the d dependence of $\mu_{2\nu}$ (restricting ourselves to even order, since odd-order moments may vanish by a symmetry). Note that the square of a Hermitian operator belongs to the same class as the operator itself.

APPENDIX P: FURTHER TRANSITION-AMPLITUDE CORRELATIONS (SEC. VII.D)

The correlations between T_{ij} and T_{ji} are not unity for non-Hermitian T's. Indeed, since, for real constants a, b, $(aT_{ij} + bT_{ji}) = (aT + bT^+)_{ij}$, we have, using (7.32),

$$\overline{T_{ij}^\nu T_{ji}^\nu} = \frac{1}{(2\nu)!} \left(\frac{\partial^{2\nu}}{\partial^\nu b \partial^\nu a} M_{2\nu}(\{aT + bT^+\}_{ij}) \right)_{a=b=0}$$

$$= \frac{1}{\nu! 2^{2\nu}} \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d-1}{2} + \nu\right)} \left(\frac{\partial^{2\nu}}{\partial^\nu b \partial^\nu a} \overline{\sigma_j^{2\nu}}(aT + bT^+) \right)_{a=b=0} , \tag{P1}$$

which gives back (7.32) for $T = T^+$. Thus, from (P1) and (7.35), we find for the linear and quadratic correlations

$$C(T_{ij}, T_{ji}) \xrightarrow[d\to\infty]{} \frac{\mathfrak{M}_2(T)}{M_1(G_T)} ,$$

$$C(T_{ij}^2, T_{ji}^2) \xrightarrow[d\to\infty]{} \frac{M_2(G_T) + \mathfrak{M}_4(T) + \mathfrak{M}_{1,3}(T) + d\mathfrak{M}_2^2(T)}{3M_2(G_T) + dM_1^2(G_T)}$$

$$\xrightarrow[d\to\infty]{\text{nonsingular } T} \left(\frac{\mathfrak{M}_2(T)}{M_1(G_T)} \right)^2 , \tag{P2}$$

where $\mathfrak{M}_{\mu,\nu}(T) \equiv \langle \tilde{T}^\mu \tilde{T}^{+\nu} \rangle$.

Moreover, since

$$\overline{\sigma_j^4} = \overline{\sum_{i \neq j} T_{ij}^2 \sum_{k \neq j} T_{jk}^2}$$

$$= (d-1)\overline{T_{ij}^2 T_{ji}^2} + (d-1)(d-2)\overline{T_{ij}^2 T_{jk}^2} ,$$

we have, using (P2) and (7.43),

$$C(T_{ij}^2, T_{jk}^2 = C(T_{ij}^2, T_{jk}^2) + (d-2)^{-1}[1 - C(T_{ij}^2, T_{ji}^2)]$$

$$\xrightarrow[d\to\infty]{} C(T_{ij}^2, T_{kj}^2) , \tag{P3}$$

as stated in the text.

APPENDIX Q: ALTERNATIVE PROOF OF ERGODICITY FOR ONE-POINT MEASURES (SEC. X.B)

The autocovariance function, Eq. (10.7), for ρ has already been given by the last form of Eq. (4.4); we see via Eqs. (4.24) and (4.25) that, for all three β values, $S^\rho(r) \xrightarrow[r\to\infty]{} 0$, as r^{-2} for $\beta = 1, 2$ and as $\cos 2\pi r / r$ for $\beta = 4$; the delta function in $S^\rho(r)$, when inserted into the last form of Eq. (10.9), contributes a term of order p^{-1} to

$\mathrm{var}_{(e)}\langle p\rangle_p$. We see then that, by the criterion given after Eq. (10.11), the variance vanishes for large p and we have the resultant ergodicity of the density for all three ensembles, as well as for the Poisson case for which $Y_2(r)$ is identically zero.

APPENDIX R: THE FORMS FOR $Y_{k,\beta}$ (SEC. X.C)

Dyson (1970) and Mehta (1971) give

$$Y_{k,\beta}=\sum_p [\sigma_\beta(r_{12})\,\sigma_\beta(r_{23})\cdots\sigma_\beta(r_{k1})]\,, \tag{R1}$$

where $\sum_p$ denotes a sum over the $(k-1)!$ distinct cyclic permutations of the indices $(1,2\ldots,k)$ and

$$\sigma_1(r)=\begin{pmatrix} s(r) & Ds(r)\\ Js(r) & s(r)\end{pmatrix},$$
$$\sigma_2(r)=s(r)\,, \tag{R2}$$
$$\sigma_4(r)=\begin{pmatrix} s(2r) & Ds(2r)\\ Is(2r) & s(2r)\end{pmatrix},$$

with $s(r)$, $Ds(r)$, $Is(r)$, and $Js(r)$ defined as in Appendix L. For $\beta=1,4$ the operation ($\frac{1}{2}$ Tr) should be inserted before the cyclic product in (R1).

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