Exact integrability in quantum field theory and statistical systems

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The properties of exactly integrable two-dimensional quantum systems are reviewed and discussed. The nature of exact integrability as a physical phenomenon and various aspects of the mathematical formalism are explored by discussing several examples, including detailed treatments of the nonlinear Schrödinger (delta-function gas) model, the massive Thirring model, and the six-vertex (ice) model. The diagonalization of a Hamiltonian by Bethe's *Ansatz* is illustrated for the nonlinear Schrödinger model, and the integral equation method of Lieb for obtaining the spectrum of the many-body system from periodic boundary conditions is reviewed. Similar methods are applied to the massive Thirring model, where the fermion-antifermion and bound-state spectrum are obtained explicitly by the integral equation method. After a brief review of the classical inverse scattering method, the quantum inverse method for the nonlinear Schrödinger model is introduced and shown to be an algebraization of the Bethe *Ansatz* technique. In the quantum inverse method, an auxiliary linear problem is used to define nonlocal operators which are functionals of the original local field on a fixed-time string of arbitrary length. The particular operators for which the string is infinitely long (free boundary conditions) or forms a closed loop around a cylinder (periodic boundary conditions) correspond to the quantized scattering data and have a special significance. One of them creates the Bethe eigenstates, while the other is the generating function for an infinite number of conservation laws. The analogous operators on a lattice are constructed for the symmetric six-vertex model, where the object which corresponds to a solution of the auxiliary linear problem is a string of vertices contracted over horizontal links (arrows). The relationship between the quantum inverse method and the transfer matrix formalism is exhibited. The inverse Gel'fand-Levitan transform which expresses the local field operator as a functional of the quantized scattering data is formulated for the nonlinear Schrödinger equation, and some interesting properties of this transformation are noted, including its reduction to a Jordan-Wigner transformation in the limit of infinitely repulsive coupling.

CONTENTS

I. INTRODUCTION

The state of an isolated system of particles or fields which is evolving in time can usually be characterized by the values of certain conserved quantities such as total energy and momentum. For most interacting systems, a more detailed kinematical description of the state will not be preserved in time. For example, the Fourier components of a field or the momentum distribution of a many-body system will generally have a complicated time dependence, often exhibiting a tendency toward thermalization of the system. A familiar exception to such stochastic behavior is the case of free fields or noninteracting particles. In this case the full momentum distribution function is conserved, and one may construct an infinite number of ordinary constants of motion (e.g., the moments of the momentum distribution). A theory which exhibits nonstochastic behavior of this sort is said to be exactly integrable. A remarkable fact which has emerged in recent years is that the phenomenon of exact integrability is not restricted to free theories, but arises in a variety of physically interesting models with nontrivial interactions. The criteria for exact integrability are not understood well enough to provide a complete list of integrable quantum field theories or a deductive way of testing a particular theory for this property. However, the list already includes some of the most interesting two-dimensional field theories, and recent speculation that gauge theories in four dimensions might be exactly integrable provides additional incentive for studying these two-dimensional theories and trying to identify the important concepts. It is possible to entertain great hopes for the future role of exact integrability in quantum field theory, but much work remains to be done. In this paper we review some of the developments which have led to our present understanding of this phenomenon.

The history of the subject is a rich one which extends over fifty years and includes some of the major achievements of mathematical physics. The seminal work of Bethe (1931) on the isotropic Heisenberg spin chain with nearest-neighbor interaction provided the first nontrivial example of an integrable quantum system. The method devised by Bethe for constructing the eigenvectors of the spin-chain Hamiltonian has since been successfully applied to a number of models in statistical mechanics and quantum field theory. A Bethe *Ansatz* was applied to the anisotropic *XXZ* spin chain by Orbach (1958) and Walker (1959) and more generally by Yang and Yang (1966). A somewhat differ-

ent application of the method was discovered by Lieb (1967a, b, c, d), who used a Bethe *Ansatz* to diagonalize the transfer matrices of the ice, F, and potassium dihydrogen phosphate (KDP) models, which are special cases of the symmetric six-vertex model. The general six-vertex model was solved by Sutherland, Yang, and Yang (1967). A separate thread of development began with Onsager's (1944) solution of the two-dimensional Ising model. As described by Schultz, Mattis, and Lieb (1964), this solution consists of a Jordan-Wigner transformation to convert Pauli matrices into fermion operators followed by a Bogoliubov rotation to diagonalize a quadratic form in the fermion operators. A similar technique was applied to the XY spin-chain Hamiltonian by the same authors (Lieb, Schultz, and Mattis, 1961). These various developments in soluble lattice models and spin chains were impressively unified and extended in the very important work of Baxter. After formulating a Bethe *Ansatz* solution of the most general symmetric six-vertex model (Baxter, 1971), he went on to solve the symmetric eight-vertex model (Baxter, 1972a, 1973a, b, c). This model includes as special cases the previously solved ice models of Lieb, as well as the two-dimensional Ising model (more precisely, two decoupled Ising models). [The completely general six-vertex model solved by Sutherland, Yang, and Yang (1967) is, however, not a special case of the Baxter model.] In addition, Baxter showed that the general anisotropic XYZ spin-chain Hamiltonian could be obtained as a logarithmic derivative of the eight-vertex model transfer matrix (Baxter, 1972b). As a result, these operators are both diagonalized by the same set of eigenvectors, and Baxter's solution of the eight-vertex model also solves the XYZ spin chain. Several excellent reviews of the various applications of Bethe's *Ansatz* in statistical mechanics are available (Kasteleyn, 1975; Gaudin, 1972, 1973; Lieb and Wu, 1972; Lieb, 1969a, b, 1970).

Direct applications of the Bethe *Ansatz* method to continuum quantum field theory began with the work of Lieb and Liniger (1963), who solved a many-body problem of bosons interacting by a two-body δ-function potential. As a field theory this is the quantum nonlinear Schrödinger model. In addition to constructing the eigenvectors of the Hamiltonian, Lieb and Liniger employed periodic boundary conditions to obtain an integral equation for the conserved ground-state density function (see Sec. II.A), and Lieb (1963) extended this integral equation method to include excitations above the ground state. More recently these methods have been used to treat certain relativistic field theories. Models which have been solved by a Bethe *Ansatz* now include the massive Thirring model (Bergknoff and Thacker, 1979a, b) and the equivalent quantum sine-Gordon equation (Sklyanin, Takhtajan, and Faddeev, 1979), as well as the Gross-Neveu (Andrei and Lowenstein, 1979, 1980a) and SU(2)-Thirring (Belavin, 1979) models. In these theories the physical vacuum is constructed as a many-body state in a manner analogous to the ground state of the finite density δ-function gas. Particle and bound-state energies are computed by Lieb's integral equation method.

There are two major lines of development which have led to our present understanding of exact integrability in quantum systems. One was the development of soluble quantum models as described above. The other was the discovery of the inverse scattering transform method for solving certain classical nonlinear wave equations [for a review see Scott, Chu, and McLaughlin (1973)]. The original work of Gardner, Greene, Kruskal, and Miura (1967) showed that the initial value problem for the nonlinear Korteweg-de Vries equation could be reduced to a sequence of linear problems. The relationship between integrability, conservation laws, and soliton behavior was clearly exhibited by this technique. Subsequent work by Zakharov and Shabat (1971) and by Ablowitz *et al.* (1973) revealed that the inverse scattering method is applicable to a variety of nonlinear equations, including the classical versions of the nonlinear Schrödinger and sine-Gordon equations. The fact that the quantum nonlinear Schrödinger equation could also be exactly solved using Bethe's *Ansatz* suggested a deep connection between inverse scattering and Bethe's *Ansatz*. Such a connection was partially confirmed by studying a higher conserved quantity as a quantum-mechanical operator (Thacker, 1978). The relationship between inverse scattering and Bethe's *Ansatz* for the nonlinear Schrödinger equation was fully realized by the development of the quantum inverse method (Sklyanin and Faddeev, 1978; Sklyanin, 1979; Thacker and Wilkinson, 1979; Honerkamp *et al.*, 1979). It was already known that the classical inverse method could be regarded as a canonical transformation to action and angle variables. These variables are constructed from the scattering data $a(k)$ and $b(k)$ of a linear eigenvalue problem and have a trivial time dependence. The analogy with Fourier transformation for linear systems has often been noted. In the quantum inverse method, the quantities $a(k)$ and $b(k)$ become operator functionals of the local fields of the system. The operator $b(k)$ is found to create eigenstates of the Hamiltonian. The states constructed in this way are identical with those of Bethe's *Ansatz*. The operator $a(k)$ commutes with the Hamiltonian and is the generator of an infinite number of conserved quantities. Thus, the quantum inverse method may be viewed as an algebraization of the Bethe *Ansatz* method. This relationship will be discussed in detail in Sec. III.

Like the Fourier transformation, the classical inverse method consists of both a direct transform and an inverse transform. The direct transform expresses the scattering data as functionals of the local fields, and the inverse transform expresses the local fields as functionals of the scattering data. The latter transform is accomplished via the Gel'fand-Levitan integral equation for the linear eigenvalue problem (Gel'fand and Levitan, 1951). In the quantum inverse method, the construction of the operators $a(k)$ and $b(k)$ comprises the direct transform. Recently the operator inverse transform for the nonlinear Schrödinger equation has been constructed by a quantum generalization of the Gel'fand-Levitan method (Creamer *et al.*, 1980a). The inverse transform expresses the field $\phi(x)$ as an expansion in powers of the quantized reflection coefficient $R(k)=b(k)a^{-1}(k)$ and its Hermitian conjugate. An expansion for the charge density $\phi^*(x)\phi(x)$ is also ob-

tained. Term-by-term analysis of these expressions has revealed some interesting properties. In the strong coupling limit $c \to \infty$ (impenetrable bosons), the operator $\tilde{R}(x)$, the Fourier transform of $R(k)$, satisfies canonical anticommutation relations. In this limit the Gel'fand-Levitan expression for $\phi(x)$ exponentiates into a Jordan-Wigner transformation relating the boson field $\phi(x)$ to the fermion field $\tilde{R}(x)$. This result sheds some light on the nature of the quantum inverse transform. The Gel'fand-Levitan expression for the charge density operator also has an interesting structure. It is closely related to the spectral integral equation for the finite temperature δ-function gas derived by Yang and Yang (1969). In the zero temperature limit this reproduces the excitation spectrum derived by Lieb. Thus, the integral equation technique for calculating the spectrum also falls naturally within the province of the quantum inverse method. These developments are discussed in Sec. III.C.

The organization of this review is neither historical nor logical, but rather, it is hoped, pedagogical. It seems likely that explicit Bethe *Ansatz* methods will eventually be subsumed by the quantum inverse method. Nevertheless, before presenting the full apparatus of the quantum inverse method we consider in Sec. II some explicit Bethe *Ansatz* solutions. This provides some insight into the structure of the eigenstates which should be helpful in understanding the more elegant algebraic methods which will be reviewed in Sec. III.

II. BETHE'S ANSATZ

In this section we will consider several soluble models which illustrate the workings of Bethe's *Ansatz*. Our first discussion in Sec. II.A deals with the nonlinear Schrödinger (δ-function gas) model. In addition to being an instructive toy model, this system provides a paradigm for other applications of Bethe's *Ansatz* in field theory and has been a central inspiration in the development of the quantum inverse method. After some elementary considerations in the two-body system, we describe the N-body *Ansatz*, periodic boundary conditions, and the integral equations for the ground state and for the excitation spectrum. In Sec. II.B similar methods are applied to the massive Thirring model. This model illustrates the application of Bethe's *Ansatz* methods to relativistic quantum field theory. The filling of the Dirac sea and the calculation of the excitation spectrum are reviewed. Field theory models with internal symmetry are briefly discussed in Sec. II.C. In Sec. II.D we introduce the soluble lattice statistics models, beginning with the six-vertex (ice or ferroelectric) model and then including the full Baxter eight-vertex model. The discussion in Sec. II.D is primarily intended to introduce the definitions and formalism of two-dimensional lattice models. Only general features of the Bethe *Ansatz* solutions are mentioned since the details are much more elegantly treated by the quantum inverse method, which will be done in Sec. III.

A. The nonlinear Schrödinger model

The nonlinear Schrödinger model in one-space dimension is defined by the Hamiltonian operator

$$H=\int (\partial_x\phi^*\partial_x\phi + c\phi^*\phi^*\phi\phi)dx, \tag{2.1}$$

where $\phi(x)$ is a nonrelativistic boson field with canonical equal time commutation relations,

$$[\phi(x), \phi^*(y)]=\delta(x-y). \tag{2.2}$$

Note that Eq. (2.1) is in the standard form of a many-body problem with the second term corresponding to a two-body delta-function potential. The Hamiltonian (2.1) commutes with the particle number operator $N=\int \phi^*\phi\, dx$, and we may therefore consider each N-body sector of the Hilbert space separately.

Before constructing the Bethe *Ansatz* solution to the model, it is amusing to study the two-body system by more conventional means. In old-fashioned time-independent perturbation theory, we may construct "in" eigenstates of H by expanding the Lippmann-Schwinger equation,

$$|\Psi(k_1,\ldots,k_N)\rangle_{\text{in}}=\sum_{i=0}^{\infty}[G_0(\omega_k)V]^N|k_1,\ldots,k_N\rangle, \tag{2.3}$$

where G_0 is the free particle Green's function operator

$$G_0(\omega)=\frac{1}{\omega-H_0+i\varepsilon}, \tag{2.4}$$

and H_0 and V are the first and second terms in (2.1), respectively. In (2.3) the state $|k_1,\ldots,k_N\rangle$ is an eigenstate of H_0 constructed by repeated application of momentum space creation operators

$$a_k^*=\int dx\, e^{ikx}\phi^*(x). \tag{2.5}$$

The energy of the state is

$$\omega_k=\sum_{i=1}^{N}k_i^2. \tag{2.6}$$

The terms in the expansion (2.3) may be represented graphically in the usual way. For $N=2$ the graphical expansion is a simple sum of bubbles as shown in Fig. 1. The first term in (2.3) is just the free state $|k_1,k_2\rangle$, which we write as

$$\int dx_1dx_2e^{i(k_1x_1+k_2x_2)}\phi^*(x_1)\phi^*(x_2)|0\rangle. \tag{2.7}$$

The second term gives

$$\int \frac{dp_1}{2\pi}\frac{dp_2}{2\pi}\left(\frac{1}{k_1^2+k_2^2-p_1^2-p_2^2+i\varepsilon}\right)(4c)(2\pi)\delta(k_1+k_2-p_1-p_2)a_{p_1}^*a_{p_2}^*|0\rangle$$

$$=\left(\frac{2ic}{k_1-k_2}\right)\int dx_1dx_2\theta(x_2-x_1)e^{i(k_1x_1+k_2x_2)}\phi^*(x_1)\phi^*(x_2)|0\rangle, \tag{2.8}$$

where we have taken $k_1 < k_2$. The higher order terms have the same structure as (2.8) with extra factors of $ic/(k_1 - k_2)$ from the loop integrations in Fig. 1,

$$2c \int \frac{dp_1/2\pi}{k_1^2 + k_2^2 - p_1^2 - p_2^2 + i\varepsilon} = \frac{ic}{k_1 - k_2} .$$

If we write $1 = \theta(x_1 - x_2) + \theta(x_2 - x_1)$ in (2.7), the coefficient of $\theta(x_2 - x_1)$ in the wave function sums up geometrically,

$$1 + 2\left(\frac{ic}{k_1 - k_2}\right) + 2\left(\frac{ic}{k_1 - k_2}\right)^2 + \cdots = \frac{k_2 - k_1 - ic}{k_2 - k_1 + ic} \equiv e^{i\Delta(k_2 - k_1)} . \tag{2.9}$$

Thus, the full two-body in-state for $k_1 < k_2$ is

$$|\Psi(k_1, k_2)\rangle_{\text{in}} = \int dx_1 dx_2 [\theta(x_1 - x_2) + \theta(x_2 - x_1) e^{i\Delta(k_2 - k_1)}] e^{i(k_1 x_1 + k_2 x_2)} \phi^*(x_1)\phi^*(x_2)|0\rangle . \tag{2.10}$$

It is not hard to show that for $k_1 > k_2$ (2.10) is an out eigenstate. From this we find that the in state and out state are related by a phase

$$|\Psi(k_1, k_2)\rangle_{\text{in}} = e^{i\Delta(k_2 - k_1)} |\Psi(k_1, k_2)\rangle_{\text{out}} , \tag{2.11}$$

and hence $e^{i\Delta}$ is the two-body S matrix. The in and out states are the properly normalized eigenstates of H, which follows from the fact that the right-hand side of (2.3) may be written as $U(0, -\infty)|k_1, \ldots k_N\rangle$, where U is the unitary time development operator in the interaction picture. The orthonormality of these states may also be checked directly (Gaudin, 1971; Thacker, 1977). Another normalization which is often convenient is obtained by multiplying (2.10) by a factor $[1 + ic/(k_2 - k_1)]$, giving the unnormalized eigenstate

$$\begin{aligned} |\Phi(k_1, k_2)\rangle &= \left(1 + \frac{ic}{k_2 - k_1}\right) |\Psi(k_1, k_2)\rangle_{\text{in}} \\ &= \int dx_1 dx_2 \left(1 - \frac{ic}{k_1 - k_2} \varepsilon(x_1 - x_2)\right) e^{i(k_1 x_1 + k_2 x_2)} \phi^*(x_1)\phi^*(x_2)|0\rangle . \end{aligned} \tag{2.12}$$

The fact that (2.10) or (2.12) is an eigenstate of H may be verified directly by applying the operator (2.1). Using integration by parts to bring the derivatives in H_0 onto the wave function, we find

$$H|\Phi(k_1, k_2)\rangle = (k_1^2 + k_2^2)|\Phi(k_1, k_2)\rangle . \tag{2.13}$$

Note that terms involving $\delta(x_1 - x_2)$ coming from kinetic-energy derivatives acting on the $\varepsilon(x_1 - x_2)$ in (2.12) cancel against terms from the interaction Hamiltonian V applied to the state.

In addition to the scattering states with k_1 and k_2 real, we may also construct a two-body bound state for the case of attractive coupling $c < 0$ by allowing the k's to be complex. To avoid a wave function which blows up when x_1 or x_2 becomes large, we keep the total momentum $K = k_1 + k_2$ real and choose $(k_1 - k_2)$ such that the quantity in the large parentheses in (2.12) vanishes for one ordering of the x's, viz., $k_1 - k_2 = ic$, giving

$$2 \int dx_1 dx_2 \theta(x_2 - x_1) e^{(1/2)c(x_1 - x_2)} \times e^{(1/2)iK(x_1 + x_2)} \phi^*(x_1)\phi^*(x_2)|0\rangle . \tag{2.14}$$

For $c < 0$ this is a bound state with a normalizable internal wave function

$$\psi_B(x_1, x_2) = e^{(1/2)c|x_1 - x_2|} . \tag{2.15}$$

The energy of the state (2.14) is $E = k_1^2 + k_2^2 = \frac{1}{2}(K^2 - c^2)$, i.e., the binding energy is $-\frac{1}{2}c^2$.

The two-body eigenstates (2.10) or (2.12) were easily constructed by conventional means. The penetrating insight of Bethe (1931), adapted to the nonlinear Schrödinger model by Lieb and Liniger (1963), was that exact eigenstates of H for the N-body system could be written down by a rather natural generalization of the two-body case. The Bethe *Ansatz* for this model is most easily written in the unnormalized form which generalizes (2.12),

$$\begin{aligned} |\Phi(k_1, \ldots, k_N)\rangle = \int \left(\prod_{i=1}^{N} e^{ik_i x_i} dx_i\right) \\ \times \prod_{i<j\leq N} \left\{1 - \frac{ic}{k_i - k_j} \varepsilon(x_i - x_j)\right\} \\ \times \phi^*(x_1) \cdots \phi^*(x_N)|0\rangle . \end{aligned} \tag{2.16}$$

By applying (2.1) to (2.16), it is again found that, by

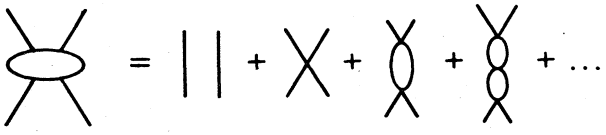

FIG. 1. Sum of graphs for the two-body wave function.

virtue of cancellations between δ-function terms from H_0 and δ-function terms from V, the state (2.16) is an eigenstate of H,

$$H|\Phi(k_1,\ldots,k_N)\rangle=\left(\sum_{i=1}^{N}k_i^2\right)|\Phi(k_1,\ldots,k_N)\rangle\,. \quad (2.17)$$

Equation (2.17) may be verified directly by showing that the wave function in (2.16) satisfies a many-body Schrödinger equation (Lieb and Liniger, 1963). The wave functions have also been obtained by summing Feynman graphs (Thacker, 1976). In Sec. III a particularly simple derivation of the result (2.17) will be obtained by the quantum inverse method.

As before, we may construct bound states for $c<0$ by letting the k_i's in (2.16) become complex. To obtain an N-body bound state we keep the total momentum $K=\sum_{i=1}^{N}k_i$ real and arrange the k_i's to be spaced by ic in the imaginary direction, i.e.,

$$\begin{aligned}k_1&=\frac{K}{N}+\tfrac{1}{2}(N-1)ic\,,\\ k_2&=\frac{K}{N}+\tfrac{1}{2}(N-3)ic\,,\\ &\vdots\\ k_N&=\frac{K}{N}-\tfrac{1}{2}(N-1)ic\,.\end{aligned} \quad (2.18)$$

Such a configuration of modes is called an "N string" and is shown in Fig. 2. By the choice (2.18) we cause one or more of the factors in the curly brackets in (2.16) to vanish unless the x_i's are arranged in one particular order, $x_1<x_2<\cdots<x_N$. After symmetrization, the N-body bound-state wave function in the rest frame is

$$\psi_B(x_1,x_2,\ldots,x_N)=\exp\left(\tfrac{1}{2}c\sum_{i<j\leq N}|x_i-x_j|\right). \quad (2.19)$$

The energy of the bound state is

$$E=\sum_{i=1}^{N}k_i^2=\frac{1}{N}K^2-\frac{N(N^2-1)}{12}c^2\,. \quad (2.20)$$

For repulsive coupling $c>0$, the wave-function *Ansatz* (2.16) may be used to study a system of particles at finite density. In the remainder of this section we discuss the periodic boundary condition method for studying the ground state and the excitation spectrum for a finite density system. Similar methods will be used for spectral calculations in the massive Thirring model in Sec. II.B. In this approach a finite density system is constructed as an N-body system in a box of length L, with periodic boundary conditions (PBC's) imposed on the N-body wave functions. In the limit $L\to\infty$ with the density N/L fixed, the PBC's reduce to linear integral equations which determine the spectrum of the theory. In Sec. III we return to the nonlinear Schrödinger model and show how these spectral techniques fit naturally into the quantum inverse formalism.

We consider the states (2.16) in a periodic box of length L, identifying the points $x=\frac{1}{2}L$ and $x=-\frac{1}{2}L$. The many-body wave function

$$\psi(x_1,x_2,\ldots,x_N)=\langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_N)|\Phi(k_1,\ldots,k_N)\rangle \quad (2.21)$$

must then satisfy periodic boundary conditions

$$\psi(-\tfrac{1}{2}L,x_2,\ldots,x_N)=\psi(\tfrac{1}{2}L,x_2\ldots,x_N)\,. \quad (2.22)$$

Periodicity in the other arguments follows from Bose symmetry. The condition (2.22) restricts the allowed values of the k_i's in a state. From (2.16) we find that (2.22) becomes

$$e^{-ik_iL/2}\prod_{j\neq i}\left(1+\frac{ic}{k_i-k_j}\right)=e^{ik_iL/2}\prod_{j\neq i}\left(1-\frac{ic}{k_i-k_j}\right) \quad (2.23)$$

or

$$e^{ik_iL}=\prod_{j\neq i}e^{i\Delta(k_j-k_i)},\quad i=1,2,\ldots,N \quad (2.24)$$

where Δ is the two-body phase shift. The most useful form of the PBC's is found by taking the logarithm of (2.24), for which we must choose a branch of the function

$$\Delta(k)=-i\ln\left(\frac{k-ic}{k+ic}\right)\,. \quad (2.25)$$

If the logarithm of (2.24) is written

$$k_iL=\sum_{j\neq i}\Delta(k_j-k_i)+2\pi n_i,\quad n_i=\text{integer} \quad (2.26)$$

then a change in the branch of (2.25) is equivalent to a redefinition of n_i's. We adopt the convention that $\Delta(k)\to 0$ as $k\to\infty$ and that $\Delta(k)$ has no discontinuity along the real axis, as shown in Fig. 3(a). Note that with this choice of branch, $\Delta(k)$ is not precisely the phase shift due to the interaction. For example, as $c\to 0$, $\Delta(k)$ goes not to zero but to a step function $-2\pi\theta(-k)$. The interaction phase shift is $\tilde{\Delta}(k)$, Fig. 3(b) which vanishes in the $c\to 0$ limit for all k, but which has a discontinuity of -2π at $k=0$ for any finite c. The physical effect of this definition of the phase shift is to intro-

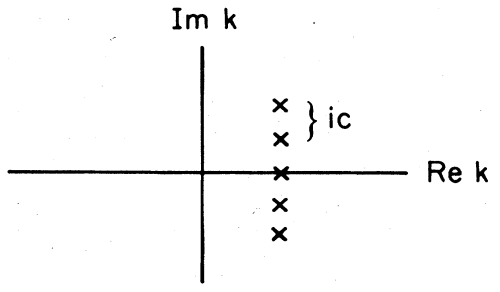

FIG. 2. An n string in the complex k plane, representing a nonlinear Schrödinger model bound state.

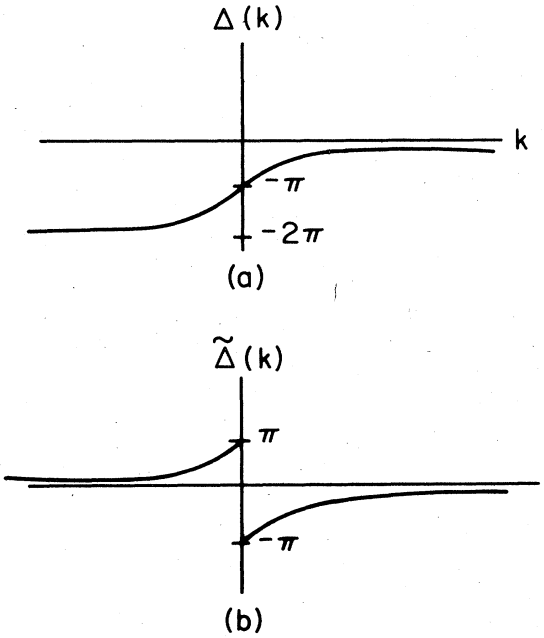

FIG. 3. (a) Phase shift used in periodic boundary conditions (2.26). (b) Interaction phase shift which vanishes for $c \to 0$.

duce a fermionic description of the spectrum in terms of the n_i's in Eq. (2.26). To see this, consider the case of two particles in a box. The two equations (2.26) may be added and subtracted to give the condition that the total momentum $k_1 + k_2$ must be $2\pi/L$ times an integer, while the relative value $k_{12} \equiv k_1 - k_2$ must satisfy

$$k_{12}L = \Delta(-k_{12}) - \Delta(k_{12}) + 2\pi(n_1 - n_2)$$
$$= -4\tan^{-1}(k_{12}/c) + 2\pi(n_1 - n_2)\,, \qquad (2.27)$$

where $-\pi/2 < \tan^{-1} < \pi/2$. Equation (2.27) has a solution k_{12} for each choice of $(n_1 - n_2)$. But for $n_1 = n_2$ the solution is $k_{12} = 0$ and the state (2.10) or (2.12) vanishes identically. Thus, the state $n_1 = n_2$ is excluded as it is in a free fermion system. In fact (2.27) shows that in the limit $c \to \infty$ (impenetrable bosons) the spectrum is identical to that of free fermions. A similar result holds for N-body states. The hard-core repulsive interaction has the same effect on the density of states as the exclusion principle has in a fermion system.

The ground-state k distribution is obtained from (2.26) by choosing the n_i's to be as closely spaced as possible, i.e., $n_{i+1} = n_i + 1$. Subtracting the PBC's for adjacent k_i's gives

$$k_{i+1} - k_i = \frac{1}{L}\sum_j \Delta(k_j - k_{i+1}) - \Delta(k_j - k_i) + \frac{2\pi}{L}\,. \qquad (2.28)$$

As $L \to \infty$, the k_i's become infinitesimally spaced, and the quantity

$$\rho(k_i) = \frac{1}{L(k_{i+1} - k_i)} \qquad (2.29)$$

approaches a continuous function. The sum in (2.28) can be replaced by an integral

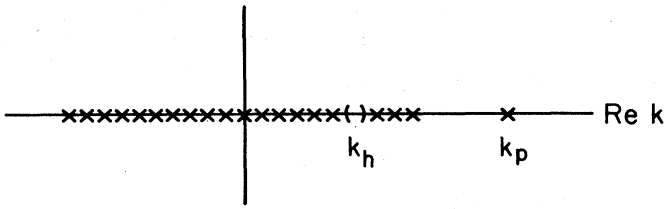

FIG. 4. Particle-hole excitation above a finite density ground state in the nonlinear Schrödinger model.

$$\frac{1}{L}\sum_j \cdots \to \int_{-k_F}^{k_F} dk\,\rho(k)\cdots\,, \qquad (2.30)$$

where k_F is determined from the particle density N/L by $\int_{-k_F}^{k_F}\rho(k)dk = N/L$. Equation (2.28) becomes an integral equation for the ground-state density $\rho(k)$,

$$2\pi\rho(k) = 1 + \int_{-k_F}^{k_F} K(k - k')\rho(k')dk'\,, \qquad (2.31)$$

where the kernel K is given by

$$K(k) = \frac{d\Delta(k)}{dk} = \frac{2c}{k^2 + c^2}\,. \qquad (2.32)$$

The ground-state energy is obtained from the solution to (2.31),

$$E_0/L = \int_{-k_F}^{k_F} k^2\rho(k)dk\,. \qquad (2.33)$$

The excitations above the ground state consist of "particles" which are filled modes above the Fermi surface and "holes" which are empty modes below the Fermi surface. For simplicity let us consider a single particle-hole excitation formed by removing a mode from below the surface at $k = k_h$ and placing it above the surface at $k = k_p$, as shown in Fig. 4. In response to such an excitation, the Fermi sea will shift slightly in a manner described by the PBC's. Each mode in the sea shifts only by an amount of order L^{-1}, but since the number of modes is of order L, the shift or "backflow" of the sea makes a finite contribution to the excitation energy. Denote the k values of the sea modes in the excited state by k_i'. These will satisfy the PBC's:

$$k_i'L = \sum_{j\neq i} \Delta(k_j' - k_i') + \Delta(k_p - k_i') - \Delta(k_h - k_i') + 2\pi n_i\,. \qquad (2.34)$$

Subtracting from this the corresponding ground-state PBC gives

$$(k_i' - k_i)L = \sum_j [\Delta(k_j' - k_i') - \Delta(k_j - k_i)]$$
$$+ \Delta(k_p - k_i') - \Delta(k_h - k_i')\,. \qquad (2.35)$$

As $L \to \infty$, the left-hand side of (2.35) approaches a continuous function which we denote by $w(k)$,

$$(k_i' - k_i)L \to w(k_i)\,. \qquad (2.36)$$

Then (2.35) reduces to

$$w(k)=\int_{-k_F}^{k_F}K(k-k')[w(k')-w(k)]\rho(k')dk' + \Delta(k_p-k)-\Delta(k_h-k)\,. \tag{2.37}$$

Using (2.31) and defining $w(k)\rho(k)\equiv F(k)$, (2.37) simplifies to

$$2\pi F(k)=\int_{-k_F}^{k_F}K(k-k')F(k')dk'+\Delta(k_p-k)-\Delta(k_h-k)\,. \tag{2.38}$$

The excitation energy $E-E_0\equiv E_1$ is found by subtracting eigenvalues,

$$E_1=k_p^2-k_h^2+\sum_{\text{sea}}(k_i'^2-k_i^2) =k_p^2-k_h^2+\int_{-k_F}^{k_F}2kF(k)dk\,. \tag{2.39}$$

Thus, the excitation energy may be regarded as the sum of the bare energy of the particle and hole and a backflow energy of the sea, the latter being written as an integral over the function $F(k)$. Equations (2.38) and (2.39) are the essential results of the Lieb analysis. A further simplification of these results may be obtained by adapting an argument of Yang and Yang (1969). We write Eq. (2.38) as

$$\int_{-k_F}^{k_F}\frac{dk'}{2\pi}[2\pi\delta(k-k')-K(k-k')]F(k')=\int_{k_h}^{k_p}K(k-k')\frac{dk'}{2\pi}\,, \tag{2.40}$$

where (2.32) has been used. Equation (2.40) can be abbreviated as

$$(1-K)F(k)=\int_{k_h}^{k_p}K(k-k')\frac{dk'}{2\pi}\,. \tag{2.41}$$

Next consider a quantity $L(k,q)$ given by

$$L(k,q)=[(1-K)^{-1}K](k,q) \equiv K(k-q)+\int_{-k_F}^{k_F}\frac{dk'}{2\pi}K(k-k')K(k'-q)+\cdots\,. \tag{2.42}$$

Then applying $(1+L)$ to both sides of (2.41) and using $(1+L)(1-K)=1$ and $(1+L)K=L$, we obtain

$$F(k)=\int_{k_h}^{k_p}L(k,k')\frac{dk}{2\pi}\,. \tag{2.43}$$

The excitation energy (2.39) can now be written

$$E_1=k_p^2-k_h^2+\int_{k_h}^{k_p}\frac{dk'}{2\pi}\int_{-k_F}^{k_F}2k\,dk\,L(k,k')\,. \tag{2.44}$$

For reasons which will become clear we define a function $\varepsilon(k)$ which is the solution to the integral equation

$$\varepsilon(k)=k^2-\mu_0+\int_{-k_F}^{k_F}\frac{dk'}{2\pi}K(k-k')\varepsilon(k')\,. \tag{2.45}$$

Here $\mu_0=$ chemical potential is a constant which is fixed by the requirement $\varepsilon(\pm k_F)=0$. [Note that $\varepsilon(k)$ is an even function of k.] Differentiating both sides of (2.45) and integrating by parts, we get an equation for $\varepsilon'(k)=d\varepsilon(k)/dk$,

$$\varepsilon'(k)=2k+\int_{-k_F}^{k_F}\frac{dk'}{2\pi}K(k-k')\varepsilon'(k')\,. \tag{2.46}$$

Using the same sort of procedure that led to (2.43), Eq. (2.46) can be solved, giving

$$\varepsilon'(k)=2k+\int_{-k_F}^{k_F}L(k,k')2k'\frac{dk'}{2\pi}\,. \tag{2.47}$$

Equation (2.44) becomes

$$E_1=k_p^2-k_h^2+\int_{k_h}^{k_p}[\varepsilon'(k)-2k]dk=\varepsilon(k_p)-\varepsilon(k_h)\,. \tag{2.48}$$

Thus, the structure of the particle-hole spectrum is completely summarized by a function $\varepsilon(k)$ which gives the physical energy (bare energy plus backflow) of a single excited mode. The function $\varepsilon(k)$ is determined by a simple integral equation, Eq. (2.45), which may be regarded as the fundamental spectral equation of the theory. The generalization of (2.45) to finite temperature was obtained by Yang and Yang (1969) using a variational method. In Sec. III we will see that the structure of the finite temperature spectral integral equation for $\varepsilon(k)$ is closely related to the structure of the Gel'fand-Levitan transform for the charge-density operator $j_0(x)=\phi^*(x)\phi(x)$.

B. The massive Thirring model

Recently it has been found that certain relativistic fermion field theories can be exactly solved by techniques which closely parallel those used in the nonlinear Schrödinger model. Early work by Thirring (1958) in the massless case and by Berezin and Sushko (1965) in the massive case considered a wave function *Ansatz* built on an unphysical reference state but did not carry out the PBC integral equation analysis needed to reach the physical Hilbert space. The problem of recovering the physical theory in the massless case was resolved by Lieb and Mattis (1965). The massive Thirring model *Ansatz* was recently rediscovered (Bergknoff and Thacker, 1979), and the construction of the physical states and eigenvalue spectrum was also carried out. This recent work was stimulated by Coleman's (1975) demonstration of the equivalence between the massive Thirring and quantum sine-Gordon models along with the emerging connection between Bethe's *Ansatz* and inverse scattering methods. This application of Bethe's *Ansatz* was also suggested by the work of Luther (1976), who pointed out that the massive Thirring model could

be regarded as the continuum limit of the XYZ Heisenberg spin chain. The method of solution we discuss in this section is related to the treatment of the XYZ spin-chain and eight-vertex model by Baxter (1972a, b; 1973a, b, c) and Johnson, Krinsky, and McCoy (1973). The connection between the spectral calculations for the two models has been discussed (Bergknoff and Thacker, 1979).

The massive Thirring model consists of a Dirac fermion field with a local $j_\mu j^\mu$ coupling, where $j_\mu = \frac{1}{2}[\bar{\psi}, \gamma_\mu \psi]$. Choosing a basis in which γ^5 is diagonal, we may write the Hamiltonian in terms of chiral components ψ_1 and ψ_2,

$$H = \int dx \Big[-i\Big(\psi_1^\dagger \frac{\partial}{\partial x}\psi_1 - \psi_2^\dagger \frac{\partial}{\partial x}\psi_2\Big) + m_0(\psi_1^\dagger\psi_2 + \psi_2^\dagger\psi_1) + 2g_0\psi_1^\dagger\psi_2^\dagger\psi_2\psi_1 \Big]. \tag{2.49}$$

Here we have chosen a certain ordering for the field operators which will facilitate the diagonalization of H. The interaction term in (2.49) differs from conventional ordering by a term proportional to the fermion number operator $N = \int (\psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2)dx$, which commutes with H. Thus, we will get the correct physical spectrum by considering energy differences between states with the same value of N, e.g., excitation energies of neutral bound states or fermion-antifermion pairs relative to the physical vacuum.

The form of the Bethe *Ansatz* which diagonalizes (2.49) can be motivated by first considering the diagonalization of the free Hamiltonian H_0, given by (2.49) with $g_0 = 0$. This also serves to introduce the idea of complex rapidity, which is extremely useful in describing the eigenstates of the full theory. To diagonalize H_0 we first construct momentum space creation operators

$$a_{1,2}^\dagger(k) = \int dx\, e^{-ikx}\psi_{1,2}^\dagger(x) \tag{2.50}$$

and mix them by a Bogoliubov rotation,

$$A_1^\dagger(k) = \cos\theta(k)a_1^\dagger(k) + \sin\theta(k)a_2^\dagger(k), \tag{2.51a}$$

$$A_2^\dagger(k) = -\sin\theta(k)a_1^\dagger(k) + \cos\theta(k)a_2^\dagger(k), \tag{2.51b}$$

where $\cot 2\theta(k) = k/m_0$. In terms of these operators, the free Hamiltonian is diagonal,

$$H_0 = \int \frac{dk}{2\pi}(k^2 + m_0^2)^{1/2}[A_1^\dagger(k)A_1(k) - A_2^\dagger(k)A_2(k)]. \tag{2.52}$$

Thus $A_1^\dagger(k)$ and $A_2^\dagger(k)$ create eigenmodes with energies $(k^2+m_0^2)^{1/2}$ and $-(k^2+m_0^2)^{1/2}$, respectively. Eigenstates of H_0 can be built upon a reference state $|0\rangle$ which is annihilated by the field operators, i.e., $\psi_1(x)|0\rangle = \psi_2(x)|0\rangle = 0$. This is an unphysical state which has all positive *and* negative energy modes empty. The physical vacuum is obtained by filling the negative energy sea. It is convenient to visualize a state as a collection of points in complex rapidity space, each point representing a filled mode. Define the rapidity ξ by $k = m_0 \sinh\xi$ and consider a local rotation of the chiral components of the field operator with $\cot 2\theta(\xi) = \sinh\xi$,

$$A^\dagger(\xi, x) = \cos\theta(\xi)\psi_1^\dagger(x) + \sin\theta(\xi)\psi_2^\dagger(x) = (2\cos\xi)^{-1/2}[e^{\xi/2}\psi_1^\dagger(x) + e^{-\xi/2}\psi_2^\dagger(x)]. \tag{2.53}$$

It is clear that the positive energy modes (2.51a) are obtained by taking $\xi = \alpha = \text{real}$,

$$A_1^\dagger(k) = \int dx\, e^{-ixm_0\sinh\alpha}A^\dagger(\alpha, x). \tag{2.54a}$$

Here and elsewhere we use ξ to denote general complex values of rapidity while α will always be real. Letting $\xi = i\pi - \alpha$ with α real, we also obtain from (2.53) the negative energy modes (2.51b),

$$A_2^\dagger(k) = \int dx\, e^{-ixm_0\sinh\alpha}A^\dagger(i\pi - \alpha, x). \tag{2.54b}$$

Thus, a positive energy mode is represented by a point along the real ξ axis, while a negative energy mode is a point along the line $\mathrm{Im}\,\xi = \pi$. The physical vacuum is the state with all modes on the $i\pi$ line filled, as shown in Fig. 5. The density of modes along this line is determined by periodic boundary conditions in a box of length L. For the free theory this gives a uniform density in k space, $k_n = 2\pi n/L$. Excited states representing fermion-antifermion pairs may be constructed by removing modes from the $i\pi$ line and placing them on the real axis.

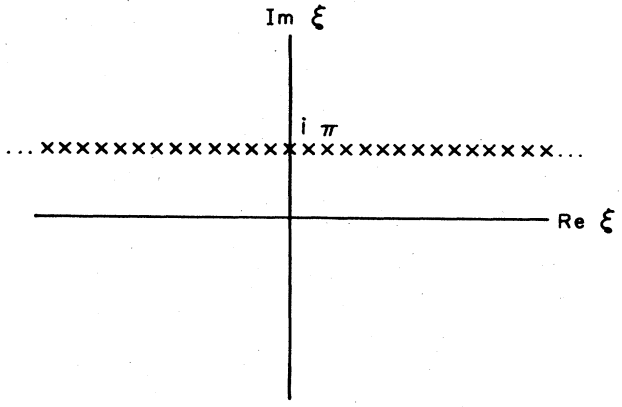

FIG. 5. Physical vacuum for the massive Thirring model.

Now let us consider the full massive Thirring Hamiltonian. We construct a Bethe *Ansatz* using the rotated local operator $A^\dagger(\xi, x)$ defined in (2.55). The *Ansatz* is structurally similar to that of Eq. (2.16) for the nonlinear Schrödinger equation,

$$|\Phi(\xi_1 \cdots \xi_N)\rangle = \int \prod_{i=1}^{N} (e^{im_0 x_i \sinh\xi_i} dx_i) \prod_{i<j\le N} \{1 - i\lambda(\xi_i - \xi_j)\varepsilon(x_i - x_j)\} A^\dagger(\xi_1, x_1)\cdots A^\dagger(\xi_N, x_N)|0\rangle. \tag{2.55}$$

The function $\lambda(\xi)$ in (2.55) is given by

$$\lambda(\xi) = \tfrac{1}{2} g_0 \tanh(\tfrac{1}{2}\xi) . \tag{2.56}$$

This corresponds to a two-body phase shift

$$\Delta(\xi) = 2\tan^{-1}\lambda(\xi) = 2\tan^{-1}(\cot\mu \tanh\tfrac{1}{2}\xi), \tag{2.57}$$

where $\mu = -\cot(\frac{1}{2}g_0)$. By applying the Hamiltonian (2.49) to the state (2.55) it can be shown that (Bergknoff and Thacker, 1979)

$$H|\Phi(\xi_1 \cdots \xi_N)\rangle = \left(\sum_{i=1}^{N} m_0 \cosh\xi_i\right)|\Phi(\xi_1 \cdots \xi_N)\rangle . \tag{2.58}$$

As with (2.17), this result follows from a cancellation between $\delta(x_i - x_j)$ terms from the kinetic energy derivatives in (2.49) acting on the ε step function in (2.55) and similar terms from the interaction Hamiltonian. Modes along the real axis give a positive contribution to the energy, while modes along the $i\pi$ line, $\mathrm{Im}\,\xi = \pi$, carry negative energy. Periodicity in $\mathrm{Im}\,\xi$ allows us to restrict our considerations to the strip $-\pi < \mathrm{Im}\,\xi \leq \pi$. In this strip, values of $\mathrm{Im}\,\xi$ aside from 0 or π generally lead to a wave function which grows exponentially as $x \to \pm\infty$. However, there are special "n-string" configurations in the complex ξ plane for which the wave function is exponentially damped, as in the case of the nonlinear Schrödinger bound states. In the present case, an n-string configuration is determined from the zeroes of the curly bracket factors in (2.55). This gives a vertical row of equally spaced points at $\xi = \alpha_s + il(\pi - \mu)$, where $l = (n-1), (n-3), \ldots, -(n-1)$, and α_s is the position of the string along the real axis as shown in Fig. 6.

To construct the physical vacuum we must fill all the negative energy modes along the $i\pi$ line just as in the free fermion theory. (The fact that this state is the correct vacuum may be inferred from the observation that all other states in the neutral sector have positive excitation energy.) The filling of the Dirac sea is accomplished by the same PBC integral equation method used for the finite density delta-function gas. Imposing

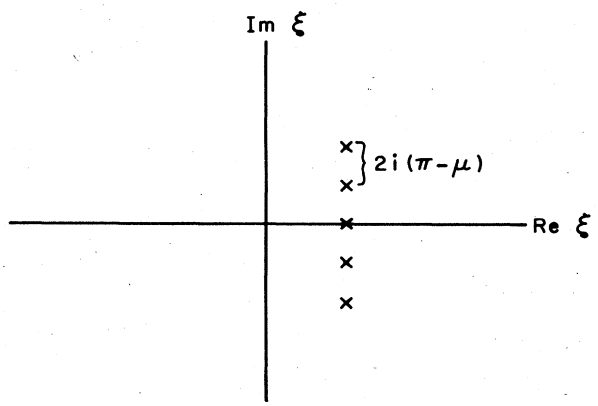

FIG. 6. An n string in the complex rapidity plane for the massive Thirring model.

PBC's on the wave function for the state (2.55) and following the same procedure which led to (2.24), we find that the set of ξ_i's in a state must satisfy

$$\exp(-\tfrac{1}{2} i m_0 L \sinh\xi_i) \prod_{j\neq i} [1 + i\lambda(\xi_i - \xi_j)]$$
$$= \exp(\tfrac{1}{2} i m_0 L \sinh\xi_i) \prod_{j\neq i} [1 - i\lambda(\xi_i - \xi_j)] . \tag{2.59a}$$

Before addressing the problem of the Dirac sea, consider first the case where ξ_i is a member of an n string. If $0 < \mathrm{Im}\,\xi_i < \pi$, there is an adjacent mode in the string with $\xi_j = \xi_i - 2i(\pi - \mu)$ for which $\lambda(\xi_i - \xi_j) = i$. Thus the left-hand side of (2.59) is zero, while the right-hand side vanishes as $L \to \infty$. Similarly, for $-\pi < \mathrm{Im}\,\xi_i < 0$ with ξ_i in an n string, there is a ξ_j which causes the right-hand side to be zero, while the left-hand side vanishes exponentially. This is just a restatement of the observation that an n-string configuration yields an asymptotically damped wave function and thus satisfies the periodic boundary conditions as $0 = 0$. Notice that this argument also restricts the length of an n string to lie within the strip $-\pi < \mathrm{Im}\,\xi_i < \pi$, since otherwise the side of (2.59a) which is not identically zero will grow exponentially instead of vanishing. Let us denote regions of the coupling μ by an integer r, where

$$\frac{r\pi}{(r+1)} < \mu < \frac{(r+1)\pi}{(r+2)} . \tag{2.60}$$

Then the length restriction on an n string imposed by (2.59a) is $n \leq r+2$. For example, the free fermion case $\mu = \pi/2$ lies on the boundary between regions $r = 0$ and $r = 1$. On both sides of this boundary, 1 and 2 strings are allowed, while 3 strings become allowed for $\mu > \pi/2$. As μ increases toward π, strings of increasing length enter the spectrum.

To calculate the energies of the physical states we must take into account the behavior of the Dirac sea. (Note that the previous discussion about the PBC's for n-string modes is unaffected by the presence of sea modes.) To determine this behavior we consider (2.59a) when ξ_i is a mode in the sea. In this case both sides of (2.59a) will generally be nonzero, so we take the log, giving

$$-m_0 L \sinh\xi_i = \sum_j \Delta(\xi_i - \xi_j) + 2\pi n_i , \tag{2.59b}$$

where Δ is given in (2.57), which can be written

$$\Delta(\xi) = -i \ln\left(-\frac{\sinh\frac{1}{2}(\xi - 2i\mu)}{\sinh\frac{1}{2}(\xi + 2i\mu)}\right) . \tag{2.61}$$

For the vacuum state, all ξ_i's are on the $i\pi$ line. As $L \to \infty$, Eq. (2.59b) reduces to an integral equation for the ground-state density $\rho(\xi)$ of modes along the $i\pi$ line,

$$2\pi\rho(\xi) = -m_0 \cosh\xi - \int_{-\Lambda+i\pi}^{\Lambda+i\pi} d\xi' K(\xi - \xi')\rho(\xi') . \tag{2.62a}$$

Since $\xi = \alpha + i\pi$ with α real, this may be written

$$2\pi\rho_1(\alpha) = m_0 \cosh\alpha - \int_{-\Lambda}^{\Lambda} d\alpha' K(\alpha - \alpha')\rho_1(\alpha') , \tag{2.62b}$$

where $\rho_1(\alpha)=\rho(\alpha+i\pi)$. [The change in relative sign of the terms in (2.62) compared with (2.31) is related to the fact that $\xi_i=\alpha_i+i\pi$ in (2.59b), requiring the n_i's to be defined as shown to form an increasing sequence.] The kernel is given by the phase shift derivative,

$$K(\alpha)=\frac{d\Delta}{d\alpha}=\frac{\sin 2\mu}{\cosh\alpha-\cos 2\mu}. \tag{2.63}$$

In (2.62) we have introduced a rapidity cutoff Λ which is needed to regularize ultraviolet divergent mode sums. These divergences will be absorbed by mass renormalization. Equation (2.62) can be solved explicitly for large Λ by Fourier transformation. When expressed in terms of the renormalized fermion mass m_F, $\rho_1(\alpha)$ is found to be finite and given by $\rho_1(\alpha)=m_F\cosh\gamma\alpha$, where γ is defined below in Eq. (2.86). I omit the details of the calculation, since they are identical to the solution of Eq. (2.76) for the spectral function $\varepsilon(\xi)$, which will be discussed in detail.

The physical particle states in the neutral sector of the theory are constructed by removing modes from the $i\pi$ line and placing them in n-string configurations about the real axis. We will consider first a simple excitation obtained by removing a mode from the $i\pi$ line at ξ_h and placing it at a point ξ_p as shown in Fig. 7. We study the response of the vacuum to such an excitation by subtracting ground-state from excited-state PBC's for a mode along the $i\pi$ line. Note that the PBC's for the vacuum modes do not require the excited modes to form an n string. (This requirement comes from the PBC's for the excited modes themselves.) This allows us to consider the vacuum response for a single excited mode at arbitrary ξ_p. The construction of multiple excitations and n strings will then be straightforward. Just as in the derivation of (2.38) for the nonlinear Schrödinger model, we define a function $w(\xi)$ which measures the shift of a sea mode due to the excitation and subtract PBC's to obtain an integral equation for the backflow function $F(\xi)=w(\xi)\rho(\xi)$,

$$2\pi F(\xi)+\int_{-\Lambda+i\pi}^{\Lambda+i\pi}K(\xi-\xi')F(\xi')d\xi'=\Delta(\xi-\xi_p)-\Delta(\xi-\xi_h). \tag{2.64}$$

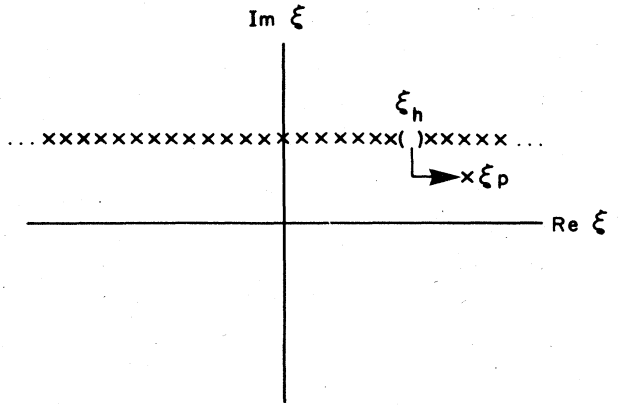

FIG. 7. Particle-hole excitation in the massive Thirring model.

In terms of $F_1(\alpha)\equiv F(\alpha+i\pi)$, the energy of the excitation relative to the ground state is given by

$$E=m_0\cosh\xi_p-m_0\cosh\xi_h+m_0\int_{-\Lambda}^{\Lambda}\sinh\alpha\,\dot{F}_1(\alpha)d\alpha. \tag{2.65}$$

Following the same line of argument that led to (2.48) we define a function $\varepsilon_0(\xi)$ as the solution to the integral equation

$$\varepsilon_0(\xi)=m_0\cosh\xi-\mu_0-\int_{-\Lambda+i\pi}^{\Lambda+i\pi}\frac{d\xi'}{2\pi}K(\xi-\xi')\varepsilon_0(\xi'), \tag{2.66}$$

where μ_0 is chosen to give $\varepsilon_0(\pm\Lambda+i\pi)=0$. In terms of this function, the energy (2.65) assumes a simple form

$$E=\varepsilon_0(\xi_p)-\varepsilon_0(\xi_h). \tag{2.67}$$

It is convenient to eliminate the constant μ_0 from (2.66), which can be done in the limit $\Lambda\to\infty$ by defining

$$\varepsilon(\xi)=\varepsilon_0(\xi)+a, \tag{2.68}$$

where a is a constant which will be chosen to simplify the integral equation for $\varepsilon(\xi)$. Substituting (2.68) into (2.66), we use the fact that

$$\int_{-\Lambda+i\pi}^{\Lambda+i\pi}K(\xi-\xi')\frac{d\xi'}{2\pi}\sim c_1+c_2\cosh\xi, \tag{2.69}$$

where c_1 and c_2 are cutoff dependent constants independent of ξ. The form of (2.69) follows from the asymptotic behavior of the kernel $K(\xi-\xi')$. Now choosing

$$a=\mu_0/(1+c_1), \tag{2.70}$$

and defining

$$m_1=m_0+ac_2, \tag{2.71}$$

we find that the function $\varepsilon(\xi)$ satisfies the integral equation

$$\varepsilon(\xi)=m_1\cosh\xi-\int_{-\Lambda+i\pi}^{\Lambda+i\pi}K(\xi-\xi')\varepsilon(\xi')\frac{d\xi'}{2\pi}. \tag{2.72}$$

Note that the energy (2.67) retains its simple form,

$$E=\varepsilon(\xi_p)-\varepsilon(\xi_h). \tag{2.73}$$

The constant m_1 in (2.71) is a rescaled bare mass which can be expressed in terms of m_0 and the finite coupling parameter μ by

$$m_1=m_0/\gamma, \tag{2.74}$$

where γ is defined below in (2.86). The easiest way to see this is to write an integral equation for $\varepsilon'(\xi)=d\varepsilon(\xi)/d\xi$. Differentiating (2.66) and using $\varepsilon_0(\pm\Lambda+i\pi)=0$, we get

$$\varepsilon'(\xi)=m_0\sinh\xi-\int_{-\Lambda+i\pi}^{\Lambda+i\pi}K(\xi-\xi')\varepsilon'(\xi')\frac{d\xi'}{2\pi}. \tag{2.75}$$

Equation (2.74) follows by comparing solutions to (2.72) and (2.75), which will be discussed below.

We wish to construct the solution to (2.72) for any complex value of ξ in the strip $-\pi<\mathrm{Im}\,\xi\leq\pi$. Our discussion will be restricted to the case $\pi/2<\mu<\pi$, i.e., g_0 positive. This is the range of coupling for which the fermion-antifermion interaction is attractive and the spectrum contains one or more bound states. In the

sine-Gordon equivalence of Coleman (1975) this corresponds to the range $0<\beta^2<4\pi$. [The case $\mu<\pi/2$ has also been discussed (Bergknoff and Thacker, 1979), although some problems remain for $\mu<\pi/3$.] To solve (2.72) for general ξ, we begin by considering the case in which ξ is along the $i\pi$ line. The continuation to general ξ will then be obtained by inserting this result back into the integral equation. Writing $\xi=\alpha+i\pi$ and defining

$$\varepsilon_1(\alpha)=\varepsilon(\alpha+i\pi), \tag{2.76}$$

Eq. (2.72) may be written

$$\varepsilon_1(\alpha)=-m_1\cosh\alpha-\int_{-\Lambda}^{\Lambda}K(\alpha-\alpha')\varepsilon_1(\alpha')\frac{d\alpha'}{2\pi}. \tag{2.77}$$

Now we apply

$$\int_{-\Lambda}^{\Lambda}\frac{d\alpha}{2\pi}e^{i\alpha y}\cdots$$

to both sides of (2.72). The limits of the α integration in the last term can be taken to infinity because the kernel $K(\alpha-\alpha')$ falls off exponentially as $\alpha\to\pm\infty$. Defining

$$\tilde{\varepsilon}_1(y)=\int_{-\Lambda}^{\Lambda}e^{i\alpha y}\varepsilon_1(\alpha)\frac{d\alpha}{2\pi}, \tag{2.78}$$

$$C(y)=\int_{-\Lambda}^{\Lambda}e^{i\alpha y}\cosh\alpha\frac{d\alpha}{2\pi}, \tag{2.79}$$

$$\tilde{K}(y)=\int_{-\infty}^{\infty}e^{i\alpha y}K(\alpha)\frac{d\alpha}{2\pi}, \tag{2.80}$$

Eq. (2.77) gives

$$\tilde{\varepsilon}_1(y)=\frac{-m_1C(y)}{1+\tilde{K}(y)}. \tag{2.81}$$

The Fourier transform of the kernel may be evaluated by contour integration from (2.63) and (2.80), which gives

$$\tilde{K}(y)=\frac{\sinh(\pi-2\mu)y}{\sinh\pi y}. \tag{2.82}$$

The function $C(y)$ given by (2.79) may be written

$$C(y)=C_+(y)+C_-(y), \tag{2.83}$$

where, for $\Lambda\to\infty$,

$$C_\pm(y)=\frac{e^{\Lambda(1\pm iy)}}{4\pi(1\pm iy)}. \tag{2.84}$$

The function $\varepsilon_1(\alpha)$ is recovered from (2.81) by inverting (2.78),

$$\begin{aligned}\varepsilon_1(\alpha)&=\int_{-\infty}^{\infty}dy\,e^{-i\alpha y}\tilde{\varepsilon}(y)\\&=\frac{m_1}{4\pi}\int_{-\infty}^{\infty}dy\,e^{-i\alpha y}\left(\frac{\sinh\pi y}{2\sinh(\pi-\mu)y\cosh\mu y}\right)\\&\quad\times\left(\frac{e^{\Lambda(1+iy)}}{1+iy}+\frac{e^{\Lambda(1-iy)}}{1-iy}\right).\end{aligned} \tag{2.85}$$

In the limit $\Lambda\to\infty$, this integral is dominated by the y-plane poles nearest the real axis. Since we are considering $\mu>\pi/2$, these poles are at $y=\pm i\gamma$, where

$$\gamma=\frac{\pi}{2\mu}. \tag{2.86}$$

From (2.85) we find

$$\varepsilon_1(\alpha)=-m_F\cosh\gamma\alpha, \tag{2.87}$$

where m_F is the renormalized fermion mass

$$m_F=\frac{m_1\gamma e^{\Lambda(1-\gamma)}}{\pi(\gamma-1)}\tan\pi\gamma. \tag{2.88}$$

To determine $\varepsilon(\xi)$ for arbitrary complex ξ, we write Eq. (2.72) using the solution (2.87), to determine the right-hand side. Note also that $\gamma<1$ for $\mu>\pi/2$, and hence with m_F held finite, $m_1\to 0$ as $\Lambda\to\infty$. Thus Eq. (2.72) may be written

$$\varepsilon(\xi)=m_F\int_{-\infty}^{\infty}K(\xi-\alpha'-i\pi)\cosh\gamma\alpha'\frac{d\alpha'}{2\pi}, \tag{2.89}$$

where we have let $\Lambda\to\infty$ in the limits of integration because the integral is convergent.

For $\xi=\alpha+i\pi$ with α real, it is easy to verify that (2.89) reproduces the result (2.87). Using (2.89) to compute $\varepsilon(\xi)$ for general values of ξ requires care in the definition of the kernel $K(\xi)$ for complex ξ. It is not correct to simply replace the real variable α in (2.63) by ξ. $K(\xi)$ is defined as the derivative of the phase shift $\Delta(\xi)$ given by (2.57) or (2.61). As in the nonlinear Schrödinger model, the choice of branch for $\Delta(\xi)$ is correlated with the choice of integers n_i in the PBC equations (2.60). The appropriate branch structure for our choice of n_i's is shown in Fig. 8. Letting $\xi=\alpha+i\sigma$ and regarding $\Delta(\alpha+i\sigma)$ as a function of α, it is seen from Fig. 8 that this function is continuous for $-2(\pi-\mu)<\sigma<2(\pi-\mu)$, while it has a step discontinuity of (-2π) at $\alpha=0$ for $-\pi<\sigma<-2(\pi-\mu)$ and for $2(\pi-\mu)<\sigma<\pi$. [The behavior of $\Delta(\xi)$ over the entire complex plane is specified by its behavior in the strip $-\pi<\mathrm{Im}\,\xi\leq\pi$ along with periodicity in $\mathrm{Im}\,\xi$.] Thus the properly defined kernel $K(\xi)$ will acquire a delta-function term for $|\mathrm{Im}\,\xi|>2(\pi-\mu)$. Defining the continuous function

$$K_c(\xi)=\frac{\sin2\mu}{\cosh\xi-\cos2\mu}, \tag{2.90}$$

we see that, in the strip $-\pi<\sigma<\pi$, the kernel is given

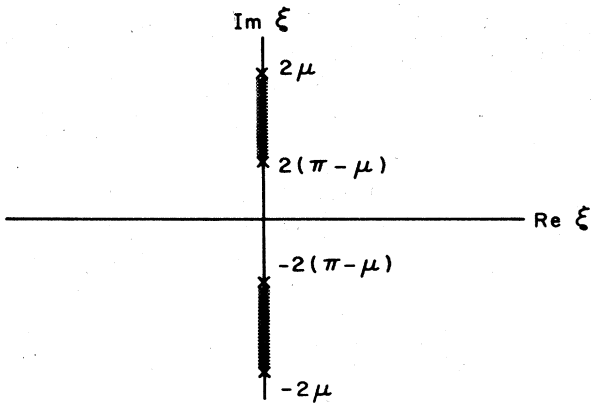

FIG. 8. Analytic structure of the two-body phase shift $\Delta(\xi)$ for the massive Thirring model.

by

$$K(\alpha+i\sigma)=K_c(\alpha+i\sigma), \quad |\sigma|<2(\pi-\mu) \tag{2.91a}$$

$$=K_c(\alpha+i\sigma)-2\pi\delta(\alpha), \quad |\sigma|>2(\pi-\mu). \tag{2.91b}$$

Values outside the strip $-\pi<\sigma\leq\pi$ are obtained by periodicity.

To provide a complete understanding of the spectrum, we must describe the phenomenon of "hole trapping," which occurs when an excited mode ξ_p enters the region

$$|\mathrm{Im}\,\xi_p|<2\mu-\pi, \tag{2.92}$$

which is the shaded region shown in Fig. 9. The unshaded regions in Fig. 9, $\pi>\mathrm{Im}\,\xi_p>2\mu-\pi$ and $\pi-2\mu>\mathrm{Im}\,\xi_p>-\pi$, may be regarded as a single region, $3\pi-2\mu>\mathrm{Im}\,\xi_p>2\mu-\pi$, by periodicity. When ξ_p is above the "threshold line" $\mathrm{Im}\,\xi_p=2\mu-\pi$, its position relative to the hole at ξ_h is unrestricted, as shown in Fig. 9(a). If the mode is pulled down below the threshold line into the shaded region [Fig. 9(b)], the associated hole is suddenly forced to be directly above it, i.e., $\mathrm{Re}\,\xi_h$ must equal $\mathrm{Re}\,\xi_p$. We will find that the bound complex (mode + hole) thus formed is just the fundamental boson of the quantum sine-Gordon model. To see the effect of crossing the threshold line, consider the PBC's (2.60) for a mode in the sea. After separating off the excited mode ξ_p from the rest of the sum, we may write the PBC's in continuum form as

$$-m_0\sinh\xi=\int_{-\Lambda+i\pi}^{\Lambda+i\pi}\Delta(\xi-\xi')\rho_p(\xi')d\xi'+\frac{1}{L}\Delta(\xi-\xi_p)+2\pi h(\xi), \tag{2.93}$$

where

$$h(\xi_i)=\frac{2\pi n_i}{L},$$

and $\rho_p(\xi)$ is the density of modes along the $i\pi$ line for the excited state. In the ground state there are no holes on the $i\pi$ line, meaning that the n_i's are closely packed, $n_{i+1}-n_i=1$. For an excited state $n_{i+1}-n_i=1+n_h$, where n_h is the number of holes between mode i and mode $i+1$. Thus

$$h(\xi_{i+1})-h(\xi_i)=\frac{2\pi}{L}+\frac{2\pi n_h}{L}, \tag{2.94}$$

which can be written in continuum notation as

$$\frac{dh(\xi)}{d\xi}=\rho_p(\xi)+\rho_h(\xi), \tag{2.95}$$

where $\rho_h(\xi)$ is the hole density. An analogous function was considered by Yang and Yang (1969) in their derivation of the thermodynamics of the delta-function gas at finite temperature. For the case of a single hole at $\xi_h=\alpha_h+i\pi$, which we consider here, the hole density may be written

$$\rho_h(\xi)=\frac{1}{L}\delta(\alpha-\alpha_h), \tag{2.96}$$

where $\xi=\alpha+i\pi$. Differentiating (2.93) and using (2.95) and (2.96), we get an equation for $\rho_p(\xi)$,

$$2\pi\rho_p(\xi)+\int_{-\Lambda+i\pi}^{\Lambda+i\pi}K(\xi-\xi')\rho_p(\xi')d\xi'$$

$$=-m_0\cosh\xi-\frac{1}{L}K(\xi-\xi_p)-\frac{1}{L}\delta(\alpha-\alpha_h). \tag{2.97}$$

By studying the singularities of this equation we will demonstrate the phenomenon of hole trapping. Consider first the case shown in Fig. 9(a) with ξ_p above the threshold line, $\mathrm{Im}\,\xi_p>2\mu-\pi$. Taking $\xi=\alpha+i\pi$ in (2.97), we see that, for this case, $K(\xi-\xi_p)$ has no delta-function piece, since $\mathrm{Im}(\xi-\xi_p)<2(\pi-\mu)$. The integral term on the left-hand side also gives no delta-function contribution. Thus the density $\rho_p(\xi)$ must be of the form

$$\rho_p(\alpha+i\pi)=f(\alpha)-\frac{1}{L}\delta(\alpha-\alpha_h), \tag{2.98}$$

where $f(\alpha)$ is continuous in the sense that the integral $\int_\alpha^{\alpha+\Delta}f(\alpha')d\alpha'$ over a small but fixed range Δ is given by a continuous function of α plus terms which fall off *faster* than $1/L$. The result (2.98) may be seen more directly by writing the formal solution of (2.97),

$$\rho_p(\xi)=\frac{1}{2\pi}(1+K)^{-1}\left(-m_0\cosh\xi-\frac{1}{L}K(\xi-\xi_p)-\frac{1}{L}\delta(\alpha-\alpha_h)\right). \tag{2.99}$$

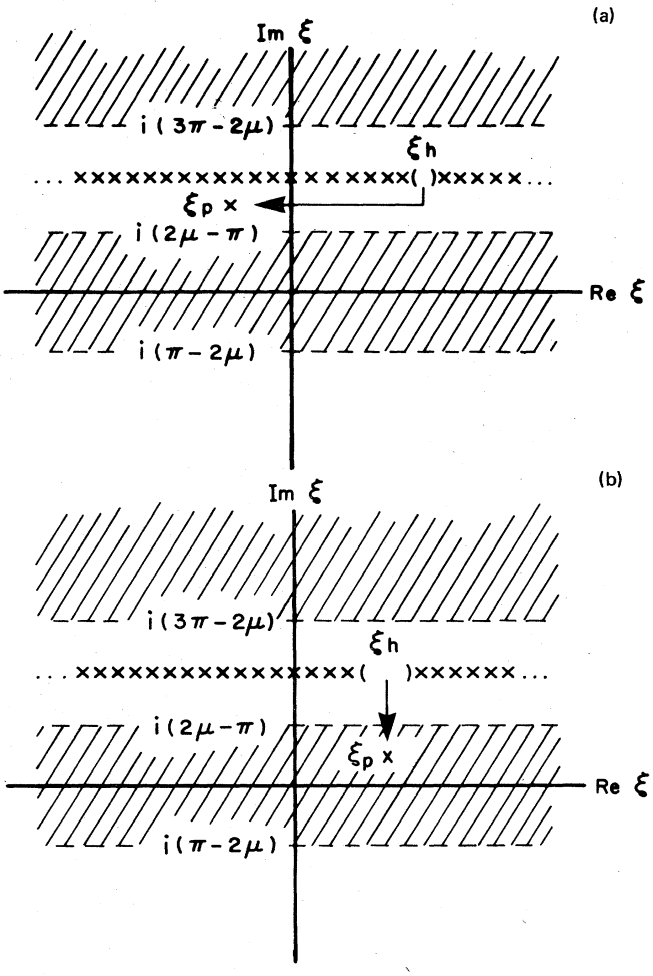

FIG. 9. (a) Excitation of an unbound particle-hole pair. (b) Excitation of a bound particle-hole pair.

The first two terms give a continuous contribution to ρ_p. For the last term we write $(1+K)^{-1}=1-K(1+K)^{-1}$, giving a delta function and a continuous piece from 1 and $K(1+K)^{-1}$, respectively, as can be seen by doing the Fourier transform implicit in (2.99). This shows that (2.98) is the correct form when ξ_p is above the threshold line. In the language of discrete modes, the second term in (2.98) simply means that there is an unoccupied mode on the $i\pi$ line at $\xi=\alpha_h+i\pi$. Now consider what happens when ξ_p moves below the threshold line (2.92). By (2.91), the kernel $K(\xi-\xi_p)$ acquires a delta-function piece,

$$K(\xi-\xi_p)=K_c(\xi-\xi_p)-2\pi\delta(\alpha-\alpha_p), \tag{2.100}$$

where $\alpha=\mathrm{Re}\,\xi$ and $\alpha_p=\mathrm{Re}\,\xi_p$. From (2.99) we find that the mode density along the $i\pi$ line becomes

$$\rho_p(\alpha+i\pi)=f(\alpha)+\frac{1}{L}\delta(\alpha-\alpha_p)-\frac{1}{L}\delta(\alpha-\alpha_h), \tag{2.101}$$

where $f(\alpha)$ is again continuous. Just as a subtracted delta function represents an unoccupied mode on the $i\pi$ line, the second term in (2.101) represents the placement of an additional mode on an already filled line. But this is not possible because of Fermi statistics. (Specifically, if two modes in the Bethe's *Ansatz* wave function (2.55) have the same ξ, the wave function vanishes identically.) The only way to avoid this problem and construct a legitimate state is to choose $\alpha_p=\alpha_h$ so that the last two terms in (2.101) cancel. The excited mode and its hole must bind together when the mode moves below the threshold line, as depicted in Fig. 9(b).

After all these preliminaries, we are now ready to evaluate the function $\varepsilon(\xi)$ from (2.89). In the unshaded region around the $i\pi$ line, $3\pi-2\mu>\mathrm{Im}\,\xi>2\mu-\pi$ [Fig. 9(a)], the kernel has no delta-function piece and is given by (2.91a) and (2.90). The evaluation of the right-hand side of (2.89) by contour integration is essentially the same as for the case $\xi=\alpha+i\pi$. This gives

$$\varepsilon(\xi)=-m_F\cosh\gamma(\xi-i\pi) \tag{2.102a}$$

for $3\pi-2\mu>\mathrm{Im}\,\xi>2\mu-\pi$. If we wish to remain in the strip $-\pi<\mathrm{Im}\,\xi\leq\pi$, periodicity of $\varepsilon(\xi)$ allows us to write (2.102) as

$$\varepsilon(\xi)=-m_F\cosh\gamma(\xi-i\pi),\quad 2\mu-\pi<\mathrm{Im}\,\xi\leq\pi \tag{2.102b}$$

$$=-m_F\cosh\gamma(\xi+i\pi),\quad -\pi<\mathrm{Im}\,\xi<\pi-2\mu. \tag{2.102c}$$

Now let us consider the case where $|\mathrm{Im}\,\xi|<2\mu-\pi$, as in Fig. 9(b). In this region, the kernel $K(\xi-\alpha'-i\pi)$ in (2.89) acquires a delta-function piece according to Eq. (2.91b). Letting $\xi=\alpha+i\lambda$, α and λ real, we write

$$K(\xi-\alpha'-i\pi)=K_c(\xi-\alpha'-i\pi)-2\pi\delta(\alpha-\alpha'). \tag{2.103}$$

Substituting this into (2.89) gives

$$\varepsilon(\xi)=m_F\int_{-\infty}^{\infty}K_c(\xi-\alpha'-i\pi)\cosh\gamma\alpha'\frac{d\alpha'}{2\pi}-m_F\cosh\gamma\alpha, \tag{2.104}$$

which, by (2.102a) may also be written as

$$\varepsilon(\alpha+i\lambda)-\varepsilon(\alpha+i\pi)=m_F\int_{-\infty}^{\infty}K_c[\alpha-\alpha'-i(\pi-\lambda)]\times\cosh\gamma\alpha'\frac{d\alpha'}{2\pi}, \tag{2.105}$$

for $|\lambda|<2\mu-\pi$. But this is also the region in which hole trapping occurs, i.e., a mode in this region at $\xi_p=\alpha+i\lambda$ must have a hole attached to it at $\xi_h=\alpha+i\pi$. The combination on the left-hand side of (2.105) is the same combination that appears in the expression for the excitation energy, Eq. (2.73), and may be regarded as the energy of an elementary boson,

$$\varepsilon_B(\alpha+i\lambda)=\varepsilon(\alpha+i\lambda)-\varepsilon(\alpha+i\pi). \tag{2.106}$$

Evaluation of the right-hand side of (2.105) is straightforward. It is interesting to compare it with the corresponding calculation in the region $|\lambda|>2\mu-\pi$ which led to the result (2.102). In either case we must evaluate the integral

$$m_F\int_{-\infty}^{\infty}K_c[\alpha-\alpha'-i(\pi-\lambda)]\cosh\gamma\alpha'\frac{d\alpha'}{2\pi}\equiv I_++I_-, \tag{2.107}$$

where

$$I_\pm=\tfrac{1}{2}m_F\int_{-\infty}^{\infty}\frac{(\sin2\mu)e^{\pm\gamma\alpha'}}{\cosh[\alpha-\alpha'+i(\pi-\lambda)]-\cos2\mu}\frac{d\alpha'}{2\pi}. \tag{2.108}$$

The integrand has two sequences of poles at $\alpha'=\alpha_n^\pm$ where

$$\alpha_n^\pm=\alpha-i(\pi-\lambda)\pm2i\mu+2in\pi. \tag{2.109}$$

When $|\lambda|>2\mu-\pi$ (the unshaded region in Fig. 9), the poles in the upper half plane are at α_n^+ for $n=0,1,2,\ldots$, and at α_n^- for $n=1,2,\ldots$. The integrals (2.108) are easily evaluated,

$$I_+=-\tfrac{1}{2}m_F e^{\gamma\alpha}e^{-i\gamma(\pi-\lambda)}[(1+e^{2i\gamma\pi}+\cdots)-(e^{2i\gamma\pi}+e^{4i\gamma\pi}+\cdots)]$$

$$=-\tfrac{1}{2}m_F e^{\gamma(\xi-i\pi)}\quad(\xi=\alpha+i\lambda), \tag{2.110}$$

with a similar result for I_-, leading to (2.102). But when the excited mode moves into the shaded region $|\lambda|<2\mu-\pi$, the pole at $\alpha_1^-=\alpha+i\lambda-i(2\mu-\pi)$ crosses from the upper to the lower half plane, with the result that

$$I_+\to-\tfrac{1}{2}m_F e^{\gamma(\xi-i\pi)}(1+e^{2i\gamma\pi})$$

$$=-m_F\cos\pi\gamma\,e^{\gamma\xi}=\tfrac{1}{2}m_B e^{\gamma\xi}, \tag{2.111}$$

where

$$m_B=-2m_F\cos\pi\gamma=2m_F\sin\left(\frac{\pi}{2}(2\gamma-1)\right), \tag{2.112}$$

which is the mass of the elementary sine-Gordon boson. A similar result holds for I_-. To summarize, the function $\varepsilon(\xi)$ in the strip $-\pi<\mathrm{Im}\,\xi\leq\pi$ is given by

$$\varepsilon(\xi)=-m_F\cosh\gamma(\xi-i\pi),\quad 2\mu-\pi<\mathrm{Im}\,\xi\leq\pi \tag{2.113a}$$

$$=-m_F\cosh\gamma(\xi+i\pi),\quad -\pi<\mathrm{Im}\,\xi<\pi-2\mu \tag{2.113b}$$

$$\varepsilon_B(\xi)=\varepsilon(\xi)-\varepsilon(\mathrm{Re}\,\xi+i\pi)$$

$$=m_B\cosh\gamma\xi,\quad |\mathrm{Im}\,\xi|<2\mu-\pi. \tag{2.113c}$$

Knowing the function $\varepsilon(\xi)$, it is an easy matter to construct the spectrum of physical states. The PBC's for the excited modes require that these modes be placed in n strings, i.e., in vertical arrangements

$$\xi_l = \alpha_s + il(\pi - \mu), \tag{2.114}$$

where $l = (n-1), (n-3), \ldots, -(n-1)$, as shown in Fig. 6. If $n \leq r$ where r is determined from the coupling by (2.60), then all of the modes in the n string lie within the region $|\mathrm{Im}\,\xi| < 2\mu - \pi$. Each of the modes has a hole stuck to it at $\alpha_s + i\pi$, and the energy of the full n-string $+n$-hole complex is

$$E_n = \sum_l m_B \cosh\gamma\xi_l = \frac{m_B \sin\left(\frac{n\pi}{2}(2\gamma-1)\right)}{\sin\left(\frac{\pi}{2}(2\gamma-1)\right)} \cosh\gamma\alpha_s$$

$$= 2m_F \sin\left(\frac{n\pi}{2}(2\gamma-1)\right)\cosh\gamma\alpha_s, \tag{2.115}$$

which is just the familiar Dashen-Hasslacher-Neveu (1975) formula first obtained by semiclassical quantization of the "breather" mode of the sine-Gordon model. [A similar formula was obtained for the XYZ spin chain spectrum by Johnson, Krinsky, and McCoy (1973). The connection between the two results was pointed out by Luther (1976).]

For the two longest strings, $n = r+1$ and $n = r+2$, the two end modes at the top and bottom of the string at $\mathrm{Im}\,\xi = \pm(n-1)(\pi-\mu)$ are within the region $|\mathrm{Im}\,\xi| > 2\mu - \pi$, as shown in Fig. 10(b). Thus two of the n holes may be located anywhere on the $i\pi$ lines. Let the positions be $\alpha_1 + i\pi$ and $\alpha_2 + i\pi$. The other $(n-2)$ holes, those associated with the modes inside the shaded region $|\mathrm{Im}\,\xi| < 2\mu - \pi$, must be directly above the string.

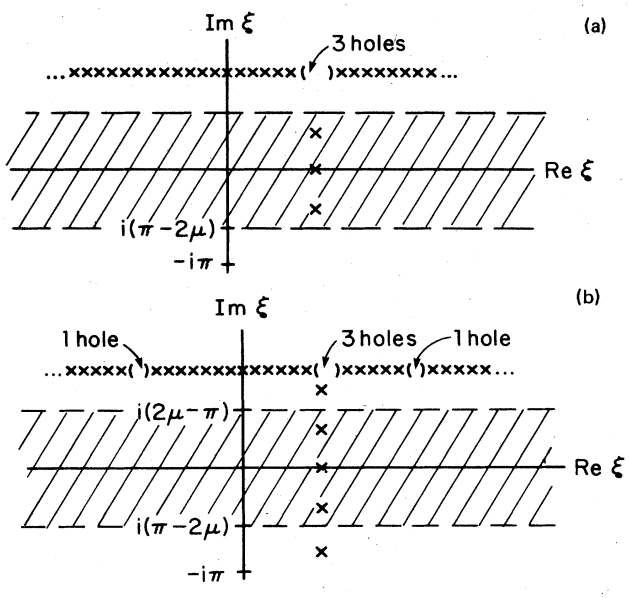

FIG. 10. (a) Bound 3-string+ 3-hole state, representing the $n = 3$ state in Eq. (2.115). (b) A 5-string state, representing an unbound fermion pair. Here and in (a), the value of the coupling is taken in the range $r = 3$.

The energy of the state may be written as the sum of three contributions. The $(n-2)$ modes inside the shaded region, along with their holes, contribute

$$2m_F \sin\left(\frac{(n-2)\pi}{2}(2\gamma-1)\right)\cosh\gamma\alpha_s, \tag{2.116}$$

just as in (2.115). The contribution from the two end modes is obtained from (2.113a) and (2.113b), and is given by

$$-m_F \cosh\gamma[\alpha_s + i(n-1)(\pi-\mu) - i\pi] - m_F \cosh\gamma[\alpha_s - i(n-1)(\pi-\mu) + i\pi] = -2m_F \sin\left(\frac{(n-2)\pi}{2}(2\gamma-1)\right)\cosh\gamma\alpha_s. \tag{2.117}$$

This exactly cancels the contribution (2.116). The net energy of the n modes and $(n-2)$ holes is zero, and the parameter α_s completely disappears from the expression for the energy of the state. The energy of the state is given entirely by the contribution of the two unbound holes,

$$E = m_F \cosh\gamma\alpha_1 + m_F \cosh\gamma\alpha_2, \quad n = r+1,\ r+2. \tag{2.118}$$

From this result we infer that the two longest n strings represent states of an unbound fermion-antifermion pair with physical rapidities $\gamma\alpha_1$ and $\gamma\alpha_2$.

Following the same steps which led to (2.115) and (2.118), one may calculate the physical values of other conserved quantities for a given state. Calculating the physical momentum in this way leads to similar expressions with $\cosh\gamma\alpha$ replaced by $\sinh\gamma\alpha$, giving a Lorentz covariant energy-momentum relation. Finally we note that the spectrum also includes compound excitations consisting of several n strings plus holes. The totality of such excitations is believed to provide a complete set of states, although this has never been demonstrated explicitly.

C. Field theories with internal symmetry

A major advance in the technology of Bethe's *Ansatz* was made by Yang (1967), who developed a method for treating the many-body problem with delta-function interaction which imposed no limitation on the symmetry of the wave function. This method provides a solution not only for the case of identical bosons discussed in Sec. II.B, but more generally for a system consisting of several different particle species (which will be referred to as "colors"). The only requirement is that all colors have identical mass and interact via a color-invariant delta-function two-body potential. In field theory, this system is described by a multicom-

ponent nonlinear Schrödinger equation, with the Hamiltonian

$$H=\int dx\left(\sum_a \partial_x\phi_a^*\,\partial_x\phi_a + c\sum_{a,b}\phi_a^*\phi_b^*\phi_a\phi_b\right), \tag{2.119}$$

where the sums run from 1 to N_c, the number of colors. Recently, a Bethe *Ansatz* solution for the chiral-invariant Gross-Neveu model has been formulated by Andrei and Lowenstein (1979, 1980), using Yang's technique to handle the internal symmetry. This model has a multicomponent fermion field interacting by a color-invariant four-fermion interaction of the form $g[(\bar\psi\psi)^2-(\bar\psi\gamma^5\psi)^2]$. It is an important model in that it has nontrivial renormalization group properties and exhibits asymptotic freedom and dynamical mass generation. A similar model, the SU(2) Thirring model, has been discussed by Belavin (1979). The study of exactly integrable field theories with internal symmetry seems likely to provide additional insights in the future. Kulish (1979) has shown that the Yang method for the multicomponent nonlinear Schrödinger model (2.119) finds a natural place in the quantum inverse method, and a similar discussion of the Gross-Neveu model has been presented by Kulish and Reshetikhin (1979). An outstanding problem of major importance is the application of Bethe *Ansatz* or quantum inverse methods to the nonlinear sigma models. Much work has been done in the investigation of exact integrability for these models, but so far they have not been analyzed by the methods we are discussing here.

In this subsection we will briefly review Yang's method in the context of the multicomponent nonlinear Schrödinger model. Relativistic fermion models are treated in a similar way, with the finite density system of the nonlinear Schrödinger model again serving as the analog of the physical vacuum in the relativistic theories. We begin by considering the two-body system for the Hamiltonian (2.119). Denoting the colors of the two particles by a_1 and a_2, we try to construct a two-body eigenstate of the form

$$|k_1,k_2\rangle=\int dx_1\,dx_2\,\psi(x_1,x_2)\phi_{a_1}^*(x_1)\phi_{a_2}^*(x_2)|0\rangle\,. \tag{2.120}$$

The Bethe *Ansatz* for this problem is a generalization of the identical boson wave function (2.12)

$$\psi(x_1,x_2)=\sum_{P,Q}[Q,P]\,\theta(x_{Q_1}<x_{Q_1})\exp i[k_{P_1}x_{Q_1}+k_{P_2}x_{Q_2}]$$

$$=\theta(x_1<x_2)\{[12,12]e^{i(k_1x_1+k_2x_2)}+[12,21]e^{i(k_2x_1+k_1x_2)}\}+\theta(x_2<x_1)\{[21,12]\,e^{i(k_1x_2+k_2x_1)}+[21,21]\,e^{i(k_2x_2+k_1x_1)}\}, \tag{2.121}$$

where $\theta(x<y)\equiv\theta(y-x)$, Q and P are permutations of (1, 2), and the coefficients $[Q,P]$ are to be chosen so that (2.120) is an eigenstate of H. Continuity of the wave function demands that

$$[12,12]+[12,21]=[21,12]+[21,21]\,. \tag{2.122}$$

The state (2.120) will be an eigenstate of (2.119), provided that delta-function terms from the kinetic energy and interaction Hamiltonians are made to cancel. (This is sometimes stated equivalently as the condition that the first derivative of the wave function should have a discontinuity of $2c$ at $x_1=x_2$.) This condition, combined with (2.122), may be written in an elegant form by collecting the coefficients $[Q,P]$ in a 2×2 matrix and denoting the columns by ξ_P. This allows us to write the stated conditions as

$$\xi_{21}=Y_{12}\,\xi_{12}\,, \tag{2.123}$$

where

$$Y_{12}=\frac{-\lambda_{12}}{1+\lambda_{12}}+\frac{1}{1+\lambda_{12}}\mathcal{P}_{12}\,. \tag{2.124}$$

Here,

$$\lambda_{12}=\frac{ic}{k_1-k_2}\,, \tag{2.125}$$

and $\mathcal{P}_{12}$ is the permutation operator which interchanges the two components of the column vector ξ_{12}. Thus (2.123) is written explicitly as

$$\begin{pmatrix}[12,21]\\[21,21]\end{pmatrix}=\frac{-\lambda_{12}}{1+\lambda_{12}}\begin{pmatrix}[12,12]\\[21,12]\end{pmatrix}+\frac{1}{1+\lambda_{12}}\begin{pmatrix}[21,12]\\[12,12]\end{pmatrix}. \tag{2.126}$$

It is easy to check that with the conditions (2.126) the continuity equation (2.122) is satisfied, and that the state (2.120) satisfies

$$H|k_1,k_2\rangle=(k_1^2+k_2^2)|k_1,k_2\rangle\,. \tag{2.127}$$

Equation (2.126) can be interpreted physically in terms of the two-body scattering process in a way which leads naturally to the generalization of this result to any number of particles and colors. Imagine the particles arranged on a line in x space. The permutations Q and P may be regarded as the arrangement from left to right of the color labels and momentum (k_i) labels, respectively. A coefficient $[Q,P]$ can be interpreted as the amplitude for an arrangement of colors and momenta, e.g., $[21,12]$ is the amplitude for finding the first particle with color a_2 and momentum k_1 and the second with color a_1 and momentum k_2. For definiteness assume that $k_1>k_2$. The time development of the system may be traced from the incoming configuration of momenta $P=(12)$ to the outgoing configuration $P=(21)$, with an elementary perturbative interaction represented by a factor of $-\lambda_{12}$, Eq. (2.125). When the ordering of the momenta interchanges, i.e., when the fast particle k_1 passes the slow particle k_2, one of two things may happen to the color. Either the colors a_1 and a_2 are exchanged between the

fast and the slow particle, in which case the color ordering Q remains the same, or else the color a_1 remains on the fast particle and a_2 remains on the slow particle. The former may happen only if the particles interact at least once, while the latter may happen whether or not they interact. The first and second terms on the right-hand side of (2.126) describe these two possibilities.

With this interpretation of (2.126), it is now easy to generalize (2.123) to the N-body case. Let Q and P be elements of the permutation group S_N and write the N-body wave function as

$$\psi = \sum_{P,Q} [Q,P]\theta(x_{Q_1} < \cdots < x_{Q_N}) \exp\left(i \sum_i k_{P_i} x_{Q_i}\right). \quad (2.128)$$

Then the state

$$|k_1 \cdots k_N\rangle = \int dx_1 \cdots dx_N \psi(x_1 \cdots x_N) \phi^*_{a_1}(x_1) \cdots \phi^*_{a_N}(x_N) |0\rangle \quad (2.129)$$

is an eigenstate of H, provided that

$$\xi_{P'} = Y^{i,i+1}_{P_i,P_{i+1}} \xi_P, \quad (2.130)$$

where ξ_P is a column vector of the $N! \times N!$ matrix $[Q,P]$ and P' is the permutation obtained from P by interchaning P_i and P_{i+1}. The Y operators are defined by

$$Y^{lm}_{ij} = \frac{-\lambda_{ij}}{1+\lambda_i} + \frac{1}{1+\lambda_{ij}} \mathcal{P}_{lm}, \quad (2.131)$$

where $\mathcal{P}_{lm}$ is the permutation operator on ξ_P which interchanges Q_l and Q_m. The mutual consistency of the equations contained in (2.130) may be established using the identities

$$Y^{ab}_{ij} Y^{ab}_{ji} = 1, \quad (2.132)$$

$$Y^{ab}_{jk} Y^{bc}_{ik} Y^{ab}_{ij} = Y^{bc}_{ij} Y^{ab}_{ik} Y^{bc}_{jk}. \quad (2.133)$$

We note the analogous roles played by the operators $Y^{i,i+1}_{P_iP_{i+1}}$ and $\mathcal{P}_{i,i+1}$ when acting on ξ_P. The first interchanges P_i and P_{i+1} (momentum labels), while the second interchanges Q_i and Q_{i+1} (color labels). The periodic boundary conditions which emerge from the wave function (2.128) can be easily understood from this point of view. The periodic boundary conditions relate the amplitude for finding a particle with a particular color and momentum label at one end of the box to the amplitude for finding a particle with the same color and momentum at the other end of the box. For an N-particle system, all N such conditions may be written in terms of a standard vector ξ_0 defined as ξ_P with P = identity. It is convenient to define an operator

$$X_{ij} = \mathcal{P}_{ij} Y^{ij}_{ij} = \frac{1 - \mathcal{P}_{ij}\lambda_{ij}}{1+\lambda_{ij}}, \quad (2.134)$$

which has the effect of interchanging both momentum and color labels. Thus, the PBC's can be written as

$$X_{j+1,j} X_{j+2,j} \cdots X_{N,j} X_{1,j} X_{2,j} \cdots X_{j-1,j} \xi_0 = \sigma_j \xi_0, \quad (2.135)$$

where

$$\sigma_j = e^{ik_jL}. \quad (2.136)$$

Equation (2.135) for $j = 1, \ldots, N$ provides N matrix eigenvalue equations for the vector ξ_0. It can be shown (Yang, 1967) that the matrices on the left-hand side of each equation commute with each other. Before proceeding to discuss (2.135) in general, we turn to two special cases. For identical bosons, $\mathcal{P}_{ij} = 1$ for all i and j (i.e., the wave function is symmetric). In this case (2.135) reduces to the usual PBC's, Eq. (2.24). If the particles are identical fermions, $P_{ij} = -1$, and (2.135) reduces to the PBC's of a noninteracting theory, $\sigma_j = 1$. This was to be expected because single-component fermions cannot interact by a delta function: their wave function vanishes when any two coordinates coincide.

One of the most remarkable aspects of Yang's method is its treatment of the matrix PBC's (2.135). As shown by Yang and extended by Sutherland (1968), the PBC's can be solved by a second application of Bethe's *Ansatz*, or more generally by a nested series of *Ansätze*. The operators $\mathcal{P}_{ij}$ acting on the vector ξ_0 form an $N! \times N!$ representation of the permutation group S_N. This may be written as a sum of irreducible representations by considering ξ_0's of various symmetry types. Choosing one particular irreducible representation reduces the PBC's (2.135) to a matrix equation with the dimension of that representation. The choice of representation is reflected in the symmetry properties of the corresponding wave function. For example, as already mentioned, the one-dimensional symmetric and antisymmetric representations produce wave functions for identical bosons and fermions, respectively. Higher dimensional representations correspond to systems with two or more species (colors) of particles. We denote an irreducible representation R by a Young tableau $[n_1, n_2, \ldots, n_l]$, where $\sum_i^l n_i = N$. Graphically this tableau has n_1 boxes in the first row, n_2 boxes in the second row, etc. For each R there is a conjugate representation $\tilde{R}$ obtained by interchanging rows and columns. Figure 11 shows the conjugate representations $[5,3]$ and $[2,2,2,1,1] \equiv [2^3, 1^2]$. If R represents a boson system with n_1 particles of color 1, n_2 particles of color 2, etc., then $\tilde{R}$ describes a fermion system with the same color content. The periodic boundary conditions for conjugate representations are related in a simple way. The eigenvalue σ_j on the right hand side of (2.135) may be regarded as a function of the representation chosen for the $\mathcal{P}_{ij}$'s and written $\sigma_j(R)$. Now write another eigenvalue equation

$$X'_{j+1,j} X'_{j+2,j} \cdots X'_{N,j} X'_{1,j} X'_{2j} \cdots X'_{j-1,j} \Phi = \mu_j \Phi, \quad (2.137)$$

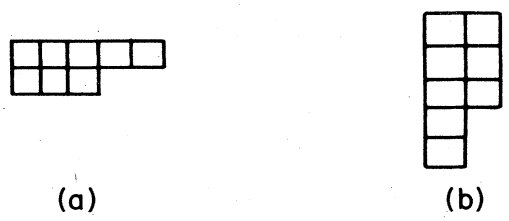

FIG. 11. (a) Representation [5, 3]. (b) Representation [2, 2, 2, 1, 1].

where

$$X'_{ij} = \frac{1+\varphi_{ij}\lambda_{ij}}{1+\lambda_{ij}}. \tag{2.138}$$

The eigenvalue μ_j is also a function of the representation. Then we see that

$$\sigma_j(R) = \mu_j(\tilde{R}). \tag{2.139}$$

This follows by noting that rows and columns in a Young tableau represent symmetrization and antisymmetrization respectively, and thus the change of sign in (2.138) compared with (2.134) is equivalent to using the conjugate representation in (2.135).

As an example we consider a system with two species of fermions, e.g., M red ones and $N-M$ blue ones. The appropriate symmetry for (2.135) is $R=[2^M,1^{N-2M}]$. By (2.139) we may instead consider (2.135) with Φ having the symmetry $\tilde{R}=[N-M,M]$. The key to Yang's method is to regard the vector Φ as a wave function describing M identical particles and $N-M$ vacancies on a lattice and to write a generalized Bethe *Ansatz* for Φ:

$$\Phi = \prod_{P\in S_M} a(P)F(\Lambda_{P_1},y_1)F(\Lambda_{P_2},y_2)\cdots F(\Lambda_{P_M},y_M), \tag{2.140}$$

where the y_i's are integers which denote the lattice site and satisfy $1\le y_1<y_2<\cdots<y_M\le N$. It is found that the *Ansatz* (2.140) can be made to simultaneously diagonalize the N operators in (2.137) (for $j=1,\ldots,N$) if we choose

$$F(\Lambda,y)=\prod_{j=1}^{y-1}\left(\frac{k_j-\Lambda+ic/2}{k_{j+1}-\Lambda-ic/2}\right). \tag{2.141}$$

The periodic boundary conditions for the new *Ansatz* (2.140) restrict the Λ's to satisfy

$$-\prod_j\left(\frac{k_j-\Lambda_\alpha+ic/2}{k_j-\Lambda_\alpha-ic/2}\right)=\prod_\beta\left(\frac{\Lambda_\alpha-\Lambda_\beta-ic}{\Lambda_\alpha-\Lambda_\beta+ic}\right). \tag{2.142}$$

The factors on the right-hand side are just the two-body phase shifts associated with the coefficients $a(P)$ in the *Ansatz* (2.140). Finally, the eigenvalue μ_j in (2.137) which is obtained from (2.140) is

$$\mu_j(\tilde{R})=\prod_\beta\left(\frac{k_j-\Lambda_\beta+ic/2}{k_j-\Lambda_\beta-ic/2}\right). \tag{2.143}$$

Using (2.139), the original matrix PBC's [Eq. (2.135)] reduce to

$$e^{ik_jL}=\prod_\beta\left(\frac{k_j-\Lambda_\beta+ic/2}{k_j-\Lambda_\beta-ic/2}\right). \tag{2.144}$$

In the limit $L\to\infty$ with N/L and M/L fixed, these equations reduce to a set of coupled integral equations for two density functions $\rho(k)$ and $\sigma(\Lambda)$:

$$2\pi\sigma(\Lambda)=-\int_{-B}^{B}K(\Lambda-\Lambda')\sigma(\Lambda')d\Lambda'+2\int_{-Q}^{Q}K(2\Lambda-2k)\rho(k)dk \tag{2.145}$$

$$2\pi\rho(k)=1+2\int_{-B}^{B}K(2k-2\Lambda)\sigma(\Lambda)d\Lambda. \tag{2.146}$$

The functions $\rho(k)$ and $\sigma(\Lambda)$ are, respectively, the total density and the density of red particles,

$$\int_{-Q}^{Q}\rho(k)dk=N/L, \tag{2.147}$$

$$\int_{-B}^{B}\sigma(\Lambda)d\Lambda=M/L. \tag{2.148}$$

Our discussion has followed Yang's original treatment of the case $\tilde{R}=[N-M,M]$. Notice that for this case, the original *Ansatz* (2.128) described a system with two particle species (colors), leading to 2×2 matrix PBC's. The second *Ansatz* (2.140) used to solve these PBC's involved only a single color (i.e., identical particles) on a lattice. Roughly speaking, the dimension of the problem was reduced by regarding the N particles as lattice sites on which the color wave function is defined, with red particles treated as occupied sites (i.e., the particles of the second *Ansatz*) and blue particles as vacancies. Sutherland (1968) has extended this method to include the treatment of an arbitrary number of colors, using a hierarchy of *Ansätze*. With each successive *Ansatz*, the dimension of the problem is reduced by treating the particles of one color as vacancies and writing a wave function for the remaining colors.

D. Vertex models, transfer matrices, and spin-chain Hamiltonians

All of the models we have discussed so far have been continuum field theories involving operators defined on continuous space and time. One of the most remarkable aspects of Bethe's *Ansatz* is that is also provides an exact treatment of certain lattice models for which operators are defined on discrete lattice sites. In fact, the original *Ansatz* of Bethe was applied to such a model, the isotropic Heisenberg spin chain. Subsequent developments included solutions to the anisotropic XXZ spin chain (Orbach, 1958; Yang and Yang, 1966) ice and ferroelectric (six-vertex) models (Lieb, 1967a, b, c; Baxter, 1971), and finally the full XYZ spin chain (Baxter, 1972b) and Baxter (eight-vertex) model (Baxter, 1972a). Solutions to the two-dimensional Ising model (Onsager, 1944; Schultz, Mattis, and Lieb, 1964) and XY spin chain (Lieb, Schultz, and Mattis, 1961) should be included in this list, since they are subsumed by the Baxter method. In the remainder of this section we will survey the subject of exactly soluble lattice models. These models are not only interesting in their own right but also as further examples of the applications of Bethe's *Ansatz*. It has recently been found that certain operators which emerge in the formulation of the quantum inverse method are directly related to operators which arise naturally in lattice models, e.g., the transfer matrix. In fact, the most elegant derivation of the algebra of scattering data operators, which is central to the quantum inverse method, is patterned after Baxter's discussion of commuting transfer matrices in the eight-vertex model. Thus the study of soluble lattice models provides important new insights into the nature of exact integrability in quantum systems.

The discussion in this section is intended to introduce

and define the soluble vertex models and spin chains, discuss the relationship between them, and briefly indicate the nature of their solution. [A more extensive review has been given by Kasteleyn (1975).] In Sec. III we will return to this subject armed with the techniques of the quantum inverse method, which considerably simplify the original solutions based on explicit Bethe *Ansätze*. All the models to be considered can be obtained in one way or another from the Baxter eight-vertex model, and we begin by defining that model. Consider a square lattice with horizontal and vertical bonds connecting nearest-neighbor lattice sites. An arrow is placed on each bond in such a way that each lattice site (vertex) has an even number of arrows entering and leaving it. Thus only the eight vertices shown in Fig. 12 are allowed. A typical configuration of arrows for a 3×3 lattice with periodic (toroidal) boundary conditions is shown in Fig. 13. To each of the eight vertex types we assign an energy ε_i, $i=1,\ldots,8$, and an associated Boltzmann weight

$$\omega_i = e^{-\beta\epsilon_i}. \tag{2.149}$$

There is no loss of generality in requiring that

$$\varepsilon_5=\varepsilon_6, \quad \varepsilon_7=\varepsilon_8. \tag{2.150}$$

To get the symmetric eight-vertex (Baxter) model, we also impose the "zero-field" condition

$$\varepsilon_1=\varepsilon_2, \quad \varepsilon_3=\varepsilon_4. \tag{2.151}$$

The vertex weights are then written

$$\begin{aligned}&\omega_1=\omega_2=a, \quad \omega_3=\omega_4=b,\\ &\omega_5=\omega_6=c, \quad \omega_7=\omega_8=d.\end{aligned} \tag{2.152}$$

The partition function is defined as the sum over all allowed configurations of arrows, with each configuration weighted by a product of Boltzmann weights (2.149) for each vertex. The calculation of the partition function is reduced to an eigenvalue problem by introduction of the transfer matrix, which describes the operation of adding an extra row of vertices to the lattice. It is convenient to introduce a matrix notation for an elementary vertex. Denoting a right- (left-) pointing arrow on a horizontal bond and an up (down) arrow on a vertical bond by + (−), and with the indices defined as shown in Fig. 14, an elementary vertex can be written as

$$L(\alpha,\beta;\lambda,\mu)=\sum_{i=1}^{4} w_i\,\sigma^i_{\alpha\beta}\,\sigma^i_{\lambda\mu}, \tag{2.153}$$

where σ^i, $i=1,2,3$ are Pauli matrices and $\sigma^4=$ identity matrix. The coefficients w_i in (2.153) are given by

$$\begin{aligned}&w_1=\tfrac{1}{2}(c+d), \quad w_2=\tfrac{1}{2}(c-d),\\ &w_3=\tfrac{1}{2}(a-b), \quad w_4=\tfrac{1}{2}(a+b).\end{aligned} \tag{2.154}$$

For our considerations, it is useful to regard the vertex (2.153) as a 2×2 matrix in the horizontal indices

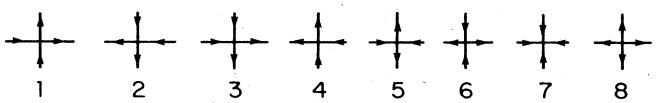

FIG. 12. Eight allowed vertices of the Baxter model.

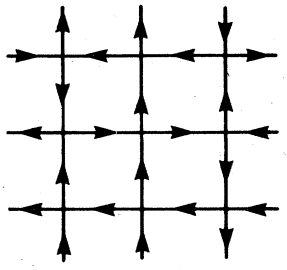

FIG. 13. Possible configuration of a 3×3 lattice with toroidal boundary conditions.

(λ,μ), the components of which are spin operators in the space of vertical arrows. Thus the vertex is written

$$L_n=\begin{pmatrix} w_3\sigma^3_n+w_4\sigma^4_n & w_1\sigma^1_n-iw_2\sigma^2_n\\ w_1\sigma^1_n+iw_2\sigma^2_n & -w_3\sigma^3_n+w_4\sigma^4_n\end{pmatrix}, \tag{2.155}$$

where the subscript n indicates that the σ-matrices act on the vertical arrow at the nth site in a row. The transfer matrix for a lattice with N sites in a row and periodic boundary conditions is given by

$$T=\mathrm{tr}(L_1L_2\cdots L_N), \tag{2.156}$$

where the trace is taken over the horizontal indices. The partition function for a lattice with M rows is given by

$$Z=\mathrm{Tr}(T^M), \tag{2.157}$$

where the trace is taken over the Hilbert space in which the transfer matrix acts, i.e., the 2^N-dimensional space spanned by vectors of the form

$$|\pm\rangle_1\otimes|\pm\rangle_2\otimes\cdots\otimes|\pm\rangle_N, \tag{2.158}$$

denoting the configuration of vertical arrows in a row. Because of (2.157), the free energy,

$$f=-\frac{1}{\beta MN}\ln Z$$

for a large lattice, is determined by the largest eigenvalue of the transfer matrix. The problem is thus reduced to that of diagonalizing the transfer matrix, which is accomplished in these models by explicitly constructing its eigenstates.

At this point it is worth considering some specific

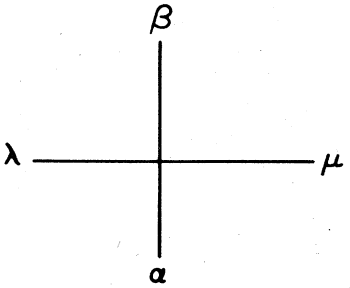

FIG. 14. Arrangement of indices for the vertex defined in Eq. (2.153).

examples of eigenstates for a special case of the Baxter model, namely, the six-vertex or generalized ice model obtained by setting the weight of vertices 7 and 8 in Fig. 11 to zero, i.e., $d=0$ or $w_1=w_2$. The basic vertex becomes

$$L_n=\begin{pmatrix} w_3\sigma_n^3+w_4\sigma_n^4 & 2w_1\sigma_n^- \\ 2w_1\sigma_n^+ & -w_3\sigma_n^3+w_4\sigma_n^4 \end{pmatrix}. \tag{2.159}$$

The six-vertex model transfer matrix has an important property which makes the construction of its eigenstates considerably simpler than in the general Baxter model. Because vertices 7 and 8 are not allowed, the number of down arrows (or up arrows) is conserved from row to row under application of the transfer matrix. This follows from the observation that, with the definition (2.156) and using (2.159), T can be written as a sum of terms each of which contains the same number of σ^+'s as σ^-'s and thus does not change the number of down arrows in a state. In particular, the state with all up arrows,

$$|0\rangle = |+\rangle_1 \otimes |+\rangle_2 \otimes \cdots |+\rangle_N , \tag{2.160}$$

(and also the state with all down arrows) is an eigenstate of the six vertex model transfer matrix. The eigenvalue associated with (2.160) is easily obtained by noting that

$$L_n|+\rangle_n=\begin{pmatrix} w_4+w_3 & 2w_1\sigma_n^- \\ 0 & w_4-w_3 \end{pmatrix}|+\rangle_n , \tag{2.161}$$

which means that

$$T|0\rangle=[(w_4+w_3)^N+(w_4-w_3)^N]|0\rangle . \tag{2.162}$$

The construction of the remaining eigenstates can be accomplished by a Bethe *Ansatz*, using (2.160) as a reference state, and treating down arrows as the "particles" of the *Ansatz*. For example, eigenstates of T with a single down arrow are given by

$$|k_1\rangle=\sum_{l=1}^{N} e^{ik_1 l}\sigma_l^-|0\rangle , \tag{2.163}$$

where k_1 must satisfy the periodic boundary condition

$$e^{ik_1 N}=1 . \tag{2.164}$$

The fact that (2.163) is an eigenvalue of T may be verified directly, using (2.156) and (2.159). It is convenient to introduce Baxter's parametrization of the vertex weights (specialized to the six-vertex case),

$$w_1=w_2=\tfrac{1}{2}\rho_0\sin 2\eta , \tag{2.165a}$$

$$w_3=\rho_0\sin\eta\cos v , \tag{2.165b}$$

$$w_4=\rho_0\cos\eta\sin v . \tag{2.165c}$$

The parameter ρ_0 enters the transfer matrix trivially as an overall normalization factor ρ_0^N. We choose it for convenience to be

$$\rho_0=[\sin(v+\eta)\sin(v-\eta)]^{-1/2} . \tag{2.166}$$

For the following discussion, it is helpful to think of v as a variable and η as a constant. It is sometimes useful to define a change of variables $v\to k$ by

$$e^{ik}=\frac{\sin(v+\eta)}{\sin(v-\eta)} . \tag{2.167}$$

This relation between k and v is a lattice analog of the relation between momentum and rapidity. The eigenfunctions of T with n down arrows are of the Bethe form,

$$|k_1,\ldots,k_n\rangle=\sum_{1\le l_1<\cdots<l_n\le N} f(l_1,\ldots,l_n)\sigma_{l_1}^-\cdots\sigma_{l_n}^-|0\rangle . \tag{2.168}$$

where

$$f(l_1,\ldots,l_n)=\sum_{P\in S_n} a(P)\exp\left(i\sum_{j=1}^{n}k_{P_j}l_j\right) . \tag{2.169}$$

In place of the variables k_i, we will often use v_i, which is related to k_i as in (2.167),

$$e^{ik_i}=\frac{\sin(v_i+\eta)}{\sin(v_i-\eta)} . \tag{2.170}$$

The coefficients $a(P)$ are defined up to an overall normalization as follows. If P and P' are permutations which are identical except for the interchange of two adjacent elements, i.e, $P=(\ldots i,j\ldots)$ and $P'=(\ldots j,i\ldots)$, then

$$\frac{a(P)}{a(P')}=\frac{\sin(v_i-v_j+2\eta)}{\sin(v_i-v_j-2\eta)} . \tag{2.171}$$

The quantity on the right-hand side is the two-body phase shift. For (2.168) to be an eigenstate of T, the k_i's or v_i's must satisfy periodic boundary conditions,

$$\left(\frac{\sin(v_i+\eta)}{\sin(v_i-\eta)}\right)^N=\prod_{\substack{j=1\\ j\neq 1}}^{n}\left(\frac{\sin(v_i-v_j+2\eta)}{\sin(v_i-v_j-2\eta)}\right) . \tag{2.172}$$

Under conditions (2.171) and (2.172), the states (2.168) satisfy the eigenvalue equation

$$T(v)|k_1,\ldots,k_n\rangle=\left[\left(\frac{\sin(v+\eta)}{\sin(v-\eta)}\right)^{N/2}\prod_{i=1}^{n}\left(\frac{\sin(v-v_i-2\eta)}{\sin(v-v_i+2\eta)}\right)+\left(\frac{\sin(v-\eta)}{\sin(v+\eta)}\right)^{N/2}\prod_{i=1}^{n}\left(\frac{\sin(v-v_i+2\eta)}{\sin(v-v_i-2\eta)}\right)\right]|k_1,\ldots,k_n\rangle . \tag{2.173}$$

The explicit demonstration of the results (2.172) and (2.173) is quite tedious, involving prodigious cancellations among "unwanted" terms (Lieb, 1967a; Baxter, 1971). Fortunately, the advent of the quantum inverse method has obviated much of this calculation. As we will see in Sec. III, the results (2.172) and (2.173) are direct consequences of a simple operator algebra. Moreover, the quantum inverse method provides an elegant treatment of the full Baxter model. We will return to this subject in Sec. III.

We conclude this section by remarking that there is a direct relationship between the Baxter model and the anisotropic (XYZ) Heisenberg spin chain (Baxter, 1972b). The vertex weights w_1, w_2, w_3, and w_4 in (2.153) can be expressed in terms of parameters ρ, η, v, and an elliptic modulus k, using an elliptic function parametrization which generalizes (2.165) (see Sec. III.D). Regarding the transfer matrix $T(v)$ as a function

of v with ρ, η, and k fixed, it may be shown that the Hamiltonian of the XYZ spin chain

$$H=-\tfrac{1}{2}\sum_{j=1}^{N}(J_x\sigma_j^x\sigma_{j+1}^x+J_y\sigma_j^y\sigma_{j+1}^y+J_z\sigma_j^z\sigma_{j+1}^z)\,, \tag{2.174}$$

can be written as a logarithmic derivative of the transfer matrix with respect to v evaluated at $v=\eta$. Since transfer matrices $T(v)$ and $T(v')$ commute for any v and v' (Baxter, 1972a), the operators H and $T(v)$ have the same v-independent set of eigenstates. The general equivalence relation between the spin-spin couplings J_x, J_y, and J_z, and the vertex weights w_i has been given by Baxter (1972b). Here we merely note two special cases of interest. (1) The six-vertex model ($d=0$ or $w_1=w_2$) is related to the XXZ spin chain ($J_x=J_y$). (2) If the vertex weights (2.154) satisfy $ab=cd$, the Baxter model reduces to two decoupled Ising models, and the corresponding spin chain Hamiltonian (2.174) is the XY chain ($J_z=0$).

III. THE QUANTUM INVERSE METHOD

The inverse scattering method was developed as a technique for solving certain nonlinear evolution equations (classical field theories). The method devised for solving the Korteweg-de Vries equation (Gardner, Greene, Kruskal, and Miura, 1967) was subsequently generalized to provide solutions of other interesting equations, including the nonlinear Schrödinger equation (Zakharov and Shabat, 1971) and the sine-Gordon equation (Ablowitz *et al.*, 1973; Takhtajan and Faddev, 1974). The Cauchy initial value problem for the nonlinear equation is reduced to a sequence of linear problems. In special cases, the method yields explicit solutions to the nonlinear equation, e.g., N-soliton formulas. The essential idea of the classical inverse scattering method is to construct a transformation from the local field variables to a new set of variables which are defined in terms of the scattering data of a linear eigenvalue problem. In this eigenvalue problem, the original field serves as the scattering potential. Thus, for example, the nonlinear Schrödinger field $\phi(x)$ at a fixed time t, is mapped into a set of scattering data $a(k)$ and $b(k)$. It is found that, for a judiciously chosen linear eigenvalue problem, the nonlinear time evolution of the field $\phi(x,t)$ translates into a trivial time dependence for the scattering data $a(k)$ and $b(k)$. The final observation which completes the solution of the Cauchy problem is that the scattering data at a given time uniquely determine the scattering potential, i.e., the field $\phi(x)$. From the values of $a(k)$ and $b(k)$ at time t', one can reconstruct the field $\phi(x,t')$ and solve the initial value problem. In Sec. III.A we review the classical inverse method for the nonlinear Schrödinger equation, concentrating on those features which are important in the treatment of the quantized theory. More thorough reviews of the classical methods are available (Scott, Chu, and McLaughlin, 1973; Ablowitz, 1978).

Recent investigations have led to the realization that the inverse scattering technique which had been developed in classical field theory could be formulated as an exact operator method for solving quantum field theory (Sklyanin and Faddeev, 1978; Sklyanin, 1979; Thacker and Wilkinson, 1979; Honerkamp *et al.*, 1979). Moreover, the quantum inverse method is closely related to the Bethe *Ansatz* technique discussed in Sec. II and provides an elegant algebraic formulation of those results. Thus, we obtain a unified understanding of methods which arose in quite different areas of physics. The remarkable connections which emerge serve to emphasize the universal nature of exact integrability.

In this section we will review the quantum inverse method and discuss some of its applications. Our main focus will be on the nonlinear Schrödinger model, which provided most of the impetus in the development of the method and remains the best-studied example. In particular, it is at present the only model for which a Gel'fand-Levitan transformation has been formulated for the quantized theory (Creamer *et al.*, 1980). In the classical theory, this is the transformation which reconstructs the local field from the scattering data, an essential step in the solution of the initial value problem. In quantum field theory it is relevant to the study of Green's functions. The formulation of the Gel'fand-Levitan transformation for other models is currently under investigation.

The quantum inverse method for the nonlinear Schrödinger model is introduced in Sec. III.B, where operator equations for both the direct and inverse transforms are obtained. In Sec. III.C, some interesting properties of the quantum Gel'fand-Levitan equation are noted and discussed. The quantum inverse method for models of lattice statistics is presented in Sec. III.D. The diagonalization of the transfer matrix for the symmetric six-vertex model (Thacker, 1980) is discussed, using a lattice version of the quantum inverse method. The solution of the full Baxter model (Faddeev, 1979; Sklyanin *et al.*, 1980) is also briefly discussed. These results illustrate the deep connection between the quantum inverse method and the transfer matrix formalism for lattice models.

A. The classical inverse method

Before discussing the quantum inverse method, we will briefly review the classical formalism. To introduce the idea of the inverse scattering transform, we consider the example of the nonlinear Schrödinger equation,

$$i\partial_t\phi=-\partial_x^2\phi+2c|\phi|^2\phi\,, \tag{3.1}$$

where $\phi(x,t)$ is a complex classical field. In this theory theory, Poisson brackets are defined for any two functionals α and β by

$$\{\alpha,\beta\}=i\int dx\left\{\frac{\delta\alpha}{\delta\phi(x)}\frac{\delta\beta}{\delta\phi^*(x)}-\frac{\delta\alpha}{\delta\phi^*(x)}\frac{\delta\beta}{\delta\phi(x)}\right\} \tag{3.2}$$

The equation of motion (3.1) may be written in Hamiltonian form as

$$\partial_0\phi=\{H,\phi\}\,, \tag{3.3}$$

where the Hamiltonian is

$$H=\int dx\,(\partial_x\phi^*\partial_x\phi+c|\phi|^4)\,. \tag{3.4}$$

The inverse method for solving the initial value prob-

lem for Eq. (3.1) was introduced by Zakharov and Shabat (1971). The method is based on a transformation at time $t=0$ from the field $\phi(x,0)=\phi(x)$ to a set of scattering data associated with the linear eigenvalue problem,

$$-i\frac{\partial}{\partial x}\Psi(x,\xi)=Q(x,\xi)\Psi(x,\xi). \tag{3.5}$$

Here $Q(x,\xi)$ is a 2×2 matrix which depends on the field variable at the point x,

$$Q(x,\xi)=\begin{pmatrix}\frac{1}{2}\xi & \sqrt{c}\phi(x)\\ -\sqrt{c}\phi^*(x) & -\frac{1}{2}\xi\end{pmatrix}. \tag{3.6}$$

For simplicity, we consider only the repulsive case $c>0$. To completely define a solution Ψ of (3.5), we must specify a boundary condition. If we assume that $|\phi(x)|\to 0$ as $x\to\pm\infty$, then for $\xi\to k=$ real, Ψ may be specified by the asymptotic behavior

$$\Psi(x,k)\underset{x\to-\infty}{\sim}V(x,k), \tag{3.7}$$

where

$$V(x,k)=\begin{pmatrix}e^{(1/2)ikx} & 0\\ 0 & e^{-(1/2)ikx}\end{pmatrix}. \tag{3.8}$$

Then it is easy to show that Ψ can be written

$$\Psi=\begin{pmatrix}\psi_1 & \tilde{\psi}_1\\ \psi_2 & \tilde{\psi}_2\end{pmatrix}, \tag{3.9}$$

where $\tilde{\psi}_1(x,k)=\psi_2^*(x,k)$ and $\tilde{\psi}_2(x,k)=\psi_1^*(x,k)$ [here we will follow essentially the notation of (Creamer, Thacker, and Wilkinson, 1980)]. The scattering data for (3.5) are defined by the asymptotic form of Ψ for $x\to+\infty$. Writing

$$\Psi(x,k)=V(x,k)G(x,k), \tag{3.10}$$

G has a finite limit as $x\to+\infty$,

$$G\underset{x\to+\infty}{\sim}\begin{pmatrix}a(k) & b^*(k)\\ b(k) & a^*(k)\end{pmatrix}. \tag{3.11}$$

Note that from (3.5) and (3.10), G satisfies the equation

$$-i\frac{\partial}{\partial x}G(x,k)=\tilde{Q}(x,k)G(x,k). \tag{3.12}$$

Denoting the asymptotic form of Q by Q_0, the matrix $\tilde{Q}$ in (3.12) is given by

$$\tilde{Q}=V^{-1}(Q-Q_0)V=\begin{pmatrix}0 & \sqrt{c}\phi e^{-ikx}\\ -\sqrt{c}\phi^* e^{ikx} & 0\end{pmatrix}. \tag{3.13}$$

Eq. (3.12) may be written as an integral equation,

$$G(x,k)=I+i\int dy\,\theta(y<x)\tilde{Q}(y,k)G(y,k), \tag{3.14}$$

where $\theta(y<x)\equiv\theta(x-y)$ is a step function. By iterating (3.14), one may generate series expansions for the components of G, and in particular, for the scattering data $a(\xi)$ and $b(\xi)$. These have the form

$$a(k)=1+c\int dx_1dy_1\theta(x_1<y_1)e^{ik(x_1-y_1)}\phi^*(x_1)\phi^*(y_1)+\cdots, \tag{3.15}$$

$$\frac{i}{\sqrt{c}}b(k)=\int dx_1e^{ikx_1}\phi^*(x_1)+c\int dx_1dx_2dy_1\theta(x_1<y_1<x_2)e^{ik(x_1+x_2-y_1)}\phi^*(x_1)\phi^*(x_2)\phi(y_1)+\cdots. \tag{3.16}$$

By constructing action and angle variables from the scattering data, it may be shown that $a(\xi)$ and $b(\xi)$ have simple Poisson brackets with the Hamiltonian (Zakharov and Manakov, 1974)

$$\{H,a(k)\}=0, \tag{3.17}$$

$$\{H,b(k)\}=ik^2b(k). \tag{3.18}$$

Thus, if $\phi(x,t)$ evolves in time according to Eq. (3.1), the scattering data of the linear problem (3.5)–(3.6) has a simple time dependence,

$$a(k,t)=a(k,0) \tag{3.19}$$

$$b(k,t)=b(k,0)e^{-ik^2t}. \tag{3.20}$$

The Poisson brackets among a, b, a^*, and b^* may also be obtained, using properties of the Wronskian for Eq. (3.5) (Zakharov and Manakov, 1974).

$$\{a(k),b(k')\}=\frac{c}{k-k'-i\epsilon}a(k)b(k'), \tag{3.21}$$

$$\{a^*(k),b(k')\}=-\frac{c}{k-k'+i\epsilon}a^*(k)b(k'), \tag{3.22}$$

$$\{a(k),a(k')\}=\{a(k),a^*(k')\}=0, \tag{3.23}$$

$$\{b(k),b(k')\}=0, \tag{3.24}$$

$$\{b(k),b^*(k')\}=2\pi i a^*(k)a(k)\delta(k-k'). \tag{3.25}$$

All these Poisson brackets may be checked order by order, using (3.4), (3.15), and (3.16).

With the field $\phi(x)$ mapped into a set of scattering data whose time dependence is given by (3.19)–(3.20), the final step in the solution of the initial value problem is the reconstruction of the field from the time-evolved scattering data. The essential method for accomplishing this was devised by Gel'fand and Levitan (1951) and adapted to this problem by Zakharov and Shabat (1971). In addition to Ψ, defined by (3.7), we define another solution $X(x,k)$ of (3.5) with asymptotic behavior

$$X(x,k)\underset{x\to+\infty}{\sim}V(x,k), \tag{3.26}$$

with components given by

$$X=\begin{pmatrix}\tilde{\chi}_1 & \chi_1\\ \tilde{\chi}_2 & \chi_2\end{pmatrix}, \tag{3.27}$$

where the column vectors χ and $\tilde{\chi}$ are related in the same way as ψ and $\tilde{\psi}$, i.e., $\tilde{\chi}_1(x,k)=\chi_2^*(x,k)$ and $\tilde{\chi}_2(x,k)=\chi_1^*(x,k)$. The Gelfand-Levitan integral equation is a

dispersion relation for an analytic function $\Phi(x,\xi)$ which is constructed from the Jost solutions of the linear eigenvalue problem. Denoting the columns of (3.9) by ψ and $\bar{\psi}$ and those of (3.27) by $\tilde{\chi}$ and χ, we see that the asymptotic conditions (3.7) and (3.26) allow continuation of $\psi(x,\xi)$ and $\chi(x,\xi)$ into the lower half ξ plane and of $\bar{\psi}$ and $\tilde{\chi}$ into the upper half ξ plane. [Note that for complex ξ, the conjugate solutions are related by $\bar{\psi}_1(x,\xi)=\psi_2^*(x,\xi^*)$, $\bar{\psi}_2(x,\xi)=\psi_1^*(x,\xi^*)$, etc.] We will construct a function Φ which is equal to $\tilde{\chi}e^{-i\xi x/2}$ in the upper half-plane. To motivate the choice of Φ in the lower half plane, we observe that, for real k, the Jost solution ψ may be written as a linear combination of χ and $\tilde{\chi}$,

$$\psi(x,k)=a(k)\tilde{\chi}(x,k)+b(k)\chi(x,k)\,. \tag{3.28}$$

Defining the reflection coefficient

$$R^*(k)=\frac{i}{\sqrt{c}}\,b(k)a^{-1}(k)\,, \tag{3.29}$$

we can write (3.28) as

$$\psi a^{-1}=\tilde{\chi}-i\sqrt{c}\,R^*\chi\,. \tag{3.30}$$

This suggests the following definition:

$$\Phi(x,\xi)=\begin{cases}\tilde{\chi}e^{-i\xi x/2}, & \mathrm{Im}\,\xi>0\\ \psi a^{-1}e^{-i\xi x/2}, & \mathrm{Im}\,\xi<0\,.\end{cases} \tag{3.31}$$

The discontinuity of Φ across the real axis is then $i\sqrt{c}R^*\chi$. For simplicity, we are considering only the case $c>0$, for which $a(\xi)$ is analytic and nonvanishing in the lower half plane. For $c<0$, $a(\xi)$ will generally have a number of zeroes which represent bound states of the linear problem and correspond to solitons of the nonlinear equation. The use of the Gel'fand-Levitan equation to construct N-soliton formulas is an interesting part of the classical treatment, but we will not discuss it further here.

Thus, for $c>0$, $\Phi(x,\xi)$ is analytic in the cut ξ plane. Since $\Phi(x,\xi)\to 1$ as $|\xi|\to\infty$, which follows from its definition (3.31) and the properties of the Jost solutions, the function Φ may be reconstructed from its discontinuity across the real axis,

$$\Phi(x,\xi)=\begin{pmatrix}1\\0\end{pmatrix}+\frac{\sqrt{c}}{2\pi}\int_{-\infty}^{\infty}dk'\,\frac{R^*(k')\chi(x,k')e^{-ik'x/2}}{k'-\xi-i\epsilon}\,. \tag{3.32}$$

Letting ξ approach the real axis from above, $\xi\to k+i\epsilon$, we obtain an equation relating the Jost solutions χ and $\tilde{\chi}$,

$$\tilde{\chi}(x,k)e^{-ikx/2}=\begin{pmatrix}1\\0\end{pmatrix}+\frac{\sqrt{c}}{2\pi}\int_{-\infty}^{\infty}dk'\,\frac{R^*(k')\chi(x,k')e^{-ik'x/2}}{k'-k-i\epsilon}\,. \tag{3.33}$$

The first component of (3.33) and the complex conjugate of the second component provide a pair of coupled integral equations for χ_1 and χ_2^*,

$$\chi_2^*(x,k)e^{-ikx/2}=1+\frac{\sqrt{c}}{2\pi}\int_{-\infty}^{\infty}dk'\,\frac{R^*(k')\chi_1(x,k')e^{-ik'x/2}}{k'-k-i\epsilon}\,, \tag{3.34a}$$

$$\chi_1(x,k)e^{ikx/2}=\frac{\sqrt{c}}{2\pi}\int_{-\infty}^{\infty}dk'\,\frac{\chi_2^*(x,k')R(k')e^{ik'x/2}}{k'-k+i\epsilon}\,. \tag{3.34b}$$

Equations (3.34) determine χ_1 and χ_2^* in terms of the reflection coefficient $R^*(k)$. The field $\phi(x)$ is then easily recovered from the asymptotic form of χ_1 as $k\to\infty$,

$$\chi_1(x,k)e^{ikx/2}\underset{k\to\infty}{\sim}-\frac{\sqrt{c}}{k}\,\phi(x)\,, \tag{3.35}$$

which follows from eigenvalue equation (3.5) for the Jost solution χ.

The direct transform and the inverse transform, defined respectively by the integral equations (3.14) and (3.34), are the central concepts in the classical inverse method. For the direct transform a Jost solution is constructed from the field $\phi(x)$ by solving Eq. (3.14), and the scattering data $a(k)$ and $b(k)$ are obtained from the asymptotic form of that solution as $x\to\infty$, Eq. (3.11). For the inverse transform a Jost solution is constructed from $a(k)$ and $b(k)$ by solving Eq. (3.34), and the field is obtained from the asymptotic form of the solution as $k\to\infty$, Eq. (3.35).

B. Quantum inverse method for the nonlinear Schrödinger model

We now formulate the quantum version of the classical methods discussed in the preceeding section and discuss their relationship with the Bethe *Ansatz* technique of Sec. II.A. The starting point is a normal ordered operator version of the Zakharov-Shabat eigenvalue equation (3.5),

$$-i\frac{\partial\Psi}{\partial x}=:\,Q\Psi\,:\,, \tag{3.36a}$$

where normal ordering for this theory means moving ϕ^*'s to the left and ϕ's to the right. In components, (3.36a) reads

$$-i\frac{\partial\psi_1}{\partial x}=\tfrac{1}{2}\xi\psi_1+\sqrt{c}\,\psi_2\phi\,, \tag{3.36b}$$

$$-i\frac{\partial\psi_2}{\partial x}=-\sqrt{c}\,\phi^*\psi_1-\tfrac{1}{2}\xi\psi_2\,, \tag{3.36c}$$

and similarly for $\bar{\psi}$, x, and $\tilde{x}$. The Jost solutions which satisfy Eq. (3.36) are operator functionals of the fields ϕ and ϕ^*. As in the classical case, a particular solution to (3.36) must be specified by a boundary condition. Before discussing the finite density gas, we will first consider the case of an unbounded system with a finite number of particles. This corresponds to the classical case $\phi(x)\to 0$ as $x\to\pm\infty$. However, in the quantum theory, one must be careful to interpret $\phi(x)\to 0$ in the sense of weak convergence, i.e., as a condition on the physical matrix elements. In dealing with operators, this means that only normal ordered products may be set to zero, e.g., $\phi^*(x+a)\phi(x)\to 0$ as $x\to\pm\infty$, but $\phi(x+a)\phi^*(x)$ must be written $\phi^*(x)\phi(x+a)+\delta(a)-\delta(a)$. In particular, the specification of an operator Jost solution by an asymptotic condition like (3.7) must be done with care.

Let us define a solution $G(x,k)$ for $\xi=k$ real, which satisfies (3.36) and the boundary condition

$$G(-L/2,k)=1\,, \tag{3.37}$$

where the limit $L\to\infty$ will be taken at an appropriate time. Note that in the $L\to\infty$ limit, the solution

$$\Psi(x,k)=G(x,k)V(-L/2,k) \tag{3.38}$$

is analogous to the classical solution $\Psi(x,k)$ defined by (3.7). Thus the scattering data operators $a(k)$ and $b(k)$ are defined by introducing

$$\tilde{G}(x,k)=V^{-1}(x,k)G(x,k)V(-L/2,K), \tag{3.39}$$

from which the a and b operators are obtained by letting $L\to\infty$ and $x\to\infty$,

$$\tilde{G}(x,k)\underset{\substack{L\to\infty\\ x\to\infty}}{\sim}\begin{pmatrix} a(k) & b^*(k)\\ b(k) & a^*(k)\end{pmatrix}\equiv T(k). \tag{3.40}$$

The operator function $\tilde{G}(x,k)$ satisfies a normal ordered integral equation of the form (3.14),

$$\tilde{G}(x,k)=I+i\int dy\,\theta(-\tfrac{1}{2}L<y<x):\tilde{Q}(y,k)\tilde{G}(y,k):, \tag{3.41}$$

with $\tilde{Q}$ defined in (3.13). Eq. (3.41) may be iterated to produce series expansions for $\tilde{G}$ and for a and b. These are identical to the classical results (3.15) and (3.16) with the terms on the right-hand side interpreted as normal ordered operators.

The central property of the scattering data operators $a(k)$ and $b(k)$ is that they satisfy simple commutation relations among themselves, their complex conjugates, and the Hamiltonian. These were obtained by two different methods, one which generalized the classical Wronskian derivation of Poisson brackets (3.23)–(3.27) (Thacker and Wilkinson, 1979), and one which adapted a technique used by Baxter in the eight-vertex model (Sklyanin, 1979). Here we will follow the approach of Sklyanin, which gives a deeper insight into the structure of the method and its relationship to transfer matrix techniques. To obtain the commutator algebra of the a and b operators, we consider a 4×4 matrix of operators which is the direct product of Jost solutions to (3.36) with two different real eigenvalues $\xi=k_1$ and k_2,

$$H_{12}(x)=G(x,k_1)\otimes G(x,k_2)\equiv G_1(x)\otimes G_2(x), \tag{3.42}$$

where $G(x,k)$ is defined by the boundary condition (3.37). Here and in the following, eigenvalues k_1 and k_2 will be indicated by subscripts. The elements of H_{12} are operator products of the elements of G_1 with the elements of G_2. The desired commutation relations are obtained by comparing (3.42) with the direct product of the same two solutions in reverse order,

$$H_{21}(x)=G_2(x)\otimes G_1(x). \tag{3.43}$$

Differentiating (3.42) and using the fact that G satisfies the Zakharov-Shabat equation (3.36), we obtain

$$-i\frac{\partial}{\partial x}H_{12}(x)=:Q_1G_1:\otimes G_2+G_1\otimes:Q_2G_2:, \tag{3.44}$$

with a similar equation for H_{21} obtained by interchanging eigenvalues $k_1\to k_2$. Normal ordering of Eq. (3.44) may be achieved by moving the ϕ in Q_1 past G_2 in the first term and the ϕ^* in Q_2 past G_1 in the second term. To do this we must know the commutators of ϕ and ϕ^* with a Jost solution $G(x,k)$. This is easily obtained from the integral equation (3.41). Writing

$$\tilde{Q}(y)=\sqrt{c}\,e^{-iky}\phi(y)\sigma^+-\sqrt{c}\,e^{iky}\phi^*(y)\sigma^-, \tag{3.45}$$

we see that

$$[\phi(x),\tilde{Q}(y)]=-\sqrt{c}\,e^{iky}\delta(x-y)\sigma^-. \tag{3.46}$$

Now commuting $\phi(x)$ with Eq. (3.41), we obtain

$$[\phi(x),\tilde{G}(x,k)]=\frac{-i\sqrt{c}}{2}e^{ikx}\sigma^-\tilde{G}(x,k), \tag{3.47}$$

where we have used the symmetric prescription $\theta(x)\delta(x)=\frac{1}{2}\delta(x)$. Finally, the commutator of ϕ with G is obtained from (3.47) and (3.39),

$$[\phi(x),G(x,k)]=\frac{-i\sqrt{c}}{2}\sigma^-G(x,k). \tag{3.48}$$

Similarly, we find

$$[\phi^*(x),G(x,k)]=\frac{-i\sqrt{c}}{2}\sigma^+G(x,k). \tag{3.49}$$

These two results can be used to normal order (3.44), which becomes

$$-i\frac{\partial}{\partial x}H_{12}(x)=:\Gamma_{12}(x)H_{12}(x):, \tag{3.50}$$

where

$$\Gamma_{12}=Q_1\otimes I+I\otimes Q_2-ic\sigma^+\otimes\sigma^-$$

$$=\begin{bmatrix} \frac{1}{2}(k_1+k_2) & \sqrt{c}\,\phi(x) & \sqrt{c}\,\phi(x) & 0\\ -\sqrt{c}\,\phi^*(x) & \frac{1}{2}(k_1-k_2) & -ic & \sqrt{c}\,\phi(x)\\ -\sqrt{c}\,\phi^*(x) & 0 & \frac{1}{2}(k_2-k_1) & \sqrt{c}\,\phi(x)\\ 0 & -\sqrt{c}\,\phi^*(x) & -\sqrt{c}\,\phi^*(x) & -\frac{1}{2}(k_1+k_2)\end{bmatrix}. \tag{3.51}$$

By interchanging k_1 and k_2 we also obtain an equation for the direct product in reverse order (3.43),

$$-i\frac{\partial H_{21}}{\partial x}=:\Gamma_{21}H_{21}:. \tag{3.52}$$

The most important property of the matrix (3.51) is that an interchange of the eigenvalues k_1 and k_2 may be accomplished by a c-number similarity transformation,

$$\Gamma_{21}(x)=\mathcal{R}\Gamma_{12}(x)\mathcal{R}^{-1}, \tag{3.53}$$

where $\mathcal{R}$ is a matrix depending only on the eigenvalues k_1 and k_2,

$$\mathcal{R}=\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & \beta & \alpha & 0\\ 0 & \alpha & \beta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \tag{3.54}$$

with

$$\alpha=\frac{k_1-k_2}{k_1-k_2-ic}, \tag{3.55}$$

$$\beta=\frac{-ic}{k_1-k_2-ic}. \tag{3.56}$$

Equation (3.53) is fundamental. It constitutes a local characterization of the exact integrability of the system.

From (3.50), (3.51), and (3.52) it is seen that $\mathcal{R}H_{12}(x)\mathcal{R}^{-1}$ satisfies the same equation as $H_{21}(x)$. These

two quantities are also equal at $x=-L$ by virtue of the boundary condition (3.37) and the definitions (3.42) and (3.43). Thus, they are equal everywhere,

$$\mathcal{R}H_{12}(x)=H_{21}(x)\mathcal{R}\,. \tag{3.57}$$

We now want to take the limits $L\to\infty$ and $x\to\infty$ in (3.57) and obtain the commutator algebra of the a and b operators defined in (3.40). But first this equation must be written in a form which is finite in these limits. This is done by introducing a function $\tilde{H}_{12}(x)$ which has a finite asymptotic behavior,

$$\tilde{H}_{12}(x)=W_{12}^{-1}(x)H_{12}(x)W_{12}(-L/2)\,. \tag{3.58}$$

Here $W_{12}(x)$ is a solution to (3.50) with the fields in Γ_{12} set equal to zero,

$$W_{12}(x)=\exp(i\Gamma_{12}^{(0)}x)\,, \tag{3.59}$$

where

$$\Gamma_{12}^{(0)}=\begin{pmatrix}\frac{1}{2}(k_1+k_2) & 0 & 0 & 0\\ 0 & \frac{1}{2}(k_1-k_2) & -ic & 0\\ 0 & 0 & \frac{1}{2}(k_2-k_1) & 0\\ 0 & 0 & 0 & -\frac{1}{2}(k_1+k_2)\end{pmatrix}. \tag{3.60}$$

The construction of $\tilde{H}_{12}$ in (3.58) is similar to that of $\tilde{G}$ in (3.39). Because of the normal ordering term $-ic$ in (3.60), W_{12} is not given simply by $V_1\otimes V_2$, i.e., the asymptotic behavior of a product of Jost solutions (in the sense of weak convergence) is not the same as the product of their asymptotic behaviors. By calculating the exponential in (3.59), we find

$$W_{12}(x)=\begin{pmatrix}e^{(i/2)(k_1+k_2)x} & 0 & 0 & 0\\ 0 & e^{(i/2)(k_1-k_2)x} & \dfrac{2c}{k_1-k_2}\sin\frac{1}{2}(k_1-k_2)x & 0\\ 0 & 0 & e^{(i/2)(k_2-k_1)x} & 0\\ 0 & 0 & 0 & e^{-(i/2)(k_1+k_2)x}\end{pmatrix}. \tag{3.61}$$

Now noting that $\mathcal{R}W_{12}=W_{21}\mathcal{R}$, we can multiply (3.57) by $W_{21}^{-1}(x)$ on the left and $W_{12}(x)$ on the right to give

$$\mathcal{R}\tilde{H}_{12}(x)=\tilde{H}_{21}(x)\mathcal{R}\,, \tag{3.62}$$

for which the limits x, $L\to\infty$ can be taken. The algebra of a and b operators is obtained by writing (3.62) in terms of the functions $\tilde{G}_1$ and $\tilde{G}_2$, using (3.39), (3.42), and (3.58),

$$\begin{aligned}&\mathcal{R}U_{12}^{-1}(x)[\tilde{G}_1(x)\otimes\tilde{G}_2(x)]U_{12}(-L/2)\\ &\quad=U_{21}(x)[\tilde{G}_2(x)\otimes\tilde{G}_1(x)]U_{21}(-L/2)\mathcal{R}\,,\end{aligned} \tag{3.63}$$

where

$$\begin{aligned}U_{12}(x)&=[V_1^{-1}(x)V_2^{-1}(x)]W_{12}(x)\\ &=I+\left(\frac{-ic}{k_1-k_2}\right)(1-e^{-i(k_1-k_2)x})(\sigma^+\otimes\sigma^-)\,,\end{aligned} \tag{3.64}$$

where I is the four-dimensional unit matrix. For $x\to\pm\infty$ and $k_1\neq k_2$, this becomes

$$U_{12}(\infty)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & \dfrac{-ic}{k_1-k_2} & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \tag{3.65}$$

Noting that $U_{12}^{-1}(\infty)=U_{21}(-\infty)$, we see that the asymptotic form of (3.63) is

$$\mathcal{R}_\infty[\mathcal{T}(k_1)\otimes\mathcal{T}(k_2)]=[\mathcal{T}(k_2)\otimes\mathcal{T}(k_1)]\mathcal{R}_\infty\,, \tag{3.66}$$

where $\mathcal{T}(k)$ is defined in (3.40), and

$$\mathcal{R}_\infty=U_{21}(\infty)\mathcal{R}U_{21}(\infty)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & \gamma & 0\\ 0 & \alpha & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \tag{3.67}$$

with

$$\alpha=\frac{k_1-k_2}{k_1-k_2-ic}\,,\quad \gamma=\frac{k_1-k_2+ic}{k_1-k_2}\,. \tag{3.68}$$

From (3.66)–(3.68) we obtain all the desired commutation relations for the scattering data operators (for $k\neq k'$):

$$a(k)b(k')=\left(1-\frac{ic}{k-k'}\right)b(k')a(k)\,, \tag{3.69}$$

$$a^*(k)b(k')=\left(1+\frac{ic}{k-k'}\right)b(k')a^*(k)\,, \tag{3.70}$$

$$b^*(k)b(k')=\frac{(k-k')^2+c^2}{(k-k')^2}b(k')b^*(k)\,, \tag{3.71}$$

$$[a(k),a(k')]=[a(k),a^*(k')]=[b(k),b(k')]=0\,. \tag{3.72}$$

The commutators of a and b with the Hamiltonian

$$[H,a(k)]=0\,, \tag{3.73}$$

$$[H,b(k)]=k^2b(k)\,, \tag{3.74}$$

may be obtained by an operator generalization of the classical argument which led to (3.52) and (3.53). Alternatively, we may expand the operator $\ln a(k)$ for $k\to\infty$,

$$\ln a(k)=\sum_{n=1}^{\infty}\frac{C_n}{k^n}\,. \tag{3.75}$$

By studying the integral equation (3.41) or equivalently the expansion (3.15), we can show that the Hamiltonian may be written in terms of the coefficients C_n in (3.75) (Faddeev, 1979), specifically,

$$H=-\frac{1}{ic}C_3+C_2-\frac{ic}{6}C_1\,. \tag{3.76}$$

The commutators (3.73) and (3.74) follow directly from this result along with (3.69) and (3.72).

From the relations (3.69)–(3.74), we conclude that $b(k)$ is a creation operator for eigenstates of H, and that $a(k)$ is the generator of an infinite number of conservation laws. The multiparticle states created by application of b operators to the vacuum,

$$|\Phi(k_1,k_2,\ldots,k_N)\rangle = b(k_1)b(k_2)\cdots b(k_N)|0\rangle , \tag{3.77}$$

are eigenstates of H because of (3.74),

$$H|\Phi(k_1,\ldots,k_N)\rangle = \left(\sum_{i=1}^{N} k_i^2\right)|\Phi(k_1,\ldots,k_N)\rangle . \tag{3.78}$$

By explicit calculation from the series (3.16), it has been shown for $N \leq 3$ that the states (3.77) are identical to the Bethe *Ansatz* states (2.16) (up to a trivial overall constant). Recently, the same result has been obtained for all N using the Gel'fand-Levitan equation (Creamer *et al.*, 1980a). Thus the quantum inverse method provides an algebraic formulation of Bethe's *Ansatz*. The operator $a(k)$ is diagonal on the states (3.77), as seen from (3.69),

$$a(k)|\Phi(k_1,\ldots,k_N)\rangle = \prod_{i=1}\left(1-\frac{ic}{k-k_i}\right)|\Phi(k_1,\ldots,k_N)\rangle . \tag{3.79}$$

The commutation relations (3.69)–(3.72) have been obtained for $k \neq k'$. A more careful analysis of the limits taken on (3.53) shows that the $(k-k')$ denominators in (3.69) and (3.70) should have infinitesimal negative imaginary parts, and that the relation (3.71) shall have an extra delta-function term (Faddeev, 1979), as already indicated by the classical result (3.27). It is convenient to define the operator reflection coefficient

$$R^*(k) = \frac{i}{\sqrt{c}} b(k)a^{-1}(k) , \tag{3.80}$$

which satisfies the commutation relations

$$R^*(k)R^*(k') = S(k',k)R^*(k')R^*(k) , \tag{3.81}$$

$$R(k)R^*(k') = S(k,k')R^*(k')R(k) + 2\pi\delta(k-k') , \tag{3.82}$$

where $S(k,k')$ is the two-body S matrix,

$$S(k,k') = \frac{k-k'-ic}{k-k'+ic} . \tag{3.83}$$

The operator $R^*(k)$ creates the same set of eigenstates as $b(k)$ but with a different normalization. The particular significance of the operator $R^*(k)$ is that

$$|\Phi(k_1,\ldots,k_N)\rangle = R^*(k_1)\cdots R^*(k_N)|0\rangle \tag{3.84}$$

is a normalized in state if $k_1 < k_2 < \cdots < k_n$ and a normalized out state if $k_1 > k_2 > \cdots > k_N$. The fact that R and R^* satisfy the simple relations (3.81) and (3.82) will be of central importance in discussing the quantum Gel'fand-Levitan transformation (see Sec. III.C).

So far only a system of particles in infinite space has been discussed, i.e., we have not introduced periodic boundary conditions in a finite box. In the conventional Bethe *Ansatz* analysis of Sec. II, the PBC's were essential to understanding the ground state and excitations of a finite density system. It might be suspected that by taking the infinite volume limit to get the algebra (3.69)–(3.72) one has lost essential information which would prevent any discussion of the finite density case. This turns out not to be the case, as we will see in Sec. III.C, where the fundamental spectral equation (2.45) is obtained directly from the infinite volume algebra without using periodic boundary conditions in a box. In the remainder of this section, we will describe an approach to the finite density problem which is closer in spirit to the usual Bethe's *Ansatz* methods. It relies on the algebra of operators $A(k)$ and $B(k)$ defined in a finite box of length L. These operators are defined in terms of the Jost solution $G(x,k)$ which obeys the boundary condition (3.37). We write

$$G(\tfrac{1}{2}L,k) = \begin{pmatrix} A(k) & C(k) \\ B(k) & D(k) \end{pmatrix} \equiv \mathcal{T}_L(k) , \tag{3.85}$$

where, for the case under consideration, $C(k) = B^*(k)$ and $D(k) = A^*(k)$. The relation (3.57), along with the definition (3.42) gives

$$\mathcal{R}[\mathcal{T}_L(k) \otimes \mathcal{T}_L(k')] = [\mathcal{T}_L(k') \otimes \mathcal{T}_L(k)]\mathcal{R} , \tag{3.86}$$

where $\mathcal{R}$ is the matrix (3.54) with

$$\alpha(k,k') = \frac{k-k'}{k-k'-ic} , \tag{3.87}$$

$$\beta(k,k') = \frac{-ic}{k-k'-ic} . \tag{3.88}$$

From (3.86) we obtain commutation relations among the elements of $\mathcal{T}_L(k)$. Of particular interest are the relations

$$[A(k),A(k')] = [B(k),B(k')] = 0 , \tag{3.89}$$

$$A(k)B(k') = \frac{1}{\alpha(k,k')} B(k')A(k) - \frac{\beta(k,k')}{\alpha(k,k')} B(k)A(k') , \tag{3.90}$$

$$D(k)B(k') = \frac{1}{\alpha(k',k)} B(k')D(k) + \frac{\beta(k',k)}{\alpha(k',k)} B(k)D(k') , \tag{3.91}$$

$$[A(k)+D(k),A(k')+D(k')] = 0 . \tag{3.92}$$

The structure of the algebra (3.86) or (3.89)–(3.92) is typical of the quantum inverse method in a finite volume. Similar results (with different functions α and β) are also found for the sine-Gordon model (Sklyanin *et al.*, 1979) and the six-vertex lattice (Thacker, 1980).

Instead of discussing the Hamiltonian, we will construct eigenstates of the operator

$$T(k) = A(k) + D(k) , \tag{3.93}$$

which is in a sense more fundamental. This operator is precisely analogous to the transfer matrix in lattice models (see Sec. III.D). We will show that the states

$$|\Phi(k_1,\ldots,k_N)\rangle = B(k_1)\cdots B(k_N)|0\rangle , \tag{3.94}$$

are exact eigenstates of $T(k)$, provided that the k_i's satisfy periodic boundary conditions. First we note that the zero-particle state $|0\rangle$ is an eigenstate of $A(k)$ and $D(k)$ separately,

$$A(k)|0\rangle = e^{ikL/2}|0\rangle , \tag{3.95}$$

$$D(k)|0\rangle = e^{-ikL/2}|0\rangle , \tag{3.96}$$

which is easily seen by writing $A(k)$ and $D(k)$ as normal ordered series expansions in ϕ and ϕ^*. Now we apply $T(k)$ to the state (3.94) and use the commutation relations (3.90) and (3.91) to commute $A(k)$ and $D(k)$ through the string of B operators. By this calculation, it is found that (3.94) is an eigenstate,

$$T(k)|\Phi(k_1,\ldots,k_N)\rangle = \Lambda(k;k_1,\ldots,k_N)|\Phi(k_1,\ldots,k_N)\rangle, \tag{3.97}$$

with eigenvalue

$$\Lambda(k;k_1,\ldots,k_N) = e^{ikL/2}\prod_{i=1}^{N}\left(1-\frac{ic}{k-k_i}\right) + e^{-ikL/2}\prod_{i=1}^{N}\left(1+\frac{ic}{k-k_i}\right), \tag{3.98}$$

provided that the k_i's satisfy the conditions

$$e^{ik_iL/2}\prod_{j\neq i}\left(1-\frac{ic}{k_i-k_j}\right) = e^{-ik_iL/2}\prod_{j\neq i}\left(1+\frac{ic}{k_i-k_j}\right), \tag{3.99}$$

for $i=1,2,\ldots,N$. Notice that Eq. (3.99) is precisely the periodic boundary conditions Eq. (2.23) obtained from the explicit Bethe wave functions. To understand how the result (3.97) follows from the relations (3.89)–(3.91), consider the action of $A(k)$ on the state (3.94). Commuting $A(k)$ through the string of B's and then using (3.95) we obtain 2^N terms, which may be grouped according to the arguments of the N B operators. Note that the second term on the right-hand side of (3.90) results in an exchange of the arguments k and k' between the A and B operators. Of the 2^N terms obtained from $A(k)$ acting on the state, one of them contains no exchanges (or equivalently, no factors of β). It is given by

$$e^{ikL/2}\prod_{i=1}^{N}\left(1-\frac{ic}{k-k_i}\right)|\Phi(k_1,k_2,\ldots,k_N)\rangle. \tag{3.100}$$

Combining this with the corresponding term from $D(k)$ gives the right-hand side of (3.97) with the eigenvalue (3.98). Thus we must show that all of the remaining terms cancel. Consider first the terms in which k is exchanged with k_1 in the state. The terms of this form are given by

$$-\frac{\beta(k,k_1)}{\alpha(k,k_1)}\left[e^{ik_1L/2}\prod_{j=2}^{N}\left(1-\frac{ic}{k_1-k_j}\right) - e^{-ik_1L/2}\prod_{j=2}^{N}\left(1+\frac{ic}{k_1-k_j}\right)\right]|\Phi(k,k_2,\cdots,k_N)\rangle, \tag{3.101}$$

which vanishes provided that the periodic boundary condition (3.99) is satisfied for $i=1$. The remaining terms in which k_i is replaced by k in the state with $i=2,3,\ldots,N$, may be calculated explicitly, but such a calculation is unnecessary. Because of the second commutator in (3.89), the state (3.94) is symmetric in the k_i's. Thus the sum of the terms in which k replaces k_i for $i>1$ are obtained from (3.101) by interchanging k_1 and k_i. All of these terms vanish by virtue of the PBC's (3.99). This completes the demonstration of Eq. (3.97).

The essential differences between the infinite volume relations (3.69)–(3.72) and the finite volume relations (3.89)–(3.92) are the presence of the second (exchange) term on the right-hand side of (3.90) and (3.91) and the fact that a and a^* commute separately, while only the combination $A+D$ commutes in the finite volume case. One result of this is that the reflection coefficient operator $R^*(k)$, Eq. (3.80), which had simple commutation relations in the infinite volume case, is not a useful operator for the finite volume system. No operator with properties like (3.81) and (3.82) has been constructed in a finite box. As we saw in Sec. III.A, the reflection coefficient plays a central role in the classical inverse transform as the kernel of the Gel'fand-Levitan integral equation. The same is true in the quantum inverse method, where the simple properties of the operators $R(k)$ and $R^*(k)$ are basic to the structure of the quantum Gel'fand-Levitan transform (see Sec. III.C). Thus the formulation of the inverse transform in the quantum theory (as it is presently understood) requires that the infinite volume limit be taken *ab initio*. This turns out to be less of a restriction than it might seem, since the finite density spectral results usually associated with periodic boundary conditions in a box can be obtained directly in an infinite volume using the Gel'fand-Levitan formalism. In this calculation the finite box is avoided by introducing a temperature parameter, and the finite density results are constructed via a fugacity expansion, each term involving matrix elements with a finite number of particles in an infinite volume.

C. Some properties of the quantum Gel'fand-Levitan transform

In Sec. III.A we found that the classical Jost solution $\chi(x,k)$, defined by (3.26) and (3.27), satisfies a Gel'fand-Levitan integral equation (3.34). This provided the basic tool for reconstructing the field configuration $\phi(x)$ from a set of scattering data. Recently it has been shown (Creamer *et al.*, 1980a; Grosse, 1979) that a corresponding integral equation for operators in the quantum theory is also valid,

$$\chi_2^*(x,k)e^{-ikx/2} = 1+\frac{\sqrt{c}}{2\pi}\int_{-\infty}^{\infty}dk'\frac{R^*(k')\chi_1(x,k')e^{-ik'x/2}}{k'-k-i\varepsilon}, \tag{3.102a}$$

$$\chi_1(x,k)e^{ikx/2} = \frac{\sqrt{c}}{2\pi}\int_{-\infty}^{\infty}dk'\frac{\chi_2^*(x,k')R(k')e^{ik'x/2}}{k'-k+i\varepsilon}, \tag{3.102b}$$

where now the ordering of the operators on the right-hand side is important. The derivation of (3.102) parallels the classical derivation in most respects, though some subtleties arise due to operator ordering [e.g., the analytic function $\Phi(x,\xi)$ is not given by (3.33) in the lower half plane]. For details we refer to the original paper. Here we confine our discussion to two interesting properties of the quantum Gel'fand-Levitan transform.

By iterating Eq. (3.102) we obtain series expansions for χ_1 and χ_2^* in terms of R and R^*. Each term in the series is normal ordered in the R operators, i.e., all R^*'s are to the left and all R's to the right. The field operator $\phi(x)$ is then obtained just as in the classical theory,

$$\chi_1(x,k)e^{ikx/2} \underset{k\to\infty}{\longrightarrow} -\frac{\sqrt{c}}{k}\phi(x)+O\left(\frac{1}{k^2}\right). \tag{3.103}$$

By taking the asymptotic behavior of the other component of the Jost solution χ_2, we also obtain an expression for the charge-density operator $j_0(x)=\phi^*(x)\phi(x)$,

$$\chi_2(x,k)e^{ikx/2} \underset{k\to\infty}{\longrightarrow} 1-\frac{ic}{k}\int_x^\infty j_0(x')dx'+O\left(\frac{1}{k^2}\right). \tag{3.104}$$

The series expansion for the field operator obtained by iterating (3.102) and taking the limit (3.103) may be written

$$\phi(x)=\int\frac{dk_1}{2\pi}R(k_1)e^{ikx}-c\int\frac{dp_1dk_1dk_2}{(2\pi)^3}\frac{R^*(p_1)R(k_1)R(k_2)e^{i(k_1+k_2-p_1)x}}{(p_1-k_1-i\varepsilon)(p_1-k_2-i\varepsilon)}+\cdots$$
$$\equiv\sum_{n=0}^{\infty}\phi^{(n)}(x), \tag{3.105}$$

where the general term is given by

$$\phi^{(n)}(x)=(-c)^n\int\left(\prod_{i=1}^{n}\frac{dp_i}{2\pi}\right)\left(\prod_{i=1}^{n+1}\frac{dk_i}{2\pi}\right)\frac{R^*(p_n)\cdots R^*(p_1)R(k_1)\cdots R(k_{n+1})e^{i(\Sigma k-\Sigma p)x}}{\prod_{m=1}^{n}[(p_m-k_m-i\varepsilon)(p_m-k_{m+1}-i\varepsilon)]} \tag{3.106}$$

Equation (3.105) describes an operator transformation from $R(k)$ to $\phi(x)$. This transformation turns out to have a remarkably simple and familiar form in the limit of infinitely repulsive coupling, $c\to\infty$. In this limit (3.105) is just a Jordan-Wigner transformation (Creamer *et al.*, 1980),

$$\phi(x)=N_R\left[\exp\left(-2\int_x^\infty\tilde{R}^*(y)\tilde{R}(y)dy\right)\tilde{R}(x)\right], \tag{3.107}$$

where $\tilde{R}(x)$ is the Fourier transform

$$\tilde{R}(x)=\int\frac{dk}{2\pi}R(k)e^{ikx} \tag{3.108}$$

and N_R represents normal ordering of R operators. Note that in the limit $c\to\infty$, the two-body S matrix (3.83) becomes -1, and the relations (3.81) and (3.82) become canonical anticommutation relations for fermion creation and annihilation operators. The Fourier transform $\tilde{R}(x)$ is a local fermion field with anticommutation relations

$$[\tilde{R}(x),\tilde{R}^*(y)]_+=\delta(x-y). \tag{3.109}$$

Thus, for the special case of the $c=\infty$ nonlinear Schrödinger model, the quantum inverse transformation becomes a Jordan-Wigner fermion-to-boson transformation. We note that (3.107) may be written in another form using the general formula for canonical fermion or boson fields $\psi(x),\psi^\dagger(x)$ (Grosse, 1979),

$$\exp\left(\int dy\,G(y)\psi^\dagger(y)\psi(y)\right)=:\exp\left(\int dy\,(e^{G(y)}-1)\psi^\dagger(y)\psi(y)\right):. \tag{3.110}$$

From (3.107), we get

$$\phi(x)=\exp\left(i\pi\int_x^\infty\tilde{R}^*(y)\tilde{R}(y)\,dy\right)\tilde{R}(x), \tag{3.111}$$

The demonstration that (3.111) or (3.107) is the $c=\infty$ limit of the Gel'fand-Levitan transformation involves an analysis of the general term ϕ of the series (3.105) (Creamer *et al.*, 1980b) which will not be repeated here. The essential idea can be seen by studying the second term of (3.105),

$$\phi^{(1)}(x)=-c\int\frac{dp_1dk_1dk_2}{(2\pi)^3}\frac{R^*(p_1)R(k_1)R(k_2)e^{i(k_1+k_2-p_1)x}}{(p_1-k_1-i\varepsilon)(p_1-k_2-i\varepsilon)}. \tag{3.112}$$

[That the first term $\phi^{(0)}$ agrees with (3.111) is obvious.] In spite of the explicit power of c in front of (3.112), $\phi^{(1)}(x)$ has a finite limit as $c\to\infty$ due to the implicit c dependence of the R operators. Symmetrizing over the integration variables k_1 and k_2 and using the commutation relation (3.81), we get

$$\phi^{(1)}(x)=-c\int\frac{dp_1dk_1dk_2}{(2\pi)^2}\frac{R^*(p_1)R(k_1)R(k_2)}{(p_1-k_1-i\varepsilon)(p_1-k_2-i\varepsilon)}\times\left(\frac{1+S(k_1,k_2)}{2}\right)e^{i(k_1+k_2-p_1)x}. \tag{3.113}$$

By partial fractioning, changing variables, and combining terms we obtain

$$\phi^{(1)}(x)=-2c\int\frac{dp_1dk_1dk_2}{(2\pi)^3}\frac{R^*(p_1)R(k_1)R(k_2)}{(k_1-k_2+ic)(p_1-k_1-i\varepsilon)}\times e^{i(k_1+k_2-p_1)x}. \tag{3.114}$$

The $c\to\infty$ limit on $\phi^{(1)}(x)$ can now be taken,

$$\phi^{(1)}(x)\underset{c\to\infty}{\longrightarrow}2i\int\frac{dp_1dk_1dk_2}{(2\pi)^3}\frac{R^*(p_1)R(k_1)R(k_2)}{(p_1-k_1-i\varepsilon)}\times e^{i(k_1+k_2-p_1)x}$$
$$=\Delta(x)\tilde{R}(x),$$

where

$$\Delta(x)=2i\int\frac{dp_1dk_1}{(2\pi)^2}\frac{R^*(p_1)R(k_1)}{(p_1-k_1-i\varepsilon)}e^{i(k_1-p_1)x}$$
$$=-2\int_x^\infty\tilde{R}^*(y)\tilde{R}(y)dy. \tag{3.115}$$

Analysis of the higher order terms in (3.105) shows that

$$\phi^{(n)}(x)=N_R\{[\Delta(x)]^n\tilde{R}(x)/n!\}, \tag{3.116}$$

leading to the result (3.107).

The transformation (3.111) may be inverted by noting that $\tilde{R}^*(x)\tilde{R}(x)=\phi^*(x)\phi(x)$, which leads to

$$\tilde{R}(x)=\exp\left(-i\pi\int_x^\infty \phi^*(y)\phi(y)dy\right)\phi(x). \tag{3.117}$$

This is a simplified form of the direct Zakharov-Shabat transform $c=\infty$. The same result should also follow directly by taking the $c\to\infty$ limit of the reflection coefficient expressed in terms of ϕ and ϕ^*, but a procedure for doing this to all orders has not been developed.

We now return to the general case of finite c and discuss the structure of the Gel'fand-Levitan transform for the charge density operator which is obtained by solving (3.112) and taking the limit (3.114). This gives

$$j_0(x)=\sum_{n=0}^{\infty} j_0^{(n)}(x), \tag{3.118}$$

where

$$j_0^{(n)}(x)=(-c)^n\int\prod_{i=1}^{n+1}\frac{dk_i dp_i}{(2\pi)^2}\frac{(\Sigma p-\Sigma k)e^{-i(\Sigma p-\Sigma k)x}}{\prod_{i=1}^{n}[(p_i-k_i-i\varepsilon)(p_i-k_{i+1}-i\varepsilon)](p_{n+1}-k_{n+1}-i\varepsilon)}$$
$$\times R^*(p_{n+1})\cdots R^*(p_1)R(k_1)\cdots R(k_{n+1}). \tag{3.119}$$

The Gel'fand-Levitan expression (3.118) is related in an interesting way to the spectral integral equation for the finite temperature delta-function gas. The finite temperature results for this model were first obtained (Yang and Yang, 1969) by a variational method using Bethe's *Ansatz* with periodic boundary conditions. More recently these results were rederived (Thacker, 1977) using a graphical formalism which had emerged from an analysis of N-particle Feynman graphs (Thacker, 1975, 1976). In the graphical derivation, it was found that the statistical mechanics of the system could be obtained by calculating certain almost-forward matrix elements of the charge-density operator. The object of interest is the partition function

$$Q(\beta,\mu)=\mathrm{Tr}\,e^{\beta(\mu N-H)}, \tag{3.120}$$

where the trace is taken over a complete set of states. The logarithm of (3.120) is an extensive quantity which may be regarded as the connected part of the partition function. It can be shown (Thacker, 1977) that this quantity is given by

$$\ln Q=\lim_{q\to 0}\mathrm{Tr}[Y(q)e^{\beta(\mu N-H)}], \tag{3.121}$$

where

$$Y(q)=e^{-iqK}N^{-1}\left(\int_{-\infty}^{\infty} j_0(x)e^{iNqx}dx\right). \tag{3.122}$$

In this expression $K=\int x\phi^*(x)\phi(x)dx$ is the Galilean boost operator which has the property

$$e^{iqK}R^*(k)=R^*(k+q)e^{iqK}. \tag{3.123}$$

Note that formally $Y(q)\to 1$ as $q\to 0$, and the right-hand side of (3.121) would naively give Q. The effect of bringing the limit outside the trace is to pick out the connected part of Q. This occurs because a diagonal matrix element of $Y(q)$ will vanish unless the state obtained by acting to the right with the operator in parentheses in (3.122) is the same as the state obtained by acting to the left with e^{-iqK}. Thus an equal amount of momentum q must be transferred to each particle in the state, which, in a graphical expansion, can only occur if a graph is fully connected.

Matrix elements of $Y(q)$, Eq. (3.122), can be computed by using the Gel'fand-Levitan series for the charge-density operator, Eq. (3.118). The pressure $\mathcal{P}$ is obtained from $\ln Q$ by simply dividing out a factor $\beta 2\pi\delta(0)$, where $2\pi\delta(0)$ is the infinite volume analog of the size of a box. Inserting the expansion (3.118) into (3.121), we get a corresponding expansion for the pressure

$$\mathcal{P}=\sum_{n=0}^{\infty}\mathcal{P}^{(n)}. \tag{3.124}$$

The trace in (3.121) is taken by summing over a complete set of Bethe *Ansatz* states

$$R^*(k_1)R^*(k_2)\cdots R^*(k_N)|0\rangle, \tag{3.125}$$

using the expressions (3.119) and the commutation rules (3.81) and (3.82) to compute the matrix elements. The first term $\mathcal{P}^{(0)}$ turns out to be the pressure of an ideal fermi gas,

$$\mathcal{P}^{(0)}=\frac{1}{\beta}\int\frac{dk}{2\pi}\ln(1+e^{\beta(\mu-k^2)}). \tag{3.126}$$

The calculation of the general term $\mathcal{P}^{(n)}$ involves a symmetrization over the k_i's and p_i's in (3.119) which leads to extra c-dependent factors in the integrand, as in the example (3.112)–(3.114) for $\phi(x)$. The first four terms in (3.124) have been obtained by direct calculation from (3.119). From these expressions the form of the general term $\mathcal{P}^{(n)}$ has been surmised. From the form of $\mathcal{P}^{(n)}$, the series (3.124) can be summed up explicitly in terms of the solution to a nonlinear integral equation. Defining $\varepsilon(k)$ as the solution to the equation

$$\varepsilon(k)=k^2-\mu-\frac{1}{\beta}\int\frac{dk'}{2\pi}K(k-k')\ln(1+e^{-\beta\varepsilon(k')}), \tag{3.127}$$

where $K(k)=2c/(k^2+c^2)$, we find that the pressure is given by

$$\mathcal{P}=\frac{1}{\beta}\int\frac{dk}{2\pi}\ln(1+e^{-\beta\varepsilon(k)}). \tag{3.128}$$

These are the same results originally obtained by Yang and Yang. By this derivation it is found that the integral equation (3.127) and the pressure (3.128) can be obtained from the Gel'fand-Levitan expression for the charge density (3.118). The expansion of (3.128) in powers of the kernel K corresponds term by term to the expansion (3.118) for $j_0(x)$. Note that if we define a Fermi momen-

tum k_F by the zeroes of $\varepsilon(k)$,

$$\varepsilon(\pm k_F)=0\,, \tag{3.129}$$

then $\varepsilon(k)\gtrless 0$ for $|k|\gtrless k_F$. The zero temperature ($\beta\to\infty$) limit of (3.127) is thus

$$\varepsilon(k)=k^2-\mu+\int_{-k_F}^{k_F}\frac{dk'}{2\pi}K(k-k')\varepsilon(k')\,, \tag{3.130}$$

which is just the spectral equation (2.45) for excitations above the ground state. In Sec. II the integral equation for $\varepsilon(k)$ was derived by studying periodic boundary conditions in a box. Here it was obtained directly from the Gel'fand-Levitan formalism using the infinite volume commutation relations for R and R^*. This emphasizes the important point that the infinite volume algebra (3.69)–(3.72) does contain the information necessary to treat the finite density system, even though it is simpler than the finite volume algebra (3.89)–(3.92). In this regard it is also worth noting that the exchange factor (β/α) in (3.91) is simply an overall factor which drops out of the periodic boundary conditions.

D. Inverse method for lattice models

One of the most intriguing aspects of the quantum inverse method is its close relationship to the transfer matrix techniques developed in soluble lattice models. This connection was first suggested by the work of Skylanin (1979), whose derivation of the commutation relations for the nonlinear Schrödinger model was inspired by Baxter's (1972a) analysis of the eight-vertex model transfer matrix. The quantum inverse method for lattice models has been studied for the XXZ spin chain (Kulish and Sklyanin, 1979) and the symmetric six-vertex model (Thacker, 1980). Recently, Takhtajan and Faddeev (1979) (see also Faddeev, 1979) have constructed the quantum inverse formalism for the full Baxter model. In each of these examples the inverse method provides an elegant algebraic derivation of results which had previously been obtained by a Bethe *Ansatz* for many-spin-wave states.

To illustrate the essential points in a fairly simple context, we will consider the quantum inverse method for the six-vertex model. The solution for the full Baxter model is briefly reviewed in the latter part of this section. In Sec. II.D we saw that the transfer matrix and partition function were constructed from an elementary vertex L_n, given by (2.155) for the eight-vertex model and (2.159) for the six-vertex model. The transfer matrix (2.156) may be regarded as a string of vertices tied together at the ends. To motivate the formulation of the quantum inverse method on a lattice, we note that the solution to a linear eigenvalue problem of the form (3.36) with specified behavior at x_0 may be written as a path-ordered exponential,

$$\Psi(x)=:P\exp\left(i\int_{x_0}^{x}Q(y)dy\right)\Psi(x_0):\,, \tag{3.131}$$

where the path ordering refers to the matrix structure, specifically,

$$P\exp i\int_{x_0}^{x}Q\,dy=\lim_{\Delta\to 0}\exp i\int_{x0}^{x_0+\Delta}Q\,dy \times\exp i\int_{x_{0+\Delta}}^{x_0+2\Delta}Q\,dy\cdots\exp i\int_{x-\Delta}^{x}Q\,dy\,. \tag{3.132}$$

In the six-vertex model, an elementary vertex is precisely analogous to one of the exponential factors in (3.132). The lattice construct which corresponds to the path-ordered exponential solution (3.131) of the linear problem (3.36) is a string of vertices contracted over horizontal arrows. Thus the analog of the Jost solution $G(x)$ with boundary condition (3.37) is given by a product of elementary vertices,

$$G(n)=L_1L_2\cdots L_n\,. \tag{3.133}$$

In particular, the matrix analogous to $\mathcal{T}_L(k)$ in (3.85) is obtained by stringing vertices all the way across the lattice,

$$\mathcal{T}(v)=L_1L_2\cdots L_N=\begin{pmatrix}A(v) & B(v)\\ C(v) & D(v)\end{pmatrix}, \tag{3.134}$$

where N is the number of sites in a row. Using a method which parallels the derivation in Sec. III B, we will obtain relations for the operators A, B, C, and D which are depicted in Fig. 15.

The algebra of the operators defined in (3.134) is based on a fundamental property of the elementary vertex. We parametrize the vertex weights by ρ, v, and η as defined in (2.165), and consider the vertex L_n as a function of v for fixed ρ and η. Then the direct product of two vertices with different values of v is related to a direct product of the same two vertices in reverse

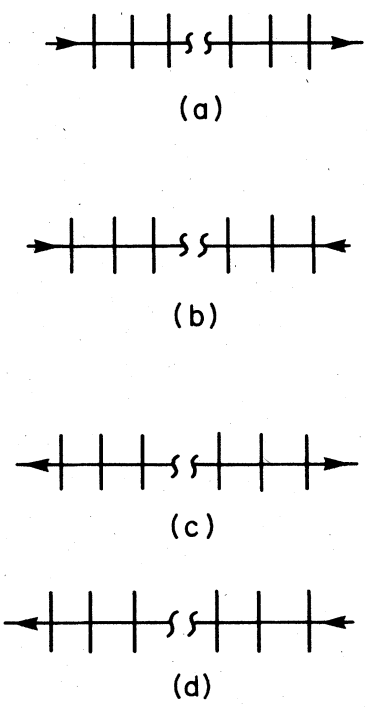

FIG. 15. Operators A, B, C, and D defined in Eq. (3.134).

order by a similarity transformation,

$$\mathcal{R}[L_n(v)\otimes L_n(v')]=[L_n(v')\otimes L_n(v)]\mathcal{R}\,, \tag{3.135}$$

where

$$\mathcal{R}=\mathcal{R}(v,v')\propto\begin{bmatrix}1&0&0&0\\0&\beta&\alpha&0\\0&\alpha&\beta&0\\0&0&0&1\end{bmatrix}, \tag{3.136}$$

with

$$\alpha\equiv\alpha(v,v')=\frac{\sin(v'-v)}{\sin(v'-v+2\eta)}, \tag{3.137}$$

$$\beta\equiv\beta(v,v')=\frac{\sin 2\eta}{\sin(v'-v+2\eta)}. \tag{3.138}$$

The relation (3.135) is analogous to the nonlinear Schrödinger result (3.53) and may be verified by direct calculation. The direct products in (3.135) may be visualized as the contraction of two vertices over a vertical bond (i.e., a spin-operator product), as shown in Fig. 16. It is interesting that the similarity transformation matrix has essentially the same structure as an elementary vertex if the latter is regarded as a four-by-four matrix.

$$L(v)=\begin{bmatrix}\rho\sin(v+\eta)&0&0&0\\0&\rho\sin(v-\eta)&\rho\sin 2\eta&0\\0&\rho\sin 2\eta&\rho\sin(v-\eta)&0\\0&0&0&\rho\sin(v+\eta)\end{bmatrix}. \tag{3.139}$$

By an appropriate choice of an overall factor in the definition (3.136) we find that

$$\mathcal{R}(v,v')=L(v'-v+\eta)\,. \tag{3.140}$$

The fundamental relation (3.135) has the structure shown in Fig. 17.

Using (3.135) we find a similar relation for the matrix of scattering data operators (3.134)

$$\mathcal{R}[\mathcal{T}(v)\otimes\mathcal{T}(v')]=[\mathcal{T}(v')\otimes\mathcal{T}(v)]\mathcal{R}\,, \tag{3.141}$$

which gives the commutation relations for A, B, C, and D defined in (3.134). These relations are identical in form to the finite volume algebra (3.89)–(3.92) for

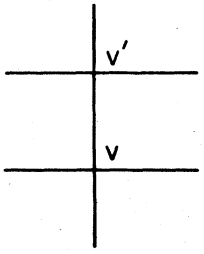

FIG. 16. Direct product of vertices $L(v)\times L(v')$.

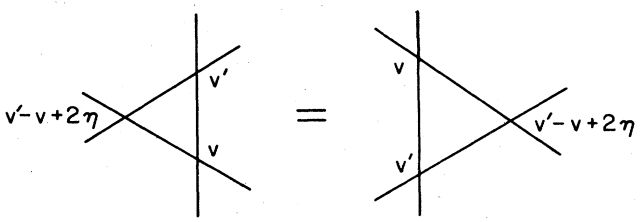

FIG. 17. Commutation relation Eq. (3.135).

the nonlinear Schrödinger model, but now with α and β given by (3.137) and (3.138). (To agree with previous conventions, we have also interchanged the roles of B and C compared with the nonlinear Schrödinger model.) From here the analysis of the six-vertex model simply repeats the steps which led from the nonlinear Schrödinger algebra (3.89)–(3.92) to the eigenvalue (3.98) and periodic boundary conditions (3.99). The operators $B(v_i)$ can be used to construct the Bethe *Ansatz* eigenstates (2.168),

$$|k_1,\ldots,k_n\rangle=B(v_1)\cdots B(v_n)|0\rangle\,, \tag{3.142}$$

where v_i and k_i are related by (2.170). It is easily seen from (2.161) that the state $|0\rangle$ with all spins up is an eigenstate of $A(v)$ and $D(v)$ separately,

$$A(v)|0\rangle=\left(\frac{\sin(v+\eta)}{\sin(v-\eta)}\right)^{N/2}|0\rangle\,, \tag{3.143}$$

$$D(v)|0\rangle=\left(\frac{\sin(v-\eta)}{\sin(v+\eta)}\right)^{N/2}|0\rangle\,. \tag{3.144}$$

By applying the transfer matrix

$$T(v)=\mathrm{tr}\mathcal{T}(v)=A(v)+D(v)\,, \tag{3.145}$$

to the states (3.142) and using the commutation relations (3.141), the eigenvalues (2.173) and periodic boundary conditions (2.172) are obtained by a straightforward procedure.

The quantum inverse method for the full Baxter eight-vertex model (Takhtajan and Faddeev, 1979) introduces some essentially new features which have not been encountered in the previously considered models. The relative simplicity of the six-vertex model resulted from the special form of the vertex matrix (2.159) or (3.139) and the commutation matrix $\mathcal{R}$, Eq. (3.136) or (3.140). By virtue of the ice rule (two arrows in and two arrows out), these matrices, written in four-by-four form, have a 1-2-1 block diagonal structure. In addition to simplifying the commutation relations, this also allows one to choose a local spin state $|+\rangle_n$ (spin up) which is annihilated by the lower left corner of the vertex matrix (2.159), viz., (2.161). For the Baxter model, the vertex is given by (2.155), which can be written in four-by-four form as

$$L(v)=\begin{bmatrix}a&0&0&d\\0&b&c&0\\0&c&b&0\\d&0&0&a\end{bmatrix}, \tag{3.146}$$

where (Baxter, 1972a)

$$a=\rho\Theta(2\eta)\Theta(v-\eta)H(v+\eta)\,, \tag{3.147a}$$

$$b=\rho\Theta(2\eta)H(v-\eta)\Theta(v+\eta)\,, \tag{3.147b}$$

$$c=\rho H(2\eta)\Theta(v-\eta)\Theta(v+\eta)\,, \tag{3.147c}$$

$$d=\rho H(2\eta)H(v-\eta)H(v+\eta)\,. \tag{3.147d}$$

Here H and Θ are Jacobi eta and theta functions with elliptic modulus k. The parametrization (3.147) expresses the vertex weights a,b,c,d in terms of ρ, η, v, and k. It reduces to the ice model form (2.165) by taking $k^{1/2}\rho=\rho_0$ and letting the modulus k go to zero. By considering the vertex (3.146) to be a function of v with ρ, η, and k fixed, it is found that a relation of the form (3.135) can be constructed for the full eight-vertex model (Baxter, 1972a) with $\mathcal{R}$ again being given by (3.140).

The fact that $d\neq 0$ (i.e., $w_1\neq w_2$) means that it is not possible to choose a local "vacuum" spin state which is annihilated by the lower left corner of (2.155). However, there is a certain gauge freedom in the choice of the vertex L_l which allows us to introduce a gauge equivalent vertex

$$L_l'=M_l^{-1}L_lM_{l+1}\,, \tag{3.148}$$

where the M_l's are a set of two-by-two c-number matrices. A $\mathcal{T}$ matrix constructed from the vertex (3.148) has elements which are linear combinations of the operators in (3.134),

$$\mathcal{T}'=L_1'L_2'\cdots L_N'=M_1^{-1}\mathcal{T}M_{N+1}\,. \tag{3.149}$$

By an appropriate choice of the transformation M_n, it is possible to find a local spin state which is annihilated by the lower left corner of the gauge-transformed vertex matrix L_n'. In fact, Baxter has shown that the eight-vertex model is equivalent to a generalized ice model where only the six vertices are allowed, but where the vertex weights depend on both the arrow configuration and on an integer l. This equivalence may be stated as a local property of the vertex (3.146) (Baxter, 1973b). It was used to construct a Bethe *Ansatz* for the transfer matrix eigenstates (Baxter, 1973c). The quantum inverse method for the Baxter model (Takhtajan and Faddeev, 1979) is also closely related to this ice-model equivalence. The relation (3.140) between the vertex matrix and the $\mathcal{R}$ matrix allows the commutation relations (3.141) to be expressed in a form which resembles the familiar case (3.89)–(3.92) in terms of operators $A_{k,l}(v)$, $B_{k,l}(v)$, $C_{k,l}(v)$, and $D_{k,l}(v)$, which are the elements of the matrix

$$\mathcal{T}_{k,l}=M_k^{-1}\mathcal{T}M_l\,. \tag{3.150}$$

From the commutation relations of these operators, the algebraic construction of eigenstates can be carried through. For a detailed discussion we refer to the work of Takhtajan and Faddeev (1979) and to the original treatment of Baxter (1972a,b; 1973a,b,c).

IV. DISCUSSION

The study of exactly integrable quantum field theories has developed rapidly over the past several years, and this development seems likely to continue apace for some time to come. In this concluding section I should like to make a few remarks about the present state of understanding and possible future directions.

An important approach to the treatment of exactly integrable quantum theories which was not discussed in this review is the study of factorizable S matrices (Zamolodchikov and Zamolodchikov, 1979). In this approach one exploits the fact that the presence of an infinite number of conservation laws in a two-dimensional theory generally will preclude the possibility of inelastic scattering. (Elastic scattering with exchange of internal quantum numbers is allowed.) This leads to a factorized S matrix which can be obtained explicitly by symmetry considerations along with analyticity and elastic unitarity. The S matrix of the sine-Gordon–massive Thirring model, which was originally obtained by this method, has recently been obtained directly from the Bethe *Ansatz* eigenstates (Korepin, 1979). Similar results for the chiral invariant Gross-Neveu model have also been obtained (Andrei and Lowenstein, 1980b). The connection between the S matrix and the eigenstates is of great interest, particularly in view of the fact that the S matrix for the $O(N)$ nonlinear sigma model is known (Zamolodchikov and Zamolodchikov, 1979), while the construction of eigenstates has not yet been accomplished.

Because of the apparently general connection between exact solubility and the existence of an infinite number of conservation laws, much effort has been devoted to the construction and study of higher conservation laws for various theories. Models for which higher quantum conservation laws have been constructed include the sine-Gordon–massive Thirring model (Flume, 1976; 1977; Kulish and Nisimov, 1976a, b; Berg *et al.*, 1976; Luscher, 1976; Flume *et al.*, 1976; Nisimov, 1977; Lowenstein and Spear, 1978), the nonlinear Schrödinger model (Thacker, 1978; Oxford, 1979), the $(\bar{\psi}\psi)^2$ Gross-Neveu model (Witten, 1978; Neveu and Papanicolao, 1979), the $O(N)$ nonlinear σ model (Pohlmeyer, 1976; Polyakov, 1977; Araf'eva *et al.*, 1978; Luscher and Pohlmeyer, 1978; Luscher, 1978; Lowenstein and Spear, 1978) and the chiral $O(N)\times O(N)$ and $SU(N)\times SU(N)$ σ models (Goldschmidt and Witten, 1980). It is reasonable to suspect that the σ models should be amenable to exact solution methods of the type we have discussed in this paper. Although a classical inverse formalism has been developed (Pohlmeyer, 1976; Zakharov and Mikhaliov, 1978), a quantum inverse method for the σ models has not yet been constructed. This is a very important unsolved problem, especially in view of the interesting analogies which can be drawn between σ models and realistic four-dimensional gauge theories.

Even within the scope of models which can presently be studied by the quantum inverse method, there are some important questions which have not yet been answered. A general method for obtaining Green's functions for integrable theories has not been devised, though progress has been made in some special cases (Wu *et al.*, 1976, Vaidya and Tracy, 1979a, b; Sato *et al.*, 1978, 1979, 1980; Jimbo *et al.*, 1980). The inverse Gel'fand-Levitan transformation has so far been formulated only for the nonlinear Schrödinger model. For other models, only the direct transformation is known. Formulation of the quantum Gel'fand-Levitan method for these

other models would be of great interest. As we saw in Sec. III.C, the Gel'fand-Levitan transformation for the nonlinear Schrödinger equation, in the limit $c \to \infty$, becomes a Jordan-Wigner transformation which expresses the interacting boson field in terms of a free fermion field, the latter being the Fourier transform of the quantized reflection coefficient. Recently, a similar connection has been found for the isotropic XY (i.e., the XX) spin chain by studying the direct transform (Fowler, 1980). The relationship between the quantum inverse method and the Jordan-Wigner fermion-boson transformation is of particular interest in the two-dimensional Ising model (which may be treated by the quantum inverse method as a special case of the Baxter model). In this model, the Jordan-Wigner transformation describes the relationship between order (spin) and disorder (kink) variables and is closely related to the self-duality between the high- and low-temperature phases of the system (Kramers and Wannier, 1941; Kadanoff and Ceva, 1971). The possibility of relating inverse scattering transformations to duality transformations is intriguing, particularly since it has recently been suggested that four-dimensional SU(N) gauge theory has a self-duality property somewhat like the Kramers-Wannier duality of the two-dimensional Ising model ('t Hooft, 1978, 1979; Mandelstam, 1979). The fermion-boson equivalence between the massive Thirring model and the quantum sine-Gordon equation (Coleman, 1975) may provide some additional insight into the question of inverse scattering *vis-à-vis* Jordan-Wigner transformations. In comparing the solution of the massive Thirring model (Bergknoff and Thacker, 1979) with that of the sine-Gordon model (Sklyanin *et al.*, 1979), the equivalence is apparent at the level of PBC's and spectral integral equations, which are essentially identical for the two theories. However, the equivalences between the operators of the two models (Coleman, 1975; Mandelstam, 1975) have not been discussed in the context of the inverse method.

The discovery of exact integrability in two-dimensional quantum field theories raises one question of central importance to the future direction of the subject: Is exact integrability an inherently two-dimensional phenomenon, or does some analogous behavior occur in realistic four-dimensional theories? The answer to this question is far from clear at present, but promising developments have been discussed by Polyakov (1979). This work draws on an analogy between four-dimensional gauge fields and two-dimensional chiral fields. (For the definition of a chiral field, see, for example, Zakharov and Mikhailov, 1978.) In gauge theory, the analog of the chiral field is a nonlocal Wilson loop operator $P \exp \oint_C A_\mu \, dx^\mu$ where C is a closed contour. The role of x space in the chiral theory is fulfilled by the space of all closed contours ("loop space") in gauge theory. The hope is that exact integrability in gauge theory will manifest itself by an infinite number of functionally conserved currents in loop space. The ideas of Polyakov fit rather well with the duality considerations of 't Hooft (1978, 1979), especially in view of the suggested connection between self-duality and exact integrability in two-dimensional models. 't Hooft proposed that the appropriate order variable in gauge theory is the Wilson loop operator, which may be interpreted as the creation operator for a loop of electric flux, and that the dual disorder variable creates a loop of magnetic flux. Whether the formulation of four-dimensional gauge theory in terms of loop variables will eventually lead to an exact solution procedure remains a matter of speculation, but the parallels with the two-dimensional formalism are amusing and encouraging.

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