Excitation dynamics in random one-dimensional systems

S. Alexander

Racah Institute of Physics, The Hebrew University, Jerusalem, Israel

J. Bernasconi and W. R. Schneider

Brown Boveri Research Center, CH-5405 Baden, Switzerland

R. Orbach

Department of Physics, University of California, Los Angeles, California 90024

The authors investigate the asymptotic (low-frequency or long-time) behavior of random one-dimensional systems described by a master equation of the form

$$C_n dP_n/dt = W_{n,n-1}(P_{n-1} - P_n) + W_{n,n+1}(P_{n+1} - P_n),$$

where either the C_n or the $W_{n,n+1}$ $(= W_{n+1,n})$ are independent positive random variables. This problem can be mapped onto a variety of physical problems, including one-dimensional particle or excitation diffusion with random spatial transfer rates or random trap depths, low-temperature properties of the random one-dimensional Heisenberg ferromagnet, one-dimensional tight-binding fermion systems with correlated diagonal and off-diagonal disorder, and electrical lines of random conductances or capacitances. Replacing dP_n/dt by d^2P_n/dt^2, the above equations also describe a harmonic chain with random force constants or random masses. Both the random W and the random C problem are shown to reduce to the same mathematical problem, and the authors review the derivation and consequences of its exact asymptotic solution for some general classes of probability densities. In particular, they are able to determine the exact $\omega \to 0$ asymptotic behavior of the single-site Green function, and thus the low-energy density of states and (for diffusion-type problems) the long-time behavior of the autocorrelation function. For diffusion-type problems, the investigators further introduce a scaling hypothesis (based on the assumption of the existence of a single characteristic length) for the time dependence of the excitation amplitude at sites other than the initially excited site. This allows the calculation of the low-frequency diffusion constant, and with the help of the Einstein relation, the low-frequency conductivity. The authors apply their results to several physical systems and, in particular, are able to account quantitatively for the complex low-frequency conductivity of the one-dimensional superionic conductor hollandite. The results may also be relevant to quasi-one-dimensional electronic systems for times (or frequencies) such that diffusion is restricted to a single dimension. The expansion of an earlier scaling approach gives additional insight into the physical meaning of the asymptotic solutions. The researchers further discuss effective-medium–type approximations, which, apart from a numerical prefactor, lead to the correct asymptotic dependences. Comparison is made with the results of alternative approaches to this type of problem, specifically with those of the continuous time random walk. The authors exhibit the remarkable satisfaction of the hyperscaling relations for the specific heat and correlation function critical exponents for the random Heisenberg ferromagnet, even though the exponents themselves are not universal. Finally, aspects of the problem which remain unsolved, or which are only partly resolved, are discussed. These include the derivation of asymptotic expansions beyond the leading term, the derivation of rigorous asymptotic expressions for the conductivity, the calculation of fluctuations in the site occupancy probabilities for the diffusion problem, and the treatment of the random one-dimensional antiferromagnet and random one-dimensional xy model. The investigators close with a short description of an application of this work to electronic conduction in highly anisotropic materials.

CONTENTS

I. INTRODUCTION

The random one-dimensional chain has been a classic test for the theoretical physicist. The first successful attack on the problem was that of Dyson (1953), followed by some important contributions by Schmidt (1957), Domb (1963), Halperin (1965, 1966), and others [for a review see also the books by Lieb and Mattis (1966) and Hori (1968)]. Dyson was able to obtain an exact solution for a special distribution of the ratio of the force constants to masses in a random harmonic chain (Dyson's Case I). This distribution has the property that it passes from a smooth decaying function, finite at zero value of its argument, to a delta function at a finite value of its argument, as a function of a parameter. Later it has been shown that Dyson's Case I relates directly to the random one-dimensional xy model (Smith, 1970) and to what is now called off-diagonal disorder for a random one-dimensional tight-binding electron band (Theodorou and Cohen, 1976).

Our detailed approach to a solution of the random one-dimensional chain is somewhat different from Dyson's because our choice of the relevant variable is not the same. The integral equation which arises in our approach therefore looks somewhat different from, but is structurally similar to, that which enters in Dyson's formalism (his Case II). The latter has been described as "... probably one of the most complicated equations of a function of a single variable in mathematical physics" (Lieb and Mattis, 1966, p. 123). Our choice of variable enables us to extract an exact solution for the single-site Green function, and for the density of states, in the low-frequency limit.

The general mathematical problem arising from the random one-dimensional harmonic chain with an arbitrary probability density for the force constants (or masses) can be mapped onto a variety of other physical problems. These include particle or excitation diffusion in a random one-dimensional material, low-temperature properties of the random one-dimensional Heisenberg ferromagnet, the one-dimensional tight-binding electron problem with correlated diagonal and off-diagonal disorder, and excitation transfer along a one-dimensional array of traps of random depth (Alexander, Bernasconi, and Orbach, 1978b; Bernasconi, Schneider, and Wyss, 1980).

In this paper we are actually mainly concerned with such diffusion-type problems, described by a master equation of the form of Eq. (2.1) or (2.2) below, rather than with the harmonic chain problem. Our asymptotic solution for the single-site Green function, however, applies to both types of systems and enables us to predict the low-energy density of states and the low-temperature specific heat of a variety of physical systems. For diffusion-type systems it determines the long-time development of the autocorrelation function, i.e., of the excitation amplitude at the site of initial excitation.

In addition, if one assumes that a single characteristic length $\xi(t)$, or $\xi(\omega)$, exists for a random one-dimensional system, one obtains expressions for the transport properties which relate them uniquely to the autocorrelation function. Though these results are not entirely rigorous, they are plausible and attractive. Their application to at least one physical system, hollandite ($K_{1.54}Mg_{0.77}Ti_{7.23}O_{16}$), a one-dimensional superionic conductor with impurity barriers of random heights, appears to explain a quite remarkable frequency dependent ionic conductivity over a wide temperature range (Bernasconi *et al.*, 1979). Further, the specific distribution of barrier heights which fits the frequency dependence of the conductivity leads to a prediction of a transition from zero-to-finite dc conductivity at a temperature in the vicinity of 400 K for this material. This remarkable property is solely a consequence of the change in the site-site transition rate probability density with temperature.

We shall consider explicitly several general classes of probability densities for the site-site coupling constants, or for the masses, respectively. The majority of probability densities which are of interest for physical problems belong to one of these classes. Not included are systems with a finite probability for a complete break, corresponding to a probability density which contains a delta function at zero coupling strength. Such "interrupted chain systems" have already been investigated by Alexander, Bernasconi, and Orbach (1978a). The corresponding autocorrelation function possesses a characteristic form, $\exp[-(\lambda t)^{1/3}]$ for long times t, where λ is a constant related to the number of breaks and to the site-site coupling between breaks. This result can also be extracted from the classic review of Lifshitz (1964) about the behavior of the density of states near a band edge. It is a consequence of complete localization of excitations within finite segments of the disordered chain.

The present paper deals with systems which are more interesting—those with zero measure for a complete break. We shall see that this may lead to a different kind of localization, but always with a diverging correlation length $\xi(t)$ as $t \to \infty$. The rate of this divergence (compared to $t^{1/2}$ in the diffusive limit) determines the "strength of localization." We wish to emphasize that all of our states are localized in the Anderson sense. Our use of the term localization rather refers to the behavior of $\xi(t)$ relative to pure diffusion as t becomes large.

To some extent, this paper has the character of a review. It presents a number of new results, but also summarizes the present status of our investigations, partly contained in previous letters and papers (Alexander, Bernasconi, and Orbach, 1978a, 1978b; Alexander and Bernasconi, 1979; Bernasconi *et al.*, 1978, 1979; Bernasconi, Schneider, and Wyss, 1980), and makes comparison with alternative methods. In Sec. II we introduce the model and the different classes of probability densities we shall consider. We then define the precise mathematical problem and discuss the quantities which are to be determined. Section III is concerned with physical realizations of the mathematical model, including an electrical analog which will be useful for the construction of our approach to the solution. In Sec. IV we derive the formal solution for the autocorrelation function, and contact is made with the classic treatment of Dyson (1953). Both random coupling constants and random masses are treated, and

the two cases are shown to reduce to the same type of integral equation. Its asymptotic solution near $\omega = 0$ (Bernasconi, Schneider, and Wyss, 1980) is presented in Sec. V, and in Sec. VI we discuss the resulting asymptotic behavior of the autocorrelation function for the different classes of probability densities. We also derive the corresponding results for the density of states and specific heat for systems onto which we can map our problem. In Sec. VII we investigate the asymptotic behavior of transport quantities, such as the mean-square displacement and the frequency-dependent conductivity $\sigma(\omega)$. A scaling hypothesis is used to relate $\sigma(\omega)$ uniquely to the single-site Green function. In Sec. VIII we summarize our principal results and discuss their implications in some detail. In Sec. IX we present an explicit scaling approach which expands the ideas of an earlier version (Alexander and Bernasconi, 1979), and in Sec. X we analyze the results of effective-medium–type approximations. Several specific applications to physical systems are discussed in Sec. XI, and comparison is made with the results of alternative approaches. In particular, we make connection with the Scher-Lax-Montrall approach (Scher and Lax, 1973; Scher and Montroll, 1975) and discuss their effective-medium–type solution and its inability to reproduce the correct asymptotic behavior of the autocorrelation function in one dimension. In Sec. XII we exhibit the remarkable satisfaction of the hyperscaling relation for the specific heat and correlation function critical exponents for the random Heisenberg ferromagnet. Finally, we outline further developments of our mathematical approach and discuss some unsolved aspects of the problem in Sec. XIII. We suggest some future applications of our solution and point out the remarkable similarities of experimental results for one-dimensional organic conductors with the predictions of our model.

We wish to point out that the reader who is only interested in the main results may skip Secs. IV through VII, which contain the technical details of our mathematical approach.

II. THE MODEL

The dynamics of our general type of one-dimensional disordered lattice systems can be described by an infinite set of coupled rate equations (a master equation). In our previous investigations (Alexander, Bernasconi, and Orbach, 1978a, 1978b; Alexander and Bernasconi, 1979; Bernasconi *et al.*, 1978, 1979; Bernasconi, Schneider, and Wyss, 1980) their form was chosen to correspond to the problem of classical diffusion in a random one-dimensional lattice, i.e.,

$$\frac{dP_n}{dt} = W_{n,n-1}(P_{n-1} - P_n) + W_{n,n+1}(P_{n+1} - P_n), \tag{2.1}$$

where n ($n = 0, \pm 1, \pm 2, \ldots$) denotes the lattice sites, $P_n(t)$ is the excitation amplitude at site n at time t, and the nearest-neighbor transfer rates (or coupling constants) $W_{n,n+1} = W_{n+1,n} \geq 0$ are assumed to be independent random variables, distributed according to a given probability density $\rho(w)$. We shall call this Case (i).

In the next section we shall show that a number of different physical problems can be reduced to an investigation of the above class of equations, with an appropriate interpretation of the quantities $W_{n,n+1}$ and P_n. In some of these realizations, one would also like to consider a generalized version of Eqs. (2.1). For an electrical line of conductances $W_{n,n+1}$ and capacitances C_n, e.g., one has

$$C_n \frac{dP_n}{dt} = W_{n,n-1}(P_{n-1} - P_n) + W_{n,n+1}(P_{n+1} - P_n), \tag{2.2}$$

where the P_n represent the node potentials. For a harmonic chain with masses M_n and force constants $W_{n,n+1}$, the equations of motion are

$$M_n \frac{d^2P_n}{dt^2} = W_{n,n-1}(P_{n-1} - P_n) + W_{n,n+1}(P_{n+1} - P_n), \tag{2.3}$$

in which case P_n denotes the displacements.

If the C_n or M_n, respectively, are independent random variables, and $W_{n,n+1}$ is a constant independent of n [we shall call this Case (ii)], then the mathematical problems corresponding to Eqs. (2.2) or (2.3), respectively, lead to the same type of integral equation as is obtained for the case of Eq. (2.1), where the $W_{n,n+1}$ are the independent random variables. This will be shown in Sec. IV.

We notice that all the above models correspond to Dyson's (1953) Case II. In Dyson's Case I the quantities λ_n (where $\lambda_{2n-1} = W_{n,n+1}/M_n$, $\lambda_{2n} = W_{n,n+1}/M_{n+1}$) are the independent random variables, and such a system is related to the random one-dimensional xy model and to a one-dimensional fermion system with off-diagonal disorder (see the introduction, Sec. I).

We shall consider the following three classes of probability densities $\rho(w)$ for the transfer rates (or coupling constants) $W_{n,n+1}$ in this paper.

Class (*a*). $\rho(w)$ such that

$$W_{av}^{-1} \equiv \int_0^\infty dw\, w^{-1}\rho(w) < \infty. \tag{2.4a}$$

Class (*b*). $\rho(w)$ such that

$$\rho(w) \to \text{const} \quad \text{as } w \to 0. \tag{2.4b}$$

Class (*c*).

$$\rho(w) = \begin{cases} (1-\alpha)w^{-\alpha}, & 0 \leq w \leq 1 \\ 0 \quad \text{otherwise}, & 0 < \alpha < 1. \end{cases} \tag{2.4c}$$

Most distributions $\rho(w)$ of interest for physical problems belong to one of these classes. An exception are $\rho(w)$ which contain a delta function at $w = 0$. Corresponding one-dimensional systems decompose into segments of finite length (interrupted chain systems) and have been investigated in various contexts (Domb *et al.*, 1959; Rice and Bernasconi, 1973; Alexander *et al.*, 1978a; Odagaki and Lax, 1980).

The corresponding classes of probability densities $r(m)$ or $r(c)$ for the distribution of the quantities M_n or C_n, respectively, are the following.

Class (*a'*). $r(m)$ such that

$$M_{av} \equiv \int_0^\infty dm\, m r(m) < \infty. \tag{2.5a}$$

Class (b'). $r(m)$ such that

$$m^2 r(m) \to \text{const} \quad \text{as } m \to \infty . \tag{2.5b}$$

Class (c').

$$r(m) = \begin{cases} (1-\alpha) m^{-(2-\alpha)}, & 1 \le m < \infty \\ 0 \quad \text{otherwise}, & 0 < \alpha < 1 . \end{cases} \tag{2.5c}$$

It should be pointed out that the above classes of probability densities are defined with respect to the asymptotic behavior of the corresponding systems. Systems belonging to the same class exhibit the same behavior only asymptotically for large times t, or small frequencies ω.

The main mathematical problem with which we shall be concerned is defined by Eq. (2.1), together with the initial condition

$$P_n(t=0) = \delta_{n,0} , \tag{2.6}$$

and the key quantity to be determined is the single-site Green function $\langle \tilde{P}_0(\omega) \rangle$, where $\tilde{P}_n$ denotes the Laplace transform of P_n,

$$\tilde{P}_n(\omega) = \int_0^\infty dt\, e^{-\omega t} P_n(t) . \tag{2.7}$$

The average $\langle \cdots \rangle$ is defined with respect to the disbtibution of the (independent) random variables $W_{n,n+1}$. Thus $\langle \tilde{P}_0(\omega) \rangle$ is the Laplace transform of the autocorrelation function $\langle P_0(t) \rangle$.

In the analogous problems corresponding to Eqs. (2.2) or (2.3), respectively, two different Green functions can be defined, related to different choices of the initial conditions. These will be discussed in later sections.

A number of physical quantities can be determined from $\langle \tilde{P}_0(\omega) \rangle$ alone. The autocorrelation function $\langle P_0(t) \rangle$, e.g., which describes the time development of the initial probability amplitude, is simply the inverse Laplace transform of $\langle \tilde{P}_0(\omega) \rangle$; and for Case (i) the corresponding eigenvalue density of states $N(\varepsilon)$ is given by

$$N(\varepsilon) = -\frac{1}{\pi} \mathrm{Im} \langle \tilde{P}_0(-\varepsilon + i0^+) \rangle , \tag{2.8}$$

with a similar expression for Case (ii). If the spectral parameter ε is related to the energy eigenvalues of the system, thermodynamic quantities (such as the specific heat) can be calculated from a knowledge of $N(\varepsilon)$.

The frequency-dependent hopping conductivity $\sigma(\omega)$, however, is a somewhat more complicated quantity of interest. It is related to the mean-square displacement $x^2(t)$ as follows (Scher and Lax, 1973; Bernasconi, Schneider, and Wyss, 1980) (the lattice constant is unity):

$$\sigma(\omega) = \frac{n_0 e^2}{k_B T} \langle D(-i\omega) \rangle , \tag{2.9}$$

where n_0 is the density, e the charge of the charge carriers, T the temperature, and $\langle D(\omega) \rangle$ a frequency-dependent mobility,

$$D(\omega) = \tfrac{1}{2}\omega^2 \int_0^\infty dt\, e^{-\omega t} x^2(t) . \tag{2.10}$$

In terms of the solutions $P_n(t)$ corresponding to Eqs. (2.1) and (2.6), $x^2(t)$ can be expressed as

$$x^2(t) = \sum_{n=-\infty}^{\infty} n^2 P_n(t) , \tag{2.11}$$

so that

$$\langle D(\omega) \rangle = \tfrac{1}{2}\omega^2 \sum_{n=-\infty}^{\infty} n^2 \langle \tilde{P}_n(\omega) \rangle . \tag{2.12}$$

Through Eqs. (2.9) and (2.12), $\sigma(\omega)$ is related to all $\langle \tilde{P}_n \rangle$, and not directly to $\langle \tilde{P}_0 \rangle$ alone. In Sec. VII, however, we shall propose and discuss a general scaling hypothesis which will allow us to relate the low-frequency behavior of $\sigma(\omega)$ uniquely to that of $\langle \tilde{P}_0(-i\omega) \rangle$. Before we develop the approach to our solution, let us first discuss some physical problems that are related to our general models of random one-dimensional lattice systems.

III. PHYSICAL REALIZATIONS

A large number of physical problems can be reduced to an investigation of the general type of equations represented by Eqs. (2.2) or (2.3) above. In the following we briefly discuss the most important of the respective physical models.

A. Diffusion and hopping conduction of classical particles (or excitations)

We consider Eqs. (2.2) and identify $C_n P_n(t)$ with the site occupation probabilities. The problem is then equivalent to a *random walk on a random lattice*, and the transfer rate from site n to site $n+1$ is given by $T_{n,n+1} = W_{n,n+1}/C_n$.

If C_n is constant, the transfer rates are symmetric, $T_{n,n+1} = T_{n+1,n}$, as the $W_{n,n+1}$ are always assumed to be symmetric. We notice that in spectral diffusion problems, for example, the transfer rates become symmetric if the temperature is sufficiently high. The anomalous asymptotic behavior associated with class (c) probability densities [Eq. (2.4c)] can be realized for an activated (symmetric) $W_{n,n+1}$,

$$W_{n,n+1} \propto \exp(-\Delta_{n,n+1}/k_B T) , \tag{3.1}$$

if the activation energies $\Delta_{n,n+1}$ are random and distributed according to a probability density

$$\bar{p}(\Delta) \propto \exp(-\Delta/T_0) , \quad \Delta \ge \Delta_0 . \tag{3.2}$$

Such a model has been applied to explain the low-frequency transport properties of the superionic conductor hollandite (Bernasconi *et al.*, 1979) (see Sec. XI), and leads to a temperature-dependent exponent, $\alpha = 1 - T/T_0$, in Eq. (2.4c).

A probability density of the type of Eq. (2.4c) can also arise from configurational disorder. At low temperatures one has $W_{n,n+1} \propto \exp(-\gamma l)$ for hopping between equivalent randomly distributed sites of density p on a one-dimensional regular lattice, and the distance l between two neighboring sites is distributed according to

$$R(l) = p(1-p)^{l-1} , \tag{3.3}$$

which leads to a density-dependent exponent, $\alpha = 1 + \ln(1-p)/\gamma$, in Eq. (2.4c). This is the one-dimensional

case of the more general d dimensional random hopping model of Scher and Lax (1973). Recently it has been investigated by Böttger *et al.* (1979) and others.

For a constant $W_{n,n+1}=W_0$ and random C_n, our model of Eq. (2.2) gives rise to a *random trapping model* if we identify C_n with $\exp(\beta\Delta_n)$, $\beta=1/k_BT$. In this case, the transfer rates are nonsymmetric,

$$T_{n,n+1}=W_0e^{-\beta\Delta_n}, \quad T_{n+1,n}=W_0e^{-\beta\Delta_{n+1}}, \tag{3.4}$$

and the quantities $C_nP_n=P_n\exp(\beta\Delta_n)$ are related to the thermal equilibrium distribution. We note that Eq. (3.4) restricts the transfer mechanism and does not represent the most general trapping model.

B. Magnetic models

The low-lying excitations of a Heisenberg chain with random ferromagnetic interactions ($J_{n,n+1}\rightarrow W_{n,n+1}$) are described by equations of the type of Eq. (2.2) (linerized equations of motion!). Configurational disorder [see above, Eq. (3.3)] again leads to a probability density for the $J_{n,n+1}$ of the type of Eq. (2.4c). Corresponding models have recently been treated by Theodorou and Cohen (1979) using a cluster approximation. Our results differ from theirs for reasons best seen in a scaling approach (Alexander and Bernasconi, 1979) (see also Sec. XI). We further note that the random C_n model would correspond to a random spin model, which, however, does not seem too interesting at the moment.

Huber and Ching (1980) have recently shown that a planar model of a classical spin glass is also equivalent (in one dimension) to our type of systems with a Class (b) probability density,

$$\rho(J)=(2/\pi)^{1/2}\exp(-J^2/2), \quad 0\leq J<\infty. \tag{3.5}$$

C. Fermion systems

A one-dimensional tight-binding model (Anderson Hamiltonian) for electrons leads to equations of the type of our Eqs. (2.1) if diagonal and off-diagonal disorder are correlated in a specific manner (site energies ε_n and transfer integrals $W_{n,n+1}$ related by $-\varepsilon_n=W_{n-1,n}+W_{n,n+1}$). The assumption of such correlations seems, however, to be somewhat artificial, and we are not aware of a physical model which would predict their existence.

D. Lattice vibrations of a harmonic chain

We consider Eqs. (2.3), where the P_n denote the site displacements, and assume that either the masses M_n or the force constants $W_{n,n+1}$ (but not both) are random variables. We show in the next section that both problems lead to the same type of integral equation which will be analyzed and discussed in subsequent sections. We note that M_ndP_n/dt obeys the same equation of motion as M_nP_n, so that we can describe momentum diffusion along the chain, as well as displacement diffusion. We shall use the Laplace transformed equations of motion in our analysis, so that we further note that ω will be replaced by ω^2 if we consider Eqs. (2.3) instead of Eqs. (2.1) or (2.2).

Since the classic treatment of Dyson (1953), the random harmonic chain problem has received a great deal of attention [see Lieb and Mattis (1966) and Hori (1968) for a review]. Nevertheless, our rigorous results for the low-energy behavior of the density of states (Bernasconi, Schneider, and Wyss, 1980) cannot be derived with previous approaches.

E. Electrical lines

Several realizations of electrical lines lead to equations of the type of Eqs. (2.2) or (2.3). In the next section we explicitly consider Eqs. (2.2), which correspond to an RC line, and discuss in some detail the cases where either the capacitances C_n or the conductances $W_{n,n+1}$ are random. This electrical analog of our general model turns out to be very useful for the construction of our approach to a solution.

IV. FORMAL SOLUTION FOR $\langle\tilde{P}_0(\omega)\rangle$

For convenience we use the language of electrical lines and consider the equations for an RC transmission line, i.e., Eqs. (2.2). The Laplace transform of these equations is

$$W_{n,n-1}(\tilde{P}_n-\tilde{P}_{n-1})+W_{n,n+1}(\tilde{P}_n-\tilde{P}_{n+1})+\omega C_n\tilde{P}_n=C_nP_n(0), \tag{4.1}$$

where $\tilde{P}_n=\tilde{P}_n(\omega)$ [see Eq. (2.7)] is the voltage at site n, C_n is a capacitance, and $W_{n,n+1}$ a conductance. For the calculation of the single-site (or diagonal) Green function $\langle\tilde{P}_0(\omega)\rangle$, we have $P_n(0)=0$ for $n\neq 0$, and it will turn out to be convenient to introduce admittances and impedances corresponding to the semi-infinite lines to the right (for $n\geq 0$) and to the left (for $n\leq 0$), respectively, of a node n. Let $G_n^{(\pm)}$ denote the line admittance to the right (left) of node n, and $R_n^{(\pm)}$ the line impedance including the capacitor C_n to the right (left) of node n (see Fig. 1). We restrict our attention in the following to $G_n\equiv G_n^{(+)}(\omega)$ and $R_n^{(+)}(\omega)$, $n\geq 0$, which are defined by

$$G_n=\tilde{I}_{n,n+1}/\tilde{P}_n=W_{n,n+1}(\tilde{P}_n-\tilde{P}_{n+1})/\tilde{P}_n, \tag{4.2a}$$

$$\phi_n\equiv\omega R_n=\omega\tilde{P}_n/\tilde{I}_{n-1,n}=C_n^{-1}(\tilde{I}_{n-1,n}-\tilde{I}_{n,n+1})/\tilde{I}_{n-1,n}, \tag{4.2b}$$

where $\tilde{I}_{n,n+1}$ are the branch currents, and where we have introduced the new variables ϕ_n to make the expressions more symmetric. The treatment of $G_n^{(-)}(\omega)$ and

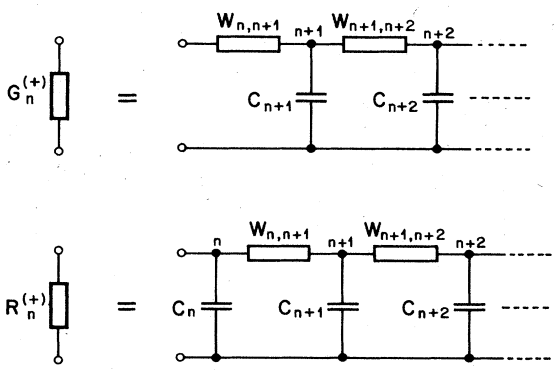

FIG. 1. Graphical illustration of the definition of the admittance $G_n^{(+)}$ and of the impedance $R_n^{(+)}$ used in connection with the electrical line described by Eq. (4.1).

$R_n^{(-)}(\omega)$ is, of course, exactly analogous.

From Eqs. (4.1) we obtain the following recursion relations for G_n and ϕ_n:

$$G_n = \frac{\omega}{\omega/W_{n,n+1} + \phi_{n+1}}, \tag{4.3a}$$

$$\phi_n = \frac{\omega}{\omega C_n + G_n}. \tag{4.3b}$$

The quantities G_n and ϕ_n can therefore be expressed as infinite continued fractions depending on ω and on the original (independent) random variables $C_n, C_{n+1}, \ldots$, and $W_{n,n+1}$, $W_{n+1,n+2}, \ldots$.

As indicated in Sec. II, we shall restrict ourselves to the following two cases.

(i) The $W_{n,n+1}$ are independent random variables, distributed according to $\rho(w)$, but the C_n are constant ($\equiv C$).

(ii) The C_n (or C_n^{-1}) are independent random variables, distributed according to $r(c)$ [or $\rho(c^{-1}) = c^2 r(c)$, respectively], but the $W_{n,n+1}$ are constant ($\equiv W$).

For case (i) we eliminate the ϕ_n, redefine ω according to $C\omega \to \omega$, and then obtain

$$G_n = \left(\frac{1}{W_{n,n+1}} + \frac{1}{\omega + G_{n+1}}\right)^{-1}. \tag{4.4a}$$

Similarly, for case (ii) we eliminate the G_n, redefine ω according to $W^{-1}\omega \to \omega$, and obtain

$$\phi_n = \left(C_n + \frac{1}{\omega + \phi_{n+1}}\right)^{-1}. \tag{4.4b}$$

In both cases we therefore obtain the same type of integral equation (Bernasconi *et al.*, 1978, 1980),

$$f_\omega(x) = \int_0^\infty dx' f_\omega(x') \int_0^\infty dy\, \rho(y) \delta\left(x - \left(\frac{1}{y} + \frac{1}{\omega + x'}\right)^{-1}\right), \tag{4.5}$$

for the probability density $f_\omega(x)$ corresponding to the random variables G_n or ϕ_n, respectively. In Eq. (4.5), $\rho(y)$ denotes the (given) probability density of the original random variables $W_{n,n+1}$ or C_n^{-1}, respectively.

As shown below, the problem of calculating the single-site Green function $\langle \tilde{P}_0(\omega) \rangle$ and related quantities reduces to the problem of solving the above integral equation. Depending on the choice of variables, there are several ways of constructing an integral equation related to a difference equation of the form of Eq. (4.1). Our approach is closely related to Dyson's (1953) original work. Dyson chooses the variables

$$\xi_{2n-1} = \omega^{-1} W_{n-1,n} \phi_n, \quad \xi_{2n} = \omega^{-1} C_n^{-1} G_n, \tag{4.6}$$

which leads to the recursion relation

$$\xi_m = \frac{\lambda_m}{\omega(1 + \xi_{m+1})}, \tag{4.7}$$

where

$$\lambda_{2n-1} = W_{n-1,n}/C_n, \quad \lambda_{2n} = W_{n,n+1}/C_n. \tag{4.8}$$

The equivalence of the random W and the random C problems is therefore obvious in Dyson's formalism and is pointed out by him. Because of the different choice of variables, his integral equation for the problems we consider (Dyson's Case II) is different from our integral equation, Eq. (4.5).

Dyson was able to solve the integral equation derived from Eq. (4.7) exactly when he treated the λ_m as independent random variables, distributed according to a particular set of probability densities. As can be seen from the definition of the λ_m this does not correspond to a meaningful realization of the RC transmission line problem (or of the random harmonic chain problem). The resulting problem can, however, be mapped on some other interesting physical models (Smith, 1970; Theodorou and Cohen, 1976) (see Sec. I).

We now show that the solution of the integral equation (4.5) enables us to calculate the statistical average of the diagonal element of the Green function, $\langle \tilde{P}_0(\omega) \rangle$ or $\langle P_0(t) \rangle$, and of the density of states.

We first consider the case of random $W_{n,n+1}$ and constant C_n ($\equiv C$ for all n), i.e., case (i) above. With the initial condition

$$P_n(t=0) = \delta_{n,0}, \tag{4.9}$$

the right-hand side of Eq. (4.1) becomes equal to $C\delta_{n,0}$, and from this, together with Eq. (4.2a), we obtain

$$\tilde{P}_0(\omega) = (\omega + G_0^{(-)} + G_0^{(+)})^{-1} \tag{4.10}$$

for a definite chain, where ω and $\tilde{P}_0$ have been normalized according to $C\omega \to \omega$ and $\tilde{P}_0/C \to \tilde{P}_0$, respectively. As $G_0^{(-)}$ and $G_0^{(+)}$ are independent random variables, both distributed according to $f_\omega(g)$, we can express the average in the following way,

$$\langle \tilde{P}_0(\omega) \rangle = \int_0^\infty dg' f_\omega(g') \int_0^\infty dg'' f_\omega(g'') \frac{1}{\omega + g' + g''}. \tag{4.11}$$

We notice that in Eqs. (4.5) and (4.11), ω is taken to be real and positive. For complex ω, $\langle \tilde{P}_0(\omega) \rangle$ is obtained by a subsequent analytic continuation, and the average time evolution of $P_0(t)$ is then given by the inverse Laplace transform,

$$\langle P_0(t) \rangle = \langle \mathfrak{L}^{-1} \tilde{P}_0(\omega) \rangle = \mathfrak{L}^{-1} \langle \tilde{P}_0(\omega) \rangle. \tag{4.12}$$

One can thus calculate the averages for each ω separately, using $f_\omega(g)$, and the fact that the $G_n(\omega)$ for different ω are correlated on any specific chain does not affect the average in Eq. (4.12).

As the matrix of the coefficients in Eqs. (2.2) is symmetric for the case (i) problem, and as the chain is moreover translationally invariant on the average, it follows that the density of states is simply given by

$$N(\varepsilon) = -\frac{1}{\pi} \mathrm{Im} \langle \tilde{P}_0(-\varepsilon + i0^+) \rangle. \tag{4.13}$$

This is equivalent to saying that $\langle P_0(t) \rangle$ is the Laplace transform of the average density of states.

The random capacitor case, case (ii) above, is slightly more complicated. For the initial condition (4.9), the analog of Eq. (4.10) becomes

$$\tilde{P}_0(\omega) = C_0[\omega C_0 + \omega/(\omega + \phi_{-1}^{(-)}) + \omega/(\omega + \phi_1^{(+)})]^{-1}, \tag{4.14}$$

and the statistical average now also involves the random variable C_0,

$$\langle\tilde{P}_0(\omega)\rangle_P=\int_0^\infty dc\, r(c)c\int_0^\infty d\phi' f_\omega(\phi')\int_0^\infty d\phi'' f_\omega(\phi'')[\omega c+\omega/(\omega+\phi')+\omega/(\omega+\phi'')]^{-1}. \tag{4.15}$$

In Eqs. (4.14) and (4.15), we have normalized ω and $\tilde{P}_0$ according to $\omega/W\to\omega$ and $W\tilde{P}_0\to\tilde{P}_0$, respectively; and the subscript P in Eq. (4.15) refers to the chosen initial condition, Eq. (4.9). The average charge at the origin is

$$\langle\tilde{Q}_0(\omega)\rangle_P=\langle C_0\tilde{P}_0(\omega)\rangle_P \tag{4.16}$$

and is therefore not proportional to $\langle\tilde{P}_0(\omega)\rangle_P$ (if the C_n are random).

We notice also that in this case (random C) the statistical averages, e.g., $\langle\tilde{P}_0(\omega)\rangle$ and $\langle\tilde{Q}_0(\omega)\rangle$, depend on the initial condition. If we impose

$$Q_n(t=0)=\delta_{n,0}, \tag{4.17}$$

e.g., instead of Eq. (4.9), this leads to

$$\langle\tilde{P}_0(\omega)\rangle_Q=\langle C_0^{-1}\tilde{P}_0(\omega)\rangle_P \tag{4.18}$$

and to

$$\langle\tilde{Q}_0(\omega)\rangle_Q=\langle\tilde{P}_0(\omega)\rangle_P. \tag{4.19}$$

Furthermore, the eigenvalue equations for the P_n (and for the Q_n) are not symmetric, so that the connection to the density of states is not immediate. One can, however, symmetrize the equations by Dyson's (1953) transfromation

$$P_n\to Y_n=C_n^{1/2}P_n. \tag{4.20}$$

It then follows that

$$N(\varepsilon)=-\frac{1}{\pi}\mathrm{Im}\langle\tilde{P}_0(-\varepsilon+i0^+)\rangle_P=-\frac{1}{\pi}\mathrm{Im}\langle\tilde{Q}_0(-\varepsilon+i0^+)\rangle_Q. \tag{4.21}$$

We have thus shown that the solution of the integral equation (4.5) allows us to calculate important properties of random linear chain systems, both for case (i)- and for case (ii)- type problems. There are, however, limitations. In particular, we have not yet been able to calculate $P_n(t)$ for general n or to evaluate the sums in Eqs. (2.11) and (2.12) in a rigorous way from the knowledge of the solution of the integral equation (4.5). The reason essentially is that, while they all have the same distribution, the $G_n(\omega)$ for different n are correlated. For the same reason one cannot obtain closed expressions for the fluctuations of $\langle P_0(t)\rangle$ [e.g., calculate $\langle P_0^2(t)\rangle$], because the implied convolutions do not commute with the averaging process. We shall return to these questions in later sections of this paper.

V. ASYMPTOTIC SOLUTION FOR $f_\omega(x)$ NEAR $\omega=0$

In this section we discuss the integral equation (4.5) in some detail and, in particular, we shall be interested in the derivation of results which are valid asymptotically as $\omega\to 0$. For convenience, we reproduce the integral equation,

$$f_\omega(x)=\int_0^\infty dx' f_\omega(x')\int_0^\infty dy\,\rho(y)\,\delta\left(x-\left(\frac{1}{y}+\frac{1}{\omega+x'}\right)^{-1}\right), \tag{5.1}$$

and, as $f_\omega(x)$ represents a probability density, we remark that an acceptable solution has to satisfy

$$f_\omega(x)\geq 0,\quad x\geq 0 \tag{5.2}$$

and

$$\int_0^\infty dx f_\omega(x)=1. \tag{5.3}$$

With respect to $\rho(y)$, we shall explicitly discuss the three classes of probability densities defined by Eqs. (2.4a)–(2.4c).

For our investigations it turns out to be convenient to study the Stieltjes transform of Eq. (5.1), which, after some manipulations, can be written as (Bernasconi, Schneider, and Wyss, 1980)

$$\hat{f}_\omega(z)=\hat{\rho}(z)+\int_0^\infty dy\,\rho(y)\left(\frac{y}{y+z}\right)^2\hat{f}_\omega\left(\omega+\frac{yz}{y+z}\right), \tag{5.4}$$

where $\hat{g}$ denotes the Stieltjes transform of a function g,

$$\hat{g}(z)=\int_0^\infty dx\,g(x)(x+z)^{-1},\quad z>0. \tag{5.5}$$

Let us first consider the trivial case of an ordered system, i.e.,

$$\rho(y)=\delta(y-Y_0), \tag{5.6}$$

for which Eq. (5.4) reduces to

$$\hat{f}_\omega(z)=\frac{1}{z+Y_0}+\left(\frac{Y_0}{z+Y_0}\right)^2\hat{f}_\omega\left(\omega+\frac{zY_0}{z+Y_0}\right). \tag{5.7}$$

The solution of Eq. (5.7) is

$$\hat{f}_\omega(z)=[z+X_0(\omega)]^{-1}, \tag{5.8}$$

i.e., we finally obtain

$$f_\omega(x)=\delta(x-X_0(\omega)), \tag{5.9}$$

where

$$X_0(\omega)=\tfrac{1}{2}[(\omega^2+4Y_0\omega)^{1/2}-\omega]. \tag{5.10}$$

From this solution of our integral equation we can then derive the well-known results for the autocorrelation function, $\tilde{P}_0(\omega)$ or $P_0(t)$, and for the density of states $N(\varepsilon)$, corresponding to an ordered system (see the following section).

For genuinely disordered systems, for which $\rho(y)$ is not a single delta function, the general solution of the integral equation (5.1) or (5.4) is not known for arbitrary values of ω. If we observe that $\hat{g}(z)\leq z^{-1}$ if g is a probability density, the following inequalities can be obtained from Eq. (5.4):

$$\hat{\rho}(z)\leq\hat{f}_\omega(z)\leq\hat{\rho}(z)+\omega^{-1}. \tag{5.11}$$

In the limit $\omega\to\infty$, this immediately leads to $\hat{f}_\infty(z)=\hat{\rho}(z)$, i.e.,

$$f_\infty(x)=\rho(x). \tag{5.12}$$

If the first $N-1$ positive moments,

$$\mu_k\equiv\int_0^\infty dx\,x^k\rho(x),\quad k=1,2,\ldots,N-1 \tag{5.13}$$

of the given probability density $\rho(x)$ exist, one can construct (Bernasconi and Schneider, 1981) an exact large ω expansion for $f_\omega(x)$ of the following form:

$$f_\omega(x)=\rho(x)+\sum_{k=1}^{N}\frac{\phi_k(x)}{\omega^k}+R_\omega^{(N)}(x). \tag{5.14}$$

The recursive determination of the $\phi_k(x)$, which in general are distributions, is rather complicated but straightforward (Bernasconi and Schneider, 1981). With Eq. (5.14) we can then derive large ω expansions for several quantities of interest, in particular for the diagonal Green function $\langle \tilde{P}_0(\omega) \rangle$.

More interesting, however, and far less trivial, is the construction of a small ω asymptotic expansion for $f_\omega(x)$. We first remark that for $\omega = 0$, Eq. (5.4) is identically satisfied by $\hat{f}_0(z) = z^{-1}$, i.e.,

$$f_0(x) = \delta(x) . \tag{5.15}$$

To investigate the approach of $f_\omega(x)$ to its limiting form at $\omega = 0$, Eq. (5.15), we have to introduce a "microscope,"

$$f_\omega(x) = \frac{1}{\varepsilon(\omega)} h_\omega\left(\frac{x}{\varepsilon(\omega)}\right), \tag{5.16}$$

which magnifies the x scale by $\varepsilon(\omega)^{-1}$. The function $\varepsilon(\omega)$ is supposed to be positive for $\omega > 0$, and to tend to zero as $\omega \to 0$. We then rewrite the integral equation for f_ω as an integral equation for h_ω, and we assume that h_ω has a limit, denoted by h, as $\omega \to 0$, and that h satisfies an integral equation obtained by a limiting procedure.

With these assumptions[1] it is possible to derive the leading asymptotic behavior of $f_\omega(x)$ near $\omega = 0$, at least for class (a) and class (c) probability densities ρ. The main steps of the rather involved mathematics are explained in the paper by Bernasconi, Schneider, and Wyss (1980), and in the following we therefore merely summarize the final results.

Asymptotically near $\omega = 0$, $f_\omega(g)$ approaches the homogeneous function representation (scaling form)

$$f_\omega(x) = \frac{1}{\varepsilon(\omega)} h\left(\frac{x}{\varepsilon(\omega)}\right), \tag{5.17}$$

where the scale factor $\varepsilon(\omega)$, as well as the limiting scaling function $h(x)$, is a functional of the given probability density $\rho(y)$. It further turns out that class (a) and class (c) systems exhibit a *qualitatively* different asymptotic behavior near $\omega = 0$. For all class (a) probability densities ρ, $h(x)$ is a universal function,

$$h(x) = \delta(x - 1) , \tag{5.18}$$

and the ω dependence of $\varepsilon(\omega)$ is also universal,

$$\varepsilon(\omega) = \mu_{-1}^{-1/2} \omega^{1/2} , \tag{5.19}$$

where μ_{-1} is the minus first moment of $\rho(y)$,

$$\mu_{-1} = \int_0^\infty dy\, y^{-1} \rho(y) . \tag{5.20}$$

Near $\omega = 0$, $f_\omega(x)$ thus approaches a delta function,

$$f_\omega(x) = \delta(x - \mu_{-1}^{-1/2}\, \omega^{1/2}) . \tag{5.21}$$

Class (a) includes the ordered system, $\rho(w) = \delta(w - W)$, for which $\mu_{-1} = W^{-1}$. Asymptotically for small ω, all class (a) systems therefore behave as an ordered system with W given by an "average transfer rate," $W_{av} = \mu_{-1}^{-1}$.

For class (c) probability densities ρ, Eq. (2.4c), $\varepsilon(\omega)$, as well as $h(x)$, are no longer universal, i.e., they depend on α. We obtain (Bernasconi, Schneider, and Wyss, 1980)

$$\varepsilon(\omega) = \omega^{1/(2-\alpha)} , \tag{5.22}$$

and $h(x) = h^{(\alpha)}(x)$, which is the solution of the following integral equation:

$$h^{(\alpha)}(x) = x^{1-\alpha} \int_x^\infty dy\, h^{(\alpha)}(y) \left(\frac{y}{y - x}\right)^{1-\alpha} . \tag{5.23}$$

$h^{(\alpha)}(x)$ can be expressed (Bernasconi, Schneider, and Wyss, 1980) in terms of a generalized hypergeometric function, the so-called H function of Fox (Fox, 1961; Gupta and Jain, 1966),

$$h^{(\alpha)}(x) = \frac{\beta\gamma}{\Gamma(\beta)} H_{12}^{20}\left[\gamma x \,\middle|\, \begin{matrix} (-1,1) \\ (-\beta,\beta) \quad (0,\beta) \end{matrix}\right], \tag{5.24}$$

where

$$\gamma = [\beta^2 \Gamma(\alpha)]^\beta , \quad \beta = 1/(2 - \alpha) . \tag{5.25}$$

For class (c) systems, $h_\omega(x)$ therefore does not approach a delta function as $\omega \to 0$, in contrast to the basic assumption of effective-medium–type approximations (Bernasconi *et al.*, 1978, 1979, 1980) (see Sec. X). Some general properties of the $h^{(\alpha)}$ functions are discussed by Bernasconi, Schneider, and Wyss (1980); and in Fig. 2, as an illustration, we plot their graph for several values of α.

The above results determine the leading $\omega \to 0$ asymptotic behavior of $f_\omega(x)$ for class (a) and class (c) systems; and in the next two sections we shall use these results to compute the leading asymptotic behavior of several physical quantities. In addition, we note that it is possible to construct exact small ω asymptotic expansions for $f_\omega(x)$ which go beyond the leading asymp-

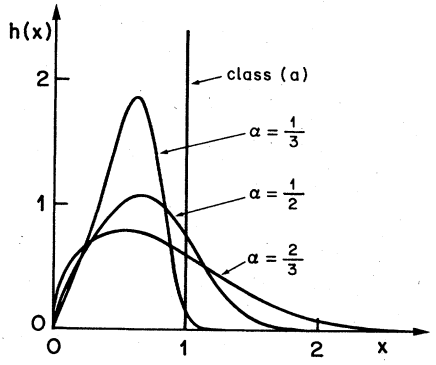

FIG. 2. Function $h^{(\alpha)}(x)$, Eq. (5.24) vs x for $\alpha = \frac{1}{3}$, $\frac{1}{2}$, and $\frac{2}{3}$. For comparison the class (a) result, $h(x) = \delta(x - 1)$, is also indicated.

[1] While we are able to solve the limiting integral equation for $h(x)$ rigorously, it remains to be proved that the integral equation for $h_\omega(x)$ has a unique solution which approaches $h(x)$ as $\omega \to 0$. It is with this proviso that we use the terms "rigorous" and "exact" in connection with the corresponding asymptotic behavior of $f_\omega(x)$, $\langle \tilde{P}_0(\omega) \rangle$, etc.

We might add that, while a rigorous mathematical proof is still missing, the above assumptions are accurately confirmed by numerically determined histograms for $f_\omega(x)$ (Bernasconi and Schneider, unpublished).

totic term (Bernasconi and Schneider, 1981).

Explicit results have been obtained for both class (*a*) and class (*c*) probability densities, and they show, for instance, that the asymptotic equivalence between class (*a*) systems and the ordered system is only valid for the leading term. See also Sec. XIII for a short preliminary account of these results.

We finally have to discuss the case of class (*b*) probability densities which represent the crossover between class (*a*) and class (*c*) systems. For such probability densities [Eq. (2.4b)], characterized by a finite value of $\rho(0)$, we have not yet been able to determine the asymptotic form of $f_\omega(x)$ near $\omega=0$ in a mathematically rigorous way. It seems obvious that logarithmic corrections to the class (*a*) behavior have to come in, but the limiting form of $f_\omega(x)$ might be different from that given by Eq. (5.17). On the other hand, there are arguments (see Sec. X) that the effective-medium result

$$f_\omega(x)\simeq \delta(x-\varepsilon(\omega)), \quad \omega\to 0 \tag{5.26}$$

where

$$\varepsilon(\omega)=\left(\frac{2}{\rho(0)}\right)^{1/2}\left(\frac{\omega}{-\ln\omega}\right)^{1/2}, \tag{5.27}$$

should give the exact leading asymptotic behavior of $\langle\tilde{P}_0(\omega)\rangle$ for class (*b*) systems, even though Eq. (5.26) may not represent the mathematically correct asymptotic solution of our integral equation (5.1).

VI. EXACT RESULTS FOR THE ASYMPTOTIC BEHAVIOR OF $\langle\tilde{P}_0(\omega)\rangle$ AND RELATED QUANTITIES

In the following, we shall use the results of the previous section to derive the asymptotic behavior of the single-site Green function $\langle\tilde{P}_0(\omega)\rangle$ as $\omega\to 0$, or $\langle P_0(t)\rangle$ as $t\to\infty$, respectively, of the density of states $N(\varepsilon)$ as $\varepsilon\to 0$, and of the specific heat $C(T)$ as $T\to 0$. For simplicity, we shall consider only the random W case. The treatment of the random C, and random M, cases is exactly analogous, and the corresponding results will be summarized and discussed in Sec. VIII.

To be definite, we explicitly consider systems described by Eq. (2.2), with constant C_n and random $W_{n,n+1}$ [distributed according to a given probability density $\rho(w)$], and shall briefly comment on the modifications for systems described by Eq. (2.3) at the end of the section. Our single-site Green function, $\langle\tilde{P}_0(\omega)\rangle$, is therefore given by Eq. (4.11), $\langle P_0(t)\rangle$ is given by its inverse Laplace transform, and $N(\varepsilon)$ can be determined according to Eq. (4.13). If ε is energy, the specific heat $C(T)$ (for Bose statistics) is given by

$$C(T)=\frac{d}{dT}\int_0^\infty d\varepsilon\,\varepsilon N(\varepsilon)[\exp(\varepsilon/k_BT)-1]^{-1}. \tag{6.1}$$

We first summarize the well-known results for the ordered systems, i.e., for

$$\rho(w)=\delta(w-W). \tag{6.2}$$

From Eqs. (4.11), (5.9), and (5.10), with $Y_0=W$, we obtain

$$\tilde{P}_0(\omega)=(4W\omega+\omega^2)^{-1/2}, \tag{6.3}$$

and by inverse Laplace transformation

$$P_0(t)=e^{-2Wt}I_0(2Wt), \tag{6.4}$$

where I_0 denotes the modified Bessel function of zeroth order. Equation (4.13), finally, leads to

$$N(\varepsilon)=\begin{cases}\frac{1}{\pi}(4W\varepsilon-\varepsilon^2)^{-1/2}, & 0\le\varepsilon\le 4W\\ 0 \quad \text{otherwise}.\end{cases} \tag{6.5}$$

The $\omega\to 0$ asymptotic behavior of $\tilde{P}_0(\omega)$ is thus given by

$$\tilde{P}_0(\omega)\approx\tfrac{1}{2}(W\omega)^{-1/2}, \tag{6.6}$$

the long-time development of $P_0(t)$ by

$$P_0(t)\approx(4\pi W)^{-1/2}t^{-1/2}, \quad t\to\infty \tag{6.7}$$

and the small ε behavior of the density of states becomes

$$N(\varepsilon)\approx\frac{1}{2\pi W^{1/2}}\varepsilon^{-1/2}, \quad \varepsilon\to 0. \tag{6.8}$$

Let us now turn to random systems. Here we first consider class (a) and class (c) probability densities, for which we have determined the exact $\omega\to 0$ asymptotic behavior of $f_\omega(x)$. We observe that for both classes one has

$$\lim_{\omega\to 0}\omega/\varepsilon(\omega)=0, \tag{6.9}$$

so that the asymptotic behavior of $\langle\tilde{P}_0(\omega)\rangle$ is simply given by

$$\langle\tilde{P}_0(\omega)\rangle\approx C_0\varepsilon(\omega)^{-1}, \quad \omega\to 0 \tag{6.10}$$

where

$$C_0=\int_0^\infty dx'h(x')\int_0^\infty dx''h(x'')\frac{1}{x'+x''}. \tag{6.11}$$

From the results of the preceding section we then immediately obtain

$$\langle\tilde{P}_0(\omega)\rangle\approx\tfrac{1}{2}\mu_{-1}^{1/2}\omega^{-1/2}, \quad \omega\to 0\ [\text{class (a)}] \tag{6.12a}$$

$$\langle\tilde{P}_0(\omega)\rangle\approx C_0^{(\alpha)}\omega^{-1/(2-\alpha)}, \quad \omega\to 0\ [\text{class (c)}] \tag{6.12c}$$

where the constant $C_0^{(\alpha)}$ is given by Eqs. (6.11) and (5.24). Its dependence on α is discussed by Bernasconi, Schneider, and Wyss (1980). With the help of general Abelian and Tauberian theorems (Doetsch, 1971) the asymptotic behavior of $\langle P_0(t)\rangle$ for $t\to\infty$ can be determined from the asymptotic behavior of $\langle\tilde{P}_0(\omega)\rangle$ for $\omega\to 0$, or of $N(\varepsilon)$ for $\varepsilon\to 0$. The resulting expressions are

$$\langle P_0(t)\rangle\approx\left(\frac{\mu_{-1}}{4\pi}\right)^{1/2}t^{-1/2}, \quad t\to\infty\ [\text{class (a)}] \tag{6.13a}$$

$$\langle P_0(t)\rangle\approx\frac{C_0^{(\alpha)}}{\Gamma\left(\frac{1}{2-\alpha}\right)}t^{-(1-\alpha)/(2-\alpha)}, \quad t\to\infty\ [\text{class (c)}]. \tag{6.13c}$$

A second quantity determined by $\langle\tilde{P}_0(\omega)\rangle$ is the eigenvalue density of states, $N(\varepsilon)$. According to Eqs. (4.13) and (6.12), its $\varepsilon\to 0$ asymptotic behavior is given by

$$N(\varepsilon)\approx\frac{\mu_{-1}^{1/2}}{2\pi}\varepsilon^{-1/2}, \quad \varepsilon\to 0\ [\text{class (a)}] \tag{6.14a}$$

$$N(\varepsilon)\approx\frac{C_0^{(\alpha)}}{\pi}\sin\left(\frac{\pi}{2-\alpha}\right)\varepsilon^{-1/(2-\alpha)}, \quad \varepsilon\to 0\ [\text{class (c)}]. \tag{6.14c}$$

If ε is energy, the corresponding specific heat, Eq. (6.1), has the following low-temperature asymptotic behavior:

$$C(T) \propto T^{1/2}, \quad T \to 0 \text{ [class (a)]} \tag{6.15a}$$

$$C(T) \propto T^{(1-\alpha)/(2-\alpha)}, \quad T \to 0 \text{ [class (c)]}. \tag{6.15c}$$

These results, Eqs. (6.12) to (6.15), apply to systems which are described by equations of the type of Eq. (2.2). The expressions for $\langle \tilde{P}_0(\omega) \rangle$ and $\langle P_0(t) \rangle$ are of interest, for example, for hopping systems and electrical lines, and Eqs. (6.14) and (6.15) apply, for example, to a random ferromagnetic Heisenberg chain (see also Sec. X for some specific applications).

For a harmonic chain with random force constants, on the other hand, which is described by Eq. (2.3), ε plays the role of the *square* of the eigenfrequency. This means that for this type of systems the exponents in Eqs. (6.14a) and (6.15c), for instance, have to be multiplied by a factor of 2. For a more extensive discussion of the random harmonic chain case see Sec. VIII.

As a general feature we observe that the exponents associated with the above asymptotic expressions all show a crossover from universal to nonuniversal (α-dependent) behavior as we pass from class (a) to class (c) probability densities $\rho(w)$. The actual crossover behavior is represented by probability densities of class (b) type, Eq. (2.4b). As mentioned in the preceding section, and discussed in Sec. X, we believe that Eqs. (5.26) and (5.27) lead to the exact $\omega \to 0$ behavior of $\langle \tilde{P}_0(\omega) \rangle$ for these crossover systems. If we observe that Eq. (6.9) also holds for this case, we immediately obtain

$$\langle \tilde{P}_0(\omega) \rangle \approx \tfrac{1}{2} \left(\frac{\rho(0)}{2} \right)^{1/2} \left(\frac{-\ln \omega}{\omega} \right)^{1/2}, \quad \omega \to 0 \text{ [class (b)]} \tag{6.12b}$$

$$\langle P_0(t) \rangle \approx \frac{1}{2\pi^{1/2}} \left(\frac{\rho(0)}{2} \right)^{1/2} \left(\frac{\ln t}{t} \right)^{1/2}, \quad t \to \infty \text{ [class (b)]} \tag{6.13b}$$

and

$$N(\varepsilon) \approx \frac{1}{2\pi} \left(\frac{\rho(0)}{2} \right)^{1/2} \left(\frac{-\ln \varepsilon}{\varepsilon} \right)^{1/2}, \quad \varepsilon \to 0 \text{ [class (b)]}. \tag{6.14b}$$

VII. ASYMPTOTIC BEHAVIOR OF $\sigma(\omega)$ AND $\langle x^2(t) \rangle$

We now turn to the determination of the asymptotic behavior of the frequency-dependent hopping conductivity, $\sigma(\omega)$, and the mean-square displacement, $\langle x^2(t) \rangle$. For convenience, we shall again restrict the discussion to the random W case, Eq. (2.2) with fixed C_n's [or Eq. (2.1), respectively].

According to Eqs. (2.9)–(2.12), we can express $\sigma(\omega)$ and $\langle x^2(t) \rangle$ in terms of the $\langle \tilde{P}_n(\omega) \rangle$, or the $\langle P_n(t) \rangle$, respectively, as

$$\sigma(\omega) = \frac{n_0 e^2}{k_B T} \langle D(-i\omega) \rangle, \tag{7.1}$$

$$\langle D(\omega) \rangle = \tfrac{1}{2} \omega^2 \sum_{n=-\infty}^{\infty} n^2 \langle \tilde{P}_n(\omega) \rangle, \tag{7.2}$$

and

$$\langle x^2(t) \rangle = \sum_{n=-\infty}^{\infty} n^2 \langle P_n(t) \rangle. \tag{7.3}$$

For the ordered system, characterized by Eq. (6.2), we can solve the problem, i.e., Eqs. (2.1) or (2.2) with the initial condition (2.6), exactly. One has

$$\tilde{P}_n(\omega) = \frac{1}{[\omega(\omega+4W)]^{1/2}} \left(\frac{2W}{\omega + 2W + [\omega(\omega+4W)]^{1/2}} \right)^{|n|}, \tag{7.4}$$

and

$$P_n(t) = e^{-2Wt} I_{|n|}(2Wt), \tag{7.5}$$

and it follows that

$$D(\omega) = W \tag{7.6}$$

and

$$x^2(t) = 2Wt. \tag{7.7}$$

I_m denotes the modified Bessel function of integer order, and we remark that Eqs. (7.4)–(7.7) are valid for all ω and for all $t \geq 0$, respectively. $\sigma(\omega)$ is therefore independent of ω, and $x^2(t)$ exhibits the well-known diffusive behavior with a diffusion coefficient $D = W$.

For disordered systems, on the other hand, an exact calculation of $\langle D(\omega) \rangle$ or $\langle x^2(t) \rangle$ does not seem possible. A large ω expansion for $\langle D(\omega) \rangle$ can, for example, be constructed from the integral equation for the spatial Fourier transform of $\langle \tilde{P}_n(\omega) \rangle$ (Bernasconi and Schneider, 1981). A straightforward iteration procedure for this integral equation leads to

$$\langle D(\omega) \rangle = \mu_1 - \frac{2}{\omega}(\mu_2 - \mu_1^2) + \frac{2}{\omega^2}(2\mu_3 - 3\mu_2\mu_1 + \mu_1^3) - \cdots, \tag{7.8}$$

where the moments μ_k of the probability density $\rho(w)$ are defined in Eq. (5.13). The determination of the more interesting $\omega \to 0$ asymptotic behavior of $\langle D(\omega) \rangle$, or, equivalently, of the $t \to \infty$ behavior of $\langle x^2(t) \rangle$, is much more difficult. In the preceding section we have calculated the exact asymptotic expressions for $\langle \tilde{P}_0(\omega) \rangle$ and $\langle P_0(t) \rangle$, but we are not able to derive the corresponding results for general $\langle \tilde{P}_n(\omega) \rangle$ or $\langle P_n(t) \rangle$ in the same rigorous way.

In a recent paper, Bernasconi, Schneider, and Wyss (1980) have therefore proposed and investigated a general *scaling hypothesis* for the n dependence of $\langle \tilde{P}_n(\omega) \rangle$ in the limit as $\omega \to 0$. We notice that a similar hypothesis can be formulated for $\langle P_n(t) \rangle$ (Alexander *et al.*, 1978; Alexander and Bernasconi, 1979; Richards and Renken, 1980), but for several reasons we prefer to work in ω space.

Let us, for the moment, restrict ourselves to positive real ω. We then assume that for $\omega \to 0$ the n dependence of $\langle \tilde{P}_n(\omega) \rangle$ can be described by a single correlation length $\xi(\omega)$, i.e., our scaling hypothesis simply becomes

$$\langle \tilde{P}_n(\omega) \rangle \approx \langle \tilde{P}_0(\omega) \rangle F(n/\xi(\omega)), \quad \omega \to 0 \tag{7.9}$$

where the scaling function $F(z)$ is normalized to satisfy $F(0) = 1$. From the normalization condition for the $\langle \tilde{P}_n(\omega) \rangle$,

$$\sum_{n=-\infty}^{\infty} \langle \tilde{P}_n(\omega)\rangle = \omega^{-1}, \tag{7.10}$$

it then follows that

$$\xi(\omega)^{-1} \approx 2\omega \langle \tilde{P}_0(\omega)\rangle, \quad \omega \to 0 \tag{7.11}$$

and one finally obtains (Bernasconi, Schneider, and Wyss, 1980)

$$\langle D(\omega)\rangle \approx D_0 \frac{1}{\omega \langle \tilde{P}_0(\omega)\rangle^2}, \quad \omega \to 0 \tag{7.12}$$

where the constant D_0 is determined by $F(z)$,

$$D_0 = \tfrac{1}{8} \int_0^\infty dz\, z^2 F(z) \Big/ \left(\int_0^\infty dz\, F(z)\right)^3. \tag{7.13}$$

For the ordered system, it immediately follows from Eq. (7.4) that $F(z) = \exp(-|z|)$, so that $D_0 = \frac{1}{4}$. In the case of disordered systems, the scaling hypothesis has been tested numerically (Bernasconi, Schneider, and Wyss, 1980) for several class (a) and class (c) probability densities. The corresponding results accurately confirmed the proposed scaling property, Eq. (7.9), and moreover indicated that $F(z)$ is in fact universal for class (a) systems, i.e.,

$$F(z) = \exp(-|z|), \quad D_0 = \tfrac{1}{4} \text{ [class (a)]}. \tag{7.14}$$

For class (c) probability densities, however, deviations from Eq. (7.14) have been observed, so that $F(z) = F^{(\alpha)}(z)$ and $D_0 = D_0^{(\alpha)}$ are expected to become nonuniversal for such systems.

Although Eq. (7.12) is derived from a hypothesis, Eq. (7.9), we believe that it gives the correct $\omega \to 0$ asymptotic behavior of $\langle D(\omega)\rangle$ for all class (a), (b), and (c) probability densities. In this connection, we remark that a rigorous lower bound for $\langle D(\omega)\rangle$ is given by

$$\langle D(\omega)\rangle \geq \text{const} \frac{1}{\omega \langle \tilde{P}_0(\omega)\rangle^2}, \quad \omega \to 0. \tag{7.15}$$

A trivial procedure (Bernasconi, Schneider, and Wyss, 1980) leads to the value $\frac{1}{24}$ for the constant on the right-hand side, but it is actually possible (Bernasconi and Schneider, 1981) to increase its value to $\frac{1}{4}$. A corresponding upper bound has, however, not yet been derived. Finally, we should also mention that an effective-medium argument (see Sec. X) also leads to Eq. (7.12), with $F_{\text{eff}}(z) = \exp(-|z|)$ and $D_0^{(\text{eff})} = \frac{1}{4}$ for all three classes of probability densities considered.

The general scaling hypothesis, Eq. (7.9), which is strongly supported by numerical simulations (Bernasconi, Schneider, and Wyss, 1980; Richards and Renken, 1980), as well as by analytical considerations (Alexander and Bernasconi, 1979; Bernasconi and Schneider, 1981) (see also Sec. IX), thus relates the $\omega \to 0$ asymptotic behavior of $\langle D(\omega)\rangle$ uniquely to that of $\langle \tilde{P}_0(\omega)\rangle$, as has been determined in the preceding section. Inserting the corresponding expressions, Eqs. (6.12a)–(6.12c), into Eq. (7.12), we obtain

$$\langle D(\omega)\rangle \approx \mu_{-1}^{-1}, \quad \omega \to 0 \text{ [class (a)]} \tag{7.16a}$$

$$\langle D(\omega)\rangle \approx \frac{2}{\rho(0)} (-\ln\omega)^{-1}, \quad \omega \to 0 \text{ [class (b)]} \tag{7.16b}$$

$$\langle D(\omega)\rangle \approx D_0^{(\alpha)} C_0^{(\alpha)-2} \omega^{\alpha/(2-\alpha)}, \quad \omega \to 0 \text{ [class (c)]} \tag{7.16c}$$

and, together with Eq. (7.1), this leads to

$$\sigma(\omega) = \frac{n_0 e^2}{k_B T} \mu_{-1}^{-1} \text{ [class (a)]} \tag{7.17a}$$

$$\sigma(\omega) = \frac{n_0 e^2}{k_B T} \frac{2}{\rho(0)} \frac{1}{-\ln(-i\omega)} \text{ [class (b)]} \tag{7.17b}$$

$$\sigma(\omega) = \frac{n_0 e^2}{k_B T} D_0^{(\alpha)} C_0^{(\alpha)-2} (-i\omega)^{\alpha/(2-\alpha)} \text{ [class (c)]} \tag{7.17c}$$

for the low-frequency asymptotic behavior of the complex ac hopping conductivity.

Asymptotically for small frequencies, class (a) systems thus again exhibit the same behavior as an ordered system, approaching a finite dc conductivity. On the other hand, $\langle D(\omega)\rangle$ for a general class (a) system changes from μ_{-1}^{-1} at $\omega = 0$ to μ_1 at $\omega = \infty$, in contrast to the ordered system where it is independent of ω.

For class (c) systems, $\sigma(\omega)$ has a completely different low-frequency behavior. It exhibits a peculiar anomalous power law behavior, Eq. (7.17c), leading to a vanishing dc conductivity. This crossover in the low-frequency behavior of $\sigma(\omega)$, as we pass from class (a) to class (c) probability densities, leads to striking predictions for some specific applications of our general models (see Sec. XI).

Finally, we briefly discuss the long-time asymptotic behavior of the mean-square displacement, $\langle x^2(t)\rangle$. According to Eqs. (7.2) and (7.3), $\langle x^2(t)\rangle$ is the inverse Laplace transform of $2\langle D(\omega)\rangle/\omega^2$, so that its $t \to \infty$ behavior can be obtained from Eqs. (7.16) using general Tauberian theorems (Doetsch, 1971). It then follows that

$$\langle x^2(t)\rangle \approx 2\mu_{-1}^{-1} t, \quad t \to \infty \text{ [class (a)]} \tag{7.18a}$$

$$\langle x^2(t)\rangle \propto t/\ln t, \quad t \to \infty \text{ [class (b)]} \tag{7.18b}$$

$$\langle x^2(t)\rangle \propto t^{2(1-\alpha)/(2-\alpha)}, \quad t \to \infty \text{ [class (c)]}. \tag{7.18c}$$

We notice that a scaling hypothesis for $\langle P_n(t)\rangle$, $t \to \infty$ (Alexander *et al.*, 1978b; Richards and Renken, 1980), which is the analog of Eq. (7.9), implies

$$\langle x^2(t)\rangle \propto \langle P_0(t)\rangle^{-2}, \quad t \to \infty \tag{7.19}$$

and therefore leads to the same asymptotic dependences, Eqs. (7.18), for the mean-square displacement.

For long times, class (a) systems thus exhibit the same diffusive behavior as an ordered system, with a well-defined finite diffusion constant

$$D = \tfrac{1}{2} \lim_{t\to\infty} \langle x^2(t)\rangle/t = \mu_{-1}^{-1} \text{ [class (a)]}. \tag{7.20}$$

For class (b) and (c) systems, on the other hand, the long-time increase of $\langle x^2(t)\rangle$ is slower than in an ordered system, and the diffusion constant is zero. In this sense we may speak of "localization," although $\langle P_0(t)\rangle$ [and, in fact, all $\langle P_n(t)\rangle$] decays to zero as $t \to \infty$.

VIII. SUMMARY OF THE PRINCIPAL GENERAL RESULTS

In this section, we summarize and briefly discuss the main *general* results concerning the asymptotic behavior of our one-dimensional random systems. Several *specific* applications of our general mathematical models will then be introduced and discussed in Sec. XI. Mathematically, our random systems are described by

Eqs. (2.1), (2.2), or (2.3), respectively, and are characterized by the probability density for the relevant random variable ($W_{n,n+1}$, C_n, or M_n). These probability densities are classified, Eqs. (2.4a), (2.4b), and (2.4c) and (2.5a), (2.5b), and (2.5c), respectively, according to the asymptotic behavior of the corresponding systems. Our main results are concerned with the limiting asymptotic behavior of the autocorrelation function, $\langle P_0(t)\rangle$, as $t\to\infty$, of its Laplace transform, $\langle\tilde{P}_0(\omega)\rangle$, as $\omega\to 0$, of the density of states, $N(\varepsilon)$, as $\varepsilon\to 0$, of the associated specific heat, $C(T)$, as $T\to 0$, of the ac hopping conductivity, $\sigma(\omega)$, as $\omega\to 0$, and of the mean-square displacement, $\langle x^2(t)\rangle$, or the correlation length, $\xi(t)=\langle x^2(t)\rangle^{1/2}$, as $t\to\infty$. Whereas in most cases the asymptotic expressions for $\langle\tilde{P}_0(\omega)\rangle$, $\langle P_0(t)\rangle$, $N(\varepsilon)$, and $C(T)$ can be derived rigorously (see Secs. V and VI), those for $\sigma(\omega)$ and $\langle x^2(t)\rangle$ rely on the validity of a scaling hypothesis (Sec. VII). In the following summary we usually indicate only the leading asymptotic dependence on the relevant variable. The corresponding prefactors are given in Secs. VI and VII for the random W case and can be determined similarly for the random C and random M cases.

A. Equation (2.2) systems, random *W* case

In Secs. V through VII we have analyzed in detail the problem corresponding to Eq. (2.2), with all C_n equal, [or Eq. (2.1), respectively] supplemented by the initial condition (4.9). This problem is, for example, appropriate to the study of particle (or excitation) diffusion on a chain with random transfer rates $W_{n,n+1}$. The probability density $\rho(w)$ for the (independent) random variables $W_{n,n+1}$ is assumed to belong to one of the three classes (a), (b), and (c), defined by Eqs. (2.4a), (2.4b) and (2.4c), and the resulting asymptotic expressions can be summarized as follows

Class (a). Such probability densities allow for the definition of an "average transfer rate," $W_{\rm av}$, where

$$W_{\rm av}^{-1}=\mu_{-1}\equiv\int_0^\infty dw\, w^{-1}\rho(w)\,, \tag{8.1}$$

and the asymptotic behavior of the diagonal or single-site Green function (or autocorrelation function) is given by

$$\langle\tilde{P}_0(\omega)\rangle\approx(4W_{\rm av}\omega)^{-1/2}\,,\quad \omega\to 0 \tag{8.2a}$$

or

$$\langle P_0(t)\rangle\approx(4\pi W_{\rm av}t)^{-1/2}\,,\quad t\to\infty \tag{8.2b}$$

respectively. The density of states behaves as

$$N(\varepsilon)\propto\varepsilon^{-1/2}\,,\quad \varepsilon\to 0 \tag{8.2c}$$

leading, if ε is energy, to a low-temperature specific heat (for Bose statistics) of the form

$$C(T)\propto T^{1/2}\,,\quad T\to 0\,. \tag{8.2d}$$

The correlation length develops as

$$\xi(t)\equiv\langle x^2(t)\rangle^{1/2}\approx(2W_{\rm av}t)^{1/2}\,,\quad t\to\infty \tag{8.2e}$$

while the low-frequency conductivity becomes independent of frequency,

$$\sigma(\omega)\propto W_{\rm av}=\text{const}\,,\quad \omega\to 0\,. \tag{8.2f}$$

The asymptotic behavior of all these quantities is thus the same as that for an ordered system described by $\rho(w)=\delta(w-W_{\rm av})$. We stress, however, that this equivalence between a general class (a) system and an ordered "averaged system" does not extend beyond the leading asymptotic term (Bernasconi and Schneider, 1981).

Class (b). These systems exhibit a crossover behavior, intermediate between the qualitatively different asymptotic behaviors of class (*a*) and class (*c*) systems, respectively. We find, not entirely rigorously,

$$\langle\tilde{P}_0(\omega)\rangle\propto(-\ln\omega/\omega)^{1/2}\,,\quad \omega\to 0 \tag{8.3a}$$

$$\langle P_0(t)\rangle\propto(\ln t/t)^{1/2}\,,\quad t\to\infty \tag{8.3b}$$

$$N(\varepsilon)\propto(-\ln\varepsilon/\varepsilon)^{1/2}\,,\quad \varepsilon\to 0 \tag{8.3c}$$

and

$$\xi(t)\propto(t/\ln t)^{1/2}\,,\quad t\to\infty \tag{8.3d}$$

$$\sigma(\omega)\propto -1/\ln(-i\omega)\,,\quad \omega\to 0\,. \tag{8.3e}$$

Class (c). These probability distributions occur in several interesting physical applications (see Sec. XI), and corresponding systems exhibit a nonconventional asymptotic behavior (compared to that of an ordered system) with nonuniversal exponents. One has

$$\langle\tilde{P}_0(\omega)\rangle\propto\omega^{-1/(2-\alpha)}\,,\quad \omega\to 0 \tag{8.4a}$$

$$\langle P_0(t)\rangle\propto t^{-(1-\alpha)/(2-\alpha)}\,,\quad t\to\infty\,, \tag{8.4b}$$

and

$$N(\varepsilon)\propto\varepsilon^{-1(2-\alpha)}\,,\quad \varepsilon\to 0\,. \tag{8.4c}$$

If ε is energy, Eq. (8.4c) leads (for Bose statistics) to a low-temperature specific heat

$$C(T)\propto T^{(1-\alpha)/(2-\alpha)}\,,\quad T\to 0\,. \tag{8.4d}$$

The long-time development of the correlation length and of the mean-square displacement are given by

$$\xi(t)=\langle x^2(t)\rangle^{1/2}\propto t^{(1-\alpha)/(2-\alpha)}\,,\quad t\to\infty \tag{8.4e}$$

and the low-frequency conductivity exhibits a peculiar power law behavior

$$\sigma(\omega)\propto(-i\omega)^{\alpha/(2-\alpha)}\,,\quad \omega\to 0\,. \tag{8.4f}$$

B. Equation (2.2) systems, random *C* case

This problem, for example, describes an electrical *RC* line with random capacitances C_n. As mentioned in Sec. III, it can also be mapped onto the problem of excitation transfer amongst a linear array of traps with random depths $[C_n\to\exp(-\Delta_n/k_BT)]$, and we shall briefly discuss the corresponding system in Sec. XI. Here we use the language of electrical lines, and the probability density $r(c)$ for the random capacitances C_n is assumed to belong to one of the classes (a'), (b'), or (c'), defined by Eqs. (2.5a)–(2.5c). In Sec. IV we have shown that the random C problem can be reduced to the same integral equation as the corresponding random W problem, with $\rho(1/c)=c^2r(c)$ playing the role of $\rho(w)$. We have, however, also shown that the calculation of $\langle\tilde{P}_0(\omega)\rangle$ and other averaged quantities of interest is somewhat more complicated than in the random W case. Furthermore, the averages depend on the chosen initial

condition. Corresponding to the two relevant variables the "node potentials" P_n and the "charges" $Q_n = C_n P_n$, we have considered two different initial conditions:

$$P_n(t=0) = \delta_{n,0} \tag{8.5}$$

and

$$Q_n(t=0) = C_n P_n(t=0) = \delta_{n,0}\,. \tag{8.6}$$

As in Sec. IV, we shall denote the corresponding averages by $\langle\cdots\rangle_P$ and $\langle\cdots\rangle_Q$, respectively. Let us discuss the results for the second initial condition, Eq. (8.6), in some detail.

Class (a'). For these probability densities $r(c)$, an "average capacitance,"

$$C_{\rm av} = \nu_1 \equiv \int_0^\infty dc\, c r(c)\,, \tag{8.7}$$

can be defined, and we have

$$\langle \tilde{P}_0(\omega)\rangle_Q \approx \tfrac{1}{2}(C_{\rm av}\omega)^{-1/2}\,, \quad \omega \to 0 \tag{8.8a}$$

$$\langle C_0 \tilde{P}_0(\omega)\rangle_Q \approx \tfrac{1}{2}(C_{\rm av}/\omega)^{1/2}\,, \quad \omega \to 0\,. \tag{8.8b}$$

The long-time behavior of the corresponding inverse Laplace transforms can again be determined from general Tauberian theorems (Doetsch, 1971), and we obtain

$$\langle P_0(t)\rangle_Q \approx (4\pi C_{\rm av})^{-1/2} t^{-1/2}\,, \quad t \to \infty \tag{8.9a}$$

$$\langle C_0 P_0(t)\rangle_Q \approx (C_{\rm av}/4\pi)^{1/2} t^{-1/2}\,, \quad t \to \infty\,. \tag{8.9b}$$

For class (a') systems, the two quantities $\langle P_0(t)\rangle_Q$ and $\langle C_0 P_0(t)\rangle_Q$ therefore exhibit the same asymptotic dependence on t. This is different for class (b') and class (c') probability densities, respectively, for which the corresponding asymptotic expressions are as follows.

Class (b').

$$\langle \tilde{P}_0(\omega)\rangle_Q \propto (-\omega \ln\omega)^{-1/2}\,, \quad \omega \to 0 \tag{8.10a}$$

$$\langle C_0 \tilde{P}_0(\omega)\rangle_Q \propto (-\ln\omega/\omega)^{1/2}\,, \quad \omega \to 0 \tag{8.10b}$$

and

$$\langle P_0(t)\rangle_Q \propto (t \ln t)^{-1/2}\,, \quad t \to \infty \tag{8.11a}$$

$$\langle C_0 P_0(t)\rangle_Q \propto (\ln t/t)^{1/2}\,, \quad t \to \infty\,. \tag{8.11b}$$

Class (c').

$$\langle \tilde{P}_0(\omega)\rangle_Q \propto \omega^{-(1-\alpha)/(2-\alpha)}\,, \quad \omega \to 0 \tag{8.12a}$$

$$\langle C_0 \tilde{P}_0(\omega)\rangle_Q \propto \omega^{-1/(2-\alpha)}\,, \quad \omega \to 0 \tag{8.12b}$$

and

$$\langle P_0(t)\rangle_Q \propto t^{-1/(2-\alpha)}\,, \quad t \to \infty \tag{8.13a}$$

$$\langle C_0 P_0(t)\rangle_Q \propto t^{-(1-\alpha)/(2-\alpha)}\,, \quad t \to \infty\,. \tag{8.13b}$$

For both class (b') and class (c') systems, $\langle P_0(t)\rangle_Q$ thus decays faster than $\langle C_0 P_0(t)\rangle_Q$. This behavior can also be expressed by a rather curious increasing in time of the average capacitance $\langle C\rangle_Q$, defined by

$$\langle C\rangle_Q \equiv \langle C_0 P_0(t)\rangle_Q / \langle P_0(t)\rangle_Q\,. \tag{8.14}$$

We obtain

$$\langle C\rangle_Q \propto \ln t\,, \quad t \to \infty \ [\text{class } (b')] \tag{8.15b}$$

$$\langle C\rangle_Q \propto t^{\alpha/(2-\alpha)}\,, \quad t \to \infty \ [\text{class } (c')] \tag{8.15c}$$

whereas for class (a') distributions $\langle C\rangle_Q$ goes to a constant

$$\langle C\rangle_Q \propto C_{\rm av}\,, \quad t \to \infty\,, \ [\text{class } (a')]\,. \tag{8.15a}$$

Similar results can be derived for the alternative initial condition, Eq. (8.5). Quite generally, we have

$$\langle \tilde{P}_0(\omega)\rangle_P = \langle C_0 \tilde{P}_0(\omega)\rangle_Q \tag{8.16a}$$

and

$$\langle P_0(t)\rangle_P = \langle C_0 P_0(t)\rangle_Q\,. \tag{8.16b}$$

The determination of the asymptotic behavior of $\langle C_0 \tilde{P}_0(\omega)\rangle_P$, on the other hand, is somewhat more complex, and we therefore refrain from giving detailed results. We might mention, however, that for class (a') probability densities $\langle C_0 \tilde{P}_0\rangle_P$ behaves as

$$\langle C_0 \tilde{P}_0(\omega)\rangle_P \propto \omega^{-1/2}\,, \quad \omega \to 0 \tag{8.17}$$

only if the second moment ν_2 also exists:

$$\nu_2 = \int_0^\infty dc\, c^2 r(c) < \infty\,. \tag{8.18}$$

Otherwise, the asymptotic ω dependence of $\langle C_0 \tilde{P}_0(\omega)\rangle_P$ explicitly depends on $r(c)$, i.e., the corresponding exponent is no longer universal.

The physical significance of the quantity $\langle C_0 \tilde{P}_0(\omega)\rangle_P$ is also peculiar. For the random capacitor chain, it would require a fixed initial voltage at the origin, but allow for the full range of capacitances. One would be in the position of having to provide, at fixed voltage, a charge for very large capacitances [$C \to \infty$ for class (c) systems]. For the random mass vibrational chain, it would correspond to a fixed initial velocity of the atom at the origin, but allow for the full range of masses. One would be in the position of having physically to provide finite velocity for very heavy ($M \to \infty$) masses. These conditions appear somewhat esoteric, and we shall not pursue the matter further here.

As discussed in Sec. IV, Eq. (4.21), the density of states, $N(\varepsilon)$, is determined by $\langle \tilde{P}_0(\omega)\rangle_P = \langle C_0 \tilde{P}_0(\omega)\rangle_Q$. According to the above results, Eqs. (8.8b), (8.10b), and (8.12b), the small ε behavior of $N(\varepsilon)$ is thus the same as for the corresponding random W case, Eqs. (8.2c), (8.3c), and (8.4c).

The mean-square displacement, $\langle x^2(t)\rangle$, and the conductivity, $\sigma(\omega)$, are obviously determined by the $\langle \tilde{P}_n(\omega)\rangle_P$. The scaling hypothesis of Sec. VII then again leads to the same asymptotic dependences as in the corresponding random W case, given by Eqs. (8.2e), (8.3d) and (8.4e), and Eqs. (8.2f), (8.3e), and (8.4f), respectively.

C. Equation (2.3) random harmonic chain systems

We now turn to the discussion of Eq. (2.3) systems, where the first derivative, dP_n/dt, is replaced by the second derivative, d^2P_n/dt^2. These equations describe the dynamics of a harmonic chain with masses M_n and force constants $W_{n\,n+1}$. The P_n denote the displacements, and as before we consider two different cases, the *random W case* (identical masses) and the *random M case* (identical force constants). By Laplace transforming Eqs. (2.3) we can again reduce both problems to an integral equation of the form of Eq. (4.5), with ω

replaced by ω^2. Because of the second-order nature of the equations, the initial conditions,

$$P_n(t=0)=\delta_{n,0} \tag{8.19}$$

or

$$M_n P_n(t=0)=\delta_{n,0}, \tag{8.20}$$

respectively, have now to be supplemented by an initial condition for the first derivatives. For simplicity, let us choose

$$\frac{dP_n}{dt}(t=0)=0 \tag{8.21}$$

in both cases. The exact $\omega \to 0$ asymptotic behavior of the quantities $\langle \tilde{P}_0(\omega)\rangle$ and $\langle M_0 \tilde{P}_0(\omega)\rangle$ can then immediately be determined from the corresponding results for the (first-order) Eq. (2.2) systems. We merely have to replace ω by ω^2 in our previous results, and then multiply this expression by ω (taking into account our choice of initial conditions in the Laplace transform of d^2P_n/dt^2). For the random W case, for instance, we then obtain

$$\langle \tilde{P}_0(\omega)\rangle \approx (4W_{\rm av})^{-1/2}, \quad \omega \to 0 \text{ [class } (a)\text{]} \tag{8.22a}$$

$$\langle \tilde{P}_0(\omega)\rangle \propto (-\ln\omega)^{1/2}, \quad \omega \to 0 \text{ [class } (b)\text{]} \tag{8.22b}$$

$$\langle \tilde{P}_0(\omega)\rangle \propto \omega^{-\alpha/(2-\alpha)}, \quad \omega \to 0 \text{ [class } (c)\text{]}. \tag{8.22c}$$

Because of the oscillating nature of $\langle \tilde{P}_0(t)\rangle$, its $t \to \infty$ asymptotic behavior can, however, not be determined from the above $\omega \to 0$ asymptotic expressions for $\langle \tilde{P}_0(\omega)\rangle$ alone. One would rather have to analyze all singularities of $\langle \tilde{P}_0(\omega)\rangle$ on the imaginary axis, about which our present treatment gives no information.

In the expressions for the density of states, ε now plays the role of the square of the eigenfrequency Ω, i.e., $N(\varepsilon) \to N(\Omega^2)$. For the harmonic chain the energy is proportional to Ω, so that for the calculation of the specific heat we have to use

$$n(\varepsilon)=2\varepsilon N(\varepsilon^2) \tag{8.23}$$

instead of $N(\varepsilon)$. From Eqs. (8.2c) and (8.4c) we have

$$n(\varepsilon) \approx \text{const}, \quad \varepsilon \to 0 \text{ [class } (a)\text{]} \tag{8.24a}$$

and

$$n(\varepsilon) \propto \varepsilon^{-\alpha/(2-\alpha)}, \quad \varepsilon \to 0 \text{ [class } (c)\text{]} \tag{8.24c}$$

and the exponent in the low-temperature specific heat is multiplied by a factor of 2,

$$C(T) \propto T, \quad T \to 0 \text{ [class } (a)\text{]} \tag{8.25a}$$

$$C(T) \propto T^{2(1-\alpha)/(2-\alpha)}, \quad T \to 0 \text{ [class } (c)\text{]}. \tag{8.25c}$$

IX. AN EXPLICIT SCALING APPROACH

It is instructive to supplement the discussion by some explicit scaling arguments which give additional insight into the physical meaning of our asymptotic solutions. These arguments expand and, to a certain extent, quantify the ideas of an earlier scaling approach (Alexander and Bernasconi, 1979). They also allow one to obtain some information on quantities which cannot be calculated directly from the solution of the integral equation (4.5).

We consider the eigenvalue equations corresponding to Eqs. (2.2) or, equivalently, the Laplace transformed version of Eqs. (2.2) with $\omega = \varepsilon$, where ε is real and positive:

$$\varepsilon C_n \tilde{P}_n + W_{n-1,n}(\tilde{P}_n - \tilde{P}_{n-1}) + W_{n,n+1}(\tilde{P}_n - \tilde{P}_{n+1}) = 0. \tag{9.1}$$

We again use the language of electrical lines and investigate the effect on the form of the solution of a change in the length scale, say, by replacing the original chain by an equivalent chain composed of segments of length N.

Let us first consider the ordered chain, i.e., $C_n = C$ and $W_{n,n+1} = W$ for all n. The basic scale for ε is then given by the "Dyson variable"

$$\lambda = W/C. \tag{9.2}$$

For sufficiently small ε, the changes in $\tilde{P}_n$ along a segment of length N can be neglected, and its response may be described by a segment conductance, W_n, and a segment capacitance, C_n, where

$$W_N = W/N, \quad C_N = NC. \tag{9.3}$$

The basic scale for ε is now given by

$$\lambda_N = \lambda/N^2, \tag{9.4}$$

and for sufficiently small ε, $\varepsilon \ll \lambda/N^2$, and on a length scale $\gg N$ the description of the chain in terms of the segments (of length N) thus becomes adequate. Alternatively, this implies that one can define a coherence length, $\xi(\varepsilon)$, by

$$\xi^2 \approx \lambda/\varepsilon. \tag{9.5}$$

We now consider a random chain. For a definite segment of length N, consisting of, say, the sites $n = 1, 2, \ldots, N$, we then have

$$W_N = \left(\sum_{n=1}^{N} 1/W_{n,n+1}\right)^{-1}, \quad C_N = \sum_{n=1}^{N} C_n, \tag{9.6}$$

and

$$\lambda_N = W_N/C_N. \tag{9.7}$$

Obviously, W_N, C_N, and λ_N, are still random, and their distribution is determined by $\rho(w)$ and $r(c)$, the probability densities for the $W_{n,n+1}$ and for the C_n, respectively.

If the moments of $1/w$ and of c, i.e.,

$$\langle w^{-m}\rangle \equiv \int_0^\infty dw\, w^{-m}\rho(w) \tag{9.8a}$$

and

$$\langle c^m\rangle \equiv \int_0^\infty dc\, c^m r(c), \tag{9.8b}$$

respectively, are all finite, the averaging in Eq. (9.6) reduces the relative fluctuations, and the distribution of λ_N becomes narrower and narrower as N increases. (Note that in this section $\langle \cdots \rangle$ is used quite generally to indicate an average over the appropriate probability distribution.) Let us, for example, consider the variance ($m=2$) and corresponding higher-order quantities ($m>2$) of $1/\lambda_N$, i.e.,

$$\eta_m(N) \equiv \left\langle \left(\frac{1}{\lambda_N} - \left\langle \frac{1}{\lambda_N} \right\rangle \right)^m \right\rangle \Big/ \left\langle \frac{1}{\lambda_N} \right\rangle^m. \tag{9.9}$$

For large N we have

$$\eta_m(N) \approx N^{-(m-1)} \left[\frac{\left\langle \left(\frac{1}{w} - \left\langle \frac{1}{w} \right\rangle \right)^m \right\rangle}{\left\langle \frac{1}{w} \right\rangle^m} + \frac{\langle (c - \langle c \rangle)^m \rangle}{\langle c \rangle^m} \right], \quad (9.10)$$

and we can define a crossover length ξ_0 such that

$$\eta_m(N) \ll 1 \text{ for } N > \xi_0 . \quad (9.11)$$

On a length scale N, $N \gtrsim \xi_0$, and for

$$\varepsilon \ll \left\langle \frac{1}{\lambda} \right\rangle^{-1} / \xi_0^2 = \left\langle \frac{1}{w} \right\rangle^{-1} \langle c \rangle^{-1} / \xi_0^2 , \quad (9.12)$$

the randomness plays no role for the dynamics of the solution. Such a statement could, of course, also be formulated as a renormalization argument. It implies, for instance, that there exists a well-defined velocity of sound, $c \propto \langle 1/\lambda \rangle^{-1/2}$, and that asymptotically for long times a particle obeys an ordinary diffusion equation. Strictly speaking, this is of course only valid if the strong inequality (9.12) is satisfied. The crossover from random to the averaged diffusive behavior does, however, occur for frequencies ε and on a length scale N given by

$$\varepsilon \approx \langle 1/\lambda \rangle^{-1} / \xi_0^2 \text{ and } N \approx \xi_0 , \quad (9.13)$$

respectively, so that at the crossover, Eq. (9.13), the averaged parameters, $\langle 1/w \rangle$ and $\langle c \rangle$, at least qualitatively describe the average properties of the system. A similar crossover argument will be used below.

A more interesting behavior is obtained for systems for which the averages in Eq. (9.10) are not defined. This is the case for our class (c) or class (c') type probability densities for which

$$\rho(w) \propto w^{-\alpha}, \quad w \to 0 \quad (0 < \alpha < 1) \quad (9.14a)$$

and

$$r(c) \propto c^{-(2-\alpha)}, \quad c \to \infty \quad (0 < \alpha < 1) \quad (9.14b)$$

respectively, so that the moments $\langle 1/\lambda^m \rangle = \langle 1/w^m \rangle \langle c^m \rangle$ diverge. A crossover length, ξ_0, or a crossover frequency, $\varepsilon = \langle 1/\lambda \rangle^{-1} / \xi_0^2$, can therefore not be defined for these systems. We will show, however, that we can define a frequency-dependent length, $\xi(\varepsilon)$, and a frequency-dependent cutoff, $C_\xi = C(\varepsilon)$ or $W_\xi = W(\varepsilon)$, respectively, in a consistent way, such that at ε and on a scale $\xi(\varepsilon)$ the system will be properly described by neglecting the effects of very large C ($C > C_\xi$) or very small W ($W < W_\xi$).

We restrict ourselves to the random C case, and first use a renormalization procedure to calculate the distribution of segment parameters. Let $r_n(c)$ denote the probability density for the distribution of the segment capacitances corresponding to segments of length ξ_n. For simplicity we choose

$$\xi_n = 2^n . \quad (9.15)$$

We decompose r_n according to

$$r_n(c) = r_n^{(0)}(c) + r_n^{(1)}(c) , \quad (9.16)$$

where $r_n^{(0)}(c)$ denotes the most divergent part of $r_n(c)$, i.e.,

$$\lim_{c \to \infty} r_n^{(1)}(c) / r_n^{(0)}(c) = 0 , \quad (9.17)$$

and, for definiteness, we write

$$r_n^{(0)}(c) = \begin{cases} A_n(1-\alpha) c^{-(2-\alpha)}, & c > C_n \\ 0, & c < C_n . \end{cases} \quad (9.18)$$

From the recursion relation

$$r_n(c) = \int dc' r_{n-1}(c') \int dc'' r_{n-1}(c'') \delta(c - c' - c'') , \quad (9.19)$$

we then obtain approximately

$$A_n \gtrsim 2A_{n-1} \gtrsim \xi_n A_0 \quad (9.20)$$

if we require that $\int dc\, r_n^{(0)}(c) \ll 1$. A lower bound for the cutoff $C_n \equiv C_\xi$ is thus defined by

$$\int dc\, r_n^{(0)}(c) = \frac{A_n}{C_n^{1-\alpha}} \approx \frac{A_0}{C_\xi^{1-\alpha}} \lesssim 1 , \quad (9.21)$$

i.e.,

$$C_\xi \gtrsim \xi^{1/(1-\alpha)} . \quad (9.22)$$

The segment capacitance distribution $r_n(c)$ (for segments of length $\xi = 2^n$) is therefore adequately described by its leading asymptotic term $r_n^{(0)}(c)$ only for segment capacitances C_s larger than $C_\xi \approx \xi^{1/(1-\alpha)}$. For $\xi < C_s < C_\xi$ there are significant contributions from $r_n^{(1)}(c)$. [Note that it follows immediately from Eq. (2.5c) that ξ is an absolute lower cutoff for $r_n(c)$.]

We can now associate a unique frequency $\varepsilon (= \varepsilon_\xi)$ with a given length scale ξ by disregarding all capacitances C with $C > C_\xi \approx \xi^{1/(1-\alpha)}$ in the original distribution [Eq. (2.5c)]. The total density p of such large C on the original chain is given by

$$p = \int_{C_\xi}^{\infty} dc\, r(c) \propto C_\xi^{-(1-\alpha)} \approx \xi^{-1} , \quad (9.23)$$

and they are separated by segments for which all capacitances are smaller than C_ξ. The length distribution of these segments is given by

$$R_\xi(N) = p(1-p)^N, \quad N = 0, 1, 2, \ldots , \quad (9.24)$$

so that their average length N is just equal to

$$\langle N \rangle_\xi = (1-p)/p \approx \xi \quad (9.25)$$

if ξ is sufficiently large. If C_N denotes the segment capacitance, we have

$$\langle C_N \rangle \propto N C_\xi^\alpha = N \xi^{\alpha/(1-\alpha)} , \quad (9.26)$$

and the average over all segment lengths becomes

$$\langle \langle C_N \rangle \rangle_\xi \propto \xi^{1/(1-\alpha)} = C_\xi . \quad (9.27)$$

The average segment capacitance, corresponding to the above segments of average length ξ, is thus proportional to C_ξ, and defines an associated frequency scale, ε_ξ, according to

$$\varepsilon_\xi \approx \left\langle \left\langle \frac{C_N}{W_N} \right\rangle \right\rangle_\xi^{-1} . \quad (9.28)$$

If we observe that $W_N \propto N^{-1}$, and that $\langle N^2 \rangle_\xi \approx \xi^2$, we immediately obtain

$$\varepsilon_\xi \propto \xi^{-(2-\alpha)/(1-\alpha)} . \quad (9.29)$$

We wish to point out that, in contrast to Eq. (9.10), the corresponding quantities

$$\eta_m(\xi) = \langle\langle (C_N - \langle\langle C_N\rangle\rangle_\xi)^m\rangle\rangle_\xi / \langle\langle C_N\rangle\rangle_\xi^m \tag{9.30}$$

remain finite as $\xi \to \infty$. The distribution of the segment capacitances C_N thus exhibits no statistical narrowing. It remains broad, with a width comparable to its mean value, C_ξ, even in the limit as $\xi \to \infty$. This is related to the exact result (Bernasconi, Schneider, and Wyss, 1980) (see Sec. V) that for class (*c*) systems $h_\varepsilon(x)$ does not go to a delta function as $\varepsilon \to 0$. On the other hand, a crossover argument nevertheless implies that at $\varepsilon = \varepsilon_\xi$ and on a length scale ξ the system is, at least qualitatively, described by the average properties of the segments which are defined through the above cutoff procedure.

With a given class (*c*) system we may now associate two auxiliary systems by either replacing the large capacitors ($C > C_\xi$) in the original chain by infinite capacitors (Case I) or by eliminating all capacitors larger than C_ξ (Case II). It is then very reasonable to conjecture that essentially any quantity we may wish to investigate is bounded by its values for these auxiliary systems, at least if ε_ξ is small enough. As a specific example we interpret our equations as representing particle diffusion and wish to calculate the mean-square distance, $\langle x^2(t_\xi)\rangle$, travelled in a time $t_\xi \sim \varepsilon_\xi^{-1}$. It then seems obvious that

$$\langle x^2(t_\xi)\rangle_{\rm I} < \langle x^2(t_\xi)\rangle < \langle x^2(t_\xi)\rangle_{\rm II}\,, \tag{9.31}$$

although we are not aware of a rigorous proof for such inequalities.

There are two contributions to $\langle x^2(t_\xi)\rangle_{\rm I}$. For short segments, $N < \xi$, the particle will reach the segment boundary in a time $t < t_\xi$. Their contribution is thus given by

$$\langle x^2(t_\xi)\rangle_{{\rm I},N<\xi} \approx \langle N^2\rangle_{N<\xi} \approx \xi^2\left[1 + 0\left(\frac{1}{\xi}\right)\right]. \tag{9.32}$$

For long segments, $N > \xi$, we can use our crossover argument to obtain

$$\langle x^2(t_\xi)\rangle_{{\rm I},N>\xi} \approx (C_\xi/\xi)^{-1} t_\xi \propto \xi^2\,, \tag{9.33}$$

so that

$$\langle x^2(t_\xi)\rangle_{\rm I} \approx A_{\rm I}\xi^2, \quad \xi \to \infty\,. \tag{9.34}$$

The same crossover argument can be applied to Case II (no large C's), so that we finally obtain

$$A_{\rm I}\xi^2 < \langle x^2(t_\xi)\rangle < A_{\rm II}\xi^2, \quad \xi \to \infty\,. \tag{9.35}$$

As $\xi \propto \varepsilon_\xi^{-(1-\alpha)/(2-\alpha)}$ and $t_\xi \sim \varepsilon_\xi^{-1}$, we conclude that for class (*c*) systems the mean-square displacement has the following long-time behavior:

$$\langle x^2(t)\rangle \propto t^{2(1-\alpha)/(2-\alpha)}, \quad t \to \infty\,. \tag{9.36}$$

This agrees with the corresponding result in Sec. VII, Eq. (7.18c), which has been derived from a general scaling hypothesis for $\langle \tilde{P}_n(\varepsilon)\rangle$. Similar arguments apply, for example, to the conductivity and density of states, and the analysis for the random W case is exactly analogous.

If α becomes negative in Eq. (9.14b), $\langle c^m\rangle$ diverges only for $m > 1-\alpha$, i.e., $\langle c\rangle$ always exists, so that we are dealing with a class (*a*) system. It follows that $C_\xi \propto \xi$, from which it can be shown that the quantities $\eta_m(\xi)$, Eq. (9.30), go to zero as $\xi \to \infty$, even for $m > 1-\alpha$. The distribution of the segment capacitances C_N thus becomes narrower and narrower as $\xi \to \infty$, and we simply have

$$\langle x^2(t)\rangle \propto \langle c\rangle^{-1} t\,, \quad t \to \infty\,. \tag{9.37}$$

The scaling arguments presented in this section are closely related to an earlier scaling approach of Alexander and Bernasconi (1979). The advantage of the present approach lies in the explicit use of a length scaling procedure which demonstrates the role of the residual fluctuations.

X. EFFECTIVE-MEDIUM-TYPE APPROXIMATIONS

In previous investigations (Bernasconi *et al.*, 1978, 1979, 1980) we have repeatedly used an effective-medium–type (EM) approximation to determine the properties of random one-dimensional systems that are related to the models defined in Sec. II. For simplicity we restrict ourselves to the random W case and consider the Laplace transformed version of Eq. (2.1), i.e.,

$$W_{n-1,n}(\tilde{P}_n - \tilde{P}_{n-1}) + W_{n,n+1}(\tilde{P}_n - \tilde{P}_{n+1}) + \omega\tilde{P}_n = P_n(0)\,. \tag{10.1}$$

In the EM description, the probability density $f_\omega(g)$ corresponding to the infinite continued fractions G_n [see Sec. IV, Eqs. (4.4) and (4.5)] is replaced by a delta function,

$$f_\omega(g) = \delta(g - g_{\rm eff}(\omega))\,. \tag{10.2}$$

This is equivalent to replacing all $W_{n,n+1}$ by an (ω dependent) effective transfer rate, $W_{\rm eff} = g_{\rm eff}(g_{\rm eff}+\omega)/\omega$, i.e., the random system is approximated by an effective ordered system. $W_{\rm eff}$ (or $g_{\rm eff}$, respectively) has to be determined by a suitably chosen self-consistency condition. Bernasconi *et al.* (1978, 1979) have, for example, used the defining equation for the G_n, Eq. (4.4a), to construct the self-consistency equation, so that $g_{\rm eff}(\omega)$ is determined by the following equation:

$$g_{\rm eff} = \int_0^\infty dw\,\rho(w)\left(\frac{1}{w} + \frac{1}{g_{\rm eff}+\omega}\right)^{-1}. \tag{10.3}$$

For all probability densities $\rho(w)$, Eq. (10.3) has a unique solution which satisfies

$$\lim_{\omega\to 0} \omega/g_{\rm eff}(\omega) = 0\,, \tag{10.4}$$

and the $\omega \to 0$ asymptotic expression for $\langle \tilde{P}_0(\omega)\rangle$ is thus simply given by [see Eq. (4.11)]

$$\langle \tilde{P}_0(\omega)\rangle \approx \tfrac{1}{2} g_{\rm eff}(\omega)^{-1}, \quad \omega \to 0\,. \tag{10.5}$$

From Eq. (10.3) we then derive the following asymptotic expressions for $\langle \tilde{P}_0(\omega)\rangle$ near $\omega = 0$ (Bernasconi, Schneider, and Wyss, 1980):

$$\langle \tilde{P}_0(\omega)\rangle \approx \tfrac{1}{2}\mu_{-1}^{1/2}\omega^{-1/2} \quad [\text{class } (a)] \tag{10.6a}$$

$$\langle \tilde{P}_0(\omega)\rangle \approx \frac{1}{2}\left(\frac{\rho(0)}{2}\right)^{1/2}\left(\frac{-\ln\omega}{\omega}\right)^{1/2} \quad [\text{class } (b)] \tag{10.6b}$$

$$\langle \tilde{P}_0(\omega)\rangle \approx \frac{1}{2}\left(\frac{\pi(1-\alpha)}{\sin\pi\alpha}\right)^{1/(2-\alpha)}\omega^{-1/(2-\alpha)} \quad [\text{class } (c)]. \tag{10.6c}$$

A comparison with the exact asymptotic expressions, Eqs. (6.12a) and (6.12c) shows that the EM approximation leads to the correct asymptotic ω dependence of $\langle \tilde{P}_0(\omega)\rangle$ for both class (a) and class (c) systems. There is, however, an important difference between the two classes. For class (a) systems we know that $h_\omega(g)$ approaches a delta function as $\omega \to 0$ (see Sec. V), i.e., the EM assumption, Eq. (10.2) is asymptotically exact. Any reasonable self-consistency equation therefore leads to the exact asymptotic expression for $\langle \tilde{P}_0(\omega)\rangle$, with respect to the prefactor, as well as with respect to the ω dependence. For class (c) systems, on the other hand, this is different. $h_\omega(g)$ does not approach a delta function asymptotically (see Sec. V), so that the EM prefactor, Eq. (10.6c), does not agree with the exact $C_0^{(\alpha)}$ value, given by Eq. (6.11). Moreover, the EM prefactor depends on the choice of the self-consistency condition. If, for instance, the usual resistor network EM theory (Kirkpatrick, 1973) is applied to the electrical network analog corresponding to Eqs. (10.1), see Fig. 3, the self-consistency equation for $W_{\rm eff} = g_{\rm eff}$ $(g_{\rm eff}+\omega)/\omega$ becomes

$$\int_0^\infty dw\, \rho(w) \frac{w - W_{\rm eff}}{w + \frac{1}{2}(g_{\rm eff}+\omega)} = 0 . \tag{10.7}$$

It follows that the asymptotic expression for $\langle \tilde{P}_0(\omega)\rangle$, Eq. (10.6c), is then replaced by

$$\langle \tilde{P}_0(\omega)\rangle \approx \frac{1}{2}\left(\frac{\pi(1-\alpha)2^\alpha}{\sin\pi\alpha}\right)^{1/(2-\alpha)} \omega^{-1/(2-\alpha)} \quad [\text{class } (c)]. \tag{10.8}$$

A comparison of the EM prefactors in Eqs. (10.6c) and (10.8) with the exact prefactor, $C_0^{(\alpha)}$, is made by Bernasconi, Schneider, and Wyss (1980). For small α, $C_0^{(\alpha)}$ can be evaluated analytically,

$$C_0^{(\alpha)} \approx \tfrac{1}{2}\alpha^{-1/2}, \quad \alpha \to 0 \tag{10.9}$$

and it follows immediately that the EM prefactors in Eqs. (10.6c) and (10.8), respectively, also reduce to $\frac{1}{2}\alpha^{-1/2}$ for $\alpha \to 0$.

For class (c) systems, our EM approximations thus reproduce the exact asymptotic expression (including the prefactor) for $\langle \tilde{P}_0(\omega)\rangle$ if we are in the limit as $\alpha \to 0$. On the other hand, the EM approximation becomes asymptotically exact for class (a) systems, as discussed above. As class (b) systems represent the crossover between class (a) systems and ($\alpha \to 0$)-class

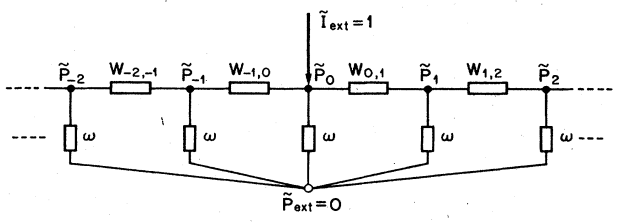

FIG. 3. Electrical network analog corresponding to Eqs. (10.1). The $W_{n,n+1}$ and ω represent conductances (or admittances), and the $\tilde{P}_n$ denote the node potentials. The "external node" is kept at zero potential, and a unit external current is flowing into node $n = 0$. In the resistor network effective-medium theory (Kirkpatrick, 1973), the $W_{n,n+1}$ are replaced by an effective conductance, $W_{\rm eff}(\omega)$.

(c) systems, it seems reasonable to conjecture that the exact class (b) asymptotic expression for $\langle \tilde{P}_0(\omega)\rangle$ is given by the EM result, Eq. (10.6b).

Within the EM approximation we can, in addition to $\langle \tilde{P}_0(\omega)\rangle$, easily evaluate the quantities $\langle \tilde{P}_n(\omega)\rangle$, and $\langle D(\omega)\rangle$ or $\sigma(\omega)$.

$\langle \tilde{P}_n(\omega)\rangle$ is simply given by Eq. (7.4), with W replaced by $W_{\rm eff}(\omega)$. This can also be written as

$$\langle \tilde{P}_n(\omega)\rangle = \langle \tilde{P}_0(\omega)\rangle \left(\frac{g_{\rm eff}}{g_{\rm eff}+\omega}\right)^{|n|}, \tag{10.10}$$

where

$$\langle \tilde{P}_0(\omega)\rangle = (2g_{\rm eff}+\omega)^{-1}, \tag{10.11}$$

and it follows that the mobility, or frequency-dependent diffusion constant, is given by

$$\langle D(\omega)\rangle = W_{\rm eff}(\omega). \tag{10.12}$$

Using $W_{\rm eff} = g_{\rm eff}(g_{\rm eff}+\omega)/\omega$, we can then relate $\langle D(\omega)\rangle$ to $\langle \tilde{P}_0(\omega)\rangle$. Asymptotically for $\omega \to 0$ we obtain

$$\langle D(\omega)\rangle \approx \frac{1}{4}\frac{1}{\omega \langle \tilde{P}_0(\omega)\rangle^2}, \quad \omega \to 0 \tag{10.13}$$

where we have used Eq. (10.4). The EM relation, Eq. (10.13), is thus qualitatively identical with our previous relation, Eq. (7.12), which has been derived on the basis of a general scaling hypothesis. The EM result for the small ω asymptotic behavior of $\langle D(\omega)\rangle$ is determined by Eqs. (10.13) and (10.6) or (10.8), respectively, and exhibits the same ω dependences as obtained in Eqs. (7.16). The resulting long-time behavior of $\langle x^2(t)\rangle$ is therefore also identical to that given in Eqs. (7.18).

The long-time behavior of $\langle P_n(t)\rangle$ can be determined from the small ω behavior of its Laplace transform, Eq. (10.10), using the methods described in Appendix C of Scher and Montroll (1975). Without presenting any details, we remark that for class (c) systems one can, for example, derive the following result:

$$\langle P_n(t)\rangle \propto \langle P_0(t)\rangle x^{-\alpha/2} \exp(-A_\alpha x^{2-\alpha}), \tag{10.14}$$

$t \to \infty$, $x \equiv |n|/t^{(1-\alpha)/(2-\alpha)} \gg 1$. Within the EM approximation, and for long times, the n dependence of $\langle P_n(t)\rangle$ is thus described by a "modified Gaussian," at least for large $|n|/t^{(1-\alpha)/(2-\alpha)}$.

A problem with the EM approximation is that it neglects fluctuations and correlations. Let us, for instance, consider the diagonal Green function

$$\tilde{P}_0(\omega) = (\omega + G^{(+)} + G^{(-)})^{-1}, \tag{10.15}$$

which is directly related to the density of states. The random quantities $G^{(+)}$ and $G^{(-)}$ are independent, and both distributed according to the same probability density, $f_\omega(g)$. In Sec. V we have shown that, for class (c) systems, at least, $f_\omega(g)$ does not approach a delta function for $\omega \to 0$, in contrast to the EM assumption of Eq. (10.2). In an ensemble of chains we would therefore find a broad distribution for $\tilde{P}_0(\omega)$, with large fluctuations from one chain to the other, even at arbitrarily low frequencies. On the other hand, one can usually only measure an average over a large number (say, N) of chains, i.e., one measures

$$\langle \tilde{P}_0(\omega)\rangle_{\rm av} \equiv \frac{1}{N}\sum_{n=1}^{N} \tilde{P}_0^{(n)}(\omega) \approx \langle \tilde{P}_0(\omega)\rangle \propto g_{\rm eff}(\omega)^{-1}\,, \quad (10.16)$$

and the variance of $\langle \tilde{P}_0(\omega)\rangle_{\rm av}$ is always of order $1/N$ and plays no role. An analogous argument applies to a conductivity measurement, for instance, in a material like the superionic conductor hollandite (Bernasconi *et al.*, 1979).

The situation is also similar for other types of experiments. In energy transport and other diffusion problems one is, for example, interested in the probability $P_0(t)$ that a particle starting out at the origin is still there after a time t. In practice one again measures an average, namely, an average over initial sites:

$$\langle P_0(t)\rangle_{\rm av} \equiv \frac{1}{N}\sum_{n=1}^{N} P_0^{(n)}(t) \approx \langle P_0(t)\rangle = \mathcal{L}^{-1}\langle \tilde{P}_0(\omega)\rangle\,, \quad (10.17)$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. The variance of $\langle P_0(t)\rangle_{\rm av}$ is again small ($\sim 1/N$) and is not important for the interpretation of the experiment. The situation would be different, however, if one were able to measure the variance of $P_0(t)$ directly, which involves the calculation of

$$\langle P_0^2(t)\rangle = \mathcal{L}^{-1}\left\langle \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\omega'\, \tilde{P}_0(\omega')\tilde{P}_0(\omega-\omega')\right\rangle . \quad (10.18)$$

If the distribution of $\tilde{P}_0(\omega)$, determined by $f_\omega(g)$, is not a delta function, the $G(\omega)$ for different ω on a definite chain are strongly correlated, so that the averaging process does not commute with the convolution, i.e.,

$$\left\langle \int d\omega'\, \tilde{P}_0(\omega')P_0(\omega-\omega')\right\rangle \neq \int d\omega'\, \langle \tilde{P}_0(\omega')\rangle\langle \tilde{P}_0(\omega-\omega')\rangle\,. \quad (10.19)$$

For such quantities the EM approximation would therefore lead to incorrect results. On the other hand, even the knowledge of $f_\omega(g)$ is not sufficient to calculate such correlated averages.

The above discussion implies that the fact that $f_\omega(g)$ does not approach a delta function asymptotically is mainly of theoretical interest, and does not play an important role in the interpretation of real experiments. It should be noted, however, that only via the rigorous results for $f_\omega(g)$ (Bernasconi, Schneider, and Wyss, 1980; and Sec. V above) have we been able to demonstrate that the EM approximation, as well as the scaling approach of Sec. IX, lead to the correct asymptotic dependences for some important averaged quantities, such as, for instance, $\langle \tilde{P}_0(\omega)\rangle$.

XI. APPLICATIONS AND COMPARISON WITH ALTERNATIVE APPROACHES

In the following, we shall apply our results to several specific realizations of our general model systems. In addition, we shall compare our approach with alternative treatments of the same type of problems.

A. A random barrier model

Bernasconi *et al.*, (1979) have introduced a random barrier model to explain the anomalous low-frequency behavior of the ionic conductivity in the one-dimensional superionic conductor hollandite ($K_{1.54}Mg_{0.77}Ti_{7.23}O_{16}$). It was assumed that the low-frequency response is dominated by thermally activated hopping of the mobile K ions over defect-induced barriers, i.e.,

$$W_{n,n+1} = f_0 \exp(-\Delta_{n,n+1}/k_B T)\,, \quad (11.1)$$

where f_0 is an attempt frequency, and n refers to the segment between the $(n-1)$st and the nth impurity barrier. A model probability density, $\bar{\rho}(\Delta)$, for the distribution of the barrier heights $\Delta_{n,n+1}$ of the following form has been proposed:

$$\bar{\rho}(\Delta) = \begin{cases} \bar{\rho}_0 \exp(-\Delta/k_B T_m)\,, & \Delta \geq \Delta_0 \\ 0\,, & \Delta < \Delta_0\,. \end{cases} \quad (11.2)$$

This leads to an expression

$$\rho(w) = \begin{cases} \rho_0(T) w^{-\alpha(T)}\,, & 0 \leq w \leq W_{\max} \\ 0 \quad \text{otherwise} \end{cases} \quad (11.3)$$

for the probability density of the transfer rates $W_{n,n+1}$, with

$$\alpha(T) = 1 - T/T_m\,. \quad (11.4)$$

For $T < T_m$, $\rho(w)$ thus belongs to class (c), and our results of Sec. VII predict a low-frequency conductivity of the form

$$\sigma(\omega) \approx c(T)(-i\omega)^{\nu(T)}\,, \quad \omega \to 0 \quad (11.5)$$

with a temperature-dependent exponent

$$\nu(T) = (T_m - T)/T_m + T)\,. \quad (11.6)$$

Above T_m, $\rho(w)$ is of class (a), leading to a frequency-independent conductivity at small ω. The model thus predicts a mobility transition at $T = T_m$. The dc conductivity and diffusion constant vanish below T_m, but have finite values above T_m. In hollandite, where $T_m \approx 400$ to 450 K, the measured (complex) conductivity $\sigma(\omega, T)$ at and below room temperature shows excellent and detailed agreement (Bernasconi *et al.*, 1979; Beyeler, 1981) with the model predictions, Eqs. (11.5) and (11.6). The mobility transition at T_m, however, has not yet been observed, and its observability turns out to be a very complex problem (Beyeler, 1981). Theoretically, its existence hinges on the specific form of the barrier height distribution, Eq. (11.2), for which we have no microscopic justification. While a modified $\bar{\rho}(\Delta)$, e.g., the introduction of an upper cutoff or a different Δ dependence, will not qualitatively change the results well below T_m, it will certainly smear out the transition at T_m. Even within our idealized model, $\alpha(T)$ goes to zero as T approaches T_m, so that our theoretical description by the leading asymptotic term is no longer adequate in the experimentally accessible frequency range (Bernasconi and Schneider, 1981; see also Sec. XIII below). An indisputable proof of the existence of the transition is therefore only obtainable from ionic dc conductivity experiments which, however, require carefully prepared "reversible" electrodes containing the mobile species. Hollandite itself is, unfortunately, unstable in contact with metallic potassium, ruling out such measurements on this material. First experimental results on a different one-dimensional superionic conductor, $Na_{0.7}Ga_{4.72}Ti_{0.29}O_8$

(Beyeler, 1981), indicate that this compound might be a better candidate than hollandite for investigations of the mobility transition.

B. Hopping model with configurational disorder

Let us now consider the one-dimensional version of a more general d-dimensional model for hopping between randomly distributed impurity sites (Scher and Lax, 1973; Scher and Montroll, 1975). Specifically, we consider hopping between equivalent, randomly distributed, sites of density p on a one-dimensional regular lattice. The distance l (measured in lattice constants) between two neighboring sites is then distributed according to

$$R(l)=p(1-p)^{l-1}, \tag{11.7}$$

and the corresponding transfer rates, $W(l)$, can at low temperatures be approximated by (Scher and Lax, 1973)

$$W(l)=W_0(T)\exp(-\gamma l). \tag{11.8}$$

This leads to a probability density, $\rho(w)$, for the transfer rates of the form of Eq. (11.3), but now with a concentration-dependent exponent,

$$\alpha(p)=1+\gamma^{-1}\ln(1-p). \tag{11.9}$$

For $p<1-\exp(-\gamma)$, $\alpha(p)$ is positive, and the distribution of the $W(l)$ is thus of class (c). Our results then lead to a low-frequency hopping conductivity of the form

$$\sigma(\omega)\propto(-i\omega)^{\nu}, \quad \nu=\alpha/(2-\alpha), \quad \omega\to 0. \tag{11.10}$$

Böttger *et al.*, (1979) have recently applied an approximate percolation approach to exactly this problem. Their treatment leads to a different conductivity exponent, $\nu=1/(3-2\alpha)$ in our notation, which is thus accurate only if α is close to one. On the other hand, this is actually the region of interest in their applications.

C. Time-dependent aspects of spectral transport

Our coupled rate equations, Eqs. (2.1) or (2.2), can be used as a simple theoretical model to discuss the time-dependent effect in fluorescent line-narrowing experiments concerned with investigations of spectral transfer within inhomogeneously broadened optical lines [see, for instance, Huber *et al.* (1977)]. Within our model, the quantity which is directly relevant to these experiments is the autocorrelation function, $\langle P_0(t)\rangle$. We note, however, that our random W model is applicable directly only when k_BT is much greater than the inhomogeneous linewidth, as only in this limit do the transfer rates become symmetric. Our results of Sec. VI show that the long-time decay of $\langle P_0(t)\rangle$ is determined by the distribution of transfer rates, $\rho(w)$. Depending on the behavior of $\rho(w)$ near $w=0$, several classes of $\rho(w)$ can be distinguished which lead to qualitatively different long-time results.

Though their paper is nominally restricted to highly anisotropic materials, Scher *et al.*, (1980) have shown how for short and intermediate times one-dimensional transport may be relevant even for three-dimensional systems. Because of fluorescent decay processes and only finite anisotropies, one can actually observe one-dimensional diffusion processes for only a finite time span. One can thus expect to observe asymptotic (long-time) one-dimensional limits only in special cases.

D. Random trapping models

As indicated in Sec. III, our random C model gives rise to a random trapping model if we identify C_n with $\exp(\Delta_n/k_BT)$. The probability density $r(c)$ is then determined by the distribution of the random trap depths Δ_n. Compared to the random W case, an additional aspect now enters into the problem. As discussed in Sec. VIII, two meaningful initial conditions can be chosen, Eqs. (8.5) or (8.6), respectively, leading to qualitatively different results for the long-time behavior of $\langle P_0(t)\rangle$ in the case of class (b) or class (c) distributions $r(c)$. The initial condition (8.5) corresponds to an optically pulsed excitation experiment where a nonequilibrium localized trap excitation distribution is created at the origin at $t=0$. Alternatively, Eq. (8.6) describes a change in temperature at the excited site caused, say, by a sudden localized heat pulse at $t=0$.

E. Localization

In our systems, almost all eigenfunctions (i.e., all with $\varepsilon\neq 0$) are localized (Goda, 1981). According to the asymptotic long-time behavior of $\langle P_0(t)\rangle$ and $\xi(t)=\langle x^2(t)\rangle^{1/2}$ we may, however, introduce a different concept of localization. For all our systems [classes (a), (b) and (c) of probability densities] the correlation length, $\xi(t)$, and the inverse autocorrelation function, $\langle P_0(t)\rangle^{-1}$, diverge as $t\to\infty$. Only for class (a) systems, however, is the rate of this divergence the same ($\propto t^{1/2}$) as in the case of pure diffusion (i.e., as in the corresponding ordered system), so that the diffusion constant and the dc conductivity are well-defined finite quantities. In class (b) and class (c) systems, the divergence is slower than $t^{1/2}$, leading to a vanishing diffusion constant and dc conductivity, respectively. In this sense we may thus speak of a localization of the excitations in class (b) and (c) systems, and the "strength of localization" is determined by the asymptotic behavior of $\xi(t)$ relative to pure diffusion.

F. Numerical simulations

A Monte Carlo study (Rich *et al.*, 1978) on time-dependent spectral transport has been performed before our EM results for the long-time decay of $\langle P_0(t)\rangle$ (Bernasconi *et al.*, 1978) were proved to be asymptotically correct. The distribution of transfer rates in this study was of class (a), and the numerical results accurately confirmed our conclusion that asymptotically the system can be described by an equivalent ordered system with an "average transfer rate",

$$W_{\rm av}^{-1}=\int_0^{\infty} dw\, w^{-1}\rho(w).$$

Bernasconi, Schneider, and Wyss (1980) have carried out numerical simulations of the $\omega\to 0$ asymptotic behavior of $\langle\tilde{P}_n(\omega)\rangle$ to test the general scaling hypothesis described in Sec. VII.

An extensive Monte Carlo study on the behavior of the mean-square displacement $\langle x^2(t)\rangle$ in class (a) and class (c) systems has recently been carried out by Richards and Renken (1980). For class (c) systems, their results raise some questions with respect to the exact asymptotic expression for $\langle x^2(t)\rangle$ and with respect to the approach to the asymptotic behavior. The corresponding problems are discussed by Richards and Renken (1980) and further clarified in a subsequent reply by Bernasconi and Beyeler (1980). In particular, the numerical simulations of Richards and Renken (1980) show that the approach to the limiting asymptotic expressions is extremely slow in the case of class (c) probability densities, which occur in several physical applications of the model. A comparison of numerical data and experimental results with analytical model predictions may therefore be possible only if correction terms beyond the leading asymptotic expression can be calculated. Richards and Renken (1980) calculate the first correction term to the asymptotic behavior of $\sigma(\omega)$ within our previous EM approach (Bernasconi *et al.*, 1978, 1979). Very recent investigations (Bernasconi and Schneider, 1981) indicate, however, that the exact asymptotic expansion may differ significantly from the corresponding EM expression.

G. Dilute ferromagnetic Heisenberg chain

Let us now consider a Heisenberg chain with random ferromagnetic nearest-neighbor interactions, $J_{n,n+1}$, which can be shown (Alexander and Holstein, 1978) to be closely related to the diffusion problem. The eigenvalue equations corresponding to low-lying excitations (for which we can use the linearized equations of motions for the spin lowering operators) are of the form of Eq. (4.1), and the low-energy density of states is thus determined by our random W results ($W_{n,n+1} \to J_{n,n+1}$) for $N(\varepsilon)$, i.e., Eqs. (6.14a), (6.14b), and (6.14c). For a dilute system, with a concentration p of magnetic sites, the distribution of the spin separation, l, is given by Eq. (11.7). The assumption of an exponentially decaying interaction, $J(l) \propto \exp(-\gamma l)$, then leads to a probability density $\rho(J)$ of class (c), provided that p is not too large, so that $\alpha = \alpha(p) > 0$. Our results of Sec. VI then predict $N(\varepsilon) \propto \varepsilon^{-1/(2-\alpha)}$ for $\varepsilon \to 0$, so that the low-temperature behavior of the corresponding specific heat is, according to Eq. (6.15c), given by

$$C(T) \propto T^{(1-\alpha)/(2-\alpha)}, \quad T \to 0. \tag{11.11}$$

Dilute Heisenberg chains of this type have recently been investigated by Theodorou and Cohen (1979) using a previously proposed cluster argument. For ferromagnetic couplings, and for a class (c) $\rho(J)$, their result for the specific heat is

$$C(T) \propto \begin{cases} -T^{(1-\alpha)} \ln T, & \alpha > \frac{1}{2} \\ T^{1/2}, & \alpha < \frac{1}{2}, \quad T \to 0 \end{cases} \tag{11.12}$$

and therefore does not agree with our asymptotic expression, Eq. (11.11). Alexander and Bernasconi (1979) have shown that a correct and internally consistent application of their cluster approach does, however, indeed lead to the correct result of Eq. (11.11).

H. Planar model of a classical spin glass

Huber and Ching (1980) have recently pointed out that the one-dimensional version of their planar model of a classical spin glass is equivalent to the type of systems considered in the present paper, with a class (b) probability density of the form

$$\rho(J) = (2/\pi)^{1/2} \exp(-J^2/2), \quad 0 \leq J < \infty. \tag{11.13}$$

Our EM approximations of Sec. I, which we believe to give the correct asymptotic expression for $N(\varepsilon)$ for class (b) systems, lead to

$$N(\varepsilon) \approx (2\pi)^{-5/4} \left(\frac{-\ln\varepsilon}{\varepsilon}\right)^{1/2}, \quad \varepsilon \to 0. \tag{11.14}$$

This result coincides with that obtained from the coherent potential approximation (CPA) theory of Huber and Ching (1980). In addition, their CPA theory actually seems to be identical (Huber, 1980) with our second version of the EM approximation which, for class (c) systems, leads to Eq. (10.8) for $\langle \tilde{P}_0(\omega)\rangle$.

I. The CTRW approach

It is of interest to compare our results with those obtained from the continuous time random walk (CTRW) approach of Scher, Lax, and Montroll (Scher and Lax, 1973; Scher and Montroll, 1975), which has been used extensively to describe the anomalous frequency- or time-dependent transport properties of disordered systems [for a review see Pfister and Scher (1978)]. The central idea of this approach is to replace the random system by an ordered system (with respect to the transfer rates) with a suitably chosen local "waiting time distribution," $\psi(t)$. Several schemes have been developed to relate $\psi(t)$ to the microscopic properties of the original model, and for some simple models it has been shown that this can be done exactly.

Applied to our type of random one-dimensional systems (random W case), the CTRW approach leads to equations of the form

$$\omega \tilde{P}_n + \lambda(\omega)(2\tilde{P}_n - \tilde{P}_{n-1} - \tilde{P}_{n+1}) = P_n(0), \tag{11.15}$$

where $\lambda(\omega)$ is related to the Laplace transform of the waiting time distribution $\psi(t)$. The CTRW approach is thus formally identical to our EM type approximations of Sec. X, with $\lambda(\omega)$ playing the role of our $W_{\text{eff}}(\omega)$. The discussion concerning the neglect of fluctuations and correlations within the EM approximation (see Sec. X) therefore also applies to the CTRW approach.

There is, however, an additional difficulty with the Scher and Lax approach. We have shown that our EM approaches lead to the correct small ω (or long-time) asymptotic description of the average properties of our random systems. In addition, it can be shown in straightforward fashion (Bernasconi and Schneider, 1981) that they also reproduce the exact averaged quantities in the opposite limit (i.e., large ω or small times, respectively). Our $W_{\text{eff}}(\omega)$ thus defines a waiting time distribution, $\psi(t)$, which gives an excellent description of our random systems. This $\psi(t)$ is, however, different from that obtained by applying the Scher and Lax (1973) procedure to our original random system. The latter leads to incorrect asymptotic expres-

sions in the $\omega \to 0$ or $t \to \infty$ limit, and the same is true (Klafter and Silbey, 1980b) for several alternative, but essentially equivalent, approaches for the determination of $\psi(t)$. As an example we may consider the exponent $\nu(\alpha)$ which describes the long-time behavior of $\langle P_0(t)\rangle \propto t^{-\nu}$ and $\langle x^2(t)\rangle \propto t^{2\nu}$ in class (c) systems. It can be shown (Klafter and Silbey, 1980b; Riseborough, 1979) that the existing waiting time approaches all lead to $\nu=(1-\alpha)/2$, whereas the correct exponent [see Eqs. (6.13c) and (7.18c)] is given by $\nu=(1-\alpha)/(2-\alpha)$. In the random W case considered here, it can be seen that the Scher and Lax average of the local function $\exp(-Wt)$ overestimates the importance of the (rare) small W at any given time. It therefore underestimates ν, as exhibited by the above explicit results.

Whereas the description of a given random system by a single $\psi(t)$, or $\lambda(\omega)$, respectively, always remains an approximation,[2] it seems important to improve on the determination of the "best" $\psi(t)$. This is particularly true in higher dimensions where an exact calculation of the average properties, even asymptotically, seems out of the question.

The Scher and Lax approach also completely neglects correlations along the paths actually traversed. As a consequence, the results do not depend on the topology of space (i.e., on the dimensionality). For dimensionalities higher than one, this deficiency becomes even more serious than in one dimension, because the small W can be avoided by the diffusing particle. It can actually be shown (Alexander, 1980) that our random W model in three dimensions always leads to conventional diffusion asymptotically.

J. Random harmonic chain systems

For random harmonic chain systems, Eq. (2.3), our asymptotic results are less complete than for diffusion-type systems, Eq. (2.2). They have been summarized and discussed at the end of Sec. VIII.

In the vast literature on disordered harmonic systems the focus is mainly on class (a) type distributions (e.g., binary mass problems). In addition, the low-frequency limit is less important than in diffusion-type problems, and other properties of the spectrum have received much wider attention [see, for instance, Hori (1968)]. Our result that for $\omega \to 0$ a class (a) system behaves as an ordered system with an average mass, $M_{\rm av}=\langle m\rangle$, or been derived several times with different methods (Domb, 1963; Maradudin and Weiss, 1968; Lu *et al.*, 1974; O'Connor and Lebowitz, 1974). On the other hand, we believe that the corresponding asymptotic expressions for class (b) and class (c) distributions are new. It seems, for example, that it is not possible to extract these asymptotic results from the integral equation in Dyson's (1953) approach. As explained in Sec. IV, this integral equation is different from ours because the relevant random variables, which are connected to the original random variables via infinite continued fractions, are chosen differently.

[2]Even our nearest-neighbor random systems involve an infinite number of "effective transfer rates," $\lambda_{|n-m|}(\omega)$, to describe their average properties exactly. Compare Klafter and Silbey (1980a).

K. Interrupted chain systems (bond percolation models

Although we have excluded them explicitly from our present treatment, we would like to add a few remarks about chains with random interruptions. In a previous paper (Alexander, Bernasconi, and Orbach, 1978a) we have calculated the exact long-time behavior of $\langle P_0(t)\rangle$ and the small ε behavior of $N(\varepsilon)$ for this type of system. The methods used were, however, entirely different from those of the present paper. With regard to the present approach we note that the integral equation for $f_\omega(g)$, Eq. (4.5), remains valid for a chain with interruptions, but we have not tried to determine its asymptotic solution for small ω. Our effective-medium (EM) approximations of Sec. X, however, seem to work quite well also for this type of system. Odagaki and Lax (1980) have recently calculated the exact frequency dependence of the hopping conductivity for such a bond percolation model, and they compare its low- and high-frequency expansion with the corresponding results of various approximate treatments. In their paper they denote our first EM approach, Eq. (10.3), by Bernasconi, Alexander, and Orbach (BAO), and their CPA is identical to our second EM approach, Eq. (10.7). The two EM approaches, as well as the CTRW, give the correct frequency dependence both in the low- and high-frequency limit, but the expansion coefficients differ with respect to each other and with respect to the exact solution. The relative merits of approximate treatments are, however, not determined by a comparison of expansion coefficients alone. In this context we remark that, in contrast to the CTRW, for instance, our EM approaches lead to the exact asymptotic frequency dependences (for small, as well as for large ω) for very general classes of probability densities, and that they give a reasonably accurate description over the entire frequency range.

XII. NONUNIVERSALITY AND HYPERSCALING

One feature of the results of this paper, exhibited in Sec. VI, is the nonuniversal character of the specific-heat exponent for class (c) probability densities $\rho(\omega)$. A similar nonuniversality is also found for the correlation length defined and derived in Sec. VII. We show in this section that though both exponents are nonuniversal, they nevertheless obey a hyperscaling relation derived for $T_c=0$ systems by Baker and Bonner (1975).[3]

Working in Laplace transform space, Eq. (6.12c) tells us that

$$\langle \tilde{P}_0(\omega)\rangle \approx C_0^{(\alpha)}\omega^{-1/(2-\alpha)}, \quad \omega \to 0. \tag{6.12c}$$

From Eq. (4.13), the density of states $N(\varepsilon)$ equals

$$N(\varepsilon) = -\frac{1}{\pi}\,\mathrm{Im}\langle \tilde{P}_0(-\varepsilon + i0^+)\rangle$$
$$= \frac{C_0^{(\alpha)}}{\pi}\sin\left(\frac{\pi}{2-\alpha}\right)\varepsilon^{-1/(2-\alpha)}. \tag{4.13}$$

The specific heat has a temperature dependence, arising from an integration of $\varepsilon N(\varepsilon)$ times a Bose factor, of

[3]This reference was pointed out to us by Professor A. Aharony during a very helpful conversation.

$$C(T) \propto T^{(1-\alpha)/(2-\alpha)} . \tag{6.15c}$$

For its part, the correlation length in ω space arises from the assumption of a scaling length for $\langle \tilde{P}_n(\omega) \rangle$, and is given by

$$\xi(\omega)^{-1} \approx 2\omega \langle \tilde{P}_0(\omega) \rangle , \quad \omega \to 0 . \tag{7.11}$$

In the diffusion problem, the inverse Laplace transform of Eq. (7.11) generates the time-dependent result, Eq. (8.4e) for class (*c*) distributions,

$$\xi(t) \propto t^{(1-\alpha)/(2-\alpha)} . \tag{8.4e}$$

This correlation length corresponds to the Green function in real time, $\langle P_n(t) \rangle$, through the scaling relation (7.9). In the thermodynamic problem we speak of the equal-time Green function, defined as the thermal and ensemble average of a product of creation and destruction operators at different sites. A comparison of the formal expressions for the two problems shows that, apart from a Bose factor in the latter, one transforms into the other upon replacement of $1/t$ by $k_B T$. We can therefore write

$$\xi(T) \propto T^{-(1-\alpha)/(2-\alpha)} . \tag{12.1}$$

As remarked previously, the exponents of T in the expression (6.15c) for $C(T)$ and in (12.1) for $\xi(T)$ are nonuniversal in that they depend on the degree of randomness through α. However, a remarkable relationship between these exponents was predicted by Baker and Bonner (1975). They considered the scaling relationships at zero-temperature critical points. In our case, T_c would certainly be zero for the one-dimensional random Heisenberg ferromagnet (it is zero even for the ordered case), and their conclusions can be applied to our results. For $T_c = 0$, convention sets

$$C \propto T^{-\alpha_s} , \tag{12.2}$$

where the subscript s refers to the singular part, and

$$\xi \propto T^{-\nu} . \tag{12.3}$$

Hyperscaling requires (Baker and Bonner, 1975)

$$-\alpha_s = d\nu , \tag{12.4}$$

where d is the dimensionality. Comparing Eqs. (8.4e) and (12.1) we see that the relation (12.4) is satisfied (since $d = 1$). We therefore have the very interesting result that our exponents satisfy hyperscaling, while at the same time they are nonuniversal. This result is unusual, though it must be noted that our expression for $\xi(T)$ does rest on an explicit scaling assumption, Eq. (7.9).[4]

XIII. CONCLUSION AND FURTHER DEVELOPMENTS

This paper has discussed in some detail the solution in the low-frequency or long-time limit of a linear master equation with random transfer rates or random trap depths. We have shown how these solutions map onto a variety of physical systems, ranging from the random vibrational chain to the random one-dimensional Heisenberg ferromagnet. We have exhibited exact asymptotic (low-frequency or long-time) expressions for the autocorrelation function, showing explicitly the nonuniversal character of the corresponding exponents. We have also demonstrated the applicability of hyperscaling between the specific heat and the correlation length exponents, even in the presence of this nonuniversality.

A number of aspects of the random one-dimensional chain problem remain, however, unsolved, or are only partly resolved. These include the following.

(a) The asymptotic $\omega \to 0$ expansions for $f_\omega(x)$ beyond the leading asymptotic term. All the expressions we have given for the asymptotic behavior represent only the leading asymptotic term, and it is not obvious *a priori* how small ω must become to make correction terms negligible. Numerical simulations (Richards and Renken, 1980), as well as an investigation of $\sigma(\omega)$ in hollandite close to T_m (Beyeler, 1981, and private communication), indicate that such corrections to the leading asymptotic behavior become increasingly important as one approaches the crossover between class (*c*) and class (*a*) behavior, i.e., for $\alpha \to 0$. It would therefore be desirable to derive asymptotic expansions which go beyond the leading term. In principle, this turns out to be possible (Bernasconi and Schneider, 1981) by writing down a small ω expansion for $f_\omega(x)$ of the form

$$f_\omega(x) = \frac{1}{\varepsilon}\left[h\left(\frac{x}{\varepsilon}\right) + \varepsilon^{k_1} h_1\left(\frac{x}{\varepsilon}\right) + \varepsilon^{k_2} h_2\left(\frac{x}{\varepsilon}\right) + \cdots\right], \tag{13.1}$$

where $\varepsilon = \varepsilon(\omega)$ is given by Eqs. (5.19) and (5.22) for class (*a*) and class (*c*) systems, respectively. Explicit results have been obtained for both class (*a*) and class (*c*) probability densities. For class (*a*) distributions for which the first N negative moments exist we have $k_n = n$ for $n < N$, while in class (*c*) systems we have, for example,

$$k_1 = \begin{cases} \alpha , & \alpha \leq \frac{1}{2} \\ 1-\alpha , & \alpha \geq \frac{1}{2} . \end{cases} \tag{13.2}$$

The calculation of the $h_n(x)$, $n \geq 1$, becomes increasingly involved (Bernasconi and Schneider, 1981), but Eq. (13.2) already indicates the importance of the correction terms when $\alpha \to 0$ or $\alpha \to 1$. We note that the corresponding correction terms for averaged quantities are of quite a different origin than the corrections calculated by Richards and Renken (1980), who worked with the effective medium results of Sec. X.

(b) The exact asymptotic results for $\sigma(\omega)$ and $\langle x^2(t) \rangle$. The results in Sec. VII of this paper all rest on the assumption of a single characteristic length, $\xi(\omega)$ or $\xi(t)$, which leads to a scaling hypothesis for the $\langle \tilde{P}_n(\omega) \rangle$ or for the $\langle P_n(t) \rangle$, respectively. Though such an assumption seems quite plausible asymptotically, it remains a conjecture, as we are unable to obtain a rigorous expression for $\langle \tilde{P}_n(\omega) \rangle$ or $\langle P_n(t) \rangle$, even in the small ω or large t limit.

(c) The fluctuations in the site occupancy probabilities. We are unable to calculate even the simplest fluctuation term $\langle P_0^2(t) \rangle$, in any time domain. This is a consequence of the lack of commutativity of the averaging process with the convolution in Laplace transform space, required for the calculation of this quantity. This would be the case even if we possessed an exact solution for

[4] This caveat was pointed out by Dr. J. E. Hirsch (private communication).

$f_\omega(x)$ for all ω.

(d) The random one-dimensional antiferromagnetic chain. Our starting point of a linear near-neighbor master equation is inappropriate for the antiferromagnetic linear chain. We require a low density of spin waves in order that the spin-wave expansion can be linearized and mapped onto the master equation for imaginary time. Very recent renormalization group and numerical calculations (Dasgupta and Ma, 1980; Hirsch and José, 1980; Bondeson and Soos, 1980) as well as earlier theoretical models (Bulaevskii *et al.*, 1972; Clark and Tippie, 1979) and a substantial amount of recent experimental work (Clark *et al.*, 1978; Sanny *et al.*, 1980; Bozler *et al.*, 1980; Miljak *et al.*, 1980; Tippie and Clark, 1981) have examined this problem. The subject has been reviewed briefly by Clark (1981). Quite interesting consequences for the limiting probability density for the random exchange couplings are found. In particular, the renormalized coupling constants are found to have a singular [type (c)] distribution whenever the initial distribution has any randomness.

(e) The random one-dimensional xy model. The random one-dimensional xy model maps onto Dyson's Case I problem (Smith, 1970; Theodorou and Cohen, 1976). Though one can transform the problem, using a pseudo-Fermion representation, the interactions which are near neighbor in the original problem turn out to be long range in the transformed representation. This destroys the possibility of mapping it onto our master equation, and we are unable to contribute to the solution of this problem using the techniques of this paper.

Very recent work of Grüner (1980a) raises the interesting possibility that the results of this paper may apply to electronic transport situations, as well as to ionic transport. It appears that random barriers in highly anisotropic electronic conductors have a major influence on charge propagation. A dc component of conductivity is present, but the ac conductivity obeys a frequency dependence much like that predicted by Eq. (7.17c) of this paper in, for example, the model compound $QN(TCNQ)_2$ (Grüner, 1980a). This may be caused by electrons trapped in regions between barriers of random heights obeying a distribution law comparable to Eq. (11.2), but with an upper cutoff, as well. Alexander *et al.* (1981) have formulated an approach to such an extension of the model, including a discussion of the electric field dependence of the ac conductivity, of the crossover to three-dimensional behavior, and of the conditions under which our treatment will be relevant to electronic conduction. Experiments are in progress (Grüner, 1980b) to test the relevance of this paper's conclusions to electronic transport in highly anisotropic materials.

ACKNOWLEDGMENTS

This work was supprted in part by the U. S. National Science Foundation, Grant No. DMR78-10312, and the U. S. Office of Naval Research, Contract No. N00014-75-C-0245. One of us (S.A.) would like to thank the Physics Department, University of California, Los Angeles, for its hospitality. Most of his contribution to this work was done there. The authors also wish to thank Professor T. Holstein for many helpful discussions which have clarified a number of points. Helpful discussions with Dr. H. Scher, Dr. J. Hirsch, Dr. A. Aharony, Dr. M. H. Cohen, Dr. P. M. Richards, Dr. H. U. Beyeler, and Dr. S. Strässler are gratefully acknowledged.

REFERENCES

Alexander, S., 1980, Jerusalem Conference on Chemical Physics (unpublished); and 1981, Phys. Rev. B **23**, 2951.
Alexander, S., and J. Bernasconi, 1979, J. Phys. C **12**, L1.
Alexander, S., J. Bernasconi, and R. Orbach, 1978a, Phys. Rev. B **17**, 4311.
Alexander, S., J. Bernasconi, and R. Orbach, 1978b, J. Phys. (Paris) Suppl. **39**, C6-706.
Alexander, S., J. Bernasconi, W. R. Schneider, and R. Orbach, 1981, in *Physics in One Dimension*, edited by J. Bernasconi and T. Schneider, Springer Series in Solid-State Sciences (Springer, Berlin), Vol. 23, p. 277.
Alexander, S., and T. Holstein, 1978, Phys. Rev. B **18**, 301.
Baker, G. A., Jr., and J. C. Bonner, 1975, Phys. Rev. B **12**, 3741.
Bernasconi, J., S. Alexander, and R. Orbach, 1978, Phys. Rev. Lett. **41**, 185.
Bernasconi, J., and H. U. Beyeler, 1980, Phys. Rev. B **21**, 3745.
Bernasconi, J., H. U. Beyeler, S. Strässler, and S. Alexander, 1979, Phys. Rev. Lett. **42**, 819.
Bernasconi, J., and W. R. Schneider, 1981, unpublished.
Bernasconi, J., W. R. Schneider, and W. Wyss, 1980, Z. Phys. B **37**, 175.
Beyeler, H. U., 1981, in *Physics in One Dimension*, edited by J. Bernasconi and T. Schneider, Springer Series in Solid-State Sciences (Springer, Berlin), Vol. 23, p. 323.
Bondeson, S. R., and Z. G. Soos, 1980, Phys. Rev. B **22**, 1793.
Böttger, H., V. V. Bryksin, and G. Yu. Yashin, 1979, J. Phys. C **12**, 2797.
Bozler, J. M., C. M. Gould, and W. G. Clark, 1980, Phys. Rev. Lett. **45**, 1303.
Bulaevskii, L. N., A. V. Zvarykina, Y. S. Karimov, R. B. Lyobovskii, and I. F. Shchegolev, 1972, Zh. Eksp. Teor. Fiz. **35**, 725 [Sov. Phys.—JETP **35**, 384].
Clark, W. G., 1981, in *Physics in One Dimension*, edited by J. Bernasconi and T. Schneider, Springer Series in Solid-State Sciences (Springer, Berlin).
Clark, W. G., J. Hammann, J. Sanny, and L. C. Tippie, 1979, in Lecture Notes in Physics No. 96, *Quasi One-Dimensional Conductors II*, edited by S. Barisic *et al.*, (Springer, New York), p. 255.
Clark, W. G., and L. C. Tippie, 1979, Phys. Rev. B **20**, 2914.
Dasgupta, C., and S. Ma, 1980, Phys. Rev. B **22**, 1305.
Doetsch, G., 1971, *Handbuch der Laplace-Transformation I und II* (Birkhauser, Basel).
Domb, C., 1963, Proc. R. Soc. London, Ser. A **276**, 418.
Domb, C., A. A. Maradudin, E. W. Montroll, and G. H. Weiss, 1959, Phys. Rev. **115**, 24.
Dyson, F. J., 1953, Phys. Rev. **92**, 1331.
Fox, C., 1961, Trans. Amer. Math. Soc. **98**, 395.
Goda, M., 1981, Suppl. Prog. Theor. Phys., in press.
Grüner, G., 1980a, Bull. Am. Phys. Soc. **25**, 255.
Grüner, G., 1980b, private communication.
Gupta, K. C., and U. C. Jain, 1966, Proc. Nat. Inst. Sci. India, Part A **36**, 594.
Halperin, B. I., 1965, Phys. Rev. **139**, A 104.
Halperin, B. I., 1966, Adv. Chem. Phys. **13**, 123.
Hirsch, J. E., and J. V. José, 1980, Phys. Rev. B **22**, 5339.
Hori, J., 1968, *Spectral Properties of Disordered Chains and Lattices* (Pergamon, Oxford).
Huber, D. L., 1980, private communication.

Huber, D. L., and W. Y. Ching, 1980, J. Phys. C **13**, 5579.
Huber, D. L., D. S. Hamilton, and B. Barnett, 1977, Phys. Rev. B **16**, 4642.
Kirkpatrick, S., 1973, Rev. Mod. Phys. **45**, 574.
Klafter, J., and R. Silbey, 1980a, Phys. Rev. Lett. **44**, 55.
Klafter, J., and R. Silbey, 1980b, J. Chem. Phys. **72**, 843.
Lieb, E. H., and D. C. Mattis, 1966, *Mathematical Physics in One Dimension* (Academic, New York).
Lifshitz, I. M., 1964. Adv. Phys. **13**, 483.
Lu, M. S., M. Nelkin, and M. Arita, 1974, Phys. Rev. B **10**, 2315.
Maradudin, A. A., and G. H. Weiss, 1968, J. Chem. Phys. **29**, 631.
Miljak, M., B. Korin, J. R. Cooper, K. Holczer, and A. Janossy, 1980, J. Phys. (Paris) **41**, 639.
O'Connor, A. J., and J. L. Lebowitz, 1974, J. Math. Phys. **15**, 692.
Odagaki, T., and M. Lax, 1980, Phys. Rev. Lett. **45**, 847.
Pfister, G., and H. Scher, 1978, Adv. Phys. **27**, 747.
Rice, M. J., and J. Bernasconi, 1973, J. Phys. F **3**, 55.
Rich, R. M., S. Alexander, J. Bernasconi, T. Holstein, S. K. Lyo, and R. Orbach, 1978, Phys. Rev. B **18**, 3048.
Richards, P. M., and R. L. Renken, 1980, Phys. Rev. B **21**, 3740.
Riseborough, P., 1979, unpublished.
Sanny, J., G. Grüner, and W. G. Clark, 1980, Solid State Commun. **35**, 657.
Scher, H., S. Alexander, and E. W. Montroll, 1980, Proc. Natl. Acad. Sci. USA **77**, 3758.
Scher, H., and M. Lax, 1973, Phys. Rev. B **7**, 4491, 4502.
Scher, H., and E. W. Montroll, 1975, Phys. Rev. B **12**, 2455.
Schmidt, H., 1957, Phys. Rev. **105**, 425.
Smith, E. R., 1970, J. Phys. C **3**, 1419.
Theodorou, G., and M. H. Cohen, 1976, Phys. Rev. B **13**, 4597.
Theodorou, G., and M. H. Cohen, 1979, Phys. Rev. B **19**, 1561.
Tippie, L. C., and W. G. Clark, 1981, Phys. Rev. B, companion papers (in press).